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Experiments giving the mass efflux of a Poiseuille flow over a naturally permeable block are reported. The efflux is greatly enhanced over the value it would have if the block were impermeable, indicating the presence of a boundary layer in the block. The velocity presumably changes across this layer from its (statistically average) Darcy value to some slip value immediately outside the permeable block. A simple theory based on replacing the effect of the boundary layer with a slip velocity proportional to the exterior velocity gradient is proposed and shown to be in reasonable agreement with experimental results.

1. Introduction

When a Newtonian fluid flows over a porous surface it is necessary, if the governing differential system is not to be underdeterminate, to specify some condition on the **tangential component of the velocity of the free fluid at the porous interface**. There exists an extensive analytical literature (e.g. see references quoted in Joseph & Tao (1966)) which describes coupled fluid motions satisfying the **Navier–Stokes equations** in the free fluid, some empirical or semi-empirical **set of equations** (typically Darcy's Law) in the permeable material, and **matching conditions at the common boundaries**. It is usual in these analyses to approximate the **fluid motion near the true boundary** by an adherence condition for the **tangential component of velocity of the free fluid at some boundary**. Of course, a certain **ambiguity is implied by the notion of a 'true' boundary for a permeable material**, and it is for this reason useful to define a nominal boundary. We fix a nominal boundary by first defining a smooth geometric surface and then assuming that the outermost perimeters of all the surface pores of the permeable material are in this surface. Thus if the surface pores were filled with solid material to the level of their respective perimeters a smooth impermeable boundary of the assumed shape would result. This definition is precise when the geometry is simple (planes, spheres, cylinder, etc.) but may not be fully adequate in more complicated situations.

Though the adherence condition is valid at an impermeable surface it is not clear that it is valid at the nominal surface of a permeable material. In the latter case, there is a migration of fluid tangent to the boundary within the porous matrix, and the requirement that there should be no migration of fluid immediately outside the boundary is approximate at best. It can in fact be argued that there is some net tangential drag due to the transfer of forward momentum

across the permeable interface. Indeed, if we were dealing with the true velocity in the porous material, there could be no slip between the free fluid and the fluid immediately within the porous boundary. In this case, a discontinuity of the tangential velocity component could not be allowed.

The experiment which we report in the following sections was designed to examine the nature of this tangential flow in the boundary region of a permeable interface. Briefly, we have used a two-dimensional Poiseuille flow above a fluid-saturated, permeable block to infer the value of the velocity at the interface. This arrangement is a model for flows in which the Navier-Stokes equations are satisfied in the free fluid, the Darcy Law is satisfied in the interior of the permeable material but not necessarily in the boundary regions, and the normal component of the velocity and the pressure are continuous at the porous boundary. The results of this experiment indicate that the effects of viscous shear appear to penetrate into the permeable material in a boundary layer region, producing a velocity distribution similar to that depicted in figure 1. The tangential component of velocity of the free fluid at the porous boundary can be considerably greater than the mean filter velocity within the body of the porous material.

2. The slip-flow boundary condition

We consider the rectilinear flow of a viscous fluid through a two-dimensional parallel channel formed by an impermeable upper wall ($y = h$) and a permeable lower wall ($y = 0$). The plane $y = 0$ defines a nominal surface for the permeable material of the type discussed above. A uniform pressure gradient is maintained in the longitudinal direction in both the channel and the permeable material. The flow through the body of the permeable material, which is homogeneous and isotropic, is assumed to be governed by Darcy's Law. This law is of the nature of a statistical result giving the empirical equivalent of the Navier-Stokes equation as averaged over a very large number of individual pores. It has been applied and tested on a very broad class of fluid flows, and its approximate validity for common viscous liquids at low Reynolds number† can scarcely be challenged (Muskat 1937, Scheidegger 1957). In the absence of body forces Darcy's Law may be written as

$$Q = -\frac{k}{\mu} \frac{dP}{dx}, \quad (1)$$

where k is the permeability of the material and Q is a volume flow rate per unit cross-sectional area. As such, Q represents the filter velocity rather than the true velocity of the fluid in the pores.

