ECON2001 Microeconomics

Lecture Notes

Term 2

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"The object of the science of Political Economy is to ascertain the laws which regulate the production, distribution and consumption of wealth, or the outward things obtained by labour, and needed or desired by man."

Statement by the Council of the University of London,
explanatory of the nature and objects of the institution, 1827

"Political economy or economics is a study of mankind in the ordinary business of life; it examines that part of individual and social action which is most closely connected with the attainment and with the use of the material requisites of wellbeing."

Alfred Marshall, *Principles of Economics*, 1890

Course Guidance

The main body of these lecture notes constitutes the examinable material for this part of the course.

Threaded through that material are:

- **Worked Examples**: These demonstrate how to use the theory in simple practical problems.
- **Case Studies**: These show examples of the theory informing applied empirical investigation.
- **History Notes**: These provide historical context to the development of the ideas.

None of these will be the subject of examination questions in themselves. Nonetheless reading them should enhance your understanding of the course and the examination will test ability to use the theory to solve problems and discuss applications.

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Topic 1 : Budget constraints and consumer demand

Summary: *Consumer choice is restricted by affordability as captured by the budget constraint and this alone restricts the nature of demand functions. Dependence on the parameters of the budget constraint are the basis for several ways to classify demands.*

Budget constraint

Consumers purchase M goods q from within a budget set B of affordable bundles. In the standard model, prices p are constant and total spending has to remain within budget so that $p'q \leq y$ where y is total budget. Maximum affordable quantity of any commodity is y/p_i and slope $\partial q_i/\partial q_j|_B = -p_j/p_i$ is constant and independent of total budget.

In practical applications budget constraints are frequently kinked or discontinuous as a consequence, for example, of taxation or non-linear pricing. If the price of a good rises with the quantity purchased (say because of taxation above a threshold) but without any discontinuity then the budget set is convex whereas if it falls (say because of a bulk buying discount) or there is a jump (say because a price change applies to units below the threshold) then the budget set is not convex.

Marshallian demands

The consumer's chosen quantities written as a function of y and p are the *Marshallian* or *uncompensated* demands $q = f(y, p)$

Consider the effects of changes in y and p on demand for, say, the ith good:

- total budget y
  - the path traced out by demands as y increases is called the *income expansion path* whereas the graph of $f_i(y, p)$ as a function of y is called the *Engel curve*
  - we can summarise dependence in the total budget elasticity

$$\epsilon_i = \frac{y}{q_i}\frac{\partial q_i}{\partial y} = \frac{\partial \ln q_i}{\partial \ln y}$$

  - if demand for a good rises with total budget, $\epsilon_i > 0$, then we say it is a *normal* good and if it falls, $\epsilon_i < 0$, we say it is an *inferior* good
  - if budget share of a good, $w_i = p_i q_i/y$, rises with total budget, $\epsilon_i > 1$, then we say it is a *luxury* or *income elastic* and if it falls, $\epsilon_i < 1$, we say it is a *necessity* or *income inelastic*

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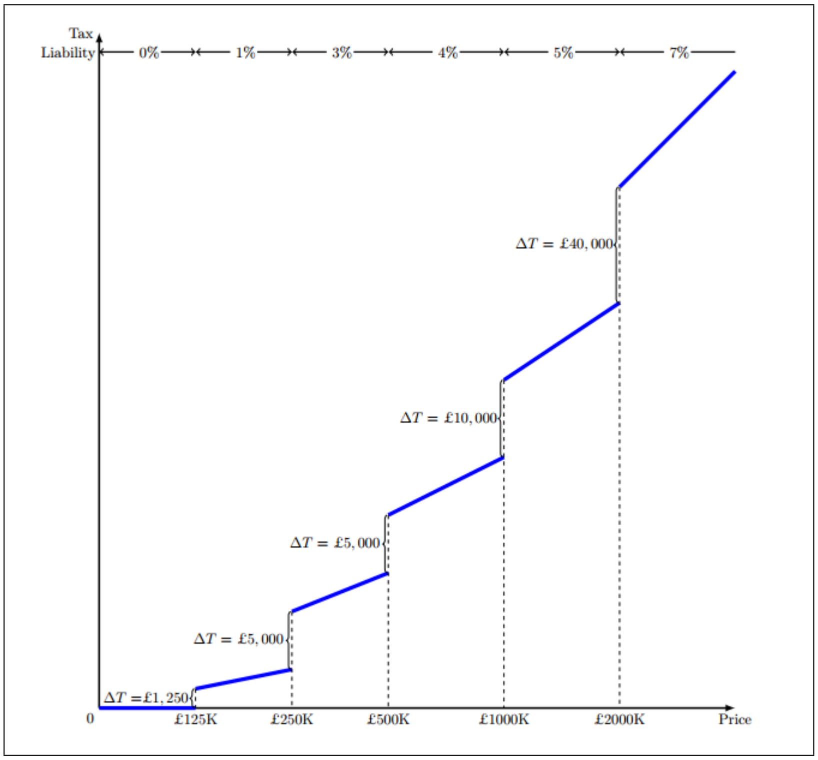
Case Study 1 : Budget constraints: Stamp Duty Land Tax

Residential property transactions in England are subject to a tax known as stamp duty land tax. Prior to late 2014, if the value of the transaction was below £125K the transaction was exempt but, once it exceeded that value, tax was due at 1% on the whole of the value of the transaction. This meant that as the value passed £125K not only did the after-tax price of owner-occupied housing increase but also the tax that was liable jumped by £1.25K. At £250K another threshold was passed at which the rate of tax increased to 3%, again on the whole of the value so that there was a jump of £5K in the liability. There were further discrete jumps for similar reasons at higher values. When translated into a budget constraint between housing and other wealth this created jumps (or "notches") at these points and there is evidence that house sales showed bunching at values just below these notches. Objections to the "badly designed" form of the tax led to the announcement of reforms smoothing out the schedule in the Autumn Statement of 2014.



[Source: M. Best and H. Kleven, 2013, Housing Market Responses to Transaction Taxes: Evidence From Notches and Stimulus in the UK, LSE Working Paper.]

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- own price p_i
  - the path traced out by demands as p_i increases is called the *offer curve* whereas the graph of $f_i(y, p)$ as a function of p_i is called the *demand curve*
  - we can summarise dependence in the (uncompensated) own price elasticity
$$\eta_{ii} = \frac{p_i}{q_i}\frac{\partial q_i}{\partial p_i} = \frac{\partial \ln q_i}{\partial \ln p_i}$$
  - if uncompensated demand for a good rises with own price, $\eta_{ii} > 0$, then we say it is a *Giffen* good
  - if budget share of a good rises with price, $\eta_{ii} > -1$, then we say it is *price inelastic* and if it falls, $\eta_{ii} < -1$, we say it is *price elastic*
- other price $p_j,\ j \neq i$
  - we can summarise dependence in the (uncompensated) cross price elasticity
$$\eta_{ij} = \frac{p_j}{q_i}\frac{\partial q_i}{\partial p_j} = \frac{\partial \ln q_i}{\partial \ln p_j}$$
  - if uncompensated demand for a good rises with the price of another, $\eta_{ij} > 0$, then we can say it is an (uncompensated) *substitute* whereas if it falls with the price of another, $\eta_{ij} < 0$, then we can say it is an (uncompensated) *complement*. These are not the best definitions of complementarity and substitutability however since they may not be symmetric, in other words q_i could be a substitute for q_j while q_j is a complement for q_j. A better definition, guaranteed to be symmetric, is one based on the concept of compensated demands to be introduced below.

Adding up

We know that demands must lie within the budget set:

$$p'f(y, p) \leq y.$$

If consumer spending exhausts the total budget then this holds as an equality,

$$p'f(y, p) = y,$$

which is known as *adding up*.

If we differentiate with respect to y then we get a property known as *Engel aggregation*

$$\sum_i p_i \frac{\partial f_i}{\partial y} = \sum_i w_i \epsilon_i = 1.$$

By adding up

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Case Study 2 : Engel Curves for Food

The extent to which nutrition responds to income in poor countries is an important issue for poverty reduction policy that is tied in with the nature of food demand. The figure below shows the estimated elasticity of calorie intake to per capita expenditure for households in rural Maharashtra, India in the mid 1980s. The mean expenditure share for food is 0.73 in the poorest 10% of households and 0.54 in the top 10% suggesting that food is a normal good and a necessity in economic terms. Compatibly with this we see a total budget elasticity for calorie intake between 0 and 1 across the range of expenditures observed with some evidence of elasticity diminishing as total outlay increases.

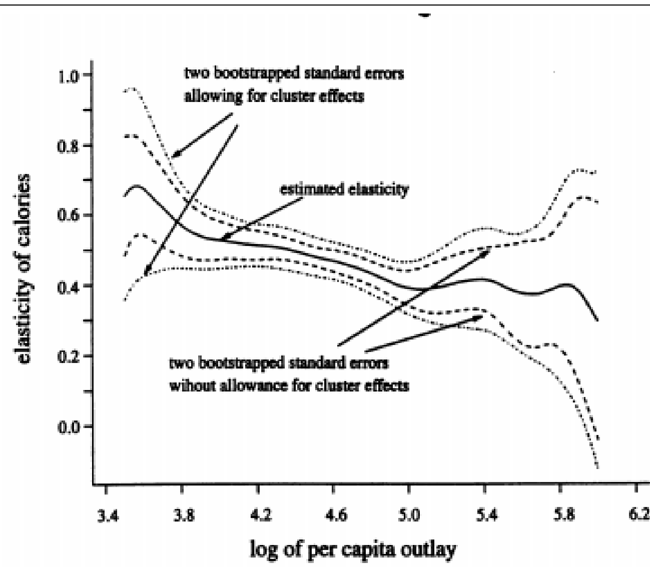

FIG. 3.—Elasticity of per capita calories to per capita expenditure, Maharashtra, India, 1983.

[Source: S. Subramaniam and A. Deaton, 1996, The Demand for Food and Calories, *Journal of Political Economy* 104, 133-162.]

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- not all goods can be inferior
- not all goods can be luxuries
- not all goods can be necessities

Also certain specifications are ruled out for *demand systems*. It is not possible, for example, for all goods to have constant income elasticities unless these elasticities are all 1. Otherwise $p_i q_i = A_i y^{\alpha_i}$, say, and $1 = \sum_i A_i y^{\alpha_i - 1}$ for all budgets y which is impossible unless all $\alpha_i = 1$. This does not rule out constant elasticities for individual goods.

There are also restrictions on price effects. The property derived by differentiating with respect to an arbitrary price p_j is known as *Cournot aggregation*

$$f_j + \sum_i p_i \frac{\partial f_i}{\partial p_j} = 0 \quad \Rightarrow \quad w_j + \sum_i w_i \eta_{ij} = 0$$

As a consequence, for example, if the price of some good goes up then purchases of some good must be reduced so no good can be a Giffen good unless it has strong complements.

Homogeneity

Multiplying y and p by the same factor does not affect the budget constraint and, if it does not affect motivations for choice within budget sets, choices should not be affected either.

Marshallian demands should therefore be *homogeneous* of degree zero:

$$f(\lambda y, \lambda p) = f(y, p)$$

for any $\lambda > 0$.

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Worked Example A : Adding up and homogeneity

Consider the demand system specified by

$$p_i f_i(y,p) = a_i y + \sum_j b_{ij} p_j + c_i \qquad i = 1,2,\ldots,M$$

for appropriate parameters a_i, b_{ij} and c_i, $i,j = 1,2,\ldots,M$. For obvious reasons this is called the *linear expenditure system*.

Adding up requires $\sum_i p_i f_i(y,p) = y$ for all possible values of y and p. Hence we need

$$\left(\sum_i a_i - 1\right) y + \sum_i \sum_j b_{ij} p_j + \sum_i c_i = 0$$

for all possible values of y and p. This can only be true if

$$\sum_i a_i = 1 \qquad \sum_i b_{ij} = 0 \qquad \sum_i c_i = 0 \qquad j = 1,2,\ldots,M.$$

These are the requirements of adding up.

To satisfy homogeneity requires $f_i(\lambda y, \lambda p) = f_i(y,p)$ for $i = 1,2,\ldots,M$ and any $\lambda > 0$. So

$$a_i \frac{\lambda y}{\lambda p} + \sum_j b_{ij} \frac{\lambda p_j}{\lambda p_i} + c_i \frac{1}{\lambda p_i} = a_i \frac{y}{p} + \sum_j b_{ij} \frac{p_j}{p_i} + c_i \frac{1}{p_i}$$

which is true if and only if $c_i = 0$ for every good.

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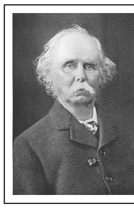
History Note I : Demand curves

The Marshallian demand curve is named after Alfred Marshall (1842-1924), a dominant figure in British economics in the late 19th and early 20th century. Marshall worked on the theory of demand in the 1890s and was the first person to use the notion of elasticity. The first person to write down such an expression for demand was probably his Irish contemporary Francis Ysidro Edgeworth (1845-1926).

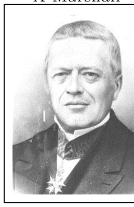
Consumer demand is however a very old notion. The earliest empirical investigation of the relationship between quantity consumed and price arguably goes back to the work of the seventeenth century political arithmetician Gregory King (1648-1712) on the price of corn. Mathematical expression of a relationship between demand and own price can be found in the work of the French engineers Augustin Cournot (1801-77) and Jules Dupuit (1804-66) in the 1830s. The term Giffen good reflects observations attributed by Marshall to the Scottish statistician Robert Giffen (1837-1910) on the demand for bread.

Engel curves are named after the German statistician Ernst Engel (1821-1896) who studied the relationship between the budget share of food and both income and demographic characteristics in household surveys in the 1850s.

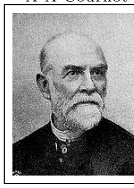
Definitions of complements and substitutes have a complex history that cannot be untangled from ideas about utility discussed below. Early ideas appear in the work of Stanley Jevons (1835-1882) in the 1870s, for example, but the most satisfactory conceptualisation came eventually from the work of John Hicks (1904-89, Nobel 1972) and Roy Allen (1906-83) at LSE in the 1950s.



A Marshall



A A Cournot



E Engel

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Topic 2 : Revealed preference and negativity

Summary: *Revealed preference restrictions can be seen as requirements of consistency for choice across different budget sets. They impose further important restrictions on the nature of consumer demands including justifying the Law of Demand.*

Revealed preference

Suppose the consumer chooses q^A at prices p^A when q^B was no more expensive than q^A:

$$p^{A\prime}q^A \geq p^{A\prime}q^B.$$

We say that q^A is (directly) revealed preferred to q^B. The *Weak Axiom of Revealed Preference (WARP)* says that the consumer would never then choose q^B at any prices p^B such that q^A was affordable:

$$p^{B\prime}q^A \leq p^{B\prime}q^B.$$

The *Strong Axiom of Revealed Preference (SARP)* says that there should be no *cycles* in revealed preference so, for example, we should never find q^A revealed preferred to q^B, q^B revealed preferred to q^C and q^C revealed preferred to q^A (or any longer cycle).

