EVOLUTIONARY APPROACH FOR ELECTROMAGNETICS IN TIME DOMAIN

Lecture Notes

Part 2: Waveguide Problem

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Chapter 1
INTRODUCTION

Summary 1 *In this chapter the following questions will be answered. Which fundamental boundary value problems are under study in Electromagnetics? Which key-point idea has been proposed for development of the Evolutionary Approach to Electromagnetics in Time Domain in the case of Cavity Problem? How can be modified this idea for development of this approach in the case of the Waveguide Problem?*

In Electromagnetics, there are three canonical boundary value problem with given initial conditions for the electromagnetic field sought, namely: *Cavity Problem, Waveguide Problem,* and *External Problem.* In this Lecture Course, *Part 1* discussed earlier has been devoted to *the Cavity Problem* when the electromagnetic field sought should be studied in *a finite subdomain* of the Euclidean background space.

The Evolutionary Approach to Electromagnetics (EAE) developed in *Part 1* for the Cavity Problem is oriented on study of the electromagnetic fields in Time Domain (TD). It means that it is free of the classical presupposing of steady-state time varying of all electromagnetic quantities at a single frequency, by definition. Though our approach is oriented on Electromagnetics in TD, most of the methods of the classical theory widely developed in Frequency Domain (FD) for the monochromatic fields (and even ready results obtained there) can be nevertheless used directly within the frame of EAE proposed. As it has been proved in *Part 1*, the electromagnetic field in a cavity can be presented as the infinite series where each term is a product of a scalar amplitude depending on time and a vector function of coordinates which is the element of a basis. These basis elements have been specified in a form of the boundary eigenvalue problems for Laplacian. Just the same mathematical problems had appeared and then studied deeply in the course of development of Electromagnetics in FD which is a century old science. Numerous results obtained there can be used in our approach with minimal modifications.

As for the time dependent amplitudes, a system of *ordinary* differential equations has been obtained for them which may be linear or nonlinear as dictated by the constitutive relations for a medium within the cavity. This old and fruitful field of mathematics opens wide possibilities for fast future progress in Electromagnetics in TD.

Second canonical problem in Electromagnetics is *the Waveguide Problem* dealt with a study of electromagnetic field in a subdomain of the Euclidean background space which is *infinite in some direction*. The same idea of splitting of the Maxwell's operator on its self-adjoint part and a remainder may be also used successfully in this case but in slightly different version.

Let us demonstrate it schematically in the simplest case of a shielded waveguide multiconnected for generality in its cross-section of rather arbitrary form but geometrically regular along Oz axis. In this case, it is convenient to split up this operator as follows

$$M = L_{z,t} - W(\mathbf{r}) + A. \tag{1.1}$$

Here, W is a linear matrix operator defined as self-adjoint due to involving in its definition of the boundary conditions over the perfect conducting surface of the waveguide. It acts on the transverse coordinates in the argument of a solution sought $X(\mathbf{r}, z, t)$, where $\mathbf{r}$ is the transverse part of the position vector $\mathbf{R} = \mathbf{r} + \mathbf{z}z$ within the waveguide, and $\mathbf{z}$ is the unit vector oriented along the waveguide axis. Linear matrix operator $L_{z,t}$ involves the partial derivatives on $z-$ coordinate and time. Operator A is a remainder of the input Maxwell's operator M . It may involve the operators from the constitutive relations for a possible medium within the waveguide, and therefore it may by nonlinear in the general case.

Since operator W should be introduced as self-adjoint one, it has a complete set of eigenvectors $\{Y_n(\mathbf{r})\}_{n=-\infty}^{\infty}$ which originates an orthonormal basis in the domain of operator M. Hence, it can be inverted analytically in the system of Maxwell's equations in the form of eigenvector series as

$$X(\mathbf{r}, z, t) = \sum_{n=-\infty}^{\infty} c_n(z, t)\, Y_n(\mathbf{r}), \tag{1.2}$$

where the scalar coefficients $c_n(z, t)$ to be sought, but the vectors $Y_n(\mathbf{r})$ are already known as the elements of the basis.

Formulation of a problem for $\{c_n(z, t)\}_{n=-\infty}^{\infty}$ can be obtained via projecting of Maxwell's equations on the same basis. This procedure results in a system of *partial* differential equations for the scalar coefficients which can be complemented with proper boundary conditions on $z-$ coordinate and initial conditions with respect to time. Thus, this way in EAE development allows to involve the theory of linear and nonlinear partial differential equations in development of Electromagnetics in TD.

The third and last canonical *External Problem* of Electromagnetics regards to study of electromagnetic field in the *unbounded* Euclidean background space. It may be considered as the limiting case of the waveguide problem when a contour of a waveguide transverse section increases and tends to infinity.

Chapter 2
FORMULATION OF THE PROBLEM

2.1 Standard Formulation of the Problem

Summary 2 *In this chapter the following questions will be answered. What is it: geometrically regular waveguide with multiconnected contour of its cross-section? What is it: given functions of impressed forces? Which form have the constitutive relations for electromagnetic field in arbitrary nonlinear dissipative medium? What is the correct formulation of Cauchy Problem in Electromagnetics? How it is possible to separate electromagnetic field vectors onto their transverse and longitudinal parts? How it is possible to separate Maxwell's equations onto their transverse and longitudinal parts? How it is possible to express the boundary conditions over the waveguide surface via transverse components only of the electromagnetic strength vectors sought* $\vec{\mathcal{E}}$ *and* $\vec{\mathcal{H}}$*? What is it: systems of differential equations of the first order for Maxwell's equations in Transversal-Longitudinal Form ? How to obtain the systems of differential equations ofSecond Order for Maxwell's equations in Transversal - Longitudinal Form? What is it: Wave-Boundary Operators? Which form can they have? Which main steps involve the program of the Evolutionary Approach development in the case of Waveguide Problem?*

2.1.1 Description of the Waveguide and Internal Medium

Shielded waveguide under consideration is a $z-$ directed cylindrical volume geometrically uniform along $Oz-$ axis. The waveguide is supposed to be multiconnected, for generality; its cross-section may be an arbitrary function of the transverse waveguide coordinates. The method presented is not only applicable to geometries which can be treated by separation of variables. The contour of cross-section of the shielding waveguide and those of the metallic inserts are closed each and denoted by L_0 and L_i where $i = 1, 2, \ldots, N$, respectively. The surface of waveguide cross-section between the contours L_0 and L_i $i = 1, 2, ..., N$ is denoted by S. The contour each is provided with the sense of rotation in such a way that the surface S is located at the left-hand-side in regard to the contours. The unit vector tangential to L_i $(i = 0, 1, \ldots, N)$ and oriented in the same sense of rotation is represented by $\mathbf{l}_i$. The unit vector normal to L_i $(i = 0, 1, \ldots, N$ as well) and oriented in the out-ward to S direction is denoted by $\mathbf{n}_i$; $\mathbf{z}$ is the unit vector in $+z$ direction. Three mutually orthogonal vectors

$$\{\mathbf{l}_i, \mathbf{n}_i, \mathbf{z}\} \tag{2.1}$$

originate the right-handed base vectors.

Position vector within the waveguide volume is denoted by $\mathbf{R}$. Further on, we will use the following equivalent its representation:

$$\mathbf{R} = \mathbf{r} + \mathbf{z}z, \tag{2.2}$$

where two-component vector $\mathbf{r}$ is the projection of the three-component position vector $\mathbf{R}$ on the waveguide cross-section S. Hence, $\mathbf{r}$ is the position vector within the domain S. Notation $\mathbf{r} \in S$ specifies a location of the position vector $\mathbf{r}$ in the "open domain" S (i.e., within S but outside its boundary L); $\mathbf{r} \in \overline{S}$ means location of $\mathbf{r}$ in the "closed domain" $\bar{S} = S + L$ (i.e., within S including the multiconnected contour L).

The problem is determination of the electromagnetic field strength vectors $\vec{\mathcal{E}}\,(\mathbf{R},t)$ and $\vec{\mathcal{H}}\,(\mathbf{R},t)$ within the waveguide volume as the functions of coordinates and time t. Later on, we will use the following equivalent presentation of the argument of these vector functions:

$$\vec{\mathcal{E}}\,(\mathbf{R},t) \equiv \vec{\mathcal{E}}\,(\mathbf{r},z,t) \qquad \vec{\mathcal{H}}\,(\mathbf{R},t) \equiv \vec{\mathcal{H}}\,(\mathbf{r},z,t)\,. \tag{2.3}$$

Waveguide volume under consideration may be filled with an arbitrary medium. A status of electromagnetic field in such a medium is specified by the constitutive relations of rather general kind like

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}(\vec{\mathcal{E}}), \quad \vec{\mathcal{B}} = \mu_0(\vec{\mathcal{H}} + \vec{\mathcal{M}}(\vec{\mathcal{H}})), \quad \vec{\mathcal{J}} \equiv \vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right), \tag{2.4}$$

where ϵ_0, μ_0 being free-space constants. Electric polarization $\vec{\mathcal{P}}(\vec{\mathcal{E}})$, magnetization $\vec{\mathcal{M}}(\vec{\mathcal{H}})$, and volume density of electric conduction current $\vec{\mathcal{J}}_\sigma(\vec{\mathcal{E}},\vec{\mathcal{H}})$ are *some given functions* of the vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ to be sought for.

Impressed sources may be also introduced within the waveguide volume. In such a case, they should be specified by given functions $\vec{\mathcal{J}}_e(\mathbf{R},t)$, and $\vec{\mathcal{J}}_h(\mathbf{r},t)$ as external parts of the volume densities of electric and magnetic currents, respectively. Volume densities of electric $\rho_e(\mathbf{r},t)$ and magnetic $\rho_h(\mathbf{r},t\,)$ charges can be found when needed via solving the equation of continuity with given their right-hand-sides, namely:

$$\partial_t\rho_{e,h} = -div\vec{\mathcal{J}}_{e,h}. \tag{2.5}$$

Volume density of electric charge of free charge carriers ρ_σ and the volume density of electric conduction current $\vec{\mathcal{J}}_\sigma$ are subjected to the same continuity equation.

2.1.2 Cauchy Problem for the System of Maxwell's Equations

We deal with the inner boundary value problem which should be solved for the system of Maxwell's equations

$$\begin{aligned} rot\vec{\mathcal{H}} &= \partial_t\vec{\mathcal{D}} + \vec{\mathcal{J}}_\sigma + \vec{\mathcal{J}}_e, \\ -rot\vec{\mathcal{E}} &= \partial_t\vec{\mathcal{B}} + \vec{\mathcal{J}}_h; \end{aligned} \tag{2.6}$$

$$\vec{\mathcal{H}}_{norm}(\mathbf{r},z,t)=0,\quad \vec{\mathcal{E}}_{tg}(\mathbf{r},z,t)=0 \quad \textit{on the waveguide surface;} \tag{2.7}$$

$$div\vec{\mathcal{D}}=\rho_\sigma+\rho_e,$$
$$div\vec{\mathcal{B}}=\rho_h \tag{2.8}$$

with respect to unknowns $\vec{\mathcal{E}}\equiv\vec{\mathcal{E}}(\mathbf{r},z,t)$, and $\vec{\mathcal{H}}\equiv\vec{\mathcal{H}}(\mathbf{r},z,t)$. The latter must satisfy the following initial conditions imposed at $t=0$:

$$\vec{\mathcal{E}}(\mathbf{r},z,0)=\vec{\mathcal{E}}_0(\mathbf{r},z),\quad \vec{\mathcal{H}}(\mathbf{r},z,0)=\vec{\mathcal{H}}_0(\mathbf{r},z), \tag{2.9}$$

where $\vec{\mathcal{E}}_0(\mathbf{r},z)$, and $\vec{\mathcal{H}}_0(\mathbf{r},z)$ are given functions within the waveguide volume.

Solution to the problem (2.4) – (2.9) should be found in the class of quadratically integrable complex-valued vector functions satisfying the condition

$$\int_{t_1}^{t_2}dt\int_{z_1}^{z_2}dz\int_{S'}(\epsilon_0\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}^*+\mu_0\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}^*)ds<\infty, \tag{2.10}$$

where $S'\subseteq\bar{S}$, the *dot* $(\cdot)$ stands for scalar product in common sense of vector algebra, the asterisk denotes complex conjugation.

2.1.3 Auxiliary Identical Reorganization

The polarization and magnetization vectors from the constitutive equations (2.10) one can separate onto their linear parts and the remainders as follows

$$\vec{\mathcal{P}}\left(\vec{\mathcal{E}}\right)=\epsilon_0\alpha(z,t)\vec{\mathcal{E}}+\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right);\qquad \vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right)=\chi(z,t)\vec{\mathcal{H}}+\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right). \tag{2.11}$$

It is evidently that the remainders

$$\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)=\vec{\mathcal{P}}\left(\vec{\mathcal{E}}\right)-\epsilon_0\alpha(z,t)\vec{\mathcal{E}};\qquad \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)=\vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right)-\chi(z,t)\vec{\mathcal{H}} \tag{2.12}$$

may involve as the nonlinear parts as some linear parts also with the coefficients $\tilde{\alpha}$ and $\tilde{\chi}$ depending on the transverse coordinates. While Eqs. (2.11) hold, the constitutive equations for the electric displacement and magnetic induction vectors acquire the forms of

$$\vec{\mathcal{D}}\left(\vec{\mathcal{E}}\right)=\epsilon_0\varepsilon(z,t)\vec{\mathcal{E}}+\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right);\qquad \vec{\mathcal{B}}\left(\vec{\mathcal{H}}\right)=\mu_0\mu(z,t)\vec{\mathcal{H}}+\mu_0\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right), \tag{2.13}$$

where

$$\varepsilon(z,t)=1+\alpha(z,t);\qquad \mu(z,t)=1+\chi(z,t). \tag{2.14}$$

In a particular case, when $\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)=\mathbf{0}$ and $\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)=\mathbf{0}$, constitutive relations (2.13) look like

$$\vec{\mathcal{D}}=\epsilon_0\varepsilon(z,t)\vec{\mathcal{E}}\quad\text{and}\quad\vec{\mathcal{B}}=\mu_0\mu(z,t)\vec{\mathcal{H}}, \tag{2.15}$$

which correspond to the waveguide filled with a linear nonstationary medium layered in the waveguide axis direction. Later on, we will suppose that

$$\varepsilon(z,t) \geq c_1 > 0, \qquad \mu(z,t) \geq c_2 > 0, \tag{2.16}$$

where c_1 and c_2 are some positive constants.

After substitution of Eqs. (2.13) in *curl* Maxwell's equations (2.6) they acquire the form of

$$\begin{aligned} rot\vec{\mathcal{H}} &= \epsilon_0 \partial_t \left\{ \varepsilon(z,t)\vec{\mathcal{E}} \right\} + \left\{ \partial_t \vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right) + \vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right) + \vec{\mathcal{J}}_e \right\}, \\ -rot\vec{\mathcal{E}} &= \mu_0 \partial_t \left\{ \mu(z,t)\vec{\mathcal{H}} \right\} + \left\{ \partial_t \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right) + \vec{\mathcal{J}}_h \right\}; \end{aligned} \tag{2.17}$$

The same substitution in *div* Maxwell's equations (2.7) results in

$$\begin{aligned} \epsilon_0 div \left\{ \varepsilon(z,t)\vec{\mathcal{E}}\left(\vec{\mathcal{R}},t\right) \right\} &= -div\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right) + \rho_\sigma + \rho_e, \\ \mu_0 div \left\{ \mu(z,t)\vec{\mathcal{H}}\left(\vec{\mathcal{R}},t\right) \right\} &= -div\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right) + \rho_h \end{aligned} \tag{2.18}$$

Right-hand-sides of Eqs. (2.17) and (2.18) suggest to introduce the equivalent densities of electric and magnetic currents and charges as follows

$$\begin{aligned} \vec{\mathcal{J}} &= \partial_t \vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right) + \vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right) + \vec{\mathcal{J}}_e, & \vec{\mathcal{I}} &= \partial_t \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right) + \vec{\mathcal{J}}_h; \\ \varrho &= -div\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right) + \rho_\sigma + \rho_e, & g &= -div\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right) + \rho_h. \end{aligned} \tag{2.19}$$

In term of these notation, *curl* equations can be written in a compact form as

$$rot\vec{\mathcal{H}} = \epsilon_0 \partial_t \left(\varepsilon\vec{\mathcal{E}}\right) + \vec{\mathcal{J}}; \qquad -rot\vec{\mathcal{E}} = \mu_0 \partial_t \left(\mu\vec{\mathcal{H}}\right) + \vec{\mathcal{I}}, \tag{2.20}$$

and *div* equations become very simple as well

$$\epsilon_0 div\left(\varepsilon\vec{\mathcal{E}}\right) = \varrho; \qquad \mu_0 div\left(\mu\vec{\mathcal{H}}\right) = g. \tag{2.21}$$

2.2 Maxwell's Equations in Transversal-Longitudinal Form

2.2.1 Separation of the Vectors on Transversal and Longitudinal Parts

Three-component electromagnetic field strength vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ each can be presented as a sum of two-component and one-component vectors as follows

$$\begin{aligned} \vec{\mathcal{E}}(\mathbf{R},t) &\equiv \vec{\mathcal{E}}(\mathbf{r},z,t) = \mathbf{E}(\mathbf{r},z,t) + \mathbf{z}E_z(\mathbf{r},z,t); \\ \vec{\mathcal{H}}(\mathbf{R},t) &\equiv \vec{\mathcal{H}}(\mathbf{r},z,t) = \mathbf{H}(\mathbf{r},z,t) + \mathbf{z}H_z(\mathbf{r},z,t). \end{aligned} \tag{2.22}$$

Vectors $\mathbf{E}$ and $\mathbf{H}$ are projections of the vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ on the waveguide cross-section, respectively, and scalars E_z and H_z are projections of the same vectors on $Oz-$ axis.

In a similar way, the vectors from Eqs. (2.19) can be presented as

$$\vec{\mathcal{J}} = \mathbf{J} + \mathbf{z}J_z, \qquad \vec{\mathcal{I}} = \mathbf{I} + \mathbf{z}I_z \tag{2.23}$$

Further on, we are going to rearrange Maxwell's equations to some form convenient for solving of the waveguide problem under consideration. To this aim, presentation of the nabla operator in the form of

$$\nabla = \nabla_\perp + \mathbf{z}\partial_z \tag{2.24}$$

will be useful, where $\nabla_\perp$ acts on space variables in the domain S of waveguide cross-section.

2.2.2 Systems of Differential Equations of the First Order

In terms of notation (2.24), the first *div* equation from Eqs. (2.21) may be rewritten as follows

$$\begin{aligned} \epsilon_0 div\left(\varepsilon\vec{\mathcal{E}}\right) &\equiv \epsilon_0\nabla\cdot\left(\varepsilon\vec{\mathcal{E}}\right) = \epsilon_0\left(\nabla_\perp + \mathbf{z}\partial_z\right)\cdot\left\{\varepsilon\left(z,t\right)\left(\mathbf{E} + \mathbf{z}E_z\right)\right\} \\ &\equiv \epsilon_0\varepsilon\left(z,t\right)\nabla_\perp\cdot\mathbf{E}\left(\mathbf{r},z,t\right) + \epsilon_0\partial_z\left\{\varepsilon\left(z,t\right)E_z\left(\mathbf{r},z,t\right)\right\} = \varrho. \end{aligned} \tag{2.25}$$

In the course of these manipulation, we have used a fact that $\nabla_\perp\varepsilon\left(z,t\right) \equiv \varepsilon\left(z,t\right)\nabla_\perp$, since operator $\nabla_\perp$ acts on transverse variables only and therefore $\varepsilon\left(z,t\right)$ looks as a constant with respect to these variables. Thus, from the last line of Eq. (2.25) one can get

$$\epsilon_0\partial_z\left(\varepsilon E_z\right) = -\epsilon_0\varepsilon\nabla_\perp\cdot\mathbf{E} + \varrho \tag{2.26}$$

The same manipulations with the second *div* equation from Eqs. (2.21) as

$$\mu_0 div\left(\mu\vec{\mathcal{H}}\right) \equiv \mu_0\mu\left(z,t\right)\nabla_\perp\cdot\mathbf{H}\left(\mathbf{r},z,t\right) + \mu_0\partial_z\left\{\mu\left(z,t\right)H_z\left(\mathbf{r},z,t\right)\right\} = g$$

result in

$$\mu_0\partial_z\left(\mu H_z\right) = -\mu_0\mu\nabla_\perp\cdot\mathbf{H} + g \tag{2.27}$$

Now, we will take up rearranging of *curl* Maxwell's equations (2.20) via projecting them on cross-section of waveguide and on waveguide axis as well. Pair of *curl* equations (2.20) can be rewritten as

$$\begin{gathered} rot\vec{\mathcal{H}} \equiv \left[\left(\nabla_\perp + \mathbf{z}\partial_z\right)\times\left(\mathbf{H} + \mathbf{z}H_z\right)\right] = \left\{\epsilon_0\partial_t\left(\varepsilon\mathbf{E}\right) + \mathbf{J}\right\} + \mathbf{z}\left\{\epsilon_0\partial_t\varepsilon E_z + J_z\right\}, \\ -rot\vec{\mathcal{E}} \equiv -\left[\left(\nabla_\perp + \mathbf{z}\partial_z\right)\times\left(\mathbf{E} + \mathbf{z}E_z\right)\right] = \left\{\mu_0\partial_t\left(\mu\mathbf{H}\right) + \mathbf{I}\right\} + \mathbf{z}\left\{\mu_0\partial_t\mu H_z + I_z\right\}. \end{gathered} \tag{2.28}$$

At this point, we need calculation of $rot\vec{\mathcal{A}}$, where three-component vector $\vec{\mathcal{A}}$ is presented as a sum of two-component vector $\mathbf{A}$ and projection of $\vec{\mathcal{A}}$ on $Oz-$axis, i.e., $\vec{\mathcal{A}} = \mathbf{A} + \mathbf{z}A_z$. Formal procedure of the calculations looks as

$$\begin{aligned} rot\vec{\mathcal{A}} &\equiv [(\nabla_\perp + \mathbf{z}\partial_z) \times (\mathbf{A} + \mathbf{z}A_z)] \\ &= [\nabla_\perp \times \mathbf{A}] + [\nabla_\perp \times \mathbf{z}A_z] + [\mathbf{z}\partial_z \times \mathbf{A}] + [\mathbf{z}\partial_z \times \mathbf{z}A_z] \,. \end{aligned}$$

However, it is evident that $[\mathbf{z}\partial_z \times \mathbf{z}A_z] \equiv 0$. One can easily verify that $[\nabla_\perp \times \mathbf{z}A_z] \equiv [\nabla_\perp A_z \times \mathbf{z}]$. Therefore $[\nabla_\perp \times \mathbf{z}A_z]$ may be rewritten as $[\nabla_\perp \times \mathbf{z}]\, A_z$ in such a sense. Later on, we will use sometimes evident identity $[\nabla_\perp \times \mathbf{z}] \equiv -[\mathbf{z} \times \nabla_\perp]$ as well. Thus,

$$rot\vec{\mathcal{A}} \equiv [(\nabla_\perp + \mathbf{z}\partial_z) \times (\mathbf{A} + \mathbf{z}A_z)] = [\nabla_\perp \times \mathbf{A}] + [\nabla_\perp \times \mathbf{z}]\, A_z + \partial_z [\mathbf{z} \times \mathbf{A}] \,. \tag{2.29}$$

With making use of procedure (2.29), *curl* equations (2.28) can be rearranged as follows

$$\begin{aligned} rot\vec{\mathcal{H}} &\equiv [\nabla_\perp \times \mathbf{H}] + [\nabla_\perp \times \mathbf{z}]\, H_z + \partial_z [\mathbf{z} \times \mathbf{H}] \\ &= \epsilon_0 \partial_t (\varepsilon \mathbf{E} + \mathbf{J}) + \mathbf{z}\,(\epsilon_0 \partial_t \varepsilon E_z + J_z)\,; \\ -rot\vec{\mathcal{E}} &\equiv -[\nabla_\perp \times \mathbf{E}] + [\mathbf{z} \times \nabla_\perp]\, E_z + \partial_z [\mathbf{E} \times \mathbf{z}] \\ &= \mu_0 \partial_t (\mu \mathbf{H} + \mathbf{I}) + \mathbf{z}\,(\mu_0 \partial_t \mu H_z + I_z)\,. \end{aligned} \tag{2.30}$$

Scalar multiplication of the first equation in Eqs. (2.30) by the unit vector $\mathbf{z}$ supplies

$$\mathbf{z} \cdot rot\vec{\mathcal{H}} = \mathbf{z} \cdot [\nabla_\perp \times \mathbf{H}] = \epsilon_0 \partial_t \varepsilon E_z + J_z \tag{2.31}$$

It means that projection of the first *curl* equation on the waveguide cross-section can be written as

$$[\nabla_\perp \times \mathbf{z}]\, H_z + \partial_z [\mathbf{z} \times \mathbf{H}] = \epsilon_0 \partial_t \varepsilon \mathbf{E} + \mathbf{J} \tag{2.32}$$

In just the same way, one can project the second equation from Eqs. (2.30) on $Oz-$ axis as

$$-\mathbf{z} \cdot rot\vec{\mathcal{E}} = -\mathbf{z} \cdot [\nabla_\perp \times \mathbf{E}] = \mu_0 \partial_t \mu H_z + I_z, \tag{2.33}$$

and obtain its projection on the waveguide cross-section in the form of

$$[\mathbf{z} \times \nabla_\perp]\, E_z + \partial_z [\mathbf{E} \times \mathbf{z}] = \mu_0 \partial_t \mu \mathbf{H} + \mathbf{I}. \tag{2.34}$$

Equations (2.32), (2.33) and (2.27) can be written as a joint system of equations with respect to the field component $H_z \equiv H_z(\mathbf{r}, z, t)$ sought for as follows

$$\boxed{\begin{array}{lr} [\nabla_\perp \times \mathbf{z}]\, H_z = \epsilon_0 \partial_t (\varepsilon \mathbf{E}) + \partial_z [\mathbf{H} \times \mathbf{z}] + \mathbf{J}, & (a) \\ \mu_0 \partial_z (\mu H_z) = -\mu_0 \mu \nabla_\perp \cdot \mathbf{H} + g, & (b) \\ \mu_0 \partial_t (\mu H_z) = \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}] - I_z. & (c) \end{array}} \tag{2.35}$$

In the course of manipulations with Eq. (2.33) resulted in equation (b) the following well known identities from vector algebra were used:

$$\begin{gathered}\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A};\\ [\mathbf{A}\times\mathbf{B}]=-[\mathbf{B}\times\mathbf{A}];\\ [\mathbf{A}\times\mathbf{B}]\cdot\mathbf{C}=[\mathbf{B}\times\mathbf{C}]\cdot\mathbf{A}=[\mathbf{C}\times\mathbf{A}]\cdot\mathbf{B}.\end{gathered} \tag{2.36}$$

Equations (2.34), (2.31) and (2.26) originate one more joint system of equations with respect to the other longitudinal field component $E_z \equiv E_z(\mathbf{r},z,t)$ sought for as

$$\boxed{\begin{aligned} &[\mathbf{z}\times\nabla_\perp]\,E_z=\mu_0\partial_t(\mu\mathbf{H})+\partial_z[\mathbf{z}\times\mathbf{E}]+\mathbf{I}, && (a)\\ &\epsilon_0\partial_t(\varepsilon E_z)=\nabla_\perp\cdot[\mathbf{H}\times\mathbf{z}]-J_z, && (b)\\ &\epsilon_0\partial_z(\varepsilon E_z)=-\epsilon_0\varepsilon\nabla_\perp\cdot\mathbf{E}+\varrho. && (c)\end{aligned}} \tag{2.37}$$

2.2.3 Systems of Differential Equations of Second Order

Left-hand-sides of equations (a) in the systems (2.35) and (2.37) involve $z-$ components of electromagnetic field sought for. These functions can be excluded from the vector equations (a) with making use of respective equations (b) and (c). Let us demonstrate it for the system of equation (2.35) in detail and exhibit then final result for the system (2.37).

Equation (a) can be shortly written as $[\nabla_\perp\times\mathbf{z}]\,H_z=\mathbf{F}_H$ with the notation

$$\mathbf{F}_H=\epsilon_0\partial_t(\varepsilon\mathbf{E})+\partial_z[\mathbf{H}\times\mathbf{z}]+\mathbf{J}. \tag{2.38}$$

Let us first subject this equation to the action of the operator $\mu_0\partial_z\mu$ as

$$\mu_0\partial_z\mu\updownarrow[\nabla_\perp\times\mathbf{z}]\,H_z=\mathbf{F}_H.$$

This procedure results in a new equation, namely:

$$\mu_0\partial_z\{\mu[\nabla_\perp\times\mathbf{z}]\,H_z\}=\mu_0\partial_z\{\mu\mathbf{F}_H\}.$$

Since $\nabla_\perp$ operator acts on the transverse coordinates only but $\mu\equiv\mu(z,t)$ is supposed*, next identity holds $\mu_0\partial_z\{\mu[\nabla_\perp\times\mathbf{z}]\,H_z\}\equiv[\nabla_\perp\times\mathbf{z}]\{\mu_0\partial_z(\mu H_z)\}$. So, we have arrived at the following equation:

$$[\nabla_\perp\times\mathbf{z}]\{\mu_0\partial_z(\mu H_z)\}=\mu_0\partial_z(\mu\mathbf{F}_H). \tag{2.39}$$

Making use of equation (b) as a direct formula, we can substitute right-hand-side of equation (b) at the left-hand-side of Eq. (2.39)

$$[\nabla_\perp\times\mathbf{z}]\{-\mu_0\mu\nabla_\perp\cdot\mathbf{H}+g\}=\mu_0\partial_z\{\mu\mathbf{F}_H\}.$$

*see Eq. (2.14)

Left-hand-side of this equation may be rewritten as

$$[\nabla_\perp \times \mathbf{z}]\{-\mu_0\mu\nabla_\perp \cdot \mathbf{H} + g\} = \mu_0\mu[\mathbf{z}\times\nabla_\perp]\nabla_\perp \cdot \mathbf{H} + [\nabla_\perp \times \mathbf{z}]\,g.$$

Final result of these manipulations are next

$$[\mathbf{z}\times\nabla_\perp]\nabla_\perp \cdot \mathbf{H} = \mu^{-1}\partial_z\{\mu\mathbf{F}_H\} + (\mu_0\mu)^{-1}[\mathbf{z}\times\nabla_\perp g]\,. \tag{2.40}$$

Let us now subject equation (a) to the action of the operator $\mu_0\partial_t\mu$ as

$$\mu_0\partial_t\mu \updownarrow [\nabla_\perp \times \mathbf{z}]\,H_z = \mathbf{F}_H.$$

This procedure results in the following equation:

$$[\nabla_\perp \times \mathbf{z}]\{\mu_0\partial_t(\mu H_z)\} = \mu_0\partial_t(\mu\mathbf{F}_H)\,.$$

At this point, we can use equation (c) as a direct formula what yields

$$[\nabla_\perp \times \mathbf{z}]\{\nabla_\perp \cdot [\mathbf{z}\times\mathbf{E}] - I_z\} = \mu_0\partial_t(\mu\mathbf{F}_H)\,.$$

This equation may be rewritten as

$$[\nabla_\perp\phi \times \mathbf{z}] = \mu_0\partial_t(\mu\mathbf{F}_H) + [\nabla_\perp I_z \times \mathbf{z}]\,, \tag{2.41}$$

where a scalar $\phi = \nabla_\perp \cdot [\mathbf{z}\times\mathbf{E}]$ is denoted. Vector multiplication of Eq. (2.41) by the unit vector $\mathbf{z}$ as

$$\mathbf{z}\times \updownarrow [\nabla_\perp\phi \times \mathbf{z}] = \mu_0\partial_t(\mu\mathbf{F}_H) + [\nabla_\perp I_z \times \mathbf{z}]$$

results in

$$\nabla_\perp\phi = \mu_0\partial_t\{\mu[\mathbf{z}\times\mathbf{F}_H]\} + \nabla_\perp I_z$$

in accordance with the identity

$$\mathbf{z}\times[\nabla_\perp\phi\times\mathbf{z}] = (\mathbf{z}\cdot\mathbf{z})\nabla_\perp\phi - \mathbf{z}(\mathbf{z}\cdot\nabla_\perp\phi) = \nabla_\perp\phi \tag{2.42}$$

where vectors $\mathbf{z}$ and $\nabla_\perp\phi$ are orthogonal. Using the identities (2.36), the scalar ϕ can be rearranged as follows

$$\phi = \nabla_\perp \cdot [\mathbf{z}\times\mathbf{E}] = [\nabla_\perp \times \mathbf{z}] \cdot \mathbf{E} = -[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E}.$$

Final result looks like

$$\nabla_\perp[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E} = -\mu_0\partial_t\{\mu[\mathbf{z}\times\mathbf{F}_H]\} - \nabla_\perp I_z. \tag{2.43}$$

Later on, we will consider a pair of two-component vector equations obtained, namely (2.40) and (2.43), as one four-component vector equation, namely:

$$\boxed{\begin{pmatrix} \epsilon_0^{-1}\left[\mathbf{z}\times\nabla_\perp\right]\nabla_\perp\cdot\mathbf{H} \\ \\ \mu_0^{-1}\nabla_\perp\left[\mathbf{z}\times\nabla_\perp\right]\cdot\mathbf{E} \end{pmatrix} = \begin{pmatrix} \mu^{-1}\partial_z\left\{\mu\epsilon_0^{-1}\mathbf{F}_H\right\}+\left(\epsilon_0\mu_0\mu\right)^{-1}\left[\mathbf{z}\times\nabla_\perp g\right] \\ \\ -\partial_t\left\{\mu\left[\mathbf{z}\times\mathbf{F}_H\right]\right\}-\mu_0^{-1}\nabla_\perp I_z \end{pmatrix}}. \tag{2.44}$$

Right-hand-side of equation (a) in the system (2.37) is denoted as

$$\mathbf{F}_E = \mu_0\partial_t\left(\mu\mathbf{H}\right)+\partial_z\left[\mathbf{z}\times\mathbf{E}\right]+\mathbf{I}. \tag{2.45}$$

Exclusion of E_z from equation (a) with using of equation (b) yields

$$\nabla_\perp\left[\nabla_\perp\times\mathbf{z}\right]\cdot\mathbf{H} = -\epsilon_0\partial_t\left\{\varepsilon\left[\mathbf{F}_E\times\mathbf{z}\right]\right\}-\nabla_\perp J_z.$$

Similar procedure with using of equation (c) supplies

$$\left[\nabla_\perp\times\mathbf{z}\right]\nabla_\perp\cdot\mathbf{E} = \varepsilon^{-1}\partial_z\left(\varepsilon\mathbf{F}_E\right)+\left(\epsilon_0\varepsilon\right)^{-1}\left[\nabla_\perp\varrho\times\mathbf{z}\right].$$

These results rearranged to the form of four-component vector equation look like

$$\boxed{\begin{pmatrix} \epsilon_0^{-1}\nabla_\perp\left[\nabla_\perp\times\mathbf{z}\right]\cdot\mathbf{H} \\ \\ \mu_0^{-1}\left[\nabla_\perp\times\mathbf{z}\right]\nabla_\perp\cdot\mathbf{E} \end{pmatrix} = \begin{pmatrix} -\partial_t\left\{\varepsilon\left[\mathbf{F}_E\times\mathbf{z}\right]\right\}-\epsilon_0^{-1}\nabla_\perp J_z \\ \\ \varepsilon^{-1}\partial_z\left(\varepsilon\mu_0^{-1}\mathbf{F}_E\right)+\left(\epsilon_0\mu_0\varepsilon\right)^{-1}\left[\nabla_\perp\varrho\times\mathbf{z}\right] \end{pmatrix}}. \tag{2.46}$$

2.2.4 Boundary Conditions

In terms of notation (2.1), the first boundary condition from Eqs. (2.7) should be read as

$$\vec{\mathcal{H}}_{norm}|_L = 0 \;\;\Rightarrow\;\; \left(\mathbf{n}_i\cdot\vec{\mathcal{H}}\right)|_{L_i} = 0, \quad i = 0,1,\ldots,N$$

in the case under consideration of the multiconnected contour L of waveguide cross-section. After substitution of the field $\vec{\mathcal{H}}$ presentation in the form of (2.22), the scalar product can be simplified to

$$\left(\mathbf{n}_i\cdot\vec{\mathcal{H}}\right) = \left(\mathbf{n}_i\cdot\left(\mathbf{H}+\mathbf{z}\mathbf{H}_z\right)\right) = \left(\mathbf{n}_i\cdot\mathbf{H}\right).$$

So, the first boundary condition from Eqs. (2.7) has the form of

$$\vec{\mathcal{H}}_{norm}|_L = 0 \;\;\Rightarrow\;\; \left(\mathbf{n}_i\cdot\mathbf{H}\right)|_{L_i} = 0, \quad i = 0,1,\ldots,N. \tag{2.47}$$

The second boundary condition from the same Eqs. (2.7) can be rewritten as

$$\vec{\mathcal{E}}_{tg}|_L = \mathbf{0} \;\;\Rightarrow\;\; \left(\mathbf{l}_i\cdot\mathbf{E}\right)|_{L_i} = 0, \quad E_z|_{L_i} = 0; \;\; i = 0,1,\ldots,N \tag{2.48}$$

in terms of notation (2.1) and (2.22). Equations (2.47) and (2.48) supply the first pair of boundary conditions for the transverse parts of the electromagnetic field sought

$$\boxed{\mathbf{n}_i \cdot \mathbf{H}(\mathbf{r}, z, t) = 0, \quad \mathbf{l}_i \cdot \mathbf{E}(\mathbf{r}, z, t) = 0; \qquad \mathbf{r} \in L_i, \quad i = 0, 1, \ldots, N.} \tag{2.49}$$

The boundary condition

$$E_z|_{L_i} = 0 \tag{2.50}$$

from Eq. (2.48) may be also rewritten in terms of the transverse field components $\mathbf{E}$ and $\mathbf{H}$. To this aim, we should impose the following condition on the function of impressed forces and the constitutive relations:

$$J_z(\mathbf{r}, z, t)\,|_{L_i} = 0, \qquad \varrho(\mathbf{r}, z, t)\,|_{L_i} = 0, \qquad \mathbf{r} \in L_i, \quad i = 0, 1, \ldots, N. \tag{2.51}$$

Under these conditions, equations (b) and (c) from the system (2.36) hold on the waveguide surface in the form of

$$\epsilon_0 \partial_t (\varepsilon E_z) = \nabla_\perp \cdot [\mathbf{H} \times \mathbf{z}], \quad \epsilon_0 \partial_z (\varepsilon E_z) = -\epsilon_0 \varepsilon \nabla_\perp \cdot \mathbf{E}; \qquad \mathbf{r} \in L_i. \tag{2.52}$$

Substitution of Eq. (2.50) in the equations (2.52) yields one more pair of the boundary conditions for the field components $\mathbf{E}$ and $\mathbf{H}$ as

$$(\nabla_\perp \cdot \mathbf{E}(\mathbf{r}))\,|_{L_i} \qquad (\nabla_\perp \cdot [\mathbf{H}(\mathbf{r}) \times \mathbf{z}])\,|_{L_i} \qquad i = 0, 1, \ldots, N. \tag{2.53}$$

Later on, we will prove that the boundary conditions for the derivatives of the transverse field components (2.53) are the sequence of the boundary conditions (2.49) for these vectors themselves.

2.2.5 Wave-Boundary Operators (WBO)

Left-hand-side of Eq. (2.44) can be rewritten as a product of a 4×4 matrix differential procedure $\hat{W}_H$ and 4-component transverse electromagnetic vector-column X

$$\begin{aligned} &\begin{pmatrix} \epsilon_0^{-1} [\mathbf{z} \times \nabla_\perp] \nabla_\perp \cdot \mathbf{H} \\ \mu_0^{-1} \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E} \end{pmatrix} = \\ &= \begin{pmatrix} \mathcal{O} & \epsilon_0^{-1} [\mathbf{z} \times \nabla_\perp] \nabla_\perp \cdot \\ \mu_0^{-1} \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot & \mathcal{O} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \equiv \hat{W}_H X, \end{aligned} \tag{2.54}$$

where $\mathcal{O}$ means 2×2 zero matrix,

$$X = col\,(\mathbf{E}, \mathbf{H}) \equiv col\,(E_1, E_2, H_1, H_2), \tag{2.55}$$

here *col* stands for "column", projections of the two-component vectors $\mathbf{E} \equiv \mathbf{E}(\mathbf{r}, z, t)$ and $\mathbf{H} \equiv \mathbf{H}(\mathbf{r}, z, t)$ on the base vectors at the domain S of waveguide cross-section are specified in subscription notation.