We now postulate that the slip velocity at the permeable interface differs from the mean filter velocity within the permeable material and that shear effects are transmitted into the body of the material through a boundary-layer region (figure 1). Across this boundary layer the velocity changes rapidly from its value u_B at the interface to the Darcy value given by equation (1). We further assume that the slip velocity for the free fluid is proportional to the shear rate at the

† The characteristic length used in defining the Reynolds number within the porous medium is of the order of a typical diameter of a channel in the medium and is ordinarily quite small. The fluid flow may be turbulent outside the porous medium and laminar within.

permeable boundary, and we relate the slip velocity to the exterior flow by the *ad hoc* boundary condition

$$\left. \frac{du}{dy} \right|_{y=O_+} = \beta(u_B - Q), \quad (2)$$

where O_+ is a boundary limit point from the exterior fluid. At the same time we retain the parallel flow assumption, which leads, through the continuity equation, to the independence of u upon x . It follows through (2) that β also does not depend on x , and so depends only on the properties of the fluid and the permeable material.

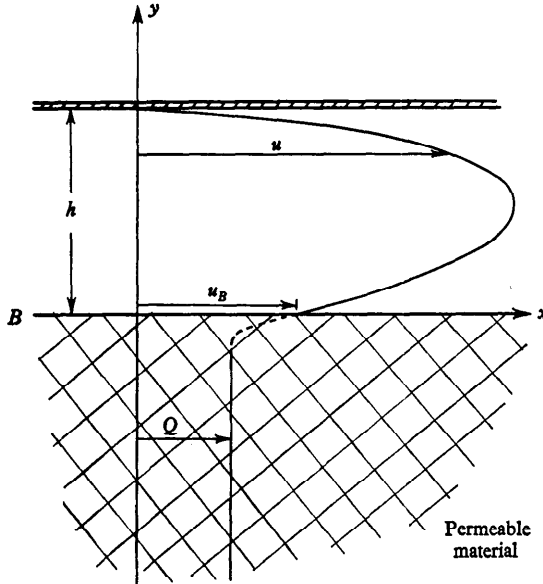


FIGURE 1. Velocity profile for the rectilinear flow in a horizontal channel formed by a permeable lower wall ($y = 0$) and an impermeable upper wall ($y = h$).

To further specify β we must consider the mechanism by which the slip velocity is induced, and we note that the pressure flow is extraneous to the boundary flow. It is convenient then, for clarity of exposition, to consider a simpler problem in which the pressure gradient is removed and the motion is induced by shear alone as in plane Couette flow. The material properties upon which β can depend are the viscosity, μ , of the fluid, the permeability, k , of the material and parameters which characterize the boundary region of the permeable material. We observe from (2) that β has dimensions of $(\text{length})^{-1}$, and we note that μ is the only known independent quantity containing the dimensions of mass and time. This suggests that β is independent of the viscosity of the fluid, a result which is strongly implied by our experimental observations. Furthermore, a length scale characterizing the permeable material is \sqrt{k} , so that we can write β as α/\sqrt{k} , where α is a dimensionless quantity depending on the material parameters which characterize the structure of permeable material within the boundary region.

We then have for the Poiseuille motion the equation

$$\frac{d^2u}{dy^2} = \frac{1}{\mu} \frac{dP}{dx}, \quad (3)$$

with the boundary conditions

$$u = 0 \quad \text{at} \quad y = h,$$

and

$$\frac{du}{dy} = \frac{\alpha}{\sqrt{k}}(u_B - Q) \quad \text{at} \quad y = 0.$$

The solution of (3) gives the velocity profile in the channel as

$$u = u_B \left(1 + \frac{\alpha}{\sqrt{k}} y \right) + \frac{1}{2\mu} (y^2 + 2\alpha y \sqrt{k}) \frac{dP}{dx},$$

where the slip velocity, u_B , is given by

$$u_B = -\frac{k}{2\mu} \left(\frac{\sigma^2 + 2\alpha\sigma}{1 + \alpha\sigma} \right) \frac{dP}{dx},$$

with

$$\sigma = h/\sqrt{k}.$$

The mass flow rate per unit width through the channel is then

$$M = -\frac{\rho h^3}{12\mu} \frac{dP}{dx} - \frac{\rho h^3}{4\mu\sigma} \left(\frac{\sigma + 2\alpha}{1 + \alpha\sigma} \right) \frac{dP}{dx}.$$