Negativity

Suppose that as prices change from p^A to p^B the consumer is compensated in the sense that their total budget is adjusted to maintain affordability of the original bundle. This is known as *Slutsky compensation*. Then choices before and after satisfy $p^{B\prime}q^A = p^{B\prime}q^B$. But the later choice cannot then have been cheaper at the initial prices or the change would violate WARP. Hence $p^{A\prime}q^A \leq p^{A\prime}q^B$. By subtraction we get *negativity*:

$$(p^B - p^A)'(q^B - q^A) \leq 0.$$

This shows a sense in which price changes and quantity changes must move, on average, in opposite directions if the consumer is compensated.

If we consider the case where the price of only one good changes then we see that this implies that Slutsky-compensated effects of own price rises must be negative. In other words, Slutsky-compensated demand curves necessarily slope down.

Note the weakness of the assumptions needed for this conclusion.

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Case Study 3 : Revealed Preference Tests

Revealed preference ideas can be applied to data on consumer outcomes to test compatibility of behaviour with economic models in ways that are not restrictively tied to any particular specification of preferences. Famulari, for example, took data on over 4000 households in 25 metropolitan districts of the US between 1982 and 1985. She grouped them into 43 demographic types, to control for taste heterogeneity, and looked for revealed preference violations in pairwise comparisons involving consumption of nine categories of items. For the median demographic type her results showed violations in only 0.70% of comparisons. However, violations were much more likely in comparisons involving similar total expenditure. When total expenditures were less than 20% apart the rate of violation was 2.67% and when less than 5% this rose to 5.13%. Violations of this sort could reflect inconsistent behaviour but could also arise from other explanations such as taste change, issues of within-household decision-making, problems of data aggregation or other inappropriateness of the model applied.

[Source: M. Famulari, 1995, A household-based, non-parametric test of demand theory, *Review of Economics and Statistics* 77, 372-382.]

Slutsky equation

The Slutsky-compensated demand function given initial bundle q^A is defined by

$$g(q^A, p) = f(p'q^A, p)$$

- that is to say it is the demand if budget is constantly adjusted to keep initial choice q^A affordable. Differentiating establishes a relationship between Slutsky-compensated and Marshallian price effects

$$\frac{\partial g_i}{\partial p_j} = q_j^A \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial p_j}.$$

The difference is the income effect $q_j^A \partial f_i/\partial y$ and it is positive if the good is normal. Hence, since the Slutsky-compensated effect has been shown to be negative, so must be the Marshallian effect for normal goods. This is the *Law of Demand*.

The Slutsky equation is highly useful. Its importance is that it allows testing of restrictions regarding compensated demands since it shows how to calculate compensated effects from the sort of uncompensated effects estimated in applied demand analysis.

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Worked Example B : Negativity and the Slutsky equation

Consider the linear expenditure system again and assume that homogeneity has been imposed so $c_i = 0$ for all goods. Demand for the ith good is

$$f_i(y,p) = \frac{a_i y}{p_i} + \sum_j b_{ij}\frac{p_j}{p_i}.$$

Using the Slutsky equation the compensated own price effect is

$$\frac{\partial g_i}{\partial p_i} = f_i \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial p_i} = \frac{b_{ii} - (1-a_i)q_i}{p_i}$$

so compensated demand curves slope down only if

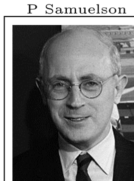
$$f_i(y,p) > \frac{b_{ii}}{1-a_i}.$$

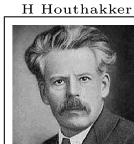
History Note II : Revealed preference, income and substitution effects

The concept of revealed preference was introduced by the the influential American economist Paul Samuelson (1915-2009, Nobel 1970) in 1938. The weak axiom having been proposed by Samuelson, the Dutch-American economist Hendrik Houthakker (1924-2008) introduced the strong axiom in 1950 leading eventually to the tying together of theories of revealed preference and theories of utility maximisation as considered below.

The separation of uncompensated price effects into income and substitution effects was established by the Russian economist and statistician, Evgeny Slutsky (1880-1948) in 1915. Although politically radical in his youth, Slutsky's pathbreaking work in economics came to a halt in the 1930s Soviet Union as he chose to concentrate on less controversial fields of statistics and meteorology.


P Samuelson


H Houthakker


E Slutsky

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Topic 3 : Consumer preferences

Summary: *Specifying which bundles the consumer prefers to which others provides the foundation for a fuller modelling of choice. The nature of these preferences can be restricted by assumptions which vary from what is minimally necessary for a viable theory to assumptions which seriously constrain the nature of choice.*

Preferences

Suppose the consumer has a preference relation $\succsim$ where $q^A \succsim q^B$ means q^A *is at least as good as* q^B. For the purpose of modelling demand this can be construed as an inclination to choose the bundle q^A over the bundle q^B. For modelling welfare effects, the interpretation needs to be strengthened to include a link to consumer wellbeing.

A weak preference relation suffices to define strict preference $\succ$ and indifference $\sim$ if we let

- $\succsim$ and $\precsim$ be equivalent to $\sim$,
- $\succsim$ and $\not\precsim$ be equivalent to $\succ$.

For any bundle q^A we can define

- the weakly preferred set $R(q^A)$ as all bundles q^B such that $q^B \succsim q^A$
- the weakly dispreferred set $L(q^A)$ as all bundles q^B such that $q^A \succsim q^B$
- the indifferent set $I(q^A)$ as all bundles q^B such that $q^B \sim q^A$ (or, in other words, as the intersection of $R(q^A)$ and $L(q^A)$)

Consumer Rationality

We want the preference relation to provide a basis to consistently identify a set of most preferred elements in any possible budget set and for this we need assumptions.

- **Completeness** Either $q^A \succsim q^B$ or $q^B \succsim q^A$. This ensures that choice is possible in any budget set.

- **Transitivity** $q^A \succsim q^B$ and $q^B \succsim q^C$ implies $q^A \succsim q^C$. This ensures that there are no *cycles* in preferences within any budget set.

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Counterexample to Completeness : Consumer preferences are incomplete if, say, there are two goods q_1 and q_2 and $q^A \succsim q^B$ if and only if both $q_1^A \geq q_1^B$ and $q_2^A \geq q_2^B$. With these prefernces, the consumer simply cannot decide between two bundles if each bundle has more of one of the two goods.

Counterexample to Transitivity : Consumer preferences are intransitive if, say, there are two goods q_1 and q_2 and $q^A \succsim q^B$ if and only if either $q_1^A \geq q_1^B$ or $q_2^A \geq q_2^B$. Take bundles q^A and q^C such that both $q_1^A > q_1^C$ and $q_2^A > q_2^C$. Clearly $q^A \succ q^C$. But if we take another bundle q^B such that $q_1^B < q_1^C$ but $q_2^B > q_2^A$ then $q^B \succsim q^A$ and $q^C \succsim q^B$. Hence transitivity is violated.

To take another example, suppose there are three goods and preferences are such that $q^A \succsim q^B$ if and only if the number of goods i for which $q_i^A > q_i^B$ is at least as many as the number of goods for which $q_i^A < q_i^B$. If we take the bundles $q^A = (0, 1, 2)$, $q^B = (2, 0, 1)$ and $q^C = (1, 2, 0)$ then we see that $q^A \succ q^B \succ q^C \succ q^A$ which clearly violates transitivity.

Together these two assumptions are sometimes referred to as *consumer rationality.* They ensure that the preference relation is a *preference ordering.*

To make preferences mathematically well behaved we make the technical assumption:

- **Continuity** If $q^A \succsim q^B$ and $q^B \succsim q^C$ then there is a bundle indifferent to q^B on any path joining q^A to q^C. This rules out discontinuous jumps in preferences.

Counterexample to Continuity : Continuity is violated by the example of *lexicographic* preferences. Say that there are two goods q_1 and q_2 and that the consumer prefers one bundle to another if and only if it either has more of q_1 or the same amount of q_1 and more of q_2. The consumer is indifferent between no bundles that differ in any way and indifferent sets are single points. Given any three different bundles it is easy to draw a path from the least to the most preferred that does not pass through any bundle indifferent to the third, since it is only necessary to avoid the path passing directly through the third bundle itself.

Utility functions

A utility function $u(q)$ is a representation of preferences such that $q_A \succsim q_B$ if and only if $u(q_A) \geq u(q_B)$. A utility function exists if preferences give a continuous ordering.

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Utility functions are not however unique since if $u(q)$ represents certain preferences then any increasing transformation $\phi(u(q))$ also represents the same preferences. We say that utility functions are *ordinal*.

Nonsatiation, monotonicity and convexity

A further assumption rules out consumers ever being fully satisfied:

- **Nonsatiation** Given any bundle there is always some direction in which changing the bundle will make the consumer better off.

If this is true then indifferent sets have no thick regions to them and we can visualise them as indifference curves.

The next assumption takes this further by specifying the direction of increasing preference:

- **Monotonicity** Larger bundles are preferred to smaller bundles.

Counterexample to Nonsatiation and monotonicity : Suppose preferences are represented by utility function $u(q) = -(q-\gamma)'(q-\gamma) = -\sum_i (q_i - \gamma_i)^2$ for some bundle γ. Indifference curves for such preferences are circles around the "bliss point" γ. Nonsatiation is violated since there is a bundle, γ, from which there is no direction in which it is possible to increase consumer satisfaction. Utility is increasing in all quantities provided that $q_i < \gamma_i$ for all i but not otherwise.

Given monotonicity, indifference curves must slope down. This slope is known as the marginal rate of substitution (MRS).

Monotonicity corresponds to increasingness of the utility function $u(q)$. An indifference curve is defined by $u(q)$ being constant and therefore the MRS is given by

$$\text{MRS} = \left.\frac{\mathrm{d}q_2}{\mathrm{d}q_1}\right|_u = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2}$$

which is obviously negative if $\partial u/\partial q_1,\ \partial u/\partial q_2 > 0$.

- **Convexity** $\lambda q^A + (1-\lambda)q^B \succsim q^B$ if $q^A \succsim q^B$ and $1 \geq \lambda \geq 0$.

This says that weakly preferred sets are convex or, equivalently, MRS is diminishing.

Convexity can be interpreted as capturing taste for variety. It says that a consumer will always prefer to mix any two bundles between which they are indifferent. The corresponding property of the utility function is known as *quasiconcavity*.

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Counterexample to Convexity : Suppose preferences are represented by utility function $u(q) = q'q = \sum_i q_i^2$. Preferences are monotonic but indifference curves are quarter-circles centred on the origin and MRS is increasing rather than diminishing.

Homotheticity and quasilinearity

Homotheticity and quasilinearity are both strong restrictions expressing different ways in which indifference curves within an indifference map can share a common shape.

- **Homotheticity** If $q^A \sim q^B$ then $\lambda q^A \sim \lambda q^B$ for any $\lambda > 0$.

Graphically this means that higher indifference curves can be constructed from lower ones by magnifying from the origin. This is a strong restriction that would rarely be made in practice but it is useful to consider as a reference case. It is not a restriction on the shape of any one indifference curve but on the relationship between indifference curves within an indifference map.

If preferences are homothetic then marginal rates of substitution are constant along rays through the origin. This is only true for homothetic preferences and this is usually an easy way to check whether given preferences are homothetic.

If there exists a homogeneous utility representation $u(q)$ where $u(\lambda q) = \lambda u(q)$ then preferences can be seen to be homothetic. Since increasing transformations preserve the properties of preferences, then any utility function which is an increasing function of a homogeneous utility function also represents homothetic preferences. In fact all utility functions representing homothetic preferences are of this form.

- **Quasilinearity (with respect to the first good)** If $\begin{pmatrix} q_1^A \\ q_2^A \\ q_3^A \\ \vdots \end{pmatrix} \sim \begin{pmatrix} q_1^B \\ q_2^B \\ q_3^B \\ \vdots \end{pmatrix}$ then $\begin{pmatrix} q_1^A + \lambda \\ q_2^A \\ q_3^A \\ \vdots \end{pmatrix} \sim \begin{pmatrix} q_1^B + \lambda \\ q_2^B \\ q_3^B \\ \vdots \end{pmatrix}$ for any λ.

Quasilinearity is another strong restriction based on a similar idea. It says that adding the same amount to one particular good preserves indifference. This means that higher indifference curves are parallel translations of lower ones. In this case, marginal rates of substitution are constant along lines parallel to axes.

If there exists a utility representation $u(q)$ such that $u(q_1, q_2, \dots) = q_1 + F(q_2, q_3, \dots)$, say, then preferences are quasilinear. This is also true of any utility functions which are increasing transformations of functions with this property.

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Worked Example C : Preferences

Some examples of preference structures illustrate some of these properties:

- **Perfect substitutes** $u(q) = \sum_i \alpha_i q_1$: The MRS between the ith and jth good is $-\alpha_i/\alpha_j$ and is constant. In two dimensions, indifference curves are parallel straight lines. Weakly preferred sets are therefore (weakly but not strictly) convex. These are the only preferences which are homothetic *and* quasilinear.

- **Perfect complements** $u(q) = \min[\alpha_1 q_1, \alpha_2 q_2, \ldots, \alpha_M q_M]$: In two dimensions, indifference curves are L-shaped with the kinks lying on a ray through the origin of slope α_1/α_2. Weakly preferred sets are convex. These preferences are homothetic but not quasilinear.

- **Cobb-Douglas**: $u(q) = \sum_i \alpha_i \ln q_i$: Preferences are homothetic, indifference curves are smooth and weakly preferred sets are convex. MRS is $-\alpha_i q_j/\alpha_j q_i$. Typically, the expression for utility would be scaled so that $\sum_i \alpha_i = 1$.

  If there are two goods and we look along the indifference curve corresponding to utility υ then $q_2 = \mathrm{e}^{\upsilon/(1-\alpha)} q_1^{-\alpha/(1-\alpha)}$. So the MRS is

  $$MRS = -\frac{\alpha}{1-\alpha}\mathrm{e}^{\upsilon/(1-\alpha)} q_1^{-1/\alpha}$$

  which is diminishing in q_1.

- **Stone-Geary**: $u(q) = \sum_i \alpha_i \ln(q_i - \gamma_i)$: This is a simple modification to Cobb-Douglas under which preferences are no longer homothetic but indifference curves remain smooth and MRS $-\alpha_i (q_j - \gamma_j)/\alpha_j (q_i - \gamma_i)$ is still diminishing.

