Markov Chains for Optimization and Information Processing

I. Basic Properties of Markov Chains

INTRODUCTION

One way to construct a Markov chain is as follows. Let $\{Y_n;\ n \geq 1\}$ be mutually independent F-valued random variables, which are globally independent of X_0. Given a mapping $f : \mathbb{N} \times E \times F \to E$, we define $\{X_n; n \geq 1\}$ recursively by

$$X_n = f(n, X_{n-1}, Y_n).$$

A Markov chain is the analogue of a deterministic sequence which is defined by a recurrence relation of the type

$$x_{n+1} = f(n, x_n),$$

as opposed to a system 'with memory', of the type

$$x_{n+1} = f(n, x_n, x_{n-1}, \ldots, x_1, x_0).$$

Here the function $f(n, \cdot)$ is replaced by the 'transition matrix'

$$P_{xy} = \mathbb{P}(X_{n+1} = y | X_n = x).$$

1. DEFINITIONS AND FIRST PROPERTIES

Definition: *The E-valued stochastic process $\{X_n; n \in \mathbb{N}\}$ is called a Markov chain whenever, for all $n \in \mathbb{N}$, the conditional law of X_{n+1} given $X_0, X_1, \ldots, X_n$ equals its conditional law given X_n, that is, for all $x_0, \ldots, x_{n+1} \in E$,*

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n).$$

A simple criteria is given by the following:

Lemma: *Let E and F be two countable sets, and let f be a mapping from $\mathbb{N} \times E \times F$ into E. Let $X_0, Y_1, Y_2, \ldots$ be mutually independent random variables, X_0 being E-valued, and the Y_n being F-valued. Let $\{X_n;\ n \geq 1\}$ be the E-valued process defined by*

$$X_{n+1} = f(n, X_n, Y_{n+1}), \quad n \in \mathbb{N}.$$

Then $\{X_n;\ n \in \mathbb{N}\}$ is a Markov chain.

So a Markov chain can be viewed as a dynamical system driven by a ‘white noise’.

Proof:

$$\begin{aligned}
\mathbb{P}(X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n) \\
&= \frac{\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n, X_{n+1} = x_{n+1})}{\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n)} \\
&= \sum_{\{z;\, f(n,x_n,z)=x_{n+1}\}} \frac{\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n, Y_{n+1} = z)}{\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n)} \\
&= \sum_{\{z;\, f(n,x_n,z)=x_{n+1}\}} \mathbb{P}(Y_{n+1} = z) \\
&= \frac{\mathbb{P}(X_n = x_n, X_{n+1} = x_{n+1})}{\mathbb{P}(X_n = x_n)}.
\end{aligned}$$

From now on, this matrix $P = (P_{xy};\ x, y \in E)$ will be assumed to be independent of the time variable n. One then says that the Markov chain is *homogeneous*.

The matrix P is called *Markovian* (or *stochastic*), in the sense that it has the property that, for all $x \in E$, the row vector $(P_{xy};\ y \in E)$ is a probability measure on E, or in other words,

$$P_{xy} \geq 0,\ \forall y \in E; \quad \sum_{y \in E} P_{xy} = 1.$$

As we will now see, the law of a Markov chain is entirely determined by the 'initial law' $(\mu_x;\ x \in E)$, which is the law of X_0, and the transition matrix of the chain.

<u>Definition:</u> *Let μ be a probability on E, and P a Markovian matrix. An E-valued random sequence $\{X_n;\ n \in \mathbb{N}\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is called a (μ, P) Markov chain (i.e. with initial law μ and transition matrix P) if:*

(i) $\mathbb{P}(X_0 = x) = \mu_x,\ \forall x \in E$;

(ii) $\mathbb{P}(X_{n+1} = y | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = P_{xy}$, $\forall x_0, \ldots, x_{n-1}, x, y \in E$.

