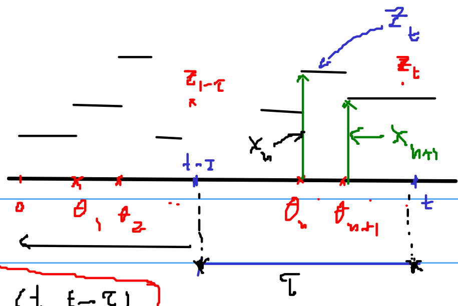


$E[X_k X_{k+h}] = \gamma_X(h)$ Wide sense stationarity

$\{\theta_n\}$ $\{X_n\}$ ← stationary

1/ $\gamma_Z(t, t-\tau)$, $\tau \geq 0$

$$= E[Z_t \cdot Z_{t-\tau}] = \sum_{n=0}^{\infty} E[Z_t Z_{t-\tau} \mid \underbrace{N(t-\tau, t)}_{\text{Number of Poisson events in } [t-\tau, t[} = n] \, P[N_\tau = n]$$

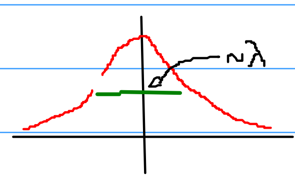
$$= \sum_{n\geq 0} E[X_0 X_n] e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} = e^{-\lambda\tau} \sum_{n \geq 0} \gamma_X(n) \frac{(\lambda\tau)^n}{n!}$$

* Suppose $\{X_n\}$ iid, $EX_n = 0$ $\rightarrow$ $\gamma_X(h) = \sigma^2 \delta(h) = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \text{ otherwise} \end{cases}$

$$\gamma_Z(t, t-\tau) = \gamma_Z(\tau) = \sigma^2 e^{-\lambda\tau} = \sigma^2 e^{-\lambda|\tau|}$$

ν = freq.

* Spectrum of Z (PSD): $\Gamma_Z(\nu) = \int_{-\infty}^{\infty} \gamma_Z(\tau) e^{-i2\pi\nu\tau} d\tau = \sigma^2 \left\{ \int_{-\infty}^{0} e^{-i2\pi\nu\tau + \lambda\tau} d\lambda + \int_0^{\infty} e^{-i2\pi\nu\tau - \lambda\tau} d\tau \right\}$



$$\frac{\sigma^2}{\lambda^2 + 4\pi^2\nu^2} \sim \Gamma_Z(\nu)$$

* $P_{t,\tau,t}(Z_{t-\tau} = z_1, Z_t = z_2) \equiv P_\tau(z_1, z_2) = \sum_{h \geq 0} P_\tau(z_1, z_2 \mid N_\tau = h) P[N_\tau = h]$

$\{X_n\}$ gaussian iid $N(0, \sigma^2)$

$$= P_\tau(z_1, z_2 \mid N_\tau > 0) P[N_\tau > 0] \quad (1)$$

$$+ P_\tau(z_1, z_2 \mid N_\tau = 0) P[N_\tau = 0] \quad (2)$$

(1) $= \underbrace{P[Z_t = z_2 \mid Z_{t-\tau} = z_1, N_\tau > 0]}_{P_X(X_0 = z_2)} \underbrace{P[Z_{t-\tau} = z_1 \mid N_\tau > 0]}_{\underbrace{P[Z_{t-\tau} = z_1]}_{P_X(X_0 = z_1)}} \underbrace{P[N_\tau > 0]}_{(1 - e^{-\lambda\tau})}$

$$= (1 - e^{-\lambda\tau}) \frac{1}{2\pi\sigma^2} e^{-\frac{z_1^2 + z_2^2}{2\sigma^2}}$$

$$① = \frac{(1-e^{-\lambda\tau})}{2\pi} \exp - \frac{z_1^2 + z_2^2}{2\sigma^2}$$

N_τ

$t-\tau$ t

$+$

$$② = \underbrace{P(Z_t = z_2 \mid Z_{t-\tau} = z_1, N_\tau = 0)}_{\delta(z_2 - z_1)} \underbrace{P(Z_{t-\tau} = z_1 \mid N_\tau = 0)}_{P_{X_0}(z_1)} \underbrace{P(N_\tau = 0)}_{e^{-\lambda\tau}} = \frac{e^{-\lambda\tau}}{\sqrt{2\pi\sigma^2}} e^{-z_1^2/2\sigma^2} \delta(z_2 - z_1)$$

$$③ \quad \gamma_Z(\tau) = e^{-\lambda\tau} \sum_{k=0}^{\infty} \gamma_X(n) \frac{(\lambda\tau)^n}{n!}$$

