ASSIGNMENT: BANACH SPACES

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Assignment Questions:

Assignment I

1. **(Theorem):** A normed space X is a Banach space iff every absolutely summable sequence in X, is summable in X.

2. **(Problem):** Let $C[a, b]$ be the linear space of all scalar valued continuous functions defined on $[a, b]$. Define $\|\cdot\|_\infty : \mathbf{C[a, b]} \to \mathbb{R}$ by

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$$

then $(C[a, b], \|\cdot\|_\infty)$ is a Banach space.

3. **(Theorem):** Prove that linear space l_n^p, $1 \leq p < \infty$ given by

$$\|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}}$$

is a Banach space.

4. **(Problem):** The real linear space $C[-1, 1]$ equipped with the norm given by

$$\|x\|_1 = \int_{-1}^{1} |x(t)|dt$$

where integral is taken in the sense of Riemann, is the incomplete normed space.

5. **(Theorem):** Prove that linear space $\mathbb{R}^n$, equipped with the norm given by

$$\|x\| = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}}$$

where, $x = (\xi_1\ \xi_2\ \cdots\cdots \xi_n) \in \mathbb{R}^n$ is a real Banach space.

6. **(Theorem):** Let X be a normed space over the field K and let M be a closed subspace of X.

Define $\|\cdot\|_q : X/M \to \mathbb{R}$ by

$$\|x + M\|_q = \inf \cdot \left\{\|x + m\| : m \in M\right\}$$

then $\left(X/M\ ,\ \|\cdot\|_q\right)$ is a normed space. Further, if X is a Banach space, then X/M is a Banach space.

Assignment II

1. **(Problem):** Give the example of linear functional on different normed linear spaces for bounded linear functional.

2. **(Problem):** Define the functional $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = x - a$$

where, $x = (\xi_1\ \xi_2\ \cdots\cdots \xi_n) \in \mathbb{R}^n$ and $x{\cdot}a$ denotes the familiar scalar product of x and a. Then f is bounded linear functional on $\mathbb{R}^n$ with

$$\|f\| = \|a\|$$

3. Give examples for unbounded linear functional.

Solutions of Assignment I

Solution of Question No.1(Theorem):

Statement:

A normed space X is a Banach space iff every absolutely summable sequence in X is summable in X.

Proof:

Assume that X is a Banach space.

Let $\{x_n\}$ be an absolutely summable sequence in X, then

$$\sum_{n=1}^{\infty} \|x_n\| = M < \infty$$

thus, for each $\epsilon > 0, \exists\ N$ such that

$$\sum_{n=N}^{\infty} \|x_n\| < \epsilon$$

Let $S_n = \sum_{k=1}^{n} x_k$ be the partial sums of the series $\sum_{k=1}^{\infty} x_k$, then

$$\begin{aligned} \|S_n - S_m\| &= \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\leq \sum_{k=m+1}^{n} \|x_k\| \\ &\leq \sum_{k=N}^{\infty} \|x_k\|, \quad n > m > N \\ &< \epsilon\ , \quad n > m > N \end{aligned}$$

$$\implies \boxed{\|S_n - S_m\| < \epsilon\ , \quad n > m > N}$$

thus, $\{S_n\}$ is a Cauchy sequence in X and must converges to some point S(say) in X. Since X is complete.

$$\implies \{x_n\} \text{ is summable in } X.$$

Conversely, Suppose that each absolutely summable sequence in X is summable in X.

We have to show that X is complete.

Let $\{x_n\}$ be a Cauchy sequence in X, then for each k, $\exists$ an integer n_k such that

$$\|x_n - x_m\| < \frac{1}{\alpha^k}\ , \quad \forall\ n, m \geq n_k$$

we may assume n_k such that $n_{k+1} > n_k$. then, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

$$
\begin{aligned}
\text{set, } y_0 &= x_{n_1} \\
y_1 &= x_{n_2} \\
y_2 &= x_{n_3} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
y_k &= x_{n_{k+1}} - x_{n_k} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

From this we note that,

- $\sum_{i=0}^{k} y_i = x_{n_{k+1}}$
- $\|y_k\| < \frac{1}{\alpha^k}$, $\quad k \geq 1$

and as such,

$$\sum_{k=0}^{\infty} \|y_k\| \leq \|y_0\| + \sum_{k=1}^{\infty} \frac{1}{\alpha^k} = \|y_0\| + 1 < \infty$$

Consequently, the sequence $\{y_k\}$ is absolutely summable to some element (say) x in X.therefore we have,

$$x_{n_k} \to x \ \in X \text{ and } k \to \infty$$

thus, the Cauchy sequence $\{x_n\}$ in X has a convergent subsequence $\{x_{n_k}\}$ converging to x.

hence,

$$\lim_{n\to\infty} x_n \to x$$

$$\text{or, } \lim_{n\to\infty} x_n = x \text{ as } n \to \infty$$

Thus X is a Banach Space.

Solution of Question No.2(Problem):

Statement:

Let $C[a,b]$ be the linear space of all scalar valued continuous functions defined on $[a,b]$. Define $\|\cdot\|_\infty : \mathbf{C[a,b]} \to \mathbb{R}$ by

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$$

then $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.

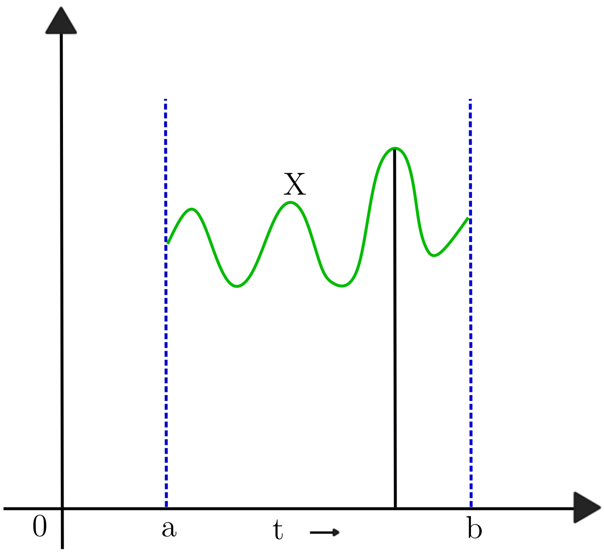
Solution:

Consider the linear space $C[a,b]$ of all scalar valued(real or complex) continuous functions, defined on $[a,b]$.

