Econometrics

Preliminaries: Matrix Algebra

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What is a Matrix?

- There are many books on Matrix Algebra
- The material in this pdf is based on: **Econometric Analysis by Greene (Appendix A)**
- A matrix is a rectangular array of numbers

$$A = [a_{ik}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & a_{2K} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nK} \end{bmatrix}$$

- a_{ik} stands for $a_{\text{row,col}}$ so that $i = 1, ..., n$ and $j = 1, ..., K$
- A is an $n \times K$ matrix
- If $K = 1$, A becomes a column vector
- If $n = 1$, A becomes a row vector

Some special matrices

- **Symmetric matrix**: a matrix that has $a_{ik} = a_{ki}$ for all i and k; e.g.,

$$A = \begin{bmatrix} 3 & 2 & 9 \\ 2 & 1 & 6 \\ 9 & 6 & 5 \end{bmatrix}$$

- **Diagonal matrix**: a square matrix that has only nonzero entries on the main diagonal; e.g.,

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Some special matrices

- **Scalar matrix**: a diagonal matrix with the same value in all diagonal elements; e.g.,

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- **Identity matrix**: a scalar matrix with ones on the diagonal; e.g.,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algebra of Matrices

- **Transposition**: The transpose of $A = [a_{ij}]$ is $A' = [a_{ji}]$; e.g.,

$$\text{if } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ then } A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

  Note if A is symmetric then $A = A'$

Algebra of Matrices

- **Addition**:

$$A + B = [a_{ij} + b_{ij}] \text{ or } A - B = [a_{ij} - b_{ij}]$$

  - Adding the zero (or null) matrix: $A + 0 = A$
  - Commutative: $A + B = B + A$
  - Associative: $(A + B) + C = A + (B + C)$

Algebra of Matrices

- **Scalar Multiplication**: Let $c \in \mathbb{R}$, then $cA = [ca_{ji}]$; e.g.,

$$\text{if } A = \begin{bmatrix} 3 & 2 & 9 \\ 2 & 1 & 6 \\ 9 & 6 & 5 \end{bmatrix} \text{ then } 2A = \begin{bmatrix} 6 & 4 & 18 \\ 4 & 2 & 12 \\ 18 & 12 & 10 \end{bmatrix}$$

Algebra of Matrices

- **Matrix Multiplication**: Let A and B be $n \times K$ and $K \times m$, respectively. (Note the dimensions!) Then,

$$C = AB = [c_{ij}] = [\sum_{k=1}^{K} a_{ik} b_{kj}]$$

- Example

$$\text{if } A = \begin{bmatrix} 3 & 2 & 9 \\ 2 & 1 & 6 \\ 9 & 6 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 2 & 7 \\ 1 & 2 & 9 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 21 & 25 & 110 \\ 13 & 16 & 71 \\ 41 & 31 & 132 \end{bmatrix}$$

Algebra of Matrices

- **Matrix Multiplication**: Let A and B be $n \times K$ and $K \times m$, respectively. Then,

$$C = AB = [c_{ij}] = [\sum_{k=1}^{K} a_{ik} b_{kj}]$$

- Some properties
  - Important: $AB \neq BA$
  - Associative: $(AB)C = A(BC)$
  - Distributive: $A(B + C) = AB + AC$
  - Transpose of a product: $(AB)' = B'A'$
  - Transpose of an extended product: $(ABC)' = C'B'A'$
  - Identity: $AI_K = I_n A = A$

Algebra of Matrices

- **Inverse of a square Matrix**: Let A be a square matrix $(n \times n)$. The inverse matrix, denoted A^{-1}, is defined such that
$$A^{-1}A = AA^{-1} = I$$
- Example: Let A be a 2×2 matrix
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
then
$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$
where
$$|A| = a_{11}a_{22} - a_{12}a_{21}$$
is the determinant of A. So if $|A| = 0$, then A is not invertible

