A Numerical Approach for Time-Optimal Control of Double Arms Robot

Reza Fotouhi-C.
Department of Mechanical Engineering
University of Saskatchewan
Saskatoon, SK S7N 0W0
306–966–5464 ref260@engr.usask.ca

Walerian Szyszkowski
Department of Mechanical Engineering
University of Saskatchewan
Saskatoon, SK S7N 0W0
306–966–5472 szyszkow@engr.usask.ca

Abstract

Time-optimal nonlinear control of rest to rest maneuvers of double arms robot is solved using Pontryagin's Minimum Principle. A numerical non-optimal solution satisfying the state equations with the initial and the final boundary conditions is found and used to obtain a good approximation of the initial values of the costates. Then the corresponding nonlinear two-point boundary-value problem is solved by applying a shooting method. A good approximation of the initial costate is usually needed to make the shooting method converge. This initial guess is generated here by the forward-backward method. A strategy that combines the shooting method and the forward-backward method to time-optimal control of the double arms robot is illustrated by the numerical examples.

1. Introduction

If only time is minimised, and the control is bounded, the optimal trajectory results in a bang-bang control. The corresponding non-singular, nonlinear two-point boundary value problem (TPBVP) can be derived using Pontryagin's Minimum Principle (PMP). This problem may be solved by the shooting method or other analytical or numerical methods to determine the switching times for the optimal sequence of the control. Unless the system is of low order (less than three), time invariant, and linear, there is little hope of determining the optimal control law analytically. Therefore, for nonlinear higher order systems, optimal control solutions are very difficult to obtain [1], [2]. The shooting method for optimal control problems was originally proposed in [3]. This method was used to solve a few simple linear examples for which a good guess for initial costates was possible to obtain intuitively. A shooting method to find the switching times for the bang-bang control by first generating the initial costates with the help of a quasilinearization technique was used in [4]. The shooting method used to solve the two-point boundary-value problem was found to be highly sensitive to the initial values of the costates. In [5] we proposed a Forward-Backward Method of integration (FBM) in which the switching times were directly approximated in each iteration from a solution of the state equations that meets the initial and final conditions. The FBM generates initial costates which then will be used in the Shooting Method (SM). The results show that the FBM is capable of generating good initial guess for Double Arms Robot (DAR) with two control forces. Cases discussed in [5] had the order of the state equation less than two. Those cases included single arm robot and any linear or nonlinear system with two state variables or less. The expected number of switches was investigated in [6]. The FBM for DAR was discussed in [7].

2. Problem Formulation

A shooting method for solving TPBVP with single control was originally proposed in [3]. We expanded the method to include multiple controls. First the formulation of time-optimal control problem for The Double Arms Robot (DAR) is presented. Then a brief description of the SM follows.

2.1. Time-Optimal Control of DAR

The equations of motion of the double arms robot are in the form:

$$\dot{x_i}(t) = a_i(x) + c_{ij}(x)u_j(t) \tag{1}$$

where x is vector of states, a and c are nonlinear functions of states, and u is vector of control forces. Here, using the notation shown in Figure 1 we have:

