Strategic Influencers and the Shaping of Beliefs

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Abstract

Influencers, from propagandists to sellers, expend vast resources targeting agents who amplify their message through word-of-mouth communication. While agents differ in network position, they also differ in their bias: agents may naturally read articles with a particular slant or buy products from a certain seller. Absent competition, an influencer prefers targeting central agents and those biased *against* it. If agents are unbiased, competition leads to influencers targeting more central agents. However, when agents have heterogeneous biases and competition is intense, the incentive to deter one's rival dominates. Influencers protect their base, targeting those with similar beliefs in equilibrium.

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1 Introduction

Strategic influencers, ranging from propagandists to sellers, expend vast resources targeting individuals, employing tools such as customized advertisements, sponsored posts, and online recommendations (Fainmesser and Galeotti [2015, 2020]). Existing technologies allow them to target recipients at a granular level, increasing direct interaction. Importantly, their message can be amplified by the peer networks of those they target. But whom should they target? Many suggest that it is best to target the most central agents so as to maximize the diffusion of one's message (e.g. Coleman et al. [1966]; Galeotti and Goyal [2009]; Banerjee et al. [2013; 2019]; Beaman et al. [2021]). However, a critical feature of the settings used to support these results is that agents only interact with each other. In reality, agents interact not only with their peers but with sources external to their peer network. Importantly, agents are often biased and thus may be naturally inclined to interact with external sources that reinforce their initial beliefs (i.e. read articles with a specific slant or buy products from a certain seller).

To understand why bias matters, consider a social network where users learn about a political event from their peers and from articles they read while browsing the internet. These users may be naturally biased in one direction or the other. That is, when they browse the internet, they will not only see articles from external news sources that specifically target them but also articles from like-minded media. Suppose a left-leaning propagandist targets the users in this social network with the goal of driving the "average" opinion regarding some event towards the left. While the propagandist will consider a user's centrality in the network, it must also consider the user's bias. Why? Because the marginal gain from targeting a user, say, who already receives persistent impressions from *other* left-leaning sources is much lower than the potential gain from targeting a user who is biased to the right. In other words, targeting users and displacing attention directed toward sources with a similar slant is not as beneficial as displacing attention that would otherwise be directed toward opposing sources.

Thus, with limited resources, influencers may need to trade-off between these two features. For example, should a politician use campaign funds to reach across the aisle or target her base, and how should she balance this decision with the benefit from targeting central agents? Likewise, sellers can benefit from word-of-mouth communication by targeting influential consumers in the network, but is this irrespective of the consumers' bias? Given limits on marketing budgets, sellers must also determine whether it is better to advertise to consumers biased towards their competitors or focus on shoring up their existing clientele (Iyer et al. [2005]).

Absent persistence of initial beliefs, DeGroot models of belief formation in social networks cannot accommodate this trade-off (Golub and Jackson [2010]; Golub and Sadler [2016]). Since initial biases are drowned out over time, influencers who care about long-run beliefs in the network are concerned solely with an agent's location in the network.

In my model, agents learn from both their neighbors and external sources of information, which include strategic influencers as well as non-strategic "private sources". The influencers push their specific messages, while the non-strategic private sources reinforce an agent's bias. Influencers have a fixed budget and target agents by spending money to increase the per-period frequency of direct engagement between themselves and the agent. I analyze settings with a single influencer and with two competing influencers. I show that there are significant differences in influencers' targeting strategies between the two settings. In the single-influencer setting, the influencer discounts an agent's centrality by their initial, persistent belief. Targeting agents biased in the influencer's favor is not valuable because one is merely displacing attention directed towards private sources already sending similar messages. Thus, the influencer favors agents biased in the opposite direction. In the competitive setting, where two influencers engage in a simultaneous move game, each still faces the same trade-off between centrality and the dissimilarity in agents' bias. However, in equilibrium, each influencer may end up targeting agents biased toward her. This is due to a deterrence incentive absent in the single-influencer setting. With competition, there is a first-order benefit from spending to reduce the influence of one's competitor. When competition is intense, the deterrence incentive dominates, and influencers expend resources to prevent agents from being 'turned' by their rivals. As a result, they focus on targeting their base: agents biased in their direction.

In the next section, I describe the model and contrast it with the relevant literature. Section 3 examines optimal targeting in a single influencer setting. Section 4 analyzes equilibrium outcomes under competition and examines the effect of network structure on equilibrium payoffs.

2 Model

There are N agents, labeled $\{1,2,\ldots,N\}$, each holding an initial belief $b_i \in [0,1]$ about a binary event. The terms "event" and "belief" can be interpreted in a number of ways. For example, b_i can be the likelihood of an agent purchasing a product from one seller in a duopoly. In a voting context, b_i reflects the likelihood of voting for a specific party in a two-party contest.

I refer to the initial belief as the bias of the agent. The agents are embedded in a *peer network*, a directed graph defined by a non-negative, row stochastic matrix P. P_{ij} represents the frequency with which agent i interacts with j or the relative level of trust agent i places in j.[1]

External to the peer network is a set of *external sources*. These include two strategic influencers and a group of "private sources". Formally, there is one influencer, M_1, with belief 1 and another influencer, M_2, with belief 0. The "belief" of the influencer is the message it desires to promote. In addition to the influencer, there is a "private source", S_i, corresponding to each agent i, with belief equal to agent i's initial belief.

2.1 Interaction and Communication

The influencers and the private sources constitute *the set of external sources*. Fix a level of external attention $\alpha_i \in [0,1)$. Each period, agent i learns from the external sources with probability $\alpha_i \in [0,1)$. With probability $1-\alpha_i$, agent i interacts with her peers according to the matrix P. When learning from the external sources, agent i receives information from M_1 and his agent-specific private source S_i. To illustrate, consider a setting where M_1 is a liberal propagandist, and M_2 is a conservative propagandist that target a moderate agent who learns outside his peer network through browsing the internet. While he reads articles from both propagandists, he also receives information from like-minded media (private source S_i) that reinforce his initial belief (bias). Importantly, targeted spending by the influencer affects the interaction rate between an agent and her private source: it *diverts* attention away from i's private sources towards the influencer. The private sources, S_i, are non-strategic with no targeting ability, allowing me to capture a passive persistence of bias.

The probability with which agent i learns from *an external source* is fixed. However, conditional on learning from external sources, the probability agent i learns from a given influencer is endogenous. If M_j targets agent i it can secure some portion of the attention that i gives to external sources. Formally, each influencer has a budget of 1 to allocate across agents in the network. The allocation decision is M_j's targeting strategy. Given a targeting strategy $a^j \in [0,1]^N$, a competition function $f : \mathbb{R}^2 \longrightarrow [0,1]$ determines the fraction of α_i that M_j wins.

[1] In the DeGroot learning model, agents update beliefs each period via a weighted average of their neighbors' beliefs. The weights are proportional to the frequency of interactions encoded in P. Therefore, beliefs in the t^{th} period are $P^t b$. This learning process is equivalent to an "article sharing process" of the following form. There is a binary event θ and two types of articles: one which pushes the message $\theta = 1$ and one that pushes $\theta = 0$. Let b_i be the initial frequency with which agent i reads and shares an article that pushes $\theta = 1$. Then $P^t b$ is the expected fraction of articles pushing $\theta = 1$ that agents have received by time t.

The fraction of α_i an influencer wins, $f(\cdot,\cdot)$, depends on her spending and her competitor's. If agent i learns from external sources, he learns from M_j with probability $f(a_i^j, a_i^{-j})$ and from S_i with probability $1 - f(a_i^1, a_i^2) - f(a_i^2, a_i^1)$.[2]

ASSUMPTION 2.1

1. $f(x,y) + f(y,x) \leq 1$
2. *f increasing and concave in its first argument*
3. *f is decreasing and convex in its second argument*

The first condition is technical, ensuring that the combined fraction of α_i won by both influencers does not exceed 1. The second condition is a standard diminishing returns property from additional spending. The third condition is a diminishing returns effect of the *opposition's spending* on one's winnings.

Each external source can be viewed as an additional node in the network that does not update its own belief. While other nodes (i.e. the agents) learn from external sources, each external source only learns from itself. For ease of exposition, define diagonal matrices D^α and D^S, where $D^\alpha_{ii} = 1 - \alpha_i$ and $D^S_{ii} = \alpha_i(1 - f(a_i^1, a_i^2) - f(a_i^2, a_i^1))$. The first two represent interaction rates between agents and external sources, and the third represents the distance of agents' initial beliefs from 1. Communication and learning can then be described via weighted-average updating according to the $(2N+2) \times (2N+2)$ matrix P^*:

$$P^* = \begin{bmatrix} D^\alpha P & \alpha f(a^1, a^2) & \alpha f(a^2, a^1) & D^S \\ \mathbf{0}_{1\times N} & 1 & 0 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & 0 & 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{I}_{N\times N} \end{bmatrix}$$

The top left block $D^\alpha P$ corresponds to peer-to-peer communication. In an abuse of notation, $\alpha f(a^j, a^{-j})$ denotes the vector of interaction frequency between agents and M_j, with the i^{th} component equal to $\alpha_i f(a_i^j, a_i^{-j})$. D^S corresponds to the direct interaction rates from the fixed private sources. The last three rows of the block matrix correspond to the external sources: each external source places weight 1 on itself.

[2]Mixed strategies would correspond to a probability measure over $\{a^1 \in [0,1]^N | \sum a_i^1 \leq 1\}$. In the single-influencer setting, the strategy space can be restricted to pure strategies because f is concave.

