Macroeconometrics

Topic 1: Macroeconomic Time Series

by Vanessa Berenguer-Rico

University of Oxford

Michaelmas 2016

Macroeconometrics

Macroeconometrics is the statistical analysis of macroeconomic data and models

Macroeconomic Time Series

- **Time** Series Data: GDP, Unemployment, Inflation, etc...
- As **time** goes by... causal/dynamic/temporal effects
- So, once upon a **time**... a **time** series lecture began!

What is a Time Series?

- A time series is a realization of a stochastic process
- A stochastic process is a family of random variables $\{X_t(\omega), t \in T, \omega \in \Omega\}$
- Examples: i.i.d., heteroscedastic, AR(p)

Stochastic Processes

- A stochastic process is a family of random variables $\{X_t\left(\omega\right), t \in T, \omega \in \Omega\}$
- Fix t, then $X_t\left(\omega\right)$: $\Omega \longrightarrow \mathbb{R}$
- Fix ω, then $X_t\left(\omega\right)$: $T \longrightarrow \mathbb{R}$
- Probabilistic framework for time series analysis

Time Series Analysis

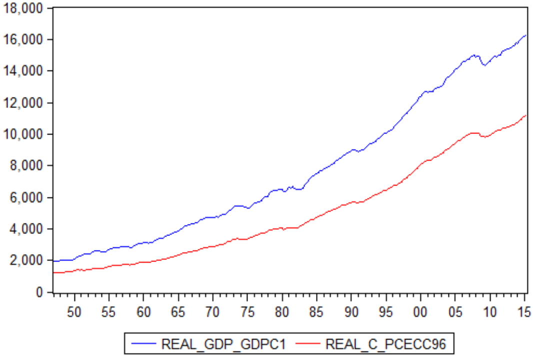
- A time series is a realization of a stochastic process
- A stochastic process is a family of random variables $\{X_t(\omega), t \in T, \omega \in \Omega\}$
- Common features of economic time series: **Dependent and heterogeneously distributed**
- Brockwell and Davis: "Introduction to time series and forecasting"

Time Series Analysis

- Common features of economic time series:
  Dependent and heterogeneously distributed
- Need to relax the i.i.d. assumption
- No identical distributions:
  e.g., trends (stochastic and/or deterministic)
- No independently distributed:
  e.g., autocorrelation

