Lecture 16: Special Functions

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restart
assume(x, real)
assume(y, real)
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1. Key points

- Hermite differential equation: $y''-2\,x\cdot y'+2\,\mathrm{n}\cdot y=0$
- Legendre's differential equation: $(1-x^2)y''-2\cdot x\cdot y'+n(n+1)y=0$
- Bessel's differential equation: $x^2y''+x\cdot y'+(x^2-n^2)y=0$
- Modified Bessel differential equation: $x^2y''+x\cdot y'-(x^2+n^2)y=0$
- Spherical Bessel differential equation: $x^2y''+2\cdot x\cdot y'+(x^2-n(n+1))y=0$
- Airy differential equation: $y''-x\cdot y=0$
- Lagguerre differential equation: $x\cdot y''+(1-x)y'+n\cdot y=0$

Maple

- HermiteH(n,x)
- LegendreP(n,x), LegendreQ(n,x)
- BesselJ(n,x), BesselY(n,x)
- HankelH1(n,x), HankelH2(n,x)
- BesselI(n,x), BesselK(n,x)
- AiryAi(x), AiryBi(x)
- LaguerreL(n,x)

Hermite equation

Hermite differential equation

$y''-2\,x\cdot y'+2\,\mathrm{n}\cdot y=0$

$$y''(x)-2\,x\,y'(x)+2\,n\,y(x)=0 \quad \textbf{(2.1)}$$

When n is integer, Hermite polynomials

$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\,e^{-x^2}$

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad \textbf{(2.2)}$$

is a solution to **(2.1)**.

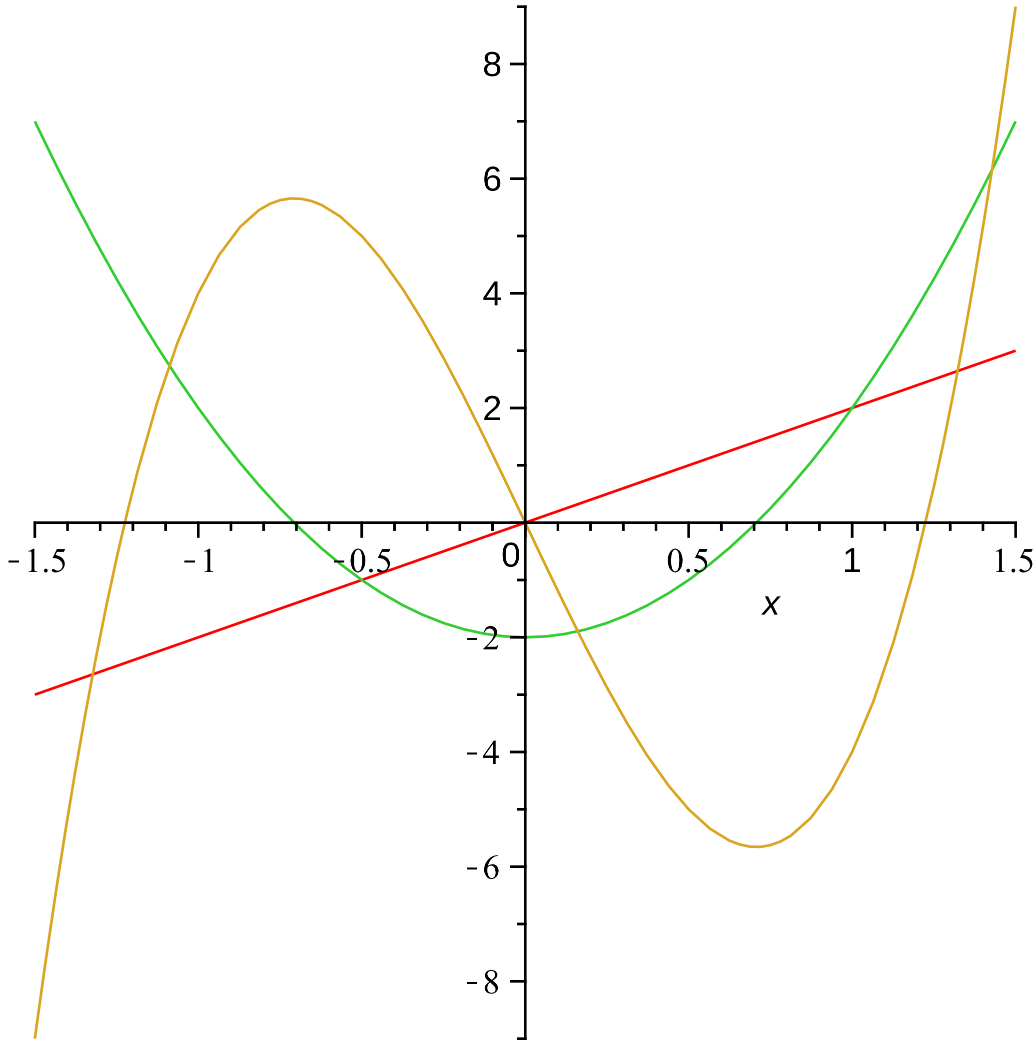
In Maple, Hermite polynomials are predefined as HermiteH(n,x)

The first few Hermite polynomials are:

$$\mathrm{HermiteH}(0,x)=H_0(x)\overset{\text{simplify}}{=}1$$
$$\mathrm{HermiteH}(1,x)=H_1(x)\overset{\text{simplify}}{=}2\,x$$
$$\mathrm{HermiteH}(2,x)=H_2(x)\overset{\text{simplify}}{=}-2+4\,x^2$$
$$\mathrm{HermiteH}(3,x)=H_3(x)\overset{\text{simplify}}{=}8\,x^3-12\,x$$

Plots

$plot([\mathrm{HermiteH}(1, x), \mathrm{HermiteH}(2, x), \mathrm{HermiteH}(3, x)], x = -1.5 ..1.5)$



Recursive relation

The Hermite polynomials $H_n(x)$ detemrined by the following recursive relation are solution to the Hermite equation.

$H_{n+1}(x) = 2\,x \cdot H_n(x) - 2\,n \cdot H_{n-1}(x)$

$$H_{n+1}(x) = 2\,x\,H_n(x) - 2\,n\,H_{n-1}(x) \qquad \mathbf{(2.3)}$$

Orthogonality

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0 & n \neq m \\ \sqrt{\pi}\, 2^n n! & n = m \end{cases}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx = \begin{cases} 0 & n \neq m \\ \sqrt{\pi}\, 2^n n! & n = m \end{cases} \quad \textbf{(2.4)}$$

The second solution to the Hermite equation is the second kind Hermite function $h_n(x)$. Since it is rarely used in physics, we don't discuss it here.

Using Maple to solve Hermite equation

For *n*=2,

$dsolve(y'' - 2\,x \cdot y' + 4 \cdot y = 0)$

$$y(x) = \_C1 \left(-2\,x\,e^{x^2} + 2\sqrt{\pi} \left(-\frac{1}{2} + x^2 \right) \operatorname{erfi}(x) \right) + \_C2\,(-1 + 2\,x^2) \quad \textbf{(2.5)}$$

The first term is the second kind Hermite function. Under typical boundary conditions in physics, the integral constant *_C1* is zero. The second term is basically the Hermite polynomial $H_2(x)$.