Define,

$$\|\cdot\|_\infty : C[a,b] \to \mathbb{R} \qquad \text{by,}$$

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$$



Now, first we have to show that $C[a,b]$ is normed linear space on $\|\cdot\|_\infty$.

$$\text{i.e.} \qquad N_1 : \quad \|x\|_\infty = \max_{t\in[a,b]} |x(t)| \geq 0$$

It is obvious thing because our function is a modulus function.

$$N_2 : \quad \|x\|_\infty = \max_{t\in[a,b]} |x(t)| = 0$$

$$\Leftrightarrow \ |x(t)| = 0 \qquad \forall\ t$$

$$\begin{aligned}
&\Leftrightarrow \; x(t) = 0 \qquad\qquad \forall\, t\\
&\Leftrightarrow \; x = (0,0,0,0,\cdots\cdots, t \text{ times}) = 0\\
&\Leftrightarrow \; x = 0
\end{aligned}$$

$$\begin{aligned}
N_3: \quad \|x+y\|_\infty &= \max_{t\in[a,b]} |x(t)+y(t)|\\
&\leq \max_{t\in[a,b]} \{|x(t)|+|y(t)|\}\\
&\leq \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |y(t)|\\
&\leq \|x\|_\infty + \|y\|_\infty
\end{aligned}$$

i.e. $\boxed{\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty}$

$N_4:$ For any scalar $\alpha \in$ (field) K.

$$\begin{aligned}
\|\alpha\cdot x\|_\infty &= \max_{t\in[a,b]} |\alpha\cdot x(t)|\\
&= \max_{t\in[a,b]} |\alpha|\cdot|x(t)|\\
&= |\alpha|\cdot \max_{t\in[a,b]} |x(t)|
\end{aligned}$$

$$\Rightarrow \boxed{\|\alpha\cdot x\|_\infty = |\alpha|\cdot\|x\|_\infty}$$

It follows that $C[a,b]$ is a **normed linear space** under the above defined norm.

Now, we have to show that $C[a,b]$ is complete under this norm.
Let $\{x_m\}$ be a Cauchy-sequence in $[a,b]$, then for each $\epsilon > 0$, $\exists$ a positive integer N such that,

$$\|x_m - x_n\|_\infty = \max_{t\in[a,b]} |x_m(t)-x_n(t)| < \epsilon \qquad \forall\, m,n \geq N \tag{1}$$

therefore for any fixed $t = t_0 \in [a,b]$, we get,

$$|x_m(t_0) - x_n(t_0)| < \epsilon \qquad \forall\, m,n \geq N$$

This shows that $x_m(t_0)$ is a Cauchy-sequence in K, but K being complete, this sequence converges.

Let $x_m(t_0) \to x(t_0)$ as $m \to \infty$, In this manner we can associate to each $t \in [a, b]$ an unique $x(t) \in K$.

This defines a pointwise function x on $[a, b]$.

Now, we have to show that $x \in C[a, b]$ and $x_m \to x$, from equation(1) we have,

$$|x_m(t) - x_n(t)| < \epsilon \qquad \forall\ m, n \geq N \text{ and } \forall\ t \in [a, b]$$

Taking $n \to \infty$ we get as,

$$|x_m(t) - x(t)| \leq \ \epsilon \qquad \forall\ m \geq N \text{ and } \forall\ t \in [a, b] \tag{2}$$

This verifies that the sequence $\{x_m\}$ of continuous functions converges uniformly to the function x on $[a, b]$. And hence the limit function x is a continuous function on $[a, b]$ such as $x \in [a, b]$.

Also, from equation(2) we have,

$$\max_{t \in [a,b]} |x_m(t) - x(t)| \leq \ \epsilon \qquad \forall\ m \geq N$$

$$\Rightarrow \ \|x_m - x\|_\infty \leq \ \epsilon \qquad \forall\ m \geq N$$

$$\Rightarrow \ x_m \to x \text{ in } C[a, b]$$

Hence, $(C[a, b], \|\cdot\|_\infty)$ is a **Banach space**.

Solution of Question No.3(Theorem):

Statement:

Prove that linear space l_n^p, $1 \leq p < \infty$ given by

$$\|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}}$$

is a Banach space.

Proof:

The linear space l_n^p equipped with the norm given by,

$$\|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}}$$

first we have to show that l_n^p is the normed linear space.

N_1 :

$$\text{Since, } \forall \quad |\xi_i| \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} |\xi_i| \geq 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}} \geq 0$$

$$\Rightarrow \|x\|_p \geq 0 \qquad \forall\ x \in l_n^p$$

N_2 :

$$\|x\|_p = 0$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} |\xi_i|^p = 0$$

$$\Leftrightarrow |\xi_i| = 0 \qquad \forall\ i = 1, 2, \cdots\cdots, n$$

$$\Leftrightarrow \xi_i = 0 \qquad \forall\ i = 1, 2, \cdots\cdots, n$$

$$\Leftrightarrow (\xi_1, \xi_2, \cdots\cdots\cdots, \xi_n) = 0$$

$$\Leftrightarrow x = 0$$

$$\|x\|_p = 0 \Leftrightarrow x = 0$$

N_3 : Let $x = (\xi_1, \xi_2, \cdots\cdots\cdots, \xi_n)$

$y = (\eta_1, \eta_2, \cdots\cdots\cdots, \eta_n)$ be any two members of l_p^n

$$\Rightarrow \|x+y\|_p = \|(\xi_1, \xi_2, \cdots\cdots\cdots, \xi_n) + (\eta_1, \eta_2, \cdots\cdots\cdots, \eta_n)\|$$

$$= \|(\xi_1+\eta_1), (\xi_2+\eta_2), \cdots\cdots\cdots, (\xi_n+\eta_n)\|$$

$$= \left(\sum_{i=1}^{n} |\xi_i+\eta_i|^p\right)^{\frac{1}{p}}$$

Using Minkowski inequaliity(finite form)

$$\leq \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |\eta_i|^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \boxed{\|x+y\|_p \leq \|x\|_p + \|y\|_p} \qquad \forall\ x, y \in l_n^p$$