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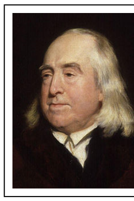
History Note III : Utility

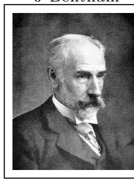
The relevance of "value in use" to demand behaviour is discussed in many works of classical economics including those of Adam Smith (1723-1790), David Ricardo (1772-1823) and Jean-Baptiste Say (1767-1832). The notion of quantifying utility appears as a solution to paradoxes of choice under uncertainty in the mid-eighteenth century writings of Daniel Bernoulli (1700-82).

Towards the end of that century, utility became the central principle in the system of ethics developed by the British philosopher, Jeremy Bentham (1748-1832) (whose dressed skeleton can be seen in UCL cloisters). The link from Benthamite ideas and associated notions of consumer utility to theories of demand is discussed in the notes on consumer choice below but is particularly explicit, for example, in Stanley Jevons' (1835-82) theory of demand.

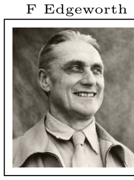
Indifference curves as a way of representing consumer preferences were introduced by the Irish economist Francis Ysidro Edgeworth (1845-1926) in the 1880s. To nineteenth century developers of these ideas, utility was thought of as cardinal - measureable and comparable. The realisation that only the preference ordering matters to demand behaviour was made by the American economist Irving Fisher (1867-1947) and Italian social philosopher Vilfredo Pareto (1848-1923) in the 1890s. The axiomatic approach to defining preferences originates in the work of the Norwegian economist Ragnar Frisch (1895-1973, Nobel 1969) in 1926. The conditions for existence of utility functions to represent preferences were explored by the Norwegian econometrician Herman Wold (1908-92) and French economist Gérard Debreu (1921-2004, Nobel 1983) in the 1940s and 1950s, Debreu for example introducing the problematic case of lexicographic preferences.



J Bentham



F Edgeworth



R Frisch

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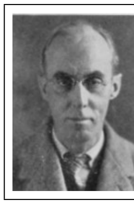
History Note IV : Preferences

The Cobb-Douglas form for preferences is an especially simple form that implies constant budget shares. It derives originally from the production function specified for work on the US by the American senator and economist Paul Douglas (1892-1976) and mathematician Charles Cobb (1875-1949). Preferences with perfect complements are sometimes referred to as Leontief preferences after their use by American economist and Russian emigré Wassily Leontief (1906-99, Nobel 1973), pioneer of input-output analysis. These are both examples of homothetic preferences, homotheticity being an idea used in the work of Ragnar Frisch (1895-1973, Nobel 1969) but first defined formally in the work on production economics of Ronald Shephard (1912-82).

Stone-Geary preferences adapt the Cobb-Douglas form to give a simple departure from homotheticity. As worked examples below show, this results in consumer demands identical to the linear expenditure system. First suggested by the Irish economist Roy Geary (1896-1983), these preference were first applied to data by the British applied economist Richard Stone (1913-91, Nobel 1984).



C Cobb

P Douglas

R Stone

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Topic 4 : Consumer choice

Summary: *If the consumer consistently chooses the most preferred among affordable bundles then it is possible to give a complete description of the properties that demands must satisfy. To see this, it is illuminating to think of consumers also as expenditure minimisers.*

Consumer optimisation

Suppose that the consumer chooses q so as to best satisfy their preferences within the budget set, which is to say to solve

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y.$$

Then choices satisfy *WARP*. In fact, the assumption of optimising behaviour is equivalent to *SARP*. Utility maximising choices therefore satisfy negativity, homogeneity and, assuming non-satiation, adding up.

Tangency condition

If all goods are chosen in positive quantities and preferences are convex then the solution to the consumer's optimisation problem is at a tangency between an indifference curve and the boundary of the budget set.

This is true even for non-linear budget sets. However if the budget set is linear (or indeed simply convex) then we know that such a tangency is unique so finding one guarantees that we have found the best choice for the consumer. If the budget set is not convex, on the other hand, then there can be multiple tangencies and the optimum can typically be found only by comparing the level of utility at each of them.

The nature of Marshallian demands can then be inferred by moving the budget constraint to capture changes in y and p and tracing out movement of the tangency.

Demand under homotheticity and quasilinearity

As income increases. slopes of budget constraints do not change. Income expenditure paths traced out by the tangencies as incomes are increased therefore all occur at points with the same MRS.

Homotheticity and quasilinearity are each characterised by the nature of paths along which MRS is constant and therefore each give rise to income expansion paths of particular shapes.

- **Homotheticity** MRS is constant along rays through the origin so income expansion paths are rays through the origin. Quantities consumed are proportional to total budget y given prices and budget shares are independent of y.

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- **Quasilinearity** MRS is constant along lines parallel to one of the axes so income expansion paths are parallel to one of the axes. Quantities demanded of all but one of the goods are independent of y (so long as all goods are chosen in positive quantities).

Constrained optimisation

Mathematically, Marshallian demands solve

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y$$

This can be solved by finding stationary points of the Lagrangean

$$u(q) - \lambda(p'q - y).$$

For interior solutions, first order conditions require

$$\frac{\partial u}{\partial q_i} = \lambda p_i$$

which imply

$$\text{MRS} = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.$$

This is a confirmation of the tangency condition - the slope of indifference curve and budget constraint are equal at interior solutions.

Expenditure minimisation

In comparison with the utility maximisation problem

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y$$

consider now the alternative problem of minimising the expenditure necessary to reach a given utility

$$\min_q p'q \qquad \text{s.t.} \qquad u(q) \geq \upsilon$$

Solutions to this problem are *Hicksian* or *compensated* demands $q = g(\upsilon, p)$.

The problem can be solved by finding stationary points of the Lagrangean

$$p'q - \mu(u(q) - \upsilon).$$

which, for interior solutions, gives first order conditions requiring

$$p_i = \mu \frac{\partial u}{\partial q_i}$$

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and therefore
$$\text{MRS} = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.$$
Note that this is exactly the same tangency condition encountered in solving the utility maximisation problem.

We can define important functions giving the values of the two problems. The value of the maximised utility function as a function of y and p can be found by substituting the Marshallian demands back into the direct utility function $u(q)$. We call this the *indirect utility function*

$$\nu(y,p) = u(f(y,p)) = \max_q u(q) \text{ s.t. } p'q \leq y$$

The value of the minimised cost as a function of υ and p can be found by costing the Hicksian demands. We call this the *expenditure function* or *cost function*
$$c(\upsilon,p) = p'g(\upsilon,p) = \min_q p'q \text{ s.t. } u(q) \geq \upsilon.$$

The link between the two problems can be expressed by noting the equality of the quantities solving the two problems

$$f(c(\upsilon,p),p) = g(\upsilon,p) \qquad f(y,p) = g(\nu(y,p),p)$$

or noting that $\nu(y,p)$ and $c(\upsilon,p)$ are inverses of each other in their first arguments
$$\nu(c(\upsilon,p),p) = \upsilon \qquad c(\nu(y,p),p) = y.$$

Expenditure function

The expenditure function $c(\upsilon,p)$ has the properties that

- it is increasing in every price in p (assuming the good is consumed) and in υ
- it is homogeneous of degree one in prices p, $c(\upsilon,\lambda p) = \lambda c(\upsilon,p)$. The Hicksian demands are homogeneous of degree zero so the total cost of purchasing them must be homogeneous of degree one
$$c(\upsilon,\lambda p) = \lambda p'g(\upsilon,\lambda p) = \lambda p'g(\upsilon,p) = \lambda c(\upsilon,p)$$
- it is concave in prices p, $c(\upsilon,\lambda p^A + (1-\lambda)p^B) \geq \lambda c(\upsilon,p^A) + (1-\lambda)c(\upsilon,p^B)$. This follows simply from the fact that the cost-minimising choices at prices p^A and p^B no longer minimise costs at $\lambda p^A + (1-\lambda)p^B$.

Indirect utility function

The properties of the indirect utility function $\nu(y,p)$ correspond exactly to those of the expenditure function given that the two are inverses of each other. In particular it is

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- increasing in y and decreasing in each element of p
- homogeneous of degree zero in y and p:

$$v(\lambda y, \lambda p) = v(y, p)$$

  This should be apparent also from the homogeneity properties of Marshallian demands.

  (The property corresponding to concavity of the expenditure function is known as *quasiconvexity* of the indirect utility function.)

Shephard's Lemma

Among the most useful features of these functions are their simple links to the associated demands. For example, it is possible to get from the expenditure function to the Hicksian demands simply by differentiating.

Since $c(\upsilon, p) = p' g(\upsilon, p)$

$$\begin{aligned} \frac{\partial c(\upsilon, p)}{\partial p_i} &= g_i(\upsilon, p) + p' \frac{\partial g(\upsilon, p)}{\partial p_i} \\ &= g_i(\upsilon, p) + \mu \sum_i \frac{\partial u}{\partial q_i} \frac{\partial g(\upsilon, p)}{\partial p_i} \\ &= g_i(\upsilon, p) \end{aligned}$$

using the first order condition for solving the cost minimisation problem and the fact that utility is held constant in that problem.

This is known as *Shephard's Lemma*. Its importance is that it allows compensated demands to be deduced simply from the expenditure function.

Roy's Identity

Since $v(c(\upsilon, p), p) = \upsilon$

$$\begin{aligned} \frac{\partial v(y, p)}{\partial p_i} + \frac{\partial v(y, p)}{\partial y} \frac{\partial c(u, p)}{\partial p_i} &= 0 \\ \Rightarrow -\frac{\partial v(y, p)/\partial p_i}{\partial v(y, p)/\partial y} &= g_i(v(y, p), p) \\ &= f_i(y, p) \end{aligned}$$

using Shephard's Lemma.

This is Roy's identity and shows that uncompensated demands can be deduced simply from the indirect utility function by differentiation.

In many ways it is easier to derive a system of demands by beginning with well specified indirect utility function $v(y, p)$ or expenditure function $c(\upsilon, p)$ and differentiating than by solving a consumer problem directly given a direct utility function $u(q)$.

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Slutsky equation, again

Since $g(\upsilon, p) = f(c(\upsilon, p), p)$

$$\begin{aligned}\frac{\partial g_i(\upsilon,p)}{\partial p_j} &= \frac{\partial f_i(y,p)}{\partial p_j} + \frac{\partial f_i(y,p)}{\partial y}\frac{\partial c(\upsilon,p)}{\partial p_j} \\ &= \frac{\partial f_i(y,p)}{\partial p_j} + \frac{\partial f_i(y,p)}{\partial y} f_j(y,p)\end{aligned}$$

Notice that this is the same as the Slutsky equation derived earlier for Slutsky-compensated demands. Hicks-compensated price derivatives are the same as Slutsky-compensated price derivatives since the two notions of compensation coincide at the margin.

This means the results derived earlier can simply be carried over. Hicksian demands therefore also satisfy *negativity* at the margin. In particular

$$\frac{\partial g_i(\upsilon,p)}{\partial p_i} \leq 0.$$

Another way to look at this is to see that it follows directly from concavity of the expenditure function and the fact, from Shephard's Lemma, that compensated demands are derivatives of the expenditure function..

(Negativity can actually be stated slightly more strongly than this, involving also restrictions on cross-price effects, but this is the most important implication).

Slutsky symmetry

There is one more property of Hicksian demands that can now be deduced. From Shephard's Lemma

$$\frac{\partial g_i(\upsilon,p)}{\partial p_j} = \frac{\partial^2 c(\upsilon,p)}{\partial p_i p_j} = \frac{\partial g_j(\upsilon,p)}{\partial p_i}$$

Compensated cross-price derivatives are therefore also *symmetric*. Holding utility constant, the effect of increasing the price of one good on the quantity chosen of another is numerically identical to the effect of increasing the price of the other good on the quantity chosen of the first good.

This shows that notions of complementarity and substitutability are consistent between demand equations if using compensated demands and provides a strong argument for defining complementarity and substitutability in such terms. This would *not* be true if using uncompensated demands because income effects are not symmetric.

Integrability

To summarise, if demands are consistent with utility maximising behaviour then they have the following properties

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- Adding up: Demands must lie on the budget constraint and therefore

$$\begin{aligned} p'f(y,p) &= y \\ p'g(\upsilon,p) &= c(\upsilon,p) \end{aligned}$$

- Homogeneity: Increasing all incomes and prices in proportion leaves the budget constraint and therefore demands unaffected

$$\begin{aligned} f_i(y,p) &= f_i(\lambda y, \lambda p) \\ g_i(\upsilon,p) &= g_i(\upsilon, \lambda p) \end{aligned}$$

- Negativity of compensated own price effects: In particular, a compensated increase in any good's price can only reduce demand for that good

$$\partial g_i/\partial p_i = \partial f_i/\partial p_i + f_i \partial f_i/\partial y \leq 0$$

- Symmetry of compensated cross price effects:

$$\partial g_i/\partial p_j = \partial g_j/\partial p_i$$

If demands satisfy these restrictions then there is a utility function $u(q)$ which they maximise subject to the budget constraint. We say demands are *integrable*. These are *all* the restrictions required by consumer optimisation.

We know a system of demands is integrable if any of the following hold

- They were derived as solutions to the utility maximisation or expenditure minimisation problem given a well specified direct utility function
- They were derived by Shephard's Lemma from a well specified cost function or they were derived by Roy's Identity from a well specified indirect utility function
- They satisfy adding up, homogeneity, symmetry and negativity.

The connections between the concepts discussed can be summarised in the diagram below:

$$\begin{array}{ccc}
\text{Utility maximisation problem} & & \text{Expenditure minimisation problem} \\
\max_q u(q) \quad s.t. \quad p'q \leq y & \longleftrightarrow & \min_q \sum p'q \quad s.t. \quad u(q) \geq \upsilon \\
\downarrow & & \downarrow \\
\text{Uncompensated demands} & & \text{Compensated demands} \\
q_i = f_i(y,p) \qquad i = 1, 2, ... & \longleftrightarrow & q_i = g_i(\upsilon,p) \qquad i = 1, 2, ... \\
\updownarrow & & \updownarrow \\
\text{Indirect utility function} & & \text{Expenditure function} \\
v(y,p) & \longleftrightarrow & c(\upsilon,p)
\end{array}$$

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Case Study 4 : Demand Systems

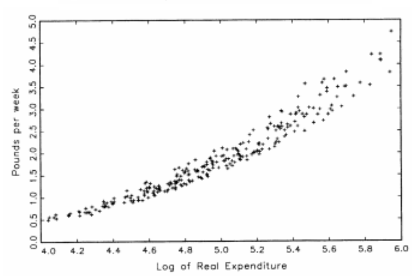
Banks, Blundell and Lewbel use UK data to estimate parameters of a particular demand specification called the Quadratic Almost Ideal demand system. This is an example of a general class of demand systems characterised by a quadratic relationship between budget shares and the log of total expenditure, a functional form that appears to fit data reasonably well. The estimates below are for a five good system (food, fuel, clothing, alcohol, other goods) for nondurable spending and are calculated for a relatively homogeneous sample of married couples in the South East over the years 1970-1986, imposing homogeneity. Slutsky symmetry and negativity are tested and found to be statistically acceptable given the estimated parameter values for all except a few high spending households.
Because the associated indirect utility function is known it is possible to calculate the exact welfare effects of reforms which change the prices of goods. The authors of the study are therefore able to calculate estimated welfare losses, measured in equivalent expenditure terms, for extension of Value Added Tax to all forms of clothing for households in the sample as in the figure below. Note that, rather than simply costing losses given existing expenditure patterns, such estimated losses allow for behavioural responses.