Example 3 *Let us calculate this* 4×4 *matrix differential procedure* $\hat{W}_H$ *for a particular case of Cartesian coordinate system.*

1. $\mathbf{H}=\mathbf{x}H_x+\mathbf{y}H_y$, $\mathbf{E}=\mathbf{x}E_x+\mathbf{y}E_y$, $\nabla_\perp=\mathbf{x}\frac{\partial}{\partial x}+\mathbf{y}\frac{\partial}{\partial y}$;

2. $\nabla_\perp\cdot\mathbf{H}=\frac{\partial}{\partial x}H_x+\frac{\partial}{\partial y}H_y\equiv\phi$;

3. $$\begin{aligned}\underline{[\mathbf{z}\times\nabla_\perp]\,\nabla_\perp\cdot\mathbf{H}}&=[\mathbf{z}\times\nabla_\perp]\,\phi=[\mathbf{z}\times\nabla_\perp\phi]=\mathbf{x}\left(-\frac{\partial}{\partial y}\phi\right)+\mathbf{y}\left(\frac{\partial}{\partial x}\phi\right)\\&=\underline{\mathbf{x}\left(-\frac{\partial^2}{\partial y\partial x}H_x-\frac{\partial^2}{\partial y^2}H\right)+\mathbf{y}\left(\frac{\partial^2}{\partial x^2}H_x+\frac{\partial^2}{\partial x\partial y}H_y\right)};\end{aligned}$$

4. $[\mathbf{z}\times\nabla_\perp]=\mathbf{x}\left(-\frac{\partial}{\partial y}\right)+\mathbf{y}\frac{\partial}{\partial x}$; $[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E}=-\frac{\partial}{\partial y}E_x+\frac{\partial}{\partial x}E_y\equiv\psi$;

5. $$\begin{aligned}\underbrace{\nabla_\perp[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E}}&=\nabla_\perp\psi=\mathbf{x}\frac{\partial}{\partial x}\psi+\mathbf{y}\frac{\partial}{\partial y}\psi\\&=\underbrace{\mathbf{x}\left(-\frac{\partial^2}{\partial x\partial y}E_x+\frac{\partial^2}{\partial x^2}E_y\right)+\mathbf{y}\left(-\frac{\partial^2}{\partial y^2}E_x+\frac{\partial^2}{\partial y\partial x}E_y\right)};\end{aligned}$$

6. $$\hat{W}_HX=\begin{pmatrix}\epsilon_0^{-1}\underline{[\mathbf{z}\times\nabla_\perp]\,\nabla_\perp\cdot\mathbf{H}}\\ \mu_0^{-1}\underbrace{\nabla_\perp[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E}}\end{pmatrix}=\begin{pmatrix}\epsilon_0^{-1}\left(-\frac{\partial^2}{\partial y\partial x}H_x-\frac{\partial^2}{\partial y^2}H_y\right)\\ \epsilon_0^{-1}\left(\frac{\partial^2}{\partial x^2}H_x+\frac{\partial^2}{\partial x\partial y}H_y\right)\\ \mu_0^{-1}\left(-\frac{\partial^2}{\partial x\partial y}E_x+\frac{\partial^2}{\partial x^2}E_y\right)\\ \mu_0^{-1}\left(-\frac{\partial^2}{\partial y^2}E_x+\frac{\partial^2}{\partial y\partial x}E_y\right)\end{pmatrix};$$

7. One can easily verify that final result of calculations obtained at p.6 is equivalent to multiplication of the following matrix differential procedure and the transverse 4-component electromagnetic vector:

$$\hat{W}_HX\equiv\begin{pmatrix}0&0&-\epsilon_0^{-1}\frac{\partial^2}{\partial y\partial x}&-\epsilon_0^{-1}\frac{\partial^2}{\partial y^2}\\ 0&0&\epsilon_0^{-1}\frac{\partial^2}{\partial x^2}&\epsilon_0^{-1}\frac{\partial^2}{\partial x\partial y}\\ -\mu_0^{-1}\frac{\partial^2}{\partial x\partial y}&\mu_0^{-1}\frac{\partial^2}{\partial x^2}&0&0\\ -\mu_0^{-1}\frac{\partial^2}{\partial y^2}&\mu_0^{-1}\frac{\partial^2}{\partial y\partial x}&0&0\end{pmatrix}\begin{pmatrix}E_x\\E_y\\H_x\\H_y\end{pmatrix}. \tag{2.56}$$

Matrix differential procedure $\hat{W}_H$ holds in the open domain S. At its boundary L, electromagnetic field sought should be subjected to the boundary conditions (2.49) and (2.53). We'll introduce now an operator which acts on the four-component vector functions of transverse coordinates in *the closed domain* $\bar{S}$ of the waveguide cross-section. To this aim, we combine the matrix differential procedure

$$\hat{W}_H=\begin{pmatrix}\mathcal{O}&\epsilon_0^{-1}[\mathbf{z}\times\nabla_\perp]\,\nabla_\perp\cdot\\ \mu_0^{-1}\nabla_\perp[\mathbf{z}\times\nabla_\perp]\cdot&\mathcal{O}\end{pmatrix} \tag{2.57}$$

introduced above in Eq. (2.54) and the first pair of the boundary conditions[†] into one *"Wave-Boundary Operator"* (*WBO*, for short) as follows

$$\boxed{W_H X = \begin{cases} \hat{W}_H X, & \textit{while } \mathbf{r} \in S; \\ \\ (\mathbf{n}_i \cdot \mathbf{H}) = 0, \;\; (\mathbf{l}_i \cdot \mathbf{E}) = 0, & \textit{while } \mathbf{r} \in L_0, L_1, \ldots, L_N. \end{cases}} \tag{2.58}$$

In terms of these notation, the vector equation (2.44) together with the boundary conditions acquire a form of the following operator equation:

$$W_H X = \begin{pmatrix} \frac{1}{\mu}\partial_z \mu \left\{ \partial_t \varepsilon \mathbf{E} + \epsilon_0^{-1} \partial_z \left[\mathbf{H} \times \mathbf{z} \right] \right\} + \left\{ \frac{1}{\epsilon_0 \mu} \partial_z \mu \mathbf{J} + \frac{1}{\epsilon_0 \mu_0 \mu} \left[\mathbf{z} \times \nabla_\perp g \right] \right\} \\ \\ -\partial_t \mu \left\{ \epsilon_0 \partial_t \varepsilon \left[\mathbf{z} \times \mathbf{E} \right] + \partial_z \mathbf{H} \right\} - \left\{ \partial_t \mu \left[\mathbf{z} \times \mathbf{J} \right] + \mu_0^{-1} \nabla_\perp I_z \right\} \end{pmatrix}, \tag{2.59}$$

where expression (2.38) for function $\mathbf{F}_H$ is substituted.

In a similar way, left-hand-side of Eq. (2.46) can be presented as the action of the other matrix differential procedure on the same four-component vector sought as follows

$$\begin{pmatrix} \epsilon_0^{-1} \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \mathbf{H} \\ \mu_0^{-1} \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \epsilon_0^{-1} \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \\ \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot & \mathcal{O} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \equiv \hat{W}_E X. \tag{2.60}$$

We can also combine the matrix differential procedure

$$\hat{W}_E = \begin{pmatrix} \mathcal{O} & \epsilon_0^{-1} \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \\ \mu_0^{-1} \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot & \mathcal{O} \end{pmatrix} \tag{2.61}$$

and the same first pair of the boundary conditions into one more *Wave-Boundary Operator* as follows

$$\boxed{W_E X = \begin{cases} \hat{W}_E X, & \textit{while } \mathbf{r} \in S; \\ \\ (\mathbf{n}_i \cdot \mathbf{H}) = 0, \;\; (\mathbf{l}_i \cdot \mathbf{E}) = 0, & \textit{while } \mathbf{r} \in L_0, L_1, \ldots, L_N. \end{cases}} \tag{2.62}$$

In terms of these notation, vector equation (2.46) together with the boundary conditions supply one more operator equation as

$$W_E X = \begin{pmatrix} -\partial_t \varepsilon \left\{ \partial_z \mathbf{E} + \mu_0 \partial_t \mu \left[\mathbf{H} \times \mathbf{z} \right] \right\} - \left\{ \partial_t \varepsilon \left[\mathbf{I} \times \mathbf{z} \right] + \epsilon_0^{-1} \nabla_\perp J_z \right\} \\ \\ \frac{1}{\varepsilon} \partial_z \varepsilon \left\{ \mu_0^{-1} \partial_z \left[\mathbf{z} \times \mathbf{E} \right] + \partial_t \mu \mathbf{H} \right\} + \left\{ \frac{1}{\mu_0 \varepsilon} \partial_z \varepsilon \mathbf{I} + \frac{1}{\epsilon_0 \mu_0 \varepsilon} \left[\nabla_\perp \varrho \times \mathbf{z} \right] \right\} \end{pmatrix}, \tag{2.63}$$

where formula (2.45) for function $\mathbf{F}_E$ is substituted.

[†]We'll prove further that boundary conditions (2.53) are a sequence of the boundary conditions (2.49).

Example 4 *Calculation of the matrix differential procedure $\hat{W}_E$ in Cartesian coordinate system.*

1. $\mathbf{H} = \mathbf{x}H_x + \mathbf{y}H_y$, $\mathbf{E} = \mathbf{x}E_x + \mathbf{y}E_y$, $\nabla_\perp = \mathbf{x}\frac{\partial}{\partial x} + \mathbf{y}\frac{\partial}{\partial y}$;

2. $[\nabla_\perp \times \mathbf{z}] = \mathbf{x}\frac{\partial}{\partial y} - \mathbf{y}\frac{\partial}{\partial x}$; $[\nabla_\perp \times \mathbf{z}] \cdot \mathbf{H} = \frac{\partial}{\partial_y} H_x - \frac{\partial}{\partial x} H_y \equiv \phi$;

3. $$\begin{aligned} \underline{\nabla_\perp [\nabla_\perp \times \mathbf{z}] \cdot \mathbf{H}} &= \nabla_\perp \phi = \mathbf{x}\frac{\partial}{\partial x}\phi + \mathbf{y}\frac{\partial}{\partial y}\phi \\ &= \underline{\mathbf{x}\left(\frac{\partial^2}{\partial x \partial y}H_x - \frac{\partial^2}{\partial x^2}H_y\right) + \mathbf{y}\left(\frac{\partial^2}{\partial y^2}H_x - \frac{\partial^2}{\partial y \partial x}H_y\right)}; \end{aligned}$$

4. $\nabla_\perp \cdot \mathbf{E} = \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y \equiv \psi$;

5. $$\begin{aligned} \underbrace{[\nabla_\perp \times \mathbf{z}] \nabla_\perp \cdot \mathbf{E}} &= [\nabla_\perp \times \mathbf{z}]\psi = [\nabla_\perp \times \mathbf{z}\psi] = \mathbf{x}\frac{\partial}{\partial y}\psi + \mathbf{y}\left(-\frac{\partial}{\partial x}\psi\right) \\ &= \underbrace{\mathbf{x}\left(\frac{\partial^2}{\partial y \partial x}E_x + \frac{\partial^2}{\partial y^2}E_y\right) + \mathbf{y}\left(-\frac{\partial^2}{\partial x^2}E_x - \frac{\partial^2}{\partial x \partial y}E_y\right)}; \end{aligned}$$

6. $$\hat{W}_E X = \begin{pmatrix} \epsilon_0^{-1} \underline{\nabla_\perp [\nabla_\perp \times \mathbf{z}] \cdot \mathbf{H}} \\ \\ \mu_0^{-1} \underbrace{[\nabla_\perp \times \mathbf{z}] \nabla_\perp \cdot \mathbf{E}} \end{pmatrix} = \begin{pmatrix} \epsilon_0^{-1}\left(\frac{\partial^2}{\partial x \partial y}H_x - \frac{\partial^2}{\partial x^2}H_y\right) \\ \epsilon_0^{-1}\left(\frac{\partial^2}{\partial y^2}H_x - \frac{\partial^2}{\partial y \partial x}H_y\right) \\ \mu_0^{-1}\left(\frac{\partial^2}{\partial y \partial x}E_x + \frac{\partial^2}{\partial y^2}E_y\right) \\ \mu_0^{-1}\left(-\frac{\partial^2}{\partial x^2}E_x - \frac{\partial^2}{\partial x \partial y}E_y\right) \end{pmatrix}.$$

   It is evident that four-component vector obtained above at the right-hand-side of p.6 can be calculated by multiplication of the next matrix differential procedure and the transverse 4-component electromagnetic vector

$$\hat{W}_E X \equiv \begin{pmatrix} 0 & 0 & \epsilon_0^{-1}\frac{\partial^2}{\partial x \partial y} & -\epsilon_0^{-1}\frac{\partial^2}{\partial x^2} \\ 0 & 0 & \epsilon_0^{-1}\frac{\partial^2}{\partial y^2} & -\epsilon_0^{-1}\frac{\partial^2}{\partial y \partial x} \\ \mu_0^{-1}\frac{\partial^2}{\partial y \partial x} & \mu_0^{-1}\frac{\partial^2}{\partial y^2} & 0 & 0 \\ -\mu_0^{-1}\frac{\partial^2}{\partial x^2} & -\mu_0^{-1}\frac{\partial^2}{\partial x \partial y} & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}. \tag{2.64}$$

It is interesting to compare the matrix differential operations $\hat{W}_H$ and $\hat{W}_E$ in particular case of Cartesian coordinate system. To make this comparison easier let us repeat here Eq. (2.56)

$$\hat{W}_H X \equiv \begin{pmatrix} 0 & 0 & -\epsilon_0^{-1}\frac{\partial^2}{\partial y \partial x} & -\epsilon_0^{-1}\frac{\partial^2}{\partial y^2} \\ 0 & 0 & \epsilon_0^{-1}\frac{\partial^2}{\partial x^2} & \epsilon_0^{-1}\frac{\partial^2}{\partial x \partial y} \\ -\mu_0^{-1}\frac{\partial^2}{\partial x \partial y} & \mu_0^{-1}\frac{\partial^2}{\partial x^2} & 0 & 0 \\ -\mu_0^{-1}\frac{\partial^2}{\partial y^2} & \mu_0^{-1}\frac{\partial^2}{\partial y \partial x} & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}.$$

As one can be seen, matrix $\hat{W}_E$ cannot be obtained via linear manipulations with matrix $\hat{W}_H$ and vice versa.

Thus, we have made in this chapter all the preparations which are needed for solution of the Waveguide Problem within the frame of the Evolutionary Approach to Electromagnetics in Time Domain. Program of our next study involves the following principal elements:

- we should prove that the Wave-Boundary Operators *(WBO)* introduced are self-adjoint in chosen space of solutions, and therefore each of them has an eigenvector set of some vector function depended on transverse waveguide coordinates;
- we should prove that the eigenvector sets of *WBO* originate a basis in chosen space of solutions, and therefore electromagnetic field sought can be presented in terms of the eigenvector series with some scalar coefficients depended on $z-$ coordinate and time;
- we should obtain a system of evolutionary equations for those scalar coefficients via projecting of Maxwell's equations in Transversal - Longitudinal Form onto the basis elements.

Chapter 3
BASIS SET IN THE DOMAIN OF WAVE-BOUNDARY OPERATORS

3.1 Eigenvalue Problems in Operator Form

Summary 5 *In this section, the following questions will be answered. What is it: Hilbert spaces $L_2^4(S)$ and $L_2^2(S)$? What is it: the domain of WBO? How to prove that the Wave-Boundary Operators (WBO) are self-adjoint in $L_2^6(V)$? Why self-adjointity of WBO is so important? Which form have the operator eigenvalue equations for operators W_H and W_E?*

3.1.1 Domain of WBO

At the definitions of the operators W_H and W_E obtained above from Maxwell's equations (see Eqs. (2.58) and (2.62)), the 4-component electromagnetic vector $X \equiv X(\mathbf{r}, z, t)$ has been put into operation which is specified by Eq. (2.55). These operators act on transverse coordinates $\mathbf{r}$ at the argument of the vector X in such a way that the vector X becomes subjected to the boundary conditions over the boundary L of the waveguide cross-section domain S. The variables z and t play a role of the parameters in regard to *WBO* and therefore they can be omitted for short.

These facts (obtained in the course of electromagnetic analysis) suggest to introduce mathematically some class of 4-component vector functions $X(\mathbf{r})$ where argument $\mathbf{r}$ varies in the closed domain $\bar{S} = S + L$. The functions should be twice differentiable in the open domain S, and additionally they should be "cut off" at the boundary L of the domain S in accordance with the electromagnetic boundary conditions required. Formally, a definition for such a class of functions can be written as follows

$$X(\mathbf{r}) = \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix}, \quad \mathbf{r} \in S; \quad \begin{array}{c} (\mathbf{l}_i \cdot \mathbf{E}(\mathbf{r})) = 0, \\ (\mathbf{n}_i \cdot \mathbf{H}(\mathbf{r})) = 0, \end{array} \quad \mathbf{r} \in L_i, \tag{3.1}$$

where $i = 0, 1, \ldots, N$ correspond to numbering of the contour L components

Formulation of the electromagnetic problem involves the condition (2.10). It compels to restrict class of the functions like (3.1) under consideration. To this aim, we use the following fragment from Eq. (2.10) :

$$\begin{aligned}&\int_{S'}(\epsilon_0\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}^* + \mu_0\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}^*)ds \equiv \\ &\equiv \int_{S'}(\epsilon_0\mathbf{E}\cdot\mathbf{E}^* + \mu_0\mathbf{H}\cdot\mathbf{H}^*)ds + \int_{S'}(\epsilon_0 E_z E_z^* + \mu_0 H_z H_z^*)\,ds < \infty,\end{aligned} \tag{3.2}$$

where ϵ_0, μ_0 are free-space constants, $S' \subseteq \bar{S}$, the dot stands for scalar product in the common sense of vector algebra, star $(^*)$ denotes complex conjugation.

The first integral at the right-hand-side in Eq. (3.2) suggests to introduce the inner product for the 4-component vectors like (3.1)as follows

$$\boxed{\langle \mathbf{X}_1, \mathbf{X}_2\rangle = \tfrac{1}{S}\int_S(\epsilon_0\mathbf{E}_1\cdot\mathbf{E}_2^* + \mu_0\mathbf{H}_1\cdot\mathbf{H}_2^*)\,ds,} \tag{3.3}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are a pair of arbitrary vector functions of transverse variables**,** each of which varies within the open domain S and satisfies the boundary conditions required over its boundary L :

$$\begin{aligned}
X_1(\mathbf{r}) &= \begin{pmatrix}\mathbf{E}_1(\mathbf{r})\\ \mathbf{H}_1(\mathbf{r})\end{pmatrix}, \quad \mathbf{r}\in S; \quad \begin{matrix}(\mathbf{l}_i\cdot\mathbf{E}_1(\mathbf{r})) = 0,\\ (\mathbf{n}_i\cdot\mathbf{H}_1(\mathbf{r})) = 0,\end{matrix} \quad \mathbf{r}\in L_i;\\
X_2(\mathbf{r}) &= \begin{pmatrix}\mathbf{E}_2(\mathbf{r})\\ \mathbf{H}_2(\mathbf{r})\end{pmatrix}, \quad \mathbf{r}\in S; \quad \begin{matrix}(\mathbf{l}_i\cdot\mathbf{E}_2(\mathbf{r})) = 0,\\ (\mathbf{n}_i\cdot\mathbf{H}_2(\mathbf{r})) = 0,\end{matrix} \quad \mathbf{r}\in L_i.
\end{aligned} \tag{3.4}$$

The integrand of (3.3) involves the free-space constants as the weighting factors.

This inner product defines a Hilbert space as the domain of the operators W_H and W_E in the problem under consideration. Let us denote this functional space as $L_2^4(V)$ which means that we deal with Hilbert space L_2 involving *4-component* vectors which vary within finite domain $\bar{S}$*.

3.1.2 Self-Adjointity of WBO and Operator Eigenvalue Equations

Initially, we introduce an auxiliary contour L' located within the domain $\bar{S}$ in the vicinity of the boundary L but not coincided with L at any point. Hence, the differential procedures $\hat{W}_H$ and $\hat{W}_E$, which has been defined in Eqs. (2.57) and (2.61), hold for the closed domain $\overline{S'}$ bounded by L'.

Self-Adjointity of Operator W_H. We will prove first in detail self-adjointity of the operator W_H. To this aim, we calculate a pair of inner products in accordance with the definition for inner product (3.3) and with using of Eq. (2.54) as follows

$$\left\langle \hat{W}_H\mathbf{X}_1, \mathbf{X}_2\right\rangle' = \frac{1}{S'}\int_{S'}(\mathbf{E}_2^*\cdot[\mathbf{z}\times\nabla_\perp]\,\nabla_\perp\cdot\mathbf{H}_1 + \mathbf{H}_2^*\cdot\nabla_\perp[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E}_1)\,ds, \tag{3.5}$$

*Further on, Hilbert space L_2 of *2-component* vectors varying in the same closed domain $\bar{S}$ will be needed for analysis. In this case, notation $L_2^2(S)$ is used to differ it from Hilbert space $L_2^4(S)$.

$$\left\langle \mathbf{X}_1, \hat{W}_H \mathbf{X}_2 \right\rangle' = \frac{1}{S'} \int_{S'} \left(\mathbf{E}_1 \cdot [\mathbf{z} \times \nabla_\perp] \nabla_\perp \cdot \mathbf{H}_2^* + \mathbf{H}_1 \cdot \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_2^* \right) ds. \quad (3.6)$$

To prove self-adjointness of the operator W_H we must carry out the following identical rearranging of all the elements at the integrands of Eqs. (3.5) and (3.6) :

- $\mathbf{E}_2^* \cdot [\mathbf{z} \times \nabla_\perp] \nabla_\perp \cdot \mathbf{H}_1 = -[\mathbf{z} \times \mathbf{E}_2^*] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_1,$

since vector operator $[\mathbf{z} \times \nabla_\perp]$ acts on scalar $(\nabla_\perp \cdot \mathbf{H}_1)$ and therefore the latter can be put at any place within $[\mathbf{z} \times \nabla_\perp]$; for our purpose , it is convenient to place it as follows $[\mathbf{z} \times \nabla_\perp] (\nabla_\perp \cdot \mathbf{H}_1) = [\mathbf{z} \times \nabla_\perp (\nabla_\perp \cdot \mathbf{H}_1)]$; further, $\mathbf{E}_2^* \cdot [\mathbf{z} \times \nabla_\perp] (\nabla_\perp \cdot \mathbf{H}_1) \equiv \mathbf{E}_2^* \cdot [\mathbf{z} \times \nabla_\perp (\nabla_\perp \cdot \mathbf{H}_1)] = [\mathbf{E}_2^* \times \mathbf{z}] \cdot \nabla_\perp (\nabla_\perp \cdot \mathbf{H}_1) = -[\mathbf{z} \times \mathbf{E}_2^*] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_1$, where identities (2.36) are used;

- $\mathbf{H}_2^* \cdot \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_1 = -\mathbf{H}_2^* \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_1]$,

since $[\mathbf{z} \times \nabla_\perp] = -[\nabla_\perp \times \mathbf{z}]$, and hence, the identities hold $\nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_1 = -\nabla_\perp [\nabla_\perp \times \mathbf{z}] \cdot \mathbf{E}_1 = -\nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_1]$, which results in $\mathbf{H}_2^* \cdot \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_1 = -\mathbf{H}_2^* \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_1]$;

- $\mathbf{E}_1 \cdot [\mathbf{z} \times \nabla_\perp] \nabla_\perp \cdot \mathbf{H}_2^* = -[\mathbf{z} \times \mathbf{E}_1] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_2^*,$

since $\mathbf{E}_1 \cdot [\mathbf{z} \times \nabla_\perp] = [\mathbf{E}_1 \times \mathbf{z}] \cdot \nabla_\perp = -[\mathbf{z} \times \mathbf{E}_1] \cdot \nabla_\perp$, and hence, $\mathbf{E}_1 \cdot [\mathbf{z} \times \nabla_\perp] \nabla_\perp \cdot \mathbf{H}_2^* = -[\mathbf{z} \times \mathbf{E}_1] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_2^*$;

- $\mathbf{H}_1 \cdot \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_2^* = -\mathbf{H}_1 \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_2^*]$,

since $[\mathbf{z} \times \nabla_\perp] = -[\nabla_\perp \times \mathbf{z}]$, and hence, $[\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_2^* = -[\nabla_\perp \times \mathbf{z}] \cdot \mathbf{E}_2^* = -\nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_2^*]$, therefore $\mathbf{H}_1 \cdot \nabla_\perp [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_2^* = -\mathbf{H}_1 \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_2^*]$;

Let us substitute now the identities marked above by the bullets at the right-hand-sides of the inner products (3.5) and (3.6)

$$\left\langle \hat{W}_H \mathbf{X}_1, \mathbf{X}_2 \right\rangle' = -\frac{1}{S'} \int_{S'} \left([\mathbf{z} \times \mathbf{E}_2^*] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_1 + \mathbf{H}_2^* \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_1] \right) ds, \quad (3.7)$$

$$\left\langle \mathbf{X}_1, \hat{W}_H \mathbf{X}_2 \right\rangle' = -\frac{1}{S'} \int_{S'} \left([\mathbf{z} \times \mathbf{E}_1] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_2^* + \mathbf{H}_1 \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_2^*] \right) ds. \quad (3.8)$$

Substraction of these equations yields

$$\begin{gathered} \left\langle \hat{W}_H \mathbf{X}_1, \mathbf{X}_2 \right\rangle' - \left\langle \mathbf{X}_1, \hat{W}_H \mathbf{X}_2 \right\rangle' = \\ = \tfrac{1}{S'} \int_{S'} \left([\mathbf{z} \times \mathbf{E}_1] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_2^* - \mathbf{H}_2^* \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_1] \right) ds + \\ + \tfrac{1}{S'} \int_{S'} \left(\mathbf{H}_1 \cdot \nabla_\perp \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_2^*] - [\mathbf{z} \times \mathbf{E}_2^*] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{H}_1 \right) ds \end{gathered} \quad (3.9)$$

This equation can be rearranged in turn with using of some elegant integral identity which regards to a pair of 2-component vector functions $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ which vary in the same domain $\mathbf{r} \in \bar{S}' = S' + L'$ bounded by the contour L'

$$\boxed{\begin{gathered}\frac{1}{S'}\int_{S'}(\mathbf{A}\cdot\nabla_\perp\nabla_\perp\cdot\mathbf{B} - \mathbf{B}\cdot\nabla_\perp\nabla_\perp\cdot\mathbf{A})\,ds = \\ = \frac{1}{S'}\oint_{L'}\{(\mathbf{n}'\cdot\mathbf{A})(\nabla_\perp\cdot\mathbf{B}) - (\mathbf{n}'\cdot\mathbf{B})(\nabla_\perp\cdot\mathbf{A})\}\,d\ell,\end{gathered}} \tag{3.10}$$

where $\mathbf{n}'$ is out-ward normal to the contour L'. This identity follows from Gauss theorem for two-dimensional case. Making use of this identity in the course of rearranging of Eq. (3.9), one can obtain

$$\begin{gathered}\left\langle \hat{W}_H\mathbf{X}_1, \mathbf{X}_2\right\rangle' - \left\langle \mathbf{X}_1, \hat{W}_H\mathbf{X}_2\right\rangle' = \\ = \frac{1}{S'}\oint_{L'}\{(\mathbf{n}'\cdot[\mathbf{z}\times\mathbf{E}_1])(\nabla_\perp\cdot\mathbf{H}_2^*) - (\mathbf{n}'\cdot\mathbf{H}_2^*)(\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}_1])\}\,d\ell + \\ + \frac{1}{S'}\oint_{L'}\{(\mathbf{n}'\cdot\mathbf{H}_1)(\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}_2^*]) - (\mathbf{n}'\cdot[\mathbf{z}\times\mathbf{E}_2^*])(\nabla_\perp\cdot\mathbf{H}_1)\}\,d\ell\end{gathered} \tag{3.11}$$

However,

$$\mathbf{n}'\cdot[\mathbf{z}\times\mathbf{E}_1] = [\mathbf{n}'\times\mathbf{z}]\cdot\mathbf{E}_1 = \mathbf{l}'\cdot\mathbf{E}_1 \quad and \quad \mathbf{n}'\cdot[\mathbf{z}\times\mathbf{E}_2^*] = [\mathbf{n}'\times\mathbf{z}]\cdot\mathbf{E}_2^* = \mathbf{l}'\cdot\mathbf{E}_2^*,$$

where $\mathbf{l}'$ is the unit vector tangential to the contour L' and oriented in the same direction as the unit vector $\mathbf{l}$ (i.e., $\mathbf{l}_1$, $\mathbf{l}_2, \ldots, \mathbf{l}_N$) on the multiconnected contour L (i.e., on L_1, $L_2, \ldots, L_N$). Hence, Eq. (3.11) can be written as

$$\begin{gathered}\left\langle \hat{W}_H\mathbf{X}_1, \mathbf{X}_2\right\rangle' - \left\langle \mathbf{X}_1, \hat{W}_H\mathbf{X}_2\right\rangle' = \\ = \frac{1}{S'}\oint_{L'}\{(\mathbf{l}'\cdot\mathbf{E}_1)(\nabla_\perp\cdot\mathbf{H}_2^*) - (\mathbf{n}'\cdot\mathbf{H}_2^*)(\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}_1])\}\,d\ell + \\ + \frac{1}{S'}\oint_{L'}\{(\mathbf{n}'\cdot\mathbf{H}_1)(\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}_2^*]) - (\mathbf{l}'\cdot\mathbf{E}_2^*)(\nabla_\perp\cdot\mathbf{H}_1)\}\,d\ell\end{gathered} \tag{3.12}$$

When L' tends to L and coincides with L at the limit, then all the superscripts "*prime*" should be cancelled (and at the inner products $\langle\circ,\circ\rangle'$ as well, since the boundary conditions may be added to $\hat{W}_H$, when L' coincides with L). To put it otherwise

$$\lim_{L'\to L}\left\{\left\langle \hat{W}_H\mathbf{X}_1, \mathbf{X}_2\right\rangle' - \left\langle \mathbf{X}_1, \hat{W}_H\mathbf{X}_2\right\rangle'\right\} = \langle W_H\mathbf{X}_1, \mathbf{X}_2\rangle - \langle \mathbf{X}_1, W_H\mathbf{X}_2\rangle \tag{3.13}$$

in accordance with the definition (2.58) for the operator W_H. At the limiting case $L' \equiv L$, Eq. (3.12) acquires the form of

$$\begin{gathered}\langle W_H\mathbf{X}_1, \mathbf{X}_2\rangle - \langle \mathbf{X}_1, W_H\mathbf{X}_2\rangle = \\ = \frac{1}{S}\sum_{i=0}^{N}\oint_{L_i}\{(\mathbf{l}_i\cdot\mathbf{E}_1)(\nabla_\perp\cdot\mathbf{H}_2^*) - (\mathbf{n}_i\cdot\mathbf{H}_2^*)(\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}_1])\}\,d\ell + \\ + \frac{1}{S}\sum_{i=0}^{N}\oint_{L_i}\{(\mathbf{n}_i\cdot\mathbf{H}_1)(\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}_2^*]) - (\mathbf{l}_i\cdot\mathbf{E}_2^*)(\nabla_\perp\cdot\mathbf{H}_1)\}\,d\ell,\end{gathered} \tag{3.14}$$

where the fact that the contour L is multiconnected in the general case has been taken into account.

Accordingly to Eq. (3.4), the boundary conditions $(\mathbf{l}_i \cdot \mathbf{E}_1) = 0$, $(\mathbf{l}_i \cdot \mathbf{E}_2^*) = 0$, and $(\mathbf{n}_i \cdot \mathbf{H}_1) = 0$, $(\mathbf{n}_i \cdot \mathbf{H}_2^*) = 0$ should be satisfied at all the components L_i of the contour L. They cancel all the integrands at the right-hand-side in Eq. (3.15), and hence, the following important identity holds

$$\boxed{\langle W_H \mathbf{X}_1, \mathbf{X}_2 \rangle - \langle \mathbf{X}_1, W_H \mathbf{X}_2 \rangle = 0.} \tag{3.15}$$

It means that *operator W_H is self-adjoint*. This identity will serve us further as a key-point for definition of the basis in the space of solutions.

Self-Adjointity of Operator W_E. In a similar way but briefly, we present now similar proof of self-adjointity of the operator W_H. To this aim, a pair of the inner products should be also calculated in accordance with the definition for inner product (3.3) and with using of Eq. (2.60), what yields

$$\left\langle \hat{W}_E \mathbf{X}_1, \mathbf{X}_2 \right\rangle' = \frac{1}{S'} \int_{S'} \left(\mathbf{E}_2^* \cdot \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \mathbf{H}_1 + \mathbf{H}_2^* \cdot \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot \mathbf{E}_1 \right) ds, \tag{3.16}$$

$$\left\langle \mathbf{X}_1, \hat{W}_E \mathbf{X}_2 \right\rangle' = \frac{1}{S'} \int_{S'} \left(\mathbf{E}_1 \cdot \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \mathbf{H}_2^* + \mathbf{H}_1 \cdot \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot \mathbf{E}_2^* \right) ds. \tag{3.17}$$

The following identities

- $\mathbf{E}_2^* \cdot \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \mathbf{H}_1 = -\mathbf{E}_2^* \cdot \nabla_\perp \nabla_\perp \cdot \left[\mathbf{H}_1 \times \mathbf{z} \right],$
- $\mathbf{H}_2^* \cdot \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot \mathbf{E}_1 = -\left[\mathbf{H}_2^* \times \mathbf{z} \right] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{E}_1,$
- $\mathbf{E}_1 \cdot \nabla_\perp \left[\nabla_\perp \times \mathbf{z} \right] \cdot \mathbf{H}_2^* = -\mathbf{E}_1 \cdot \nabla_\perp \nabla_\perp \cdot \left[\mathbf{H}_2^* \times \mathbf{z} \right],$
- $\mathbf{H}_1 \cdot \left[\nabla_\perp \times \mathbf{z} \right] \nabla_\perp \cdot \mathbf{E}_2^* = -\left[\mathbf{H}_1 \times \mathbf{z} \right] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{E}_2^*$

can be proved and used then for rearranging of the integrands in Eqs. (3.16) and (3.17). Substraction of these equations with the integrands rearranged yields

$$\begin{gathered} \left\langle \hat{W}_E \mathbf{X}_1, \mathbf{X}_2 \right\rangle' - \left\langle \mathbf{X}_1, \hat{W}_E \mathbf{X}_2 \right\rangle' = \\ = \tfrac{1}{S'} \int_{S'} \left\{ \mathbf{E}_1 \cdot \nabla_\perp \nabla_\perp \cdot \left[\mathbf{H}_2^* \times \mathbf{z} \right] - \left[\mathbf{H}_2^* \times \mathbf{z} \right] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{E}_1 \right\} ds + \\ + \tfrac{1}{S'} \int_{S'} \left\{ \left[\mathbf{H}_1 \times \mathbf{z} \right] \cdot \nabla_\perp \nabla_\perp \cdot \mathbf{E}_2^* - \mathbf{E}_2^* \cdot \nabla_\perp \nabla_\perp \cdot \left[\mathbf{H}_1 \times \mathbf{z} \right] \right\} ds. \end{gathered} \tag{3.18}$$

Making use of the identity (3.10), Eq. (3.18) can be written in the limiting case $L' \to L$ as follows

$$\langle W_E \mathbf{X}_1, \mathbf{X}_2 \rangle - \langle \mathbf{X}_1, W_E \mathbf{X}_2 \rangle =$$

$$= \frac{1}{S'} \sum_{i=0}^{N} \oint_{L_i} \left\{ (\mathbf{n}_i \cdot \mathbf{E}_1) \underline{(\nabla_\perp \cdot [\mathbf{H}_2^* \times \mathbf{z}])} + (\mathbf{l}_i \cdot \mathbf{H}_2^*) \underline{(\nabla_\perp \cdot \mathbf{E}_1)} \right\} d\ell -$$
$$- \frac{1}{S'} \sum_{i=0}^{N} \oint_{L_i} \left\{ (\mathbf{l}_i \cdot \mathbf{H}_1) \underline{(\nabla_\perp \cdot \mathbf{E}_2^*)} + (\mathbf{n}_i \cdot \mathbf{E}_2^*) \underline{(\nabla_\perp \cdot [\mathbf{H}_1 \times \mathbf{z}])} \right\} d\ell. \tag{3.19}$$

where the fact $[\mathbf{n}_i \times \mathbf{z}] = \mathbf{l}_i$ is taken into account (see Eq. (2.1)).

In this case, we can avail ourself of the second form of the boundary conditions (2.53) though they have not been inserted in the definition (2.62) for the operator W_E. Later on, we'll demonstrate that the vectors satisfy automatically the boundary conditions of the first form (2.49) when they have satisfied the boundary conditions of the second form (2.53). So, at the integrands of Eq. (3.19), the following boundary conditions of the second form hold:

$$(\nabla_\perp \cdot \mathbf{E}_1) = 0, \qquad (\nabla_\perp \cdot [\mathbf{H}_1(\mathbf{r}) \times \mathbf{z}]) = 0;$$
$$(\nabla_\perp \cdot \mathbf{E}_2^*(\mathbf{r})) = 0, \qquad (\nabla_\perp \cdot [\mathbf{H}_2^*(\mathbf{r}) \times \mathbf{z}]) = 0, \tag{3.20}$$

while $\mathbf{r} \in L_i$, $i = 0, 1, \ldots, N$. These equations cancel the right-hand-side of Eq. (3.19), and one more important identity holds

$$\boxed{\langle W_E \mathbf{X}_1, \mathbf{X}_2 \rangle - \langle \mathbf{X}_1, W_E \mathbf{X}_2 \rangle = 0,} \tag{3.21}$$

which means that *operator W_E is self-adjoint* as well. This identity will be used also as a key-point for definition of the basis in the space of solutions.

3.2 Equivalent Vector and Scalar Boundary Eigenvalue Problems

Summary 6 *In this section, the following questions will be answered. Which fundamental properties of WBO eigensolutions follow from the fact of their self-adjointity? Can you prove some of them? How to recast* WBO *eigenvalue equations into equivalent **vector** boundary eigenvalue problems? Which properties of* WBO *eigensolutions follow from the symmetry of these operators? How to recast the **vector** eigenvalue problems for operators W_H and W_E into some **scalar** eigenvalue problems for transverse Laplacian? Which forms have the eigenvalue problems for the scalar potentials of* WBO *eigenvectors? How many* WBO *eigenvectors may be corresponded to **non-zero valued** their eigenvalues? How many* WBO *eigenvectors may be corresponded to **zero valued** their eigenvalues? Which possible forms they can have? Why the harmonic functions do appear at the equivalent scalar eigenvalue problems for the* *potentials of* WBO *eigenvectors? What is the difference (in the general case) between the potential harmonic functions, corresponding to the cases of W_H and W_H operators?*

Wave-boundary operators W_H and W_E act on the vectors which vary in the same closed domain $\bar{S} = S + L$ as it specified by Eq. (3.1). Class of these vector functions is restricted by the requirements of (i) their twice differentiability in the open domain S, and (ii) their quadratic integrability in accordance with definition (3.3) for the inner product. So, Hilbert space is the domain of these operators. Operators W_H and W_E are self-adjoint in the domain, so each has an eigenvector set in Hilbert space. Eigenvectors of the operators W_H and W_E are the eigensolutions of the operator eigenvalue problems

$$\boxed{W_H Y_m(\mathbf{r}) = p_m Y_m(\mathbf{r}); \qquad W_E Z_n(\mathbf{r}) = p_n Z_n(\mathbf{r}),} \tag{3.22}$$

which follow immediately from the identities (3.15) and (3.21), respectively. Here, the numbers p_m and q_n are the eigenvalues of the operators, and the vectors $Y_m(\mathbf{r})$ and $Z_n(\mathbf{r})$ are the eigensolutions corresponding to them, respectively.

Self-adjointity of *the Wave-Boundary Operators* W_H and W_E **is the fact of especial importance since it permits various general statements.**

Some general facts, which regard to any self-adjoint operator, will be given below without proof. One can prove them as the exercises in just the same way as it was made in the first part of this course, i.e., in *Part1: Cavity Problem.*

- The eigenvalues p_m and q_n are real.
- Spectrums as the eigenvalues p_m as the eigenvalues q_n are discrete, and each has a single point of condensation which is located at infinity.
- The eigenvalues $p_m \neq 0$ and $q_n \neq 0$ may be degenerated, but power of their degeneracy is always finite. It means that there is *a finite number* of linearly independent eigenvectors which corresponds to the same eigenvalue. The eigenvectors corresponded to the same eigenvalue can be orthonormalized via relevant orthogonal transformation and normalization.
- The subscription notation of the eigenvalues $p_m \neq 0$ and $q_n \neq 0$ can be considered as numbering them in order of their increasing values with taking into account possible their degeneracy.
- Operators W_H and W_E both have *zero* as their eigenvalue: i.e., $p_0 = 0$ and $q_0 = 0$ are the eigenvalues of the operators W_H and W_E, respectively. However, the eigenvalues $p_0 = 0$ and $q_0 = 0$ each have *infinite* power of degeneracy, in contrast to the eigenvalues $p_m \neq 0$ and $q_n \neq 0$, degeneration of which is always finite.

- Each pair of the eigenvectors corresponding to *distinct* eigenvalues of a self-adjoint operator is orthogonal in the sense of inner product (3.3).

In the course of discussion of the eigenvalue problems (3.22) in detail, we'll dwell on these points specially.

3.2.1 Boundary Eigenvalue Problems for Operator W_H

Equivalent *Vector* Boundary Eigenvalue Problem.

Consider in detail the first operator eigenvalue equation of the system (3.22). To obtain equivalent vector boundary eigenvalue problem we should substitute the eigenvector $Y_m(\mathbf{r})$ (in the form of (3.1)) at the first equation (3.22). Nominally, this operation may be written as follows

$$Y_m(\mathbf{r}) \equiv \begin{pmatrix} \mathbf{E}_m(\mathbf{r}) \\ \mathbf{H}_m(\mathbf{r}) \end{pmatrix}, \quad \begin{matrix} (\mathbf{l} \cdot \mathbf{E}_m(\mathbf{r}))\,|_L = 0 \\ (\mathbf{n} \cdot \mathbf{H}_m(\mathbf{r}))\,|_L = 0 \end{matrix} \quad \Rightarrow \quad W_H Y_m(\mathbf{r}) = p_m Y_m(\mathbf{r}).$$

As a result, operator eigenvalue equation $W_H Y_m(\mathbf{r}) = p_m Y_m(\mathbf{r})$ turns into boundary eigenvalue problem for a system of *vector* differential equations as

$$\boxed{\begin{cases} \frac{1}{\epsilon_0}\left[\mathbf{z} \times \nabla_\perp\right] \nabla_\perp \cdot \mathbf{H}_m(\mathbf{r}) = p_m \mathbf{E}_m(\mathbf{r}), \qquad (\mathbf{n} \cdot \mathbf{H}_m(\mathbf{r}))\,|_L = 0; \qquad (a) \\ \frac{1}{\mu_0}\nabla_\perp \left[\mathbf{z} \times \nabla_\perp\right] \cdot \mathbf{E}_m(\mathbf{r}) = p_m \mathbf{H}_m(\mathbf{r}), \qquad (\mathbf{l} \cdot \mathbf{E}_m(\mathbf{r}))\,|_L = 0. \qquad (b) \end{cases}} \tag{3.23}$$

Degeneration of the eigenvalues $p_m \neq 0$. In the general case, solution to the problem (3.23) is a pair of complex-valued vector functions

$$\mathbf{E}_m(\mathbf{r}) = \mathbf{E}_m^{'}(\mathbf{r}) + i\mathbf{E}_m^{''}(\mathbf{r}), \qquad \mathbf{H}_m(\mathbf{r}) = \mathbf{H}_m^{'}(\mathbf{r}) + i\mathbf{H}_m^{''}(\mathbf{r}), \tag{3.24}$$

where $i = \sqrt{-1}$, and $p_m \neq 0$ is supposed. When the functions (3.24) are a solution to the problem (3.23), corresponding to some eigenvalue $p_m \neq 0$, then the complex conjugated functions $\mathbf{E}_m^* = \mathbf{E}_m^{'} - i\mathbf{E}_m^{''}$ and $\mathbf{H}_m^* = \mathbf{H}_m^{'} - i\mathbf{H}_m^{''}$ are also solution to the problem (3.23), corresponding to the same eigenvalue. This property means that *each eigenvalue* $p_m \neq 0$ has the following pair of 4-component eigenvectors corresponding to it:

$$Y_m = \begin{pmatrix} \mathbf{E}_m(\mathbf{r}) \\ \mathbf{H}_m(\mathbf{r}) \end{pmatrix}, \qquad Y_m^* = \begin{pmatrix} \mathbf{E}_m^*(\mathbf{r}) \\ \mathbf{H}_m^*(\mathbf{r}) \end{pmatrix}. \tag{3.25}$$

It is evident, that the eigenvectors (3.25), corresponding to the *same eigenvalue* $p_m \neq 0$, can be also presented as

$$Y_m' = \tfrac{1}{2}\left(Y_m + Y_m^*\right) = \begin{pmatrix} \mathbf{E}_m' \\ \mathbf{H}_m' \end{pmatrix}, \qquad Y_m'' = \tfrac{1}{2}\left(Y_m - Y_m^*\right) = i\begin{pmatrix} \mathbf{E}_m'' \\ \mathbf{H}_m'' \end{pmatrix} \tag{3.26}$$

Block symmetry of the matrix differential procedure $\hat{W}_H$, which has been involved in the definition of the operator W_H (see Eqs. (2.57) and (2.58)), induces some additional useful properties of its eigenvalues and eigenvectors. However, the cases of the eigenvalues $p_m \neq 0$ and $p_m|_{m=0} \equiv p_0 = 0$ require different ways for analysis. We start here the case $p_m \neq 0$, but the case $p_0 = 0$ is treated below once the eigenvalues $p_m \neq 0$ are analyzed.

