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A Global Performance Index for the Kinematic Optimization of Robotic Manipulators

In this paper, a novel performance index for the kinematic optimization of robotic manipulators is presented. The index is based on the condition number of the Jacobian matrix of the manipulator, which is known to be a measure of the amplification of the errors due to the kinematic and static transformations between the joint and Cartesian spaces. Moreover, the index proposed here, termed global conditioning index (CGI), is meant to assess the distribution of the aforementioned condition number over the whole workspace. Furthermore, the concept of a global index is applicable to other local kinematic or dynamic indices. The index introduced here is applied to a simple serial two-link manipulator, to a spherical three-degree-of-freedom serial wrist, and to three-degree-of-freedom parallel planar and spherical manipulators. Results of the optimization of these manipulators, based on the GCI, are included.

1 Introduction

Several criteria have been used in the past for the design of robotic manipulators. Most of the robots currently in use have been designed considering kinematic invertibility as a constraint, i.e., requiring that the solution to the inverse kinematic problem be available in closed-form. Many authors (Gupta and Roth, 1982; Cwiakala and Lee, 1985; Kohli and Spanos, 1985; Lin and Freudenstein, 1986; Gupta, 1986; Kumar and Patel, 1986) have also analyzed the workspace of manipulators and have sometimes used it as a design criterion. Other authors (Vijaykumar et al., 1986; Yang and Lai, 1985; Yoshikawa, 1985) have investigated the possibility of defining dexterity or manipulability indices which could be used for optimization. A review of these is given in Klein and Blaho (1987).

The recent development of numerical algorithms (Tsai and Morgan, 1985; Takano, 1985; Angeles, 1985), capable of kinematically inverting serial manipulators of arbitrary architecture, allows designers to relax the aforementioned invertibility constraint, thereby easing the introduction of new design criteria.

The condition number of the manipulator Jacobian matrix has attracted the attention of some researchers (Salisbury and Craig, 1982; Angeles and Rojas, 1987; Angeles and López-Cajún, 1988). In fact, the condition number of a matrix is used in numerical analysis to estimate the error generated in the solution of a linear system of equations by the error on the data (Strang, 1976). Hence, when applied to the Jacobian matrix, the condition number will give a measure of the accuracy of (*a*) the Cartesian velocity of the end effector that is produced by the joint rates calculated from Jacobian inversion; and (*b*) the static load—force and moment—acting on the end effector upon measurements by torque cells at the joint axes. The condition number has already been used for the kinematic optimization of closed-loop manipulators (Gosselin and Angeles, 1988 and 1989; Stoughton and Kokkinis, 1987).

The said condition number, which can be regarded as a measure of the Jacobian invertibility, is of great interest for the planning of optimum trajectories of given robots. However, for the problem at hand, i.e., the optimization of the kinematic design of a manipulator, one may be interested in an index that represents a global property of the manipulator. The performance index presented here, which is termed the global conditioning index (GCI), is based on the distribution of the condition number of the Jacobian matrix over the entire manipulator workspace. It is thus a measure of the Jacobian invertibility over the whole workspace. After having defined this index, we will apply it to four different cases. The first two cases consist of a simple planar, serial, two-link and a spherical, serial, three-link manipulators which can be optimized in closed form. The last two cases are concerned in turn with the optimization of a planar and a spherical three-degree-of-freedom parallel manipulator, whose optimization requires a numerical procedure. The results obtained for the spherical parallel manipulator are compared with the results obtained when maximizing its workspace.

2 Definition of the Global Conditioning Index

The Jacobian matrix of a serial-type manipulator is defined as the matrix representing the transformation mapping the joint rates into the Cartesian velocities. This transformation is written as:

$$\mathbf{J}\dot{\theta} = \dot{\mathbf{x}} \tag{1}$$

where $\dot{\theta}$ is the vector of joint rates and $\dot{\mathbf{x}}$ is the vector of

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Cartesian velocities. However, since in the case of closed-loop manipulators the roles of the direct and inverse kinematic problems are exchanged (Hunt, 1983), it is more convenient to define the Jacobian matrix, for these manipulators, in terms of the inverse transformation, i.e.:

$$\mathbf{K}\dot{\mathbf{x}} = \dot{\theta} \tag{2}$$

The accuracy of the control of the manipulator is dependent on the condition number of the Jacobian matrix (Salisbury and Craig, 1982; Angeles and Rojas, 1987) as explained above. This number is to be kept as small as possible, the smallest value that can be attained being unity, which is a value associated with isotropic matrices. The condition number of the manipulator is defined as that of its Jacobian, namely

$$\kappa = \|\mathbf{J}\| \ \|\mathbf{J}^{-1}\| \tag{3}$$

where $\|\cdot\|$ denotes any norm of its matrix argument. In this paper, the following frame-invariant norm is adopted throughout:

$$\|\mathbf{J}\| = \sqrt{\operatorname{tr}(\mathbf{J}\mathbf{W}\mathbf{J}^T)} \tag{4}$$