Agents update their beliefs each period according to a DeGroot learning rule: beliefs at time t are $P^{*t}b$.[3] The influencers want the average limiting belief in the network to match their own. The average limiting belief in the network is:

$$B(a^1,a^2) = \lim_{t\to\infty}\frac{1}{N}e^T P^{*t}b, \text{ where } e^T \text{ is a row vector of 1's.}$$

Influencer M_1 wishes to maximize $B(a^1,a^2)$ while M_2 wishes to minimize it.[4]

The choice to use long-run limiting beliefs as the objective is standard in the literature. In a marketing context, limiting beliefs can be interpreted as long-run reputation or long-run purchase frequency (Bimpikis et al. [2016]). In a political context, each influencer represents a political party, and the long-run beliefs can be interpreted as the steady-state probability that a given agent casts his vote for a particular party. I discuss other influencer objectives in Sections 2.2 and 5.

2.2 Relation to the Literature

My paper fits into the theoretical literature on opinion dynamics (see Golub and Sadler [2016] for a survey). It offers a model of learning that incorporates both DeGroot learning and the persistence of agents' initial beliefs.[5] The learning process in my model is related to that described in Friedkin and Johnsen (1999), where agents are embedded in a network, learn from their peers, and are heterogenous in their susceptibility to interpersonal influence. A lack of consensus in the limit arises when some agents are not susceptible to peer influence (e.g. when some agents are "stubborn", as in Acemoglu et al. [2010] and Yildiz et al. [2013]). My model can be viewed similarly by interpreting $1-\alpha_i$ as agent i's susceptibility to his peers and α_i as his susceptibility to the external sources. Moreover, I also have stubborn nodes: the influencers and private sources. However, the stubborn influencers can choose whom to link to and the intensity of that link. In effect, agent susceptibility toward the external sources is endogenous.

[3] If $\alpha_i = 0$, which means agents do not interact with any external sources, then beliefs at time t are $P^t b$ as in the classic DeGroot learning models.

[4] I extend Theorem 3.1 in the proof of Theorem 4.3 to show this quantity is well-defined.

[5] Chandrasekhar et al. (2020) provide empirical evidence demonstrating that simple DeGroot learning mirrors observed patterns of behavior. Molavi, Tahbaz-Salehi, and Jadbabaie (2018) and Dasaratha, Hak, and Golub (2019) provide microfoundations. Criticisms of the rule are highlighted by DeMarzo, Vayanos, and Zwiebel (2003), who show that agents do not account for repetition of information under DeGroot learning. However, they show that accounting for this bias requires significant computing power. Thus, there are bounded-rationality arguments in favor of the learning rule.

The idea of influencer targeting to spread information in a network is related to the research on "seeding" a network (e.g. Kempe, Kleinberg, and Tardos [2003, 2005]; Banerjee et al. [2013, 2019]; Kim et al. [2015]). However, my paper incorporates *strategic competition in diffusion*. The addition of competition in seeding causes influencers to consider how seeding reduces the influence of their rivals, a force absent in a single-influencer seeding setting. My model of competition between influencers contrasts with Lever (2010) and Grabisch et al. (2018), which also look at a competitive setting where influencers attempt to shape beliefs in a network. In Lever (2010), politicians spend money to influence voters embedded in a network. However, spending in his model has a one-time effect on voters' *initial* beliefs. Thus, voters' importance is dictated entirely by their effect within the peer network. In my paper, agents interact with influencers repeatedly. As a result, agents that are influential within the peer network do not have the same importance.[6] In Grabisch et al. (2018), influencer strategies are restricted to the formation of a single link in the network, and the effect of this link formation is fixed. I allow influencers to choose both the breadth and intensity of their targeting. This permits an understanding of how the characteristics of competition affect equilibrium targeting behavior along the bias and network centrality dimensions.

Mostagir et al. (2022) also examine how a single influencer manipulates long-run beliefs of agents in a network when agents receive impressions across multiple external sources. A crucial distinction is that the influencer is not budget constrained and only benefits from targeting if an agent's belief reaches a particular threshold. One can view Mostagir et al. (2022) through the lens of my model by changing the influencer's objective function from the average limiting belief to a threshold objective and replacing the budget constraint with a marginal cost associated with spending. Then, when there is no competition, an influencer may find it optimal not to spend resources targeting (e.g. the targeting necessary to push beliefs above the threshold is too costly). However, my model highlights the importance of competition even with the threshold objective. While an influencer may not engage in targeting in the absence of competition, the presence of competition can encourage an influencer to spend to reduce *the rival's ability to manipulate beliefs.*

The most closely related works are Bimpikis et al. (2016) and Goyal et al. (2019), which study competitive diffusion between two firms on a network. However, agent bias is not persistent, and firms only care about the average fraction of "impressions" generated. Specifically,

[6] In fact, Lever (2010) is a special case of the model in this paper when α approaches zero. See Section 3.2.

Bimpikis et al. (2016) is a special case of my framework when the influencer's objective is to maximize the long-run weights agents place on her (i.e. M_1 maximizing the average of the elements in the $(N+1)^{th}$ column of $\lim_{t\to\infty} P^{*t}$). An influencer with this objective is agnostic about how agents interact with other external sources. In my model, influencers must be concerned with the *distribution* of impressions generated and the *distribution* of long-run weights across *all external sources*. This distinction is significant not just at a technical level but in terms of applications. For example, suppose agents make binary choices based on their beliefs about a state. An agent's limiting belief is based on the entire distribution of messages he receives, not just the fraction of messages received directly and indirectly from an influencer.

Finally, the qualitative features of the results about targeting behavior in the presence of competition fit within the general literature on competitive targeted advertising (see Bergemann and Bonatti [2011] for a survey). Iyer et al. (2005) find that firms always target consumers already biased towards their products. However, their model is one of price competition with no social learning. In my model, such behavior depends on how targeted spending by an influencer affects the attention agents pay to her competitor. When influencers target like-minded agents in my model, it is due to an incentive to deter one's competitor and protect one's base. This particular economic force also contrasts my result with Sadler (2020), which examines opinion dynamics in a single-influencer setting. Sadler (2020) identifies conditions under which the influencer targets her base, but this is due to risk-aversion on the part of the influencer. In my model, influencers are risk-neutral, but appeals to the base occur due to competition. Van Zandt (2004) and Johnson (2013) look at strategic targeting when consumers pay selective attention to advertising messages. I incorporate a reduced-form version of selective attention through the competition function f: influencers cannot control all of an agent's attention. However, consumers are still learning in my model when they are not directly interacting with the influencers. Thus, an influencer must consider the messages agents receive when they are not giving her their attention.

3 Optimal Targeting: Single-Influencer

3.1 Targeting Strategy

I begin by considering a setting with a single strategic influencer, M_1. To "convert" the model to this setting, I simply eliminate the second influencer and use a single-variable competition function f. Communication and learning can then be described via weighted-average updating according to the $(2N+1)\times(2N+1)$ matrix P^*:

$$P^* = \begin{bmatrix} D^\alpha P & \alpha f(a^1) & D^S \\ \mathbf{0}_{1\times N} & 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times N} & 0 & \mathbf{I}_{N\times N} \end{bmatrix}$$

The goal of M_1 is to target agents to drive the average limiting belief in the network as close to 1 as possible. Her optimization problem is:

$$\max_{a_i^1, i=1,\ldots,n} \quad B(a^1)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} a_i^1 \leq 1 \; and \; a_i^1 \geq 0 \; for \; all \; i$$

The optimal targeting strategy takes into account the following features:

1. The agent's bias.
2. The frequency with which an agent interacts with external sources.
3. The agent's position within the network.

The last two quantities are given by b_i and $\alpha_i f(a_i^1)$, respectively. Regarding the first, how does one quantify the importance of an agent in the network? In each period, each agent i receives a message from outside their peer network with probability α_i. From the influencer's perspective, it must quantify how much her message gets dispersed through the network once a given agent i receives said message. Consider the matrix $\sum_{t=0}^{\infty}(D^\alpha P)^t$. The $(j,i)^{th}$ entry represents the time-discounted expected number of paths between j and i. In other words, the long-run influence i has on j. The matrix $\sum_{t=0}^{\infty}(D^\alpha P)^t$ can be written succinctly as $(I-D^\alpha P)^{-1}$, where I is the $N\times N$ identity matrix. Denote the vector $e^T(I-D^\alpha P)^{-1}$ as $\hat{q}$. Each component $\hat{q}_i = [e^T(I-D^\alpha P)^{-1}]_i = \sum_{j=1}^{N}(\sum_{t=0}^{\infty}(D^\alpha P)^t)_{ji}$ quantifies the total long-run influence agent i has on the rest of the network. However, with probability α_i, each agent i interacts with a source outside his peer network. Thus, the network centrality is scaled-down by α_i. Let q denote the vector of the scaled-down network centrality measures. That is, $q_i = \alpha_i \hat{q}_i$. Call q **the attention-adjusted centrality vector**.[7]

[7] Related is Katz-Bonacich centrality (Bonacich [1987]; Bloch et al. [2017]). Also related is the DeGroot centrality measure in Mostagir et al. (2022). In that paper, influencer targeting is a binary strategy (to target or not to target), and the frequency of interaction conditional on targeting is fixed. As a result, the amount of attention agents devote to external sources is endogenous. In mine, the available attention toward external sources is fixed,

The average limiting belief can be decomposed into a linear sum of each of these features.

