Dual-Number Transformation and Its Applications to Robotics

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Abstract—In robotics, to deal with coordinate transformation in three-dimensional (3D) Cartesian space, the homogeneous transformation is usually applied. It is defined in the four-dimensional space, and its matrix multiplication performs the simultaneous rotation and translation. The homogeneous transformation, however, is a point transformation. In contrast, a line transformation can also naturally be defined in 3D Cartesian space, in which the transformed element is a line in 3D space instead of a point. In robotic kinematics and dynamics, the velocity and acceleration vectors are often the direct targets of analysis. The line transformation will have advantages over the ordinary point transformation, since the combination of the linear and angular quantities can be represented by lines in 3D space. Since a line in 3D space is determined by four independent parameters, finding an appropriate type of "number representation" which combines two real variables is the first key prerequisite. The dual number is chosen for the line representation, and lemmas and theorems indicating relavent properties of the dual number, dual vector, and dual matrix are proposed. This is followed by the transformation and manipulation for the robotic applications. The presented procedure offers an algorithm which deals with the symbolic analysis for both rotation and translation. In particular, it can effectively be used for direct determination of Jacobian matrices and their derivatives. It is shown that the proposed procedure contributes a simplified approach to the formulation of the robotic kinematics, dynamics, and control system modeling.

I. Introduction

IN ROBOTICS, dealing with coordinate transformation in the Cartesian space is usually done by means of homogeneous transformation. It associates two points $\boldsymbol{x}, \boldsymbol{y} \in R^3$ with a 3×3 orthogonal matrix A and a 3×1 column vector $\boldsymbol{b}$ to form a linear equation as

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{b}. \tag{1}$$

It represents that the matrix A performs a rotation on the initial point $\boldsymbol{x}$ to reach the final point $\boldsymbol{y}$ with a translation corresponding to $\boldsymbol{b}$. In an augmented space, alternatively, we say that a point $\boldsymbol{x}$ is represented by a 4×1 column vector as $\boldsymbol{x}_1 = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$. Likewise, the final point becomes $\boldsymbol{y}_1 = \left[\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right]$. With a 4×4 matrix

$$A_1 = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \tag{2}$$

(1) is equivalent to

$$\boldsymbol{y}_1 = A_1 \boldsymbol{x}_1, \tag{3}$$

and (3) is called a homogeneous transformation equation. Therefore, the homogeneous transformation is a point transformation, since vectors $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ represent the points in the augmented space.

Similarly, we can define a line transformation. Suppose for the basic transformation form (3) that $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ were two lines, i.e., each of $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ would have two independent conditions (totally including four independent parameters), since they uniquely determine a line in $\boldsymbol{R}^3$. What would the matrix A_1 then be so that (3) would become a line transformation equation which could transform an initial line $\boldsymbol{x}_1$ to the final line $\boldsymbol{y}_1$?

Because a velocity vector, which is a derivative of the point vector with respect to time, may be represented by a line in $\boldsymbol{R}^3$, a line transformation matrix can directly operate on the velocity vector. This is a consequence of the fact that, for any differential motion equation, the highest order derivative appearing in any equation derived by using the line transformation is one order lower than in an equation derived by a point transformation. For instance, in robotics, using the homogeneous transformation one may formulate a Lagrangian equation with second-order derivatives, whereas using the line transformation only the first derivatives appear in the Lagrangian formulation [1]. For the robot kinematic equation, the Jacobian matrix includes first-order derivatives, or cross product operators if one uses the homogeneous transformation. When a line transformation is applied, however, the Jacobian matrix has neither derivatives, nor cross products.

Now the question is how to define such a line transformation? Since a line is uniquely determined by two independent conditions, we should look for some type of "number representation" that can combine two real variables. From Rooney [2], there are three types of such numbers:

1) the complex number $a + ib$, where $i^2 = -1$,
2) the dual number $a + \epsilon b$, where $\epsilon^2 = 0$, and
3) the double number $a + jb$, where $j^2 = +1$,

each of which contains two ordered real numbers a and b with a different multiplication rule for the special numbers i, ϵ, and j. In fact, the first type is the well-known complex number which, under the usual operations of addition, subtraction, and multiplication, forms an algebraic field. The second type, the dual number which, under the usual operations, forms an

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algebraic ring [3], may be used as the line representation and transformation.

The dual number, originally introduced by Clifford in 1873 [4], was further developed by Study in 1901 [5] to represent the dual angle in spatial geometry. If a distance between two lines in three-dimensional (3D) space is d and an angle between their directions is α, then the dual angle can be defined by

$$\bar{\alpha}=\alpha+\epsilon d \tag{4}$$

which combines two relative parameters of two arbitrary lines in 3D space into a dual number with $\epsilon^2 = 0$.

Like the usual functions over the field of complex numbers, a function of dual number $f(a + \epsilon b)$ can be expanded into a formal Taylor series:

$$f(a+\epsilon b)=f(a)+\epsilon bf'(a)+\epsilon^2 \frac{b^2}{2} f''(a)+\cdots=f(a)+\epsilon bf'(a) \tag{5}$$

where f', f'' are the first- and second-order derivatives of f, respectively. It is seen that, due to the definition $\epsilon^2 = 0$, the dual number function has a very simple power series expansion. According to (5), one obtains

$$(a+\epsilon b)^{-1}=\frac{1}{a}-\epsilon \frac{b}{a^2}, \qquad \text{where } a \neq 0 \tag{6}$$

$$e^{a+\epsilon b}=e^a \cdot e^{\epsilon b}=e^a(1+\epsilon b)=e^a+\epsilon be^a. \tag{7}$$

Likewise, for the dual angle $\bar{\alpha} = \alpha + \epsilon d$, one gets

$$\sin(\alpha+\epsilon d)=\sin\alpha+\epsilon d\cos\alpha \tag{8}$$

$$\cos(\alpha+\epsilon d)=\cos\alpha-\epsilon d\sin\alpha \tag{9}$$

and so forth.

Note that for (6), if $a = 0$, then $a + \epsilon b$ has no inverse. This is why the collection of all dual numbers composes a ring but not a field. One may also define a conjugate dual number of $a + \epsilon b$ by $a - \epsilon b$, so that

$$(a+\epsilon b)(a-\epsilon b)=a^2. \tag{10}$$

Like the complex number, a is called the modulus of the dual number, due to (10). Also, b/a is called the parameter of the dual number, since if $a \neq 0$, then by (7),

$$a+\epsilon b=a\left(1+\epsilon \frac{b}{a}\right)=a \cdot e^{\epsilon b/a}. \tag{11}$$

The dual number has a geometrical meaning which is discussed in detail in [6]. We will utilize the dual vector and dual matrix, which are constructed by dual numbers, to deal with the spatial coordinate transformation in robotics. The concept is not so familiar to researchers outside the field of mechanical engineering. We will extend the known results [12]–[14] to the formulation of Jacobian for the simplicity of symbolic manipulation and computation of robotic kinematics and dynamics.

II. Dual Vector and Dual Matrix

A vector or a matrix in complex space is treated the same as the real vector or real matrix with the only difference that each element is thought to be a complex number. While a dual vector or a dual matrix, each element of which is a dual number, is handled by dividing it into two parts, the real part and the dual part, each of which is to be separately considered.

