Distance Metrics on the Rigid-Body Motions with Applications to Mechanism Design

F. C. Park
Department of Mechanical and Aerospace Engineering,
University of California, Irvine,
Irvine, CA 92717

In this article we examine the problem of designing a mechanism whose tool frame comes closest to reaching a set of desired goal frames. The basic mathematical question we address is characterizing the set of distance metrics in SE(3), the Euclidean group of rigid-body motions. Using Lie theory, we show that no bi-invariant distance metric (i.e., one that is invariant under both left and right translations) exists in SE(3), and that because physical space does not have a natural length scale, any distance metric in SE(3) will ultimately depend on a choice of length scale. We show how to construct left- and right-invariant distance metrics in SE(3), and suggest a particular left-invariant distance metric parametrized by length scale that is useful for kinematic applications. Ways of including engineering considerations into the choice of length scale are suggested, and applications of this distance metric to the design and positioning of certain planar and spherical mechanisms are given.

1 Introduction

In this article we address the following version of the mechanism synthesis problem: given a set of goal frames in physical space, select a mechanism (from some suitably parametrized kinematic class) whose tool frame can reach this set or, if this is not possible, comes "closest" to reaching these goal frames. Assuming an inertial reference frame and length scale for physical space have been chosen, each frame can be assigned an element of the Euclidean group SE(3) (also known in the kinematics literature as the *homogeneous transformations*, or *rigid-body motions*). The problem of precisely measuring "closeness" between frames then reduces to the equivalent mathematical problem of defining a distance metric in SE(3).

Clearly any number of arbitrary distance metrics can be defined, but certain features make the metric more physically meaningful. For example, one would like the metric to be *left-invariant*: if $d: \mathrm{SE}(3) \times \mathrm{SE}(3) \rightarrow \mathbb{R}$ denotes the distance metric, then given two frames $F_1, F_2 \in \mathrm{SE}(3)$, one would like $d(F_1, F_2) = d(LF_1, LF_2)$ for all $L \in \mathrm{SE}(3)$. Physically left-invariance reflects the invariance of the metric with respect to choice of inertial frame. Similarly, one may seek *right-invariance* in the distance metric, i.e., $d(F_1, F_2) = d(F_1 R, F_2 R)$ for any fixed $R \in \mathrm{SE}(3)$. Right-invariance may be desirable in certain grasping operations, in which the measured distances should be independent of where the body-fixed frame is attached to the rigid body. Finally, since any distance metric in SE(3) combines position and orientation, one would like the metric to be *scale-invariant*: the distance between two frames should be invariant (up to a constant scaling factor) with respect to choice of length scale for physical space.

In this article we characterize the distance metrics in SE(3) using the coordinate-free methods of Riemannian geometry. The cornerstone of our approach is to regard SE(3) as a Lie group equipped with a Riemannian metric. We show that no bi-invariant distance metric (i.e., one that is both left- and right-invariant) can be constructed in SE(3), and that because no natural length scale exists for physical space, any distance metric must necessarily be parametrized by a choice of length scale. Explicit formulas for general left- and right-invariant distance metrics in SE(3) are given. We then discuss applications of the distance metrics to problems in mechanism design. A particular left-invariant distance metric suitable for kinematic applications is proposed, and ways of choosing the length scale that take into account engineering aspects of the problem are suggested. Motivated in part by the work of Ravani [9] and Bodduluri [1], we present examples in which the kinematic parameters of a mechanism are optimized so that its workspace is a best fit to the set of goal frames.

The main results put forth in this paper confirm what many kinematicians have suspected all along, namely, that distances on the Euclidean group cannot be measured in a way that is invariant with respect to choice of both fixed and moving frames. Some careful thought suggests that it cannot be otherwise. What we have done is to rigorously prove this conjecture using the methods of Riemannian geometry. The geometric approach moreover leads to a systematic way of formulating distance metrics in SE(3) that reflects engineering considerations in a natural and straightforward way. There are, of course, certain special cases where bi-invariant distance metrics can be formulated (e.g., measuring distances in SO(3), or measuring distances between two configurations

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of the same rigid body—we discuss these special cases later). However, one of our main points is that it is futile to attempt to formulate a general bi-invariant distance metric on SE(3). Rather, it would be more desirable to focus on how to formulate a sensible distance metric for the given problem, exploiting any special structures that arise, and addressing questions on left versus right invariance, choice of length scale, etc.

2 The Geometry of SE(3)

2.1 Canonical Coordinates. For our purposes it is sufficient to think of SE(3) as consisting of matrices of the form $\begin{bmatrix} \theta & b \\ 0 & 1 \end{bmatrix}$, where $\theta \in SO(3)$ and $b \in \mathbb{R}^3$. SE(3) has the structure of both a differentiable manifold and an algebraic group, and is an example of a Lie group.[1] Elements of SE(3) will alternatively be denoted by the pair (θ, b), with group multiplication understood to be $(\theta_1, b_1) \cdot (\theta_2, b_2) = (\theta_1\theta_2, \theta_1 b_2 + b_1)$. SO(3) and $\mathbb{R}^3$ are also Lie groups under matrix multiplication and vector addition, respectively, as is their Cartesian product $\mathbb{R}^3 \times SO(3)$. This latter product space should not be confused with SE(3), as group multiplication is defined differently in each case. Some other well-known examples of matrix Lie groups include Gl(n), the general linear group of $n \times n$ nonsingular matrices, and Sl(n), the special linear group of $n \times n$ nonsingular matrices with unit determinant.