$\mathbf{W}$ being defined as $w\mathbf{1}$ where $w = 1/n$, and n is the dimension of the square matrix $\mathbf{J}$. Of course, the same definition applies to $\mathbf{K}$. Other definitions for the norm of a matrix could be used as well. For instance, a definition of the form

$$\|\mathbf{A}\| = \max_{x \neq 0} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}$$

would be equally valid. Moreover, if the foregoing vector norms are understood to be the Euclidean norm, then this definition will lead to a condition number measured as the ratio of the largest to the smallest singular value of $\mathbf{J}$. These two definitions of the condition number being frame-invariant, they are equivalent.

It is important to notice that, since the Jacobian is configuration-dependent, its condition number is a local property of the manipulator and therefore bears information on the accuracy of the control in this particular configuration only. This criterion was used in Gosselin and Angeles (1988 and 1989), where the condition number was minimized over the space of manipulator parameters together with the space of configurations. Isotropic configurations were then isolated and the corresponding designs were termed isotropic. However, since isotropy is a property of a limited subset of the workspace, either a curve or a surface within the manipulator's workspace (Salisbury and Craig, 1982; Angeles and Rojas, 1987), it does not guarantee, in general, that the overall conditioning of the manipulator is optimum.

To obtain a measure of the global behavior of the manipulator condition number, the following global conditioning index η is proposed:

$$\eta = \frac{A}{B} \tag{5}$$

where

$$A = \int_W \left(\frac{1}{\kappa}\right) \mathrm{d}W \tag{6}$$

and

$$B = \int_W \mathrm{d}W \tag{7}$$

in which κ is the condition number at a particular point of W, the manipulator's workspace, and the denominator B is the volume of the workspace. The reciprocal of the condition number has been used for it is better behaved than κ itself over the overall workspace. In fact, it is bounded as follows:

$$0 \leq \left(\frac{1}{\kappa}\right) \leq 1 \tag{8}$$

which thus produces a bounded performance index, i.e.,

$$0 < \eta < 1 \tag{9}$$

An alternative definition of A can also be given as:

$$A = \int_W \left(\frac{1}{\kappa}\right)^2 \mathrm{d}W \tag{10}$$

The squaring of the reciprocal of the condition number is not necessary here, since the condition number is a positive definite quantity. However, the definition given in equation (10) can sometimes simplify the algebra since the condition number, as defined in equation (3), is given by the square root of a product. Both definitions are acceptable.

It is also pointed out that the concept of a global index, as defined in equations (5–7) could also be used with another argument than the condition number. Hence the global optimization of other kinematic or dynamic properties would be possible by replacing the factor $(1/\kappa)$ by the desired local index in equation (6). The main point here is to be able to obtain a global property of the manipulator.

The index defined in equation (5) is to be maximized over the space of manipulator parameters. Thus, the closer to unity the index is, the better the overall behavior of the condition number and hence, of the manipulator. The normality condition necessary for a stationary value of η is given by:

$$\frac{\partial \eta}{\partial \mathbf{h}} = 0 \tag{11}$$

where $\mathbf{h}$ is defined as the vector containing the Hartenberg-Denavit parameters (Hartenberg and Denavit, 1964) of the manipulator, i.e.:

$$\mathbf{h} = [a_1, b_1, \alpha_1, \ldots, a_n, b_n, \alpha_n]^T \tag{12}$$

Application of this condition to equations (5–7) leads to the normality condition given below:

$$\int_W \frac{\partial}{\partial \mathbf{h}} \left(\frac{1}{\kappa}\right) \mathrm{d}W - \eta \frac{\partial B}{\partial \mathbf{h}} = 0 \tag{13}$$

