Assignment

Wavelet Analysis

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1 Assignment Questions

1. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{1}{50\sqrt{t}} & , \quad t \in (0,1); \\ 0 & , \quad \text{Otherwise} \end{cases}$$

   Then show that $f \in L^1(\mathbb{R})$ but $f \notin L^2(\mathbb{R})$.

2. Find the Fourier transform of the following functions:

   (a) $f(t) = \begin{cases} -20e^{-t} & , \quad t \geq 0; \\ 20e^{t} & , \quad t < 0 \end{cases}$

   (b) $f(t) = \begin{cases} 10t & , \quad 0 \leq t < 1; \\ 0 & , \quad \text{Otherwise} \end{cases}$

3. Define a function $\psi(t)$ by

$$\psi(t) = \begin{cases} 3 & , \quad 0 \leq t < \frac{1}{2}; \\ -3 & , \quad \frac{1}{2} \leq t < 1; \\ 0 & , \quad \text{Otherwise} \end{cases}$$

   Then show that (i) $||\psi||_2 = 3$. (ii) $\int_{-\infty}^{\infty} \psi(t)dt = 0$.

4. Let $\psi \in L^2(\mathbb{R})$ be a basic wavelet and ϕ be a bounded, integrable function, then $\psi * \phi$ is also a basic wavelet.

5. Define Mexican Hat wavelet $\psi_M(t)$ and prove the following

   (i) $\int_{-\infty}^{\infty} \psi_M(t)dt = 0$.

   (ii) $C_{\psi_M} < \infty$.

6. Let $\delta(x)$ denotes the Dirac delta function.Then find the value of the following integrals:

   (i) $\int_{-\infty}^{\infty} \delta(x-1)\left\{\cos^2(\pi x) + \sin^2(\pi x) + e^{\pi x}\right\}dx$.

   (ii) $\int_{-\infty}^{\infty} \delta(x)\left\{x^2 + x + 1\right\}(e^x)dx$.

7. Let us define a function $g_\alpha(x)$ by

$$g_\alpha(x) = \frac{e^{\frac{-x^2}{4\alpha}}}{2\sqrt{\pi\alpha}}, \alpha > 0.$$

Then prove the following:

(i) $g \in L^1(\mathbb{R})$.

(ii) $g \in L^2(\mathbb{R})$.

(iii) $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(iv) $\|g_\alpha\|_1 = 1 \quad \forall\, \alpha > 0$.

Does $g \in L^4(\mathbb{R})$? Explain.

2 Solution of Assignment Questions

2.1 Solution of Question No. 1

For inclusivity of a function $f(t)$ in $L^1(\mathbb{R})$, we have to show that

$$\|f\|_1^2 = \int_{-\infty}^{\infty} |f(t)|\, dt < \infty. \qquad \forall\, t \in \mathbb{R}$$

Since in our problem

$$f(t) = \begin{cases} \frac{1}{50\sqrt{t}} & , \quad t \in (0,1); \\ 0 & , \quad \text{Otherwise} \end{cases}$$

$$\begin{aligned}
\text{So,} \qquad \|f\|_1^2 = \int_{-\infty}^{\infty} |f(t)|\, dt &= \int_{-\infty}^{0} |f(t)|\, dt + \int_{0}^{1} |f(t)|\, dt + \int_{1}^{\infty} |f(t)|\, dt \\
&= \int_{-\infty}^{0} |0|\, dt + \int_{0}^{1} \left|\frac{1}{50\sqrt{t}}\right| dt + \int_{1}^{\infty} |0|\, dt \\
&= \int_{0}^{1} \frac{1}{50|\sqrt{t}|} dt = \frac{1}{50} \int_{0}^{1} \frac{1}{\sqrt{t}} dt \\
&= \frac{1}{50} \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} \frac{1}{\sqrt{t}} dt \\
&= \frac{1}{50} \lim_{\epsilon \to 0^+} \left[2\sqrt{t}\right]\Big|_{\epsilon}^{1} \\
&= \frac{2}{50} \lim_{\epsilon \to 0^+} \left[1 - \sqrt{\epsilon}\right] \\
&= \frac{1}{25} \left[1 - \lim_{\epsilon \to 0^+} \sqrt{\epsilon}\right] \\
&= \frac{1}{25} [1 - 0] \\
&= \frac{1}{25} < \infty
\end{aligned}$$

$$\text{i.e,} \qquad \|f\|_1^2 = \int_{-\infty}^{\infty} |f(t)|\, dt = \frac{1}{25} < \infty$$

$$\text{thus,} \qquad f \in L^1(\mathbb{R}).$$

Again, For inclusivity of a function $f(t)$ in $L^2(\mathbb{R})$, we have to show that

