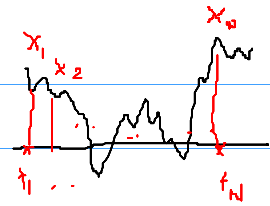
1/

$\overline{(X+Y)^2} = \overline{x^2} + \overline{y^2} + 2\,\overline{xy}$

$Var(x) = \overline{x^2} - \bar{x}^2 = \mathbb{E}\,(x-\bar{x})^2$

$\underline{X} : \Omega \to \mathbb{R}^N$ $\qquad \underline{X} = \begin{bmatrix} x_1 \\ \cdots \\ x_N \end{bmatrix}$



$\underline{Y} : \Omega \to \mathbb{R}^N$

$$\mathbb{E}\left[\underline{X}\cdot\underline{Y}^T\right] = \begin{array}{c|c} & m \\ \hline n & \begin{bmatrix} & \vdots \\ \cdots & \overline{x_n y_m} \end{bmatrix} \end{array}$$

$$\overline{(X+Y)^2} = \overline{X^2} + \overline{Y^2} + 2\,\overline{XY}$$

$$= (\overline{X^2} - \bar{X}^2) + (\overline{y^2} - \bar{y}^2) + 2\,\overline{xy} - 2\bar{x}\bar{y}$$

$$+ \underbrace{\bar{x}^2 + \bar{y}^2 + 2\bar{x}\bar{y}}_{(\bar{x}+\bar{y})^2}$$

* Weak law of large Numbers

$\underbrace{Z_n = \frac{1}{n}(X_1 + \dots + X_n)}$ $\qquad x_n$ indep. identically distributed iid

$\bar{X} = \bar{x}_n \qquad \sigma^2 = \overline{x_n^2}$

$\mathbb{E}\,Z_n = \frac{1}{n}\sum \mathbb{E}X_i = \frac{1}{n}\times n\bar{x} = \bar{X}$

Chebyshev. Th.

$$\mathbb{P}\left\{|Z_n - \bar{X}| > \varepsilon\right\} = \mathbb{P}\left\{|Z_n - \bar{X}|^2 > \varepsilon^2\right\} \leqslant \frac{1}{\varepsilon^2}\underbrace{\mathbb{E}\left[|Z_n - \bar{X}|^2\right]} \longrightarrow 0$$

$\varepsilon > 0$

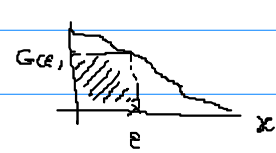
But: $\mathbb{E}\,|Z_n - \bar{x}|^2 = Var(Z_n) = \frac{1}{n^2}(n \times Var(X_i)) = \frac{n\cdot\sigma^2}{n^2} = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$

Proof of chebyshev.

Let $X \geqslant 0$, $\mathbb{E}X < \infty$

$$\mathbb{E}X = \int_0^\infty x\, P_X(x)\,dx = \int_0^\infty G_X(x)\,dx \qquad G_X(x) = \mathbb{P}[X > x]$$

$$\geqslant \varepsilon\, G(\varepsilon)$$



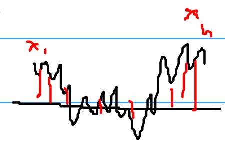
Characteristic function: $X \sim P_X(x)$

$$\varphi_X(u) \triangleq \mathbb{E}\, \underbrace{e^{iuX}} = \mathbb{E}\sum_n \frac{(iu)^n}{n!} X^n = \sum \frac{(iu)^n}{n!}\overline{X^n}$$

$$\varphi_X(0)=1 \qquad \varphi'_X(0) = i\overline{X} \qquad \varphi''_X(0) = -\frac{\overline{X^2}}{2}$$

$$\varphi^{(n)}_X(0) = \frac{i^n}{n!}\overline{X^n}$$

* $X_1,\dots,X_n$



$\boxed{\text{iid}}$, $\mathrm{Var}\,X_n = \overline{X_n^2} = \sigma^2$

$z_n = \frac{1}{\sqrt n}(X_1+\dots+X_n)$ CLT

$$\psi_{z_n}(u) = \log \varphi_{z_n}(u) \quad ; \quad \varphi_{z_n}(u) = \mathbb{E}\, e^{iu\frac{(X_1+\dots+X_n)}{\sqrt n}} = \mathbb{E}\left[e^{iu\frac{X_1}{\sqrt n}} \times\dots\times e^{iu\frac{X_n}{\sqrt n}}\right]$$

indep.

