Planning of Minimum- Time Trajectories for Robot Arms

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The International Journal of Robotics Research 1986 5: 90
DOI: 10.1177/027836498600500305

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Abstract

The minimum-time path for a robot arm has been a long-standing and unsolved problem of considerable interest. We present a general solution to this problem which involves joint-space tessellation, a dynamic time-scaling algorithm, and a graph search. The solution incorporates full dynamics of movement and actuator constraints, and can easily be extended for joint limits and workspace obstacles. It was found that optimal paths tend to be nearly straight lines in joint space. We discuss implementation difficulties due to the tessellation and to combinatorial proliferation of paths.

1. Introduction

The determination of the time-optimal path for manipulators is an important problem in robot trajectory planning. This paper presents a general solution to this problem involving *joint-space* tessellation and a graph search that is made relatively efficient through use of time-scaling properties of dynamics. The minimum-time requirement imposes constraints on the velocities at each point on our grid, so that tessellation in velocity space is not needed. Our algorithm takes into account a full dynamic model of a manipulator and actuator torque limits in arriving at the time-optimal trajectory (i.e., the path and the time dependence along the path) from any given starting state to any final state. Kinematic constraints due to joint limits and the presence of obstacles are quite easily incorporated into the solution.

Attempts in the past at solving this problem have fallen short of a general solution. Most commonly, researchers have linearized the dynamics in order to apply standard techniques of linear optimal control theory to the solution. In the earliest attempts along these lines (Kahn 1969; Kahn and Roth 1971), the expected bang-bang solution with multiple switching points was derived. Approaches based on dynamics linearization have recently been cast into doubt due to time-scaling properties of dynamics (Hollerbach 1983*a*; 1983*b*; 1984), since it can be shown that the velocity product terms have the same significance relative to the acceleration dynamic terms for all speeds of movement. Thus the main presumption used to justify linearization, namely that the velocity product terms can be ignored because they are only significant at higher movement speeds, is fundamentally wrong.

Purely kinematic approaches were used by Luh and Walker (1977); Luh and Lin (1981); and Lin, Chang, and Luh (1983). They attempt to find the sequence of time intervals that minimize the total time spent on moving between two points in space. The three approaches differ in the details of the algorithms, but they all suffer from the same shortcomings: the dynamics of the arm are not considered at all, so that the constraints consist of sets of bounds on position, velocity, and acceleration. These bounds are imposed by the weakest configuration of the arm, so that the motion in other areas of the workspace is suboptimal. In addition, all three algorithms require as input a set of knot points that define a path. This path is not necessarily the best possible one.

Another class of solutions does take into account the full dynamics but presumes a bang-coast-bang form of control, again by analogy to the linear optimal case. Scheinman and Roth (1985) assume a bang-coast-bang control form for each joint where (1) each

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This paper describes research done at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology. Support for the laboratory's research is provided in part by the Systems Development Foundation, in part by the Office of Naval Research under contract N00014-81-0494, and in part by the Defense Advanced Research Projects Agency under Office of Naval Research contracts N00014-80-C-0505 and N00014-82-K-0334.

The International Journal of Robotics Research,
Vol. 5, No. 3, Fall 1986,

control has a maximum of two switching points and (2) the total positive torque time is equal to the total negative torque time. Their solution is an iterative one, where they set a time, solve for the distances covered and the velocities attained, and adjust the switching times until a satisfactory solution is found. Our results indicate that these presumptions about control are in general not valid for the true time-optimal solution.

Brown (1984) and Kornhauser and Brown (1985) independently developed an approach based on a state-space tessellation and a graph search, which nevertheless differs from our approach in critical ways. They presume a bang-coast-bang solution with fixed switching points; all possible combinations of bangs and coasts yield nine distinct torque patterns. At a given point in state space all nine torque patterns are applied for a set period of time, and the dynamics are integrated to find the resultant state-space point. This point is rounded to the nearest neighbor in the tessellation. The network is searched via standard procedures to find the fastest trajectory. Again, the presumption on control makes these solutions nonoptimal.

Kim and Shin (1985) present a heuristic solution to the time-optimal trajectory along essentially fixed parameterized paths in joint space, adapted from Taylor's (1979) algorithm. The paths consist of straight-line segments connected at via points, and a curved transition region from one straight-line segment to the next with a bounded deviation from the knot point. The via points are given, and the only leeway in modifying the path is the transition path. It is presumed that there is constant acceleration and deceleration to a maximum cruise velocity along each straight-line segment, and that the transition is made with constant acceleration. Dynamics are introduced into the transition points as constraints on accelerations. The transition points are adjusted to minimize time, given all the constraints. Because of the heuristic nature of this method, it is possible that the solution is far from optimal. There is also no guarantee that the solution does not violate torque bounds in intermediate regions.

The new development that facilitates the approach presented in this paper is the discovery of a fundamental time-scaling property of manipulator dynamics. This property was utilized to solve for the time-optimal trajectory along a predetermined path by Bobrow (1982); Bobrow, Dubowsky, and Gibson (1983); Dubowsky and Shiller (1985); and Shiller and Dubowsky (1985). The equations of motion are written in terms of the distance along the Cartesian path taken by the arm, and the bounds on the torques/forces obtainable from the motors are used to construct a curve in distance-velocity space that constitutes an upper limit on the performance of the arm. A set of switching points is then found that moves the arm as close as possible to the limit curve. In his Ph.D. thesis, Bobrow (1982) proves the optimality of this algorithm. One actuator is always saturated, and the others adjust their torques so that some constraints on the motion are not violated. The only problem with it is that the path has to be known beforehand. A very similar approach was developed later by Shin and McKay (1983; 1985*a*), with the only difference being a parameterization of a path in joint space instead of in Cartesian space.