N_4 : Let α be any scalar and x is an arbitrary element of l_n^p. then,

$$\|\alpha \cdot x\|_p = \|\alpha \cdot (\xi_1, \xi_2, \cdots\cdots\cdots, \xi_n)\|$$

$$= \|(\alpha\xi_1, \alpha\xi_2, \cdots\cdots\cdots, \alpha\xi_n)\|$$

$$= \left(\sum_{i=1}^{n} |\alpha \cdot \xi_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} |\alpha|^p \cdot |\xi_i|^p\right)^{\frac{1}{p}}$$

$$= (|\alpha|^p)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \boxed{\|\alpha \cdot x\|_p = |\alpha| \cdot \|x\|_p} \qquad \forall\ x \in l_n^p$$

Thus l_n^p togather with the norm $\|\cdot\|_p$ i.e $\left(l_n^p, \|\cdot\|_p\right)$ is a **normed linear space**.

Now, in order to show that l_n^p are Banach space, we have to prove their completenness, with respect to the norm defined above.

Let $\{x_m\}$ be a Cauchy-sequence in l_n^p.
where $x_m = \left\{\xi_1^{(m)}, \xi_2^{(m)}, \cdots\cdots\cdots, \xi_n^{(m)}\right\} \in K^n$
Then for each $\epsilon > 0$, $\exists$ a positive integer N such that

$$\|x_m - x_k\| = \left(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(k)}|^p\right)^{\frac{1}{p}} < \epsilon \qquad \forall\ m, k \geq N$$

$$= \sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(k)}|^p < \epsilon^p \qquad \forall\ m, k \geq N$$

$$= |\xi_i^{(m)} - \xi_i^{(k)}| < \epsilon \qquad \forall\ m, k \geq N;\ i = 1, 2, \cdots\cdots\cdots, n$$

This shows that for fixed $(1 \leq i \leq n)$ the sequence $\left\{\xi_i^{(m)}\right\}_{m=1}^{\infty}$ is a Cauchy-sequence in K.
Since K is complete, it is converges in K.
Let $\xi_i^{(m)} \to \xi_i$ as $m \to \infty$
Using these, we define n limits as

$$x = (\xi_1, \xi_2, \cdots\cdots\cdots, \xi_n) \in l_n^p$$

Now,

$$\|x_m - x_k\| = \left(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(k)}|^p\right)^{\frac{1}{p}} < \epsilon \qquad \forall\ m, k \geq N$$

Taking $K \to \infty$ we have,

$$\|x_m - x\|_p < \epsilon \qquad \forall\ m \geq N$$

$$\Rightarrow\ x_m \to x \text{ in } l_n^p$$

Hence, l_n^p is complete and therefore it is a **Banach Space**.

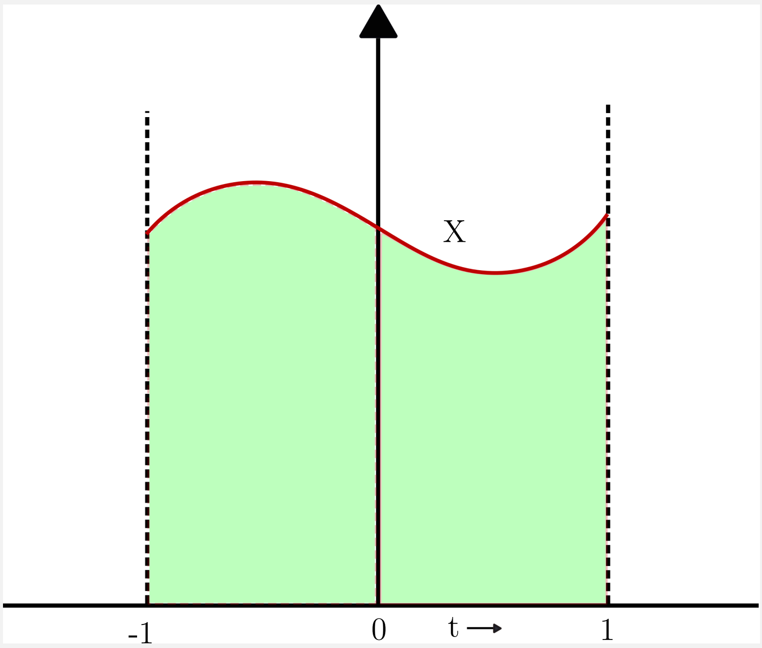
Solution of Question No.4(Problem):

Statement:

The real linear space $C[-1,1]$ equipped with the norm given by

$$\|x\|_1 = \int_{-1}^{1} |x(t)|dt$$

where integral is taken in the sense of Riemann, is the incomplete normed space.



