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1 Assignment Questions

Instruction: Attempt any four of the following questions -

1. Give examples of families of sets that are not an algebra or a $\sigma-$ algebra.

2. Prove that each closed subset of $\mathbb{R}^d$ is a G_δ and each open subset of $\mathbb{R}^d$ is $F_\sigma-$ set.

3. Prove that Lebesgue outer measure on $\mathbb{R}^d$ is an outer measure, and it assigns to each $d-$ dimensional interval its volume.

4. Let $(X, \mathscr{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathscr{A}$. For a function $f : A \to \mathbb{R}$, the conditions

   (a) f is measurable with respect to $\mathscr{A}$,

   (b) for each open subset U of $\mathbb{R}$ the set $f^{-1}(U)$ belongs to $\mathscr{A}$,

   (c) for each closed subset C of $\mathbb{R}$ the set $f^{-1}(C)$ belongs to $\mathscr{A}$, and

   (d) for each Borel subset B of $\mathbb{R}$ the set $f^{-1}(B)$ belongs to $\mathscr{A}$

   are equivalent.

5. Let $(X, \mathscr{A}, \mu)$ be a measure space.Let $(Y, \mathscr{B})$ be a measurable space and let $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be measurable.Let g be an extended real valued $\mathscr{B}$ measurable function on Y then g is μf^{-1} integrable if and only if gof is $\mu-$integrable. If these are integrable then

$$\int_Y g d(\mu f^{-1}) = \int_X gof du$$

2 Solution of Assignment Questions

2.1 Solution of Question No. 1

Let us define σ-algebra;

A collection $\mathscr{A}$ of subsets of X is a σ-algebra[1] on X If

a) $X \in \mathscr{A}$,

b) for each set A that belongs to $\mathscr{A}$, the set A^c belongs to $\mathscr{A}$.

c) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathscr{A}$, the set $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathscr{A}$, and

d) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathscr{A}$, the set $\bigcap_{i=1}^{\infty} A_i$ belongs to $\mathscr{A}$

Thus a σ-algebra on X is a family of subsets of X that contains X and is closed under complementation, under the formation of countable unions, and under the formation of countable intersections.

Examples of Families of sets that are not σ-algebra :

1) Let X be an infinite set, and let $\mathscr{A}$ be the collection of all finite subsets of X. Then $\mathscr{A}$ does not contain X and is not closed under complementation; hence it is not an algebra (or a σ-algebra) on X.

2) Let X be an infinite set, and let $\mathscr{A}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathscr{A}$ is an algebra on X but is not closed under the formation of countable unions; hence it is not a σ-algebra.

3) Let X be an uncountable set, and let $\mathscr{A}$ be the collection of all countable (i.e., finite or countably infinite) subsets of X. Then $\mathscr{A}$ does not contain X and is not closed under complementation; hence it is not an algebra.

4) Let L be the collection of all finite disjoint unions of all intervals of the form:

   $(-\infty, a], (a, b], (b, \infty), \emptyset, \mathbb{R}$.

   Then L is an algebra over $\mathbb{R}$, but not a σ-algebra because union of sets $\left\{(0, \frac{i-1}{i}]\right\}$ for all $i \geq 1 = (0, 1) \notin L$.

[1] The terms field and σ-field are sometimes used in place of algebra and σ-algebra.

2.2 Solution of Question No. 2

Suppose that F is a closed subset of $\mathbb{R}^d$. We need to construct a sequence $\{U_n\}$ of open subsets of $\mathbb{R}^d$ such that $F = \cap_n U_n$. For this define U_n by

$$U_n = \left\{ x \in \mathbb{R}^d : \|x - y\| < \frac{1}{n} \qquad \text{for some } y \text{ in } F \right\}$$

(Note that U_n is empty if F is empty.) It is clear that each U_n is open and that $F \subseteq \cap_n U_n$. The reverse inclusion follows from the fact that F is closed (note that each point in $\cap_n U_n$ is the limit of a sequence of points in F). Hence each closed subset of $\mathbb{R}^d$ is a G_δ.