General solution by Maple

$assume(n, integer)$

$dsolve(y'' - 2\,x \cdot y' + 2\,n \cdot y = 0)$

$$y(x) = \_C1\, M\left(\frac{1}{2} - \frac{n}{2}, \frac{3}{2}, x^2 \right) x + \_C2\, U\left(\frac{1}{2} - \frac{n}{2}, \frac{3}{2}, x^2 \right) x \quad \textbf{(2.6)}$$

Legendre equation

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Legendre's differential equation of degree *n* (0th order)

$(1 - x^2) y'' - 2 \cdot x \cdot y' + n(n+1) y = 0$

$$(1 - x^2)\, y''(x) - 2\,x\,y'(x) + n(n+1)\, y(x) = 0 \quad \textbf{(3.1)}$$

Two linearly independent solutions to **(3.1)** is known as the first kind of Legendre polynomials $P_n(x)$ and the second kind of Legendre function $Q_n(x)$.

First kind Legendre polynomials

$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{d x^n} (x^2 - 1)^n$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{d x^n} (x^2 - 1)^n \quad \textbf{(3.2)}$$

In Maple, Legendre polynomials are predefined as $\mathrm{LegendreP}(n, x)$.

$\mathrm{LegendreP}(0, x) = 1$

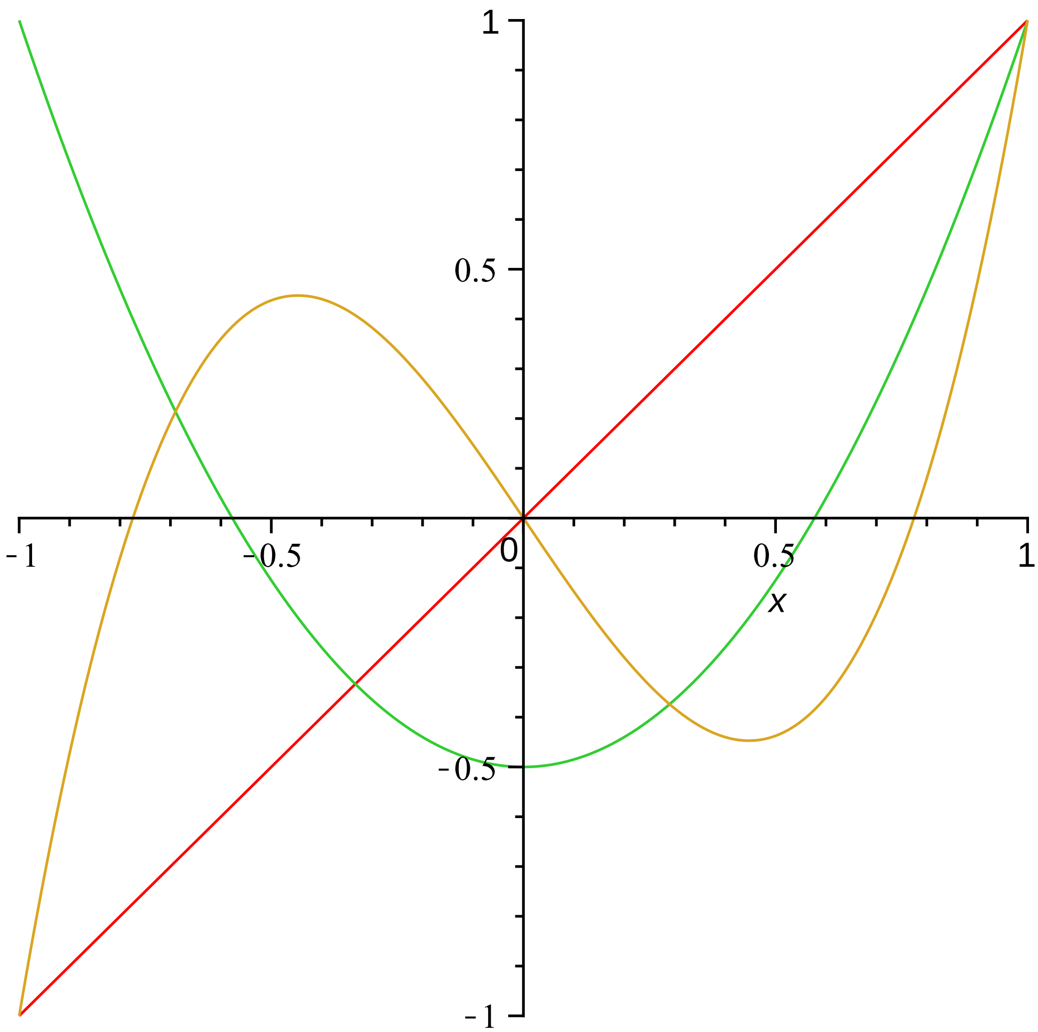
$\mathrm{LegendreP}(1, x) = x$

$\mathrm{LegendreP}(2, x) = P_2(x) \overset{\text{simplify}}{=} -\frac{1}{2} + \frac{3\,x^2}{2}$

$\mathrm{LegendreP}(3, x) =$

$$P_3(x) \overset{\text{simplify}}{=} \frac{5}{2}x^3 - \frac{3}{2}x$$

$plot([\mathrm{LegendreP}(1, x), \mathrm{LegendreP}(2, x), \mathrm{LegendreP}(3, x)], x = -1..1)$



Orthogonality

$\sqrt{n + \frac{1}{2}}\, P_n(x)$ forms an orthonormal basis set for $x \in [-1, 1]$.

$$\int_{-1}^{1} P_n(x) \cdot P_m(x)\, \mathrm{d}x = \begin{cases} 0 & \mathrm{n} \neq \mathrm{m} \\ \dfrac{2}{2n+1} & \mathrm{n} = \mathrm{m} \end{cases}$$

$$\int_{-1}^{1} P_n(x) \cdot P_m(x)\, \mathrm{d}x = \begin{cases} 0 & n \neq m \\ \dfrac{2}{2n+1} & n = m \end{cases} \tag{3.3}$$

Recursive equation

$$(n+1)P_{n+1}(x) - (2n+1)x\,P_n(x) + n\,P_{n-1}(x) = 0$$

$$(n+1)\,P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0 \qquad (3.4)$$

Second kind Legendre function is not common in physics problem.

$$\mathrm{LegendreQ}(0,x) = Q_0(x) \overset{\text{simplify}}{=} \frac{\ln(x+1)}{2} - \frac{\ln(x-1)}{2}$$

$$\mathrm{LegendreQ}(1,x) = Q_1(x) \overset{\text{simplify}}{=} \frac{x\ln(x+1)}{2} - \frac{x\ln(x-1)}{2} - 1$$

Solution by Maple

$n := 2:$

$dsolve\left((1-x^2)y''-2\cdot x\cdot y'+n(n+1)y=0\right)$

$$y(x) = \_C1\,x + \_C2\left(-\frac{\ln(1+x)\,x}{2} + 1 + \frac{\ln(-1+x)\,x}{2}\right) \qquad (3.5)$$

$unassign('n'):$

General solution by Maple

$dsolve\left((1-x^2)y''-2\cdot x\cdot y'+n(n+1)y=0\right)$

$$y(x) = \_C1\,P_{\frac{\sqrt{1+4n(1+n)}}{2}-\frac{1}{2}}(x) + \_C2\,Q_{\frac{\sqrt{1+4n(1+n)}}{2}-\frac{1}{2}}(x) \qquad (3.6)$$

General Legendre equation

$assume(n, integer)$

$assume(m, integer)$

Legendre's differential equation

$$(1-x^2)y''-2\cdot x\cdot y'+\left(n(n+1)-\frac{m^2}{1-m^2}\right)y=0$$

$$(1-x^2)\,y''(x) - 2\,x\,y'(x) + \left(n(n+1)-\frac{m^2}{1-m^2}\right)y(x) = 0 \qquad (4.1)$$

where m and n are integers and $m \le n$. (mathematically speaking non-integer values are allowed but not popular in physics.)

