Robot Kinematic Modeling and Control Based on Dual Quaternion Algebra — Part I: Fundamentals.

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Abstract. This is the first part of a three-part tutorial on dual quaternion algebra applied to robotics, whose goal is to help readers grasp the main ideas behind dual quaternions and their application to robot kinematic modeling and control. Part I presents the fundamentals of dual quaternion algebra, starting from the basic definitions of complex numbers and rotations in the plane and then extending the idea to rigid motions in the tridimensional case by means of quaternions and dual quaternions. In Part II, the dual quaternion algebra is applied to the kinematic modeling of different types of robots, such as mobile robots, robot manipulators, cooperative systems, mobile manipulators, and humanoids. Finally, Part III presents several control laws that are useful to control robots such as the ones modeled in Part II. Along with a relatively formal presentation, all three parts present several examples and exercises that may help readers in the comprehension of this increasingly important topic.

Contents

CHAPTER 1

Introduction

In robotics textbooks, one of the first chapters is typically dedicated to the presentation of the theory of rigid body motion. In general, the representations of translation and rotation are introduced separately, and then they are grouped together to form the homogeneous transformation.

Typically, rotation matrices are used to represent rotation, whereas translation is represented by the Cartesian position. Grouping them together leads to homogeneous transformation matrices (HTM). Even in cases where different parameterizations of rigid motions are shown for the case of completeness, the homogenous transformation matrices are used throughout the text. Examples of this kind of exposition are found in the textbooks of Paul (1981), Spong, Hutchinson, and Vidyasagar (2006), Dombre and Khalil (2007) and Siciliano et al. (2009).

An alternative is presented by Murray, Z. Li, and Sastry (1994). In their textbook, a complete presentation of robot modeling and control is made in the light of screw theory. However, this kind of presentation seems to be much more an exception than the rule.

Although HTM are quite common to represent kinematic motion, they impose some additional work to control the end-effector. More specifically, a very common technique is kinematic control in task space. Such control techniques take into consideration the relationship between the task space variables and the joint variables. This relationship is typically given by

$$\dot{\boldsymbol{x}} = \boldsymbol{J}\dot{\boldsymbol{q}}, \tag{1.0.1}$$

where $\boldsymbol{x}$ is the vector of the task space variables, $\boldsymbol{q}$ is the vector of the joint variables, and $\boldsymbol{J}$ is the Jacobian matrix relating the joints velocities and the end-effector velocities.

The parametrization of the task space variables is usually given by a vector and not directly by the HTM. Hence, it is necessary to choose the parametrization and also to extract these variables from the HTM that represents the end-effector pose.

Although one can consider these problems as secondary issues, the better theory is usually the one that explains more using fewer—and preferably simpler—arguments. Hence, a theory capable of eliminating intermediate steps between modeling and control would be better according to the aforementioned criterion. Moreover, from an engineering point of view, fewer intermediate steps between modeling and control can potentially reduce errors in implementation and development, leading to safer robots.

Murray, Z. Li, and Sastry (1994) present a more mathematical approach to robot modeling and control based on the screw theory and matrix exponentials, whereas J. McCarthy (1990) presents an approach based on dual quaternions. Despite the fact that the latter is not a robotics textbook, McCarthy and his collaborators consistently developed a theory for robot modeling using dual quaternion theory (Dooley and J. McCarthy, 1993; Perez and J. M. McCarthy, 2004). Also, as they are more focused on mechanism analysis, they have not provided a complete treatment on robot control.

This tutorial presents another point of view on kinematic modeling and control based on dual quaternions. The advantage of using unit dual quaternions is that they are more compact than HTM, as the former has eight elements whereas the latter has sixteen. In addition, dual quaternions have strong algebraic properties and can be used to represent rigid motions, twists, wrenches and several geometrical primitives—e.g., Plücker lines, planes, etc.—in a very straightforward way. Moreover, it is easy to extract geometric parameters from a given unit dual quaternion (e.g., translation, axis of rotation, angle of rotation) and dual quaternions multiplications are less expensive than HTM multiplications (Adorno, 2011, p. 42). Also, unit dual quaternions do not have representational singularities (although this feature is also present in HTM) and are easily mapped into a vector structure, which can be particularly convenient when controlling a robot, as there is no need to extract parameters from the dual quaternion to perform such task. Thanks to the aforementioned advantages, dual quaternion algebra will be used throughout the tutorial as the main mathematical tool for both robot modeling and control.

Although this material is far from being complete, it is comprehensive and mature enough for those wishing to obtain an initial background on dual quaternion algebra and the main techniques of robot kinematic modeling and control. Should readers have any suggestions and/or constructive criticisms, please email them to `adorno@ufmg.br`.

CHAPTER 2

Rigid motions: from complex numbers to dual quaternions

Before entering into the world of quaternions and dual quaternions, first we are going to take a look at complex numbers and how they can be very useful in the representation of rigid motions in the plane. For instance, let us consider the complex plane in figure 2.0.1. A point (x, y) is represented by the complex number $\boldsymbol{p}' = x + \hat{\imath}y$, where $\hat{\imath}$ is the standard imaginary unit; that is, $\hat{\imath}^2 = -1$. Since $x + \hat{\imath}y = x\,(1 + \hat{\imath}\cdot 0) + y\,(0 + \hat{\imath}\cdot 1)$, therefore 1 and $\hat{\imath}$ constitute an orthogonal basis in the complex plane.

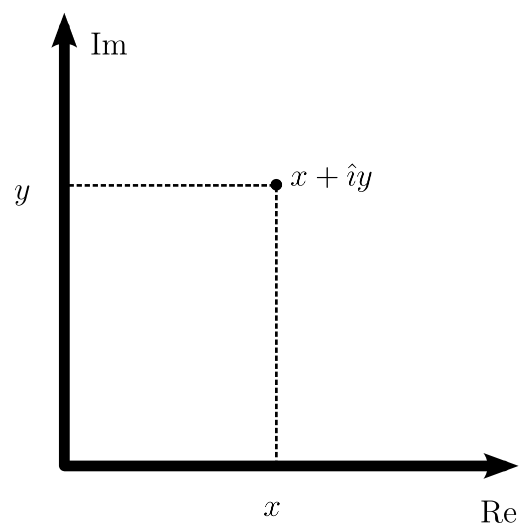


FIGURE 2.0.1. A point in the complex plane.

If the complex plane is rotated by θ, as shown in figure 2.0.2, then 1 and $\hat{\imath}$ are mapped into a new orthonormal basis:

$$
\begin{aligned}
1 &\to \cos\theta + \hat{\imath}\sin\theta \\
\hat{\imath} &\to -\sin\theta + \hat{\imath}\cos\theta.
\end{aligned}
$$

As the basis has changed, a point $\boldsymbol{p}' = x + \hat{\imath}y$—given initially in the rotated complex plane—when expressed in the fixed complex plane is given by

$$
\begin{aligned}
\boldsymbol{p} &= x\,(\cos\theta + \hat{\imath}\sin\theta) + y\,(-\sin\theta + \hat{\imath}\cos\theta) \\
&= (x\cos\theta - y\sin\theta) + \hat{\imath}\,(x\sin\theta + y\cos\theta)\,.
\end{aligned}
$$

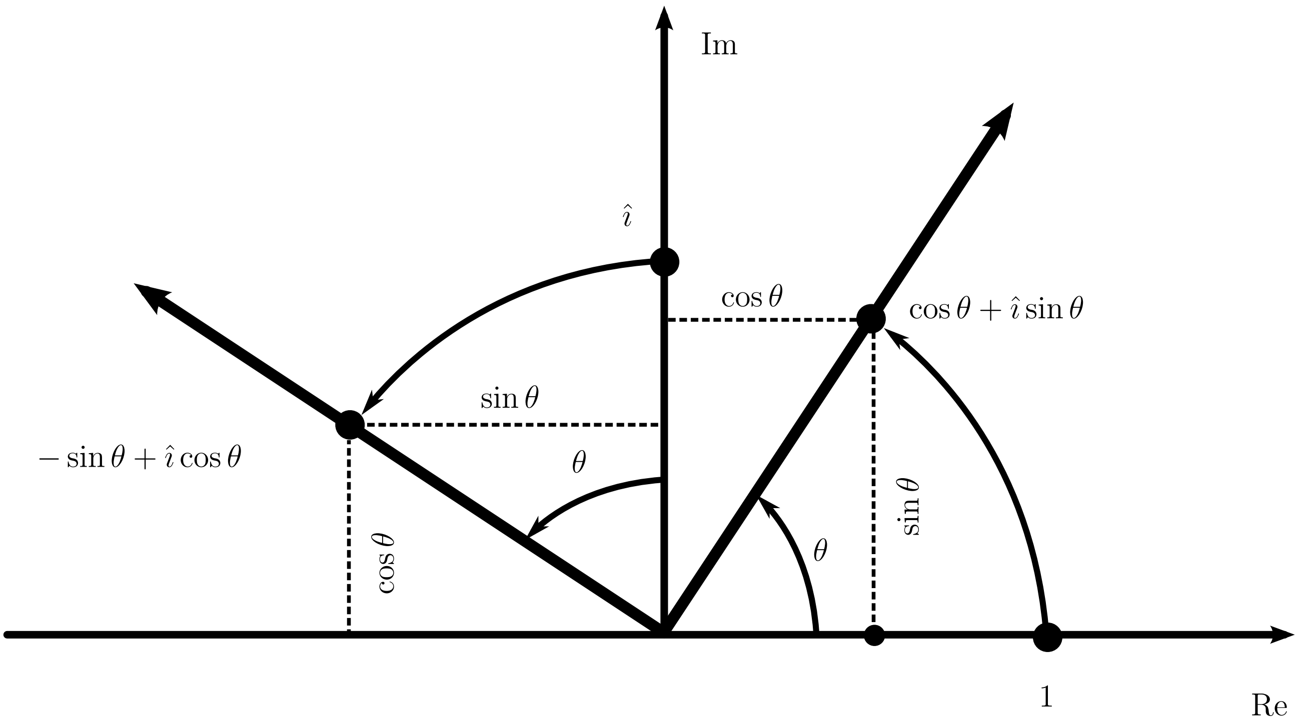


FIGURE 2.0.2. Rotated complex plane.

Remarkably, if $\boldsymbol{r} \triangleq \cos\theta + \hat{\imath}\sin\theta$, then we can verify by direct calculation that

$$\boldsymbol{p} = \boldsymbol{r}\boldsymbol{p}'. \tag{2.0.1}$$

EXAMPLE 2.0.1 Consider a point $\boldsymbol{p}' = 1 + \hat{\imath}$ rigidly attached to a complex plane $\mathcal{F}'$ (that is, $\boldsymbol{p}'$ never changes with respect to $\mathcal{F}'$). This complex plane is rotated by $\pi/4$ and consequently $\boldsymbol{p}'$ rotates accordingly with respect to the fixed frame $\mathcal{F}$. The new coordinates of the rotated point $\boldsymbol{p}$ with respect to the fixed frame is

$$\begin{aligned}\boldsymbol{p} = \boldsymbol{r}\boldsymbol{p}' &= \left(\cos\frac{\pi}{4} + \hat{\imath}\sin\frac{\pi}{4}\right)(1+\hat{\imath})\\ &= \left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) + \hat{\imath}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)\\ &= \hat{\imath}\sqrt{2}.\end{aligned}$$

It is interesting to note that two successive rotations given by angles θ_1 and θ_2 are represented by the multiplication $\boldsymbol{r}_1\boldsymbol{r}_2$. This is easily shown if we use the Euler formula $e^{\hat{\imath}\theta} = \cos\theta + \hat{\imath}\sin\theta$, since $\boldsymbol{r} = e^{\hat{\imath}\theta}$ and hence

$$\begin{aligned}\boldsymbol{r}_1\boldsymbol{r}_2 &= e^{\hat{\imath}\theta_1}e^{\hat{\imath}\theta_2}\\ &= e^{\hat{\imath}(\theta_1+\theta_2)}\\ &= \cos(\theta_1+\theta_2) + \hat{\imath}\sin(\theta_1+\theta_2).\end{aligned}$$

Example 2.0.2 Consider the rotated point of example 2.0.1. If another rotation of $\pi/4$ rad is performed, the final rotation is $\pi/2$. Thus, after the second rotation the rotated point is given by

$$\left(\cos\frac{\pi}{4} + \hat{\imath}\sin\frac{\pi}{4}\right)\hat{\imath}\sqrt{2} = -1 + \hat{\imath}.$$

If we rotate the original complex plane by $\pi/2$, the point $\boldsymbol{p}' = 1 + \hat{\imath}$ will be rotated accordingly and the final rotated point, with respect to the fixed frame, is

$$\left(\cos\frac{\pi}{2} + \hat{\imath}\sin\frac{\pi}{2}\right)(1+\hat{\imath}) = -1 + i.$$

It should be clear by now that two rotations of $\pi/4$ correspond to a single rotation of $\pi/2$, as expected.

A rotation of $-\theta$ is given by $e^{-\hat{\imath}\theta}$; that is,

$$\begin{aligned}e^{-\hat{\imath}\theta} &= \cos(-\theta) + \hat{\imath}\sin(-\theta)\\ &= \cos\theta - \hat{\imath}\sin\theta\\ &= \boldsymbol{r}^*,\end{aligned}$$

where $\boldsymbol{r}^*$ is the complex conjugate of $\boldsymbol{r}$. Since $\boldsymbol{r}^*\boldsymbol{r} = \boldsymbol{r}\boldsymbol{r}^* = 1$, we conclude that $\boldsymbol{r}^*$ is the inverse operation of $\boldsymbol{r}$. This is true only for complex numbers $\boldsymbol{r}$ such that $\|\boldsymbol{r}\| = 1$, which is the case of complex numbers representing rotations in the plane.

Let us recall that a rotation $\boldsymbol{r}$ relates the coordinates of the point $\boldsymbol{p}'$ in the rotated complex plane $\mathcal{F}'$ to the coordinates of the point given in the fixed frame $\mathcal{F}$ by means of the equation $\boldsymbol{p} = \boldsymbol{r}\boldsymbol{p}'$. If a point is given in the fixed frame $\mathcal{F}$ and we want to find the coordinates of this point in the rotated frame $\mathcal{F}'$, we can exploit the fact that $\boldsymbol{r}^*$ is the inverse of $\boldsymbol{r}$ to find $\boldsymbol{p}'$. More specifically,

$$\begin{aligned}\boldsymbol{r}\boldsymbol{p}' &= \boldsymbol{p}\\ \boldsymbol{r}^*\boldsymbol{r}\boldsymbol{p}' &= \boldsymbol{r}^*\boldsymbol{p}\\ \boldsymbol{p}' &= \boldsymbol{r}^*\boldsymbol{p}. \qquad (2.0.2)\end{aligned}$$

Example 2.0.3 Consider a rotated frame $\mathcal{F}'$, which suffered a rotation of $\pi/2$ with respect to the fixed frame $\mathcal{F}$, and a point $\boldsymbol{p} = 1 + \hat{\imath}$ given with respect to $\mathcal{F}$. This point, represented in the rotated frame $\mathcal{F}'$, is given by

$$\begin{aligned}\boldsymbol{p}' &= \boldsymbol{r}^*\boldsymbol{p}\\ &= \left(\cos\frac{\pi}{2} - \hat{\imath}\sin\frac{\pi}{2}\right)(1+\hat{\imath})\\ &= 1 - \hat{\imath}.\end{aligned}$$

From example 2.0.3, we realize that the complex number $\boldsymbol{r} = \cos\theta + \hat{\imath}\sin\theta$ also provides a mapping of the same point between two coordinate systems. If we denote the fixed frame as $\mathcal{F}_0$ and the rotated frame as $\mathcal{F}_1$, the complex number $\boldsymbol{r}$ represents the rotation of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$, and it is appropriate to explicitly indicate this rotation as $\boldsymbol{r}_1^0$. In this notation, the superscript always represents the reference frame, and hence $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ represent the coordinates of point $\boldsymbol{p}$ with respect to frames $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. This way,

$$\boldsymbol{p}^0 = \boldsymbol{r}_1^0\boldsymbol{p}^1 \tag{2.0.3}$$

provides the mapping of point $\boldsymbol{p}$ from frame $\mathcal{F}_1$ to frame $\mathcal{F}_0$. The inverse mapping is given by

$$\boldsymbol{p}^1 = \boldsymbol{r}_0^1\boldsymbol{p}^0, \tag{2.0.4}$$

where $\boldsymbol{r}_0^1 = \left(\boldsymbol{r}_1^0\right)^*$.

2.1. Basic facts and definitions of quaternions

Quaternions were introduced by Hamilton in the nineteenth century and can be regarded as an extension of complex numbers, where the three imaginary (also called quaternionic) units $\hat{\imath}, \hat{\jmath}, \hat{k}$ are defined and have the following properties (Hamilton, 1844):

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1. \tag{2.1.1}$$

This way, the set $\mathbb{H}$ of quaternions is defined formally as

$$\mathbb{H} \triangleq \left\{h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4 \;:\; h_1, h_2, h_3, h_4 \in \mathbb{R}\right\}. \tag{2.1.2}$$

From (2.1.2) we see that $\mathbb{C} \subset \mathbb{H}$, that is, a complex number is a particular case of a quaternion where the coefficients related to the imaginary units $\hat{\jmath}$ and $\hat{k}$ are both equal to zero.

Using an analogy with standard complex numbers, quaternions can be divided in real and imaginary parts. The real part of $\boldsymbol{h}$, denoted by $\operatorname{Re}(\boldsymbol{h})$, is the scalar h_1. The imaginary part, denoted by $\operatorname{Im}(\boldsymbol{h})$ is the one containing the imaginary components; that is, $\operatorname{Im}(\boldsymbol{h}) \triangleq \hat{\imath}h_2+\hat{\jmath}h_3+\hat{k}h_4$. Hence, $\boldsymbol{h} = \operatorname{Re}(\boldsymbol{h}) + \operatorname{Im}(\boldsymbol{h})$.

The addition/subtraction and multiplication of quaternions use the properties of the imaginary units and are given by the next two definitions.