Algebra of Matrices

- **Determinant of a square Matrix**: A practical definition (Laplace): Let A be a $K \times K$ matrix. For any row i,

$$|A| = \sum_{k=1}^{K} a_{ik}(-1)^{i+k}|A_{ik}|$$

  where $|A_{ik}|$ is a **minor** of A, i.e., the determinant of A_{ik} where A_{ik} is the matrix obtained after deleting row i and column k
- $C_{ik} = (-1)^{i+k}|A_{ik}|$ is known as a **cofactor**
- Example: If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } |A| = a_{11}a_{22} - a_{12}a_{21}$$

Algebra of Matrices

- Example: If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  then

$$\begin{aligned} |A| &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} \\ &\quad -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{21}a_{12}a_{33} \end{aligned}$$

Algebra of Matrices

- **Inverse of a matrix**. General Case:

$$A^{-1} = \frac{1}{|A|} C'$$

where

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1K} \\ C_{21} & C_{22} & \cdots & C_{2K} \\ & & \cdots & \\ C_{n1} & C_{n2} & \cdots & C_{nK} \end{bmatrix}$$

is the matrix of cofactors

- Equivalently: The $ik-$th element of A^{-1}, say a^{ik}, is

$$a^{ik} = \frac{|C_{ki}|}{|A|}.$$

Algebra of Matrices

- Example:

$$\text{if } A = \begin{bmatrix} 3 & 2 & 9 \\ 2 & 1 & 6 \\ 9 & 6 & 5 \end{bmatrix} \text{ then } A^{-1} = \frac{1}{22} \begin{bmatrix} -31 & 44 & 3 \\ 44 & -66 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

Algebra of Matrices

- **Rank of a Matrix**: The rank of a matrix, denoted $rank(A)$, is the maximum number of linearly independent rows or columns
- $rank(A) = rank(A') \leq \min(\#\text{rows}, \#\text{cols})$
- A matrix A is full rank if its rank is equal to the number of columns it contains
- $rank(AB) \leq \min(rank(A), rank(B))$
- If A is $M \times n$ and B is a square matrix with $rank(B) = n$, then $rank(AB) = rank(A)$
- Important: $rank(A) = rank(A'A) = rank(AA')$
- **Proposition**: $|A| \neq 0$ if and only if it has full rank

Algebra of Matrices

- **Rank of a Matrix**: Example: Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

  - if $|A| = a_{11}a_{22} - a_{12}a_{21} \neq 0$, then $rank(A) = 2$
  - if $|A| = 0$ because $A = 0$, then $rank(A) = 0$
  - otherwise (that is if $A \neq 0$ and $|A| = 0$), then $rank(A) = 1$

Algebra of Matrices

- **Quadratic forms and Definite Matrices**

- A quadratic form: $q = x'Ax$ where A is a symmetric matrix

- **Positive Definite**: A is positive definite if $x'Ax > 0$ for all nonzero x
- **Negative Definite**: A is positive definite if $x'Ax < 0$ for all nonzero x

- **Positive Semidefinite**: A is positive semidefinite if $x'Ax \geq 0$ for all nonzero x (var-cov matrices)
- **Negative Semidefinite**: A is negative semidefinite if $x'Ax \leq 0$ for all nonzero x

Algebra of Matrices

- **Calculus and Matrix Algebra**
- First rule:
$$\frac{\partial(Ax)}{\partial x} = A'$$
- Second rule:
$$\frac{\partial(x'Ax)}{\partial x} = (A + A')x$$

Algebra of Matrices

- **Calculus and Matrix Algebra**

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Example: First rule

$$\frac{\partial(Ax)}{\partial x} = \frac{\partial \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 4x_2 \end{bmatrix}}{\partial x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A'$$

- Second rule:

$$\begin{aligned} \frac{\partial(x_1^2 + 5x_1x_2 + 4x_2^2)}{\partial x} &= \begin{bmatrix} 2x_1 + 5x_2 \\ 5x_1 + 8x_2 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= (A + A')x \end{aligned}$$