0-7803-2550-8/95$4.00©1995 IEEE

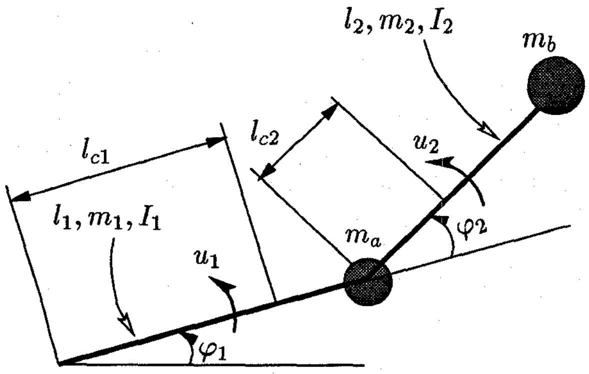

Figure 1: Physical Parameters of Double Arms Robot

$$
\begin{aligned}
a_1(x) &= x_2 \\
a_2(x) &= \tfrac{+a_{13}}{\Delta} \times \left[a_{22}(2x_2 + x_4)x_4 + a_{12}x_2^2\right] \\
&\quad - \tfrac{g}{\Delta} \times (a_{14}a_{22} - a_{24}a_{12}) \\
a_3(x) &= x_4 \\
a_4(x) &= \tfrac{-a_{13}}{\Delta} \times \left[a_{12}(2x_2 + x_4)x_4 + a_{11}x_2^2\right] \\
&\quad + \tfrac{g}{\Delta} \times (a_{14}a_{12} - a_{24}a_{11})
\end{aligned}
$$

$$
c(x) = \frac{1}{\Delta}\begin{bmatrix} 0 & 0 \\ +a_{22} & -a_{12} \\ 0 & 0 \\ -a_{12} & +a_{11} \end{bmatrix} \qquad u(t) = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}
$$

where:

$$
\begin{aligned}
a_{11} &= c_1 + c_2 + 2c_3\cos(x_3) \\
a_{12} &= c_2 + c_3\cos(x_3) \\
a_{13} &= c_3\sin(x_3) \\
a_{22} &= c_2 \\
a_{14} &= c_4\cos(x_1) + c_5\cos(x_1 + x_3) \\
a_{24} &= c_5\cos(x_1 + x_3) \\
c_1 &= m_1 l_{c1}^2 + I_1 + (m_2 + m_a + m_b)l_1^2 \\
c_2 &= m_2 l_{c2}^2 + I_2 + m_b l_2^2 \\
c_3 &= (m_2 l_{c2} + m_b l_2)l_1 \\
c_4 &= m_1 l_{c1} + (m_2 + m_a + m_b)l_1 \\
c_5 &= (m_2 l_{c2} + m_b l_2) \\
\Delta &= a_{11}a_{22} - a_{12}^2 = c_1c_2 - c_3^2\cos^2(x_3)
\end{aligned}
$$

Here, $x_1 = \varphi_1, x_2 = \dot{\varphi_1}$ are rotation and angular velocity of the first arm, and $x_3 = \varphi_2, x_4 = \dot{\varphi_2}$ are rotation and angular velocity of the second arm as it is shown in Figure 1, and g is the gravitational acceleration. The control forces are bounded as

$$M_i^- \leq u_i(t) \leq M_i^+ \tag{2}$$

The mathematical statement of time-optimal control of DAR with two control forces u_1, u_2 which minimises the maneuver time t_f of the arms from x_0 to x_f is as follows. First define the Hamiltonian H as:

$$H(x, u, p) = 1 + p^T[a(x) + c(x)u(t)] \tag{3}$$

where the states x, the costates p, and H must satisfy the necessary conditions for optimality. After applying the Pontryagin's Minimum Principle (PMP) we have:

$$
\begin{aligned}
\dot{x}^* &= \frac{\partial H}{\partial p} \\
\dot{p}^* &= -\frac{\partial H}{\partial x} \\
H(x^*, u^*, p^*) &\leq H(x^*, u, p^*)
\end{aligned} \tag{4}
$$

The first set of equations (4) is identical to equations (1). Using (3) the costate equations can be derived as:

$$\dot{p_i}(t) = p_j \times f_{ij}(x, u, g) \tag{5}$$

or:

$$
\begin{aligned}
\dot{p_1} &= p_2 \times f_{12}(x, g) + p_4 \times f_{14}(x, g) \\
\dot{p_2} &= -p_1 + p_2 \times f_{22}(x) + p_4 \times f_{24}(x) \\
\dot{p_3} &= p_2 \times f_{32}(x, u, g) + p_4 \times f_{34}(x, u, g) \\
\dot{p_4} &= -p_3 + p_2 \times f_{42}(x) + p_4 \times f_{44}(x)
\end{aligned}
$$