If U is open, then U^c is closed and so is a G_δ. Thus there is a sequence $\{U_n\}$ of open sets such that $U^c = \cap_n U_n$. The sets U_n^c are then closed, and $U = \cup_n U_n^c$; hence U is an F_σ.

2.3 Solution of Question No. 3

We begin by verifying that m^* is an outer measure. The relation $m^*(\emptyset) = 0$ holds, since for each positive number ϵ there is a sequence $\{(a_i, b_i)\}$ of open intervals (whose union necessarily includes $\emptyset$) such that $\sum_i (b_i - a_i) < \epsilon$. For the monotonicity of m^*, note that if $A \subseteq B$, then each sequence of open intervals that covers B also covers A, and so $m^*(A) \leq m^*(B)$. Now consider the countable subadditivity of m^*. Let $\{A_n\}_{n=1}^\infty$ be an arbitrary sequence of subsets of $\mathbb{R}$. If $\sum_n m^*(A_n) = +\infty$, then $m^*(\cup_n A_n) \leq \sum_n m^*(A_n)$ certainly holds.

So suppose that $\sum_n m^*(A_n) < +\infty$, and let ϵ be an arbitrary positive number. For each n choose a sequence $\{(a_{n,i}, b_{n,i})\}_{i=1}^\infty$ that covers A_n and satisfies

$$\sum_{i=1}^{\infty} (b_{n,i}, a_{n,i}) < m^*(A_n) + \frac{\epsilon}{2^n}.$$

If we combine these sequences into one sequence $\{(a_j, b_j)\}$, then the combined sequence satisfies

$$\cup_n A_n \subseteq U_j (a_j, b_j)$$

and

$$\sum_j (b_j - a_j) < \sum_n \left(m^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_n m^*(A_n) + \epsilon$$

These relations, together with the fact that ϵ is arbitrary, imply that $m^*(\cup_n A_n) \leq \sum_n m^*(A_n)$. Thus m^* is an outer measure.

Now, Suppose that if K is a compact d-dimensional interval and if $\{\mathbb{R}_i\}_{i=1}^{\infty}$ is a sequence of bounded and open d-dimensional intervals for which $K \subseteq \cup_{n=1}^{\infty} \mathbb{R}_i$, then there is a positive integer n such that $K \subseteq \cup_{n=1}^{\infty} \mathbb{R}_i$, and K can be decomposed into a finite collection $\{K_j\}$ of d-dimensional intervals that overlap only on their boundaries and are such that for each j the interior of K_j is included in some $\mathbb{R}_i$ (where $i \leq n$). From this it follows that

$$\boxed{\text{vol}(K) = \sum_j \text{vol}(K_j) \leq \sum_i \text{vol}(\mathbb{R}_i)}$$

and hence that $\text{vol}(K) \leq m^*(K)$.
Overall, we shown that "Lebesgue outer measure on $\mathbb{R}^d$ is an outer measure, and it assigns to each d-dimensional interval its volume".

2.4 *Solution of Question No. 4*

Let $\mathscr{F} = \{B \subseteq \mathbb{R} : f^{-1}(B) \in \mathscr{A}\}$. Then the fact that $f^{-1}(\mathbb{R}) = A$ and the identities

$$f^{-1}(B^c) = A - f^{-1}(B)$$

and

$$f^{-1}\left(\bigcup_n B_n\right) = \bigcup_n f^{-1}(B_n)$$

imply that $\mathscr{F}$ is a σ-algebra on $\mathbb{R}$. To require that f be measurable is to require that $\mathscr{F}$ contain all the intervals of the form $(-\infty, b]$ or equivalently (since $\mathscr{F}$ is a σ-algebra) to require that $\mathscr{F}$ include the σ-algebra on $\mathbb{R}$ generated by these intervals. Since the σ-algebra generated by these intervals is the σ-algebra of Borel subsets of $\mathbb{R}$ so it is obvious, conditions (a) and (d) are equivalent. However the σ-algebra of Borel subsets of $\mathbb{R}$ is also generated by the collection of all open subsets of $\mathbb{R}$ and by the collection of all closed subsets of $\mathbb{R}$, and so conditions (b) and (c) are equivalent to the others.

3 References

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- *For document related info visit* https://github.com/akhlak919/LaTeX_Stuffs