Solution:

First of all we must show that $C[-1,1]$ is a normed linear space with respect to norm $\|\cdot\|_1$.

$$N_1 : \qquad \|x\|_1 = \int_{-1}^{1} |x(t)|dt \geq 0 \qquad \forall\ x \in C[-1,1]$$

Since our integrand is modulus function, so it's an obvious case and hence there is nothing to do more.

$$N_2 : \qquad \|x\|_1 = \int_{-1}^{1} |x(t)|dt = 0 \ \Leftrightarrow \ 0 \qquad \forall\ x \in C[-1,1]$$

It is also very straightforward thing and there nothing to do more.

N_3 :
$$\|x+y\|_1 = \int_{-1}^{1} |x(t)+y(t)|dt \qquad \forall\ x,y \in C[-1,1]$$
$$\leq \int_{-1}^{1} |x(t)|dt + \int_{-1}^{1} |y(t)|dt$$

$$\boxed{\|x+y\|_1 \leq \|x\|_1 + \|y\|_1} \qquad \forall\ x,y \in C[-1,1]$$

N_4 : Let $\alpha \in K$ where K is a field and $x \in C[a,b]$ be any element then

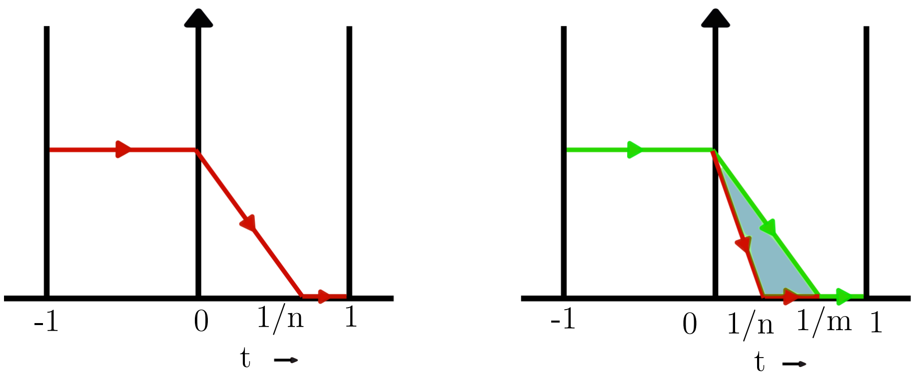
$$\|\alpha \cdot x\|_1 = \int_{-1}^{1} |\alpha x \cdot (t)|dt$$
$$= |\alpha| \cdot \int_{-1}^{1} |x(t)|dt$$
$$\|\alpha \cdot x\|_1 = |\alpha| \cdot \|x\|_1$$

Hence, all the four condition to be a normed linear space is satisfied, thus $(C[-1,1], \|\cdot\|_1)$ is a **normed linear space**.

Now, Let's check the **completeness**.
Consider a sequence $\{x_n\}$ whose terms are defined as,

$$x_n(t) = \begin{cases} 1 & -1 \leq t \leq 0 \\ 1 - nt & 0 < t < \frac{1}{n} \\ 0 & \frac{1}{n} < t \leq 1 \end{cases}$$

Let's draw picture of above defined piecewise function for our convenience,



It may be observed that $\{x_n\}$ is a Cauchy sequence.Geometrically the function x_n is shown in above figure.

And $\|x_n - x_m\|$ represents the area of the triangle shown in the figure.

Clearly, each $x_n(t)$ is continuous on $[-1,1]$, also $\{x_n\}$ is a Cauchy sequence in $C[-1,1]$.

If $n > m$ then,

$$\|x_n - x_m\| = \int_{-1}^{1} |x_n(t) - x_m(t)|dt$$

Let if possible, $x_n \to x$ in $[-1,1]$.

But,

$$\begin{aligned}\|x_n - x\| &= \int_{-1}^{1} |x_n(t) - x(t)|dt \\ &= \int_{-1}^{0} |1 - x(t)|dt + \int_{0}^{\frac{1}{n}} |x_n(t) - x(t)|dt + \int_{\frac{1}{n}}^{1} |x(t)|dt\end{aligned}$$

Since integrands are non-negative,so our each integral on the RHS also.

Hence $\|x_n - x\| \to 0$ would imply that each integral on RHS approaches to zero as $n \to \infty$. So we have,

$$\begin{cases} \lim\limits_{n\to\infty} \int_{-1}^{0} |1 - x(t)|dt = 0 \\ \\ \lim\limits_{n\to\infty} \int_{0}^{\frac{1}{n}} |x_n(t) - x(t)|dt = 0 \\ \\ \lim\limits_{n\to\infty} \int_{\frac{1}{n}}^{1} |x(t)|dt = 0 \end{cases}$$

Now, extracting the valu of $x(t)$ from each integral we have,

$$x(t) = \begin{cases} 1 & -1 \le t \le 0 \\ 0 & 0 < t \le 1 \end{cases}$$

But, here we see that the function is breaks at $t = 0$, so that function is not continuous in $[-1,1]$.So as such $x \notin C[-1,1]$.

So that, it voilates the criteria for completeness.Eventually, we say that $(C[-1,1], \|\cdot\|_1)$ is **incomplete normed linear space** i.e. **not a Banach Space**.

Solution of Question No.5(Theorem):

Statement:

Prove that linear space $\mathbb{R}^n$, equipped with the norm given by

$$\|x\| = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}}$$

where, $x = (\xi_1 \ \xi_2 \ \cdots\cdots \xi_n) \in \mathbb{R}^n$ is a real Banach space.