where f_{ij} are known algebraic functions of states x, gravity g or controls u. This is a two-point boundary-value problem since the solution x must satisfy the initial conditions :

$$x^*(t_0) = x_0 \tag{6}$$

and the final conditions :

$$x^*(t_f) = x_f \tag{7}$$

For Hamiltonian the extra condition is:

$$H(x^*, u^*, p^*) = 0 \qquad \text{for all } t \tag{8}$$

The control forces obtained from PMP are

$$u_i^*(t) = \begin{bmatrix} M_i^+ & for & G_i < 0 \\ M_i^- & for & G_i > 0 \end{bmatrix} \tag{9}$$

$$
\begin{aligned}
G_1 &= \tfrac{1}{\Delta} \times (+p_2a_{22} - p_4a_{12}) \\
G_2 &= \tfrac{1}{\Delta} \times (-p_2a_{12} + p_4a_{11})
\end{aligned} \tag{10}
$$

Where $G_i = p_j^T c_{ji}$ is the switch functions corresponding to the control u_i, and c_{ji} is the ith column of c. It is assumed that any trajectory [solution of equations (4), (6), (7), (8), (9)] remains bounded on a closed, bounded interval containing t_0, t_f. The (*) indicates optimal solution. From now on we omit the superscript (*) for simplicity.

2.2. the Shooting Method for DAR

Following is the brief description of the SM which can be applicable to the time-optimal control problem for double arms robot. We can rewrite this problem as defined by equations (3) to (10) for the DAR with 4 state variables x, and 4 costate variables p in the form,

$$\dot{X} = F[X(t), u(t)]$$
$$X(t_0) = \left\{ \begin{array}{c} x_{i0} \\ B_i \end{array} \right\} \quad i = 1, \cdots, m$$
$$L[x(t_f), B, t_f] = \left\{ \begin{array}{c} [x_i(t_f) - x_{if}] \\ H(t_f) \end{array} \right\} = 0$$
$$u_i(t) = \left[\begin{array}{lll} M_i^+ & if & G_i(X) < 0 \\ M_i^- & if & G_i(X) > 0 \end{array} \right] \tag{11}$$

Where, $X = [x, p]^T$ is vector of n variables which includes both states x and costates p. $F[X, u]$ is vector of nonlinear functions which define the variation of states and costates and is completely defined by (1), (5). $X(t_0)$ is vector of initial states x_{i0} and costates B_i, where B is an $m-$vector of unknown initial costates. $L[x(t_f), B, t_f]$ is an $l-$vector representing the error at the target point, where $l \geq m$. This vector includes the initial conditions of costates, and the final conditions of states, and extra condition for Hamiltonian (8) to be met at the target point. The calculated final state at each iteration is denoted by $x_i(t_f)$, and x_{if} is the final desired condition of the state. If $B^{(k)}$, $t_f^{(k)}$ are values of B, t_f in iteration k, we have to correct them in the next iteration to reduce $L[x(t_f), B, t_f]$ to zero. So, at each iteration we have:

$$\left[\begin{array}{c} B^{(k+1)} \\ t_f^{(k+1)} \end{array} \right] = \left[\begin{array}{c} B^{(k)} \\ t_f^{(k)} \end{array} \right] + \left[\begin{array}{c} \Delta B^{(k)} \\ \Delta t_f^{(k)} \end{array} \right] \tag{12}$$