Let p be a point on a matrix Lie group **G**, and $X(t)$ a smooth curve on **G** defined over some open interval of 0 such that $X(0) = p$. The derivative $\dot{X}(0)$ is said to be a *tangent vector* to **G** at p; the set of all tangent vectors at p, denoted $\mathbf{T}_p\mathbf{G}$, forms a vector space, called the *tangent space to* **G** *at* p. The tangent space at the identity $p = I$ is given a special name, called the *Lie algebra* of **G**, and denoted by a lower-case **g**. On SO(3) it is easily seen that the Lie algebra so(3) consists of the 3×3 skew-symmetric matrices: if $\theta(t)$ is a curve on SO(3) such that $\theta(0) = I$, then differentiating both sides of $\theta(t)\theta^T(t) = I$, it follows that $\dot{\theta}(0) + \dot{\theta}^T(0) = 0$, so that elements of so(3) are matrices of the form

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \triangleq [\omega] \tag{1}$$

where $\omega \in \mathbb{R}^3$. Where no confusion arises an element $[\omega] \in so(3)$ will also be written as $\omega \in \mathbb{R}^3$. Similarly, the Lie algebra of SE(3), denoted se(3), consists of the 4×4 matrices of the form $\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$ where $[\omega] \in so(3)$ and $v \in \mathbb{R}^3$. Elements of se(3) will alternatively be represented as (ω, v). Observe that so(3) and se(3) are vector spaces that can be identified with $\mathbb{R}^3$ and $\mathbb{R}^6$, respectively.

Defined on each Lie algebra is the exponential mapping into the corresponding Lie group. On matrix groups the exponential mapping corresponds to the usual matrix exponential, i.e., if A is an element of the Lie algebra, then $\exp A = I + A + A^2/2! + \ldots$ is an element of the Lie group. On so(3) and se(3) the exponential mapping is *onto*, i.e., for every $\theta \in SO(3)$ (resp., $X \in SE(3)$) there exists a $[\omega] \in so(3)$ (resp., $x \in se(3)$) such that $\exp[\omega] = \theta$ (resp., $\exp x = X$). On so(3) and se(3) explicit formulas for the exponential mapping exist.

[1]See, e.g., Boothby [2] for a background on Lie groups and Riemannian geometry.

Lemma 1 *Given* $[\omega] \in so(3)$,

$$\exp[\omega] = I + \frac{\sin\|\omega\|}{\|\omega\|} \cdot [\omega] + \frac{1 - \cos\|\omega\|}{\|\omega\|^2} \cdot [\omega]^2$$

where $\|\omega\|$ *is the Euclidean norm.*

Lemma 2 *Let* $(\omega, v) \in se(3)$. *Then*

$$\exp \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \exp[\omega] & Av \\ 0 & 1 \end{bmatrix}$$

where

$$A = I + \frac{1 - \cos\|\omega\|}{\|\omega\|^2} \cdot [\omega] + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3} \cdot [\omega]^2$$

Note that if A is an element of some Lie algebra, the set $\{e^{At} | t \in \mathbb{R}\}$ itself forms a group, in this case a subgroup of the Lie group. Such groups are called *one-parameter subgroups* of a Lie group. We shall see shortly that the one-parameter subgroups play an important role in defining distance metrics on SO(3) and SE(3).

One of the attractive features of the exponential mapping is that it provides a convenient set of local coordinates for SE(3): elements of SE(3) can be represented in a unique and continuous fashion as points in $\mathbb{R}^6$, in much the same way that Euler angles act as local coordinates for SO(3) (although Euler angles lack many of the advantages provided by the exponential map, as we shall see shortly). We now give a more precise description of these coordinates. First, it is well known that the exponential map is a homeomorphism[2] over a neighborhood of the identity element of a Lie group. On SO(3) and SE(3) this neighborhood includes essentially the entire group except for a subset of measure zero. Hence, over this neighborhood the inverse of the exponential, or *logarithm*, is a well-defined continuous mapping. Explicit formulas for the logarithm on SO(3) and SE(3) can also be derived.

Lemma 3 *Given* $\theta \in SO(3)$ *such that* $\mathrm{Tr}(\theta) \neq -1$. *Then*

$$\log \theta = \frac{\phi}{2 \sin \phi}(\theta - \theta^T)$$

where ϕ satisfies $1 + 2\cos\phi = \mathrm{Tr}(\theta)$, $|\phi| < \pi$. Furthermore, $\|\log \theta\|^2 = \phi^2$.

Lemma 4 *Suppose* $\theta \in SO(3)$ *such that* $\mathrm{Tr}(\theta) \neq -1$, *and let* $b \in \mathbb{R}^3$. *Then*

$$\log \begin{bmatrix} \theta & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} [\omega] & A^{-1}b \\ 0 & 0 \end{bmatrix}$$

where $[\omega] = \log \theta$, *and*

$$A^{-1} = I - \frac{1}{2} \cdot [\omega] + \frac{2\sin\|\omega\| - \|\omega\|(1 + \cos\|\omega\|)}{2\|\omega\|^2 \sin\|\omega\|} \cdot [\omega]^2$$

Remark 1 Lemmas 1 and 3 suggest the standard visualization of SO(3) as a ball of radius π, centered at the origin with the antipodal points identified; a point ω in the ball represents a rotation by an angle $\|\omega\|$ about the line passing from the origin through ω. Similarly, for any $\theta \in SO(3)$, the point $\log\theta$ in the solid ball represents the rotation θ. Thus, the image of the logarithm mapping on SO(3) is $\{\omega \in \mathbb{R}^3 | \|\omega\| < \pi\}$. The rotations whose traces equal -1 have a rotation angle of π, and their logarithms are points on the boundary of the solid ball. In this case $\log\theta$ can have two possible values: if $\hat{\omega}$ is a unit length eigenvector of θ associated with

[2]A *homeomorphism* is a continuous 1-1 mapping whose inverse is also continuous.

the eigenvalue 1, then a simple calculation shows that log $\theta = \pm\pi[\hat{\omega}]$.