Thus a dual vector in 3D dual space

$$\bar{\boldsymbol{a}}=\begin{bmatrix} a_1+\epsilon b_1 \\ a_2+\epsilon b_2 \\ a_3+\epsilon b_3 \end{bmatrix}$$

may be rewritten as

$$\bar{\boldsymbol{a}}=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}+\epsilon\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}=\boldsymbol{a}+\epsilon \boldsymbol{b} \tag{12}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^3$. Likewise, a 3×3 dual matrix $\bar{\boldsymbol{A}}$ may also be divided into a real part $\boldsymbol{A}$ and a dual part $\boldsymbol{B}$, both of which are 3×3 real matrices as

$$\bar{\boldsymbol{A}}=\boldsymbol{A}+\epsilon \boldsymbol{B}. \tag{13}$$

The multiplication of the dual vectors and the dual matrices is similar to that for dual numbers, i.e., for dual vectors $\bar{\boldsymbol{a}} = \boldsymbol{a} + \epsilon \boldsymbol{b}$ and $\bar{\boldsymbol{c}} = \boldsymbol{c} + \epsilon \boldsymbol{d}$, and for dual matrices $\bar{\boldsymbol{A}} = \boldsymbol{A} + \epsilon \boldsymbol{B}$ and $\bar{\boldsymbol{C}} = \boldsymbol{C} + \epsilon \boldsymbol{D}$,

$$\text{inner product } \bar{\boldsymbol{a}}^T\bar{\boldsymbol{c}}=\boldsymbol{a}^T\boldsymbol{c}+\epsilon(\boldsymbol{b}^T\boldsymbol{c}+\boldsymbol{a}^T\boldsymbol{d}) \tag{14}$$

$$\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}=\boldsymbol{AC}+\epsilon(\boldsymbol{BC}+\boldsymbol{AD}) \tag{15}$$

$$\bar{\boldsymbol{A}}\bar{\boldsymbol{a}}=\boldsymbol{Aa}+\epsilon(\boldsymbol{Ab}+\boldsymbol{Ba}) \tag{16}$$

where $(\cdot)^T$ = transpose of $(\cdot)$. In addition, they have the following important properties based on their multiplication rules and the Taylor series expansion [7], [8].

1) If $\boldsymbol{A}$ is nonsingular, then the inverse of $\bar{\boldsymbol{A}}$, denoted as $\bar{\boldsymbol{A}}^{-1}$, satisfies $\bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^{-1} = \boldsymbol{I}$, where $\boldsymbol{I}$ is a 3×3 real identity matrix, and

$$\bar{\boldsymbol{A}}^{-1}=(\boldsymbol{A}+\epsilon\boldsymbol{B})^{-1}=\boldsymbol{A}^{-1}-\epsilon\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}. \tag{17}$$

2) If $\det \boldsymbol{A} \neq 0$, then

$$\det\bar{\boldsymbol{A}}=\det(\boldsymbol{A}+\epsilon\boldsymbol{B})=\det\boldsymbol{A} \cdot e^{\epsilon \operatorname{tr}(\boldsymbol{A}^{-1}\boldsymbol{B})} = \det\boldsymbol{A} \cdot (1+\epsilon \operatorname{tr}(\boldsymbol{A}^{-1}\boldsymbol{B})) \tag{18}$$

where det means determinant and tr means trace. Also, if $\det \boldsymbol{A} = 0$, then $\det \bar{\boldsymbol{A}} = 0$.

3) We define that $\bar{\boldsymbol{A}}$ is orthonormal if $\bar{\boldsymbol{A}}^T = \bar{\boldsymbol{A}}^{-1}$. For orthonormal dual matrix $\bar{\boldsymbol{A}} = \boldsymbol{A} + \epsilon\boldsymbol{B}$, we have

a) $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^T = \bar{\boldsymbol{A}}^T\bar{\boldsymbol{A}} = \boldsymbol{I}$.
b) $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$, i.e., its real part $\boldsymbol{A}$ is also orthonormal.
c) $\boldsymbol{A}^T\boldsymbol{B} + \boldsymbol{B}^T\boldsymbol{A} = \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{B}^T + \boldsymbol{B}\boldsymbol{A}^T = \boldsymbol{0}$, i.e., $\boldsymbol{A}^T\boldsymbol{B}$ and $\boldsymbol{A}\boldsymbol{B}^T$ are skew-symmetric matrices.

These properties can be easily verifed. From them, we see that the dual matrix has many simple formulas representing a variety of properties. This is the reason why we divide each of the dual vectors and dual matrices into two parts, then treat them separately. The rank of the dual part matrix has an interesting property that can be stated as in the following lemma.

Lemma 1: For any 3×3 orthonormal dual matrix $\bar{A} = A + \epsilon B$, if $B \neq 0$, then rank $(B) = 2$.

Proof: Using the property c) and b) of 3), $A^T B + B^T A = 0$, then $B = -AB^T A$, since A is orthonormal. It yields det $B = (-1)^3 \det(AB^T A) = -(\det A)^2 \cdot \det B$, and $(\det A)^2 = 1$. Hence det $B = 0$, i.e., rank $(B) < 3$.

Moreover, $A^T B \neq 0$ if $B \neq 0$, and is a skew-symmetric matrix which can be expressed as

$$A^T B = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

for some x, y, z with at least one of them being nonzero. Thus rank $(A^T B) = 2$ and $2 = \text{rank}(A^T B) \leqq \text{rank}(B) < 3$, therefore, rank $(B) = 2$ always for $B \neq 0$.

Apparently, we further have the following corollary.

Corollary 1: For the same condition as Lemma 1,

$$\dim \mathfrak{N}(B) = \dim \mathfrak{N}(B^T) = 1$$

where dim $\mathfrak{N}(B)$ means the dimension of null space of the matrix B.

This corollary seems to be simple, but it tells us an important fact that there must be a nontrivial vector contained in the null space of B. The next section will describe the vector that lies in the null space.

Definition 1: A dual vector $\bar{a} = a + \epsilon b$, where $a, b \in R^3$, is called a unit screw [9], if $\|a\|^2 = a^T a = 1$ and $a^T b = 0$. We refer to a unit screw as $\bar{a}_0 = a_0 + \epsilon b_0$.

Clearly, $\bar{a}_0^T \bar{a}_0 = a_0^T a_0 + \epsilon(a_0^T b_0 + b_0^T a_0) = 1$, which leads to the name of unit screw for $\bar{a}_0$. In Definition 1, the length of b_0 is not restricted. In the general case, we may choose b_0 so that the unit screw $\bar{a}_0 = a_0 + \epsilon b_0$ uniquely represents a directed line in 3D space, i.e., lets its real part a_0 be a unit vector along this line and its dual part b_0 be a moment vector of the line with respect to the origin. The moment vector is defined as a cross product $b_0 = p \times a_0$, where p is a radial vector whose tail is at the origin of the coordinates, and whose head is at somewhere on the line.