It follows that the fractional increase in mass flow rate through the channel with a permeable lower wall over what it would be if the wall were impermeable is

$$\Phi = \frac{3(\sigma + 2\alpha)}{\sigma(1 + \alpha\sigma)}. \quad (4)$$

The quantity Φ takes the value 3 when $\sigma = \sqrt{2}$, independent of α . This occurs when the velocity at the permeable wall of the channel is equal to the Darcy value within the permeable material, and the velocity profile in the channel has a zero gradient at the permeable wall. In most applications, h will be considerably greater than $\sqrt{(2k)}$. It is probable, therefore, that for values of σ near $\sqrt{2}$ the average size of the individual pores within the material is at least equal to the height of the channel, and the assumption of rectilinear flow in the channel breaks down.

3. Experiments

The equipment is designed to provide accurate simultaneous measurements of the flow rate through a long porous block and the flow rate through a small uniform gap immediately above this block. The porous block is inserted into an open rectangular channel which connects an upstream reservoir with two downstream reservoirs (figure 2). The downstream end of the block is positioned against an adjustable divider plate which is set to the same height as the top of the block and is designed to separate the efflux from the gap from that through the porous block. The porous blocks have non-porous vertical aluminium sides which are used to support small spacers. The top of the channel rests on these spacers and is held down firmly by means of adjusting screws. In this manner, a gap of any desired size can be created by using the appropriate spacers. The channel top is made of plate glass and has pressure orifices at intervals of 1 in. along the centre line.

A constant head is maintained in the upstream reservoir by means of an adjustable overflow weir. Fluid flows into the stratified test section through a smooth converging entrance, and on leaving the channel flows into one of the two downstream reservoirs. The fluid leaves these reservoirs over weirs which can be adjusted independently, so that the pressure at the exit plane of the porous block is the same in both the porous material and in the gap above the material. Solenoid valves direct the fluid either to measuring vessels or to a large collecting tank, from where it is returned through a filter to the upstream reservoir.

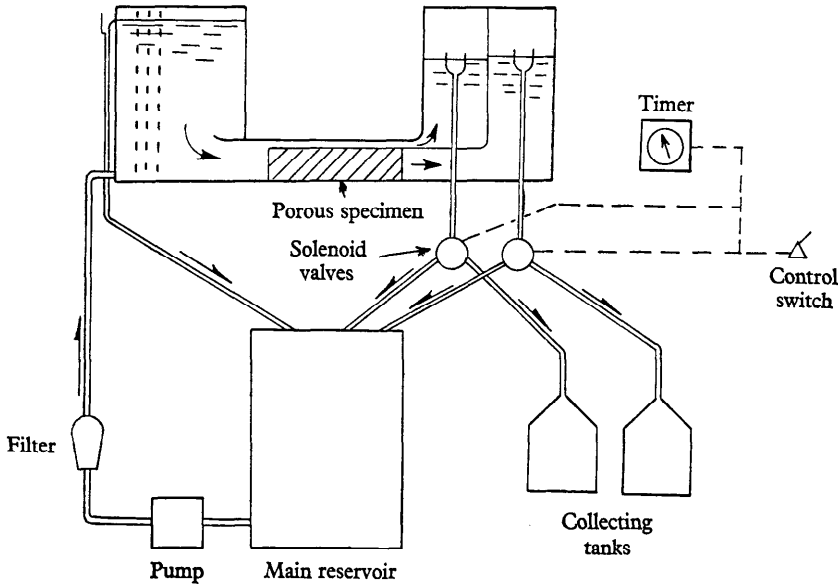


FIGURE 2. Experimental arrangement.

Various samples of two types of permeable material have been used. These are low-density nickel foametal manufactured by the General Electric Company, and aloxite manufactured by the Carborundum Company. These materials were chosen on account of their basic difference in structure. Foametal has a cellular structure consisting of irregularly-shaped interconnected pores formed by a lattice construction, whereas aloxite is made from fused crystalline aluminium oxide grains held together with a ceramic bond.

All specimens were 8 in. long, with an effective flow area of 3.5 by 1.5 in.