Remark 2 From the previous lemmas and the above remark it can be seen that on SO(3) the preimage of the exponential mapping is multiple-valued: for any $\theta \in SO(3)$ such that $[\omega] = \log\theta$, $\|\omega\| \leq \pi$, it follows that $\theta = e^{[\omega]+2\pi n[\hat{\omega}]}$ for any integer n, where $\hat{\omega} = \omega/\|\omega\|$. This is akin to the situation in the complex plane, where if $e^{i\phi}$ is a point on the unit circle for some $0 \leq \phi \leq 2\pi$, then $e^{i(\phi+2\pi n)}$ corresponds to the same point for any integer n.

Remark 3 Local coordinates can be defined over some neighborhood of the identity element for general Lie groups in the same way using the exponential map; these local coordinates can also be extended by left or right multiplication to cover the entire group. Chevalley [4] calls coordinates obtained from the exponential map *canonical coordinates*, or *coordinates of the first kind*.

2.2 Riemannian Metrics on SO(3) and SE(3). A *Riemannian metric* on a differentiable manifold is a smooth assignment of an inner product to the tangent space at each point on the manifold. It is important to note the distinction between Riemannian versus distance metrics. Riemannian metrics are the means by which familiar Euclidean concepts like lengths, angles, and volumes can be extended to abstract differentiable manifolds. Recall, for example, that the length of a space curve $(x(t), y(t), z(t))$ in $\mathcal{R}^3$, where $a \leq t \leq b$, is given by $\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$. Similarly, the length of a curve $p(t)$ lying on a manifold $\mathcal{M}$, where t is defined over the range $a \leq t \leq b$, can be defined in terms of a Riemannian metric $\langle \cdot, \cdot \rangle$ as

$$L = \int_a^b \langle \dot{p}(t), \dot{p}(t) \rangle_p^{1/2} dt \tag{2}$$

where $\langle \cdot, \cdot \rangle_p$ denotes the inner product on $T_p\mathcal{M}$, the tangent space to $\mathcal{M}$ at p. In cases where there is no confusion the subscript p will be omitted from the Riemannian metric symbol.

We now consider the representation of velocities of rigid-body motions. Let $X(t) = (\theta(t), p(t))$ be a curve in SE(3) describing the trajectory of a rigid body relative to an inertial frame. The tangent vector $\dot{X}(t)$ can then be identified with an element of se(3) in one of two ways: it is easily verified that both $\dot{X}X^{-1} = (\dot{\theta}\theta^{-1}, \dot{p} - \dot{\theta}\theta^{-1}p)$ and $X^{-1}\dot{X} = (\theta^{-1}\dot{\theta}, \theta^{-1}\dot{p})$ are elements of se(3). The latter is referred to as the *body-fixed velocity* representation of $\dot{X}$, since $\theta^{-1}\dot{\theta}$ is the angular velocity of the rigid body in body-fixed frame coordinates. Similarly, $\dot{X}X^{-1}$ is known as the *space velocity* representation of $\dot{X}$, since $\dot{\theta}\theta^{-1}$ is the angular velocity of the rigid body in inertial frame coordinates.

Since any tangent vector on SE(3) can be expressed as an element of se(3) by either the space or body-fixed velocity representation, it follows that any inner product on se(3) will define two special classes of Riemannian metrics on SE(3). Specifically, let $X_1(t)$ and $X_2(t)$ denote two smooth curves in SE(3) passing through p at $t = 0$. Denote their body-fixed velocities at p by $X_1^{-1}\dot{X}_1 = (\omega_{1b}, v_{1b})$ and $X_2^{-1}\dot{X}_2 = (\omega_{2b}, v_{2b})$, respectively. Now, define an inner product on se(3) by the symmetric positive-definite quadratic form

$$Q = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \tag{3}$$

so that $\langle \dot{X}_1(0), \dot{X}_2(0) \rangle_p = \omega_{1b}^T A \omega_{2b} + \omega_{1b}^T B^T v_{2b} + v_{1b}^T B \omega_{2b} + v_{1b}^T C v_{2b}$. A Riemannian metric generated in this way is said to be *left-invariant*; if instead we had used the space velocity representation $\dot{X}_1 X_1^{-1}$ and $\dot{X}_2 X_2^{-1}$ with the quadratic form Q, the metric would then be called *right-invariant*.

The physical significance of the left- and right-invariant Riemannian metrics is that they are invariant with respect to translations of the inertial and body-fixed reference frames, respectively. That is, under a change of inertial frame the trajectory of a rigid body $X(t)$ is transformed according to $X(t) \to TX(t)$ for some constant $T \in SE(3)$. Since $(TX)^{-1}(T\dot{X}) = X^{-1}\dot{X}$ it follows that under the left-invariant metric $\langle \dot{X}(t), \dot{X}(t) \rangle = \langle T\dot{X}(t), T\dot{X}(t) \rangle$ for any $T \in SE(3)$. Similarly, under the right-invariant metric the quantity $\langle \dot{X}(t), \dot{X}(t) \rangle$ is preserved under any transformation of the form $X(t) \to X(t)T$, which amounts to a change of body-fixed frame. Note that the left- and right-invariant Riemannian metrics can be generalized in the obvious way to general matrix Lie groups.