One has then the

<u>Proposition:</u> *A necessary and sufficient condition for an E-valued random sequence $\{X_n;\ n \in \mathbb{N}\}$ to be a (μ, P) Markov chain is that, for all $n \in \mathbb{N}$, the law of the random variable $(X_0, X_1, \ldots, X_n)$ be given by*

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \mu_{x_0} P_{x_0 x_1} \times \cdots \times P_{x_{n-1} x_n}.$$

Proof For the necessary condition, if $\mathbb{P}(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) > 0$, then

$$\begin{aligned}\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n) =& \mathbb{P}(X_n = x_n | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) \\ & \times \cdots \times \mathbb{P}(X_1 = x_1 | X_0 = x_0)\mathbb{P}(X_0 = x_0),\end{aligned}$$

and the above identity follows from the definition. Otherwise, both sides of the identity in the statement are zero (consider the smallest index k such that $\mathbb{P}(X_0 = x_0, \ldots, X_k = x_k) = 0$).

We now turn to the sufficient condition. (i) The identity in the statement follows from the definition. Let us prove more than (ii):

$$\begin{aligned}\mathbb{P}(X_{n+1} = x_{n+1}, \ldots, X_{n+p} = x_{n+p} | X_0 = x_0, \ldots, X_n = x_n) \\ = \frac{\mu_{x_0} P_{x_0 x_1} \times \cdots \times P_{x_{n+p-1} x_{n+p}}}{\mu_{x_0} P_{x_0 x_1} \times \cdots \times P_{x_{n-1} x_n}}\end{aligned}$$

and (ii) now follows if we choose $p = 1$. □

We have also the following result:

If $\{X_n;\ n \in \mathbb{N}\}$ *is a* (μ, P) *Markov chain, then for all* $n, p, x_0, \dots,$ x_{n+p},

$$\mathbb{P}(X_{n+1} = x_{n+1}, \dots, X_{n+p} = x_{n+p} | X_0 = x_0, \dots, X_n = x_n)$$
$$= P_{x_n x_{n+1}} \times \cdots \times P_{x_{n+p-1} x_{n+p}}.$$

A probability μ on E is considered to be a row vector, a mapping $g : E \to \mathbb{R}$ as a column vector, which justifies the notation

$$(\mu P)_y = \sum_{x \in E} \mu_x P_{xy},$$
$$(Pg)_x = \sum_{y \in E} P_{xy} g_y,$$

and the integral of a function g with respect to a measure μ is written (whenever the sum converges absolutely) as the product of a row vector on the left with a column vector on the right:

$$\mu g = \sum_{x \in E} \mu_x g_x.$$

Proposition: *Let $\{X_n; n \in \mathbb{N}\}$ be a (μ, P) Markov chain. Then*

(i) $\mathbb{P}(X_n = y | X_0 = x) = \mathbb{P}(X_{n+p} = y | X_p = x) = (P^n)_{xy}$,

(ii) $\mathbb{P}(X_n = y) = (\mu P^n)_y$,

(iii) $\mathbb{E}[g(X_n) | X_0 = x] = (P^n g)_x$.

Proof:

(i)

$$\mathbb{P}(X_n = y | X_0 = x) = \sum \mathbb{P}(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 | X_0 = x)$$

(ii) We note that

$$\begin{aligned}\mathbb{P}(X_n = y) &= \sum_{x \in E} \mathbb{P}(X_n = y, X_0 = x) \\ &= \sum_{x \in E} \mathbb{P}(X_n = y | X_0 = x) \mu_x,\end{aligned}$$

and we use (i).

(iii) We again use (i) starting from

$$\mathbb{E}[g(X_n) | X_0 = x] = \sum_{y \in E} g_y \mathbb{P}(X_n = y | X_0 = x).$$

2. EXAMPLES

1. Random Walk in $\mathbb{Z}^d$

Let $\{Y_n; n \in \mathbb{N}^*\}$ denote an i.i.d. $\mathbb{Z}^d$-valued random sequence, with the common law λ, and let X_0 be a $\mathbb{Z}^d$-valued random variable, independent of the Y_n. Then the random sequence $\{X_n; n \geq 0\}$ defined by

$$X_{n+1} = X_n + Y_{n+1}, \quad n \in \mathbb{N},$$

is a (μ, P) Markov chain, with μ the law of X_0, and $P_{xy} = \lambda_{y-x}$. The classical case is that of the symmetric random walk starting from 0, that is,

$$\mu = \delta_0, \quad \lambda_{\pm e_i} = \frac{1}{2d},$$

where $(e_1, \ldots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$.