Of course, a general dual vector is not necessarily a unit screw. It is called a *motor*, the abbreviation of "moment and vector" [9]. A motor $\bar{a} = a + \epsilon b$ can always be represented as $\bar{a} = \bar{\alpha} \cdot \bar{a}_0$, where $\bar{a}_0$ is a unit screw, and $\bar{\alpha}$ is a dual number. If $\bar{\alpha} = \alpha + \epsilon\beta$ and $\bar{a}_0 = a_0 + \epsilon b_0$, then

$$\bar{a} = a + \epsilon b = (\alpha + \epsilon\beta)(a_0 + \epsilon b_0) = \alpha a_0 + \epsilon(\alpha b_0 + \beta a_0),$$

so that $a = \alpha a_0$ and $b = \alpha b_0 + \beta a_0$. Since $\|a\| = \alpha \|a_0\| = \alpha$, and $a^T b = \alpha^2 a_0^T b_0 + \alpha\beta a_0^T a_0 = \alpha\beta$, thus for a given motor $\bar{a} = a + \epsilon b$, we can always find a dual number $\bar{\alpha} = \alpha + \epsilon\beta$ in which $\alpha = \|a\|$ and $\beta = a^T b / \|a\|$, and a unit screw $\bar{a}_0 = a_0 + \epsilon b_0$ in which $a_0 = \alpha^{-1} a = a/\|a\|$, and $b_0 = \alpha^{-1}(b - \beta a_0) = \|a\|^{-1}[b - (a^T b/\|a\|^2)a]$ such that $\bar{a} = \bar{\alpha}\bar{a}_0$. Actually, a_0 is the unit vector of a, and b_0 is a component of b, which is perpendicular to a. The other component of b is βa_0 that is parallel to a. Therefore, like the modulus for a dual number, the motor also has its modulus $\bar{\alpha}$ which is, however, a dual number instead of a real number.

Lemma 2: For any 3×3 orthonormal dual matrix $\bar{A} = A + \epsilon B$, each column or each row of $\bar{A}$ is a unit screw.

Proof: Suppose $\bar{a}_i = a_i + \epsilon b_i$ is the ith column (or row) of $\bar{A}$, $i = 1, 2, 3$, then a_i and b_i must be the ith column (or row) of A and B, respectively. Since A is orthonormal, so $\|a_i\| = 1$. Since $A^T B$ or AB^T is a skew-symmetric matrix, each diagonal element of $A^T B$ or AB^T is zero, namely $a_i^T b_i = 0$ if $\bar{a}_i$ is the ith column of $\bar{A}$, or $a_i b_i^T = 0$ if $\bar{a}_i$ is the ith row of $\bar{A}$. Hence $\bar{a}_i$ is a unit screw.

It is seen that the sum of two unit screws is not a unit screw in general. However, for the transformation of a unit screw, we have the following lemma.

Lemma 3: The unit screw is invariant under the orthonormal dual-number transformation in 3D space.

Proof: Suppose a unit screw $\bar{a}_0 = a_0 + \epsilon b_0$, with $a_0, b_0 \in R^3$, is operated on by a 3×3 orthonormal dual matrix $\bar{A} = A + \epsilon B$, i.e., $\bar{A}\bar{a}_0 = Aa_0 + \epsilon(Ab_0 + Ba_0)$. Since A is orthonormal, $\|Aa_0\|^2 = a_0^T A^T A a_0 = 1$. While

$$(Aa_0)^T(Ab_0 + Ba_0) = a_0^T A^T A b_0 + a_0^T A^T B a_0,$$

in which the first term is equal to $a_0^T b_0 = 0$. Since the second term is a scaler,

$$a_0^T A^T B a_0 = (a_0^T A^T B a_0)^T = a_0^T B^T A a_0.$$

However, $B^T A = -A^T B$, thus

$$a_0^T A^T B a_0 = a_0^T B^T A a_0 = -a_0^T A^T B a_0,$$

which implies that the second term is also zero. Hence $\bar{A}\bar{a}_0$ is still a unit screw.

These lemmas are used to build a theoretical base for the future applications of the dual-number transformation.

III. Dual-Number Transformation Procedure

The homogeneous transformation has been widely used in robotics. It involves multiplications of 4×4 matrices to transform coordinate frames in symbolic forms. In contrast, dual-number transformation requires multiplications of 3×3 matrices for the same operation, so long as the dual angle is adopted. After the coordinate transformation, constructing the dynamic and kinematic equations is also convenient using the dual-number transformation procedure.

The relationship between these two transformations in robotics are as follows. Consider the standard defining relationship between coordinate frames of two adjacent links of a robot, as shown in Fig. 1, given in [10, p.53]:

rotate about z_{k-1} (axis), an angle θ_k;
translate along z_{k-1}, a distance d_k;
translate along rotated $x_{k-1} = x_k$ (axis), a length a_k;
rotate about x_k, the twist angle α_k.

The homogeneous transformation that decribes these four

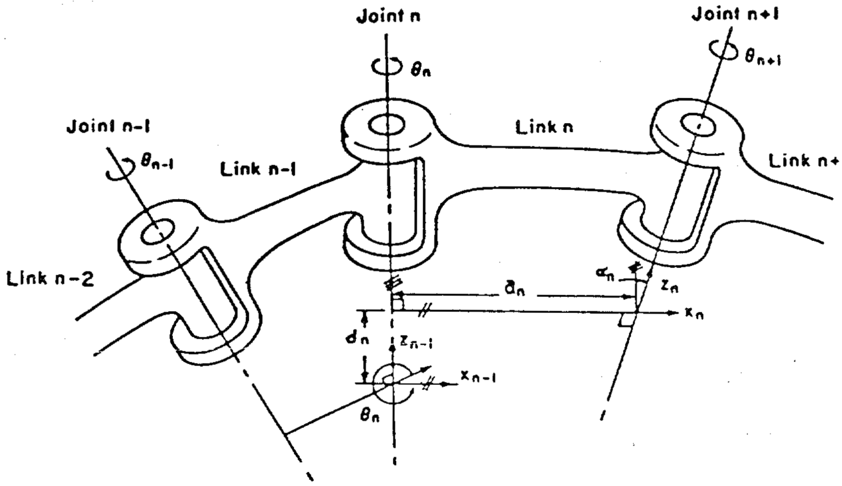

Fig. 1. Link parameters θ, d, α, and a.

steps is

$$A_{k-1}^{k}=\begin{bmatrix} c\theta_k & -s\theta_k & 0 & 0\\ s\theta_k & c\theta_k & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_k\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\cdot\begin{bmatrix} 1 & 0 & 0 & a_k\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & c\alpha_k & -s\alpha_k & 0\\ 0 & s\alpha_k & c\alpha_k & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{19}$$

where $c\theta_k = \cos\theta_k$, $s\theta_k = \sin\theta_k$, etc., for abbreviation. Note that the first two matrices represent a rotation about and a translation along the same axis, hence they are commutable. This is also true for the last two matrices. The transformation A_{k-1}^{k} transforms any four-dimensional vector with reference to the kth link coordinate frame to the k-1st link coordinate frame. To apply the dual-number algebra to robot kinematics, one defines a dual-displacement scalar

$$\bar{\theta}_k = \theta_k + \epsilon d_k \tag{20}$$

in which, as mentioned earlier, θ_k and d_k are displacements with reference to the same axis. In a way, $\bar{\theta}_k$ describes the displacements of a "screw" after it is turned, which gives rise to the name of "calculus of screws." Likewise, one defines

$$\bar{\alpha}_k = \alpha_k + \epsilon a_k. \tag{21}$$

Equation (20) or (21) is called the dual angle and is shown in Fig. 2.

