Advanced Topics in Machine Learning: Convolutional Networks (Part 1)

Laurens van der Maaten and Anton Bakhtin

Image classification






Image classification

tench

alligator lizard

bullet train

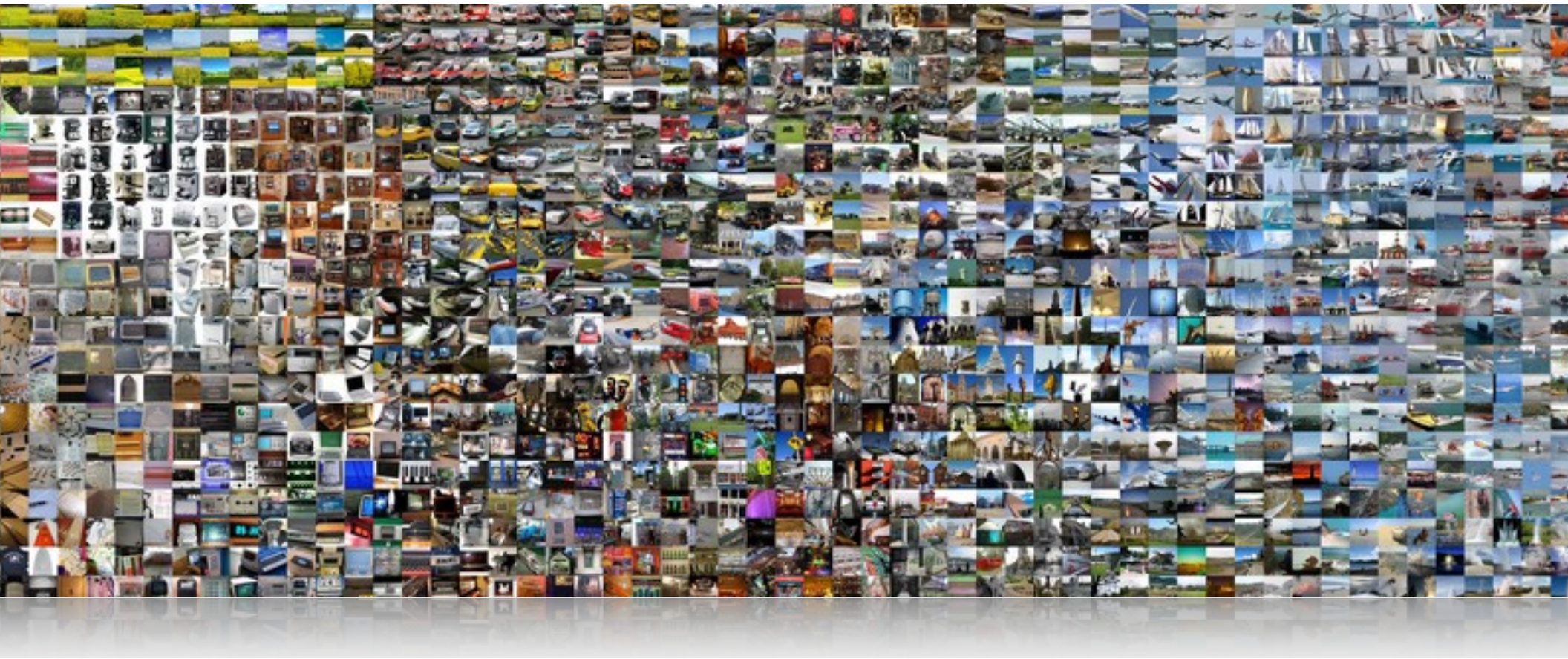
agaric

artichoke

bride and groom

Image classification

- Train a **convolutional network** on **millions** of **annotated images**



Empirical Risk Minimization

- Learn model based on training data $\mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{N}$ by minimizing:

$$\mathbb{E}[\ell(f(\mathbf{x}; \mathbf{w}), y)]_{p(\mathbf{x},y)}$$

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input data

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label

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model parameters

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model output

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loss function

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expected loss

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data distribution

Empirical Risk Minimization

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$$\mathbb{E}[\ell(f(\mathbf{x}; \mathbf{w}), y)]_{p(\mathbf{x},y)} \approx \frac{1}{N} \sum_{n=1}^{N} \ell(f(\mathbf{x}_n; \mathbf{w}), y_n)$$

input sample

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sample label

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empirical loss

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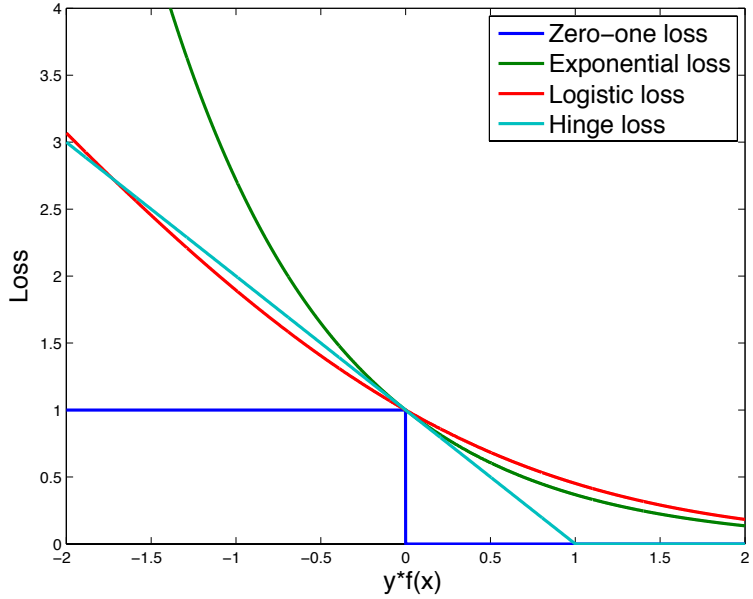
- Examples of **loss functions**:



* We assume the label y is -1 or +1.

Empirical Risk Minimization

- A **logistic regressor** is an ERM model with two properties:
  - The model is linear: $y' = f(\mathbf{x}; \mathbf{w}) = \mathbf{w}^\top \mathbf{x}$
  - The loss function is the logistic loss*: $\ell(y', y) = \log(1 + \exp(-y'y))$

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- In the **multi-class** setting, logistic regression can be written as**:

$$\ell(f(\mathbf{x}; \mathbf{w}), y) = -\log \frac{\exp(\mathbf{y}^\top f(\mathbf{x}; \mathbf{w}))}{\sum_{\mathbf{y}'} \exp(\mathbf{y}'^\top f(\mathbf{x}; \mathbf{w}))}$$

$$= -\mathbf{y}^\top f(\mathbf{x}; \mathbf{w}) + \log \sum_{\mathbf{y}'} \exp(\mathbf{y}'^\top f(\mathbf{x}; \mathbf{w}))$$

* We assume the label y is -1 or +1.
** We assume the label y is a one-hot vector.

Empirical Risk Minimization

- How do we **minimize** the empirical loss?

Empirical Risk Minimization

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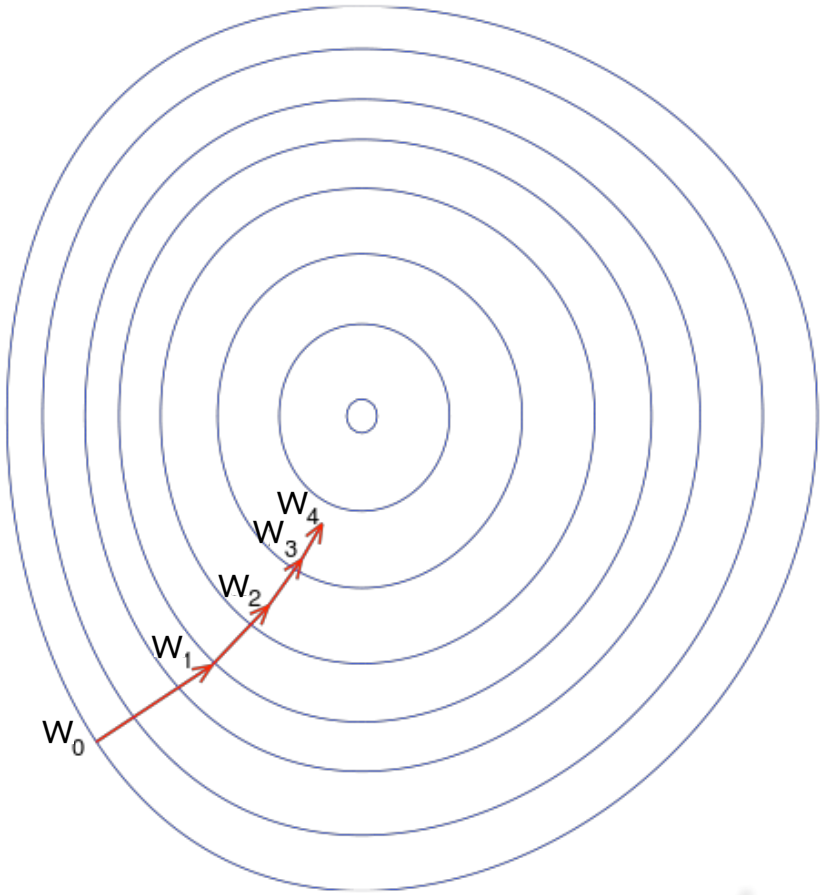
- **Gradient descent**! At each iteration, do a small step in the direction of the loss gradient:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \frac{\eta_t}{N} \sum_{n=1}^{N} \frac{\partial \ell(f(\mathbf{x}_n), y_n)}{\partial \mathbf{w}}$$