4. Results

Initial experiments were performed using demineralized water and gaps formed by impermeable upper and lower walls. The measured mass flow rates are compared with the predicted Poiseuille flow rates in figure 3, where the data are plotted as $M_p/\mu b$ against $-(\rho h^3/12\mu^2) dP/dx$. These quantities correspond, respectively, to measured and predicted Reynolds numbers based on the mean velocity in the gap and the height of the gap. The agreement between the measured values and the predicted values is good over the whole range of measured Reynolds

numbers, with over 90 % of the experimental values having errors of less than 2 %. The repeatability of the data coupled with the linearity of the pressure gradients recorded during the individual tests demonstrates that our method of setting up the experiment produces uniform gaps very close to a nominal size defined by the thickness of the spacers.

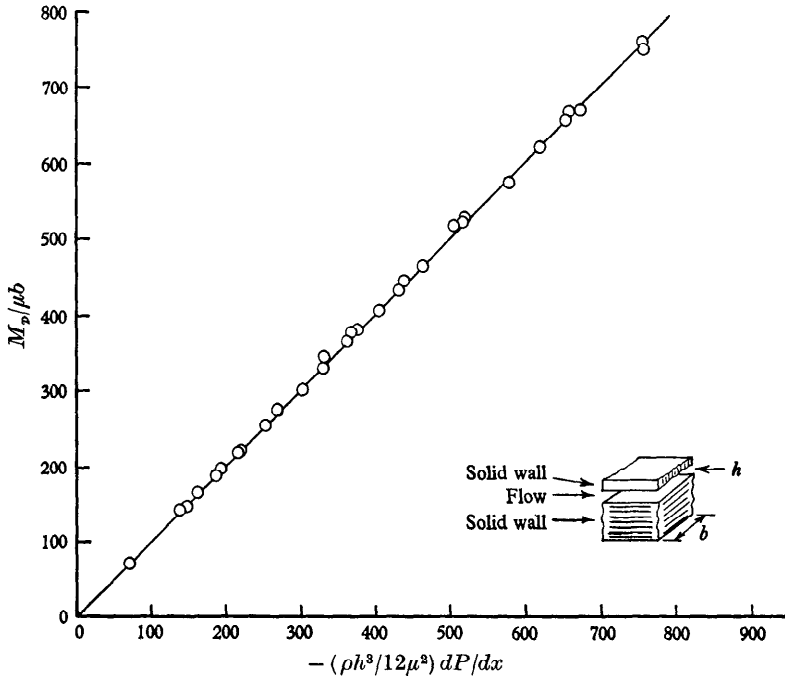


FIGURE 3. Calibration of the gap for Poiseuille flow using demineralized water. $M_p/\mu b$ is the Reynolds number computed from the measured mass flow rate, and $-(\rho h^3/12\mu^2) dP/dx$ is the Reynolds number computed from the measured pressure gradient.

Figure 4 shows the measured fractional increase in mass flow rate through the gap plotted against the parameter h/\sqrt{k} for the flow of demineralized water over a foametal block. Also plotted on this figure are two members of the family of curves given by (4), with assumed values for α of 0.8 and 1.2. These experimental data for water show considerably more scatter than the data obtained from the experiments using oil, described below and in figures 6 and 7, and we are thus not able to fix a value for α . The data, however, appear to be consistent with the assumed form for the boundary condition at the permeable wall. The scatter occurs mainly as a result of errors involved in adjusting and measuring the pressure gradients along the gap and the block. The equipment is designed primarily for use with thin oils, so that water, having a much smaller dynamic viscosity requires a correspondingly smaller pressure gradient to produce a given mass flow rate. With a small pressure gradient, the flows through the gap and the block are very sensitive to inaccurate adjustments of the downstream pressure heads.

