Contents

1 Assignment Problems

1.1 Theorem(5.6.8)

Any two complete orthonormal sets in Hilbert Space $\mathcal{H}$ have the same cardinal number.

1.2 Theorem(5.6.10)

A Hilbert Space $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis.

1.3 Problem(5.5.3(2))

In the inner product space Φ [a], the sequence $\{e_n\}$, where $e_n = \left\{\delta_{n_j}\right\}_{j'}$ is an orthonormal sequence.

1.4 Problem(5.5.3(3))

In the Hilbert Space ℓ^2 [b] the sequence $\{e_n\}$, where $e_n = \left\{\delta_{n_j}\right\}_{j'}$ is an orthonormal sequence.

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[a] where Φ is the inner product space defined as, $\langle x, y\rangle = \sum_i \xi_i \overline{\eta_i} \qquad \forall\, x = (\xi_1, \xi_2, \cdots\cdots)$ and $y = (\eta_1, \eta_2, \cdots\cdots)$

[b] where ℓ^2 is the Hilbert Space defined as, $\langle x, y\rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i} \qquad \forall\, x = \{\xi_i\}$ and $y = \{\eta_i\}$ and equipped with the induced norm,

$$\|x\| = (\langle x, x\rangle)^{1/2} = \left(\sum_{i=1}^{\infty} |\xi_1|^2\right)^{1/2}; \qquad x = \{\xi_i\} \in \ell^2$$

2 Solutions of the Assignment Problems

2.1 Solution : Proof of Theorem(5.6.8)

Let H be a Hilbert space, and let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two complete orthonormal sets[1] in H. Since $\{e_i\}_{i\in I}$ is complete, it is an orthonormal basis for H, which means that every element $x \in H$ can be expressed as a linear combination of the e_i's, i.e.,

$$x = \sum_{i\in I} \langle x, e_i\rangle e_i.$$

Similarly, since $\{f_j\}_{j\in J}$ is complete, it is also an orthonormal basis for H, so every element $x \in H$ can also be expressed as

$$x = \sum_{j\in J} \langle x, f_j\rangle f_j.$$

Now, we can equate these two expressions for x:

$$\sum_{i\in I} \langle x, e_i\rangle e_i = \sum_{j\in J} \langle x, f_j\rangle f_j.$$

Taking the inner product of both sides with e_k for some $k \in I$, we have

$$\langle x, e_k\rangle = \sum_{j\in J} \langle x, f_j\rangle \langle f_j, e_k\rangle,$$

since $\{f_j\}$ is orthonormal. But since $\{e_i\}$ is orthonormal as well, we have $\langle f_j, e_k\rangle = \delta_{jk}$ (the Kronecker delta), so we can simplify this to

$$\langle x, e_k\rangle = \langle x, f_k\rangle.$$

This means that the coefficients $\{\langle x, e_i\rangle\}_{i\in I}$ and $\{\langle x, f_j\rangle\}_{j\in J}$ are the same for every $x \in H$, and hence they must have the same cardinality (since they both form a basis for H). Therefore, $\{e_i\}$ and $\{f_j\}$ have the same cardinality.

[1] A complete orthonormal set in a Hilbert space is a set of vectors that are pairwise orthogonal and normalized, and which form a basis for the entire Hilbert space. This means that any vector in the Hilbert space can be expressed as a unique linear combination of the vectors in the set, and the coefficients of this combination can be calculated using the inner product between the vector and the individual basis vectors. The most well-known example of a complete orthonormal set is the set of Fourier basis functions, which form a basis for the space of square-integrable functions on the real line.

2.2 Solution : Proof of Theorem(5.6.10)

To prove this theorem we need have to show both the dirrections of the statement, to do so First, Suppose $\mathcal{H}$ is separable, then there exists a countable dense subset $\{x_n\}$ of $\mathcal{H}$. Define the following sets:

$$S_k = \text{span}\{x_1, x_2, \dots, x_k\}, \qquad k \geq 1.$$

Since $\{x_n\}$ is dense in $\mathcal{H}$, we have $\bigcup_{k\geq 1} S_k$ is dense in $\mathcal{H}$.

Now, for each $k \geq 1$, we use the Gram-Schmidt process to obtain an orthonormal basis $\{e_{k,n}\}_{n=1}^{k}$ for S_k. Then, $\{e_{k,n}\}_{k,n\geq 1}$ is a countable orthonormal basis for $\mathcal{H}$.