Theorem 3.1 *The average limiting belief in the network is* $\frac{1}{N}\sum_{i=1}^{n} q_i[(1-b_i)f(a_i^1)+b_i]$*. The influencer targets more agents with high attention-adjusted centrality and those with opposite bias. Formally, given optimal targeting strategy* a^{1*}*:*

1. *If* $(1-b_i)q_i > (1-b_j)q_j$ *then either* $a_i^{1*} > a_j^{1*}$ *OR* $a_i^{1*} = a_j^{1*} = 0$*.*
2. *If* $a_i^{1*} > a_j^{1*}$ *then* $(1-b_i)q_i > (1-b_j)q_j$*.*

Proof: See Appendix. ■

The sharp characterization of the average limiting belief in the network highlights the fundamental forces at work. Unlike traditional DeGroot learning models, the average limiting belief will not be each agent's limiting belief. A consensus will not emerge because the influencer and private sources act as "stubborn nodes" in the network that never update their beliefs.

All things equal, agents with a higher attention-adjusted centrality are more valuable to target. The attention-adjusted centrality measure $q_i = \alpha_i \hat{q}_i$ is a weighted network centrality measure: each agent i's contribution to the limiting belief is scaled by the amount of direct attention that the agent gives to external sources each period. Notice, though, that the influencer considers the messages agents receive from the private sources: q_i is weighted by $1-b_i$. An influencer must consider the agent's initial beliefs as those are reinforced via the residual attention paid to the private sources; the influencer must consider the agent's bias. Agents with an initial belief farther away from 1 are more important for targeting. Since the influencer faces no competition, there is less need to target agents who are already biased towards her message: such agents will receive similar messages anyways! Within the single-influencer setting, an influencer prefers targeting agents with initial beliefs *farther* from her message. In the extreme case where agents have either belief 1 or 0, all agents with a belief of 1 are ignored. That is, $a_i^1 = 0$ when $b_i = 1$.

The single-influencer setting should be interpreted as an environment where the strategic influencer faces passive competition, and so her targeting can displace the attention agents pay to their private sources. For example, when faced with passive competition, a seller will target

but the distribution of attention is not. Hence, my centrality measure is static: it takes into account transmission within the peer network scaled by the amount of available attention directed outside the peer network.

agents biased towards its competitor; a propagandist will target those biased in the opposite direction; a political candidate will target prospective voters leaning towards the rival candidate.

3.2 Attention-Adjusted Centrality versus Peer Influence

The attention-adjusted centrality vector q that the influencer uses to quantify the targeting value of an agent differs from other common measures of network influence, such as eigenvector centrality. If there were no external sources and agents only interacted with one another, the limiting beliefs in the network would be given by wb, where w is the unique unit left-hand eigenvector of P corresponding to the eigenvalue 1.[8] In Lever (2010), influencers target agents based on eigenvector centrality because of the limited ability of influencers to interact with agents. If influencers had a one-time ability to perturb the initial beliefs of agents, then each would target agents according to w. In my setting, influencers interact with agents repeatedly, leading to the attention-adjusted centrality q becoming the vector of interest. As the vector of attention paid towards external sources, α, approaches 0, q approaches the span of w.

Proposition 3.2 *Consider any strictly decreasing positive sequence $\left\{\alpha^{(j)}\right\}_{j=1}^{\infty}$, $\lim_{j\to\infty}\alpha_i^j = 0$ for each i. For any $\varepsilon > 0$, there exists L, such that for $j > L$:*

$$\left|\left|\frac{1}{N}q - w\right|\right|_2 < \varepsilon$$

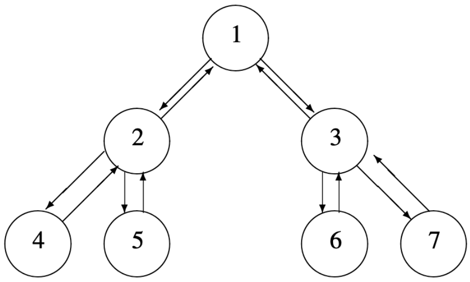
Proof: See Appendix.

In particular, when all agents are constrained to interact with their peers at the same rate $\alpha_i = \alpha$, Proposition 3.2 states that there is a cutoff $\bar{\alpha} > 0$ such that for $\alpha_i < \bar{\alpha}$ the *rank ordering* of the agents according to q corresponds to that of w. For $\alpha > \bar{\alpha}$, these measures may diverge. Observe that as $\alpha \longrightarrow 1^-$ for each i, $\sum_{j=0}^{\infty}(1-\alpha)^j P^j$ puts more weight on the early terms. To illustrate, consider the following network and centrality measures for different values of α:

[8] Each component, w_i, is the relative impact that agent i has on the rest of the agents *when there is only peer-to-peer learning*. See Jackson (2010) for a discussion.

Example 1

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0 & 0.1 & 0.1 & 0 & 0 \\ 0.4 & 0 & 0.4 & 0 & 0 & 0.1 & 0.1 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \end{pmatrix}$$



$$\alpha \to 0: \quad w = \begin{bmatrix} 0.32 & 0.24 & 0.24 & 0.05 & 0.05 & 0.05 & 0.05 \end{bmatrix}$$

$$\alpha = 0.2: \quad q = \begin{bmatrix} 8.747 & 7.73 & 7.73 & 2.70 & 2.70 & 2.70 & 2.70 \end{bmatrix}$$

$$\alpha = 0.6: \quad q = \begin{bmatrix} 1.99 & 2.12 & 2.12 & 1.36 & 1.36 & 1.36 & 1.36 \end{bmatrix}$$

As α increases, agents 2 and 3 become more central because they are separated from agents $4-7$ by a single edge. The probability of the influencer's message being received by those outside nodes indirectly from agent 1 decreases as agents pay less attention to their peers. Agent 1 is most influential when only considering peer effects, but it influences peripheral agents **through** *agents 2 and 3. As α increases, these middlemen become more important.*

Example 1 highlights the tension between direct and indirect targeting. In the case of the tree, there is a bottleneck effect where the root node transmits its beliefs slowly through other agents. Thus, changes in α will have a greater effect on the centrality measure of the root. As α increases, it makes targeting peripheral agents more beneficial.

4 Competition

To incorporate competition, I add a second influencer, M_2, with belief 0. Influencer M_1 wishes *to maximize* the average belief, while M_2 wishes *to minimize* it. To provide an interpretation, consider two firms competing for customers. The initial belief represents an agent's natural, passive bias towards each firm's products. The long-run beliefs represent the long-run frequency of purchases from a given firm. In a political context, each influencer represents a rival political party. The long-run beliefs represent the long-run probability with which a given agent sides with a particular party.

When both influencers are strategic, the optimization problems for each, fixing the targeting decision of her competitor, are as follows:

M_1	M_2
$\max_{a_1^1,\ldots,a_N^1} B(a^1,a^2)$	$\max_{a_1^2,\ldots,a_N^2} 1-B(a^1,a^2)$
s.t. $\sum_{i=1}^N a_i^1 \leq 1,\ \ a_i^1 \geq 0$ for each i	s.t. $\sum_{i=1}^N a_i^2 \leq 1,\ \ a_i^2 \geq 0$ for each i

The influencers engage in a simultaneous move game where each selects a targeting strategy.

Definition 4.1 *A* **pure strategy equilibrium** *is a profile of pure strategies* (a^1,a^2) *such that each influencer is best-responding to her competitor's targeting strategy.*

I focus on pure strategy equilibria. Not only are they guaranteed to exist, but mixed-strategy equilibria do not. I show this in the proof of Theorem 4.3 in the appendix. Extending the proof of Theorem 3.1 to incorporate competing influencers, the average limiting belief in the network under any targeting profile is:

$$B(a^1,a^2) = \underbrace{\frac{1}{N}\sum_{i=1}^N q_i f(a_i^1,a_i^2)(1-b_i)}_{\substack{\text{Gain from direct interaction} \\ \text{“Single-Influencer” Component}}} - \underbrace{\frac{1}{N}\sum_{i=1}^N q_i f(a_i^2,a_i^1) b_i}_{\substack{\text{Gain From Reducing} \\ \text{Competitor's Influence}}} + \underbrace{\frac{1}{N}\sum_{i=1}^N q_i b_i}_{\substack{\text{Avg. Belief} \\ \text{w/ no Influencers}}}$$

Looking at the expression for the average limiting belief highlights important incentives in the competition game. Both influencers weigh centrality in the same manner. That is, all else equal, the more central an agent, the higher the marginal gain from targeting that agent. How influencers treat agents based on their initial, persistent beliefs is not immediate. An influencer's spending on agent i has two effects: it *increases* direct interaction with the agent and *decreases* her competitor's direct interaction with the agent. The former is scaled by $1-b_i$ while the latter is scaled by b_i. Hence, there are benefits from targeting those with beliefs far from 1 and those with beliefs close to 1.

The influencer objective functions are reminiscent of Colonel Blotto games. In fact, my model offers a microfoundation for such games. In traditional Blotto games, “winning" is discrete. One can interpret this game as a Blotto game where winning is continuous, battlefields are of size q_i, and each influencer has advantages on some battlefields over others. The size of battlefield i is the share of information the agents in the network receive due to direct and indirect connection to agent i.

4.1 Value of Optimal Targeting

Before analyzing equilibrium dynamics when both influencers are strategic, it is important to quantify the value of strategic targeting. How much better does optimal targeting do versus merely targeting each agent equally? Furthermore, in what networks does optimal targeting lead to maximal gain over uniform targeting?

