Lecture Notes 9

Linear Programming; Standard LP; Simplex Algorithm; Two Phase Simplex Method; Duality

Linear Program (LP)

A linear program is an optimization problem of the form

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

where $c \in R^n$, b$\in R^m$, A$\in R^{m\times n}$.

Notes:

- $x \geq 0$ means that each component is non-negative
- Objective function is linear
- Constraint set consists of linear equations/inequalities

(convex set)

LP Example: Production Scheduling

Woodworking shop data:

Input	Product		Input Availability
	Table	Chair	
Labour	5	6	30
Materials	3	2	12
Production Levels	x_1	x_2	
Unit price	1	5	

- Goal: Schedule production to maximize total revenue $x_1 + 5x_2$
- Production constraints:
  - Labour constraint: $5x_1 + 6x_2 \leq 30$
  - Materials constraint: $3x_1 + 2x_2 \leq 12$
  - Physical constraints: $x_1 \geq 0,\ x_2 \geq 0$

LP in Matrix Form

- LP in matrix form

maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

where $A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 30 \\ 12 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

- Equivalent form

minimize $-c^T x$

subject to $Ax \leq b$

$x \geq 0$

LP Example: Optimal Diet

Nutrition table:

Vitamin	Milk	Eggs	Daily Requirement
F	3	7	45
G	4	2	60
Intake	x_1	x_2	
Unit cost	2	5	

- Goal: Minimize total cost $2x_1 + 5x_2$
- Dieatary constraints:
  - Vitamin F constraint: $3x_1 + 7x_2 \geq 45$
  - Vitamin G constraint: $4x_1 + 2x_2 \geq 60$
  - Physical constraints: $x_1 \geq 0,\ x_2 \geq 0$

Optimal Diet LP in Matrix Form

- LP in matrix form

minimize $c^T x$

subject to $Ax \geq b$

$x \geq 0$

where $A = \begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 45 \\ 60 \end{bmatrix}$, $c = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

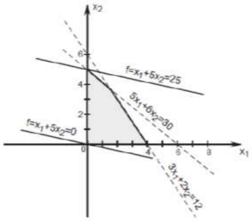
Geometric Method for LP in 2-D

- Reconsider the LP in matrix form

maximize $c^T x$

subject to $Ax \le b$

$x \ge 0$



where $A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 30 \\ 12 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

- Level sets of the objective function

$c^T x = f$ (parallel lines; moving up as f increases)

- The maximum value 25 is reached when the level set is exiting the constraint set at the corner point $[0\ 5]^T$
- The minimum value 0 is reached at the corner point $[0\ 0]^T$
- It turns out that solution of an LP problem (if it exists) always lies on a vertex of the constraint set (focus on vertices)

Standard Form LP

- LP in standard form:

minimize $c^T x$

subject to $Ax = b$

$x \ge 0$

where $c \in R^n$, $b \in R^m$, $A \in R^{m \times n}$, $b \ge 0$, rank(A)=m, $m < n$

- Any LP problem can be converted into an equivalent standard form LP problem
- If it is a maximization, multiply the objective function by -1
- If A is not of full row rank, remove one or more rows
- If the i-th component of b is negative, multiply that row by -1
- Inequality constraints: Add slack (surplus) variables for $\le$ ($\ge$) constraints

Converting to Standard Form LP

- Slack Variables: Suppose we have $\le$ constraint as $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$. Introduce a slack variable x_{n+1} to get the equivalent constraint:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1$$
$$x_{n+1} \ge 0$$

- Surplus Variables: Suppose we have $\ge$ constraint as $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ge b_1$. Introduce a surplus variable x_{n+1} to get the equivalent constraint:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - x_{n+1} = b_1$$
$$x_{n+1} \ge 0$$

Converting to Standard Form LP

- Non-positive Variables: Suppose one of the variables (say, x_1) has the constraint we have $x_1 \le 0$
  - Let $x'_1 = -x_1$ and replace every occurance of x_1.
- Free Variables: Suppose one of the variables (say, x_1) does not have any constraint.
  - Split the variable into two, i.e., replace $x_1 = u_1 - v_1$ and use the constraints $u_1 \ge 0,\ v_1 \ge 0$.