Proof:

The linear space $\mathbb{R}^n$, equipped with the norm given by

$$\|x\| = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}}$$

where, $x = (\xi_1 \ \xi_2 \ \cdots\cdots \xi_n) \in \mathbb{R}^n$

Now, first of all we must prove that $\mathbb{R}^n$ togather with the norm $\|\cdot\|$ is a normed linear space.

$$N_1 : \qquad \text{Since } \forall \ |\xi_i| \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} |\xi_i| \geq 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}} \geq 0$$

$$\Rightarrow \|x\| \geq 0 \qquad \forall \ x \in \mathbb{R}^n$$

$$N_2 : \qquad \|x\| = 0$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} |\xi_i|^2 = 0$$

$$\Leftrightarrow \xi_i = 0 \qquad \forall \ i = 1, 2, \cdots\cdots\cdots, n$$

$$\Leftrightarrow (\xi_1 \ \xi_2 \ \cdots\cdots\cdots \ \xi_n) = 0$$

$$\Leftrightarrow x = 0$$

$$\Rightarrow \|x\| = 0 \Leftrightarrow x = 0$$

N_3 : Let $x = (\xi_1\ \xi_2\ \cdots\cdots\cdots\ \xi_n)$ and $y = (\eta_1\ \eta_2\ \cdots\cdots\cdots\ \eta_n)$ be any two members of $\mathbb{R}^n$ then we have,

$$\begin{aligned}
\|x+y\| &= \|(\xi_1\ \xi_2\ \cdots\cdots\cdots\ \xi_n) + (\eta_1\ \eta_2\ \cdots\cdots\cdots\ \eta_n)\| \\
&= \|(\xi_1+\eta_1)\ \ (\xi_2+\eta_2)\ \ \cdots\cdots\cdots (\xi_n+\eta_n)\| \\
&= \left(\sum_{i=1}^{n} |\xi_i+\eta_i|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Using Minkowski inequality (finite form)

$$\begin{aligned}
&\le \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} |\eta_i|^2\right)^{\frac{1}{2}} \\
&\le \|x\| + \|y\|
\end{aligned}$$

$$\Rightarrow \boxed{\|x+y\| \le \|x\| + \|y\|} \qquad \forall\ x, y \in \mathbb{R}^n$$

N_4 : Let α be any scalar from field K and x is an arbitrary element of $\mathbb{R}^n$. Then we have,

$$\begin{aligned}
\|\alpha \cdot x\| &= \left(\sum_{i=1}^{n} |\alpha \cdot \xi_i|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n} |\alpha|^2 \cdot |\xi_i|^2\right)^{\frac{1}{2}} \\
&= \left(|\alpha|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\Rightarrow \boxed{\|\alpha \cdot x\| = |\alpha| \cdot \|x\|} \qquad \forall\ x \in \mathbb{R}^n$$

Hence $\mathbb{R}^n$ are normed linear space with above defined norm.

Now, in order to show that $\mathbb{R}^n$ are Banach Spaces, we have to prove their completeness with respect to the norm defined above.

Let $\{x_m\}$ be a Cauchy sequence in $\mathbb{R}^n$, where, $x_m = \left(\xi_1^{(m)}\ \xi_2^{(m)}\ \cdots\cdots\cdots\ \xi_n^{(m)}\right) \in \mathbb{R}^n$

Then for each $\epsilon > 0$, $\exists$ a positive integer N such that

$$\begin{aligned}
\|x_m - x_p\| = \left(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(p)}|^2\right)^{\frac{1}{2}} &< \epsilon \qquad\qquad \forall\ m, p \geq N \\
= \sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(p)}|^2 &< \epsilon^2 \qquad\qquad \forall\ m, p \geq N \\
= |\xi_i^{(m)} - \xi_i^{(p)}| &< \epsilon \qquad\qquad \forall\ m, p \geq N;\ i = 1, 2, \cdots\cdots\cdots, n
\end{aligned}$$

This shows that for fixed $(1 \leq i \leq n)$ the sequence $\left\{\xi_i^{(m)}\right\}_{m=1}^{\infty}$ is a Cauchy-sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, it is converges in $\mathbb{R}$.

Let $\xi_i^{(m)} \to \xi_i$ as $m \to \infty$

Using these, we define n limits as

$$x = (\xi_1, \xi_2, \cdots\cdots\cdots, \xi_n) \in \mathbb{R}^n$$

Now,

$$\|x_m - x_p\| = \left(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(p)}|^2\right)^{\frac{1}{2}} < \epsilon \qquad\qquad \forall\ m, p \geq N$$

Taking $p \to \infty$ we have,

$$\|x_m - x\| < \epsilon \qquad\qquad \forall\ m \geq N$$

$$\Rightarrow\ x_m \to x \text{ in } \mathbb{R}^n$$

Hence, $\mathbb{R}^n$ is complete and therefore it is a **Banach Space**.

Solution of Question No.6(Theorem):

Statement:

Let X be a normed space over the field K and let M be a closed subspace of X.

Define $\|\cdot\|_q : X/M \to \mathbb{R}$ by

$$\|x+M\|_q = \inf\cdot\left\{\|x+m\| : m \in M\right\}$$

then $\left(X/M\ ,\ \|\cdot\|_q\right)$ is a normed space. Further, if X is a Banach space, then X/M is a Banach space.

Proof:

We verify all the postulates for a norm.

N_1 : Since $\|x+m\|$ is a non-negative real number and every set of non-negative real number is bounded below.

It follows that the $\inf\cdot\left\{\|x+m\| : m \in M\right\}$ exists and is non-negative that is,

$$\|x+m\|_q \geq 0 \qquad ;\forall\ x \in M \in X/M$$

N_2 : Let $x+m=m$ (zero element of X/M) $\forall\ x \in M$. Hence,

$$\|x+M\|_q = \inf\cdot\left\{\|x+m\| : m \in M\right\}$$

$$= \inf\cdot\left\{\|y\| : y \in M\right\} = 0$$

$\left[\because\ M \text{ being a subspace contains zero vector whose norm is real number } 0.\right]$

thus,

$$\|x+M\|_q = \|0+M\|_q = \|0\| = 0$$
$$\Rightarrow\ x+m=m\ \Rightarrow\ \|x+M\|_q = 0$$