$$x_n = \alpha x_{n-1} + v_n \qquad \{v_n\} \text{ iid.} \quad \mathbb{E}[v_n] = 0, \quad \mathbb{E}[v_n^2] = \beta$$

$$|\alpha| < 1$$

$$\gamma_X(n) = \mathbb{E}[x_k \cdot x_{k-n}]$$

$$\gamma_X(0) = \mathbb{E}[x_n^2] = \alpha^2 \underbrace{\mathbb{E}[x_{n-1}^2]}_{\gamma_X(0)} + \underbrace{\mathbb{E}[v_n^2]}_{\beta} + 2\alpha \underbrace{\mathbb{E}[x_{n-1} \cdot v_n]}_{0} \rightarrow \gamma_X(0) = \frac{\beta}{1-\alpha^2}$$

$$\underbrace{\mathbb{E}[x_n \cdot x_{n-k}]}_{\gamma_X(k)} = \alpha \underbrace{\mathbb{E}[x_{n-1} x_{n-k}]}_{\gamma_X(k-1)} + \underbrace{\mathbb{E}[v_n x_{n-k}]}_{0} \rightarrow \gamma_X(k) = \alpha \gamma_X(k-1)$$

$$\gamma_X(k) = \frac{\beta}{1-\alpha^2} \cdot \alpha^k \qquad |\alpha| < 1$$

$$\gamma_Z(\tau) = e^{-\lambda\tau} \underbrace{\frac{\beta}{1-\alpha^2}}_{\sigma_X^2} \sum_{k \geq 0} \frac{(\lambda\tau)^k}{k!} \alpha^k = \sigma_X^2 e^{-\lambda(1-\alpha)\tau}$$

- Conditioning on gaussian vectors
- Lin. Mean square estimation of parameters
- → Antenna Processing
- → Spectrum Estimation

* GAUSSIAN CONDITIONING

$$\underline{Z} = \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow n \end{matrix} \qquad z \in \mathbb{R}^{n+m} \qquad E[\underline{Z}] = \underline{0} \qquad R_z = E[\underline{z}\,\underline{z}^T]$$

$$R_z = \begin{bmatrix} E\,\underline{x}\,\underline{x}^T & E(XY^T) \\ E(YX^T) & E[YY^T] \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{bmatrix} \qquad R_{yx} = R_{xy}^T$$

$$Z \sim N(0, R_z) \rightarrow p_z(\underline{z}) = \frac{1}{|2\pi R_z|^{1/2}} \exp -\frac{1}{2}\left(\underline{z}^T R_z^{-1} \underline{z}\right) \leftarrow$$

$\hat{X}(Y) = E[X \mid Y]$ = Best. m.s. estim of X observing Y

$$= \int x \, p_{x|y}(x \mid y)\, dx = \frac{\int x \, p_z(x, y)\, dx}{p_y(y)}$$

→ Calculate R_z^{-1}

$$\text{let } M = R_z^{-1} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \; ; \; \begin{bmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{bmatrix} \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$$

$$R_{yx} A + R_y B = 0 \Rightarrow \underline{B = -R_y^{-1} R_{yx} A} \rightarrow \underline{B^T = -A R_{xy} R_y^{-1}}$$

$$R_x A + R_{xy} B = I$$

$$* \quad z^T R_z^{-1} z - y^T R_y^{-1} y = [x^T \; y^T] \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - y^T R_y^{-1} y = x^T A x + x^T B^T y + y^T B x + y^T C y - y^T R_y^{-1} y$$

$$= (x + A^{-1}B^T y)^T A (x + A^{-1} B^T y) - y^T \left[B A^{-1} B^T + R_y^{-1} \right] Y$$

↓

Extremum at

$$X^* = -A^{-1} B^T y = \underbrace{R_{xy} R_y^{-1}}_{L_{x|y}} y = \hat{X}(y) = \underline{E[X \mid Y]}$$

$$\mathrm{cov}(X \mid Y) = \mathrm{cov}\,(\underbrace{X - \hat{X}(y)}_{\text{error}}) = A^{-1}$$

$$A^{-1} = R_x + R_{xy} B A^{-1} =$$

$$= R_x - R_{xy} R_y^{-1} R_{yx} = \underline{R_x - L_{x|y} R_y L_{x|y}^T}$$

* Now minimize directly

$$\min_{L} \mathbb{E}\{\|\underbrace{x - L y}_{\varepsilon}\|^2\}$$

for any (non necessarily joint gaussians) random vectors X, Y

Reminder: for any square matrix M $\quad tr(M) = \sum_{k=1}^{n} M_{kk} = \sum$ eigen values $= \sum_{k=1}^{n} \lambda_k$