Definition 2.1.1 (Quaternion addition/subtraction)
Let $\boldsymbol{h} = h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4$ and $\boldsymbol{h}' = h_1' + \hat{\imath}h_2' + \hat{\jmath}h_3' + \hat{k}h_4'$ be two quaternions. The quaternion sum (the subtraction is analogous) is

$$\boldsymbol{h} + \boldsymbol{h}' = h_1 + h_1' + \hat{\imath}\left(h_2 + h_2'\right) + \hat{\jmath}\left(h_3 + h_3'\right) + \hat{k}\left(h_4 + h_4'\right)$$

Definition 2.1.2 (Quaternion multiplication)
Let $\boldsymbol{h} = h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4$ and $\boldsymbol{h}' = h_1' + \hat{\imath}h_2' + \hat{\jmath}h_3' + \hat{k}h_4'$ be two quaternions. The quaternion multiplication is

$$\begin{aligned}
\boldsymbol{h}\boldsymbol{h}' &= (h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4)(h_1' + \hat{\imath}h_2' + \hat{\jmath}h_3' + \hat{k}h_4') \\
&= (h_1h_1' - h_2h_2' - h_3h_3' - h_4h_4') + \\
&\quad\ \hat{\imath}(h_1h_2' + h_2h_1' + h_3h_4' - h_4h_3') + \\
&\quad\ \hat{\jmath}(h_1h_3' - h_2h_4' + h_3h_1' + h_4h_2') + \\
&\quad\ \hat{k}(h_1h_4' + h_2h_3' - h_3h_2' + h_4h_1').
\end{aligned}$$

The set of quaternions $\mathbb{H}$ forms a group under quaternion multiplication (Murray, Z. Li, and Sastry, 1994). It is easy to show that quaternions are associative and distributive, but non-commutative. The next two definitions refer to the conjugate and norm of quaternions.

Definition 2.1.3 (Quaternion conjugate)
The conjugate $\boldsymbol{h}^*$ of a quaternion $\boldsymbol{h} = h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4$ is analogous to complex numbers; that is,

$$\begin{aligned}
\boldsymbol{h}^* &= \operatorname{Re}(\boldsymbol{h}) - \operatorname{Im}(\boldsymbol{h}) \\
&= h_1 - \left(\hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4\right).
\end{aligned}$$

Definition 2.1.4 (Quaternion norm)
The norm of a quaternion $\boldsymbol{h}$ is defined as

$$\|\boldsymbol{h}\| = \sqrt{\boldsymbol{h}\boldsymbol{h}^*} = \sqrt{\boldsymbol{h}^*\boldsymbol{h}}.$$

It is easy to show that the norm of a quaternion is a non-negative real number (see exercise 2.8.4).

FACT 2.1.1

For a non-zero quaternion $\boldsymbol{h}$, its inverse is given by

$$\boldsymbol{h}^{-1} = \frac{\boldsymbol{h}^*}{\|\boldsymbol{h}\|^2}.$$

Proof. By definition, if $\boldsymbol{h}^{-1}$ is the inverse of $\boldsymbol{h}$, then

$$\boldsymbol{h}^{-1}\boldsymbol{h} = 1 = \boldsymbol{h}\boldsymbol{h}^{-1}.$$

Thus, starting from the first equality we have

$$\begin{aligned} \boldsymbol{h}^{-1}\boldsymbol{h} &= 1 \\ \boldsymbol{h}^{-1}\boldsymbol{h}\boldsymbol{h}^* &= \boldsymbol{h}^* \\ \boldsymbol{h}^{-1}\|\boldsymbol{h}\|^2 &= \boldsymbol{h}^*. \end{aligned}$$

Since $\|\boldsymbol{h}\|^2$ is a scalar, then $\boldsymbol{h}^{-1} = \boldsymbol{h}^*/\|\boldsymbol{h}\|^2$. The reasoning for the second equality is the same and yields, as expected, the same result. ■

Sometimes it is useful to perform the multiplication between matrices and quaternions. Some authors prefer to use an implicit notation for this operation; that is, they implicitly consider the parametrization of the quaternion into a vector before doing the multiplication. However, this type of notation can lead to confusion, mainly if several complex operations are performed mixing quaternions and matrices. In the sequel, the vec_4 operator is introduced, followed by the definition of the multiplication between four-by-four matrices and quaternions.

Definition 2.1.5
Given a quaternion $\boldsymbol{h} = h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4$, the vec_4 operator performs the one-by-one mapping $\operatorname{vec}_4 : \mathbb{H} \to \mathbb{R}^4$; that is,

$$\operatorname{vec}_4 \boldsymbol{h} \triangleq \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}.$$

The inverse operation is given by $\underline{\mathbf{vec}}_4 : \mathbb{R}^4 \to \mathbb{H}$; that is, let $\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T$,

$$\begin{aligned} \boldsymbol{h} &= \underline{\mathbf{vec}}_4 \, \boldsymbol{u}, \\ \operatorname{Re}(\boldsymbol{h}) &= u_1, \\ \operatorname{Im}(\boldsymbol{h}) &= u_2\hat{\imath} + u_3\hat{\jmath} + u_4\hat{k}. \end{aligned}$$

The vec_4 operator just takes the coefficients of the quaternion and stacks them in a vector. The inverse operation just takes a four-dimensional vector and maps its elements to the coefficients of a quaternion.

Example 2.1.1 Let $\boldsymbol{h} = a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d$, then $\operatorname{vec}_4 \boldsymbol{h} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$.
Let $\boldsymbol{v} = \begin{bmatrix} 0 & a & 0 & -b \end{bmatrix}^T$, then $\underline{\mathbf{vec}}_4 \, \boldsymbol{v} = \hat{\imath}a - \hat{k}b$.

Definition 2.1.5 leads to an important result in terms of quaternion multiplication, shown next.

Definition 2.1.6

Using definition 2.1.2, it is easy to verify by direct calculation that, for $\boldsymbol{h}, \boldsymbol{h}' \in \mathbb{H}$,

$$\begin{aligned}\operatorname{vec}_4(\boldsymbol{h}\boldsymbol{h}') &= \overset{+}{\boldsymbol{H}}_4(\boldsymbol{h}) \operatorname{vec}_4 \boldsymbol{h}' \\ &= \overset{-}{\boldsymbol{H}}_4(\boldsymbol{h}') \operatorname{vec}_4 \boldsymbol{h},\end{aligned}$$

where

$$\overset{+}{\boldsymbol{H}}_4(\boldsymbol{h}) = \begin{bmatrix} h_1 & -h_2 & -h_3 & -h_4 \\ h_2 & h_1 & -h_4 & h_3 \\ h_3 & h_4 & h_1 & -h_2 \\ h_4 & -h_3 & h_2 & h_1 \end{bmatrix}, \qquad \overset{-}{\boldsymbol{H}}_4(\boldsymbol{h}') = \begin{bmatrix} h'_1 & -h'_2 & -h'_3 & -h'_4 \\ h'_2 & h'_1 & h'_4 & -h'_3 \\ h'_3 & -h'_4 & h'_1 & h'_2 \\ h'_4 & h'_3 & -h'_2 & h'_1 \end{bmatrix},$$

and $\overset{+}{\boldsymbol{H}}_4(\cdot)$ and $\overset{-}{\boldsymbol{H}}_4(\cdot)$ are called Hamilton operators.[a]

[a]The term Hamilton operator is not commonly used, at least in the robotics literature. But since it seemed appropriate, I borrowed the term from Akyar (2008).

This latter definition states that, even though the quaternion multiplication is not commutative, the Hamilton operators commute between them. As it will become evident in the next chapters, this property is quite useful. For a more complete account on the properties of Hamilton operators, see the works of Chou (1992) and Akyar (2008).[1]

2.1.1. Pure imaginary quaternions. Quaternions with only imaginary part are called pure imaginary quaternions or simply pure quaternions. They form the set

$$\mathbb{H}_p \triangleq \{\boldsymbol{p} \in \mathbb{H} : \operatorname{Re}(\boldsymbol{p}) = 0\} \tag{2.1.3}$$

and are particularly interesting in the sense that they have close connections to $\mathbb{R}^3$. For example, given two pure quaternions we can define the cross product and dot product using only quaternion operations.

Definition 2.1.7

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two pure quaternions. The cross product is defined as

$$\boldsymbol{u} \times \boldsymbol{v} \triangleq \frac{\boldsymbol{u}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{u}}{2}.$$

It is easy to show that the previous definition corresponds to the standard cross product between two vectors in $\mathbb{R}^3$ (see exercise 2.8.5).

[1]It is important to note, however, that Akyar's definition of Hamilton operators is somewhat different from the one presented here.

Definition 2.1.8

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two pure quaternions. The dot product is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} \triangleq -\frac{(\boldsymbol{u}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{u})}{2}.$$

As in the case of the cross product, it is easy to show that the previous definition corresponds to the standard dot product between two vectors in $\mathbb{R}^3$ (see exercise 2.8.6).

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are pure quaternions, using directly the definitions of dot product and cross product we can verify that

$$\boldsymbol{u}\boldsymbol{v} = -\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{v}. \tag{2.1.4}$$

This is particularly interesting because if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, then $\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{v}$.

Analogously to complex numbers, which represent rigid motions in the plane, quaternions may be used to represent rigid motions in three dimensions.

2.1.2. Translation. Since pure quaternions are directly related to vectors in $\mathbb{R}^3$, they represent translations in three dimensions. For instance, consider figure 2.1.1. The translation quaternion $\boldsymbol{p} = p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k}$ has each one of its coordinates along the orthogonal axes represented by the imaginary units. In addition, analogously to complex numbers, the imaginary units and the real part form an orthogonal basis but since we can visualize only three coordinates, the real axis is omitted.

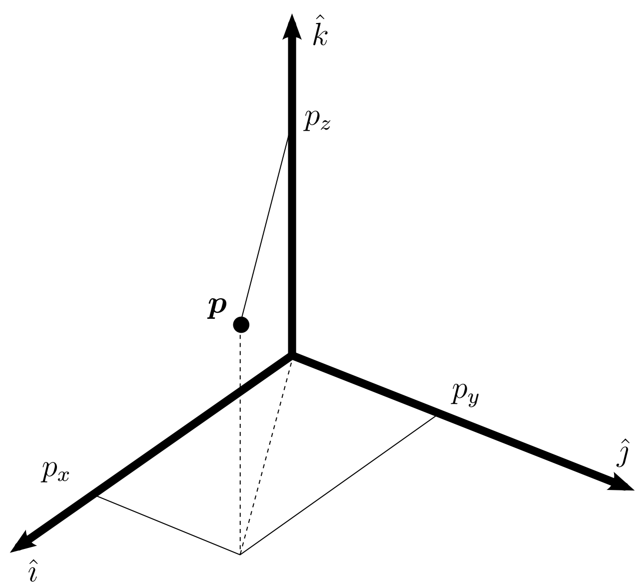
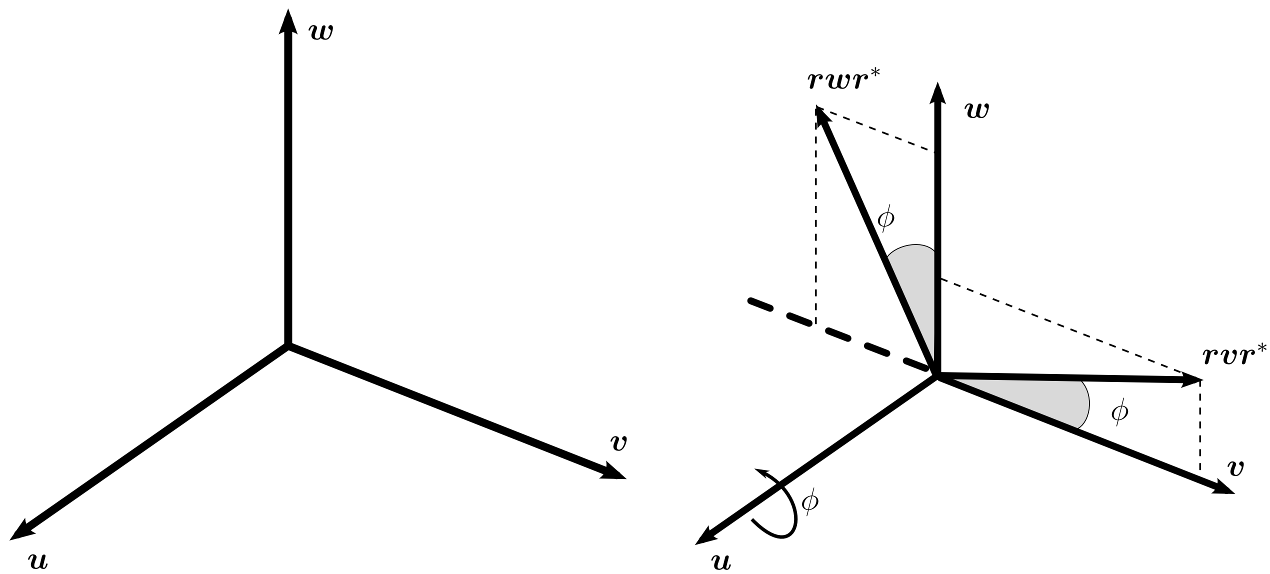


Figure 2.1.1. Translation quaternion $\boldsymbol{p} = p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k}$. Note that both the real and the imaginary axes form an orthogonal basis, but the real axis is omitted.

2.1.3. Rotation. Let us consider the unit quaternions $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}_p$ such that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$. From (2.1.4), $\boldsymbol{w} = \boldsymbol{u}\boldsymbol{v}$. This way, $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis, and hence $\boldsymbol{u}\boldsymbol{v} = \boldsymbol{w}$, $\boldsymbol{w}\boldsymbol{u} = \boldsymbol{v}$, and $\boldsymbol{v}\boldsymbol{w} = \boldsymbol{u}$ (exercise 2.8.8). The orthonormal basis $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is shown in figure 2.1.2a.



(A) Coordinate system represented by the orthonormal basis $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

(B) Projection of $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$, after a rotation, onto the fixed coordinate system.

FIGURE 2.1.2. Rotation ϕ around axis $\boldsymbol{u}$.

Now we are ready to show that a unit quaternion $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{u}\sin(\phi/2)$ (see exercise 2.8.10) represents a rotation of ϕ around the rotation axis $\boldsymbol{u}$. Let us first consider the transformation $\boldsymbol{r}\boldsymbol{u}\boldsymbol{r}^*$:

$$\begin{aligned}
\boldsymbol{r}\boldsymbol{u}\boldsymbol{r}^* &= \left(\cos\left(\frac{\phi}{2}\right) + \boldsymbol{u}\sin\left(\frac{\phi}{2}\right)\right)\boldsymbol{u}\left(\cos\left(\frac{\phi}{2}\right) - \boldsymbol{u}\sin\left(\frac{\phi}{2}\right)\right) \\
&= \left(\cos\left(\frac{\phi}{2}\right)\boldsymbol{u} - \sin\left(\frac{\phi}{2}\right)\right)\left(\cos\left(\frac{\phi}{2}\right) - \boldsymbol{u}\sin\left(\frac{\phi}{2}\right)\right) \\
&= \cos^2\left(\frac{\phi}{2}\right)\boldsymbol{u} + \cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + \boldsymbol{u}\sin^2\left(\frac{\phi}{2}\right) \\
&= \boldsymbol{u}. \qquad (2.1.5)
\end{aligned}$$

If $\boldsymbol{r}$ is really a rotation around axis $\boldsymbol{u}$, then the projection of $\boldsymbol{u}$ in the new rotated coordinate system should be unchanged, as expected (see figure 2.1.2b). Now, let us proceed with the transformations of $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$
\begin{aligned}
\boldsymbol{r}\boldsymbol{w}\boldsymbol{r}^* &= \left(\cos\left(\frac{\phi}{2}\right) + \boldsymbol{u}\sin\left(\frac{\phi}{2}\right)\right)\boldsymbol{w}\left(\cos\left(\frac{\phi}{2}\right) - \boldsymbol{u}\sin\left(\frac{\phi}{2}\right)\right)\\
&= \left(\cos\left(\frac{\phi}{2}\right)\boldsymbol{w} - \boldsymbol{v}\sin\left(\frac{\phi}{2}\right)\right)\left(\cos\left(\frac{\phi}{2}\right) - \boldsymbol{u}\sin\left(\frac{\phi}{2}\right)\right)\\
&= \left(\cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)\boldsymbol{w} - 2\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)\boldsymbol{v}\\
&= \boldsymbol{w}\cos\phi - \boldsymbol{v}\sin\phi. \qquad (2.1.6)
\end{aligned}
$$

The projection of the rotated $\boldsymbol{w}$ axis onto the fixed coordinate system is given by (2.1.6), as shown in figure 2.1.2b. The same reasoning is applied to $\boldsymbol{v}$ to obtain

$$\boldsymbol{r}\boldsymbol{v}\boldsymbol{r}^* = \boldsymbol{v}\cos\phi + \boldsymbol{w}\sin\phi, \qquad (2.1.7)$$

which represents the projection of the rotated $\boldsymbol{v}$ axis onto the fixed coordinate system, as shown in figure 2.1.2b. Thus, we conclude that $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{u}\sin(\phi/2)$ is a rotation of ϕ around rotation axis $\boldsymbol{u}$, which is equivalent to say that the plane normal to $\boldsymbol{u}$ is rotated by ϕ. In this sense, a general rotation ϕ around an arbitrary unit norm rotation axis $\boldsymbol{n} = n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k}$ (see figure 2.1.3) is given by

$$\boldsymbol{r} = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\boldsymbol{n}. \qquad (2.1.8)$$

Furthermore, as shown by (2.1.5)–(2.1.7), a point $\boldsymbol{p}^i$ in frame $\mathcal{F}_i$ may be projected onto another frame $\mathcal{F}_j$ by means of

$$\boldsymbol{p}^j = \boldsymbol{r}_i^j\boldsymbol{p}^i\left(\boldsymbol{r}_i^j\right)^*, \qquad (2.1.9)$$

where the same notation for complex numbers was used; that is, the quaternion $\boldsymbol{r}_j^i$ represents the rotation of $\mathcal{F}_j$ with respect to $\mathcal{F}_i$ and, conversely, $\boldsymbol{r}_i^j$ represents the rotation of $\mathcal{F}_i$ with respect to $\mathcal{F}_j$.