In my model, communication occurs repeatedly, and so the message sent by the influencer will always be received. What the influencer cares about is the distribution of the fraction of messages received by each agent. Since agents receive messages from other sources (e.g. private sources), influencers will also be concerned with the content of the other messages an agent receives. These features amplify the significance of indirect learning in the network. As a result, knowledge of the network is critical for an influencer. To illustrate, consider the following example using the Tullock competition function, $f(x,y) = \frac{x}{x+y+\delta}$ for $\delta > 0$.

Example 2 *Given a peer network P, suppose the competition function is* $f(x,y) = \frac{x}{x+y+\delta}$ *for* $\delta > 0$*.* M_2 *spreads its budget uniformly, which means* $a_i^2 = \frac{1}{N}$ *for each i. As a result,* M_1*'s optimization problem is the following:*

$$\max_{a_1^1,\ldots,a_N^1} \frac{1}{N}\sum_{i=1}^{N} q_i[(1-b_i)\frac{a_i^1}{a_i^1+\frac{1}{N}+\delta} - b_i\frac{\frac{1}{N}}{a_i^1+\frac{1}{N}+\delta} + b_i]$$

$$subject\ to \sum_{i=1}^{N} a_i^1 \leq 1\ and\ a_i^1 \geq 0\ for\ all\ i$$

A complete derivation of the optimal solution to the optimization problem can be found in the Appendix. Conditional on the optimal targeting strategy having full support (i.e. influencer spends a non-zero amount targeting each agent), the average limiting belief in the network under the optimal targeting strategy is:

$$1 - \frac{1}{N(2+N\delta)}\Big(\sum_{i=1}^{N}\sqrt{q_i\beta_i}\Big)^2,\ where\ \beta_i = (1-b_i)\delta + \frac{1}{N}$$

If M_1 *targets each agent equally, the average limiting belief is* $\frac{1+\sum_{i=1}^{N} q_i b_i \delta}{2+N\delta}$*.*

The example above illustrates the benefits of strategic targeting over simple uniform targeting. Notice that the gain in the payoff from strategic targeting over simple uniform targeting depends on the network structure (i.e. the attention-adjusted centrality vector q). Thus, for what networks and attention-adjusted centrality vectors q is the gain maximal?

Proposition 4.2 *Suppose $b_i = b$ for each i, and M_2 targets uniformly. Under optimal targeting, M_1's payoff is bounded above by $b + (1-b)f(1, \frac{1}{N}) - bf(\frac{1}{N}, 1)$. Moreover, for any $\varepsilon > 0$, there exists a peer network P such that M_1's payoff is within ε of $b + (1-b)f(1, \frac{1}{N}) - bf(\frac{1}{N}, 1)$.*

Proof: See Appendix. ∎

Proposition 4.2 provides a closed-form expression for the maximal payoff M_1 can achieve from knowledge of the network and targeting optimally. In particular, if $b_i = \frac{1}{2}$ for all i, Proposition 4.2 implies that for some networks, strategic targeting can provide a gain of approximately $\frac{f(1,\frac{1}{N}) - f(\frac{1}{N},1)}{2}$. The proof of the proposition reveals which networks yield the highest benefit from strategic targeting. Holding the α_i's fixed, consider a network with attention-adjusted centrality vector q, where $q_1 \geq q_2 \geq \ldots \geq q_N$. M_1 earns a higher payoff from strategic targeting when facing a network with attention-adjusted centrality q' such that $q'_1 > q_1$ and $q'_j < q_j$ for all $j \geq 2$. The influencer prefers if the network centralities are concentrated amongst a few individuals. In other words, the strategic influencer receives the highest payoff when facing a star network: a network where a single agent has significant influence over all peers. Likewise, the minimum payoff is achieved when the network is complete, as the influencer is forced to distribute her budget equally across agents. The intuition behind this is that when the attention-adjusted-centralities are dispersed, peer-to-peer learning is not as significant. Thus, targeting becomes less beneficial. In networks with highly dispersed centralities (e.g. star networks), the influencer, not worried about competition, can expend all her resources targeting the most central agents and benefit tremendously from peer-to-peer learning.

4.2 Equilibrium: Unbiased Agents

When both influencers are strategic and compete, what do targeting dynamics look like? To isolate the interaction between centrality and competition, suppose agents are unbiased: $b_i = \frac{1}{2}$ for all i. Such a setting may represent a duopoly that is initially undifferentiated or a political contest between two parties where agents are not biased toward any particular party.

Theorem 4.3 *Suppose $b_i = \frac{1}{2}$ for all i. Then there are only pure-strategy symmetric equilibria. Moreover, if f satisfies $\frac{\partial f(a,c)}{\partial x \partial y} \leq \frac{\partial f(c,a)}{\partial x \partial y}$ for $a < c$, then in any equilibrium, influencers spend more targeting central agents in the network than non-central agents.*

Proof: See Appendix. ∎

When agents are unbiased, the game is symmetric zero-sum, and competition leads to targeting agents symmetrically. Moreover, while a pure-strategy equilibrium is guaranteed to exist, Theorem 4.3 highlights that these are the *only* equilibria.

Theorem 4.3 also reveals that for a large class of competition functions f, all equilibria involve influencers targeting more central agents in the network. Competing influencers target symmetrically and focus their targeting on agents with high-attention-adjusted centrality.

To provide intuition regarding the condition on f, suppose an influencer spends a on an agent, and her competitor spends $c > a$. The term $\frac{\partial f(a,c)}{\partial x \partial y}$ represents the effect of the competitor's spending on the marginal return in direct interaction. Now, $\frac{\partial f(c,a)}{\partial x \partial y}$ can be interpreted as the effect of the competitor's spending on one's marginal return of deterrence (i.e. how the competitor's spending affects $\frac{\partial f(c,a)}{\partial y}$). Thus, $\frac{\partial f(a,c)}{\partial x \partial y} \leq \frac{\partial f(c,a)}{\partial x \partial y}$ for $a < c$ reflects the idea that overspending disincentivizes one's opponent from spending on that agent. To see this more clearly, consider M_1's objective, which is to maximize:

$$\frac{1}{N}\sum_{i=1}^{N} q_i \cdot \frac{1}{2} \underbrace{[f(a_i^1, a_i^2) - f(a_i^2, a_i^1)]}_{\text{Gain from targeting agent } i}$$

The function $h(x,y) = f(x,y) - f(y,x)$ represents the "normalized gain" from targeting agent i ("normalized" because it does not include the scaling via the attention-adjusted centrality). The partial derivative of h with respect to x is the sum of the marginal return in direct interaction and the positive externality created by reducing the competitor's direct interaction. The condition on f implies that $\frac{\partial h(a,c)}{\partial x \partial y} \leq 0$ for $c \geq a$. In other words, influencer spending decisions are partially strategic substitutes. Consequently, $\frac{\partial h(a,a)}{\partial x} > \frac{\partial h(c,c)}{\partial x}$: when influencers are spending small amounts on an agent, there is a greater gain to increased spending. Many competition functions satisfy this property, including the classical Tullock competition function, $f(x,y) = \frac{x}{x+y+\delta}$.[9]

4.3 Biased Agents

When agents have varying biases, the game is no longer symmetric, and asymmetric equilibria emerge. Do the insights from the single-influencer setting carry over? In equilibrium, do influencers focus on targeting those with dissimilar beliefs? The answer is not obvious because

[9] This competition function has been employed in a number of areas, including the economics of advertising, tournaments, and political economy. See Corchón (2007) for a survey.

the competitive setting introduces a benefit from targeting those with similar beliefs to reduce the competitor's influence.

To determine how influencers incorporate agents' biases in the presence of competition, I consider a class of networks that I call *balanced*.

Definition 4.4 *A network of N agents, N even, is said to be* **balanced** *if there exists a bijective map* $G:\{1,\ldots,N\}\longrightarrow\{1,\ldots,N\}$ *with* $b_i = 1-b_{G(i)}$ *and* $q_i = \alpha_i\hat{q}_i = \alpha_{G(i)}\hat{q}_{G(i)} = q_{G(i)}$.

In a balanced network, for each agent i with belief b_i and attention-adjusted centrality q_i, there is a unique agent j such that $b_j = 1-b_i$ and $\alpha_j\hat{q}_j = \alpha_i\hat{q}_i$. Individual agents may be biased in one direction or another, but there is no bias *on average*. Many networks have this structure, including the popular dumbbell network. However, a balanced network does not require "symmetry" of shape, merely symmetry of the attention-adjusted centrality $q_i = \alpha_i\hat{q}_i$, which is a significantly weaker condition. Considering such networks allows me to isolate the effect of the characteristics of competition on the incentive to target like-minded agents in equilibrium.

Recall the interpretation of the competing influencers as a model of duopoly competition between two firms fighting for customers. A balanced network is an environment with two groups of customers, those leaning towards one of the firms and the other leaning towards the second firm. Within each group, agents have differing intensities of bias. The constraint on centralities ensures that no one set of customers has a dominant influence over the other.

In the example below, I describe a game over a two-agent balanced network. The competition function is the classical Tullock competition function.