If the left- and right-invariant metrics defined by Q are identical, the metric is then said to be *bi-invariant*. On SO(3) it is well-known that a family of bi-invariant metrics exists, of the form $Q = cI$, where c is a positive scale factor and I is the 3×3 identity matrix. It is also well-known that SE(3) has no bi-invariant metric (see, e.g., Loncaric [7]). To see this, note that the space and body-fixed velocities are related by

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ [b]\theta & \theta \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \tag{4}$$

for some $(\theta, b) \in SE(3)$. Hence, if the Riemannian metric defined by Q were bi-invariant, it must follow that

$$\begin{bmatrix} \theta & 0 \\ [b]\theta & \theta \end{bmatrix}^T \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \theta & 0 \\ [b]\theta & \theta \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \tag{5}$$

However, it can be shown that the only choice of A, B, and C for which this can hold is $A = \alpha I$, $B = \beta I$, and $C = 0$, where α, β are positive scalars, in which case Q is no longer positive definite. If we allow positive semidefinite Q as possible candidates for a metric, then the above calculation shows that there exists a two-parameter family of bi-invariant *pseudo-Riemannian* metrics.

For mechanism problems requiring inertial frame invariance, a reasonable choice of Riemannian metric is the left-invariant metric given by $A = cI$, $C = dI$, and $B = 0$ in the quadratic form Q above; here c and d denote positive scalar constants that act as scale factors for orientation and position, respectively. For this reason we shall refer to this metric as the *scale-dependent left-invariant metric on* SE(3). This particular choice of Q makes sense physically for several reasons: the metric restricted to SO(3) is bi-invariant, and the isotropy of $\mathcal{R}^3$ is also ensured. Thus, if $X(t)$ is a curve in SE(3) and $X^{-1}\dot{X} = (\omega_b, v_b)$ its body-fixed velocity, the left-invariant metric applied to the tangent vector of X is

$$\langle \dot{X}, \dot{X} \rangle = c\,\omega_b^T \omega_b + d v_b^T v_b \tag{6}$$

The same arguments can be applied for right-invariant Riemannian metrics generated by this choice of Q, although now one gives up inertial frame invariance in favor of body-fixed frame invariance. Note that in either case angular and rectilinear velocities are being combined, so that the choice of c and d determines the relative emphases on position and orientation. Since there is no natural choice for these scales, one that is suitable for the application at hand must be chosen. This point is worth emphasizing: any distance metric on SE(3) will ultimately depend on a choice of length and angle scales, because of the fundamental geometry of SE(3). Later we suggest some ways of choosing scales that are appropriate for the given class of tasks.

2.3 SO(3) and SE(3) as Metric Spaces. Before constructing the class of left- and right-invariant distance metrics on SE(3), we first prove that no bi-invariant $\mathcal{C}^2$ distance metric exists on a Lie group that does not admit a bi-invariant Riemannian metric. Recall that a distance metric $d(\cdot, \cdot)$ on a Lie group $\mathbf{G}$ is bi-invariant if $d(x, y) = d(pxq, pyq)$ for all $p, q \in \mathbf{G}$. A general distance metric need not even be differentiable at points $(x, y) \in \mathbf{G} \times \mathbf{G}$ where $x = y$. In the statement of the following theorem a $\mathcal{C}^2$ distance metric is understood to mean $\mathcal{C}^2$ everywhere except over the set $\{(x, y) \in \mathbf{G} \times \mathbf{G} | x = y\}$.

Theorem 1 *Let* $\mathbf{G}$ *be a Lie group that does not admit a bi-invariant Riemannian metric. Then there exists no bi-invariant* $\mathcal{C}^2$ *distance metric in* $\mathbf{G}$.

Proof: Suppose $d: \mathbf{G} \times \mathbf{G} \to \mathbb{R}$ is a $\mathcal{C}^2$ bi-invariant distance metric in $\mathbf{G}$ (i.e., given $x, y \in \mathbf{G}$, $d(x, y) = d(pxq, pyq)$ for any $p, q \in \mathbf{G}$). For some sufficiently large integer k the map $d^k(e, \cdot): G \to \mathbb{R}$, where $e \in \mathbf{G}$ is the identity, will be $\mathcal{C}^2$ at e. Denote this map by $f: \mathbf{G} \to \mathbb{R}$, i.e., $f(x) = d^k(e, x)$. From the bi-invariance of $d(\cdot, \cdot)$ it is clear that $f(x) = f(pxp^{-1})$ for any $p \in \mathbf{G}$. The proof now proceeds by constructing a bi-invariant Riemannian metric on $\mathbf{G}$ in terms of f, which leads to a contradiction. Let H_p be the Hessian of f at $T_p\mathbf{G}$ (see, e.g., Milnor [8]). Since $f \geq 0$, and $f(x) = 0$ if and only if $x = e$, H_e is symmetric positive-definite and defines a Riemannian metric on $\mathbf{G}$. Moreover, since $f(x) = f(pxp^{-1})$ it follows that

$$H_e(v, w) = H_e(pvp^{-1}, pwp^{-1})$$

for any $p \in \mathbf{G}$. However, this inner product on $T_e\mathbf{G}$ extends to a bi-invariant Riemannian metric on $\mathbf{G}$, contradicting our hypothesis. □

The above theorem shows that it is not possible to define a smooth distance metric in SE(3) that is invariant with respect to both left and right translations. We now show how distance metrics on Riemannian manifolds can be induced from the Riemannian metric; this is a standard construction whose description is available in any Riemannian geometry text.