The integration over the workspace can be performed in the Cartesian space providing that its boundary is known. This will be done in the examples presented here which involve parallel manipulators. However, for current serial manipulators, the workspace is not always known in the Cartesian space and is, in general, much easier to describe in the joint space. If we want the GCI to still be a measure based on the Cartesian space metric, the transformation from one coordinate system to the other can be introduced in the integral where we have to include the absolute value of the determinant of the Jacobian matrix, Δ. The normality condition, equation (13), then becomes:

$$\int_R \frac{\partial}{\partial \mathbf{h}} \left(\frac{1}{\kappa}\right) |\Delta| \mathrm{d}\theta_n \ldots \mathrm{d}\theta_2 \mathrm{d}\theta_1 - \eta \frac{\partial B}{\partial \mathbf{h}} = 0 \tag{14}$$

where R denotes the workspace (in joint coordinates), and each of A and B are computed accordingly, i.e., as:

$$A = \int_R \left(\frac{1}{\kappa}\right) |\Delta| \mathrm{d}\theta_n \ldots \mathrm{d}\theta_2 \mathrm{d}\theta_1 \tag{15}$$

and

$$B = \int_R |\Delta| \mathrm{d}\theta_n \ldots \mathrm{d}\theta_2 \mathrm{d}\theta_1 \tag{16}$$

It is pointed out that an alternative definition of the GCI based on the joint space metric would take away the determinant of the Jacobian from the above integrals. This GCI would have

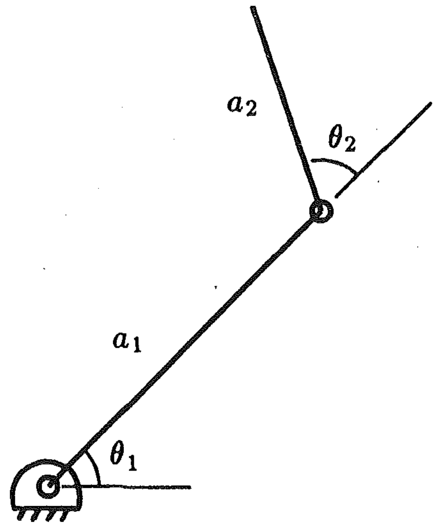

Fig. 1 Planar two-link manipulator

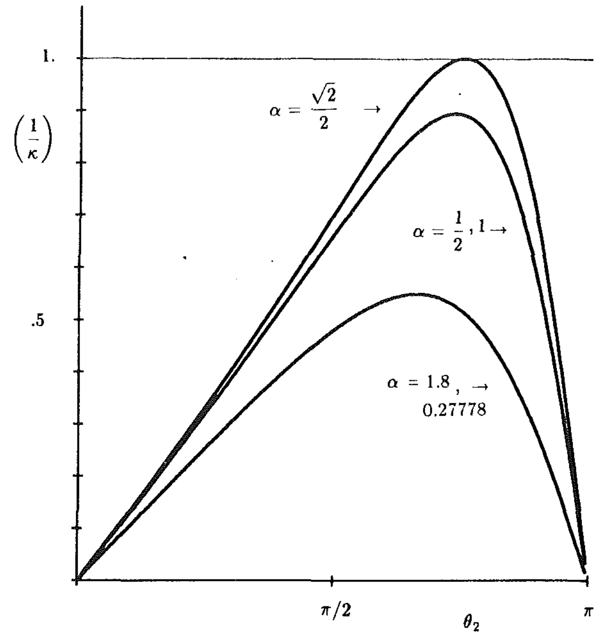

Fig. 2(*a*) Reciprocal of the condition number of the planar two-link manipulator as a function of θ_2 for three different values of α

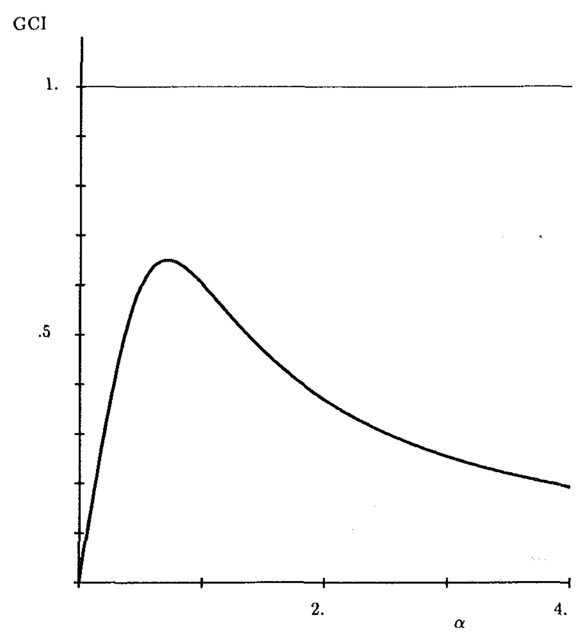

Fig. 2(*b*) Global conditioning index of the planar two-link manipulator for different values of α

a slightly different, but also meaningful interpretation and in many instances it may be easier to handle mathematically, when serial manipulators are considered.