The dual-number transformation that describes the four steps is given by [7]

$$\bar{A}_{k-1}^{k}=\begin{bmatrix} c\bar{\theta}_k & -s\bar{\theta}_k & 0\\ s\bar{\theta}_k & c\bar{\theta}_k & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 0 & c\bar{\alpha}_k & -s\bar{\alpha}_k\\ 0 & s\bar{\alpha}_k & c\bar{\alpha}_k\end{bmatrix}$$

$$=\begin{bmatrix} c\bar{\theta}_k & -s\bar{\theta}_k c\bar{\alpha}_k & s\bar{\theta}_k s\bar{\alpha}_k\\ s\bar{\theta}_k & c\bar{\theta}_k c\bar{\alpha}_k & -c\bar{\theta}_k s\bar{\alpha}_k\\ 0 & s\bar{\alpha}_k & c\bar{\alpha}_k\end{bmatrix}. \tag{22}$$

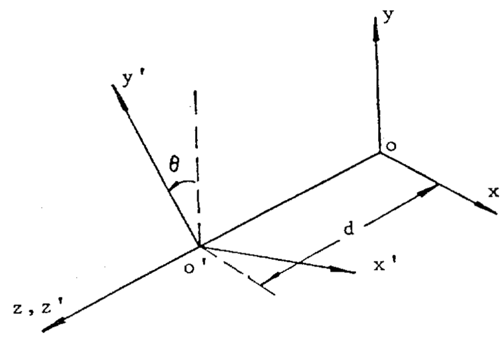

Fig. 2. Dual angle $\bar{\theta} = \theta + \epsilon d$.

It is seen that $\bar{A}_{k-1}^{k}$ is orthonormal so that

$$(\bar{A}_{k-1}^{k})^{-1} = (\bar{A}_{k-1}^{k})^{T} = \bar{A}_{k}^{k-1}. \tag{23}$$

In fact, (22) can be interpreted as a rotation of an angle of $\bar{\theta}_k$ about the z_{k-1} axis of the dual coordinates, and then a rotation of an angle of $\bar{\alpha}_k$ about the x_k axis of the dual coordinates. This leads to the mechanism of the dual-number transformation procedure which is given below. First, however, let us introduce the following definition.

Defintion 2: The dual-number transformation from the ith coordinates to jth coordinates is a 3×3 orthonormal dual matrix $\bar{A}_j^i = R_j^i + \epsilon S_j^i$, in which the real part R_j^i, the dual part S_j^i, and $\bar{A}_j^i$ itself satisfy, in addition to all properties of dual matrices given in the last section, the following multiplication rules:

1) $\bar{A}_j^i = \bar{A}_j^k \bar{A}_k^i$, for any $i, j, k = 0, 1, 2, \cdots, n$;
2) $R_j^i = R_j^k R_k^i$, for any $i, j, k = 0, 1, 2, \cdots, n$;
3) $S_j^i = R_j^k S_k^i + S_j^k R_k^i$, for any $i, j, k = 0, 1, 2, \cdots, n$.

These rules can be easily verified. Particularly, when $i = j$, $\bar{A}_i^i = R_i^i = I$, an identity, and $S_i^i = \mathbf{0}$.

Now the dual-number transformation procedure can be stated as follows. Execute all multiplications between 3×3 real rotation (orthonormal) matrices until the objective symbolic result is obtained. Then replace each revolute angle θ_i and each twist angle α_i ($i = 1, \cdots, n$) by corresponding dual angles $\bar{\theta}_i = \theta_i + \epsilon d_i$ and $\bar{\alpha}_i = \alpha_i + \epsilon a_i$, and using (8) and (9) to separate the real part and dual part of the result matrix. The entire information is contained in these two parts.

The theoretical justification of this procedure is included in the discussion on the principle of transference [7], [8]. It is clear that after using the dual-number transformation procedure, the resulting dual matrix is always orthonormal, since it is a product of several orthonormal dual matrices which are given by (22).

This procedure includes two major steps: multiplications of 3×3 matrices and separation of real and dual parts. Let us take the PUMA 560 robot as an illustrative example.

The PUMA 560 robot has a Danavit-Hartenberg table shown in Table I.

Suppose that the objective matrix is $\bar{A}_0^2$. First we have

$$R_0^1=\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_1 & -s\alpha_1 \\ 0 & s\alpha_1 & c\alpha_1 \end{bmatrix}$$

where $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s\alpha_i = \sin\alpha_i$, and $c\alpha_i = \cos\alpha_i$. Since we have a special case of $\bar{\alpha}_1 = \alpha_1 = -90°$, R_0^1 can be presimplified as

$$R_0^1=\begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix}.$$

This step of presimplification, however, is not valid whenever $\bar{\alpha}_1 \neq \alpha_1$ even if $\alpha_1 = -90°$. Now

$$R_1^2=\begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_2 & -s\alpha_2 \\ 0 & s\alpha_2 & c\alpha_2 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2c\alpha_2 & s_2s\alpha_2 \\ s_2 & c_2c\alpha_2 & -c_2s\alpha_2 \\ 0 & s\alpha_2 & c\alpha_2 \end{bmatrix}.$$

so that

$$R_0^2=R_0^1R_1^2=\begin{bmatrix} c_1c_2 & -c_1s_2c\alpha_2-s_1s\alpha_2 & c_1s_2s\alpha_2-s_1c\alpha_2 \\ s_1c_2 & -s_1s_2c\alpha_2+c_1s\alpha_2 & s_1s_2s\alpha_2+c_1c\alpha_2 \\ -s_2 & -c_2c\alpha_2 & c_2s\alpha_2 \end{bmatrix}.$$

This is the intermediate objective result. For the procedure of separation, replace R_0^2 by $\bar{A}_0^2$ and substitute each dual angle into corresponding real angle and separating real part and dual part of the result matrix, i. e., $\bar{\theta}_i = \theta_i + \epsilon d_i$, $\bar{\alpha}_i = \alpha_i + \epsilon a_i$ for $i = 1, 2$; and use (8) and (9) to obtain $s\bar{\alpha}_2 = \epsilon a_2$ and $c\bar{\alpha}_2 = 1$. We finally have

$$\bar{A}_0^2=R_0^2+\epsilon\begin{bmatrix} -d_2c_1s_2 & -d_2c_1c_2-a_2s_1 & a_2c_1s_2 \\ -d_2s_1s_2 & -d_2s_1c_2+a_2c_1 & a_2s_1s_2 \\ -d_2c_2 & d_2s_2 & a_2c_2 \end{bmatrix}.$$

At first glance, the dual part S_0^2 of $\bar{A}_0^2$ seems as complicated as the real part R_0^2. In practice, such as the analysis of kinematics or dynamics, however, we require only to know its third row or third column. This will be discussed in detail later.

The real part of any dual-number transformation describes

TABLE I
DENAVIT-HARTENBERG TABLE FOR PUMA 560 ROBOT

Variable	θ_i	d_i	α_i	a_i	$c\bar{\alpha}_i$	$s\bar{\alpha}_i$
θ_1	θ_1	0	$-90°$	0	0	-1
θ_2	θ_2	d_2	$0°$	a_2	1	ϵa_2
θ_3	θ_3	0	$90°$	0	0	1
θ_4	θ_4	d_3	$-90°$	0	0	-1
θ_5	θ_5	0	$90°$	0	0	1
θ_6	θ_6	0	$0°$	0	1	0

the rotation while the dual part contains the complete information about the translation. Now the question is what is the relation between the dual part and the translation vector $\boldsymbol{b}$ appearing in (1)? To answer this question, consider the following definition and theorem.