The length of a curve on a Riemannian manifold $\mathfrak{M}$ can be defined in terms of the Riemannian metric as follows. Given a curve $X: [a, b] \to \mathfrak{M}$, with Riemannian metric $\langle \cdot, \cdot \rangle$, its length is defined as

$$L = \int_a^b \langle \dot{X}(t), \dot{X}(t) \rangle^{1/2} dt \tag{7}$$

By this definition the length of the curve is invariant under reparametrizations of $[a, b]$; L can therefore be regarded as the arc-length of the curve. $\mathfrak{M}$ then admits the structure of a metric space, in which the distance between two points X_1 and X_2 is defined as the infimum of the lengths of piecewise-differentiable curves from X_1 to X_2. Unfortunately the square root in the integrand complicates the Euler-Lagrange equations for the length functional. It is more convenient to consider the *energy* functional

$$E = \int_a^b \langle \dot{X}(t), \dot{X}(t) \rangle dt \tag{8}$$

Critical points of E are known as *geodesics*, and it is well-known that the geodesics are also critical points of L. While E is clearly dependent upon the parametrization of the curve, it turns out that the geodesics are automatically parametrized proportional to arc-length—said another way, if $X(t)$ is a geodesic, then $\langle \dot{X}(t), \dot{X}(t) \rangle$ is constant along the entire length of $X(t)$. In short, the minimum-length curve between two points on a Riemannian manifold (i.e., the curve minimizing L) corresponds to the minimum-length geodesic[3] (i.e., the curve minimizing E) between these two points. For this reason the minimum-length curves are also referred to as the *minimal geodesics*.

To regard SO(3) and SE(3) as metric spaces the corresponding minimal geodesics on these Lie groups need to be determined. The geodesics on SO(3) (with respect to the bi-invariant metric) are given by left and right translations of the one-parameter subgroups $e^{[\omega]t}$, where $[\omega] \in so(3)$ and $t \in \mathbb{R}$ (see, e.g., Boothby [2]). While in general there may exist more than one minimal geodesic between two points on an arbitrary Riemannian manifold (consider, for example, two antipodal points on the sphere), on SO(3) the minimal geodesics are especially easy to identify in terms of the canonical coordinates. Specifically, the inverse images of the one-parameter subgroups correspond to straight lines in $\mathbb{R}^3$ passing through the origin; this is immediate from the solid ball analogy of SO(3). Moreover, the length of the minimal geodesic is especially simple to compute:

Theorem 2 *Let* $\theta_1, \theta_2 \in$ SO(3). *Then the distance* $L = d(\theta_1, \theta_2)$ *induced by the standard bi-invariant metric on* SO(3) *is*

$$d(\theta_1, \theta_2) = \| \log (\theta_1^{-1}\theta_2) \|$$

where $\| \cdot \|$ *denotes the standard Euclidean norm.*

Proof: SO(3), being a compact simply-connected Lie group, has a bi-invariant Riemannian metric corresponding to the negative of its Killing form; it is well-known that the geodesics of Lie groups with bi-invariant metric are formed by left- and right-translations of the one-parameter subgroups. Let $\langle \cdot, \cdot \rangle$ denote the standard bi-invariant Riemannian metric on SO(3). Then given a curve $\theta: [0, 1] \to$ SO(3), with $\theta(0) = \theta_1$ and $\theta(1) = \theta_2$,

$$\begin{aligned} \langle \dot{\theta}, \dot{\theta} \rangle^{1/2} &= \| \theta^{-1}\dot{\theta} \| \\ &= \sqrt{\frac{1}{2}\mathrm{Tr}\left((\theta^{-1}\dot{\theta})(\theta^{-1}\dot{\theta})^T \right)} \end{aligned}$$

The minimal geodesic from θ_1 to θ_2 is then $\theta(t) = \theta_1 e^{[\omega]t}$, where $[\omega] = \log(\theta_1^{-1}\theta_2)$. Its length is therefore $d(\theta_1, \theta_2) = \int_0^1 \langle \dot{\theta}, \dot{\theta} \rangle^{1/2} dt = \sqrt{1/2\mathrm{Tr}([\omega][\omega]^T)} = \|\omega\|$ as claimed. □

Remark 4 A simple calculation shows that this distance measure is invariant with respect to both left and right translations; that is, $d(T\theta_1, T\theta_2) = d(\theta_1 T, \theta_2 T) = d(\theta_1, \theta_2)$ for any $T \in$ SO(3). Also, from the solid ball analogy of SO(3) there exists two minimal geodesics when $\mathrm{Tr}(\theta_1^{-1}\theta_2) = -1$, given in canonical coordinates by the two line segments from the origin to the two antipodal points representing $\log(\theta_1^{-1}\theta_2)$. Clearly the lengths of the two geodesics are identical, so that the distance formula is valid for either value of $\log(\theta_1^{-1}\theta_2)$.

The situation in SE(3) is unfortunately more complicated, primarily because the one-parameter subgroups e^{At}, $A \in$ se(3) and $t \in \mathbb{R}$, are no longer geodesics on SE(3). However, the geodesics can be obtained from the geodesics on the product space $\mathbb{R}^3 \times$ SO(3) as follows. To determine the minimal geodesics (with respect to the scale-dependent left-invariant Riemannian metric) between two points (θ_1, b_1), $(\theta_2, b_2) \in$ SE(3), we first determine the minimal geodesic between θ_1 and θ_2 on SO(3), denoted $\theta^*(t)$, and the minimal geodesic in $\mathbb{R}^3$ between b_1 and b_2, denoted $b^*(t)$; note that $b^*(t)$ is simply the straight line connecting b_1 and b_2, and $\theta^*(t)$ is found with respect to the usual bi-invariant metric as before. The minimal geodesic on SE(3) is then $(\theta^*(t), b^*(t))$, and the geodesic distance on SE(3) is given by the following simple formula:

