TD 1 et 2 : Fourier transform of 2D discrete functions, filtering in the frequency space

M1 E3A international track, site Evry

UE "Image and Signal processing", Univ. Paris-Saclay / Univ Evry

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Exercise 1 : Properties of the Fourier transform

Let $f(x)$ a summable function, its Fourier transform is given by :
$\Im\{f(x)\} = F(u) = \int f(x)\exp(-2\pi jux)dx$

1. Prove the linearity of the Fourier Transform (FT).
2. Computer the FT of the transposed function $f(-x)$.
3. Scale change : consider $a \neq 0$, compute the FT of $f(ax)$.
4. Compute the FT of the translated function $f(x-a)$.
5. Compute the FT of the modulated function $f(x)\exp(2\pi ju_0 x)$.
6. Compute the FT of the the derivative function $f'(x)$.
7. Find the function $g(x)$ with a FT equal to $\frac{\partial F(u)}{\partial u}$.

Exercise 2 : Fourier Transform and convolution

Let $f(x)$ and $g(x)$ two summable functions and their Fourier Transform $F(u)$ and $G(u)$.

1. Show that the FT of $f \star g$ is $F(u)G(u)$.

Exercise 3 : Fourier transform of a Gaussian

1. Determine the Fourier Transform $F(u)$ of $f(x) = \exp(-\pi x^2)$.
2. Using a change of variable, determine the Fourier transform of $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})$
3. Comment the relation between the standard deviation of the Gaussian and its Fourier transform.

Exercise 4 : 2D Fourier transform

Consider : $F(u,v) = \int\int f(x,y)\exp\left[-2\pi i(ux+vy)\right]dxdy$

1. Show that we can reduce this expression to a composition of two monodimensional transformations.

Exercise 5 : Fourier Transform of a Laplacian

Calculate the Fourier of a Laplacian : $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Exercise 6 : Discrete Fourier Transform

1. Consider the function f defined for 4 samples :
   $f(0) = 2\,;\ f(1) = 3\,;\ f(2) = 4\,;\ f(3) = 4$
   Compute its discrete Fourier Transform.

2. Consider the filter : $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$.

   Compute its discrete Fourier Transform (DFT).

Exercise 7 : Fast Fourier Transform

The objective is to reveal some properties that could be useful to compute fast the Fourier Transform in few operations.

$F(u) = \sum_{x=0}^{N-1} f(x) \exp(\frac{-2\pi iux}{N})$

where N is the number of samples.

Let write $w_N = \exp(\frac{-2\pi i}{N})$ and $N = 2M$,

1. Show that the previous expression could be written :

   $F(u) = \frac{1}{2}\left[\frac{1}{M}\sum_{x=0}^{M-1} f(2x).w_M^{u_x} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)w_M^{u_x}.w_{2M}^{u}\right]$

   We will use :

   $F_P(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)w_M^{u_x}$ and $F_I(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)w_M^{u_x}$

   Then :

   $F(u) = F_P(u) + F_I(u)w_{2M}^{u}$.
2. Show that : $F(u+M) = F_P(u) - F_I(u)w_{2M}^{u}$
3. Explain why this implementation is faster.
4. Compute the FFT for $N = 4$. Show how the partial results are re-used for upper dimensions.

Exercise 8 : Isotropy

1. Show that the Laplacian $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is an isotropic operator (invariant toward rotation).
2. Show that the amplitude of Gradient $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$ is isotropic.
3. Show that the amplitude of Gradient $\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$ is not isotropic.

Exercise 9 : Average Filter

Let's consider the average filter 3×3 excluding from calculation the pixel in the middle.

1. Find the filter $H(u, v)$ that is equivalent in the frequency space.
2. Show that the resulting filter is a low pass filter.

Exercise 10 : Laplacian Filter

Let's condider the Laplacien 3×3 filter : $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

1. Find the filter $H(u, v)$ that is equivalent in the frequency space.

2. Show that the resulting filter is a high pass filter.

Exercise 11 : Approximation of a Laplacian

Show that $f(x, y) - \nabla^2 f(x, y)$ could be approximated by $f(x, y) - \overline{f}(x, y)$ where $\overline{f}$ is the average f in a given neighborhood.

Recall :

- If $y = -x$ then $dy = -dx$ and $\int_{-\infty}^{+\infty} dx = -\int_{+\infty}^{-\infty} dy = \int_{-\infty}^{+\infty} dy$
- $\exp(x + y) = \exp(x)\exp(y)$
- $\exp(x)^a = \exp(ax)$
- $\frac{1}{\exp(x)} = \exp(-x)$
- Gauss Integral : $\int \exp(-\alpha x^2) dx = \sqrt{\frac{\pi}{\alpha}}$
- $\int \exp(ax) dx = \frac{1}{a}\exp(ax)$
- $\exp(i\theta) = \cos(\theta) + j\sin(\theta)$
- $\cos(\theta) = \cos(-\theta)$ and $\sin(\theta) = -\sin(-\theta)$
- $\cos(\theta) = \frac{1}{2}(\exp(j\theta) + \exp(-j\theta))$
- $\sin(\theta) = -\frac{j}{2}(\exp(j\theta) - \exp(-j\theta))$