Definition 3: The radial vector $p_j^i \in R^3$ is defined as a vector whose tail is at the origin of jth coordinates and whose arrow points to the origin of ith frame, and it is projected onto the jth frame. Also, for radial vector $p_j^i = (p_1\ p_2\ p_3)^T$, its cross-product operator can be defined by a skew-symmetric matrix

$$P_j^i=\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}.$$

Namely, for any vector $a \in R^3$, $P_j^i a = p_j^i \times a$.

Theorem 1: For any 3×3 orthonormal dual matrix $\bar{A}_j^i = R_j^i + \epsilon S_j^i$ ($i, j = 0, 1, \cdots, n$ and $i \neq j$) which is derived by the dual-number transformation procedure, and for a radial vector p_j^i, its cross-product operator satisfies

$$P_j^i=S_j^iR_j^{iT}=S_j^iR_i^j. \tag{24}$$

The proof of this theorem is quite lengthy and is given in the Appendix.

It is clear from Theorem 1 that each column of $\bar{A}_j^i$ not only is a unit screw, but also has a moment vector which is the corresponding column of S_j^i. In other words,

$$s_j^i=p_j^i\times r_j^i \tag{25}$$

where s_j^i is either one of three columns of S_j^i, and r_j^i is the corresponding column of R_j^i. Therefore, each column of $\bar{A}_j^i$ uniquely determines the corresponding axis of the ith coordinate frame with respect to the jth frame in 3D space. According to Theorem 1, we have the following corollary.

Corollary 2: Under the same condition as Theorem 1, the radial vector

$$p_j^i \in \mathfrak{N}(S_i^j) \tag{26}$$

where $\mathfrak{N}(\cdot)$ is the null space of $(\cdot)$, and $S_i^j = S_j^{iT}$.

Proof: Invoking (24), we have

$$-P_j^i=P_j^{iT}=R_j^iS_i^j,$$

and since $p_j^i \times p_j^i = o$, thus

$$o=-P_j^ip_j^i=R_j^iS_i^jp_j^i.$$

Since R_j^i is an orthonormal matrix, $S_j^i p_j^i = o$ or $p_j^i \in \mathfrak{N}(S_i^j)$.

This corollary answers the question of what vector lies in the one-dimensional null space of S_i^j. It may also be interpreted as follows: because the radial vector p_j^i passes through two origins of the ith and jth coordinate frames, its moment vector about either one of the two origins is zero so that $\bar{A}_i^j p_j^i$ should be equal to $R_i^j p_j^i$ and thus $S_i^j p_j^i = o$. Therefore, Corollary 2 gives a clear geometrical meaning that all row vectors of S_i^j (or all column vectors of S_j^i) lie on a common plane which passes through the origin of jth frame and whose normal vector is colinear with p_j^i. In addition, this common plane provides an explanation of why the rank of S_i^j is always equal to 2 in Lemma 1.

Theorem 1 and Corollary 2 provide us with two approaches to finding the radial position vectors in dual-number transformation. The dual-number transformation can easily be converted to the homogeneous transformation in both symbolic and numeric forms, and vice versa, by applying these two approaches.

IV. Construction of Jacobian Matrix

For any robot with n joints, its joint velocity and the end-effector Cartesian velocity are related by the Jacobian matrix [10]. In a physical three-dimensional Cartesian space, the end-effector velocity is usually represented in terms of six independent variables: three for position and three for orientation. Therefore, the Jacobian matrix is a $6 \times n$ matrix.

Since the dual-number transformation is of line transformation which operates on dual vectors, and transforms them into new dual vectors, the derivation of the Jacobian matrix and its equation becomes more concise by using dual-number transformation than by differential homogeneous transformation.

Let $\bar{v}_j = \omega_j + \epsilon v_j$ be the dual Cartesian velocity, where ω_j and v_j are, respectively, the angular and linear velocities of jth coordinate frame with respect to the base coordinates, but projected onto their own coordinate system. Then the dual velocity has a following recursive formula [1]:

$$\begin{cases} \bar{v}_j = \bar{A}_j^{j-1}(\dot{\bar{q}}_j + \bar{v}_{j-1}), & j = 1, 2, \cdots, n \\ \bar{v}_0 = o \end{cases} \tag{27}$$

where $\bar{A}_j^{j-1}$ is the dual-number transformation from j-1st coordinate frame to jth coordinates, and $\dot{\bar{q}}_j$ is the dual joint velocity of jth joint, conventionally rotating about or sliding along j-1st z axis, i.e.,

$$\dot{\bar{q}}_j = \dot{\bar{\theta}}_j \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (\dot{\theta}_j + \epsilon \dot{d}_j) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{cases} \dot{\theta}_j \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & \text{if } j\text{th joint is revolute} \\ \epsilon \dot{d}_j \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & \text{if } j\text{th joint is prismatic} \end{cases},$$

in which $\dot{\theta}_j$ and $\dot{d}_j$ are, respectively, the angular and linear velocities of the joint j with respect to j-1st coordinate frame. For a robot having n joints, the velocity is obtained by setting $j = n$ in (27) and solving this recursive equation so that

$$\bar{v}_n = \sum_{i=1}^{n} \bar{A}_n^{i-1} \dot{\bar{q}}_i = \sum_{i=1}^{n} \bar{t}_n^{i-1}(\dot{\theta}_i + \epsilon \dot{d}_i) = \bar{J}\dot{\bar{\theta}} \tag{28}$$

where $\bar{t}_n^{i-1}$ is the third column of $\bar{A}_n^{i-1}$, $\bar{J} = (\bar{t}_n^0 \cdots \bar{t}_n^{n-1})$ is the $3 \times n$ dual matrix, and $\dot{\bar{\theta}} = (\dot{\bar{\theta}}_1 \dot{\bar{\theta}}_2 \cdots \dot{\bar{\theta}}_n)^T$ with $\dot{\bar{\theta}}_i = \dot{\theta}_i + \epsilon \dot{d}_i$. Since each $\dot{\bar{\theta}}_i$ is either $\dot{\theta}_i$ or $\epsilon \dot{d}_i$, namely, a joint has only one variable corresponding to either a revolute or a prismatic joint, then we define $\dot{\theta} = (\dot{\theta}_1 \cdots \dot{d}_i \cdots \dot{\theta}_n)^T \in R^n$, whenever ith joint is prismatic, and others are revolute. Since each $\bar{t}_n^{i-1} = r_n^{i-1} + \epsilon s_n^{i-1}$ and each term in the sum of (28) is $\bar{t}_n^{i-1}(\dot{\theta}_i + \epsilon \dot{d}_i) = r_n^{i-1}\dot{\theta}_i + \epsilon(s_n^{i-1}\dot{\theta}_i + r_n^{i-1}\dot{d}_i)$, (28) can be written in an augmented form as

$$V_n = \begin{bmatrix} v_n \\ \omega_n \end{bmatrix} = \begin{bmatrix} J_S \\ J_R \end{bmatrix} \dot{\theta} = J\dot{\theta} \tag{29}$$

where v_n and ω_n are, respectively, the dual part and real part of the dual velocity $\bar{v}_n$, i.e., $\bar{v}_n = \omega_n + \epsilon v_n$, and $J_R = (r_n^0 r_n^1 \cdots r_n^{n-1})$ and $J_S = (s_n^0 s_n^1 \cdots s_n^{n-1})$ in which r_n^{i-1} and s_n^{i-1} are the real part and dual part of $\bar{t}_n^{i-1}$ respectively. In other words, if $\bar{A}_n^{i-1} = R_n^{i-1} + \epsilon S_n^{i-1}$, then r_n^{i-1} is the third column of R_n^{i-1} and s_n^{i-1} is the third column of S_n^{i-1}, and $J = [{}^{J_S}_{J_R}]$ is known as the $6 \times n$ Jacobian matrix. Note that if the ith joint is prismatic, since $\dot{\bar{\theta}}_i = \epsilon \dot{d}_i$ in (28), then in J_S, s_n^{i-1} should be replaced by r_n^{i-1}; and in J_R, r_n^{i-1} should be replaced by the naught vector o. Equation (29) is called the Jacobian equation in robot kinematics.