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty. \qquad \forall\, t \in \mathbb{R}$$

So, $$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{0} |f(t)|^2 dt + \int_{0}^{1} |f(t)|^2 dt + \int_{1}^{\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{0} |0|^2 dt + \int_{0}^{1} \left|\frac{1}{50\sqrt{t}}\right|^2 dt + \int_{1}^{\infty} |0|^2 dt$$

$$= \frac{1}{2500}\int_{0}^{1} \frac{1}{|\sqrt{t}|^2} dt$$

$$= \frac{1}{2500}\int_{0}^{1} \frac{1}{t} dt$$

$$= \frac{1}{2500}\lim_{\epsilon\to 0^+}\int_{\epsilon}^{1} \frac{1}{t} dt$$

$$= \frac{1}{2500}\lim_{\epsilon\to 0^+}[\ln t]\Big|_{\epsilon}^{1}$$

$$= \frac{1}{2500}\lim_{\epsilon\to 0^+}[\ln(1) - \ln(\epsilon)]$$

$$= \frac{1}{2500}\left[0 - \lim_{\epsilon\to 0^+}\ln(\epsilon)\right]$$

$$= \frac{1}{2500}[-(-\infty)]$$

$$= \frac{1}{2500}[\infty] = \infty$$

i.e, $$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2500}[\infty] = \infty$$

thus, $f \notin L^2(\mathbb{R})$.

Thus we have shown that $f \in L^1(\mathbb{R})$ but $f \notin L^2(\mathbb{R})$.

2.2 Solution of Question No. 2(a)

To find the Fourier transform of the function $f(t)$.Let's first check is it in $L^1(\mathbb{R})$ or not ? So,

$$\begin{aligned}
\|f\|_1^2 = \int_{-\infty}^{\infty} |f(t)|\,dt &= \int_{-\infty}^{0} |f(t)|\,dt + \int_{0}^{\infty} |f(t)|\,dt \\
&= \int_{-\infty}^{0} |20e^t|\,dt + \int_{0}^{\infty} |-20e^{-t}|\,dt \\
&= 20\left[\int_{-\infty}^{0} |e^t|\,dt + \int_{0}^{\infty} |e^{-t}|\,dt\right] \\
&= 20\left[\int_{-\infty}^{0} e^t dt + \int_{0}^{\infty} e^{-t} dt\right] \\
&= 20\left[\lim_{M_1\to-\infty}\int_{M_1}^{0} e^t dt + \lim_{M_2\to\infty}\int_{0}^{M_2} e^{-t} dt\right] \\
&= 20\left[\lim_{M_1\to-\infty}\left(e^t\right)\Big|_{M_1}^{0} + \lim_{M_2\to\infty}\left(-e^{-t}\right)\Big|_{0}^{M_2}\right] \\
&= 20\left[\lim_{M_1\to-\infty}\left(1-e^{M_1}\right) + \lim_{M_2\to\infty}\left(1-e^{-M_2}\right)\right] \\
&= 20\left[1-\lim_{M_1\to-\infty} e^{M_1} + 1 - \lim_{M_2\to\infty} e^{-M_2}\right] \\
&= 20\,[1-0+1-0]
\end{aligned}$$

$$\int_{-\infty}^{\infty} |f(t)|\,dt = 40 < \infty$$

i.e, $$\|f\|_1^2 = \int_{-\infty}^{\infty} |f(t)|\,dt = 40 < \infty$$

thus, $f(t) \in L^1(\mathbb{R})$.