Hollerbach (1983*a*; 1983*b*; 1984) independently developed the time-scaling property of dynamics in joint space and used a simplified form of this property to scale a fixed-velocity profile for maximum speed given actuator constraints. This general formulation has been discretized for our optimization algorithm.

Rajan (1985) developed a parameterized path method to solve the full minimum-time path problem based on the Bobrow, Dubowsky, and Gibson (1983) procedure. Cubic spline paths were set at each joint, and the parameters of the cubic splines were locally optimized by finding the minimum time along a particular path. This solution was augmented by adding a second spline and a knot point to the single joint spline, and the optimization procedure was repeated with the larger number of parameters. Splines and knot points continued to be added until there was little change in the minimum time. This elegant procedure could serve as a useful point of comparison to the present algorithm, although direct comparisons are not available. Potential difficulties with this approach are the presence of local minima and the handling of obstacles.

Lin and Chang (1985) also present a cubic spline solution similar to Rajan's, except for the method of evaluating the minimum time along a particular path. They do not make use of the recent time-scaling algorithms for dynamics. Instead, their own method for

scaling of acceleration and dynamics appears incorrect because of the omission of a term proportional to the generalized momentum.

The next section introduces the problem in a more rigorous fashion and describes the way with which we manage to work in *state space,* yet tessellate only in *joint space.* Section 3 introduces the *dynamic scaling algorithm,* which is used to calculate the times of travel along each segment of the grid, along with the required torques and the possible velocities at the end of the segment, given initial conditions on position and velocity at the beginning of the segment. Section 4 presents results of a two-dimensional simulation and Section 5 is a discussion of extensions to this work. Portions of this research were reported in Sahar and Hollerbach (1985).

2. The General Solution

The objective is to move the arm from point A to point B in state space in minimum time. Given are:

1. the kinematic and inertial parameters of the manipulator;
2. the dynamic equation

$$\tau = \mathbf{H}(\theta(t))\ddot{\theta}(t) + \dot{\theta}(t)^T\mathbf{C}(\theta(t))\dot{\theta}(t) + \mathbf{g}(\theta(t)), \quad (1)$$

   where τ is the vector of joint torques, θ is the n-dimensional joint-space position vector, $\mathbf{H}$ is an $n \times n$ symmetric inertia matrix, $\mathbf{C}$ is an $n \times n \times n$ tensor, n is the number of joints, and $\mathbf{g}$ is a gravity-dependent vector, and
3. constant bounds on the torques/forces available from the motors,

$$\tau_l \leq \tau \leq \tau_u, \quad (2)$$

   which can be made state-dependent without modifying the algorithm

The main idea behind our algorithm is to tessellate joint space into a grid. The requirement for time optimality, together with the dynamics of the arm, constrains the number of velocities possible at each position node, given the position and velocity at the previous node, so that there is no need to tessellate velocity space. A tree of all possible state-space paths can be constructed and then searched for the minimum-time path.

In terms of assigning velocity tessellations to each position node, the results of Bobrow (1982) show that one actuator is always at a torque bound. Therefore, there are at most $2n$ possible transitions from a given state to a neighboring position. The time-scaling algorithm presented in the next section determines for each transition the velocity at this neighboring position and the transition time. Many of these $2n$ transitions can be discarded immediately, due either to saturation in one of the actuators causing the torque in one of the other motors to exceed its bound or to a contradictory situation where the velocity is in the direction opposite to the position increment. In our two-dimensional implementation, the number of solutions at each state-space point was usually two or less.

2.1. Graph Search

The graph search can be made considerably more efficient through the use of some general search techniques: (1) best first search, which chooses to expand the best path so far; and (2) branch and bound, which uses an upper bound on the travel time to eliminate any path with a partial cost larger than the upper bound. The choice here was to first run the algorithm with a very sparse grid and use the time found, rounded up in some suitable manner depending on the magnitude of the result, as the upper bound for a second run with a finer grid.

The procedure for finding the optimal path follows.

1. Form a queue of paths, containing an empty path (one with only the starting node).
2. Until the goal is reached (success), or the queue is empty (failure):
   a. Remove the first path from the queue.
   b. Calculate the positions of the next points reachable from the current (last in path) point.
   c. For each one of the next points, the dynamic scaling algorithm (described in the next section) is used to find the possible

velocities at the next points and the times of travel.

d. New paths are formed from the old path and from all admissible new points. (A point is inadmissible if one of the calculated torques exceeds its bound, if the velocity of one joint is in the opposite direction to its position increment, or if the current total time of travel exceeds the upper bound.)
e. Add new paths to the queue and sort it such that the path with the smallest cost is on top.