It is easy to see that, since $\boldsymbol{r}$ has unit norm, its inverse is given by

$$
\begin{aligned}
\boldsymbol{r}^* &= \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\boldsymbol{n}\\
&= \cos\left(\frac{-\phi}{2}\right) + \sin\left(\frac{-\phi}{2}\right)\boldsymbol{n},
\end{aligned}
$$

which means that $\boldsymbol{r}^*$ represents the rotation of $-\phi$ around axis $\boldsymbol{n}$. Hence, $\left(\boldsymbol{r}_j^i\right)^* = \boldsymbol{r}_i^j$. Also, since $\boldsymbol{r}^*$ is the inverse of $\boldsymbol{r}$, then

$$\boldsymbol{r}\boldsymbol{r}^* = \boldsymbol{r}^*\boldsymbol{r} = 1.$$

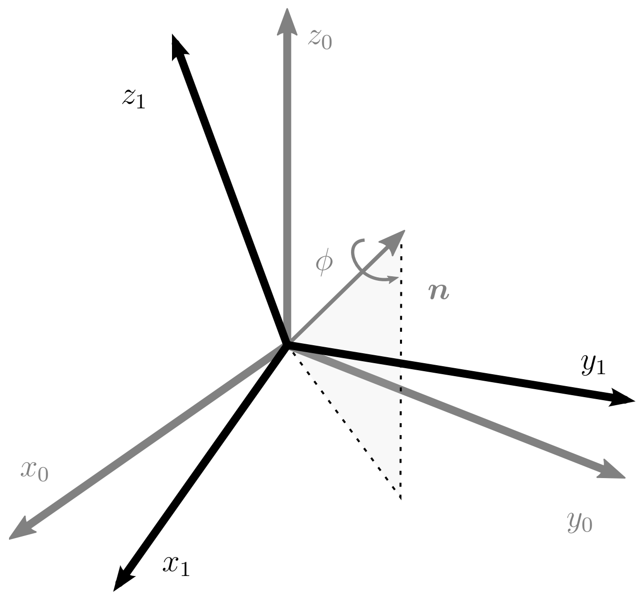


FIGURE 2.1.3. Rotation angle ϕ around the unit rotation axis $\boldsymbol{n}$, represented by $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$.

It is important to note that $\boldsymbol{r} = 1$ represents absence of rotation, because $\cos 0 - \boldsymbol{n}\sin 0 = 1$.

EXAMPLE 2.1.2 The point $\boldsymbol{p}^1$ in coordinate system $\mathcal{F}_1$ projected onto the coordinate system $\mathcal{F}_0$ is given by

$$\boldsymbol{p}^0 = \boldsymbol{r}_1^0 \boldsymbol{p}^1 \left(\boldsymbol{r}_1^0\right)^*. \tag{2.1.10}$$

Conversely, the projection of $\boldsymbol{p}^0$ onto coordinate system $\mathcal{F}_1$ yields

$$\boldsymbol{p}^1 = \boldsymbol{r}_0^1 \boldsymbol{p}^0 \left(\boldsymbol{r}_0^1\right)^*. \tag{2.1.11}$$

Alternatively, (2.1.11) can be found from (2.1.10) using the fact that the inverse of $\boldsymbol{r}_1^0$ is $(\boldsymbol{r}_1^0)^*$:

$$\begin{aligned}
\boldsymbol{r}_1^0 \boldsymbol{p}^1 \left(\boldsymbol{r}_1^0\right)^* &= \boldsymbol{p}^0 \\
\left(\boldsymbol{r}_1^0\right)^* \boldsymbol{r}_1^0 \boldsymbol{p}^1 \left(\boldsymbol{r}_1^0\right)^* \boldsymbol{r}_1^0 &= \left(\boldsymbol{r}_1^0\right)^* \boldsymbol{p}^0 \boldsymbol{r}_1^0 \\
\boldsymbol{p}^1 &= \left(\boldsymbol{r}_1^0\right)^* \boldsymbol{p}^0 \boldsymbol{r}_1^0 \\
&= \boldsymbol{r}_0^1 \boldsymbol{p}^0 \left(\boldsymbol{r}_0^1\right)^*.
\end{aligned}$$

Both projections are shown in figure 2.1.4.

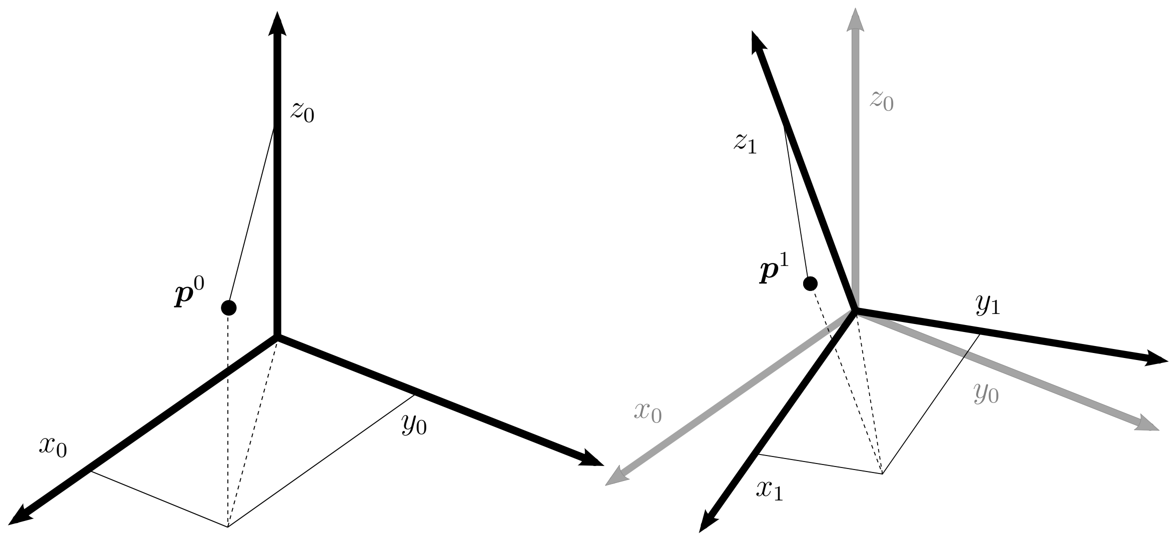


FIGURE 2.1.4. The point $\boldsymbol{p}^0$ in frame $\mathcal{F}_0$ (*left*) may be projected onto the rotated frame $\mathcal{F}_1$ (*right*) by means of $\boldsymbol{p}^1 = \boldsymbol{r}_0^1\boldsymbol{p}^0\left(\boldsymbol{r}_0^1\right)^*$.

An important fact is that the product of two unit quaternions is also a unit quaternion, and hence a rotation. This is easy to see if we consider three frames $\mathcal{F}_i$, $\mathcal{F}_j$, and $\mathcal{F}_k$. Thus,

$$\boldsymbol{p}^j = \boldsymbol{r}_i^j\boldsymbol{p}^i\left(\boldsymbol{r}_i^j\right)^*, \tag{2.1.12}$$

$$\boldsymbol{p}^k = \boldsymbol{r}_j^k\boldsymbol{p}^j\left(\boldsymbol{r}_j^k\right)^*. \tag{2.1.13}$$

Replacing (2.1.12) in (2.1.13) yields $\boldsymbol{p}^k = \boldsymbol{r}_j^k\boldsymbol{r}_i^j\boldsymbol{p}^i\left(\boldsymbol{r}_i^j\right)^*\left(\boldsymbol{r}_j^k\right)^*$. As $(\boldsymbol{ab})^* = \boldsymbol{b}^*\boldsymbol{a}^*$, then $\left(\boldsymbol{r}_i^j\right)^*\left(\boldsymbol{r}_j^k\right)^* = \left(\boldsymbol{r}_j^k\boldsymbol{r}_i^j\right)^*$, thus $\boldsymbol{p}^k = \boldsymbol{r}_j^k\boldsymbol{r}_i^j\boldsymbol{p}^i\left(\boldsymbol{r}_j^k\boldsymbol{r}_i^j\right)^*$. But we already know that the projection of $\boldsymbol{p}^i$ on $\mathcal{F}_k$ is $\boldsymbol{p}^k = \boldsymbol{r}_i^k\boldsymbol{p}^i\left(\boldsymbol{r}_i^k\right)^*$. Consequently, we conclude that $\boldsymbol{r}_i^k = \boldsymbol{r}_j^k\boldsymbol{r}_i^j$. Conversely,

$$\begin{aligned}
\boldsymbol{r}_k^i &= \left(\boldsymbol{r}_i^k\right)^* \\
&= \left(\boldsymbol{r}_j^k\boldsymbol{r}_i^j\right)^* \\
&= \left(\boldsymbol{r}_i^j\right)^*\left(\boldsymbol{r}_j^k\right)^* \\
&= \boldsymbol{r}_j^i\boldsymbol{r}_k^j.
\end{aligned}$$

EXAMPLE 2.1.3 Let a unit quaternion $\boldsymbol{r}_0^0 = 1$ represent the initial orientation of a reference frame $\mathcal{F}_0$. After n rotations, the final orientation is given by $\boldsymbol{r}_n^0 = \boldsymbol{r}_1^0 \ldots \boldsymbol{r}_n^{n-1}$. Note that the superscript and subscript represent the original and final frames, respectively.

Given transformations such as (2.1.9), there is an equivalent—but more compact—representation given by the adjoint transformation, as shown in the next definition.

DEFINITION 2.1.9
Given a unit quaternion $\boldsymbol{r}$ and a pure quaternion $\boldsymbol{p}$, the adjoint transformation is defined as

$$\mathrm{Ad}\left(\boldsymbol{r}\right)\boldsymbol{p} \triangleq \boldsymbol{r}\boldsymbol{p}\boldsymbol{r}^{*}.$$

It is easy to see, by direct calculation, that

$$\mathrm{Ad}\left(\boldsymbol{r}_1\boldsymbol{r}_2\right)\boldsymbol{p} = \mathrm{Ad}\left(\boldsymbol{r}_1\right)\mathrm{Ad}\left(\boldsymbol{r}_2\right)\boldsymbol{p}.$$

Unit norm quaternions form, under multiplication, the algebraic group Spin (3) (Selig, 2005).

2.1.4. Rigid Motions in three dimensions. The complete rigid motion between $\mathcal{F}_a$ and $\mathcal{F}_b$ can be represented by a translation $\boldsymbol{p}^a_{ab}$ and a rotation $\boldsymbol{r}^a_b$, as illustrated in figure 2.1.5.[2]

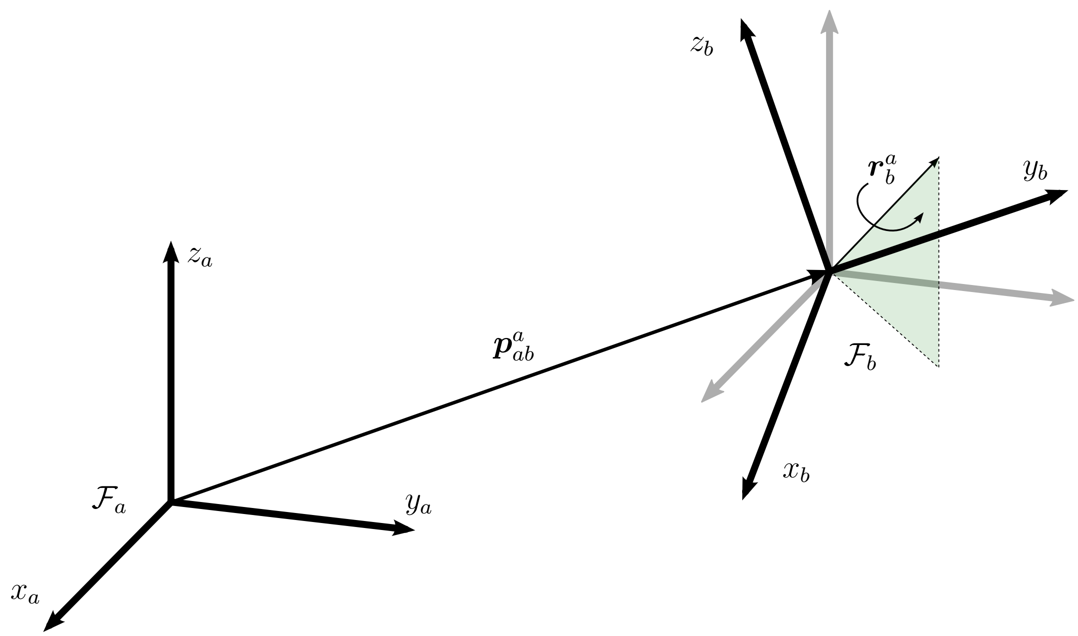


FIGURE 2.1.5. Rigid motion represented by quaternions: first a translation $\boldsymbol{p}^a_{ab}$ is performed, followed by a rotation $\boldsymbol{r}^a_b$.

The sequence of rigid motions using quaternions is given by the ring operations (addition and multiplication) of $\mathbb{H}$. For instance, let the tuple $(\boldsymbol{p}^a_{ab}, \boldsymbol{r}^a_b)$ represent the rigid motion from $\mathcal{F}_a$ to $\mathcal{F}_b$ and $\left(\boldsymbol{p}^b_{bc}, \boldsymbol{r}^b_c\right)$ represent the rigid motion from $\mathcal{F}_b$ to $\mathcal{F}_c$ (see fig. 2.1.6). The resultant rigid motion

[2]Note that, sometimes, it is convenient in a sequence of rigid motions to represent the original and final frames in the subscript of translation quaternions; for instance, $\boldsymbol{p}^a_{ab}$ is a translation from $\mathcal{F}_a$ to $\mathcal{F}_b$, represented in $\mathcal{F}_a$. On the other hand, $\boldsymbol{p}^b_{ab}$ is the same translation from $\mathcal{F}_a$ to $\mathcal{F}_b$, but represented in $\mathcal{F}_b$.

from $\mathcal{F}_a$ to $\mathcal{F}_c$ is given by

$$\begin{aligned}
\left(\boldsymbol{p}^a_{ac}, \boldsymbol{r}^a_c\right) &= \left(\boldsymbol{p}^a_{ab} + \boldsymbol{p}^a_{bc},\ \boldsymbol{r}^a_b \boldsymbol{r}^b_c\right) \\
&= \left(\boldsymbol{p}^a_{ab} + \boldsymbol{r}^a_b \boldsymbol{p}^b_{bc} \boldsymbol{r}^{a*}_b,\ \boldsymbol{r}^a_b \boldsymbol{r}^b_c\right) \\
&= \left(\boldsymbol{p}^a_{ab} + \operatorname{Ad}\left(\boldsymbol{r}^a_b\right) \boldsymbol{p}^b_{bc},\ \boldsymbol{r}^a_b \boldsymbol{r}^b_c\right) \qquad (2.1.14)
\end{aligned}$$

that is, the final rotation is given by the composition between the intermediate rotations, whereas the final position takes into consideration the frame-rotation movement; that is, $\boldsymbol{p}^b_{bc}$ is projected onto $\mathcal{F}_a$ by a frame transformation $\boldsymbol{p}^a_{bc} = \boldsymbol{r}^a_b \boldsymbol{p}^b_{bc} \boldsymbol{r}^b_a = \operatorname{Ad}\left(\boldsymbol{r}^a_b\right) \boldsymbol{p}^b_{bc}$ (recall that $\boldsymbol{r}^b_a = \boldsymbol{r}^{a*}_b$), and then $\boldsymbol{p}^a_{ab} + \boldsymbol{p}^a_{bc} = \boldsymbol{p}^a_{ac}$.

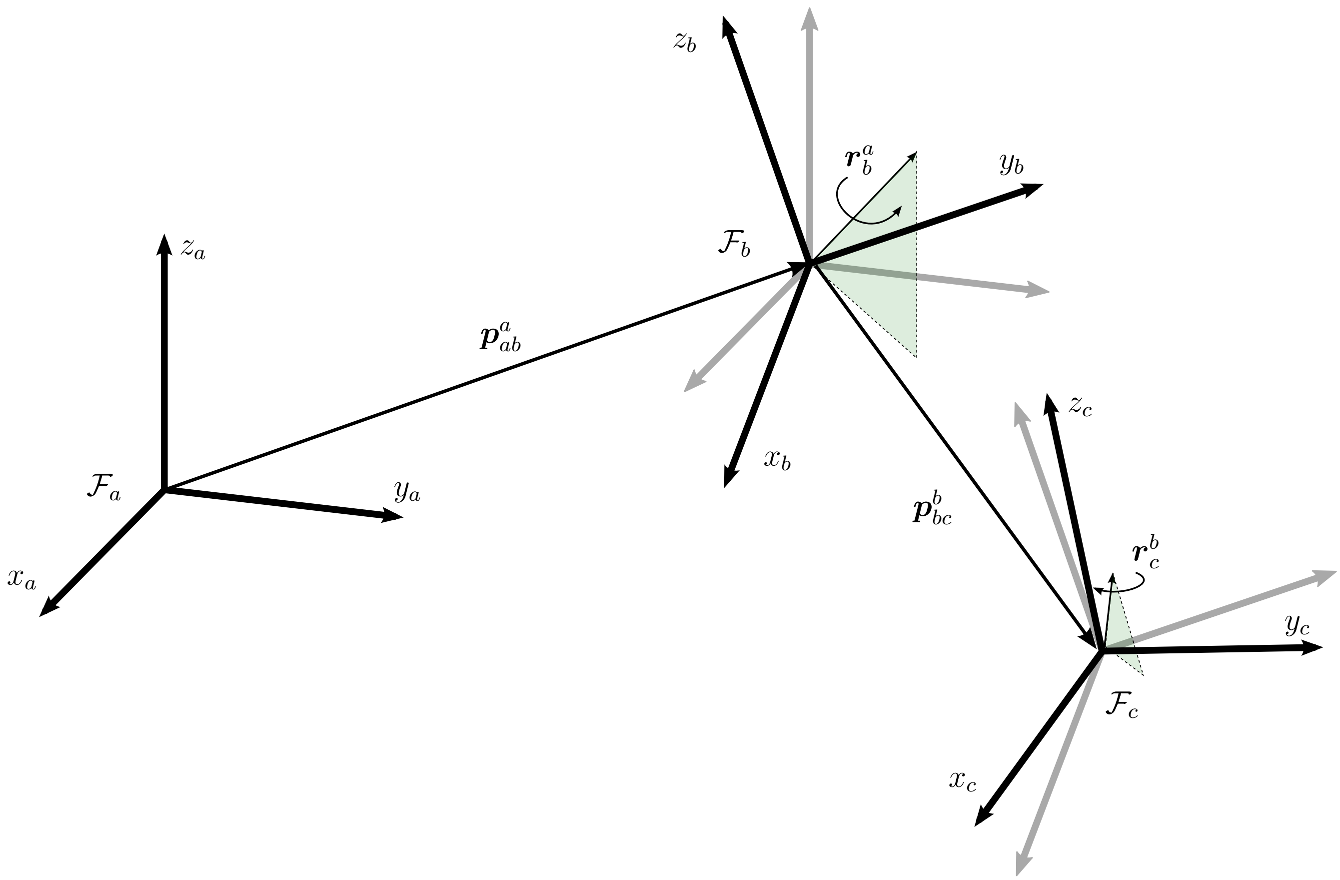


FIGURE 2.1.6. Sequence of rigid motions represented by quaternions.