The correction $\Delta B^{(k)}$, $\Delta t_f^{(k)}$ can be computed by minimising in each iteration a norm of L:

$$\| L \| = \left(\sum_{i=1}^{l=5} L_i^2 \right)^{1/2}$$

In [3] the Newton's method was used for this purpose. We followed the same method with some modification. The vector of corrections is defined as:

$$\left[\begin{array}{c} \Delta B^{(k)} \\ \Delta t_f^{(k)} \end{array} \right] = -\alpha_k \left[\begin{array}{c} \delta B^{(k)} \\ \delta t_f^{(k)} \end{array} \right] \tag{13}$$

where $\delta B^{(k)}$ and $\delta t_f^{(k)}$ are calculated from:

$$\left[\begin{array}{cc} \frac{\partial \mathcal{L}[B^{(k)}, t_f^{(k)}]}{\partial B} & \frac{\partial \mathcal{L}[B^{(k)}, t_f^{(k)}]}{\partial t_f} \end{array} \right] \left[\begin{array}{c} \delta B^{(k)} \\ \delta t_f^{(k)} \end{array} \right] = \mathcal{L}[B^{(k)}, t_f^{(k)}] \tag{14}$$

The scalar α_k is chosen between $0 \leq \alpha_k \leq 1$, and $\mathcal{L}[B^{(k)}, t_f^{(k)}] \equiv w_i \times L_i[x(t_f), B, t_f], \quad i = 1, \cdots, l$. Here, w_i are weight functions to accommodate the difference of the magnitudes of the states as well as of the Hamiltonian. Components of the matrix $\left[\begin{array}{cc} \frac{\partial \mathcal{L}}{\partial B} & \frac{\partial \mathcal{L}}{\partial t_f} \end{array} \right]_{5\times5}$ can be calculated as follows:

$$\frac{\partial \mathcal{L}}{\partial B} = \left(\frac{\partial L}{\partial X} \frac{\partial X}{\partial B} + \frac{\partial L}{\partial B} \right)_{t_f}, \quad \frac{\partial \mathcal{L}}{\partial t_f} = \left(\frac{\partial L}{\partial X} \frac{dX}{dt} + \frac{\partial L}{\partial t_f} \right)_{t_f}$$

The choice of w_i is quite important for convergence as it is shown in the next section.

3. Numerical Examples

Two examples will be examined here to show that the radius of convergence of the SM is very narrow. For starting the iteration we need to guess the initial values for $B^{(0)}$ and $t_f^{(0)}$. If these initial guesses are close enough to their optimal values, the SM works very well. However, usually it is very difficult to have a good guess for these initial costates and the final time. The Forward-Backward Method of integration (FBM) we proposed in [5] generate a sufficiently good guess for initial costates and final time. These values then can be used to run the SM. With the following examples we will show the usefulness of FBM. The details of FBM for DAR can be found in [7]. The first example is a rest to rest motion from straight to straight configurations ($\varphi_2(0) = 0, \varphi_2(t_f) = 0$) with four state variables and two controls, taken from [8], [9]. Its optimal solution has three switch times. The second example is a rest to rest motion from straight to broken configurations ($\varphi_2(0) = 0, \varphi_2(t_f) \neq 0$) with the same physical properties of the DAR. Its optimal solution has also three switch times. For starting the SM we have to integrate the state with initial conditions x_0, and the costate equations with initial conditions $B_i = p_i(0)$ from t_0 to t_f. The target error vector $L[x(t_f), B, t_f]$ should vanish at final time t_f. The components of this vector are:

$$\{L\} = \left\{ \begin{array}{c} w_i \times [x_i(t_f) - x_{if}] \\ w_5 \times H(t_f) \end{array} \right\} \quad i = 1, \cdots, m$$

and the target error norm RLC is:

$$RLC = \| L \| = \sqrt{L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_5^2}$$

When this norm is less than a small positive prespecified value ϵ, the iterations in the shooting method are terminated. For the following examples the convergence criteria is set to $\epsilon = 1.0E-11$. The following norm defines the closeness of the initial and