3. The Dynamic Scaling Algorithm

The previous section described the global solution to finding the optimal path. This section describes the local solution for the minimum traveling time between nodes, the achieved velocity for each permissible transition, and the corresponding torques. We rely on a reformulation of the dynamics of a manipulator, which indicates how the underlying dynamics change when the time dimension of a trajectory changes (Hollerbach 1984). Suppose the time dependence along a fixed path is changed from $\theta(t)$ to $\theta(r(t))$, where the time scaling factor $r(t)$ is a strictly increasing function of time with $r(0) = 0$ and $r(t_1) = t_f$ for some $t_1 > 0$. Then the joint torques $\tau'(t)$ for the time-scaled movement are related to the original movement torques $\tau(t)$ by:

$$\tau'(t) = \dot{r}^2\{\tau(r) - \mathbf{g}(\theta(r))\} + \ddot{r}\mathbf{H}(\theta(r)) \cdot \frac{d\theta(r)}{dr} + \mathbf{g}(\theta(t)). \tag{3}$$

This general formulation is now discretized, with known position $\theta(i-1)$ and $\theta(i)$, respectively, at the beginning and end of the segment, and known velocity $\dot{\theta}(i-1)$ at the beginning of the segment. Instead of looking at the beginning of the segment (i.e., point $i-1$), consider the midpoint. Define an average velocity and acceleration

$$\dot{\theta}\left(i-\frac{1}{2}\right) = \frac{\Delta\theta(i)}{\Delta t}, \tag{4}$$

$$\ddot{\theta}\left(i-\frac{1}{2}\right) = \frac{\dot{\theta}(i) - \dot{\theta}(i-1)}{\Delta t}, \tag{5}$$

where the distance $\Delta\theta(i) = \theta(i) - \theta(i-1)$ is covered in time Δt. The important point here is that no functional relationship is assumed between $\dot{\theta}(i-\frac{1}{2})$ and $\ddot{\theta}(i-\frac{1}{2})$, but rather that these relationships are used as predictors that could be subject to correction. By assuming a linear velocity (constant acceleration) we can find the velocity at the end of the segment:

$$\dot{\theta}(i) = 2\frac{\Delta\theta(i)}{\Delta t} - \dot{\theta}(i-1). \tag{6}$$

Substituting this into our definition of the average acceleration, we obtain:

$$\ddot{\theta}\left(i-\frac{1}{2}\right) = 2\frac{\Delta\theta(i)}{\Delta t^2} - 2\frac{\dot{\theta}(i-1)}{\Delta t}. \tag{7}$$

Suppose the time of travel was Δs instead of Δt. The new average velocity and acceleration at the midpoint are related to the old by:

$$\dot{\theta}'\left(i-\frac{1}{2}\right) = \frac{\Delta t}{\Delta s}\dot{\theta}\left(i-\frac{1}{2}\right), \tag{8}$$

$$\ddot{\theta}'\left(i-\frac{1}{2}\right) = \frac{\Delta t^2}{\Delta s^2}\ddot{\theta}\left(i-\frac{1}{2}\right) + 2\frac{\Delta t - \Delta s}{\Delta s^2}\dot{\theta}(i-1). \tag{9}$$

We call these quantities "scaled." If the scaled velocity and acceleration are substituted into the equation of motion (Eq. 1), then an equation for τ', the torque required to cover the distance in Δs, in terms of τ, the torque that was used when the distance was covered in Δt, is obtained:

$$\tau' = \frac{\Delta t^2}{\Delta s^2}(\tau - \mathbf{g}) + 2\frac{\Delta t - \Delta s}{\Delta s^2}\mathbf{H} \cdot \dot{\theta}(i-1) + \mathbf{g}. \tag{10}$$

Rewriting Eq. (10) as a quadratic equation for Δs:

$$\Delta s^2(\tau' - \mathbf{g}) + 2\,\Delta s\,\mathbf{H} \cdot \dot{\theta}(i-1) - [\Delta t^2(\tau - \mathbf{g}) + 2\,\Delta t\,\mathbf{H} \cdot \dot{\theta}(i-1)] = 0. \tag{11}$$

These are n equations for the unknown Δs (one for each joint), which are found by substituting the torque

bounds for τ'. They all have to be satisfied simultaneously in order to achieve optimality without violating one of the constraints. This development leads to a simple procedure to calculate the minimum time to travel from point $i-1$ to point i, and the combination of the torques that will do it:

1. Calculate $\theta(i-\frac{1}{2})$, $\dot{\theta}(i-\frac{1}{2})$, and $\ddot{\theta}(i-\frac{1}{2})$, using the known $\theta(i-1)$, $\Delta\theta(i)$, and $\dot{\theta}(i-1)$, and an arbitrarily chosen Δt (setting $\Delta t = 1$ is the most efficient).
2. Solve the equation of motion (Eq. 1) for the corresponding τ.
3. For each joint, use Eq. (11) to find Δs, where τ' is the vector of given torque bounds. Solve for every one of the bounds.
4. Use all the Δs found in step 3, together with the corresponding torque bound, to calculate the torques for the other joints from Eq. (10). Retain only those time/torque combinations that do not exceed the bounds.
5. The permitted velocities at the end of the segment can be calculated from Eq. (6), with Δt replaced by Δs.