$$= \underbrace{\mathbb{E}\left(e^{iu\frac{X_1}{\sqrt n}}\right)}_{\varphi_{X_1}(u)}\cdots \underbrace{\mathbb{E}\left(e^{iu\frac{X_n}{\sqrt n}}\right)}_{\varphi_{X_N}(u)} = \varphi_X^n\left(\frac{u}{\sqrt n}\right) \quad ; \quad \overline{X}=0$$

$$\underbrace{\psi_{z_n}(u)} = \sum_{k=1}^n \log\varphi_{X_k}\left(\frac{u}{\sqrt n}\right) = n\log\left\{1-\frac{u^2}{2n}\overline{X^2} + O\left(\frac{1}{n^{3/2}}\right)\right\} \quad ; \quad \log(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2}+\dots+\frac{\epsilon^n}{n}(-)^{n-1}+\dots$$

$$= -\frac{u^2}{2}\overline{X^2} + O(1/\sqrt n) \xrightarrow[n\to\infty]{} -\frac{u^2\overline{X^2}}{2}$$

$$\left\{\varphi_{z_n}(u) \xrightarrow[n\infty]{} \underbrace{e^{-u^2\overline{X^2}/2}}\right.$$

= ch. funct. of Gaussian (Normal) R.V.

* In general:

$$\underline{X}\sim N(0,R) \Leftrightarrow P_X(x) = \underbrace{\frac{1}{|2\pi R|^{1/2}}}_{c} \exp\left\{-\frac12 x^T R^{-1} x\right\}$$

$$\varphi_{\underline X}(u) = \mathbb{E}\, e^{i\underline{u}^T\cdot\underline{x}} = \mathbb{E}\,e^{\overset{\sqrt{-1}}{i}(u_1x_1+\dots+u_nx_n)}$$

$$= c\int \exp(-)\frac12\Big[\underbrace{x^TR^{-1}x - 2iu^Tx}_{(x-iRu)^TR^{-1}(x-iRu)+u^TRu}\Big]dx = e^{-\frac{u^TRu}{2}}\cdot \underbrace{c\int e^{-\frac12(x-iRu)^TR^{-1}(\dots)}dx}_{1}$$

$$= e^{-\frac{u^TRu}{2}}$$

3/ X Poisson $X \in \{0,1,2,\dots\} \equiv \mathbb{N}$ $\mathbb{P}(X=n) = e^{-\lambda} \frac{\lambda^n}{n!}$; $\lambda = \mathbb{E}(X)$ (parameter)

$$\varphi_X(u) = \mathbb{E}\, e^{iuX} = \sum_n e^{iun} \cdot \mathbb{P}[X=n]$$

$$= \sum_{n\geq 0} \frac{\lambda^n}{n!} e^{-\lambda + inu} = e^{\lambda(e^{iu}-1)}$$

$Z = X + Y$
X, Y indep & Poisson

$$\varphi_Z(u) = \varphi_X(u) \cdot \varphi_Y(u) = e^{(\lambda+\mu)(e^{iu}-1)}$$

* $X_1, \dots, X_n$: $X_k \sim N(0, \sigma_k^2)$ & indep

$$Z = \alpha_1 X_1 + \dots + \alpha_N X_N \qquad \varphi_Z(u) = \mathbb{E}\left[e^{iu(\alpha_1 X_1 + \dots + \alpha_n X_n)}\right] = \varphi_{X_1}(\alpha_1 u) \cdots \varphi_{X_n}(\alpha_n u)$$

$$= \exp - \frac{1}{2} (\alpha_1^2 \sigma_1^2 + \dots + \alpha_n^2 \sigma_n^2) u^2 \Rightarrow Z \sim \text{Gaussian}$$

4/ $Z = X + Y$, X, Y Poisson indep. λ_1, λ_2

$$\mathbb{P}[X=k \mid Z=n] \overset{?}{=} \text{Binomial}$$

$X \in \{0,1,2,\dots,N\}$

$$\mathbb{P}(X=k) = \binom{N}{k} \alpha^k (1-\alpha)^{N-k}$$

$0 \leq \alpha \leq 1$

$$\frac{N!}{k!\,(N-k)!}$$

$$= \mathbb{P}[A \mid B] \overset{?}{=} \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

$$= \frac{\mathbb{P}[Z_n = n \mid X=k]\, \mathbb{P}(X=k)}{\mathbb{P}[Z=n]} = \frac{\mathbb{P}[Y=n-k]\, \mathbb{P}[X=k]}{\mathbb{P}[Z=n]} = \frac{e^{\lambda_2} \lambda_2^{n-k}\, e^{\lambda_1} \lambda_1^k}{(n-k)!\, k!\, (\lambda_1+\lambda_2)^n}\, n!$$