Example 3 *Let* $f(x,y) = \frac{x}{x+y+\delta}$*, with* $\delta\in(0,1)$*. Suppose the network has two agents with initial beliefs* $b_1 = 1$ *and* $b_2 = 0$*. Assume the attention-adjusted centrality measures satisfy* $q_1 = q_2 = q$*. Using the Karush-Kuhn-Tucker conditions of optimality, the following system of equations must be satisfied:*

$$\frac{a_1^2}{a_1^1+a_1^2+\delta} = \frac{a_2^2+\delta}{a_2^1+a_2^2+\delta} \text{ and } \frac{a_2^1}{a_2^1+a_2^2+\delta} = \frac{a_1^1+\delta}{a_1^1+a_1^2+\delta}$$

$$a_1^1+a_2^1 = a_1^2+a_2^2 = 1$$

Solving yields the unique equilibrium: $a^1 = (\frac{1-\delta}{2},\frac{1+\delta}{2})$ *and* $a^2 = (\frac{1+\delta}{2},\frac{1-\delta}{2})$.

Consistent with the single-influencer setting, the influencers spend more of their budget targeting the agent with a differing initial belief. The particular competition function used in Example 3 incentivizes targeting agents with different beliefs, aligning with the findings in the single-influencer setting. However, this will not always be true. The characteristics of the equilibria are sensitive to the properties of the competition function f. The Tullock competition function incentivizes influencers to "reach across the aisle" because it does not incentivize deterrence. To formalize this, I introduce the following definition:

Definition 4.5 *Competition is said to be* **intense** *if the following holds:*

1. $-\frac{\partial f(c,a)}{\partial y} > \frac{\partial f(c,a)}{\partial x}$ *whenever* $a < c$.
2. $\frac{\partial f(a,c)}{\partial x} - \frac{\partial f(c,a)}{\partial y} \geq \frac{\partial f(c,a)}{\partial x} - \frac{\partial f(a,c)}{\partial y}$ *whenever* $a < c$.

To provide intuition behind the definition, suppose an influencer is underspending on one agent and overspending on another relative to her competitor. The first condition represents the *deterrence incentive*: spending more on the agent she is underspending on will hurt her competitor more than spending on the one she is overspending on will help herself. The second condition represents a competitive incentive: there are weakly larger gains to be had from spending on agents one is underspending on than from continuing to spend on those one is overspending on. This condition is satisfied by numerous classical competition functions, such as the Tullock competition function from Example 3. The key property is the first, which the Tullock competition function does not satisfy.[10] To see this, notice that $\frac{\partial f(c,a)}{\partial x} = \frac{a+\delta}{(c+a+\delta)^2}$ and $-\frac{\partial f(c,a)}{\partial y} = \frac{c}{(c+a+\delta)^2}$. When M_1 and M_2 spend similar amounts on two agents, the marginal gain from direct interaction is greater than that from deterrence. Therefore, M_1 would spend more on the agent biased in the opposite direction.

Theorem 4.6 *Given a balanced network, if competition is intense, influencers spend more targeting agents with conforming beliefs.*

Proof: See Appendix. ■

[10]One can satisfy the definition with an extension of the Tullock competition function that incorporates the notion that agents are already aware of each influencer (i.e. $f(0,0) > 0$). For example, suppose $f(x,y) = \frac{x}{x+y+\delta} + \varepsilon(1-y)$ where $\delta \geq \frac{1+\sqrt{5}}{2}$ and $\varepsilon \in [\frac{1}{\delta}, \frac{\delta}{1+\delta}]$. Under f, competition is intense.

The proof of the theorem shows that for any pair of agents i and $G(i)$, each influencer spends more targeting the agent who is already biased towards her message. When competition is intense, the gain from protecting conforming agents outweighs the loss from reducing spending on agents with dissimilar beliefs. Each influencer benefits more from targeting agents that are more valuable to her competitor. Such agents are precisely the ones that are biased toward the influencer. The most powerful incentive is deterring the opposition and *protecting* one's conforming agents from being altered. Such a finding informs some of the applications highlighted in the introduction. The reason why political groups or opinion segments on TV direct resources to target their base, rather than making in-roads to others, may result from deterrence incentives. In a duopoly where firms compete for customers, firms will spend money targeting customers who are already biased towards purchasing their product.

The conditions needed in the definition of "intense competition" are to ensure that Theorem 4.6 holds *independent* of the magnitude of the biases and distribution of centralities q in the network. If one had more network information, such conditions could be relaxed. For example, if network centralities are not too dispersed so that each influencer targets each agent with a fraction $\varepsilon > 0$ of her budget, then $-\frac{\partial f(c,a)}{\partial y} > \frac{\partial f(c,a)}{\partial x}$ need only hold for $\varepsilon \leq a < c$.

Intense competition is sufficient but not necessary for influencers to spend more targeting like-minded agents. However, the first condition specified in the definition of intense competition is critical.

Proposition 4.7 *If in any balanced network, influencers spend more targeting those with conforming beliefs, then* $-\frac{\partial f(c,a)}{\partial y} > \frac{\partial f(c,a)}{\partial x}$ *whenever* $a < c$.

Proof: See Appendix. ■

Proposition 4.7 demonstrates that the deterrence incentive must be strong for equilibrium targeting to favor like-minded agents, *independent* of the magnitude of the biases and distribution of centralities q in the network. Importantly, if $b_i \in \{0, 1\}$ for each i, then this deterrence property is both necessary and sufficient for Theorem 4.6 to hold. The crucial feature that leads to targeting like-minded agents is whether targeted spending can reduce the competitor's ability to influence an agent. It is not so much whether spending leads to more direct interaction with a given agent but whether it can reduce direct interaction with one's competitor.

4.4 Biased Agents and Network Structure

In sections 4.2 and 4.3, the equilibrium limiting beliefs in the targeting game are $\frac{1}{2}$ due to the symmetry at play. In unbalanced networks with biased agents, this will not be the case. When networks are not balanced, and agents are biased, pinning down the explicit strategies is difficult. However, one can get a handle on comparative statics related to the effect of network structure on influencer equilibrium payoffs. Specifically, when agent bias is heterogenous, what networks would an influencer prefer to face when its competitor is also strategic? When one's competitor is "passive" (e.g. targeting uniformly), facing a star network is preferable. Does this remain the case when one's competitor is not passive?

Let $\mathscr{E}(\mathbf{q},\mathbf{b})$ denote the set of equilibrium strategies with attention-adjusted centrality measure $\mathbf{q}$ and initial bias vector $\mathbf{b}$.[11] For any $(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b})$, the quantity of interest is the payoff to M_j in the equilibrium (σ_1,σ_2), denoted $\Pi_j(\sigma_1,\sigma_2)$. Fix an initial bias vector $\mathbf{b}$, and without loss of generality, assume $b_1\geq b_2\geq\ldots\geq b_N$.

Proposition 4.8

$$\max_{\mathbf{q}}\max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b})}\Pi_1(\sigma_1,\sigma_2)\geq\max_{j\in\{1,\ldots,N\}} b_j+(1-2b_j)f(\tfrac{1}{j},\tfrac{1}{j})$$

Proof: See Appendix. ■

The expression $\max_{\mathbf{q}}\max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b})}\Pi_1(\sigma_1,\sigma_2)$ is the maximum possible payoff that M_1 can achieve in any network given initial bias vector $\mathbf{b}$. Proposition 4.8 provides a lower bound on this value. This lower bound is found by looking at equilibrium payoffs when influencers compete across dumbbell networks with t central nodes where the t central nodes share a *common* initial bias of b_t.

COROLLARY 4.9 *Suppose there are $K\leq N$ agents with initial bias $b>\frac{1}{2}$. M_1 prefers a "dumbbell" network with $t\leq K$ central nodes over a star network if and only if $f(\frac{1}{t},\frac{1}{t})<f(1,1)$.*[12]

Corollary 4.9 stands in stark contrast to Proposition 4.2. When an influencer's competitor is passive (i.e. uniform targeting), the influencer prefers to face a star network. That is not necessarily the case when both influencers are strategic. When centralities are more dispersed,

[11] Two networks with the same attention-adjusted centrality vector are indistinguishable from the perspective of the influencers.

[12] The condition on function f is not stringent. For instance, any function of the form $f(x,y)=\frac{c(x)}{c(x)+c(y)+\delta}$ for $c(\cdot)$ increasing and $c(0)=0$, satisfies the property in the proposition. Also, f submodular is a sufficient condition.

one's competitor must spread out its budget, allowing M_1 to take more advantage of agents' favorable bias. The result is particularly salient when all agents share a common bias towards M_1. Such a setting would arise if M_1 were an incumbent (e.g. a firm or political party) that has acted as a single-influencer for a length of time, resulting in a homogenous bias towards it. Proposition 4.8 and its corollary imply that if the incumbent were to face an unexpected competitor, the incumbent would prefer to compete over a complete network, where all agents have equal influence, rather than a star network.

5 Conclusion

I study how influencers target agents in a network to shape beliefs. I develop a model of competition between influencers over a network to demonstrate how competition affects targeting behavior. Without competition, an influencer trades off an agent's centrality with the dissimilarity of the agent's belief. With competition, this does not necessarily carry over, as there is a first-order benefit from deterring one's competitor. If agents are unbiased, this deterrence effect is weak. As a result, influencers target symmetrically according to agents' position in the network; they favor those with high centrality. However, when agents have varying biases and the deterrence incentive is strong, equilibria arise where influencers focus their efforts on targeting like-minded agents.

At a technical level, this paper provides a framework to analyze targeting behavior when influencers have objectives beyond maximizing the average belief of the network. For example, if an influencer is focused on maximizing long-run awareness, the objective would be to maximize the number of impressions generated. That is, influencers would care about maximizing the long-run weights agents place on them. Similarly, in other contexts and applications, it might be that the influencer only cares about agent beliefs in so far as those beliefs lead to specific actions. Thus, she is concerned with the number of agents whose limiting beliefs cross a specific threshold. Crucially, the model in this paper is sufficiently tractable to analyze *competitive* settings between influencers with such objectives, thereby allowing for an understanding of how competition affects targeting dynamics.