Associate Legendre function $P_n^m(x) = (-1)^m\,(1-x^2)^{\frac{m}{2}}\,\frac{d^m}{dx^m}P_n(x)$

$$P_n^m(x) = (-1)^m\,(1-x^2)^{\frac{m}{2}}\,\frac{d^m}{dx^m}P_n(x) \qquad (4.2)$$

$$Q_n^m(x) = (x^2-1)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_n(x)$$

$$Q_n^m(x) = (x^2-1)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_n(x) \tag{4.3}$$

is a solution to the general Legendre differential equation. The second kind of assocate Legendre function

$$\mathrm{LegendreP}(1, 1, x) = P_1^1(x) \overset{\text{simplify}}{=} \sqrt{x-1}\,\sqrt{x+1}$$
$$\mathrm{LegendreP}(2, 1, x) = P_2^1(x) \overset{\text{simplify}}{=} 3\sqrt{x-1}\,\sqrt{x+1}\,x$$
$$\mathrm{LegendreP}(2, 2, x) = P_2^2(x) \overset{\text{simplify}}{=} 3x^2-3$$
$$\mathrm{LegendreQ}(1, 1, x) = Q_1^1(x) \overset{\text{simplify}}{=} \frac{\ln(x+1)\,x^2 - \ln(x+1) - \ln(x-1)\,x^2 + \ln(x-1) - 2x}{2\sqrt{x-1}\,\sqrt{x+1}}$$
$$\mathrm{LegendreQ}(2, 1, x) = Q_2^1(x) \overset{\text{simplify}}{=}$$
$$\frac{3x^3\ln(x+1) - 3x\ln(x+1) - 3x^3\ln(x-1) + 3x\ln(x-1) + 4 - 6x^2}{2\sqrt{x-1}\,\sqrt{x+1}}$$
$$\mathrm{LegendreQ}(2, 2, x) = Q_2^2(x) \overset{\text{simplify}}{=}$$
$$\frac{1}{2(x^2-1)}\big(3x^4\ln(x+1) - 3x^4\ln(x-1) - 6\ln(x+1)\,x^2 + 6\ln(x-1)\,x^2 + 3\ln(x+1) - 3\ln(x-1) - 6x^3 + 10x\big)$$

$$\int_{-1}^{1} P_n^m(x)\,P_\ell^m(x)\,dx = \begin{cases} 0 & n \neq \ell \\ \dfrac{2(n+m)!}{(n-m)!\,(2n+1)} & n = \ell \end{cases}$$

$$\int_{-1}^{1} P_n^m(x)\,P_\ell^m(x)\,dx = \begin{cases} 0 & n \neq \ell \\ \dfrac{2\,(n+m)!}{(n-m)!\,(2n+1)} & n = \ell \end{cases} \tag{4.4}$$

$$\int_{-1}^{1} \frac{P_m^n(x)\,P_m^\ell(x)}{1-x^2}\,dx = \begin{cases} 0 & n \neq \ell \\ \dfrac{(m+n)!}{(m-n)!\,n} & n = \ell \end{cases}$$

$$\int_{-1}^{1} \frac{P_m^n(x)\,P_m^\ell(x)}{1-x^2}\,dx = \begin{cases} 0 & n \neq \ell \\ \dfrac{(n+m)!}{(m-n)!\,n} & n = \ell \end{cases} \tag{4.5}$$

Solution by Maple

$m := 2 :$
$n := 3 :$

$$dsolve\left(\left(1-x^2\right)y''-2\cdot x\cdot y'+\left(n(n+1)-\frac{m^2}{1-m^2}\right)y=0\right)$$

$$y(x) = \_C1\ P_{\frac{\sqrt{165}}{6}-\frac{1}{2}}(x) + \_C2\ Q_{\frac{\sqrt{165}}{6}-\frac{1}{2}}(x) \tag{4.6}$$

$unassign('n')$:
$unassign('m')$:

$$dsolve\left(\left(1-x^2\right)y''-2\cdot x\cdot y'+\left(n(n+1)-\frac{m^2}{1-m^2}\right)y=0\right)$$

$$y(x) = \_C1\ P_{-\frac{-\sqrt{-1+5m^2-4n(1+n)+4n(1+n)m^2}+\sqrt{-1+m^2}}{2\sqrt{-1+m^2}}}(x) + \_C2\ Q_{-\frac{-\sqrt{-1+5m^2-4n(1+n)+4n(1+n)m^2}+\sqrt{-1+m^2}}{2\sqrt{-1+m^2}}}(x) \tag{4.7}$$

Bessel equation

Bessel's differential equation

$x^2y''+x\cdot y'+\left(x^2-n^2\right)y=0$

$$x^2\,y''(x)+x\,y'(x)+\left(x^2-n^2\right)y(x)=0 \tag{5.1}$$

Two linearly independent solutions are known as Bessel function, $J_n(x)$ and Weber function $Y_n(x)$.
Hence, a geneal solution is given by

$y(x)=C_1J_n(x)+C_2\,Y_n(x)$

$$y(x)=C_1\,J_n(x)+C_2\,Y_n(x) \tag{5.2}$$

The second solution is also called Neuman function and denoted as $N_n(x)=Y_n(x)$.

General solution by Maple

$dsolve\left(x^2y''+x\cdot y'+\left(x^2-n^2\right)y=0\right)$

$$y(x)=\_C1\ J_n(x)+\_C2\ Y_n(x) \tag{5.3}$$

Hankel functions $H_n^{(1)}(x)=J_n(x)+\mathrm{i}N_n(x)$, $H_n^{(2)}(x)=J_n(x)-\mathrm{i}N_n(x)$ are also a pair of linearly independent solutions.

In Maple, these functions are predefined as BesselJ(n,x), BesselY(n,x), BesselH1(n,x), and BesselH2(n,x).
Since they can be expressed only by infinte series, Maple cannot express them in closed forms.

$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+n)}\left(\frac{x}{2}\right)^{2k+n}$