2.1.5. Relationship between the unit quaternion derivative and angular velocity. One very useful result provided by quaternion algebra is the relationship between the unit quaternion derivative and the angular velocity of a frame, given by

$$\dot{\boldsymbol{r}}_b^a = \frac{1}{2}\boldsymbol{\omega}_{ab}^a \boldsymbol{r}_b^a, \tag{2.1.15}$$

where $\boldsymbol{r}_b^a$ is the rotation of frame $\mathcal{F}_b$ with respect to frame $\mathcal{F}_a$ and $\boldsymbol{\omega}_{ab}^a$ is the angular velocity of frame $\mathcal{F}_b$ with respect to frame $\mathcal{F}_a$, expressed in frame $\mathcal{F}_a$.

Proof. First let us consider a point $\boldsymbol{p}^b$ rigidly attached to $\mathcal{F}_b$, such that $\dot{\boldsymbol{p}}^b = 0$. Since $\boldsymbol{p}^a = \boldsymbol{r}_b^a \boldsymbol{p}^b \boldsymbol{r}_a^b$, then

$$\dot{\boldsymbol{p}}^a = \dot{\boldsymbol{r}}_b^a \boldsymbol{p}^b \boldsymbol{r}_a^b + \boldsymbol{r}_b^a \boldsymbol{p}^b \dot{\boldsymbol{r}}_a^b. \tag{2.1.16}$$

Isolating $\boldsymbol{p}^b$ and substituting in (2.1.16) yields

$$\dot{\boldsymbol{p}}^a = \dot{\boldsymbol{r}}_b^a \boldsymbol{r}_a^b \boldsymbol{p}^a + \boldsymbol{p}^a \boldsymbol{r}_b^a \dot{\boldsymbol{r}}_a^b.$$

Let $\boldsymbol{\alpha} = \dot{\boldsymbol{r}}_b^a \boldsymbol{r}_a^b$, thus $\operatorname{Re}(\boldsymbol{\alpha}) = 0$ (see fact 2.7.2) and $\boldsymbol{\alpha}^* = \boldsymbol{r}_b^a \dot{\boldsymbol{r}}_a^b$ (see exercise 2.8.16). Hence,

$$\begin{aligned}
\dot{\boldsymbol{p}}^a &= \boldsymbol{\alpha}\boldsymbol{p}^a + \boldsymbol{p}^a \boldsymbol{\alpha}^* \\
&= \boldsymbol{\alpha}\boldsymbol{p}^a - \boldsymbol{p}^a \boldsymbol{\alpha} \\
&= 2\left(\boldsymbol{\alpha} \times \boldsymbol{p}^a\right).
\end{aligned} \tag{2.1.17}$$

Since the linear velocity is the cross product between the angular velocity and the position—that is, $\dot{\boldsymbol{p}}^a = \boldsymbol{\omega}_{ab}^a \times \boldsymbol{p}^a$—it follows that

$$2\left(\boldsymbol{\alpha} \times \boldsymbol{p}^a\right) = \boldsymbol{\omega}_{ab}^a \times \boldsymbol{p}^a,$$

which means that

$$\begin{aligned}
\boldsymbol{\omega}_{ab}^a &= 2\boldsymbol{\alpha} \\
&= 2\dot{\boldsymbol{r}}_b^a \boldsymbol{r}_a^b.
\end{aligned}$$

Finally, isolating $\dot{\boldsymbol{r}}_b^a$ yields the quaternion propagation equation:

$$\dot{\boldsymbol{r}}_b^a = \frac{1}{2}\boldsymbol{\omega}_{ab}^a \boldsymbol{r}_b^a.$$

■

2.2. Dual quaternions

The previous section showed how quaternions are used to provide a complete description of rigid motions. However, as we can note from (2.1.14), complete rigid motions—i.e., rotations and translations—are represented by using ring operations on set $\mathbb{H}$; that is, we must use both quaternion additions and quaternion multiplications. Moreover, operations are done on pairs of quaternions, $(\boldsymbol{p}, \boldsymbol{r})$, which is not a problem *per se*, but it may be desireable to have a more compact representation. One such representation is the unit dual quaternion, which requires only group operations on set $\mathcal{H}$—i.e., only dual quaternion multiplications—and which are done on single elements. Such compact representation leads to a cleaner presentation and the algebra involved turns out to be simpler, allowing straightforward algebraic manipulations.

In order to define dual quaternions, in addition to the imaginary units $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we must add a new algebraic unit—namely the dual unit ε—to form a new algebraic structure. The algebra induced by the dual unit was introduced by Clifford (1871) and is called algebra of dual numbers. In this algebra, ε is nilpotent and has the following properties:

$$\varepsilon \neq 0, \quad \varepsilon^2 = 0.$$

For a dual number $\underline{\boldsymbol{a}} = \boldsymbol{a} + \varepsilon \boldsymbol{a}'$, the number $\boldsymbol{a}$ is the primary part whereas $\boldsymbol{a}'$ is the dual part. Also, the primary and dual parts can be extracted by using the operators $\mathcal{P}\left(\underline{\boldsymbol{a}}\right)$ and $\mathcal{D}\left(\underline{\boldsymbol{a}}\right)$, respectively. Hence,

$$\underline{\boldsymbol{a}} = \mathcal{P}\left(\underline{\boldsymbol{a}}\right) + \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{a}}\right).$$

Typically, the primary and dual parts are composed of the same type of elements, that is, scalars, complex numbers or quaternions. In the more general case, when the primary and dual parts consist of quaternions, dual numbers are usually called dual quaternions and form the set $\mathcal{H}$, which is formally defined as follows:

$$\mathcal{H} \triangleq \left\{\boldsymbol{h}_1 + \varepsilon \boldsymbol{h}_2 \,:\, \boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbb{H},\, \varepsilon \neq 0,\, \varepsilon^2 = 0\right\}. \tag{2.2.1}$$

The attentive reader will notice that $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathcal{H}$. More specifically, a real number is a complex number with imaginary part equal to zero (e.g., $a = a + 0\hat{\imath}$); a complex number is a quaternion in which the coefficients related to the imaginary numbers $\hat{\jmath}$ and $\hat{k}$ are equal to zero (e.g., $a + b\hat{\imath} = a + b\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$); finally, a quaternion is a dual quaternion with dual part equal to zero (e.g., $a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} + \varepsilon(0 + 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k})$).

The usual operations—sum/subtraction, multiplication—of dual quaternions take into account the ε operator and are defined bellow.

Definition 2.2.1

Let $\underline{\boldsymbol{a}}_1$ and $\underline{\boldsymbol{a}}_2$ be dual quaternions. The sum/subtraction between them is

$$\underline{\boldsymbol{a}}_1 \pm \underline{\boldsymbol{a}}_2 = \mathcal{P}\left(\underline{\boldsymbol{a}}_1\right) \pm \mathcal{P}\left(\underline{\boldsymbol{a}}_2\right) + \varepsilon\big(\mathcal{D}\left(\underline{\boldsymbol{a}}_1\right) \pm \mathcal{D}\left(\underline{\boldsymbol{a}}_2\right)\big).$$

Definition 2.2.2
Let $\underline{\boldsymbol{a}}_1$ and $\underline{\boldsymbol{a}}_2$ be dual quaternions. The multiplication between them is

$$\begin{aligned}\underline{\boldsymbol{a}}_1\underline{\boldsymbol{a}}_2 &= \left(\mathcal{P}\left(\underline{a}_1\right) + \varepsilon\,\mathcal{D}\left(\underline{a}_1\right)\right)\left(\mathcal{P}\left(\underline{a}_2\right) + \varepsilon\,\mathcal{D}\left(\underline{a}_2\right)\right)\\ &= \mathcal{P}\left(\underline{a}_1\right)\mathcal{P}\left(\underline{a}_2\right) + \varepsilon\left(\mathcal{P}\left(\underline{a}_1\right)\mathcal{D}\left(\underline{a}_2\right) + \mathcal{D}\left(\underline{a}_1\right)\mathcal{P}\left(\underline{a}_2\right)\right).\end{aligned}$$

Note that the nilpotent property of the ε operator is used in the multiplication operation.

The operators and properties derived for quaternions in section 2.1 are easily extended to dual quaternions. For instance, the real and imaginary parts of a dual quaternion are given by

$$\begin{aligned}\operatorname{Re}\left(\underline{\boldsymbol{h}}\right) &\triangleq \operatorname{Re}\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) + \varepsilon\operatorname{Re}\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right),\\ \operatorname{Im}\left(\underline{\boldsymbol{h}}\right) &\triangleq \operatorname{Im}\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) + \varepsilon\operatorname{Im}\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right).\end{aligned}$$

This way, given $\underline{\boldsymbol{h}} \in \mathcal{H}$, it is straightforward to show that $\underline{\boldsymbol{h}} = \operatorname{Re}\left(\underline{\boldsymbol{h}}\right) + \operatorname{Im}\left(\underline{\boldsymbol{h}}\right)$, since

$$\begin{aligned}\operatorname{Re}\left(\underline{\boldsymbol{h}}\right) + \operatorname{Im}\left(\underline{\boldsymbol{h}}\right) &= \operatorname{Re}\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) + \varepsilon\operatorname{Re}\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right) + \operatorname{Im}\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) + \varepsilon\operatorname{Im}\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right)\\ &= \underbrace{\operatorname{Re}\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) + \operatorname{Im}\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right)}_{\mathcal{P}(\underline{\boldsymbol{h}})} + \varepsilon\underbrace{\left(\operatorname{Re}\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right) + \operatorname{Im}\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right)\right)}_{\mathcal{D}(\underline{\boldsymbol{h}})}\\ &= \mathcal{P}\left(\underline{\boldsymbol{h}}\right) + \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{h}}\right) = \underline{\boldsymbol{h}}.\end{aligned}$$

Definition 2.2.3
The conjugate of the dual quaternion $\underline{\boldsymbol{h}} = \operatorname{Re}\left(\underline{\boldsymbol{h}}\right) + \operatorname{Im}\left(\underline{\boldsymbol{h}}\right)$ is

$$\underline{\boldsymbol{h}}^* \triangleq \operatorname{Re}\left(\underline{\boldsymbol{h}}\right) - \operatorname{Im}\left(\underline{\boldsymbol{h}}\right).$$

It is easy to show that $\underline{\boldsymbol{h}}^* = \mathcal{P}\left(\underline{\boldsymbol{h}}\right)^* + \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{h}}\right)^*$ (see exercise 2.8.18).

The set $\mathcal{H}_p$ of pure dual quaternions is defined formally as

$$\mathcal{H}_p \triangleq \left\{\underline{\boldsymbol{h}} \in \mathcal{H} \,:\, \operatorname{Re}\left(\underline{\boldsymbol{h}}\right) = 0\right\}. \tag{2.2.2}$$

The cross product and dot product are also defined for pure dual quaternions; that is, given $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}} \in \mathcal{H}_p$, the cross product is defined as

$$\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} \triangleq \frac{\underline{\boldsymbol{a}}\underline{\boldsymbol{b}} - \underline{\boldsymbol{b}}\underline{\boldsymbol{a}}}{2}, \tag{2.2.3}$$

and the dot product is defined as

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} \triangleq -\frac{\underline{\boldsymbol{a}}\underline{\boldsymbol{b}} + \underline{\boldsymbol{b}}\underline{\boldsymbol{a}}}{2}. \tag{2.2.4}$$

When the primary and dual parts of a dual number are composed of real scalars, usually we call it a real dual number or, when there is no ambiguity, simply dual number. The set $\mathbb{D}$ of real dual numbers is formally defined as

$$\mathbb{D} \triangleq \{\underline{\boldsymbol{h}} \in \mathcal{H} \,:\, \operatorname{Im}(\underline{\boldsymbol{h}}) = 0\}. \tag{2.2.5}$$

FACT (Taylor expansion). *The Taylor expansion of the function* $f(\underline{x}) = f\big(\mathcal{P}(\underline{x}) + \varepsilon\,\mathcal{D}(\underline{x})\big)$ *at the point* $\underline{x}_0 = \mathcal{P}(\underline{x})$*, where* $\underline{x} \in \mathbb{D}$*, is (Gu and Luh, 1987):*

$$\begin{aligned}
f(\underline{x}) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\underline{x}_0)}{n!}(\underline{x} - \underline{x}_0)^n \\
&= f(\underline{x}_0) + f'(\underline{x}_0)(\underline{x} - \underline{x}_0) + \frac{f''(\underline{x}_0)}{2}(\underline{x} - \underline{x}_0)^2 + \cdots \\
&= f\big(\mathcal{P}(\underline{x})\big) + f'\big(\mathcal{P}(\underline{x})\big)\varepsilon\,\mathcal{D}(\underline{x}) + \frac{f''\big(\mathcal{P}(\underline{x})\big)}{2}\big(\varepsilon\,\mathcal{D}(\underline{x})\big)^2 + \cdots \\
&= f\big(\mathcal{P}(\underline{x})\big) + \varepsilon f'\big(\mathcal{P}(\underline{x})\big)\,\mathcal{D}(\underline{x}),
\end{aligned} \tag{2.2.6}$$

where $f^{(n)}(\underline{x})$ *denotes the* n*-th derivative of the function* f *at* $\underline{x}$*.*

DEFINITION 2.2.4

Analogously to quaternions, the norm of the dual quaternion $\underline{\boldsymbol{h}}$ is given by

$$\|\underline{\boldsymbol{h}}\| = \sqrt{\underline{\boldsymbol{h}}\underline{\boldsymbol{h}}^*} = \sqrt{\underline{\boldsymbol{h}}^*\underline{\boldsymbol{h}}}.$$

FACT 2.2.1

The inverse of the dual number $\underline{a} \in \mathbb{D}$ is

$$\underline{a}^{-1} = \frac{1}{\mathcal{P}(\underline{a})} - \varepsilon\frac{\mathcal{D}(\underline{a})}{\mathcal{P}(\underline{a})^2},\ \mathcal{P}(\underline{a}) \neq 0.$$

Proof. Let $f(\underline{x}) = \underline{x}^{-1}$. Since $f(\mathcal{P}(\underline{x})) = \frac{1}{\mathcal{P}(\underline{x})}$ and $f'(\mathcal{P}(\underline{x})) = -\frac{1}{\mathcal{P}(\underline{x})^2}$, using (2.2.6) with $\underline{x} = \underline{a}$ leads directly to the expression of $\underline{a}^{-1}$. ■

FACT 2.2.2

For a non-zero dual quaternion $\underline{\boldsymbol{h}}$, its inverse is given by

$$\underline{\boldsymbol{h}}^{-1} = \frac{\underline{\boldsymbol{h}}^*}{\|\underline{\boldsymbol{h}}\|^2}.$$

Proof. By definition, if $\underline{\boldsymbol{h}}^{-1}$ is the inverse of $\underline{\boldsymbol{h}}$, then

$$\underline{\boldsymbol{h}}^{-1}\underline{\boldsymbol{h}} = 1 = \underline{\boldsymbol{h}}\underline{\boldsymbol{h}}^{-1}.$$

Thus, starting from the first equality we have

$$\begin{aligned}
\underline{\boldsymbol{h}}^{-1}\underline{\boldsymbol{h}} &= 1 \\
\underline{\boldsymbol{h}}^{-1}\underline{\boldsymbol{h}}\underline{\boldsymbol{h}}^{*} &= \underline{\boldsymbol{h}}^{*} \\
\underline{\boldsymbol{h}}^{-1}\left\|\underline{\boldsymbol{h}}\right\|^{2} &= \underline{\boldsymbol{h}}^{*}.
\end{aligned}$$

Since $\left\|\underline{\boldsymbol{h}}\right\|^{2}$ is a dual scalar, then $\underline{\boldsymbol{h}}^{-1} = \underline{\boldsymbol{h}}^{*}/\left\|\underline{\boldsymbol{h}}\right\|^{2}$. The reasoning for the second equality is the same and yields, as expected, the same result. ■

The vec operator and its inverse can be extended to dual quaternions, analogously to quaternions, as shown below.

Definition 2.2.5

Given a dual quaternion $\underline{\boldsymbol{h}} \in \mathcal{H}$, the vec_8 operator performs the one-by-one mapping $\mathrm{vec}_8 : \mathcal{H} \to \mathbb{R}^8$; that is,

$$\mathrm{vec}_8\,\underline{\boldsymbol{h}} = \begin{bmatrix} \mathrm{vec}_4\left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) \\ \mathrm{vec}_4\left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right) \end{bmatrix}.$$

The inverse operation is given by $\underline{\mathbf{vec}}_8 : \mathbb{R}^8 \to \mathcal{H}$; that is, let $\boldsymbol{u} = \begin{bmatrix} u_1 & \cdots & u_8 \end{bmatrix}^T$,

$$\begin{aligned}
\underline{\boldsymbol{h}} &= \underline{\mathbf{vec}}_8\,\boldsymbol{u}, \\
\mathrm{Re}\left(\underline{\boldsymbol{h}}\right) &= u_1 + \varepsilon u_5, \\
\mathrm{Im}\left(\underline{\boldsymbol{h}}\right) &= \hat{\imath}u_2 + \hat{\jmath}u_3 + \hat{k}u_4 + \varepsilon\left(\hat{\imath}u_6 + \hat{\jmath}u_7 + \hat{k}u_8\right).
\end{aligned}$$

Example 2.2.1 Let $\boldsymbol{u} = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}^T$, then $\underline{\mathbf{vec}}_8\,\boldsymbol{u} = \hat{\imath}2 + \varepsilon\left(1 + \hat{\imath}2 + \hat{\jmath}3 + \hat{k}4\right)$.