Macroeconomic Time Series

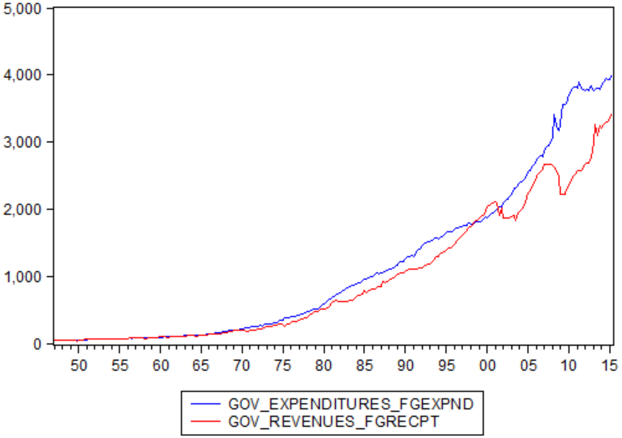
- Real Consumption and Real GDP



- U.S. Quarterly Data from the Federal Reserve Bank of St. Louis: 1947Q1-2015Q2

Macroeconomic Time Series

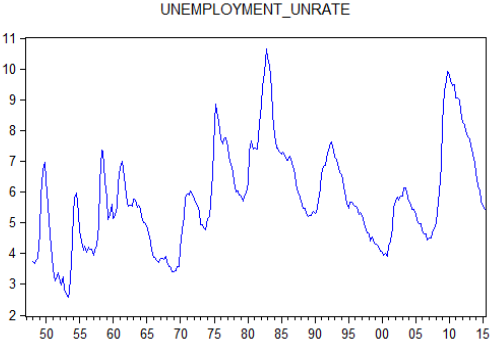
- Government Expenditures and Revenues



- U.S. Quarterly Data from the Federal Reserve Bank of St. Louis: 1947Q1-2015Q2

Macroeconomic Time Series

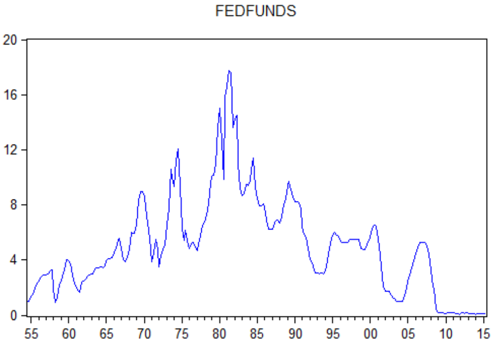
- Unemployment Rate



- U.S. Quarterly Data from the Federal Reserve Bank of St. Louis: 1948Q1-2015Q2

Macroeconomic Time Series

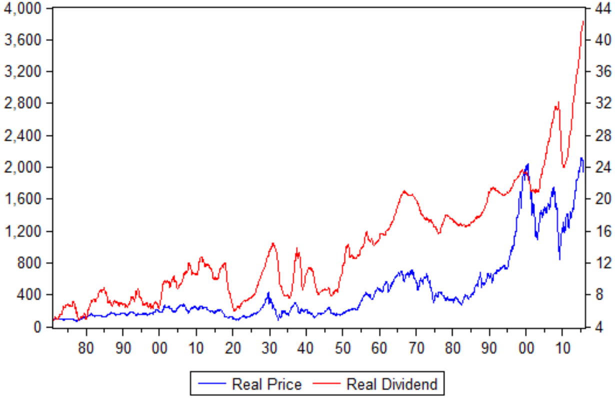
- FED Funds



- U.S. Quarterly Data from the Federal Reserve Bank of St. Louis: 1954Q3-2015Q2

Macroeconomic Time Series

- Real Stock Prices and Real Dividends



- U.S. Monthly Data from Robert Shiller: 1871m1-2012m6

Time Series Analysis

- Common features of economic time series:
  Dependent and heterogeneously distributed
- Need to relax the i.i.d. assumption
- No identical distributions:
  e.g., trends (stochastic and/or deterministic)
- No independently distributed:
  e.g., autocorrelation

Autocovariance and Autocorrelation Functions

- The autocovariance function of a process X_t is

$$\gamma_X(h) = Cov(X_t, X_{t-h}) = E[\{X_t - E(X_t)\}\{X_{t-h} - E(X_{t-h})\}]$$

- The autocorrelation function of a process X_t is

$$\rho_X(h) = Corr(X_t, X_{t-h}) = \frac{Cov(X_t, X_{t-h})}{\sqrt{Var(X_t)}\sqrt{Var(X_{t-h})}}$$

Sample Autocovariance and Autocorrelation Functions

- The sample autocovariance is

$$\hat{\gamma}_X(h) = \frac{1}{T}\sum_{t=1}^{T-h}(X_{t+h} - \bar{X}_T)(X_t - \bar{X}_T)$$

- The sample autocorrelation is

$$\hat{\rho}_X(h) = \frac{\hat{\gamma}_X(h)}{\sqrt{\hat{V}(X_t)}\sqrt{\hat{V}(X_{t-h})}}$$

Time Series Analysis

- Common features of economic time series:
  Dependent and heterogeneously distributed

- Need to relax the i.i.d. assumption

- Two key concepts (to start with):
  Stationarity and Ergodicity

Relaxing identical distributions: Stationarity

- **Strict vs Weak Stationarity**
- Strict Stationarity: refers to join finite dimensional distributions
- Weak Stationarity: refers to first and second moments (only): i.e., mean, variance/autocovariance

Strict Stationarity

- **Strict Stationarity:** The time series $\{X_t, t \in \mathbb{Z}\}$ is said to be strictly stationary if the joint distributions of $(X_{t_1}, ..., X_{t_k})'$ and $\left(X_{t_{1+h}}, ..., X_{t_{k+h}}\right)'$ are the same for all positive integers k and for all $t_1, ..., t_k, h \in \mathbb{Z}$.

