A PROOFS OF THEOREM

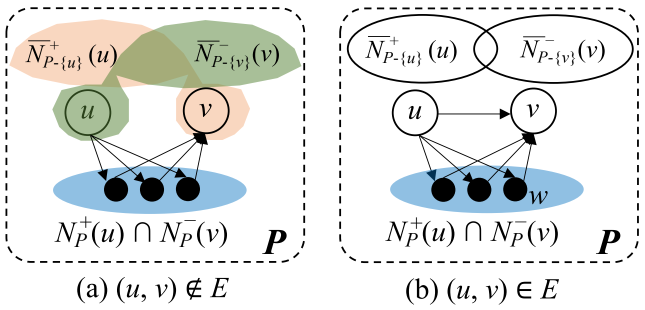


Figure 8: Second-Order Pruning

A.1 Proof of Second-Order Pruning Rules

Theorem 4.2 is proven as follows and we can similarly prove Theorems 4.3–4.5.

Proof. This can be seen from Figure 8. Since P is a (k_1, k_2)-plex, we have $\overline{d^+_{P-\{u\}}}(u) \le k_1 - 1$ (since $\overline{d^+_P}(u) \le k_1$) and $\overline{d^-_{P-\{v\}}}(v) \le k_2 - 1$ (since $\overline{d^-_P}(u) \le k_2$). In Case (a) where $(u, v) \notin E$, any vertex $w \in P$ can only fall in the following 3 scenarios: (1) $w \in \overline{N^+_{P-\{u\}}}(u)$ (which contains v), (2) $w \in \overline{N^-_{P-\{v\}}}(v)$ (which contains u), and (3) $w \in N^+_P(u) \cap N^-_P(v)$. Note that w may be in both (1) and (2). So we have:

$$\begin{aligned}
|P| &= |N^+_P(u) \cap N^-_P(v)| + |\overline{N^+_{P-\{u\}}}(u) \cup \overline{N^-_{P-\{v\}}}(v)| \\
&\le |N^+_P(u) \cap N^-_P(v)| + |\overline{N^+_{P-\{u\}}}(u)| + |\overline{N^-_{P-\{v\}}}(v)| \\
&\le |N^+_P(u) \cap N^-_P(v)| + (k_1 - 1) + (k_2 - 1) \\
&= |N^+_P(u) \cap N^-_P(v)| + k_1 + k_2 - 2,
\end{aligned}$$

so $|N^+_P(u) \cap N^-_P(v)| \ge |P| - k_1 - k_2 + 2 \ge q - k_1 - k_2 + 2$.

In Case (ii) where $(u, v) \in E$, any vertex $w \in P$ can only be in one of the following 4 scenarios: (1) $w = u$, (2) $w = v$, (3) $w \in N^+_P(u) \cap N^-_P(v)$, and (4) $w \in \overline{N^+_{P-\{u\}}}(u) \cup \overline{N^-_{P-\{v\}}}(v)$, so:

$$\begin{aligned}
|P| &= 2 + |N^+_P(u) \cap N^-_P(v)| + |\overline{N^+_{P-\{u\}}}(u) \cup \overline{N^-_{P-\{v\}}}(v)| \\
&\le 2 + |N^+_P(u) \cap N^-_P(v)| + |\overline{N^+_{P-\{u\}}}(u)| + |\overline{N^-_{P-\{v\}}}(v)| \\
&\le 2 + |N^+_P(u) \cap N^-_P(v)| + (k_1 - 1) + (k_2 - 1) \\
&= |N^+_P(u) \cap N^-_P(v)| + k_1 + k_2,
\end{aligned}$$

so $|N^+_P(u) \cap N^-_P(v)| \ge |P| - k_1 - k_2 \ge q - k_1 - k_2$. □

A.2 Proof of Theorem 5.1

Proof. Consider any two vertices u, v in a (k_1, k_2)-plex P where $k_1, k_2 \le \frac{q+1}{2}$, we show that they cannot be more than 2 hops apart (i.e., cannot fall out of the 6 cases in Figure 9).