4. Implementation

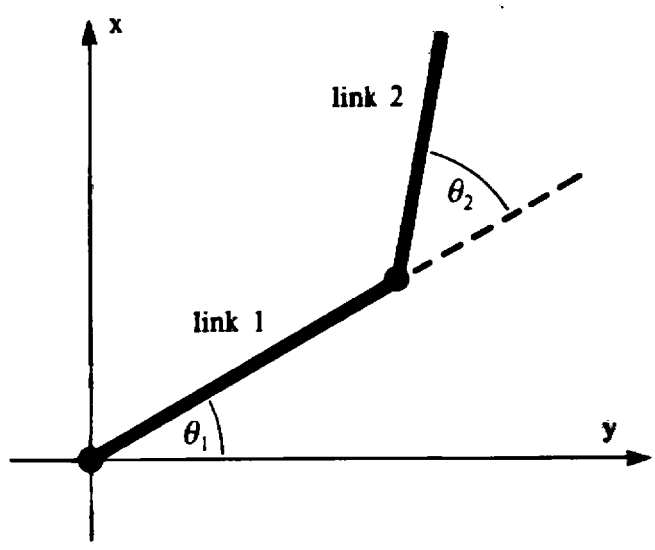
The minimum-time algorithm is illustrated by simulation with a two-degree-of-freedom, planar manipulator (Horn 1975; Brady et al. 1982) on a Symbolics 3600 Lisp Machine. The equations of motion are

$$\begin{aligned}\tau_1 = {} & [I_1 + I_2 + \tfrac{1}{4}(m_1 l_1^2 + m_2 l_2^2) + m_2 l_1^2 + m_2 l_1 l_2 c_2]\ddot{\theta}_1 \\ & + (I_2 + \tfrac{1}{4}m_2 l_2^2 + \tfrac{1}{2}m_2 l_1 l_2 c_2)\ddot{\theta}_2 - \tfrac{1}{2}m_2 l_1 l_2 s_2 \dot{\theta}_2^2 \\ & - m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ & + [\tfrac{1}{2}m_2 l_2 c_{1+2} + l_1(\tfrac{1}{2}m_1 + m_2)c_1]g,\end{aligned} \tag{12}$$

$$\begin{aligned}\tau_2 = {} & (I_2 + \tfrac{1}{4}m_2 l_2^2 + \tfrac{1}{2}m_2 l_1 l_2 c_2)\ddot{\theta}_1 \\ & + (I_2 + \tfrac{1}{4}m_2 l_2^2)\ddot{\theta}_2 + \tfrac{1}{2}m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \\ & + \tfrac{1}{2}m_2 l_2 c_{1+2} g,\end{aligned} \tag{13}$$

where θ_i, l_i, m_i, and I_i are the joint angle, length, mass, and moment of inertia, respectively, of link i; $s_i(c_i)$ is the sine(cosine) of the joint i variable; and c_{1+2} is $\cos(\theta_1 + \theta_2)$. The values for the link lengths, masses, and moments of inertia were taken from Asada (1984)

Fig. 1. A two-degree-of-freedom planar arm. $l_1 = l_2 = 0.5\ m$, $m_1 = 50\ kg$, $m_2 = 30\ kg$, $I_1 = 5\ kg \cdot m^2$, $I_2 = 3\ kg \cdot m^2$.



(Fig. 1). The torque bounds were chosen as

$$\begin{aligned} -350Nm \le \tau_{\theta_1} \le 350Nm, \\ -100Nm \le \tau_{\theta_2} \le 100Nm. \end{aligned} \tag{14}$$

4.1. Tessellation of Joint Space

The manner of tessellation is illustrated in Fig. 2 for this two-link manipulator. The joint positions are first tessellated by connecting the beginning and final nodes by a straight line and dividing this line into k equal intervals. The remainder of the position grid is defined normal to this line by translating the tessellated points at integer multiples of the point spacing. From a given position node three paths are allowed: one path is parallel to the original straight line and skips one grid point (from A to C), the other two paths connect diagonally relative to the first path in the direction of the goal (from A to B and A to D). See Fig. 3 for an example of a 3×3 grid.

The number of paths leaving each node is limited to three as a compromise between representation of path curvature and combinatorial proliferation of possible paths. This did create difficulties due to discontinuities, which were handled with special measures described later. Given the influence of the choice of the

Fig. 2. The basic grid unit in joint space for the two-link manipulator.

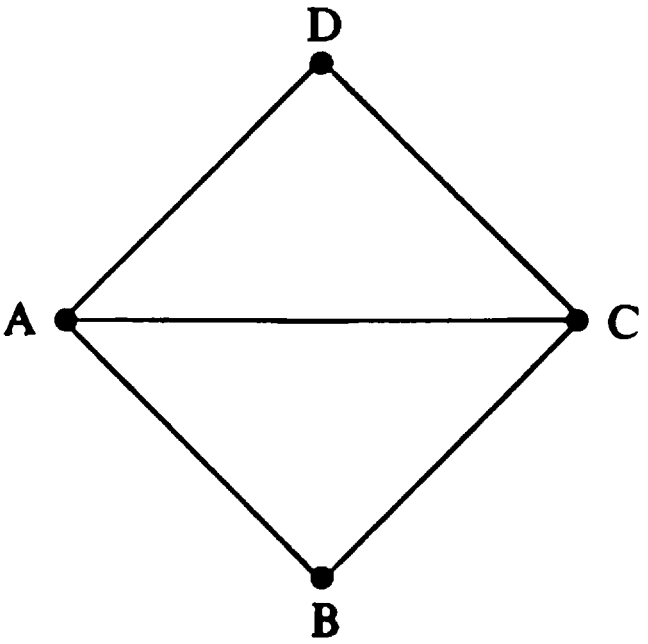
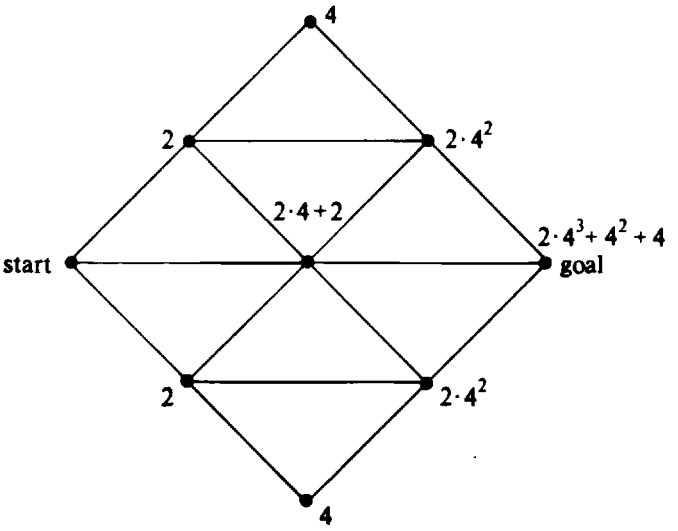