Example: Converting to Standard LP

- Consider the LP Problem which is not in standard form

maximize $3x_1 + 5x_2 - x_3$

subject to $x_1 + 2x_2 + 4x_3 \le -4$

$-5x_1 - 3x_2 + x_3 \ge 15$

$x_2 \le 0\ ,\ x_3 \ge 0$

- Equivalent LP in Standard Form:

minimize $-3(x_6 - x_7) + 5x_2' + x_3$

subject to $-(x_6 - x_7) + 2x_2' - 4x_3 - x_4 = 4$

$-5(x_6 - x_7) + 3x_2' + x_3 - x_5 = 15$

$x_2', x_3, x_4, x_5, x_6, x_7 \ge 0$

Basic Solutions

- Consider LP problem in standard form:

minimize $c^T x$

subject to $Ax = \mathrm{b}$

$x \ge 0$

where A is a m x n matrix, m < n; $rank(A) = m$; $b \ge 0$

- Feasible points: x that satisfy $Ax = \mathrm{b}$ and $x \ge 0$
- Recall the three types of elementary row operations
  1) Inter-change two rows of the matrix
  2) Multiply one row of the matrix by a (nonzero) constant
  3) Add a constant multiple of one row to another row

Basic Solutions

- To solve $Ax = b$ using elementary row operations
- Form the augmented matrix $[A\ b]$
- Reduce $[A\ b]$ by elemantary row operations (using Gauss-Jordan algorithm) to a unique reduced Echelon form (called Canonical augmented matrix)

$$[I_m\ Y_{m,n-m}\ y_0]$$

- $x^* = \begin{bmatrix} y_0 \\ 0_{n-m,1} \end{bmatrix}$ is a basic solution (it is called a degenerate basic solution if any component of y_0 is zero)

- Basic columns: First m columns of A
- Basic variables: First m variables x_1, x_2, ..., x_m
- Non-basic variables: Last n-m variables x_{m+1},x_{m+2}, ..., x_n

Basic Solutions-Example

- Pick any m linearly independent columns of A: a_{k_1},a_{k_2}, ...,a_{k_m}, then x_{k_1},x_{k_2}, ...,x_{k_m} are the basic variables
- There are at most $\binom{n}{m}$ basic solutions

Example: Consider $Ax = b$ with $\mathrm{A} = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

- For basis $\mathrm{B} = [a_1\ a_2]$, $\mathrm{x} = [6\ 2\ 0\ 0]^T$ is a basic solution
- For basis $\mathrm{B} = [a_2\ a_3]$, $\mathrm{x} = [0\ 2\ -6\ 0]^T$ is a basic solution
- For basis $\mathrm{B} = [a_3\ a_4]$, $\mathrm{x} = [0\ 0\ 0\ 2]^T$ is a basic solution
- For basis $\mathrm{B} = [a_1\ a_4]$, $\mathrm{x} = [0\ 0\ 0\ 2]^T$ is a basic solution
- For basis $\mathrm{B} = [a_2\ a_4]$, $\mathrm{x} = [0\ 0\ 0\ 2]^T$ is a basic solution
- For basis $\mathrm{B} = [a_2\ a_4]$, $\mathrm{x} = [0\ 0\ 0\ 2]^T$ is a basic solution
- For $\mathrm{B} = [a_1\ a_3]$; inconsistent system of equations
- Note that $\mathrm{x} = [\ 3\ 1\ 0\ 1]^T$ is a solution but it is not basic

Basic Feasible Solutions

- Basic Feasible Solution (BFS): A basic solution that satisfies the constraints $\mathrm{A}x = \mathrm{b}$ and $x \geq 0$

Example: Consider $Ax = b$ with $\mathrm{A} = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

- $\mathrm{x} = [6\ 2\ 0\ 0]^T$ is a BFS
- $\mathrm{x} = [0\ 2\ -6\ 0]^T$ is a basic solution but not feasible
- $\mathrm{x} = [0\ 0\ 0\ 2]^T$ is a degenerate BFS
- $\mathrm{x} = [\ 3\ 1\ 0\ 1]^T$ is feasible but not basic

- Geometrically, a BFS corresponds to a 'corner' point (vertex) of the constraint set
- In the sequel, we assume non-degenerate BFS for simplicity of derivations (results can also be extended to degenerate BFS)

Optimal Solutions

- A feasible point that minimizes the objective function is called an optimal feasible solution
- A BFS that is also optimal is called an optimal basic feasible solution

Fundamental Theorem of LP:

Consider an LP problem in standard form.

1. If there exists a feasible solution, then there exists a BFS.

2. If there exists an optimal feasible solution, then there exists an optimal BFS.

Fundamental Theorem Consequences

- Suppose that a solution exists for the LP. To find the solution, search among the set of BFS.

- The fundamental theorem of LP reduces the search from infinite points to among finite number of points
- It is still computationally expensive; e.g., consider n=50, m=5, we have to search among 2.118.760 possibilities
- Therefore, the brute force approach of exhaustively comparing all the BFSs is impractical

- Simplex method: an organized way of going from one BFS to another to search for the global minimizer.