Without loss of generality, we only consider the path from v to u where the first edge is outbound from v, i.e., Cases 1(a)–(c). Cases 2(a)–(c) are symmetric and can be similarly proved.

If v directly points to u, we are done since Case 1(a) occurs. Now assume that edge $(v, u) \notin E$, and we show that:

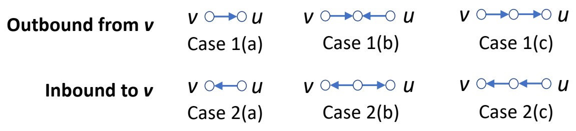


Figure 9: Cases for Two-Hop Diameter Upper Bound

- **Case (I): $(\mathbf{u}, \mathbf{v}) \notin \mathbf{E}$**, then both Case 1(b) and Case 1(c) should be satisfied. **(i) We first prove Case 1(b).** Note that $u \notin N^+(v)$ and $v \notin N^+(u)$. Since $k_1 \le \frac{q+1}{2} \le \frac{|P|+1}{2}$, u and v each points to at least $|P| - k_1 \ge \frac{|P|-1}{2}$ other vertices in P, so they must share an out-neighbor; otherwise, there exist $2 \cdot \frac{|P|-1}{2} = |P| - 1$ vertices other than u and v, leading to a contradiction since there will be at least $(|P| + 1)$ vertices in P when adding u and v. **(ii) We next prove Case 1(c).** Note that $v \notin N^-(u)$ and $u \notin N^+(v)$. Since $k_1 \le \frac{q+1}{2} \le \frac{|P|+1}{2}$, v points to at least $|P| - k_1 \ge \frac{|P|-1}{2}$ other vertices in P; also since $k_2 \le \frac{q+1}{2} \le \frac{|P|+1}{2}$, u is pointed to by at least $|P| - k_2 \ge \frac{|P|-1}{2}$ other vertices in P. So, $N^+(v)$ and $N^-(u)$ must intersect as illustrated by Figure 9 Case 1(c); otherwise, there will be $(|P| + 1)$ vertices in P when adding u and v.
- **Case (II): $(\mathbf{u}, \mathbf{v}) \in \mathbf{E}$,** then Case 1(c) should be satisfied. The proof is the same as (ii) above. Note that we cannot guarantee Case 1(b) anymore, since $v \in N^+(u)$, i.e., v can be one of the at least $|P| - k_1 \ge \frac{|P|-1}{2}$ neighbors of u, invalidating the prove for (i) above.

Symmetrically, consider the path from v to u where the first edge is inbound to v, i.e., Cases 2(a)–(c). If u directly points to v, we are done since Case 2(a) occurs. If edge $(u, v) \notin E$:

- **Case (III): edge (v, u) does not exist in G,** then both Case 2(b) and Case 2(c) should be satisfied. The proof is symmetric to Case (I) above and thus omitted.
- **Case (IV): edge (v, u) exists in G,** then Case 2(c) should be satisfied. The proof is symmetric to Case (II) above.

Putting the above discussions together, we obtain the following 4 cases, for each of which we explain how to exclude an impossible candidate u from $ext(S)$ given a vertex $v \in S$, where $ext(S)$ denotes the set of candidate vertices that can extend S into a valid (k_1, k_2)-plex.