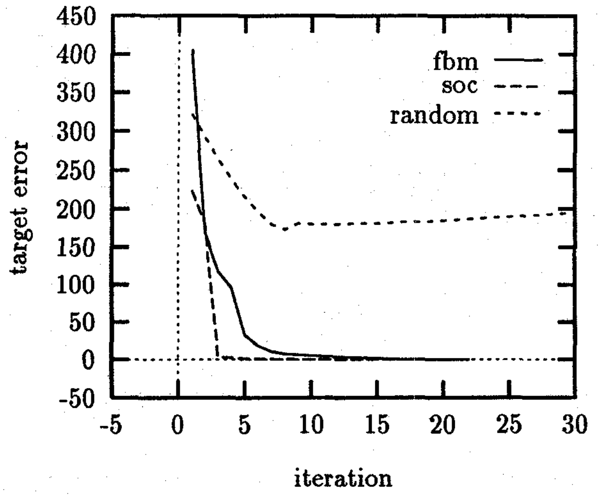

Figure 2: Target error RLC for various starting values (example 1).

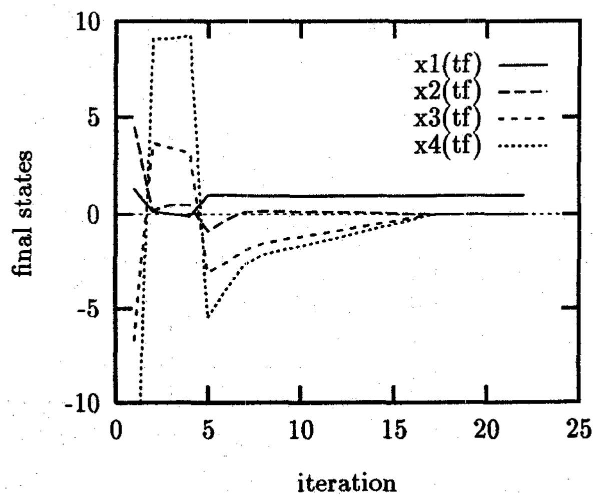

Figure 3: Convergence of the final states x_i with initial values generated by FBM (example 1).

the converged values of the calculated parameters:

$$d = \sqrt{\left(\sum_{i=1}^{m=4}(B_i^{iv} - B_i^{ov})^2\right) + (t_f^{iv} - t_f^{ov})^2}$$

where B_i^{iv}, B_i^{ov} are initial values and optimal values for the initial costates. t_f^{iv}, t_f^{ov} are initial values and optimal values for the final time.

3.1. Example 1: Straight to Straight Arms

This example is a rest to rest motion of double arms robot from straight to straight configurations. The physical parameters are as it was reported in [8] and [9]:

$$\begin{array}{lll} l_1 = 2l_{c1} = 0.40m & m_1 = 29.58kg & m_a = 0kg \\ l_2 = 2l_{c2} = 0.25m & m_2 = 15.00kg & m_b = 6kg \\ I_1 = 0.416739kg.m^2 & M_1^{\mp} = \mp 25Nm & g = 0 \\ I_2 = 0.205625kg.m^2 & M_2^{\mp} = \mp 9Nm & \end{array}$$

where, l_1 and l_2 are lengths of the arms (Figure 1), l_{c1} and l_{c2} centres of gravity of the arms, m_1 and m_2 mass of the arms, m_a and m_b mass at the end of the arms, I_1 and I_2 mass moment of inertia of the arms with respect to the centre of gravity, $M_1^{\mp}$ and $M_2^{\mp}$ are bounds of controls, and g is the gravitational acceleration.

The initial and the final conditions of the states in $[rad]$ and $[rad/s]$ are:

$$x(0) = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}^T$$

$$x(t_f) = \begin{bmatrix} 0.975 & 0.0 & 0.0 & 0.0 \end{bmatrix}^T$$

This example has been executed several times with different initial values for B_i, t_f. Table 1 summarises the results of these executions. The initial costates SOC (Semi Optimal Control) are calculated using some information for optimal solution reported in [8] and [9]. Namely, knowing the switch times t_{si}, the final time t_f, and forcing the following mixed boundary conditions for the costates, ($H(t_{s1}) = 0$, $H(t_{s2}) = 0$, $G_2(t_{s1}) = 0$, $G_1(t_{s2}) = 0$) we can get a set of the initial costates named here as SOC.