Since the function $f(t) \in L^1(\mathbb{R})$, thus the Fourier transform exists and given as

$$\begin{aligned}
\mathcal{F}[f(t)](\omega) = \hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt \qquad \forall\ \omega \in \mathbb{R}. \\
&= \int_{-\infty}^{0} e^{-i\omega t} f(t)dt + \int_{0}^{\infty} e^{-i\omega t} f(t)dt
\end{aligned}$$

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{0} e^{-i\omega t}(20e^{t})dt + \int_{0}^{\infty} e^{-i\omega t}(-20e^{-t})dt \\
&= 20\left[\int_{-\infty}^{0} e^{-i\omega t}(e^{t})dt - \int_{0}^{\infty} e^{-i\omega t}(e^{-t})dt\right] \\
&= 20\left[\int_{-\infty}^{0} e^{-(-1+i\omega)t}dt - \int_{0}^{\infty} e^{-(1+i\omega)t}dt\right] \\
&= 20\left[\lim_{M_1\to-\infty}\int_{M_1}^{0} e^{-(-1+i\omega)t}dt - \lim_{M_2\to\infty}\int_{0}^{M_2} e^{-(1+i\omega)t}dt\right] \\
&= 20\left[\left(-\frac{1}{(-1+i\omega)}\right)\lim_{M_1\to-\infty}\left\{e^{-(-1+i\omega)t}\Big|_{M_1}^{0}\right\} - \frac{1}{1+i\omega}\lim_{M_2\to\infty}\left\{e^{-(1+i\omega)t}\Big|_{0}^{M_2}\right\}\right] \\
&= 20\left[\frac{1}{1-i\omega}\lim_{M_1\to-\infty}\left\{e^{-(-1+i\omega)t}\Big|_{M_1}^{0}\right\} - \frac{1}{1+i\omega}\lim_{M_2\to\infty}\left\{e^{-(1+i\omega)t}\Big|_{0}^{M_2}\right\}\right] \\
&= 20\left[\frac{1}{1-i\omega}\lim_{M_1\to-\infty}\left\{1-e^{-(-1+i\omega)M_1}\right\} - \frac{1}{1+i\omega}\lim_{M_2\to\infty}\left\{e^{-(1+i\omega)M_2}-1\right\}\right] \\
&= 20\left[\frac{1}{1-i\omega}\left\{1-\lim_{M_1\to-\infty} e^{-(-1+i\omega)M_1}\right\} - \frac{1}{1+i\omega}\left\{\lim_{M_2\to\infty} e^{-(1+i\omega)M_2}-1\right\}\right] \qquad (1)
\end{aligned}$$

Since, $\lim_{M_1\to-\infty}\left|e^{-(-1+i\omega)M_1}\right|$

$$= \lim_{M_1\to-\infty}\left\{|e^{M_1}|\cdot|e^{-i\omega M_1}|\right\}$$

$$= \lim_{M_1\to-\infty} e^{M_1} = 0$$

Therefore, $\lim_{M_1\to-\infty} e^{-(-1+i\omega)M_1} = 0$

Similarly, $\lim_{M_2\to\infty}\left|e^{-(1+i\omega)M_2}\right|$

$$= \lim_{M_1\to\infty}\left\{|e^{-M_2}|\cdot|e^{-i\omega M_2}|\right\}$$

$$= \lim_{M_2\to\infty} e^{-M_2} = 0$$

Therefore, $\lim_{M_2\to\infty} e^{-(1+i\omega)M_2} = 0$

Using these values in equation (1) we get as

$$\hat{f}(\omega) = 20\left[\frac{1}{1-i\omega}\{1-0\} - \frac{1}{1+i\omega}\{0-1\}\right]$$
$$= 20\left[\frac{1}{1-i\omega} + \frac{1}{1+i\omega}\right]$$
$$= 20\left[\frac{2}{1+\omega^2}\right]$$
$$\boxed{\hat{f}(\omega) = \frac{40}{1+\omega^2}}$$

2.3 *Solution of Question No. 2(b)*

Since, $f(t) = \begin{cases} 10t & , \quad 0 \le t < 1; \\ 0 & , \quad \text{Otherwise} \end{cases}$

To find the Fourier transform of the function $f(t)$, let's first check wheather is it in $L^1(\mathbb{R})$ or not ?

