Econometrics

Preliminaries: Asymptotic Theory

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Outline

Asymptotic Theory

- The Law of Large Numbers
- The Central Limit Theorem

Asymptotic Theory

Setup: Population and Unknown parameter

Example: Mean starting salary of newly graduated PPE students last year from the University of Oxford. Collect a random sample of n PPE students graduated last year

Population	Random Sample
X	$\{x_1, x_2, ..., x_n\}$
μ, σ^2	$\bar{x}_n$ (sample analog)

Random sample: independent and identically distributed random variables $\{x_i\} \sim iid\left(\mu, \sigma^2\right)$ (both μ and σ^2 are assumed to be finite, exist)

Asymptotic Theory

LLN and CLT are about the behaviour of the sample mean

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

as the number of observations, n, gets large

Note: since x_i are random variables, $\bar{x}_n$ is also a random variable

Asymptotic Theory

A first look at the random variable: $\bar{x}_n$

(i) Expected Value

$$E\left[\bar{x}_n\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E\left[x_i\right] = \mu$$

Asymptotic Theory

A first look at the random variable: $\bar{x}_n$

(ii) Variance

$$\begin{aligned}
V[\bar{x}_n] &= V\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} V[x_i] + \sum_{i=1}^{n}\sum_{j\neq i}^{n} Cov[x_i, x_j]\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} V[x_i] \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Asymptotic Theory

A first look at the random variable: $\bar{x}_n$

(iii) Standard Deviation

$$s.d.\left[\bar{x}_n\right] = \frac{\sigma}{\sqrt{n}}$$

Asymptotic Theory

- If $\{x_i\} \sim iidN\left(\mu, \sigma^2\right)$ then $\bar{x}_n \sim N\left(\mu, \sigma^2/n\right)$ for each n
- Unfortunately: if $\{x_i\}$ are not normal, exact sampling distribution of $\bar{x}_n$ can be very complicated
- Fortunately, LLN and CLT: large sample ($n \to \infty$) approximations to the sampling distribution of $\bar{x}_n$. Normality is not required: Very powerful!
- LLN and CLT: very important results in probability theory + play a crucial role in statistics (estimation and inference)

Law of Large Numbers

- LLN: Let $\{x_i\} \sim iid\left(\mu, \sigma^2\right)$ (both μ and σ^2 are assumed to be finite, exist), then $\bar{x}_n \xrightarrow{p} \mu$ as $n \to \infty$. **(Chebyshev)**
- Convergence in probability means that: For any $c > 0$ $P\left(|\bar{x}_n - \mu| > c\right) \longrightarrow 0$ as $n \to \infty$
- LLN: conditions for $\bar{x}_n$ to be close to μ with high probability when n is large

Law of Large Numbers

- Very useful result will help us to show the LLN: Convergence in mean square implies convergence in probability! (Due to Chebyshev)

- Convergence in mean square: $\lim_{n\to\infty} E\left[(\bar{x}_n - \mu)^2\right] = 0$

- Note: $E\left[(\bar{x}_n - \mu)^2\right] = V\left[\bar{x}_n\right]$; hence,

$$\lim_{n\to\infty} E\left[(\bar{x}_n - \mu)^2\right] = \lim_{n\to\infty} V\left[\bar{x}_n\right] = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0,$$

which shows that $\bar{x}_n \xrightarrow{p} \mu$ as $n \to \infty$, i.e., the sample mean is a consistent estimator of the population mean

The Central Limit Theorem

- Approximate distribution of $\bar{x}_n$ when n is large?
- Recall: $\bar{x}_n \xrightarrow{p} \mu$ or equivalently that $\bar{x}_n - \mu \xrightarrow{p} 0$ since $\lim_{n\to\infty} V[\bar{x}_n] = 0$. This means $\bar{x}_n$ has a degenerate distribution in the limit (takes only a single value!)
- Lets consider

$$z_n \equiv = \frac{\bar{x}_n - E[\bar{x}_n]}{\sqrt{V[\bar{x}_n]}} = \frac{\bar{x}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}\,(\bar{x}_n - \mu)}{\sigma}$$

The Central Limit Theorem

- Expected Value

$$E\left[z_n\right] = E\left[\frac{\bar{x}_n - E\left[\bar{x}_n\right]}{\sqrt{V\left[\bar{x}_n\right]}}\right] = \frac{1}{\sqrt{V\left[\bar{x}_n\right]}} E\left[\bar{x}_n - E\left[\bar{x}_n\right]\right] = 0$$

- Variance:

$$\begin{aligned} V\left[z_n\right] &= V\left[\frac{\bar{x}_n - E\left[\bar{x}_n\right]}{\sqrt{V\left[\bar{x}_n\right]}}\right] \\ &= \frac{1}{V\left[\bar{x}_n\right]} V\left[\bar{x}_n - E\left[\bar{x}_n\right]\right] \\ &= \frac{1}{V\left[\bar{x}_n\right]} V\left[\bar{x}_n\right] \\ &= 1 \end{aligned}$$

The Central Limit Theorem

- CLT: **(Lindeberg-Levy)**: Let $\{x_i\} \sim iid\left(\mu, \sigma^2\right)$ (both μ and σ^2 are assumed to be finite, exist), then $z_n \xrightarrow{d} N\left(0, 1\right)$

- Conditions under which z_n converges in distribution to a standard normal random variable

- Asymptotic distribution of z_n is $N\left(0, 1\right)$ or $z_n \overset{A}{\sim} N\left(0, 1\right)$ or $\bar{x}_n \overset{A}{\sim} N\left(\mu, \sigma^2/n\right)$

The Central Limit Theorem

- $\xrightarrow{d}$ means that the sample or empirical cumulative distribution function of $\bar{x}_n$ converge (as $n \to \infty$) to the cumulative distribution function of a standard normal:

$$\lim_{n \to \infty} F_n \left(\bar{x}_n \right) = F \left(x \right),$$

where

$$F \left(x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

The Central Limit Theorem

- Lindeberg-Levy CLT can proved via the characteristic function of a random variable: $E\left[e^{i\lambda x}\right] = \int_{-\infty}^{\infty} e^{i\lambda x} dF(x)$, it completely defines its probability distribution

- Proof of the (Lindeberg-Levy) CLT uses the following result:
  If $E\left[e^{i\lambda y_n}\right] \to E\left[e^{i\lambda y}\right]$ for every λ and $E\left[e^{i\lambda y}\right]$ is continuous at $\lambda = 0$, then $y_n \xrightarrow{d} y$

- In the CLT, $y_n = \bar{x}_n$ and y is a standard normal. Idea: show that the characteristic function of $\bar{x}_n$ converges to that of a $N(0, 1)$, see, for instance, Amemiya (1985) p.91

LLN and CLT: Uses

- **LLN**: Consistent estimates

- **CLT**: Inferences. For instance, confidence intervals

$$z_n = \frac{\bar{x}_n - \mu}{s.d.\left(\bar{x}_n\right)} \overset{A}{\sim} N\left(0, 1\right)$$

Hence,

$$P\left(-1.96 \leq \frac{\bar{x}_n - \mu}{s.d.\left(\bar{x}_n\right)} \leq 1.96\right) = 0.95$$

Or

$$P\left(\bar{x}_n - 1.96 s.d.\left(\bar{x}_n\right) \leq \mu \leq \bar{x}_n + 1.96 s.d.\left(\bar{x}_n\right)\right) = 0.95$$

Therefore, $CI : \bar{x}_n \pm 1.96 s.d.\left(\bar{x}_n\right)$