Symmetry of the eigenvalues and eigenvectors in the case $p_m \neq 0$. All the eigenvalues $p_m \neq 0$ of the operator W_H lie symmetrically on real axis Op with respect to the point $p = 0$ and may be put in order as

$$p_{+m} > 0, \qquad p_{-m} = -p_{+m} < 0. \tag{3.27}$$

When the vectors

$$Y_{+m} = \begin{pmatrix} \mathbf{E}_{+m} \\ \mathbf{H}_{+m} \end{pmatrix}, \qquad Y^*_{+m} = \begin{pmatrix} \mathbf{E}^*_{+m} \\ \mathbf{H}^*_{+m} \end{pmatrix} \tag{3.28}$$

correspond to the eigenvalue $p_{+m} > 0$, then the vectors

$$Y_{-m} = \begin{pmatrix} \mathbf{E}_{+m} \\ -\mathbf{H}_{+m} \end{pmatrix}, \qquad Y^*_{-m} = \begin{pmatrix} \mathbf{E}^*_{+m} \\ -\mathbf{H}^*_{+m} \end{pmatrix} \tag{3.29}$$

correspond to the eigenvalue p_{-m}, where the star means complex conjugation.

To prove this claim, we first write the boundary value problem

$$\begin{cases} \frac{1}{\epsilon_0}\left[\mathbf{z} \times \nabla_\perp\right] \nabla_\perp \cdot \mathbf{H}_{+m} = p_{+m}\mathbf{E}_{+m}, \qquad \left(\mathbf{n} \cdot \mathbf{H}_{+m}\right)|_L = 0; \qquad (a) \\ \frac{1}{\mu_0}\nabla_\perp\left[\mathbf{z} \times \nabla_\perp\right] \cdot \mathbf{E}_{+m} = p_{+m}\mathbf{H}_{+m}, \qquad \left(\mathbf{l} \cdot \mathbf{E}_{+m}\right)|_L = 0, \qquad (b) \end{cases} \tag{3.30}$$

which is equivalent to operator equation $W_H Y_{+m} = p_{+m} Y_{+m}$. In the system of equations (3.30), line (a) we multiply by (-1), but line (b) we rearrange identically and write a result as follows

$$\begin{cases} \frac{1}{\epsilon_0}\left[\mathbf{z} \times \nabla_\perp\right] \nabla_\perp \cdot (-\mathbf{H}_{+m}) = (-p_{+m})\mathbf{E}_{+m}, \qquad \big(\mathbf{n} \cdot (-\mathbf{H}_{+m})\big)|_L = 0; \qquad (a) \\ \frac{1}{\mu_0}\nabla_\perp\left[\mathbf{z} \times \nabla_\perp\right] \cdot \mathbf{E}_{+m} = (-p_{+m})(-\mathbf{H}_{+m}), \qquad \left(\mathbf{l} \cdot \mathbf{E}_{+m}(\mathbf{r})\right)|_L = 0. \qquad (b) \end{cases} \tag{3.31}$$

Let us write now formally the boundary eigenvalue problem which is equivalent to operator equation $W_H Y_{-m} = p_{-m} Y_{-m}$

$$\begin{cases} \frac{1}{\epsilon_0}\left[\mathbf{z} \times \nabla_\perp\right] \nabla_\perp \cdot \mathbf{H}_{-m} = p_{-m}\mathbf{E}_{-m}, \qquad \left(\mathbf{n} \cdot \mathbf{H}_{-m}\right)|_L = 0; \qquad (a) \\ \frac{1}{\mu_0}\nabla_\perp\left[\mathbf{z} \times \nabla_\perp\right] \cdot \mathbf{E}_{-m} = p_{-m}\mathbf{H}_{-m}, \qquad \left(\mathbf{l} \cdot \mathbf{E}_{-m}\right)|_L = 0. \qquad (b) \end{cases} \tag{3.32}$$

Comparison of the problems (3.31) and (3.32) give rise in a conclusion that

$$p_{-m} = -p_{+m}, \quad \textit{and} \quad \mathbf{E}_{-m} = \mathbf{E}_{+m}, \quad \mathbf{H}_{-m} = -\mathbf{H}_{+m}. \tag{3.33}$$

Hence, the first statement in Eq. (3.29) holds. The second one is a consequence of the general statement (3.25).

The symmetry pointed out in Eq. (3.33) is useful in the course of practical calculations of the eigenvalues and eigenvectors corresponding to $p_m \neq 0$, since it makes volume of the calculations twice less. Indeed, it is enough to calculate in a some way all the eigenvalues $p_{+m} > 0$ and the vectors $\mathbf{E}_{+m}$ and $\mathbf{H}_{+m}$. The eigenvalues $p_{-m} < 0$, and the vectors $\mathbf{E}_{-m}$ and $\mathbf{H}_{-m}$ can be then found out easily using Eq. (3.33).

***Scalar* Eigenvalue Problem for the Case $p_{+m} > 0$.**

The form of the vector boundary eigenvalue problem (3.30)

$$\begin{cases} \frac{1}{\epsilon_0}\left[\mathbf{z} \times \nabla_\perp\right] \nabla_\perp \cdot \mathbf{H}_{+m} = p_{+m}\mathbf{E}_{+m}, & \left(\mathbf{n} \cdot \mathbf{H}_{+m}\right)|_L = 0; \qquad (a) \\ \frac{1}{\mu_0}\nabla_\perp \left[\mathbf{z} \times \nabla_\perp\right] \cdot \mathbf{E}_{+m} = p_{+m}\mathbf{H}_{+m}, & \left(\mathbf{l} \cdot \mathbf{E}_{+m}\right)|_L = 0, \qquad (b) \end{cases}$$

and the boundary conditions (2.53)

$$\left(\nabla_\perp \cdot \left[\mathbf{H}_{+m} \times \mathbf{z}\right]\right)|_L = 0, \qquad \left(\nabla_\perp \cdot \mathbf{E}_{+m}\right)|_L = 0$$

suggest to present vector unknowns $\mathbf{E}_{+m}$ and $\mathbf{H}_{+m}$ via some scalar potential $\Psi_m(\mathbf{r})$ in the following way:

$$\mathbf{E}_{+m} = \sqrt[-2]{\epsilon_0}\left[\nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z}\right], \qquad \mathbf{H}_{+m} = \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m(\mathbf{r}). \tag{3.34}$$

Indeed, substitution of Eqs. (3.34) in Eqs. (3.30) and (2.53) yields

$$\begin{cases} \sqrt[-2]{\epsilon_0}\left[\mathbf{z} \times \nabla_\perp\right] \nabla_\perp \cdot \nabla_\perp \Psi_m = p_{+m}\left[\nabla_\perp \Psi_m \times \mathbf{z}\right], & \left(\mathbf{n}\cdot\nabla_\perp \Psi_m\right)|_L = 0; \\ \sqrt[-2]{\epsilon_0}\nabla_\perp \left[\mathbf{z} \times \nabla_\perp\right] \cdot \left[\nabla_\perp \Psi_m \times \mathbf{z}\right] = p_{+m}\nabla_\perp \Psi_m, & \left(\mathbf{l} \cdot \left[\nabla_\perp \Psi_m \times \mathbf{z}\right]\right)|_L = 0, \end{cases} \tag{3.35}$$

$$\left(\nabla_\perp \cdot \left[\nabla_\perp \Psi_m \times \mathbf{z}\right]\right)|_L = 0, \qquad \left(\nabla_\perp \cdot \left[\nabla_\perp \Psi_m \times \mathbf{z}\right]\right)|_L = 0. \tag{3.36}$$

In such a case, boundary conditions (3.36) are satisfied identically

$$\begin{aligned} \left(\nabla_\perp \cdot \left[\nabla_\perp \Psi_m \times \mathbf{z}\right]\right)|_L &\equiv div\left[\nabla_\perp \Psi_m \times \mathbf{z}\right] \\ &= \mathbf{z}\cdot\underbrace{rot\nabla\Psi_m(\mathbf{r})}_{\equiv\mathbf{0}} - \nabla_\perp \Psi_m \cdot \underbrace{rot\mathbf{z}}_{\equiv\mathbf{0}} \equiv 0, \end{aligned}$$

when $\Psi_m(\mathbf{r})$ is an arbitrary twice differentiable function†. Boundary conditions at the problem (3.35) acquire the form of Neumann condition for function $\Psi_m(\mathbf{r})$ as

$$\frac{\partial}{\partial n_i}\Psi_m(\mathbf{r})|_{L_i} = 0, \qquad i = 0, 1, \ldots, N, \tag{3.37}$$

†So, chosen form (3.34) for the vectors sought allow us to exclude the boundary conditions (2.53) from subsequent analysis of the boundary eigenvalue problem for operator W_H.

since the following identities hold

$$(\mathbf{n}\cdot\nabla_\perp\Psi_m)\,|_L = \left(\tfrac{\partial}{\partial n}\Psi_m\right)|_L \equiv \left(\tfrac{\partial}{\partial n_i}\Psi_m\right)|_{L_i} = 0,$$

$$\begin{gathered}(\mathbf{l}\cdot[\nabla_\perp\Psi_m\times\mathbf{z}])\,|_L = ([\mathbf{z}\times\mathbf{l}]\cdot\nabla_\perp\Psi_m)\,|_L = \\ = (\mathbf{n}\cdot\nabla_\perp\Psi_m)\,|_L \equiv \left(\tfrac{\partial}{\partial n_i}\Psi_m\right)|_{L_i} = 0,\end{gathered}$$

where $i = 0, 1, \ldots, N$.

Vector differential equations both at the problem (3.35) can be rearranged to the following form:

$$\nabla_\perp\left(\Delta_\perp\Psi_m + p_{+m}\sqrt{\epsilon_0\mu_0}\Psi_m\right) = 0, \tag{3.38}$$

where $\Delta_\perp \equiv \nabla_\perp\cdot\nabla_\perp = \Delta - \partial^2/\partial z^2$ is *transverse part of Laplacian.* Indeed, equation $[\mathbf{z}\times\nabla_\perp]\,\nabla_\perp\cdot\nabla_\perp\Psi_m = p_{+m}\,[\nabla_\perp\Psi_m\times\mathbf{z}]$ is equivalent to $[\mathbf{z}\times\nabla_\perp(\Delta_\perp\Psi_m)] = [\mathbf{z}\times\nabla_\perp(-p_{+m}\Psi_m)]$, which results in Eq. (3.38). In the second equation of the system (3.35), element $[\mathbf{z}\times\nabla_\perp]\cdot[\nabla_\perp\Psi_m\times\mathbf{z}]$ can be rearranged as follows

$$\begin{aligned}[\mathbf{z}\times\nabla_\perp]\cdot[\nabla_\perp\Psi_m\times\mathbf{z}] &= -[\nabla_\perp\times\mathbf{z}]\cdot[\nabla_\perp\Psi_m\times\mathbf{z}] \\ &= -\nabla_\perp\cdot\nabla_\perp\Psi_m \equiv -\Delta_\perp\Psi_m\end{aligned}$$

what results in Eq. (3.38) as well.

Vector equation (3.38) holds identically when scalar potential $\Psi_m(\mathbf{r})$ is a solution of two-dimensional Helmholtz equation, namely: $\Delta_\perp\Psi_m + p_{+m}\Psi_m = 0$. This equation originates simultaneously with the boundary condition (3.37) well-studied Neumann boundary eigenvalue problem for Laplacian as

$$\boxed{\left(\Delta_\perp + p_{+m}\sqrt{\epsilon_0\mu_0}\right)\Psi_m(\mathbf{r}) = 0, \qquad \tfrac{\partial}{\partial n_i}\Psi_m(\mathbf{r})|_{L_i} = 0, \qquad i = 0, 1, \ldots, N.} \tag{3.39}$$

The set $\{\Psi_m(\mathbf{r})\}_{m=1}^{\infty}$ is complete in the class of functions subjected to Neumann boundary condition. It is well known that arbitrary scalar function of $\mathbf{r}$, which satisfies Neumann boundary condition, can be presented as Fourier series in terms of the eigenfunctions $\Psi_m(\mathbf{r})$ normalized in the proper way.

Uniqueness of the potentials in the case $p_m \neq 0$. Now, we are going to prove that presentation (3.34) for the vector sought is unique. To make it sure, let us come back one more to the vector boundary eigenvalue problem (3.30), namely:

$$\begin{cases}\frac{1}{\epsilon_0}\,[\mathbf{z}\times\nabla_\perp]\,\nabla_\perp\cdot\mathbf{H}_{+m} = p_{+m}\mathbf{E}_{+m}, & (\mathbf{n}\cdot\mathbf{H}_{+m})\,|_L = 0; \qquad (a) \\ \frac{1}{\mu_0}\nabla_\perp\,[\mathbf{z}\times\nabla_\perp]\cdot\mathbf{E}_{+m} = p_{+m}\mathbf{H}_{+m}, & (\mathbf{l}\cdot\mathbf{E}_{+m})\,|_L = 0, \qquad (b)\end{cases}$$

The second boundary conditions (2.53), which have the form of

$$(\nabla_\perp\cdot\mathbf{E})\,|_{L_i} \qquad (\nabla_\perp\cdot[\mathbf{H}\times\mathbf{z}])\,|_{L_i} \qquad i = 0, 1, \ldots, N,$$

allow to present the vectors $\mathbf{E}_{+m}$ and $\mathbf{H}_{+m}$ at the vector boundary problem via *two* scalar potentials $\Psi'_m(\mathbf{r})$ and $\Psi''_m(\mathbf{r})$ sought as follows

$$\mathbf{E}_{+m} = \sqrt[-2]{\epsilon_0}\left[\nabla_\perp \Psi'_m(\mathbf{r}) \times \mathbf{z}\right] \qquad \mathbf{H}_{+m} = \sqrt[-2]{\mu_0}\nabla_\perp \Psi''_m(\mathbf{r}).$$

Substitution of these presentations in the vector boundary eigenvalue problem yields

$$\begin{cases} \nabla_\perp\left(\Delta_\perp \Psi''_m + p_{+m}\sqrt{\epsilon_0\mu_0}\Psi'_m\right) = \mathbf{0}, & (a) \\ & \frac{\partial}{\partial n}\Psi'_m|_L = 0, \quad \frac{\partial}{\partial n}\Psi''_m|_L = 0, \\ \nabla_\perp\left(\Delta_\perp \Psi'_m + p_{+m}\sqrt{\epsilon_0\mu_0}\Psi''_m\right) = \mathbf{0}, & (b) \end{cases}$$

Substraction and summation of the equations (a) and (b) yield a pair of new vector boundary value problems with Neumann boundary conditions for the potentials $\Psi'_m(\mathbf{r})$ and $\Psi''_m(\mathbf{r})$ as:

$$\begin{aligned} &\nabla_\perp\left\{\left(\Delta_\perp - p_{+m}\sqrt{\epsilon_0\mu_0}\right)\left(\Psi'_m - \Psi''_m\right)\right\} = \mathbf{0}, \quad \frac{\partial}{\partial n}\left(\Psi'_m - \Psi''_m\right)|_L = 0; \quad (c) \\ &\nabla_\perp\left\{\left(\Delta_\perp + p_{+m}\sqrt{\epsilon_0\mu_0}\right)\left(\Psi'_m + \Psi'_m\right)\right\} = \mathbf{0}, \quad \frac{\partial}{\partial n}\left(\Psi'_m + \Psi'_m\right)|_L = 0; \quad (d) \end{aligned} \tag{3.40}$$

Let us denote

$$\begin{aligned} \left(\Delta_\perp - p_{+m}\sqrt{\epsilon_0\mu_0}\right)\left(\Psi'_m - \Psi''_m\right) \equiv F_m^{(-)}, \\ \left(\Delta_\perp + p_{+m}\sqrt{\epsilon_0\mu_0}\right)\left(\Psi'_m + \Psi'_m\right) \equiv F_m^{(+)}. \end{aligned} \tag{3.41}$$

The a pair of the problems (3.40) may be rewritten as follows

$$\nabla_\perp F_m^{(\pm)}(\mathbf{r}) = \mathbf{0}; \qquad \frac{\partial}{\partial n}\left(\Psi'_m - \Psi''_m\right)|_L = 0, \quad \frac{\partial}{\partial n}\left(\Psi'_m + \Psi'_m\right)|_L = 0. \tag{3.42}$$

Here, we need to address to Neumann boundary eigenvalue problem (3.39) one more, which we rewrite as follows

$$\left(\Delta_\perp + p_k\sqrt{\epsilon_0\mu_0}\right)\psi_k(\mathbf{r}) = 0, \qquad \frac{\partial}{\partial n_i}\psi_k(\mathbf{r})|_{L_i} = 0, \qquad i = 0, 1, \ldots, N. \tag{3.43}$$

The following integral over domain S bounded the multiconnected contour L

$$\int_S \nabla_\perp \psi_k(\mathbf{r}) \cdot \nabla_\perp F_m^{(\pm)}(\mathbf{r})\, ds = 0 \tag{3.44}$$

is equal to zero, since functions $\nabla_\perp F_m^{(\pm)}(\mathbf{r})$ are identical zero accordingly to Eq. (3.42) . In vector analysis, well-known identity $\nabla_\perp \cdot (\varphi \nabla_\perp \psi) = \nabla_\perp \psi \cdot \nabla_\perp \varphi + \varphi \Delta_\perp \psi$ holds. Using it, the integrand at Eq. (3.44) can be rearranged as follows

$$\begin{aligned} \nabla_\perp \psi_k \cdot \nabla_\perp F_m^{(\pm)} &= -F_m^{(\pm)}\Delta_\perp \psi_k + \nabla_\perp \cdot \left(F_m^{(\pm)}\nabla_\perp \psi_k\right) \\ &= p_k\sqrt{\epsilon_0\mu_0}\psi_k F_m^{(\pm)} + \nabla_\perp \cdot \left(F_m^{(\pm)}\nabla_\perp \psi_k\right) \end{aligned}$$

since $-\Delta_{\perp}\psi_k = p_k\sqrt{\epsilon_0\mu_0}\psi_k$ in accordance with the eigenvalue problem (3.43). After substitution of this equation at the right-hand-side of Eq. (3.44) and using then two-dimensional version of Gauss' theorem, one can obtain

$$\int_S \nabla_{\perp}\psi_k \cdot \nabla_{\perp}F_m^{(\pm)}ds \equiv p_k\sqrt{\epsilon_0\mu_0}\int_S \psi_k F_m^{(\pm)}ds + \oint_L F_m^{(\pm)}\frac{\partial}{\partial n}\psi_k d\ell = 0 \tag{3.45}$$

However, the contour integral at the right-hand-side of Eq. (3.45) should be cancelled due to the boundary condition at the problem (3.43). Hence, the following equation

$$p_k\sqrt{\epsilon_0\mu_0}\int_S \psi_k(\mathbf{r})\,F_m^{(\pm)}(\mathbf{r})\,ds = 0,$$

where $p_k \neq 0$, should hold for *all the functions* from the set $\{\psi_k(\mathbf{r})\}$. It means that

$$F_m^{(-)}(\mathbf{r}) = 0, \qquad F_m^{(+)}(\mathbf{r}) = 0, \tag{3.46}$$

since the set $\{\psi_k(\mathbf{r})\}$ is complete. The first equation yields

$$F_m^{(-)}(\mathbf{r}) \equiv \left(\Delta_{\perp} - p_{+m}\sqrt{\epsilon_0\mu_0}\right)(\Psi'_m - \Psi''_m) = 0. \tag{3.47}$$

Hence, $(\Psi'_m - \Psi''_m) = 0$, since $\Delta_{\perp}(\Psi'_m - \Psi''_m) < 0$ always and $p_{+m} > 0$ in Eq. (3.47). Therefore

$$\Psi'_m(\mathbf{r}) = \Psi''_m(\mathbf{r}) \equiv \Psi_m(\mathbf{r}),$$

since the set $\{\psi_k(\mathbf{r})\}$ is complete. The second equation yields Helmholtz equation

$$\begin{aligned} F_m^{(+)}(\mathbf{r}) &\equiv \left(\Delta_{\perp} + p_{+m}\sqrt{\epsilon_0\mu_0}\right)(\Psi'_m + \Psi''_m) \\ &= 2\left(\Delta_{\perp} + p_{+m}\sqrt{\epsilon_0\mu_0}\right)\Psi_m(\mathbf{r}) = 0, \end{aligned}$$

which repeat given at the eigenvalue problem (3.39).

Case $p_0 = 0$.

Let us return to the question we were deferred. Consider first possible formally case $p_0 = 0$ at the scalar boundary eigenvalue problem (3.39) for the potential $\Psi_0(\mathbf{r})$, which can be written as

$$\Delta_{\perp}\Psi_0(\mathbf{r}) = 0, \qquad \tfrac{\partial}{\partial n_i}\Psi_0(\mathbf{r})|_{L_i} = 0. \tag{3.48}$$

Since potential $\Psi_0(\mathbf{r})$ satisfies Laplace equation, it is a harmonic function. In accordance with the minimum-maximum theorem from the theory of harmonic functions, solution to the problem (3.48) in the closed domain $\bar{S} = S + L$ should be some constant C. The constant C can be chosen as $C = 1$ without loss of generality. So,

$$\boxed{\Psi_0(\mathbf{r}) = 1, \qquad \mathbf{r} \in S + L.} \tag{3.49}$$

Thus, the eigenvalue $p_0 = 0$ results in zero-valued vectors of kind (3.34) : $\mathbf{E}_0 = \mathbf{0}$ and $\mathbf{H}_0 = \mathbf{0}$, since $\nabla_\perp \Psi_0(\mathbf{r}) = \nabla_\perp 1 \equiv \mathbf{0}$.

When the eigenvalue $p_0 = 0$ is substituted at the first operator equation (3.22), it acquires the form of $W_H Y_0(\mathbf{r}) = 0$, where $Y_0(\mathbf{r}) = col\,(\mathbf{E}_0(\mathbf{r}), \mathbf{H}_0(\mathbf{r}))$. Vector boundary value problem (3.23), which is equivalent to this operator equation, acquires the form of

$$\begin{aligned} &[\mathbf{z} \times \nabla_\perp]\, \underline{\nabla_\perp \cdot \mathbf{H}_0(\mathbf{r})} = 0, \qquad (\mathbf{n} \cdot \mathbf{H}_0(\mathbf{r}))\,|_L = 0; \qquad (a) \\ &\nabla_\perp \underline{[\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_0(\mathbf{r})} = 0, \qquad (\mathbf{l} \cdot \mathbf{E}_0(\mathbf{r}))\,|_L = 0. \qquad (b) \end{aligned} \tag{3.50}$$

So, we have now a pair of *independent* problems (see lines (a) and (b) in Eq. (3.50)) instead of the pair of *simultaneous* equations which we had before when $p_m \neq 0$ (see Eq. (3.23)). Left-hand-sides at the differential equations (3.50) can turn into zero when

$$\nabla_\perp \cdot \mathbf{H}_0(\mathbf{r}) = 0 \quad \text{and} \quad [\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_0(\mathbf{r}) = 0. \tag{3.51}$$

We are interested in the nontrivial solutions, when $\mathbf{E}_0(\mathbf{r}) \neq \mathbf{0}$ and $\mathbf{H}_0(\mathbf{r}) \neq \mathbf{0}$. The forms of differential equations (3.51) suggest to present their solutions as

$$\boxed{\mathbf{H}_0(\mathbf{r}) = \sqrt[-2]{\mu_0}\,[\mathbf{z} \times \nabla_\perp \phi'(\mathbf{r})] \quad \text{and} \quad \mathbf{E}_0(\mathbf{r}) = \sqrt[-2]{\epsilon_0}\, \nabla_\perp \phi''(\mathbf{r}),} \tag{3.52}$$

where $\phi'(\mathbf{r})$ and $\phi''(\mathbf{r})$ are some twice differentiable potential functions. Indeed,

$$\begin{gathered} \sqrt{\mu_0}\nabla_\perp \cdot \mathbf{H}_0 = \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \phi'(\mathbf{r})] = \\ = div\,[\mathbf{z} \times grad\phi'] = grad\phi' \cdot \underbrace{rot\mathbf{z}}_{\equiv 0} - \mathbf{z} \cdot \underbrace{rotgrad\phi'}_{\equiv 0} \equiv 0, \end{gathered}$$

$$\begin{gathered} \sqrt{\epsilon_0}\,[\mathbf{z} \times \nabla_\perp] \cdot \mathbf{E}_0 = -\,[\nabla_\perp \times \mathbf{z}] \cdot \mathbf{E}_0 = \\ = -\nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}_0] = -\nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \phi''(\mathbf{r})] \equiv 0. \end{gathered}$$

In addition to the double differentiability of the functions $\phi'(\mathbf{r})$ and $\phi''(\mathbf{r})$, which required the differential equations (3.50), the boundary conditions

$$\begin{gathered} \sqrt{\mu_0}\,(\mathbf{n} \cdot \mathbf{H}_0)\,|_L = (\mathbf{n} \cdot [\mathbf{z} \times \nabla_\perp \phi'])\,|_L \\ = ([\mathbf{n} \times \mathbf{z}] \cdot \nabla_\perp \phi')\,|_L = (\mathbf{l} \cdot \nabla_\perp \phi')\,|_L = \left(\tfrac{\partial}{\partial l}\phi'\right)|_L = 0, \end{gathered}$$

$$\sqrt{\epsilon_0}\,(\mathbf{l} \cdot \mathbf{E}_0)\,|_L = (\mathbf{l} \cdot \nabla_\perp \phi'')\,|_L = \left(\tfrac{\partial}{\partial l}\phi''\right)|_L = 0$$

impose one more constraint on the potentials. Over the contour L, potentials $\phi'(\mathbf{r})$ and $\phi''(\mathbf{r})$ both must have some constant values. In the case of multiconnected contour under consideration, the potential each may have *constant but different values* on different components of the multiconnected contour L. So, the boundary conditions for the potentials should be read as follows

$$\boxed{\phi'(\mathbf{r})\,|_{L_i} = C_i', \quad \phi''(\mathbf{r})\,|_{L_i} = C_i'', \quad i = 0, 1, \ldots, N,} \tag{3.53}$$

where C_i' and C_i'' are some constants.

Properties of the operator W_H does not permit to obtain more information about the eigenvectors which are presented via potentials $\phi'(\mathbf{r})$ and $\phi''(\mathbf{r})$. This problem can be solved completely in the course of parallel analysis of the eigensolutions, which correspond to zero-valued eigenvalues of the operators W_H and W_E (see end of the next section).

3.2.2 Boundary Eigenvalue Problems for Operator W_E

Equivalent *Vector* Boundary Eigenvalue Problem.

Now, we can consider briefly the second operator eigenvalue equation from the system (3.22), since this analysis is similar to previous. Boundary eigenvalue problem, which is equivalent to operator equation $W_E Z_n(\mathbf{r}) = p_n Z_n(\mathbf{r})$, is the following:

$$\boxed{\begin{cases} \frac{1}{\epsilon_0}\nabla_\perp\left[\nabla_\perp \times \mathbf{z}\right]\cdot\mathbf{H}_n(\mathbf{r}) = q_n\mathbf{E}_n(\mathbf{r}), & (\mathbf{n}\cdot\mathbf{H}_n(\mathbf{r}))\,|_L = 0; \quad (a) \\ \frac{1}{\mu_0}\left[\nabla_\perp \times \mathbf{z}\right]\nabla_\perp\cdot\mathbf{E}_n(\mathbf{r}) = q_n\mathbf{H}_n(\mathbf{r}), & (\mathbf{l}\cdot\mathbf{E}_n(\mathbf{r}))\,|_L = 0. \quad (b) \end{cases}} \tag{3.54}$$

where two-component vectors $\mathbf{E}_n(\mathbf{r})$ and $\mathbf{H}_n(\mathbf{r})$ are the constituents of the four-component eigenvector of the operator W_E

$$Z_n(\mathbf{r}) = \begin{pmatrix} \mathbf{E}_n(\mathbf{r}) \\ \mathbf{H}_n(\mathbf{r}) \end{pmatrix}, \qquad \begin{matrix} (\mathbf{n}\cdot\mathbf{H}_n(\mathbf{r}))\,|_L = 0; \\ (\mathbf{l}\cdot\mathbf{E}_n(\mathbf{r}))\,|_L = 0, \end{matrix} \tag{3.55}$$

which satisfies the same boundary conditions over L as the eigenvectors $Y_m(\mathbf{r})$.

Degeneration of the eigenvalues $q_n \neq 0$. In the general case, solution to the problem (3.54) is a pair of complex-valued vector functions

$$\mathbf{E}_n(\mathbf{r}) = \mathbf{E}_n'(\mathbf{r}) + i\mathbf{E}_n''(\mathbf{r}), \qquad \mathbf{H}_n(\mathbf{r}) = \mathbf{H}_n'(\mathbf{r}) + i\mathbf{H}_n''(\mathbf{r}). \tag{3.56}$$

Complex conjugated functions $\mathbf{E}_n^* = \mathbf{E}_n' - i\mathbf{E}_m''$ and $\mathbf{H}_n^* = \mathbf{H}_n' - i\mathbf{H}_n''$ are a solution to this problem as well, which correspond to the same eigenvalue $q_n \neq 0$. It means that each eigenvalue $q_n \neq 0$ is double, since there is a pair of eigenvectors

$$Z_n = \begin{pmatrix} \mathbf{E}_n(\mathbf{r}) \\ \mathbf{H}_n(\mathbf{r}) \end{pmatrix}, \qquad Z_n^* = \begin{pmatrix} \mathbf{E}_n^*(\mathbf{r}) \\ \mathbf{H}_n^*(\mathbf{r}) \end{pmatrix}, \tag{3.57}$$

which correspond to the same eigenvalue. In practical situations, one can use another pair of eigenvectors corresponding to the same eigenvalue $q_n \neq 0$ as

$$Z_n' = \begin{pmatrix} \mathbf{E}_n' \\ \mathbf{H}_n' \end{pmatrix}, \qquad Z_n'' = i\begin{pmatrix} \mathbf{E}_n'' \\ \mathbf{H}_n'' \end{pmatrix}, \tag{3.58}$$

or combinations of the linearly independent vectors from Eqs. (3.57) and (3.58).

Matrix differential procedure $\hat{W}_E$ has the same block symmetry as the procedure $\hat{W}_H$. It induces respective symmetry of the eigenvalues and eigenvectors of the operator W_E. The cases of the eigenvalues $q_n \neq 0$ and $q_n|_{n=0} \equiv q_0 = 0$ should be analyzed individually as well.

Symmetry of the eigenvalues and eigenvectors in the case $q_n \neq 0$. All the eigenvalues $q_n \neq 0$ of the operator W_E lie symmetrically on real axis Oq with respect to the point $q = 0$ and may be put in order as

$$q_{+n} = -q_{-n} > 0. \tag{3.59}$$

When the vectors

$$Z_{+n} = \begin{pmatrix} \mathbf{E}_{+n} \\ \mathbf{H}_{+n} \end{pmatrix}, \qquad Z^*_{+n} = \begin{pmatrix} \mathbf{E}^*_{+n} \\ \mathbf{H}^*_{+n} \end{pmatrix} \tag{3.60}$$

correspond to the eigenvalue $q_{+n} > 0$, then the vectors

$$Z_{-n} = \begin{pmatrix} \mathbf{E}_{+n} \\ -\mathbf{H}_{+n} \end{pmatrix}, \qquad Z^*_{-n} = \begin{pmatrix} \mathbf{E}^*_{+n} \\ -\mathbf{H}^*_{+n} \end{pmatrix} \tag{3.61}$$

correspond to the eigenvalue $q_{-n} = -q_{+n}$, where the star means complex conjugation. It can be proved in the same way as in the case of operator W_H. When a pair of eigenvectors corresponding to the eigenvalues $q_{+n} > 0$, is chosen in the form of (3.58), then the pair of eigenvectors, corresponding to the symmetric eigenvalue $q_{-n} = -q_{+n}$, should be get as

$$Z'_n = \begin{pmatrix} \mathbf{E}'_n \\ -\mathbf{H}'_n \end{pmatrix}, \qquad Z''_n = i \begin{pmatrix} \mathbf{E}''_n \\ -\mathbf{H}''_n \end{pmatrix}. \tag{3.62}$$

***Scalar* eigenvalue problem for the case $q_n \geq 0$.**

We start here analysis of the cases $q_0 = 0$ and $q_{+n} > 0$ simultaneously, but separation them will be made further on. Before the separation, subscript $n \equiv n \geq 0$ will be used.

The form of the differential equations at vector boundary eigenvalue problem (3.54) suggest to present vectors $\mathbf{E}_n$ and $\mathbf{H}_n$ via some scalar potential $\Phi(\mathbf{r})$ in the following way:

$$\mathbf{E}_n = \sqrt[-2]{\epsilon_0} \nabla_\perp \Phi(\mathbf{r}), \qquad \mathbf{H}_n = \sqrt[-2]{\mu_0} \left[\mathbf{z} \times \nabla_\perp \Phi(\mathbf{r})\right]. \tag{3.63}$$

where $\Phi(\mathbf{r})$ should be a twice differentiable function. Substitution of these presentations in vector differential equations (3.54) yields

$$\nabla_\perp \left(\Delta_\perp \Phi + q_n \sqrt{\epsilon_0 \mu_0} \Phi\right) = \mathbf{0}, \tag{3.64}$$

where $q_n \geq 0$, and $\Delta_\perp$ is transverse part of Laplacian as before. Substitution of the vectors (3.63) at the boundary conditions in Eqs. (3.54) results in

$$\tfrac{\partial}{\partial l_i}\Phi\left(\mathbf{r}\right)|_{L_i} = 0, \qquad i = 0, 1, \ldots, N.$$

It means that function $\Phi\left(\mathbf{r}\right)$ must have some constant values C_i at the components L_i of the contour L, where $i = 0, 1, \ldots, N$. In other words, the contours L_i are the lines of some constant levels of the function $\Phi\left(\mathbf{r}\right)$.

Equality $(\Delta_\perp\Phi + q_n\Phi) = 0$ satisfies Eq. (3.64) identically. This equality, and the boundary conditions for $\Phi\left(\mathbf{r}\right)$ on the contours L_i originate simultaneously the following *scalar* boundary eigenvalue problem for transverse Laplacian:

$$\left(\Delta_\perp + q_n\sqrt{\epsilon_0\mu_0}\right)\Phi\left(\mathbf{r}\right) = 0, \qquad \Phi\left(\mathbf{r}\right)|_{L_i} = C_i, \qquad i = 0, 1, \ldots, N. \tag{3.65}$$

Potential $\Phi\left(\mathbf{r}\right)$ sought we separate on two parts as $\Phi\left(\mathbf{r}\right) = \Phi_n\left(\mathbf{r}\right) + \Phi_0\left(\mathbf{r}\right)$, each of which should be specified. To eliminate somewhat an arbitrariness from this separation, we impose different boundary conditions on these parts over the contour L as follows

$$\Phi\left(\mathbf{r}\right) = \Phi_n\left(\mathbf{r}\right) + \Phi_0\left(\mathbf{r}\right): \qquad \Phi_n\left(\mathbf{r}\right)|_{L_i} = 0. \quad \Phi_0\left(\mathbf{r}\right)|_{L_i} = C_i, \tag{3.66}$$

where the constants C_i are the same as in Eq. (3.65). Further, to eliminate an arbitrariness from this separation completely, we subject function $\Phi_0\left(\mathbf{r}\right)$ to Helmholtz equation (3.65) with admissible eigenvalue $q_0 = 0$. Hence, function $\Phi_n\left(\mathbf{r}\right)$ must satisfy then Helmholtz equation (3.65) with all the other positive eigenvalues of q_n which we denote as $q_{+n} > 0$. Thus, we have two different problems for different parts of $\Phi\left(\mathbf{r}\right)$ from separation (3.66), which specify them completely in the form of the boundary eigenvalue problems for Laplacian $\Delta_\perp$ in the following ways:

$$\boxed{\Delta_\perp\Phi_0\left(\mathbf{r}\right) = 0, \qquad \Phi_0\left(\mathbf{r}\right)|_{L_i} = C_i, \qquad i = 0, 1, \ldots, N;} \tag{3.67}$$

$$\boxed{\left(\Delta_\perp + q_{+n}\sqrt{\epsilon_0\mu_0}\right)\Phi_n\left(\mathbf{r}\right) = 0, \qquad \Phi_n\left(\mathbf{r}\right)|_{L_i} = 0, \qquad i = 0, 1, \ldots, N,} \tag{3.68}$$

where $q_{+n} > 0$. Hence, potential $\Phi\left(\mathbf{r}\right)$ in the form of Eq. (3.66) is specified completely.

Presentation of the vector fields (3.63)via potential $\Phi_0\left(\mathbf{r}\right)$ may be rewritten as

$$\mathbf{E}_n = \mathbf{E}_{+n} + \mathbf{E}^{(0)}, \qquad \mathbf{H}_n = \mathbf{H}_{+n} + \mathbf{H}^{(0)}, \tag{3.69}$$

where the following notation is introduced:

$$\begin{aligned} \mathbf{E}_{+n} &= \sqrt[-2]{\epsilon_0}\nabla_\perp\Phi_n, & \mathbf{H}_{+n} &= \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\Phi_n\right]; \\ \mathbf{E}^{(0)} &= \sqrt[-2]{\epsilon_0}\nabla_\perp\Phi_0, & \mathbf{H}^{(0)} &= \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\Phi_0\right]. \end{aligned} \tag{3.70}$$

Uniqueness of this presentation can be proved in the same way as it was made above in the case of the operator W_H.

Case $q_0 = 0$.

Let us first finish discussion concerning the problem (3.67). When the contour L is singly-connected (i.e., $N = 0$), its solution is very simple: $\Phi_0(\mathbf{r}) = C_0$ in accordance with the minimum-maximum theorem for harmonic functions. Hence, $\mathbf{E}^{(0)}(\mathbf{r}) = \mathbf{H}^{(0)}(\mathbf{r}) = \mathbf{0}$, since $\nabla_\perp \Phi_0(\mathbf{r}) = \nabla_\perp C_0 \equiv \mathbf{0}$. One can put $C_0 = 1$ without loss in generality.

Let it be now $N > 0$ but finite. In this case, harmonic function $\Phi_0(\mathbf{r})$ can be presented as a combination of some linearly independent harmonic functions $u_k(\mathbf{r})$ in the following way:

$$\Phi_0(\mathbf{r}) = \sum_{k=0}^{N} u_k(\mathbf{r}), \tag{3.71}$$

where $u_k(\mathbf{r})$'*s* are solutions to the following boundary value problems for Laplace equation:

$$\begin{aligned} k = 0: \qquad & \Delta_\perp u_0(\mathbf{r}) = 0, \quad u_0(\mathbf{r})|_{L_i} = C_0; \quad i = 0, 1, \ldots, N; \\ k = 1, \ldots, N: \qquad & \Delta_\perp u_k(\mathbf{r}) = 0, \quad u_k(\mathbf{r})|_{L_i} = \delta_{ki} C_i; \quad i = 0, 1, \ldots, N. \end{aligned} \tag{3.72}$$

Here, δ_{ki} is Kronecker's delta, and constants C_0, C_1, ..., C_N are the same as at the boundary conditions (3.67). It is evident that

$$u_0(\mathbf{r}) = C_0 \qquad \textit{while} \quad \mathbf{r} \in S + L \tag{3.73}$$

for arbitrary form of the multi-connected contour L. Constant C_0 can be chosen as $C_0 = 1$ without the loss in generality. Functions $u_k(\mathbf{r})$ with $k = 1, \ldots, N$ can be easily found when the forms of the contour components L_0, L_1, ... L_N are given. So, Eq. (3.71) can be rewritten as

$$\boxed{\Phi_0(\mathbf{r}) = 1 + \textstyle\sum_{k=1}^{N} u_k(\mathbf{r}).} \tag{3.74}$$

Let us return one more to the vector fields $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$ from Eqs. (3.70) generating by harmonic function $\Phi_0(\mathbf{r})$. Its form given in Eq. (3.74) suggests to introduce a set the vectors generating by functions $u_k(\mathbf{r})$ as follows

$$\mathbf{E}^{(0)} \equiv \mathbf{E}_k^{(0)} = \sqrt[-2]{\epsilon_0}\,\nabla_\perp u_k; \quad \mathbf{H}^{(0)} \equiv \mathbf{H}_k^{(0)} = \sqrt[-2]{\mu_0}\,[\mathbf{z} \times \nabla_\perp u_k]; \quad k = 1, \ldots, N. \tag{3.75}$$

Subscript $k = 0$ is omitted, since $\nabla_\perp u_0 = \nabla_\perp 1 \equiv \mathbf{0}$ in the general case of arbitrary multi-connected contour.