The concept of global conditioning index will now be applied to four different manipulators for which optimum designs will be obtained.

3 Examples

3.1 Planar, Serial, Two-Link Manipulator. The serial, two-link manipulator under study is shown in Fig. 1. This manipulator is capable of positioning a point on its plane. The Jacobian matrix, as defined in equation (1), can be written in a coordinate frame attached to link 1 as:

$$\mathbf{J} = \begin{bmatrix} -a_2\sin\theta_2 & -a_2\sin\theta_2 \\ a_1 + a_2\cos\theta_2 & a_2\cos\theta_2 \end{bmatrix} \tag{17}$$

Therefore, we have:

$$\mathbf{J}^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_2\cos\theta_2 & a_2\sin\theta_2 \\ -(a_1 + a_2\cos\theta_2) & -a_2\sin\theta_2 \end{bmatrix} \tag{18}$$

where

$$\Delta = a_1a_2\sin\theta_2 \tag{19}$$

The frame-invariant condition number of $\mathbf{J}$, can then be computed from equations (3) and (4) (Angeles and Rojas, 1987), and is given by:

$$\kappa = (a_1^2 + 2a_2^2 + 2a_1a_2\cos\theta_2)/2a_1a_2\sin\theta_2 \tag{20}$$

or

$$\kappa = (1/\alpha + 2\alpha + 2\cos\theta_2)/2\sin\theta_2 \tag{21}$$

where

$$\alpha = a_2/a_1 > 0 \tag{22}$$

Some plots of the condition number against θ_2 are shown in Fig. 2(*a*) for a few values of α. It can be seen that the only value that leads to an isotropic manipulator is $\alpha = \sqrt{2}/2$, a fact that was pointed out in Salisbury and Craig (1982). However, it is interesting to notice that, from a figure presented in the latter reference, it is not obvious that this value of α gives the best GCI. The curves shown in Fig. 2(*a*) are plotted against θ_2 and this way of presenting the curves allows us to see that the isotropic manipulator should lead to an optimum GCI since the value of the condition number for this value of α is always the lowest. Now, in order to compute the manipulator's GCI, we have to integrate the reciprocal of κ over the workspace. Since we have expressed the condition number as a function of joint angle θ_2 and the linkage parameter α only, it is convenient to evaluate the integrals described in equations (5–7) in the joint space, i.e., to use the formulation developed in equations (15–16). We cover the workspace of the manipulator by integrating on one of the two branches of the manipulator. The behavior of the condition number will be symmetric on the other branch and integrating over the whole joint space would lead to the same result provided we use the absolute value of the determinant. We will use only one of the branches here to avoid absolute values. For example, letting angle θ_2 vary between 0 and π, we have:

$$B=\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{\pi}a_1a_2\sin\theta_2 d\theta_2 d\theta_1=4\pi a_1a_2 \tag{23}$$

which leads to:

$$\eta=\frac{1}{4\pi a_1a_2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{2\sin\theta_2}{1/\alpha+2\alpha+2\cos\theta_2}\right)a_1a_2\sin\theta_2 d\theta_2 d\theta_1 \tag{24}$$

and can be further simplified to:

$$\eta=\int_0^{\pi}\left(\frac{\sin^2\theta_2}{1/\alpha+2\alpha+2\cos\theta_2}\right)d\theta_2 \tag{25}$$

Then, taking the derivative with respect to the only parameter involved, i.e., α, and setting it equal to zero, one obtains:

$$(2-1/\alpha^2)\int_0^{\pi}\frac{\sin^2\theta_2 d\theta_2}{(1/\alpha+2\alpha+2\cos\theta_2)^2}=0 \tag{26}$$

The integral in equation (26) is a positive definite quantity. Therefore, this equation is satisfied if, and only if:

$$\alpha=\frac{\sqrt{2}}{2} \tag{27}$$

The investigation of the denominator of the integrand in equation (26) shows that the integrand does not suffer from any singularity. In fact, the condition under which the denominator vanishes can be written as:

$$\cos\theta_2=-\left(\alpha+\frac{1}{2\alpha}\right) \tag{28}$$

which leads to:

$$1+4\alpha^4<0 \tag{29}$$

and cannot be satisfied for α real.

In this case, the optimum design in the sense of the global conditioning index is found to lead to the isotropic manipulator already discussed in Salisbury and Craig (1982); Angeles and Rojas (1987). The global conditioning index of the two-link manipulator as a function of α is shown in Fig. 2(*b*). Its maximum value is $\eta_{max}=0.6506$, for $\alpha=\sqrt{2}/2$.