Gradient descent



Note how the gradient is orthogonal to the isolines.

Gradient descent

- Recall the **logistic loss** function for linear models:

$$\mathcal{L}(\mathcal{D}; \mathbf{w}) = \frac{1}{N} \sum_{n=1}^{N} \log(1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n))$$

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- Derivation of the **gradient**:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = -\frac{1}{N} \sum_{n=1}^{N} \frac{1}{1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n)} \exp(-y_n \mathbf{w}^\top \mathbf{x}_n) \mathbf{x}_n y_n$$

$$= -\frac{1}{N} \sum_{n=1}^{N} \frac{\exp(-y_n \mathbf{w}^\top \mathbf{x}_n)}{1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n)} \mathbf{x}_n y_n$$

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$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{w}} &= -\frac{1}{N} \sum_{n=1}^{N} \frac{1}{1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n)} \exp(-y_n \mathbf{w}^\top \mathbf{x}_n) \mathbf{x}_n y_n \\
&= -\frac{1}{N} \sum_{n=1}^{N} \frac{1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n) - 1}{1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n)} \mathbf{x}_n y_n \\
&= -\frac{1}{N} \sum_{n=1}^{N} (1 - p(y_n = 1 | \mathbf{x}_n)) \mathbf{x}_n y_n
\end{aligned}$$

relation to maximum likelihood interpretation

The need for speed

Is gradient descent efficient?

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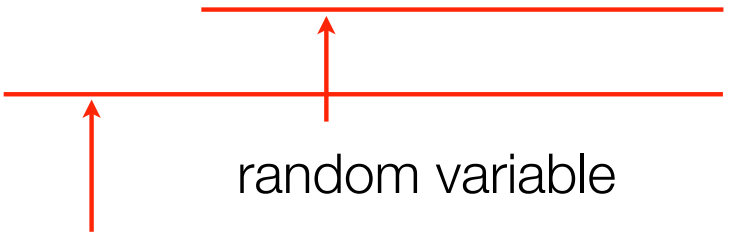

Stochastic gradient descent

- Selects a **random** data point *n*, and updates parameters using:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \eta_t \frac{\partial L(\mathbf{x}_n, y_n; \mathbf{w})}{\partial \mathbf{w}}$$

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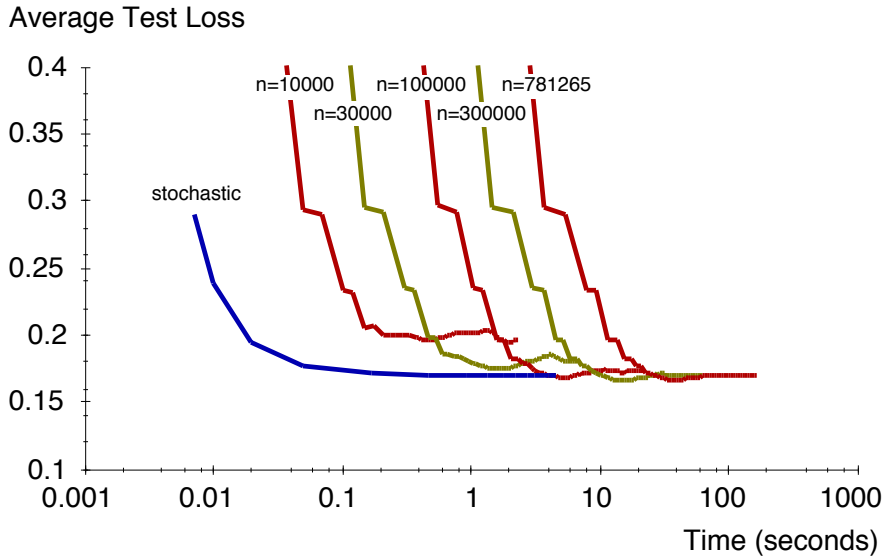
- This minimization technique converges **much faster** in practice:



Stochastic gradient descent

- In practice, we often use **mini-batch SGD**. Why?

Stochastic gradient descent

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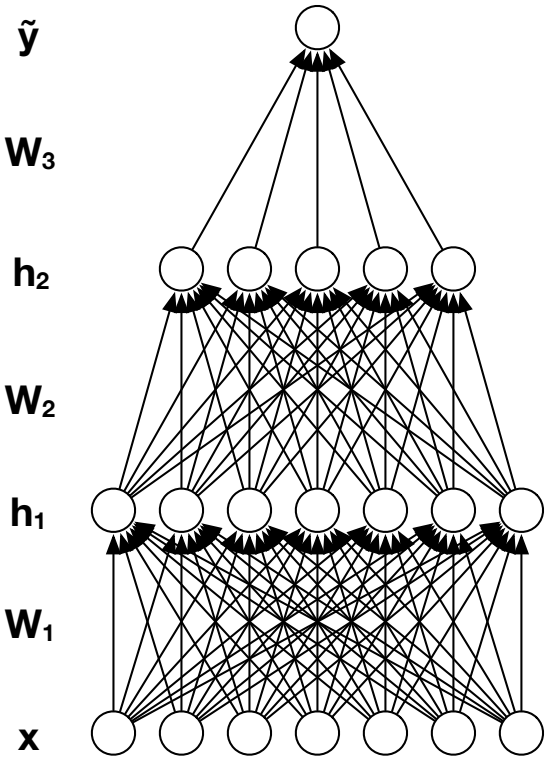
Multilayer networks

- Multilayer networks repeatedly apply linear and nonlinear transforms:



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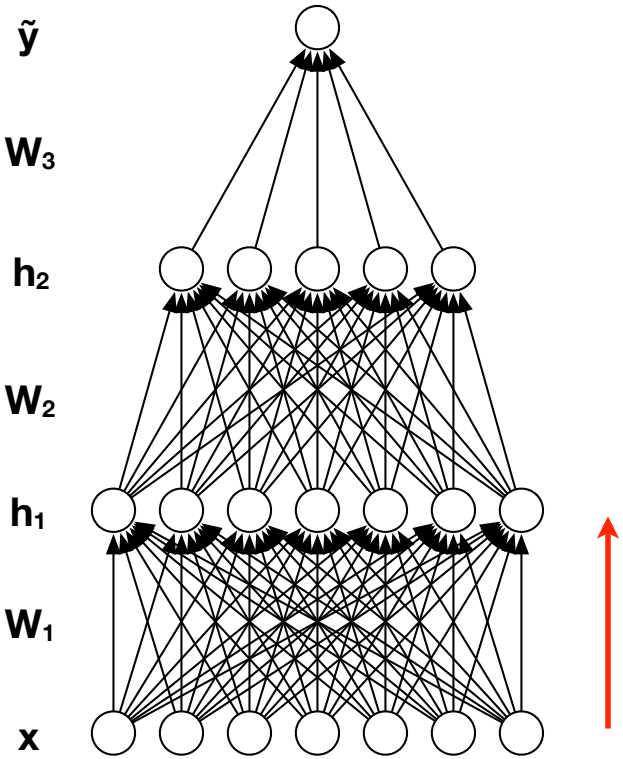

$$\mathbf{h}_1 = f_1(\mathbf{W}_1^{\mathrm{T}}\mathbf{x}) = \frac{1}{1+\exp(-\mathbf{W}_1^{\mathrm{T}}\mathbf{x})}$$