A Appendix

Lemma A.1 *If $\hat{P} = D^{\alpha}P$, for some $\alpha \neq \mathbf{0}$, then* $\lim_{t\to\infty} \hat{P}^t = 0$.

Proof: $\hat{P}$ is substochastic, and at least one row has sum strictly less than 1. Since P is aperiodic and strongly connected, $\hat{P}$ is as well. Therefore, $\hat{P}$ is *irreducible* and there exists $n \in \mathbb{N}$ such that $\hat{P}^n$ has all positive entries. By the Perron-Frobenius theorem, there exists $\lambda > 0$ such that λ is the largest eigenvalue of $\hat{P}$ and the associated unit *left* eigenvector v of $\hat{P}$ is strictly positive.

Let $\Psi \in \mathbb{R}^N_+$ be the positive vector such that $\Psi_i = \frac{1}{N}(1 - \sum_{j=1}^N \hat{P}_{ij})$. Thus, the i^{th} component of Ψ is the number that when added to *each* element of the i^{th} row of $\hat{P}$ ensures that the row sum is 1 $\implies \hat{P}' = \hat{P} + \Psi^T$ is stochastic. Now:

$$\lambda v = v\hat{P} \implies \lambda v_i = \sum_{j=1}^N \hat{P}_{ji} v_j \text{ for each } i$$

$$\implies \lambda = \sum_{i=1}^N \sum_{j=1}^N \hat{P}_{ji} v_j = \sum_{i=1}^N \sum_{j=1}^N (\hat{P}_{ji} - \Psi_j + \Psi_j) v_j = \sum_{i=1}^N \sum_{j=1}^N (\hat{P}_{ji} + \Psi_j) v_j - \sum_{i=1}^N \sum_{j=1}^N \Psi_j v_j$$

$$= \sum_{i=1}^N \sum_{j=1}^N (\hat{P}_{ij} + \Psi_i) v_j - N(\Psi \cdot v) = |\hat{P}' v| - N(\Psi \cdot v) \text{ where } |\cdot| \text{ denotes the standard L1 norm}$$

$$= 1 - N(\Psi \cdot V) < 1 \text{ because } \Psi \text{ is non-zero, non-negative vector}$$

As a result, $v\hat{P} = \lambda v \implies v\hat{P}^t = \lambda^t v \implies \lim_{t\to\infty} v\hat{P}^t = 0$. Since v is positive and $\hat{P}$ is non-negative, $\lim_{t\to\infty} \hat{P}^t = 0$. ■

Proof of Theorem 3.1: The top left block, $D^{\alpha}P$, represents the weightings on agents within the network, *excluding* the influencer and private sources. By Lemma A.1, the left block of P^{*t} converges precisely to the zero matrix as $t \longrightarrow \infty$. The bottom $N+1$ rows are $e_{N+1}, e_{N+2}, \ldots, e_{2N+1}$, respectively.[13] I only need to calculate what happens to the first N entries of the last $N+1$ columns of $\lim_{t\to\infty} P^{*t}$. Let $V = [\alpha f(a^1) \;\; D^S]$:

$$\lim_{t\to\infty} \sum_{i=0}^t \hat{P}^i V = (I - D^{\alpha}P)^{-1} V$$

$$\implies \lim_{t\to\infty} P^{*t} = \begin{bmatrix} \mathbf{0} & (I - D^{\alpha}P)^{-1} V \\ \mathbf{0}_{(N+1)\times N} & \mathbf{I}_{(N+1)\times(N+1)} \end{bmatrix}$$

The expression for $B(a^1)$ follows. Using the Karush-Kuhn-Tucker conditions for optimality:

[13] e_k is a row vector of zeros with a 1 in the k^{th} component.

$$a_i^{1*} = \begin{cases} a_i^1 & \text{s.t. } \frac{1}{N}(1-b_i)q_i \frac{\partial f(a_i^1)}{\partial x} = \mu \\ 0 \ \text{ if } \ \frac{1}{N}(1-b_i)q_i \frac{\partial f(0)}{\partial x} = \mu - \lambda_i, \ \ \lambda_i \geq 0 \end{cases}$$

From the closed-form expression of the optimal response:

$$\frac{1}{N}(1-b_i)q_i \frac{\partial f(a_i^{1*})}{\partial x} \geq \frac{1}{N}(1-b_j)q_j \frac{\partial f(a_j^{1*})}{\partial x}$$

Equality holds if and only if $a_j > 0$. Since f is concave, the result follows. ■

Proof of Proposition 3.2: Define W to be an $N \times N$ matrix where each row is equal to the social influencer vector w (e.g. the left-hand Perron vector of P). Since $\left|\left|\frac{e^T(I-D^{\alpha^{(j)}}P)^{-1}}{e^T(I-D^{\alpha^{(j)}}P)^{-1}e} - w\right|\right|_2$ is continuous for all non-zero $\alpha^{(j)}$, it suffices to look at sequences $\left\{\alpha^{(j)}\right\}$, $\lim_{j\to\infty}\alpha^{(j)} = 0$ where for each j, $\alpha_i^j = \alpha_{i'}^{(j)}$ for all agents i and i'. That is, each agents interacts with external sources with the same frequency. Thus, without loss, consider any real sequence $\left\{\alpha^{(j)}\right\}$ where $\alpha^{(j)} \in \mathbb{R}$, $\alpha^{(j)} < 1$ and $\lim_{j\to\infty}\alpha^{(j)} = 0$. It follows that:

$$\begin{aligned}
\left|\left|\frac{e^T(I-D^{\alpha^{(j)}}P)^{-1}}{N/\alpha^{(j)}} - w\right|\right|_2 &= \frac{1}{N}\left|\left|\alpha^{(j)}e^T(I-(1-\alpha^{(j)})P)^{-1} - \alpha^{(j)}\frac{N}{\alpha^{(j)}}w\right|\right|_2 \\
&= \frac{1}{N}\left|\left|\alpha^{(j)}e^T(I-(1-\alpha^{(j)})P)^{-1} - \alpha^{(j)}e^T(I-(1-\alpha^{(j)}W)^{-1}\right|\right|_2 \\
&= \frac{1}{N}\left|\left|\alpha^{(j)}e^T\sum_{t=0}^{L-1}(1-\alpha^{(j)})^t(P^t-W) + \alpha^{(j)}e^T\sum_{t=L}^{\infty}(1-\alpha^{(j)})^t(P^t-W)\right|\right|_2
\end{aligned}$$

For any $\varepsilon > 0$, take $\varepsilon' < \varepsilon$. There exists L sufficiently large such that each element of P^t is within ε' of each element of $W \implies$

$$\begin{aligned}
&\frac{1}{N}\left|\left|\alpha^{(j)}e^T\sum_{t=0}^{L-1}(1-\alpha^{(j)})^t(P^t-W) + \alpha^{(j)}e^T\sum_{t=L}^{\infty}(1-\alpha^{(j)})^t(P^t-W)\right|\right|_2 \\
&\leq \frac{1}{N}\left|\left|\alpha^{(j)}e^T\sum_{t=0}^{L-1}(1-\alpha^{(j)})^t(P^t-W)\right|\right|_2 + \frac{1}{N}\left|\left|\alpha^{(j)}e^T\sum_{t=L}^{\infty}(1-\alpha^{(j)})^t(P^t-W)\right|\right|_2 \\
&\leq \frac{1}{N}\left|\left|\alpha^{(j)}e^T\sum_{t=0}^{L-1}(1-\alpha^{(j)})^t(P^t-W)\right|\right|_2 + (1-\alpha^{(j)})^L\varepsilon' \\
&\leq \varepsilon \text{ for } \alpha^{(j)} \text{ sufficiently close to } 0
\end{aligned}$$

■

Lemma A.2 *Given network centrality $\hat{q} = e^T(I - D^\alpha P)^{-1}$, where e^T is a row-vector of* 1*'s, let* q *denote the attention-adjusted centrality. Then* $\sum_{i=1}^N q_i = N$.