Canonical Augmented Matrix

- Given m linearly independent columns of A: a_{k_1},a_{k_2}, ...,a_{k_m}, then x_{k_1},x_{k_2}, ...,x_{k_m} are the basic variables
- To solve $Ax = b$ using elementary row operations
- Reduce the augmented matrix $[A\ b]$to canonical form where

the k_1-th column is $\mathrm{e}_1 = [1\ 0\ \ldots\ 0]^T$, k_2-th column is $\mathrm{e}_2 = [0\ 1\ \ldots\ 0]^T$, ..., k_m-th column is $\mathrm{e}_m = [0\ 0\ \ldots\ 1]^T$, and the last column becomes, say y_0

- Since x_{k_1},x_{k_2}, ...,x_{k_m} are the basic variables, we have $x_{k_i} = y_{0i}$, i = 1,2, ..., m; i.e., y_0 is the coordinates of b wrt the basis $\{a_{k_1}$,a_{k_2}, ...,$a_{k_m}\}$

- Example: If the first m columns is a basis, we have

$$[I_m\ Y_{m,n-m}\ y_0]$$

i.e., $x^* = \begin{bmatrix} y_0 \\ 0_{n-m,1} \end{bmatrix}$ is a basic solution

Pivoting

- Given $A = [a_{ij}]$, b, to pivot $[A \;\; b]$ about the (p,q)-th element a_{pq} means:
  - Divide the p-th row by a_{pq}
  - Then, for each row $i \neq p$, subtract a_{iq} times the p-th row from the i-th row, i.e.,

$$\text{row}_i \to \text{row}_i - \frac{a_{iq}}{a_{pq}} \text{row}_p$$

- In the resulting matrix, the q-th column is e_p .

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{p,q-1} & a^*_{pq} & a_{p,q+1} & \cdots & b_p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,q-1} & a_{iq} & a_{i,q+1} & \cdots & b_q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \to$$

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{a_{p,q-1}}{a_{pq}} & 1 & \frac{a_{p,q+1}}{a_{pq}} & \cdots & \frac{b_p}{a_{pq}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{iq-1} - \frac{a_{iq}}{a_{pq}} a_{p,q-1} & 0 & a_{i,q+1} - \frac{a_{iq}}{a_{pq}} a_{p,q+1} & \cdots & b_q - \frac{a_{iq}}{a_{pq}} b_p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{bmatrix}$$

Updating the Augmented Matrix

- Consider $Ax = b$ where $A = [a_1 \;\; a_2 \; \ldots \; a_n]$
- For simplicity, assume $\{a_1, a_2, \ldots, a_m\}$ be a basis, i.e., $x_1, x_2, \ldots, x_m$ be the basic variables
- The canonical augmented matrix can be derived by pivoting about the element (1,1), then (2,2), ..., (m,m)

- Suppose we want to replace the basis vector a_p, with a new basis vector a_q, $q > m$:
  new basis $\{a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_m, a_q\}$
- a_p , x_p: departing basis vector, departing basic variable (DV)
- a_q , x_q: entering basis vector, entering basic variable (EV)

- The canonical augmented matrix for the new basis can be obtained by pivoting about (p,q)

Example

Consider $Ax = b$ with $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

- Let the basis be $B = [a_1 \;\; a_2]$, then

$$[A \;\; b] \to \begin{bmatrix} 1 & 0 & -1 & 3 & 6 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

$x = [6 \;\; 2 \;\; 0 \;\; 0]^T$ is a basic feasible solution

- Let the new basis be $B = [a_2 \;\; a_3]$,
- x_1: DV, x_3: EV; pivot about (1,3) component

- $[A \;\; b] \to \begin{bmatrix} 1 & 0 & -1 & 3 & 6 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 & -3 & -6 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$

$x = [0 \;\; 2 \;\; -6 \;\; 0]^T$ is a basic solution but not feasible

Simplex Method: Basic Idea

- Start with an initial basis corresponding to a BFS
- Then move to an adjacent BFS (adjacent vertex) in such a way that the objective function decreases
- If the stopping criterion is satisfied, we stop; otherwise, we repeat the process.

- To move from one basic solution to an adjacent one:

If a_p leaves the basis and a_q enters the basis, we pivot about the (p,q) position of the canonical augmented matrix

- Two questions that arise:

1. How do we ensure that the adjacent basic solution we move to is feasible?
2. How do we choose EV x_q to ensure the objective function at the new basic solution is smaller than the previous value?