3.2 Spherical, Serial, Three-Degree-of-Freedom Wrist. A spherical wrist is shown in Fig. 3. Since the axes of the three joints intersect as a common point, the parameters defining the architecture of the wrist are reduced to the angles α_1 and α_2. We then have:

$$\mathbf{h}=[\alpha_1,\alpha_2]^T \tag{30}$$

If we denote by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ the three unit vectors along the kinematic pairs of the wrist, we can write the Jacobian matrix as:

$$\mathbf{J}=[\mathbf{e}_1\quad \mathbf{e}_2\quad \mathbf{e}_3] \tag{31}$$

This matrix is now represented in a coordinate frame attached to the second link, i.e.,

$$\mathbf{J}=\begin{bmatrix}0 & 0 & \sin\theta_2\sin\alpha_2\\ \sin\alpha_1 & 0 & -\cos\theta_2\cos\alpha_2\\ \cos\alpha_1 & 1 & \cos\alpha_2\end{bmatrix} \tag{32}$$

from which we can write:

$$\Delta=\sin\alpha_1\sin\alpha_2\sin\theta_2 \tag{33}$$

Using equations (3) and (4) one can then derive an expression for the condition number (Angeles and Rojas, 1987), which gives:

$$\left(\frac{1}{\kappa}\right)^2=\frac{3N}{D} \tag{34}$$

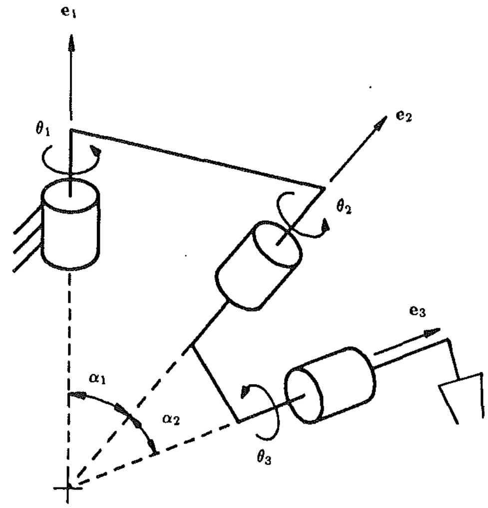

Fig. 3 Open loop 3-dof spherical wrist

where

$$N=\sin^2\alpha_1\sin^2\alpha_2\sin^2\theta_2 \tag{35}$$

and

$$D=\sin^2\alpha_1(1+\cos^2\alpha_2)+\sin^2\alpha_2(1+\sin^2\theta_2) + \cos^2\alpha_1\sin^2\alpha_2\cos^2\theta_2+2\cos\alpha_1\sin\alpha_1\cos\alpha_2\sin\alpha_2\cos\theta_2 \tag{36}$$

As in the case of the first example, the spherical wrist has two branches and the integration can be performed on one of them to avoid the use of absolute values. For instance, we can choose the branch for which the determinant of the Jacobian matrix is positive, i.e., integrate over θ_2 from 0 to π. Now, equation (13) will lead to two equations since $\mathbf{h}$ is of dimension 2. The integrand of the first term of each of these equations can be written as:

$$\frac{\partial}{\partial\alpha_1}\left[\left(\frac{1}{\kappa}\right)^2\Delta\right]=\frac{3}{D^2}[3\sin^2\alpha_1\cos\alpha_1\sin^3\alpha_2\sin^3\theta_2 D - 2\sin^3\alpha_1\sin^3\alpha_2\sin^3\theta_2 D'] \tag{37}$$

where

$$D'=\sin\alpha_2\cos\alpha_2\cos\theta_2\cos2\alpha_1+\sin\alpha_1\cos\alpha_1(1 + \cos^2\alpha_2-\sin^2\alpha_2\cos^2\theta_2) \tag{38}$$

and

$$\frac{\partial}{\partial\alpha_2}\left[\left(\frac{1}{\kappa}\right)^2\Delta\right]=\frac{3}{D^2}[3\sin^3\alpha_1\sin^2\alpha_2\cos\alpha_2\sin^3\theta_2 D - 2\sin^3\alpha_1\sin^3\alpha_2\sin^3\theta_2 D''] \tag{39}$$

where

$$D''=\sin\alpha_1\cos\alpha_1\cos\theta_2\cos2\alpha_2+\sin\alpha_2\cos\alpha_2\cos^2\alpha_1 + \sin\alpha_2\cos\alpha_2(\sin^2\theta_2+\cos^2\alpha_1\cos^2\theta_2) \tag{40}$$

