ASSIGNMENT

Fourier Analysis and Summability Theory

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Assignment Questions:

1. Prove Euler-Fourier formula for Fourier coefficients.

2. A necessary and sufficient condition for the Fourier series,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

should converge to the sum s is that,

$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right) \cdot u \ \phi(u)}{\sin\left(\frac{u}{2}\right)} = 0.$$

3. A function is defined by $f(x) = \begin{cases} x + x^2 & , \ -\pi < x < \pi \\ \pi^2 & , \ x = \pm\pi \end{cases}$

Find the Fourier series of f.

4. State and Prove Fejer-Lebesgue Theorem.

5. Discuss the Uniqueness of trigonometric series.

Solutions

Solution of Question No 1.

- **Proof of Euler-Fourier formula for Fourier coefficients:**

Since we know that Fourier series for $f(x)$ is given by,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

- Evaluation of a_0 :

Integrating equation(1) term by term within the integration limit 0 to 2π we have like this,

$$\int_0^{2\pi} f(x)\ dx = \frac{1}{2}\int_0^{2\pi} a_0\ dx + \sum_{n=1}^{\infty}\left[\int_0^{2\pi} (a_n \cos nx + b_n \sin nx)\, dx\right]$$

Since,

$$\int_0^{2\pi} \cos nx\ dx = 0$$

and

$$\int_0^{2\pi} \sin nx\ dx = 0$$

$$\therefore \int_0^{2\pi} f(x)\ dx = \frac{1}{2}a_0 \int_0^{2\pi}\ dx = \frac{1}{2}a_0\ x\Big|_0^{2\pi} = \frac{1}{2}a_0\,[2\pi - 0]$$

OR,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx$$

- Evaluation of a_n :

Multiplying equation(1) by $\cos mx$, $\forall m \in \mathbb{Z}$ and integrating term by term from 0 to 2π we have,

$$\int_0^{2\pi} f(x)\ \cos mx\ dx = \frac{1}{2}a_0 \int_0^{2\pi} \cos mx\ dx + \sum_{n=1}^{\infty}\left[\int_0^{2\pi} a_n \cos mx \cos nx\ dx + \int_0^{2\pi} b_n \cos mx \sin nx\ dx\right]$$

If $m = n$, then we have,

$$\int_0^{2\pi} f(x)\cos nx\ dx = 0 + a_n \cdot \pi + 0$$

OR,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx$$

- Evaluation of b_n :

Multiplying equation(1) by $\sin mx$, $\forall m \in \mathbb{Z}$ and integrating term by term from 0 to 2π we have,

$$\int_0^{2\pi} f(x)\ \sin mx\ dx = \frac{1}{2}a_0\int_0^{2\pi} \sin mx\ dx + \sum_{n=1}^{\infty}\left[\int_0^{2\pi} a_n \sin mx \cos nx\ dx + \int_0^{2\pi} b_n \sin mx \sin nx\ dx\right]$$

If $m = n$, then above expression can be written as,

$$\int_0^{2\pi} f(x)\sin nx\ dx = 0 + 0 + b_n \cdot \pi$$

OR,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx$$

These above a_0, a_n, b_n are the required Euler-Fourier formula for the Fourier coefficients.

Solution of Question No 2.

Since we know that, By Dirichlet integral,

$$(s_n - s) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)\cdot u\ \phi(u)}{\sin\left(\frac{u}{2}\right)} du \tag{1}$$

where, $\phi(u) = f(x+u) + f(x-u) - 2s$.

Necessary Condition:

Suppose that given Fourier series, converges to $'s'$ as $n \to \infty$

$$\lim_{n\to\infty} (s_n f)(x) = s$$

or,

$$\lim_{n\to\infty} [(s_n f)(x) - s] \to 0.$$

So, according to the equation(1) above can be written as,

$$\lim_{n\to\infty} \left[\frac{1}{2\pi}\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)\cdot u\ \phi(u)}{\sin\left(\frac{u}{2}\right)} du\right] \to 0, \text{ as } n \to \infty.$$

or,

$$\lim_{n\to\infty} \left[\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)\cdot u\ \phi(u)}{\sin\left(\frac{u}{2}\right)} du\right] \to 0, \text{ as } n \to \infty.$$

