Mathematical Morphology

Image and signal processing

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M1 E3A - UEVE/Upsay

Introduction

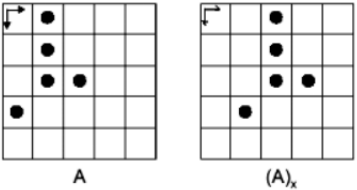
- **History**
  - 1965 : first publication of Georges Matheron & Jean Serra using the name « mathematical morphology»
  - 1965 : 2D Binary Morphology
  - 1973 : Iterative binary algorithms (skeleton, thinning)
  - 1978 : gray level processing, watershed transforms, level sets

Introduction

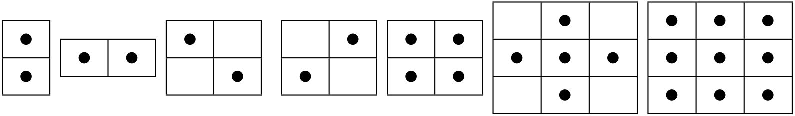
Let's consider :

- A, B = set of pixels = set of coordinates
- $A \cup B = \{x \,|x \in A \text{ or } x \in B\} \rightarrow$ Union
- $A \cap B = \{x \,|x \in A \text{ and } x \in B\} \rightarrow$ Intersection
- $A^c = \{x \,|x \notin A \ \} \rightarrow$ Complement
- $A - B = \{x \,|x \in A \text{ and } x \notin B\} \rightarrow$ Difference
- $(A)_c = \{x \,|x = a + c, a \in A \ \} \rightarrow$ Translation
- $\widetilde{A} = \{x \,|-x \in A \ \} \rightarrow$ Inversion

- **Example : translation**
  $A = \{(1,0),(1,1),(1,2),(2,2),(0,3)\}$
  $\mathbf{x} = (1,0)$
  $(A)_\mathbf{x} = \{(2,0),(2,1),(2,2),(3,2),(1,3)\}$

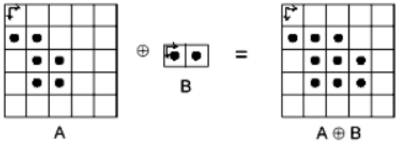


- **Structuring element**
  - Allows local analysis of an image
  - The shape and size of the structuring element give the neighborhood points that are taken into account in the analysis.
  - Most common :



$$D_B(A) = \left\{ x / \widetilde{B_x} \cap A \neq \oslash \right\} = A \oplus B$$

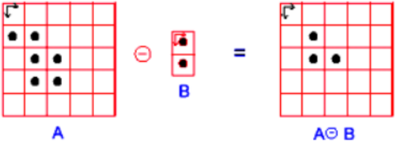
$$A \oplus B = \bigcup (A)_b$$



- **Commutative** : $A \oplus B = B \oplus A$
- **Associative** : $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
- **Monotony** : $A \subseteq B \Longrightarrow C \oplus A \subseteq C \oplus B$
- **Distributive** : $(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$ and $(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$

$$E_B(A) = \{x/B_x \subset A\} = A \ominus B$$

$$A \ominus B = \bigcap (A)_{-b}$$



- **Monotony** : $A \subseteq B \Longrightarrow A \ominus C \subseteq B \ominus C$
- **Monotony** : $A \supseteq B \Longrightarrow C \ominus A \subseteq C \ominus B$
- **Duality erosion/dilation** :

$$(A \ominus B)^c = A^c \oplus \widetilde{B}$$

$$(A \oplus B)^c = A^c \ominus \widetilde{B}$$

$(A \ominus B)^c = \{x/B_x \subset A\}^c = \{x/B_x \cap A^c = \oslash\}^c =$
$\{x/B_x \cap A^c \neq \oslash\} = A^c \oplus \widetilde{B}$

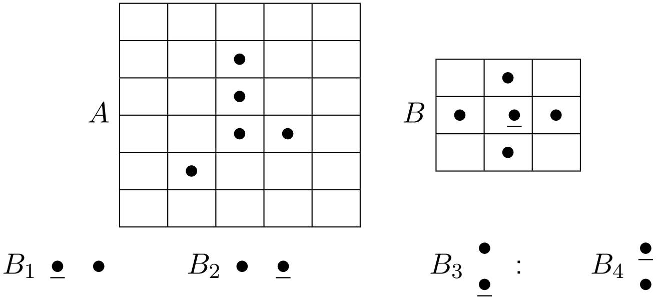
Basic operations in mathematical morphology

Erosion and dilation : union of structuring elements

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$$

- **Example** :



Original binary image, eroded image, dilated image

Original binary image, eroded image, dilated image

- **Remarks :**
  - Dilatation makes small holes disappear and makes objects bigger.
  - Erosion makes small objects disappear and thins the remaining objects.
  - Dilation and erosion are non-reversible operations.