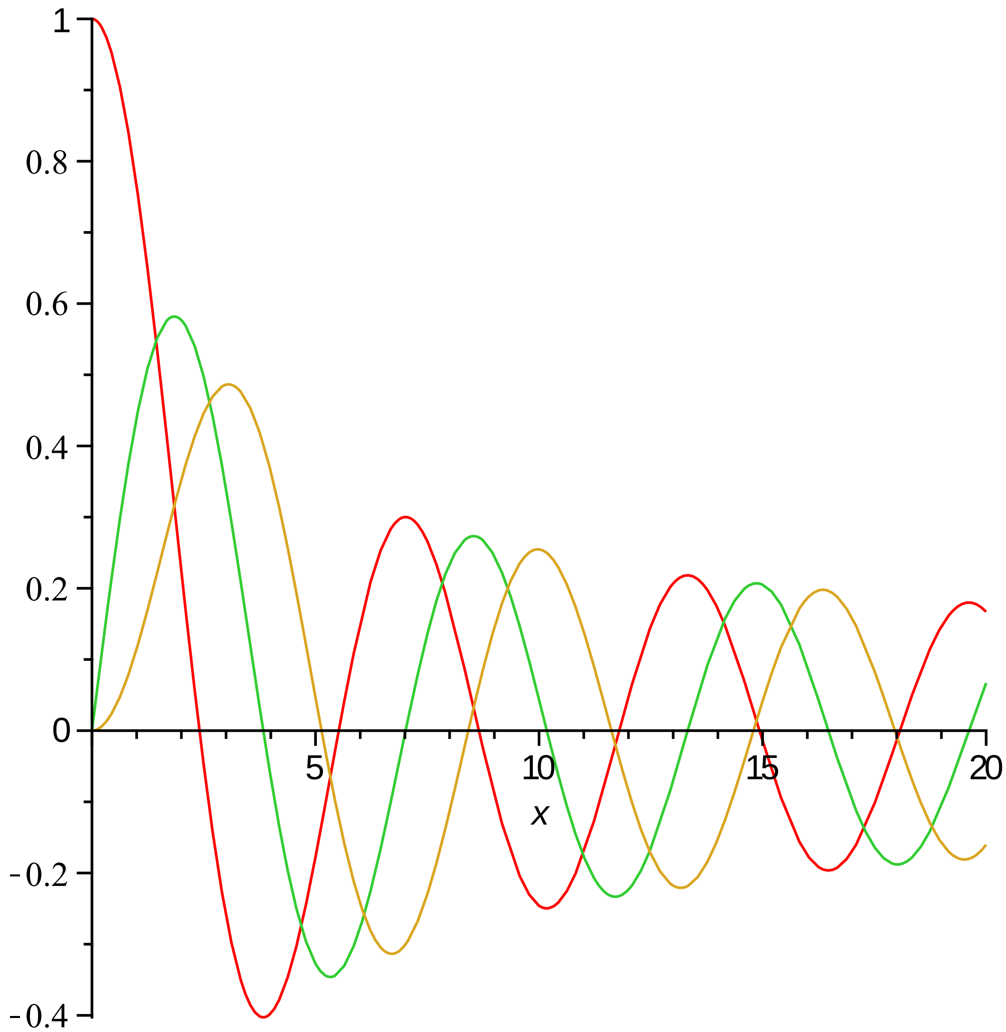
$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+n)}\left(\frac{x}{2}\right)^{2k+n}$$

$$Y_n(x) = \lim_{k \to n} \frac{\cos(k \cdot \pi) J_k(x) - J_{-k}(x)}{\sin(k \cdot \pi)}$$

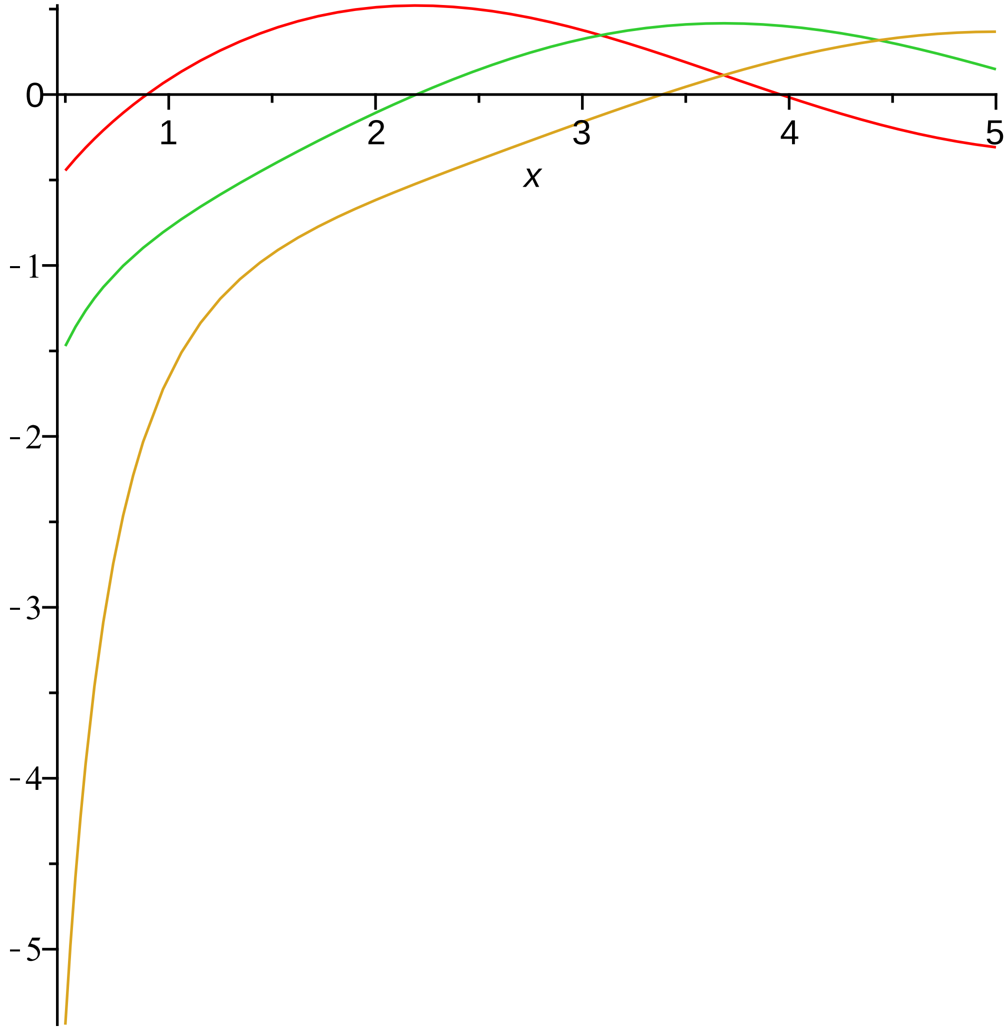
$$Y_n(x) = \lim_{k \to n} \frac{\cos(k \cdot \pi) J_k(x) - J_{-k}(x)}{\sin(k \cdot \pi)} \qquad \textbf{(5.5)}$$

Plot

$plot([\text{BesselJ}(0, x), \text{BesselJ}(1, x), \text{BesselJ}(2, x)], x = 0..20)$



$plot([\text{BesselY}(0, x), \text{BesselY}(1, x), \text{BesselY}(2, x)], x = 0.5..5)$



Bessel functions are not orthogonal!

Modified Bessel equation

Modified Bessel differential equation

$x^2 y'' + x \cdot y' - (x^2 + n^2) y = 0$

$$x^2 y''(x) + x y'(x) - (x^2 + n^2) y(x) = 0 \quad (6.1)$$

Two linearly independent solutions are the first kind and second kind of modeified Bessel functions, $I_n(x)$ and $K_n(x)$, respectively.

General solution by Maple

$dsolve(x^2 y'' + x \cdot y' - (x^2 + n^2) y = 0)$

$$y(x) = \_C1\, I_n(x) + \_C2\, K_n(x) \quad (6.2)$$

They are related to the Bessel functions as follows:

$I_n(x) = \mathrm{i}^{-n} J_n(\mathrm{i}x),\ K_n(x) = \frac{\pi}{2}\mathrm{i}^{n+1} H_n^{(1)}(\mathrm{i}x)$