$X \sim \lambda_1$
$Y \sim \lambda_2$

$$= \binom{n}{k} q^k (1-q)^{n-k}, \quad q \triangleq \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

* Conditional expectations : (X, Y) $P_{X,Y}(x,y)$

observable.

$$\mathbb{E}[X \mid Y=y] = \int x\, p_{XY}(x/y)\, dx$$

$$= \int x \frac{p_{XY}(x,y)}{p_Y(y)} dx = \frac{1}{p_Y(y)} \int x\, p_{XY}(x,y)\, dx$$

x

* Cond. Expectation as a information processing tool

X, Y

$$\min_{\forall h} \mathbb{E} \|X - h(Y)\|^2 \Rightarrow h(Y) = \mathbb{E}[X|Y]$$

$$\mathbb{E}\left\{ \mathbb{E}\left[\underbrace{\|X - h(Y)\|^2}_{} \,\Big|\, Y \right] \right\}$$

$$X^2 + h(Y)^2 - X\,h(Y)$$

$$\mathbb{E}\left\{ \mathbb{E}[X^2|Y] - \mathbb{E}\big((X - h(Y)) \,|\, Y\big).h(Y) \right\}$$

$\Rightarrow$ Min. for $\boxed{h(Y) = \mathbb{E}(X|Y)}$

* Exercise: $\underline{X}, \underline{Y}$ gaussian Random vectors.

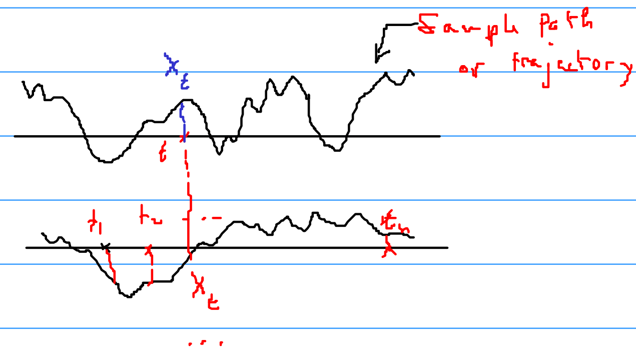
Show that $\mathbb{E}[\underline{X}|\underline{Y}] = L\underline{Y}$ $\quad \left\{ L = \underbrace{\mathbb{E}[\underline{X}\cdot\underline{Y}^T]}_{R_{XY}} \cdot \underbrace{\mathbb{E}[\underline{Y}\underline{Y}^T]^{-1}}_{R_Y} \right.$

(X, Y) gaussian:

$$\underline{Z} = \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}, \quad R_Z = \left[\begin{array}{c|c} R_X & R_{XY} \\ \hline R_{YX} & R_Y \end{array}\right]; \quad R_{YX} = R_{XY}^T$$

$$\mathbb{E}[X|y] = \int x \frac{P_{XY}(x,y)}{P_Y(y)}\,dx$$

10/ Random Process: $\{X_t, \; t \in \mathbb{R}\}$



Independent increment process:

$\forall n, t_1, \ldots, t_n$

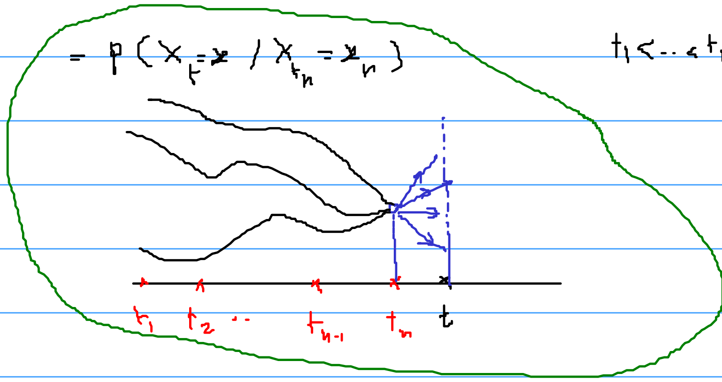
$(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \ldots, (X_{t_n} - X_{t_{n-1}})$ Independent