2. Galton-Watson Process

This is a branching process $\{Z_n; n \in \mathbb{N}\}$ where Z_n denotes the number of males in the nth generation with a certain name, those individuals being all descendants of a common ancestor, the unique male in generation 0 ($Z_0 = 1$ almost surely). We assume that the ith male from the nth generation has ξ_i^n male children ($1 \leq i \leq Z_n$), in such a way that

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^n.$$

Our main assumption is that the random variables $\{\xi_i^n;\, i = 1, 2, \ldots, n = 0, 1, 2, \ldots\}$ are i.i.d., so that in particular Z_n and $\{\xi_1^n, \ldots, \xi_p^n, \ldots\}$ are independent.

The random sequence $\{Z_n;\, n \in \mathbb{N}\}$ is a (μ, P) $\mathbb{N}$-valued Markov chain, with $\mu = \delta_1$ and

$$P_{xy} = (p^{*x})_y,$$

where p^{*x} denotes the xth convolution power of the joint law p on $\mathbb{N}$ of the ξ_n^k, that is, the law of the sum of x i.i.d. random variables, all having the law p.

3. Discrete time queue

We consider a queue at a counter. X_n denotes the number of customers who are either waiting or being served at time n. Between time n and time $n+1$, Y_{n+1} new customers join the queue, and whenever $X_n > 0$, Z_{n+1} customers leave the queue (with $Z_{n+1} = 0$ or 1). We assume that $X_0, Y_1, Z_1, Y_2, Z_2 \ldots$ are mutually independent, with $0 < \mathbb{P}(Y_n = 0) < 1$, and moreover $\mathbb{P}(Z_n = 1) = p = 1 - \mathbb{P}(Z_n = 0)$. We have

$$X_{n+1} = X_n + Y_{n+1} - \mathbf{1}_{\{X_n > 0\}} Z_{n+1}.$$

3. STOPPING TIMES AND STRONG MARKOV PROPERTY

We wish to extend the Markov property, replacing the fixed time *n* by a random time.

Let us first reformulate the Markov property. Let $\{X_n; n \in \mathbb{N}\}$ be an E-valued Markov chain defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given a probability measure μ on E, we shall use the notation $\mathbb{P}_\mu$ to denote any probability on $(\Omega, \mathcal{F})$ such that under $\mathbb{P}_\mu$ the sequence $\{X_n; n \geq 0\}$ is a Markov chain with initial law μ; in other words, μ is the law of X_0, that is,

$$\mathbb{P}_\mu(X_0 = x) = \mu_x, \; x \in E.$$

Whenever $\mu = \delta_x$, we shall write $\mathbb{P}_x$ instead of $\mathbb{P}_{\delta_x}$. $\mathbb{P}_x$ can be interpreted as the conditional law of X, given that $X_0 = x$. For any $n \geq 0$, we define $\mathcal{F}_n$ to be the sigma-algebra of those events which are 'determined by $X_0, X_1, \ldots, X_n$', that is,

$$\mathcal{F}_n = \left\{\{\omega; (X_0(\omega), \ldots, X_n(\omega)) \in B_n\}; \; B_n \in \mathcal{P}(E^{n+1})\right\},$$

where $\mathcal{P}(F)$ denotes the collection of all the subsets of F.