Fig. 3. Number of possible trajectories at each node, defined in joint space for the two-link manipulator.



tessellation number k on the number of paths and hence on computation time, a usual value of $k = 15$ was used. The largest value of k used was 20. To further reduce the search, the two corners of the rectangular grid not on the diagonal connecting the start and the goal were cut away, as it was noticed that the optimal paths generated by the planner tended to concentrate around the diagonal.

4.2. The Problem of Corners

One implication of tessellating space into a rectangular grid is that the motion may be discontinuous: when moving from point A to C through B (Fig. 2), the velocity has to drop to zero at point B or change direction instantaneously, which implies infinite accelerations. Therefore, the corner has to be smoothed out, for example, by fitting a circle between points A and C. Since our local algorithm is defined for a straight line, the circle would have to be approximated by a sequence of straight lines. This solution is impractical because of the combinatorial expansion of states along the straight-line sequence from start to goal, given the possible new states with each application of the time-scaling algorithm.

Without corner smoothing, the velocity vector at the corner point B (Fig. 2), for example, is not aligned with the direction vector A to C. Globally, the arm has to decelerate sharply after the corner B to reach point C, and the effect on the algorithm is to discard any nonstraight-line path as too costly. This technical problem with our tessellation is unacceptable since it is unlikely that a joint-space, straight-line path is always optimal.

A heuristic approximation was used for redefining the direction but not the magnitude of the velocity vector at the corner point B to lie along B-C. For the rationale, assume that the true path between A and C is indeed a circular arc and that the velocity along the arc is approximately constant. Now let the radius of the circle shrink gradually until the circle becomes a point at B. The accelerations required to keep the arm on the circle grow, until in the limit of the point they become infinite. The magnitude of the velocity, however, does not change, so that the magnitude of the velocity going into the circular arc is the same as the velocity magnitude coming out of the other side of the arc. In the limit, the same is true of point B.

4.3. Complexity

By counting the possible number of paths on a grid, such as the one in Fig. 3, the worst-case complexity for

Fig. 4. Moving from [−0.5, −1.0] to [0.5, 1.0]: the path in joint and Cartesian spaces.

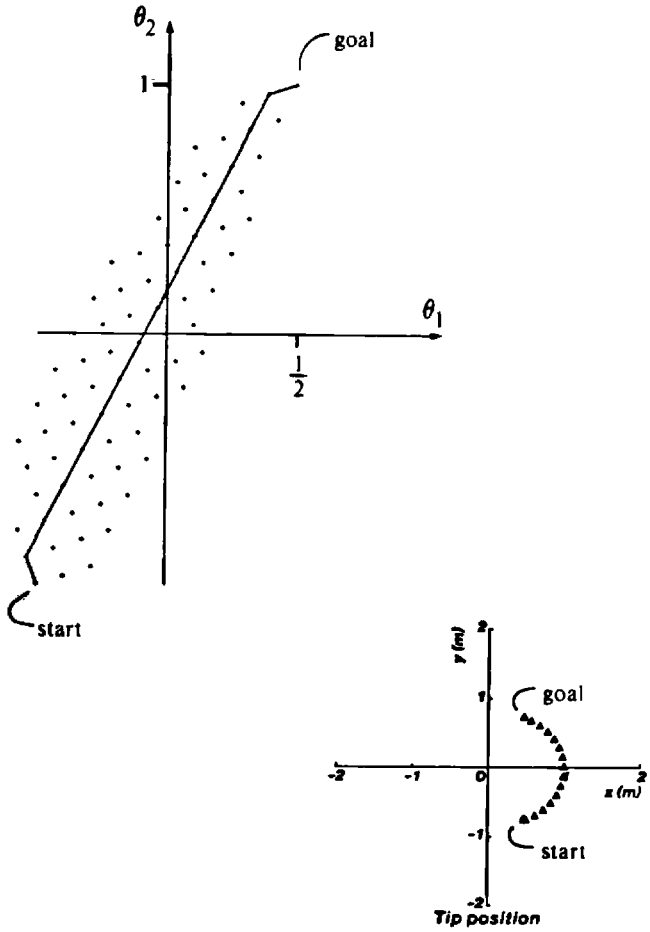
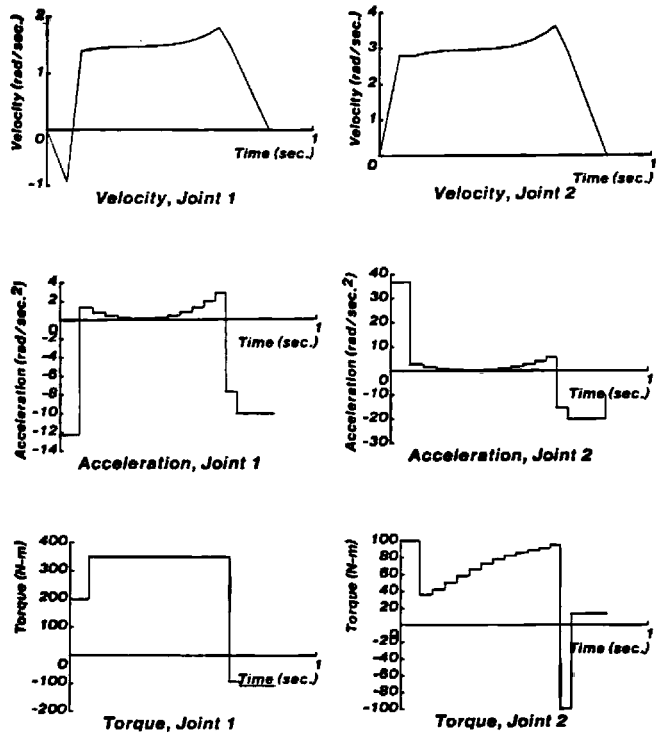


