Examination on Statistical Signal Processing

June 2021

PROBLEM A

Consider the scalar (real) valued hidden Markov process defined by the following equations.

$$
\begin{aligned}
X_n &= a_n X_{n-1} + U_n \\
Y_n &= c_n X_n + \sqrt{\beta} W_n
\end{aligned}
$$

where for integer $n \geq 0$, $\{X_0, U_n, W_n\}$ are independent $\mathcal{N}(0,1)$, $\beta > 0$.

A.1 Recall Kalman equations in the context of this simplified model. Notation conventions are

$$
\begin{aligned}
X_n^- &= E\left(X_n | Y_{0:n-1}\right) \\
\hat{X}_n &= E\left(X_n | Y_{0:n}\right) \\
P_n^- &= cov\left(X_n - X_n^-\right) \\
P_n &= cov\left(X_n - \hat{X}_n\right)
\end{aligned}
$$

All these quantities are scalars. (Matlab notation is used, for example $Y_{0:n} \equiv (Y_0, ..., Y_n)$).

A.2 Let $a_n = a$, $0 < a < 1$, $c_n = c$. From the recursion relations between P_n and P_n^- , P_n^- and P_{n-1} deduce a relation between P_n and P_{n-1} .

A.3 To calculate the asymptotic expression of P_n take $P_n = P = const.$ in this equation and solve for P.

A.4 Describe the behavior of P with respect to the noise variance β and the dynamics $a_n = a$ and comment the result.

PROBLEM B

In this problem we view the beginning of the EM algorithm for the identification of a Markov chain. $(X_n)\,, n \geq 0$ is a Markov sequence with values in $\{1, -1\}$and $p(X_0 = 1) = 1/2$. For $n \geq 1$, the transition probabilities $p(x_n|x_{n-1})$ are given by the matrix

$$A = \begin{bmatrix} \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \\ \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} \end{bmatrix}$$

where $0 < \varepsilon < 1$ is a parameter to be estimated. The obszerved signal is $Y_n = X_n + \sqrt{\beta} W_n$, $n = 0 : N$, such that $W_n \sim \mathcal{N}(0, 1)$ is an additive noise independent of (X_n) and $\beta > 0$ is to be estimated. The vector of parameters is $\theta = (\varepsilon, \beta)$.

B.1. Check that $p(x_n|x_{n-1}) = (1 + \varepsilon x_{n-1} x_n)/2$.

B.2 Let $f_{0:N}(x_{0:N}) \equiv f_{0,\ldots,N}(x_0, \ldots, x_N)$ be *any* probability density on $\{x_{0:N}\}$. Argue from Jensen's inequality to show that

$$\log p\left(y_{0:N}|\theta\right) \geq H(\theta) + C_0$$

where

$$H(\theta) = \sum_{x_{0:N}} f_{0:N}(x_{0:N}) \log p(x_{0:N}, y_{0:N}|\theta)$$

and C_0 is a term independent of θ .

B.3 By conditioning $y_{0:N}$ on $x_{0:N}$ and using properties of Markov sequences ans independent sequences show that

$$\begin{aligned}\log p\left(x_{0:N}, y_{0:N}|\theta\right) &= -\frac{1}{2\beta}\sum_{n=0}^{N}\left(y_n - x_n\right)^2 - \frac{1}{2}\beta \\ &+ \sum_{n=1}^{N}\log\left(1+\varepsilon x_{n-1}x_n\right) + C_1\end{aligned}$$

where C_1 is a term independent of θ.

B.4 Combinate this relation with the expression of $H(\theta)$ to show that $H(\theta)$ is maximized by the equations

$$\beta = \sum_{n=0}^{N}\sum_{\{x_n\}} f_n\left(x_n\right)\left(y_n - x_n\right)^2$$

and

$$\sum_{n=0}^{N}\sum_{\{x_{n-1}x_n\}} f_{n-1,n}(x_{n-1}, x_n)\frac{x_{n-1}x_n}{1+\varepsilon x_{n-1}x_n} = 0$$

B.5 Solve this last equation in ε (consider that $x_{n-1}x_n$ takes only 2 values).

PROBLEM C

Let $(S_n), n \geq 0$ a random sequence with ± 1 values and probability distributions p_N such that for $J_n > 0$

$$\begin{aligned}p_N(s_{1:N}) &= q_N(s_{1:N})/Z_N \\ &= \frac{\exp -\frac{1}{2}\sum_{n=1}^{N} J_n s_{n-1}s_n}{\sum_{\{s_{1:N}\}} \exp -\frac{1}{2}\sum_{n=1}^{N} J_n s_{n-1}s_n}\end{aligned}$$

C.1 Compute the conditional distribution

$$p_N(s_n|\tilde{s}_n)$$

where $\tilde{s}_n$ is the vector $(s_i, i \neq n)$, deduce that (S_n) is a Markov sequence and write its transitions matrix.

C2. Let $(Y_n), n \geq 0$ be a sequence of random observations on S_n such that

$$Y_n = S_n + W_n$$

where the W_n are i.i.d $\mathcal{N}(0, \beta)$. Define the *Hamiltonian*

$$H(s_{1:N}) = -\frac{1}{2}\sum_{n=1}^{N} J_n s_{n-1} s_n + \sum_{n=1}^{N} y_n s_n$$

Show that maximizing (in $s_{1:N}$) the joint probability

$$p(s_{1:N}, y_{1:N})$$

is equivalent to compute the limit for $t \to \infty$ of the *finite temperature estimates* defined by

$$\hat{s}_n(t) = \frac{\sum_{s_n} s_n \exp tH(s_{1:N})}{\sum_{s_n} \exp tH(s_{1:N})}$$

C3. For fixed t calculate the expression for $\hat{s}_n(t)$ show that it takes the following form and interpret this result:

$$\hat{s}_n(t) = \tanh\left\{t\left(y_n - \varphi_n\right)\right\}$$