Par.	*SOC*	*FBM*	*Random*	*converged*
B_1	-1.086330	-0.990	2.0	-0.456692
B_2	-0.419652	-0.366	-2.0	-0.298622
B_3	-0.323267	-0.280	2.0	-0.093548
B_4	-0.111253	-0.949	-2.0	-0.074258
t_f	1.085000	0.795	2.0	1.083378
d	0.6821	1.0826	4.2260	0.0

Table 1: Initial and final values of parameters for the SM (example 1).

The optimal control solution has three switches, ($t_{s1} = 0.08751933, t_{s2} = 0.5416889, t_{s3} = 0.5872438sec.$), and the final time $t_f = 1.083378sec.$, with the following sequence for controls:

$$u(t) = \begin{bmatrix} u_1 = -25 & if & 0 \le t < t_{s2} \\ u_1 = +25 & if & t_{s2} < t \le t_f \\ u_2 = -9 & if & 0 \le t < t_{s1} \\ u_2 = +9 & if & t_{s1} < t < t_{s3} \\ u_2 = -9 & if & t_{s3} < t \le t_f \end{bmatrix} \quad (15)$$

As it can be seen from table 1, if the initial costates are not close enough to their optimal values, the SM fails to converge ($d \approx 4.22$ for this example). The initial costates obtained from the first iteration of FBM

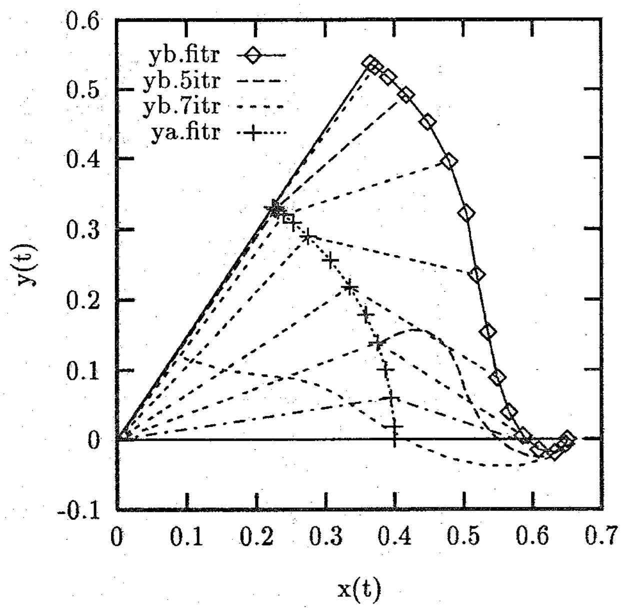

Figure 4: Trajectories of end of the elbow arm (y_b, x_b) and end of the shoulder arm (y_a, x_a) (example 1).

(d=1.0826) are sufficiently close to their optimal values and the SM converges.

Figure 2 shows the iterations of the SM with different initial values, the SOC, and the FBM that converged and the random guess which did not converge. Figure 3 shows the convergence of the final states in the SM with initial values generated by the FBM. Figure 4 shows the trajectories of end of the elbow arm (y_b, x_b) and end of the shoulder arm (y_a, x_a) at final iteration ($k = 22$). The path $yb.5itr$ is (y_b, x_b) at 5th iteration, and the path $yb.7itr$ is (y_b, x_b) at 7th iteration. Also the initial, intermediate, and final configurations of the DAR are indicated.