DV Choice-Feasibility

- Feasibility: the last column should be non-negative
- Consider $Ax = b$ where $A = [a_1 \;\; a_2 \; \ldots \; a_n]$
- Suppose we have a set of basic variables corresponding to a BFS; For simplicity, assume it is $x_1, x_2, \ldots, x_m$ with the corresponding basis $\{a_1, a_2, \ldots, a_m\}$

- We want to replace one of the basis vector a_p, with a new basis vector a_q, $q > m$:
- Suppose the canonical augmented matrix for the original basis is ($y_{i0} \geq 0$, $i = 1, 2, \ldots, m$, due to initial feasibilitiy)

$$[A, b] = \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}$$

DV Choice-Feasibility

- If for example, a_{m+1} replaces a_1, then the canonical augmented matrix becomes, where the last column is required to be non-negative for feasible basic solution

$$[A, b] \to \begin{bmatrix} \frac{1}{y_{1m+1}} & 0 & \cdots & 0 & 1 & \cdots & \frac{y_{1n}}{y_{1m+1}} & \frac{y_{10}}{y_{1m+1}} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}$$

$$\to \begin{bmatrix} \frac{1}{y_{1m+1}} & 0 & \cdots & 0 & 1 & \cdots & \frac{y_{1n}}{y_{1m+1}} & \frac{y_{10}}{y_{1m+1}} \\ -\frac{y_{2m+1}}{y_{1m+1}} & 1 & \cdots & 0 & 0 & \cdots & y_{2n} - \frac{y_{1n}}{y_{1m+1}} y_{2m+1} & y_{20} - \frac{y_{10}}{y_{1m+1}} y_{2m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ \frac{y_{mm+1}}{y_{1m+1}} & 0 & \cdots & 1 & 0 & \cdots & y_{mn} - \frac{y_{1n}}{y_{1m+1}} y_{mm+1} & y_{m0} - \frac{y_{10}}{y_{1m+1}} y_{mm+1} \end{bmatrix}$$

DV Choice-Feasibility

- In general, if basis vector a_p is replaced with a new basis vector a_q, then by pivoting about (p,q), the last column has the following components:

  p-th element: $\frac{y_{p0}}{y_{pq}}$, all other elements: $y_{i0} - \frac{y_{p0}}{y_{pq}} y_{iq}$, $i \neq p$
- If $y_{pq} < 0$, then the feasibility set is unbounded
- To ensure feasibility, choose p s.t. $y_{pq} > 0$ and all components are non-negative:

$$y_{i0} - \frac{y_{p0}}{y_{pq}} y_{iq} = y_{iq} \left(\frac{y_{i0}}{y_{iq}} - \frac{y_{p0}}{y_{pq}}\right) \geq 0$$

$$\rightarrow \ p = argmin \left\{\frac{y_{i0}}{y_{iq}}\right\}$$

EV Choice-Optimality

- The goal is to choose the new basis vector a_q so that the new BFS has lower objective function value
- Suppose the canonical augmented matrix for the original basis is:

$$[A, b] = \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}$$

- Basic cost function coefficients are those associated with the basic variables: $c_0 = [c_1 \ \ c_2 \ ... c_m]^T$
- The objective function for the BFS $x^* = [y_{10} \ \ ... \ \ y_{m0} \ \ 0 \ \ ... 0]^T$

$$z_0 = c^T x^* = \sum_{i=1}^{m} c_i y_{i0}$$

EV Choice-Optimality

- Now, consider a new basis where a_q enters the basis and a_p leaves the basis
- The new BFS is

$$x^{**} = \left[y_{10} - \frac{y_{p0}}{y_{pq}} \ y_{1q}, \ldots, 0, y_{m0} - \frac{y_{p0}}{y_{pq}} \ y_{mq} \ \ , 0, \ldots, 0 \ \ \frac{y_{p0}}{y_{pq}} \ \ \ 0 \ldots \ \ 0\right]^T$$

- The objective function for the new BFS x^{**}

$$z = c^T x^{**} = \sum_{i=1, i\neq p}^{m} c_i \left(y_{i0} - \frac{y_{p0}}{y_{pq}} \ y_{iq}\right) + c_q \frac{y_{p0}}{y_{pq}}$$

$$= z_0 + (c_q - \sum_{i=1}^{m} c_i y_{iq}) \frac{y_{p0}}{y_{pq}}$$

- Note that the q-th column of A is $[y_{1q} \ \ y_{2q} \ \ ... y_{mq}]^T$
- The q-th component of $c_0^T A$: $z_q = \sum_{i=1}^{m} c_i y_{iq}$
- Therefore, $z = z_0 + (c_q - z_q) \frac{y_{p0}}{y_{pq}}$; $z < z_0$ if $c_q < z_q$

EV Choice-Optimality

- Define the relative cost coefficients (RCC)

$$r_i = \begin{cases} 0, & for \ i = 1,2, \ldots, m \ (basic) \\ c_i - z_i, & for \ i = m+1, , \ldots, n \ (non-basic) \end{cases}$$

- Note: the RCC for a basic variable is always 0.
- If $r_q < 0$, then the new BFS is better.
- RCC vector r is $r = c^T - c_0^T A$
- Optimality conditions: $r = c^T - c_0^T A \geq 0$
- In summary, we choose a variable x_q that has negative RCC value as the entering variable.