Weak Stationarity

- **Weak Stationarity:** The time series $\{X_t, t \in \mathbb{Z}\}$ is said to be weakly stationary if:

  (i) $E\left[X_t\right] = m \quad$ for all t
  (ii) $E\left[X_t^2\right] < \infty \quad$ for all t
  (iii) $Cov\left(X_t, X_s\right) = Cov\left(X_{t+h}, X_{s+h}\right) \quad$ for all $t, s, h \in \mathbb{Z}$

Stationarity

- Two notions of stationarity

- **Strict Stationarity:** The time series $\{X_t, t \in \mathbb{Z}\}$ is said to be strictly stationary if the joint distributions of $(X_{t_1}, ..., X_{t_k})'$ and $(X_{t_{1+h}}, ..., X_{t_{k+h}})'$ are the same for all positive integers k and for all $t_1, ..., t_k, h \in \mathbb{Z}$.

- **Weak Stationarity**: The time series $\{X_t, t \in \mathbb{Z}\}$ is said to be weakly stationary if:

  (i) $E[X_t] = m \quad$ for all t
  (ii) $E[X_t^2] < \infty \quad$ for all t
  (iii) $Cov(X_t, X_s) = Cov(X_{t+h}, X_{s+h}) \quad$ for all $t, s, h \in \mathbb{Z}$

Strict vs Weak Stationarity

- Strict Stationarity: refers to join finite dimensional distributions
- Weak Stationarity: refers to first and second moments (only): i.e., mean, variance/autocovariance
- In principle, neither concept implies each other; but...
- If the first and second moments exist, then strict stationarity implies weak stationarity
- The converse is not generally true but...

Strict vs Weak Stationarity

- Under Gaussianity both concepts coincide!
- Definition: The process X_t is a Gaussian time series if and only if the distribution functions of X_t are all multivariate normal
- If X_t is stationary Gaussian, then it is also strictly stationary

Some Stationary Processes

- iid
- White Noise
- MA(1)
- AR(1)

Some Non-stationary Processes

- Deterministic Trends
- Stochastic Trends:
  e.g., Random Walk
- Deterministic and Stochastic Trends:
  e.g., Random Walk with Drift
- Breaks

Some Examples

- Linear time trend

$$x_t = \mu + \beta t + u_t; \quad u_t \sim i.i.d.\,(0,1)$$

- i.i.d., AR(1)

$$u_t \sim i.i.d.\,(0,1)\,; \quad x_t = \phi x_{t-1} + u_t; \quad |\phi| < 1$$

- Random Walk

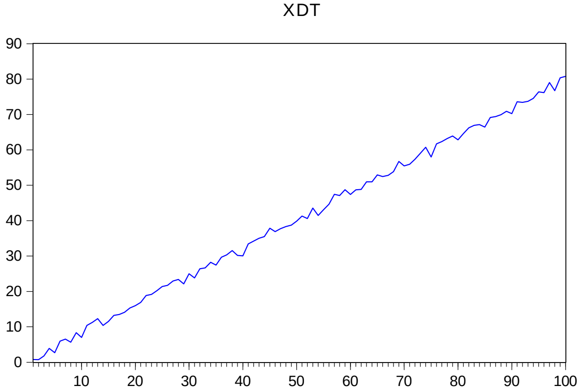
$$x_t = x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)\,; \quad x_0 = 0$$

- Random Walk with Drift

$$x_t = \alpha + x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)\,; \quad x_0 = 0$$