The sets $\left\{\mathbf{E}_k^{(0)} = \sqrt[-2]{\epsilon_0}\,\nabla_\perp u_k\right\}_{k=1}^{N}$ and $\left\{\mathbf{H}_k^{(0)} = \sqrt[-2]{\mu_0}\,[\mathbf{z} \times \nabla_\perp u_k]\right\}_{k=1}^{N}$ each involve linearly independent vectors only, since generating them harmonic functions are linearly independent. We may name the vectors (3.75) as *harmonic vectors.* The constants $C_1, C_2, \ldots, C_N$, which specify the harmonic potentials via problems (3.72), can be then found while the sets of the harmonic vectors will be subjected to orthonormalization‡.

In closing, we should substitute eigenvalue $q_0 = 0$ of operator W_E at the second operator equation (3.22) . In such a case, it acquires the form of $W_E Z_0(\mathbf{r}) = 0$, where $Z_0(\mathbf{r}) = col\,(\mathbf{E}_0(\mathbf{r}), \mathbf{H}_0(\mathbf{r}))$. Operator equation $W_E Z_0(\mathbf{r}) = 0$ supplies equivalent vector boundary value problem as

$$\begin{aligned} &\nabla_\perp \underline{[\nabla_\perp \times \mathbf{z}] \cdot \mathbf{H}_0(\mathbf{r})} = \mathbf{0}, \qquad (\mathbf{n}\cdot\mathbf{H}_0(\mathbf{r}))\,|_L = 0; \qquad (a)\\ &[\nabla_\perp \times \mathbf{z}]\, \underline{\nabla_\perp \cdot \mathbf{E}_0(\mathbf{r})} = \mathbf{0}, \qquad (\mathbf{l}\cdot\mathbf{E}_0(\mathbf{r}))\,|_L = 0. \qquad (b)\end{aligned} \tag{3.76}$$

So, we have again a pair of independent problems (see lines (a) and (b) in Eq. (3.76)) instead of the pair of simultaneous equations which we had when $q_n \neq 0$ (see Eq. (3.54)). Left-hand-sides at the differential equations (3.76) become identical zero when the following equalities hold:

$$[\nabla_\perp \times \mathbf{z}] \cdot \mathbf{H}_0(\mathbf{r}) = 0 \qquad \nabla_\perp \cdot \mathbf{E}_0(\mathbf{r}) = 0.$$

These equations have nontrivial solutions $\mathbf{E}_0(\mathbf{r}) \neq \mathbf{0}$ and $\mathbf{H}_0(\mathbf{r}) \neq \mathbf{0}$, which can be presented via some twice differentiable potentials $\psi'(\mathbf{r})$ and $\psi''(\mathbf{r})$ as follows

$$\boxed{\mathbf{E}_0(\mathbf{r}) = \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \psi''(\mathbf{r}) \times \mathbf{z}] \qquad \mathbf{H}_0(\mathbf{r}) = \sqrt[-2]{\mu_0}\nabla_\perp \psi'(\mathbf{r})\,.} \tag{3.77}$$

Subjecting of the vectors (3.77) to the boundary conditions at the problem (3.76) supplies equivalent Neumann boundary conditions for the scalar potentials as

$$\boxed{\tfrac{\partial}{\partial n_i}\psi'(\mathbf{r})\,|_{L_i} = 0, \quad \tfrac{\partial}{\partial n_i}\psi''(\mathbf{r})\,|_{L_i} = 0, \quad i = 0, 1, \ldots, N.} \tag{3.78}$$

Properties of the operator W_E does not furnish with more information about these potentials However, understanding of this problem can be achieved via parallel analysis of the eigensolutions, which correspond to zero-valued eigenvalues of the operators W_H and W_E.

3.3 Main Properties of the Eigenvalues and Eigenvectors of *WBO*

Summary 7 *At the first two following subsections, all main results (obtained before for the eigensolutions of the* WBO W_E *and* W_H*) are concentrated. Further on, they will be* ***essentially needed*** *for development of our approach. Therefore the content of these subsections one must know and can use further as a hand-book. The following questions regard to the third subsection. How to prove that all the eigenvectors of operator* W_H*, corresponding to* ***infinite number*** *of its eigenvalues* $p_{\pm m} \neq 0$*, are the*

‡We'll consider this question specially later on in a subsection titled as *"Orthonormalization of the Harmonic Eigenvectors"*.

eigenvectors of operator W_E corresponding to its ***single*** *eigenvalue $q_0 = 0$? How to prove that all the eigenvectors of operator W_E, corresponding to* ***infinite number*** *of its eigenvalues $q_{\pm n} \neq 0$, are the eigenvectors of operator W_H corresponding to its* ***single*** *eigenvalue $p_0 = 0$? Why one may consider the harmonic eigenvectors of operator W_E as the intersection of W_H and W_E eigenvector sets?*

3.3.1 Properties of Eigenvalues and Eigenvectors of Operator W_E.

1. Spectrum of operator W_E is real and discrete.

2. All nonzero-valued eigenvalues $q_n \neq 0$ are located on real axis symmetrically with respect to the pint $q_0 = 0$ and may put in order as

$$q_{+n} = -q_{-n} > 0. \tag{3.79}$$

   Hence, it is enough to find the positive eigenvalues only, since negative ones can be then found as the mirror reflection (3.79) of the positive eigenvalues on real axis with respect to the point $q_0 = 0$.

3. All the *positive* eigenvalues coincide with squared eigenvalues of so-called "membrane" functions, which are the eigensolutions of scalar Dirichlet's boundary eigenvalue problem for the transverse Laplacian $\Delta_\perp$:

$$\begin{gathered}(\Delta_\perp + \varkappa_n^2)\,\Phi_n(\mathbf{r}) = 0, \quad \Phi_n(\mathbf{r})\,|_{L_i} = 0, \quad i = 0, 1, \ldots, N; \\ q_{+n} = \varkappa_n^2/\sqrt{\epsilon_0\mu_0} > 0.\end{gathered} \tag{3.80}$$

   The membrane functions $\Phi_n(\mathbf{r})$ serve as the potentials for eigenvectors $Z_{\pm n}(\mathbf{r})$ of operator W_E, which correspond to its *non-zero* valued eigenvalues $q_{\pm n} \neq 0$.

4. Eigenvectors of the operator W_E, corresponding to the positive eigenvalues $q_{+n} > 0$, are specified via the membrane functions (3.80) as follows

$$Z_{+n}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\Phi_n(\mathbf{r}) \\ \sqrt[-2]{\mu_0}\,[\mathbf{z}\times\nabla_\perp\Phi_n(\mathbf{r})] \end{pmatrix}. \tag{3.81}$$

5. When the membrane function is real, then the eigenvalues $q_{+n} > 0$ are simple: i.e., eigenvector (3.81), corresponding to this eigenvalue, is single.

6. When the membrane functions are obtained as a complex valued function, i.e.,

$$\Phi_n(\mathbf{r}) = \Phi'_n(\mathbf{r}) + i\Phi''_n(\mathbf{r}), \tag{3.82}$$

   then the eigenvalue $q_{+n} > 0$ is double. In such a case, there is a pair of linearly independent eigenvectors, which correspond to the same eigenvalue $q_{+n} > 0$. They can be presented via the same membrane function (3.82) either as

$$Z_{+n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\Phi_n \\ \sqrt[-2]{\mu_0}\,[\mathbf{z}\times\nabla_\perp\Phi_n] \end{pmatrix}, \quad Z^*_{+n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\Phi^*_n \\ \sqrt[-2]{\mu_0}\,[\mathbf{z}\times\nabla_\perp\Phi^*_n] \end{pmatrix}, \tag{3.83}$$

or as follows

$$Z'_{+n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi'_n \\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi'_n\right] \end{pmatrix}, \quad Z''_{+n} = i\begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi''_n \\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi''_n\right] \end{pmatrix}. \tag{3.84}$$

7. Eigenvectors, corresponding to the symmetrical negative eigenvalues $q_{-n} = -q_{+n} < 0$, are expressible via the same membrane function (3.82). When the function $\Phi_n(\mathbf{r})$ is real, then the eigenvalue q_{-n} is simple, and there is a single eigenvector

$$Z_{-n}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi_n(\mathbf{r}) \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi_n(\mathbf{r})\right] \end{pmatrix}, \tag{3.85}$$

which corresponds to this eigenvalue.

8. Otherwise, when $\Phi_n(\mathbf{r})$ is a complex-valued function, then there is a pair of linearly independent eigenvectors, which corresponds to each eigenvalue $q_{-n} < 0$. One the pair can be presented as

$$Z_{-n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi_n \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi_n\right] \end{pmatrix}, \quad Z^*_{-n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi^*_n \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi^*_n\right] \end{pmatrix}, \tag{3.86}$$

or the other equivalent pair can be introduced as follows

$$Z'_{-n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi'_n \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi'_n\right] \end{pmatrix}, \quad Z''_{-n} = i\begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi''_n \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi''_n\right] \end{pmatrix}. \tag{3.87}$$

9. Operator W_E has *zero* as its eigenvalue $q_0 = 0$ as well. All infinite number of the eigenvectors of operator W_H (see below), which correspond to the *non-zero valued* its eigenvalues $p_{\pm m} \neq 0$ of operator W_H, correspond to the single eigenvalue $q_0 = 0$ of operator W_E. So, the eigenvalue $q_0 = 0$ of operator W_E has infinite power of degeneration. This assertion will be proved at the end of this subsection.

10. When the contour L of the waveguide cross-section is simple (i.e., $L \equiv L_0$ is a singly-connected contour), there is not other eigenvectors which correspond to the eigenvalue $q_0 = 0$ of operator W_E.

11. When the contour L is multiconnected (i.e., it consists of the components $L_0, L_1, \ldots, L_N$), then some different from constant harmonic potential function $\Phi_0(\mathbf{r})$ exists, which generates additionally a *finite number* of the eigenvectors corresponding to the eigenvalue $q_0 = 0$. This harmonic function $\Phi_0(\mathbf{r})$ can be presented as a combination of the other harmonic functions as follows

$$\Phi_0(\mathbf{r}) = 1 + \sum_{k=1}^{N} u_k(\mathbf{r}). \tag{3.88}$$

Here, the constant 1 is the harmonic function $u_0(\mathbf{r}) = 1$, and the other linearly independent harmonic functions $u_1(\mathbf{r}), \ldots, u_N(\mathbf{r})$ can be found as the solutions to Laplace equation which satisfy some inhomogeneous boundary conditions as

$$k = 1, \ldots, N: \qquad \Delta_\perp u_k(\mathbf{r}) = 0, \quad u_k(\mathbf{r})|_{L_i} = \delta_{ki} C_i; \quad i = 0, 1, \ldots, N, \tag{3.89}$$

where $C_i : C_1, \ldots, C_N$ are some constants, δ_{ki} -Kronecker's delta.

12. Each harmonic function from the set $\{u_k(\mathbf{r})\}_{k=1}^{N}$ generates a pair of the following linearly independent harmonic eigenvectors:

$$Z_{+k}^{(u)} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp u_k \\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp u_k\right] \end{pmatrix}, \quad Z_{-k}^{(u)} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp u_k \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp u_k\right] \end{pmatrix} \tag{3.90}$$

which correspond to the eigenvalue $q_0 = 0$ of the operator W_E. Since the harmonic functions $u_k(\mathbf{r})$ serve as the potentials for the eigenvectors (3.90) it is naturally to name them as *harmonic eigenvectors of WBO.* These eigenvectors are linearly independent since their potentials $u_k(\mathbf{r})$ are linearly independent.

13. The constants $C_1, \ldots, C_N$, which specify the harmonic functions $u_1, \ldots, u_N$, can be found via subjecting of finite subset of the harmonic eigenvectors

$$\left\{ Z_{\pm k}^{(u)}(\mathbf{r}) \right\}_{k=1}^{N}$$

to orthonormalization with using the proper orthogonal transformation and normalization[§].

3.3.2 Properties of Eigenvalues and Eigenvectors of Operator W_H.

1. Spectrum of operator W_H is real and discrete.

2. All nonzero-valued eigenvalues $p_m \neq 0$ are located on real axis symmetrically with respect to the pint $p_0 = 0$ and may put in order as

$$p_{+m} = -p_{-m} > 0. \tag{3.91}$$

So, it is enough to find the positive eigenvalues only, since all the negative ones can be found via Eq.(3.91).

3. All the *positive* eigenvalues coincide with the squared eigenvalues of the "membrane" functions, which are specified by Neumann scalar boundary eigenvalue problem for transverse Laplacian $\Delta_\perp$ as follows

$$\begin{gathered} \left(\Delta_\perp + \nu_m^2\right)\Psi_m(\mathbf{r}) = 0, \quad \frac{\partial}{\partial n_i}\Psi_m(\mathbf{r})|_{L_i} = 0, \quad i = 0, 1, \ldots, N; \\ p_{+m} = \nu_m^2 / \sqrt{\epsilon_0 \mu_0} > 0. \end{gathered} \tag{3.92}$$

[§]See subsection *"Orthonormalization of the Harmonic Eigenvectors"*.

Remind that $\partial/\partial n_i$ means derivative in the direction of outward normal $\mathbf{n}_i$ over the component L_i $(i = 0, 1, \ldots N)$ of the multi-connected contour L of the waveguide cross-section S.

4. Eigenvectors of the operator W_H, corresponding to its positive eigenvalues $p_{+m} > 0$, are specified via the membrane functions (3.92) as follows

$$Y_{+m}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m(\mathbf{r}) \end{pmatrix}. \tag{3.93}$$

5. When the membrane function $\Psi_m(\mathbf{r})$ is real, then the eigenvalues $p_{+m} > 0$ are simple: i.e., eigenvector (3.93), corresponding to this eigenvalue, is single.

6. When the membrane function is complex valued , i.e.,

$$\Psi_m(\mathbf{r}) = \Psi'_m(\mathbf{r}) + i\Psi''_m(\mathbf{r}), \tag{3.94}$$

then the eigenvalue $p_{+m} > 0$ is double: i.e., there is a pair of linearly independent eigenvectors, which correspond to the same eigenvalue. They can be presented via the same membrane function (3.92) either as

$$Y_{+m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix}, \quad Y^*_{+m}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi^*_m \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi^*_m \end{pmatrix}, \tag{3.95}$$

or as follows

$$Y'_{+m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi'_m \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi'_m \end{pmatrix}, \quad Y''_{+m} = i\begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi''_m \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi''_m \end{pmatrix}. \tag{3.96}$$

7. Eigenvectors, corresponding to the symmetrical negative eigenvalues $p_{-m} = -p_{+m} < 0$, are expressible via the same membrane function (3.82). When the function $\Psi_m(\mathbf{r})$ is real, then the eigenvalue p_{-m} is simple, and there is a single eigenvector

$$Y_{-m}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi_m(\mathbf{r}) \end{pmatrix}, \tag{3.97}$$

which corresponds to this eigenvalue.

8. Otherwise, when $\Psi_m(\mathbf{r})$ is complex-valued, then there is a pair of linearly independent eigenvectors, which corresponds to each eigenvalue $p_{-m} < 0$. One the pair can be presented as

$$Y_{-m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi'_m \end{pmatrix}, \quad Y^*_{-n} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi^*_m \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi^*_m \end{pmatrix}, \tag{3.98}$$

or the other equivalent pair can be introduced as follows

$$Y'_{-m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi'_m \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi'_m \end{pmatrix}, \quad Y''_{-n} = i\begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi''_m \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi''_m \end{pmatrix}, \tag{3.99}$$

9. Operator W_H has *zero* as its eigenvalue $p_0 = 0$ as well. All infinite number of the eigenvectors of operator W_E (see above), which correspond to the *non-zero valued* its eigenvalues $q_{\pm n} \neq 0$, correspond to the single eigenvalue $p_0 = 0$ of operator W_H. So, the eigenvalue $p_0 = 0$ of operator W_H has infinite power of degeneration: the same as the eigenvalue $q_0 = 0$ of the operator W_E. This assertion will be also proved at the end of this subsection.

10. In the course of analysis of the eigenvalue problem for operator W_H, the constant-valued harmonic function was found as

$$\Psi_0(\mathbf{r}) = 1, \tag{3.100}$$

which is available as for the simple contour $L \equiv L_0$ as for the multi-connected one. In contrast with the function Φ_0 given in Eq. (3.88), function Ψ_0 can not generate the harmonic eigenvectors, because of $\nabla_\perp \Psi_0 \equiv \mathbf{0}$ in the general case.

11. However, one can make sure that the subset of harmonic eigenvectors $\left\{ Z_{\pm k}^{(u)} \right\}_{k=1}^{N}$, which was found as corresponding to the eigenvalue $q_0 = 0$ of operator W_E, corresponds to the eigenvalue $p_0 = 0$ of operator W_H as well. In other words, the following identity holds:

$$Y_{\pm k}^{(u)}(\mathbf{r}) \equiv Z_{\pm k}^{(u)}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp u_k(\mathbf{r}) \\ \pm \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp u_k(\mathbf{r})\right] \end{pmatrix}. \tag{3.101}$$

Harmonic Eigenvectors as an Intersection of *WBO* Eigenvector Sets.

Thus, one can conclude that this finite number of the harmonic eigenvectors lie within a domain of *intersection* of *WBO* W_E and W_H eigenvector sets¶, since they correspond to the same zero-valued eigenvalue $p_0 = q_0 = 0$ of these operators both.

To make sure that identity $Y_{\pm k}^{(u)}(\mathbf{r}) \equiv Z_{\pm k}^{(u)}(\mathbf{r})$ is true we remind of two facts obtained above. The first one is Eq. (3.90), where harmonic eigenvectors $Z_{\pm k}^{(u)}(\mathbf{r})$ were obtained as a result of *calculation* of the eigenvectors corresponding to the eigenvalue $q_0 = 0$ of operator W_E. The second fact is concerned with Eqs. (3.52) and(3.53) which regard to analysis of eigensolutions corresponding to the eigenvalue $p_0 = 0$ at the eigenvalue problem corresponding to operator W_H. Using Eqs. (3.52) and(3.53), one may conclude that a set of eigenvectors corresponding to the eigenvalue $p_0 = 0$ of operator W_H can have the following *possible form:*

$$Y_0(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \phi''(\mathbf{r}) \\ \pm \sqrt[-2]{\mu_0}\left[\mathbf{z} \times \nabla_\perp \phi'(\mathbf{r})\right] \end{pmatrix}, \qquad \begin{matrix} \phi'(\mathbf{r})\,|_{L_i} = C_i', \\ \phi''(\mathbf{r})\,|_{L_i} = C_i'', \end{matrix} \tag{3.102}$$

¶So, *WBO* in the forms of W_E and W_H both have a subset with *finite number* of the harmonic eigenvectors in the general case. In mathematics, a subset with a finite number of eigenvectors of some operator is sometimes called as *a defect* of this operator. So, operators W_E and W_H both have the subset of harmonic eigenvectors as their defect.

where $i = 0, 1, \ldots, N$; C_i', C_i'' are *arbitrary* constants, and $\phi'(\mathbf{r}), \phi''(\mathbf{r})$ are *arbitrary* twice differentiable functions. In view of this arbitrariness, one can put $C_i' = C_i'' = C_i$ and $\phi'(\mathbf{r}) = \phi''(\mathbf{r}) = u_k(\mathbf{r})$ where C_i and $u_k(\mathbf{r})$ are the same as in Eq. (3.89). In such a case, Eq. (3.102) should be rewritten as

$$Y_0(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp u_k(\mathbf{r}) \\ \pm \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp u_k(\mathbf{r})\right] \end{pmatrix} = Z_{\pm k}^{(u)}(\mathbf{r}) \equiv Y_{\pm k}^{(u)}(\mathbf{r}) \tag{3.103}$$

where $k = 1, \ldots, N$. Hence, identity $Y_{\pm k}^{(u)}(\mathbf{r}) \equiv Z_{\pm k}^{(u)}(\mathbf{r})$ holds,.

Parallel Analysis of *WBO* Eigenvectors.

Let us put now $C_i' = C_i'' = 0$ and $\phi'(\mathbf{r}) = \phi''(\mathbf{r}) = \Phi_n(\mathbf{r})$ in Eq. (3.102), where $\Phi_n(\mathbf{r})$ is the same as in Eq.(3.80). Hence, Eq. (3.102) should be rewritten as follows

$$Y_0(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi_n(\mathbf{r}) \\ \pm \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp \Phi_n(\mathbf{r})\right] \end{pmatrix} \equiv Z_{\pm n}(\mathbf{r}), \tag{3.104}$$

where $n = 1, 2, \ldots$, and other notation given in Eqs. (3.81) and (3.85) are used. It means that all the eigenvectors $Z_{\pm n}(\mathbf{r})$ of operator W_E, which correspond to *infinite number* of non-zero valued eigenvalues $q_{\pm n} \neq 0$, are the eigenvectors of operator W_H corresponding to its *single* zero-valued eigenvalue $p_0 = 0$.

In the course of analysis of eigensolutions corresponding to the eigenvalue $q_0 = 0$ of operator W_E, Eqs. (3.77) and (3.78) have been obtained. They suggest that operator W_E has a set of eigenvectors, corresponding to *zero-valued* eigenvalue $q_0 = 0$ of the following *possible form only:*

$$Z_0(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\left[\nabla_\perp \psi'(\mathbf{r}) \times \mathbf{z}\right] \\ \pm \sqrt[-2]{\mu_0}\nabla_\perp \psi''(\mathbf{r}) \end{pmatrix}, \qquad \begin{array}{l} \frac{\partial}{\partial n_i}\phi'(\mathbf{r})\,|_{L_i} = 0, \\ \frac{\partial}{\partial n_i}\phi''(\mathbf{r})\,|_{L_i} = 0, \end{array} \tag{3.105}$$

where $i = 0, 1, \ldots, N$; $\psi'(\mathbf{r}), \psi''(\mathbf{r})$ are *arbitrary* twice differentiable functions, and $\partial/\partial n_i$ means derivative in the direction of the outward normal to the component L_i of the multi-connected contour L of the waveguide cross-section S.

Each function from the set $\{\Psi_m(\mathbf{r})\}$, which is specified by Eq. (3.92), satisfies Neumann boundary conditions: the same as the functions $\psi'(\mathbf{r})$ and $\psi''(\mathbf{r})$ in Eq. (3.105). Therefore we may put $\psi'(\mathbf{r}) = \psi''(\mathbf{r}) = \Psi_m(\mathbf{r})$ in Eq. (3.105). It yields

$$Z_0(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\left[\nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z}\right] \\ \pm \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m(\mathbf{r}) \end{pmatrix} \equiv Y_{\pm m}(\mathbf{r}), \tag{3.106}$$

where $m = 1, 2, \ldots$, and other notation given in Eqs. (3.93) and (3.97) are used. It means that all the eigenvectors $Y_{\pm m}(\mathbf{r})$ of operator W_H, which correspond to *infinite number* of non-zero valued eigenvalues $p_{\pm m} \neq 0$, are the eigenvectors of operator W_E corresponding to its *single* zero-valued eigenvalue $q_0 = 0$.

3.4 Orthonormalization of *WBO* Eigenvector Set

Summary 8 *How it is possible to present* WBO *eigenvectors in terms of "pure electric" and "pure magnetic"* 4*−component vectors? Why a normalization of* WBO *eigenvectors is equivalent to a normalization of their scalar potential functions? Which normalization for the potential functions is used? Which orthonormality condition satisfies* 4*−component eigenvectors of* WBO*? Which orthonormality condition satisfies "pure electric" and "pure magnetic"* 4*−component constituents of* WBO *eigenvectors? Is it possible to orthonormalize the harmonic eigenvectors? Which procedure can be used for this purpose? How to use this procedure practically? What is the content of Weyl theorem in the two-dimensional case? How to prove that the complete eigenvector set of* WBO *originates a basis in the Hilbert space* $L_2^4(S)$*?*

3.4.1 Orthormalization of Eigenvectors in Subset $\{Y_{\pm m}\}_{m=1}^{\infty}$

Four-component eigenvectors $Y_{\pm m}$ were originally introduced as follows

$$Y_{+m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix}, \quad Y_{-m}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix}. \tag{3.107}$$

For orthonormalization and other purposes needed further on, it has sense to introduce some new presentation for these vectors. To this aim, we rewrite vectors (3.107) as follows

$$Y_{+m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix};$$

$$Y_{-m} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ -\sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix},$$

where $\mathbf{0} = col\,(0,0)$ is 2-component vector with zero-valued components. These equations suggest to introduce a pair of new 4-component vectors as follows

$$Y_m^{\uparrow}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \mathbf{0} \end{pmatrix}, \quad Y_m^{\downarrow}(\mathbf{r}) = \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix}, \tag{3.108}$$

which may be named as "pure electric" and "pure magnetic" vectors, respectively. In terms of notation (3.108), eigenvectors $Y_{\pm m}$ can be written as

$$Y_{\pm m} = Y_m^{\uparrow} \pm Y_m^{\downarrow}. \tag{3.109}$$

In terms of the same notation (3.109), inner products[‖] for different combinations of the eigenvectors $Y_{+m}$and Y_{-m} can be written as follows

$$\langle Y_{+m}, Y_{+m}\rangle = \left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle + \left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle; \tag{3.110}$$

[‖]Definition of the inner product see in Eq. (3.3).

$$\langle Y_{-m}, Y_{-m}\rangle = \left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle + \left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle; \tag{3.111}$$

$$\langle Y_{+m}, Y_{-m}\rangle = \left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle - \left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle, \tag{3.112}$$

since the following identities hold:

$$\begin{aligned} \left\langle Y_m^{\uparrow}, Y_m^{\downarrow}\right\rangle &= \tfrac{1}{S}\textstyle\int_S \left\{\sqrt{\epsilon_0}\mathbf{0}\cdot[\nabla_\perp \Psi_m \times \mathbf{z}] + \sqrt{\mu_0}\mathbf{0}\cdot\nabla_\perp\Psi_m^*\right\} ds \equiv 0; \\ \left\langle Y_m^{\downarrow}, Y_m^{\uparrow}\right\rangle &= \tfrac{1}{S}\textstyle\int_S \left\{\sqrt{\epsilon_0}\mathbf{0}\cdot[\nabla_\perp \Psi_m^* \times \mathbf{z}] + \sqrt{\mu_0}\mathbf{0}\cdot\nabla_\perp\Psi_m\right\} ds \equiv 0. \end{aligned} \tag{3.113}$$

Inner products $\left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle$ and $\left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle$ coincide, namely:

$$\left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle = \frac{1}{S}\int_S \nabla_\perp\Psi_m\cdot\nabla_\perp\Psi_m^* ds \equiv \frac{1}{S}\int_S \left|\nabla_\perp\Psi_m\right|^2 ds; \tag{3.114}$$

$$\begin{aligned} \left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle &= \frac{1}{S}\int_S [\nabla_\perp\Psi_m \times \mathbf{z}]\cdot[\nabla_\perp\Psi_m^* \times \mathbf{z}]\, ds \\ &= \frac{1}{S}\int_S \nabla_\perp\Psi_m\cdot\nabla_\perp\Psi_m^* ds = \left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle. \end{aligned} \tag{3.115}$$

We can specify a numerical value for inner product

$$\left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle = \left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle = \frac{1}{S}\int_S \left|\nabla_\perp\Psi_m\right|^2 ds$$

via subjecting potentials $\Psi_m(\mathbf{r})$ to the proper normalization condition. Let us normalize the eigenfunctions $\Psi_m(\mathbf{r})$ in such a way that the following equality holds

$$\frac{1}{S}\int_S \nabla_\perp\Psi_m\cdot\nabla_\perp\Psi_m^* ds \equiv \frac{1}{S}\int_S \left|\nabla_\perp\Psi_m\right|^2 ds = 1 \tag{3.116}$$

for all the values $m = 1, 2, \ldots$. To turn this normalization for gradients of $\Psi_m(\mathbf{r})$ into normalization of these potentials themselves, we can use known identity

$$\nabla_\perp\cdot\left(\Psi_m^*\nabla_\perp\Psi_m\right) = \nabla_\perp\Psi_m^*\cdot\nabla_\perp\Psi_m + \Psi_m^*\Delta_\perp\Psi_m, \tag{3.117}$$

and the eigenvalue equation (3.92) as a direct formula, which yields

$$\Delta_\perp\Psi_m = -p_{+m}\sqrt{\epsilon_0\mu_0}\Psi_m. \tag{3.118}$$

After simple manipulations with Eqs. (3.116), (3.117) and using of two-dimensional version of Gauss' theorem, one can get

$$\frac{1}{S}\int_S \left|\nabla_\perp\Psi_m\right|^2 ds = p_{+m}\sqrt{\epsilon_0\mu_0}\frac{1}{S}\int_S \left|\Psi_m\right|^2 ds + \sum_{i=0}^{N}\oint_{L_i}\Psi_m^*\frac{\partial}{\partial n_i}\Psi_m d\ell_i = 1.$$

Since $\frac{\partial}{\partial n_i}\Psi_m|_{L_i} = 0$ in accordance with the formulation of the eigenvalue problem (3.92), the contour integrals cancel, and one can get the normalization condition needed for the potentials Ψ_m as

$$p_{+m}\sqrt{\epsilon_0\mu_0}\frac{1}{S}\int_S |\Psi_m|^2\, ds = 1. \tag{3.119}$$

We can substitute now the normalization conditions obtained as

$$\boxed{\left\langle Y_m^{\uparrow}, Y_m^{\uparrow}\right\rangle = \left\langle Y_m^{\downarrow}, Y_m^{\downarrow}\right\rangle = p_{+m}\sqrt{\epsilon_0\mu_0}\tfrac{1}{S}\int_S |\Psi_m|^2\, ds = 1.} \tag{3.120}$$

in Eqs. (3.110) – (3.112). As a result, one can get that all the 4-component eigenvectors corresponding to the eigenvalues $p_{\pm m} \neq 0$ are normalized as follows

$$\langle Y_{+m}, Y_{+m}\rangle = \langle Y_{-m}, Y_{-m}\rangle = 2, \tag{3.121}$$

and an arbitrary pair of the eigenvectors Y_{+m} and Y_{-m} is orthogonal in the sense of the inner product as

$$\langle Y_{+m}, Y_{-m}\rangle = 0. \tag{3.122}$$

This is true, since these eigenvectors correspond to *distinct* eigenvalues $p_m > 0$ and $p_{-m} = -p_{+m} < 0$, respectively, of self-adjoint operator W_H.

Let us consider now a pair of the eigenvectors Y_{+m} and $Y_{+m'}$, corresponding to *distinct* eigenvalues $p_{+m} > 0$ and $p_{+m'} \neq p_{+m} > 0$, respectively. They should be orthogonal on the same reason as

$$\langle Y_{+m}, Y_{+m'}\rangle = \left\langle Y_m^{\uparrow}, Y_{m'}^{\uparrow}\right\rangle + \left\langle Y_m^{\downarrow}, Y_{m'}^{\downarrow}\right\rangle = 0. \tag{3.123}$$

In such a case, eigenvectors Y_{+m} and $Y_{-m'}$ are also orthogonal as

$$\langle Y_{+m}, Y_{-m'}\rangle = \left\langle Y_m^{\uparrow}, Y_{m'}^{\uparrow}\right\rangle - \left\langle Y_m^{\downarrow}, Y_{m'}^{\downarrow}\right\rangle = 0, \tag{3.124}$$

since they correspond to distinct eigenvalues $p_{+m} > 0$ and $p_{-m'} = -p_{+m'} < 0$ of the self-adjoint operator. From Eqs. (3.123) and (3.17), it follows immediately that

$$\left\langle Y_m^{\uparrow}, Y_{m'}^{\uparrow}\right\rangle = 0, \qquad \left\langle Y_m^{\downarrow}, Y_{m'}^{\downarrow}\right\rangle = 0. \tag{3.125}$$

One can joint Eqs. (3.120) and (3.125) in the form of condition

$$\left\langle Y_m^{\uparrow}, Y_{m'}^{\uparrow}\right\rangle = \delta_{mm'}, \qquad \left\langle Y_m^{\downarrow}, Y_{m'}^{\downarrow}\right\rangle = \delta_{mm'}, \tag{3.126}$$

where $\delta_{mm'}$ is Kronecker's delta, and m, m' can get arbitrary values $1, 2, \ldots$ independently. Normalization condition (3.121) and orthogonality condition (3.126) supply one more orthonormality condition

$$\langle Y_{\pm m}, Y_{\pm m'}\rangle = 2\delta_{\pm m, \pm m'}, \tag{3.127}$$

which regards to complete 4 $-$component eigenvectors of the operator W_H.

3.4.2 Orthormalization of Eigenvectors in Subset $\{Z_{\pm n}\}_{n=1}^{\infty}$

In a similar way, 4−component eigenvectors $Z_{\pm n}$ of operator W_E can be presented as

$$Z_{\pm n} = Z_n^{\uparrow} \pm Z_n^{\downarrow}, \tag{3.128}$$

where "pure electric" and "pure magnetic" 4−component vectors are again introduced as

$$Z_n^{\uparrow} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\mathbf{z} \times \nabla_{\perp}\Phi_n] \\ \mathbf{0} \end{pmatrix}, \qquad Z_n^{\downarrow} = \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\nabla_{\perp}\Phi_n \end{pmatrix}. \tag{3.129}$$

Inner products can be obtained for the same combinations of the eigenvectors Z_{+n} and Z_{-n} in the form of

$$\begin{aligned} \langle Z_{+n}, Z_{+n}\rangle &= \left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle + \left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle, \\ \langle Z_{-n}, Z_{-n}\rangle &= \left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle + \left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle, \\ \langle Z_{+n}, Z_{-n}\rangle &= \left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle - \left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle. \end{aligned} \tag{3.130}$$

Inner products $\left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle$ and $\left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle$ coincide, as one can be seen below

$$\left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle = \frac{1}{S}\int_S \nabla_{\perp}\Phi_n \cdot \nabla_{\perp}\Phi_n^* ds = \frac{1}{S}\int_S \left|\nabla_{\perp}\Phi_n\right|^2 ds,$$

$$\begin{aligned} \left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle &= \frac{1}{S}\int_S [\mathbf{z} \times \nabla_{\perp}\Phi_n] \cdot [\mathbf{z} \times \nabla_{\perp}\Phi_n^*]\, ds \\ &= \frac{1}{S}\int_S \nabla_{\perp}\Phi_n \cdot \nabla_{\perp}\Phi_n^* ds = \left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle. \end{aligned}$$

Normalization condition is used in the form of

$$\left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle = \left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle = \frac{1}{S}\int_S \left|\nabla_{\perp}\Phi_n\right|^2 ds = 1. \tag{3.131}$$

With making use of identity (3.117) , eigenvalue equation (3.80) as a direct formula

$$\Delta_{\perp}\Phi_n = -q_{+n}\sqrt{\epsilon_0\mu_0}\Phi_n, \tag{3.132}$$

and two-dimensional version of Gauss' theorem, one can get

$$\frac{1}{S}\int_S \left|\nabla_{\perp}\Phi_n\right|^2 ds = q_{+n}\sqrt{\epsilon_0\mu_0}\frac{1}{S}\int_S \left|\Phi_n\right|^2 ds + \sum_{i=0}^{N}\oint_{L_i} \Phi_n \frac{\partial}{\partial n_i}\Phi_n^* d\ell_i. \tag{3.133}$$

Boundary condition $\Phi_n|_{L_i} = 0$ from the eigenvalue problem (3.80) cancels the contour integrals at the right-hand-sides of Eq. (3.133) . Substitution then of Eq. (3.133) at Eq. (3.131) yields normalization required for the potential Φ_n as

$$\left\langle Z_n^{\uparrow}, Z_n^{\uparrow}\right\rangle = \left\langle Z_n^{\downarrow}, Z_n^{\downarrow}\right\rangle = q_{+n}\sqrt{\epsilon_0\mu_0}\frac{1}{S}\int_S \left|\Phi_n\right|^2 ds = 1. \tag{3.134}$$

A set of Eqs. (3.130) and (3.134) yields orthonormality conditions for the constituents $Z_n^{\uparrow}$ and $Z_n^{\downarrow}$ of the eigenvectors $Z_{\pm n}$ as follows

$$\left\langle Z_n^{\uparrow}, Z_{n'}^{\uparrow} \right\rangle = \delta_{nn'}, \qquad \left\langle Z_n^{\downarrow}, Z_n^{\downarrow} \right\rangle = \delta_{nn'}, \tag{3.135}$$

and for the eigenvectors $Z_{\pm n}$ themselves as well

$$\langle Z_{\pm n}, Z_{\pm n'} \rangle = 2\delta_{\pm n, \pm n'}. \tag{3.136}$$

3.4.3 Orthormalization in Subset of Harmonic Eigenvectors

Remind that the 4−component harmonic eigenvectors were specified in Eq. (3.89) as

$$Y_{\pm k}^{(u)}(\mathbf{r}) \equiv Z_{\pm k}^{(u)}(\mathbf{r}) = \left(\begin{array}{c} \sqrt[-2]{\epsilon_0} \nabla_\perp u_k(\mathbf{r}) \\ \pm \sqrt[-2]{\mu_0} \left[\mathbf{z} \times \nabla_\perp u_k(\mathbf{r})\right] \end{array} \right), \tag{3.137}$$

where the potential harmonic functions $u_k(\mathbf{r})$ are the solutions to Laplace equation satisfying inhomogeneous boundary conditions as

$$k = 1, \ldots, N: \qquad \Delta_\perp u_k(\mathbf{r}) = 0, \quad u_k(\mathbf{r})|_{L_i} = \delta_{ki} C_i; \quad i = 0, 1, \ldots, N, \tag{3.138}$$

and $C_1, C_2, \ldots, C_N$ are some constants.

A pair of the eigenvectors $Y_{+k}^{(u)}$ and $Y_{-k}^{(u)}$ with the same subscript k are already orthogonal in the sense of inner product, namely:

$$\begin{aligned} \left\langle Y_{+k}^{(u)}, Y_{-k}^{(u)} \right\rangle &= \frac{1}{S} \int_S \left\{ \nabla_\perp u_k \cdot \nabla_\perp u_k^* - [\mathbf{z} \times \nabla_\perp u_k] \cdot [\mathbf{z} \times \nabla_\perp u_k^*] \right\} ds \\ &= \frac{1}{S} \int_S \nabla_\perp u_k \cdot \nabla_\perp u_k^* ds - \frac{1}{S} \int_S \nabla_\perp u_k \cdot \nabla_\perp u_k^* ds = 0. \end{aligned} \tag{3.139}$$

All the vectors at the subset $\left\{ Y_{+k}^{(u)} \right\}_{k=1}^{N}$ are linearly independent**, but they are not orthogonal in the general case of arbitrary form of waveguide cross-section. Now, our goal is to calculate new *orthonormal* subset of the harmonic eigenvectors

$$\{U_{\pm k}\}_{k=1}^{N}, \tag{3.140}$$

which may be available for the general case. To this aim, we apply an orthogonal transformation to the subset $\left\{ Y_{+k}^{(u)} \right\}_{k=1}^{N}$ and introduce then the proper normalization. For this goal, standard Gramme-Shmidt procedure will be used.

**Since potential harmonic functions u_k are linearly independent, as one can be seen in Eq. (3.138).

Calculation of the eigenvectors $U_{+1}(\mathbf{r})$ and $U_{-1}(\mathbf{r})$. We start with calculation of $U_{+1}(\mathbf{r})$ as

$$U_{+1}(\mathbf{r}) = C_1 \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_{\perp}\omega_1(\mathbf{r}) \\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_{\perp}\omega_1(\mathbf{r})\right] \end{pmatrix}, \tag{3.141}$$

where C_1 is the same unknown constant as in Eq. (3.138), but harmonic function $\omega_1(\mathbf{r})$ is specified completely as follows

$$\Delta_{\perp}\omega_1(\mathbf{r}) = 0, \quad \omega_1(\mathbf{r})\,|_{L_i} = \delta_{1i}; \qquad i = 0, 1, \ldots, N. \tag{3.142}$$

The constant C_1 we can find via subjecting of the eigenvector U_{+1} to the same normalization

$$\langle U_{+1}, U_{+1}\rangle = 2, \tag{3.143}$$

which was used for the eigenvectors $Y_{\pm m}$ and $Z_{\pm n}$ in Eqs. (3.121) and (3.136), respectively. Therefore, one can get

$$\langle U_{+1}, U_{+1}\rangle = 2C_1^2\frac{1}{S}\int_S \nabla_{\perp}\omega_1\cdot\nabla_{\perp}\omega_1^* ds = 2, \tag{3.144}$$

where the identity $[\mathbf{z}\times\nabla_{\perp}\omega_1(\mathbf{r})]\cdot[\mathbf{z}\times\nabla_{\perp}\omega_1^*(\mathbf{r})] = \nabla_{\perp}\omega_1\cdot\nabla_{\perp}\omega_1^*$ is taken into account.