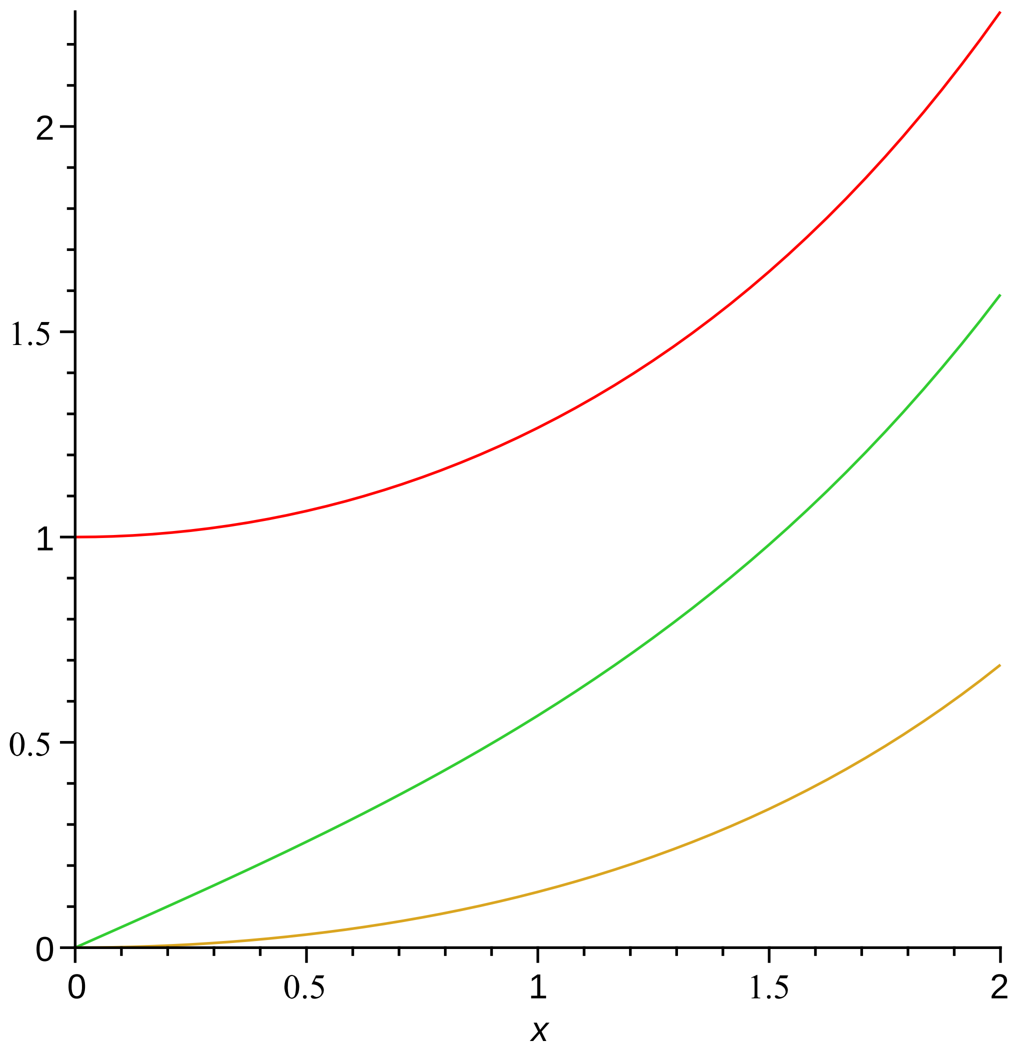
$$I_n(x) = \mathrm{i}^{-n} J_n(\mathrm{i}x),\ K_n(x) = \frac{\pi}{2}\mathrm{i}^{n+1} H_n^{(1)}(\mathrm{i}x) \tag{6.3}$$

In Maple, the modified Bessel functions are predefined as BesselI(*n*,*x*) and BesselK(*n*,*x*)

Plot

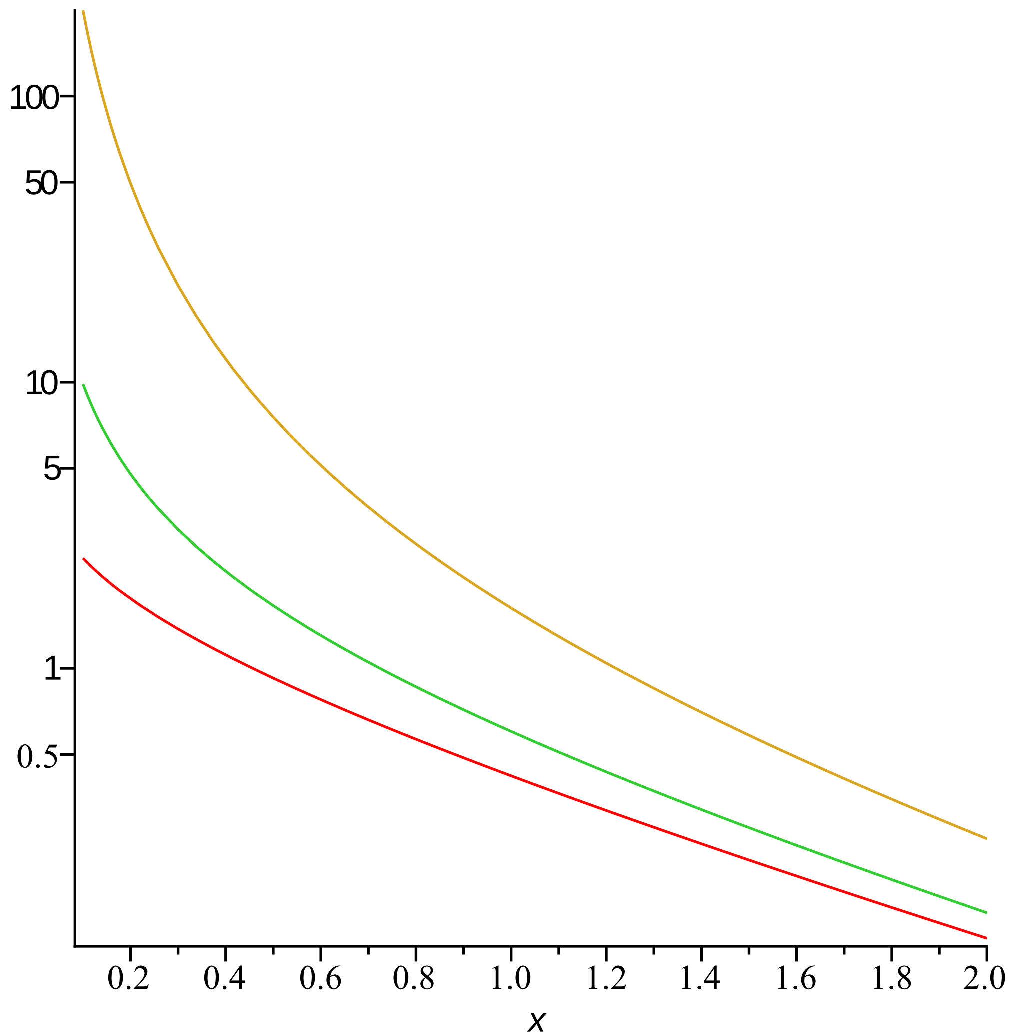
Modified Bessel function of the first kind

$plot(\,[\mathrm{BesselI}(0, x), \mathrm{BesselI}(1, x), \mathrm{BesselI}(2, x)\,], x = 0\,..2\,)$



Modified Bessel function of the second kind

$logplot(\,[\mathrm{BesselK}(0, x), \mathrm{BesselK}(1, x), \mathrm{BesselK}(2, x)\,], x = 0.1\,..2\,)$



Spherical Bessel equation

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Spherical Bessel differential equation

$x^2 y'' + 2 \cdot x \cdot y' + \left(x^2 - n(n+1)\right) y = 0$

$$x^2 y''(x) + 2\, x\, y'(x) + \left(x^2 - n(n+1)\right) y(x) = 0 \tag{7.1}$$

Two linearly independent solutions are spherical Bessel and spherical Nuemann functions, $j_n(x)$ and $n_n(x)$.