An investigation of the effects of surface roughness on Poiseuille flow was

performed in order to eliminate this as the major mechanism producing the observed increase in the mass efflux through the gap. For this experiment the lower wall of the gap consisted of an impermeable rough surface formed from a slab of resinoid-bonded aluminium oxide, having an average pore size which corresponded closely to that of the permeable material used for the experiments described above. The surface was not rough in the sense that asperities projected

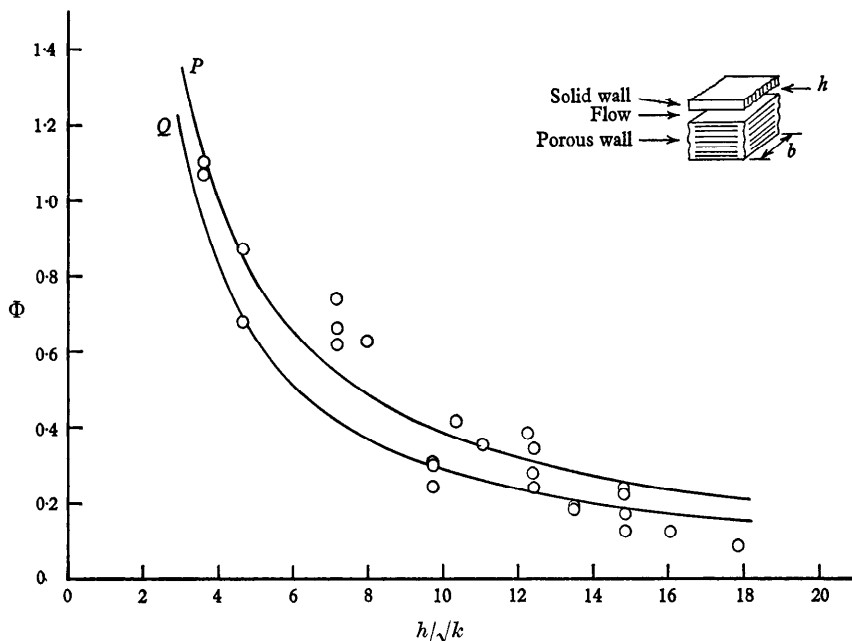


FIGURE 4. Φ as a function of h/\sqrt{k} for foametal porous specimens using demineralized water. $k = 1.1 \times 10^{-6}$ in.². Curve P, $\alpha = 0.8$; curve Q, $\alpha = 1.2$.

into the flow, but was flat with a large number of surface pores which penetrated into the material, producing a surface similar in texture to that of the foametal block. The results of this experiment are shown in figure 5, where the resistance factor, λ , is plotted against Reynolds number based on the height of the gap and the average velocity in the gap. Also shown in this figure is the theoretical result for rectilinear flow, $\lambda = 24/Re$. The experimental points agree closely with the theoretical result except at the very low Reynolds numbers and in the region where transition from laminar to turbulent flow begins to occur. The data at the low Reynolds numbers were obtained using gap sizes very close to the average pore size of the rough surface, so that the assumption of rectilinear flow within the gap was probably not valid for these tests. We conclude from this figure that, for gap sizes greater than an average pore diameter, the surface roughness caused by the pores has a negligible effect on the flow rate, provided the flow is in the laminar regime. It follows that the substantial increase in the mass flow rate through the gap produced by the presence of the permeable wall is a consequence of the shearing action in the boundary layer within the wall.