Jacobian matrix J in (29) can have $n + 1$ different forms of its expression, because each r and s in J have $n + 1$ choices of projection coordinate frames, although the reference frame is fixed on the base coordinate frame and each s, as a moment vector, is always about the origin of the last (nth) coordinate frame. In other words, each r in J has $n + 1$ choices of its subscript, from 0 to n. These $n + 1$ different forms of J can be orthonormally transformed to each other, namely, for any choice of subscript k ($0 \leq k \leq n$),

$$J_{(k)} = \begin{bmatrix} R_k^n & 0 \\ 0 & R_k^n \end{bmatrix} J_{(n)} \tag{30}$$

where $J_{(n)}$ is the J of (29) and $J_{(k)}$ is the J projected onto the kth coordinates.

Intuitively, if the subscript is selected at the halfway points, i.e., the nearest integer of $n/2$, then the Jacobian matrix J has the simplest symbolic form. Let us take the PUMA 560 as an illustrative example to build the simplest form of its Jacobian matrix.

Since the PUMA 560 has six joints, and the last three joints have the coordinates frames with a same origin, we may choose the coordinates of the third link but translated onto the last common origin position, to be the projection coordinates. We call this translated coordinates the 3′-th frame. Thus according to Table I, we have

$$\bar{A}_2^{3'} = \begin{bmatrix} c_3 & -\epsilon d_3 c_3 & s_3 \\ s_3 & -\epsilon d_3 s_3 & -c_3 \\ \epsilon d_3 & 1 & 0 \end{bmatrix}$$

$$\bar{A}_1^{3'} = \begin{bmatrix} \bar{c}_{23} & -\epsilon d_3\bar{c}_{23} + \epsilon a_2 s_2 & \bar{s}_{23} \\ \bar{s}_{23} & -\epsilon d_3 \bar{s}_{23} - \epsilon a_2 c_2 & -\bar{c}_{23} \\ \epsilon a_2 s_3 + \epsilon d_3 & 1 & -\epsilon a_2 c_3 \end{bmatrix}$$

$$\bar{A}_0^{3'} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -\bar{s}_{23} & \epsilon d_3 \bar{s}_{23} + \epsilon a_2 c_2 & \bar{c}_{23} \end{bmatrix}$$

where $\bar{s}_{23} = \sin(\bar{\theta}_2 + \bar{\theta}_3) = s_{23} + \epsilon d_2 c_{23}$ and $\bar{c}_{23} = \cos(\bar{\theta}_2 + \bar{\theta}_3) = c_{23} - \epsilon d_2 s_{23}$. The dual parts of the third row of these matrices are, respectively, $(d_3\ 0\ 0)$, $(a_2 s_3 + d_3\ 0\ -a_2 c_3)$, and $(-d_2 c_{23}\ d_3 s_{23} + a_2 c_2\ -d_2 s_{23})$. Likewise, the real parts are $(0\ 1\ 0)$, $(0\ 1\ 0)$, and $(-s_{23}\ 0\ c_{23})$.

Then, these dual parts should be the corresponding columns of the upper left 3×3 submatrix S_I of J, and these real parts should be the corresponding columns of the lower left submatrix R_I of J, i.e.,

$$S_I = \begin{bmatrix} -d_2 c_{23} & a_2 s_3 + d_3 & d_3 \\ d_3 s_{23} + a_2 c_2 & 0 & 0 \\ -d_2 s_{23} & -a_2 c_3 & 0 \end{bmatrix},$$

$$R_I = \begin{bmatrix} -s_{23} & 0 & 0 \\ 0 & 1 & 1 \\ c_{23} & 0 & 0 \end{bmatrix}.$$

While the 3×3 lower right submatrix R_{II} of J also consists of three columns which are individually the real parts of third columns of $\bar{A}_{3'}^3$, $\bar{A}_{3'}^4$, and $\bar{A}_{3'}^5$. The result for the PUMA 560 robot is

$$R_{II} = \begin{bmatrix} 0 & -s_4 & c_4 s_5 \\ 0 & c_4 & s_4 s_5 \\ 1 & 0 & c_5 \end{bmatrix}.$$

Thus the matrix $J_{(3')}$ is composed by these three submatrices and a zero upper right submatrix because of zero offsets for the last three joints, i.e.,

$$J_{(3')} = \begin{bmatrix} S_I & 0 \\ R_I & R_{II} \end{bmatrix}. \tag{31}$$

After constructing $J_{(3')}$, to find $J_{(k)}$ for any k other than $3'$ becomes a simple job which is first to compute the matrix $R_k^{3'}$, then replace n in (30) by $3'$, and substitute $J_{(3')}$ and $R_k^{3'}$ into (30).

V. Derivatives of Jacobian Matrix

In robotic dynamics and control system modeling, the time-derivative of the Jacobian matrix $\dot{J}$ is often required in the equations and their solutions. The Jacobian matrix can be constructed by using dual-number transformation procedure; the question is whether we can take advantage of the dual-number transformation to derive an explicit expression of $\dot{J}$? The following theorem answers this question [9].

Theorem 2: Let $F(\theta)$ be any first-differentiable scaler, vector, or matrix function of n-dimensional vector θ. If θ is replaced by a dual vector $\theta + \epsilon\dot{\theta}$, where $\dot{\theta}$ is time derivative of θ, then

$$F(\theta + \epsilon\dot{\theta}) = F(\theta) + \epsilon\dot{F}(\theta). \tag{32}$$

Proof: According to the Taylor series expansion, we have

$$F(\theta + \epsilon\dot{\theta}) = F(\theta) + \epsilon \left(\frac{\partial F(\theta)}{\partial \theta}\right)^T (\dot{\theta} \otimes I)$$

where $\otimes$ is Kronecker product and I is an $n \times n$ identity matrix if $F(\theta)$ is an $m \times n$ matrix, or $I = 1$ if $F(\theta)$ is a vector or scalar. However,

$$\dot{F}(\theta) = \left(\frac{\partial F(\theta)}{\partial \theta}\right)^T (\dot{\theta} \otimes I). \tag{33}$$

Hence, this theorem is proven.