Proof: Expanding as a power series:

$$q = \hat{q}(I - D^\alpha) = e^T \Big[I + D^\alpha P + (D^\alpha P)^2 + \dots\Big](I - D^\alpha)$$

$$= e^T \Big[I + D^\alpha(P - I) + (D^\alpha)^2 P(P - I) + (D^\alpha)^3 P^2(P - I) + \dots\Big]$$

$$\Longrightarrow \sum_{i=1}^N q_i = e^T \Big[I + D^\alpha(P - I) + (D^\alpha)^2 P(P - I) + (D^\alpha)^3 P^2(P - I) + \dots\Big] e$$

Since the sum of the rows of $P - I$ are all 0, the above expression reduces to $e^T I e = N$. ■

Proof of Example 2: For notational convenience, let $\beta_i = (1 - b_i)\delta + \frac{1}{N}$. Let μ denote the multiplier associated with the binding budget constraint. Applying the Karush-Kuhn-Tucker conditions yields the following (for now, ignore the non-negativity constraints):

$$\Longrightarrow \mu = \frac{1}{N} q_i \frac{\beta_i}{(a_i^{1*} + \frac{1}{N} + \delta)^2} \text{ for each } i \Longrightarrow (a_i^{1*} + \frac{1}{N} + \delta)^2 = \frac{q_i \beta_i}{N\mu}$$

$$\Longrightarrow a_i^{1*} = \sqrt{\frac{q_i \beta_i}{N\mu}} - \delta - \frac{1}{N}$$

The budget constraint binds at optimum $\Longrightarrow \sqrt{\frac{1}{N\mu}} \sum_{j=1}^N \sqrt{q_j \beta_j} = 2 + N\delta$:

$$\Longrightarrow a_i^{1*} = \frac{(2 + N\delta)\sqrt{q_i \beta_i}}{\sum_{j=1}^N \sqrt{q_j \beta_j}} - \delta - \frac{1}{N}$$

If $\frac{(2 + N\delta)\sqrt{q_i \beta_i}}{\sum_{j=1}^N \sqrt{q_j \beta_j}} - \delta - \frac{1}{N} < 0$, set $a_i^{1*} = 0$. Let $\mathscr{I} = \left\{ i : \frac{(2 + N\delta)\sqrt{q_i \beta_i}}{\sum_{j=1}^N \sqrt{q_j \beta_j}} - \delta - \frac{1}{N} > 0 \right\}$:

$$\Longrightarrow a_i^{1*} = \begin{cases} \frac{(2 + |\mathscr{I}|\delta)\sqrt{q_i \beta_i}}{\sum_{j \in \mathscr{I}} \sqrt{q_j \beta_j}} - \delta - \frac{1}{N} & i \in \mathscr{I} \\ 0 & i \notin \mathscr{I} \end{cases}$$

The expression may still be negative for some $i \in \mathscr{I}$. If so, repeat but with a new $\mathscr{I}' \subset \mathscr{I}$. This iterative procedure will converge, leading to an optimal solution. For mathematical simplicity, assume that each agent i is targeted with a positive fraction of the budget at the optimum. The limiting average belief in the network is:

$$\frac{1}{N}\sum_{i=1}^{N} q_i[(1-b_i)\frac{a_i^{1*}}{a_i^{1*}+\frac{1}{N}+\delta}-\frac{\frac{1}{N}}{a_i^{1*}+\frac{1}{N}+\delta}b_i+b_i]$$

$$=\frac{1}{N}\sum_{i=1}^{N} q_i[(1-b_i)\frac{a_i^{1*}}{a_i^{1*}+\frac{1}{N}+\delta}+\frac{a_i^{1*}+\delta}{a_i^{1*}+\frac{1}{N}+\delta}b_i]=\frac{1}{N}\sum_{i=1}^{N} q_i\frac{a_i^{1*}+b_i\delta}{a_i^{1*}+\frac{1}{N}+\delta}$$

$$=\frac{1}{N}\sum_{i=1}^{N} q_i\frac{\frac{(2+N\delta)\sqrt{q_i\beta_i}}{\sum_{j=1}^{N}\sqrt{q_j\beta_j}}-\beta_i}{\frac{(2+N\delta)\sqrt{q_i\beta_i}}{\sum_{j=1}^{N}\sqrt{q_j\beta_j}}}=\frac{1}{N}\sum_{i=1}^{N} q_i-\frac{1}{N(2+N\delta)}\sum_{i=1}^{N}\sqrt{q_i\beta_i}\Big(\sum_{j=1}^{N}\sqrt{q_j\beta_j}\Big)$$

$$=1-\frac{1}{N(2+N\delta)}\Big(\sum_{i=1}\sqrt{q_i\beta_i}\Big)^2$$

Now, if M_1 chose to spend uniformly on each agent, the average limiting belief would be:

$$\frac{1}{N}\sum_{i=1}^{N} q_i[(1-b_i)\frac{a_i^{1*}}{a_i^{1*}+\frac{1}{N}+\delta}-\frac{\frac{1}{N}}{a_i^{1*}+\frac{1}{N}+\delta}b_i+b_i]$$

$$=\frac{1}{N(2+N\delta)}\sum_{i=1}^{N} q_i(1+Nb_i\delta)=\frac{1+\sum_{i=1}^{N} q_ib_i\delta}{2+N\delta} \qquad \blacksquare$$

Proof of Proposition 4.2: Let $h(a^1)=bf(a_i^1,\frac{1}{N})-(1-b)f(\frac{1}{N},a_i^1)+b$. Fix an attention-adjusted centrality vector $q=(q_1,\dots,q_N)$. Without loss, assume $q_1\geq q_2\geq\dots\geq q_N>0$. Denote the optimal targeting strategy by a^{1*}. Consider a sufficiently small $\delta>0$ and a perturbed vector $q'=(q'_1,\dots,q'_N)$ such that $q'_1=q_1+\delta$, $0<q'_i\leq q_i$ for $i\geq 2$, and $\sum q'_i=N$. Essentially, q_1 increases by δ and centralities of the other nodes are lowered. This can be done without changing the α_i's and just by perturbing the peer network P.[14] Let $\hat{a}^{1*}$ denote the optimal targeting strategy when the attention-adjusted-centralities are given by q'. Since $q'_1>q'_2\geq\dots\geq q'_N$, by Theorem 3.1:

$$\frac{1}{N}\sum_{i=1}^{N} q'_ih(\hat{a}_i^{1*})>\frac{1}{N}\sum_{i=1}^{N} q'_ih(a_i^{1*})=\frac{1}{N}\sum_{i=1}^{N} q_ih(a_i^{1*})+\frac{1}{N}\left(\delta h(a_1^{1*})-\sum_{i=2}^{N}(q_i-q'_i)h(a_i^{1*})\right)\geq\frac{1}{N}\sum_{i=1}^{N} q_ih(a_i^{1*})$$

Thus, the influencer prefers the network with attention-adjusted centrality vector q'. $\blacksquare$

[14]Consider $\widetilde{P}$ defined by $\widetilde{P}_{i1}=1-\gamma$ for each i and $\widetilde{P}_{ij}=0$ for $i\geq 1$ and $j>1$. As $\gamma\to 0$, the network centrality vector $e^T(I-D^\alpha\widetilde{P})^{-1}$ approaches $(\frac{N}{\alpha_1},0,\dots,0)$. Since attention-adjusted centrality is a continuous function of the peer network and the space of row-stochastic matrices is path-connected, the intermediate value theorem holds. Hence, such a centrality vector q' can be constructed.

Lemma A.3 *There is no mixed-strategy equilibrium.*

Proof: Define $h_i(x,y) = (1-b_i)f(x,y) - b_i f(y,x)$ for any $x,y \in [0,1]$. Given pure strategies $a^1, a^2 \in \{z : z \in \mathbb{R}^N, z_i \geq 0, \sum_{i=1}^N z_i = 1\}$, the payoff to M_1 is $B(a^1,a^2) = \frac{1}{N}\sum_{i=1}^N q_i[h_i(a_i^1,a_i^2) + b_i]$, while the payoff to M_2 is $1 - B(a^1,a^2)$. The game is obviously zero-sum.

Suppose there is a mixed-strategy equilibrium and M_2 uses mixed strategy σ_2 over the simplex. It must be that M_1 is indifferent between all actions in the support of her strategy and prefers the actions in the support to those outside of it. Suppose M_1 plays a pure strategy a^1 where $a_i^1 = \mathbb{E}_{\sigma_2}[a_i^2]$. Then by Jensen's inequality, M_1's payoff is:

$$\frac{1}{N}\int \sum_{i=1}^N q_i h_i(a_i^1,a_i^2)d\sigma_2 = \frac{1}{N}\sum_i q_i \int h_i(a_i^1,a_i^2)d\sigma_2(a_i^2) > \frac{1}{N}\sum_i q_i h_i(a_i^1,a_i^1) = \frac{1}{N}\sum_{i=1}^N q_i b_i$$

Thus, any mixed strategy equilibrium must guarantee M_1 a payoff strictly greater than $\frac{1}{N}\sum_{i=1}^N q_i b_i$. By a symmetric argument, M_2 must be guaranteed a payoff strictly greater than $1 - \frac{1}{N}\sum_{i=1}^N q_i b_i$. However, the sum of their payoffs would then be strictly greater than 1, which is impossible. Thus, no mixed-strategy equilibrium exists. ■

Proof of Theorem 4.3: The closed form for the average limiting belief follows from a simple extension of Theorem 3.1. Let $V = [\alpha f(a^1,a^2) \;\; \alpha f(a^2,a^1) \;\; D^S]$:

$$\lim_{t\to\infty}\sum_{i=0}^t \hat{P}^i V = (I - D^\alpha P)^{-1}V$$

$$\Longrightarrow \lim_{t\to\infty} P^{*t} = \begin{bmatrix} \mathbf{0} & (I-D^\alpha P)^{-1}V \\ \mathbf{0}_{(N+2)\times N} & \mathbf{I}_{(N+2)\times(N+2)} \end{bmatrix}$$

The expression for $B(a^1,a^2)$ follows. Given $B(a^1,a^2)$ is concave in its first argument and $-B(a^1,a^2)$ is concave in its second argument, a pure strategy equilibrium exists. By Lemma A.3, no mixed strategy equilibrium exists. Since all equilibria are symmetric, I suppress dependence of the targeting strategy on the index of the influencer. Thus, consider any equilibrium a (both influencers select strategy a). Suppose $a_i \leq a_j$ and $q_i > q_j$. It follows that:

$$q_i\Big(\frac{\partial f(a_i,a_i)}{\partial x} - \frac{\partial f(a_i,a_i)}{\partial y}\Big) > q_i\Big(\frac{\partial f(a_i,a_j)}{\partial x} - \frac{\partial f(a_j,a_i)}{\partial y}\Big) > q_i\Big(\frac{\partial f(a_j,a_j)}{\partial x} - \frac{\partial f(a_j,a_j)}{\partial y}\Big)$$

$$> q_j\Big(\frac{\partial f(a_j,a_j)}{\partial x} - \frac{\partial f(a_j,a_j)}{\partial y}\Big)$$

This violates the Karush-Kuhn-Tucker conditions of optimality unless $a_i = a_j = 0$. ■

Lemma A.4 *In a balanced network, if* (a^1,a^2) *is a pure strategy equilibrium, then* $a_i^1 = a_{G(i)}^2$.