Definition 2.2.6
For $\underline{\boldsymbol{h}}, \underline{\boldsymbol{h}'} \in \mathcal{H}$, using definitions 2.2.5 and 2.1.6 it is easy to verify, by direct calculation, that

$$\operatorname{vec}_8 \left(\underline{\boldsymbol{h}}\underline{\boldsymbol{h}'}\right) = \overset{+}{\boldsymbol{H}}_8 \left(\underline{\boldsymbol{h}}\right) \operatorname{vec}_8 \underline{\boldsymbol{h}}' = \overset{-}{\boldsymbol{H}}_8 \left(\underline{\boldsymbol{h}}'\right) \operatorname{vec}_8 \underline{\boldsymbol{h}},$$

where

$$\overset{+}{\boldsymbol{H}}_8 \left(\underline{\boldsymbol{h}}\right) = \begin{bmatrix} \overset{+}{\boldsymbol{H}}_4 \left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) & \boldsymbol{0}_4 \\ \overset{+}{\boldsymbol{H}}_4 \left(\mathcal{D}\left(\underline{\boldsymbol{h}}\right)\right) & \overset{+}{\boldsymbol{H}}_4 \left(\mathcal{P}\left(\underline{\boldsymbol{h}}\right)\right) \end{bmatrix}, \qquad \overset{-}{\boldsymbol{H}}_8 \left(\underline{\boldsymbol{h}}'\right) = \begin{bmatrix} \overset{-}{\boldsymbol{H}}_4 \left(\mathcal{P}\left(\underline{\boldsymbol{h}}'\right)\right) & \boldsymbol{0}_4 \\ \overset{-}{\boldsymbol{H}}_4 \left(\mathcal{D}\left(\underline{\boldsymbol{h}}'\right)\right) & \overset{-}{\boldsymbol{H}}_4 \left(\mathcal{P}\left(\underline{\boldsymbol{h}}'\right)\right) \end{bmatrix},$$

and $\overset{+}{\boldsymbol{H}}_8$ and $\overset{-}{\boldsymbol{H}}_8$ are the Hamilton operators extended to dual quaternions.

2.3. Unit dual quaternions

Unit dual quaternions (i.e., dual quaternions possessing unit norm) play a very important role in the representation of rigid motions, and they belong to the set

$$\underline{\boldsymbol{\mathcal{S}}} \triangleq \left\{\underline{\boldsymbol{h}} \in \mathcal{H} \,:\, \left\|\underline{\boldsymbol{h}}\right\| = 1\right\}.$$

Elements of $\underline{\boldsymbol{\mathcal{S}}}$ equipped with the multiplication operation represent elements of $\text{Spin}(3) \ltimes \mathbb{R}^3$, the group of rigid motions that double covers $SE(3)$ (Selig, 2005). Considering $\boldsymbol{r} \in \text{Spin}(3)$ and $\boldsymbol{p} \in \mathbb{H}_p$, the unit dual quaternion corresponding to the translation $\boldsymbol{p}$ followed by the rotation $\boldsymbol{r}$ is given by

$$\underline{\boldsymbol{x}} = \boldsymbol{r} + \varepsilon \frac{1}{2} \boldsymbol{p}\boldsymbol{r}. \tag{2.3.1}$$

First, it is straightforward to see that $\underline{\boldsymbol{x}} \in \underline{\boldsymbol{\mathcal{S}}}$ because

$$\begin{aligned} \underline{\boldsymbol{x}}\underline{\boldsymbol{x}}^* &= \left[\boldsymbol{r} + \varepsilon \frac{1}{2}\boldsymbol{p}\boldsymbol{r}\right]\left[\boldsymbol{r}^* + \varepsilon\frac{1}{2}\boldsymbol{r}^*\boldsymbol{p}^*\right] \\ &= \boldsymbol{r}\boldsymbol{r}^* + \varepsilon\left(\frac{1}{2}\boldsymbol{r}\boldsymbol{r}^*\boldsymbol{p}^* + \frac{1}{2}\boldsymbol{p}\boldsymbol{r}\boldsymbol{r}^*\right) \\ &= 1, \end{aligned}$$

where we used the fact $\boldsymbol{p}^* = -\boldsymbol{p}$ and $\boldsymbol{r}\boldsymbol{r}^* = 1$. In addition, $\mathcal{P}\left(\underline{\boldsymbol{x}}\right) = \boldsymbol{r}$ and $\mathcal{D}\left(\underline{\boldsymbol{x}}\right) = (1/2)\boldsymbol{p}\boldsymbol{r}$, so the primary part of $\underline{\boldsymbol{x}}$ represents the rotation and the dual part of $\underline{\boldsymbol{x}}$ contains the information of translation. This way, for any $\underline{\boldsymbol{x}} \in \underline{\boldsymbol{\mathcal{S}}}$ the rotation is given by $\boldsymbol{r} = \mathcal{P}\left(\underline{\boldsymbol{x}}\right)$ and the translation is given by

$$\boldsymbol{p} = 2\,\mathcal{D}\left(\underline{\boldsymbol{x}}\right)\mathcal{P}\left(\underline{\boldsymbol{x}}\right)^*. \tag{2.3.2}$$

Remarkably, the composition of rigid transformations is given by a sequence of dual quaternion multiplications. More specifically, consider $\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2 \in \underline{\boldsymbol{\mathcal{S}}}$ such that $\underline{\boldsymbol{x}}_1 = \boldsymbol{r}_1 + \varepsilon\left(1/2\right)\boldsymbol{p}_1\boldsymbol{r}_1$ and $\underline{\boldsymbol{x}}_2 = \boldsymbol{r}_2 + \varepsilon\left(1/2\right)\boldsymbol{p}_2\boldsymbol{r}_2$, thus

$$\underline{\boldsymbol{x}}_3 \triangleq \left[\boldsymbol{r}_1 + \varepsilon\frac{1}{2}\boldsymbol{p}_1\boldsymbol{r}_1\right]\left[\boldsymbol{r}_2 + \varepsilon\frac{1}{2}\boldsymbol{p}_2\boldsymbol{r}_2\right] = \boldsymbol{r}_1\boldsymbol{r}_2 + \varepsilon\frac{1}{2}\left(\boldsymbol{r}_1\boldsymbol{p}_2\boldsymbol{r}_2 + \boldsymbol{p}_1\boldsymbol{r}_1\boldsymbol{r}_2\right). \tag{2.3.3}$$

Clearly, $\underline{\boldsymbol{x}}_3 \in \underline{\boldsymbol{\mathcal{S}}}$ because

$$\begin{aligned}
\underline{\boldsymbol{x}}_3\underline{\boldsymbol{x}}_3^* &= \left[\boldsymbol{r}_1\boldsymbol{r}_2 + \varepsilon\frac{1}{2}\left(\boldsymbol{r}_1\boldsymbol{p}_2\boldsymbol{r}_2 + \boldsymbol{p}_1\boldsymbol{r}_1\boldsymbol{r}_2\right)\right]\left[\boldsymbol{r}_2^*\boldsymbol{r}_1^* + \varepsilon\frac{1}{2}\left(\boldsymbol{r}_2^*\boldsymbol{p}_2^*\boldsymbol{r}_1^* + \boldsymbol{r}_2^*\boldsymbol{r}_1^*\boldsymbol{p}_1^*\right)\right] \\
&= \boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_2^*\boldsymbol{r}_1^* + \varepsilon\frac{1}{2}\left(\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_2^*\boldsymbol{p}_2^*\boldsymbol{r}_1^* + \boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_2^*\boldsymbol{r}_1^*\boldsymbol{p}_1^* + \boldsymbol{r}_1\boldsymbol{p}_2\boldsymbol{r}_2\boldsymbol{r}_2^*\boldsymbol{r}_1^* + \boldsymbol{p}_1\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_2^*\boldsymbol{r}_1^*\right) \\
&= 1 + \varepsilon\frac{1}{2}\left(\boldsymbol{r}_1\boldsymbol{p}_2^*\boldsymbol{r}_1^* + \boldsymbol{p}_1^* + \boldsymbol{r}_1\boldsymbol{p}_2\boldsymbol{r}_1^* + \boldsymbol{p}_1\right) \\
&= 1 + \varepsilon\frac{1}{2}\left[\boldsymbol{r}_1\left(\boldsymbol{p}_2^* + \boldsymbol{p}_2\right)\boldsymbol{r}_1^* + \boldsymbol{p}_1^* + \boldsymbol{p}_1\right] \\
&= 1.
\end{aligned}$$

In addition,

$$\mathcal{P}\left(\underline{\boldsymbol{x}}_3\right) = \boldsymbol{r}_1\boldsymbol{r}_2 \tag{2.3.4}$$

and

$$\begin{aligned}
\boldsymbol{p}_3 &= 2\,\mathcal{D}\left(\underline{\boldsymbol{x}}_3\right)\mathcal{P}\left(\underline{\boldsymbol{x}}_3\right)^* \\
&= \left(\boldsymbol{r}_1\boldsymbol{p}_2\boldsymbol{r}_2 + \boldsymbol{p}_1\boldsymbol{r}_1\boldsymbol{r}_2\right)\boldsymbol{r}_2^*\boldsymbol{r}_1^* \\
&= \boldsymbol{p}_1 + \operatorname{Ad}\left(\boldsymbol{r}_1\right)\boldsymbol{p}_2. \qquad (2.3.5)
\end{aligned}$$

Comparing (2.3.4) and (2.3.5) to (2.1.14) we conclude that multiplications of unit dual quaternions represent indeed the composition of rigid transformations.

Last, given $\underline{\boldsymbol{x}} \in \underline{\boldsymbol{\mathcal{S}}}$ the inverse transformation is given by the dual quaternion conjugate $\underline{\boldsymbol{x}}^*$, which corresponds to the group inverse of $\mathrm{Spin}(3)\ltimes\mathbb{R}^3$ because $\underline{\boldsymbol{x}}^*\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}\underline{\boldsymbol{x}}^* = 1$ (Selig, 2005). (Consequently, 1 is the group identity of $\mathrm{Spin}(3)\ltimes\mathbb{R}^3$ because $\underline{\boldsymbol{x}}1 = 1\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$.) In order to see that $\underline{\boldsymbol{x}}^*$ is the inverse rigid transformation of $\underline{\boldsymbol{x}}$, first one must realize that if $\boldsymbol{p} = 0$ and $\boldsymbol{r} = 1$, then $\underline{\boldsymbol{x}} = \boldsymbol{r} + \varepsilon\left(1/2\right)\boldsymbol{p}\boldsymbol{r} = 1$, which can be regarded as "absence" of rigid transformation as there is no translation ($\boldsymbol{p} = 0$) and no rotation (i.e., $\boldsymbol{r} = 1 = \cos\left(\phi/2\right) + \boldsymbol{n}\sin\left(\phi/2\right)$, which implies $\phi = 0$). By the own definition of $\underline{\boldsymbol{\mathcal{S}}}$, its elements have unit norm and hence $\forall\underline{\boldsymbol{x}} \in \underline{\boldsymbol{\mathcal{S}}} \implies \underline{\boldsymbol{x}}\underline{\boldsymbol{x}}^* = \underline{\boldsymbol{x}}^*\underline{\boldsymbol{x}} = 1$, which implies that the transformation given by $\underline{\boldsymbol{x}}$ is "canceled" by $\underline{\boldsymbol{x}}^*$ and vice-versa. Hence $\underline{\boldsymbol{x}}^*$ corresponds to the inverse transformation of $\underline{\boldsymbol{x}}$.

The next three statements are related to unit dual quaternions and will be useful in the next sections.

Proposition 2.3.1
Let $\underline{\boldsymbol{h}} = \boldsymbol{r} + \varepsilon\frac{1}{2}\boldsymbol{p}\boldsymbol{r}$ be a unit dual quaternion with $\boldsymbol{r} = \cos(\phi/2) + \sin(\phi/2)\,\boldsymbol{n}$, where $\boldsymbol{n} = \hat{\imath}n_x + \hat{\jmath}n_y + \hat{k}n_z$ and $\boldsymbol{p} = \hat{\imath}p_x + \hat{\jmath}p_y + \hat{k}p_z$. The logarithm of $\underline{\boldsymbol{h}}$ is

$$\log \underline{\boldsymbol{h}} = \frac{\phi\boldsymbol{n}}{2} + \varepsilon\frac{\boldsymbol{p}}{2}.$$

Proof. See (Han, Wei, and Z.-X. Li, 2008). ■

Note that $\log \underline{\boldsymbol{h}} \in \mathcal{H}$ and $\operatorname{Re}(\log \underline{\boldsymbol{h}}) = 0$; hence, $\log \underline{\boldsymbol{h}} \in \mathcal{H}_p$.

Proposition 2.3.2
Let $\underline{\boldsymbol{g}} \in \mathcal{H}_P$, the exponential of $\underline{\boldsymbol{g}}$ is

$$\exp \underline{\boldsymbol{g}} = \mathcal{P}\left(\exp \underline{\boldsymbol{g}}\right) + \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{g}}\right)\mathcal{P}\left(\exp \underline{\boldsymbol{g}}\right) \tag{2.3.6}$$

$$\mathcal{P}\left(\exp \underline{\boldsymbol{g}}\right) = \begin{cases} \cos\left\|\mathcal{P}\left(\underline{\boldsymbol{g}}\right)\right\| + \frac{\sin\left\|\mathcal{P}\left(\underline{\boldsymbol{g}}\right)\right\|}{\left\|\mathcal{P}\left(\underline{\boldsymbol{g}}\right)\right\|}\mathcal{P}\left(\underline{\boldsymbol{g}}\right) & \text{if } \left\|\mathcal{P}\left(\underline{\boldsymbol{g}}\right)\right\| \neq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.3.7}$$

Proof. Eq. (2.3.6) is a direct consequence of (2.2.6) (the extension of the Taylor expansion to $\mathcal{H}$ is left as an exercise to the reader) and (2.3.7) is shown by M.-J. Kim, M.-S. Kim, and Shin (1996). ■

Definition 2.3.1
Given propositions 2.3.1 and 2.3.2, the *geometrical* exponentiation of the unit dual quaternion $\underline{\boldsymbol{h}}$—i.e., $\underline{\boldsymbol{h}}$ raised to the λ-th power—is

$$\underline{\boldsymbol{h}}^{\{\lambda\}} \triangleq \exp(\lambda \log \underline{\boldsymbol{h}}).$$

If the unit dual quaternion $\underline{\boldsymbol{h}}$ is written in terms of its geometrical components, that is, $\underline{\boldsymbol{h}} = \boldsymbol{r} + \varepsilon\,(1/2)\,\boldsymbol{p}\boldsymbol{r}$, where $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$, then

$$\underline{\boldsymbol{h}}^{\{\lambda\}} = \boldsymbol{r}^{\{\lambda\}} + \varepsilon\frac{1}{2}\lambda\boldsymbol{p}\boldsymbol{r}^{\{\lambda\}},$$

where $\boldsymbol{r}^{\{\lambda\}} = \cos(\lambda\phi/2) + \boldsymbol{n}\sin(\lambda\phi/2)$.

2.4. Plücker lines

Section 2.1.3 has shown that quaternions are used to perform *point* transformations; that is, given a rotation quaternion $\boldsymbol{r}_b^a$ between frames $\mathcal{F}_a$ and $\mathcal{F}_b$, a point $\boldsymbol{p}^b$ is expressed in frame

$\mathcal{F}_a$ by means of the transformation $\boldsymbol{p}^a = \mathrm{Ad}\left(\boldsymbol{r}_b^a\right)\boldsymbol{p}^b$. As we shall see in the following sections, dual quaternions perform *line* transformations, that is, transformations of lines between different frames. In order to proceed with this idea, first we must learn how lines are represented in a three dimensional space by means of Plücker lines (Bottema and Roth, 1979).

Consider figure 2.4.1, where $\boldsymbol{l} \in \mathbb{H}_p$ with $\|\boldsymbol{l}\| = 1$ represents the direction of the line and $\boldsymbol{p} \in \mathbb{H}_p$ is a point in the line. A Plücker line is represented by the dual quaternion (Daniilidis, 1999)

$$\underline{\boldsymbol{l}} = \boldsymbol{l} + \varepsilon \boldsymbol{m},$$

where $\boldsymbol{m}$ is the line moment; that is,

$$\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{l}.$$

Whereas $\boldsymbol{l}$ represents the direction of the line, the quaternion $\boldsymbol{m}$ determines its spatial location independently of $\boldsymbol{p}$. To see that, consider an arbitrary point $\boldsymbol{p}_\lambda = \boldsymbol{p} + \lambda \boldsymbol{l}$, where $\lambda \in \mathbb{R}$, which is clearly on $\underline{\boldsymbol{l}}$. The new line moment, $\boldsymbol{m}_\lambda$, is given by

$$\begin{aligned}
\boldsymbol{m}_\lambda = \boldsymbol{p}_\lambda \times \boldsymbol{l} &= \frac{\boldsymbol{p}_\lambda \boldsymbol{l} - \boldsymbol{l}\boldsymbol{p}_\lambda}{2} \\
&= \frac{(\boldsymbol{p} + \lambda\boldsymbol{l})\,\boldsymbol{l} - \boldsymbol{l}\,(\boldsymbol{p} + \lambda\boldsymbol{l})}{2} = \frac{\boldsymbol{p}\boldsymbol{l} + \lambda\boldsymbol{l}\boldsymbol{l} - \boldsymbol{l}\boldsymbol{p} - \lambda\boldsymbol{l}\boldsymbol{l}}{2} \\
&= \frac{\boldsymbol{p}\boldsymbol{l} - \boldsymbol{l}\boldsymbol{p}}{2} = \boldsymbol{m}.
\end{aligned}$$

In addition, since $\mathrm{Re}\left(\boldsymbol{l}\right) = \mathrm{Re}\left(\boldsymbol{m}\right) = 0$, then $\mathrm{Re}\left(\underline{\boldsymbol{l}}\right) = 0$; this way, Plücker lines are pure dual quaternions and form the set $\mathcal{H}_p \subset \mathcal{H}$. Furthermore, since $\|\boldsymbol{l}\| = 1$ and $\boldsymbol{l} \cdot \boldsymbol{m} = 0$, an arbitrary Plücker line has four degrees of freedom (Daniilidis, 1999).

2.4.1. Transformation of Plücker lines. In the next paragraphs, first a Plücker line is represented with respect to two different frames; then, the result is used to show the existence of a unit dual quaternion (i.e., a dual quaternion with unit norm) that corresponds to a transformation between these two frames.

