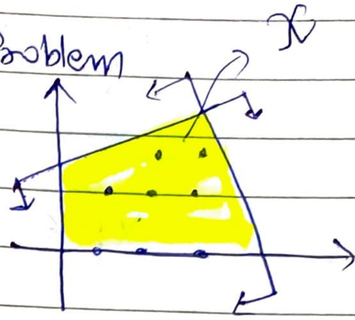
BENDERS DECOMPOSITION

Consider the following problem

$$\text{Max } Z = C^T X + h^T Y$$
$$AX + Gy \leq b$$
$$X \in \mathcal{X} \cap \mathbb{Z}^n$$
$$Y \geq 0$$

Bounded Problem



We say that variable set X is the set of "complicating" variables i.e without X the resulting problem is easy

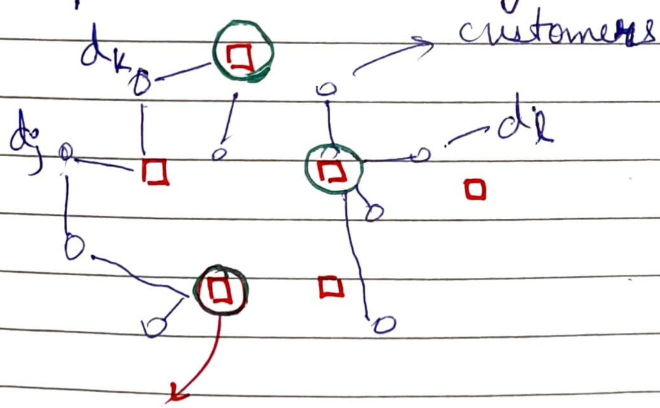
Suppose that variables X are fixed ($\bar{X}$)

$$LP(X): \text{Max } Z_{LP}(\bar{X}) = h^T Y$$
$$\text{s.t} \quad Gy \leq b - A\bar{X}$$
$$Y \geq 0$$

easy

Example:

Uncapacitated Facility Location Problem (UFL)



$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is open} \\ 0 & \text{o/w} \end{cases}$$

y_{ij} = % of demand of j that is covered by facility i

$x_i \in \{0, 1\}$ $y_{ij} \in [0, 1]$

$$\text{Max } Z = -\sum_{i \in F} F_i x_j + \sum_{i \in F} \sum_{j \in N} c_{ij} y_{ij}$$

$$\sum_{i \in F} y_{ij} = 1 \quad j \in N$$

$\rightarrow G$ $\rightarrow b$

$$\cancel{\sum_{j \in N} y_{ij} \le x_i |N|} \text{ or } y_{ij} \le x_i \quad i \in F,\ j \in N$$

$$\boxed{-} x_i + \boxed{1} y_{ij} \le \boxed{0}$$

$$x_i \in \{0, 1\} \quad i \in F$$

A

$$y_{ij} \ge 0 \quad i \in F,\ j \in N$$

$LP(\bar{x})$: $\text{Max } Z_{LP}(\bar{x}) = h^T y$

s.t

$$Gy \leq b - A\bar{x} \quad \rightarrow u$$

$$y \geq 0$$

$D(\bar{x})$ $\text{Min } Z_d(\bar{x}) = u^T(b - A\bar{x})$

s.t

$$u^T G \geq h^T$$

$$u \geq 0$$

Original problem

$$\text{Max } Z = C^T x + h^T y$$

$$Ax + Gy \leq b$$

$$x \in \mathcal{X} \subseteq \mathbb{Z}^n$$

$$y \geq 0$$

$\Rightarrow$

$$\text{Max } Z = C^T x + \gamma(x)$$

s.t $\gamma(x) = \text{Max } hy$

s.t

$$Gy \leq b - A\bar{x}$$

$$y \geq 0$$

$$x \in \mathcal{X} \subseteq \mathbb{Z}^n$$

$$\text{Max } Z = C^T x + \gamma(x)$$

$$\gamma(x) = \text{Min } U^T(b - Ax)$$

s.t $u^T G \geq h^T$

$$u \geq 0$$

$$x \in \mathcal{X} \cap \mathbb{Z}$$

Assumption 1: Original Problem is bounded implies that $LP(x)$ is also bounded

We can use $D(x)$ to characterize the solutions of $LP(x)$

Minkowski's characterization theorem for $D(x)$

$$\boxed{\begin{aligned} \text{Min} \quad & Z_d(\bar{x}) = u^T(b - A\bar{x})^T \\ \text{s.t} \quad & u^T G \geq h^T \\ & u \geq 0 \end{aligned}}$$

always feasible

Let u^i for $i = 1, \_\_, k$ is the set of extreme points of $D(x)$ and v^j for $j = 1, \_\_, l$ is the set of extreme rays of $D(x)$ then $D(x)$ is equivalent to

$$\boxed{\begin{aligned} \text{Min} \quad & Z_d(\bar{x}) = (b - A\bar{x})^t \left(\sum_{i=1}^{k} \lambda_i u^i + \underbrace{\sum_{j=1}^{l} \mu_j v^j}_{\mu = 0} \right) \\ \text{s.t} \quad & \sum_{i=1}^{k} \lambda_i = 1 \\ & \lambda_i, \mu_{ij} \geq 0 \end{aligned}}$$