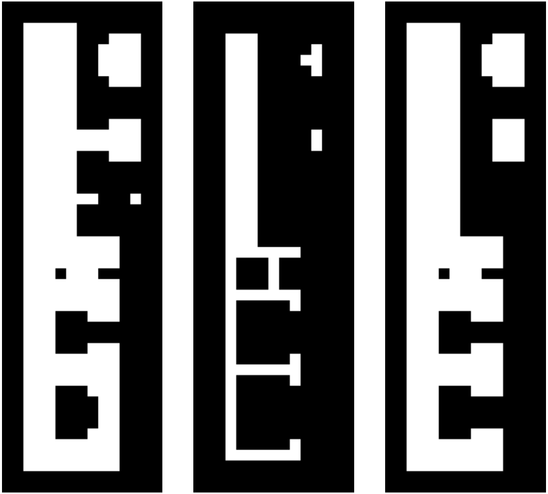
Opening

$$A \circ B = (A \ominus B) \oplus B$$

- **Purpose :**
  - Separation of "objects" whose connection is smaller than the size of the structuring element
  - Removal of "objects" whose size is smaller than the structuring element

Opening

Example



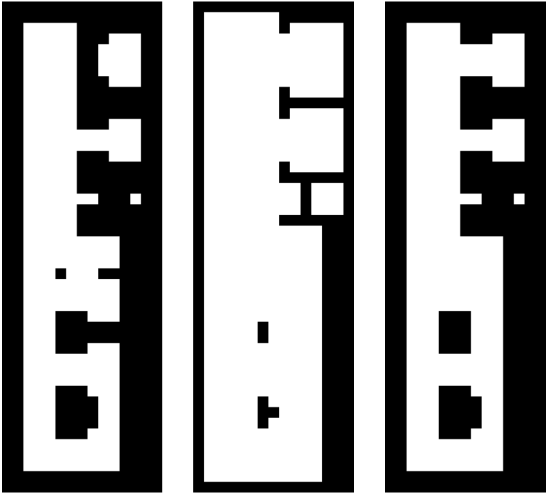
Closing

$$A \bullet B = (A \oplus B) \ominus B$$

- **Purpose :**
  - Union of components whose distance is smaller than size of the structuring element
  - Fill holes smaller than the structuring element

Closing

Example



- Opening and closing duality : $(A \bullet B)^c = A^c \circ \widetilde{B}$.
- Idempotency : $(A \circ B) \circ B = A \circ B$ and $(A \bullet B) \bullet B = A \bullet B$
- $A \circ B \subseteq A$
- $A \subseteq B \Longrightarrow A \circ C \subseteq B \circ C$
- $A \bullet B \supseteq A$
- $A \subseteq B \Longrightarrow A \bullet C \subseteq B \bullet C$

Skeleton

- The skeleton is the set of $S_0, S_1, ..., S_n$ so that :

$$S_n = (A \ominus_n B) - (A \ominus_n B) \circ B$$

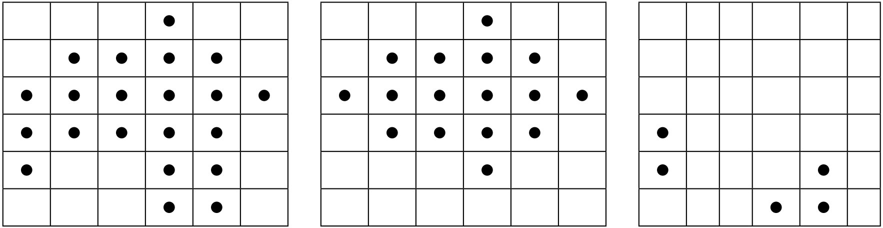
$$(A \ominus_0 B) = A$$

- We can come back to the initial image using the skeleton in applying the reverse operation :

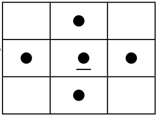
$$A = \bigcup_{n=0}^{N} S_n \oplus_n B$$