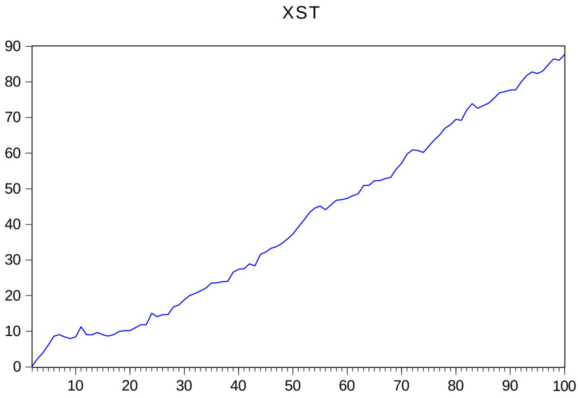
Deterministic Trend

$$x_t = \mu + \beta t + u_t; \quad u_t \sim i.i.d.N\left(0, 1\right)$$



Random Walk with Drift

$$x_t = \alpha + x_{t-1} + u_t; \;\; u_t \sim i.i.d.N\left(0,1\right); \;\; x_0 = 0$$



An i.i.d. process

- Let $u_t \sim i.i.d. \left(0, \sigma^2\right)$
- No trends. Example:

$$x_t = \mu + u_t$$

- Stochastic Properties

$$E\left[x_t\right] = \mu$$

$$V\left[x_t\right] = \sigma^2$$

$$Cov\left[x_t, x_s\right] = 0$$

AR(1)

- Let $u_t \sim i.i.d. \left(0, \sigma^2\right)$
- No trends. Example:

$$x_t = \mu + \phi x_{t-1} + u_t; \quad |\phi| < 1$$

- Stochastic Properties

$$E\left[x_t\right] = \frac{\mu}{(1-\phi)}$$

$$V\left[x_t\right] = \frac{\sigma^2}{\left(1-\phi^2\right)}$$

$$Cov\left[x_t, x_s\right] = \phi^{|t-s|} \frac{\sigma^2}{\left(1-\phi^2\right)}$$

Linear Deterministic Trend

- Let $u_t \sim i.i.d. \left(0, \sigma^2\right)$
- Deterministic trends. Example:

$$x_t = \mu + \beta t + u_t$$

- Stochastic Properties

$$E\left[x_t\right] = \mu + \beta t$$

$$V\left[x_t\right] = \sigma^2$$

$$Cov\left[x_t, x_s\right] = 0$$

AR(1) with a Deterministic Trend

- Let $u_t \sim i.i.d. \left(0, \sigma^2\right)$
- Deterministic trends. Example:

$$x_t = \mu + \beta t + \phi x_{t-1} + u_t; \quad |\phi| < 1$$

- Stochastic Properties

$$E\left[x_t\right] = \frac{\mu}{(1-\phi)} - \frac{\phi\beta}{(1-\phi)^2} + \frac{\beta}{(1-\phi)} t$$

$$V\left[x_t\right] = \frac{\sigma^2}{\left(1-\phi^2\right)}$$

$$Cov\left[x_t, x_s\right] = \phi^{|t-s|} \frac{\sigma^2}{\left(1-\phi^2\right)}$$

Random Walk

- Let $u_t \sim i.i.d.\,(0, \sigma^2)$
- Stochastic Trend. Example:

$$x_t = x_{t-1} + u_t; \quad x_0 = 0$$

- Solving Backwards

$$\begin{aligned} x_t &= x_{t-1} + u_t \\ &= x_{t-2} + u_{t-1} + u_t \\ &= x_{t-3} + u_{t-2} + u_{t-1} + u_t \\ &= \ldots \\ &= x_0 + \sum_{j=1}^{t} u_j \end{aligned}$$

Random Walk

- Stochastic Trend. Example:

$$x_t = x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)\,; \quad x_0 = 0$$

$$x_t = x_0 + \sum_{j=1}^{t} u_j$$

- Stochastic Properties

$$E\left[x_t\right] = x_0 = 0$$

$$V\left[x_t\right] = \sigma^2 t$$

$$Cov\left[x_t, x_s\right] = \min\left\{t, s\right\}\sigma^2$$

Random Walk with Drift

- Let $u_t \sim i.i.d. \, (0, \sigma^2)$
- Deterministic & Stochastic Trend. Example:

$$x_t = \beta + x_{t-1} + u_t; \quad x_0 = 0$$

- Solving Backwards

$$\begin{aligned}
x_t &= \beta + x_{t-1} + u_t \\
&= 2\beta + x_{t-2} + u_{t-1} + u_t \\
&= 3\beta + x_{t-3} + u_{t-2} + u_{t-1} + u_t \\
&= \ldots \\
&= t\beta + x_0 + \sum_{j=1}^{t} u_j
\end{aligned}$$