Conversely, we have

$$\|x+M\|_q = 0\ \Rightarrow\ \inf\cdot\left\{\|x+m\| : m \in M\right\} = 0 \qquad \text{for some } x \in X$$

then $\exists$ a sequencene $\{m_k\}_{k=1}^{\infty} \subset M$ such that,

$$\|x+m_k\| = 0 \text{ as } k \to \infty$$

$$\text{or,} \qquad \lim_{k\to\infty}\|x+m_k\| = 0$$

$$\Rightarrow \lim_{k\to\infty} m_k = -x$$

$$\Rightarrow -x \in M$$

$$\left\{\text{Since, } M \text{ is closed and } \{m_k\}_{k=1}^{\infty} \text{ is a sequence in } M \text{ converging to } -x\right\}$$

$$\Rightarrow x + m \qquad \left\{\because M \text{ is a subspace}\right\}$$

$$\Rightarrow x + m = m \qquad \left\{\because \text{The zero element of } X/M\right\}$$

Thus we have shown that

$$\boxed{\|x+m\|_q = 0 \Leftrightarrow x+m = m} \qquad (\text{the zero of } X/M)$$

N_3 : Let $x+m, y+m \in X/M$ then,

$$\|(x+m)+(y+m)\|_q = \|(x+y)+m\|_q$$

[By definition of addition of coset]

$$= \inf\cdot\left\{\|(x+y)+m\| : m \in M\right\}$$

$$\leq \inf\cdot\left\{\|x+m_1\| + \|y+m_2\| : m_1, m_2 \in M\right\}$$

$$\leq \inf\cdot\left\{\|x+m_1\| : m_1 \in M\right\} + \inf\cdot\left\{\|y+m_2\| : m_2 \in M\right\}$$

$$\boxed{\|(x+m)+(y+m)\|_q \leq \|x+m\|_q + \|y+m\|_q} \qquad \forall\, x, y \in X/M$$

This proves the triangle inequaliity.

N_4 : For $x \in X$ and $\alpha \in K$ with $\alpha \neq 0$, we have

$$\begin{aligned}\|\alpha\cdot(x+m)\|_q &= \|\alpha\cdot x+m\|_q\\ &= \inf\cdot\Big\{\|\alpha\cdot x+m\| : m\in M\Big\}\\ &= \inf\cdot\Big\{\|\alpha\cdot x+\alpha\cdot m'\| : m'=\frac{m}{\alpha}\in M\Big\}\end{aligned}$$

$$\begin{aligned}&= |\alpha|\cdot\inf\cdot\Big\{\|x+m'\| : m'\in M\Big\}\\ &= |\alpha|\cdot\|x+m\|_q\\ &\boxed{\|\alpha\cdot(x+m)\|_q=|\alpha|\cdot\|x+m\|_q}\end{aligned}$$

$$\forall\ x\in X/M \text{ and } \alpha\in K$$

Thus we conclude that $\left(X/M, \|\cdot\|_q\right)$ is a normed space over field K.

First assume that X is a Banach space.Then we have to show that X/M is a Banach space.

Let $\{x_n+m\}$ be a Cauchy sequence in X/M. We shall first construct a convergent subsequence of $\{x_n+m\}$ in X/M.

Evidently, it is possible to find a subsequence $\{x_{n_1}+m\}$ of the sequence $\{x_n+m\}$ such that

$$\begin{aligned}&\|(x_{n_2}+m)-(x_{n_1}+m)\|_q<\frac{1}{2}\\ &\|(x_{n_3}+m)-(x_{n_2}+m)\|_q<\frac{1}{2^2}\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\|(x_{n_k+1}+m)-(x_{n_k}+m)\|_q<\frac{1}{2^k}\end{aligned}$$

Choose any vector $y_1\in x_{n_1}+m$ and $y_2\in x_{n_2}+m$ such that

$$\|y_2-y_1\|<\frac{1}{2}$$

We then find $y_3\in x_{n_3}+m$ such that

$$\|y_3-y_2\|<\frac{1}{2^2}$$

Proceeding in this way, we get the sequence $\{y_k\}$ in X such that

$$x_{n_k} + m = y_k + m$$

$$\text{and} \qquad \|y_{k+1} - y_k\| < \frac{1}{\alpha^k} \qquad (k = 1, 2, \cdots\cdots\cdots)$$

Let $k > r$ then,

$$\begin{aligned}
\|y_k - y_r\| &= \|(y_k - y_{k-1}) + (y_{k-1} - y_{k-2}) + \cdots\cdots\cdots + (y_{r+1} - y_r)\| \\
&\leq \ \|y_k - y_{k-1}\| + \|y_{k-1} - y_{k-2}\| + \cdots\cdots\cdots + \|y_{r+1} - y_r\| \\
&< \frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \cdots\cdots\cdots + \frac{1}{2^r} < \frac{1}{2r-1}
\end{aligned}$$

$$\Rightarrow \ \|y_k - y_r\| < \frac{1}{2^{r-1}}$$

Therefore, it follows that $\{y_k\}$ is a Cauchy sequence in X but X is being complete $\exists\ y \in X$ such that

$$\lim_{k\to\infty} \|y_k - y\| = 0$$

Since,

$$\begin{aligned}
\|(x_{n_k} - m) - (y + m)\|_q &= \|(y_k + m) - (y + m)\|_q \\
&= \|(y_k - y) + m\|_q \\
&\leq \ \|y_k - y\|
\end{aligned}$$

It follows that,

$$\lim_{k\to\infty} (x_{m_k} + m) = (y + m) \in X/M$$

Thus we have proved that the Cauchy sequence $\{x_n + m\}$ has a convergent subsequence in X/M.

Since we know that if the subsequence of a Cauchy sequence converges, the sequence itself converges.

Hence, the Cauchy sequence $\{x_n + m\}$ converges in X/M and thus X/M is complete.

Thus X/M is a **Banach Space**.