ESTIMATED DEMAND ELASTICITIES

	Food	Fuel	Clothing	Alcohol	Other
			Compensated		
Food	−0.7817 (0.15)	0.1068 (0.06)	0.2919 (0.13)	0.2914 (0.08)	0.0888 (0.11)
Fuel	0.4653 (0.29)	−0.7669 (0.22)	−0.4590 (0.29)	0.6330 (0.19)	0.1274 (0.29)
Clothing	1.0423 (0.41)	−0.3805 (0.17)	−0.9606 (0.42)	0.0571 (0.23)	0.2419 (0.34)
Alcohol	1.5341 (0.42)	0.7588 (0.21)	0.0580 (0.41)	−1.6492 (0.44)	−0.7017 (0.59)
Other	0.0490 (0.06)	0.0163 (0.03)	0.0390 (0.07)	−0.0769 (0.06)	−0.0277 (0.10)

Note: Standard errors are in parentheses.

WELFARE LOSS: QUAI MODEL, 17.5% TAX ON CLOTHING



[Source: J. Banks, R. Blundell and A. Lewbel, 1997, Quadratic Engel Curves and Consumer Demand, *Review of Economics and Statistics* 79, 527-539.]

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Worked Example D : Demands under Stone-Geary preferences

Suppose the direct utility function is $u(q) = \sum_i \alpha_i \ln(q_i - \gamma_i)$ with $\sum_i \alpha_i = 1$. This is Stone-Geary preferences as introduced earlier.
The tangency condition defining optimum consumer choice is

$$MRS = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{\alpha_i}{\alpha_j}\left(\frac{q_j - \gamma_j}{q_i - \gamma_i}\right) = -\frac{p_i}{p_j} \qquad i, j = 1, 2, \ldots, M$$

Hence for each good $p_i q_i = p_i \gamma_i + \frac{\alpha_i}{\alpha_1}(p_1 q_1 - p_1 \gamma_1)$ and substituting into the budget constraint

$$y = \sum_i p_i q_i = \sum_i p_i \gamma_i + \frac{1}{\alpha_1}(p_1 q_1 - p_1 \gamma_1).$$

So $p_1 q_1 = p_1 \gamma_1 + \alpha_1 (y - \sum_i p_i \gamma_i)$ and by a similar argument we establish all Marshallian demands

$$f_i(y, p) \quad = \quad \gamma_i + \frac{\alpha_i}{p_i}\left(y - \sum_j p_j \gamma_j\right) \qquad i = 1, 2, \ldots, M.$$

Budget shares are $w_i = \alpha_i + \left(p_i \gamma_i - \alpha_i \sum_j p_j \gamma_j\right)/y$ so those goods are necessities for which $p_i \gamma_i$ is greater than $\alpha_i \sum_j p_j \gamma_j$.
Substituting into the direct utility function gives the indirect utility function

$$\begin{aligned} v(y, p) &= \sum_i \alpha_i \ln(f_i(y, p) - \gamma_i) \\ &= \ln\left(y - \sum_i p_i \gamma_i\right) - \sum_i \alpha_i \ln p_i + \sum_i \alpha_i \ln \alpha_i \end{aligned}$$

Inverting $v(y, p)$ in y then gives the expenditure function

$$c(\upsilon, p) = \sum_i p_i \gamma_i + \mathrm{e}^{\upsilon} \Pi_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$$

The compensated demands are then most easily found by differentiating $c(\upsilon, p)$ (using Shephard's Lemma) or by substituting $c(\upsilon, p)$ into the uncompensated demands

$$g_i(\upsilon, p) \quad = \quad \gamma_i + \frac{\alpha_i}{p_i}\mathrm{e}^{\upsilon} \Pi_j \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j} \qquad i = 1, 2, \ldots, M.$$

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Worked Example E : Demands under quasilinear preferences

Suppose direct utility takes the quasilinear form

$$u(q_1, q_2) = \alpha \ln q_1 + q_2$$

The tangency condition is

$$MRS \quad = \quad -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} = -\frac{\alpha}{q_1} = -\frac{p_1}{p_2} \qquad \Rightarrow p_1 q_1 = \alpha p_2$$

This defines an interior optimum assuming $y > \alpha p_2$.
Hence, directly and by substituting into the budget constraint, uncompensated demands are

$$\begin{aligned} f_1(y,p) &= \alpha p_2/p_1 \\ f_2(y,p) &= (y/p_2) - \alpha \end{aligned}$$

Uncompensated demand for the first good is independent of total budget y.
Substituting into the direct utility function gives the indirect utility function

$$\begin{aligned} v(y,p) &= \alpha \ln f_1(y,p) + f_2(y,p) \\ &= \alpha \ln(\alpha p_2/p_1) + (y/p_2) - \alpha \end{aligned}$$

Inverting in y gives the expenditure function

$$c(\upsilon, p) = p_2(\upsilon - \alpha \ln(\alpha p_2/p_1) + \alpha)$$

Differentiating or substituting then gives the compensated demands

$$\begin{aligned} g_1(\upsilon,p) &= \frac{\partial c(\upsilon,p)}{\partial p_1} = f_1(c(\upsilon,p),p) = \alpha p_2/p_1 \\ g_2(\upsilon,p) &= \frac{\partial c(\upsilon,p)}{\partial p_2} = f_2(c(\upsilon,p),p) = \upsilon - \alpha \ln(\alpha p_2/p_1) \end{aligned}$$

The compensated demand for the first good is independent of υ.

If $y \leq \alpha p_2$ then nothing is spent on the second good, $f_1(y,p) = y/p_1$ and $f_2(y,p) = 0$. Hence $v(y,p) = \alpha \ln(y/p_1)$ and $c(\upsilon,p) = p_1 e^{\upsilon/\alpha}$.

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Worked Example F : Demands under perfect complements

Suppose direct utility is

$$u(q_1, q_2) = \min[a_1 q_1, a_2 q_2]$$

Goods are perfect complements and at the optimum

$$\begin{aligned} a_1 q_1 &= a_2 q_2 \\ \Rightarrow a_1 f_1(y, p) &= a_2 f_2(y, p) = \frac{a_1 a_2 y}{a_2 p_1 + a_1 p_2} \end{aligned}$$

Substituting into the direct utility function gives the indirect utility function

$$\begin{aligned} \upsilon(y, p) &= \min[f_1(y, p), f_2(y, p)] \\ &= \frac{a_1 a_2 y}{a_2 p_1 + a_1 p_2} \end{aligned}$$

Inverting in y gives the expenditure function

$$c(\upsilon, p) = \frac{1}{a_1 a_2} \upsilon(a_2 p_1 + a_1 p_2)$$

Differentiating or substituting gives the compensated demands

$$g_i(\upsilon, p) = \frac{\partial c(\upsilon, p)}{\partial p_i} = \upsilon / a_i \qquad i = 1, 2$$

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Worked Example G : Integrability

Conditions for adding up, homogeneity and negativity for the linear expenditure system have been discussed above. Now consider symmetry. Using the Slutsky equation again

$$\frac{\partial g_i}{\partial p_j} = f_j \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial p_j} = \frac{b_{ij} + a_i q_j}{p_i} \qquad i \neq j.$$

Hence $\partial g_i/\partial p_j = \partial g_i/\partial p_j$ requires

$$p_j b_{ij} + a_i a_j y + a_i \sum_k b_{jk} p_k = p_i b_{ji} + a_j a_i y + a_j \sum_k b_{ik} p_k.$$

By equating terms on each side for each price, this can be seen to be true if

$$\begin{aligned} (a_i - 1) b_{ji} &= a_j b_{ii} \qquad i \neq j \\ a_i b_{jk} &= a_j b_{ik} \qquad i \neq j \neq k \end{aligned}$$

Introduce a new notation γ_i for $b_{ii}/(1 - a_i)$. Then the first condition implies $b_{ij} = -a_i \gamma_j$ for all $i \neq j$ in which case the second set of conditions are all automatically satisfied. So the system reduces to

$$p_i f_i(y, p) = p_i \gamma_i + a_i \left(y - \sum_j p_j \gamma_j \right) \qquad i = 1, 2, \ldots, M$$

It is clear that the linear expenditure system with integrability restrictions imposed is exactly the same as the Stone-Geary demand system with $a_i = \alpha_i$.

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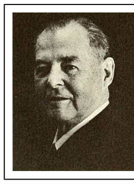
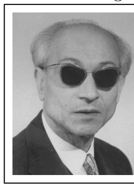
History Note V : Consumer Choice

The first utility-based derivation of a demand curve is in the work in the 1840s of the French engineer Jules Dupuit (1804-66) on the valuation of bridges where he identifies willingness to pay with marginal utility. The Prussian economist Hermann Gossen (1810-1858) published work in the 1850s which went almost unnoticed despite recognising the fundamental principle that optimal budgetary allocation would equate marginal utilities of spending across goods. It was in the 1860s and 1870s that the economists Stanley Jevons (1835-1882) working in England (and latterly Professor at UCL), Carl Menger (1840-1921) in Austria and Léon Walras (1834-1910) in Switzerland all arrived independently at an understanding of marginalist principles. Jevons and Walras were enthusiastic advocates of mathematical formalisation whereas Menger was determinedly opposed.

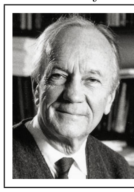
Manipulation of indirect utility and expenditure functions is mainly a twentieth century approach, though the Italian engineer Giovanni Antonelli (1858-1944) appears to have explored ideas more or less unnoticed in the 1880s, including effectively defining an indirect utility function. These notions reappeared in the 1930s and 1940s in the work of American economist Harold Hotelling (1895-1973) and French economist René Roy (1894-1977) and mathematician Jean Ville (1910-1989). The expenditure function was introduced in the 1950s by American economist Lionel McKenzie (1919-2010), the parallel between expenditure minimisation and cost minimisation in production economics allowing him to pull across the lemma proved by Ronald Shephard (1912-82) establishing the link to cost-minimising demands. These are designated as Hicksian demands after John Hicks (1904-89, Nobel 1972) who was instrumental in replacing Slutsky's notion of compensation with that of holding utility constant.



H Hotelling



R Roy



L McKenzie

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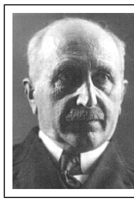
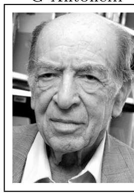


G Antonelli



L Hurwicz

H Uzawa

History Note VI : Integrability

Negativity and symmetry as required properties of demands were proved by Evgeny Slutsky (1880-1948) in 1915. The question of what properties of demand are sufficient for compatibility with utility maximisation and of how to work back from demands to preferences was first raised by Giovanni Antonelli (1858-1944) in the 1880s who recognised that the answer would involve integration from implied marginal rates of substitution.

The conclusive solution to the question establishing adding up, homogeneity, negativity and symmetry as jointly sufficient conditions for integrability came in the 1960s work of the Polish-American economist Leonid Hurwicz (1917-2008, Nobel 2007), the Japanese economist Hirofumi Uzawa (1928-2014) and American economist Marcel Richter (1932-2014).

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Topic 5 : Consumer welfare

Summary: *The view of consumers as choosing in their own best interests suggests natural ways to measure the effects of prices on consumer welfare.*

Consumer surplus

By Roy's identity the effect of a small change in price p_i on utility is proportional to the quantity consumed

$$\frac{\partial v}{\partial p_i} = -\frac{\partial v}{\partial y} q_i.$$

The horizontal distance of the demand curve from the vertical axis is therefore an indicator of the marginal welfare cost of increasing price. If $\partial v/\partial y$ is constant then the effect of increasing the price to the point where none of the good is demanded is therefore a triangular area underneath a demand curve. We call this *consumer surplus*.

What sort of demand curve do we need to use to keep $\partial v/\partial y$ constant? If preferences are quasilinear then this will be true of the Hicksian or Marshallian demand curve. Consumer surplus arguments are therefore rigorously justified if preferences are quasilinear.

Even if preferences are not quasilinear the general idea behind calculating consumer surplus as the area under a demand curve usually still gives a reasonable approximation to a good measure of welfare.

Along a compensated demand curve utility is held constant by construction and $q_i = \partial c(\upsilon, p)/\partial p_i$. The increase in the cost of reaching that given utility as a price changes can be regarded as a measure of the cost to the consumer and can be equated to an area under a demand curve

$$c(\upsilon, p^A) - c(\upsilon, p^B) \;=\; \int_{p^B}^{p^A} \frac{\partial c(\upsilon, p)}{\partial p_i} \mathrm{d}p_i \;=\; \int_{p^B}^{p^A} g_i(\upsilon, p) \mathrm{d}p_i$$

If the fixed level of utility is initial utility then this is the *compensating variation*. If the fixed level of utility is final utility then this is the *equivalent variation*. Both are recognised measures of welfare

Cost of living indices

The expenditure function is the ideal concept for comparing cost of living. We can define a (true or Konüs) cost of living index as the ratio of the minimum cost of reaching a given utility in two periods. Say that we are comparing

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current prices p^A with prices in a base period p^B. Then the cost of living index is the ratio of expenditure functions

$$T(\upsilon, p^A, p^B) = \frac{c(\upsilon, p^A)}{c(\upsilon, p^B)}$$

Notice that such a cost of living index depends on the utility level υ at which we make the comparison. Must this be so? There are only two case in which not:

- If prices are proportional $p^A = \lambda p^B$ then the cost of living index is equal to λ at all υ, whatever preferences,
- If preferences are homothetic then the cost of living index is equal at all υ, whatever prices.

Two common approximations to the true index are used. Both compare the cost of purchasing a fixed bundle of goods.

The Laspeyres index is the ratio of the costs of purchasing the base period bundle q^B

$$L(p^A, p^B) = \frac{p^{A\prime} q^B}{p^{B\prime} q^B} = \sum_i w_i^B \left(\frac{p_i^A}{p_i^B}\right)$$

where the second expression shows that the Laspeyres index can be conveniently written as a budget-share-weighted average of price ratios.

If $L(p^A, p^B) < 1$ and consumer's total expenditure is unchanged then the consumer can afford the base bundle and cannot be worse off.

We also know that $p^{B\prime} q^B = c(\upsilon^B, p^B)$ where υ^B is the base period utility and also that $p^{A\prime} q^B$ cannot be less than the minimum cost of attaining u^B in the current period (since q^B gives utility υ^B but not necessary most cheaply at current prices). Hence the Laspeyres index is greater than the true cost of living index at base period utility,

$$L(p^A, p^B) \geq T(\upsilon^B, p^A, p^B).$$

This is so because consumers are free to substitute away from goods which become more expensive and therefore evaluating the cost at a fixed bundle exaggerates the impact on cost of living.