Random Walk with Drift

- Deterministic & Stochastic Trend. Example:

$$x_t = \beta + x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)$$

$$x_t = x_0 + \beta t + \sum_{j=1}^{t} u_j$$

- Stochastic Properties

$$E\left[x_t\right] = x_0 + \beta t$$

$$V\left[x_t\right] = \sigma^2 t$$

$$Cov\left[x_t, x_s\right] = \min\left\{t, s\right\}\sigma^2$$

Some Non-stationary processes

- Non-stationarities: Examples

- Non-stationary **in mean:**

$$x_t = \mu + \beta t + u_t; \quad u_t \sim i.i.d.N\left(0,1\right) \quad \text{or} \quad x_t = \begin{cases} u_t, & t < k \\ \mu + u_t, & t \geq k \end{cases}$$

- Non-stationary **in variance**:

$$x_t = x_{t-1} + u_t; \quad u_t \sim i.i.d.\left(0,1\right)$$

- Non-stationary **in mean and variance**:

$$x_t = \beta + x_{t-1} + u_t; \quad u_t \sim i.i.d.\left(0,1\right)$$

Time Series Analysis

- Common features of economic time series:
  Dependent and heterogeneously distributed
- Need to relax the i.i.d. assumption
- Two key concepts (to start with):
  stationarity and ergodicity

Dependence: Ergodicity

- Ergodicity is a tricky business
- Technically, it is a highly abstract concept
- These technicalities are beyond the scope of these lectures
- We will intuitively discuss ergodicity

Dependence: Ergodicity

- The idea is to allow as much dependence/memory as the Law of Large Numbers allows
- Law of Large Numbers: For a process X_t, we want to estimate $E(X_t) = \mu$
- Stationarity is not enough (example: $Y_t = Z + U_t$ where $Z \sim N(0,1)$, $U_t \sim i.i.d. N(0,1)$ and Z is independent of U_t)
- Ergodicity $\sim$ Asymptotic independence (today's events have no impact on sufficiently distant events)

Dependence: Ergodicity

- The idea is to allow as much dependence/memory as the Law of Large Numbers allows
- Law of Large Numbers: For a process X_t, we want to estimate $E(X_t) = \mu$
- Ensemble Average: (cross-section-like)

$$\frac{1}{N}\sum_{i=1}^{N} X_{it}$$

- Temporal Average: (time-series-like)

$$\frac{1}{T}\sum_{t=1}^{T} X_{it}$$

Dependence: Ergodicity

- Ensemble Average: (cross-section-like)

$$\frac{1}{N}\sum_{i=1}^{N} X_{it}$$

- Temporal Average: (time-series-like)

$$\frac{1}{T}\sum_{t=1}^{T} X_{it}$$

- In time series we have to work with the temporal average. Under which conditions this is a good choice?

- Ergodicity: does the temporal average converge to the same limit as the ensemble average $E(X_t) = \mu$?

Dependence: Ergodicity

- Ergodicity for the mean: A covariance stationary process is ergodic for the mean if

$$\bar{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{p} E(X_t) = \mu$$

- Recall: Mean square convergence implies convergence in probability

$$Var(\bar{X}_T) = \frac{1}{T^2}\left\{\sum_{t=1}^{T} Var(X_t) + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} Cov(X_t, X_s)\right\}$$

- Sufficient condition for ergodicity of a weakly stationary process: $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$ (see Hamilton, p. 47)

Asymptotic Theory

- Limit Theorems for dependent and/or heterogeneously distributed observations
- Law of Large Numbers
- Central Limit Theorem
- See for instance White (1984): "Asymptotic Theory for Econometricians"