We can simplify this integral with making use of well-known identity $\nabla_{\perp}\cdot(\varphi\nabla_{\perp}\psi) = \nabla_{\perp}\varphi\cdot\nabla_{\perp}\psi + \varphi\Delta_{\perp}\psi$. When φ and ψ are the harmonic functions then $\Delta_{\perp}\psi = \Delta_{\perp}\varphi = 0$. Hence,

$$\nabla_{\perp}\varphi\cdot\nabla_{\perp}\psi = \nabla_{\perp}\cdot(\varphi\nabla_{\perp}\psi). \tag{3.145}$$

In the case of Eq. (3.144), we should put $\varphi = \omega_1$ and $\psi = \omega_1^*$; then

$$\nabla_{\perp}\omega_1\cdot\nabla_{\perp}\omega_1^* = \nabla_{\perp}\cdot(\omega_1\nabla_{\perp}\omega_1^*). \tag{3.146}$$

Substitution of Eq. (3.146) in Eq. (3.144) and using then Gauss' theorem yields

$$\begin{aligned} C_1^2\frac{1}{S}\int_S \nabla_{\perp}\omega_1\cdot\nabla_{\perp}\omega_1^* ds &= C_1^2\frac{1}{S}\int_S \nabla_{\perp}\cdot(\omega_1\nabla_{\perp}\omega_1^*)ds \\ &= C_1^2\sum_{i=0}^{N}\frac{1}{S}\int_{L_i}\delta_{1i}\frac{\partial\omega_1^*}{\partial n_i}d\ell = C_1^2\frac{1}{S}\int_{L_1}\frac{\partial\omega_1^*}{\partial n_1}d\ell = 1, \end{aligned} \tag{3.147}$$

where the boundary condition $\omega_1|_{L_i} = \delta_{1i}$ from Eq. (3.142) was used as well. So, the constant C_1 can be obtained as

$$\boxed{C_1 = \{J_{11}\}^{-1/2}, \qquad J_{11} \equiv \tfrac{1}{S}\int_{L_1}\tfrac{\partial\omega_1^*}{\partial n_1}d\ell.} \tag{3.148}$$

Thus, we can write the pair of mutually orthogonal 4− component harmonic eigenvectors sought as follows

$$U_{+1} = \sqrt[-2]{J_{11}} \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\omega_1 \\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\omega_1\right] \end{pmatrix}, \quad U_{-1} = \sqrt[-2]{J_{11}} \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\omega_1 \\ -\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\omega_1\right] \end{pmatrix}. \tag{3.149}$$

They satisfy the normalization condition both

$$\langle U_{+1}(\mathbf{r}), U_{+1}(\mathbf{r})\rangle = 2, \qquad \langle U_{-1}(\mathbf{r}), U_{-1}(\mathbf{r})\rangle = 2, \tag{3.150}$$

and they are orthogonal as well in the sense of the inner product, i. e.

$$\langle U_{+1}(\mathbf{r}), U_{-1}(\mathbf{r})\rangle = 0. \tag{3.151}$$

Calculation of the eigenvectors $U_{+2}(\mathbf{r})$ and $U_{-2}(\mathbf{r})$. Harmonic eigenvector $U_{+2}(\mathbf{r})$ we present as a linear combination of the vector $U_{+1}(\mathbf{r})$ obtained above in Eq. (3.149), and the other linearly independent harmonic vector $\tilde{U}_{+2}(\mathbf{r})$, namely:

$$U_{+2}(\mathbf{r}) = \alpha U_{+1}(\mathbf{r}) + C_2\tilde{U}_{+2}(\mathbf{r}), \tag{3.152}$$

where α and C_2 are unknown constants[††], end vector $\tilde{U}_{+2}(\mathbf{r})$ is specified as follows

$$\tilde{U}_{+2}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\omega_2(\mathbf{r}) \\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\omega_2(\mathbf{r})\right] \end{pmatrix}. \tag{3.153}$$

Harmonic function $\omega_2(\mathbf{r})$ is given below in the same form of the boundary value problem for Laplace equation as

$$\Delta_\perp\omega_2(\mathbf{r}) = 0, \quad \omega_2(\mathbf{r})|_{L_i} = \delta_{2i}; \qquad i = 0, 1, \ldots, N. \tag{3.154}$$

At the first step, we subject a pair of the vectors $U_{+2}(\mathbf{r})$ and $U_{+1}(\mathbf{r})$ to the condition of orthogonality, i.e.,

$$\begin{aligned} \langle U_{+2}, U_{+1}\rangle &= \left\langle \alpha U_{+1} + C_2\tilde{U}_{+2}, U_{+1}\right\rangle \\ &= \alpha\langle U_{+1}, U_{+1}\rangle + C_2\left\langle \tilde{U}_{+2}, U_{+1}\right\rangle = 2\alpha + C_2\left\langle\tilde{U}_{+2}, U_{+1}\right\rangle = 0. \end{aligned} \tag{3.155}$$

Inner product $\left\langle\tilde{U}_{+2}, U_{+1}\right\rangle$ can be calculated explicitly as follows

$$\begin{aligned} \left\langle\tilde{U}_{+2}, U_{+1}\right\rangle &= 2C_1\frac{1}{S}\int_S \nabla_\perp\omega_2\cdot\nabla_\perp\omega_1^* ds = 2C_1\frac{1}{S}\int_S\nabla_\perp\cdot\left(\omega_2\nabla_\perp\omega_1^*\right) \\ &= 2C_1\sum_{i=0}^{N}\frac{1}{S}\int_{L_i}\delta_{2i}\frac{\partial\omega_1^*}{\partial n_i}d\ell = 2C_1\frac{1}{S}\int_{L_2}\frac{\partial\omega_1^*}{\partial n_2}d\ell \end{aligned} \tag{3.156}$$

[††]The constant C_2 is the same as at the boundary condition in Eq. (3.138).

where coefficient C_1 is given in Eq. (3.148). So, Eq. (3.155) acquires the form of

$$\alpha + C_2 C_1 J_{21} = 0, \qquad J_{21} = \frac{1}{S}\int_{L_2} \frac{\partial \omega_1^*}{\partial n_2} d\ell, \tag{3.157}$$

which supplies one of the unknown constants α and C_2 as expressed via another one, par example,

$$\alpha = -C_2 C_1 J_{21}. \tag{3.158}$$

We can substitute now Eq. (3.158) in Eq. (3.152). As a result, we have definition of the vector U_{+2} sought with only one unknown constant C_2, namely:

$$U_{+2}(\mathbf{r}) = C_2 \left\{ \tilde{U}_{+2}(\mathbf{r}) - C_1 J_{21} U_{+1}(\mathbf{r}) \right\} \tag{3.159}$$

At the second and last step, constant C_2 can be calculated via subjecting of $U_{+2}(\mathbf{r})$ to normalization under the condition of

$$\langle U_{+2}(\mathbf{r}), U_{+2}(\mathbf{r}) \rangle = 2. \tag{3.160}$$

Left-hand-side at Eq. (3.160) can be recast as follows

$$\begin{aligned} \langle U_{+2}(\mathbf{r}), U_{+2}(\mathbf{r}) \rangle &= \\ &= C_2^2 \left\langle \tilde{U}_{+2} - (J_{21}/J_{11})\, \tilde{U}_{+1}, \tilde{U}_{+2} - (J_{21}/J_{11})\, \tilde{U}_{+1} \right\rangle = 2 \end{aligned} \tag{3.161}$$

Simple algebraic calculations yield coefficient C_2 needed as

$$\boxed{C_2 = \left\{ J_{22} + |J_{21}/J_{11}|^2 J_{11} - (J_{21}/J_{11}) J_{12} - (J_{21}/J_{11})^* J_{21} \right\}^{-1/2},} \tag{3.162}$$

where the following notation are introduced:

$$\begin{aligned} J_{22} &= \frac{1}{S}\int_{L_2} \frac{\partial \omega_2^*}{\partial n_2} d\ell, \qquad J_{11} = \frac{1}{S}\int_{L_1} \frac{\partial \omega_1^*}{\partial n_1} d\ell, \\ J_{21} &= \frac{1}{S}\int_{L_2} \frac{\partial \omega_1^*}{\partial n_2} d\ell, \qquad J_{12} = \frac{1}{S}\int_{L_1} \frac{\partial \omega_2^*}{\partial n_1} d\ell. \end{aligned} \tag{3.163}$$

When contour L has only three components, i.e., $L \in \{L_0, L_1, L_2\}$, then procedure of orthonormalization is finished. Otherwise, it should be continued on the same scheme.

It has sense to introduce new harmonic functions $\Omega_1(\mathbf{r})$ and $\Omega_2(\mathbf{r})$ as the following linear combinations of $\omega_1(\mathbf{r})$ and $\omega_2(\mathbf{r})$:

$$\begin{aligned} \Omega_1(\mathbf{r}) &= C_1 \omega_1(\mathbf{r}) \equiv \omega_1(\mathbf{r}) / \sqrt{J_{11}}; \\ \Omega_2(\mathbf{r}) &= C_2 \left[\omega_2(\mathbf{r}) - (J_{21}/J_{11})\, \omega_1(\mathbf{r}) \right]. \end{aligned} \tag{3.164}$$

In such a case, the harmonic eigenvectors orthonormalized can be presented as follows

$$U_{\pm 1} = \begin{pmatrix} \sqrt[-2]{\epsilon_0} \nabla_\perp \Omega_1 \\ \pm \sqrt[-2]{\mu_0} \left[\mathbf{z} \times \nabla_\perp \Omega_1 \right] \end{pmatrix}, \qquad U_{\pm 2} = \begin{pmatrix} \sqrt[-2]{\epsilon_0} \nabla_\perp \Omega_2 \\ \pm \sqrt[-2]{\mu_0} \left[\mathbf{z} \times \nabla_\perp \Omega_2 \right] \end{pmatrix}. \tag{3.165}$$

It is possible also to introduce "pure electric" and "pure magnetic" eigenvector components as

$$U_1^{\uparrow} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\Omega_1 \\ \mathbf{0} \end{pmatrix}, \qquad U_1^{\downarrow} = \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\,[\mathbf{z}\times\nabla_\perp\Omega_1] \end{pmatrix};$$
$$U_2^{\uparrow} = \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp\Omega_2 \\ \mathbf{0} \end{pmatrix} \qquad U_2^{\downarrow} = \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\,[\mathbf{z}\times\nabla_\perp\Omega_2] \end{pmatrix}. \tag{3.166}$$

Then the vectors from Eqs. (3.165) can be shortly written as follows

$$U_{\pm1} = U_1^{\uparrow} \pm U_1^{\downarrow}, \qquad U_{\pm2} = U_2^{\uparrow} \pm U_2^{\downarrow}. \tag{3.167}$$

One can easily obtain evident orthogonality condition as

$$\left\langle U_k^{\uparrow}, U_l^{\downarrow}\right\rangle = \left\langle U_k^{\downarrow}, U_l^{\uparrow}\right\rangle = 0 \tag{3.168}$$

when $k \neq l : k, l = 1, 2$. Series of orthonormality conditions obtained above for 4− component harmonic eigenvectors can be now written as

$$\begin{aligned}
\langle U_{+1}, U_{+1}\rangle &= \langle U_{-1}, U_{-1}\rangle = \left\langle U_1^{\uparrow}, U_1^{\uparrow}\right\rangle + \left\langle U_1^{\downarrow}, U_1^{\downarrow}\right\rangle = 2,\\
\langle U_{+1}, U_{-1}\rangle &= \langle U_{-1}, U_{+1}\rangle = \left\langle U_1^{\uparrow}, U_1^{\uparrow}\right\rangle - \left\langle U_1^{\downarrow}, U_1^{\downarrow}\right\rangle = 0;\\
\langle U_{+2}, U_{+2}\rangle &= \langle U_{-2}, U_{-2}\rangle = \left\langle U_2^{\uparrow}, U_2^{\uparrow}\right\rangle + \left\langle U_2^{\downarrow}, U_2^{\downarrow}\right\rangle = 2,\\
\langle U_{+2}, U_{-2}\rangle &= \langle U_{-2}, U_{+2}\rangle = \left\langle U_2^{\uparrow}, U_2^{\uparrow}\right\rangle - \left\langle U_2^{\downarrow}, U_2^{\downarrow}\right\rangle = 0
\end{aligned} \tag{3.169}$$

They result in simple orthonormality conditions for the 4− component constituents of the harmonic eigenvectors

$$\left\langle U_k^{\uparrow}, U_l^{\uparrow}\right\rangle = \delta_{kl}, \qquad \left\langle U_k^{\downarrow}, U_l^{\downarrow}\right\rangle = \delta_{kl}; \qquad k, l = 1, 2. \tag{3.170}$$

where δ_{kl} is Kronecker's delta.

3.4.4 Orthogonality of the Eigenvectors Taken from Different Subsets

Eigenvector set of *WBO* involves three subsets, namely:

$$\{Y_{\pm m}\}_{m=0}^{\infty}, \qquad \{Z_{\pm n}\}_{n=0}^{\infty}, \qquad \{U_{\pm k}\}_{k=1}^{N}. \tag{3.171}$$

The elements within the subset each were denoted as follows

$$Y_{\pm m} = Y_m^{\uparrow} \pm Y_m^{\downarrow}; \qquad Z_{\pm n} = Z_n^{\uparrow} \pm Z_n^{\downarrow}; \qquad U_{\pm k} = U_k^{\uparrow} \pm U_k^{\downarrow}, \tag{3.172}$$

where the 4− component vectors marked with arrow up are "pure electric vectors", and marked with arrow down are "pure magnetic vectors. They are presented via respective potential functions in the following forms:

$$
\begin{aligned}
Y_m^{\uparrow} &= \begin{pmatrix} \sqrt[-2]{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \\ \mathbf{0} \end{pmatrix}, \qquad & Y_m^{\downarrow} &= \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\nabla_\perp \Psi_m \end{pmatrix}; \\
Z_n^{\uparrow} &= \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Phi_n \\ \mathbf{0} \end{pmatrix}, \qquad & Z_n^{\downarrow} &= \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\,[\mathbf{z} \times \nabla_\perp \Phi_n] \end{pmatrix}; \\
U_k^{\uparrow} &= \begin{pmatrix} \sqrt[-2]{\epsilon_0}\nabla_\perp \Omega_k \\ \mathbf{0} \end{pmatrix}, \qquad & U_k^{\downarrow} &= \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0}\,[\mathbf{z} \times \nabla_\perp \Omega_k] \end{pmatrix}.
\end{aligned}
\tag{3.173}
$$

All inner products of the vectors marked with the arrows oriented in opposite directions are orthogonal. This is conditioned by the structure of these vectors shown in Eq. (3.173) , in particular,

$$
\left\langle Y_m^{\uparrow}, Y_m^{\downarrow} \right\rangle = \frac{1}{S} \int_S \{\sqrt{\epsilon_0}\,[\nabla_\perp \Psi_m \times \mathbf{z}] \cdot \mathbf{0} + \sqrt{\mu_0}\mathbf{0} \cdot \nabla_\perp \Psi_m^*\} ds \equiv 0.
$$

In the same way one can get the following series orthogonality conditions:

$$
\begin{aligned}
\left\langle Y_m^{\uparrow}, Y_m^{\downarrow} \right\rangle &= \left\langle Y_m^{\downarrow}, Y_m^{\uparrow} \right\rangle = 0, \\
\left\langle Z_n^{\uparrow}, Z_n^{\downarrow} \right\rangle &= \left\langle Z_n^{\downarrow}, Z_n^{\uparrow} \right\rangle = 0, \\
\left\langle U_k^{\uparrow}, U_k^{\downarrow} \right\rangle &= \left\langle U_k^{\downarrow}, U_k^{\uparrow} \right\rangle = 0,
\end{aligned}
\tag{3.174}
$$

and one more series as well

$$
\begin{aligned}
\left\langle Y_m^{\uparrow}, Z_n^{\downarrow} \right\rangle &= \left\langle Y_m^{\downarrow}, Z_n^{\uparrow} \right\rangle = \left\langle Y_m^{\uparrow}, U_k^{\downarrow} \right\rangle = \left\langle Y_m^{\downarrow}, U_k^{\uparrow} \right\rangle = 0, \\
\left\langle Z_n^{\uparrow}, Y_m^{\downarrow} \right\rangle &= \left\langle Z_n^{\downarrow}, Y_m^{\uparrow} \right\rangle = \left\langle Z_n^{\uparrow}, U_k^{\downarrow} \right\rangle = \left\langle Z_n^{\downarrow}, U_k^{\uparrow} \right\rangle = 0, \\
\left\langle U_k^{\uparrow}, Y_m^{\downarrow} \right\rangle &= \left\langle U_k^{\downarrow}, Y_m^{\uparrow} \right\rangle = \left\langle U_k^{\uparrow}, Z_n^{\downarrow} \right\rangle = \left\langle U_k^{\downarrow}, Z_n^{\uparrow} \right\rangle = 0.
\end{aligned}
\tag{3.175}
$$

Further, each element from subset[‡‡] $\{Y_{\pm m}\}_{m=1}^{\infty}$ corresponds to some *non-zero valued* eigenvalue of the operator W_H. Meanwhile, all the elements from the subsets $\{Z_{\pm n}\}_{n=0}^{\infty}$ and $\{U_{\pm k}\}_{k=1}^{N}$ correspond to the eigenvalue $p_0 = 0$ of this operator. In other words, they play a role of the eigenvector Y_0 of this operator. Since the operator is

[‡‡]Please note, that the element with the subscript $m = 0$ (i.e., Y_0) is excluded here.

self-adjoint, each a pair of its eigenvectors corresponding to *distinct* eigenvalues are orthogonal in the sense of inner product. For example,

$$\begin{aligned} \langle Y_{+m}, Z_{+n}\rangle &= \left\langle Y_m^{\uparrow} + Y_m^{\downarrow}, Z_n^{\uparrow} + Z_n^{\downarrow}\right\rangle = \left\langle Y_m^{\uparrow}, Z_n^{\uparrow}\right\rangle + \left\langle Y_m^{\downarrow}, Z_n^{\downarrow}\right\rangle = 0, \\ \langle Y_{+m}, Z_{-n}\rangle &= \left\langle Y_m^{\uparrow} + Y_m^{\downarrow}, Z_n^{\uparrow} - Z_n^{\downarrow}\right\rangle = \left\langle Y_m^{\uparrow}, Z_n^{\uparrow}\right\rangle - \left\langle Y_m^{\downarrow}, Z_n^{\downarrow}\right\rangle = 0, \end{aligned} \tag{3.176}$$

where orthogonality conditions (3.175) are used. Summation and substraction of Eqs. (3.176) result in the first pair of the following orthogonality conditions. The other pointed out there can be obtained in a similar way as

$$\begin{aligned} &\left\langle Y_m^{\uparrow}, Z_n^{\uparrow}\right\rangle = \left\langle Y_m^{\downarrow}, Z_n^{\downarrow}\right\rangle = \left\langle Y_m^{\uparrow}, U_k^{\uparrow}\right\rangle = \left\langle Y_m^{\downarrow}, U_k^{\downarrow}\right\rangle = 0, \\ &\left\langle Z_n^{\uparrow}, Y_m^{\uparrow}\right\rangle = \left\langle Z_n^{\downarrow}, Y_m^{\downarrow}\right\rangle = \left\langle Z_n^{\uparrow}, U_k^{\uparrow}\right\rangle = \left\langle Z_n^{\downarrow}, U_k^{\downarrow}\right\rangle = 0, \\ &\left\langle U_k^{\uparrow}, Y_m^{\uparrow}\right\rangle = \left\langle U_k^{\downarrow}, Y_m^{\downarrow}\right\rangle = \left\langle U_k^{\uparrow}, Z_n^{\uparrow}\right\rangle = \left\langle U_k^{\downarrow}, Z_n^{\downarrow}\right\rangle = 0. \end{aligned} \tag{3.177}$$

These conditions of orthogonality and the conditions of orthonormality obtained above in this section are needed essentially for further development of our approach.

3.5 Basis Set for 3− Component Vector Functions of Transverse Variables

Summary 9 *Why the* WBO *eigenvector set originates a basis in $L_2^4(S)$, in principle? How to prove this fact? What is the matter of two-dimensional version of Weyl theorem? How to use this theorem for proof of completeness of the* WBO *eigenvector set in $L_2^4(S)$? Which eigenfunction set can be used for presentation of scalar unknowns E_z and H_z in terms of eigenvector series?*

Three-component vectors of lectromagnetic field sought

$$\vec{\mathcal{E}}(\mathbf{r},z,t) = \mathbf{E}(\mathbf{r},z,t) + \mathbf{z}E_z(\mathbf{r},z,t); \qquad \vec{\mathcal{H}}(\mathbf{r},z,t) = \mathbf{H}(\mathbf{r},z,t) + \mathbf{z}H_z(\mathbf{r},z,t) \tag{3.178}$$

in the waveguide under consideration we are going to present in a form of eigenfunction of transverse coordinates $\mathbf{r}$ series. We'll prove that *WBO* eigenvector originates a basis in the domain of *WBO,* i.e., Hilbert space $L_2^4(S)$. Then we'll discuss why scalar functions sought E_z and H_z can be presented in terms of series potential functions for *WBO* eigenvectors.

3.5.1 Basis Set for Transverse Vector Components

Summarizing previous results, conclusion can be drawn that the eigenvector set of *WBO* is a linear manifold $\mathbb{M}$ in Hilbert space $L_2^4(S)$. Since it involves three mutually orthogonal subspaces, it can be presented as

$$\mathbb{M} = \mathbb{N} \oplus \mathbb{D} \oplus \mathbb{U}, \tag{3.179}$$

where $\oplus$ stands for direct summation of the orthogonal subspaces. Notation $\mathbb{N}, \mathbb{D}$, and $\mathbb{U}$ of the subspaces mentioned is explained below in detail. We are going to prove that $\mathbb{M}$ coincides with $L_2^4(S)$, and the elements of $\mathbb{M}$ (i.e., the eigenvector set of *WBO*) originate a basis in $L_2^4(S)$.

To prove it we should first point out that the subspaces $\mathbb{N}$, $\mathbb{D}$, and $\mathbb{U}$ each, in their turn, are made of the some subspaces of Hilbert space $L_2^2(S)$ of 2− component vector functions as

$$\mathbb{N} = \begin{pmatrix} \mathbb{J}^2_{\mathcal{N}} \\ \mathbb{G}^2_{\mathcal{N}} \end{pmatrix}; \qquad \mathbb{D} = \begin{pmatrix} \mathbb{G}^2_{\mathcal{D}} \\ \mathbb{J}^2_{\mathcal{D}} \end{pmatrix}; \qquad \mathbb{U} = \begin{pmatrix} \mathbb{G}^2_{\Omega} \\ \mathbb{J}^2_{\Omega} \end{pmatrix}. \tag{3.180}$$

Subscripts $\mathcal{D}$ and $\mathcal{N}$ mean here that potential functions for the vector elements of the subspaces, respectively, satisfy either Dirichlet's or Neumann boundary condition over boundary L of domain S.

Elements of $\mathbb{J}^2_{\mathcal{N}}(S)$ and $\mathbb{G}^2_{\mathcal{N}}(S)$ are generated by the same potential Ψ_m as follows

$$\mathbb{J}^2_{\mathcal{N}} : \{[\nabla_\perp \Psi_m \times \mathbf{z}]\}_{m=1}^{\infty}; \qquad \mathbb{G}^2_{\mathcal{N}} : \{\nabla_\perp \Psi_m\}_{m=1}^{\infty}. \tag{3.181}$$

Potential functions $\Psi_m(\mathbf{r})$, $\mathbf{r} \in S$ are eigensolutions normalized of well known Neumann boundary eigenvalue problem for Laplacian, namely:

$$(\Delta_\perp + \nu_m^2)\Psi_m = 0, \quad \frac{\partial}{\partial n_i}\Psi_m|_{L_i} = 0, \quad i = 0, 1, \ldots, N; \quad \frac{\nu_m^2}{S}\int_S |\Psi_m|^2 \, ds = 1. \tag{3.182}$$

Elements of subspaces $\mathbb{J}^2_{\mathcal{D}}(S)$ and $\mathbb{G}^2_{\mathcal{D}}(S)$ are generated in a similar way by the other potential functions $\Phi_n(\mathbf{r})$, $\mathbf{r} \in S$ as follows

$$\mathbb{G}^2_{\mathcal{D}} : \{\nabla_\perp \Phi_n\}_{n=1}^{\infty}; \qquad \mathbb{J}^2_{\mathcal{D}} : \{[\mathbf{z} \times \nabla_\perp \Phi_n]\}_{n=1}^{\infty}. \tag{3.183}$$

Potentials $\Phi_n(\mathbf{r})$, $\mathbf{r} \in S$ are eigensolutions of Dirichlet's boundary eigenvalue problem for Laplacian, i.e.,

$$(\Delta_\perp + \varkappa_n^2)\Phi_n = 0, \quad \Phi_n|_{L_i} = 0, \quad i = 0, 1, \ldots, N; \quad \frac{\varkappa_m^2}{S}\int_S |\Phi_n|^2 \, ds = 1. \tag{3.184}$$

Subspaces $\mathbb{G}^2_{\Omega}$ and $\mathbb{J}^2_{\Omega}$ each involve a finite number of so-called harmonic vectors. When the contour is singly connected (i.e., $L \equiv L_0$), then the subspaces each involve the only zero-valued vector, i.e., $\mathbb{G}^2_{\Omega} \in \mathbf{0}$ and $\mathbb{J}^2_{\Omega} \in \mathbf{0}$. When the contour involves N components (i.e., $L \in L_0, L_1, \ldots L_N$) then these subspaces involve the following elements:

$$\mathbb{G}^2_{\Omega} = \{\nabla_\perp \Omega_k\}_{k=1}^{N}; \qquad \mathbb{J}^2_{\Omega} = \{[\mathbf{z} \times \nabla_\perp \Omega_k]\}_{k=1}^{N}, \tag{3.185}$$

where $\Omega_k(\mathbf{r})$ are the harmonic functions. They are composed as the linear combinations of the following linearly independent harmonic functions:

$$\Omega_k = \sum_{j=1}^{k} C_j \omega_j : \quad \Delta_\perp \omega_j = 0, \quad \omega_j|_{L_i} = \delta_{ji}; \quad j = 1, \ldots, N; \quad i = 0, 1, \ldots, N. \tag{3.186}$$

Constants C_j should be specified in the course of subjecting of $\nabla_\perp \Omega_k$ to orthonormality conditions as

$$C_j: \qquad \tfrac{1}{S}\int_S \nabla_\perp \Omega_k \cdot \nabla_\perp \Omega_{k'} ds = \delta_{kk'}. \tag{3.187}$$

In terms of notation (3.180), equation (3.179) can be rewritten as

$$\mathbb{M} = \begin{pmatrix} \mathbb{J}^2_{\mathcal{N}} \\ \mathbb{G}^2_{\mathcal{N}} \end{pmatrix} \oplus \begin{pmatrix} \mathbb{G}^2_{\mathcal{D}} \\ \mathbb{J}^2_{\mathcal{D}} \end{pmatrix} \oplus \begin{pmatrix} \mathbb{G}^2_{\Omega} \\ \mathbb{J}^2_{\Omega} \end{pmatrix} \equiv \begin{pmatrix} \mathbb{J}^2_{\mathcal{N}} \oplus \mathbb{G}^2_{\mathcal{D}} \oplus \mathbb{G}^2_{\Omega} \\ \mathbb{J}^2_{\mathcal{D}} \oplus \mathbb{G}^2_{\mathcal{N}} \oplus \mathbb{J}^2_{\Omega} \end{pmatrix}. \tag{3.188}$$

To prove that $\mathbb{M} \equiv L^4_2(S)$ we need to refer to 2− dimensional version of Weyl Theorem which concerns to Hilbert space $L^2_2(S)$ of 2−component vector functions varying in a finite domain S of Euclidean background space.

Originally, H. Weyl had studied Hilbert space $L^3_2(V)$ of 3− component vector functions from general statements of function-theoretical analysis in connection with boundary eigenvalue problems for Laplacian. (See H. Weyl, *The method of orthogonal projection in potential theory,* Duke Math. Journal, vol. 7, 1940, 411-444.) Now his main result is known as Weyl Theorem about orthogonal splitting of Hilbert space. The same theorem has been then easily proved for Hilbert space $L^2_2(S)$, which can be shortly written as follows

$$L^2_2(S) = \mathbb{J} \oplus \mathbb{G} \oplus \mathbb{U} \tag{3.189}$$

Here, the subspace $\mathbb{J}$ is the closure of a lineal of 2−component vectors $[\mathbf{z} \times \nabla_\perp \varphi]$, which are solenoidal (since $\nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \varphi] \equiv 0$) and so smooth that potential function φ satisfy either Dirichlet's or Neumann boundaty condition over the boundary L of the domain S :

$$\mathbb{J}: \qquad \{[\mathbf{z} \times \nabla_\perp \varphi]: \ (\triangle + k^2)\,\varphi = 0, \qquad \textit{either } \varphi|_L = 0, \ \ \textit{or } \tfrac{\partial}{\partial n}\varphi|_L = 0\}. \tag{3.190}$$

The subspace $\mathbb{G}$ is the closure of a lineal of $\nabla_\perp \varphi$, i.e.,

$$\mathbb{G}: \qquad \{\nabla \varphi: \ (\triangle + k^2)\,\varphi = 0, \qquad \textit{either } \varphi|_S = 0, \ \ \textit{or } \tfrac{\partial}{\partial N}\varphi|_S = 0\}. \tag{3.191}$$

The subspace $\mathbb{U}$ is the closure of a lineal of gradients of harmonic functions continuously differentiable everywhere in the closed domain $\overline{S}$:

$$\mathbb{U}: \qquad \{\nabla u: \qquad \triangle u = 0 \ \ \}. \tag{3.192}$$

Coming back to our equation (3.188) with Weyl Theorem (3.189) at hand we can conclude

$$\mathbb{M} = \begin{pmatrix} \mathbb{J}^2_{\mathcal{N}} \oplus \mathbb{G}^2_{\mathcal{D}} \oplus \mathbb{G}^2_{\Omega} \\ \mathbb{J}^2_{\mathcal{D}} \oplus \mathbb{G}^2_{\mathcal{N}} \oplus \mathbb{J}^2_{\Omega} \end{pmatrix} = \begin{pmatrix} L^2_2(S) \\ L^2_2(S \end{pmatrix} \equiv L^4_2(S). \tag{3.193}$$

Hence, the *WBO* eigenvector set originates a basis in their domain. Therefore thansverse part of electromagnetic field sought can be presented in terms of the eigenvector series.

3.5.2 Basis Set for Longitudinal Field Components

Besides the 2− component transverse vectors **E** and **H**, electromagnetic field sought (3.178) involves two scalar functions E_z and H_z as well, which bespeak a longitudinal part of the field towards waveguide axis. Component $E_z(\mathbf{r}, z, t)$ must satisfy Dirichlet's boundary condition

$$E_z(\mathbf{r}, z, t)\,|_{L_i} = 0, \quad \mathbf{r} \in L_i, \quad i = 0, 1, \ldots, N \tag{3.194}$$

in time and position over the waveguide surface. We have obtained above a complete set of orthonormed potentials $\{\Phi_n\}_{n=1}^{\infty}$, which are specified in the form of boundary eigenvalue problem for Laplacian as

$$(\Delta_\perp + \varkappa_n^2)\,\Phi_n(\mathbf{r}) = 0, \quad \Phi_n(\mathbf{r})\,|_{L_i} = 0, \quad \frac{\varkappa_n^2}{S}\int_S \Phi_n \Phi_{n'}^* ds = \delta_{nn'}, \tag{3.195}$$

where $i = 0, 1, \ldots, N$, and $\varkappa_n^2 = q_{+n}\sqrt{\epsilon_0\mu_0} > 0$. Since the boundary conditions in Eqs. (3.194) and (3.195) coincide, field component $E_z(\mathbf{r}, z, t)$ as a function transverse coordinates $\mathbf{r}$ can be presented as the series in terms of potential functions $\Phi_n(\mathbf{r})$.

Meanwhile, longitudinal field component $H_z(\mathbf{r}, z, t)$, as a function of transverse coordinates, can be presented in terms of the other orthonormal complete set of potentials, namely: $\{\Psi_m\}_{m=0}^{\infty}$. They have been specified in the form of Neumann boundary eigenvalue problem for Laplacian as

$$\begin{array}{lll} m = 0: & \Psi_0(\mathbf{r}) = 1, & \mathbf{r} \in \bar{S}; \\ (\Delta_\perp + \nu_m^2)\,\Psi_m(\mathbf{r}) = 0, & \frac{\partial}{\partial n_i}\Psi_m(\mathbf{r})\,|_{L_i} = 0, & \frac{\nu_m^2}{S}\int_S \Psi_m \Psi_{m'}^* ds = \delta_{mm'}, \end{array} \tag{3.196}$$

Here, $i = 0, 1, \ldots, N$; subscript $m = 0$ identifies the first solution corresponding to eigenvalue $p_0 = \nu_0 = 0$, and $m > 0$ stands for $\nu_m^2 = p_{+m}\sqrt{\epsilon_0\mu_0} > 0$.

It is fitting to note, that potential $\Psi_0(\mathbf{r}) = 1$ identifies very special waveguide mode which has the only longitudinal field component $\vec{\mathcal{H}}_0(\mathbf{r}, z, t) = \mathbf{z}H_{z0}(z, t)$. Transversal components $\mathbf{E}_0$ and $\mathbf{H}_0$ of this mode are absent, since they should be specified by the gradient of Ψ_0 but $\nabla_\perp\Psi_0 \equiv \mathbf{0}$.

Chapter 4
PROJECTING OF VECTOR FIELD ONTO THE BASIS

Summary 10 *Which presentation for electromagnetic field sought is named as transversally-longitudinal? Which basis should be used for the spectral decomposition of* $4-$ *component electromagnetic field vector consisting of transverse electric and magnetic field components? How to get the spectral decompositions for the transverse electric and magnetic vectors individually? Which basis should be used for the spectral decomposition of the longitudinal component of electric field? Why? Which basis should be used for the spectral decomposition of the longitudinal component of magnetic field? Which spectral coefficients should be found out in the course further analysis? Which part of Maxwell's equations can be used for this goal?*

4.1 Projecting of Transverse Electromagnetic Vectors onto the Basis

Remind once again that electromagnetic field sought has been presented in so-called transverse-longitudinal form as follows

$$\vec{\mathcal{E}}\left(\mathbf{r},z,t\right)=\mathbf{E}\left(\mathbf{r},z,t\right)+\mathbf{z}E_z\left(\mathbf{r},z,t\right);\qquad \vec{\mathcal{H}}\left(\mathbf{r},z,t\right)=\mathbf{H}\left(\mathbf{r},z,t\right)+\mathbf{z}H_z\left(\mathbf{r},z,t\right). \tag{4.1}$$

Solution to the waveguide problem should be found out in the class of quadratically integrable functions, i.e., in Hilbert space.

Two-component strength vectors $\mathbf{E}\left(\mathbf{r},z,t\right)$ and $\mathbf{H}\left(\mathbf{r},z,t\right)$ has been joint in a unified four-component electromagnetic vector as

$$\begin{pmatrix} \mathbf{E}\left(\mathbf{r},z,t\right) \\ \mathbf{H}\left(\mathbf{r},z,t\right) \end{pmatrix}=\mathcal{X}\left(\mathbf{r},z,t\right). \tag{4.2}$$

In Hilbert space $L_2^4\left(S\right)$, an orthonormal basis on the waveguide cross-section was specified. It consists of three mutually orthogonal subsets of $4-$ component vector functions of transverse coordinates varying in closed domain $\bar{S}$

$$\left\{Y_{\pm m}\left(\mathbf{r}\right)\right\}_{m=1}^{\infty};\quad \left\{Z_{\pm n}\left(\mathbf{r}\right)\right\}_{n=1}^{\infty};\quad \left\{U_{\pm k}\left(\mathbf{r}\right)\right\}_{k=1}^{N}, \tag{4.3}$$

each element of which satisfies requisite electromagnetic boundary conditions. The completeness of the basis implies that vector field $\mathcal{X}\left(\mathbf{r},z,t\right)$ may be approximated by the series with respective expansion coefficients depending on variables z and t as follows

$$\mathcal{X}(\mathbf{r},z,t)=\sum_{m=1}^{\infty}A_{\pm m}(z,t)\,Y_{\pm m}(\mathbf{r})+\sum_{n=1}^{\infty}B_{\pm n}(z,t)\,Z_{\pm n}(\mathbf{r})+\sum_{k=1}^{N}C_{\pm k}(z,t)\,U_{\pm k}(\mathbf{r})\,. \tag{4.4}$$

We need to recast Eq. (4.4) for facilitating further analysis. To this aim, it is fitting to remind of presentations of the basis elements via their potentials, namely:

$$Y_{\pm m}=Y_m^{\uparrow}\pm Y_m^{\downarrow},\quad Z_{\pm n}=Z_n^{\uparrow}\pm Z_n^{\downarrow},\quad U_{\pm k}=U_k^{\uparrow}\pm U_k^{\downarrow}, \tag{4.5}$$

where the following notation were introduced:

$$\begin{array}{llr} Y_m^{\uparrow}=\begin{pmatrix}\sqrt[-2]{\epsilon_0}\left[\nabla_{\perp}\Psi_m(\mathbf{r})\times\mathbf{z}\right]\\ \mathbf{0}\end{pmatrix}, & Y_m^{\downarrow}=\begin{pmatrix}\mathbf{0}\\ \sqrt[-2]{\mu_0}\nabla_{\perp}\Psi_m(\mathbf{r})\end{pmatrix}; & (a)\\ Z_n^{\uparrow}=\begin{pmatrix}\sqrt[-2]{\epsilon_0}\nabla_{\perp}\Phi_n(\mathbf{r})\\ \mathbf{0}\end{pmatrix}, & Z_n^{\downarrow}=\begin{pmatrix}\mathbf{0}\\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_{\perp}\Phi_n(\mathbf{r})\right]\end{pmatrix}; & (b)\\ U_k^{\uparrow}=\begin{pmatrix}\sqrt[-2]{\epsilon_0}\nabla_{\perp}\Omega_k(\mathbf{r})\\ \mathbf{0}\end{pmatrix}, & U_k^{\downarrow}=\begin{pmatrix}\mathbf{0}\\ \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_{\perp}\Omega_k(\mathbf{r})\right]\end{pmatrix}. & (c)\end{array} \tag{4.6}$$

Let us recast the first series from the right-hand-side of Eq. (4.4). To this aim, we substitute here the presentation for vectors $Y_{\pm m}(\mathbf{r})$ given in Eq. (4.5) what results in

$$\sum_{m=1}^{\infty}A_{\pm m}(z,t)\,Y_{\pm m}(\mathbf{r})=\sum_{m=1}^{\infty}\left\{A_{+m}(z,t)\,Y_{+m}(\mathbf{r})+A_{-m}(z,t)\,Y_{-m}(\mathbf{r})\right\}.$$

Each term of the series obtained at the right-hand-side can be then rearranged as follows

$$\begin{aligned}A_{+m}Y_{+m}+A_{-m}Y_{-m}&=A_{+m}\left(Y_m^{\uparrow}+Y_m^{\downarrow}\right)+A_{-m}\left(Y_m^{\uparrow}-Y_m^{\downarrow}\right)\\ &=\left(A_{+m}+A_{-m}\right)Y_m^{\uparrow}+\left(A_{+m}-A_{-m}\right)Y_m^{\downarrow}\end{aligned}$$

Let us denote

$$A_{+m}(z,t)+A_{-m}(z,t)\equiv V_m^h(z,t)\,,\qquad A_{+m}(z,t)-A_{-m}(z,t)\equiv I_m^h(z,t)\,. \tag{4.7}$$

Utilizing line (a) in Eqs. (4.6), the resulting expression for the first series from Eq.

(4.4) can be written as follows

$$\begin{aligned}\sum_{m=1}^{\infty} A_{\pm m} Y_{\pm m} &= \sum_{m=1}^{\infty} V_m^h(z,t)\, Y_m^{\uparrow}(\mathbf{r}) + \sum_{m=1}^{\infty} I_m^h(z,t)\, Y_m^{\downarrow}(\mathbf{r}) \\ &= \sum_{m=1}^{\infty} V_m^h \begin{pmatrix} \sqrt[-2]{\epsilon_0}\left[\nabla_\perp \Psi_m \times \mathbf{z}\right] \\ \mathbf{0} \end{pmatrix} + \sum_{m=1}^{\infty} I_m^h \begin{pmatrix} \mathbf{0} \\ \sqrt[-2]{\mu_0} \nabla_\perp \Psi_m \end{pmatrix} \\ &= \begin{pmatrix} \sqrt[-2]{\epsilon_0} \sum_{m=1}^{\infty} V_m^h(z,t)\left[\nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z}\right] \\ \\ \sqrt[-2]{\mu_0} \sum_{m=1}^{\infty} I_m^h(z,t)\, \nabla_\perp \Psi_m(\mathbf{r}) \end{pmatrix}. \end{aligned} \tag{4.8}$$

After similar manipulations, the second series from Eq. (4.4) assumes the form

$$\sum_{n=1}^{\infty} B_{\pm n}(z,t)\, Z_{\pm n}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0} \sum_{n=1}^{\infty} V_n^e(z,t)\, \nabla_\perp \Phi_n(\mathbf{r}) \\ \\ \sqrt[-2]{\mu_0} \sum_{n=1}^{\infty} I_n^e(z,t)\left[\mathbf{z} \times \nabla_\perp \Phi_n(\mathbf{r})\right] \end{pmatrix}, \tag{4.9}$$

on introducing a new pair of independent coefficients depending on z and t variables

$$V_n^e(z,t) = B_{+n}(z,t) + B_{-n}(z,t), \qquad I_n^e(z,t) = B_{+n}(z,t) - B_{-n}(z,t). \tag{4.10}$$

The third and last series from Eq. (4.4) can be rearranged analogously to the series

$$\sum_{k=1}^{N} C_{\pm k}(z,t)\, U_{\pm k}(\mathbf{r}) = \begin{pmatrix} \sqrt[-2]{\epsilon_0} \sum_{k=1}^{N} V_k(z,t) \nabla_\perp \Omega_k(\mathbf{r}) \\ \\ \sqrt[-2]{\mu_0} \sum_{k=1}^{N} I_k(z,t)\left[\mathbf{z} \times \nabla_\perp \Omega_k(\mathbf{r})\right] \end{pmatrix}, \tag{4.11}$$

with some unknown multipliers in front, i.e.,

$$V_k(z,t) = C_{+k}(z,t) + C_{-k}(z,t) \qquad I_k(z,t) = C_{+k}(z,t) - C_{-k}(z,t). \tag{4.12}$$

It is also apparent that Eq. (4.4) can be convert to the outcome

$$\begin{aligned}\mathcal{X} &= \sum_{m=1}^{\infty} A_{\pm m} Y_{\pm m} + \sum_{n=1}^{\infty} B_{\pm n} Z_{\pm n} + \sum_{k=1}^{N} C_{\pm k} U_{\pm k} \\ &= \begin{pmatrix} \sqrt[-2]{\epsilon_0} \left\{ \sum_{m=1}^{\infty} V_m^h \left[\nabla_\perp \Psi_m \times \mathbf{z}\right] + \sum_{n=1}^{\infty} V_n^e \nabla_\perp \Phi_n + \sum_{k=1}^{N} V_k \nabla_\perp \Omega_k \right\} \\ \\ \sqrt[-2]{\mu_0} \left\{ \sum_{m=1}^{\infty} I_m^h \nabla_\perp \Psi_m + \sum_{n=1}^{\infty} I_n^e \left[\mathbf{z} \times \nabla_\perp \Phi_n\right] + \sum_{k=1}^{N} I_k \left[\mathbf{z} \times \nabla_\perp \Omega_k\right] \right\} \end{pmatrix}. \end{aligned} \tag{4.13}$$

Inasmuch as

$$\mathcal{X} = \begin{pmatrix} \mathbf{E}(\mathbf{r}, z, t) \\ \mathbf{H}(\mathbf{r}, z, t) \end{pmatrix},$$

the following equality holds

$$\begin{pmatrix} \mathbf{E} \\ \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \sqrt[-2]{\epsilon_0} \left\{ \sum\limits_{m=1}^{\infty} V_m^h \left[\nabla_\perp \Psi_m \times \mathbf{z} \right] + \sum\limits_{n=1}^{\infty} V_n^e \nabla_\perp \Phi_n + \sum\limits_{k=1}^{N} V_k \nabla_\perp \Omega_k \right\} \\ \\ \sqrt[-2]{\mu_0} \left\{ \sum\limits_{m=1}^{\infty} I_m^h \nabla_\perp \Psi_m + \sum\limits_{n=1}^{\infty} I_n^e \left[\mathbf{z} \times \nabla_\perp \Phi_n \right] + \sum\limits_{k=1}^{N} I_k \left[\mathbf{z} \times \nabla_\perp \Omega_k \right] \right\} \end{pmatrix}.$$

This equality for the 4− component vectors implies equality of the 2− component its constituents as

$$\boxed{\begin{array}{c} \mathbf{E} = \sqrt[-2]{\epsilon_0} \left\{ \sum_{m=1}^{\infty} V_m^h \left[\nabla_\perp \Psi_m \times \mathbf{z} \right] + \sum_{n=1}^{\infty} V_n^e \nabla_\perp \Phi_n + \sum_{k=1}^{N} V_k \nabla_\perp \Omega_k \right\}; \\ \\ \mathbf{H} = \sqrt[-2]{\mu_0} \left\{ \sum_{m=1}^{\infty} I_m^h \nabla_\perp \Psi_m + \sum_{n=1}^{\infty} I_n^e \left[\mathbf{z} \times \nabla_\perp \Phi_n \right] + \sum_{k=1}^{N} I_k \left[\mathbf{z} \times \nabla_\perp \Omega_k \right] \right\}. \end{array}} \tag{4.14}$$

Such a presentation for the transverse components of electromagnetic field sought amounts to decompositions of the vectors in terms of the known vector waveguide modal functions depending on transverse coordinates $\mathbf{r}$ with unknown yet scalar coefficient depending on variables z and t.