Sufficient Condition:

Given that,

$$\lim_{n\to\infty} \left[\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)\cdot u\ \phi(u)}{\sin\left(\frac{u}{2}\right)} du\right] = 0$$

or,

$$\lim_{n\to\infty} \left[\frac{1}{2\pi}\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)\cdot u\ \phi(u)}{\sin\left(\frac{u}{2}\right)} du\right] \to 0, \text{ as } n \to \infty.$$

Now, by equation(1) we get as,

$$\lim_{n\to\infty} [(s_n f)(x) - s] \to 0.$$

or,

$$\lim_{n\to\infty} (s_n f)(x) = s.$$

Solution of Question No 3.

Since, we know that, If a function $f(x)$ is 2π periodic then, the Fourier series of the function $f(x)$ is given by,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

where, the Euler-Fourier coefficients a_0, a_n, b_n can be computed using the formula,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

Given, function is $f(x) = \begin{cases} x + x^2 & , -\pi < x < \pi \\ \pi^2 & , x = \pm\pi \end{cases}$, so the Fourier series of the function can be written as,

$$x + x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2}$$

Computation of $\mathbf{a_0}$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\ dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

Computation of $\mathbf{a_n}$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx \ dx$$

$$= \frac{1}{\pi}\left[\left(x+x^2\right)\left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left\{(1+2x)\left(\frac{\sin nx}{n}\right)\right\}dx\right]$$

$$= \frac{1}{\pi}\left[(1+2x)\left(\frac{\cos nx}{n^2}\right)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2\cdot\left(\frac{\cos nx}{n^2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{(1+2\pi)\cos n\pi}{n^2} - \frac{(1-2\pi)\cos(-n\pi)}{n^2}\right\} + 2\cdot\left(\frac{\sin nx}{n^3}\Big|_{-\pi}^{\pi}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\{(1+2\pi)-(1-2\pi)\}\cos n\pi}{n^2} + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi\cos n\pi}{n^2}\right]$$

$$= \frac{4\cos n\pi}{n^2}$$

$$a_n = \frac{4\,(-1)^n}{n^2}$$

Computation of $\mathbf{b_n}$:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx \ dx$$

$$= \frac{1}{\pi}\left[\left(x+x^2\right)\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(1+2x)\left(\frac{\cos nx}{n}\right)dx\right]$$

or,

$$b_n = \frac{1}{\pi}\Bigg[\left(\pi+\pi^2\right)\left(-\frac{\cos n\pi}{n}\right) - \left(-\pi+\pi^2\right)\left(-\frac{\cos(-n\pi)}{n}\right) + \left\{(1+2x)\left(\frac{\sin nx}{n^2}\right)\Big|_{-\pi}^{\pi}\right\} - \int_{-\pi}^{\pi} 2\cdot\left(\frac{\sin nx}{n^2}\right)dx\Bigg]$$

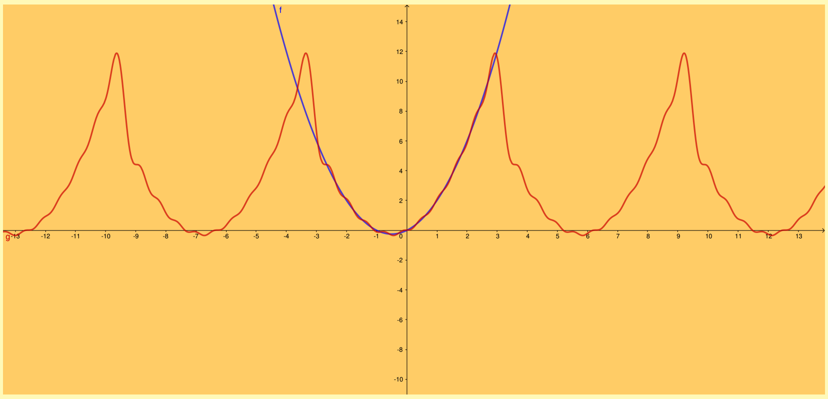

Figure 1: Fourier Series Approx.