Consider two frames $\mathcal{F}_a$ and $\mathcal{F}_b$, a reference frame $\mathcal{F}$, and the Plücker line $\underline{\boldsymbol{l}} = \boldsymbol{l} + \varepsilon\boldsymbol{m}$ shown in figure 2.4.2. The Plücker line with respect to $\mathcal{F}_a$ is $\underline{\boldsymbol{l}}^a = \boldsymbol{l}^a + \varepsilon\boldsymbol{m}_a^a$ and with respect to $\mathcal{F}_b$ is $\underline{\boldsymbol{l}}^b = \boldsymbol{l}^b + \varepsilon\boldsymbol{m}_b^b$. From the definition of a Plücker line, $\boldsymbol{m}_a^a = \boldsymbol{p}_a^a \times \boldsymbol{l}^a$ and $\boldsymbol{m}_b^b = \boldsymbol{p}_b^b \times \boldsymbol{l}^b$. Hence,

$$\boldsymbol{m}_b^b = \frac{\boldsymbol{p}_b^b\boldsymbol{l}^b - \boldsymbol{l}^b\boldsymbol{p}_b^b}{2}.$$

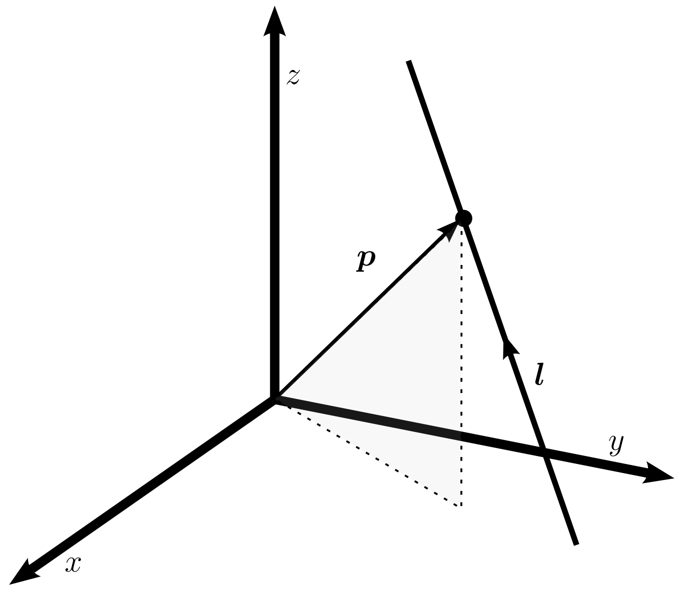


FIGURE 2.4.1. Plücker line.

From figure 2.4.2, one can see that $\boldsymbol{p}_b^b = \boldsymbol{p}_{ba}^b + \operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a$, and thus

$$
\begin{aligned}
\boldsymbol{m}_b^b &= \frac{\left(\boldsymbol{p}_{ba}^b + \operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a\right)\boldsymbol{l}^b - \boldsymbol{l}^b\left(\boldsymbol{p}_{ba}^b + \operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a\right)}{2} \\
&= \frac{\boldsymbol{p}_{ba}^b\boldsymbol{l}^b - \boldsymbol{l}^b\boldsymbol{p}_{ba}^b}{2} + \frac{\operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a\boldsymbol{l}^b - \boldsymbol{l}^b\operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a}{2} \\
&= \boldsymbol{p}_{ba}^b \times \boldsymbol{l}^b + \frac{\operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a\boldsymbol{l}^b - \boldsymbol{l}^b\operatorname{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}_a^a}{2}.
\end{aligned}
$$

Using the transformation $\boldsymbol{l}^b = \boldsymbol{r}_a^b\boldsymbol{l}^a\boldsymbol{r}_b^a$,

$$
\begin{aligned}
\boldsymbol{m}_b^b &= \boldsymbol{p}_{ba}^b \times \boldsymbol{l}^b + \frac{\boldsymbol{r}_a^b\boldsymbol{p}_a^a\boldsymbol{r}_b^a\boldsymbol{r}_a^b\boldsymbol{l}^a\boldsymbol{r}_b^a - \boldsymbol{r}_a^b\boldsymbol{l}^a\boldsymbol{r}_b^a\boldsymbol{r}_a^b\boldsymbol{p}_a^a\boldsymbol{r}_b^a}{2} \\
&= \boldsymbol{p}_{ba}^b \times \boldsymbol{l}^b + \frac{\boldsymbol{r}_a^b\left(\boldsymbol{p}_a^a\boldsymbol{l}^a - \boldsymbol{l}^a\boldsymbol{p}_a^a\right)\boldsymbol{r}_b^a}{2} \\
&= \boldsymbol{p}_{ba}^b \times \boldsymbol{l}^b + \boldsymbol{r}_a^b\boldsymbol{m}_a^a\boldsymbol{r}_b^a \\
&= \boldsymbol{p}_{ba}^b \times \boldsymbol{l}^b + \boldsymbol{m}_a^b. \qquad (2.4.1)
\end{aligned}
$$

Since dual quaternions perform line transformations analogously to the way quaternions perform point transformations, the goal is to find the dual quaternion $\underline{\boldsymbol{h}}_a^b$ that satisfies $\underline{\boldsymbol{l}}^b = \underline{\boldsymbol{h}}_a^b\underline{\boldsymbol{l}}^a\underline{\boldsymbol{h}}_b^a$,

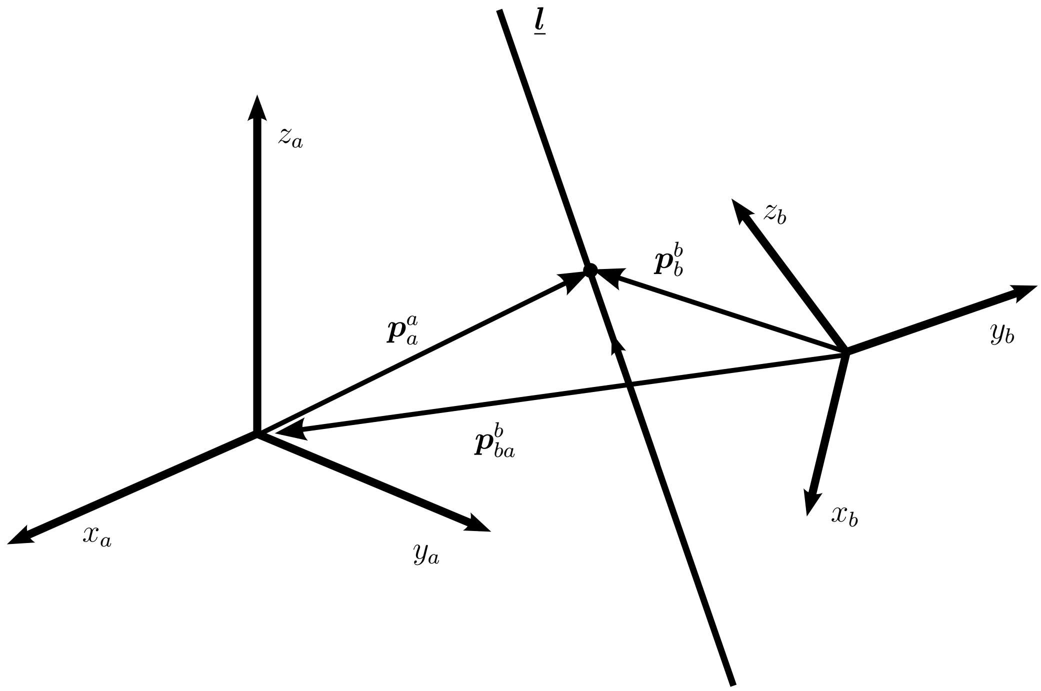


FIGURE 2.4.2. Plücker line with respect to frames $\mathcal{F}_a$ and $\mathcal{F}_b$.

where $\underline{\boldsymbol{h}}_b^a = \left(\underline{\boldsymbol{h}}_a^b\right)^*$, and $\underline{\boldsymbol{l}}^a$ and $\underline{\boldsymbol{l}}^b$ are the Plücker lines with respect to frames $\mathcal{F}_a$ and $\mathcal{F}_b$, respectively; thus,

$$\underline{\boldsymbol{l}}^b = \underline{\boldsymbol{h}}_a^b \underline{\boldsymbol{l}}^a \underline{\boldsymbol{h}}_b^a \tag{2.4.2}$$

$$\boldsymbol{l}^b + \varepsilon \boldsymbol{m}_b^b = \left(\mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) + \varepsilon\, \mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right)\right)\left(\boldsymbol{l}^a + \varepsilon \boldsymbol{m}_a^a\right)\left(\mathcal{P}\left(\underline{\boldsymbol{h}}_b^a\right) + \varepsilon\, \mathcal{D}\left(\underline{\boldsymbol{h}}_b^a\right)\right). \tag{2.4.3}$$

Expanding the previous equation yields

$$\boldsymbol{l}^b = \mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{l}^a\, \mathcal{P}\left(\underline{\boldsymbol{h}}_b^a\right) \tag{2.4.4}$$

and

$$\begin{aligned} \boldsymbol{m}_b^b = \mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{l}^a\, \mathcal{D}\left(\underline{\boldsymbol{h}}_b^a\right) + \mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{m}_a^a\, \mathcal{P}\left(\underline{\boldsymbol{h}}_b^a\right) & \\ + \mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{l}^a\, \mathcal{P}\left(\underline{\boldsymbol{h}}_b^a\right). & \end{aligned} \tag{2.4.5}$$

From (2.4.4), $\boldsymbol{l}^b = \mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{l}^a\, \mathcal{P}\left(\underline{\boldsymbol{h}}_b^a\right)$, and hence $\mathcal{P}\left(\underline{\boldsymbol{h}}_b^a\right) = \boldsymbol{r}_b^a$ and $\mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) = \boldsymbol{r}_a^b$. Compare (2.4.1) and (2.4.5) to obtain\c

$$\begin{aligned} \boldsymbol{p}_{ba}^b \times \boldsymbol{l}^b &= \boldsymbol{r}_a^b \boldsymbol{l}^a\, \mathcal{D}\left(\underline{\boldsymbol{h}}_b^a\right) + \mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{l}^a \boldsymbol{r}_b^a, \\ \frac{\boldsymbol{p}_{ba}^b \boldsymbol{r}_a^b \boldsymbol{l}^a \boldsymbol{r}_b^a - \boldsymbol{r}_a^b \boldsymbol{l}^a \boldsymbol{r}_b^a \boldsymbol{p}_{ba}^b}{2} &= \boldsymbol{r}_a^b \boldsymbol{l}^a\, \mathcal{D}\left(\underline{\boldsymbol{h}}_b^a\right) + \mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right) \boldsymbol{l}^a \boldsymbol{r}_b^a \end{aligned}$$

which implies

$$\mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right) = \frac{1}{2}\boldsymbol{p}_{ba}^b\boldsymbol{r}_a^b$$

and

$$\mathcal{D}\left(\underline{\boldsymbol{h}}_b^a\right) = -\frac{1}{2}\boldsymbol{r}_b^a\boldsymbol{p}_{ba}^b. \tag{2.4.6}$$

It is straightforward to show that $\mathcal{D}\left(\underline{\boldsymbol{h}}_b^a\right) = \mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right)^*$.

Since $\underline{\boldsymbol{h}}_a^b = \mathcal{P}\left(\underline{\boldsymbol{h}}_a^b\right) + \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{h}}_a^b\right)$, hence

$$\underline{\boldsymbol{h}}_a^b = \boldsymbol{r}_a^b + \varepsilon\frac{1}{2}\boldsymbol{p}_{ba}^b\boldsymbol{r}_a^b. \tag{2.4.7}$$

It is easy to see that $\left\|\underline{\boldsymbol{h}}_a^b\right\| = 1$. From the dual quaternion norm, if $\left\|\underline{\boldsymbol{h}}_a^b\right\| = 1$, hence $\underline{\boldsymbol{h}}_a^b\underline{\boldsymbol{h}}_a^{b*} = 1$. This is verified by direct calculation:

$$\begin{aligned}
\underline{\boldsymbol{h}}_a^b\underline{\boldsymbol{h}}_a^{b*} &= \left(\boldsymbol{r}_a^b + \varepsilon\frac{1}{2}\boldsymbol{p}_{ba}^b\boldsymbol{r}_a^b\right)\left(\boldsymbol{r}_b^a - \varepsilon\frac{1}{2}\boldsymbol{r}_b^a\boldsymbol{p}_{ba}^b\right) \\
&= \boldsymbol{r}_a^b\boldsymbol{r}_b^a + \varepsilon\frac{1}{2}\left(\boldsymbol{p}_{ba}^b\boldsymbol{r}_a^b\boldsymbol{r}_b^a - \boldsymbol{r}_a^b\boldsymbol{r}_b^a\boldsymbol{p}_{ba}^b\right) \\
&= 1.
\end{aligned}$$

2.5. Planes in dual quaternion space

A plane can be completely described by the unit norm vector normal to the plane and the perpendicular distance from the origin of a given coordinate system (Selig, 2000), as shown in Fig. 2.5.1. When written in dual quaternion form, the unit norm vector is given by $\boldsymbol{n} \in \mathbb{H}_p$, such that $\|\boldsymbol{n}\| = 1$ and the perpendicular distance d is given by $d = \boldsymbol{q}\cdot\boldsymbol{n}$, where $\boldsymbol{q} \in \mathbb{H}_p$ is an arbitrary point in the plane; thus, the plane $\underline{\boldsymbol{\pi}}$ is given by

$$\underline{\boldsymbol{\pi}} \triangleq \boldsymbol{n} + \varepsilon d = \boldsymbol{n} + \varepsilon\boldsymbol{q}\cdot\boldsymbol{n}. \tag{2.5.1}$$

Proposition 2.5.1

Let the plane be represented in frame $\mathcal{F}_a$ by $\underline{\boldsymbol{\pi}}^a = \boldsymbol{n}^a + \varepsilon d^a$, where $d^a = \boldsymbol{q}^a\cdot\boldsymbol{n}^a$. The same plane, represented in frame $\mathcal{F}_b$, is given by

$$\underline{\boldsymbol{\pi}}^b = \mathrm{Ad}_\sharp\left(\underline{\boldsymbol{x}}_a^b\right)\underline{\boldsymbol{\pi}}^a, \tag{2.5.2}$$

where $\underline{\boldsymbol{x}}_a^b = \boldsymbol{r}_a^b + \varepsilon(1/2)\boldsymbol{p}_{ba}^b\boldsymbol{r}_a^b$ is the rigid transformation from $\mathcal{F}_b$ to $\mathcal{F}_a$,

$$\mathrm{Ad}_\sharp\left(\underline{\boldsymbol{x}}\right)\underline{\boldsymbol{y}} \triangleq \underline{\boldsymbol{x}}^\sharp\underline{\boldsymbol{y}}\underline{\boldsymbol{x}}^* \tag{2.5.3}$$

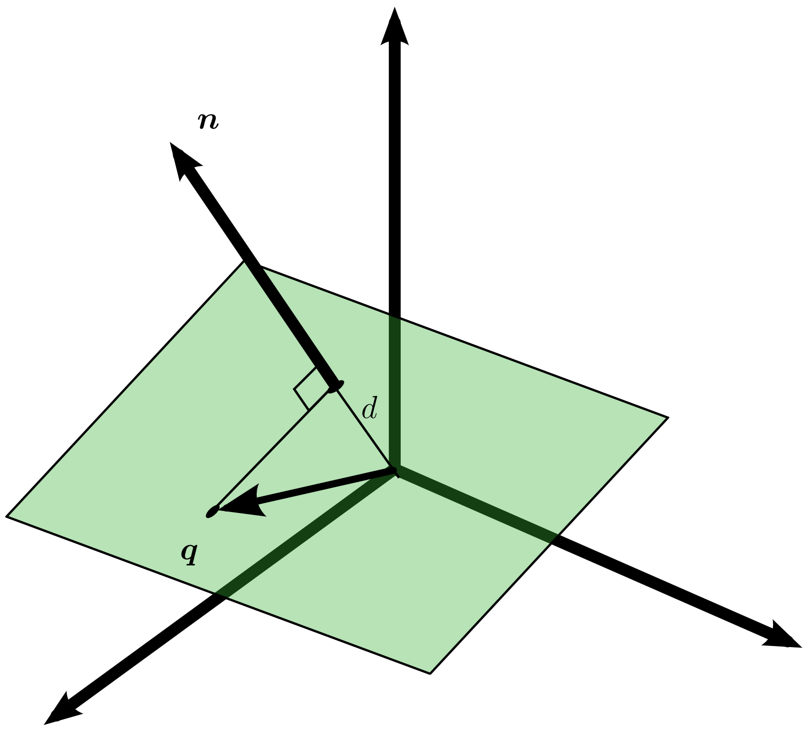


FIGURE 2.5.1. Plane with respect to frames $\mathcal{F}_a$ and $\mathcal{F}_b$.

and

$$\underline{\boldsymbol{x}}^{\sharp} \triangleq \mathcal{P}\left(\underline{\boldsymbol{x}}\right) - \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{x}}\right). \tag{2.5.4}$$

Proof. The plane, with respect to $\mathcal{F}_b$, is given by $\underline{\boldsymbol{\pi}}^b = \boldsymbol{n}^b + \varepsilon d^b$, where

$$\boldsymbol{n}^b = \operatorname{Ad}\left(\mathcal{P}\left(\underline{\boldsymbol{x}}_a^b\right)\right)\boldsymbol{n}^a \tag{2.5.5}$$

and $d^b = \boldsymbol{t}^b \cdot \boldsymbol{n}^b$, with $\boldsymbol{t}^b = \boldsymbol{p}_{ba}^b + \boldsymbol{q}^b$ (see Fig 2.5.2). Thus,

$$\begin{aligned}
\boldsymbol{t}^b \cdot \boldsymbol{n}^b &= -\frac{\boldsymbol{t}^b\boldsymbol{n}^b + \boldsymbol{n}^b\boldsymbol{t}^b}{2} \\
&= -\frac{\left(\boldsymbol{p}_{ba}^b + \boldsymbol{q}^b\right)\boldsymbol{n}^b + \boldsymbol{n}^b\left(\boldsymbol{p}_{ba}^b + \boldsymbol{q}^b\right)}{2} \\
&= -\frac{\boldsymbol{p}_{ba}^b\boldsymbol{n}^b + \boldsymbol{n}^b\boldsymbol{p}_{ba}^b}{2} - \frac{\boldsymbol{q}^b\boldsymbol{n}^b + \boldsymbol{n}^b\boldsymbol{q}^b}{2} \\
&= \boldsymbol{p}_{ba}^b \cdot \boldsymbol{n}^b + \boldsymbol{q}^b \cdot \boldsymbol{n}^b.
\end{aligned}$$