Moreover, the second term of each of the normal equations, equation (13), contains a factor $\partial B/\partial\mathbf{h}$ whose integrand, for the first equation, is given by:

$$\frac{\partial\Delta}{\partial\alpha_1}=\cos\alpha_1\sin\alpha_2\sin\theta_2 \tag{41}$$

and, for the second equation, by:

$$\frac{\partial\Delta}{\partial\alpha_2}=\sin\alpha_1\cos\alpha_2\sin\theta_2 \tag{42}$$

By inspection of equations (37–42), it becomes obvious that the normality condition is verified if $\alpha_1=\alpha_2=\pi/2$. Therefore, these angles constitute an optimum design in the sense of the global conditioning index. Again, this design is found to be

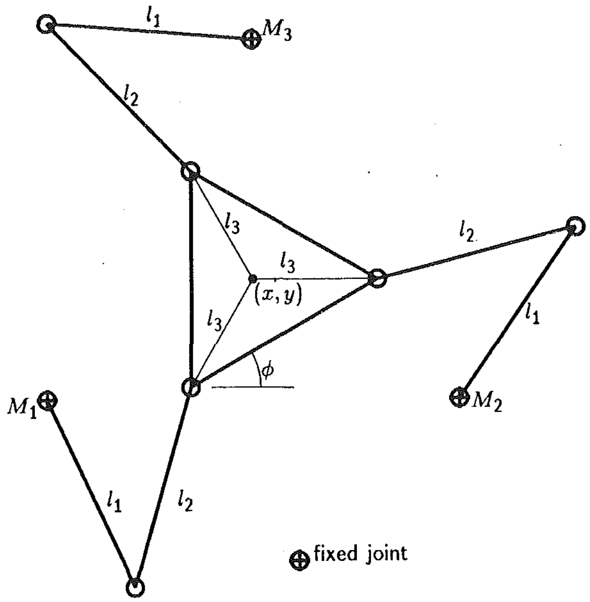

Fig. 4 Planar 3-dof closed-loop manipulator

the isotropic manipulator found using a local conditioning index (Angeles and Rojas, 1987). The results obtained for these first two examples can be reproduced using a GCI based on the joint space metric, which actually leads to simpler integrals. The procedure is identical to the one described above.

3.3 Three-Degree-of-Freedom, Parallel, Planar Manipulator. A planar three-degree-of-freedom, parallel, manipulator is shown in Fig. 4. This manipulator is used to position and orient a gripper in the plane. The parallel architecture allows us to fix all the motors, whose axes are denoted here by M_1, M_2, and M_3, to the manipulator's base. The manipulator is assumed to be symmetric and the parameters are then given by l_1, l_2, and l_3. This manipulator has been studied in detail in Gosselin and Angeles (1988) where its workspace is described as a volume in the (x,y,ϕ) space, ϕ being the angle that defines the orientation of the gripper. Isotropic designs were also given and it was shown that the Jacobian matrix of this manipulator, as defined in equation (2), can be written as:

$$\mathbf{K}=\begin{bmatrix} -a_1/c_1 & -b_1/c_1 & -d_1/c_1 \\ -a_2/c_2 & -b_2/c_2 & -d_2/c_2 \\ -a_3/c_3 & -b_3/c_3 & -d_3/c_3 \end{bmatrix} \tag{43}$$

where

$$a_i=-g_1g_2(x-x_{oi})+g_2\cos\theta_i+g_1\cos\phi_i \tag{44}$$
$$b_i=-g_1g_2(y-y_{oi})+g_2\sin\theta_i+g_1\sin\phi_i \tag{45}$$
$$c_i=g_2[(y-y_{oi})\cos\theta_i-(x-x_{oi})\sin\theta_i]+\sin(\theta_i-\phi_i) \tag{46}$$
$$d_i=g_1[(y-y_{oi})\cos\theta_i-(x-x_{oi})\sin\phi_i]-\sin(\theta_i-\phi_i) \tag{47}$$

and

$$g_1=1/l_1 \quad g_2=1/l_3 \tag{48}$$

(x_{oi},y_{oi}) being the coordinates of the point of the axis of the ith motor intersecting the plane of the manipulator and θ_i is the angle associated with the ith powered joint. Moreover, angles ϕ_i are defined as:

$$\phi_1=\phi+\pi/6 \tag{49}$$
$$\phi_2=\phi+5\pi/6 \tag{50}$$
$$\phi_3=\phi-\pi/2 \tag{51}$$

The complexity of the expression of the condition number, and the fact that the Jacobian is written as a function of both the Cartesian and the joint coordinates, forces us to resort to a numerical integration in order to evaluate the GCI. The integration has been carried out over the workspace in the Cartesian space. The algorithm to compute the volume of this workspace, B, developed in Gosselin and Angeles (1988), was used and a triple numerical integration was introduced to compute the numerator of η i.e.:

$$A=\int_\phi\int_y\int_x\left(\frac{1}{\kappa}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}\phi \tag{52}$$

Table 1

Planar 3-dof parallel manipulators having an optimum GCI.