Because the Jacobian matrix J derived from the kinematic relation is always a first-differentiable matrix function of θ, it obviously has

$$J(\theta + \epsilon\dot{\theta}) = J(\theta) + \epsilon\dot{J}(\theta). \tag{34}$$

It means that if we know the symbolic form of J, and want to find $\dot{J}$, then we just replace each θ_i by $\theta_i + \epsilon\dot{\theta}_i$ in $J(\theta)$ ($i = 1, \cdots, n$) and extract its dual part from the dual matrix $J(\theta + \epsilon\dot{\theta})$. Let us take the upper left submatrix $S_I(\theta)$ of the Jacobian matrix J in (31) as an example for testing the validity of Theorem 2. Since

$$S_I(\theta) = \begin{bmatrix} -d_2 c_{23} & a_2 s_3 + d_3 & d_3 \\ d_3 s_{23} + a_2 c_2 & 0 & 0 \\ -d_2 s_{23} & -a_2 c_3 & 0 \end{bmatrix}$$

and c_2 is replaced by $\cos(\theta_2 + \epsilon\dot{\theta}_2) = c_2 - \epsilon\dot{\theta}_2 s_2$, s_3 is replaced by $\sin(\theta_3 + \epsilon\dot{\theta}_3) = s_3 + \epsilon\dot{\theta}_3 c_3$, c_3 is replaced by $c_3 - \epsilon\dot{\theta}_3 s_3$, and s_{23} and c_{23} are, respectively, replaced by $s_{23} + \epsilon(\dot{\theta}_2 + \dot{\theta}_3)c_{23}$ and $c_{23} - \epsilon(\dot{\theta}_2 + \dot{\theta}_3)s_{23}$, then

$$S_I(\theta + \epsilon\dot{\theta}) = S_I(\theta) + \epsilon \begin{bmatrix} d_2(\dot{\theta}_2 + \dot{\theta}_3)s_{23} & a_2\dot{\theta}_3 c_3 & 0 \\ d_3(\dot{\theta}_2 + \dot{\theta}_3)c_{23} - a_2\dot{\theta}_2 s_2 & 0 & 0 \\ -d_2(\dot{\theta}_2 + \dot{\theta}_3)c_{23} & a_2\dot{\theta}_3 s_3 & 0 \end{bmatrix}.$$

Obviously, the dual part of $S_I(\theta + \epsilon\dot{\theta})$ is the time derivative of $S_I(\theta)$.

It is seen that if the addition and the multiplication of dual numbers are implemented in computer program in a same manner as handling the complex numbers in Fortran or any other available programming language, then according to Theorem 2, the time derivative of any function could numerically be calculated.

Furthermore, in Theorem 2 the first derivative of a given function can be taken with respect to any scalar parameter other then time. For example, suppose we choose θ_i, the ith joint variable, as the scalar parameter, then for $F(\theta)$, θ will be replaced by

$$\theta + \epsilon \frac{\partial\theta}{\partial\theta_i} = \theta + \epsilon e_i, \qquad i = 1, \cdots, n \tag{35}$$

where $\boldsymbol{e}_i = (0 \cdots 0\ 1\ 0 \cdots 0)^T$ in which only the ith element is one and others are zero. Based on the proof of Theorem 2 the dual part of $\boldsymbol{F}(\boldsymbol{\theta} + \epsilon \boldsymbol{e}_i)$ should be $\partial F/\partial \theta_i$.

In robotic dynamic equations, one used to find the explicit form of kinetic energy K in terms of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$, then to substitute it into the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\boldsymbol{\theta}}}\right)-\frac{\partial K}{\partial \boldsymbol{\theta}}=\tau+\tau_g \tag{36}$$

where $\tau \in \boldsymbol{R}^n$ is the generalized joint torque and τ_g is the gravitational joint torque.

According to the analysis of robot dynamic model [11], we have the kinetic energy

$$K=\frac{1}{2}\,\dot{\boldsymbol{\theta}}^T \boldsymbol{W}\dot{\boldsymbol{\theta}} \tag{37}$$

and the generalized momentum

$$p=\frac{\partial K}{\partial \dot{\boldsymbol{\theta}}}=\boldsymbol{W}\dot{\boldsymbol{\theta}} \tag{38}$$

where $\boldsymbol{W}$ is an $n \times n$ total inertial matrix which only depends on $\boldsymbol{\theta}$. Equation (36) can then be rewitten as

$$\boldsymbol{W}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{W}}\dot{\boldsymbol{\theta}}-\frac{\partial K}{\partial \boldsymbol{\theta}}=\tau+\tau_g. \tag{39}$$

In the second term of (39), we may calculate it based on Theorem 2 as

$$\dot{\boldsymbol{W}}\dot{\boldsymbol{\theta}}=(\text{dual part of } \boldsymbol{W}(\boldsymbol{\theta}+\epsilon\dot{\boldsymbol{\theta}}))\dot{\boldsymbol{\theta}} \tag{40}$$

if the explicit form of $\boldsymbol{W}$ has been known. While for the third term of (39), each component is $\partial K/\partial \theta_i$, $i = 1, \cdots, n$, which can be found by first replacing θ_i in the explicit expression of K by $\theta_i + \epsilon$ and all other θ_j $(j \neq i)$ have no charge, and then taking the dual part.

In addition, three interesting relations may be derived based on Theorem 2. Suppose we define

$$\hat{\boldsymbol{V}}=\boldsymbol{V}+\epsilon\dot{\boldsymbol{V}} \qquad \hat{\boldsymbol{J}}=\boldsymbol{J}+\epsilon\dot{\boldsymbol{J}} \qquad \hat{\boldsymbol{\theta}}=\boldsymbol{\theta}+\epsilon\dot{\boldsymbol{\theta}},$$

then by (29),

$$\hat{\boldsymbol{V}}=\boldsymbol{J}\dot{\boldsymbol{\theta}}+\epsilon(\dot{\boldsymbol{J}}\dot{\boldsymbol{\theta}}+\boldsymbol{J}\ddot{\boldsymbol{\theta}})=(\boldsymbol{J}+\epsilon\dot{\boldsymbol{J}})(\dot{\boldsymbol{\theta}}+\epsilon\ddot{\boldsymbol{\theta}})=\hat{\boldsymbol{J}}\dot{\hat{\boldsymbol{\theta}}} \tag{41}$$

since $\epsilon^2 = 0$. Obviously, (41) is consistent with the results obtained by replacing $\boldsymbol{\theta}$ in the Jacobian equation (29) by $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta} + \epsilon\dot{\boldsymbol{\theta}}$ and applying Theorem 2. Similarly, if we define the dual kinetic energy $\hat{K} = K + \epsilon\dot{K}$, dual generalized momentum $\hat{p} = p + \epsilon\dot{p}$, and dual total inertial matrix $\hat{\boldsymbol{W}} = \boldsymbol{W} + \epsilon\dot{\boldsymbol{W}}$, then according to (37) and (38), we have

$$\hat{K}=\frac{1}{2}\,\dot{\hat{\boldsymbol{\theta}}}^T \hat{\boldsymbol{W}}\dot{\hat{\boldsymbol{\theta}}} \tag{42}$$

and

$$\hat{p}=\hat{\boldsymbol{W}}\dot{\hat{\boldsymbol{\theta}}} \tag{43}$$

by the same reason.