Theorem: A BFS is optimal if and only if the corresponding RCC values (components of $r = c^T - c_0^T A$) are all non-negative.

The Simplex Algorithm

1. Form a canonical augmented matrix corresponding to an initial basic feasible solution
2. Calculate the RCCs $r = c^T - c_0^T A$ corresponding to the non-basic variables
3. If $r_j > 0$ for all j, then stop; the current BFS is optimal
4. Select a q such that $r_q < 0$
5. If no $y_{iq} > 0$, stop-the problem is unbounded; else calculate $p = argmin \left\{\frac{y_{i0}}{y_{iq}} : \ y_{iq} > 0\right\}$
6. Update the canonical augmented matrix by pivoting about the (p,q)-th element
7. Go to step 2.

Simplex Tableau

- Consider LP problem in standard form:

  minimize $c^T x$

  subject to $Ax = b$

  $x \geq 0$

where A is a $m \times n$ matrix; $rank(A) = m$; $b \geq 0$

- Suppose that we already know one set of basic variables x_0; Let c_0 be the basic cost coefficient in c associated with basic variable x_0
- Ex: if $x_0 = \{x_1, x_4, x_7\}$ are the set of basic variables, then

$$c_0 = [c_1 \ \ c_4 \ \ c_7]^T$$

Simplex Tableau

	x^T c^T	
x_0, c_0	A	b
	$r = c^T - c_0^T A$	$-c_0^T b$

- 2nd row and column can be deleted to obtain the final simplex tableau:

	x^T	
x_0	A	b
	$c^T - c_0^T A$	$-c_0^T b$

Example

- minimize $-120x_1 - 80x_2$
  subject to $2x_1 + x_2 \le 6$
  $7x_1 + 8x_2 \le 28$
- Write in standard form
  minimize $-120(y_1 - z_1) - 80(y_2 - z_2)$
  subject to $2(y_1 - z_1) + (y_2 - z_2) + w_1 = 6$
  $7(y_1 - z_1) + 8(y_2 - z_2) + w_2 = 28$
  $y_1, z_1, y_2, z_2, w_1, w_2 \ge 0$
- The initial feasible solution x^*
- $x_0 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $c_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $c^T - c_0^T A = c^T$

$$x^* = \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 28 \end{bmatrix}.$$

Example

- Simplex Tableau

		y_1	z_1	y_2	z_2	w_1	w_2	
		-120	120	-80	80	0	0	
w_1	0	2	-2	1	-1	1	0	6
w_2	0	7	-7	8	-8	0	1	28
		-120	120	-80	80	0	0	0

- Final Simplex Tableau

	y_1	z_1	y_2	z_2	w_1	w_2	
w_1	2	-2	1	-1	1	0	6
w_2	7	-7	8	-8	0	1	28
	-120^*	120	-80	80	0	0	0

Simplex Method

1. Locate the most negative number in the bottom row of the simplex tableau, excluding the last column. The variable associated with this column is E.V. We also call it the work column. If more than one candidate for the most negative numbers exist, pick any one.
2. Form ratios by dividing each positive number in the working column, excluding the last row, into the element in the same row and last column. Designate the element in the work column that yields the smallest ratio as the pivot element. The basic variable associated with this pivot element is D.V. If no element in the work column is positive, stop. The problem has no solution.
3. Pivoting about the pivot element: Use elementary row operations to convert the pivot element to 1 and then to reduce all other elements in the work column to 0

Simplex Method

4. Swap E.V. with D.V. Replace the variable in the pivot row and the first column by the variable in the first row and work column. The new first column is the current set of basic variables.
5. Repeat Step 1 through Step 4 until there are no negative numbers in the last row, excluding the last column. The basic feasible solution derived from the very last set of variable is a minimal point: assign to each basic variable (variables in the first column) the number in the same row and last column, and all others are zero. The negative number of the number in the last row and last column is the minimum value.