$$b_n = \frac{1}{\pi}\left[\left\{\left(\pi+\pi^2\right)-\left(-\pi+\pi^2\right)\right\}\left(-\frac{\cos n\pi}{n}\right)+2\cdot\left(\frac{\cos nx}{n^3}\Bigg|_{-\pi}^{\pi}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}-\frac{2\cos\left(-n\pi\right)}{n^3}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi\cos n\pi}{n}\right)$$

$$b_n = -\frac{2\left(-1\right)^n}{n}$$

Using the values of a_0, a_n, b_n in equation(1) we have,

$$f(x) = \frac{2\pi^2}{2\cdot 3}+\sum_{n=1}^{\infty}\left[\left\{\frac{4\left(-1\right)^n}{n^2}\right\}\cdot\cos nx-\left\{\frac{2\left(-1\right)^n}{n}\right\}\sin nx\right]$$

or,

$$f(x) = \frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\cdots\right] \\ -2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\cdots\right]$$

Hence, this is the Fourier series of the given problem.

Answer of Question No. 4:

Statement of Fejer-Lebesgue Theorem:

The Fourier series of $f(x)$, where $x \in [0, 2\pi]$ is $(C, 1)$ summable to sum $f(x)$ for every value of x, for which

$$\int_0^t \left| f(x+u) - f(x-u) \right| du = o(t)$$

In particular, the Fourier series of $f(x)$ is $(C, 1)$ summable almost everywhere, for every value of x.

Proof of Fejer-Lebesgue Theorem:

The Fourier series of $f(x)$ is given by,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

Let, $(s_n f)(x)$ be n^{th} partial sum of the Fourier series mentioned in equation(1) then,

$$(s_n f)(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

The $(C, 1)$ means of above is given by,

$$(\sigma_n f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} (s_k f)(x)$$

by, using Fejer's integral we get as,

$$(\sigma_n f)(x) = \frac{1}{2\pi n} \int_0^{\pi} \frac{\sin^2\left(\frac{nu}{2}\right)}{\sin^2\left(\frac{u}{2}\right)} \left[f(x+u) + f(x-u) \right] du \tag{2}$$

If $f(x) = 1$, then

$$(\sigma_n f)(x) = 1$$

$$1 = \frac{1}{2\pi n} \int_0^{\pi} \left(\frac{\sin^2\left(\frac{nu}{2}\right)}{\sin^2\left(\frac{u}{2}\right)} \right) \cdot 2 \, du \tag{3}$$

Multiplying equation(3) by $f(x)$ and subtracting from equation(2), we have,

$$(\sigma_n f)(x) - f(x) = \frac{1}{2\pi n}\int_0^{\pi}\left(\frac{\sin^2\left(\frac{nu}{2}\right)}{\sin^2\left(\frac{u}{2}\right)}\right)\Big[f(x+u)+f(x-u)-2f(x)\Big]du$$

$$= \frac{1}{2\pi n}\int_0^{\pi}\left(\frac{\sin^2\left(\frac{nu}{2}\right)}{\sin^2\left(\frac{u}{2}\right)}\right)\phi(u)\,du$$

where, $\phi(u) = \Big[f(x+u)+f(x-u)-2f(x)\Big]$.
or,

$$(\sigma_n f)(x) - f(x) = \frac{2}{n\pi}\int_0^{\delta}\left\{\left(\frac{\sin^2\frac{nu}{2}}{u^2}\right)\cdot\phi(u)\right\}du + \frac{1}{2n\pi}\int_0^{\delta}\sin^2\frac{nu}{2}\left[\frac{1}{\sin^2\frac{u}{2}} - \frac{1}{\left(\frac{u}{2}\right)^2}\right]\phi(u)\,du$$
$$+\frac{1}{2n\pi}\int_{\delta}^{\pi}\left\{\left(\frac{\sin^2\frac{nu}{2}}{\sin^2\frac{u}{2}}\right)\cdot\phi(u)\right\}du$$

$$(\sigma_n f)(x) - f(x) = I_1 + I_2 + I_3 \text{ (say)},\ \forall\ \ 0<\delta<\pi \tag{4}$$

then, $|I_2| \to 0$ as $n\to\infty$.
&,

$$|I_3| \le \frac{1}{2n\pi}\int_{\delta}^{\pi}\frac{|\phi(u)|}{\sin^2\frac{u}{2}}du \to 0 \text{ as } n\to\infty.$$