Solutions of Assignment II

Solution of Question No.1(Problem):

Problem

Give the example of linear functional on different normed linear spaces for bounded linear functional.

Since, we know that if X be a normed space over the field K, a mapping $f : X \to K$ is said to be a linear functional on X if,

$$\boxed{f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)} \qquad \forall\ x, y \in X \text{ and } \alpha, \beta \in K$$

A linear functional is said to be real or complex according as the field K is $\mathbb{R}$ or $\mathbb{C}$ respectively.

Example:

Let the Banach space $(l^1, \|\cdot\|_1)$, define the linear functional $f : l^1 \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \xi_i \qquad x = \{\xi_i\}$$

then f is bounded linear functional on l^1 with $\|f\| = 1$.

Solution:

Let f be a function from l^1 into $\mathbb{R}$ defined by,

$$f(x) = \sum_{i=1}^{\infty} \xi_i \qquad \text{where } x = \{\xi_1\} \in l^1 \tag{I}$$

Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in l^1$ and,

$$\begin{aligned} \text{let} \quad x &= \alpha_1 x_1, \cdots\cdots\cdots, \alpha_n x_n \\ \text{and } y &= \beta_1 x_1, \cdots\cdots\cdots, \beta_n x_n \end{aligned}$$

then,

$$\begin{aligned} f(\alpha x + \beta y) &= f\left[\alpha\left(\alpha_1 x_1 + \cdots\cdots\cdots + \alpha_n x_n\right) + \beta\left(\beta_1 x_1 + \cdots\cdots\cdots + \beta_n x_n\right)\right] \\ &= f\left[\left(\alpha\alpha_1 + \beta\beta_1\right) x_1 + \cdots\cdots\cdots + \left(\alpha\alpha_n + \beta\beta_n\right) x_n\right] \\ &= \left(\alpha\alpha_1 + \beta\beta_1\right)\xi_1 + \cdots\cdots\cdots + \left(\alpha\alpha_n + \beta\beta_n\right)\xi_n \\ &= \alpha\left(\alpha_1\xi_1 + \cdots\cdots\cdots + \alpha_n\xi_n\right) + \beta\left(\beta_1\xi_1 + \cdots\cdots\cdots + \beta_n\xi_n\right) \\ &= \alpha f(x) + \beta f(y) \\ f(\alpha x + \beta y) &= \alpha f(x) + \beta f(y) \qquad \forall\ x, y \in l^1 \ \&\ \alpha, \beta \in \mathbb{R} \end{aligned}$$

Therefore, f is a linear functional on l^1 over the field $\mathbb{R}$.

Now, we have to show that f is bounded.
Since, the linear functional $f : l^1 \to \mathbb{R}$ defined by,

$$f(x) = \sum_{i=1}^{\infty} \xi_i \qquad\qquad x = \{\xi_i\} \in l^1$$

we have,

$$f(x) = \sum_{i=1}^{\infty} |\xi_i| = \sum_{i=1}^{\infty} |\xi_i \cdot 1|$$

By virtue of Cauchy-Schwartz inequality, we have

$$f(x) \leq \ |1| \sum_{i=1}^{\infty} |\xi_i| \ \leq \ \sum_{i=1}^{\infty} |\xi_i| = |x|_1 \quad \forall\ x \in l^1$$

Since, we know that $\|x\| = |x|$
then, we have

$$\boxed{f(x) \leq \ \|x\|_1}, \quad \forall\ x \in l^1$$

$\Rightarrow$ f is bounded on l^1.

Furthermore, we have

$$\|f\| = \text{sup.}\Big\{|f(x)| : x \in X, \|x\| \leq \ 1\Big\} \cdot \|1\|$$

$$\Rightarrow \ \|f\| \leq \ \|1\|$$

$$\Rightarrow \ \|f\| \leq \ 1 \tag{A}$$

Since, f is bounded then we have,

$$\|f\| \geq \frac{|f(x)|}{\|x\|_1} \tag{II}$$

Because, if f is bounded then $f(x) \leq \ \|f\|\|x\| \quad \forall\ x \in X$ equivalently, if $x \neq 0$ then

$$\|f\| \geq \frac{|f(x)|}{\|x\|}$$

For, $x = e_1 \in l^1$,
Now, using equation (I) , we have

$$f(x) = x \ \Rightarrow \ |f(x)| = |x|$$

$$\Rightarrow \ |f(x)| = \|x\|_1$$

$$\Rightarrow \ |f(e_1)| = \|e_1\|_1$$

$$\Rightarrow \ \boxed{|f(e_1)| = \|e_1\|_1}$$

Using this in equation (II) , we get as

$$\|f\| \geq \ \frac{|f(e_1)|}{\|e_1\|_1} = \frac{\|e_1\|_1}{\|e_1\|_1} = 1$$

$$\Rightarrow \ \|f\| \ \geq \ 1 \tag{B}$$

from equation(A) and equation(B) we get as,

$$\boxed{\|f\| = 1}$$

Hence, f **is a bounded linear functional with** $\|f\| = 1$.

Solution of Question No.2(Problem):

Problem

Define the functional $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = x - a$$

where, $x = (\xi_1\ \xi_2\ \cdots\cdots\xi_n) \in \mathbb{R}^n$ and $x \cdot a$ denotes the familiar scalar product of x and a. Then f is bounded linear functional on $\mathbb{R}^n$ with

$$\|f\| = \|a\|$$

Solution:

