DANTZIG-WOLFE DECOMPOSITION

Min CX
s.t $AX = b$
$X \in \mathcal{X}$

Suppose that $\mathcal{X}$ is bounded and polyhedron then

$$X \in \mathcal{X} \Leftrightarrow X = \sum_{j=1}^{k} \lambda_j x^j$$

$$\sum_{j=1}^{k} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad \forall j$$

Equivalent Problem

$$\text{Min} \sum_{j=1}^{k} (cx^j)\lambda_j$$

$$\text{s.t} \sum_{j=1}^{k} (Ax^j)\lambda_j = b \qquad \pi_1$$

$$\sum_{j=1}^{k} \lambda_j = 1 \qquad \pi_0$$

$$\lambda_j \geq 0 \quad j = 1, 2, \ldots, k$$

KUHN-TUCKER OPTIMALITY CONDITIONS

[1] PRIMAL FEASIBILITY

$$\sum_{j=1}^{k} (Ax^j)\lambda_j = b$$

$$\sum_{j=1}^{k} \lambda_j = 1 \qquad \lambda_j \geq 0 \quad \forall j$$

[2] DUAL FEASIBILITY

$$cx_j - \pi_1 Ax^j - \pi_0 \geq 0.$$

c

[3] COMPLEMENTARY SLACKNESS

$$\lambda_j (cx^j - \pi_1 Ax^j - \pi_0) = 0 \qquad j = 1, 2, \ldots, k$$

Suppose only ν extreme points are generated say $X^1, X^2, \ldots, X^\nu$. Solve the following problem

$$\text{Min} \sum_{j=1}^{\nu} (cX^j)\lambda_j$$

$$\text{s.t} \sum_{j=1}^{\nu} (AX^j)\lambda_j = b$$

$$\sum_{j=1}^{\nu} \lambda_j = 1, \quad \lambda_j \geq 0 \quad \forall j = 1, 2, \ldots, \nu$$

Take the optimal solution and adjoin to it $\lambda_{\nu+1} = \lambda_K = 0$. Now K-T conditions [1] & [2] hold.

The only source of violation is [2] because $cX^j - \pi_1 AX^j - \pi_0 < 0$ for some $j > \nu$

We must find out whether $cX^j - \pi_1 AX^j - \pi_0 \geq 0$ for all j or not. This can be done by solving the following subproblem:

SUBPROBLEM:

$$\text{Min} (c - \pi_1 A) X = \theta$$
$$\text{s.t} \quad X \in \underline{X}$$

Let the optimal extreme point be $X^{\nu+1}$.
Calculate $cX^{\nu+1} - \pi_1 AX^{\nu+1} - \pi_0 \ (\theta^* - \pi_0)$
If ≥ 0 then we are optimal and finished.
If not, then repeat the original problem with $\nu+1$ extreme points

D-W Decomposition (Prof Notes)

SUMMARY

MASTER PROBLEM

$$\text{Min} \sum_{j=1}^{\nu} (cx^j)\lambda_j$$

Dual Variables

$$\text{s.t} \sum_{j=1}^{\nu} (Ax^j)\lambda_j = b \qquad \pi_1$$

$$\sum_{j=1}^{\nu} \lambda_j = 1 \qquad \pi_0$$

$$\lambda_j \geq 0 \qquad j = 1, 2, \ldots, \nu$$

SUBPROBLEM

$$\text{Min } (c - \pi_1 A)x = \Theta$$

$$\text{s.t } x \in X$$

Let Optimal be $x^{\nu+1}$, Θ^*.

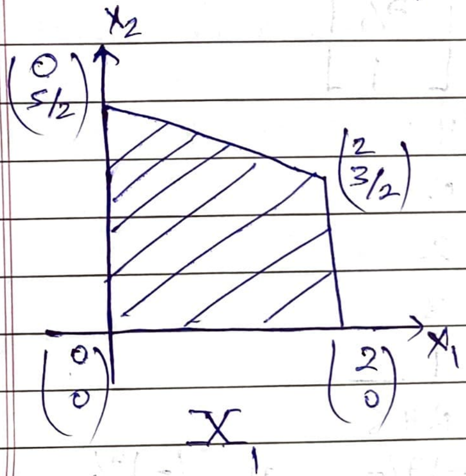
If $\Theta^* - \pi_0 \geq 0$ stop [redacted]