The Paasche index is the ratio of the costs of purchasing the current period bundle q^A

$$P(p^A, p^B) = \frac{p^{A\prime} q^A}{p^{B\prime} q^A}.$$

By similar arguments the consumer must be worse off if $P(p^A, p^B) > 1$ and total expenditure is unchanged. Likewise the Paasche index can be shown to be less than the true cost of living index at current utility υ^A,

$$P(p^A, p^B) \leq T(\upsilon^A, p^A, p^B).$$

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If preferences are homothetic then the true index T is the same at all utilities so we can write simply

$$P \leq T \leq L.$$

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Worked Example H : Price indices

Take the example of Stone-Geary preferences $u(q) = \sum_i \alpha_i \ln(q_i - \gamma_i)$ as analysed earlier.
Suppose that the first good is a necessity at base prices, so $s_1^B > \alpha_1$ where we define $s_i^B = p_i^B \gamma_i / \sum_j p_j^B \gamma_j$. Base period budget shares are

$$w_i^B \;=\; \alpha_i + \left(s_i^B - \alpha_i\right) \sum_j p_j^B \gamma_j / y^B \qquad i = 1, 2, \ldots, M$$

as shown earlier.
Say that between the base and final period the price of good 1 doubles and other prices stay the same, so we can express the Laspeyres index as

$$\begin{aligned} L(p^A, p^B) &= \sum_i w_i^B \left(\frac{p_i^A}{p_i^B}\right) \\ &= \sum_i w_i^B + w_1^B \\ &= 1 + \alpha_1 + \left(s_1^B - \alpha_1\right) \sum_j p_j^B \gamma_j / y^B. \end{aligned}$$

The Laspeyres index is therefore higher for poorer households, those with higher y^B.
The true index evaluated at base utility is

$$\begin{aligned} T(\upsilon^B, p^A, p^B) &= \frac{c(\upsilon^B, p^A)}{c(\upsilon^B, p^B)} \\ &= \frac{\sum_i p_i^A \gamma_i + \mathrm{e}^{\upsilon^B} \Pi_i \left(\frac{p_i^A}{\alpha_i}\right)^{\alpha_i}}{\sum_i p_i^B \gamma_i + \mathrm{e}^{\upsilon^B} \Pi_i \left(\frac{p_i^B}{\alpha_i}\right)^{\alpha_i}} \\ &= \left(\sum_i p_i^B \gamma_i + p_1^B \gamma_1 + \left(y^B - \sum_i p_i^B \gamma_i\right) 2^{\alpha_1}\right) / y^B \\ &= 2^{\alpha_1} + \left(1 + s_1^B - 2^{\alpha_1}\right) \sum_i p_i^B \gamma_i / y^B \end{aligned}$$

Since $s_1^B > \alpha_1$ and $1 + \alpha_1 - 2^{\alpha_1} > 0$ for all $0 < \alpha_1 < 1$ the true index is also higher for poorer households, those with higher y^B.
Note also that the Laspeyres index does indeed always exceed this true index

$$L(p^A, p^B) - T(\upsilon^B, p^A, p^B) = (1 + \alpha_1 - 2^{\alpha_1}) \left(1 - \sum_i p_i^B \gamma_i / y^B\right) > 0$$

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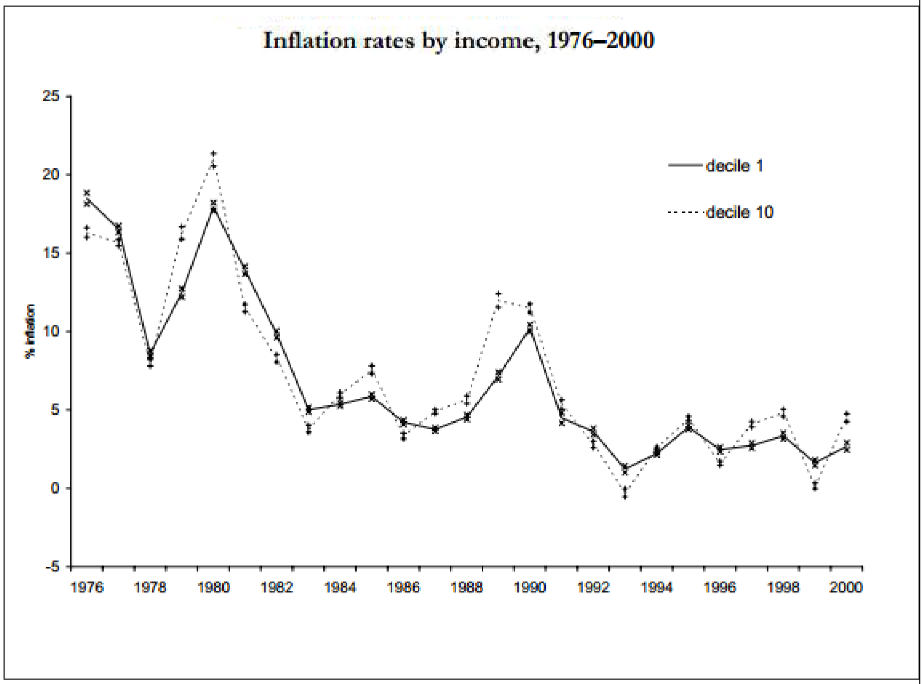
Case Study 5 : Price Indices

If preferences are not homothetic and different goods change prices at different rates then inflation will differ for households at different points in the income distribution. The figure below shows rates of inflation for households in the highest and lowest decile groups in the UK over a quarter of a century. These are calculated from the UK Family Expenditure Survey using Laspeyres indices. Inflation is not systematically higher for one group than the other but in individual years it can differ by as much as 5 per cent. It has been higher for the rich more often but the biggest differences have been in years when the poor were hardest hit.



[Source: I. Crawford and Z. Smith, 2002, Distributional Aspects of Inflation, Institute for Fiscal Studies Commentary 90.]

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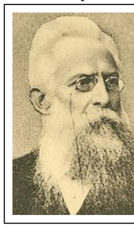
History Note VII : Consumer welfare

Ideas for measurement of consumer welfare precede modern consumer theory. Jules Dupuit (1804-66) introduced the notion of consumer surplus in his 1840s work on bridges, mentioned above, and the idea was refined by Alfred Marshall (1842-1924). Compensating variation as a measure of the welfare effect of a price change was proposed by John Hicks (1904-89, Nobel 1972) in the 1930s.

The notion of a price index as an average of relative prices for different goods can be traced back at least to the English bishop William Fleetwood (1656-1723) writing in the early 1700s and as a ratio of costs of representative bundles at least to the English economist John Lowe (d. 1831) in the 1820s. The Laspeyres and Paasche indices are attributable to the ideas of fixing this bundle at either the initial or final bundle as proposed by the German statisticians Etienne Laspeyres (1834-1913) and Hermann Paasche (1851-1925) respectively. The suggestion that it would be appropriate to take different bundles optimally delivering a fixed utility and derivation of results about such an index's properties is owed to the Russian statistician Alexandr Konüs (1895-1990) writing in the 1920s.

J Dupuit



E Laspeyres

A A Konüs

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Topic 6 : Endowments and labour supply

Summary: *Extending the analysis to recognise the fact that consumers may begin with endowments and buy as well as sell goods leads to a richer model. In particular, it suggests a way to model supply of labour and to understand why responses to wage changes may not be as simple as responses to other prices.*

Buying and selling

Suppose an individual has endowments $\omega = (\omega_1, \omega_2, ...)$ of goods. The consumer problem becomes

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y + p'\omega$$

Demands are now

$$q_i = f_i(y + p'\omega, p) \qquad i = 1, 2, \ldots$$

where $f_i(.)$ is the standard uncompensated demand function. It is sometimes convenient to draw a distinction between *gross demands* q and *net demands* or *excess demands* $z = q - \omega$. Let $\phi(y, p, \omega) = f(y + p'\omega, p) - \omega$ define a new function giving net demands as a function of y, p and ω.

Changes in endowments have effects like income effects. Changes in prices have the usual effects plus an effect due to the change in the value of the individual's endowment - the *endowment income effect.* Specifically

$$\begin{aligned} \frac{\partial \phi_i}{\partial p_j} &= \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial y}\omega_j \\ &= \frac{\partial g_i}{\partial p_j} - (q_j - \omega_j)\frac{\partial f_i}{\partial y} \end{aligned}$$

where $g_i(v, p)$ is the usual compensated demand function. This extends the Slutsky equation to the case of demand with endowments. Note that, written in terms of net demands, the equation is unchanged

$$\frac{\partial \phi_i}{\partial p_j} = \frac{\partial g_i}{\partial p_j} - z_j \frac{\partial \phi_i}{\partial y}.$$

Notice that the sign of the income effect depends upon whether the individual is a net buyer ($q_i > \omega_i$) as in the usual case or a net seller ($q_i < \omega_i$). An increase in the price of a normal good can now increase demand if the individual is a net seller and the endowment income effect is strong enough.

Note that there is an important revealed preference argument relating to uncompensated demands in this situation. A seller will never become a buyer if the price rises and a buyer will never become a seller if the price falls. In each case, such a change is not possible since it would involve consuming a bundle available before the change when the bundle then chosen remains affordable.

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Labour supply

The prime example of the importance of considering demand with endowments is the analysis of labour supply. Suppose an individual has preferences over hours not working ("leisure") h and consumption c. They have unearned income of m and endowment of time T. The price of consumption is p and the nominal wage is w. The individual's budget constraint is

$$pc + wh = m + wT \equiv M$$

which may appear more familiar if written in terms of hours worked $l = T - h$:

$$pc = m + wl.$$

The value of endowments in this context $M = m + wT$ is referred to as *full income.*

Suppose the individual is free to choose any hours of work, subject only to their budget constraint. Demand for leisure can be written as an uncompensated demand function, dependent on full income, wage and output price

$$h = \phi(m, w, p, T) + T = f(M, w, p)$$

or as a compensated demand function

$$h = g(\upsilon, w, p).$$

The Slutsky equation for leisure is

$$\frac{\partial \phi}{\partial w} = \frac{\partial f}{\partial w} + T\frac{\partial f}{\partial M} = \frac{\partial g}{\partial w} - (h - T)\frac{\partial f}{\partial M}.$$

Since the individual *sells* time ($h < T$) the income effect of a wage change is opposed to the compensated effect if leisure is normal.

Rephrasing in the more familiar terms of labour supply l we can define an uncompensated labour supply function

$$l = L^u(m, w, p) \equiv T - f(m + wT, w, p)$$

and an compensated labour supply function

$$l = L^c(\upsilon, w, p) \equiv T - g(\upsilon, w, p)$$

and it follows that

$$\frac{\partial L^u}{\partial w} = \frac{\partial L^c}{\partial w} + l\frac{\partial L^u}{\partial m}.$$

The opposition of income and substitution effects is still there and the direction of the uncompensated wage effect on chosen hours of work depends upon the balance between the two. Since that balance can differ at different wage rates it is quite possible for a labour supply curve to slope upwards at some wages and downwards at others as in a so-called backward-bending labour supply curve.

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Worked Example I : Labour supply

Suppose the utility function defined over leisure h and consumption c has a Cobb-Douglas form
$$u(c, h) = \alpha \ln h + (1 - \alpha) \ln c.$$
The budget constraint is $wh + pc = wT + m$ which we can rewrite in terms of labour supplied $l = T - h$ to express the consumer's problem as
$$\max_l \alpha \ln(T - l) + (1 - \alpha) \ln \frac{wl + m}{p}$$
.
The first order condition for solution $\alpha/(T - l) = (1 - \alpha)w/(wl + m)$ is solved by
$$l = (1 - \alpha)T - \alpha m/w.$$
Labour supply is increasing in w for all values of $m > 0$ so the substitution effect always dominates the income effect for such preferences.

Case Study 6 : Labour Supply

Practical study of labour supply is complicated by a number of factors. Taxation and benefit systems make actual budget constraints linking hours of work to labour income much more complicated than the simple linear budget constraint considered here. Corner solutions at zero hours may be optimal for certain types of households with low wages and participation responses to changes in budget constraints can be as important as adjustments in hours worked. Also individual labour supply decisions of individuals within households need to be modelled jointly to capture interdependence.
Lone mothers are a particularly common focus of policy interest. Over the 1980s and 1990s several changes in US tax policy were aimed at encouraging them to work and these were accompanied by unprecedented increases in employment and hours. Meyer and Rosenbaum compare the behaviour of single women with and without children to assess the contribution of different policy changes using a large population survey. They find that over 60% of the change was accounted for by responses to changes in tax credits, a smaller but still significant proportion to changes in welfare programs and little to changes in Medicaid, training and child care programs. They conclude that policies aimed at "making work pay" rather than penalising not working are most effective in achieving the desired effects.

[Source: B. Meyer and D. Rosenbaum, 2001, Welfare, the Earned Income Tax Credit, and the labor supply of single mothers, *Quarterly Journal of Economics* 116, 1063-1114.]

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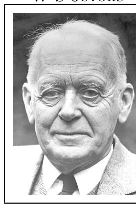
History Note VIII : Labour supply

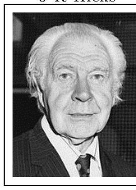
Recognition that labour supply can be analysed as a consumer choice problem involving the weighing of disutility of labour against the utility of consumption can be found in marginalist works, for example, of Stanley Jevons (1835-1882), Alfred Marshall (1842-1924) and even Hermann Gossen (1810-1858).

The possibility that the opposition of income and substitution effects given the endowment of time might create ambiguity in the slope of labour supply curves and even backward-bending behaviour was clarified by work of British economists Lionel Robbins (1898-1984) and John Hicks (1904-89, Nobel 1972) in the 1930s.

W S Jevons



J R Hicks



L Robbins

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Topic 7 : Intertemporal choice

Summary: *Allocation of spending over time can be seen as a further example of the theory of demand with endowments. Particular features of the intertemporal setting suggest restrictions on preferences which put natural structure on intertemporal demand behaviour. If we allow a variety of assets then in the simplest models their investment decision is separable from the decision about when to consume the returns.*

Intertemporal choice

Another example of demand with endowments is analysis of intertemporal choice. Suppose an individual has preferences over consumption when young c_0 and consumption when old c_1. They have endowed income of y_0 and $y_1$in the two periods. (If necessary, bequests received can be treated as part of y_0 and bequests given as part of c_1). Assume no uncertainty about the future. If the real interest rate on bonds linking the two periods is equal to r for both lending and borrowing then the budget constraint is

$$c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r}.$$

which implies that the present value of consumption must equal the present value of income.