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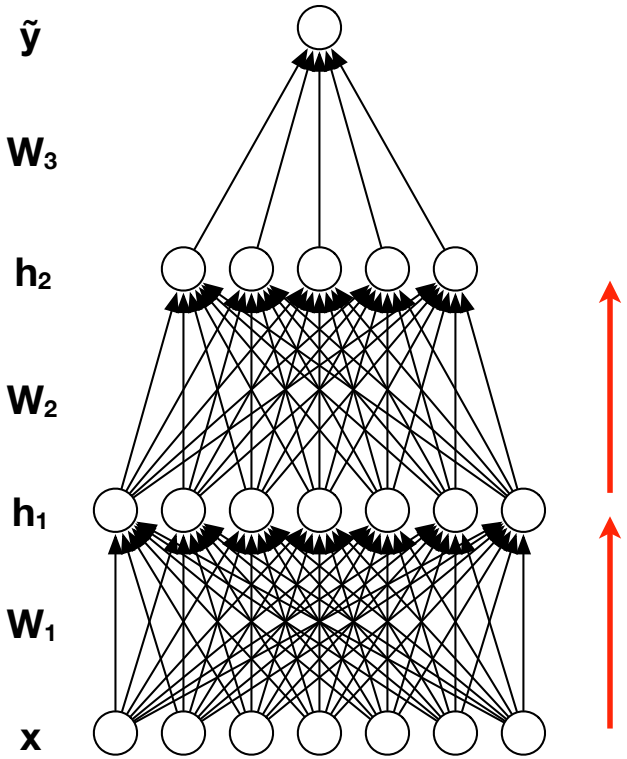

$$\mathbf{h}_2 = f_2(\mathbf{W}_2^{\mathrm{T}}\mathbf{h}_1) = \frac{1}{1+\exp(-\mathbf{W}_2^{\mathrm{T}}\mathbf{h}_1)}$$

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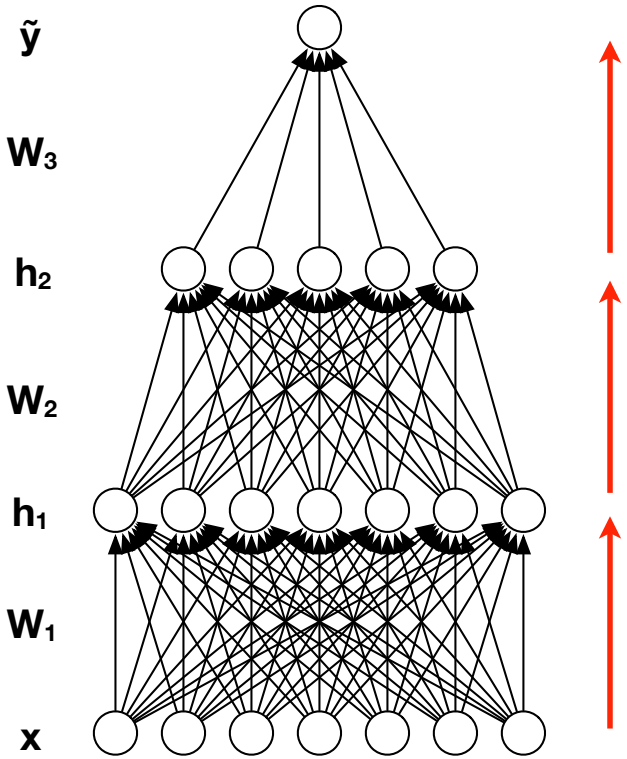

$$\tilde{y} = f_3(\mathbf{W}_3^{\mathrm{T}}\mathbf{h}_2) = \frac{1}{1+\exp(-\mathbf{W}_3^{\mathrm{T}}\mathbf{h}_2)}$$

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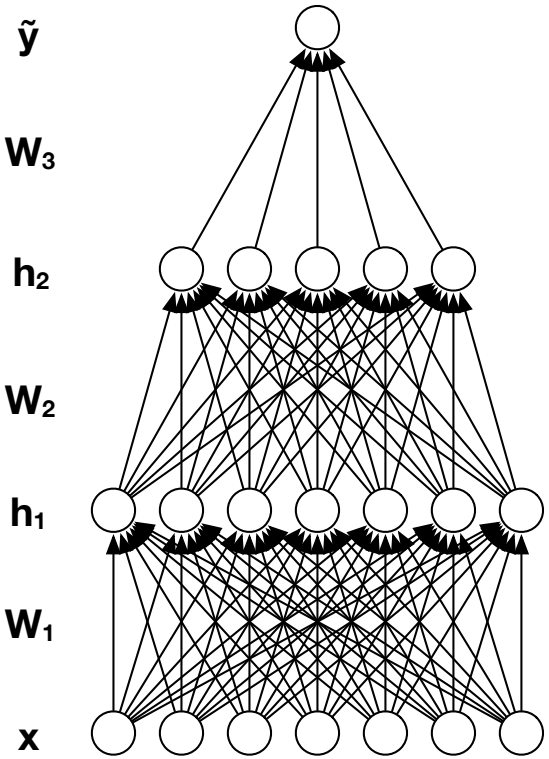
Backpropagation

- **Backprogation** uses **chain rule** to compute **loss gradient**:



Assume quadratic loss: $C = \frac{1}{2}(y - \tilde{y})^2$

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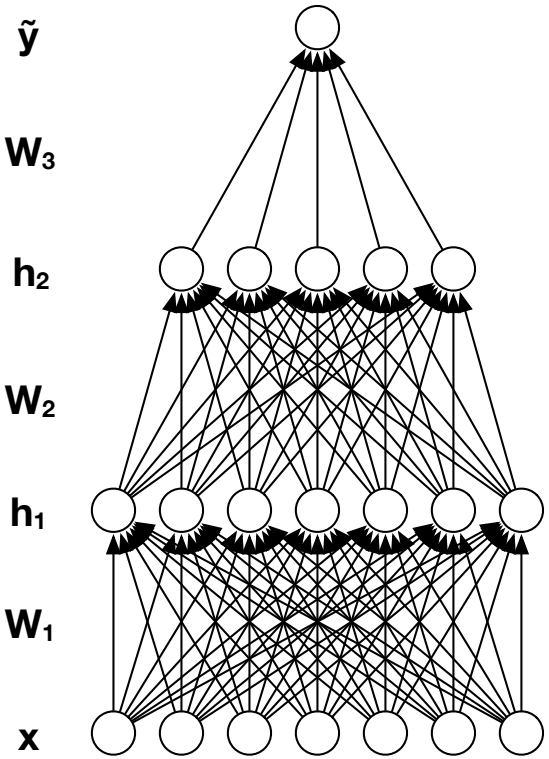

Assume quadratic loss: $C = \frac{1}{2}(y - \tilde{y})^2$

Backpropagation proceeds in two steps:

1. **Propagate** the error signal

2. Compute the associated **weight gradient**

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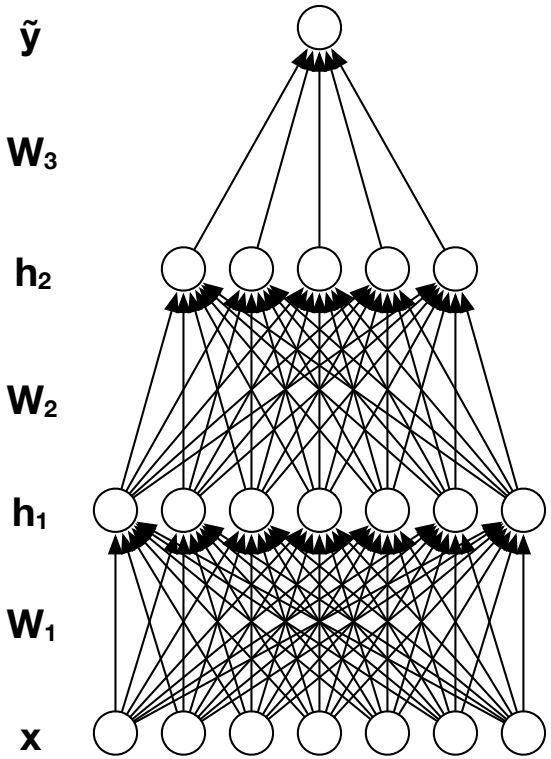