	Case 1	Case 2	Case 3
l_1	0.9940	1.1855	0.9968
l_2	1.3274	4.5987	0.7838
l_3	2.6293	5.1739	0.9719
η	0.79156	0.69691	0.42961

Case 1: Unconstrained
Case 2: Workspace constrained to be simply connected
Case 3: Workspace constrained to be simply connected and link lengths constrained to be less than the distance between the motors i.e. $0<l_i<1$, $i=1,2,3$.

The optimization was then performed using the complex method (Box, 1965). Optimum results are shown in Table 1, where three cases are reported. The first one represents the solution obtained when no constraints are imposed on the maximization of the GCI. However, the manipulator then obtained has a rather limited workspace. Therefore, a second optimization was conducted with a constraint on the workspace. This presents no particular problem since the optimization method used is well suited for handling inequality constraints. The manipulator was then forced to have a nonvanishing workspace for every angle ϕ of the gripper, a criterion that was introduced in Gosselin and Angeles (1988). The associated inequality constraints were derived in the foregoing reference. The solution obtained for this problem is identified as case 2 in Table 1. The corresponding optimum manipulator now has a much larger workspace. However, the link lengths are quite long, which may induce major mechanical interference problems. A new optimization problem can be set up by imposing additional inequality constraints in order to remedy this situation. For instance, case 3 of Table 1 shows the solution obtained when the link lengths are forced to be less than the distance between the motors, i.e., $(0<l_i<1,\ i=1,2,3)$ and the constraint on the workspace used in case 2 is imposed. Notice that the introduction of the constraints has led to a reduction of the GCI. The three cases reported here are obviously not the only possible designs and they are shown to illustrate how one can use the GCI to optimize a manipulator while meeting other design constraints associated with a particular problem.

3.4 Three-Degree-of-Freedom, Parallel, Spherical Manipulator. A spherical, three-degree-of-freedom, parallel manipulator is shown in Fig. 5. This manipulator is used to orient a gripper in space. Again, the parallel architecture allows us to fix all the motors, represented here as cylinders, to the manipulator's base. The manipulator is assumed to be symmetric and the design parameters are given by the link angles α_1 and α_2. This manipulator has been studied in detail in Gosselin and Angeles (1989), where its workspace is described as a subset of the unit sphere centered at the origin of the (q_0,q_1,q_2) space. These quantities are defined in terms of the scalar and vector linear invariants of the rotation tensor $\mathbf{Q}$ which represents the rotation of the gripper. We have:

$$q_0=\frac{\mathrm{tr}(\mathbf{Q})-1}{2} \tag{53}$$

and

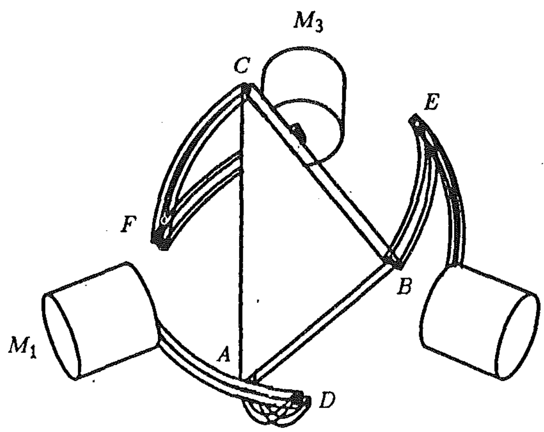

Fig. 5 Spherical three-degree-of-freedom closed-loop manipulator

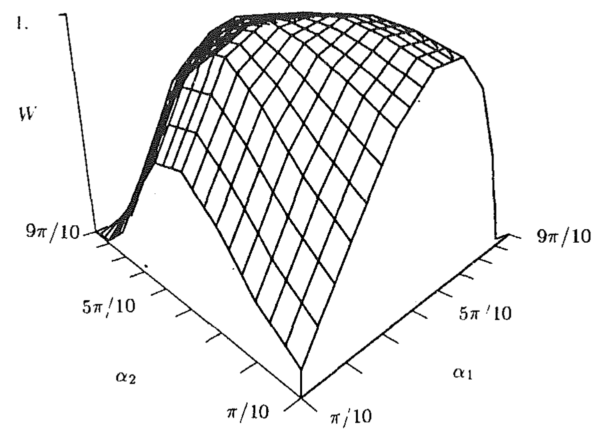

Fig. 7 Normalized workspace of the spherical parallel manipulator as a function of α_1 and α_2