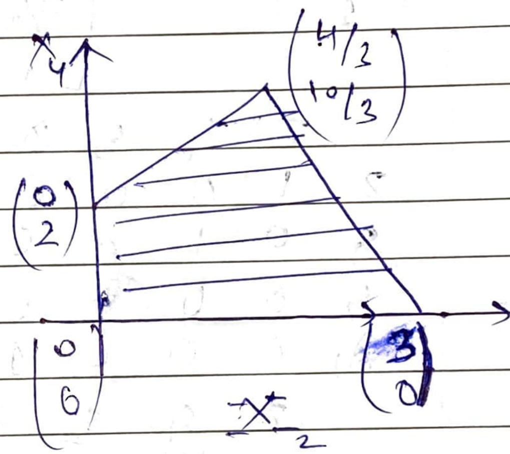
O/W Replace ν by $\nu+1$ and repeat the master problem

EXAMPLE MIN $-2x_1 - x_2 - x_3 + x_4$

$$
\begin{aligned}
\text{s.t}\quad & x_1 + x_3 \le 2 \\
& x_1 + x_2 + 2x_4 \le 3
\end{aligned}
\Bigg\}\; AX \le b
$$

$$
\left.
\begin{aligned}
& x_1 \le 2 \\
& x_1 + 2x_2 \le 5 \\
& -x_3 + x_4 \le 2 \\
& 2x_3 + x_4 \le 6 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
\right\}\; X
$$





Iteration 1

$(x^{1})^T = (0, 0, 0, 0)$

Master Problem

$\text{Min } (cx^1)\lambda_1$

$\text{s.t } (Ax^1)\lambda_1 \le b \quad \pi_1$

$\lambda_1 = 1 \quad \pi_0$

$\lambda_1 \ge 0$

$\pi_1 = (0, 0) \qquad \pi_0 = 0$

	Z	S_1 (π_{11})	S_2 (π_{12})	λ_1 (π_0)	RHS
Z	1	0	0	0	0
S_1	0	1	0	0	2
S_2	0	0	1	0	3
λ_1	0	0	0	1	1

Subproblem M_1

$\text{Min } (c - \pi_1 A)x$

$\text{s.t } x \in X$

i.e. $\text{Min } -2x_1 - x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$

Optimal $\begin{pmatrix} 2 \\ 3/2 \end{pmatrix}$

$\text{Min } -x_3 + x_4$

$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in X$

Optimal $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$(c-\pi_1 A)x - \pi_0 = -4 - 3/2 - 3 - 0 = -17/2 < 0$

Iteration 2

$$Ax^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3/2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7/2 \end{bmatrix}$$

$$\begin{pmatrix} Ax^2 \\ 1 \end{pmatrix} = B^{-1} \begin{bmatrix} 5 \\ 7/2 \\ 1 \end{bmatrix} = I \begin{bmatrix} 5 \\ 7/2 \\ 1 \end{bmatrix}$$

	z	s_1	s_2	λ_1	λ_2	RHS
z	1	0	0	0	17/2	0
s_1	0	1	0	0	5	2
s_2	0	0	1	0	7/2	3
λ_1	0	0	0	1	1	1

	z	s_1	s_2	λ_1	λ_2	RHS
z	1	$-17/10$	0	0	0	$-17/5$
s_1	0	1/5	0	0	1	2/5
s_2	0	$-7/10$	1	0	0	8/5
λ_1	0	$-1/5$	0	1	0	3/5

$\pi_1 = (-17/10, 0)$

$\pi_0 = 0$

Subproblem

$\text{Min } (c - \pi_1 A)x \Rightarrow \text{Min } -3/10\, x_1 - x_2$ st $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1$ $\quad$ Min $7/10\, x_3 + x_4$ st $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in X_2$

optimal $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5/2 \end{pmatrix}$ $\quad$ optimal $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(c - \pi_1 A)x - \pi_0 = -5/2 < 0$ so continue

Iteration 3

$$Ax^3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \end{bmatrix}$$

$$\begin{pmatrix} Ax^3 \\ 1 \end{pmatrix} = B^{-1} \begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 0 \\ -7/10 & 1 & 0 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix}$$

Master Problem

	z	s_1	s_2	λ_1	λ_2	λ_3	RHS
z	1	$-17/10$	0	0	0	$5/2$	$-17/5$
λ_2	0	$1/5$	0	0	1	0	$2/5$
s_2	0	$-7/10$	1	0	0	$5/2$	$8/5$
λ_1	0	$-1/5$	0	1	0	1	$3/5$

	z	s_1	s_2	λ_1	λ_2	λ_3	RHS	
z	1	$-6/5$	0	$-5/2$	0	0	$-49/10$	
λ_2	0	$1/5$	0	0	1	0	$2/5$	$\pi_1 = (-6/5, 0)$
s_2	0	$-1/5$	1	$-5/2$	0	0	$1/10$	$\pi_0 = -5/2$
s_3	0	$-1/5$	0	1	0	1	$3/5$	