3.2. Example 2: Straight to Broken Arms

This example is a rest to rest motion of double arms robot from straight to broken configurations. The physical parameters are the same as in example 1. The initial and the final conditions of the states in $[rad]$ and $[rad/s]$ are:

$$x(0) = \left[\begin{array}{cccc} 0.0 & 0.0 & 0.0 & 0.0 \end{array}\right]^T$$

$$x(t_f) = \left[\begin{array}{cccc} 0.76 & 0.0 & 0.261799 & 0.0 \end{array}\right]^T$$

This example also has been executed several times with different initial values for B_i, t_f. Table 2 summarises the results of these executions. The optimal control solution has three switches, ($t_{s1} = 0.07291017, t_{s2} = 0.511852, t_{s3} = 0.561220 sec.$), and the final time ($t_f = 1.023704 sec.$), with the same sequence for controls as (15).

Par.	*SOC*	*FBM*	*Random*	*converged*
B_1	-1.086330	-0.997	2.0	-0.534711
B_2	-0.419652	-0.369	-2.0	-0.310833
B_3	-0.323267	-0.285	2.0	-0.117210
B_4	-0.111253	-0.959	-2.0	-0.077990
t_f	1.085000	1.085	2.0	1.023704
d	0.602867	1.01251	4.29044	0.0

Table 2: Initial and final values of parameters for the SM (example 2).

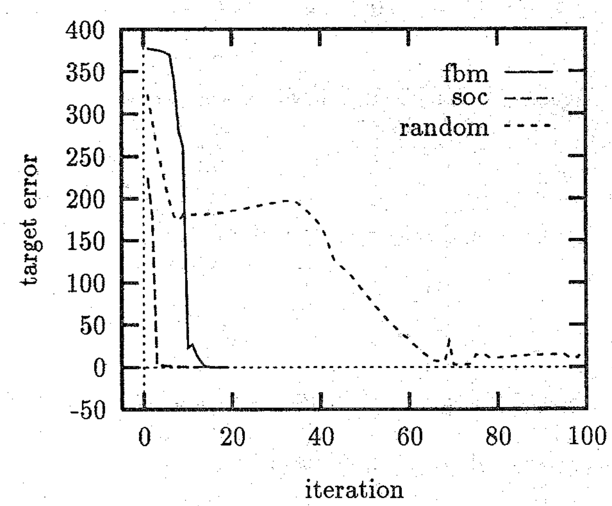

Figure 5: Target error RLC for various starting values (example 2).

As it can be seen from table 2, if the initial costates are not close enough to their optimal values, the SM fails to converge ($d \approx 4.29$ for this example). The initial costates obtained from the first iteration of FBM (d=1.01251) are again close enough to their optimal values making the SM converge.

Figure 5 shows the iterations of the SM with different initial values, the SOC, and the FBM that converged and the random guess which did not converge. Figure 6 shows the convergence of the final states in the SM with initial values generated by the FBM. Figure 7 shows the trajectories of end of the elbow arm (y_b, x_b) and end of the shoulder arm (y_a, x_a) at final iteration ($k = 20$). The path $yb.10itr$ is (y_b, x_b) at 10th iteration, and the path $yb.12itr$ is (y_b, x_b) at 12th iteration. Again, the initial, intermediate, and final configurations of the DAR are indicated.

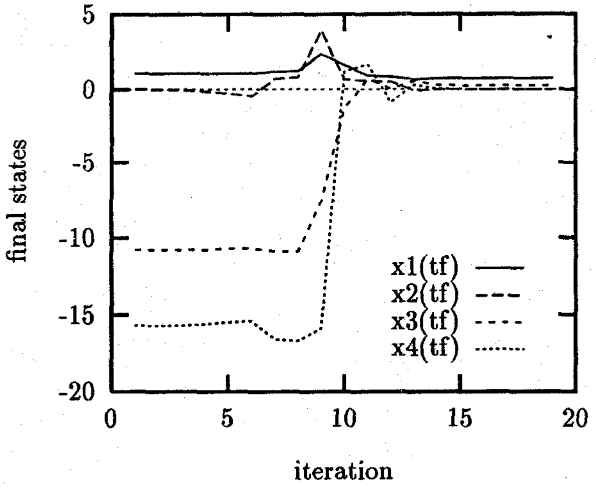

Figure 6: Convergence of the final states x_i with initial values generated by FBM (example 2).