Example

- Consider

	y_1	z_1	y_2	z_2	w_1	w_2	
w_1	2^*	-2	1	-1	1	0	6
w_2	7	-7	8	-8	0	1	28
	-120	120	-80	80	0	0	0

- Step 1: The work column is the first (-120 is the most negative).
- Step 2: The ratios are $\begin{bmatrix} 6 \\ 28 \end{bmatrix} : \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

So 2^* is the pivot; w_1 is DV; y_1 is EV
- Step 3: Perform row operations to obtain the updated simplex table

	y_1	z_1	y_2	z_2	w_1	w_2	
y_1	1	-1	$1/2$	$-1/2$	$1/2$	0	3
w_2	0	0	$9/2^{**}$	$-9/2$	$-7/2$	1	7
	0	0	-20^*	20	60	0	360

Example

- Repeat the above steps; work with column that includes -20
- The ratios are $\begin{bmatrix}3\\7\end{bmatrix}:\begin{bmatrix}1/2\\9/2\end{bmatrix}=\begin{bmatrix}6\\14/9\end{bmatrix}$

So $9/2^{**}$ is the pivot; w_2 is DV; y_2 is EV

- Perform row operations to obtain the updated simplex table

	y_1	z_1	y_2	z_2	w_1	w_2	
y_1	1	-1	0	0	$\frac{8}{9}$	$-\frac{1}{9}$	$\frac{20}{9}$
y_2	0	0	1	-1	$-7/9$	$2/9$	$14/9$
	0	0	0	0	$\frac{400}{9}$	$40/9$	$\frac{3520}{9}$

- Since last row is non-negative, we are done!
- Optimal solution: $\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}20/9\\14/9\end{bmatrix}$, $z_1=z_2=\mathrm{w}_1=w_2=0$
- Optimal cost: -3520/9; original variables: $x_1=20/9$, $x_2=14/9$

Two Phase Simplex Method

- The simplex method requires an initial basis
- Brute force approach: Arbitrarily choose m basic columns and transform the augmented matrix for the problem into canonical form. If right-most column is positive, then we have a legitimate (initial) BFS. Otherwise, try again
- Brute force approach not practical as it requires potentially $\binom{n}{m}$ tries
- Consider LP problem in standard form:

$$\begin{aligned}&\text{minimize} \quad c^T x\\&\text{subject to} \quad Ax=\mathrm{b}\\&\qquad\qquad\quad x\geq 0\end{aligned}$$

where A is a $m\times n$ matrix; $\mathrm{r}ank(A)=m$; $b\geq 0$

Artificial LP

- LP problem in standard form:

$$\begin{aligned}&\text{minimize} \quad c^T x\\&\text{subject to} \quad Ax=\mathrm{b}\\&\qquad\qquad\quad x\geq 0\end{aligned}$$

where A is a $m\times n$ matrix; $\mathrm{r}ank(A)=m$; $b\geq 0$

- Associated Artificial LP problem:

$$\begin{aligned}&\text{minimize} \quad \mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_\mathrm{m}\\&\text{subject to} \quad [A \quad Im]\begin{bmatrix}x\\y\end{bmatrix}=\mathrm{b}\\&\qquad\qquad\quad \begin{bmatrix}x\\y\end{bmatrix}\geq 0\end{aligned}$$

where $y=[\mathrm{y}_1 \ \ \mathrm{y}_2 \ \ldots \ \mathrm{y}_\mathrm{m}]^T$ is the vector of artifical variables

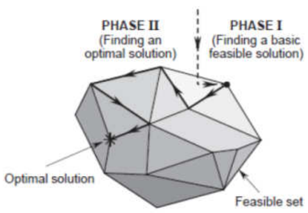
- Note that Artificial LP has an obvious BFS: $\begin{bmatrix}0\\b\end{bmatrix}$

Artificial LP

- Proceed with the simplex algorithm to solve the artificial LP
- Note that for $\mathrm{y}\geq 0$, we have $\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_\mathrm{m}\geq 0$ and the strict inequality holds if one $\mathrm{y}_\mathrm{i}>0$.
- If the minimum value of the artificial LP=0, then y_i must be non-basic, i.e., $\mathrm{y}_\mathrm{i}=0, \mathrm{i}=1,2,\ldots,\mathrm{m}$.
- Proposition: The original LP problem has a BFS if and only if the associated artificial problem has an optimal feasible solution with objective function value zero.
- Note:In this case, the optimal solution for the artificial LP serves as a BFS for the original LP.

Two Phase Simplex Method

- Phase I: Solve the artificial problem using simplex method
- Phase II: Use the BFS resulting from phase I to initialize the simplex algorithm to solve the original LP problem.