Theorem 3 *Let* $X_1 = (\theta_1, b_1)$ *and* $X_2 = (\theta_2, b_2)$ *be two points in* SE(3). *Then the distance* $L = d(X_1, X_2)$ *induced by*

[3]There is another technical condition that the manifold must be *geodesically complete* in order for this to be true; since the manifolds considered in this paper all satisfy this condition, we shall not dwell on this point any further.

the scale-dependent left-invariant metric on SE(3) *is*

$$d(X_1, X_2) = \sqrt{c\|\log(\theta_1^{-1}\theta_2)\|^2 + d\|b_2 - b_1\|^2}$$

where $\|\cdot\|$ *denotes the Euclidean norm.*

Proof: Denote the left-invariant metric on SE(3) by $\langle \cdot, \cdot \rangle$. Now if ∇ is the Levi-Civita connection associated with $\langle \cdot, \cdot \rangle$ (see, e.g., Gallot et al. [5] for an introduction to connections), then any curve $\mathcal{C}(t)$ is a geodesic if and only if $\nabla_{\dot{\mathcal{C}}}\dot{\mathcal{C}} = 0$. However, for any two left-invariant vector fields X and Y the Levi-Civita connection is defined by

$$\nabla_X Y = \frac{1}{2}\{[X, Y] - \mathrm{ad}_X^*(Y) - \mathrm{ad}_Y^*(X)\}$$

where ad_X is the linear transformation corresponding to the adjoint operator on the Lie algebra, and ad_X^* is its adjoint (see e.g., Cheeger and Ebin [3]). Therefore the one-parameter subgroups are geodesics if and only if $\mathrm{ad}_X^*(X) = 0$ for all X, which in fact holds if $\langle \cdot, \cdot \rangle$ were bi-invariant. We now show that $\mathrm{ad}_X^*(X) \neq 0$. If $X = (\omega, v) \in \mathrm{se}(3)$, then a simple calculation shows that $\mathrm{ad}_X : \mathrm{se}(3) \to \mathrm{se}(3)$, in terms of the standard basis on se(3), is given by

$$\mathrm{ad}_X = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}$$

and $\mathrm{ad}_X^*(X) = (0, v \times w)$, which is nonzero.

Now define $\pi : \mathrm{SE}(3) \to \mathrm{SO}(3) \times \mathbb{R}^3$ to be the projection map $(\theta, b) \to (\theta, b)$. Let the Riemannian metric on $\mathrm{SO}(3) \times \mathbb{R}^3$ be the product of the bi-invariant metric on SO(3) and the Euclidean metric on $\mathbb{R}^3$, and consider the scale-dependent left-invariant Riemannian metric on SE(3). The map π is a smooth covering map and a local isometry, or a Riemannian covering map. Since Riemannian covering maps simultaneously project and lift geodesics to geodesics (see, for example, Gallot et al. [5]), it follows that the geodesics on SE(3) are the liftings of the geodesics on $\mathrm{SO}(3) \times \mathbb{R}^3$. □

Because the Riemannian metric is only left-invariant, the above distance measure on SE(3) is invariant with respect to only left-translations: $d(X_1, X_2) = d(TX_1, TX_2)$ for any $T \in \mathrm{SE}(3)$, so that in particular the distance between X_1 and X_2 is the same as the distance between I and $X_1^{-1}X_2$. In general, however, $d(X_1, X_2) \neq d(X_1T, X_2T)$ for $T \in \mathrm{SE}(3)$. Of course, use of the right-invariant Riemannian metric would lead to a distance measure that was invariant with respect to right rather than left translations. The explicit formula for the distance metric induced by the scale-dependent right-invariant Riemannian metric is given by the following:

Theorem 4 *Let* $X_1 = (\theta_1, b_1)$ *and* $X_2 = (\theta_2, b_2)$ *be two points in* SE(3). *Then the distance* $L = d(X_1, X_2)$ *induced by the scale-dependent right-invariant metric on* SE(3) *is*

$$d(X_1, X_2) = \sqrt{c\|\log(\theta_1\theta_2^T)\|^2 + d\|b_1 - \theta_1\theta_2^T b_2\|^2}$$

where $\|\cdot\|$ *denotes the Euclidean norm.*

The proof is analogous to that of Theorem 3. Note that in both the right- and left-invariant distance metrics the bi-invariance of SO(3) and the isotropy of $\mathbb{R}^3$ are preserved. The question of which metric is more suitable necessarily depends on the application. For example, in mechanism positioning problems inertial frame invariance is generally regarded as being more important, whereas for certain grasping problems in which a natural inertial frame already exists right-invariance may be more desirable. Similar statements can be made about the choice of scale factors c and d: since no natural length scale exists for physical space, engineering considerations typically must be applied to determine a suitable set of scale factors. An alternative way of introducing engineering considerations into the problem is to replace the choice of scale factors with a single rigid body representing the manipulated object. This method, it turns out, defines a distance metric on the configuration space of a given rigid body (as opposed to a distance metric on SE(3)), and can be interpreted as taking the average of our proposed distance metric over the rigid body. In the following section these and other issues are further discussed via examples.

3 Applications to Mechanism Design

The workspace fitting problem of mechanism design can now be stated as follows. Let $\mathfrak{M}$ denote the joint-space manifold of the mechanism in question, $\mathcal{A}$ its space of kinematic parameters, and $f: \mathfrak{M} \times \mathcal{A} \to \mathrm{SE}(3)$ the forward kinematic map. Denote local coordinates on $\mathfrak{M}$ and $\mathcal{A}$ by x and a, respectively, so that $f = f(x, a)$. Given a set of desired goal frames $\{X_1, X_2, \ldots, X_p\} \subseteq \mathrm{SE}(3)$, the general form of the workspace fitting problem can now be stated as finding the kinematic parameters a and the location of the base and tool frames, $B, T \in \mathrm{SE}(3)$, that minimize

$$J(a, B, T) = \sum_{i=1}^{p} \min_x d(Bf(x, a)T, X_i) \qquad (9)$$

We now consider some examples of the workspace fitting problem to illustrate our design methodology.