Skeleton

Example

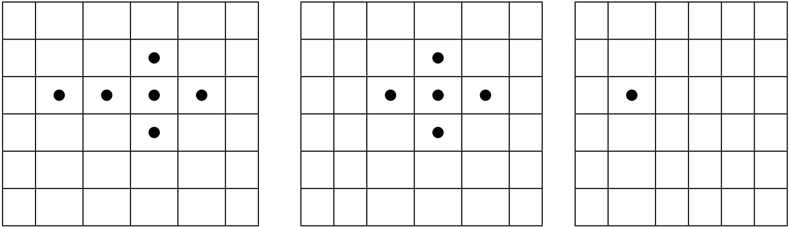


$A \;\; ; \;\; A \circ B$ and S_0

B 

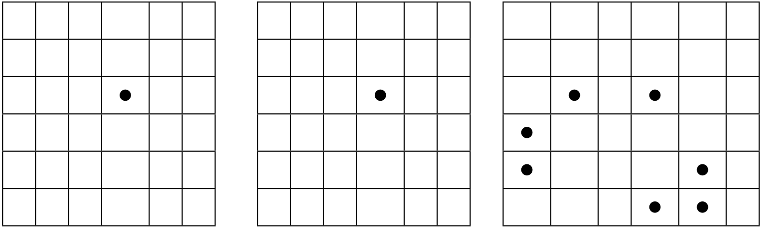
$$S_0 = (A \ominus_0 B) - (A \ominus_0 B) \circ B = A - A \circ B$$

Skeleton
Example



$A \ominus B$; $A \ominus B \circ B$ and S_1

$$S_1 = (A \ominus B) - (A \ominus B) \circ B$$



$A \ominus_2 B$; S_2 and $S = \cup S_1, S_2, S_3$

$$S_2 = (A \ominus_2 B) - (A \ominus_2 B) \circ B$$

- The skeleton is obtained by successive thinning and each thinning is obtained in the following way :

$$A - A \otimes V_x.$$

V_x represents the neighborhood configuration centered at x.

$$A \otimes V_x = \left\{ \begin{array}{c} 1 \text{ if } V_x(A) \in V \\ 0 \text{ else} \end{array} \right\}$$

.

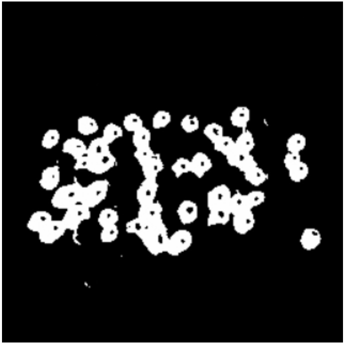
V is the set of eligible configurations :

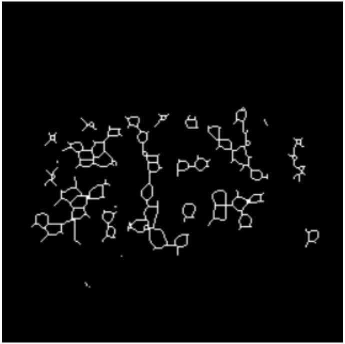
$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \times & 1 & \times \\ \hline 1 & 1 & 1 \\ \hline \end{array} \text{ et } \begin{array}{|c|c|c|} \hline \times & 0 & \times \\ \hline 1 & 1 & 0 \\ \hline 1 & 1 & \times \\ \hline \end{array}$$

and their rotations.

Skeleton

Example



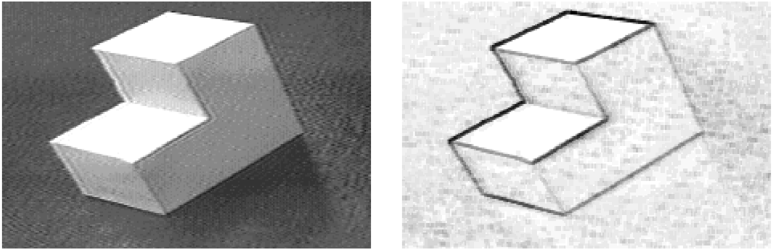


Morphological edges

Let's define :

- The internal and external contours of the shape : $D_B(A) - E_B(A)$
- The outer contour of the shape : $D_B(A) - A$
- The internal contours of the shape : $A - E_B(A)$

Morphological edges
Example



Example of morphological gradient obtained with a square structuring element.

Symmetric Difference

$$X \Delta Y = (X \cup Y) - (X \cap Y)$$

- The symmetric difference between two sets X and Y is the set of elements that belong only to X or to à Y.
- Application : motion detection

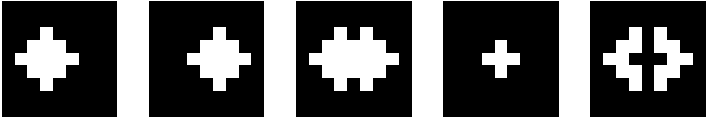


Image 1 ; image 2 ; union ; intersection ; difference between union and intersection.