Fig. 5. Moving from [−0.5, −1.0] to [0.5, 1.0]: velocity, acceleration, and torque at the joints.



our planar example is estimated as $O(4^k)$, where k is the number of points along one dimension of the grid. This calculation assumes that only two velocities are possible at the end of each segment, given some initial conditions at the beginning of the segment. This assumption is based on observations during the simulation of many examples. It is a conservative assumption, inasmuch as in many segments only one possible velocity was retained, whereas there were never more than two.

Another complexity factor is the sorting time for the least costly path in the queue. With queues of thousands of paths the sorting operation requires minutes, which makes this algorithm too impractical. Based on this experience, the sorting operating was eliminated and the new path was inserted in the appropriate place in the already sorted queue. In our Lisp Machine implementation the improvement by simple merging was somewhat offset by creation of extraneous lists, leading to address space and garbage collection problems that increased running overhead by a half.

4.4. Results

The results of two simulated runs of the algorithm are shown, with initial and final velocities taken to be zero. Figures 4–6 show the path for moving from [−0.5, −1.0] to [0.5, 1.0] in joint space, which took 0.836 second of simulated time. The figures illustrate the tendency of the algorithm to find a path that is close to a straight line in joint space. In this example there is a slight reconfiguration of the arm at the beginning and end that somehow is dynamically easier to move. Figure 6 shows the magnitudes of the accelera-

Fig. 6. Moving from [−0.5, −1.0] to [0.5, 1.0]: magnitude of tip velocity and acceleration in Cartesian space.

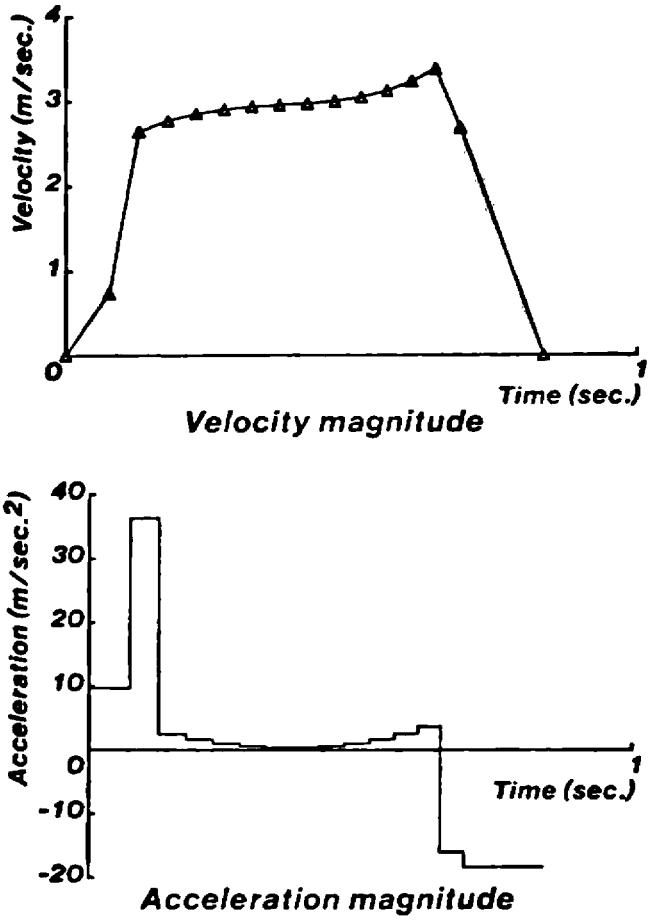
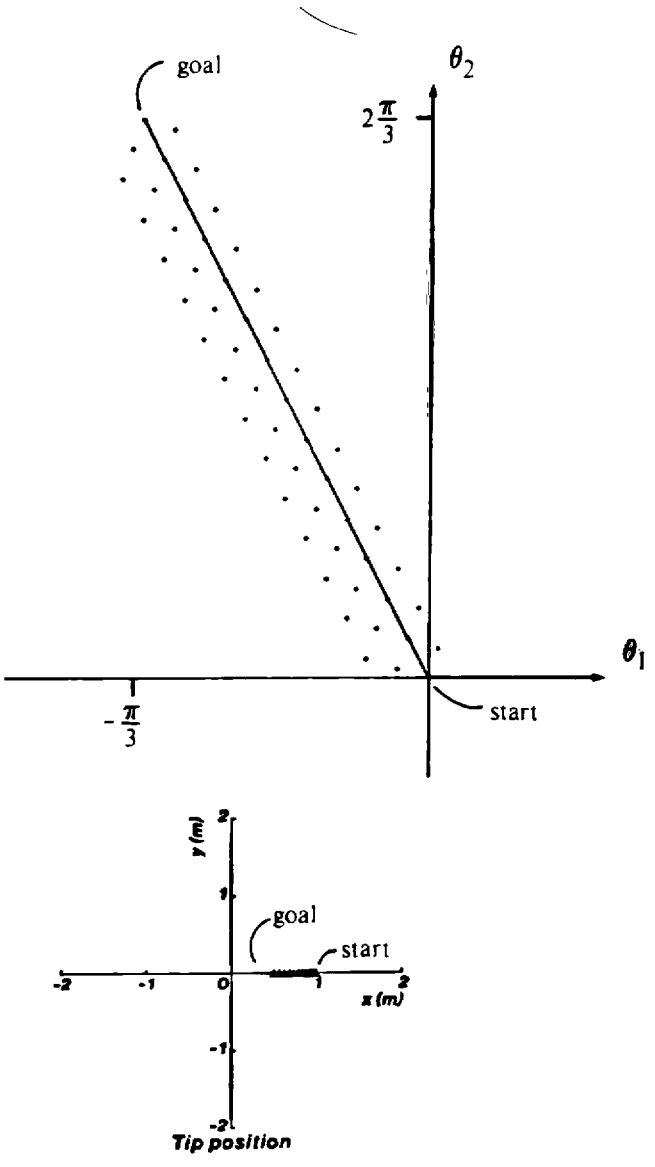