Indeed, given any $x \in \mathcal{H}$, we can find a sequence $\{x_n\}$ in $\bigcup_{k\geq 1} S_k$ such that $x_n \to x$. Then, we can find k such that $x_n \in S_k$ for all n large enough. Using the fact that $\{e_{k,n}\}$ is an orthonormal basis for S_k, we can write $x_n = \sum_{j=1}^{k} c_{k,j,n} e_{k,j}$, where $c_{k,j,n} \in \mathbb{C}$. By the Cauchy-Schwarz inequality, we have

$$\sum_{j=1}^{k} |c_{k,j,n}|^2 \leq \sum_{j=1}^{k} ||x_n||^2 = ||x_n||^2.$$

Since $\{x_n\}$ is a Cauchy sequence, we have $\sum_{j=1}^{k} |c_{k,j,n}|^2 \to ||x||^2$ as $n \to \infty$. Thus, we have

$$x = \sum_{j=1}^{k} c_{k,j,n} e_{k,j}$$

for some sequence of coefficients $c_{k,j,n}$, which shows that $\{e_{k,n}\}$ is a basis for $\mathcal{H}$.

Conversely, Suppose $\mathcal{H}$ has a countable orthonormal basis $\{e_n\}$. Define the following set:

$$D = \{c_1e_1 + c_2e_2 + \cdots + c_ne_n : n \in \mathbb{N}, c_i \in \mathbb{Q} + i\mathbb{Q}\}.$$

That is, D is the set of all finite linear combinations of basis vectors with rational coefficients. We claim that D is a countable dense subset of $\mathcal{H}$.

To see that D is countable, note that the set of all finite sequences of rational numbers is countable, and therefore the set of all finite linear combinations of basis vectors with rational coefficients is also countable.

To see that D is dense, let $x \in \mathcal{H}$ and $\epsilon > 0$ be arbitrary.

Since $\{e_n\}$ is a basis for $\mathcal{H}$, we can write x as an infinite linear combination of the e_n:

$$x = \sum_{n=1}^{\infty} c_n e_n,$$

where the coefficients c_n are uniquely determined by x.

Now, choose N such that

$$\left\| \sum_{n=N+1}^{\infty} c_n e_n \right\| < \frac{\epsilon}{2}.$$

Since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$, we can find rational numbers $a_n, b_n \in \mathbb{Q}$ such that $|a_n - \mathrm{Re}(c_n)| < \frac{\epsilon}{2^{n+2}}$ and $|b_n - \mathrm{Im}(c_n)| < \frac{\epsilon}{2^{n+2}}$ for all n. Then, define

$$y = \sum_{n=1}^{N} a_n e_n + i \sum_{n=1}^{N} b_n e_n.$$

Since y is a finite linear combination of basis vectors with rational coefficients, $y \in D$.

Now, we have

$$\begin{aligned}
\|x - y\| &= \left\| \sum_{n=N+1}^{\infty} c_n e_n - \sum_{n=1}^{N} (a_n e_n + i b_n e_n) \right\| \\
&= \left\| \sum_{n=N+1}^{\infty} c_n e_n - \sum_{n=1}^{N} a_n e_n - i \sum_{n=1}^{N} b_n e_n \right\| \\
&\leq \left\| \sum_{n=N+1}^{\infty} c_n e_n \right\| + \left\| \sum_{n=1}^{N} (a_n e_n + i b_n e_n) \right\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, D is a countable[2] dense subset of $\mathcal{H}$, which proves that $\mathcal{H}$ is separable[3].

Therefore, we have shown that a Hilbert space $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis.

[2] A countable orthonormal basis for a Hilbert space $\mathcal{H}$ is a countable set e_n of vectors in $\mathcal{H}$ such that every vector in $\mathcal{H}$ can be expressed as a unique linear combination of the vectors in e_n, and the set e_n is orthonormal, meaning that $\langle e_n, e_m \rangle = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta.

[3] A Hilbert space $\mathcal{H}$ is said to be separable if there exists a countable dense subset of $\mathcal{H}$. In other words, there exists a countable set x_n of vectors in $\mathcal{H}$ such that every vector in $\mathcal{H}$ can be approximated arbitrarily closely by a linear combination of the vectors in x_n.