VI. Conclusion

Properties, lemmas, and theorems for the dual number, dual vector, and dual matrix have been discussed. The dual-number transformation procedure, based on these properties and the principle of transference, can be used for finding Jacobian matrices in robotic kinematics and their derivatives in robotic dynamics and control modeling. This approach leads to convenience in symbolic manipulation. It also provides a concise way for numeric computation if the additions and the multiplications of dual numbers are implemented in computer programs. In particular, in dealing with the line-transformation problems, the dual-number approach manifests its advantage significantly.

Appendix

Proof of Theorem 1

Based on the dual-number transformation procedure, we have

$$\bar{\boldsymbol{A}}_j^i=\boldsymbol{R}_j^i(\theta+\epsilon d,\ \alpha+\epsilon a) \tag{A1}$$

where θ, α, d, and $a \in \boldsymbol{R}^n$. Applying the Taylor series expansion to (A1), the dual part of $\bar{\boldsymbol{A}}_j^i$ should be

$$\boldsymbol{S}_j^i=\sum_{k=j+1}^{i}\left(\frac{\partial \boldsymbol{R}_j^i}{\partial \theta_k}\,d_k+\frac{\partial \boldsymbol{R}_j^i}{\partial \alpha_k}\,a_k\right), \qquad j<i. \tag{A2}$$

Note that in (A2) the subscript j is assumed to be less than superscript i. If $j > i$, the index k of the sum in (A2) is changed from $k = i + 1$ to j. Thus

$$\boldsymbol{S}_j^i\boldsymbol{R}_i^j=\sum_{k=j+1}^{i}\left(d_k\,\frac{\partial \boldsymbol{R}_j^i}{\partial \theta_k}\,\boldsymbol{R}_i^j+a_k\,\frac{\partial \boldsymbol{R}_j^i}{\partial \alpha_k}\,\boldsymbol{R}_i^j\right), \qquad j<i. \tag{A3}$$

We denote $(\partial \boldsymbol{R}_j^i/\partial \theta_k)\boldsymbol{R}_i^j = \boldsymbol{P}_{zk}$ and $\partial \boldsymbol{R}_j^i/\partial \alpha_k)\boldsymbol{R}_i^j = \boldsymbol{P}_{xk}$ for $k = j + 1$ to i. $\boldsymbol{P}_{zk}$ and $\boldsymbol{P}_{xk}$ are 3×3 skew-symmetric matrices, since $\boldsymbol{R}_j^i\boldsymbol{R}_i^j = \boldsymbol{I}$, an identity so that $(\partial \boldsymbol{R}_j^i/\partial \theta_k)\boldsymbol{R}_i^j + \boldsymbol{R}_j^i(\partial \boldsymbol{R}_i^j/\partial \theta_k) = \boldsymbol{0}$.

For each $\boldsymbol{P}_{zk}$, we may modify it as

$$\boldsymbol{P}_{zk}=\frac{\partial(\boldsymbol{R}_j^{k-1}\boldsymbol{R}_{k-1}^k\boldsymbol{R}_k^i)}{\partial \theta_k}\,\boldsymbol{R}_i^j=\boldsymbol{R}_j^{k-1}\,\frac{\partial \boldsymbol{R}_{k-1}^k}{\partial \theta_k}\,\boldsymbol{R}_k^j, \qquad j<k \leqq i \tag{A4}$$

since $\boldsymbol{R}_j^{k-1}$ and $\boldsymbol{R}_k^i$ do not contain θ_k. Furthermore,

$$\boldsymbol{P}_{zk}=\boldsymbol{R}_j^{k-1}\,\frac{\partial \boldsymbol{R}_{k-1}^k}{\partial \theta_k}\,\boldsymbol{R}_k^{k-1}\boldsymbol{R}_{k-1}^j, \qquad j<k \leqq i. \tag{A5}$$

Likewise,

$$\boldsymbol{P}_{xk}=\boldsymbol{R}_j^k\boldsymbol{R}_k^{k-1}\,\frac{\partial \boldsymbol{R}_{k-1}^k}{\partial \alpha_k}\,\boldsymbol{R}_k^j, \qquad j<k \leqq i. \tag{A6}$$

In (A5) and (A6), matrix $\boldsymbol{R}_{k-1}^k$ is the real part of the four steps dual-number transformation (22). It can be shown [1] that

$$\frac{\partial \boldsymbol{R}_{k-1}^{k}}{\partial \theta_k} \boldsymbol{R}_{k}^{k-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A7}$$

$$\boldsymbol{R}_{k}^{k-1} \frac{\partial \boldsymbol{R}_{k-1}^{k}}{\partial \alpha_k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{A8}$$

whenever $k = 1, \cdots, n$.

Denoting the skew-symmetric matrices in (A7) and (A8) as δ_z and δ_x, respectively, we have

$$\boldsymbol{P}_{zk} = \boldsymbol{R}_{j}^{k-1} \delta_z \boldsymbol{R}_{k-1}^{j}, \qquad j < k \leqq i \tag{A9}$$

$$\boldsymbol{P}_{xk} = \boldsymbol{R}_{j}^{k} \delta_x \boldsymbol{R}_{k}^{j}, \qquad j < k \leqq i. \tag{A10}$$

With careful derivation and noticing that $\boldsymbol{R}_{k-1}^{j}$ and $\boldsymbol{R}_{k}^{j}$ are the orthonormal matrices, we obtain that $\boldsymbol{P}_{zk}$ and $\boldsymbol{P}_{xk}$ are, respectively, the cross- product operators of the third column $\boldsymbol{r}_{jz}^{k-1}$ of $\boldsymbol{R}_{j}^{k-1}$ and the first column $\boldsymbol{r}_{jx}^{k}$ of $\boldsymbol{R}_{j}^{k}$, namely

$$\boldsymbol{P}_{zk} = (\boldsymbol{r}_{jz}^{k-1} \times) \tag{A11}$$

$$\boldsymbol{P}_{xk} = (\boldsymbol{r}_{jx}^{k} \times), \qquad j < k \leqq i. \tag{A12}$$

On the other hand, the radial position vector $\boldsymbol{p}_j^i$ may be expressed by means of resultant vector of all links between the jth and ith frames, i.e.,

$$\boldsymbol{p}_j^i = \sum_{k=j+1}^{i} (d_k \boldsymbol{r}_{jz}^{k-1} + a_k \boldsymbol{r}_{jx}^{k}), \qquad j < i. \tag{A13}$$

Substituting (A11) and (A12) into (A3) and comparing it with (A13), we finally obtain

$$\begin{aligned} \boldsymbol{S}_j^i \boldsymbol{R}_i^j &= \sum_{k=j+1}^{i} [d_k(\boldsymbol{r}_{jz}^{k-1} \times) + a_k(\boldsymbol{r}_{jx}^{k} \times)] \\ &= \left(\sum_{k=j+1}^{i} (d_k \boldsymbol{r}_{jz}^{k-1} + a_k \boldsymbol{r}_{jx}^{k}) \times \right) \\ &= (\boldsymbol{p}_j^i \times) = \boldsymbol{P}_j^i, \qquad j < i. \end{aligned} \tag{A14}$$

When $j > i$, the index k of the sum in (A14) is changed by $k = i + 1$ to j, and the result $\boldsymbol{S}_j^i \boldsymbol{R}_i^j = \boldsymbol{P}_j^i$ is still valid.

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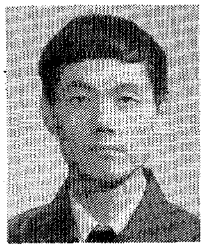

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