So,

$$\|f\|_1^2 = \int_{-\infty}^{\infty} |f(t)|\,dt = \int_{-\infty}^{0} |f(t)|\,dt + \int_{0}^{1} |f(t)|\,dt + \int_{1}^{\infty} |f(t)|\,dt$$
$$= \int_0^1 |10t|\,dt$$
$$= 10\int_0^1 t dt = 10\frac{t^2}{2}\Big|_0^1 = 5t\Big|_0^1 = 5 < \infty$$
$$\int_{-\infty}^{\infty} |f(t)|\,dt = 5 < \infty$$

So $f \in L^1(\mathbb{R})$.Thats why Fourier transform exists and given by

$$\mathcal{F}[f(t)](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt \qquad \forall\ \omega \in \mathbb{R}.$$
$$= \int_{-\infty}^{0} e^{-i\omega t} f(t)dt + \int_0^1 e^{-i\omega t} f(t)dt + \int_1^{\infty} e^{-i\omega t} f(t)dt$$
$$= 10\int_0^1 te^{-i\omega t}dt$$
$$= 10\left[\frac{te^{-i\omega t}}{-i\omega}\Big|_0^1 - \int_0^1 \frac{e^{-i\omega t}}{-i\omega}dt\right]$$

$$\hat{f}(\omega) = 10\left[\frac{e^{-i\omega}}{-i\omega} - \frac{e^{0}}{-i\omega} - \int_0^1 \frac{e^{-i\omega t}}{-i\omega}dt\right]$$

$$= 10\left[\frac{1}{-i\omega}(e^{-i\omega} - 1) - \int_0^1 \frac{e^{-i\omega t}}{-i\omega}dt\right]$$

$$= 10\left[\frac{1}{-i\omega}(e^{-i\omega} - 1) - \frac{e^{-i\omega t}}{(-i\omega)^2}\bigg|_0^1\right]$$

$$= 10\left[\frac{1}{-i\omega}(e^{-i\omega} - 1) - \frac{e^{-i\omega}}{(-i\omega)^2} + \frac{1}{(-i\omega)^2}\right]$$

$$= 10\left[\frac{1}{-i\omega}(e^{-i\omega} - 1) + \frac{e^{-i\omega}}{\omega^2} + \frac{1}{\omega^2}\right]$$

$$= 10\left[\frac{1}{-i\omega}(e^{-i\omega} - 1) + \frac{e^{-i\omega}}{\omega^2} + \frac{1}{\omega^2}\right]$$

So, the Fourier transform of the function, $f(t)$ is

$$\boxed{\hat{f}(\omega) = 10\left[\frac{1}{-i\omega}(e^{-i\omega} - 1) + \frac{e^{-i\omega}}{\omega^2} + \frac{1}{\omega^2}\right]}$$

2.4 *Solution of Question No. 3*

Given function is,

$$\psi(t) = \begin{cases} 3 \quad , & 0 \le t < \frac{1}{2}; \\ -3 \quad , & \frac{1}{2} \le t < 1; \\ 0 \quad , & \text{Otherwise} \end{cases}$$

(i) $||\psi||_2$ is given by

$$||\psi||_2^2 = \int_{-\infty}^{\infty} |\psi(t)|^2 dt$$

$$= \int_{-\infty}^{0} |\psi(t)|^2 dt + \int_{0}^{\frac{1}{2}} |\psi(t)|^2 dt + \int_{\frac{1}{2}}^{1} |\psi(t)|^2 dt + \int_{1}^{\infty} |\psi(t)|^2 dt$$

$$= \int_0^{\frac{1}{2}} |3|^2 dt + \int_{\frac{1}{2}}^{1} |-3|^2 dt$$

$$||\psi||_2^2 = \int_0^1 3^2 dt = 9t\bigg|_0^1 = 9 \implies ||\psi||_2 = 3$$

(ii) $\int_{-\infty}^{\infty} \psi(t)dt$ is evaluated as

$$\int_{-\infty}^{\infty} \psi(t)dt = \int_{-\infty}^{0} \psi(t)dt + \int_{0}^{\frac{1}{2}} \psi(t)dt + \int_{\frac{1}{2}}^{1} \psi(t)dt + \int_{1}^{\infty} \psi(t)dt$$

$$= \int_{0}^{\frac{1}{2}} 3dt + \int_{\frac{1}{2}}^{1} (-3)dt$$

$$= \int_{0}^{1} (3-3)dt$$

$$\int_{-\infty}^{\infty} \psi(t)dt = 0$$

2.5 *Solution of Question No. 4*

It is provided in our problem that $\psi \in L^2(\mathbb{R})$ be a basic wavelet. i.e

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

where $\hat{\psi}(\omega)$ is the Fourier transform of the function $\psi \in L^2(\mathbb{R})$.