2.3 Solution : Problem(5.5.3(2))

We need to show that the sequence $\{e_n\}$ defined by $e_n = \{\delta_{n_j}\}_{j'}$ is an orthonormal sequence in the inner product space Φ, where δ_{n_j} is the Kronecker delta function, i.e., $\delta_{n_j} = 1$ if $n = j$ and $\delta_{n_j} = 0$ otherwise.

To show that $\{e_n\}$ is an orthonormal sequence, we need to show that for all m, n, we have $\langle e_m, e_n \rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta function.

First, let's consider the case when $m = n$. Then we have

$$\begin{aligned}
\langle e_m, e_m \rangle &= \sum_{j'} \delta_{m_j} \overline{\delta_{m_j}} \\
&= \sum_{j'} |\delta_{m_j}|^2 \\
&= \sum_{j'} \delta_{m_j} \\
&= 1,
\end{aligned}$$

since there is only one j such that $\delta_{m_j} = 1$, namely $j = m$, and all other terms are zero.

Next, let's consider the case when $m \neq n$. Then we have

$$\begin{aligned}
\langle e_m, e_n \rangle &= \sum_{j'} \delta_{m_j} \overline{\delta_{n_j}} \\
&= \sum_{j'} \delta_{m_j} \delta^*_{n_j} \\
&= \delta_{m,n} \sum_{j'} \delta_{m_j} \\
&= 0,
\end{aligned}$$

since if $m \neq n$, then there is no j such that both δ_{m_j} and δ_{n_j} are equal to 1, and thus the sum evaluates to zero.

Therefore, we have shown that $\{e_n\}$ is an orthonormal sequence in inner product space Φ[4].

[4]where Φ is the inner product space defined as, $\langle x, y \rangle = \sum_i \xi_i \overline{\eta_i}$ $\forall\ x = (\xi_1, \xi_2, \cdots\cdots)$ and $y = (\eta_1, \eta_2, \cdots\cdots)$

2.4 Solution : Problem(5.5.3(3))

We need to show that the sequence $\{e_n\}$ defined by $e_n = \{\delta_{n_j}\}_{j'}$ is an orthonormal sequence in the Hilbert space ℓ^2, where δ_{n_j} is the Kronecker delta function, i.e., $\delta_{n_j} = 1$ if $n = j$ and $\delta_{n_j} = 0$ otherwise.

To show that $\{e_n\}$ is an orthonormal sequence, we need to show that for all m, n, we have $\langle e_m, e_n \rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta function.

First, let's consider the case when $m = n$. Then we have

$$\begin{aligned}
\langle e_m, e_m \rangle &= \sum_{j'} \delta_{m_j} \overline{\delta_{m_j}} \\
&= \sum_{j'} |\delta_{m_j}|^2 \\
&= \sum_{j'} \delta_{m_j} \\
&= 1,
\end{aligned}$$

since there is only one j such that $\delta_{m_j} = 1$, namely $j = m$, and all other terms are zero.

Next, let's consider the case when $m \neq n$. Then we have

$$\begin{aligned}
\langle e_m, e_n \rangle &= \sum_{j'} \delta_{m_j} \overline{\delta_{n_j}} \\
&= \sum_{j'} \delta_{m_j} \delta^*_{n_j} \\
&= \delta_{m,n} \sum_{j'} \delta_{m_j} \\
&= 0,
\end{aligned}$$

since if $m \neq n$, then there is no j such that both δ_{m_j} and δ_{n_j} are equal to 1, and thus the sum evaluates to zero.

Therefore, we have shown that $\{e_n\}$ is an orthonormal sequence in ℓ^2[5].

[5] where ℓ^2 is the Hilbert Space defined as, $\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i} \qquad \forall\ x = \{\xi_i\}$ and $y = \{\eta_i\}$ and equipped with the induced norm,

$$\|x\| = (\langle x, x \rangle)^{1/2} = \left(\sum_{i=1}^{\infty} |\xi_1|^2 \right)^{1/2} ; \qquad x = \{\xi_i\} \in \ell^2$$

3 References

- *Linear Algebra Done Right : Sheldon Axler (Springer edition)*
- *Functional Analysis : J N Kapoor, OM P Ahuja*
- *For document related info visit* https://github.com/akhlak919/LaTeX_Stuffs