Since $\boldsymbol{q}^b \cdot \boldsymbol{n}^b = \boldsymbol{q}^a \cdot \boldsymbol{n}^a$ (see exercise 2.8.15), hence

$$d^b = \boldsymbol{t}^b \cdot \boldsymbol{n}^b = \boldsymbol{p}_{ba}^b \cdot \boldsymbol{n}^b + d^a. \tag{2.5.6}$$

From (2.5.1), (2.5.3), and (2.5.4), we obtain

$$
\begin{aligned}
\underline{\boldsymbol{\pi}}^b &= \mathrm{Ad}_\sharp\left(\underline{\boldsymbol{x}}_a^b\right)\underline{\boldsymbol{\pi}}^a \\
&= \left(\underline{\boldsymbol{x}}_a^b\right)^\sharp \underline{\boldsymbol{\pi}}^a \left(\underline{\boldsymbol{x}}_a^b\right)^* \\
&= \left(\boldsymbol{r}_a^b - \varepsilon\frac{1}{2}\boldsymbol{p}_{ba}^b\boldsymbol{r}_a^b\right)\left(\boldsymbol{n}^a + \varepsilon d^a\right)\left(\boldsymbol{r}_a^{b*} - \varepsilon\frac{1}{2}\boldsymbol{r}_a^{b*}\boldsymbol{p}_{ba}^b\right) \\
&= \boldsymbol{n}^b + \varepsilon\left(d^a + \boldsymbol{p}_{ba}^b \cdot \boldsymbol{n}^b\right).
\end{aligned}
\tag{2.5.7}
$$

Comparing (2.5.7) to (2.5.5) and (2.5.6), we conclude the proof. ■

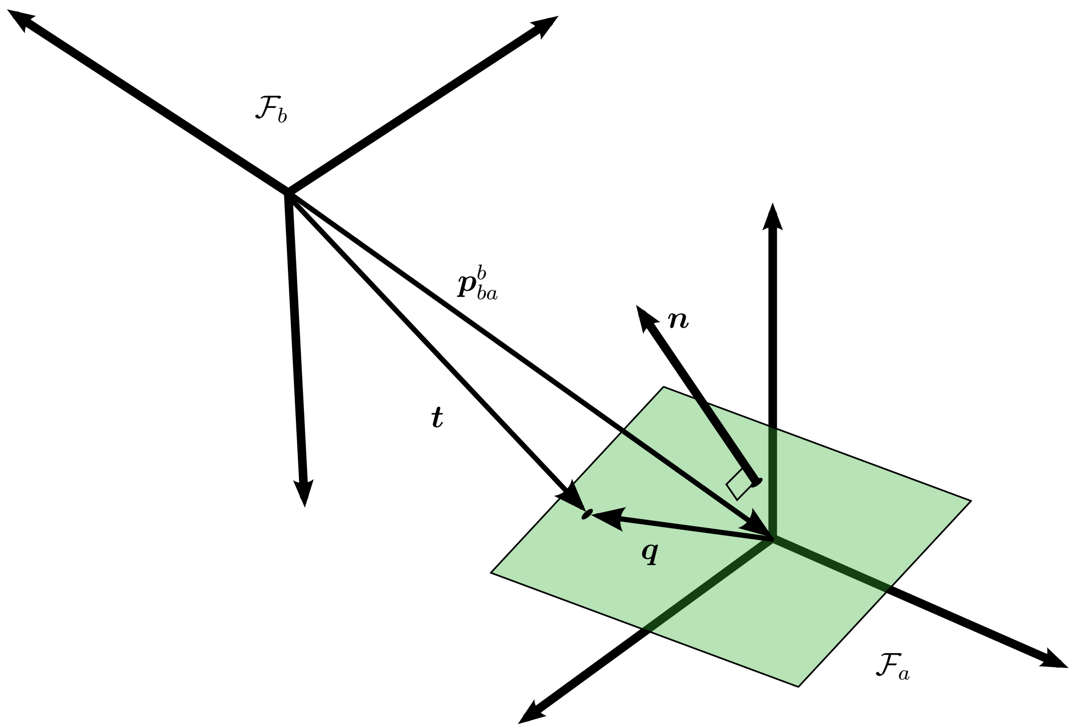


FIGURE 2.5.2. Plane with respect to frames $\mathcal{F}_a$ and $\mathcal{F}_b$.

2.6. Relationship between the derivative of unit dual quaternions and twists

Let us consider the unit dual quaternion $\underline{\boldsymbol{x}}_b^a = \boldsymbol{r}_b^a + \varepsilon\,(1/2)\,\boldsymbol{p}_{ab}^a\boldsymbol{r}_b^a$. Then

$$
\dot{\underline{\boldsymbol{x}}}_b^a = \frac{1}{2}\underline{\boldsymbol{\xi}}_{ab}^a\underline{\boldsymbol{x}}_b^a,
\tag{2.6.1}
$$

where $\underline{\boldsymbol{\xi}}_{ab}^a = \boldsymbol{\omega}_{ab}^a + \varepsilon\left(\boldsymbol{v}_{ab}^a + \boldsymbol{p}_{ab}^a \times \boldsymbol{\omega}_{ab}^a\right)$.

Proof. Taking the first derivative of $\underline{\boldsymbol{x}}_b^a$, we obtain

$$
\dot{\underline{\boldsymbol{x}}}_b^a = \dot{\boldsymbol{r}}_b^a + \varepsilon\frac{1}{2}\left(\dot{\boldsymbol{p}}_{ab}^a\boldsymbol{r}_b^a + \boldsymbol{p}_{ab}^a\dot{\boldsymbol{r}}_b^a\right)
$$

such that, using equation (2.1.15) and $\dot{\boldsymbol{p}}_{ab}^{a} = \boldsymbol{v}_{ab}^{a}$, we get

$$\begin{aligned} \mathcal{P}\left(\dot{\underline{\boldsymbol{x}}}_{b}^{a}\right) &= \dot{\boldsymbol{r}}_{b}^{a} = \frac{1}{2}\boldsymbol{\omega}_{ab}^{a}\boldsymbol{r}_{b}^{a}, \\ \mathcal{D}\left(\dot{\underline{\boldsymbol{x}}}_{b}^{a}\right) &= \frac{1}{2}\left(\boldsymbol{v}_{ab}^{a} + \frac{1}{2}\boldsymbol{p}_{ab}^{a}\boldsymbol{\omega}_{ab}^{a}\right)\boldsymbol{r}_{b}^{a}. \end{aligned} \tag{2.6.2}$$

Expanding (2.6.1) we obtain

$$\begin{aligned} \dot{\underline{\boldsymbol{x}}}_{b}^{a} &= \frac{1}{2}\left[\mathcal{P}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right) + \varepsilon\mathcal{D}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right)\right]\left[\boldsymbol{r}_{b}^{a} + \varepsilon\frac{1}{2}\boldsymbol{p}_{ab}^{a}\boldsymbol{r}_{b}^{a}\right] \\ &= \frac{1}{2}\left[\mathcal{P}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right)\boldsymbol{r}_{b}^{a} + \varepsilon\left(\mathcal{D}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right)\boldsymbol{r}_{b}^{a} + \frac{1}{2}\mathcal{P}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right)\boldsymbol{p}_{ab}^{a}\boldsymbol{r}_{b}^{a}\right)\right] \\ &= \frac{1}{2}\mathcal{P}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right)\boldsymbol{r}_{b}^{a} + \varepsilon\frac{1}{2}\left(\mathcal{D}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right) + \frac{1}{2}\mathcal{P}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right)\boldsymbol{p}_{ab}^{a}\right)\boldsymbol{r}_{b}^{a}. \end{aligned} \tag{2.6.3}$$

Comparing equation (2.6.2) with equation (2.6.3) we obtain $\mathcal{P}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right) = \boldsymbol{\omega}_{ab}^{a}$ and

$$\begin{aligned} \mathcal{D}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right) + \frac{1}{2}\boldsymbol{\omega}_{ab}^{a}\boldsymbol{p}_{ab}^{a} &= \boldsymbol{v}_{ab}^{a} + \frac{1}{2}\boldsymbol{p}_{ab}^{a}\boldsymbol{\omega}_{ab}^{a} \\ \mathcal{D}\left(\underline{\boldsymbol{\xi}}_{ab}^{a}\right) &= \boldsymbol{v}_{ab}^{a} + \boldsymbol{p}_{ab}^{a}\times\boldsymbol{\omega}_{ab}^{a}. \end{aligned}$$

■

2.6.1. Numerical integration of unit dual quaternions. Consider the unit dual quaternion $\underline{\boldsymbol{x}}(t) = \boldsymbol{r}(t) + \varepsilon(1/2)\,\boldsymbol{p}(t)\,\boldsymbol{r}(t)$—where $\boldsymbol{r}(t) = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$, with $\phi \triangleq \phi(t)$ and $\boldsymbol{n} = \boldsymbol{n}(t)$—and its corresponding time derivative $\dot{\underline{\boldsymbol{x}}}(t)$. When performing the numerical integration of $\dot{\underline{\boldsymbol{x}}}$, one may be tempted to use the simple first-order approximation

$$\dot{\underline{\boldsymbol{x}}} \approx \frac{\underline{\boldsymbol{x}}_{k} - \underline{\boldsymbol{x}}_{k-1}}{T},$$

to obtain

$$\underline{\boldsymbol{x}}_{k} = \underline{\boldsymbol{x}}_{k-1} + T\dot{\underline{\boldsymbol{x}}}, \tag{2.6.4}$$

where T is the integration step. The problem with such approach is that $\underline{\boldsymbol{x}}$ has unit norm and belongs to a multiplicative group, but $\underline{\boldsymbol{x}}_{k}$ usually does not have unit norm due to the approximation error of (2.6.4) and the additive operation. This is particularly troubling, since rigid motions are represented by unit dual quaternions, thus (2.6.4) not only has intrinsic approximation errors, but from an algebraic point of view, *does not make sense.* Approximative methods should at least respect the group operation of unit dual quaternions. In order to derive such method first consider the angular velocity $\boldsymbol{\omega} = \boldsymbol{n}\dot{\phi}$ and the linear velocity $\boldsymbol{v} = \dot{\boldsymbol{p}}$ associated to $\dot{\underline{\boldsymbol{x}}}$, thus

$$\dot{\phi} \approx \frac{\phi_k - \phi_{k-1}}{T} \qquad \Longrightarrow \Delta\phi \approx T\dot{\phi} \tag{2.6.5}$$

and

$$\dot{\boldsymbol{p}} \approx \frac{\boldsymbol{p}_k - \boldsymbol{p}_{k-1}}{T} \qquad \Longrightarrow \Delta\boldsymbol{p} \approx T\dot{\boldsymbol{p}}, \tag{2.6.6}$$

where $\Delta\phi = \phi_k - \phi_{k-1}$ and $\Delta\boldsymbol{p} = \boldsymbol{p}_k - \boldsymbol{p}_{k-1}$. These approximations related to the linear velocity and angular are embedded into the twist $\underline{\boldsymbol{\xi}} = \boldsymbol{\omega} + \varepsilon\left(\boldsymbol{v} + \boldsymbol{p} \times \boldsymbol{\omega}\right)$—where $\boldsymbol{\omega} \triangleq \boldsymbol{\omega}(t)$, $\boldsymbol{p} \triangleq \boldsymbol{p}(t)$, and $\boldsymbol{v} \triangleq \boldsymbol{v}(t)$—in the following manner:

$$\begin{aligned}
\frac{T\underline{\boldsymbol{\xi}}}{2} &= \frac{\boldsymbol{\omega}T}{2} + \varepsilon\left(\frac{\boldsymbol{v} + \boldsymbol{p} \times \boldsymbol{\omega}}{2}\right)T \\
&= \frac{\boldsymbol{n}T\dot{\phi}}{2} + \varepsilon\left(\frac{\dot{\boldsymbol{p}}T + \boldsymbol{p} \times \boldsymbol{n}T\dot{\phi}}{2}\right).
\end{aligned}$$

Using (2.6.5) and (2.6.6) we obtain

$$\begin{aligned}
\frac{T\underline{\boldsymbol{\xi}}}{2} &= \boldsymbol{n}\frac{\Delta\phi}{2} + \varepsilon\left(\frac{\Delta\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{n}\Delta\phi}{2}\right) \\
\Longrightarrow \exp\left(\frac{T\underline{\boldsymbol{\xi}}}{2}\right) &= \boldsymbol{r}_\Delta + \varepsilon\frac{1}{2}\boldsymbol{p}_\Delta\boldsymbol{r}_\Delta,
\end{aligned}$$

where $\boldsymbol{p}_\Delta = \Delta\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{n}\Delta\phi$ and $\boldsymbol{r}_\Delta = \cos\left(\Delta\phi/2\right) + \boldsymbol{n}\sin\left(\Delta\phi/2\right)$. Since $\underline{\boldsymbol{\xi}}$ is given with respect to the inertial frame, thus the associated transformation must be performed using the inertial frame as reference; that is,

$$\underline{\boldsymbol{x}}_k = \exp\left(\frac{T\underline{\boldsymbol{\xi}}(t)}{2}\right)\underline{\boldsymbol{x}}_{k-1}. \tag{2.6.7}$$

EXAMPLE 2.6.1 Consider the time-varying trajectory given by

$$\underline{\boldsymbol{x}}(t) = \boldsymbol{r}(t) + \varepsilon\frac{1}{2}\boldsymbol{p}(t)\boldsymbol{r}(t),$$

where $\boldsymbol{p}(t) = \hat{\imath}\cos t + \hat{\jmath}\sin t + \hat{k}t$ and

$$\boldsymbol{r}(t) = \cos\left(\frac{\phi(t)}{2}\right) + \boldsymbol{n}(t)\sin\left(\frac{\phi(t)}{2}\right)$$

with $\phi(t) = 2t$ and $\boldsymbol{n}(t) = \hat{\imath}\cos(t)/\sqrt{2} + \hat{\jmath}\sin t + \hat{k}\cos(t)/\sqrt{2}$. (Note that $\|\boldsymbol{n}\| = 1$, as expected.) Figure 2.6.1 shows the trajectory for the time interval $t = [0, 5]s$. In order to illustrate the numerical integration given by (2.6.7), we must first find the twist $\underline{\boldsymbol{\xi}}(t)$ associated to $\dot{\underline{\boldsymbol{x}}}(t)$. Thus,

$$\boldsymbol{v}(t) = \dot{\boldsymbol{p}}(t) = -\sin(t)\,\hat{\imath} + \cos(t)\,\hat{\jmath} + \hat{k}.$$

From the quaternion propagation equation (2.1.15),

$$\begin{aligned}\boldsymbol{\omega}(t) &= 2\dot{\boldsymbol{r}}(t)\,\boldsymbol{r}(t)^* \\ &= 2\left(\boldsymbol{n}(t) + \dot{\boldsymbol{n}}(t)\sin t\cos t - \dot{\boldsymbol{n}}(t)\,\boldsymbol{n}(t)\sin^2 t\right).\end{aligned}$$

The resultant twist is given by

$$\underline{\boldsymbol{\xi}}(t) = \boldsymbol{\omega}(t) + \varepsilon\left(\boldsymbol{v}(t) + \boldsymbol{p}(t)\times\boldsymbol{\omega}(t)\right).$$

The trajectory obtained by numerical integration is given by

$$\underline{\boldsymbol{x}}_I(t+T) = \exp\left(\frac{T\underline{\boldsymbol{\xi}}(t)}{2}\right)\underline{\boldsymbol{x}}_I(t), \tag{2.6.8}$$

$$\underline{\boldsymbol{x}}_I(0) = \underline{\boldsymbol{x}}(0).$$

The effect of the integration step T in the trajectory approximation is shown in figure 2.6.2. For $T = 0.01\,s$, the numerical approximation is very accurate as the trajectory generated by (2.6.8) (black curve) is very close to the trajectory generated analytically (red curve with red circles). On the other hand, for increasing values of T the numerical approximation starts to have large deviations from the analytical trajectory, which is expected, since the larger the value of the integration step, the more inaccurate is the numerical approximation.

2.7. Additional facts and properties of quaternions and dual quaternions

This section presents additional facts and properties of quaternions and dual quaternions which were ommited from the main text in order to ensure a smoother reading.

2.7.1. Quaternions.

Fact 2.7.1

Consider $\boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbb{H}$. From definition 2.1.2 it is easy to verify, by direct calculation, that $\operatorname{Re}(\boldsymbol{h}_1\boldsymbol{h}_2) = (\operatorname{vec}_4 \boldsymbol{h}_1)^T \operatorname{vec}_4 \boldsymbol{h}_2^*$.

Fact 2.7.2

Let $\boldsymbol{h}$ be a quaternion with unit norm; hence $\operatorname{Re}\left(\dot{\boldsymbol{h}}\boldsymbol{h}^*\right) = \operatorname{Re}\left(\boldsymbol{h}\dot{\boldsymbol{h}}^*\right) = 0$.