4.2 Projecting of Longitudinal Components onto the Basis

The projection procedure for $E_z(\mathbf{r}, z, t)$ field component starts from the choice of a complete set of potential functions suitable for decomposition. Inasmuch as tangential to waveguide surface E_z- component must satisfy the boundary condition

$$E_z(\mathbf{r}, z, t)\,|_{L_i} = 0, \qquad i = 0, 1, \ldots, N, \tag{4.15}$$

orthonormal complete set of functions depending on transverse coordinates

$$\left\{ \Phi_n(\mathbf{r}) : \;\; \left(\Delta_\perp + \varkappa_n^2 \right) \Phi_n = 0, \;\; \Phi_n|_{L_i} = 0, \;\; \varkappa_n^2 = q_{+n}\sqrt{\epsilon_0 \mu_0} > 0 \; \right\}_{n=1}^{\infty} \tag{4.16}$$

obtained before is proper as a known basis. Hence, $E_z(\mathbf{r}, z, t)$, as a function of transverse variables, may be written in the form of a decomposition in terms of basis functions $\Phi_n(\mathbf{r})$ with some unknown coefficients depending on z and t variables as follows

$$\boxed{E_z(\mathbf{r}, z, t) = \sqrt[-2]{\epsilon_0} \sum_{n=1}^{\infty} e_n(z, t)\, \varkappa_n^2 \Phi_n(\mathbf{r}).} \tag{4.17}$$

Here, multiplier $1/\sqrt{\epsilon_0}$ is introduced to provide the same physical dimension for $e_n(z, t)$ which has similar coefficients depending on z and t variables in Eqs. (4.14). The multipliers $\varkappa_n^2$ are introduced in the series terms with the same purpose and for facilitating of further calculations as well.

For a similar decomposition of $H_z(\mathbf{r}, z, t)$ component, the other orthonormal complete set of potential functions is suited, namely:

$$\left\{ \ \Psi_0(\mathbf{r}) = 1, \qquad \Psi_m(\mathbf{r}) \ \right\}_{m=1}^{\infty}, \tag{4.18}$$

where

$$\Psi_m(\mathbf{r}) : \ \ \left(\Delta_\perp + \nu_m^2\right)\Psi_m = 0, \ \ \tfrac{\partial}{\partial n_i}\Psi_m|_{L_i} = 0, \ \ \nu_m^2 = p_{+m}\sqrt{\epsilon_0\mu_0} > 0. \tag{4.19}$$

Expansion of $H_z(\mathbf{r}, z, t)$ in terms of basis functions (4.18) can be done in a similar way with the outcome

$$\boxed{H_z(\mathbf{r}, z, t) = \ \sqrt[-2]{\mu_0}\left\{h_0(z, t) + \textstyle\sum_{m=1}^{\infty} h_m(z, t)\,\nu_m^2\Psi_m(\mathbf{r})\right\},} \tag{4.20}$$

where coefficients $h_0(z, t)$ and $h_m(z, t)$ are to be found in the course of further analysis.

Thus, we have obtained presentations of electromagnetic field sought as the series (4.14) for the field projections on the waveguide cross-section and the series (4.17) , (4.21) for the axial components. Mathematically, these series amount to some spectral decompositions of the field sought in terms of known eigensolutions of the boundary value problem for Laplacian as the elements of a basis in Hilbert space. Physically, they have a sense of the field presentations in terms of natural waveguide modes. All the series involve the spectral coefficients which are unknown functions of z and t. Physically, they have a sense of the amplitudes of the waveguide modes.

These amplitudes, depending on z and t variables, may be separated onto three sets as follows

$$\boxed{\begin{array}{cc} h_0(z, t)\,; & (H_{\text{hom}}) \\[2ex] \left\{h_m(z, t)\,, V_m^h(z, t)\, I_m^h(z, t)\right\}_{m=1}^{\infty}; & (H) \\[2ex] \left\{e_n(z, t)\,, V_n^e(z, t)\,, I_n^e(z, t)\right\}_{n=1}^{\infty}. & (E) \\[2ex] \left\{V_k(z, t)\,, I_k(z, t)\right\}_{k=1}^{N} & (T) \end{array}} \tag{4.21}$$

The *first* set involves single element h_0: see line (H_{hom}) in Eq. (4.21) . Function $h_0(z, t)$ is an amplitude of a very special waveguide mode which has the following field components:

$$\vec{\mathcal{E}}_0^{(H)}(\mathbf{r}, z, t) = \mathbf{0}, \qquad \vec{\mathcal{H}}_0^{(H)}(\mathbf{r}, z, t) = \mathbf{z}h_0(z, t)\,. \tag{4.22}$$

So, electromagnetic field of this mode has distinct from zero *magnetic* field only which, in turn, consists of single longitudinal component. This field is *homogeneous* in the waveguide cross-section. The *second* set, shown in (H)- line in Eq. (4.21) , corresponds to standard $TE-$ modes in waveguide. Structure of these modes is specified as follows

$$\begin{aligned} \vec{\mathcal{E}}_m^{(H)}(\mathbf{r},z,t) &= \sqrt[-2]{\epsilon_0}V_m^h(z,t)\left[\nabla_\perp\Psi_m(\mathbf{r})\times\mathbf{z}\right]; \\ \vec{\mathcal{H}}_m^{(H)}(\mathbf{r},z,t) &= \sqrt[-2]{\mu_0}\left\{I_m^h(z,t)\nabla_\perp\Psi_m(\mathbf{r})+\mathbf{z}h_m(z,t)\nu_m^2\Psi_m(\mathbf{r})\right\}. \end{aligned} \tag{4.23}$$

The *third* set marked as $(E)-$ line in Eq. (4.21) specifies standard $TM-$ modes with the following structure of electromagnetic field:

$$\begin{aligned} \vec{\mathcal{E}}_n^{(E)}(\mathbf{r},z,t) &= \sqrt[-2]{\epsilon_0}\left\{V_n^e(z,t)\nabla_\perp\Phi_n(\mathbf{r})+\mathbf{z}e_n(z,t)\varkappa_n^2\Phi_n(\mathbf{r})\right\}; \\ \vec{\mathcal{H}}_n^{(E)}(\mathbf{r},z,t) &= \sqrt[-2]{\mu_0}I_n^e(z,t)\left[\mathbf{z}\times\nabla_\perp\Phi_n(\mathbf{r})\right]. \end{aligned} \tag{4.24}$$

The *forth*, i.e., $(T)-$ line in Eq. (4.21), specifies $TEM-$ modes which can exist in a waveguide when its cross section is multiconnected. Number of these modes is always finite and it depends on the number N of components. Structure of the modal fields looks as follows

$$\begin{aligned} \vec{\mathcal{E}}_k^{(T)}(\mathbf{r},z,t) &= \sqrt[-2]{\epsilon_0}V_k(z,t)\nabla_\perp\Omega_k(\mathbf{r}); \\ \vec{\mathcal{H}}_k^{(T)}(\mathbf{r},z,t) &= \sqrt[-2]{\mu_0}I_k(z,t)\left[\mathbf{z}\times\nabla_\perp\Omega_k(\mathbf{r})\right]. \end{aligned} \tag{4.25}$$

In the general case, electromagnetic field sought is a superposition of all the waveguide modes with the outcome

$$\begin{aligned} \vec{\mathcal{E}} &= \sum_m^\infty \vec{\mathcal{E}}_m^{(H)}(\mathbf{r},z,t)+\sum_{n=1}^\infty \vec{\mathcal{E}}_n^{(E)}(\mathbf{r},z,t)+\sum_{k=1}^N \vec{\mathcal{E}}_k^{(T)}(\mathbf{r},z,t); \\ \vec{\mathcal{H}} &= \mathbf{z}h_0(z,t)+\sum_m^\infty \vec{\mathcal{H}}_m^{(H)}(\mathbf{r},z,t)+\sum_{n=1}^\infty \vec{\mathcal{H}}_n^{(E)}(\mathbf{r},z,t)+\sum_{k=1}^N \vec{\mathcal{H}}_k^{(T)}(\mathbf{r},z,t), \end{aligned} \tag{4.26}$$

where $\vec{\mathcal{E}}\equiv\vec{\mathcal{E}}(\mathbf{r},z,t)$ and $\vec{\mathcal{H}}\equiv\vec{\mathcal{H}}(\mathbf{r},z,t)$. In order to use Eqs. (4.26) for calculations of the field in concrete practical situations, we should to learn to calculate the modal coefficients enumerated in Eq. (4.21). We start scrutiny of this problem in the following chapter.

Chapter 5
PROJECTING OF MAXWELL'S EQUATIONS ONTO THE BASIS

Summary 11 *Why is it necessary to project Maxwell's equations on elements of the basis? Which form of Maxwell's equations is chosen for projecting? Why? Which elements of basis should be used for projecting of* ***the scalar*** *equations with respect to H_z and E_z field components? Why? Which elements of basis should be used for projecting of* ***the vector*** *equations? Why? How many partial differential equations can supply each of the vector equations? Why? Are all of them independent equations? Which form have the waveguide evolutionary equations in the simplest case of the problem under consideration? Which ways exist to solution of the waveguide evolutionary equations?*

5.1 Projecting of the First Order Differential Equations for H_z

5.1.1 Projecting of Scalar Differential Equations

remind of a pair of scalar differential equations which are written in lines (a) and (b) in the system of differential equations of the first order (2.35) with respect to function $H_z(\mathbf{r}, z, t)$:

$$\begin{aligned} \mu_0 \partial_z (\mu H_z) &= -\mu_0 \mu \nabla_\perp \cdot \mathbf{H} + g, \quad (b) \\ \mu_0 \partial_t (\mu H_z) &= \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}] - I_z. \quad (c) \end{aligned} \tag{5.1}$$

We need to calculate first $\nabla_\perp \cdot \mathbf{H}$ and $\nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}]$ which are present at these equations. In Eq. (4.14) , the expression for vector $\mathbf{H}$ is given in the form of decomposition as

$$\mathbf{H} = \sqrt[-2]{\mu_0} \left\{ \sum_{m=1}^{\infty} I_m^h \nabla_\perp \Psi_m + \sum_{n=1}^{\infty} I_n^e [\mathbf{z} \times \nabla_\perp \Phi_n] + \sum_{k=1}^{N} I_k [\mathbf{z} \times \nabla_\perp \Omega_k] \right\}$$

Hence, $\nabla_\perp \cdot \mathbf{H}$ can be calculated as follows

$$\begin{aligned} \nabla_\perp \cdot \mathbf{H} &= \sqrt[-2]{\mu_0} \left\{ \begin{array}{c} \sum_{m=1}^{\infty} I_m^h \nabla_\perp \cdot \nabla_\perp \Psi_m \\ + \sum_{n=1}^{\infty} I_n^e \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Phi_n] \\ + \sum_{k=1}^{N} I_k \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Omega_k] \end{array} \right\} \\ &= \sqrt[-2]{\mu_0} \sum_{m=1}^{\infty} I_m^h \Delta_\perp \Psi_m = - \sqrt[-2]{\mu_0} \sum_{m'=1}^{\infty} I_{m'}^h \nu_{m'}^2 \Psi_{m'}, \end{aligned} \tag{5.2}$$

since

$$\Delta_\perp \Psi_m = -\nu_m^2 \Psi_m; \quad \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Phi_n] \equiv 0, \quad \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Omega_k] = 0.$$

Further, in the same equation (4.14) vector $\mathbf{E}$ is given as

$$\mathbf{E} = \sqrt[-2]{\epsilon_0} \left\{ \sum_{m=1}^{\infty} V_m^h [\nabla_\perp \Psi_m \times \mathbf{z}] + \sum_{n=1}^{\infty} V_n^e \nabla_\perp \Phi_n + \sum_{k=1}^{N} V_k \nabla_\perp \Omega_k \right\}.$$

Hence,

$$[\mathbf{z} \times \mathbf{E}] = \sqrt[-2]{\epsilon_0} \left\{ \sum_{m=1}^{\infty} V_m^h \nabla_\perp \Psi_m + \sum_{n=1}^{\infty} V_n^e [\mathbf{z} \times \nabla_\perp \Phi_n] + \sum_{k=1}^{N} V_k [\mathbf{z} \times \nabla_\perp \Omega_k] \right\}$$

inasmuch as $[\mathbf{z} \times [\nabla_\perp \Psi_m \times \mathbf{z}]] = (\mathbf{z} \cdot \mathbf{z}) \nabla_\perp \Psi_m - \mathbf{z} (\mathbf{z} \cdot \nabla_\perp \Psi_m) = \nabla_\perp \Psi_m$. So,

$$\begin{aligned} \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}] &= \sqrt[-2]{\epsilon_0} \left\{ \begin{array}{c} \sum_{m=1}^{\infty} V_m^h \nabla_\perp \cdot \nabla_\perp \Psi_m \\ + \sum_{n=1}^{\infty} V_n^e \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Phi_n] \\ + \sum_{k=1}^{N} V_k \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Omega_k] \end{array} \right\} \\ &= \sqrt[-2]{\epsilon_0} \sum_{m=1}^{\infty} V_m^h \nabla_\perp \cdot \nabla_\perp \Psi_m = - \sqrt[-2]{\epsilon_0} \sum_{m'=1}^{\infty} V_{m'}^h \nu_{m'}^2 \Psi_{m'}, \end{aligned} \tag{5.3}$$

because of

$$\nabla_\perp \cdot \nabla_\perp \Psi_m = \Delta_\perp \Psi_m = -\nu_m^2 \Psi_m; \quad \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Phi_n] = 0, \quad \nabla_\perp \cdot [\mathbf{z} \times \nabla_\perp \Omega_k] = 0.$$

Now, we can substitute outcomes (5.2) and (5.3) in Eqs. (5.1) what yields

$$\begin{aligned} \mu_0 \partial_z (\mu H_z) &= \sqrt{\mu_0} \mu \textstyle\sum_{m'=1}^{\infty} I_{m'}^h \nu_{m'}^2 \Psi_{m'} + g, \quad (b) \\ \mu_0 \partial_t (\mu H_z) &= - \sqrt[-2]{\epsilon_0} \textstyle\sum_{m'=1}^{\infty} V_{m'}^h \nu_{m'}^2 \Psi_{m'} - I_z. \quad (c) \end{aligned} \tag{5.4}$$

Remind that the presentation for $H_z(\mathbf{r}, z, t)$ is given in Eq. (4.20) as

$$H_z(\mathbf{r}, z, t) = \sqrt[-2]{\mu_0} \left\{ h_0(z,t) + \sum_{m'=1}^{\infty} h_{m'}(z,t) \nu_{m'}^2 \Psi_{m'}(\mathbf{r}) \right\}. \tag{5.5}$$

Substitution of Eq. (4.20) at the right-hand-sides of Eqs. (5.4) yields

$$\begin{aligned} \mu_0 \partial_z (\mu H_z) &= \sqrt{\mu_0} \left\{ \partial_z (\mu h_0) + \textstyle\sum_{m'=1}^{\infty} \partial_z (\mu h_{m'}) \nu_{m'}^2 \Psi_{m'}(\mathbf{r}) \right\}; \\ \mu_0 \partial_t (\mu H_z) &= \sqrt{\mu_0} \left\{ \partial_t (\mu h_0) + \textstyle\sum_{m'=1}^{\infty} \partial_t (\mu h_{m'}) \nu_{m'}^2 \Psi_{m'}(\mathbf{r}) \right\}, \end{aligned} \tag{5.6}$$

where $\mu \equiv \mu(z,t) > 0$ is an arbitrary differentiable function on z and t variables.

Equations (5.4) and (5.6) supply jointly the following system of series equations:

$$\begin{aligned} &\partial_z\left(\mu h_0 \Psi_0\right)+\sum_{m'=1}^{\infty} \partial_z\left(\mu h_{m'}\right) \nu_{m'}^2 \Psi_{m'}=\mu \sum_{m'=1}^{\infty} I_{m'}^h \nu_{m'}^2 \Psi_{m'}+\sqrt[-2]{\mu_0} g ; \\ &\partial_{ct}\left(\mu h_0 \Psi_0\right)+\sum_{m'=1}^{\infty} \partial_{ct}\left(\mu h_{m'}\right) \nu_{m'}^2 \Psi_{m'}=-\sum_{m'=1}^{\infty} V_{m'}^h \nu_{m'}^2 \Psi_{m'}-\sqrt{\epsilon_0} I_z, \end{aligned} \tag{5.7}$$

Here, $\Psi_0 \equiv \Psi_0(\mathbf{r})=1$, and operation ∂_{ct} means

$$\partial_{ct}=\sqrt{\epsilon_0 \mu_0} \partial_t=\frac{1}{c} \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial c t}, \tag{5.8}$$

where $c=1 / \sqrt{\epsilon_0 \mu_0}$ is free-space light velocity. Set of functions

$$\left\{\Psi_0, \Psi_m\right\}_{m=1}^{\infty}$$

satisfies the following conditions of orthonormality:

$$\begin{aligned} &\tfrac{1}{S} \int_S\left|\Psi_0\right|^2 d s=\tfrac{1}{S} \int_S d s=1 ; \\ &\tfrac{1}{S} \int_S \Psi_{m'} \Psi_0^* d s=\tfrac{1}{S} \int_S \Psi_{m'} d s=0 ; \\ &\nu_m^2 \tfrac{1}{S} \int_S \Psi_{m'} \Psi_m^* d s=\delta_{m m'} . \end{aligned} \tag{5.9}$$

We first multiply equations (5.7) both on $(1/S)$ and integrate then them over waveguide cross-section S with making use of orthonormality conditions (5.9). The first pair of partial differential equations holds as an outcome:

$$\boxed{\begin{aligned} &\partial_z\{\mu(z, t) h_0(z, t)\}=\sqrt[-2]{\mu_0} \tfrac{1}{S} \int_S g d s ; \\ &\partial_{ct}\{\mu(z, t) h_0(z, t)\}=-\sqrt{\epsilon_0} \tfrac{1}{S} \int_S I_z d s . \end{aligned}} \tag{5.10}$$

Notation g and $I_z=\mathbf{z} \cdot \vec{\mathcal{I}}$ are explained in the subsection titled as "*Auxiliary Identical Reorganization*". We multiply now the same equations (5.7) both on $\left(\nu_m^2 / S\right) \Psi_m^*$ and integrate them over waveguide cross-section S. With using orthonormality conditions (5.9) in the course of calculations, we obtain one more set of partial differential equations, namely:

$$\boxed{\begin{aligned} &\partial_z\left\{\mu(z, t) h_m(z, t)\right\}=\mu(z, t) I_m^h(z, t)+\sqrt[-2]{\mu_0} \tfrac{1}{S} \int_S g \Psi_m^* d s ; \\ &\partial_{ct}\left\{\mu(z, t) h_m(z, t)\right\}=-V_m^h(z, t)-\sqrt{\epsilon_0} \tfrac{1}{S} \int_S I_z \Psi_m^* d s, \end{aligned}} \tag{5.11}$$

where $m=1,2, \ldots$ .This is a final result of projecting of scalar equations (2.35) on the basis elements.

5.1.2 Projecting of Vector Differential Equation

Now, we should study vector equation from the system (2.35) which is placed in line (a) there. Premultiplying with the unit vector $\mathbf{z}$ as $[\mathbf{z}\times$, it can be written as follows

$$\nabla_\perp H_z = \epsilon_0 \partial_t \left(\varepsilon \left[\mathbf{z} \times \mathbf{E}\right]\right) + \partial_z \mathbf{H} + \left[\mathbf{z} \times \mathbf{J}\right], \tag{5.12}$$

since $[\mathbf{z}\times [\nabla_\perp H_z \times \mathbf{z}]] = \nabla_\perp H_z$, and $[\mathbf{z}\times [\mathbf{H} \times \mathbf{z}]] = \mathbf{H}$.

After substitution of H_z given in Eq. (5.5), its left-hand-side acquires the following presentation:

$$\nabla_\perp H_z = \sqrt[-2]{\mu_0} \sum_{m'=1}^{\infty} h_{m'}\left(z,t\right) \nu_{m'}^2 \nabla_\perp \Psi_{m'}. \tag{5.13}$$

At the right-hand-side, we have two another vectors which are also decomposed in terms of basis elements accordingly to Eqs. (4.14) as follows

$$[\mathbf{z} \times \mathbf{E}] = \sqrt[-2]{\epsilon_0} \left\{ \begin{array}{c} \sum_{m'=1}^{\infty} V_{m'}^h \nabla_\perp \Psi_{m'} + \sum_{n'=1}^{\infty} V_{n'}^e \left[\mathbf{z} \times \nabla_\perp \Phi_{n'}\right] \\ + \sum_{k'=1}^{N} V_{k'} \left[\mathbf{z} \times \nabla_\perp \Omega_{k'}\right] \end{array} \right\}; \tag{5.14}$$

$$\mathbf{H} = \sqrt[-2]{\mu_0} \left\{ \begin{array}{c} \sum_{m'=1}^{\infty} I_{m'}^h \nabla_\perp \Psi_{m'} + \sum_{n;=1}^{\infty} I_{n'}^e \left[\mathbf{z} \times \nabla_\perp \Phi_{n'}\right] \\ + \sum_{k'=1}^{N} I_{k'} \left[\mathbf{z} \times \nabla_\perp \Omega_{k'}\right] \end{array} \right\}. \tag{5.15}$$

Substitution of these decompositions in Eq. (5.12) suggests that this equation should be projected on the basis elements $\nabla_\perp \Psi_m$, $[\mathbf{z} \times \nabla_\perp \Phi_n]$, and $[\mathbf{z} \times \nabla_\perp \Omega_k]$.

Projecting on $\nabla_\perp \Psi_m$. Let us make dot product of Eq. (5.13) with vector $\nabla_\perp \Psi_m^*$ and integrate outcome obtained over waveguide cross-section S as follows

$$\frac{1}{S}\int_S \nabla_\perp H_z \cdot \nabla_\perp \Psi_m^* ds = \sqrt[-2]{\mu_0} \sum_{m'=1}^{\infty} h_{m'}\left(z,t\right) \nu_{m'}^2 \frac{1}{S}\int_S \nabla_\perp \Psi_{m'} \cdot \nabla_\perp \Psi_m^* ds$$

Using then orthonormality condition

$$\frac{1}{S}\int_S \nabla_\perp \Psi_{m'} \cdot \nabla_\perp \Psi_m^* ds = \delta_{mm'}, \tag{5.16}$$

one can get

$$\frac{1}{S}\int_S \nabla_\perp H_z \cdot \nabla_\perp \Psi_m^* ds = \sqrt[-2]{\mu_0} \sum_{m'=1}^{\infty} h_{m'}\left(z,t\right) \nu_{m'}^2 \delta_{mm'} = \sqrt[-2]{\mu_0} \nu_m^2 h_m\left(z,t\right). \tag{5.17}$$

The same integration of dot product $[\mathbf{z} \times \mathbf{E}] \cdot \nabla_\perp \Psi_m^*$ with using of Eq. (5.14) yields

$$\frac{1}{S}\int_S [\mathbf{z} \times \mathbf{E}] \cdot \nabla_\perp \Psi_m^* ds = \sqrt[-2]{\epsilon_0} \left\{ \begin{array}{c} \sum_{m'=1}^{\infty} V_{m'}^h \frac{1}{S}\int_S \nabla_\perp \Psi_{m'} \cdot \nabla_\perp \Psi_m^* ds \\ + \sum_{n'=1}^{\infty} V_{n'}^e \frac{1}{S}\int_S \left[\mathbf{z} \times \nabla_\perp \Phi_{n'}\right] \cdot \nabla_\perp \Psi_m^* ds \\ + \sum_{k'=1}^{N} V_{k'} \frac{1}{S}\int_S \left[\mathbf{z} \times \nabla_\perp \Omega_{k'}\right] \cdot \nabla_\perp \Psi_m^* ds \end{array} \right\}.$$

The following pair of orthogonality conditions

$$\tfrac{1}{S}\int_S [\mathbf{z}\times\nabla_\perp\Phi_{n'}]\cdot\nabla_\perp\Psi_m^* ds = 0; \qquad \tfrac{1}{S}\int_S [\mathbf{z}\times\nabla_\perp\Omega_{k'}]\cdot\nabla_\perp\Psi_m^* ds = 0 \tag{5.18}$$

cancels two series. Using then orthonormality condition, one can get

$$\frac{1}{S}\int_S [\mathbf{z}\times\mathbf{E}]\cdot\nabla_\perp\Psi_m^* ds = \sqrt[-2]{\epsilon_0}\sum_{m'=1}^{\infty} V_{m'}^h \delta_{mm'} = \sqrt[-2]{\epsilon_0} V_m^h(z,t)\,. \tag{5.19}$$

Integration of dot product $\mathbf{H}\cdot\nabla_\perp\Psi_m^*$ looks as follows

$$\frac{1}{S}\int_S \mathbf{H}\cdot\nabla_\perp\Psi_m^* ds = \sqrt[-2]{\mu_0}\left\{\begin{array}{c} \sum_{m'=1}^{\infty} I_{m'}^h \frac{1}{S}\int_S \nabla_\perp\Psi_{m'}\cdot\nabla_\perp\Psi_m^* ds \\ +\sum_{n'=1}^{\infty} I_{n'}^e \frac{1}{S}\int_S [\mathbf{z}\times\nabla_\perp\Phi_{n'}]\cdot\nabla_\perp\Psi_m^* ds \\ +\sum_{k'=1}^{N} I_{k'} \frac{1}{S}\int_S [\mathbf{z}\times\nabla_\perp\Omega_{k'}]\cdot\nabla_\perp\Psi_m^* ds \end{array}\right\}.$$

Orthonormality conditions (5.16) and (5.18) furnish the following simple outcome

$$\frac{1}{S}\int_S \mathbf{H}\cdot\nabla_\perp\Psi_m^* ds = \sqrt[-2]{\mu_0}\sum_{m'=1}^{\infty} I_{m'}^h \delta_{mm'} = \sqrt[-2]{\mu_0} I_m^h(z,t)\,. \tag{5.20}$$

Direct substitution of results obtained in Eqs. (5.17), (5.19) and (5.20) supplies

$$\sqrt[-2]{\mu_0}\nu_m^2 h_m(z,t) = \sqrt{\epsilon_0}\partial_t\left(\varepsilon V_m^h(z,t)\right) + \partial_z \sqrt[-2]{\mu_0} I_m^h(z,t) + \frac{1}{S}\int_S [\mathbf{z}\times\mathbf{J}]\cdot\nabla_\perp\Psi_m^* ds.$$

After simple manipulations, final result of projecting of Eq. (5.12) on basis element $\nabla_\perp\Psi_m^*$ acquires a form of the following equation:

$$\boxed{-\partial_{ct}\left\{\varepsilon(z,t)V_m^h(z,t)\right\} - \partial_z I_m^h(z,t) + \nu_m^2 h_m(z,t) = \sqrt{\mu_0}\tfrac{1}{S}\int_S [\mathbf{z}\times\mathbf{J}]\cdot\nabla_\perp\Psi_m^* ds.}$$

where

$$\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t = \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial(ct)}, \tag{5.21}$$

and $c = 1/\sqrt{\epsilon_0\mu_0}$ is free-space light velocity.

Projecting on $[\mathbf{z}\times\nabla_\perp\Phi_n]$. In the course of further calculations the following conditions of orthonormality will be needed:

$$\frac{1}{S}\int_S [\mathbf{z}\times\nabla_\perp\Phi_{n'}]\cdot[\mathbf{z}\times\nabla_\perp\Phi_n^*]\, ds = \frac{1}{S}\int_S \nabla_\perp\Phi_{n'}\cdot\nabla_\perp\Phi_n^* = \delta_{nn'};$$

$$\tfrac{1}{S}\int_S \nabla_\perp\Psi_m\cdot[\mathbf{z}\times\nabla_\perp\Phi_n^*]\, ds = 0; \qquad \tfrac{1}{S}\int_S [\mathbf{z}\times\nabla_\perp\Omega_{k'}]\cdot[\mathbf{z}\times\nabla_\perp\Phi_n^*]\, ds = 0.$$

Using these identities, projections of the vectors from Eq. (5.12) on basis element $[\mathbf{z} \times \nabla_\perp \Phi_n]$ can be easily convert to the following outcomes

$$\frac{1}{S}\int_S \nabla_\perp H_z \cdot [\mathbf{z} \times \nabla_\perp \Phi_n^*]\, ds = 0; \tag{5.22}$$

$$\frac{1}{S}\int_S [\mathbf{z} \times \mathbf{E}] \cdot [\mathbf{z} \times \nabla_\perp \Phi_n^*]\, ds = \sqrt[-2]{\epsilon_0} \sum_{n'=1}^{\infty} V_{n'}^e \delta_{nn'} = \sqrt[-2]{\epsilon_0} V_n^e\left(z,t\right); \tag{5.23}$$

$$\frac{1}{S}\int_S \mathbf{H} \cdot [\mathbf{z} \times \nabla_\perp \Phi_n^*]\, ds = \sqrt[-2]{\mu_0} \sum_{n'=1}^{\infty} I_{n'}^e \delta_{nn'} = \sqrt[-2]{\mu_0} I_n^e\left(z,t\right). \tag{5.24}$$

Direct substitution them in Eq. (5.12) yields

$$0 = \sqrt{\epsilon_0}\partial_t\left(\varepsilon V_n^e\left(z,t\right)\right) + \sqrt[-2]{\mu_0}\partial_z I_n^e\left(z,t\right) + \frac{1}{S}\int_S [\mathbf{z} \times \mathbf{J}] \cdot [\mathbf{z} \times \nabla_\perp \Phi_n^*]\, ds.$$

Simple manipulations supply one more partial differential equation as

$$\boxed{\partial_{ct}\left\{\varepsilon\left(z,t\right)V_n^e\left(z,t\right)\right\} + \partial_z I_n^e\left(z,t\right) = -\sqrt{\mu_0}\tfrac{1}{S}\int_S \mathbf{J} \cdot \nabla_\perp \Phi_n^* ds,}$$

where $\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t$.

Projecting on $[\mathbf{z} \times \nabla_\perp \Omega_k]$. Projecting should be carried out on the same scheme; in the course of calculations the following conditions of orthonormality are needed:

$$\frac{1}{S}\int_S [\mathbf{z} \times \nabla_\perp \Omega_{k'}] \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds = \delta_{kk'};$$

$$\tfrac{1}{S}\int_S \nabla_\perp \Psi_{m'} \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds = 0; \qquad \tfrac{1}{S}\int_S [\mathbf{z} \times \nabla_\perp \Phi_{n'}] \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds = 0.$$

Final results of projecting of the vectors from Eq. (5.12) on basis element $[\mathbf{z} \times \nabla_\perp \Omega_k]$ are

$$\frac{1}{S}\int_S \nabla_\perp H_z \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds = 0; \tag{5.25}$$

$$\frac{1}{S}\int_S [\mathbf{z} \times \mathbf{E}] \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds = \sqrt[-2]{\epsilon_0} \sum_{k'=1}^{N} V_{k'} \delta_{kk'} = \sqrt[-2]{\epsilon_0} V_k\left(z,t\right); \tag{5.26}$$

$$\frac{1}{S}\int_S \mathbf{H} \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds = \sqrt[-2]{\mu_0} \sum_{k'=1}^{N} I_{k'} \delta_{kk'} = \sqrt[-2]{\mu_0} I_k\left(z,t\right). \tag{5.27}$$

Direct substitution of Eqs. (5.25), (5.26) and (5.27) in Eq. (5.12) yields

$$0 = \sqrt{\epsilon_0}\partial_t\left\{\varepsilon\left(z,t\right)V_k\left(z,t\right)\right\} + \sqrt[-2]{\mu_0}\partial_z I_k\left(z,t\right) + \frac{1}{S}\int_S [\mathbf{z} \times \mathbf{J}] \cdot [\mathbf{z} \times \nabla_\perp \Omega_k^*]\, ds.$$

After simple manipulations, this result can be recast in the following partial differential equation:

$$\boxed{\partial_{ct}\left\{\varepsilon\left(z,t\right)V_k\left(z,t\right)\right\}+\partial_z I_k\left(z,t\right)=-\sqrt{\mu_0}\tfrac{1}{S}\int_S \mathbf{J}\cdot\nabla_\perp\Omega_k^* ds,}$$

where $\partial_{ct}=\sqrt{\epsilon_0\mu_0}\partial_t$.

5.2 Projecting of the First Order Differential Equations for E_z

5.2.1 Projecting of Scalar Differential Equations

Remind that the pair of scalar differential equations in system (2.37) looks as follows

$$\begin{aligned} \epsilon_0\partial_t\left(\varepsilon E_z\right) &= \nabla_\perp\cdot\left[\mathbf{H}\times\mathbf{z}\right]-J_z, \quad (b)\\ \epsilon_0\partial_z\left(\varepsilon E_z\right) &= -\epsilon_0\varepsilon\nabla_\perp\cdot\mathbf{E}+\varrho. \quad (c)\end{aligned} \tag{5.28}$$

Function $E_z\left(\mathbf{r},z,t\right)$ sought is given in Eq. (4.17) in the form of eigenfunction series as

$$E_z\left(\mathbf{r},z,t\right)=\sqrt[-2]{\epsilon_0}\sum_{n'=1}^{\infty}e_{n'}\left(z,t\right)\varkappa_{n'}^2\Phi_{n'}\left(\mathbf{r}\right). \tag{5.29}$$

Calculations of $\nabla_\perp\cdot\mathbf{E}$ and $\nabla_\perp\cdot\left[\mathbf{H}\times\mathbf{z}\right]$ can be obtained in explicit form with making use of identities

$$\nabla_\perp\cdot\left[\nabla_\perp\Psi_m\times\mathbf{z}\right]=0;\quad \nabla_\perp\cdot\nabla_\perp\Omega_k\equiv\Delta_\perp\Omega_k=0$$

and Helmholtz equation for function Φ_n in the form of a direct formula as

$$\nabla_\perp\cdot\nabla_\perp\Phi_n\equiv\Delta_\perp\Phi_n=-\varkappa_n^2\Phi_n.$$

For example,

$$\begin{aligned}\nabla_\perp\cdot\mathbf{E} &= \sqrt[-2]{\epsilon_0}\left\{\begin{array}{c}\sum_{m=1}^{\infty}V_m^h\nabla_\perp\cdot\left[\nabla_\perp\Psi_m\times\mathbf{z}\right]\\ +\sum_{n=1}^{\infty}V_n^e\nabla_\perp\cdot\nabla_\perp\Phi_n+\sum_{k=1}^{N}V_k\nabla_\perp\cdot\nabla_\perp\Omega_k\end{array}\right\}\\ &= \sqrt[-2]{\epsilon_0}\sum_{n=1}^{\infty}V_n^e\Delta_\perp\Phi_n=-\sqrt[-2]{\epsilon_0}\sum_{n'=1}^{\infty}V_{n'}^e\varkappa_{n'}^2\Phi_{n'},\end{aligned} \tag{5.30}$$

and in a similar way,

$$\nabla_\perp\cdot\left[\mathbf{H}\times\mathbf{z}\right]=\sqrt[-2]{\mu_0}\sum_{n=1}^{\infty}I_n^e\Delta_\perp\Phi_n=-\sqrt[-2]{\mu_0}\sum_{n'=1}^{\infty}I_{n'}^e\varkappa_{n'}^2\Phi_{n'}. \tag{5.31}$$

All the functions given in Eqs. (5.29), (5.30) and (5.31) are the eigenfunctions series in terms of complete set of functions $\left\{\Phi_n\left(\mathbf{r}\right)\right\}$. Formally, procedure of projecting of Eqs. (5.28) on basis elements $\Phi_n\left(\mathbf{r}\right)$ looks as follows

$$\begin{aligned}\epsilon_0\partial_t\left(\varepsilon\tfrac{1}{S}\int_S E_z\Phi_n^* ds\right) &= \tfrac{1}{S}\int_S\nabla_\perp\cdot\left[\mathbf{H}\times\mathbf{z}\right]\Phi_n^* ds-\tfrac{1}{S}\int_S J_z\Phi_n^* ds, \quad (b)\\ \epsilon_0\partial_z\left(\varepsilon\tfrac{1}{S}\int_S E_z\Phi_n^* ds\right) &= -\epsilon_0\varepsilon\tfrac{1}{S}\int_S\nabla_\perp\cdot\mathbf{E}\Phi_n^* ds+\tfrac{1}{S}\int_S\varrho\Phi_n^* ds. \quad (c)\end{aligned} \tag{5.32}$$

We should substitute Eqs. (5.29) , (5.30) and (5.31) in Eq. (5.32) and make use of the orthonormality condition

$$\frac{\varkappa_n^2}{S}\int_S \Phi_{n'}\Phi_n^* ds = \delta_{nn'}$$

in the course of calculations. An outcome is a pair of partial differential equations with respect to the spectral coefficients

$$\boxed{\begin{aligned}\partial_{ct}\left\{\varepsilon\left(z,t\right)e_n\left(z,t\right)\right\} &= -I_n^e\left(z,t\right) - \sqrt{\mu_0}\tfrac{1}{S}\textstyle\int_S J_z\Phi_n^* ds;\\ \partial_z\left\{\varepsilon\left(z,t\right)e_n\left(z,t\right)\right\} &= \varepsilon\left(z,t\right)V_n^e\left(z,t\right) + \sqrt[-2]{\epsilon_0}\tfrac{1}{S}\textstyle\int_S \varrho\Phi_n^* ds,\end{aligned}}$$

where $\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t$.

5.2.2 Projecting of Vector Differential Equation

Now, we should project on basis elements vector equation from the system (2.37) which is placed in line (a) there. First we scalar multiply this equation by z_0 from the right what yields

$$\nabla_\perp E_z = \mu_0\partial_t\left\{\mu\left[\mathbf{H}\times\mathbf{z}\right]\right\} + \partial_z\mathbf{E} + \left[\mathbf{I}\times\mathbf{z}\right]. \tag{5.33}$$

Spectral decompositions of the vectors, which are involved in this equation, look as follows

$$\nabla_\perp E_z\left(\mathbf{r},z,t\right) = \sqrt[-2]{\epsilon_0}\sum_{n'=1}^{\infty} e_{n'}\left(z,t\right)\varkappa_{n'}^2\nabla_\perp\Phi_{n'}\left(\mathbf{r}\right); \tag{5.34}$$

$$\mathbf{H}\times\mathbf{z} = \sqrt[-2]{\mu_0}\left\{\sum_{m'=1}^{\infty} I_{m'}^h\left[\nabla_\perp\Psi_{m'}\times\mathbf{z}\right] + \sum_{n'=1}^{\infty} I_{n'}^e\nabla_\perp\Phi_{n'} + \sum_{k'=1}^{N} I_{k'}\nabla_\perp\Omega_{k'}\right\}; \tag{5.35}$$

$$\mathbf{E} = \sqrt[-2]{\epsilon_0}\left\{\sum_{m'=1}^{\infty} V_{m'}^h\left[\nabla_\perp\Psi_{m'}\times\mathbf{z}\right] + \sum_{n'=1}^{\infty} V_{n'}^e\nabla_\perp\Phi_{n'} + \sum_{k'=1}^{N} V_{k'}\nabla_\perp\Omega_{k'}\right\} \tag{5.36}$$

(see original formulas (4.17) , and (4.14)). Inasmuch as these vectors are decomposed in terms of basis elements $\left[\nabla_\perp\Psi_m\times\mathbf{z}\right]$, $\nabla_\perp\Phi_n$ and $\nabla_\perp\Omega_k$, we should project Eq. (5.33) on all these vectors.