Let,

$$I_2 = \int_0^{\delta}\left\{\left(\frac{\sin^2\frac{nu}{2}}{u^2}\right)\cdot\phi(u)\right\}du$$

$$\phi(t) = \int_0^t |\phi(u)|\,du$$

$$\le \int_0^t |f(x+u)-f(x)|\,dx + \int_0^t |f(x-u)-f(x)|\,du$$

$$\therefore\ \phi(t) = o(t)$$

Rough idea of Manipulation of above expression

$$\text{Since, } \phi(u) = f(x+u) + f(x-u) - f(x)$$

$$\text{It can be written as, } = \{f(x+u) - f(x)\} + \{f(x-u) - f(x)\}$$

$$|\phi(u)| = |\{f(x+u) - f(x)\} + \{f(x-u) - f(x)\}|$$

or,

$$\int_0^t |\phi(u)|\, du \leq \int_0^t |f(x+u) - f(x)|\, du + \int_0^t |f(x-u) - f(x)|\, du$$

$$= o(t) + o(t) = o(t)$$

For, $\epsilon > 0,\ \exists$ a positive integer $n > 0,\ \forall\ n \in \mathbb{Z}$ such that,

$$\phi(t) < \epsilon \cdot t \ , \text{ whenever } t < n.$$

We know that,

$$\phi(t) = o(t) \implies \lim_{t \to 0} \frac{\phi(t)}{t} = 0$$

For, $\epsilon > 0,\ \exists\ \eta > 0,$ such that,

$$\left| \frac{\phi(t)}{t} - 0 \right| < \epsilon \ , \ |t| < \eta \ , \ t > 0$$

$$\frac{\phi(t)}{t} < \epsilon \ , \ t < \eta$$

$$\therefore\ \phi(t) < \epsilon \cdot t \ , \ t < \eta$$

Choose, $n > \frac{1}{\eta}$

$$I_1 = \int_0^{\delta} \left\{ \left(\frac{\sin^2 \frac{nu}{2}}{u^2} \right) \cdot \phi\,(u) \right\} du$$

$$= \int_0^{\frac{1}{n}} \left\{ \left(\frac{\sin^2 \frac{nu}{2}}{u^2} \right) \cdot \phi\,(u) \right\} du + \int_{\frac{1}{n}}^{\eta} \left\{ \left(\frac{\sin^2 \frac{nu}{2}}{u^2} \right) \cdot \phi\,(u) \right\} du + \int_{\eta}^{\delta} \left\{ \left(\frac{\sin^2 \frac{nu}{2}}{u^2} \right) \cdot \phi\,(u) \right\} du$$

Next,

$$\begin{aligned}
|I'_{1\cdot 1}| &= \left| \int_0^{\frac{1}{n}} \left\{ \left(\frac{\sin^2 \frac{nu}{2}}{u^2} \right) \cdot \phi\left(u\right) \right\} du \right| \\
&\leq \int_0^{\frac{1}{n}} \frac{\left(\frac{nu}{2}\right)^2}{u^2} \cdot \phi\left(u\right) du \\
&\leq \frac{n^2}{4} \int_0^{\frac{1}{n}} |\phi(u)|\, du \\
&\leq \frac{n^2}{4} \left(\frac{\epsilon}{n}\right) \\
|I'_{1\cdot 1}| &\leq \frac{n\epsilon}{4}
\end{aligned}$$

$$\begin{aligned}
\text{Now,}\ \ I'_{1\cdot 2} &= \int_{\frac{1}{n}}^{\eta} \frac{\sin^2\left(\frac{nu}{2}\right)}{u^2} \phi(u) du \\
|I'_{1\cdot 2}| &\leq \int_{\frac{1}{n}}^{\eta} \frac{\left|\sin^2\left(\frac{nu}{2}\right)\right|}{u^2} |\phi(u)|\, du \\
&\leq \int_{\frac{1}{n}}^{\eta} \frac{|\phi(u)|}{u^2} du \\
&= \left\{ \frac{\phi(u)}{u^2} \right\} \Bigg|_{\frac{1}{n}}^{\eta} - \int_{\frac{1}{n}}^{\eta} \left\{ \frac{-2}{u^3} \phi(u) \right\} du
\end{aligned}$$