- **Case A: $(v, u) \in E$ and $(u, v) \in E$.** In this case, we always have $u \in ext(S)$.
- **Case B: $(v, u) \notin E$ and $(u, v) \in E$.** Based on Case (II) above, we have $u \in ext(S)$ only if a path $u \leftarrow w \leftarrow v$ exists in G for some $w \in V$ ($w \ne u, v$).
- **Case C: $(v, u) \in E$ and $(u, v) \notin E$.** Based on Case (IV) above, we have $u \in ext(S)$ only if a path $u \longrightarrow w \longrightarrow v$ exists in G for some $w \in V$ ($w \ne u, v$).
- **Case D: $(v, u) \notin E$ and $(u, v) \notin E$.** Based on Case (I) above, we have Condition (C1): $u \in ext(S)$ only if both Case 1(b) and Case 1(c) in Figure 9 are satisfied. Similarly, based on Case (III) above, we have Condition (C2): $u \in ext(S)$ only if both Case 2(b) and Case 2(c) are satisfied. Combining both conditions, $u \in ext(S)$ only if there exist $w_1, w_2, w_3, w_4 \in$

$V - \{u, v\}$ such that $u \leftarrow w_1 \leftarrow v$ and $u \leftarrow w_2 \rightarrow v$ and $u \rightarrow w_3 \leftarrow v$ and $u \rightarrow w_4 \rightarrow v$.

Once we have applied the above rules to prune $ext(S)$ to exclude invalid candidates u, let us abuse the notation to use G again to denote the resulting graph induced by $S \cup ext(S)$ after pruning. Note that we can apply this diameter-based pruning on the pruned G again, since some vertex w in Case B (resp. Case C) could have been pruned by Case C (resp. Case B) in the previous iteration, causing some required paths to disappear, further invalidating more vertices u from $ext(S)$. This pruning can be iteratively run over G.

Based on the above idea, Algorithm 2 computes the set of vertices in $ext(S)$ that are not 2-hop pruned by a vertex $v \in S$. Specifically, Line 1 computes O (resp. I) as the set of v's out-neighbors (resp. in-neighbors) u that belong to Case B (resp. Case C). Then, Line 3 recovers S_O (resp. S_I) as the set of v's all non-pruned out-neighbors (resp. in-neighbors) w in Case B (resp. Case C) with path $v \rightarrow w \rightarrow u$ (resp. $v \leftarrow w \leftarrow u$). Note that $N^{\pm}(v) \subseteq ext(S)$ based on Case A so its vertices cannot be further pruned, so the iterative pruning is contributed by the shrink of sets O and I.

Next, Line 4 prunes away those vertices $u \in O$ (resp. $u \in I$) that cannot find a path $u \rightarrow w$ (resp. $u \leftarrow w$) for some non-pruned $w \in N^-(v)$ (resp. $w \in N^+(v)$), which is based on Case C (resp. Case B). Note that if O or I shrinks in Line 4, Line 5 will trigger another iteration of pruning. When the loop of Lines 2–5 exits, we have O (resp. I) being the remaining vertices $u \in ext(S)$ in Case B (resp. Case C) after iterative pruning. Finally, Line 6 computes the set B of vertices where u satisfies Case D w.r.t. v, and Line 7 unions the 4 disjoint candidate sets that correspond to Cases A, B, C and D, respectively, to obtain the final 2-hop pruned $ext(S)$ for a vertex $v \in S$. We denote this set as $\mathbb{B}(v)$, which is returned by Line 7. $\square$

B INCREMENTAL MAINTENANCE METHOD

B.1 Incremental Set Maintenance

We maintain three global containers for P, C and X, respectively, using the data structure shown in Figure 10. Specifically, two arrays with capacity $|V|$ are maintained for each set S: (1) *list*[] which keeps the list of S's elements for scanning and adding new elements; (2) *pos*[] with which maps vertex ID back to its position in *list*[], to facilitate element search and removal. Note that element addition, search and removal can all be done in $O(1)$ time. To illustrate using Figure 10, $S.add(10)$ appends 10 to *list*[] at position 3, and records $pos[10] = 3$; while $S.remove(3)$ first finds the element position in *list*[] as $pos[3] = 1$, and then erases it in *list*[] by moving the last element 10 to *list*[1], and moving 1 from *pos*[3] to *pos*[10]. In this way, we can incrementally populate $\langle P, C, X\rangle$ for calling BK2(.) with minimal cost.