Example 1. In this example we illustrate how the choice of length scale affects the solution to mechanism design problems, by determining the base location of a $1R$, unit link length open chain with tool frame as shown (see Fig. 1). The two goal frames are

Table 1

Frame	x_d	y_d	θ_d
1	0.0	1.0	0°
2	0.0	−1.0	0°

Using the scale-dependent left-invariant distance metric with $c = 0$ and $d = 1$ the optimal base location is the origin —the mechanism is able to reach both goal positions (0, 1) and (0, −1), with orientation errors ignored since $c = 0$. As $d/c \to 0$ the base location asymptotically approaches (−1, 0) along the x-axis, with numerical difficulties encountered as d/c becomes very small. Of course, when $d = 0$ no unique solution exists; any base location will result in a zero total distance, since the mechanism can always achieve the desired goal frame orientations. The shifting of the optimal base location along the x-axis intuitively makes sense, since we are increasingly emphasizing orientation accuracy over position accuracy as $d/c \to 0$.

Example 2. Now consider the problem of finding the optimal base location of a $2R$ planar open chain, given 9 goal frames in the plane. Suppose each link is of unit length, and that the scale parameters are set to $c = d = 1$. The nine desired goal frames of the end-effector are given by (x_d, y_d, θ_d) in the table below. The base location that minimizes the distance between the workspace and the desired frames is $(x, y) = (0.583, 0.584)$. The points in the workspace closest to the goal frames are given by (x_a, y_a, θ_a) below:

Table 2

Frame	x_d	y_d	θ_d	x_a	y_a	θ_a
1	0.0	-2.0	−90°	0.17	-1.37	−94.0°
2	1.0	-1.0	−90°	1.09	-1.10	−96.8°
3	2.0	0.0	0°	2.10	-0.17	10.1°
4	1.0	1.0	0°	0.88	1.13	−5.85°
5	1.0	1.0	90°	1.16	0.81	98.2°
6	0.0	2.0	90°	-0.11	2.06	84.2°
7	3.0	0.0	0°	2.53	0.14	−3.83°
8	0.0	3.0	90°	0.18	2.52	98.2°
9	2.0	2.0	45°	1.99	2.00	46.0°

Figure 2 shows the optimal placement of the planar chain relative to the goal frames. The three depicted configurations of the chain are the locations where the distance is minimized between the desired and achieved end-effector frames for frames 2, 6, and 9.

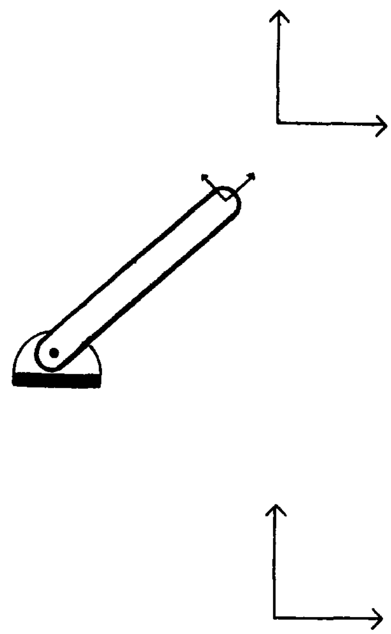

Fig. 1 Finding the optimal base location for a 1*R* open chain

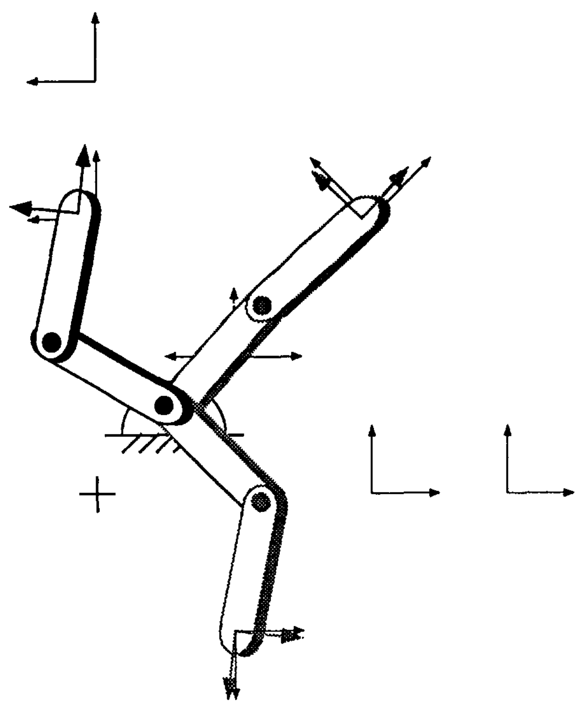

Fig. 2 Finding the optimal base location for a 2*R* planar chain

Example 3. In this example we determine the optimal joint-axis direction for the base link of a $2R$ spherical mechanism, given seven goal frames. The link twists are set to 90 degrees, and the seven desired goal frames are specified by their *x-y-z* Euler angles $(\theta_d, \phi_d, \psi_d)$ in the table below. The optimal direction for the base joint axis (measured in the inertial frame) is $(-0.113, 0.966, 0.231)$. The points in the workspace closest to the seven goal frames are given by $(\theta_a, \phi_a, \psi_a)$ below:

Table 3

Frame	θ_d	ϕ_d	ψ_d	θ_a	ϕ_a	ψ_a
1	90°	0°	0°	92.78°	−3.149°	−1.175°
2	130°	10°	−65°	126.4°	−0.176°	−59.99°
3	90°	0°	90°	100.4°	9.721°	90.42°
4	125°	30°	−10°	128.3°	25.74°	−8.527°
5	145°	25°	−15°	142.5°	29.61°	−18.96°
6	105°	35°	55°	99.08°	34.41°	55.92°
7	110°	5°	0°	106.1°	8.982°	0.035°

Distances in this example are measured only in SO(3), which, recall, has a natural (bi-invariant) Riemannian metric. Therefore, for purely spherical mechanisms the solutions are independent of the scale parameter c.