Fig. 7. Moving from [0.0, 0.0] to [−π/3, 2π/3]: the path in joint and Cartesian spaces.



tion and velocity in Cartesian space. Note the similarity to a bang-coast-bang profile, in terms of acceleration.

In contrast, moving from a fully extended position ([0.0, 0.0] in joint space) along the Cartesian x-axis, to the point $[x, y] = [0.5, 0.0]$ ($[-\pi/3, 2\pi/3]$ in joint space), results in a straight line, both in Cartesian space and in joint space (Figs. 7–9). This is the result of the particular configuration of the arm, where the elbow is pointing down, and gravity does not have much effect on the motion. Therefore, the fastest path is also the shortest one, a straight line in joint space, which for this configuration happens also to be a straight line in Cartesian space. This motion was accomplished in 0.525 second of simulated time.

From these and many more results emerges a pattern: usually, the fastest path is close to a straight line in joint space. Although the grid was not fine enough so that we can state this conclusively, it seems that the path is symmetric about the midpoint between the starting point and the destination. The torques tend to switch twice at the most, so one heuristic used by Scheinman and Roth (1985) seems to be justified (see Section 1), but contrary to their other heuristics, only one joint torque is at a maximum at a time, and the total positive torque time is not equal to the total negative torque time.

Fig. 8. Moving from [0.0, 0.0] to [−π/3, 2π/3]: velocity, acceleration, and torque at the joints.

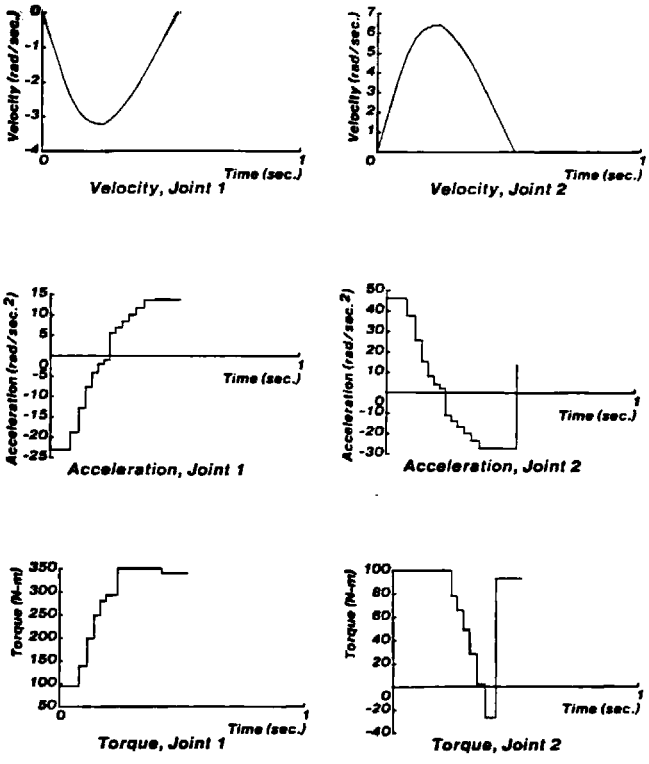
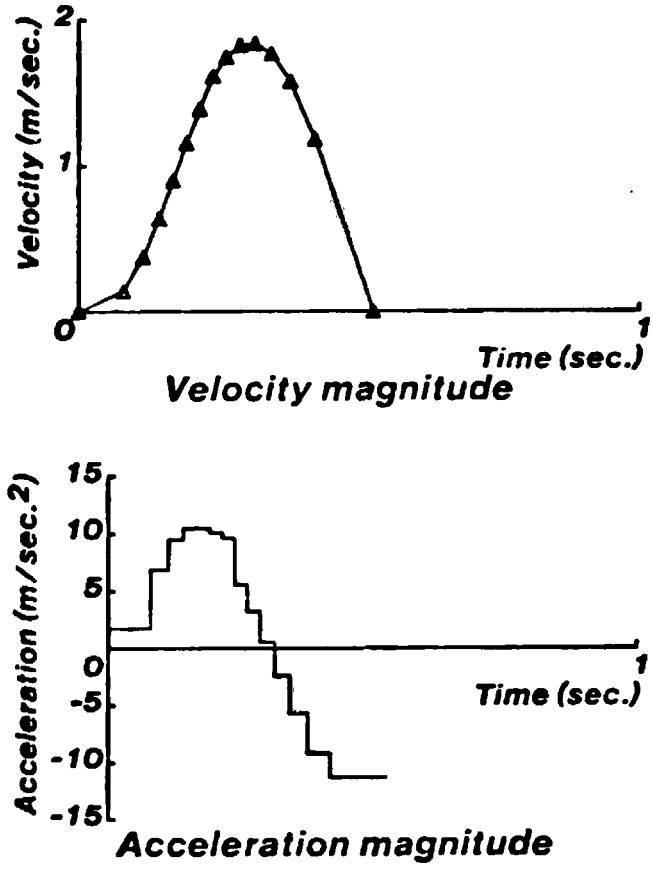