Again, it is also provided that ϕ is a bounded, integrable function.

Then we have to show that $C_{\psi*\phi} < \infty$. i.e $\psi * \phi$ is also a basic wavelet.

To do this, since we have

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

and we also know that,

$$(\psi * \phi)(t) = \int_{-\infty}^{\infty} \psi(t-u)\phi(u)du$$

thus,

$$\left(\psi \hat{*} \phi\right)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}(\psi * \phi)(t)dt = \hat{\psi}(\omega) \cdot \hat{\phi}(\omega)$$

then,

$$C_{\psi*\phi} = \int_{-\infty}^{\infty} \frac{|(\psi \hat{*} \phi)(\omega)|^2}{|\omega|} d\omega$$

$$= \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega) \cdot \hat{\phi}(\omega)|^2}{|\omega|} d\omega$$

$$C_{\psi*\phi} = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2 \cdot |\hat{\phi}(\omega)|^2}{|\omega|} d\omega$$

Since $\psi \in L^2(\mathbb{R})$ is a basic wavelet and the ϕ is a bounded, integrable function then the overall integration should be less than ∞.i.e.

$$C_{\psi*\phi} = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2 \cdot |\hat{\phi}(\omega)|^2}{|\omega|} d\omega < \infty$$

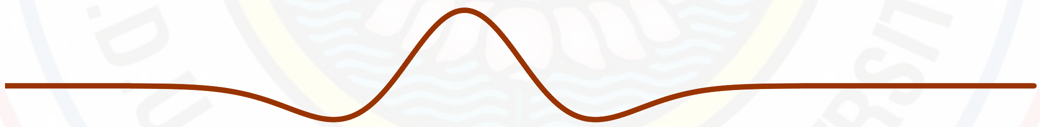
Hence, $C_{\psi*\phi} < \infty$. So, $\left(\psi * \phi\right)$ is a basic wavelet.

2.6 *Solution of Question No. 5*

Mexican Hat wavelet(Definition)

The **Mexican Hat wavewlet/Ricker wavelet/Second derivative wavelet** is denoted by $\psi_{\mathbf{M}}(\mathbf{t})$ and defined by

$$\boxed{\psi_M(t) = -\frac{d^2}{dt^2}\left(e^{-t^2/2}\right) = \left(1-t^2\right)e^{-t^2/2}}$$



$\psi_M(t) = (1-t^2)e^{-t^2/2}$

(I) We have to show $\int_{-\infty}^{\infty} \psi_M(t)dt = 0$.To do so

$$\int_{-\infty}^{\infty} \psi_M(t)dt = \int_{-\infty}^{\infty} (1-t^2)e^{-t^2/2}dt$$

$$\int_{-\infty}^{\infty} \psi_M(t)dt = \int_{-\infty}^{\infty} e^{-t^2/2}dt - \int_{-\infty}^{\infty} t^2e^{-t^2/2}dt$$

$$\int_{-\infty}^{\infty} \psi_M(t)dt = I_1 + I_2 \tag{1}$$

$$\text{where, } I_1 = \int_{-\infty}^{\infty} e^{-t^2/2}dt \text{ and } I_2 = \int_{-\infty}^{\infty} t^2 e^{-t^2/2}dt.$$

Let's evaluate I_1,

$$I_1 = \int_{-\infty}^{\infty} e^{-t^2/2}dt = 2\int_{0}^{\infty} e^{-t^2/2}dt$$

$$\boxed{\text{Let, } \frac{t^2}{2} = u \implies t = \sqrt{2u} \implies dt = \frac{1}{\sqrt{2}}u^{-\frac{1}{2}}du}$$

$$\begin{aligned}
I_1 &= 2\int_{0}^{\infty} e^{-u}\frac{1}{\sqrt{2}}u^{-\frac{1}{2}}du \\
&= \frac{2}{\sqrt{2}}\int_{0}^{\infty} e^{-u}u^{-\frac{1}{2}}du \\
&= \sqrt{2}\int_{0}^{\infty} e^{-u}u^{\frac{1}{2}-1}du \\
&= \sqrt{2}\Gamma\frac{1}{2} \\
&= \sqrt{2}\cdot\sqrt{\pi} \\
I_1 &= \sqrt{2\pi} \qquad (2)
\end{aligned}$$