Projecting on $\nabla_\perp\Phi_n$. Technically, this procedure looks as follows

$$\begin{aligned}\frac{1}{S}\int_S \nabla_\perp E_z\cdot\nabla_\perp\Phi_n^* ds &= \mu_0\partial_t\left\{\mu\frac{1}{S}\int_S\left[\mathbf{H}\times\mathbf{z}\right]\cdot\nabla_\perp\Phi_n^* ds\right\} \\ &+\partial_z\left\{\frac{1}{S}\int_S\mathbf{E}\cdot\nabla_\perp\Phi_n^* ds\right\} + \frac{1}{S}\int_S\left[\mathbf{I}\times\mathbf{z}\right]\cdot\nabla_\perp\Phi_n^* ds.\end{aligned} \tag{5.37}$$

All integrals can be calculated explicitly with making use of orthonormality conditions

$$\frac{1}{S}\int_S \nabla_\perp \Phi_{n'} \cdot \nabla_\perp \Phi_n^* ds = \delta_{nn'};$$

$$\tfrac{1}{S}\int_S \left[\nabla_\perp \Psi_{m'} \times \mathbf{z}\right] \cdot \nabla_\perp \Phi_n^* ds = 0; \qquad \tfrac{1}{S}\int_S \nabla_\perp \Omega_{k'} \cdot \nabla_\perp \Phi_n^* ds = 0.$$

Calculations yield the following outcomes:

$$\frac{1}{S}\int_S \nabla_\perp E_z \cdot \nabla_\perp \Phi_n^* ds = \sqrt[-2]{\epsilon_0} \sum_{n'=1}^{\infty} e_{n'}(z,t)\, \varkappa_{n'}^2 \delta_{nn'} = \sqrt[-2]{\epsilon_0}\, e_n(z,t)\, \varkappa_n^2; \tag{5.38}$$

$$\frac{1}{S}\int_S \left[\mathbf{H} \times \mathbf{z}\right] \cdot \nabla_\perp \Phi_n^* ds = \sqrt[-2]{\mu_0} \sum_{n'=1}^{\infty} I_{n'}^e(z,t)\, \delta_{nn'} = \sqrt[-2]{\mu_0}\, I_n^e(z,t); \tag{5.39}$$

$$\left\{\frac{1}{S}\int_S \mathbf{E} \cdot \nabla_\perp \Phi_n^* ds\right\} = \sqrt[-2]{\epsilon_0} \sum_{n'=1}^{\infty} V_{n'}^e(z,t)\, \delta_{nn'} = \sqrt[-2]{\epsilon_0}\, V_n^e(z,t). \tag{5.40}$$

Direct substitution of Eqs. (5.38), (5.39) and (5.40) in Eq. (5.37) yields

$$\sqrt[-2]{\epsilon_0}\, e_n(z,t)\, \varkappa_n^2 = \sqrt{\mu_0}\,\partial_t \left\{\mu(z,t)\, I_n^e(z,t)\right\} + \sqrt[-2]{\epsilon_0}\,\partial_z V_n^e(z,t) + \frac{1}{S}\int_S \left[\mathbf{I} \times \mathbf{z}\right] \cdot \nabla_\perp \Phi_n^* ds.$$

After evident algebraic manipulations, we get partial differential equations

$$\boxed{-\partial_{ct}\left\{\mu(z,t)\, I_n^e(z,t)\right\} - \partial_z V_n^e(z,t) + \varkappa_n^2 e_n(z,t) = \sqrt{\epsilon_0}\tfrac{1}{S}\int_S \left[\mathbf{I} \times \mathbf{z}\right] \cdot \nabla_\perp \Phi_n^* ds,}$$

where $\partial_{ct} = \sqrt{\epsilon_0 \mu_0}\,\partial_t$.

Projecting on $\left[\nabla_\perp \Psi_m \times \mathbf{z}\right]$. Formal this procedure looks as follows

$$\begin{aligned} \frac{1}{S}\int_S \nabla_\perp E_z \cdot \left[\nabla_\perp \Psi_m^* \times \mathbf{z}\right] ds \;\; = \;\; & \mu_0 \partial_t \left\{\mu \frac{1}{S}\int_S \mathbf{H} \cdot \nabla_\perp \Psi_m^* ds\right\} \\ & + \partial_z \left\{\frac{1}{S}\int_S \mathbf{E} \cdot \left[\nabla_\perp \Psi_m^* \times \mathbf{z}\right] ds\right\} + \frac{1}{S}\int_S \mathbf{I} \cdot \nabla_\perp \Psi_m^* ds. \end{aligned} \tag{5.41}$$

In the course of integration the following conditions of orthonormality will be useful:

$$\frac{1}{S}\int_S \left[\nabla_\perp \Psi_{m'} \times \mathbf{z}\right] \cdot \left[\nabla_\perp \Psi_m^* \times \mathbf{z}\right] ds = \frac{1}{S}\int_S \nabla_\perp \Psi_{m'} \cdot \nabla_\perp \Psi_m^* ds = \delta_{mm'};$$

$$\tfrac{1}{S}\int_S \nabla_\perp \Phi_{n'} \cdot \left[\nabla_\perp \Psi_m^* \times \mathbf{z}\right] ds = 0; \qquad \tfrac{1}{S}\int_S \nabla_\perp \Omega_{k'} \cdot \left[\nabla_\perp \Psi_m^* \times \mathbf{z}\right] ds = 0.$$

Calculations of these integrals supply the following results:

$$\frac{1}{S}\int_S \nabla_\perp E_z \cdot \left[\nabla_\perp \Psi_m^* \times \mathbf{z}\right] ds = 0; \tag{5.42}$$

$$\frac{1}{S}\int_S \mathbf{H}\cdot\nabla_\perp\Psi_m^* ds = \sqrt[-2]{\mu_0}\sum_{m'=1}^{\infty} I_{m'}^h(z,t)\,\delta_{mm'} = \sqrt[-2]{\mu_0} I_m^h(z,t)\,; \tag{5.43}$$

$$\frac{1}{S}\int_S \mathbf{E}\cdot[\nabla_\perp\Psi_m^*\times\mathbf{z}]\,ds = \sqrt[-2]{\epsilon_0}\sum_{m'=1}^{\infty} V_{m'}^h(z,t)\,\delta_{mm'} = \sqrt[-2]{\epsilon_0} V_m^h(z,t)\,. \tag{5.44}$$

Substitution of Eqs. (5.42), (5.43) and (5.44) in Eq. (5.41) yields

$$0 = \sqrt{\mu_0}\partial_t\left\{\mu I_m^h(z,t)\right\} + \sqrt[-2]{\epsilon_0}\partial_z\left\{V_m^h(z,t)\right\} + \frac{1}{S}\int_S \mathbf{I}\cdot\nabla_\perp\Psi_m^* ds.$$

Hence, one more partial differential equation holds

$$\boxed{\partial_{ct}\left\{\mu(z,t)\,I_m^h(z,t)\right\} + \partial_z V_m^h(z,t) = -\sqrt{\epsilon_0}\tfrac{1}{S}\int_S \mathbf{I}\cdot\nabla_\perp\Psi_m^* ds,}$$

where $\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t$.

Projecting on $\nabla_\perp\Omega_k$. This procedure applied to Eq. (5.33) converts it to the following form:

$$\begin{aligned}\frac{1}{S}\int_S \nabla_\perp E_z\cdot\nabla_\perp\Omega_k^* ds &= \mu_0\partial_t\left\{\mu\frac{1}{S}\int_S[\mathbf{H}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds\right\} \\ &\quad+\partial_z\frac{1}{S}\int_S \mathbf{E}\cdot\nabla_\perp\Omega_k^* ds + \frac{1}{S}\int_S[\mathbf{I}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds\end{aligned} \tag{5.45}$$

Integrals obtained can be calculated with using the orthonormality conditions as

$$\frac{1}{S}\int_S \nabla_\perp\Omega_{k'}\cdot\nabla_\perp\Omega_k^* ds = \delta_{kk'};$$

$$\tfrac{1}{S}\int_S[\nabla_\perp\Psi_{m'}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds = 0; \qquad \tfrac{1}{S}\int_S\nabla_\perp\Phi_{n'}(\mathbf{r})\cdot\nabla_\perp\Omega_k^* ds = 0.$$

Calculations similar to carried out above yield

$$\frac{1}{S}\int_S \nabla_\perp E_z\cdot\nabla_\perp\Omega_k^* ds = 0; \tag{5.46}$$

$$\frac{1}{S}\int_S[\mathbf{H}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds = \sqrt[-2]{\mu_0}\sum_{k'=1}^{N} I_{k'}(z,t)\,\delta_{kk'} = \sqrt[-2]{\mu_0} I_k(z,t)\,; \tag{5.47}$$

$$\frac{1}{S}\int_S \mathbf{E}\cdot\nabla_\perp\Omega_k^* ds = \sqrt[-2]{\epsilon_0}\sum_{k'=1}^{N} V_{k'}(z,t)\,\delta_{kk'} = \sqrt[-2]{\epsilon_0} V_k(z,t)\,. \tag{5.48}$$

Substitution of Eqs. (5.46), (5.47) and (5.48) in Eq. (5.45) supplies ultimate equation

$$0 = \sqrt{\mu_0}\partial_t\left\{\mu(z,t)\,I_k(z,t)\right\} + \sqrt[-2]{\epsilon_0}\partial_z V_k(z,t) + \frac{1}{S}\int_S[\mathbf{I}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds,$$

which can be easily recast to the form

$$\boxed{\partial_{ct}\left\{\mu(z,t)\,I_k(z,t)\right\} + \partial_z V_k(z,t) = -\sqrt{\epsilon_0}\tfrac{1}{S}\int_S[\mathbf{I}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds,}$$

where $\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t$.

5.3 Set of Partial Differential Equations

Before exhibiting the set of partial differential equations obtained, remind of a position of unknown functions of (z, t) in the presentation of electromagnetic field sought. Three-component vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ each have been written as the sum of their transverse $2-$ component projections on waveguide cross-section: $\mathbf{E}$ and $\mathbf{H}$, respectively, and longitudinal components E_z and H_z, namely:

$$\vec{\mathcal{E}}\left(\mathbf{r},z,t\right) = \mathbf{E}\left(\mathbf{r},z,t\right) + \mathbf{z}E_z\left(\mathbf{r},z,t\right); \quad \vec{\mathcal{H}}\left(\mathbf{r},z,t\right) = \mathbf{H}\left(\mathbf{r},z,t\right) + \mathbf{z}H_z\left(\mathbf{r},z,t\right).$$

Basis has been specified as the eigenfunction set of some self-adjoint part of Maxwell's operator, which acts on transverse coordinates $\mathbf{r}$ only. Hence, the components of vectors sought were presented as Fourier series in terms of the eigensolutions as

$$\begin{aligned} E_z &= \sqrt[-2]{\epsilon_0}\sum_{n=1}^{\infty} e_n\left(z,t\right)\varkappa_n^2\Phi_n\left(\mathbf{r}\right), \\ \mathbf{E} &= \sqrt[-2]{\epsilon_0}\left\{\sum_{m=1}^{\infty} V_m^h\left[\nabla_\perp\Psi_m\times\mathbf{z}\right] + \sum_{n=1}^{\infty} V_n^e\nabla_\perp\Phi_n + \sum_{k=1}^{N} V_k\nabla_\perp\Omega_k\right\}; \end{aligned} \tag{5.49}$$

$$\begin{aligned} H_z &= \sqrt[-2]{\mu_0}\left\{h_0\left(z,t\right) + \sum_{m=1}^{\infty} h_m\left(z,t\right)\nu_m^2\Psi_m\left(\mathbf{r}\right)\right\}, \\ \mathbf{H} &= \sqrt[-2]{\mu_0}\left\{\sum_{m=1}^{\infty} I_m^h\nabla_\perp\Psi_m + \sum_{n=1}^{\infty} I_n^e\left[\mathbf{z}\times\nabla_\perp\Phi_n\right] + \sum_{k=1}^{N} I_k\left[\mathbf{z}\times\nabla_\perp\Omega_k\right]\right\}, \end{aligned} \tag{5.50}$$

where scalar potentials $\Psi_m\left(\mathbf{r}\right), \Phi_n\left(\mathbf{r}\right), \Omega_k\left(\mathbf{r}\right)$ are known as eigensolutions of Dirichlet's and Neumann boundary value problems for Laplacian. Multipliers $e\ h\ I,\ V$ in front are unknown functions of z and t variables. Partial differential equations for the multipliers $e\ h\ I,\ V$ are just the ones obtained via projecting of Maxwell's equations on the same basis.

5.3.1 Partial Differential Equations for the General Case

All the partial differential equations obtained above for the general case **are boxed.** Now, we select independent equations and collect up them. To make this system of simultaneous equations visible, we separate it onto three subsets as follows

$$\begin{aligned} -\partial_{ct}\left(\mu I_n^e\right) - \partial_z V_n^e + \varkappa_n^2 e_n &= \sqrt{\epsilon_0}\tfrac{1}{S}\textstyle\int_S\left[\mathbf{I}\times\mathbf{z}\right]\cdot\nabla_\perp\Phi_n^* ds; && (a) \\ \partial_z\left(\varepsilon e_n\right) &= \varepsilon V_n^e + \sqrt[-2]{\epsilon_0}\tfrac{1}{S}\textstyle\int_S\varrho\Phi_n^* ds, && (b) \\ \partial_{ct}\left(\varepsilon e_n\right) &= -I_n^e - \sqrt{\mu_0}\tfrac{1}{S}\textstyle\int_S J_z\Phi_n^* ds; && (c) \end{aligned} \tag{5.51}$$

and

$$
\begin{aligned}
-\partial_{ct}\left(\varepsilon V_m^h\right) - \partial_z I_m^h + \nu_m^2 h_m &= \sqrt{\mu_0}\tfrac{1}{S}\textstyle\int_S [\mathbf{z}\times\mathbf{J}]\cdot\nabla_\perp\Psi_m^* ds, && (a)\\
\partial_z\left(\mu h_m\right) &= \mu I_m^h + \sqrt[-2]{\mu_0}\tfrac{1}{S}\textstyle\int_S g\Psi_m^* ds, && (b)\\
\partial_{ct}\left(\mu h_m\right) &= -V_m^h - \sqrt{\epsilon_0}\tfrac{1}{S}\textstyle\int_S I_z\Psi_m^* ds, && (c)\\
\partial_z\left(\mu h_0\right) &= \sqrt[-2]{\mu_0}\tfrac{1}{S}\textstyle\int_S g ds, && (d)\\
\partial_{ct}\left(\mu h_0\right) &= -\sqrt{\epsilon_0}\tfrac{1}{S}\textstyle\int_S I_z ds; && (e)
\end{aligned}
\tag{5.52}
$$

and as well

$$
\begin{aligned}
\partial_{ct}\left(\mu I_k\right) + \partial_z V_k &= -\sqrt{\epsilon_0}\tfrac{1}{S}\textstyle\int_S [\mathbf{I}\times\mathbf{z}]\cdot\nabla_\perp\Omega_k^* ds, && (a)\\
\partial_{ct}\left(\varepsilon V_k\right) + \partial_z I_k &= -\sqrt{\mu_0}\tfrac{1}{S}\textstyle\int_S \mathbf{J}\cdot\nabla_\perp\Omega_k^* ds. && (b)
\end{aligned}
\tag{5.53}
$$

Remind of notation used: $\partial_{ct}=\sqrt{\epsilon_0\mu_0}\partial_t$, and $\varepsilon \equiv \varepsilon(z,t) > 0$, $\mu \equiv \mu(z,t) > 0$ may be piecewise differentiable functions of their arguments both, superscript $(^*)$ means complex conjugation, as usually.

Left-hand-sides at the group of equations (5.51) involve only coefficients which correspond to $TE-$ waveguide modes. Left-hand-sides of equations at the group (5.52) include only coefficients corresponding to $TH-$ modes. And left-hand-sides of equations (5.53) involve the coefficients corresponding to $TEM-$ modes. Integrands at the right-hand-sides of Eqs. (5.51) - (5.53) involve constitutive relations of the general kind as $\vec{\mathcal{P}}\left(\vec{\mathcal{E}}\right)$, $\vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right)$ and $\vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right)$ which are included in notation of the current and charge quantities. Therefore, Eqs. (5.51) - (5.53) should be considered as a system of simultaneous equations. In some particular cases, groups of Eqs. (5.51), (5.52), (5.53) become independent from each other. This situation will be discussed later on in detail.

Remind of notation of the current and charge quantities which were introduced at the initial stage of formulation of our problem. Let us first discuss current quantities, i.e.,

$$J_z = \mathbf{z}\cdot\vec{\mathcal{J}},\quad \mathbf{J} = \vec{\mathcal{J}} - \mathbf{z}J_z;\qquad I_z = \mathbf{z}\cdot\vec{\mathcal{I}},\quad \mathbf{I} = \vec{\mathcal{I}} - \mathbf{z}I_z, \tag{5.54}$$

where $\vec{\mathcal{J}}$ and $\vec{\mathcal{I}}$ are densities of complete electric and magnetic currents, respectively. They have the following constituents:

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}_e + \vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right) + \partial_t\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right);\qquad \vec{\mathcal{I}} = \vec{\mathcal{J}}_h + \partial_t\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right). \tag{5.55}$$

Here, $\vec{\mathcal{J}}_e \equiv \vec{\mathcal{J}}_e(\mathbf{r},z,t)$, $\vec{\mathcal{J}}_h = \vec{\mathcal{J}}_h(\mathbf{r},z,t)$ are given functions of impressed forces; $\vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right)$ density of free carriers current which is induced by electromagnetic field in a medium inside a waveguide under consideration. In the general case, it is a nonlinear function of its arguments both. Classical Ohm's law

$$\vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right)=\sigma\vec{\mathcal{E}} \tag{5.56}$$

is a linearized version of this dependence. Further, $\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)$ and $\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)$ are the reminders of complete vectors of polarization $\vec{\mathcal{P}}\left(\vec{\mathcal{E}}\right)$ and magnetization $\vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right)$ which are introduced as follows

$$\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)=\vec{\mathcal{P}}\left(\vec{\mathcal{E}}\right)-\epsilon_0\alpha\left(z,t\right)\vec{\mathcal{E}},\qquad \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)=\vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right)-\chi\left(z,t\right)\vec{\mathcal{H}}. \tag{5.57}$$

Functions $\alpha\left(z,t\right)$ and $\chi\left(z,t\right)$ specify permittivity ε and permeability μ as

$$\varepsilon\equiv\varepsilon\left(z,t\right)=1+\alpha\left(z,t\right)\qquad \mu\equiv\mu\left(z,t\right)=1+\chi\left(z,t\right), \tag{5.58}$$

which are involved in Eqs. (5.51) - (5.53) . Constitutive relations of classical macroscopic electromagnetic theory

$$\vec{\mathcal{D}}=\epsilon_0\varepsilon\left(z,t\right)\vec{\mathcal{E}},\qquad \vec{\mathcal{B}}=\mu_0\mu\left(z,t\right)\vec{\mathcal{H}},\qquad \vec{\mathcal{J}}_\sigma=\sigma\vec{\mathcal{E}} \tag{5.59}$$

corresponds to a particular case

$$\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)=\mathbf{0},\qquad \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)=\mathbf{0}. \tag{5.60}$$

In Eqs. (5.51) , (5.52) , densities of complete electric and magnetic charges, ϱ and g, respectively, are present as well. They have been determined as follows

$$\varrho=\rho_e+\rho_\sigma-div\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right),\qquad g=\rho_h-div\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right), \tag{5.61}$$

where ρ_e and ρ_h are the charge densities produced by the functions of impressed forces. When respective current densities $\vec{\mathcal{J}}_e=\mathbf{J}_e+\mathbf{z}J_z^e$ and $\vec{\mathcal{J}}_h=\mathbf{J}_h+\mathbf{z}J_z^h$ are given as some functions of position and time, then ρ_e and ρ_h can be found using the equation of continuity as a differential equation with given right-hand-sides, i.e.,

$$\partial_t\rho_e=-\partial_zJ_z^e-\nabla_\perp\cdot\mathbf{J}_e,\qquad \partial_t\rho_h=-\partial_zJ_z^h-\nabla_\perp\cdot\mathbf{J}_h \tag{5.62}$$

When equations of continuity (5.62) hold, then complete charge and current densities satisfy the same equation automatically, i.e.,

$$\partial_t\varrho+\partial_zJ_z+\nabla_\perp\cdot\mathbf{J}=0,\qquad \partial_tg+\partial_zI_z+\nabla_\perp\cdot\mathbf{I}=0. \tag{5.63}$$

Besides the partial differential equations included in the system of Eqs. (5.51) - (5.53) , we have obtained one more pair of equations, namely*:

$$\partial_{ct}\left(\varepsilon V_n^e\right)+\partial_zI_n^e=-\sqrt{\mu_0}\frac{1}{S}\int_S\mathbf{J}\cdot\nabla_\perp\Phi_n^*ds \tag{5.64}$$

*See boxed equations in subsections *"Projecting of Vector Differential Equations"* both.

and

$$\partial_{ct}\left(\mu I_m^h\right)+\partial_z V_m^h=-\sqrt{\epsilon_0}\frac{1}{S}\int_S \mathbf{I}\cdot\nabla_\perp\Psi_m^* ds. \tag{5.65}$$

However, they are not independent equations. In particular, Eq. (5.64) is a corollary of Eqs. (5.51) placed in lines (b) and (c). Indeed, let us apply operation ∂_{ct} at the equation in line (b) and operation ∂_z at the equation in line (c). Substraction of the results obtained yields

$$0=\partial_{ct}\left(\varepsilon V_n^e\right)+\partial_z I_n^e+\sqrt{\mu_0}\frac{1}{S}\int_S\left(\partial_t\varrho+\partial_z J_z\right)\Phi_n^* ds. \tag{5.66}$$

However, $\partial_t\varrho+\partial_z J_z=-\nabla_\perp\cdot\mathbf{J}$ accordingly to Eq. (5.63), therefore integral from Eq. (5.66) may be rewritten as

$$\int_S\left(\partial_t\varrho+\partial_z J_z\right)\Phi_n^* ds=-\int_S\Phi_n^*\left(\nabla_\perp\cdot\mathbf{J}\right)ds \tag{5.67}$$

By virtue of known identity

$$\nabla_\perp\cdot\left(\Phi_n^*\mathbf{J}\right)=\mathbf{J}\cdot\nabla_\perp\Phi_n^*+\Phi_n^*\left(\nabla_\perp\cdot\mathbf{J}\right),$$

one can get

$$-\int_S\Phi_n^*\left(\nabla_\perp\cdot\mathbf{J}\right)ds=\int_S\mathbf{J}\cdot\nabla_\perp\Phi_n^* ds-\int_S\nabla_\perp\cdot\left(\Phi_n^*\mathbf{J}\right)ds$$

The second integral at the right-hand-side can be recast by applying of Gauss theorem and cancelled as follows

$$\int_S\nabla_\perp\cdot\left(\Phi_n^*\mathbf{J}\right)ds=\int_L\Phi_n^*\left(\mathbf{n}\cdot\mathbf{J}\right)dl=0,$$

since $\Phi_n|_L=\Phi_n^*|_L=0$. Hence,

$$\int_S\left(\partial_t\varrho+\partial_z J_z\right)\Phi_n^* ds=-\int_S\Phi_n^*\left(\nabla_\perp\cdot\mathbf{J}\right)ds=\int_S\mathbf{J}\cdot\nabla_\perp\Phi_n^* ds. \tag{5.68}$$

Substitution of Eq. (5.68) in Eq. (5.66) yields Eq. (5.64). Further, Eq. (5.65) is a corollary of Eqs. (5.52) placed in lines (b) and (c), and boundary condition

$$\left(\mathbf{n}\cdot\mathbf{I}\right)|_L=0$$

for the transverse component of magnetic current which is orthogonal to perfect conducting waveguide surface by definition.

5.3.2 Initial Conditions and External Sources for Waveguide Waves

Statement of Cauchy problem for system of Maxwell's equations provides for electromagnetic field at initial moment of time should be given. Inasmuch as the field is sought for in the form of natural waveguide mode decompositions like (5.49) and (5.50), convenient form for the initial condition is the same series. For example, let $t = 0$ is the initial instant, and electromagnetic field is specified as

$$\vec{\mathcal{E}}\left(\mathbf{r}, z, t\right)|_{t=0} = \mathbf{E}\left(\mathbf{r}, z, 0\right) + \mathbf{z}E_z\left(\mathbf{r}, z, 0\right), \quad \vec{\mathcal{H}}\left(\mathbf{r}, z, t\right)|_{t=0} = \mathbf{H}\left(\mathbf{r}, z, 0\right) + \mathbf{z}H_z\left(\mathbf{r}, z, 0\right),$$

where the functions of $\mathbf{r}$ and t at the right-hand-sides should be known. Within the frame of our approach, they should be specified as

$$\begin{aligned} E_z\left(\mathbf{r}, z, 0\right) &= \sqrt[-2]{\epsilon_0} \sum_{n=1}^{\infty} e_n\left(z, 0\right) \varkappa_n^2 \Phi_n\left(\mathbf{r}\right), \\ \mathbf{E}\left(\mathbf{r}, z, 0\right) &= \sqrt[-2]{\epsilon_0} \left\{ \begin{array}{c} \sum_{m=1}^{\infty} V_m^h\left(z, 0\right)\left[\nabla_{\perp}\Psi_m \times \mathbf{z}\right] \\ +\sum_{n=1}^{\infty} V_n^e\left(z, 0\right)\nabla_{\perp}\Phi_n + \sum_{k=1}^{N} V_k\left(z, 0\right)\nabla_{\perp}\Omega_k \end{array} \right\}; \end{aligned} \tag{5.69}$$

$$\begin{aligned} H_z\left(\mathbf{r}, z, 0\right) &= \sqrt[-2]{\mu_0} \left\{ h_0\left(z, 0\right) + \sum_{m=1}^{\infty} h_m\left(z, 0\right) \nu_m^2 \Psi_m\left(\mathbf{r}\right) \right\}, \\ \mathbf{H}\left(\mathbf{r}, z, 0\right) &= \sqrt[-2]{\mu_0} \left\{ \begin{array}{c} \sum_{m=1}^{\infty} I_m^h\left(z, 0\right)\nabla_{\perp}\Psi_m + \sum_{n=1}^{\infty} I_n^e\left(z, 0\right)\left[\mathbf{z} \times \nabla_{\perp}\Phi_n\right] \\ +\sum_{k=1}^{N} I_k\left(z, 0\right)\left[\mathbf{z} \times \nabla_{\perp}\Omega_k\right] \end{array} \right\}, \end{aligned} \tag{5.70}$$

where the scalar multipliers depending on $(z, 0)$ can be calculated via projecting of known functions from the left-hand-sides on the respective elements of basis.

Given functions of impressed forces have been introduced from the very beginning as $\vec{\mathcal{J}}_e\left(\mathbf{r}, z, t\right)$ and $\vec{\mathcal{J}}_h\left(\mathbf{r}, z, t\right)$. They are present at the right-hand-sides of the evolutionary equations exhibited in previous section. However, there is one more form of introducing of impressed forces. They may be given in the form of some boundary conditions at $Oz-$axis in the course of solution of the evolutionary equations. We will consider this possibility below when needed.

Chapter 6
LINEAR WAVEGUIDE EVOLUTIONARY EQUATIONS

6.1 Homogeneous Equations for Nonstationary Layered Lossless Media

Summary 12 *Which conditions correspond to linear layered in $Oz-$ direction nonstationary lossless media? Which equations are called homogeneous? Which formulation of the evolutionary equations corresponds to the problem for free $TM-$ waveguide modes? How components of electromagnetic field of $TM-$ modes can be expressed via solutions of Klein-Gordon equation? Which formulation of the evolutionary equations corresponds to the problem for free $TE-$ waveguide modes? How components of electromagnetic field of $TE-$ modes can be expressed via solutions of Klein-Gordon equation? How to introduce the functions of impressed forces in homogeneous Klein-Gordon equation and wave equation? How to obtain general solution of classical homogeneous wave equation? What is it: formula of d'Alembert for classical wave equation? How to generalize of d'Alembert formula for classical Klein-Gordon equation? Which combinations of variables z and t allow to separate them in classical Klein-Gordon equation? How to solve a problem for homogeneous Klein-Gordon equation with inhomogeneous boundary conditions applied on $z-$ variable using separation of variables? What does it mean "modal basis" and "evolutionary basis"?*

Let us first consider the simplest particular case when

$$\vec{\mathcal{J}}_e = \mathbf{0}, \quad \vec{\mathcal{J}}_h = \mathbf{0}; \quad \vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right) = \mathbf{0}, \quad \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right) = \mathbf{0}; \quad \vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}}, \vec{\mathcal{H}}\right) = \mathbf{0}. \tag{6.1}$$

First pair of conditions, i.e., $\vec{\mathcal{J}}_e = \mathbf{0}$, $\vec{\mathcal{J}}_h = \mathbf{0}$ (and hence, $\rho_e = 0$, $\rho_h = 0$ accordingly to the equation of continuity) means that electric and magnetic charges and currents as the functions of impressed sources are absent. The second pair, i.e., $\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right) = \mathbf{0}$, and $\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right) = \mathbf{0}$, means that a waveguide under consideration is filled with a medium electromagnetic field in which satisfies the following constitutive relations:

$$\vec{\mathcal{D}}(\mathbf{r}, z, t) = \epsilon_0 \varepsilon(z, t)\, \vec{\mathcal{E}}(\mathbf{r}, z, t), \qquad \vec{\mathcal{B}}(\mathbf{r}, z, t) = \mu_0 \mu(z, t)\, \vec{\mathcal{H}}(\mathbf{r}, z, t). \tag{6.2}$$

So, a waveguide is filled with a layered along its axis nonstationary medium*. We have formerly supposed that $\varepsilon(z, t) \geq c_1 > 0$ and $\mu(z, t) \geq c_2 > 0$ are differentiable

*A medium is called layered when its electromagnetic parameters depend on one direction. A medium is called nonstationary when its electromagnetic parameters depend on time.

functions almost everywhere. The last condition

$$\vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right)=\mathbf{0} \tag{6.3}$$

means that free charge curriers are absent within the media. In particular, it means that a medium under consideration does not have ohmic losses. This limitation has introduced here for simplicity only. It is not principle restriction for our approach, and we'll take it away elsewhere later on.

Thus, complete currents and charges within a waveguide under consideration are equal to zero, i.e.,

$$\vec{\mathcal{J}}\overset{def}{=}\vec{\mathcal{J}}_e+\vec{\mathcal{J}}_\sigma\left(\vec{\mathcal{E}},\vec{\mathcal{H}}\right)+\partial_t\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)=\mathbf{0};\qquad \vec{\mathcal{I}}\overset{def}{=}\vec{\mathcal{J}}_h+\partial_t\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)=\mathbf{0}. \tag{6.4}$$

$$\varrho\overset{def}{=}\rho_e+\rho_\sigma-div\vec{\mathcal{P}}'\left(\vec{\mathcal{E}}\right)=0,\qquad g\overset{def}{=}\rho_h-div\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right)=0. \tag{6.5}$$

Since

$$\vec{\mathcal{J}}\overset{def}{=}\mathbf{J}+\mathbf{z}J_z,\qquad \vec{\mathcal{I}}\overset{def}{=}\mathbf{I}+\mathbf{z}I_z,$$

it means that

$$\mathbf{J}=\mathbf{0},\quad J_z=0;\qquad \mathbf{I}=\mathbf{0},\quad I_z=0. \tag{6.6}$$

Under conditions (6.5) and (6.6), all the integrands cancel in Eqs. (5.51) – (5.53) what makes these differential equations homogeneous[†]. When these equations are homogeneous, then we have three independent systems of partial differential equations each of which is closed[‡]. Physically, it means that $TM-$, $TE-$, and $TEM-$ modes evolve individually in the waveguide with such a medium: i.e., they do not suffer intermodal transformations.

6.1.1 Formulation of the Problem for Free and Forced $TM-$ Modes

After substitution of Eqs. (6.5) and (6.6) in Eqs. (5.51), one can get

$$\begin{aligned} &-\partial_{ct}\left(\mu I_n^e\right)-\partial_z V_n^e+\varkappa_n^2 e_n=0, && (a)\\ &V_n^e=\varepsilon^{-1}\partial_z\left(\varepsilon e_n\right), && (b)\\ &I_n^e=-\partial_{ct}\left(\varepsilon e_n\right). && (c)\end{aligned} \tag{6.7}$$

where $n=1,2,\ldots$. Functions $e_n(z,t)$, $V_n^e(z,t)$, and $I_n^e(z,t)$ are amplitudes of the components of electromagnetic field sought, namely:

$$\begin{aligned} E_{zn}&=\sqrt[-2]{\epsilon_0}\varkappa_n^2\Phi_n(\mathbf{r})\,e_n(z,t), & \mathbf{E}_n&=\sqrt[-2]{\epsilon_0}\nabla_\perp\Phi_n(\mathbf{r})\,V_n^e(z,t),\\ H_{zn}&=0, & \mathbf{H}_n&=\sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\Phi_n(\mathbf{r})\right]I_n^e(z,t). \end{aligned} \tag{6.8}$$

[†]Differential equation is called homogeneous when a force function (placed usually at the right-hand-side of the equation) is equal to zero.

[‡]i.e., groups of equations (5.51), (5.52), and (5.53) originate each individually closed system of equations, where number of functions sought and number of equations given coincide.

Using formulas placed in lines (a) and (b), we can eliminate functions $V_n^e(z,t)$ and $I_n^e(z,t)$ from the equation placed in line (a) and get

$$\partial_{ct}\left[\mu\partial_{ct}(\varepsilon e_n)\right] - \partial_z\left[\varepsilon^{-1}\partial_z(\varepsilon e_n)\right] + \varkappa_n^2 e_n = 0. \tag{6.9}$$

Instead of function $e_n(z,t)$ sought, it has sense to introduce new unknown function as follows

$$D_n(z,t) = \varepsilon(z,t)\, e_n(z,t). \tag{6.10}$$

When $\varepsilon \equiv \varepsilon(z,t)$ and $\mu \equiv \mu(z,t)$, this function satisfies differential equation with variable coefficients, i.e.

$$\left(\varepsilon\partial_{ct}\mu\partial_{ct} - \varepsilon\partial_z\varepsilon^{-1}\partial_z + \varkappa_n^2\right) D_n(z,t) = 0. \tag{6.11}$$

This equation is more simple then original Eq. (6.9) obtained with respect to function $e_n(z,t)$. Solutions of Eq. (6.11), supplemented with the proper initial conditions, exhibit evolution of free $TM-$ waveguide modes in space and time from given their initial states to observable ones.

When $\varepsilon = const.$ and $\mu = const.$, Eq. (6.11) is well known: (i) in Radio Engineering, this is one of the forms of Telegrafist's Equation; (ii) in Quantum Mechanics, it named as Klein-Gordon equation; (iii) in Theory of Oscillations, it describes movements of a string placed in an elasitc medium; (iv) in Electrical Engineering, it describes mode propagation in hollow free-space waveguides. In our case, a medium inside the waveguides may have varying parameters, i.e., $\varepsilon \equiv \varepsilon(z,t)$ and $\mu \equiv \mu(z,t)$. Hence, Eq. (6.11) may be considered as some generalization of this classical equation.

Despite Eq. (6.11) is homogeneous, we can nevertheless introduce given function of impressed forces. It can be made via supplementation it with some boundary conditions on $z-$ coordinate as follows, for example,

$$\boxed{\begin{gathered}\left(\varepsilon\partial_{ct}\mu\partial_{ct} - \varepsilon\partial_z\varepsilon^{-1}\partial_z + \varkappa_n^2\right) D_n(z,t) = 0,\\ D_n(z,t)\,|_{z=0} = \varphi(t)\,, \partial_z D_n(z,t)\,|_{z=0} = \psi(t)\,,\end{gathered}} \tag{6.12}$$

where $\varphi(t)$ and $\psi(t)$ should be given functions. In the sense of waveguide theory presented, it means that amplitudes of longitudinal and transversal components of $\vec{\mathcal{E}}(\mathbf{r},z,t)$ at the origin $z=0$ are given as

$$e_n(z,t)\,|_{z=0} = \varphi(t)/\varepsilon(0,t)\,, \qquad V_n^e(z,t)\,|_{z=0} = \psi(t)/\varepsilon(0,t)\,,$$

see Eqs. (6.8). In closing, we present all the field components of $TM-$ waveguide modes expressed in terms of solution to problem (6.12) which look as follows

$$\boxed{\begin{aligned} E_{zn} &= \sqrt[-2]{\epsilon_0}\varkappa_n^2\Phi_n(\mathbf{r})\,\varepsilon^{-1}D_n(z,t)\,, & \mathbf{E}_n &= \sqrt[-2]{\epsilon_0}\nabla_\perp\Phi_n(\mathbf{r})\,\varepsilon^{-1}\partial_z D_n(z,t)\,;\\ H_{zn} &= 0, & \mathbf{H}_n &= \sqrt[-2]{\mu_0}\left[\nabla_\perp\Phi_n(\mathbf{r})\times\mathbf{z}\right]\partial_{ct}D_n(z,t)\,,\end{aligned}} \tag{6.13}$$

where $n = 1, 2, \dots$, and Eqs. (6.8) are used. When $\varphi = \psi = 0$ at the boundary conditions in problem (6.12) , then its solution exhibits in Eqs. (6.13) electromagnetic field of free $TM-$ modes propagating along the waveguide. Otherwise, solution obtained has physical sense of $TM-$ mode excited by a source placed in waveguide cross-section $z = 0$.

6.1.2 Formulation of the Problem for Free and Forced $TE-$ Modes

Substitution of Eqs. (6.5) and (6.6) in Eqs. (5.52) , yields

$$
\begin{aligned}
-\partial_{ct}\left(\varepsilon V_m^h\right) - \partial_z I_m^h + \nu_m^2 h_m = 0, &\quad (a)\\
I_m^h = \mu^{-1}\partial_z\left(\mu h_m\right), &\quad (b)\\
V_m^h = -\partial_{ct}\left(\mu h_m\right), &\quad (c)\\
\partial_z\left(\mu h_0\right) = 0, &\quad (d)\\
\partial_{ct}\left(\mu h_0\right) = 0, &\quad (e)
\end{aligned}
\tag{6.14}
$$

where $\varepsilon \equiv \varepsilon(z,t) \geq c_1 > 0$ and $\mu \equiv \mu(z,t) \geq c_2 > 0$ are functions differentiable almost everywhere. First we solve equations placed in lines (d) and (e) ; analysis of the others can be made analogously to previous case.

Equation (e) suggests that function μh_0 may be depended on variable z only But equation (d) convince us of the function is a constant. If this waveguide mode is equal to zero when $t = -\infty$, for example, it means that

$$h_0(z,t) \equiv 0 \qquad while \quad -\infty \leq z, t \leq \infty.$$

Equations placed in lines $(a) - (c)$ at Eq. (6.14) involve functions $h_m(z,t)$, $V_m^h(z,t)$, and $I_m^h(z,t)$ which are amplitudes of appropriate component of the field of $TE-$ mode, namely:

$$
\begin{aligned}
&E_{zm} = 0, && \mathbf{E}_m = \sqrt[-2]{\epsilon_0}\left[\nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z}\right] V_m^h(z,t),\\
&H_{zm} = \sqrt[-2]{\mu_0}\,\nu_m^2 \Psi_m(\mathbf{r})\, h_m(z,t), && \mathbf{H}_m = \sqrt[-2]{\mu_0}\,\nabla_\perp \Psi_m(\mathbf{r})\, I_m^h(z,t).
\end{aligned}
\tag{6.15}
$$

Substitution of formulas placed in lines (b) and (c) in equation placed in line (a) eliminates functions $V_m^h(z,t)$ and $I_m^h(z,t)$ there. Instead of function $h_m(z,t)$ sought, it is convenient to introduce new function

$$B_m(z,t) = \mu(z,t)\, h_m(z,t).$$

After simple manipulations, we obtain one more form of generalized Klein-Gordon equation as

$$\left(\mu\partial_{ct}\varepsilon\partial_{ct} - \mu\partial_z\mu^{-1}\partial_z + \nu_m^2\right) B_m(z,t) = 0. \tag{6.16}$$

This equation is similar to Eq. (6.11) exhibited above. We may supplement it with the boundary conditions at $z = 0$ in a similar way as it was made in the case of Eq. (6.12) what yields

$$\boxed{\begin{gathered}\left(\mu\partial_{ct}\varepsilon\partial_{ct} - \mu\partial_z\mu^{-1}\partial_z + \nu_m^2\right) B_m\left(z,t\right) = 0, \\ B_m\left(z,t\right)|_{z=0} = \varphi\left(t\right), \partial_z B_m\left(z,t\right)|_{z=0} = \psi\left(t\right).\end{gathered}} \tag{6.17}$$

Using Eqs. (6.15), one can present all the components of $TE-$ waveguide modes in terms of function $B_m\left(z,t\right)$ as follows

$$\boxed{\begin{array}{ll} E_{zm} = 0, & \mathbf{E}_m = \sqrt[-2]{\epsilon_0}\left[\mathbf{z}\times\nabla_\perp\Psi_m\left(\mathbf{r}\right)\right]\partial_{ct}B_m\left(z,t\right), \\ H_{zm} = \sqrt[-2]{\mu_0}\nu_m^2\Psi_m\left(\mathbf{r}\right)\mu^{-1}B_m\left(z,t\right), & \mathbf{H}_m = \sqrt[-2]{\mu_0}\nabla_\perp\Psi_m\left(\mathbf{r}\right)\mu^{-1}\partial_z B_m\left(z,t\right), \end{array}} \tag{6.18}$$

where $m = 1, 2, \ldots$. In terms of notation exhibited in Eqs. (6.15) for amplitudes of longitudinal and transversal components of $\vec{\mathcal{H}}\left(\mathbf{r},z,t\right)$, the boundary conditions at Eq. (6.17) look as follows

$$h_m\left(z,t\right)|_{z=0} = \varphi\left(t\right)/\mu\left(0,t\right), \qquad I_m^h\left(z,t\right)|_{z=0} = \psi\left(t\right)/\mu\left(0,t\right).$$

6.1.3 Formulation of the Problem for Free and Forced $TEM-$ Modes

Under conditions given in Eqs. (6.5) and (6.6), pair of Eqs. (5.53) becomes as

$$\begin{gathered}\partial_{ct}\left(\mu I_k\right) + \partial_z V_k = 0, \quad (a) \\ \partial_{ct}\left(\varepsilon V_k\right) + \partial_z I_k = 0. \quad (b)\end{gathered} \tag{6.19}$$

Functions $V_k\left(z,t\right)$ and $I_k\left(z,t\right)$ are the amplitudes of transverse field components of $TEM-$ modes, namely:

$$\begin{array}{ll} \mathbf{E}_k\left(\mathbf{r},z,t\right) = \sqrt[-2]{\epsilon_0}\nabla_\perp\Omega_k\left(\mathbf{r}\right)V_k\left(z,t\right), & E_{zk}\left(\mathbf{r},z,t\right) = 0, \\ \mathbf{H}_k\left(\mathbf{r},z,t\right) = \sqrt[-2]{\mu_0}\left[\mathbf{z}\times\nabla_\perp\Omega_k\left(\mathbf{r}\right)\right]I_k\left(z,t\right), & H_{zk}\left(\mathbf{r},z,t\right) = 0. \end{array} \tag{6.20}$$

Such the modes can exist when the contour of waveguide cross-section is multiconnected. Here, $k = 1, 2, \ldots, N$, where $N \geq 1$ is a number of components of the contour.