For example, for the else-branch in Algorithm 4 Lines 25–28, we can first conduct $C.remove(v_p)$ and $X.add(v_p)$ before calling BK2(.) at Line 25, and then recover C and X for use by Lines 26–28 by conducting $C.add(v_p)$ and $X.remove(v_p)$. Line 26 (resp. Line 27) then removes from C (resp. X) those elements v such that $P \cup \{v_p, v\}$ is a not (k_1, k_2)-plex, and let us denote the set of removed elements by C'' (resp. X''). Then, we can conduct $P.add(v_p)$ and call BK(.) with (P, C, X) in Line 28, after which we recover P, C and X for backtracking by conducting $P.remove(v_p)$, $C.add(C'')$ and $X.add(X'')$. Note that C'' and X'' are kept in global buffers whose

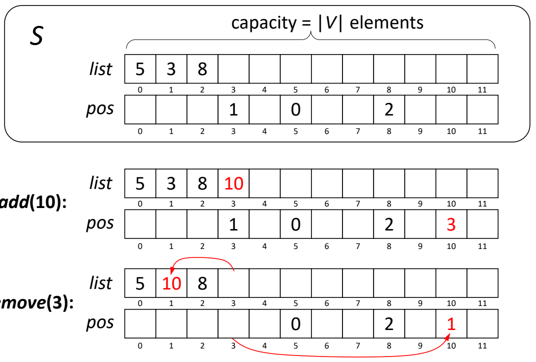


Figure 10: Data Structure for Set Maintenance

spaces are initialized at the beginning, and reused throughout the recursive execution.

In case of parallel execution, we let each thread maintain its own global containers for P, C and X as well as buffers C'' and X'', which are reused when the thread processes its fetched tasks one after another. However, when adding a task that computes BK2(.) for $\langle P, C, X\rangle$ to Q, we cannot directly copy the two arrays *list*[] and *pos*[] shown in Figure 10, since *pos*[] is sparse which basically requires copying the entire array of capacity $|V|$ that is very costly. We propose to only copy the elements in *list*[] from the current thread's global container to the task object to be added to Q; and when the task is fetched by a thread for processing, the thread reconstructs *list*[] and *pos*[] in its global containers from the saved elements. In this way, for the example shown in Figure 10, we only need to store the three elements 5, 3 and 8 with the created task, the cost of which is negligible.

B.2 Incremental Degree Maintenance

We also incrementally maintain (1) $d^+_{P\cup C}(.)$ and $d^-_{P\cup C}(.)$ which are needed in Lines 20, 26 and 27; and (2) $d^+_P(.)$ and $d^-_P(.)$ which are needed in Lines 8, 26 and 27, as we have explained at the end of Section 3. For example, whenever we remove a vertex v from C, we need to decrement $d^-_{P\cup C}(u)$ for all $u \in N^+_{P\cup C}(v)$ and $d^+_{P\cup C}(u)$ for all $u \in N^-_{P\cup C}(v)$.

C ADDITIONAL EXPERIMENTS

C.1 Complete Results in Parallel Executions

We implemented the parallel version of Ours based on the description in Section 7. Table 4 reports the (k_1, k_2)-plex enumeration time of parallel Ours on all our datasets in Table 1 when running with 2, 4, 8, 16 and 32 threads, respectively, where we use the default timeout threshold $\tau_{time} = 0.1$ ms is adopted, which is tested to work consistently well on various datasets. As Table 4 shows, our parallel algorithm is efficient and scales up well with the number of CPU cores on all the datasets.