Three different samples of foametal and two samples of aloxite were used in

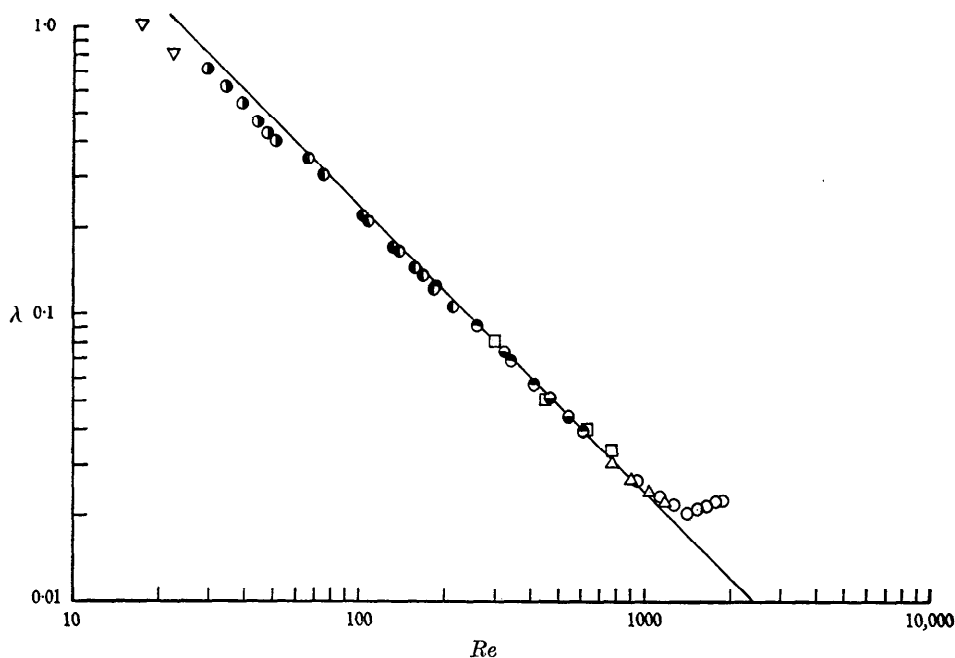


FIGURE 5. Resistance coefficient as a function of Reynolds number for a rough wall using demineralized water. ∇ , $h = 0.0104$ in.; \bullet , $h = 0.0156$ in.; \odot , $h = 0.0236$ in.; \ominus , $h = 0.0313$ in.; \oplus , $h = 0.0340$ in.; \square , $h = 0.0405$ in.; \triangle , $h = 0.0469$ in.; \circ , $h = 0.0641$ in. —, $\lambda = 24/Re$.

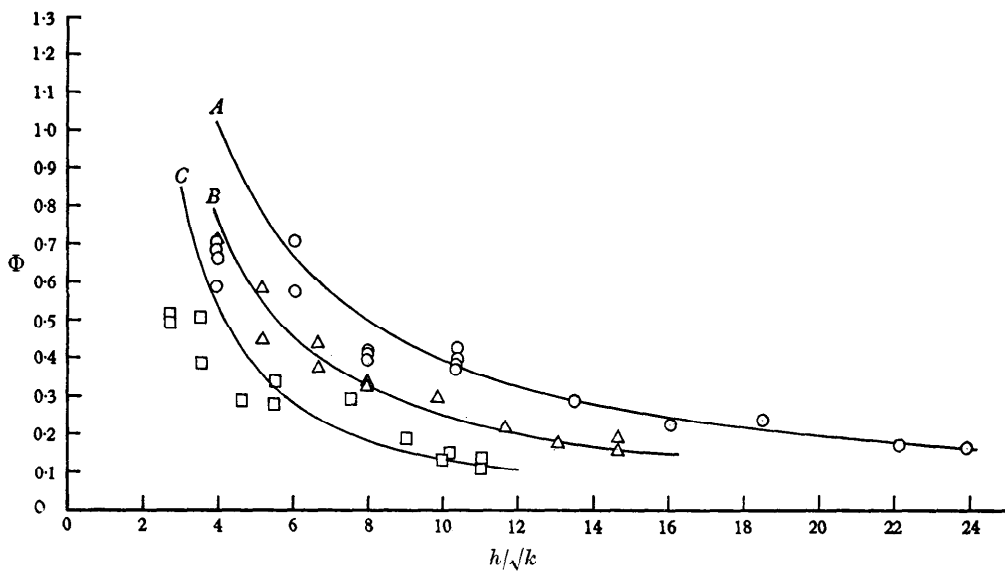


FIGURE 6. Φ as a function of h/\sqrt{k} for foametal porous specimens using Sinclair 100-Grade Duro oil. \circ , $k = 1.5 \times 10^{-5}$ in.²; \triangle , $k = 6.1 \times 10^{-5}$ in.²; \square , $k = 12.7 \times 10^{-5}$ in.². Curve A, $\alpha = 0.78$; curve B, $\alpha = 1.45$; curve C, $\alpha = 4.0$.

experiments with Sinclair 100-Grade Duro oil. The fractional increase in mass flow rate through the gap for each of the foametal blocks is compared in figure 6 with a member of the family of curves given by (4), where the values of α have been chosen such that the curves appear to give the best fits to the experimental points. Comparison of figures 4 and 6 (curve A) suggests that for a given permeable material, the fractional increase in the mass flow through the gap is essentially