Proof: Let $A_1 = \left\{a \in [0,1]^N | \sum a_i \leq 1\right\}$ denote M_1's strategy set. Similarly, let $A_2 = A_1$ denote M_2's strategy set. Define the function $\pi_j : A_1 \times A_2 \longrightarrow [0,1]$ to be M_j's payoff function. Notice that $\pi_1(x,y) = 1 - \pi_2(x,y)$, and so the game is trivially zero-sum. Since M_j's payoff function is concave in her strategy, it follows that there is at least one pure-strategy equilibrium.

Given a balanced network, let G denote the corresponding function that maps each agent to her counterpart. One can view G as a permutation on $\{1,\ldots,N\}$. In an abuse of notation, given any vector $x \in \mathbb{R}^N$, define $G(x) = (x_{G(1)},\ldots,x_{G(N)})$. Recognize that $G \circ G$ is the identity operator and $\pi_1(x,y) = \pi_2(G(y),G(x))$. Thus, if (x,y) is an equilibrium, $(G(y),G(x))$ must also be an equilibrium. Furthermore, $\pi_j(x,G(x)) = \frac{1}{2}$ for any x, which means that any pure-strategy equilibrium must yield payoffs of $\frac{1}{2}$ to each influencer.

Suppose there is an equilibrium (x,y) such that $y \neq G(x)$. This implies that $\pi_1(x,y) = \pi_1(x,G(x)) = \frac{1}{2} \implies \pi_2(x,y) = \pi_2(x,G(x)) = \frac{1}{2}$. However, π_2 is concave in both arguments $\implies \pi_2(x,\lambda y + (1-\lambda)G(x)) > \frac{1}{2}$ for some $\lambda \in (0,1)$. This contradicts the assumption that (x,y) is an equilibrium. Thus, any equilibrium must be of the form $(x,G(x))$. ■

Proof of Theorem 4.6: Lemma A.3 implies that there will only be pure strategy equilibria. Now, consider any equilibrium (a^1,a^2). By Lemma A.4, $a^2 = G(a^1)$. Suppose there is an agent i with $b_i > \frac{1}{2}$ such that $a_i^1 < a_j^1$ where $j = G(i)$. Then by the Karush-Kuhn-Tucker conditions of optimality, it follows that:

$$(1-b_i)\frac{\partial f(a_i^1,a_i^2)}{\partial x} - b_i\frac{\partial f(a_i^2,a_i^1)}{\partial y} \leq (1-b_j)\frac{\partial f(a_j^1,a_j^2)}{\partial x} - b_j\frac{\partial f(a_j^2,a_j^1)}{\partial y}$$

$$\implies (1-b_i)\frac{\partial f(a_i^1,a_i^2)}{\partial x} - b_i\frac{\partial f(a_i^2,a_i^1)}{\partial y} \leq b_i\frac{\partial f(a_i^2,a_i^1)}{\partial x} - (1-b_i)\frac{\partial f(a_i^1,a_i^2)}{\partial y}$$

$$\implies b_i\Big(-\frac{\partial f(a_i^2,a_i^1)}{\partial y} - \frac{\partial f(a_i^2,a_i^1)}{\partial x}\Big) \leq (1-b_i)\Big(-\frac{\partial f(a_i^1,a_i^2)}{\partial y} - \frac{\partial f(a_i^1,a_i^2)}{\partial x}\Big)$$

Now, $-\frac{\partial f(a_i^2,a_i^1)}{\partial y} - \frac{\partial f(a_i^2,a_i^1)}{\partial x} > 0$ and $-\frac{\partial f(a_i^2,a_i^1)}{\partial y} - \frac{\partial f(a_i^2,a_i^1)}{\partial x} + \frac{\partial f(a_i^1,a_i^2)}{\partial y} + \frac{\partial f(a_i^1,a_i^2)}{\partial x} \geq 0$. Since $b_i > \frac{1}{2}$, that means $b_i > 1 - b_i$:

$$\implies b_i\Big(-\frac{\partial f(a_i^2,a_i^1)}{\partial y} - \frac{\partial f(a_i^2,a_i^1)}{\partial x}\Big) > (1-b_i)\Big(-\frac{\partial f(a_i^1,a_i^2)}{\partial y} - \frac{\partial f(a_i^1,a_i^2)}{\partial x}\Big)$$

This is a contradiction given the conditions on f as a result of competition being intense.

Thus, it must be that $a_i^1 > a_{G(i)}^1$ in equilibrium. A symmetric argument demonstrates that $a_{G(i)}^2 > a_i^2$. Each influencer spends more targeting the agent with a similar belief. ■

Proof of Proposition 4.7: Since the influencer targeted like-minded agents in all networks, then, in particular, they must do so when the initial beliefs of agents are such that $b_i \in \{0,1\}$ for all i. Thus, let us consider only balanced networks with such initial beliefs. Consider two agents i and j such that $j = G(i)$ and $b_i = 1$. Let total spending $T(i,j)$ on these agents by an influencer in equilibrium (since the network is balanced both influencers spend the same total amount on each of these agents). Now, for networks where the centrality of agents i and j are sufficiently small, $T(i,j) = 0$. Likewise, when the centralities are sufficiently high, $T(i,j) = 1$. Hence, there exists networks such that for any $D \in [0,1]$, $T(i,j) = D$.

For any network with $T(i,j) = D$, the KKT conditions imply that in equilibrium:

$$\Longrightarrow b_i\Big(-\frac{\partial f(a_i^2,a_i^1)}{\partial y} - \frac{\partial f(a_i^2,a_i^1)}{\partial x}\Big) \leq 0$$

In any equilibrium the influencers spend more on conforming agents $\Longrightarrow$ expression in the parentheses is negative whenever $a_i^1 < a_i^2$. Since $a_i^2 = a_j^1 = D - a_i^1$, we have that:

$$-\frac{\partial f(a_i^2,a_i^1)}{\partial y} - \frac{\partial f(a_i^2,a_i^1)}{\partial x} < 0 \text{ for } a_i^1 < a_i^2 \Longleftrightarrow \frac{\partial f(D-a,a)}{\partial y} - \frac{\partial f(D-a,a)}{\partial x} < 0 \text{ for } a < \frac{D}{2}$$

Given this must hold for all $D \in [0,1]$, the result follows. ■

Proof of Proposition 4.8: Without loss of generality, assume $b_1 \geq b_2 \geq \ldots \geq b_K > \frac{1}{2}$. For each $t \in \{1,\ldots,K\}$, consider the following network G_ε with attention-adjusted centrality vector $\mathbf{q}'$ and belief vector $\mathbf{b}'$:

1. $q_1' = q_2' = \ldots = q_t' = \frac{N}{t} - \varepsilon$
2. $q_{t+1}' = q_{t+2}' = \ldots = q_N' = \dfrac{\varepsilon}{N-t}$
3. $b_1' = b_2' = \ldots = b_t' = b_t$
4. $b_j' = b_j$ for $j > t$

For ε small, $\mathscr{E}(\mathbf{q}',\mathbf{b}')$ is a dumbbell network with t nodes: a network where all centrality is concentrated evenly across t agents. When the two influencers compete over $\mathscr{E}(\mathbf{q}',\mathbf{b}')$, as

$\varepsilon \to 0$, the lone equilibrium strategy becomes $a_i^1 = a_i^2 = \frac{1}{t}$ for $i \in \{1,\dots,t\}$ and $a_i^1 = a_i^2 = 0$ for $i \in \{t+1,\dots,N\}$. Therefore:

$$\lim_{\varepsilon\to 0} \max_{(\sigma_1,\sigma)\in\mathscr{E}(\mathbf{q}',\mathbf{b}')} \Pi_1(\sigma_1,\sigma_2) = \lim_{\varepsilon\to 0} \frac{1}{N}\sum_{i=1}^{N} q_i' b_i' + \frac{1}{N}\sum_{i=1}^{t} q_i'(1-2b_t) f(\frac{1}{t},\frac{1}{t}) + \frac{1}{N}\sum_{i=t+1}^{N} q_i'(1-2b_i) f(0,0)$$

$$= b_t - (1-2b_t) f(\tfrac{1}{t},\tfrac{1}{t})$$

Now, since $\max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b})} \Pi_1(\sigma_1,\sigma_2) \geq \max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b}')} \Pi_1(\sigma_1,\sigma_2)$, whenever $\mathbf{b}' \leq \mathbf{b}$, it follows that $\max_q \max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b})} \Pi_1(\sigma_1,\sigma_2) \geq \max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q}'(\varepsilon),\mathbf{b}')} \Pi_1(\sigma_1,\sigma_2)$ for every ε.

$$\implies \max_q \max_{(\sigma_1,\sigma_2)\in\mathscr{E}(\mathbf{q},\mathbf{b})} \Pi_1(\sigma_1,\sigma_2) \geq b_t - (1-2b_t) f(\tfrac{1}{t},\tfrac{1}{t}) \qquad \blacksquare$$

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