Since, $\phi(t) < \epsilon \cdot t$ then,

$$\begin{aligned}
|I'_{1\cdot 2}| &\leq \frac{1}{n^2}\phi(n) - \eta^2 \frac{1}{2}\left(\frac{1}{n}\right) + 2\int_{\frac{1}{n}}^{\eta} \frac{\epsilon u}{u^3} du \\
\text{So,}\ \ |I'_{1\cdot 2}| &< \frac{\epsilon\eta}{n^2} - \frac{\eta^2\epsilon}{n} + 2\epsilon \left(\frac{-1}{u}\Bigg|_{\frac{1}{n}}^{\eta} \right) \\
&< \frac{\epsilon}{\eta} - n\epsilon + 2\epsilon\left(\eta - \frac{1}{\eta}\right) \\
&< \frac{-\epsilon}{\eta} + n\epsilon \\
&< \frac{\epsilon}{\eta} + 2n\epsilon
\end{aligned}$$

$$
\begin{aligned}
|I'_{1\cdot 2}| &< \ \epsilon n + 2n\epsilon \\
&= 3n\epsilon \text{ Since, } \frac{1}{\eta} < n \\
\text{So, } \ |I'_{1\cdot 2}| &< \ 3n\epsilon
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I'_{1\cdot 3} &= \int_\eta^\delta \frac{\sin^2\left(\frac{nu}{2}\right)}{u^2}\phi(u)du \\
|I'_{1\cdot 3}| &\leq \ \int_\eta^\delta \frac{\sin^2\left(\frac{nu}{2}\right)}{u^2}\left|\phi(u)\right| du \\
&\leq \ \frac{1}{\eta^2}\int_\eta^\delta \left|\phi(u)\right| du \\
&= \frac{A}{\eta^2} \ \left(\because \frac{1}{u^2} \leq \ \frac{1}{\eta^2}, \ \forall \ u \in [\eta, \delta]\right)
\end{aligned}
$$

$$
\text{where, } \ A = \int_\eta^\delta \left|\phi(u)\right| du
$$

$$
\therefore \ |I'_{1\cdot 3}| < \ \frac{A}{\eta^2}.
$$

$$
\begin{aligned}
\text{Now, } \ I_1 &= \frac{2}{n\pi}\int_0^\delta \frac{\sin^2\left(\frac{nu}{2}\right)}{u^2}\phi(u)du = \frac{2}{n\pi}\left[I'_{1\cdot 1} + I'_{1\cdot 2} + I'_{1\cdot 3}\right] \\
|I_1| \ &\leq \ \frac{2}{n\pi}\left[n \cdot \frac{\epsilon}{4} + 3n\epsilon + \frac{A}{\eta^2}\right] \\
&= \frac{2}{\pi}\left[\frac{\epsilon}{4} + 3\epsilon + \frac{A}{\eta^2 n}\right]
\end{aligned}
$$

So, $|I_1| \to 0$ as $n \to \infty$, ϵ is an arbitrary choosen positive number.

$$
\therefore \ I_1 \to 0 \text{ as } n \to \infty
$$

Then by equation no.(4),

$$\lim_{n\to\infty}\left[\left(\sigma_n f\right)(x) - f(x)\right] = 0$$

The Fourier series of $f(x)$ is $(C,1)$ summable to the sum $f(x)$ i.e
The Fourier series of $f(x)$ is $(C,1)$ summable to $f(x)$ almost everywhere.

Answer of Question No 5.

Uniqueness of Trigonometric Series:

Statement:

If two trigonometric series converges to the same sum in $(0, 2\pi)$ with possible exception of a finite no. of points , then correspondent coefficients of these series are equal. i.e. The series are identical.