Example

- $$\begin{aligned}&\text{minimize} \quad 2x_1+3x_2\\&\text{subject to} \quad 4x_1+2x_2\geq 12\\&\qquad\qquad\quad x_1+4x_2\geq 6\\&\qquad\qquad\quad x_1,x_2\geq 0\end{aligned}$$
- Standard form:

$$\begin{aligned}&\text{minimize} \quad 2x_1+3x_2\\&\text{subject to} \quad 4x_1+2x_2-x_3=12\\&\qquad\qquad\quad x_1+4x_2-x_4=6\\&\qquad\qquad\quad x_1,x_2,x_3,x_4\geq 0\end{aligned}$$

- No obvious BFS

Example: Phase I

- Phase I:

  minimize $x_5 + x_6$

  subject to $4x_1 + 2x_2 - x_3 + x_5 = 12$

  $x_1 + 4x_2 - x_4 + x_6 = 6$

  $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
- $x = [0\ \ 0\ \ 0\ \ 0\ 12\ \ 6]^T$ is a BFS
- Simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	4	2	−1	0	1	0	12
x_6	1	4 *	0	−1	0	1	6
	− 5	−6*	1	1	0	0	−18

- Pivoting position: (2,2); DV: x_6 EV: x_2

Example: Phase I

- Updated tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	$\frac{7}{2}$*	0	−1	$\frac{1}{2}$	1	$-\frac{1}{2}$	9
x_2	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$
	$-\frac{7}{2}$*	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	−9

- Pivoting position: (1,1); DV: x_5 EV: x_1

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{18}{7}$
x_2	0	1	$\frac{1}{14}$	$-\frac{2}{7}$	$-\frac{1}{14}$	$\frac{2}{7}$	$\frac{6}{7}$
	0	0	0	0	1	1	0

- Minimum value zero is achieved
- $x = [18/7\ \ 6/7\ \ 0\ \ 0]^T$ is a BFS

Example: Phase II

- Phase II tableau:

	x_1	x_2	x_3	x_4	
x_1	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{18}{7}$
x_2	0	1	$\frac{1}{14}$	$-\frac{2}{7}$	$\frac{6}{7}$
	0	0	$\frac{5}{14}$	$\frac{4}{7}$	$-\frac{54}{7}$

- Already optimal
- Minimum value of 54/7 is achieved
- $x = [18/7\ \ 6/7\ \ 0\ \ 0]^T$ is the minimizer

Duality Example

- Given the nutrition table:

Vitamin	Milk	Eggs	Daily Requirement
V	2	4	40
W	3	2	50
Intake	x_1	x_2	
Unit cost	3	5/2	

- LP formulation:

  minimize $3x_1 + \frac{5}{2}x_2$

  subject to $2x_1 + 4x_2 \geq 40$

  $3x_1 + 2x_2 \geq 50$

  $x_1, x_2 \geq 0$
- LP problem in matrix form:

  minimize $c^T x$

  subject to $Ax \geq \mathrm{b}$

  $x \geq 0$

Duality Example

- LP problem in matrix form:

  minimize $c^T x$

  subject to $Ax \geq \mathrm{b}$

  $x \geq 0$

where $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$; $b = \begin{bmatrix} 40 \\ 50 \end{bmatrix}$; $c = \begin{bmatrix} 3 \\ 5/2 \end{bmatrix}$

- Optimal solution is $x = \begin{bmatrix} 15 & \frac{5}{2} \end{bmatrix}^T$ with cost 51.25
- Consider the related problem where the health store owner sells vitamin pills V and W; he has to set their unit prices λ_1 and λ_2
- Since the nutritional requirements for vitamins V and W are 40 and 50, the total daily revenue is $40\lambda_1 + 50\lambda_2$
- The store owner wants to maximize his revenue

Duality Example

- To be competitive, he cannot set his prices to be higher than the price of obtaining the nutritional equivalent from milk and eggs, i.e., the milk and egg prices must satisfy (for milk): $2\lambda_1 + 3\lambda_2 \leq 3$ (for eggs): $4\lambda_1 + 2\lambda_2 \leq 5/2$
- The store owner's LP problem:

  maximize $40\lambda_1 + 50\lambda_2$

  subject to $2\lambda_1 + 3\lambda_2 \leq 3$

  $4\lambda_1 + 2\lambda_2 \leq \frac{5}{2}$

  $\lambda_1, \lambda_2 \geq 0$
- Optimal cost is same 51.25 with $\lambda = \begin{bmatrix} \frac{3}{16} & \frac{7}{8} \end{bmatrix}^T$

Dual Problem

- The store owner's LP problem in matrix form:

  maximize $\lambda^T b$
  subject to $\lambda^T A \le c^T$
  $\lambda \ge 0$

The matrices A, b and c are the same as in

  minimize $c^T x$
  subject to $Ax \ge b$
  $x \ge 0$

- Note that the store owner's LP problem can be deduced from the Optimal Diet LP by

maximize → minimize, $\ge \to \le$, $c \to b$, $b \to c$, $A^T \to A$

- Store owner's LP problem is the dual problem of Optimal Diet LP

Duality in LP

- Duality, the study of dual LP problems, is important
- The solution to one gives information about the solution to the other
- Duality can be used to improve the performance of the simplex algorithm (Primal-Dual algorithm)
- Duality is useful in the design of new algorithms (e.g., Karmarkar's algorithm)
- Duality is used in sensitivity analysis (how much will the solution to an LP problem change if we slightly change the numbers in the problem data?)
- Duality is the basis for studying matrix games.