Next, Let's evaluate I_2,

$$I_2 = \int_{-\infty}^{\infty} t^2 e^{-t^2/2}dt = 2\int_{0}^{\infty} t^2 e^{-t^2/2}dt$$

$$\boxed{\text{Let, } \frac{t^2}{2} = u \implies t^2 = 2u \implies dt = \frac{1}{\sqrt{2}}u^{-\frac{1}{2}}du}$$

$$\begin{aligned}
I_2 &= 2\int_{0}^{\infty} 2ue^{-u}\frac{1}{\sqrt{2}}u^{-\frac{1}{2}}du \\
&= \frac{2\cdot 2}{\sqrt{2}}\int_{0}^{\infty} ue^{-u}u^{-\frac{1}{2}}du
\end{aligned}$$

$$I_2 = \frac{2\cdot 2}{\sqrt{2}} \int_0^\infty e^{-u} u^{\frac{1}{2}} du$$

$$= \frac{2\cdot 2}{\sqrt{2}} \int_0^\infty e^{-u} u^{\frac{3}{2}-1} du$$

$$= \frac{2\cdot 2}{\sqrt{2}} \Gamma\frac{3}{2}$$

$$= \frac{2\cdot 2}{\sqrt{2}} \cdot \frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{2\cdot 2}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2}$$

$$I_2 = \sqrt{2\pi} \qquad (3)$$

Using equation (2) and (3) in equation (1), we get as

$$\boxed{\int_{-\infty}^{\infty} \psi_M(t)dt = \sqrt{2\pi} - \sqrt{2\pi} = 0}$$

(II) We have to show that $C_{\psi_M} < \infty$. To do so

Since we know that

$$\hat{\psi}_M(\omega) = \sqrt{2\pi}\omega^2 e^{-\omega^2/2}$$

Next,

$$C_{\psi_M} = \int_{-\infty}^{\infty} \frac{|\hat{\psi}_M(\omega)|^2}{|\omega|} d\omega$$

$$= \int_{-\infty}^{\infty} \frac{|2\pi\omega^4 e^{-\omega^2}|}{|\omega|} d\omega$$

$$= 4\pi \int_0^{\infty} \frac{|\omega^4 e^{-\omega^2}|}{|\omega|} d\omega$$

$$= 4\pi \int_0^{\infty} \frac{\omega^4 e^{-\omega^2}}{\omega} d\omega$$

$$= 4\pi \int_0^{\infty} \omega^3 e^{-\omega^2} d\omega$$

$$\boxed{\text{Let } \omega^2 = u \implies \omega = \sqrt{u} \text{ and } d\omega = \frac{1}{2}u^{-\frac{1}{2}}du}$$

$$C_{\psi_M} = 4\pi \int_0^\infty u^{3/2} \cdot e^{-u}\frac{1}{2}u^{-1/2}du$$

$$= 2\pi \int_0^\infty e^{-u}udu$$

$$= 2\pi \cdot 1 = 2\pi$$

Hence,

$$\boxed{C_{\psi_M} = 2\pi < \infty}$$

Thus, ψ_M is a basic wavelet.

2.7 *Solution of Question No. 6*

The Dirac Delta function $\delta(x)$ is very cool in the sense that

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Its unique characteristics do not end there though, because when integrating the Dirac Delta function we would get

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Or, if we have another function $f(x)$ multiplied to the Dirac Delta function and integrating them we would get

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(x)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

Since

$$\int_{-\infty}^{\infty} \delta(x)\,dx = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Therefore in the previous integral we would have

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty}^{\infty} f(0)\delta(0)\,dx = f(0)$$

What if we have $\delta(x-a)$? It's the same thing! The only thing here is we need to satisfy the condition $x-a=0$ such that $\int_{-\infty}^{\infty} \delta(x-a)dx = 1$. If the bounds of our integral though is not infinity, we need to make sure that the if we let $x=a$, a would be in the bounds of the integral or else the integral would evaluate into zero.