They are related to the Bessel functions as follows:

$j_n(x) = \sqrt{\dfrac{\pi}{2\,x}}\, J_{n+\frac{1}{2}}(x) = x^n \left(-\dfrac{1}{x}\, \dfrac{d}{d\,x}\right)^n \left(\dfrac{\sin(x)}{x}\right)$

$$j_n(x) = \sqrt{\frac{\pi}{2\,x}}\,J_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{\sin(x)}{x}\right) \tag{7.2}$$

$$n_n(x) = \sqrt{\frac{\pi}{2\,x}}\,N_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{\cos(x)}{x}\right)$$

$$n_n(x) = \sqrt{\frac{\pi}{2\,x}}\,N_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{\cos(x)}{x}\right) \tag{7.3}$$

Spherical Bessel functions can be expressed in simple form. For example, $j_0(x) = \frac{\sin(x)}{x}$ and $n_0(x) = -\frac{\cos(x)}{x}$.

$$j_0(x) = \sqrt{\frac{\pi}{2\,x}}\,\mathrm{BesselJ}\left(\frac{1}{2}, x\right) = \frac{\sqrt{\frac{\pi}{x}}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}} \overset{\text{combine}}{=} \frac{\sin(x)\,\sqrt{\frac{1}{x}}}{\sqrt{x}} \xrightarrow{\text{simplify symbolic}} \frac{\sin(x)}{x}$$

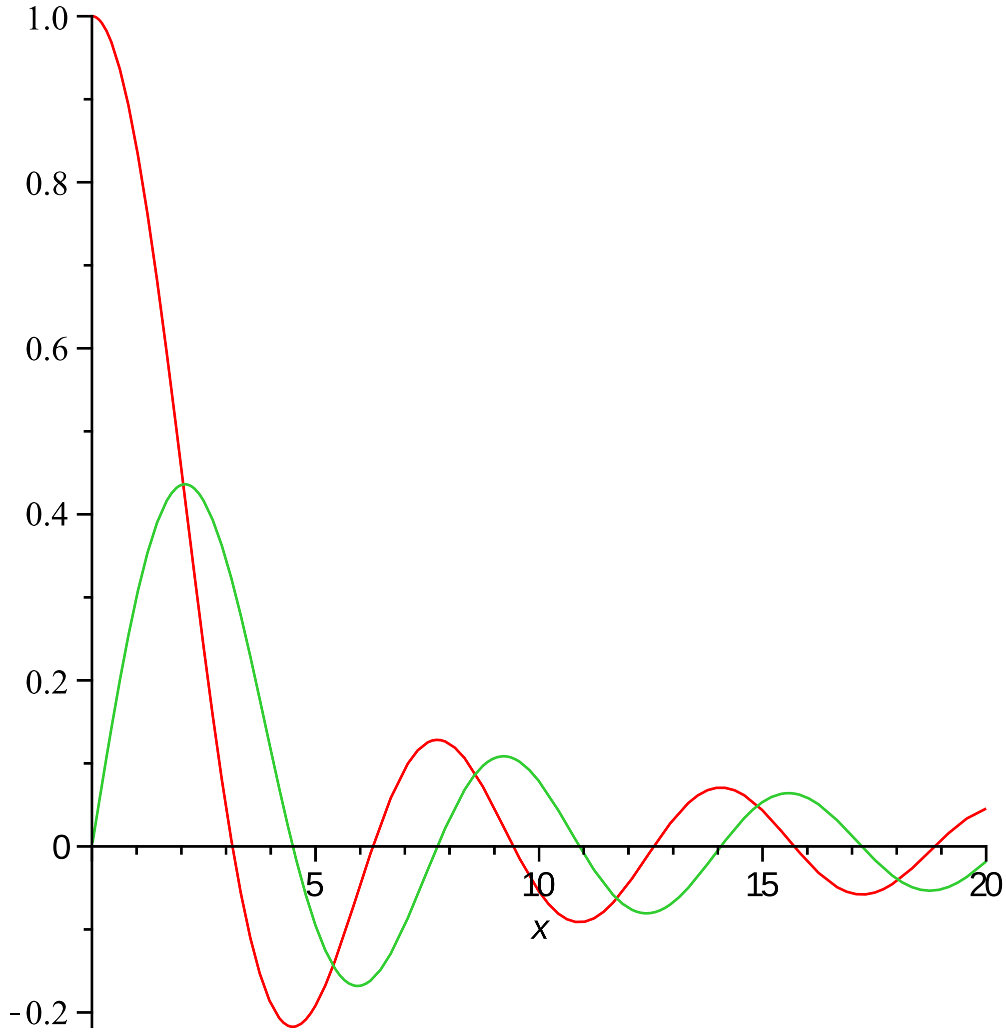
$$j_1(x) = \sqrt{\frac{\pi}{2\,x}}\,\mathrm{BesselJ}\left(\frac{3}{2}, x\right) = \frac{\sqrt{\frac{\pi}{x}}\,(-\cos(x)\,x + \sin(x))}{\sqrt{\pi}\,x^{3/2}} \overset{\text{combine}}{=}$$

$$\frac{(-\cos(x)\,x + \sin(x))\,\sqrt{\frac{1}{x}}}{x^{3/2}} \xrightarrow{\text{simplify symbolic}} \frac{-\cos(x)\,x + \sin(x)}{x^2}$$

Plot

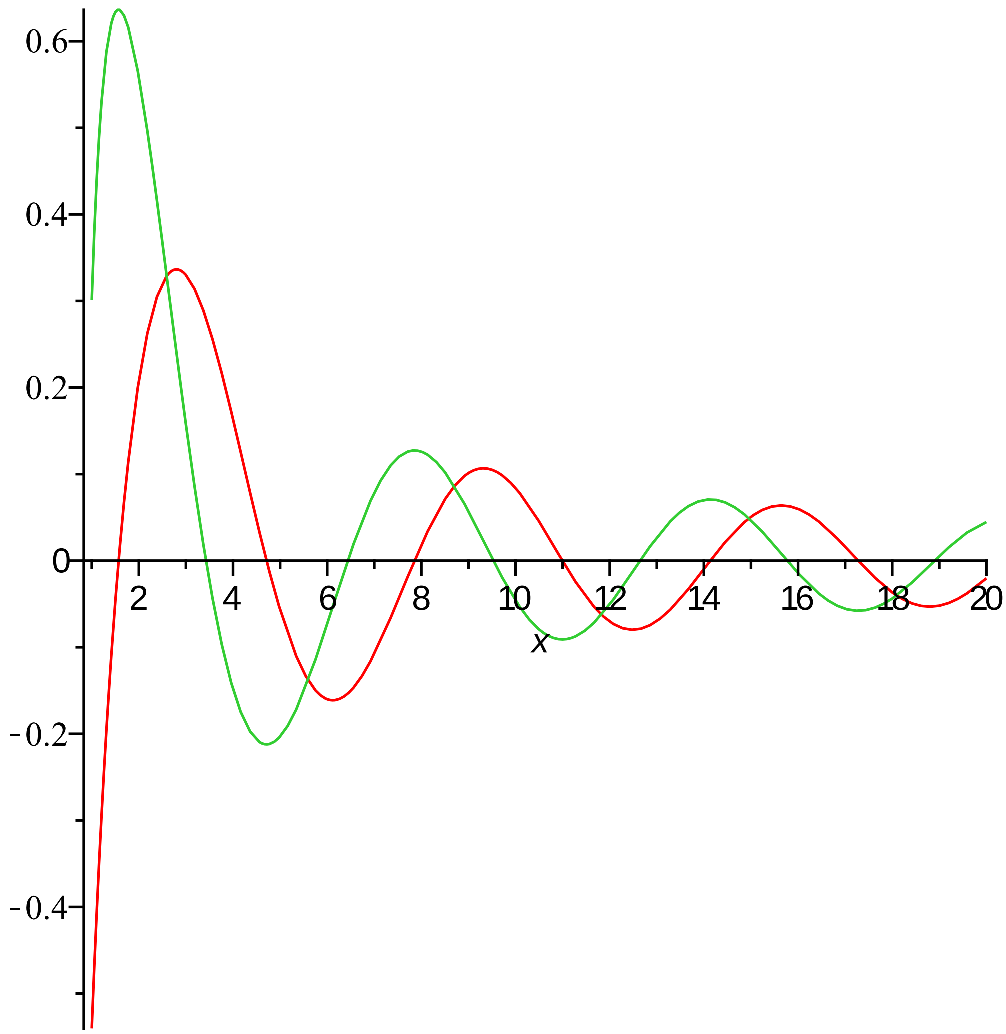
Spherical Bessel $j_0(x)$ and $j_1(x)$.