x, y vectors $\quad \underbrace{x^T \cdot y}_{\sum x_k y_k} = tr(y \cdot x^T)$

$$Var(\varepsilon) = \mathbb{E}\{\|\varepsilon\|^2\} = \mathbb{E}(\varepsilon^T \cdot \varepsilon) = \mathbb{E}[tr(\varepsilon \cdot \varepsilon^T)] = tr\, \mathbb{E}[\varepsilon \cdot \varepsilon^T] \qquad \mathbb{E}[W] = \overline{W}$$

$$= tr\left\{ \underbrace{\mathbb{E}[x \cdot x^T]}_{R_X} - L \underbrace{\overline{YX^T}}_{R_{YX}} - \underbrace{\overline{XY^T}}_{R_{XY}} L^T + L \underbrace{\overline{YY^T}}_{R_Y} L^T \right\}$$

$$= tr\left\{ (L - R_{XY} R_Y^{-1}) R_Y (L - R_{XY} R_Y^{-1})^T \right\} + tr\left\{ R_X - R_{XY} R_Y^{-1} R_{YX} \right\}$$

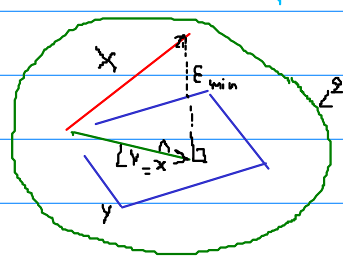
positive definite: $\forall \vec{w}: \vec{w}^T R_Y w > 0$

$Var(\varepsilon)$ is minimized by

$$L = R_{XY} \cdot R_Y^{-1}$$

$$\min_{L} Var(\varepsilon) = tr\left\{ R_X - \underbrace{R_{XY} R_Y^{-1} R_{YX}}_{L R_Y L^T} \right\}$$

geometrical Interpretation:



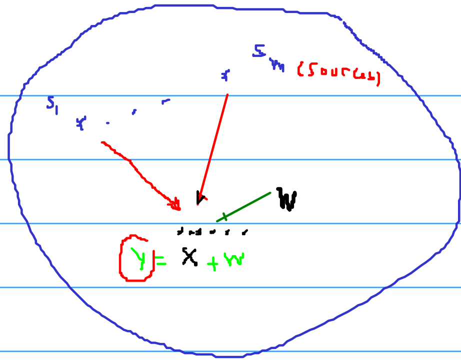
$$\mathbb{E}[\varepsilon^T \cdot Y] = 0$$

$$tr\, \mathbb{E}[(X - LY) Y^T] = 0$$

$$\underbrace{\mathbb{E}[X \cdot Y^T]}_{R_{XY}} - L \underbrace{\mathbb{E}[Y Y^T]}_{R_Y} = 0 \qquad \text{(Normal Equation)}$$

$$cov(\varepsilon) = cov(X) - cov(LY)$$

APPLICATION TO ANTENNA PROCESSING (Music Algorithm)



VAN TREES : Estimation, Detection & Modulation
(2 VOL.)
LIBGEN

$$\left\{ \begin{array}{l} \vec{X} = \sum_{i=1}^{m} \theta_i \underline{S}_i \quad \text{Antenna} = \underset{\sim}{S} \cdot \underline{\theta} \\ \vec{Y} = \vec{X} + \vec{W} \end{array} \right.$$

θ_i: Random; $\underline{S}_i$: Deterministic; $\vec{W}$: Noise.

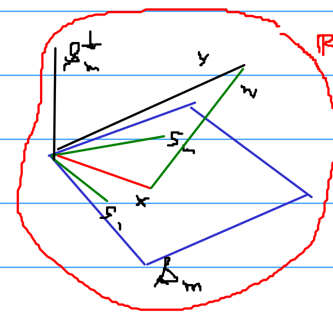
$\overline{W} = 0 \qquad \overline{WW^T} = R_W \quad , \; R_{XW} = E[XW^T] = 0$

Optimal Receiver

$$\boxed{L = R_{XY} R_Y^{-1}} = \overline{X(X+W)^T} \left\{ \overline{(X+W)(X+W)^T} \right\}^{-1}$$

$$= \underbrace{R_X}\underbrace{[R_X + R_W]}_{R_Y}^{-1} = \underset{\sim}{S} \underbrace{\overline{\theta\theta^T}}_{C_\theta} \underset{\sim}{S} \, [\underset{\sim}{S} C_\theta \underset{\sim}{S}^T + R_W]^{-1}$$

$\underset{\sim}{S} = [\underline{S}_1, \ldots, \underline{S}_m] . \quad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$



* Hypothesis: Noise is 'white', isotropic $\rightarrow$ $R_W = \beta I$

$\rightarrow$ $L = R_X R_Y^{-1} = (R_Y - R_W) R_Y^{-1} = I - R_W R_Y^{-1} = I - \beta R_Y^{-1}$