Let f be a function from $\mathbb{R}^n$ into $\mathbb{R}$ defined by,

$$f(x) = x - a$$

$$\text{or,}\quad f(x) = (\xi_1\ \xi_2\ \cdots\cdots\cdots\ \xi_n)\cdot(a_1\ a_2\ \cdots\cdots\cdots\ a_n)$$

$$\text{or,}\quad f(x) = (a_1\ \xi_1\ a_2\ \xi_2\ \cdots\cdots\cdots\ a_n\ \xi_n)$$

$$\text{where, } a = (a_1\ a_2\ \cdots\cdots\cdots a_n) \in \mathbb{R}^n$$

$$\text{and, } x = (\eta_1\ \eta_2\ \cdots\cdots\cdots\eta_n) \in \mathbb{R}^n$$

If $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} f(\alpha x + \beta y) &= f\left[\alpha\left(\xi_1\ \xi_2\ \cdots\cdots\cdots\xi_n\right) + \beta\left(\eta_1\ \eta_2\ \cdots\cdots\cdots\eta_n\right)\right] \\ &= f\left[\alpha\xi_1 + \beta\eta_1, \cdots\cdots\cdots, +\alpha\xi_n + \beta\eta_n\right] \\ &= \alpha_1\left(\alpha\xi_1 + \beta\eta_1\right) + \cdots\cdots\cdots + \alpha_n\left(\alpha\xi_n + \beta\eta_n\right) \\ &= \alpha\left(\alpha_1\ \xi_1 + \cdots\cdots\cdots\ \alpha_n\ \xi_n\right) + \beta\left(\alpha_1\ \eta_1 + \cdots\cdots\cdots\ \alpha_n\ \eta_n\right) \\ &= \alpha f\left(\xi_1\cdots\cdots\cdots\xi_n\right) + \beta f\left(\eta_1\cdots\cdots\cdots\eta_n\right) \\ f(\alpha x + \beta y) &= \alpha f(x) + \beta f(y) \qquad \forall\ x, y \in \mathbb{R}^n\ \&\ \alpha, \beta \in \mathbb{R} \end{aligned}$$

therefore, f is a linear function on $\mathbb{R}^n$ over the field $\mathbb{R}$.
Now, we have to show that f is bounded.
Since, the linear function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = x - a \tag{1}$$

where, $x = (\xi_1 \ \xi_2 \cdots\cdots\cdots \xi_n)$ and $a = (a_1 \ a_2 \cdots\cdots\cdots a_n)$, both x & $a \in \mathbb{R}^n$

we have,

$$|f(x)| = |x - a|$$

Since, we know that $\|x\| = |x|$

$$\Rightarrow \ |f(x)| = \|x - a\|$$

By virtue of Cauchy-Schwartz inequality we have,

$$|f(x)| = |x - a| \leq \|x\| \cdot \|a\|$$

let, $\|a\| = K$ then,

$$\boxed{|f(x)| \leq K \cdot \|x\|} \qquad\qquad \forall \ x \in \mathbb{R}^n$$

Hence f is **bounded**.

Furthermore, we have $\|f\| = \sup.\left\{|f(x)| : x \in X, \ \|x\| \leq 1\right\} \leq \|a\|$

$$\Rightarrow \ \|f\| \leq \|a\| \tag{A}$$

Since, f is bounded, then we have,

$$\|f\| \geq \frac{|f(a)|}{\|a\|} \tag{2}$$

Because, if f is bounded then $f(x) \leq \ \|f\|\|x\| \quad \forall \ x \in X$ equivalently, if $x \neq 0$ then,

$$\|f\| \geq \frac{|f(x)|}{\|x\|}$$

Now, using equation(1) we have,

$$f(x) = x \cdot a$$

$$\text{put } x = a$$

$$f(a) = a \cdot a$$

$$|f(a)| = |a| \cdot |a|$$

$$|f(a)| = \|a\| \cdot \|a\| = \|a\|^2$$

$$\Rightarrow \ \boxed{|f(a)| = \|a\|^2}$$

Using this in equation(2) we have,

$$\|f\| \geq \frac{\|a\|^2}{\|a\|} = \|a\|$$
$$\Rightarrow \ \|f\| \geq \|a\| \tag{B}$$

From equation(A) and equation(B), we have

$$\boxed{\|f\| = \|a\|}$$

Solution of Question No.3:

Example of unbounded linear functional:

Let $X = (C[a,b], \|\cdot\|_1)$ be the normed space and let $\delta_{t_0} : C[a,b] \to \mathbb{R}$ be the linear functional then, δ_{t_0} is unbounded in X.

Solution:

Let $\delta_{t_0} : C[a,b] \to \mathbb{R}$ be a function defined by,

$$\delta_{t_0}(x) = x(t_0), \quad x \in C[a,b]$$

then we have to show that δ_{t_0} is linear functional.
Let $\alpha, \beta \in \mathbb{R}$ & $x, y \in X$ then we have,

$$\delta_{t_0}(\alpha x + \beta y) = \alpha x(t_0) + \beta y(t_0), \quad t \in [a,b]$$
$$= \alpha\delta_{t_0}(x) + \beta\delta_{t_0}(x), \quad t \in [a,b]$$
$$\delta_{t_0}(\alpha x + \beta y) = \alpha\delta_{t_0}(x) + \beta\delta_{t_0}(x), \quad t \in [a,b]$$

Now, we have to show that δ_{t_0} is unbounded.

Let $x_n(t) = t^n, \qquad \forall\ n \in \mathbb{N}$ then,

$$\|x_n\|_\infty = \sup.\left\{|t^n|\right\} = 1 \qquad \forall\ t \in [a,b]$$

$$\text{and, } \delta_{t_0}(x_n) = x_n(t_0) = n$$
$$\Rightarrow \ \delta_{t_0}(x_n) = n$$

therefore, $\|\delta_{t_0}(x_n)\| = n = n \cdot 1 = n \cdot \|x_n\|_\infty \quad \forall\ n \in \mathbb{N}$

thus, there is no fixed real number $K > 0$ such that

$$\Rightarrow \ \|\delta_{t_0}(x_n)\|_\infty \leq K \cdot \|x_n\|_\infty$$

Hence, δ_{t_0} is **unbounded** in X.

- For more info about the document, please visit: https://github.com/akhlak919