$$e_3 = (y - \tilde{y}) \frac{\partial f_3}{\partial \mathbf{W}_3^T \mathbf{h}_2} = (y - \tilde{y}) \tilde{y} (1 - \tilde{y})$$

$$\frac{\partial C}{\partial \mathbf{W}_3} = e_3 \mathbf{h}_2$$

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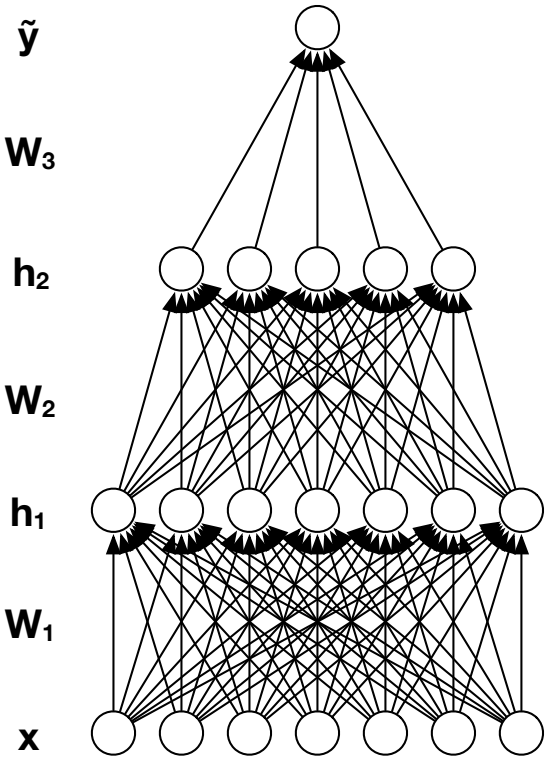

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$$\mathbf{e}_2 = \mathbf{h}_2 \circ (1 - \mathbf{h}_2) \circ (\mathbf{W}_3 e_3)$$

$$\frac{\partial C}{\partial \mathbf{W}_2} = \mathbf{h}_1 \mathbf{e}_2^\top$$

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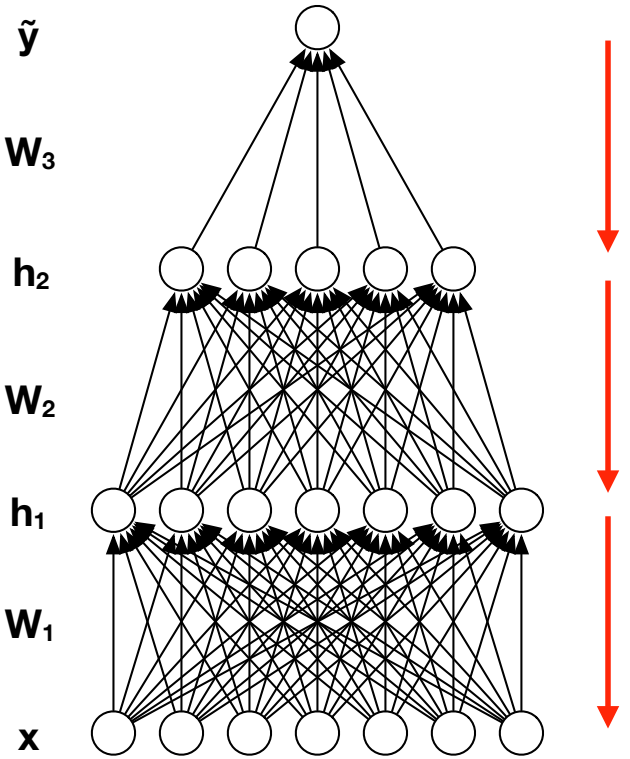

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$$\frac{\partial C}{\partial \mathbf{W}_1} = \mathbf{x} \mathbf{e}_1^\top$$

Multilayer networks

- Suppose we wanted to train a multilayer network to recognize images

- What would go wrong?

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Suppose the image has 256x256 RGB pixels,
and we have 1,000 classes.
How many parameters would a single layer have?

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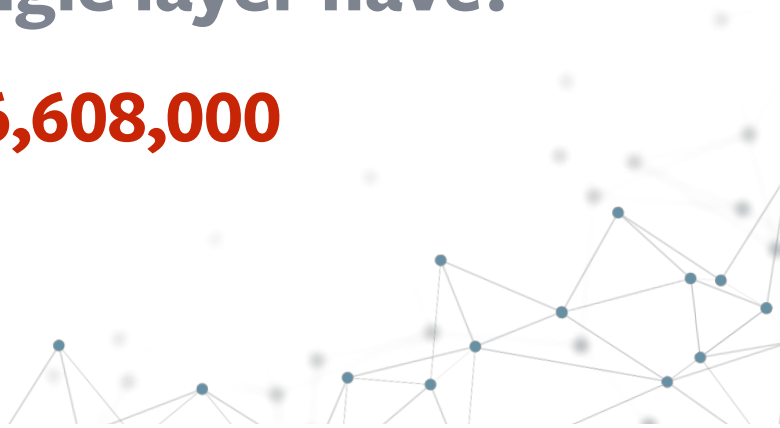
3 x 256 x 256 x 1,000 = 196,608,000



Convolutional networks

- You can think of each column in a linear layer as a "filter" for the image
- Do different parts of an image have to be filtered differently?

Convolutional networks

- You can think of each column in a linear layer as a "filter" for the image
- Do different parts of an image have to be filtered differently? **No!**

Convolution

- Discrete convolution*: $(f \star g)[i] = \sum_{k=1}^{K} f[i+k]g[k]$

* Technically, this operation is called **cross-correlation**.
Most machine learning work uses the two interchangeably.

Convolution

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value in input signal

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Convolution

- Discrete convolution*: $(f \star g)[i] = \sum_{k=1}^{K} f[i+k]g[k]$

value in filter ("kernel")

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Convolution

- Discrete convolution*: $(f \star g)[i] = \sum_{k=1}^{K} f[i+k]g[k]$

value in output

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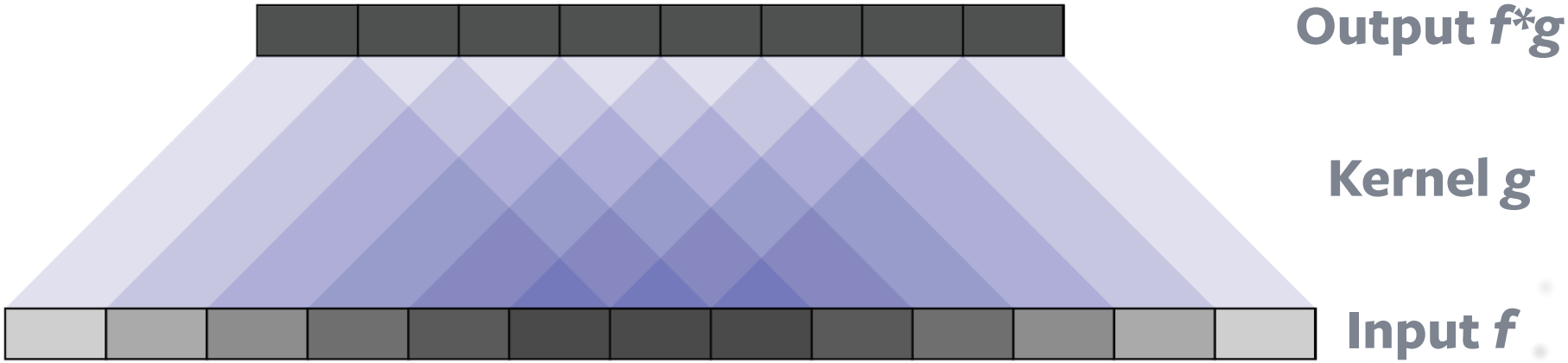
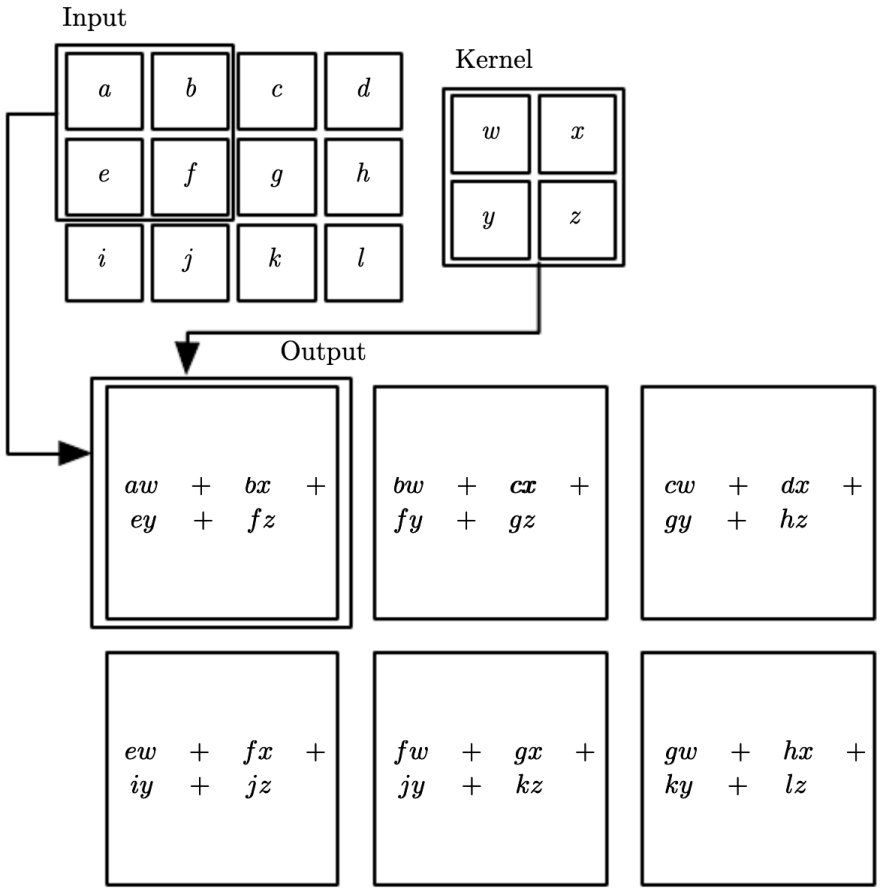