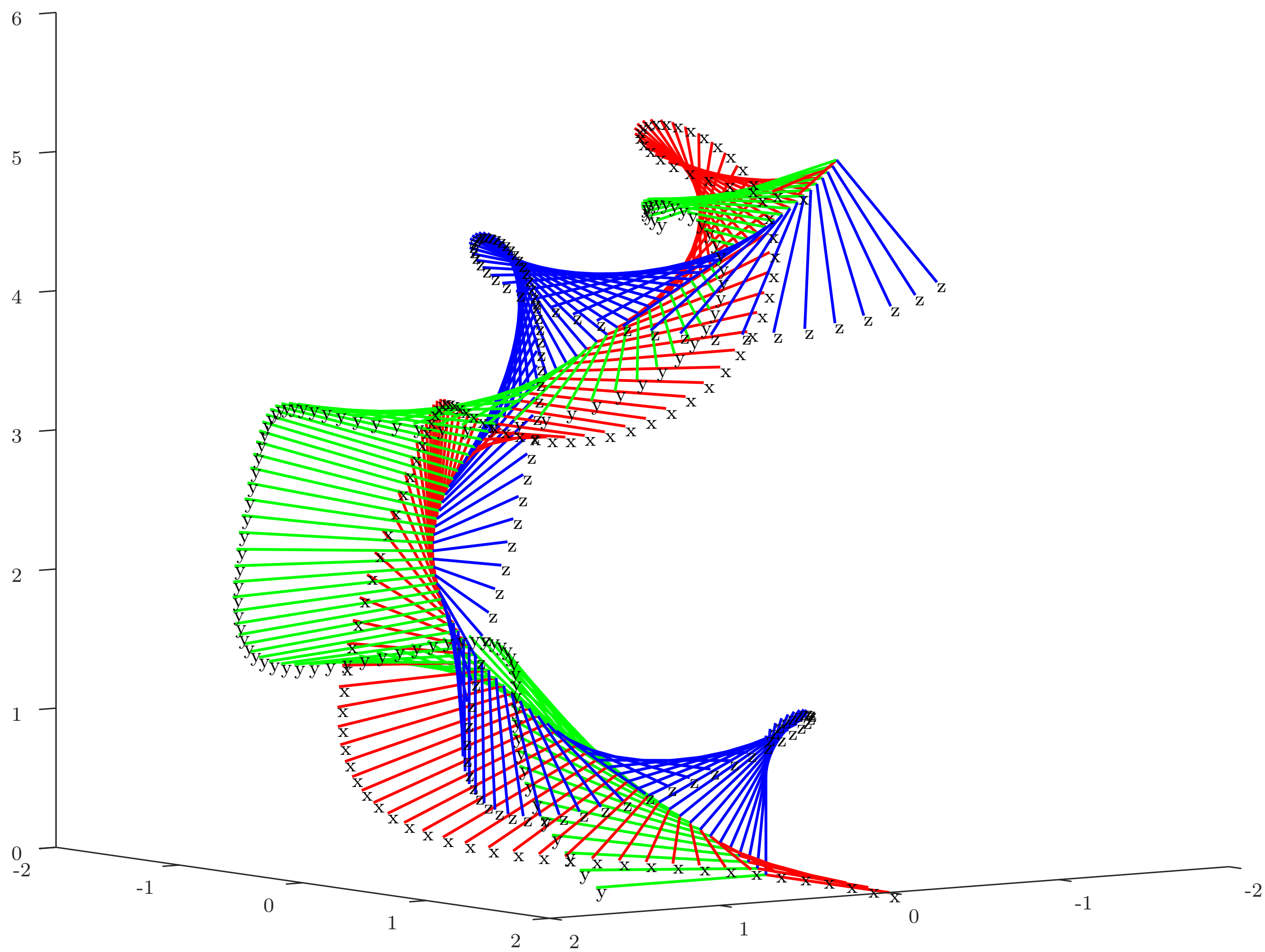


FIGURE 2.6.1. Time varying trajectory in $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$ of example 2.6.1.

Proof. Taking the first derivative of $\boldsymbol{h}\boldsymbol{h}^* = 1$, we get $\dot{\boldsymbol{h}}\boldsymbol{h}^* + \boldsymbol{h}\dot{\boldsymbol{h}}^* = 0$, which means that $\mathrm{Re}\left(\dot{\boldsymbol{h}}\boldsymbol{h}^* + \boldsymbol{h}\dot{\boldsymbol{h}}^*\right) = 0$. Thus,

$$
\begin{aligned}
0 &= \mathrm{Re}\left(\dot{\boldsymbol{h}}\boldsymbol{h}^*\right) + \mathrm{Re}\left(\boldsymbol{h}\dot{\boldsymbol{h}}^*\right)\\
&= \left(\mathrm{vec}_4\,\dot{\boldsymbol{h}}\right)^T \mathrm{vec}_4\,\boldsymbol{h} + \left(\mathrm{vec}_4\,\boldsymbol{h}\right)^T \mathrm{vec}_4\,\dot{\boldsymbol{h}}\\
&= 2\left(\mathrm{vec}_4\,\dot{\boldsymbol{h}}\right)^T \mathrm{vec}_4\,\boldsymbol{h}.
\end{aligned}
$$

Hence, $\left(\mathrm{vec}_4\,\dot{\boldsymbol{h}}\right)^T \mathrm{vec}_4\,\boldsymbol{h} = 0$, which implies $\mathrm{Re}\left(\dot{\boldsymbol{h}}\boldsymbol{h}^*\right) = \mathrm{Re}\left(\boldsymbol{h}\dot{\boldsymbol{h}}^*\right) = 0$. ■

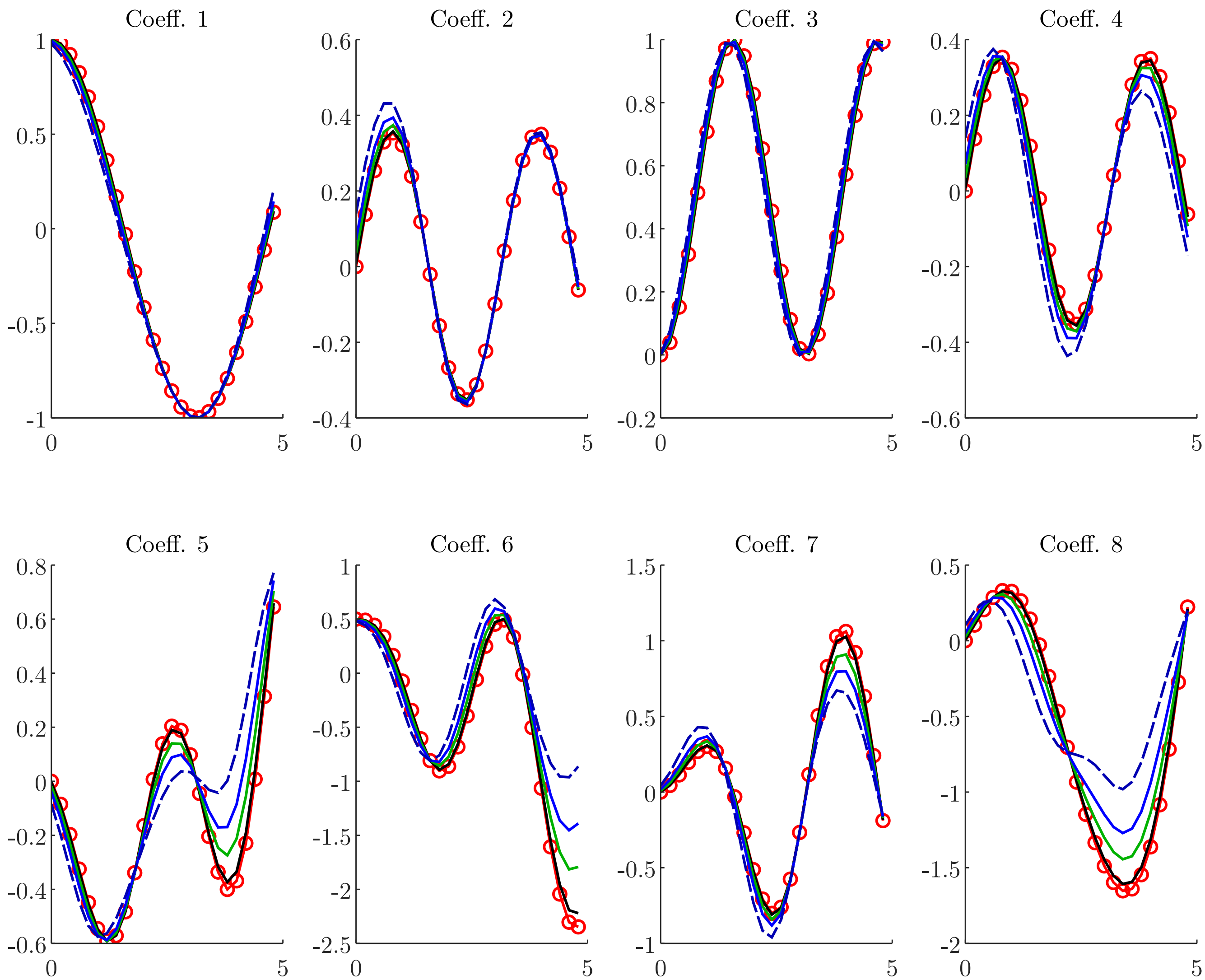


FIGURE 2.6.2. Effect of the integration step in the trajectory approximation. Each figure corresponds to a coefficient of the dual quaternion. The curve with *red circles* corresponds to the trajectory obtained analytically, whereas the *black* ($T = 0.01\,s$), *dark green* ($T = 0.05$), *blue* ($T = 0.1$) and *dashed blue* ($T = 0.2\,s$) curves currespond to the trajectories obtained numerically.

PROPOSITION 2.7.1

Given a constant λ and a unit quaternion $\boldsymbol{r}$ with constant rotation axis, such that $\dot{\boldsymbol{r}} = (1/2)\,\boldsymbol{\omega}\boldsymbol{r}$, the angular velocity $\boldsymbol{\omega}_\lambda$ associated to the time derivative of $\boldsymbol{r}^{\{\lambda\}}$ is given by

$$\boldsymbol{\omega}_\lambda = \lambda\boldsymbol{\omega}.$$

Proof. Consider $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$, then[3]

$$\boldsymbol{r}^{\{\lambda\}} = \cos\left(\frac{\lambda\phi}{2}\right) + \boldsymbol{n}\sin\left(\frac{\lambda\phi}{2}\right)$$
$$\implies \frac{d}{dt}\boldsymbol{r}^{\{\lambda\}} = \left[-\sin\left(\frac{\lambda\phi}{2}\right) + \boldsymbol{n}\cos\left(\frac{\lambda\phi}{2}\right)\right]\frac{\lambda\dot{\phi}}{2}.$$

From the quaternion progagation equation,

$$\begin{aligned}\boldsymbol{\omega}_\lambda &= 2\frac{d}{dt}\boldsymbol{r}^{\{\lambda\}}\boldsymbol{r}^{\{\lambda\}*}\\ &= \lambda\dot{\phi}\boldsymbol{n}\\ &= \lambda\boldsymbol{\omega}.\end{aligned}$$

■

Proposition 2.7.2

Given $\lambda \in \mathbb{R}$ and a unit quaternion $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$, then $\boldsymbol{r}^{\{-\lambda\}} = \left(\boldsymbol{r}^{\{\lambda\}}\right)^* = (\boldsymbol{r}^*)^{\{\lambda\}}$.

Proof. The proof can be verified by direct calculation. ■

Proposition 2.7.3

Given $\lambda \in \mathbb{R}$ and a unit quaternion $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$,

$$\frac{d}{dt}\boldsymbol{r}^{\{\lambda\}} = \lambda\dot{\boldsymbol{r}}\boldsymbol{r}^{\{\lambda-1\}}.$$

Proof. From propositions 2.7.1 and 2.7.2, and the fact that $\boldsymbol{r}^{\{t\}}\boldsymbol{r}^{\{s\}} = \boldsymbol{r}^{\{t+s\}}$ (see exercise 2.8.20)

$$\begin{aligned}\frac{d}{dt}\boldsymbol{r}^{\{\lambda\}} &= \frac{1}{2}\boldsymbol{\omega}_\lambda\boldsymbol{r}^{\{\lambda\}}\\ &= \frac{1}{2}\lambda\boldsymbol{\omega}\boldsymbol{r}^{\{\lambda\}}\\ &= \left(\frac{1}{2}\boldsymbol{\omega}\boldsymbol{r}\right)\left(\lambda\boldsymbol{r}^{\{\lambda-1\}}\right)\\ &= \lambda\dot{\boldsymbol{r}}\boldsymbol{r}^{\{\lambda-1\}}.\end{aligned}$$

■

[3]Note that the geometrical exponentiation, as presented in Definition 2.3.1, is defined only for unit dual quaternions. This way, the notation $\dot{\boldsymbol{r}}^{\{\lambda\}}$ is ambiguous and thus we use $d\boldsymbol{r}^{\{\lambda\}}/dt$ instead.

2.8. Exercises

EXERCISE 2.8.1. Show that a complex number $\boldsymbol{r} = \cos\theta + \hat{\imath}\sin\theta$ has unit norm.

EXERCISE 2.8.2. Show that rotations in the plane are commutative.

EXERCISE 2.8.3. Show that the following relations

$$-\hat{\jmath}\hat{\imath} = \hat{\imath}\hat{\jmath} = \hat{k}$$
$$-\hat{k}\hat{\jmath} = \hat{\jmath}\hat{k} = \hat{\imath}$$
$$-\hat{\imath}\hat{k} = \hat{k}\hat{\imath} = \hat{\jmath}$$

are a direct consequence of (2.1.1).

EXERCISE 2.8.4. Given a quaternion $\boldsymbol{h} = h_1 + \hat{\imath}h_2 + \hat{\jmath}h_3 + \hat{k}h_4$, show that $\|\boldsymbol{h}\| = \|\mathrm{vec}_4\,\boldsymbol{h}\| = \sqrt{h_1^2 + h_2^2 + h_3^2 + h_4^2}$.

EXERCISE 2.8.5. Given the pure quaternions $\boldsymbol{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\boldsymbol{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, show that the cross product defined in definition 2.1.7 corresponds to the cross product between two vectors $\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$.

EXERCISE 2.8.6. Given the pure quaternions $\boldsymbol{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\boldsymbol{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, show that the dot product defined in definition 2.1.8 corresponds to the dot product between two vectors $\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$.

EXERCISE 2.8.7. Show that $\mathbb{H}_p$ is isomorphic to $\mathbb{R}^3$ under the addition operation.

EXERCISE 2.8.8. Show that if $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis, then $\boldsymbol{uv} = \boldsymbol{w}$, $\boldsymbol{wu} = \boldsymbol{v}$, and $\boldsymbol{vw} = \boldsymbol{u}$.

EXERCISE 2.8.9. Show that, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}$, the following equality holds: $(\boldsymbol{ab})^* = \boldsymbol{b}^*\boldsymbol{a}^*$.

EXERCISE 2.8.10. Show that if $\boldsymbol{n}$ is a pure quaternion with unit norm, then quaternions of the form $\boldsymbol{r} = \cos(\phi/2) + \sin(\phi/2)\boldsymbol{n}$ also have unit norm.

EXERCISE 2.8.11. Show that if $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are rotation quaternions, then $\boldsymbol{r} = \boldsymbol{r}_1\boldsymbol{r}_2$ has unit norm.

EXERCISE 2.8.12. Show that if $\boldsymbol{r}$ is a rotation quaternion, then $-\boldsymbol{r}$ represents the same rotation.

EXERCISE 2.8.13. Show, by means of quaternion operations, that a rotation angle ϕ around rotation axis $\boldsymbol{n}$ is equivalent to the rotation angle $-\phi$ around rotation axis $-\boldsymbol{n}$.

EXERCISE 2.8.14. Show that, if $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$ and $\boldsymbol{s} = \cos(\pi - \phi/2) - \boldsymbol{n}\sin(\pi - \phi/2)$, then $\boldsymbol{s} = -\boldsymbol{r}$.

EXERCISE 2.8.15. Given $\boldsymbol{p}^a, \boldsymbol{t}^a \in \mathbb{H}_p$ and $\boldsymbol{r}_a^b \in \mathrm{Spin}(3)$, show that $\boldsymbol{p}^a \cdot \boldsymbol{t}^a = \boldsymbol{p}^b \cdot \boldsymbol{t}^b$, where $\boldsymbol{p}^b = \mathrm{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{p}^a$ and $\boldsymbol{t}^b = \mathrm{Ad}\left(\boldsymbol{r}_a^b\right)\boldsymbol{t}^a$.

EXERCISE 2.8.16. Given the unit quaternion $\boldsymbol{r}_b^a$, show that

$$\left(\frac{d}{dt}\boldsymbol{r}_b^a\right)^* = \frac{d}{dt}\left(\boldsymbol{r}_b^{a*}\right) = \frac{d}{dt}\boldsymbol{r}_a^b.$$

EXERCISE 2.8.17. Show that, given two frames $\mathcal{F}_a$ and $\mathcal{F}_b$, and a rotation $\boldsymbol{r}_b^a$, if the angular velocity is expressed with respect to $\mathcal{F}_b$, then the quaternion propagation equation is given by $\dot{\boldsymbol{r}}_b^a = (1/2)\,\boldsymbol{r}_b^a\boldsymbol{\omega}_{ab}^b$.

EXERCISE 2.8.18. Show that $\mathrm{Re}\left(\underline{\boldsymbol{h}}\right) - \mathrm{Im}\left(\underline{\boldsymbol{h}}\right) = \mathcal{P}\left(\underline{\boldsymbol{h}}\right)^* + \varepsilon\,\mathcal{D}\left(\underline{\boldsymbol{h}}\right)^*$.

EXERCISE 2.8.19. Given a constant $\lambda \in \mathbb{R}$ and a unit dual quaternion $\underline{\boldsymbol{x}} = \boldsymbol{r} + \varepsilon(1/2)\boldsymbol{p}\boldsymbol{r}$, such that $\dot{\underline{\boldsymbol{x}}} = (1/2)\underline{\boldsymbol{\xi}}\underline{\boldsymbol{x}}$ with $\underline{\boldsymbol{\xi}} = \boldsymbol{\omega} + \varepsilon\left(\dot{\boldsymbol{p}} + \boldsymbol{p}\times\boldsymbol{\omega}\right)$ and $\dot{\boldsymbol{r}} = (1/2)\,\boldsymbol{\omega}\boldsymbol{r}$, show that the twist $\underline{\boldsymbol{\xi}}_\lambda$ related to $d\underline{\boldsymbol{x}}^{\{\lambda\}}/dt$ is given by $\underline{\boldsymbol{\xi}}_\lambda = \lambda\left[\boldsymbol{\omega} + \varepsilon\left(\dot{\boldsymbol{p}} + \lambda\boldsymbol{p}\times\boldsymbol{\omega}\right)\right]$.

EXERCISE 2.8.20. Given the unit quaternion $\boldsymbol{r} = \cos(\phi/2) + \boldsymbol{n}\sin(\phi/2)$, the unit dual quaternion $\underline{\boldsymbol{x}} = \boldsymbol{r} + \varepsilon\,(1/2)\,\boldsymbol{p}\boldsymbol{r}$, and $s, t \in \mathbb{R}$, show that $\boldsymbol{r}^{\{s\}}\boldsymbol{r}^{\{t\}} = \boldsymbol{r}^{\{s+t\}}$ and $\underline{\boldsymbol{x}}^{\{s\}}\underline{\boldsymbol{x}}^{\{t\}} \neq \underline{\boldsymbol{x}}^{\{s+t\}}$.

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