Fig. 9. Moving from [0.0, 0.0] to [−π/3, 2π/3]: magnitude of tip velocity and acceleration in Cartesian space.



5. Discussion and Conclusions

A method for finding minimum-time paths for manipulator arms, given dynamic, kinematic, and geometric constraints was developed. The method is based on the creation of a state-space search tree representing all possible solutions and searching it for the best one. The search tree is smaller than one would initially expect, because velocity space was not actually tessellated. Instead, it was recognized that the requirement for time optimality limited the number of possible velocities at the end of each segment.

The algorithm as currently formulated and implemented is not suitable for routine off-line trajectory planning, as even a 15 × 15 grid required minutes to hours of computation. We see the current primary benefit as providing the first idea of what the minimal time path looks like, however long the computation time, which was not possible by any means before. From this knowledge it would be possible to compare the performance of more efficient trajectory planning methods, and perhaps develop a method that is faster but sufficiently close to optimal. It might be possible to improve the algorithm itself by new graph search methods or hierarchical methods that zero in on the optimal path while restricting the state-space expansion. The Lisp Machine itself is quite slow for numerical operations, and a more algebraically tuned system might speed the algorithm considerably. Finally, it may be possible to recast the algorithm into a faster parallel computation.

Several methods were examined for pruning the search tree of possible paths, but further improvements are required. Ideally, one would like a lower bound estimate of the cost from any point in state space to the goal in order to aid graph pruning (Winston 1984), by a process of comparing the sum of this estimate and the current cost at the state-space node to the upper

bound. If a path can be thrown out early, then no time is wasted on its descendents. We had considered as a lower bound the time to travel from the present node to the goal while accelerating only and without endpoint velocity constraint, but we cannot prove this bound is valid. An attempt at finding such a bound is described in Shin and McKay (1985*b*), but it is unclear what effect their restrictions have on the validity of the results. It seems that a true lower bound would presume some form of analytic solution to the very problem we are trying to solve.

The present tessellation, which allows only three directions of motion from each node, may exclude better paths. It would be desirable to find better tessellation techniques that allow finer resolution of direction and distance, yielding better approximation of curves by straight-line segments, yet curbing the combinatorial proliferation of nodes and retaining efficient management of the grid. One possibility is to solve the problem iteratively, beginning with a coarse tessellation and then building a finer tessellation around subsequent solutions. A finer tessellation could be built in a restricted band around the coarse solution to limit the combinatorics, and the directions could be made less sharp to mitigate the corner problem. Additionally, a finer tessellation could automatically be imposed around apparent corners in joint space.

Lacking that, better solutions to the corner problem are required, since the present approximation implies unrealizable infinite accelerations at the corner point. If an additional penalty were placed on the redefinition of the velocity at a corner point, the tendency toward straight-line joint paths would be enhanced. The current restriction to directions, where at least one joint moves closer to the goal, should be relaxed, since conceivably a solution might involve temporarily moving diametrically away from the goal.

The algorithm is easily extended to handle configuration constraints due to joint excursion limits and to workspace obstacles, when the latter are converted to obstacles in joint space (e.g., Lozano-Perez 1982). Since these translate into forbidden regions in joint space, the grid nodes within the forbidden regions are excluded from the tessellation. The ease of incorporating obstacle avoidance is a general feature of joint-space tessellation, as also noted by Brown (1984) and Kornhauser and Brown (1985), and is an advantage over trajectory representation by parameterized paths.

Finally, the algorithm presented here indicates that the fastest motion between two points is close to a straight line in joint space. The curvature at the beginning and end of the path in Fig. 4 may represent an interaction between the shortest path in joint space and the natural dynamics of the movement. Presently, it cannot be completely ruled out that tessellation problems contributed to some extent in forcing a nearly straight-line solution in joint space. The curvature at the beginning and end of the path in Fig. 4 can be taken as some indication that the tessellation problems did not always dominate. Preliminary results with a parameterized path method have verified the essential linearity of the optimal paths in joint space.

The complexity of the state-space search may currently rule out six-degree-of-freedom implementations, but an implementation on a three-joint manipulator seems possible. Conceivably, a sparser grid might yield knot points for conventional polynomial interpolation that would nevertheless give superior paths. In addition, our algorithm can serve as a starting point for the development of faster, less exact algorithms, and for the verification of their results.

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