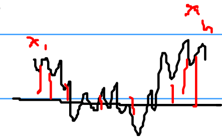
Characteristic Function: $X \sim P_X(x)$

$$\varphi_X(u) \triangleq \mathbb{E}\, e^{iux} = \mathbb{E}\sum_n \frac{(iu)^n}{n!} x^n = \sum \frac{(iu)^n}{n!}\overline{x^n}$$

$\varphi_X(0)=1 \qquad \varphi'_X(0) = i\bar{x} \qquad \varphi''_X(0) = -\frac{\overline{x^2}}{2}$

$$\varphi_X^{(n)}(0) = \frac{i^n}{n!}\overline{x^n}$$

* $x_1,\dots,x_n$



$\boxed{\text{iid}}$, $\mathrm{Var}\, x_n = \overline{x_n^2} = \sigma^2$

$z_n = \frac{1}{\sqrt n}(x_1+\dots+x_n)$ CLT

$\psi_{z_n}(u) = \log \varphi_{z_n}(u)$; $\varphi_{z_n}(u) = \mathbb{E}\, e^{iu\frac{(x_1+\dots+x_n)}{\sqrt n}} = \mathbb{E}\left[e^{iu\frac{x_1}{\sqrt n}} \times\dots\times e^{iu\frac{x_n}{\sqrt n}}\right]$

indep.

$$= \underbrace{\mathbb{E}\left(e^{iu\frac{x_1}{\sqrt n}}\right)}_{\varphi_{x_1}(u)} \cdots \underbrace{\mathbb{E}\left(e^{iu\frac{x_n}{\sqrt n}}\right)}_{\varphi_{x_N}(u)} = \varphi_X^n\left(\frac{u}{\sqrt n}\right) \quad ; \quad \bar x = 0$$

$$\underline{\psi_{z_n}(u)} = \sum_{k=1}^n \log\varphi_{x_k}\left(\frac{u}{\sqrt n}\right) = n\log\left\{1-\frac{u^2}{2n}\overline{x^2} + O\left(\frac{1}{n^{3/2}}\right)\right\} \quad ; \quad \log(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2}+\dots+\frac{\varepsilon^n}{n}(-)^{n-1}+\dots$$

$$= -\frac{u^2}{2}\overline{x^2} + O(1/\sqrt n) \xrightarrow[n\to\infty]{} -\frac{u^2\overline{x^2}}{2}$$

$$\left\{\varphi_{z_n}(u) \xrightarrow[n\infty]{} e^{-u^2\overline{x^2}/2}\right.$$ = ch. funct. of Gaussian (Normal) R.V.

* In general:

$$\underline{X} \sim N(0, R) \Leftrightarrow P_X(x) = \underbrace{\frac{1}{|2\pi R|^{1/2}}}_{c} \exp\left\{-\frac12 x^T R^{-1} x\right\}$$

$$\varphi_{\underline X}(u) = \mathbb{E}\, e^{i\underline{u}^T\cdot\underline{x}} = \mathbb{E}\, e^{\overset{\sqrt{-1}}{i}(u_1x_1+\dots+u_nx_n)}$$

$$= c\int \exp(-)\frac12\Big[\underbrace{x^T R^{-1}x - 2iu^Tx}_{(x-iRu)^TR^{-1}(x-iRu)+u^TRu}\Big]dx = e^{-\frac{u^TRu}{2}}\cdot \underbrace{c\int e^{-\frac12(x-iRu)^TR^{-1}(\dots)}dx}_{1}$$

$$= e^{-\frac{u^TRu}{2}}$$