* Credits: Chris Olah

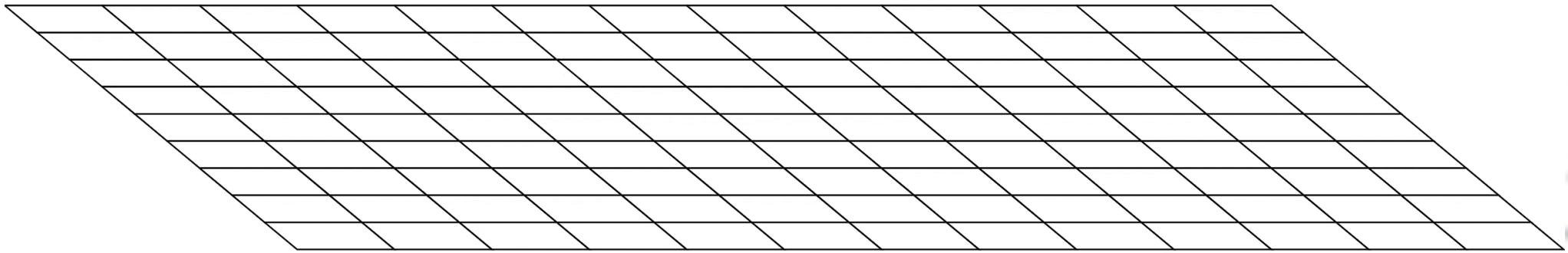
Convolution

- Input and kernel can have two (or more) dimensions:



Convolution

- Illustration of 2D convolution:



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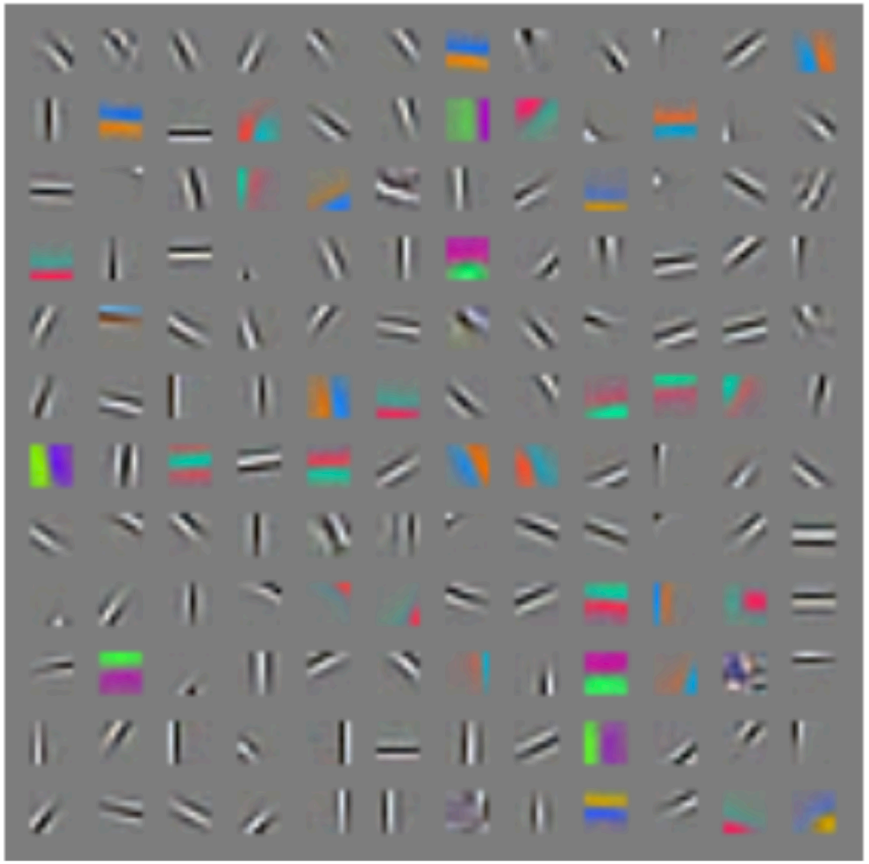
Convolution

- **Visualization** of convolution filters learned in first layer of network:

$$(f \star g)[i] = \sum_{k=1}^{K} f[i+k]g[k]$$



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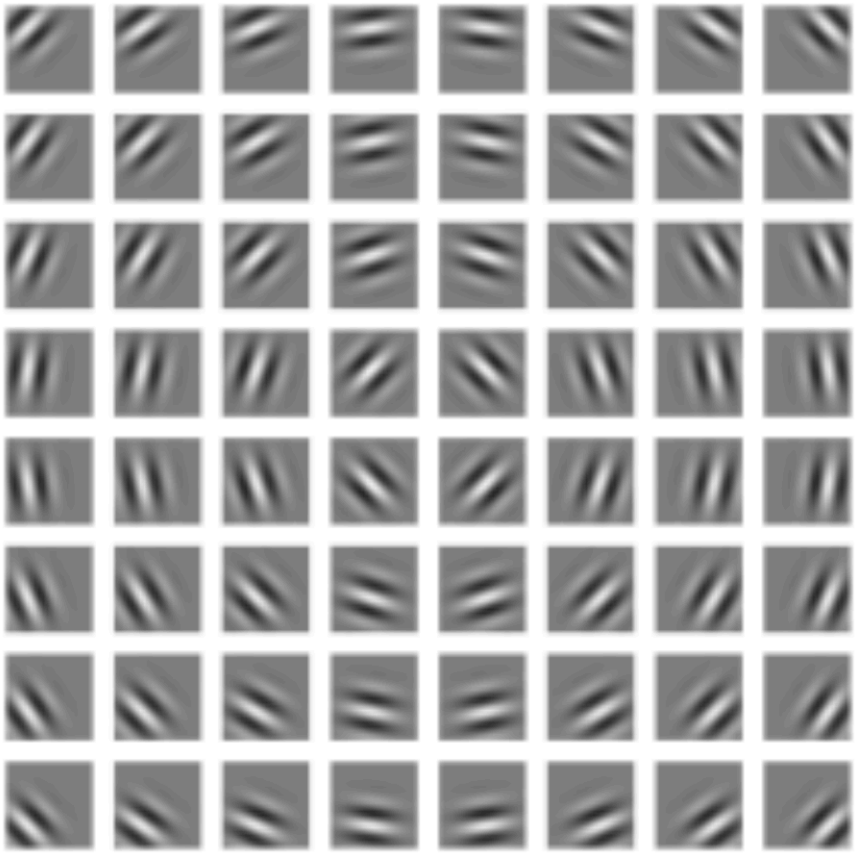
Convolution