$$plot\left(\left[\frac{\sin(x)}{x}, \frac{\sin(x) - x\cdot\cos(x)}{x^2}\right], x = 0..20\right)$$



Spherical Bessel $n_0(x)$ and $n_1(x)$.

$$plot\left(\left[-\frac{\cos(x)}{x}, -\frac{\cos(x) - x\cdot\sin(x)}{x^2}\right], x = 1..20\right)$$



Airy equation

Airy differential equation

$y'' - x \cdot y = 0$

$$y''(x) - x\, y(x) = 0 \qquad \textbf{(8.1)}$$

Two linearly independent solutions are the first and second kind of Airy functions, Ai(x) and Bi(x), respecctively. They are related to modified Bessel functions as follows:

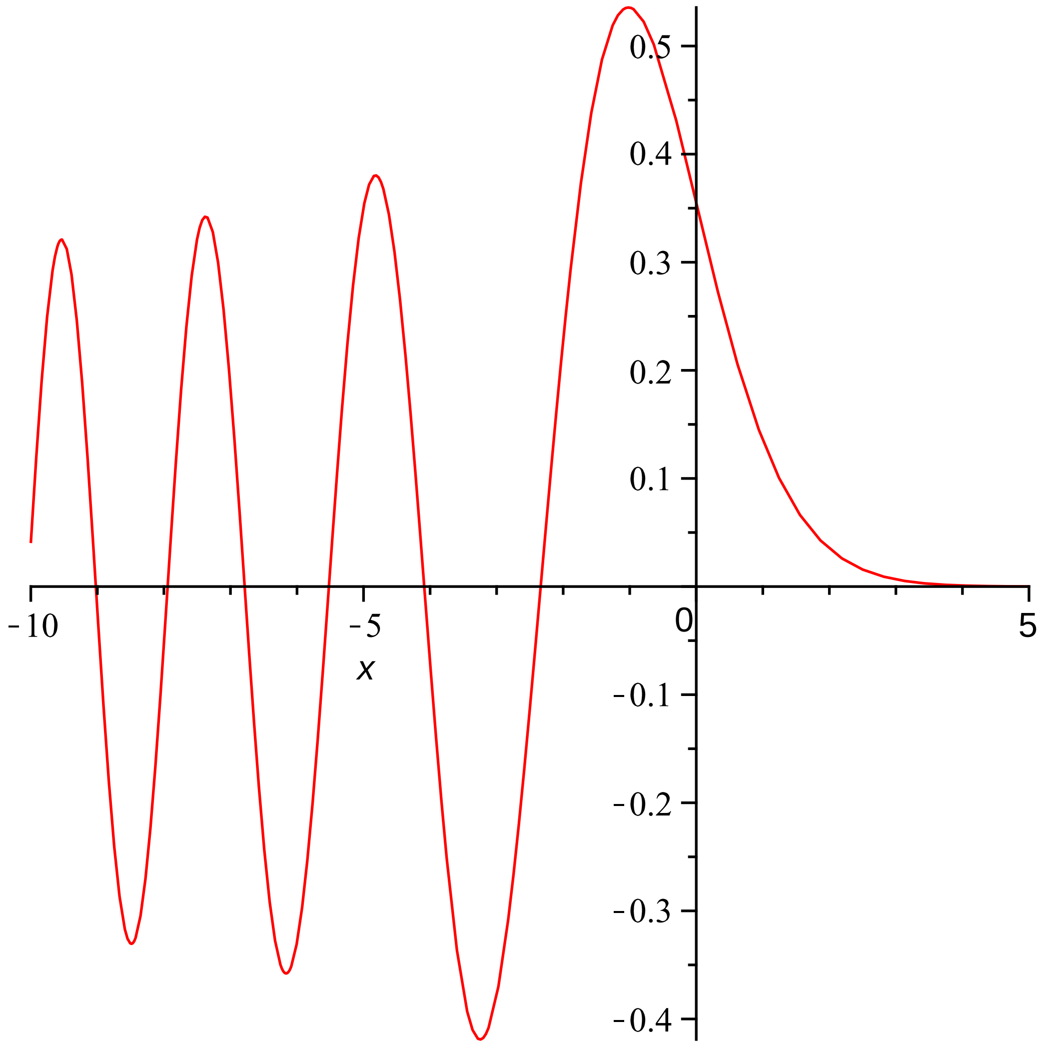
$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\,K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right), \quad \mathrm{Bi}(x) = \sqrt{\frac{x}{3}}\left[I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right]$$

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\,K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right),\quad Bi(x) = \sqrt{\frac{x}{3}}\left[I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right] \tag{8.2}$$

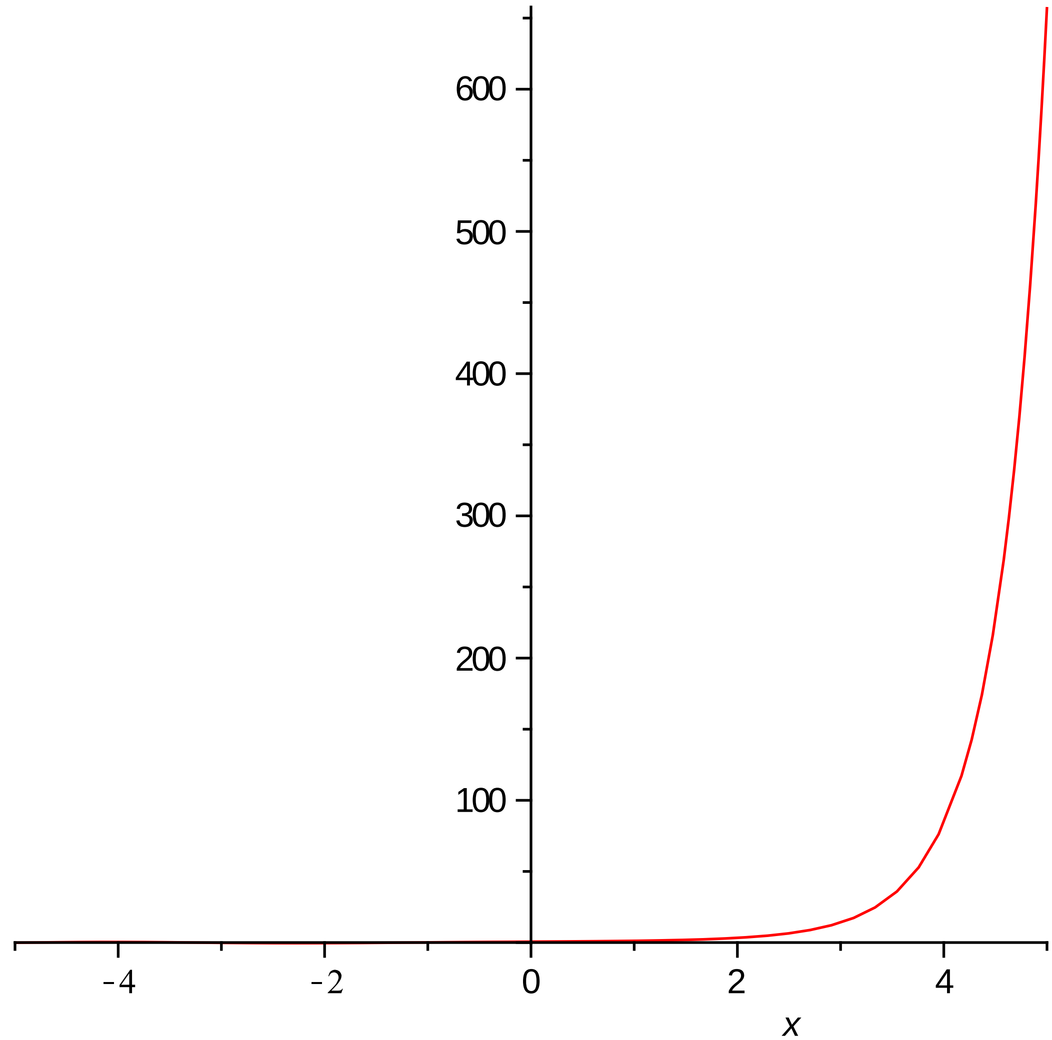
In Maple, the Airy functions are predefined as AiryAi(*x*) and AiryBi(*x*).