Proof:

Let us consider two trigonometric series ,

$$\frac{1}{2}a_0' + \sum_{n=1}^{\infty}\left(a_n' \cos nx + b_n' \sin nx\right) \tag{1}$$

$$\frac{1}{2}a_0'' + \sum_{n=1}^{\infty}\left(a_n'' \cos nx + b_n'' \sin nx\right) \tag{2}$$

defined on interval $(0, 2\pi)$. Suppose both the series converges to the same sum ,

$$f(x) = s \text{ in interval } (0, 2\pi)$$

except at finite no. of points. Then,

$$\frac{1}{2}\left(a_0' - a_0''\right) + \sum_{n=1}^{\infty}\left\{\left(a_n' - a_n''\right)\cos nx + \left(b_n' - b_n''\right)\sin nx\right\} \to \ 0 \text{ in } (0, 2\pi)$$

Let, $a_0 = a_0' - a_0'',\ a_n = a_n' - a_n'',\ b_n = b_n' - b_n''$ Then,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \text{ converges to } 0 \text{ in } (0, 2\pi)$$

By Cantor's Lemma , we have,

$(a_n \cos nx + b_n \sin nx) \to 0$ as $n \to \infty \implies a_n \to 0$ & $b_n \to 0$ as $n \to \infty$.

Now,

$$F(x) = \frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty} \frac{(a_n \cos nx + b_n \sin nx)}{n^2}$$

therefore, $F(x)$ is a continuous function of x in $[0, 2\pi]$. Now, by Riemann's first theorem , we have,

$$G(x, h) = \frac{F(x + 2h) + F(x - 2h) - 2F(x)}{4h^2} \to 0 \text{ as } h \to \infty \ \forall \ x \in (0, 2\pi)$$

$\implies$ Second generalized derivative of $F(x)$ is zero. Since $F(x)$ is continuous and $a_n \to 0$, $b_n \to 0$ as $n \to \infty$. This implies that $F(x)$ is a linear function in $(0, 2\pi)$.*i.e*

$$F(x) = ax + b \ , \text{ where } a, b \in \mathbb{R}$$

$$\implies ax + b = \frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty} \left(\frac{a_n \cos nx + b_n \sin nx}{n^2} \right)$$

$$\text{or, } \sum_{n=1}^{\infty} \left(\frac{a_n \cos nx + b_n \sin nx}{n^2} \right) = \frac{1}{4}a_0x^2 - ax - b \qquad (3)$$

This series, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{a_n \cos nx + b_n \sin nx}{n^2} \right)$ is a periodic function of period 2π.

$\therefore \ \frac{1}{4}a_0x^2 - ax - b$ is also periodic with period 2π. $\implies a_0 = 0$, $a = 0$.

$$\text{so, } \sum_{n=1}^{\infty} \left(\frac{a_n \cos nx + b_n \sin nx}{n^2} \right) = -b \qquad (4)$$

Multiplying equation(4) by $\cos(mx)$ and integrating term by term in $(0, 2\pi)$, we have,

$$\sum_{n=1}^{\infty} \left\{ \int_0^{2\pi} \frac{a_n}{n^2} (\cos mx \cdot \cos nx) \, dx + \int_0^{2\pi} \frac{b_n}{n^2} (\cos mx \cdot \sin nx) \, dx \right\} = -b \int_0^{2\pi} \cos mx \, dx$$

$$\implies \frac{a_m}{m^2} \times \pi = 0$$

$$\implies a_m = 0$$

$$\implies a_n = 0$$

$$\implies a'_n - a''_n = 0$$

$$\implies a'_n = a''_n.$$

Multiplying equation(4) by $\sin(mx)$ and integrating term by term in $(0, 2\pi)$, we have,

$$\sum_{n=1}^{\infty}\left\{\int_0^{2\pi}\frac{a_n}{n^2}\left(\sin mx\cdot\cos nx\right)dx+\int_0^{2\pi}\frac{b_n}{n^2}\left(\sin mx\cdot\sin nx\right)dx\right\}=-b\int_0^{2\pi}\sin mx\,dx$$

$$\implies \frac{b_m}{m^2}\times\pi = 0$$

$$\implies b_m = 0$$

$$\implies b_n = 0$$

$$\implies b'_n - b''_n = 0$$

$$\implies b'_n = b''_n.$$

This is the required proof of uniqueness of trigonometric series.