$$\int_{-b}^{b} f(x)\delta(x-a)\,dx = \begin{cases} 0, & \text{if } b < a \text{ or } -b > a \text{ such that we cannot let } x \neq a \\ f(0), & \text{if } -b < a < b \text{ such that we can let } x = a \end{cases}$$

(i) We have to evaluate, $\int_{-\infty}^{\infty} \delta(x-1)\left\{\cos^2(\pi x) + \sin^2(\pi x) + e^{\pi x}\right\}dx$.

In the view of above discussion about Dirac Delta function we can evaluate above integral easialy, So

$$\int_{-\infty}^{\infty} \delta(x-1)\left\{\cos^2(\pi x) + \sin^2(\pi x) + e^{\pi x}\right\}dx = f(1)$$

where, $f(x) = \left\{\cos^2(\pi x) + \sin^2(\pi x) + e^{\pi x}\right\}$.

Thus

$$\int_{-\infty}^{\infty} \delta(x-1)\left\{\cos^2(\pi x) + \sin^2(\pi x) + e^{\pi x}\right\}dx = \left\{\cos^2(\pi) + \sin^2(\pi) + e^{\pi}\right\} = 1 + e^{\pi}$$

(ii)We have to evaluate, $\int_{-\infty}^{\infty} \delta(x)\left\{x^2 + x + 1\right\}(e^x)dx$.

As, similar to the above solution, we can found this one also like,

$$\int_{-\infty}^{\infty} \delta(x)\left\{x^2 + x + 1\right\}(e^x)dx = f(0)$$

where, $f(x) = \left\{x^2 + x + 1\right\}(e^x)$.

Thus

$$\int_{-\infty}^{\infty} \delta(x)\left\{x^2 + x + 1\right\}(e^x)dx = \left\{0^2 + 0 + 1\right\}(e^0) = 1.$$

2.8 Solution of Question No. 7

We have given a function $g_\alpha(x)$ defined as

$$g_\alpha(x) = \frac{e^{\frac{-x^2}{4\alpha}}}{2\sqrt{\pi\alpha}}, \alpha > 0.$$

(i) To show that $g \in L^1(\mathbb{R})$,we have to show $||g_\alpha||_1^2 = \displaystyle\int_{-\infty}^{\infty} |g_\alpha(x)|\,dx < \infty$. Thus

$$\begin{aligned}
||g_\alpha||_1^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|\,dx &= \int_{-\infty}^{\infty} \left|\frac{e^{\frac{-x^2}{4\alpha}}}{2\sqrt{\pi\alpha}}\right| dx \\
&= \int_{-\infty}^{\infty} \frac{|e^{-\frac{x^2}{4\alpha}}|}{|2\sqrt{\pi\alpha}|} dx \\
&= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\alpha}} dx \\
&= \frac{1}{\sqrt{\pi\alpha}} \int_{0}^{\infty} e^{-\frac{x^2}{4\alpha}} dx
\end{aligned}$$

$$\boxed{\text{Let, } u = \frac{x^2}{4\alpha} \implies dx = \sqrt{\alpha}u^{-\frac{1}{2}}du}$$

$$\begin{aligned}
||g_\alpha||_1^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|\,dx &= \frac{1}{\sqrt{\pi\alpha}} \int_{0}^{\infty} e^{-u}\sqrt{\alpha}u^{-\frac{1}{2}}du \\
&= \frac{\sqrt{\alpha}}{\sqrt{\pi\alpha}} \int_{0}^{\infty} e^{-u}u^{-\frac{1}{2}}du \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u}u^{\frac{1}{2}-1}du \\
&= \frac{1}{\sqrt{\pi}}\Gamma\frac{1}{2} \\
&= \frac{\sqrt{\pi}}{\sqrt{\pi}}
\end{aligned}$$

$$||g_\alpha||_1^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|\,dx = 1 < \infty$$

Hence, $g_\alpha \in L^1(\mathbb{R})$.