Dual LP Problem

- Primal LP:

  minimize $c^T x$
  subject to $Ax \ge b$
  $x \ge 0$

- Dual LP:

  maximize $\lambda^T b$
  subject to $\lambda^T A \le c^T$
  $\lambda \ge 0$

- This pair of problems is called the symmetric form of duality (diet problem is an example of this)

- The primal and dual problems are related by

maximize → minimize, $\ge \to \le$, $c \to b$, $b \to c$, $A^T \to A$

Remarks on Duality I

- The dual of the dual problem is the primal problem
- Primal LP: Dual LP:

  minimize $c^T x$ maximize $\lambda^T b$
  subject to $Ax \ge b$ subject to $\lambda^T A \le c^T$
  $x \ge 0$ $\lambda \ge 0$

- The Dual LP is equivalent to:

  minimize $-\lambda^T b$
  subject to $-\lambda^T A \ge -c^T$
  $\lambda \ge 0$

- The Dual of equivalent Dual LP is equivalent to:

  maximize $-c^T y$
  subject to $-Ay \le -b$
  $y \ge 0$

- The last problem is equivalent to the original problem

Asymmetrric Dual LP Form

- Primal LP (in standard form):

  minimize $c^T x$
  subject to $Ax = b$
  $x \ge 0$

- Dual LP:

  maximize $\lambda^T b$
  subject to $\lambda^T A \le c^T$

- This pair of problems is called the asymmetric form of duality
- To derive the asymmetric form of duality above from the symmetric form, write the equality constraint $Ax = b$ as

$$\begin{bmatrix} A \\ -A \end{bmatrix} x \ge \begin{bmatrix} b \\ -b \end{bmatrix}$$

Remarks on Duality II

- Dual LP Primal LP (in standard form):

  maximize $[u^T \quad v^T] \begin{bmatrix} b \\ -b \end{bmatrix}$
  subject to $[u^T \quad v^T] \begin{bmatrix} A \\ -A \end{bmatrix} \le c^T$
  $u, v \ge 0$

- Let $\lambda = u - v$ to obtain the asymmetric form Dual LP:

  maximize $\lambda^T b$
  subject to $\lambda^T A \le c^T$

- Both primal and dual problems are defined by the same data A, b and c
- Next we discuss some fundamental properties of dual LP problems (all these properties hold for both symmetric and asymmetric forms)

Weak Duality Lemma

- Primal LP: Dual LP:

minimize $c^T x$ maximize $\lambda^T b$

subject to $Ax \geq b$ subject to $\lambda^T A \leq c^T$

$x \geq 0$ $\lambda \geq 0$

- Lemma: Suppose that x and λ are feasible solutions to primal and dual LP problems, respectively. Then,

$$c^T x \geq \lambda^T b$$

- Proof: $\lambda^T b \leq \lambda^T Ax \leq c^T x$.
- Interpretation: The objective function value of any feasible solution to one problem is a bound for the optimal objective function value for the other
- maximum of dual $\leq$ minimum of primal
- If one problem is unbounded, then the other has no feasible solution (infeasible)

Duality Properties

- Primal LP: Dual LP:

minimize $c^T x$ maximize $\lambda^T b$

subject to $Ax \geq b$ subject to $\lambda^T A \leq c^T$

$x \geq 0$ $\lambda \geq 0$

- From the Weak Duality Lemma, we have the following result:
- Theorem: Suppose that x and λ are feasible solutions to the primal and dual, respectively. If $c^T x = \lambda^T b$, then x and λ are optimal solutions to their respective problems.

- Duality Theorem: If the primal problem has an optimal solution (say x_0), then so does the dual (say λ_0), and the optimal values of their respective objective functions are equal, i.e., $c^T x_0 = \lambda_0^T b$.

Summary

- Primal LP: Dual LP:

minimize $c^T x$ maximize $\lambda^T b$

subject to $Ax \geq b$ subject to $\lambda^T A \leq c^T$

$x \geq 0$ $\lambda \geq 0$

- Primal unbounded → dual infeasible

- Primal bounded → dual bounded, and there is no gap between them

- Primal infeasible → dual is either unbounded or infeasible