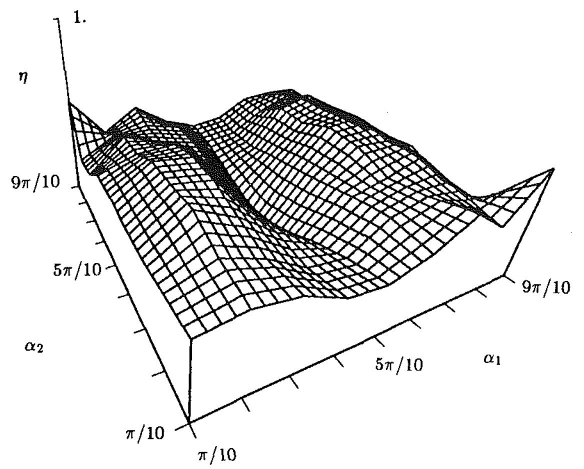

Fig. 6 GCI of the spherical parallel manipulator as a function of α_1 and α_2

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathrm{vect}(\mathbf{Q}) \equiv \frac{1}{2} \begin{bmatrix} Q_{32} - Q_{23} \\ Q_{13} - Q_{31} \\ Q_{21} - Q_{12} \end{bmatrix} \tag{54}$$

where q_i, $i = 1,2,3$ are the three components of the vector linear invariant of tensor $\mathbf{Q}$ and Q_{ij} is the (i,j) entry the matrix representation of tensor $\mathbf{Q}$. The Jacobian matrix of this manipulator, defined in equation (2), can be written as:

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \\ \mathbf{k}_3 \end{bmatrix} \tag{55}$$

with

$$\mathbf{k}_i = \frac{(\mathbf{w}_i \times \mathbf{v}_i)^T}{(\mathbf{u}_i \times \mathbf{w}_i) \cdot \mathbf{v}_i} \tag{56}$$

where $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i)$ are the unit vectors along the motor axes, the joints attached to the gripper, and the intermediate joints, respectively, for the ith leg.

Again, a numerical integration can be carried out on the workspace of the manipulator in order to evaluate the GCI. This is done as:

$$\eta = \frac{\mathrm{A}}{\mathrm{B}} \tag{57}$$

with

$$\mathrm{A} = \int_{q_2} \int_{q_1} \int_{q_0} \left(\frac{1}{\kappa}\right) \mathrm{d}q_0 \mathrm{d}q_1 \mathrm{d}q_2 \tag{58}$$

and

$$\mathrm{B} = \int_{q_2} \int_{q_1} \int_{q_0} \mathrm{d}q_0 \mathrm{d}q_1 \mathrm{d}q_2 \tag{59}$$

These integrations were performed for different values of α_1 and α_2 and the results are given in Fig. 6 where the GCI is plotted as a function of α_1 and α_2. It is pointed out that the maximum GCI ($\eta = 0.52$) is obtained for $\alpha_1 = 7\pi/30$ and $\alpha_2 = 13\pi/30$ approximately. A symmetry about the central point $\alpha_1 = \alpha_2 = \pi/2$ has also been observed. This point was found to be the one having the minimum GCI ($\eta = 0.056$) which indicates that, for this manipulator, the optimization of the GCI conflicts seriously with the maximization of the workspace. Indeed, it was shown in Gosselin and Angeles (1989) that the central point of the $\alpha_1 - \alpha_2$ region of interest, shown in Figs. 6 and 7, is the one having the maximum workspace. This can be clearly seen in Fig. 7 where the workspace of the manipulator is plotted as a function of α_1 and α_2.

4 Conclusions

A global conditioning index for the kinematic optimization of manipulators was proposed. This index represents a global performance of the condition number of the Jacobian matrix over the whole workspace. The examples presented here show that closed-form solutions can be obtained for simple manipulators, whereas more general cases require the introduction of numerical methods. The index was also shown to be applicable to serial or parallel manipulators of arbitrary architecture. The results obtained for the spherical parallel manipulator show that the optimization of its GCI is seriously conflicting with the maximization of its workspace. This situation is different from the one encountered in the case of the spherical serial wrist for which the optima of these criteria coincide. It is also important to remember that the concept of a global index to characterize the kinematic or dynamic properties of manipulators could be used with local indices different than the condition number. Finally, the applicability of the GCI to more complex manipulator architectures should call for numerical optimization methods.

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