- These look a lot like **Gabor filters** commonly used for edge detection:



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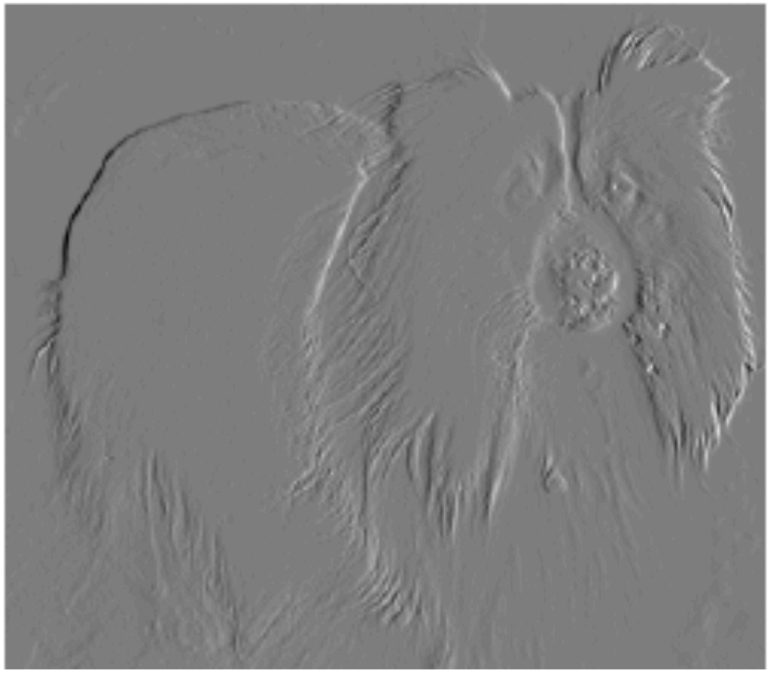

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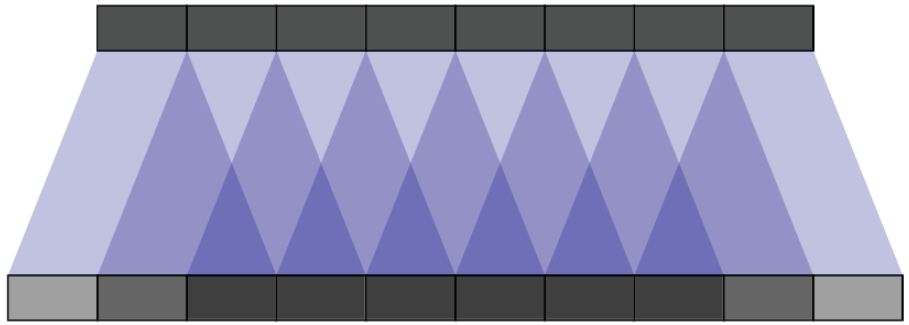
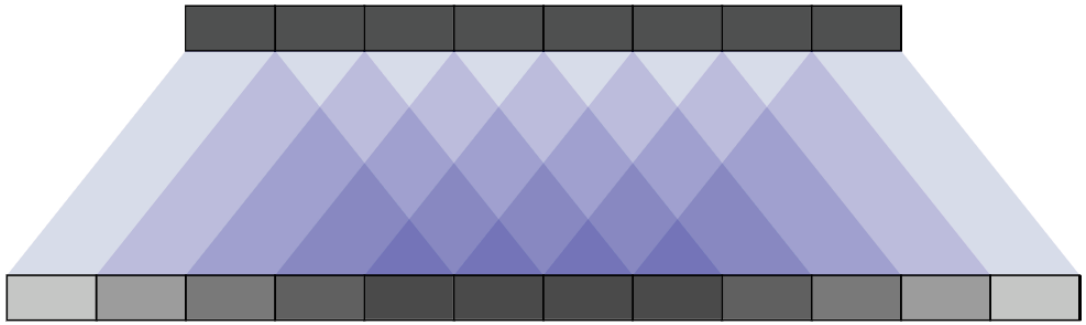
Convolution

- Convolutional kernels have a **size**:

kernel size = 3:



kernel size = 5:

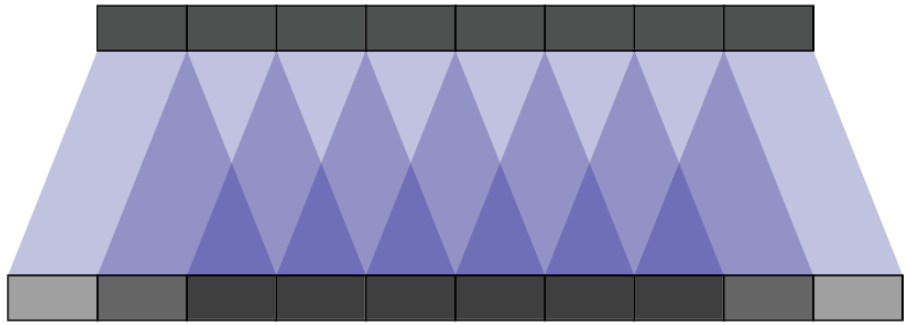


* Credits: Chris Olah

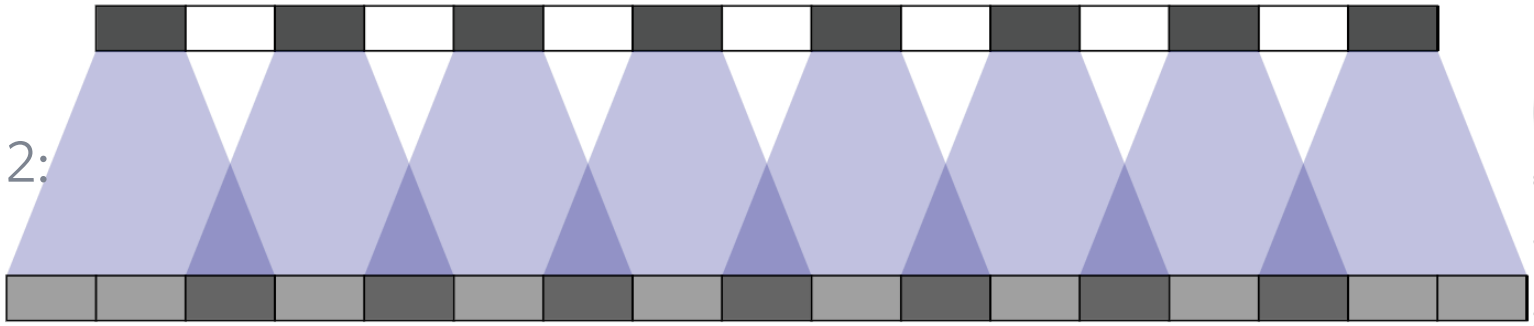
Convolution

- Convolutional kernels have a **stride**:

kernel stride = 1:



kernel stride = 2:



* Credits: Chris Olah

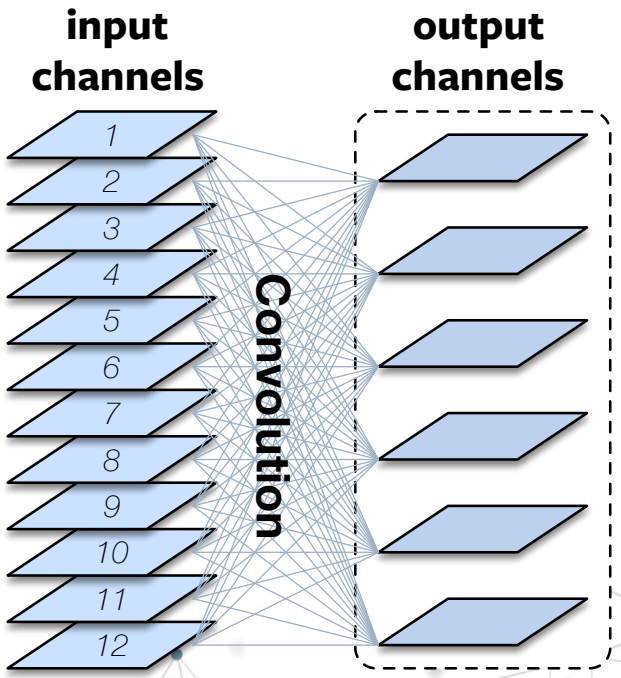
Convolutional networks

- Convolutional network layers generally have **multiple input channels** (for example, RGB) and **multiple output channels**