Ex: Brownian process

10/ Markov. processes:

$$P(X_t = x \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n)$$

$$= P(X_t = x \mid X_{t_n} = x_n) \qquad t_1 < \ldots < t_n.$$



given any X_t with indep increments:

$$P[X_t = x \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n] = P[X_t - X_{t_n} = x - x_n \mid X_{t_n} = x_n,\ X_{t_i} - X_{t_{i-1}} = x_i - x_{i-1},\ i = 2:n]$$

$$= P[X_t - X_{t_n} = x - x_n \mid X_{t_n} = x_n]$$

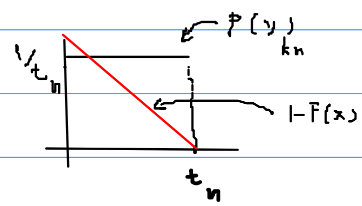
$$= P[X_t = x \mid X_{t_n} = x_n]$$

$\rightarrow \{X_t\}$ is Markovian

11/ $Y_{1n}, Y_{2n}, \ldots, Y_{nn}$ triangular array of R.V.

iid, uniformly distributed on $(0, t_n)$

$Z_n = \min(Y_{1n}, \ldots, Y_{nn})$



a/ $P(Z_n > x) = P(Y_{1n} > x, \ldots, Y_{nn} > x)$

$= P(Y_{1n} > x) \ldots P(Y_{nn} > x)$

$= [1 - F(x)]^n \qquad F(x) = P[Y_{kn} \leq x] \quad \forall k$

$= \exp n \log(1 - F(x))$

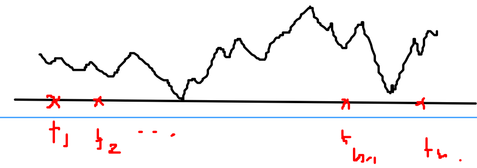
b/ $\log P(Z_n > x) = n \log(1 - x/t_n) = n\left\{-\frac{x}{t_n} + o\left(\frac{x}{t_n}\right)\right\} = -x\left(\frac{n}{t_n} + o\left(\frac{n}{t_n}\right)\right)$

$\frac{n}{t_n} \rightarrow \lambda$

$\left\{ P[Z_n > x] \xrightarrow[n\infty]{} e^{-\lambda x} \right.$

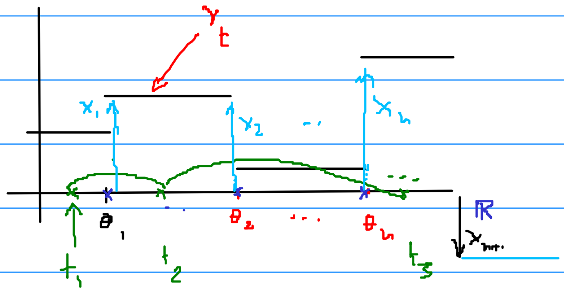
Exercise on correlation functions

A Random Process is defined by all dist. functions



$$P(x_{t_1}, \ldots, x_{t_n}) \quad \forall n, \quad \forall\, t_1 < t_2 < \cdots < t_n$$

Corr. function $\quad \gamma_x(t_1, t_2) = \mathbb{E}[X_{t_1} \cdot X_{t_2}] = \gamma_x(t_2 - t_1)$ $\boxed{\text{in stationary case}}$



$\{\theta_n\}$ is Poisson Process if:

1/ For any times $t-\tau$, t, $\tau \geq 0$

$N_{t,t-\tau}$ = nb. of random pulses θ_i occuring in $]t-\tau, t]$.

is a poisson R.V. with parameter $\lambda\tau$

$$\mathbb{P}[N_{t,t-\tau} = n] = e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$$

2/ for any times $t_1, t_2, \ldots, t_N$.

$N_{t_i, t_{i+1}}$, are indep.

3/ $\{X_n\}$ independent, 0 mean, gaussian with variance σ^2

Ex: compute $\gamma_y(t, t-\tau) = \mathbb{E}[X_t \cdot X_{t-\tau}]$, $\tau \geq 0$

$$\sim e^{-\sigma^2 \cdot \tau}$$

Hint:

$$\left\{ \begin{array}{l} \mathbb{E}[Y_t \cdot Y_{t-\tau} \mid N_{t,t-\tau} = n] = ? \\ \underbrace{\mathbb{E}[Z]}_{} = \mathbb{E}\{\mathbb{E}[Z \mid U]\} = \sum_n \underbrace{\mathbb{E}[Z \mid U = n]}_{} \cdot \mathbb{P}[U = n] \end{array} \right.$$