Plot

$plot(\mathrm{AiryAi}(x), x = -10..5)$



$plot(\mathrm{AiryBi}(x), x = -5..5)$



General Solutions by Maple

$dsolve(y''-x\cdot y=0)$

$$y(x) = \_C1\ Ai(x) + \_C2\ Bi(x) \quad \textbf{(8.3)}$$

Since $Bi(x) \Rightarrow \infty$ *as* $x \Rightarrow \infty$, the second term is usually elimnaed by physical boundary condition.

Lagguerre equation

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Lagguerre differential equation

$x\cdot y''+(1-x)y'+n\cdot y=0$

$$x\,y''(x) + (1-x)\,y'(x) + n\,y(x) = 0 \quad \textbf{(9.1)}$$

Laguerre's polynomial $L_n(x)$ is a solution.

$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right) \quad \textbf{(9.2)}$$

In Maple, Laguerre polynomials are predefined as LaguerreL(n,x)

The first few Laguerre polynomials are:

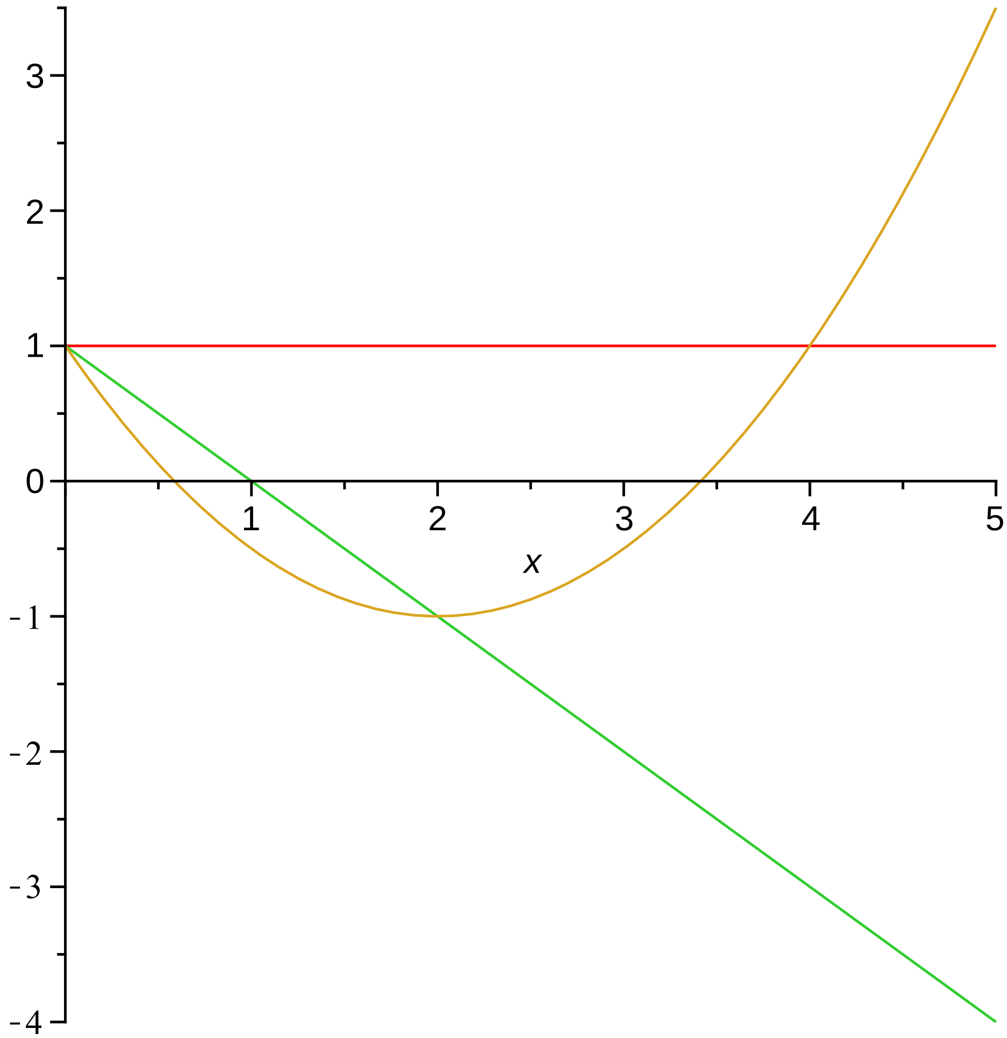
$$\mathrm{LaguerreL}(0, x) = L_0(x) \overset{\text{simplify}}{=} 1$$

$$\mathrm{LaguerreL}(1, x) = L_1(x) \overset{\text{simplify}}{=} 1 - x$$

$$\mathrm{LaguerreL}(2, x) = L_2(x) \overset{\text{simplify}}{=} 1 - 2x + \frac{1}{2} x^2$$

Plot

$plot([\mathrm{LaguerreL}(0, x), \mathrm{LaguerreL}(1, x), \mathrm{LaguerreL}(2, x)], x = 0..5)$



Orthogonality

$e^{-\frac{x}{2}} L_n(x)$ forms an orthonormal basis set for $x \in [0, \infty)$:

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \tag{9.3}$$

General Solutions by Maple

assume$(n, integer)$

$$y(x) = \_C1\, M(-n, 1, x) + \_C2\, U(-n, 1, x) \tag{9.4}$$

$n := 2 :$

dsolve$(x \cdot y'' + (1 - x) y' + n \cdot y = 0)$

$$y(x) = \_C1\, (2 - 4x + x^2) + \_C2 \left(\frac{(2 - 4x + x^2)\, \mathrm{Ei}_1(-x)}{4} + \frac{e^x (-3 + x)}{4} \right) \tag{9.5}$$

The first term is $2\, L_2(x)$. The second term is usually elimnated by physical boundary conditions since it diverges as.