Demand for current consumption is

$$c_0 = f_0\left(y_0 + \frac{y_1}{1+r}, r\right)$$

The effect of interest rate changes clearly depend upon whether the individual is a saver or a borrower since this determines the sign of the income effect. Note that an interest rate rise will never induce a saver to become a borrower and an interest rate fall will never induce a borrower to become a saver.

Often it is assumed that the utility function can be written as the sum of utility contributions from the different periods with similar within-period utility functions but with future utility discounted. Thus

$$u(c_0, c_1) = \nu(c_0) + \frac{1}{1+\delta}\,\nu(c_1)$$

where $\nu(.)$ is the within-period utility function and δ is a *subjective discount rate*. Convexity of preferences, which amounts here to a desire to smooth consumption over the life-cycle, requires $\nu(.)$ to be concave, $\nu''(.) < 0$.

Maximising such a utility function subject to the lifetime budget constraint

$$\max_{c_0} \quad \nu(c_0) + \frac{1}{1+\delta}\,\nu(y_1 + (y_0 - c_0)(1+r))$$

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gives first order condition
$$\frac{\nu'(c_0)}{\nu'(c_1)} = \frac{1+r}{1+\delta}.$$
This is known as the consumption Euler equation. Given concavity of $\nu(.)$, if $r = \delta$, so that subjective discounting matches the market interest rate and impatience cancels out the market incentive to save, then $c_0 = c_1$ and the consumption stream is flat. If $r > \delta$ then $c_0 < c_1$ and if $r < \delta$ then $c_0 > c_1$. In all of these cases c_0 and c_1 will both be increasing functions of lifetime resources $y_0 + y_1/(1+r)$.

The responsiveness of consumption decisions to changes in the interest rate r depends critically in such a context on the concavity in the within-period utility function $\nu(.)$. The more concave is $\nu(.)$ the more sensitive is the intertemporal marginal rate of substitution to differences in consumption between periods and therefore the less dramatically do consumption decisions need to respond to bring the MRS into harmony with the interest rate.

Asset choice

Suppose that as well as investing in bonds with fixed return of r the individual can also invest in another asset - say a family enterprise. If X is placed in the family enterprise in the first period then suppose $F(X)$ is returned in the second period.

The optimisation problem now has two dimensions
$$\max_{c_0, X} \quad \nu(c_0) + \frac{1}{1+\delta}\,\nu(y_1 + F(X) + (y_0 - c_0 - X)(1+r))$$
and first order conditions (assuming an interior solution) are
$$\begin{aligned} \frac{\nu'(c_0)}{\nu'(c_1)} &= \frac{1+r}{1+\delta} \\ F'(X) &= (1+r). \end{aligned}$$
Note that the solution to the financial decision is independent of intertemporal preferences. The individual invests in the family enterprise until the marginal rate of return falls to the market interest rate. This maximises the present value of the individual's asset portfolio and the first order condition for optimum consumption choice, given that present value, is as in the simpler problem above.

The simplicity of the investment decision is a consequence of assuming away issues concerning risk, liquidity and so on.

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Case Study 7 : Intertemporal Choice

The top two panels of the figure below show how mean consumption and income vary over the life cycles of American households. The figure is made by taking data from the US Consumer Expenditure Survey over the 1980s and early 1990s, grouping observed households into generational cohorts according to their date of birth and plotting the means of these variables against age. The humped shape to both profiles reflects to some extent the pattern of household size as captured in the profiles for numbers of adults and children below.
Regressing mean changes in log consumption on interest rates (with controls for seasonality and demographic change) gives an estimate of the Euler equation suggesting an intertemporal elasticity of substitution of 0.637 (with a standard error of 0.333).

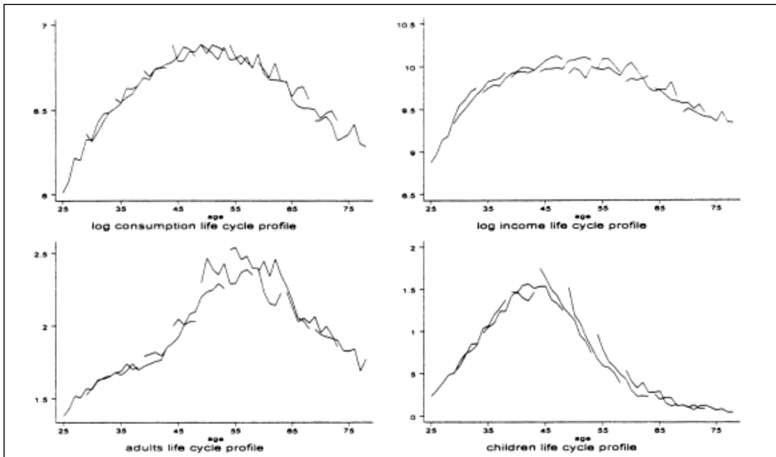

Figure 1. Life-Cycle Profiles.

[Source: O. Attanasio, J. Banks, C. Meghir and G. Weber, 1999, Humps and bumps in lifetime consumption, Journal of Business and Economic Statistics 17, 22-35.]

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Worked Example J : Life-cycle spending

Suppose an individual lives for two periods with life-time preferences

$$u(c_0, c_1) = \ln(c_0 - \gamma_0) + \frac{1}{1+\delta}\ln(c_1 - \gamma_1)$$

where γ_0 and γ_1 are period-specific consumption needs. This is similar to the additive model considered above but with within-period utility functions allowed to vary with needs of the period, $\nu_t(c_t) = \ln(c_t - \gamma_t)$. Note that this is an intertemporal version of Stone-Geary preferences. Utility-maximising demands take the form

$$\begin{aligned} c_0 &= \gamma_0 + \frac{1+\delta}{2+\delta}\left[y_0 - \gamma_0 + \frac{y_1 - \gamma_1}{1+r}\right] \\ c_1 &= \gamma_1 + \frac{1+r}{2+\delta}\left[y_0 - \gamma_0 + \frac{y_1 - \gamma_1}{1+r}\right] \end{aligned}$$

The life-cycle path for consumption is independent of the path of incomes given the discounted present value of lifetime income. A higher value for the interest rate r leads to a more steeply rising path and a higher value of the impatience parameter δ to a less steeply rising one. Even if $r = \delta$ the consumption path will still not be flat if $\gamma_0 \neq \gamma_1$.

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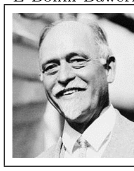
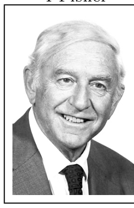
History Note IX : Intertemporal Choice

The application of marginalist ideas to questions of intertemporal allocation was taken up in depth in the 1870s by the Austrian civil servant and economist Eugen von Böhm-Bawerk (1851-1914), a follower of Carl Menger (1840-1921). His ideas led to development of the two period model of income and consumption outlined here, laid out as a model of intertemporal choice in the work of the American economist Irving Fisher (1867-1947) in the early twentieth century and making clear the links between interest rates, intertemporal rates of substitution, intertemporal rates of return and consumer time preference.

Relationships between paths of income and consumption were a prominent aspect of the life-cycle model of Franco Modigliani (1918-2003, Nobel 1985) and permanent income model of Milton Friedman (1912-2006, Nobel 1975) developed in the US in the 1950s and 1960s.


E Böhm-Bawerk


I Fisher


F Modigliani

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Topic 8 : Choice under uncertainty

Summary: *Choice under uncertainty can be modelled similarly in many ways to intertemporal choice, with the distinction between states of the world mirroring that between periods of time and aversion to risk mirroring the desire to smooth consumption over time. The theory is able to describe important aspects of behaviour but also fails to capture certain anomalies.*

Uncertainty

Extending the standard analysis to the case of uncertainty involves regarding quantities consumed in different uncertain states of the world as different goods. Preferences will depend on perceived probabilities of states of the world occurring. Budget constraints depend on the mechanisms available for managing risk.

Some examples of budget constraints in circumstances involving risk are:

- An individual with an asset worth A faces a probability π of losing it. He can purchase insurance of K at a cost of γK. Consumption in case of loss is $c_1 = (1-\gamma)K$ and in the case of no loss is $c_0 = A - \gamma K$. The budget constraint for the individual is $c_0 = A - \frac{\gamma}{1-\gamma}c_1$.
- An individual with income of m has a true tax liability of T but tries to evade an amount D by underdeclaration. There is probability π of being audited in which case he pays the full liability T plus a fine fD. Consumption in case of audit is $c_1 = m - T - fD$ and in the case of no audit is $c_0 = m - T + D$. The budget constraint for the individual is $c_1 = (1+f)(m-T) - fc_0$.

In both of these cases the budget constraint is linear, downward sloping and independent of the probability π.

Preferences are defined over quantities consumed in the different states $(c_0, c_1 ...)$ and depend on perceived probabilities of the states occurring $(\pi_0, \pi_1, ...)$. Under certain assumptions it may be reasonable to regard the consumer as maximising *expected utility*

$$u(c_0, c_1 ..., \pi_0, \pi_1, ...) = \sum \pi_i \nu(c_i)$$

for some state-specific utility function $\nu(.)$. We refer to $u(\cdot)$ as a *von-Neumann-Morgenstern expected utility function.*

The most controversial assumption required to justify an expected utility formulation is the *sure thing principle* (or the closely related *strong independence axiom*). Consider the following two choices:

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$$\text{Choice 1}: \left[\begin{array}{lccc} & & \text{Probability} & \\ & \pi_1 & \pi_2 & 1-\pi_1-\pi_2 \\ \text{Option } A_1 & \alpha_1 & \alpha_2 & \gamma \\ \text{Option } B_1 & \beta_1 & \beta_2 & \gamma \end{array}\right]$$

$$\text{Choice 2}: \left[\begin{array}{lccc} & & \text{Probability} & \\ & \pi_1 & \pi_2 & 1-\pi_1-\pi_2 \\ \text{Option } A_2 & \alpha_1 & \alpha_2 & \delta \\ \text{Option } B_2 & \beta_1 & \beta_2 & \delta \end{array}\right]$$

In each case the two options deliver the same outcome as each other with probability $1-\pi_1-\pi_2$ (though these outcomes differ between the two choices). It might therefore be argued that the choice should be driven only by the different outcomes occurring in the other columns. However these outcomes are the same in the two choices. Therefore if A_1 is preferred to B_1 it is argued that A_2 should be preferred to B_2. This is the sure thing principle. Combined with other less controversial axioms extending choice to uncertain situations with multiple outcomes it implies that the MRS between consumption in any two states is independent of outcomes in any other state.

Note that the function $\nu(.)$ is *not* ordinal. Preferences are changed by arbitrary increasing transformations of $\nu(.)$. However $u(.)$ is still ordinal.

Risk aversion

To capture risk aversion we need to capture the fact that risk averse individuals prefer to receive the expected value of any gamble with certainty to undertaking the gamble. Thus

$$\nu((1-\pi)c_0+\pi c_1) > (1-\pi)\nu(c_0)+\pi\nu(c_{1)}.$$

For this always to be true requires that $\nu(.)$ be a concave function. The degree of concavity is an indicator of the strength of aversion to risk.

Consider the insurance case again. An expected utility maximising consumer chooses K to maximise

$$(1-\pi)\nu(A-\gamma K)+\pi\nu((1-\gamma)K).$$

The first order condition requires

$$(1-\pi)\gamma\nu'(A-\gamma K) = (1-\gamma)\pi\nu'((1-\gamma)K).$$

If insurance is actuarially fair then $\pi=\gamma$ and therefore $\nu'(A-\gamma K) = \nu'((1-\gamma)K)$. If the individual is risk averse then $\nu'(.)$ is a decreasing function and therefore $A=K$ so there is full insurance. This is a typical illustration of behaviour under risk. The fairness of insurance means that risk can be eliminated without compromising expected consumption and a risk averse individual chooses therefore to eliminate risk.

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Case Study 8 : Choice under Uncertainty

Theoretical consumers behaving as expected utility maximisers seem to show a perhaps unnaturally high proficiency in their understanding and ability to work with probabilities. In practice there is abundant evidence that individuals understand probabilities poorly. Even where probabilities are described clearly their behaviour often fails to satisfy axioms upon which the theory is based. They attach greater importance to the status quo than is explicable and their decisions depend upon how the situation is framed when described to them.
A particularly well known example in which behaviour fails to fit is the so-called Allais paradox. Consider the following two choices:

$$\text{Choice 1 :}\quad \left[\begin{array}{cccc} & & \text{Probability} & \\ & 0.01 & 0.33 & 0.66 \\ \text{Option } A_1 & 2400 & 2400 & 2400 \\ \text{Option } B_1 & 0 & 2500 & 2400 \end{array}\right]$$

$$\text{Choice 2 :}\quad \left[\begin{array}{cccc} & & \text{Probability} & \\ & 0.01 & 0.33 & 0.66 \\ \text{Option } A_2 & 2400 & 2400 & 0 \\ \text{Option } B_2 & 0 & 2500 & 0 \end{array}\right]$$

Kahneman and Tversky report the outcome of presenting these to 72 students. In the first choice 82% chose A_1 whereas in the second choice 83% chose B_2. 61% of students made the modal choice in both cases but this combination of choices violates the sure thing principle. Presumably the attraction of certainty in the first choice has something to do with this.
Expected utility theory captures some features of behaviour that we would want to include, such as aversion to risk, and continues to be widely assumed in much work but it has clear inadequacies in many contexts and exploration of such deviations is an important part of ongoing research.

[Source: D. Kahneman and A. Tversky, 1979, Prospect theory: an analysis of decision under risk, *Econometrica* 47, 263-292.]

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Worked Example K : Insurance

Suppose that an individual is an expected utility maximiser with within-state utility function given by $\nu(c) = \ln c$. Since this is concave the individual is risk-averse. Their objective actually has the form of Cobb-Douglas utility with the probabilities taking the place of preference coefficients

$$u(c_0, c_1) = (1 - \pi) \ln c_0 + \pi \ln c_1.$$

The first order condition for optimal insurance choice

$$\frac{(1-\pi)\gamma}{A - \gamma K} = \frac{(1-\gamma)\pi}{(1-\gamma)K}$$

implies $K = \pi A/\gamma$. As expected this is less than A unless $\gamma \leq \pi$. Wealth levels in the two states are $c_0 = (1 - \pi)A$ and $c_1 = (1 - \gamma)\pi A/\gamma$.

If insurance is less than fair $\pi < \gamma$ then there is underinsurance. The extent of underinsurance depends on the concavity of $\nu(.)$.

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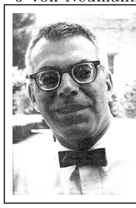
History Note X : Choice under Uncertainty

Development of ideas of probability in the eighteenth century was tied closely to development of ideas about choice under uncertainty. The St Petersburg paradox which seemed to suggest an implausible case in which it would be rational to gamble unlimited sums of money was resolved by the postulation of logarithmic utility by Daniel Bernoulli (1700-82), alluded to in earlier sections.