C.2 Ablation Study

To verify the effectiveness of the various techniques we proposed, we conduct ablation study with Ours by disabling one of the technique at a time. This gives four variants Ours\SOP, Ours\ITP, Ours\BRA, Ours\LAP. For example, Ours\SOP is the variant when second-order

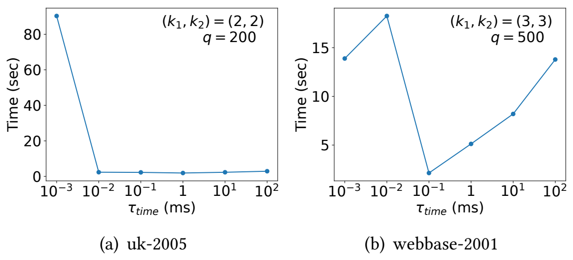
Table 4: The Running Time (sec) in Parallel Executions

Dataset	k_1	k_2	q	#(k_1, k_2)-plexes	Serial	2 Threads	4 Threads	8 Threads	16 Threads	32 Threads
bitcoin	2	5	10	144.00	0.06	0.04	0.02	0.01	0.01	0.02
			12	0	0.01	0.00	0.00	0.00	0.00	0.00
	3	5	10	24,261	8.35	4.78	2.46	1.29	0.66	0.36
			12	286	1.05	0.62	0.33	0.17	0.09	0.05
wiki-vote	3	5	10	474	1.85	1.08	0.55	0.29	0.15	0.08
			12	0	0.08	0.05	0.02	0.01	0.01	0.01
	4	5	10	30,222	349.32	197.94	102.45	52.57	26.57	16.44
			12	94	16.63	9.76	4.99	2.61	1.32	0.91
mathoverflow	2	2	10	523,633	28.55	18.48	9.48	4.93	2.50	1.34
			12	150,888	13.35	9.01	4.56	2.38	1.22	0.69
	2	3	10	1,762,917	63.16	47.81	24.41	12.71	6.45	3.52
			12	602,862	32.44	24.92	12.61	6.63	3.36	1.86
as-caida	2	3	10	23,314	7.99	0.34	0.19	0.10	0.08	0.15
			15	185	0.93	0.02	0.01	0.01	0.01	0.02
	3	4	15	17,303	14.87	0.63	0.33	0.18	0.10	0.10
			20	0	0.19	0.01	0.00	0.00	0.00	0.00
epinions	2	3	10	773,095	23.85	18.34	9.48	4.94	2.56	1.41
			12	252,511	10.33	8.06	4.18	2.23	1.19	0.80
	3	4	12	24,599,029	2,053.04	1,174.05	605.69	314.25	158.78	85.62
			15	2,382,084	381.99	229.27	118.32	61.53	31.62	18.13
email-euall	2	3	10	17,438	1.69	1.26	0.65	0.34	0.18	0.19
			12	1442	0.58	0.39	0.21	0.11	0.07	0.12
	3	4	12	196,878	80.58	46.44	23.49	12.18	6.14	3.38
			15	1351	9.19	5.20	2.68	1.40	0.72	0.44
amazon0505	3	5	10	4,696	0.27	0.08	0.04	0.02	0.02	0.01
			12	24	0.14	0.01	0.00	0.00	0.00	0.00
	5	3	10	1,532	0.20	0.04	0.02	0.01	0.01	0.00
			12	1	0.13	0.00	0.00	0.00	0.00	0.00
web-google	2	3	15	997	0.29	0.05	0.02	0.01	0.01	0.00
			20	57	0.21	0.01	0.01	0.00	0.00	0.00
	3	4	15	1,993	0.32	0.06	0.03	0.02	0.01	0.01
			20	61	0.22	0.02	0.01	0.00	0.00	0.00
soc-pokec-relationships	2	3	15	17,980	2.11	0.92	0.49	0.25	0.21	0.39
			20	17	0.95	0.03	0.01	0.01	0.01	0.02
	3	3	15	508,884	54.91	32.24	16.52	8.65	4.41	2.41
			20	648	1.66	0.49	0.25	0.13	0.07	0.06
wiki-talk	2	2	15	22,219	287.28	173.50	87.73	44.75	22.50	11.48
			20	0	69.70	38.84	19.89	10.27	5.17	2.67
	2	3	15	165,156	525.13	376.12	191.71	97.82	48.99	25.01
			20	0	129.02	77.30	39.23	20.22	10.16	5.21
arabic-2005	2	2	1000	106,895	3,906.70	1,897.35	1,017.31	527.40	269.04	140.58
	3	3	3000	3,500	111.60	60.20	29.56	15.10	8.69	4.64
uk-2005	2	2	200	117,255	301.09	162.70	84.11	43.34	21.82	11.38
	3	3	400	66,639	168.98	91.51	46.34	23.95	12.11	6.34
it-2004	2	2	1000	15,087	1,170.05	621.80	321.12	166.53	85.40	52.04
	3	3	3000	1,034	77.90	41.35	21.72	10.21	6.40	3.94
webbase-2001	2	2	300	327,882	646.41	320.84	164.46	90.02	53.56	30.75
	3	3	500	88,777	184.44	95.16	52.07	30	18.15	8.75
clue-web	2	2	10	1,215,977	26.05	14.15	7.66	3.82	1.75	1.69
	3	3	20	95,383	52.92	31.76	16.46	8.56	4.42	2.37