If unbounded. ?

$$(b - A\bar{x}) v^j < 0$$

(1) We need to ensure that any $\bar{x}$ that is given to $LP(x)$

$$\underbrace{(b - A\bar{x}) v^j \geq 0,} \quad j = 1, \_\_, l$$

↳ these gurantee that $LP(x)$ is always feasible. Feasibility Benders Cuts

Benders optimality cuts $\leftarrow$ $\begin{cases} r(x) = \min\limits_{i=1,\dots,k} \{(b-Ax)u^i\} \\ \eta = r(x) \leq (b-Ax)u^i \quad i=1,\dots,k \end{cases}$

classmate

Date

Page

(2) $\text{Max } Z = C^t x + r(x)$

$$\left| \begin{array}{l} r(x) = \min \; u^T(b-Ax) \\ \text{s.t.} \quad u^t G \geq h^t \\ \quad\quad u \geq 0 \end{array} \right|$$

$x \in \mathcal{X} \cap Z$

$$\left| \begin{array}{l} \text{Max } Z = C^T x + \overset{\eta}{\overset{\parallel}{r(x)}} \\ \text{s.t. } (b-Ax)v^j \geq 0 \quad j=1,\dots,l \\ \\ \eta = r(x) \leq (b-Ax)u^i \quad i=1,\dots,k \\ x \in \mathcal{X} \cap Z^n \end{array} \right|$$

Benders reformulation

Do implementation on Gurobi

Example

$$\text{Max } 5x_1 - 2x_2 + 9x_3 + 2y_1 - 3y_2 + 4y_3$$

s.t

$$5x_1 - 3x_2 - 7x_3 + 2y_1 + 3y_2 + 6y_3 \leq 2$$
$$4x_1 + 2x_2 + 4x_3 + 3y_1 - y_2 + 3y_3 \leq 10$$
$$x_i \leq 5 \quad i = 1, \dots, 3$$
$$x_i \in \mathbb{Z} \quad i = 1, \dots, 3$$
$$y_i \geq 0 \quad i = 1, \dots, 3$$

$LP(\bar{x})$

$$\text{Max } 2y_1 - 3y_2 + 4y_3$$

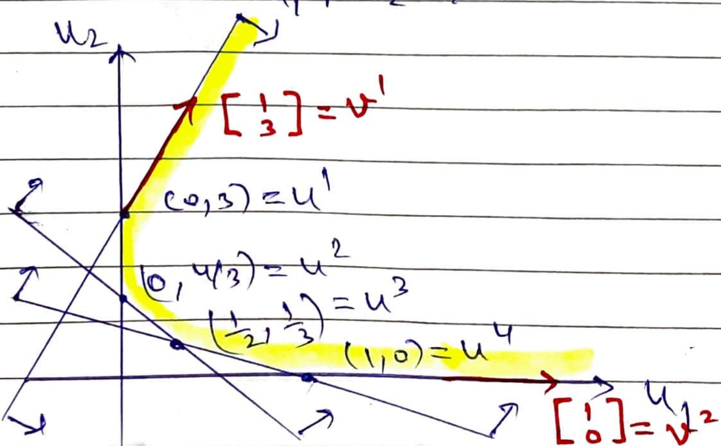
$$\text{s.t } u_1 \leftarrow 2y_1 + 3y_2 + 6y_3 \leq -2 - (5\bar{x}_1 - 3\bar{x}_2 - 7\bar{x}_3)$$
$$u_2 \leftarrow 3y_1 - y_2 + 3y_3 \leq 10 - (4\bar{x}_1 + 2\bar{x}_2 + 4\bar{x}_3)$$

$$y_i \geq 0 \quad i = 1, \dots, 3$$

$D(\bar{x})$

$$\text{Min } \left((-2 - (5\bar{x}_1 - 3\bar{x}_2 - 7\bar{x}_3))u_1 + (10 - (4\bar{x}_1 + 2\bar{x}_2 + 4\bar{x}_3))u_2\right)$$

$$\text{s.t } 2u_1 + 3u_2 \geq 2$$
$$3u_1 - u_2 \geq -3$$
$$6u_1 + 3u_2 \geq 4$$
$$u_1, u_2 \geq 0$$



only one decision variables,

Bender's reformulation

$$\text{Max} \quad Z = 5x_1 - 2x_2 + 9x_3 + \eta$$

st

Feasibility cuts

$$\begin{cases} (-2-(5x_1-3x_2-7x_3))1 + (10-(4x_1+2x_2+4x_3))3 \geq 0 \\ (-2-(5x_1-3x_2-7x_3))1 + (10-(4x_1+2x_2+4x_3))0 \geq 0 \end{cases}$$

optimality cut

$$\begin{cases} \eta \leq -2-5x_1+3x_2+7x_3 & (1,0) \\ \eta \leq \frac{1}{2}(-2-5x_1+3x_2+7x_3) + \frac{1}{3}(10-(4x_1+2x_2+4x_3)) & (\frac{1}{2}, \frac{1}{3}) \\ \eta \leq \frac{4}{3}(10-4x_1-2x_2-4x_3) & (0, 4/3) \\ \eta \leq 3(10-4x_1-2x_2-4x_3) & (0, 3) \end{cases}$$

↳ Put this in Gurobi

Only X and η variables
What happened with y?