* $R_Y = R_X + \beta I$ $\rightarrow$ 1/ Eigenvectors of R_X are eigenvectors of R_Y
2/ $\mathcal{S}$ and $\mathcal{S}^\perp$ are eigenspaces of R_Y

- $R_Y = \beta I + \underset{\sim}{S} C_\theta \underset{\sim}{S}^T$
- $R_Y \underline{S}_i = \beta \underline{S}_i + \underbrace{\underset{\sim}{S} C_\theta \underset{\sim}{S}^T \cdot \underline{S}_i}_{\in \mathcal{S}} \in \mathcal{S} \quad \therefore \; R_Y \mathcal{S} \subseteq \mathcal{S}$
- If $\vec{V} \in \mathcal{S}^\perp$, $R_Y \vec{V} = \beta \vec{V} + \underset{\sim}{S} C_\theta \underbrace{\underset{\sim}{S}^T \vec{V}}_{0} = \beta \vec{V} \in \mathcal{S}^\perp$
- Let $R_X = E\, XX^T = \underbrace{\sum_{i=1}^{m}}_{} \lambda_i \underline{u}_i \underline{u}_i^T$: $\left\{ \begin{array}{l} \{u_i\} \text{ are orthogonal for } R_X = R_X^T \\ \lambda_i \geq 0 \quad (R_X \text{ is positive definite}) \end{array} \right.$

$u_i \perp u_j \quad i \neq j$
$\{u_i\}$ span $\mathcal{S}$

$$R = \underset{\sim}{U} \Lambda \underset{\sim}{U}^{-1} = \underset{\sim}{U} \Lambda \underset{\sim}{U}^{+} = [\underline{u}_1, \ldots, \underline{u}_N] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_N \end{bmatrix} \begin{bmatrix} \underline{u}_1^+ \\ \cdots \\ \underline{u}_N^+ \end{bmatrix}$$

* Let $u_{m+1}, \ldots, u_N$ span $\mathcal{S}^\perp$ and $u_i \perp u_j = 0 \quad i \neq j$, $i, j = m+1 : N$

$$I = \sum_{i=1}^{N} u_i u_i^+ \Rightarrow \underbrace{R_W} = \beta I = \beta \underbrace{\sum_{i=1}^{m} \underline{u}_i \underline{u}_i^+}_{\text{Proj on } \mathcal{S}} + \beta \underbrace{\sum_{i=m+1}^{N} \underline{u}_i \underline{u}_i^+}_{\text{Proj on } \mathcal{S}^\perp}$$

$\underline{u}_i \underline{u}_i^+$: Projector in direction u_i

$$R_Y = \sum_{i=1}^{m} \mu_i \underline{u}_i \underline{u}_i^+ + \beta \sum_{m+1}^{N} u_i u_i^+$$

* $\underbrace{R_y} = \sum \underbrace{(\lambda_i+\beta)}_{\mu_i} \underline{u}_i \underline{u}_i^{+} + \beta \sum_{i=m+1}^{N} u_i u_i^{+}$, $\lambda_i \geq 0 \Rightarrow \beta = \underline{\text{least eigenvalue of } R_y}$

μ_i = 'Dominant' eigenvalues of R_y

$$R_y^{-1} = \sum \mu_i^{-1} \underline{u}_i \underline{u}_i^{+} + \frac{1}{\beta} \sum u_i u_i^{+}$$

$$L = I - \beta R_y^{-1} = \sum_{i=1}^{m} \left(1 - \frac{\beta}{\mu_i}\right) \underline{u}_i \underline{u}_i^{+}$$

$$1 - \frac{\beta}{\mu_i} = 1 - \frac{\beta}{\lambda_i+\beta} = \frac{\lambda_i}{\lambda_i+\beta} = \frac{\lambda_i/\beta}{1+\frac{\lambda_i}{\beta}}$$

if $\beta/\mu_i \searrow 0$: $L \to P_{[S]_\beta} = \sum_{i=1}^{m} \underline{u}_i \underline{u}_i^{+}$

* EXERCISE

$$\mathrm{Var}(\varepsilon_{min}) = \mathrm{tr}\{R_x - R_{xy} R_y^{-1} R_{yx}\}$$

$$= \beta \sum_{i=1}^{m} \frac{\lambda_i}{\beta+\lambda_i} \leq m\beta$$

$$\mathrm{Var}(\underline{w}) = N\beta$$

$$\frac{\mathrm{Var}(\varepsilon_{min})}{\mathrm{Var}(w)} \leq \frac{m}{N}$$

* VAN TREES

* SHELDON ROSS

Applied probability