The expected utility theorem was proved by the Hungarian-American polymath John von Neumann (1903-57) and German-American collaborator Oskar Morgenstern (1902-77) in connection with their foundational work on game theory. The sure thing principle was put forward by the American statistician Leonard Savage (1917-71) as part of a proposed axiomatisation of subjective decision theory. Measurement and comparison of risk aversion in an expected utility framework was explored in the 1960s by American economists Kenneth Arrow (1921-2017, Nobel 1972) and John Pratt (b.1931).

The Allais paradox, attributed to French economist Maurice Allais (1911-2010, Nobel 1988), was one of several departures of behaviour from the postulates of expected utility theory that have been apparent since the 1950s. Paradoxes such as these have done much to encourage the growth of the field of behavioural economics associated, for example, with the Israeli-American and Israeli psychologists, Daniel Kahneman (b. 1934, Nobel 2002) and Amos Tversky (1937-96).

J von Neumann



L J Savage

M Allais

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Topic 9 : General equilibrium

Summary: *If we put consumers together and allow them to trade then we can ask whether prices will exist that lead all markets to clear and, if so, whether this provides a sensible or illuminating theory of price determination. Adding profit-maximising producers to the picture turns out not to complicate matters unmanageably.*

Exchange equilibrium

Suppose the economy consists of H households. The hth household has endowment ω^h and consumes a bundle q^h. An allocation of goods is said to be *feasible* if the aggregate amount consumed of each good equals the aggregate endowment

$$\sum_h q_i^h = \sum_h \omega_i^h \quad i = 1, 2, ..., H$$

The initial endowments obviously constitute one feasible allocation.

Demands if prices are p are $q_i^h = f_i^h(p'\omega^h, p)$, $i = 1, 2, ..., H$ where $p'\omega^h$ is the value of the individual's endowment. Note that there is no assumption that different households have the same preferences. Market demand is found by adding the demands across individuals

$$Q_i(p'\omega^1, p'\omega^2, ..., p) = \sum_h f_i^h(p'\omega^h, p) \quad i = 1, 2, ...$$

Note the dependence on the complete distribution of endowments.

Let z_i^h denote the excess demand from the ith household. The aggregate excess demand z_i is given by the excess of market demand over the sum of endowments

$$Z_i(p) = \sum_h z_i^h = \sum_h [f_i^h(p'\omega^h, p) - \omega_i^h] \quad i = 1, 2, ...$$

General equilibrium - referred to also as *market equilibrium*, *competitive equilibrium* or *Walrasian equilibrium* - is a set of prices such that aggregate excess demand is zero on all markets.

$$Z_i = \sum_h z_i^h = 0 \quad i = 1, 2, ...$$

A competitive allocation, if it exists, is another example of a feasible allocation.

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Walras' Law

If there are M goods then this seems to define M equations in M unknown prices. However this is misleading. The fact that each household must be on its budget constraint implies that the value of that household's excess demand is zero

$$\sum_i p_i q_i^h = \sum_i p_i \omega_i^h \quad \Rightarrow \quad \sum_i p_i z_i^h = 0.$$

Adding this equation over households establishes that the value of aggregate excess demand is also zero

$$\sum_i p_i Z_i(p) = 0.$$

This is *Walras' law* and is true for any prices (not only the equilibrium prices). It implies that the M excess demands are not independent - in fact there are only $M-1$ independent excess demands to set to zero.

However since demands are homogeneous multiplying all prices by any positive number will give the same excess demands. If any prices constitute a Walrasian equilibrium, then so therefore do any positive multiple of those prices. It is therefore only relative prices which are determined by the equilibrium conditions.

There are therefore actually $M-1$ independent equations determining $M-1$ relative prices.

A common diagrammatic representation of general equilibrium for a two-person two-good exchange economy is the *Edgeworth-Bowley box.* This is a rectangular box whose horizontal and vertical dimensions are set to the economy-wide endowments of the two goods. Consumptions of the two individuals are read from opposite corners of the box and any point in the box represents a feasible allocation in the economy. Preferences of the two individuals can be represented by drawing indifference curves. A price vector for the economy defines a common budget constraint passing through the endowment point representing trading possibilities for the two individuals. Individuals' desired trades are defined by tangencies between this budget constraint and indifference curves for the two individuals. Equilibrium exists if and only if these desired trades coincide.

Existence, uniqueness and stability

Nothing said so far ensures existence of a Walrasian equilibrium but if aggregate demands vary continuously as a function of prices then it can be proved that at least one equilibrium must exist.

There is no guarantee however that the equilibrium will be unique without further assumptions on preferences and indeed economies with multiple equilibria are easily illustrated.

Whether or not prices in an economy out of equilibrium will tend to move so as to take it towards equilibrium is a question that cannot be answered without

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a theory of what happens out of equilibrium. If not all demands can be met from the economy's endowments then what happens and how do prices adjust? It is possible to tell artificial and simple stories demonstrating the ability of a hypothetical auctioneer to find equilibrium prices under appropriate assumptions about preferences but these arguably tell us little about the stability of equilibrium in actual economies with trading taking place out of equilibrium.

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Worked Example L : Exchange equilibrium

Demands if prices are p are $q_i^h = f_i^h(Y^h, p)$, $i = 1, 2, ..., M$ where $Y^h = p'\omega^h$ is the value of the individual's endowment. We need to find a price vector p solving the market clearing equations

$$\sum_h f_i^h(p'\omega^h, p) = \sum_h \omega_i^h \quad i = 1, 2, ..., M$$

From Walras' law we need only solve for $M-1$ relative prices achieving market clearing on $M-1$ of the M markets.
As an example, suppose there are only two goods so we need to find only one relative price to clear one market. Since we can solve only for the relative price we normalise the price of good 2 to be 1 and let P be the price of the first. Let there be two consumers A an B who have Cobb-Douglas demands over the two goods. Individual h therefore has demands

$$q_1^h = \alpha^h Y^h/P \qquad q_2^h = (1-\alpha^h)Y^h \qquad h = A, B$$

where α^h is an individual-specific taste parameter.
Individual endowments are $\omega^h = (\omega_1^h, \omega_2^h)$. Therefore, by substituting the value of the endowments, demands are

$$q_1^h = \alpha^h(P\omega_1^h + \omega_2^h)/P \qquad q_2^h = (1-\alpha^h)(P\omega_1^h + \omega_2^h)$$

To find equilibrium, we know from Walras' law that we need only find the price to clear one market. Take the first. Market clearing requires

$$\omega_1^A + \omega_1^B = q_1^B + q_1^B = \alpha^A(P\omega_1^A + \omega_2^A)/P + \alpha^B(P\omega_1^B + \omega_2^B)/P$$

Solving for P gives

$$P = \frac{\alpha^A\omega_2^A + \alpha^B\omega_2^B}{(1-\alpha^A)\omega_1^A + (1-\alpha^B)\omega_1^B}.$$

Note that this is increasing in endowments of the second good and decreasing in endowments of the first. Note also that it is increasing in the demand parameters α^A and α^B. These are readily intelligible demand and supply effects for this example.

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Equilibrium in economies with production

We can introduce production into the economy by assuming the existence of K firms, each taking prices as given and choosing production plans so as to maximise profits given their own technology

$$\pi^k = \max_{y^k} p'y^k \qquad \text{s.t.} \qquad F^k(y^k) \leq 0$$

for $k = 1, 2, ..., K$. Note that firms are not being assumed to share a common technology.

Inputs to production come from consumers' net supply of endowments (and particularly labour supply). Net outputs of firms supplement consumers' endowments as sources of resources for consumption.

Firms are assumed to be owned by consumers so that profits are returned to them. If the hth household owns a share θ_{hk} of the kth firm then its budget constraint is therefore

$$p'q^h \leq p'\omega^h + \sum_k \theta_{hk}\pi^k$$

. All profits are assumed to be distributed in this way so that $\sum_h \theta_{hk} = 1$ for all $k = 1, 2, \ldots$.

We can now redefine aggregate excess demand for any good as the excess of aggregate consumption over endowments *and* production:

$$Z_i(p) = \sum_h \left(q_i^h - \omega_i^h\right) - \sum_k y_i^k.$$

Given that $\sum_h \theta_{hk} = 1$, note that Walras' law still holds since

$$\sum_i p_i Z_i(p) = 0.$$

Walrasian equilibrium is defined in the same way as a price vector which ensures excess demands are zero on all markets. Existence of at least one equilibrium is guaranteed if aggregate net demands are continuous functions of prices, which in such an economy is assured by strict convexity of preferences *and* production possibilities.

Limitations of general equilibrium analysis

Models of general equilibrium describe idealised economies without market power for individual agents, without frictions which might prevent clearing of markets and in which all interdependence between agents can be accommodated through the price mechanism. They are interesting and important benchmarks because of their interesting welfare properties, discussed below, but many of the most interesting questions in economics are about the properties of economies which do not fit such a simplified description.

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Worked Example M : General equilibrium with production

We now need to find a price vector p solving the more complicated market clearing equations

$$\sum_h f_i^h(p'\omega^h + \sum_k \theta_{hk}\pi^k(p), p) = \sum_h \omega_i^h \ + \ \sum_k y_i^k(p) \quad i = 1, 2, ..., M$$

where $\pi^k(p)$ and $y_i^k(p)$ are functions giving profits and net supplies given p. From Walras' law we still only need only solve for $M-1$ relative prices achieving market clearing on $M-1$ of the M markets.
To take a typically simple example, suppose there is one firm, one consumer, one output and one input which is labour time. Suppose the only endowment is one unit of time held by the consumer. The consumer receives all profits, $\theta_{11} = 1$.
We can solve only for the relative price so we normalise the price of output to be 1 and let W be the price of labour (which is to say, the real wage). Full income for the consumer is $W + \pi(W)$ where $\pi(W)$ is the firm's profits returned to the consumer as unearned income.
Let the consumer have Cobb-Douglas preferences so that their demand for time is $h = \alpha(W + \pi(W))/W$ where α is a preference parameter. Supply of labour is therefore $l = 1 - h = (1-\alpha) - \alpha\pi(w)/W$.
Let the firm have a technology which allows it to produce output $\sqrt{L}$ with labour input L. Profit is therefore $\sqrt{L} - WL$. Maximising this gives labour demand $L = 1/4W^2$ and maximised profit $\pi(W) = 1/4W$. Equating supply of labour to demand for labour requires

$$1/4W^2 \ = \ (1-\alpha) - \alpha/4W^2$$

which implies equilibrium real wage

$$W \ = \ \frac{1}{2}\sqrt{\frac{1+\alpha}{1-\alpha}}.$$

A stronger preference for leisure tightens supply of labour and increases the equilibrium real wage in this simple economy.

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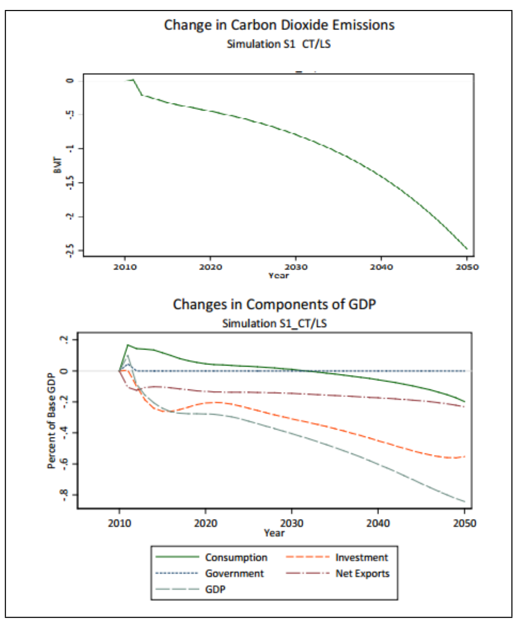
Case Study 9 : Computable General Equilibrium

One of the things to be learned from general equilibrium models is the way in which changes in conditions in one market can ramify into effects on others. This can be particularly important, for example, in analysing the impact of tax changes which affect goods which are inputs into production of many others. General equilibrium effects therefore seem particularly important in considering the effect of carbon taxes proposed as policy responses to global warming. Several institutions have developed computable multiple sector models to assist in analysis. An example is G-Cubed, a nine-geographical-region twelve-industrial-sector model of the world economy developed at the Brookings Institution. The figure below shows projections for the impact of a gradually introduced fossil fuel tax on the US economy (with revenue returned to households in the form of lump sum rebates). by raising the price of coal, natural gas and oil, the policy reform achievements significant abatement of carbon dioxide emissions. Underlying this is a complex pattern of sectoral adjustments which feed through into changing composition of GDP as illustrated in the lower panel. Scenarios in which the tax revenue is used differently can look quite different. The virtue of this sort of modelling is that it can deliver a sophisticated picture of the effects of nuanced reforms. The weakness is the sometimes opaque complex dependence on details of the model's formulation, estimation of its parameters and assumptions about smoothness of equilibriation.



[Source: W. McKibbin, A. Morris, P. Wilcoxen and Y. Cai, 2012, The Potential Role of a Carbon Tax in US Fiscal Reform, Brookings Institution, Climate and Energy Economics Discussion Paper.]

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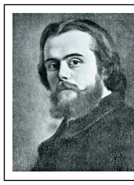
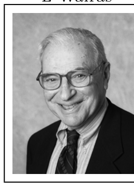
History Note XI : General equilibrium

Determination of prices in exchange is a topic that has been of long-standing interest to economics. Léon Walras (1834-1910) was particularly successful in formulating a mathematical description of a competitive equilibrium with multiple markets and with consumption and production of goods, inspired in part by notions of equilibrium in physics. The Edgeworth-Bowley box used to illustrate equilibrium geometrically was a construction introduced by the Irish economist Francis Ysidro Edgeworth (1845-1926) and popularised by the English statistician Arthur Bowley (1869-1918) (both sometime lecturers at UCL) after refinement by the Italian social thinker Vilfredo Pareto (1848-1923).

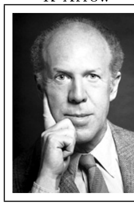
Rigorous proofs of theorems on existence and uniqueness of equilibrium for an economy in which complete markets are hypothesised to cover all uncertain contingencies were made in the work of the 1950s by the American economists Lionel McKenzie (1919-2010) and Kenneth Arrow (1921-2017, Nobel 1972) and his French coauthor Gérard Debreu (1921-2004, Nobel 1983), work widely regarded as a highpoint of its brand of mathematical economics.



L Walras



K Arrow



G Debreu

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Topic 10 : Welfare theorems and public goods

Summary: *Competitive equilibrium exhausts the scope for mutual gains from trade. Outcomes are efficient in the sense that it is impossible to make everyone better off. Moreover any efficient allocation is supportable as a competitive equilibrium if government can redistribute endowments appropriately. If individual demands are interdependent, as there are when there are public goods or externalities, then efficiency can be promoted by public intervention.*

Fundamental welfare theorems for exchange equilibria

Walrasian equilibrium in exchange economies have the general property of *Pareto efficiency*. This means that there is no feasible allocation such that all consumers are better off (or some are better off without any being any worse off).