Two unknown functions $V_k\left(z,t\right)$ and $I_k\left(z,t\right)$ in Eqs. (6.19) we can express via one potential. Furthermore, it is possible to do it in two different ways as follows.

1st manner. Let function $f_k\left(z,t\right)$ be a potential for functions $V_k\left(z,t\right)$ and $I_k\left(z,t\right)$ which are expressible via this potential as follows

$$V_k = \varepsilon^{-1}\partial_z f_k, \qquad I_k = -\partial_{ct}f_k. \tag{6.21}$$

Then line (b) in Eqs. (6.19) turns into identity $0 = 0$, but line (a) supplies differential equation for the potential sought as

$$\boxed{\partial_{ct}\mu\partial_{ct}f_k - \partial_z\varepsilon^{-1}\partial_z f_k = 0.} \tag{6.22}$$

In terms of notation (6.21) , the components of electromagnetic field look as follows

$$\boxed{\begin{array}{ll} \mathbf{E}_k(\mathbf{r},z,t) = \sqrt[-2]{\epsilon_0}\nabla_\perp\Omega_k(\mathbf{r})\,\varepsilon^{-1}\partial_z f_k(z,t)\,, & E_{zk}(\mathbf{r},z,t) = 0, \\ \mathbf{H}_k(\mathbf{r},z,t) = \sqrt[-2]{\mu_0}\left[\nabla_\perp\Omega_k(\mathbf{r}) \times \mathbf{z}\right]\partial_{ct}f_k(z,t)\,, & H_{zk}(\mathbf{r},z,t) = 0. \end{array}} \tag{6.23}$$

2nd manner. Let functions $V_k(z,t)$ and $I_k(z,t)$ sought be expressible via some other potential $F_k(z,t)$ in the following way:

$$V_k = \partial_{ct}F_k, \qquad I_k = -\mu^{-1}\partial_z F_k. \tag{6.24}$$

In such a case, equation in line (a) turns into identity $0 = 0$, but equation in line (b) furnishes differential equation for the potential sought as

$$\boxed{\partial_{ct}\varepsilon\partial_{ct}F_k - \partial_z\mu^{-1}\partial_z F_k = 0.} \tag{6.25}$$

In terms of this potential, field components look as follows

$$\boxed{\begin{array}{ll} \mathbf{E}_k(\mathbf{r},z,t) = \sqrt[-2]{\epsilon_0}\nabla_\perp\Omega_k(\mathbf{r})\,\partial_{ct}F_k(z,t)\,, & E_{zk}(\mathbf{r},z,t) = 0, \\ \mathbf{H}_k(\mathbf{r},z,t) = \sqrt[-2]{\mu_0}\left[\nabla_\perp\Omega_k(\mathbf{r}) \times \mathbf{z}\right]\mu^{-1}\partial_z F_k(z,t)\,, & H_{zk}(\mathbf{r},z,t) = 0. \end{array}} \tag{6.26}$$

Equations (6.23) and (6.25) are generalizations of so-called unidirectional wave equation. Solutions of these equations themselves exhibit free $TEM-$ waveguide modes propagating in some nonstationary media layered along waveguide axis. However, these equations can be also provided with a pair of appropriate boundary conditions on $Oz-$ coordinate. In such a case, solution obtained will be corresponded to the mode enforced by some impressed source which is specified by the boundary conditions.

6.2 Classical Wave Equation for $TEM-$ Modes

Let a waveguide under consideration be filled with a homogeneous stationary medium that means that

$$\varepsilon(z,t) = \varepsilon \equiv const., \qquad \mu(z,t) = \mu \equiv const. \tag{6.27}$$

Under this condition, equations (6.22) and (6.25) coincide, i.e.,

$$\begin{gathered} \left(\epsilon_0\mu_0\varepsilon\mu\partial_t^2 - \partial_z^2\right) f_k(z,t) = 0, \\ \left(\epsilon_0\mu_0\varepsilon\mu\partial_t^2 - \partial_z^2\right) F_k(z,t) = 0, \end{gathered} \tag{6.28}$$

since $\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t$. Here the coefficient factor

$$\sqrt{\epsilon_0\mu_0\varepsilon\mu} = \frac{1}{c} \tag{6.29}$$

has the dimensions of a velocity and corresponds physically to light velocity in a homogeneous stationary medium with electromagnetic parameters mentioned above in Eqs. (6.27) . So $\epsilon_0\mu_0\varepsilon\mu\partial_t^2$ may be denoted as

$$\epsilon_0\mu_0\varepsilon\mu\partial_t^2 = \frac{1}{c^2}\partial_t^2 \equiv \partial_{ct}^2 \tag{6.30}$$

As long as equations (6.28) coincide, we may denote the functions sought as

$$f(z,t) \quad \equiv \quad f_k(z,t) \quad or \quad F_k(z,t), \tag{6.31}$$

and rewrite equations (6.28) both as follows

$$\left(\partial_{ct}^2 - \partial_z^2\right) f = 0. \tag{6.32}$$

6.2.1 Periodic Wave Functions.

Waves of a periodic type play an important role in elaborating the theory of linear partial differential equations, since they offer the advantages of a specific nature and ready superposition for the purposes of generating broad class of solutions. The representative, for the one-dimensional wave equation (6.32) , of a simply periodic wave motion that proceeds in the direction of increasing z can be written as

$$f(z,t) = A\cos\left[k(z-ct)+\delta\right] \tag{6.33}$$

or

$$f(z,t) = A\cos\left[kz-\omega t+\delta\right] \tag{6.34}$$

if $A, k,$ and δ are constants, and moreover

$$\omega = kc. \tag{6.35}$$

At any given point, z, a wave of preceding variety undergoes an oscillation, of period $T = 2\pi/\omega$ and within the amplitude limits $\pm A$, whose phase depends on the magnitude of δ. Furthermore, it appears that the quantity $\lambda = 2\pi/k$ defines the spatial periodicity, or the so-called wave length, which is revealed by the overall wave profile at any instant of time.

To gauge the forward process of this periodic wave motion, let us suppose that reference is made to a definite phase of displacement, such that

$$kz - \omega t + \delta = \text{cons}\tan\text{t}$$

in order to keep pace with any individual value of the phase (and thus retain an invariable aspect of the wave) a unique rate of advance along the direction of wave propagation must be maintained, namely:

$$v = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\omega}{k} = c, \tag{6.36}$$

which may be termed the phase velocity. It is also apparent that the ratio $c/\lambda = \omega/2\pi = 1/T = \nu$, or radian frequency, can be identified with the number of wave crests (i.e., wave maxima) which pass any point in a unit time interval; and that $\nu/c = 1/\lambda = k/2\pi$ designates the number of waves per unit length, though k itself will henceforth be called the wave number.

Inasmuch as actual conditions of excitation preclude the possibility of realizing an exactly periodic wave (or regular train that can be continued without substantive change to the respective limits of space and time) the latter must be considered as a mathematical abstraction. There is a considerable scope for the manipulation of such waves, principally by superposition, in the closer simulation of actual states of wave motion. The combination of any number of wave trains (6.33) with varying amplitudes and phases, but fixed wave length, is again a form of the same type. However, if we merely superpose a pair of trains of equal amplitude A and vanishing phase, δ, though with distinct angular frequencies ω_1, ω_1 and corresponding wave numbers k_1, k_2 the result

$$\begin{aligned} f(z,t) &= A\cos(k_1 z - \omega_1 t) + A\cos(k_2 z - \omega_2 t) \\ &= 2A\cos\left(\frac{k_1 - k_2}{2}z - \frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{k_1 + k_2}{2}z - \frac{\omega_1 + \omega_2}{2}t\right) \end{aligned} \tag{6.37}$$

is a form whose respective trigonometric factors characterize progressive motions at two new sets of values for these parameters. When the primary values of the frequency and wave numbers are nearly equal, $\omega_1 \approx \omega_2$ and $k_1 \approx k_2$, the first factor varies much more slowly in time and position then does the second; and it may viewed as superposing gradual changes on the latter, amounts to the conservation of a regular train into a series of uniformly spaced wave packets that are comprised between adjacent nulls in the complete amplitude function.

If the phase velocities of the component waves are equal, that is

$$\omega_1/k_1 = \omega_2/k_2 = c,$$

their combination, or the system of wave packets, advances without change of form at the same velocity, for both factors in (6.37) are synchronous in their rate advance, viz.,

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = c.$$

A quite different state of affair obtains, on the other hand, if the phase velocities for the component waves are distinct, a possibility which is ruled out in the case of solutions to the wave equation (6.32) though not for other equations; as then the two factors in (6.37) possess separate rates of forward progress and an unsteady or aperiodic wave pattern results. Furthermore, isolated disturbances or wave packets of limited spatial extension can be fashioned at any instant of time by a sufficiently general combination of periodic waves with different wave lengths and phase velocities, though these are found to undergo continued broadening or dispersion in the course of time.

There is a procedural advantage which accrues, in the context of linear wave equations, to the representation of an inherently real and time-periodic wave functions in terms of complex valued forms of solution. Since real and imaginary parts of the complex exponential function

$$e^{i(kz-\omega t)}$$

are individual solutions of the homogeneous wave equation (6.32) if (6.35) holds, we may, in particular, express the progressive wave (6.34) by

$$f(z,t) = \mathrm{Re}\left\{Ae^{i(kz-\omega t)}\right\}, \tag{6.38}$$

where the amplitude A is allowed a complex nature, thus obviating the need for separate phase factor $\exp(i\delta)$. The merit for such representation, or its analogue fashioned with the imaginary part, is that the operations of addition differentiation and multiplication can be performed expeditiously with the exponential functions, leaving the real or imaginary part to be extracted afterwards. In the subsequent presentation this final step is usually omitted, so that the equations ordinarily feature the exponential function themselves.

6.2.2 Cauchy Problem for Wave Equation

The representative (6.32) of second order partial differential equations is atypical insofar as its general solution can be exhibited, namely

$$f(x,t) = f_-(z-ct) + f_+(z+ct) \tag{6.39}$$

where f_-, f_+ designate any twice differentiable functions of one variable; this conclusion is directly foreshadowed by the form which the differential equation assumes on introducing the independent variables $\xi = z - ct$, $\eta = z + ct$, viz.,

$$\frac{\partial^2 f}{\partial\xi\partial\eta} = 0. \tag{6.40}$$

The functional character of the term $f_-(z-ct)$ in (6.39) is such as to imply a progressive motion of wave with invariable aspect, inasmuch as its envelope (or numerical spectrum for all values of z) at the instant t is a replica of that at an earlier time, say $t=0$, in the sense of equality between the values of this term at the pair of locations z, $z+ct$. Evidently the constant c defines the speed with which the wave advances. The opposite signs of $\partial f_-/\partial z$ and $\partial f_-/\partial t$ manifest by relation

$$\frac{\partial f_-}{\partial z} = \frac{-1}{c}\frac{\partial f_-}{\partial t} \tag{6.41}$$

are in keeping with a sense of progression towards the right (that is, increasing values of z). Likewise, we may verify that the second term in (6.39), $f_+(z+ct)$, represents a progressive motion which travels towards the left (or negative $z-$ direction) with the same rate of advance c and satisfy the first order differential equation

$$\frac{\partial f_+}{\partial z} = \frac{1}{c}\frac{\partial f_+}{\partial t}. \tag{6.42}$$

A fixed value for the magnitude of these oppositely directed motions obtains when the locus of coordinate and time variables is a straight line, viz.:

$$z \mp ct = \mathrm{cons}\tan \mathrm{t}, \tag{6.43}$$

respectively; the distinct families of parallel lines generated by assigning different values to the constant in (6.43) are so-called characteristics of the wave equation (6.32) and find joint specification through the differential relation

$$\mathrm{d}z^2 - c^2 \mathrm{d}t^2 = 0 \tag{6.44}$$

whose solutions are other than straight lines in the circumstance that c is a function of z and t when $\varepsilon \equiv \varepsilon(z,t)$, and/or $\mu \equiv \mu(z,t)$.

Both terms of (6.39) are involved when the wave is sought in the aftermath ($t > 0$) of arbitrary given *initial conditions* (at $t = 0$) relating to the wave and velocity, say

$$\boxed{f(z,0) = f_0(z), \qquad \frac{\partial}{\partial t} f(z,0) = v_0(z).} \tag{6.45}$$

In the case of a wave with unlimited span, the general solution (6.39) then takes the explicit form attributable to d'Alembert

$$\boxed{f(z,t) = \frac{1}{2}\left[f_0(z-ct) + f_0(z+ct) + \frac{1}{c}\int_{z-ct}^{z+ct} v_0(\xi)\, d\xi\right], \quad -\infty < t < \infty, \quad t > 0.} \tag{6.46}$$

The latter representation may also be derived by employing Riemann's method for integration of the wave equation with auxiliary data (here, of the initial variety), in which the characteristics (6.43) are prominently featured.

6.3 Classical Klein-Gordon Equation for $TM-$ and $TE-$ Modes

Let us impose the same conditions on electromagnetic parameters of the media within a waveguide as it was made in Eq. (6.27), i.e.,

$$\varepsilon(z,t) = \varepsilon \equiv const., \qquad \mu(z,t) = \mu \equiv const.$$

Then we substitute these values in general linear Klein-Gordon equations obained as

$$(\varepsilon\partial_{ct}\mu\partial_{ct} - \varepsilon\partial_z\varepsilon^{-1}\partial_z + \varkappa_n^2)\, D_n(z,t) = 0,$$

$$(\mu\partial_{ct}\varepsilon\partial_{ct} - \mu\partial_z\mu^{-1}\partial_z + \nu_m^2)\, B_m(z,t) = 0,$$

which were written in Eqs. (6.11) and (6.16). It yields

$$\left(\epsilon_0\mu_0\varepsilon\mu\partial_t^2 - \partial_z^2 + \varkappa_n^2\right) D_n(z,t) = 0,$$
$$\left(\epsilon_0\mu_0\varepsilon\mu\partial_t^2 - \partial_z^2 + \nu_m^2\right) B_m(z,t) = 0. \tag{6.47}$$

As for operation $\epsilon_0\mu_0\varepsilon\mu\partial_t^2$, we use the same notation which was introduced in Eq. (6.30), as follows

$$\epsilon_0\mu_0\varepsilon\mu\partial_t^2 = \frac{1}{c^2}\partial_t^2 \equiv \partial_{ct}^2.$$

Since equations (6.47) coincide, let us introduce a pair of new notation as

$$\kappa^2 \;\Rightarrow\; \textit{either}\;\; \varkappa_n^2 \;\; \textit{or} \;\; \nu_m^2,$$
$$F \;\Rightarrow\; \textit{either}\;\; D_n \;\; \textit{or} \;\; B_m. \tag{6.48}$$

Then Eqs. (6.47) both may be written as one equation, i.e.,

$$\left(\epsilon_0\mu_0\varepsilon\mu\partial_t^2 - \partial_z^2 + \kappa^2\right) F(z,t) = 0. \tag{6.49}$$

6.3.1 Cauchy Problem for Klein-Gordon Equation

To obtain the formal solution of an initial value problem for the preceding equation, which generalizes the d'Alembert result (6.46) and makes clear the unsteadying role of the parameter κ, we may employ a method of synthesis that is based on periodic constituents. Let the initial wave and velocity distributions (6.45) have the Fourier integral representations

$$F(z,0) = F_0(z) = \int_{-\infty}^{\infty} a(k)\, e^{ikz} dk, \qquad \tfrac{\partial}{\partial t} F(z,0) = v_0(z) = \int_{-\infty}^{\infty} b(k)\, e^{ikz} dk$$

and observe that the time-periodic wave functions $\exp\{i(kz \mp \omega t)\}$ are compatible with (6.49) provided that

$$\omega(k) = c\sqrt{k^2+\kappa^2} \tag{6.50}$$

and the affiliated phase velocity

$$v = c\sqrt{\left(1+(\kappa/k)^2\right)} \tag{6.51}$$

manifests a dependence of the wave number. A solution of (6.49) can thus be expressed in the form

$$F(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kz-\omega(k)t)} dk + \int_{-\infty}^{\infty} B(k)\, e^{i(kz+\omega(k)t)} dk \tag{6.52}$$

and imposition of the conditions (**??**) yields the relations

$$A(k) = \tfrac{1}{2}\left[a(k) + \tfrac{i}{\omega(k)} b(k)\right], \qquad B(k) = \tfrac{1}{2}\left[a(k) - \tfrac{i}{\omega(k)} b(k)\right]$$

whence

$$F(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikz} \cos\left(\sqrt{(k^2+\kappa^2)}ct\right) dk + \frac{1}{c}\int_{-\infty}^{\infty} e^{ikz} \frac{\sin\left(\sqrt{(k^2+\kappa^2)}ct\right)}{\sqrt{(k^2+\kappa^2)}} dk. \tag{6.53}$$

If the expression for the Fourier transforms $a(k)$, $b(k)$ of the initial wave distribution and velocity, namely

$$a(k) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} F_0(\xi)\, e^{-ik\xi} d\xi$$

$$b(k) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} v_0(\xi)\, e^{-ik\xi} d\xi$$

are introduced in (6.53) we obtain

$$F(z,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} F_0(\xi)\, d\xi \int_0^{\infty} \cos k(z-\xi) \cos\left(\sqrt{(k^2+\kappa^2)}ct\right) dk + \frac{1}{\pi c}\int_{-\infty}^{\infty} v_0(\xi)\, d\xi \int_0^{\infty} \cos k(z-\xi)\, \frac{\sin\left(\sqrt{(k^2+\kappa^2)}ct\right)}{\sqrt{(k^2+\kappa^2)}} dk,$$

or

$$F(z,t) = \frac{1}{\pi c}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathcal{G}(z-\xi, ct; \kappa)\, F_0(\xi)\, d\xi + \frac{1}{\pi c}\int_{-\infty}^{\infty} \mathcal{G}(z-\xi, ct; \kappa)\, v_0(\xi)\, d\xi \tag{6.54}$$

where

$$\mathcal{G}(x,\tau;\kappa) = \int_0^{\infty} \cos\zeta x \frac{\sin\left(\sqrt{\zeta^2+\kappa^2}\tau\right)}{\sqrt{\zeta^2+\kappa^2}} d\zeta. \tag{6.55}$$

The latter function, whose appearance in (6.54) signifies that it represent a particular solution of the original differential equation (6.49), has a discontinuous behavior in relation to the argument variables x, τ. For details bearing on this effect, which is connected with the finite speed of propagation of disturbances governed by the equation (6.49), let us first rewrite (6.55) in the sequential versions

$$\mathcal{G}(x,\tau;\kappa) = \frac{1}{2}\int_{-\infty}^{\infty} e^{\zeta x} \frac{\sin\left(\sqrt{\zeta^2+\kappa^2}\tau\right)}{\sqrt{\zeta^2+\kappa^2}} d\zeta = \frac{1}{2}\int_{-\infty}^{\infty} e^{\kappa x \sin\varphi} \sin(\kappa\tau\cosh\varphi)\, d\varphi \tag{6.56}$$

where a transformation of variables, $\zeta = \kappa\sinh\varphi$, figures. On next resolving the sine factor into complex exponentials and setting

$$\alpha = \kappa x = \mu\sinh\varphi, \qquad \beta = \kappa\tau = \mu\cosh\bar{\varphi} \tag{6.57}$$

so that

$$\beta^2 - \alpha^2 = \kappa^2 \left(\tau^2 - x^2\right) = \mu^2$$

we obtain

$$\begin{aligned} \mathcal{G} &= \frac{1}{4i} \int_{-\infty}^{\infty} \left[\exp\left(i\mu \cosh\left(\varphi + \bar{\varphi}\right)\right) - \exp\left(-i\mu \cosh\left(\varphi - \bar{\varphi}\right)\right)\right] \qquad (6.58) \\ &= \frac{1}{4i} \cdot \pi i \left[H_0^{(1)}(\mu) + H_0^{(2)}(\mu)\right] = \frac{\pi}{2} J_0(\mu), \;\; \mu > 0 \end{aligned}$$

with the customary notations for Hankel functions of the first and second kinds, $H_0^{(1)}$, $H_0^{(2)}$ and the (zero order) Bessel function, $J_0(\mu) = \frac{1}{2}\left[H_0^{(1)}(\mu) + H_0^{(2)}(\mu)\right]$. when the hyperbolic functions in (6.57) are exchanged, or

$$\begin{aligned} \alpha &= \nu \cosh \bar{\varphi}, \\ \beta &= \nu \sinh \bar{\varphi} \qquad (6.59) \\ \alpha^2 - \beta^2 &= \kappa^2 \left(x^2 - \tau^2\right) = \nu^2 = -\mu^2 \end{aligned}$$

it appears that

$$\begin{aligned} \mathcal{G} &= \frac{1}{4i} \int_{-\infty}^{\infty} \left[\exp\left(i\nu \cosh\left(\varphi + \bar{\varphi}\right)\right) - \exp\left(-i\nu \cosh\left(\varphi - \bar{\varphi}\right)\right)\right] \qquad (6.60) \\ &= 0, \;\; \nu > 0 \end{aligned}$$

since the infinite limits permit the removal of $\pm\bar{\varphi}$ from the respective terms,on suitable shifts of the integration variable, and thus imply their mutual cancellation. In accordance with (6.54) and the representation which follows from (6.57) – (6.60), viz.

$$\mathcal{G}(x, \tau; \kappa) = \begin{cases} \frac{\pi}{2} J_0\left(\kappa\sqrt{\tau^2 - x^2}\right), & \tau^2 > x^2 \\ & \qquad\qquad \kappa > 0 \\ 0 & \tau^2 < x^2 \end{cases} \qquad (6.61)$$

the solution of an initial value problem for the differential equation (6.49) achieves the explicit characterizations

$$\begin{aligned} F(z, t) &= \frac{1}{2c} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} J_0\left[\kappa\sqrt{c^2t^2 - (z - \xi)^2}\right] F_0(\xi)\, d\xi \qquad (6.62) \\ &\quad + \frac{1}{2c} \int_{-\infty}^{\infty} J_0\left[\kappa\sqrt{c^2t^2 - (z - \xi)^2}\right] v_0(\xi)\, d\xi \\ &= \frac{1}{2}\left[F_0(z - ct) + F_0(z + ct)\right] \\ &\quad - \frac{1}{2}\kappa ct \int_{z-ct}^{z+ct} \frac{J_1\left[\kappa\sqrt{c^2t^2 - (z - \xi)^2}\right]}{\sqrt{c^2t^2 - (z - \xi)^2}} F_0(\xi)\, d\xi \\ &\quad + \frac{1}{2} \int_{z-ct}^{z+ct} J_0\left[\kappa\sqrt{c^2t^2 - (z - \xi)^2}\right] v_0(\xi)\, d\xi, \end{aligned}$$

where of the second are linked with the specific properties of the Bessel functions

$$J_0(0) = 1, \qquad \tfrac{d}{dx} J_0(x) = -J_1(x)$$

Two features stand out in this generalization of d'Alembert's formula (6.46) which is, fittingly recovered if the parameter κ vanishes. The first term at right-hand-side of Eq. (6.62) reflects a (previously assumed) role for c as the speed involved in transmitting the influence of (initial) data to other locations at later times. The second (integral) terms demonsrate existence of other solutions then functions of $z \pm ct$. It means a consequence of the latter feature that **the evolution of wave forms subject to Eq.** (6.49) **cannot be realized through the direct superposition of oppositely moving profiles which retain a permanent form.**

6.3.2 Separation of Variables at Klein-Gordon Equation

Equation (6.49) , rewritten in the form of

$$\left(\partial_{ct}^2 - \partial_z^2 + \kappa^2\right) F(z,t) = 0, \tag{6.63}$$

is a generalization of the wave equation due to the presence of the eigenvalue κ^2 of WBO at the operator which has put in parenthesis at Eq. (6.63). It coincides with the unidimensional wave equation if $\kappa^2 \equiv 0$, i.e.,

$$\left(\partial_{ct}^2 - \partial_z^2\right) f(z,t) = 0. \tag{6.64}$$

Solution of the latter is the superposition of oppositely moving profiles each of which retain a permanent form as

$$f(z,t) = f_-(z - ct) + f_+(z + ct),$$

where $f_\pm$ are arbitrary twice differentiable functions.

However, presence of the numerical coefficient κ^2 in the differential operator, which is given in Eq. (6.63) in parenthesis, changes its features basically. In previous section, one can be seen it in the course of solving of one particular example, which regards to this operator. In Eq. (6.63) , the function F sought is treated as depended on two variables, i. e., z and $t : F \equiv F(z,t)$. In the connection of separation of variables, a question arises: is it possible to find *some combinations* of these variables as $u(z,t)$ and $v(z,t)$, which (i) we may consider as new independent variables provided that (ii) these new variables can be separated in Klein-Gordon equation. In the other words, we consider separation of variables in regard to the function F as follows

$$F \equiv F(z,t) = F\left[u(z,ct), v(z,ct)\right] \equiv F(u,v) = U(u)\, V(v). \tag{6.65}$$

To answer this question, we should first see, which form acquires Eq. (6.63) in terms of new variables as some functions of the original ones, i.e., in terms of $u \equiv u(z,t)$ and $v \equiv v(z,t)$. For recasting of Eq. (6.63) in terms of u and v, we must make preliminaries as follows

$$\frac{1}{c}\frac{\partial}{\partial t}F(u,v)=\frac{1}{c}\left\{\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial t}\right\};$$

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}F(u,v) &= \frac{1}{c^2}\left\{\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 F}{\partial v^2}\left(\frac{\partial v}{\partial t}\right)^2+\frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial t^2}\right\}\\ &\quad+\frac{1}{c^2}\left\{\frac{\partial^2 F}{\partial u\partial v}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}+\frac{\partial^2 F}{\partial v\partial u}\frac{\partial v}{\partial t}\frac{\partial u}{\partial t}\right\}.\end{aligned}$$

Since u and v are the independent variables, the latter equation can be simplified to

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}F(u,v)=\frac{1}{c^2}\left\{\begin{array}{c}\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{\partial^2 F}{\partial v^2}\left(\frac{\partial v}{\partial t}\right)^2\\ +\frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial t^2}+\frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial t^2}+2\frac{\partial^2 F}{\partial u\partial v}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\end{array}\right\}. \tag{6.66}$$

In a similar way, one can obtain

$$\frac{\partial^2}{\partial t^2}F(u,v)=\left\{\begin{array}{c}\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial z}\right)^2+\frac{\partial^2 F}{\partial v^2}\left(\frac{\partial v}{\partial z}\right)^2\\ +\frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial z^2}+\frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial z^2}+2\frac{\partial^2 F}{\partial u\partial v}\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\end{array}\right\}. \tag{6.67}$$

Substitution of Eqs. (6.66) and (6.67) in Eq. (6.63) supplies

$$\begin{aligned}&\left[\left(\tfrac{1}{c}\tfrac{\partial u}{\partial t}\right)^2-\left(\tfrac{\partial u}{\partial z}\right)^2\right]\tfrac{\partial^2 F}{\partial u^2}+\left[\left(\tfrac{1}{c}\tfrac{\partial v}{\partial t}\right)^2-\left(\tfrac{\partial v}{\partial z}\right)^2\right]\tfrac{\partial^2 F}{\partial v^2}+\\ &+\left[\tfrac{1}{c^2}\tfrac{\partial^2 u}{\partial t^2}-\tfrac{\partial^2 u}{\partial z^2}\right]\tfrac{\partial F}{\partial u}+\left[\tfrac{1}{c^2}\tfrac{\partial^2 v}{\partial t^2}-\tfrac{\partial^2 v}{\partial z^2}\right]\tfrac{\partial F}{\partial v}+\\ &+2\left[\tfrac{1}{c}\tfrac{\partial u}{\partial t}\tfrac{1}{c}\tfrac{\partial v}{\partial t}-\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}\right]\tfrac{\partial^2 F}{\partial u\partial v}+\kappa^2 F=0\end{aligned} \tag{6.68}$$

At this point, the following comments may be needed:

- Equation (6.68) is equivalent to Eq. (6.63) but written in terms of some new variables $u(z,t)$ and $v(z,t)$;
- Equation (6.68) has variable coefficients, which are selected with square brackets;
- In order to separate the variables u and v and get a solution of Eq. (6.68) in the form of $F(u,v)=U(u)V(v)$, one should define the dependences $u(z,t)$ and $v(z,t)$ in a very special way. Simple look at Eq. (6.68) can not suggest the definition needed.

Deep analysis of Klein-Gordon equation has been made in the mathematical book[§] from the statements of separation of variables. Using group theory approach,

[§]Willard Miller, Jr., *Symmetry and Separation of Variables,* Addison-Wesley Pub. Co., Massachusets, 1977.

11 *orthogonal* coordinate systems were found, in which separation of new variables is possible. Besides, it was elucidated¶ that it is possible to obtain another set of *nonorthogonal* coordinate systems which are available for separation of the variables as well. Below, we cite the list of complete set of the *orthogonal coordinate systems,* which is borrowed from the book just cited.

Group theory gives positive answer on this question. Below, 11 versions of variables ct and z as the functions of new variables u and v are listed. Inverse functional dependences, i.e., $u \equiv u(z, ct)$ and $v \equiv v(z, ct)$, can be found easily.

Coordinate systems for separation of variables in Klein-Gordon Equation

1. $ct = u$, $z = v : U(u)\, V(v)$ is a product of the exponentials.

2. $ct = u \cosh v$, $z = u \sinh v$; $0 \le u < \infty, -\infty < v < \infty : U(u)\, V(v)$ is a product of Bessel function and an exponential.

3. $ct = \frac{1}{2}\left(u^2 + v^2\right)$, $z = uv$; $-\infty < u < \infty$, $0 \le v < \infty : U(u)\, V(v)$ is a product of parabolic cylinder functions.

4. $ct = uv$, $z = \frac{1}{2}\left(u^2 + v^2\right)$; $-\infty < u < \infty$, $0 \le v < \infty : U(u)\, V(v)$ is a product of parabolic cylinder functions.

5. $ct = \frac{1}{2}(u - v)^2 + u + v$, $z = -\frac{1}{2}(u - v)^2 + u + v$; $-\infty < u, v < \infty : U(u)\, V(v)$ is a product of Airy functions.

6. $2ct = \cosh \frac{u-v}{2} + \sinh \frac{u+v}{2}$, $2z = \cosh \frac{u-v}{2} - \sinh \frac{u+v}{2}$; $-\infty < u,\ v < \infty :$ $U(u)\, V(v)$ is a product of Mathieu functions.

7. $ct = \sinh(u - v) + \frac{1}{2}e^{u+v}$, $z = \sinh(u - v) - \frac{1}{2}e^{u+v}$, $-\infty < u, v < \infty : U(u)\, V(v)$ is a product of Bessel functions.

8. $ct = \cosh(u - v) + \frac{1}{2}e^{u+v}$, $z = \cosh(u - v) - \frac{1}{2}e^{u+v}$, $-\infty < u, v < \infty :$ $U(u)\, V(v)$ is a product of Bessel functions.

9. $ct = \sinh u \cosh v$, $z = \cosh u \sinh v$; $-\infty < u,\ v < \infty : U(u)\, V(v)$ is a product of Mathieu functions.

10. $ct = \cosh u \cosh v$, $z = \sinh u \sinh v$; $-\infty < u < \infty$, $0 \le v < \infty : U(u)\, V(v)$ is a product of Mathieu functions.

11. $ct = \cos u \cos v$, $z = \sin u \sin v$; $0 < u < 2\pi$, $0 \le v < \pi : U(u)\, V(v)$ is a product of Mathieu functions.

¶following the same technique of group theory

Case 1: $ct = u,\ z = v.$ It is evidently that function $F \equiv U(u)V(v)$ is a product of the following exponentials:

$$U(u) = e^{i\frac{\omega}{c}u} = e^{i\omega t}, \qquad V(v) = e^{i\gamma v} = e^{i\gamma z}, \tag{6.69}$$

where ω and γ are the constants of separation of the variables u and v; constant ω has physical dimension of frequency, γ has the dimension of propagation constant; c is free-space light velocity. Substitution of the product

$$U(u)V(v) = e^{i(\omega t + \gamma z)}$$

in Klein-Gordon equation (6.63) yields

$$\left(\partial_{ct}^2 - \partial_z^2 + \kappa^2\right) e^{i(\omega t+\gamma z)} = \left(-\frac{\omega^2}{c^2} + \gamma^2 + \kappa^2\right) e^{i(\omega t+\gamma z)} = 0,$$

which means that

$$-\frac{\omega^2}{c^2} + \gamma^2 + \kappa^2 = 0. \tag{6.70}$$

Using Eq. (6.70), one can express the constant of separation ω via another one, γ, as

$$\omega = \pm\sqrt{\kappa^2 + \gamma^2}, \tag{6.71}$$

or wise versa as follows

$$\gamma = \pm\sqrt{\frac{\omega^2}{c^2} - \kappa^2}. \tag{6.72}$$

Let the constants ω and γ be real provided that $-\infty < \omega < \infty$ and $-\infty < \gamma < \infty$. When the constant γ^2 has possible minimal value $\gamma^2 = 0$, then the frequency constant ω has values $\omega \equiv \omega_c = \pm\kappa$, accordingly to Eq. (6.71). In Electromagnetics, the value ω_c is called as *cut-off frequency.* Let us substitute now the values from Eqs. (6.71) and (6.72) in the solution given as the product of the exponentials, i.e.,

$$U(u)V(v) = F[u(z,ct), v(z,ct)] = e^{i(\omega t+\gamma z)} = e^{i\omega t} e^{\pm iz\sqrt{\frac{\omega^2}{c^2}-\kappa^2}}, \tag{6.73}$$

where $\gamma = \pm\sqrt{\frac{\omega^2}{c^2} - \kappa^2}$. When $\omega < \omega_c$, then $\sqrt{\frac{\omega^2}{c^2} - \kappa^2} = i\sqrt{\kappa^2 - \frac{\omega^2}{c^2}}$, where $\sqrt{-1} = +i$ is chosen. It means that the solution of Klein-Gordon equation decreases or increases along $z-$ coordinate depending on sign $(+)$ or $(-)$ chosen at the second exponential in Eq. (6.73).

Case 2: $ct = u\cosh v,\ z = u\sinh v; 0 \le u < \infty, -\infty < v < \infty.$ First we should find the dependences needed as $u(z,t)$ and $v(z,t)$. Since $c^2t^2 = u^2\cosh^2 v$, and $z^2 = u^2\sinh^2 v$, then

$$c^2t^2 - z^2 = u^2\left(\cosh^2 v - \sinh^2 v\right) = u^2,$$

and hence,

$$\boxed{u = \sqrt{c^2t^2 - z^2},} \tag{6.74}$$

since $0 \leq u < \infty$, by definition. Further,

$$\frac{z}{ct} = \frac{u \sinh v}{u \cosh v} = \tanh v,$$

and hence,

$$\boxed{v = \text{arctanh}\, \tfrac{z}{ct} \equiv \tfrac{1}{2} \ln \tfrac{ct+z}{ct-z}.} \tag{6.75}$$

Now, we calculate the partial derivatives of $u\,(z,t)$ and $v\,(z,t)$ which are placed in square brackets in Eq. (6.68).

$$\frac{1}{c}\frac{\partial u}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\sqrt{c^2t^2 - z^2} = \frac{ct}{\sqrt{c^2t^2 - z^2}} = \frac{u \cosh v}{u} = \cosh v; \tag{6.76}$$

$$\frac{1}{c}\frac{\partial v}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\ln\frac{ct+z}{ct-z}\right) = -\frac{z}{c^2t^2 - z^2} = -\frac{u \sinh v}{u^2} = -\frac{\sinh v}{u}; \tag{6.77}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}\cosh v = \frac{1}{c}\frac{\partial v}{\partial t}\sinh v = -\frac{\sinh^2 v}{u}; \tag{6.78}$$

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial v}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{\sinh v}{u}\right) = \frac{\sinh 2v}{u^2}; \tag{6.79}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\sqrt{c^2t^2 - z^2} = -\frac{z}{\sqrt{c^2t^2 - z^2}} = -\frac{u \sinh v}{u} = -\sinh v; \tag{6.80}$$

$$\frac{\partial v}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{2}\ln\frac{ct+z}{ct-z}\right) = \frac{ct}{c^2t^2 - z^2} = \frac{u \cosh v}{u^2} = \frac{\cosh v}{u}; \tag{6.81}$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\cosh^2 v}{u}; \tag{6.82}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\sinh 2v}{u^2}. \tag{6.83}$$

Now, we substitute the results obtained in Eqs. (6.76) $-$ (6.83) in Klein-Gordon equation (6.68) written formally in terms of u, v variables. It supplies

$$\left[\cosh^2 v - \sinh^2 v\right] \tfrac{\partial^2 F}{\partial u^2} + \left[\tfrac{\sinh^2 v}{u^2} - \tfrac{\cosh^2 v}{u^2}\right] \tfrac{\partial^2 F}{\partial v^2} +$$

$$+ \left[-\tfrac{\sinh^2 v}{u} + \tfrac{\cosh^2 v}{u}\right] \tfrac{\partial F}{\partial u} + \left[\tfrac{\sinh 2v}{u^2} - \tfrac{\sinh 2v}{u^2}\right] \tfrac{\partial F}{\partial v} +$$

$$+2\left[-\tfrac{\sinh v \cosh v}{u} + \tfrac{\sinh v \cosh v}{u}\right] \tfrac{\partial^2 F}{\partial u \partial v} + \kappa^2 F = 0$$

Evident algebraic manipulations with the coefficients placed in square brackets furnish

$$\boxed{\frac{\partial^2 F}{\partial u^2} + \frac{1}{u}\frac{\partial F}{\partial u} + \kappa^2 F - \frac{1}{u^2}\frac{\partial^2 F}{\partial v^2} = 0.} \tag{6.84}$$

It is obviously that solution of this equation can be obtained by separation of the variables (u, v). Indeed, when

$$F(u, v) = U(u) V(v) \tag{6.85}$$

is chosen, then Eq. (6.84) can be rewritten as follows

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \kappa^2 - \frac{1}{u^2}\frac{V''(v)}{V(v)}\right) U(u) = 0. \tag{6.86}$$

Function $V(v)$ we may subject to equation

$$V''(v) = \alpha^2 V(v),$$

where free parameter α is a constant of separation of the variables u, v in Eq. (6.86). This equation supplies the solution

$$V(v) = e^{\pm \alpha v}. \tag{6.87}$$

Hence, function $U(u)$ satisfies Bessel equation

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \kappa^2 - \frac{\alpha^2}{u^2}\right) U(u) = 0, \tag{6.88}$$

which has two linearly independent solutions, in particular,

$$U(u) = A_\alpha J_\alpha(\kappa u) + B_\alpha Y_\alpha(\kappa u), \tag{6.89}$$

where A_α and B_α are arbitrary constants.

Example 13 *Let it be*

$$V(v) = e^{-\alpha v}, \qquad U(u) = J_\alpha(\kappa u),$$

hence,

$$F(z, t) = U(u) V(v) = e^{-\alpha v} J_\alpha(\kappa u) \tag{6.90}$$

Since[‖] $u = \sqrt{c^2 t^2 - z^2}$, $v = \frac{1}{2}\ln\frac{ct+z}{ct-z}$, *and*

$$e^{\alpha v} = e^{-\frac{\alpha}{2}\ln\frac{ct+z}{ct-z}} = e^{\ln\left(\frac{ct-z}{ct+z}\right)^{\alpha/2}} = \left(\frac{ct-z}{ct+z}\right)^{\alpha/2},$$

[‖]see Eqs. (6.74) and (6.75)

then right-hand-side of Eq. (6.90) *can be written as follows*

$$F(z,t) = \left(\frac{ct-z}{ct+z}\right)^{\alpha/2} J_\alpha\left(\kappa\sqrt{c^2t^2-z^2}\right) \equiv F_\alpha(z,t), \tag{6.91}$$

where α *is a free numerical parameter. At the origin of* $Oz-$ *coordinate (i.e., when* $z=0$*) this solution becomes very simple, namely:*

$$F_\alpha(z,t) = J_\alpha(\kappa ct). \tag{6.92}$$

We may choose free parameter α *as integer, i.e.,* $\alpha = 0, 1, 2, \ldots$. *In such a case, a set of functions*

$$\{J_n(\kappa ct)\}_{n=0}^{\infty} \tag{6.93}$$

is complete. It means that arbitrary function of time $\varphi(t)$ *can be presented in the form of so-called Neumann series, i.e.,*

$$\varphi(t) = \sum_{n=0}^{\infty} C_n J_n(\kappa ct), \tag{6.94}$$

where C_n*'s are appropriate constants.*

Example 14 *Let* $\varphi(t) = H(t)$ *is the unit-step function, i.e.,*

$$H(t) = \begin{cases} 1, & \text{while } t \geq 0, \\ 0, & \text{while } t < 0. \end{cases}$$

Then $C_0 = 1$, $C_{2n+1} = 0$, *and* $C_{2n} = 2$, *i.e.,*

$$H(t) = J_0(\kappa ct) + 2\sum_{n=1}^{\infty} J_{2n}(\kappa ct). \tag{6.95}$$

So, when $F(z,t)|_{z=0} = H(t)$, *then*

$$F(z,t) = J_0\left(\kappa\sqrt{c^2t^2-z^2}\right) + 2\sum_{n=1}^{\infty}\left(\frac{ct-z}{ct+z}\right)^n J_{2n}\left(\kappa\sqrt{c^2t^2-z^2}\right). \tag{6.96}$$