The first two examples require a choice of scale factors c and d, and in general the outcome will depend on this choice. It should be emphasized that this scale dependence arises not from any idiosyncracies of the design algorithm, but rather from the underlying geometry of SE(3); *any* general distance metric on SE(3) will have this feature. Engineering considerations are therefore necessary to determine an appropriate set of scales. One way of selecting values for c and d relies on an *a priori* assumption about a mechanism's Cartesian workspace volume. For example, if one wishes to weight the orientation and position equally in some sense, c and d can be chosen such that the position and orientation volumes of the resulting workspace are the same. Since the volume of SO(3) with respect to the bi-invariant volume form is $8\pi^2$ (where $c = 1$), d can be chosen such that the Cartesian workspace volume is also $8\pi^2$. Because one usually has some idea about the extent of a mechanism's workspace, the above method is a practical and intuitively straightforward means of including workspace constraints into the mechanism design problem.

A somewhat different approach to addressing the issue of scale selection is to consider the size and shape of the typical manipulated object rather than the mechanism's workspace.

If a mechanism is required to manipulate only a certain rigid object (e.g., a grappling device) then the goal frames can be replaced instead by goal configurations of the object. A distance metric on the configurations of the rigid object can now used to determine a mechanism design. This object distance metric, originally proposed by Kazerounian and Rastegar [6], has the form

$$d_{avg} = \frac{\int_{\text{object}} \|x - x'\| dV}{\int_{\text{object}} dV} \tag{10}$$

where x is a point on the object in the original configuration, x' is the corresponding point in the new configuration, and $\|\cdot\|$ is some suitably defined norm in $\mathbb{R}^3$ that is invariant under Euclidean transformations. This object distance metric can be interpreted within our geometric framework as the normalized average of the left-invariant distance metric (with d set to zero) when the body-fixed frame is attached at all points making up the rigid object. One of the attractive features of this object distance metric is that it is invariant with respect to both inertial and body-fixed reference frames, and also does not require an explicit selection of scale factors for positions and orientations. These advantages are acquired via the identification of some specific rigid object on which the entire mechanism design process is based.

Strictly speaking, the object distance metric is not a distance metric on SE(3), but rather a means of measuring distances between two configurations of the same rigid body. One of the difficulties with using the object distance metric for workspace fitting is the excessive computational requirement for even simple objects; even with polyhedral approximations of the rigid object, computing the above integrals is time-consuming. More fundamentally, a standard rigid object may not exist for a given mechanism synthesis problem, in which case ad hoc choices based on engineering considerations will have to be made. The choice of rigid object in defining the object distance metric can even be viewed in some sense as a means of choosing scale factors c and d. Hence, for mechanism design problems in which the object to be manipulated can be clearly identified, the synthesis procedure based on the object distance metric is, although computationally demanding, a physically meaningful choice. For synthesis problems in which it is more reasonable to bound the mechanism's Cartesian workspace, one can choose the scale factors c and d to reflect this bound (one method has been suggested above), and then apply the left-invariant distance metric on SE(3).

One way in which to account for the inertial properties of the object being manipulated is to set Q in Eq. (3) to the object's generalized inertia. Specifically, if Q is the generalized inertia of the object expressed in body-fixed frame coordinates, then the distances between frames can be defined as the length of the minimal geodesics in terms of this metric. As is well known, obtaining these geodesics require the solution of Euler's equations. Therefore, this method of computing distances will also be computationally demanding.

4 Conclusion

In this paper we have formulated, using ideas from Riemannian geometry and Lie theory, a class of translation-invariant distance metrics in SE(3). We have shown that no smooth distance metric exists in SE(3) that is invariant with respect to both left and right translations, but that metrics can be constructed that are invariant with respect to one or the other. Explicit closed-form formulas for these distance metrics are derived. These metrics not only preserve the isotropy of physical space, but also reflect the fact that no natural length scale for physical space exists. Also, when restricted to SO(3) explicit formulas for bi-invariant distance metrics are provided. One of the interesting related results of our analysis is that screw motions are not geodesics with respect to the translation-invariant Riemannian metrics as commonly believed.

Applications of this class of distance metrics to mechanism design problems have been presented. Because physical space does not admit a natural length scale, some ways of choosing scale factors for the distance metric have been presented that take into account engineering considerations. These methods rely on some *a priori* bound for the Cartesian workspace of the mechanism. The object distance metric of Kazerounian and Rastegar [6] is also interpreted within our geometric framework, and the circumstances under which each of the various metrics are more suitable are discussed. The selection of an arbitrary object in constructing the object distance metric, and the choice of scale factors in the distance metric in SE(3), appear to be somehow related, and investigating these connections might lead to computationally efficient formulas for the object distance metric. Fundamentally, our results should be useful for kinematic applications in which the notion of SE(3) as a metric space is required.

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