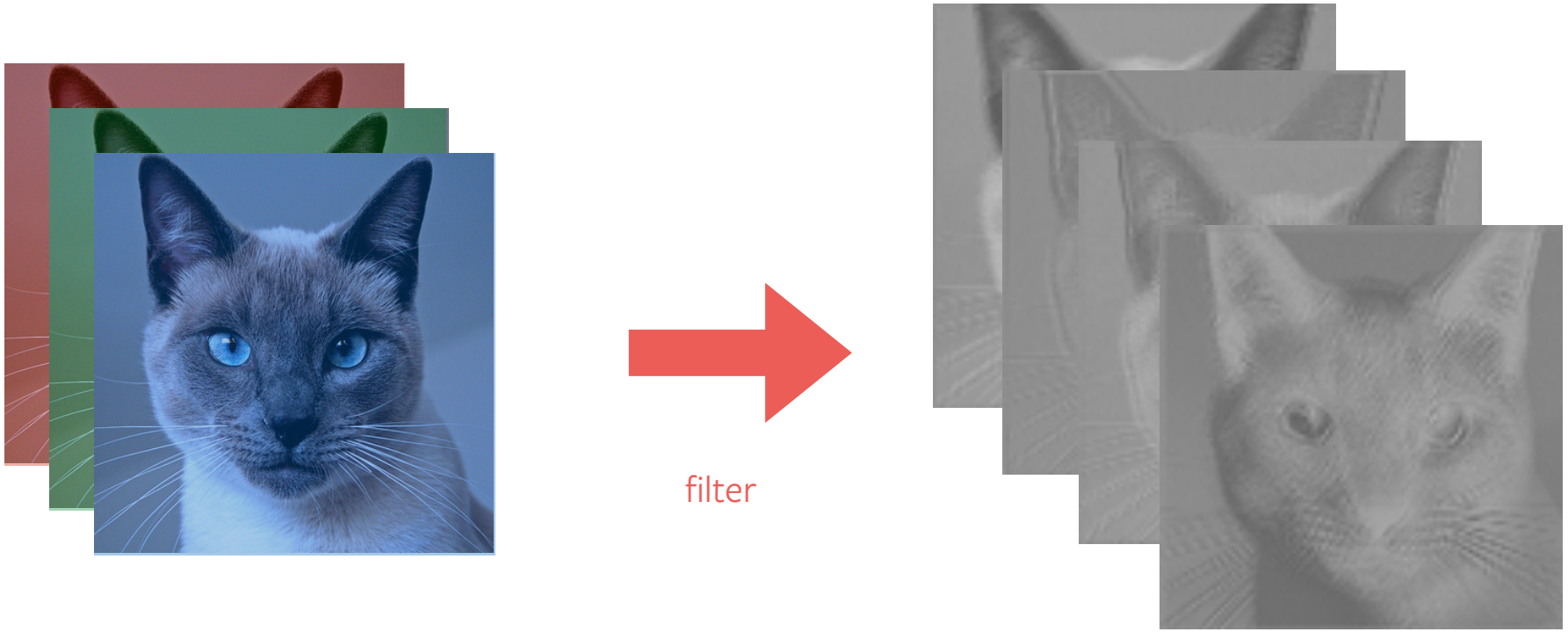
Convolutional networks

- Convolutional network layers generally have **multiple input channels** (for example, RGB) and **multiple output channels**

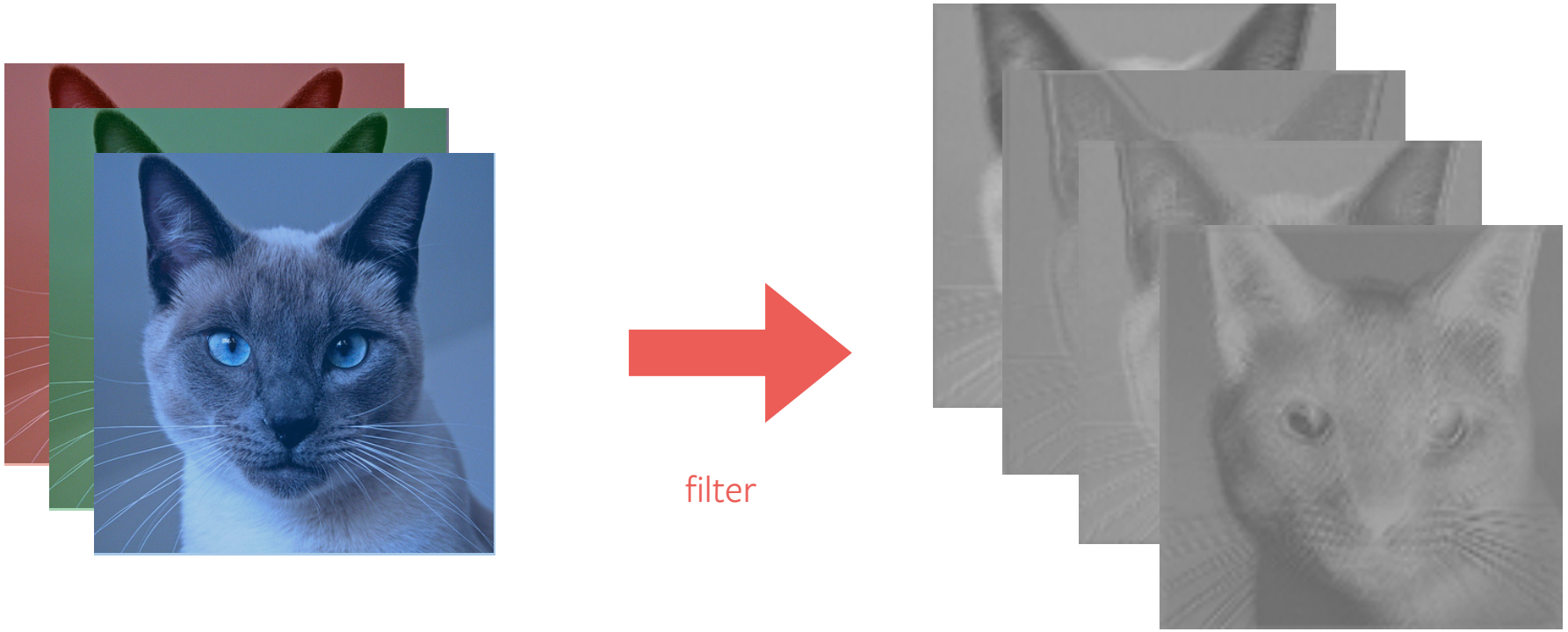
- To produce a single output channel:
  - **Convolve** each input channel with some kernel
  - Note that the kernel is **different** for each input channel
  - **Sum** the convolved input channels to produce the output channel