Table 5: Ablation Study.

Network	k_1	k_2	q	Ours\SOP	Ours\ITP	Ours\BRA	Ours\LAP	Ours
bitcoin	2	5	10	92.57	0.07	0.08	0.07	**0.06**
			12	48.56	0.02	0.02	0.02	**0.01**
	3	5	10	98.05	10.37	11.49	8.82	**8.35**
			12	79.78	1.13	1.49	1.09	**1.05**
wiki-vote	3	5	10	1,685.38	7.73	2.55	1.98	**1.85**
			12	1,423.06	0.17	0.14	0.11	**0.08**
	4	5	10	2,363.89	6,192.11	447.41	369.35	**349.32**
			12	1,859.31	81.95	20.68	17.53	**16.63**
arabic-2005	2	2	1,000	8436.63	>6 hrs	>6 hrs	>6 hrs	**3906.70**
	3	3	3,000	133.37	>6 hrs	215.30	>6 hrs	**111.60**
uk-2005	2	2	200	>6 hrs	386.27	>6 hrs	>6 hrs	**301.09**
	3	3	400	412.13	189.50	>6 hrs	>6 hrs	**168.98**

pruning is turned off in Ours. Table 5 shows the results of our ablation study with serial execution for the two small datasets and the first two large datasets. The other results are similar but omitted due to space limit. It can be observed that Ours is the clear winner, and all of the techniques are beneficial. In particular, SOP is essential on the two small graphs to achieve orders of magnitude speedup, while ITP is also essential in a number of cases such as on wiki-vote with $k_1 = 4$ and $k_2 = 5$. The four techniques are even more important on large graphs since the program frequently runs beyond 6 hours even when missing one technique. In particular, LAP is essential to allow our four large-graph experiments to finish within 6 hours, followed by BRA for 3 experiments to be within 6 hours, then by ITP for 2 experiments and finally by SOP for 1 experiment. We can see that BK2 with BRA and LAP becomes more important than the graph pruning techniques SOP and ITP on large graphs, though SOP and ITP are more important on small graphs.



(a) uk-2005 (b) webbase-2001

Figure 11: The Running Time (sec) When τ_{time} is Varied.

C.3 Experiments on Load Balancing

Load Balancing. We also conducted experiments to study the impact of τ_{time}. Recall from Section 7 that if a task runs for a period of more than τ_{time}, it will decompose itself so that the decomposed tasks can be processed in parallel. Figure 11 shows the impact of τ_{time} on the job running time, when using 32 threads and varying τ_{time} as 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 100 for two representative settings on our large graphs uk-2005 and webbase-2001. We can see that our default setting of $\tau_{time} = 0.1$ ms does work the best in general. Also, Figure 11(a) shows that if we set τ_{time} too small, too many tasks will be created so the task creation time dominates and the running time can be significantly increased, and Figure 11(b) shows that if we set τ_{time} too large, load balancing suffers so the running time can be many times longer.