Subproblem

$\text{Min } (c - \pi_1 A)x$
s.t. $x \in X$

$\Rightarrow \text{Min } -\frac{4}{5}x_1 - x_2$ s.t. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1$ | $\text{Min } \frac{1}{5}x_3 + x_4$ s.t. $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in X_2$

Optimal $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \end{pmatrix}$ | $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(c - \pi_1 A)x - \pi_0 = 2(-4/5) - 3/2 - 5/2 = -3/5 < 0$$

Iteration 4

$$AX^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 7/2 \end{bmatrix}$$

$$B^{-1}\begin{pmatrix} AX^4 \\ 1 \end{pmatrix} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/5 & 1 & -5/2 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \\ 3/5 \end{bmatrix}$$

	z	s_1	s_2	λ_1	λ_2	λ_3	λ_4	RHS
z	1	$-6/5$	0	$-5/2$	0	0	$3/5$	$-49/10$
λ_2	0	$1/5$	0	0	1	0	$2/5$	$2/5$
s_2	0	$-1/5$	1	$-5/2$	0	0	$3/5$	$1/10$
s_3	0	$-1/5$	0	1	0	1	$3/5$	$3/5$

	z	s_1	s_2	λ_1	λ_2	λ_3	λ_4	RHS	
z	1	-1	-1	0	0	0	0	-5	
λ_2	0	$1/3$	$-2/3$	$5/3$	1	0	0	$1/3$	$\pi_1 = (-1,-1)$
λ_4	0	$-1/3$	$5/3$	$-25/6$	0	0	1	$1/6$	$\pi_0 = 0$
λ_3	0	0	-1	$7/2$	0	1	0	$1/2$	

Subproblem

$\text{Min } (c - \pi_1 A)X \Rightarrow \text{Min } 0x_1 + 0x_2$ s.t. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1$ | $\text{Min } 0x_3 + 0x_4$ s.t. $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in X_2$

optimal not unique $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(c - \pi_1 A)X - \pi_0 = 0 \Rightarrow$ Alternate optima if X^5 not already in master problem

X^* is given by $\lambda_2 X^2 + \lambda_3 X^3 + \lambda_4 X^4$ $\quad z^* = -5$

$$X^* = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 3/2 \\ 3 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 5/2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 2 \\ 3/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

LOWER BOUNDS

$$\begin{aligned} &\text{Min } CX \\ &\text{s.t } AX = b \\ &X \in \mathcal{X} \end{aligned}$$

Let $\hat{X}$ be any feasible solution i.e $A\hat{X} = b$, $\hat{X} \in \mathcal{X}$

Let π_1 be any dual vector

Consider the following problem:

$$\begin{aligned} &\text{Min } (C - \pi_1 A)X \\ &\text{s.t } \quad X \in \mathcal{X} \end{aligned}$$

we must have $\quad C\hat{X} \geq \pi_1 b + \underset{X \in \mathcal{X}}{\text{Min}} (C - \pi_1 A)X$

Proof: $\hat{X} \in \mathcal{X} \Rightarrow (C - \pi_1 A)\hat{X} \geq \underset{X \in \mathcal{X}}{\text{Min}} ((C - \pi_1 A)X$

$AX = b$ so $(C - \pi_1 A)\hat{X} = C\hat{X} - \pi_1 A\hat{X} = C\hat{X} - \pi_1 b$

giving us $\quad C\hat{X} \geq \pi_1 b + \underset{X \in \mathcal{X}}{\text{Min}} (C - \pi_1 A)X$

At each iteration of the DW decomposition, we get an improved primal feasible solution.
Also a lower bound is provided by $\pi_1 b + \underset{X \in \mathcal{X}}{\text{Min}} (C - \pi_1 A)X$

Therefore rather than stopping when
$\underset{X \in \mathcal{X}}{\text{Min}} (C - \pi A)X - \pi_0 \geq 0$ (optimal)

We may stop when we are close enough to optimal i.e if we have a primal feasible soln $\hat{x}$ with $c\hat{x} - (\pi_i b + \min_{x \in X} (c - \pi A) x) < \epsilon$ for some $\epsilon > 0$