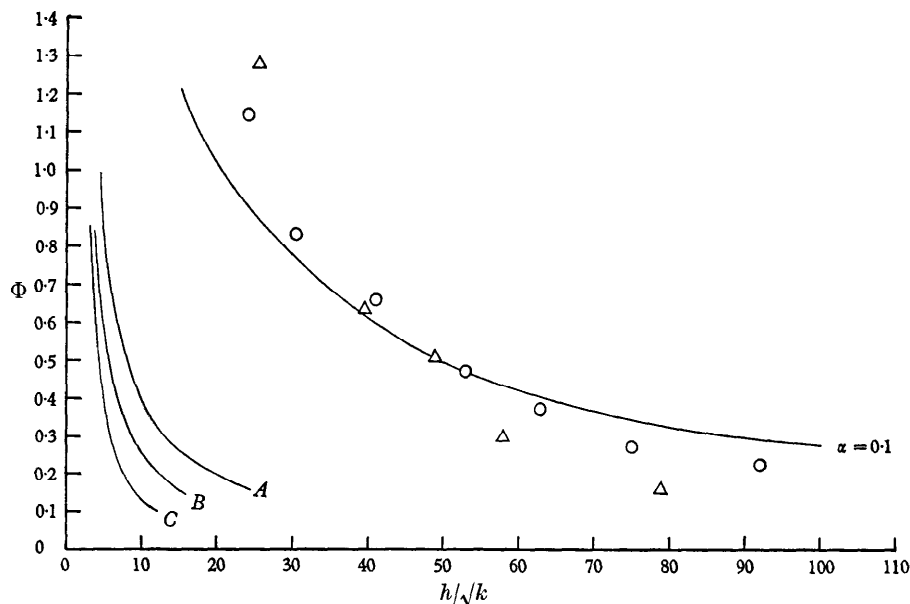


FIGURE 7. Φ as a function of h/\sqrt{k} for aloxite porous specimens using Sinclair 100-Grade Duro oil. O, $k = 1.0 \times 10^{-6}$ in.²; Δ , $k = 2.48 \times 10^{-6}$ in.².

the same for oil and water. The ratio of the dynamic viscosity of the 100-Grade oil to that of water is roughly 30, and it would seem, on this account, very unlikely that the fractional increase in mass flow could depend strongly on the viscosity.

The results for the aloxite blocks are shown in figure 7, which includes, for comparison, the curves presented in figure 6. It is not possible to find values of α for these two blocks which will allow members of equation (4) to be chosen such that they pass through all the experimental points. A value of α of 0.1 appears to give the best fit for both specimens. The most striking feature about figure 7 is that the granular material (aloxite) appears to have a much greater effect on the exterior flow than the 'lattice-type' material (foametal). A possible reason for this is to be found in the structures of the two materials. The foametal has a uniform pore distribution throughout the whole of the material, whereas the aloxite is a compacted granular material, and so probably has a greater porosity very close to the edges of the material than in the main body of the material. This would result in a higher value of the permeability in the boundary layer region than the value measured for the block. Consequently, the effect of this type of material on the rectilinear flow in the gap would be that of a material

with a somewhat higher permeability than the measured value. The slip coefficient is thus dependent on the structure of the material at the interface, and materials having roughly the same permeability or even bulk porosity may have quite different slip coefficients.

Since α depends on the structural constants of the material, an attempt was made to correlate α with the 'average pore diameter' of the material. Pore 'diameters' were measured over a large part of the upper surface of each foametal block and averaged. It turned out that for each sample the range of pore sizes about the mean was not great, so that the concept of an 'average pore diameter' for this type of material is probably not too unrealistic. The identification of

Block	$k(\text{in.}^2)$	α	Average pore size (in.)
Foametal A	1.5×10^{-5}	0.78	0.016
Foametal B	6.1×10^{-5}	1.45	0.034
Foametal C	12.7×10^{-5}	4.0	0.045
Aloxite	1.0×10^{-6}	0.1	0.013
Aloxite	2.48×10^{-6}	0.1	0.027

TABLE 1

individual pores at the surface of the aloxite, however, was not possible because of the construction of the material. An estimate of the effective pore size for each block has been obtained from the manufacturers specifications of the theoretical effective pore size, this being the size of the largest spherical particle that can just pass through the material. These results are summarized in table 1, which shows that α for the foametal material depends directly on the 'average pore diameter' at the interface.