(ii) To show that $g \in L^2(\mathbb{R})$,we have to show $\|g_\alpha\|_2^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^2 dx < \infty$. Thus

$$\|g_\alpha\|_2^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^2 dx = \int_{-\infty}^{\infty} \left| \frac{e^{\frac{-x^2}{4\alpha}}}{2\sqrt{\pi\alpha}} \right|^2 dx$$

$$= \int_{-\infty}^{\infty} \frac{|e^{-\frac{x^2}{4\alpha}}|^2}{|2\sqrt{\pi\alpha}|^2} dx$$

$$= \frac{1}{4\pi\alpha} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha}} dx$$

$$= \frac{1}{2\pi\alpha} \int_{0}^{\infty} e^{-\frac{x^2}{2\alpha}} dx$$

$$\boxed{\text{Let, } u = \frac{x^2}{2\alpha} \implies dx = \sqrt{\frac{\alpha}{2}} u^{-\frac{1}{2}} du}$$

$$\|g_\alpha\|_2^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^2 dx = \frac{1}{2\pi\alpha} \int_{0}^{\infty} e^{-u} \sqrt{\frac{\alpha}{2}} u^{-\frac{1}{2}} du$$

$$= \frac{1}{2\pi\alpha} \cdot \sqrt{\frac{\alpha}{2}} \int_{0}^{\infty} e^{-u} \cdot u^{\frac{1}{2}-1} du$$

$$= \frac{1}{2\pi\alpha} \cdot \sqrt{\frac{\alpha}{2}} \Gamma \frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2\pi\alpha} \cdot \sqrt{\frac{\alpha}{2}}$$

$$\|g_\alpha\|_2^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^2 dx = \frac{1}{\sqrt{8\pi\alpha}} < \infty$$

Hence, $g_\alpha \in L^2(\mathbb{R})$.

(iii) We have to show that $g_\alpha \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Since from above discussion we see that $g_\alpha \in L^1(\mathbb{R})$ as well as $g_\alpha \in L^2(\mathbb{R})$.

Thus, $g_\alpha \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(iv) We have to show that $\|g_\alpha\|_1 = 1$.

Since from (i) , it is obvious that $\|g_\alpha\|_1 = 1$. Hence, there is nothing to do more.

Yes, $g_\alpha \in L^4(\mathbb{R})$, Here is the explaination,

Explaination that $g_\alpha \in L^4(\mathbb{R})$

To show the inclusivity of $g_\alpha \in L^4(\mathbb{R})$.We must show that $||g_\alpha||_4^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^4 dx < \infty$. Thus,

$$||g_\alpha||_4^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^4 dx = \int_{-\infty}^{\infty} \left| \frac{e^{\frac{-x^2}{4\alpha}}}{2\sqrt{\pi\alpha}} \right|^4 dx$$

$$= \int_{-\infty}^{\infty} \frac{|e^{-\frac{x^2}{4\alpha}}|^4}{|2\sqrt{\pi\alpha}|^4} dx$$

$$= \frac{1}{16\pi^2\alpha^2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha}} dx$$

$$= \frac{1}{8\pi^2\alpha^2} \int_0^{\infty} e^{-\frac{x^2}{\alpha}} dx$$

$$\boxed{\text{Let, } u = \frac{x^2}{\alpha} \implies dx = \sqrt{\alpha} \cdot \frac{1}{2} u^{-\frac{1}{2}} du}$$

$$||g_\alpha||_4^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^4 dx = \frac{1}{8\pi^2\alpha^2} \int_0^{\infty} e^{-u} \sqrt{\alpha} \cdot \frac{1}{2} u^{-\frac{1}{2}} du$$

$$= \frac{\sqrt{\alpha}}{16\pi^2\alpha^2} \int_0^{\infty} e^{-u} \cdot u^{\frac{1}{2}-1} du$$

$$= \frac{\sqrt{\alpha}}{16\pi^2\alpha^2} \Gamma\frac{1}{2}$$

$$= \frac{\sqrt{\pi\alpha}}{16\pi^2\alpha^2}$$

$$||g_\alpha||_4^2 = \int_{-\infty}^{\infty} |g_\alpha(x)|^4 dx = \frac{\sqrt{\pi\alpha}}{16\pi^2\alpha^2} < \infty$$

Hence, $g_\alpha \in L^4(\mathbb{R})$. It completes the explaination.