Hit and Miss transform

$$HitMiss(A, B1, B2) = (A \ominus B_1) \cap (A^c \ominus B_2)$$

- $B_1 \cap B_2 = \oslash$
- Operation equivalent to a template matching $\rightarrow$ B_1 : template (Hit) and B_2 : background (Miss)
- Erosion = 1 if the image is locally equal to the structuring element (1 inside the structuring element). With the hit and miss transform, zeros (background) should be similar.
- Example : detecting isolated points (8-connectivity)

$$B_1 \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \qquad B_2 \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 0 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

Hit and Miss transform

Example

- Detecting corners (up/right)

A

			●	●			
		●	●	●	●		
		●	●	●	●		
			●	●			
			●				

A^c

●	●	●	●	●	●	●	●
●	●	●			●	●	●
●	●					●	●
●	●					●	●
●	●	●			●	●	●
●	●	●		●	●	●	●
●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●

B_1

0	0	0
●	●	0
0	●	0

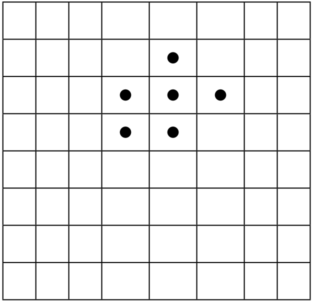
B_2

0	●	●
0	0	●
0	0	0

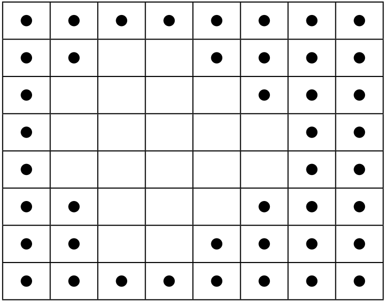
Hit and Miss transform

Example

$A \ominus B_1$



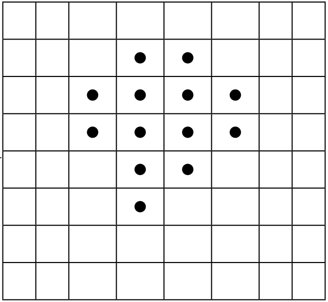
$A^c \ominus B_2$



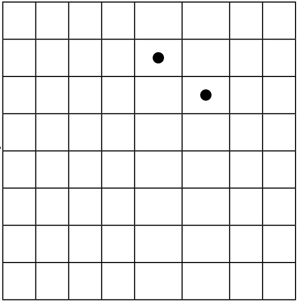
Hit and Miss transform

Example

A



$HitAndMiss$



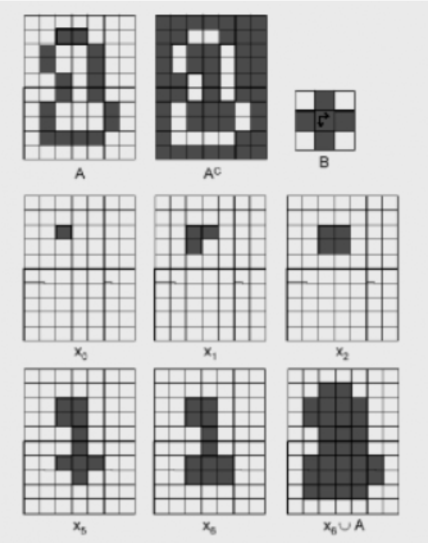
Shape filling

$$\begin{cases} x_{k+1} = (x_k \oplus B) \cap A^C \\ k = 0, .., n \\ K = x_n \cup A \end{cases}$$

Where A is the initial shape, B the structuring element and x_k lthe intermediate shapes at each iteration. K is the resulting image.

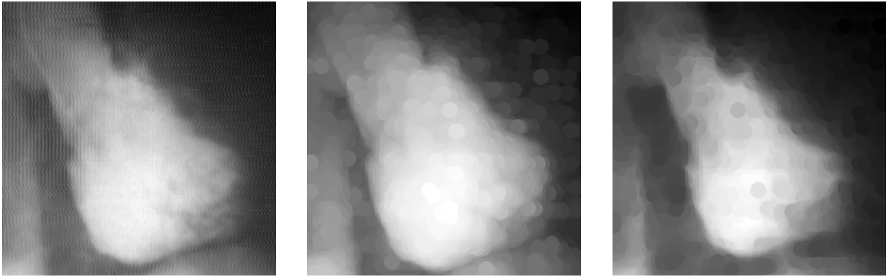
- Choose a seed inside the shape$\rightarrow x_0$
- Iterative process

Shape filling



Mathematical morphology for gray scale images

- Dilation $\rightarrow$ sup : $A \oplus B = \sup\{I(u)/u \in B_x\}$
- Erosion $\rightarrow$ inf : $A \ominus B = \inf\{I(u)/u \in B_x\}$



Input Image, dilatation and erosion