Simple Geometric Derivation of the Euler Lagrange Equations

Aki Anandam

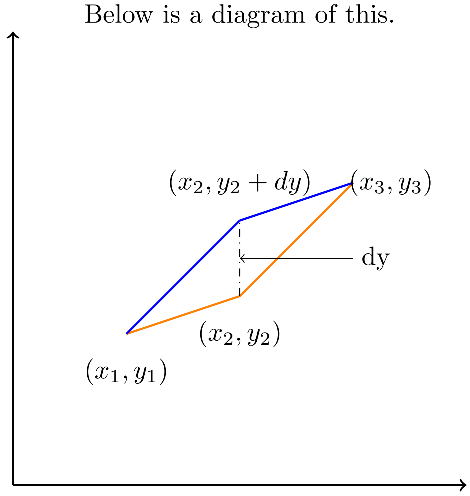
November 20, 2020

The Euler Lagrange equations are differential equations used for finding extrema in functionals. Their most common application is in Lagrangian mechanics to find the path of stationary action. Here, I will show a simple geometric derivation for them.

We begin with a functional, I which equals this expression

$$\int_{x_0}^{x_1} F(x, y, y')\, dx$$

where F is some function of x, y, and y'. We want this function to be stationary, because then, so will the integral. We will assume we have our optimal path, y(x), and if we deviate the path slightly, then the ratio of the corresponding deviation to F and that ratio should be 0. We will then break down our smooth curve y(x), into many discrete piecewise line segments. Below is a diagram of this.



We have our initial path from (x_1, y_1) to (x_2, y_2) to (x_3, y_3) in orange, and we have our modified path changing (x_2, y_2) to $(x_2, y_2 + dy)$ in blue. The difference in height is labeled dy. We want to find the ratio between corresponding change to F caused by the change in y and that change in y and set it equal to zero. Now, because F is a function of both y and y', the change in y changes F both directly and indirectly. It directly affects F because F is a function of y, and the ratio is simply $\frac{\partial f}{\partial y}$

Now, we have to find how the change in y affects y' and how that affects F. Going back to our diagram, let's look at the slopes of each line segment. Before the change in y, the slope from x_1 to x_2 was $\frac{y_2-y_1}{x_2-x_1}$ After the change, it became $\frac{y_2+dy-y_1}{x_2-x_1}$

So the dy' at that point is the difference of these two expressions

$$dy'|_{x_1} = \frac{dy}{x_2 - x_1}$$

Similarly, the original slope from x_2 to x_3 was $\frac{y_3-y_2}{x_3-x_2}$

After the change in y, it became $\frac{y_3-y_2-dy}{x_3-x_2}$

The difference is dy' at that point, so

$$dy'|_{x_2} = \frac{-dy}{x_3 - x_2}$$

These are the ratios of how y affects y'. We need to multiply by the ratio of how y' affects F. This will be the second way of how y affects F. However, we need to make sure we are multiplying by $\frac{\partial f}{\partial y'}$ at that point, because that value may change as x varies. Therefore, the second small change in F is

$$\partial F = \frac{\partial F}{\partial y'}|_{x_1} dy'|_{x_1} + \frac{\partial F}{\partial y'}|_{x_2} dy'|_{x_2}$$

$$\partial F = -dy(\frac{\partial F}{\partial y'}|_{x_2} \frac{1}{\Delta x|_{x_2}} - \frac{\partial F}{\partial y'}|_{x_1} \frac{1}{\Delta x|_{x_1}})$$

We'll let $\Delta x|_{x_1}$ equal$\Delta x|_{x_2}$, and since $x_2 = x_1 + \Delta x$, we can say

$$\frac{\partial F}{\partial y} = -\frac{\frac{\partial F}{\partial y'}|_{x_1+\Delta x} - \frac{\partial F}{\partial y'}|_{x_1}}{\Delta x}$$

This is simply the derivative at x_1, but our choice of x_1 was arbitrary, and this relationship should hold throughout their entire function. Therefore

$$\frac{\partial F}{\partial y} = -\frac{d}{dx}\frac{\partial F}{\partial y'}$$

We have found our two ratios that tell how F changes due to a small change in y. We have to add these two ratios to get the net change of F. If y is an extrema of F, then this change should equal zero. This means that if we slightly alter y, then F will not change (to the first order). Adding these two and setting them equal to zero, we get the well known equations.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$