To prove this suppose it were not the case. Then there would exist a feasible allocation r^1, r^2,... such that r^1 was preferred to q^1, r^2 was preferred to q^2 and so on. But then these bundles could not be affordable at the equilibrium prices p or the consumers would have purchased them. Thus

$$\sum_i p_i r_i^h \geq \sum_i p_i q_i^h = \sum_i p_i \omega_i^h \quad h = 1, 2, ..., H$$

with at least one of these inequalities being strict.

Adding across consumers gives

$$\sum_i p_i \sum_h r_i^h > \sum_i p_i \sum_h \omega_i^h.$$

But this conflicts with feasibility which requires $\sum_h r_i^h = \sum_h \omega_i^h$, $i = 1, 2, ..., M$ Hence there can be no such alternative allocation.

This is the *First Fundamental Theorem of Welfare Economics*. Walrasian equilibrium is always Pareto efficient.

Note that this says nothing about desirability in other respects. There are many Pareto efficient allocations and any Walrasian equilibrium that results from particular initial endowments will be just one. The locus of Pareto efficient allocations is known as the *contract curve*. Considerations such as distributional equity may give good grounds for regarding certain allocations on the contract curve to be socially preferred to others and whether or not these are attained will inevitably depend, for example, upon the equity in the allocation of initial endowments.

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The *Second Fundamental Theorem of Welfare Economics* tells us conversely that, under certain further assumptions, any Pareto efficient allocation can be sustained as a Walrasian equilibrium given the right allocation of initial endowments. In particular, this is true if we assume all agents have convex preferences.

Efficiency of exchange equilibria

Take an economy with total endowments $\Omega = \sum_h \omega^h$ and consider choosing the allocation across consumers to maximise the utility of one consumer, say $h = 1$, subject to specified utilities, $\bar{u}^h,\ h = 2, 3, \ldots, H$, for all of the others:

$$\max_{q^2,\, q^3,\, \ldots} u^1\left(\Omega - \sum_{h=2,3,\ldots,H} q^h\right)$$
$$\text{s.t.} \qquad u^h(q^h) \ = \ \bar{u}^h,\ h = 2, 3, \ldots, H$$

The set of feasible solutions to problems of this nature constitute the set of Pareto optima for the economy. First order conditions for internal optima require

$$\frac{\partial u^1}{\partial q_i^1} \ = \ \lambda^h \frac{\partial u^h}{\partial q_i^h},\ h = 2, 3, \ldots, H$$

where λ^h is the Lagrange multiplier on the utility constraint for the jth household. These imply

$$\frac{\partial u^h / \partial q_i^h}{\partial u^h / \partial q_j^h} \ = \ \frac{\partial u^g / \partial q_i^g}{\partial u^g / \partial q_j^g}$$

for every pair of households $\{g, h\}$ and every pair of goods $\{i, j\}$. Unless marginal rates of substitution are equated across consumers then a reallocation can improve some individuals utilities without harming others and the allocation is not Pareto optimal. Competitive equilibrium achieves Pareto efficiency because consumer choice ensures all individuals equate their MRS to a common price ratio.

Fundamental welfare theorems for equilibria with production

Both fundamental theorems still hold for economies with production if assumptions are appropriately extended. In particular Walrasian equilibria are still Pareto efficient.

To prove this, assume, as when proving efficiency in an exchange economy, that it were not so. Then there would exist productions plans x^1, x^2,... and a feasible allocation r^1, r^2,... such that r^1 was preferred to q^1, r^2 was preferred to q^2 and so on. But, as in the earlier proof, these bundles could not be affordable

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at the equilibrium prices p, given equilibrium profits or the consumers would have purchased them. Thus

$$\sum_i p_i r_i^h \geq \sum_i p_i \omega_i^h + \sum_k \theta_{hk} \sum_i p_i y_i^k \quad h = 1, 2, ..., H$$

with at least one of these inequalities being strict.

Adding across consumers gives

$$\sum_i p_i \sum_h r_i^h > \sum_i p_i \left[\sum_h \omega_i^h + \sum_k y_i^k \right]$$

since $\sum_h \theta_{hk} = 1$.

But if this alternative allocation is feasible under the alternative production plans then

$$\sum_h r_i^h = \left[\sum_h \omega_i^h + \sum_k x_i^k \right].$$

and therefore

$$\sum_i p_i \sum_k x_i^k > \sum_i p_i \sum_k y_i^k$$

which means that, contrary to assumption, the equilibrium production plans cannot be maximising profits.

Efficiency of equilibria with production

Extending the earlier reasoning, consider choosing both consumption allocation and production plans to maximise the utility of one consumer, again say $h = 1$, subject to specified utilities, $\bar{u}^h$, $h = 2, 3, \ldots$, for all of the others but also subject to technical feasibility:

$$\begin{aligned} \max_{q^2,\, q^3,\, \ldots} u^1 &\left(\sum_k y^k - \sum_{h=2,3,\ldots,H} q^h \right) \\ \text{s.t.} \qquad u^h(q^h) &= \bar{u}^h, \; h = 2, 3, \ldots, H \\ F^k(y^k) &= 0, \; k = 1, 2, \ldots, K \end{aligned}$$

First order conditions for internal optima now require

$$\begin{aligned} \frac{\partial u^1}{\partial q_i^1} &= \lambda^h \frac{\partial u^h}{\partial q_i^h}, \; h = 2, 3, \ldots, H \\ &= \mu^k \frac{\partial F^k}{\partial y_i^k}, \; k = 1, 2, \ldots, K \end{aligned}$$

where μ^k is the Lagrange multiplier on the technology constraint for the kth firm These imply

$$\frac{\partial u^h / \partial q_i^h}{\partial u^h / \partial q_j^h} = \frac{\partial F^k / \partial y_i^k}{\partial F^k / \partial y_j^k}$$

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Worked Example N : Contract Curve

Consider the two-person two-good Cobb-Douglas exchange economy described earlier. Assume that both individuals have the same preferences, $\alpha^A = \alpha^B = \alpha$. Pareto efficiency requires that the two individuals have the same MRS

$$\left(\frac{\alpha}{1-\alpha}\right)\frac{q_2^A}{q_1^A} = \left(\frac{\alpha}{1-\alpha}\right)\frac{q_2^A}{q_1^A}$$

which means both consume the two goods in the same ratio. Plainly that has to be the same as the ratio of economy-wide endowments Ω_2/Ω_1 where $\Omega_i = \omega_i^A + \omega_i^B$. The contract curve is therefore the diagonal of the Edgeworth-Bowley box defined by $q_i^A = \phi\Omega_i$, $q_i^B = (1-\phi)\Omega_i$, $i = 1, 2$, for all values of $0 \leq \phi \leq 1$.

Every one of the allocations of the contract curve is sustainable as a competitive equilibrium. Suppose, for example, that egalitarian principles mean that the government favours the equal allocation $q_i^A = q_i^B = \frac{1}{2}\Omega_i$, $i = 1, 2$. If it intervenes to reallocate endowments to this point so that $\omega_i^A = \omega_i^B = \frac{1}{2}\Omega_i$, $i = 1, 2$ then individuals will not trade away from consumption of those endowments.

for every household h, every firm k and every pair of goods $\{i, j\}$. Marginal rates of substitution need not only to be equated across consumers but also to be equated to common marginal rates of transformation among firms. Competitive equilibrium still achieves this because firms face the same price ratios as consumers.

Public goods

Public goods are goods that are usually characterised as *nonrival* and *nonexcludable*. Nonrivalry means that the benefits of consumption enjoyed by one person do not compromise those enjoyed by another. Nonexcludability means that noone can be prevented from enjoying the benefits once the good is provided. These characteristics are conceptually distinct and there are nonrivalrous goods that are excludable (broadcasting) and rivalrous goods from the benefits of which it is difficult to exclude people (common-pool resources like fisheries). Provision of public goods raises problems both at the national level but also at much smaller levels such as within individual households.

For simplicity, let us assume there is only a single private good, q^h, and extend the specification of utilities for individuals within the population $h = 1, 2, \ldots, H$ so as to include dependence not only on this good but also a collectively consumed public good Q, $u^h = u^h(q^h, Q)$. Let us also simplify production technology so that there is a constant marginal rate of transformation between

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the two goods at which P units of the private good can be converted into one unit of the public good. In other words $\Omega = \sum_h q^h + PQ$ for some fixed Ω.

Efficient supply of public goods

Finding the condition for efficient supply of the public good and allocation of private goods requires solving

$$\begin{aligned}\max_{q^2,\, q^3,\, \ldots} u^1\left(\Omega - \sum_{h=2,3,\ldots,H} q^h - PQ, Q\right) \\ \text{s.t.} \qquad u^h(q^h, Q) \;=\; \bar{u}^h,\ h = 2, 3, \ldots, H\end{aligned}$$

First order conditions for internal optima now require

$$\begin{aligned}\frac{\partial u^1}{\partial q^1} &= \lambda^h \frac{\partial u^h}{\partial q^h},\ h = 2, 3, \ldots, H \\ P\frac{\partial u^1}{\partial q^1} &= \frac{\partial u^h}{\partial Q} + \sum_{h=2,3,\ldots,H} \lambda^h \frac{\partial u^h}{\partial Q}\end{aligned}$$

which together imply

$$P \;=\; \sum_h \frac{\frac{\partial u^h}{\partial Q}}{\frac{\partial u^h}{\partial q^h}}$$

Optimal provision requires not that the marginal rate of substitution between the public and private good be equated for each consumer to P but rather that the sum be equated to P. This is sometimes known as the *Samuelson condition*. Provision of the public good benefits all consumers and these benefits need to be totalled up when considering efficiency of provision not considered separately.

Private and public provision of public goods

How then should public goods be provided? Suppose that individuals start with endowments of the private good from which they can contribute individually towards a common pool to be used for provision of the public good. Demands are now interdependent and we need to take that into account in deciding how to model decision making. Suppose that individuals take the contributions of others as given and consider the Nash equilibrium in private contributions G^h. Each individual solves

$$max_{G^h} u^h\left(\omega^h - PG^h, G^h + \sum_{g\neq h} G^g\right)$$

Some will not contribute at all but those who do will choose such that

$$P \;=\; \frac{\partial u^h/\partial Q}{\partial u^h/\partial q^h},$$

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equating individual MRS to P. It is clear that the condition for efficiency will not be satisfied. The sum of MRS will typically be much greater than P. The public good will therefore be underprovided as individuals fail to take account of the benefits to others from private provision.

This is usually taken as a case for collective public provision, financed by taxes and determined by some mechanism of collective choice. The appropriate nature of taxes and the elicitation of individual valuations of public provision is a topic pursued in courses on public economics.

It is the dependence of individual wellbeing on levels provided by others that generates the inefficiency in private provision. Similar problems will arise from other forms of interdependence. One person's consumption of a good can affect other's enjoyment negatively (for example, noise pollution) and there can be positive or negative interdependence between activities of different producers or between consumers and producers. These are all examples of the more general phenomenon of externalities.

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Worked Example O : Public Goods

Consider an economy of N individuals consuming a public good Q and a private good q^h, $h = 1, 2, \ldots, N$. All have preferences described by Cobb-Douglas utility $u(q^h, Q) = \alpha \ln q^h + (1-\alpha)\ln Q$. Individuals begin with endowments of the private good ω^h but there is technology through which endowments can be transformed into the public good. The economy has a constant marginal rate of transformation P between the two goods such that its production possibilities are $\sum_h q^h + PQ = \Omega$ where $\Omega = \sum_h \omega^h$.

Pareto optimal supply requires the sum of individual marginal rates of substitution equal the marginal rate of transformation

$$\left(\frac{\alpha}{1-\alpha}\right)\frac{\sum_h q^h}{Q} = P$$

Substituting into the production possibility frontier gives $PQ = \alpha\Omega$.

If the public good is supplied privately then, assuming each individual contributes positively, each individual supplies only up to the point where

$$\left(\frac{\alpha}{1-\alpha}\right)\frac{q^h}{Q} = P.$$

Each individual consumes the same quantity of the private good, $q^h = (1-\alpha)PQ/\alpha$, and substitution of this into the production possibility frontier gives $PQ = \alpha\Omega/(\alpha + N - \alpha N)$. The public good is inefficiently undersupplied.

If any individual has endowment below the common private consumption so that $\omega^h < (1-\alpha)\Omega/(\alpha + N - \alpha N)$ then they will not contribute. Equilibrium provision will be determined by decisions among those who are contributors.

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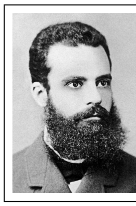
History Note XII : Welfare in General Equilibrium

The ethical defensibility of market outcomes has been a persistent matter of political contention that has attracted the interest of economists for centuries. The interpretation of the welfare theorems as vindicating the observations of Adam Smith (1723-90) on the workings of an "invisible hand" in market economies is common, if arguably strained.

The Pareto criterion for welfare comparison is owed to the Italian social thinker Vilfredo Pareto (1848-1923) and fits naturally with the withdrawal from willingness to make interpersonal comparisons of utility associated with his recognition that consumer theory relies only on ordinal description of preferences. Pareto himself understood that exchange equilibrium exhausted the scope for Pareto improvements. Proof of the first and second fundamental welfare theorems are associated principally with work in the 1930s and 1940s by Abba Lerner (1903-82) and Oskar Lange (1904-65), extended in the 1950s by Kenneth Arrow (1921-2017, Nobel 1972). Lange and Lerner were both socialists by inclination concerned with the possibility for decentralising efficient decision making under public ownership, Lerner a Russian-born economist who worked in the UK then the US and Lange a Polish economist who worked also in the UK and US before returning to communist Poland after the Second World War.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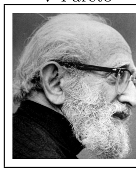
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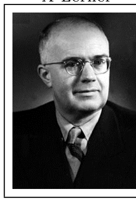
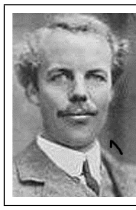

O Lange

History Note XIII : Public Goods, Externalities and Collective Action

Recognition that collective consumption creates a case for state intervention can be found in Adam Smith (1723-90). Conditions for optimum supply of a public good were proved in the 1950s by American economist Paul Samuelson (1915-2009, Nobel 1970). The economic concept of externalities, the difficulties posed by them for market allocation and the possibility of remedies through taxation were discussed by English economist Arthur Cecil Pigou (1877-1959) in the 1920s. The possibility that well-defined property rights over externality-producing activities would nonetheless permit the achievement of efficient solutions through bargaining was discussed by the British economist Ronald Coase (1910-2013, Nobel 1991) in the 1960s. Elinor Ostrom (1933-2012, Nobel 2009) has studied the practice of economic governance of common property resources.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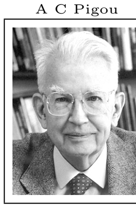
R Coase


E Ostrom

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