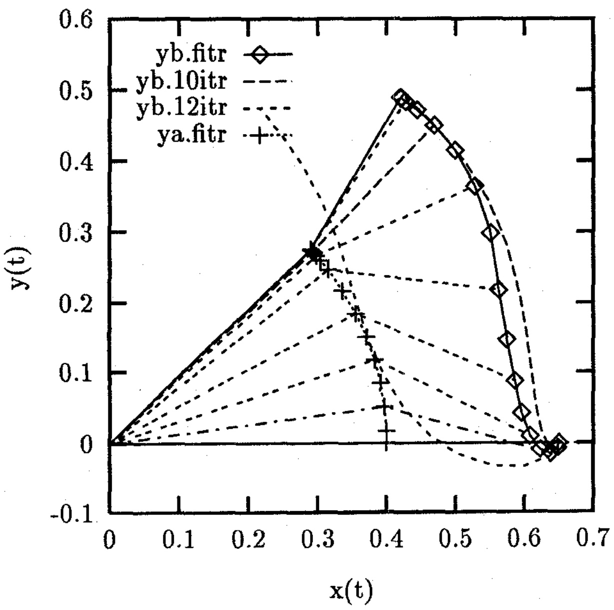

Figure 7: Trajectories of end of the elbow arm (y_b, x_b) and end of the shoulder arm (y_a, x_a) (example 2).

4. Conclusion

A method of solving time-optimal control problems for the double arms robot is presented. This method does not require any somewhat blind assumptions for the initial costates. The FBM generates a set of the initial costate to be used in the SM. The initial costates generated by the FBM are sufficiently close to their optimal values. Usefulness of the FBM for generation of the initial costate for nonlinear systems of the DAR with double controls is demonstrated through the numerical examples. Since the FBM does not use any linearization in the costate generation process, it can be used in any nonlinear optimal control problem.

References

[1] D. E. Kirk, *Optimal Control Theory An Introduction.* Englewood Cliffs New Jersey: Prentice-Hall Inc., 1970.

[2] E. R. Pinch, *Optimal Control and the Calculus of Variations.* New York: Oxford University Press Inc., 1993.

[3] G. J. Lastman, "A shooting method for solving two-point boundary-value problems arising from non-singular bang-bang optimal control problems," *International Journal Control,* vol. 27, no. 4, pp. 513–524, 1978.

[4] F. Li and P. M. Bainum, "An improved shooting method for solving minimum-time maneuver problems," *Advances in Dynamics and Control of Flexible Spacecraft and Space-based Manipulators,* vol. 20, pp. 13–18, 1990.

[5] R. Fotouhi-C. and W. Szyszkowski, "An algorithm for time-optimal control problems," *Journal of Dynamic Systems, Measurement, and Control,* 1995. submitted.

[6] W. Szyszkowski and R. Fotouhi-C, "On the number of switches in time-optimal control of manipulators," *International Journal of Optimal Control, Applications & Methods,* 1994. submitted.

[7] W. Szyszkowski and R. Fotouhi-C, "A numerical method for time-optimal control of double arms robot," *IEEE Transactions on Automatic Control,* 1995. submitted.

[8] L. G. Van Willigenburg and R. P. H. Loop, "Computation of time-optimal controls applied to rigid manipulators with friction," *International Journal of Control,* vol. 54, no. 5, pp. 1097–1117, 1991.

[9] H. P. Geering, L. Guzzella, S. A. R. Hepner, and C. H. Onder, "Time-optimal motions of robots in assembly tasks," *IEEE Transactions on Automatic Control,* vol. AC-31, no. 6, pp. 512–518, 1986.