Finally, our experiments have shown consistently that the increase in mass flow rate through the gap is accompanied by an increase in mass flow rate through the permeable block. This increase, expressed as a fraction of the flow rate predicted by Darcy's Law, is a relatively much smaller effect than the corresponding effect in the free fluid, and results have not yet sufficient precision to justify quantitative conclusions. Our experiments, however, indicate that this fractional increment in mass flow increases with gap size. These results are consistent with the hypothesis that a boundary layer in the permeable block may strongly influence the rectilinear flow in the gap above the block.

5. Conclusions

The data presented here appear to indicate that the rectilinear flow of a viscous fluid over the surface of a permeable material induces a boundary layer region within the material. The effects of this boundary layer can be such as to greatly alter the nature of the tangential motion near the nominal boundary. A slip-flow boundary condition with one experimentally-determined parameter is not inconsistent with the experimental data. This parameter would appear to be

largely independent of viscosity, but it does seem to depend on material parameters, other than the permeability, and in particular on structural parameters characterizing the nature of the porous surface.

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REFERENCES

- JOSEPH, D. D. & TAO, L. N. 1966 Lubrication of a porous Bearing-Stokes' solution. *J. App. Mech. Trans. ASME*, 753-760.
- MUSKAT, M. 1937 *The Flow of Fluids Through Porous Media*. New York: McGraw-Hill.
- SCHNEIDER, M. 1957 *The Physics of Flow Through Porous Media*. New York: Mac-Millan.

REVIEWS

Modern Developments in the Mechanics of Continua. Edited by S. ESKINAZI. Academic Press, 1966. 203 pp. 80s.

Advances in Applied Mechanics, Volume 9. Edited by G. KUERTI. Academic Press, 1966. 387 pp. 120s.

The first book is the complete proceedings of an international conference on rheology at Syracuse University in mid-1965. Papers on solids predominate and are exclusively theoretical. They reflect, however, just one approach to solid mechanics, as favoured by the American 'rationalist' school, represented at this conference by B. D. Coleman, C. Truesdell and C. C. Wang, among others. The main subjects were basic thermodynamics, the structure of constitutive laws, and the decay of acceleration waves. The analysis is for the most part concerned with very simple deformations of extremely general (hypothetical) materials. By contrast the papers on fluids have to do with technically significant problems, and in consequence are specific and largely experimental. The main topics are the Toms phenomenon in turbulent flow of dilute polymers, measurements on wave propagation in birefringent viscoelastic liquids, and the use of the Galerkin technique to estimate the critical Taylor number for stability of Couette flow of second-order fluids. There seems to be little justification for juxtaposing research papers, differing so much in spirit, outlook and context, within a single small volume.

The other book under review contains four substantial survey articles. P. G. Drazin and L. N. Howard give a systematic and unified account of the theory of hydrodynamic stability in relation to parallel flows of inviscid fluids. The subject-matter is expounded in minute detail and is closely argued. N. N. Moiseev and A. A. Petrov deal with 'sloshing' of inviscid incompressible liquid in a partly full container. Numerical results for variously shaped containers are obtained by the Ritz variational method, as outlined by Moiseev in Volume 8 for a more extensive class of oscillation problems. R. M. Rosenberg presents a closely knit and logically developed study of periodic solutions in the non-linear vibration of systems with many degrees of freedom. Rather than duplicate Minorski's book, he has chosen to concentrate on the classical geometric approach, as extended by Synge, and shows its effectiveness even for strongly non-linear systems (in which the potential energy is not a quadratic form). Finally P. Perzyna writes on viscoplastic theory, especially stress waves, from the standpoint that all apparently dynamic phenomena in the properties of metals are attributable to viscosity (this view probably needs substantial qualification after J. F. Bell's recent experimental work). This is a useful article, the first of its kind on this subject; it tends, however, to be somewhat indiscriminating in the choice of material and references, and does not attain the balance of the three other contributions.

R. HILL