Convolutional networks

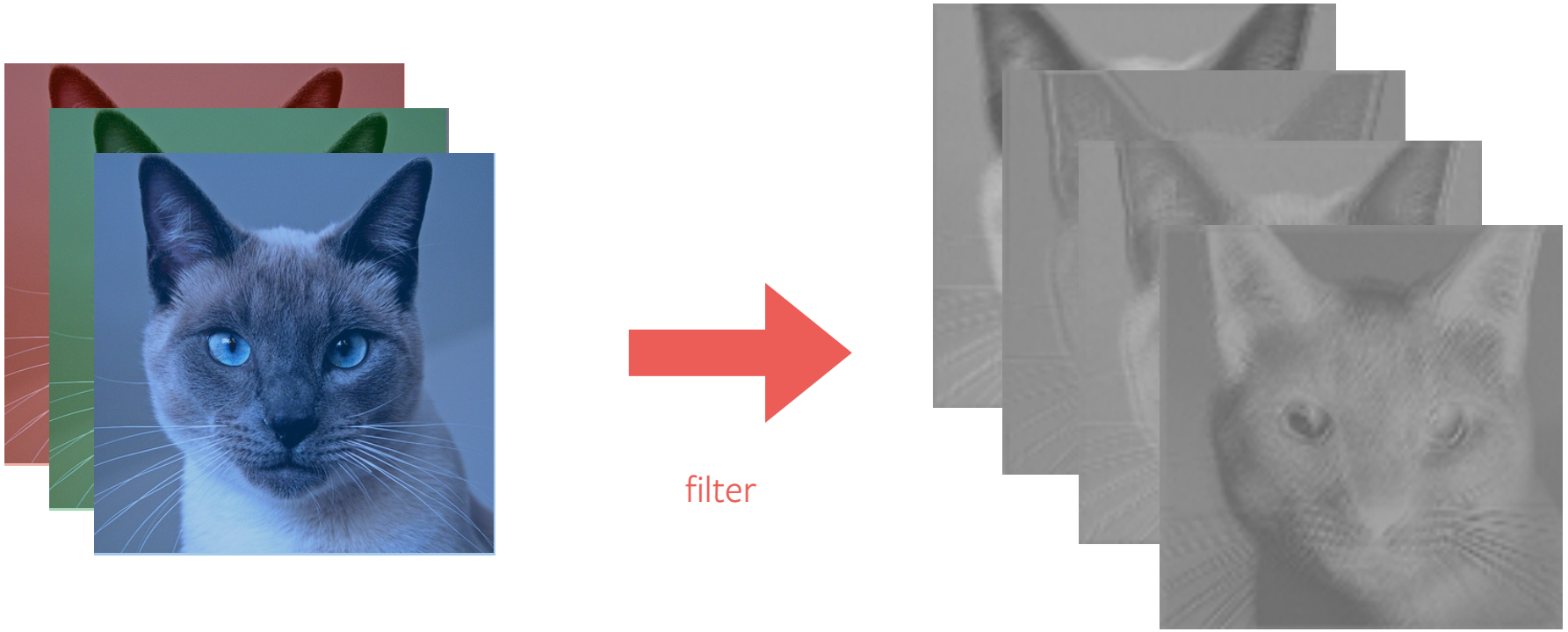


Convolutional networks



How many parameters does this convolutional layer have?

Convolutional networks



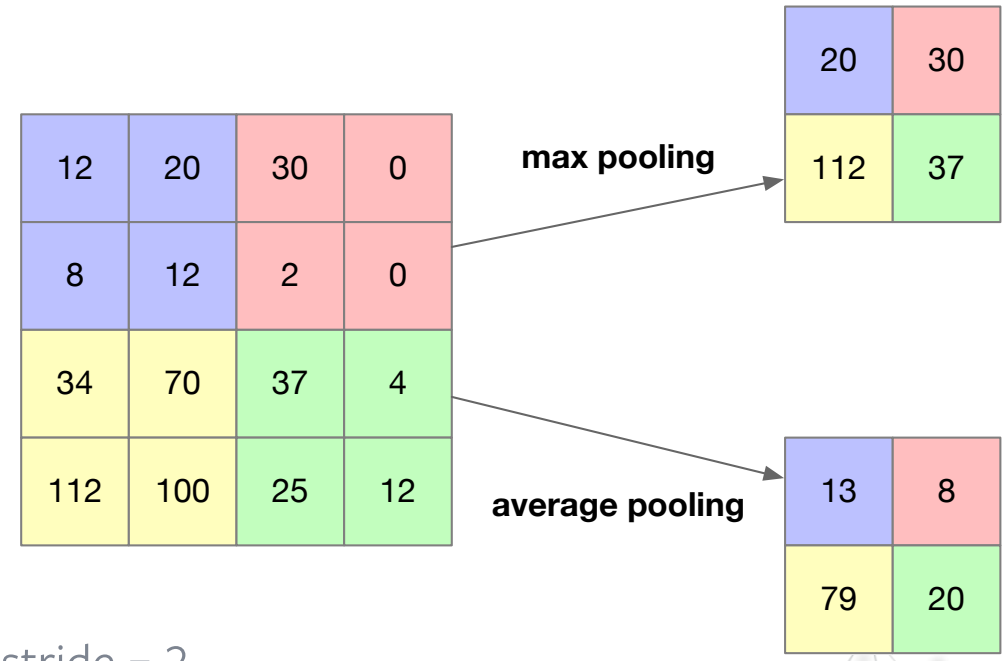
How many parameters does this convolutional layer have?

3 x 4 x kernel height x kernel width

Pooling

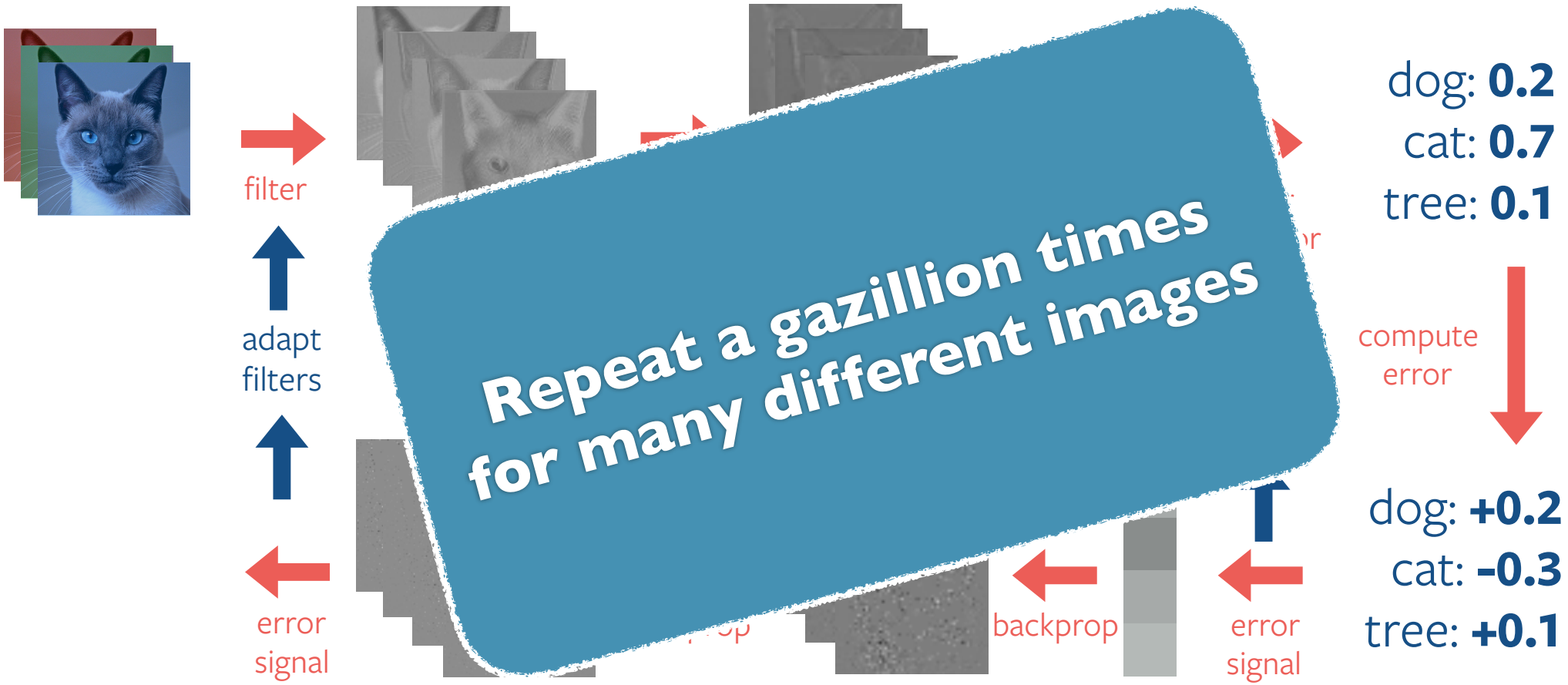
- Input and output of a convolution layer both have **spatial structure**

- We gradually remove the spatial structure via **pooling**:



* This pool uses size = 2, stride = 2.

Convolutional networks



Summary

- Loss measures discrepancy between a prediction and the true label
- ERM minimizes the loss averaged over the training data
- Logistic regression is an linear ERM model that uses logistic loss
- Multi-layer networks iteratively apply learned linear functions and fixed non-linear functions: gradients computed by chain rule (backprop)
- Convolutional networks exploit spatial structure in inputs

Reading material

- V. Vapnik. **The Nature of Statistical Learning Theory**, 1999:
  - Chapter 1.

- I. Goodfellow, Y. Bengio, and A. Courville. **Deep Learning**, 2016.
  - Chapter 6, 8, and 9.

Questions?