<u>Theorem:</u> *Let $\{X_n; n \geq 0\}$ be a (μ, P) Markov chain. Then for any $n \in \mathbb{N}$, $x \in E$, conditionally upon $\{X_n = x\}$, $\{X_{n+p}; p \geq 0\}$ is a (δ_x, P) Markov chain, which is independent of $(X_0, \ldots, X_n)$. In other words, for all $A \in \mathcal{F}_n$ and any $m > 0$, $x_1, \ldots, x_m \in E$,*

$$\begin{aligned}&\mathbb{P}\left(A \cap \{X_{n+1} = x_1, \ldots, X_{n+m} = x_m\} | X_n = x\right)\\&\quad = \mathbb{P}(A|X_n = x)\mathbb{P}_x(X_1 = x_1, \ldots, X_m = x_m).\end{aligned}$$

<u>Proof:</u> It suffices to prove the result in the case where $A = \{X_0 = y_0, X_1 = y_1, \ldots, X_n = y_n\}$ (A is a finite or countable union of disjoint sets of that form, and the result in the general case will then follow from the σ-additivity of $\mathbb{P}$). It suffices to consider the case $y_n = x$, since otherwise both sides of the equality vanish. The left-hand side of the identity in the statement equals

$$\frac{\mathbb{P}(X_0 = y_0, \ldots, X_n = x, X_{n+1} = x_1, \ldots, X_{n+m} = x_m)}{\mathbb{P}(X_n = x)},$$

Which, applying the proposition in §1, equals:

$$\frac{\mathbb{P}(A)}{\mathbb{P}(X_n = x)} \times P_{xx_1} \times P_{x_1x_2} \times \cdots \times P_{x_{m-1}x_m},$$

or, in other words,

$$\mathbb{P}(A|X_n = x)\mathbb{P}_x(X_1 = x_1, \ldots, X_m = x_m).$$

<u>Definition:</u> *A random variable T taking values in the set $\mathbb{N} \cup \{+\infty\}$ is called a stopping time if, for all $n \in \mathbb{N}$,*

$$\{T = n\} \in \mathcal{F}_n.$$

In other words, the observation of $X_0, X_1, \ldots, X_n$, the trajectory of the chain up to time n, is enough to decide whether or not T equals n.

Example: *(i)For all $x \in E$, the first passage time at state x,*

$$S_x = \begin{cases} \inf\{n \geq 0;\ X_n = x\} & \textit{if such an } n \textit{ exists,} \\ +\infty, & \textit{otherwise,} \end{cases}$$

and the time of the first return to state x,

$$T_x = \begin{cases} \inf\{n \geq 1;\ X_n = x\} & \textit{if such an } n \textit{ exists,} \\ +\infty, & \textit{otherwise,} \end{cases}$$

are stopping times. (With the convention that the infimum of the empty set is $+\infty$, it is sufficient to write: $T_x = \inf\{n \geq 1;\ X_n = x\}$.) In the case of T_x this is because

$$\{T_x = n\} = \{X_1 \neq x\} \cap \ldots \cap \{X_{n-1} \neq x\} \cap \{X_n = x\}.$$

(ii) For all $A \subset E$, the time of the first visit to the set A,

$$T_A = \inf\{n \geq 1;\ X_n \in A\},$$

is a stopping time.

(iii) *On the other hand, the time of the last visit to* A,

$$L_A = \sup\{n \geq 1; X_n \in A\},$$

is not a stopping time, since we need to know the trajectory after time n *in order to decide whether or not* $L_A = n$.

We shall denote by $\mathcal{F}_T$ the σ-algebra of events which are 'determined by $X_0, X_1, \dots, X_T$', which is defined as the σ-algebra of those events $B \in \mathcal{F}$ which are such that for all $n \in \mathbb{N}$,

$$B \cap \{T = n\} \in \mathcal{F}_n.$$

Theorem: (Strong Markov property) *Let* $\{X_n; n \geq 0\}$ *be a* (μ, P) *Markov chain, and* T *a stopping time. Conditionally upon* $\{T < \infty\} \cap \{X_T = x\}$, $\{X_{T+n}; n \geq 0\}$ *is a* (δ_x, P) *Markov chain, which is independent of* $\mathcal{F}_T$. *In other words, for all* $A \in \mathcal{F}_T$ *and all* $m > 0$, $x_1, \dots, x_m \in E$,

$$\mathbb{P}(A \cap \{X_{T+1} = x_1, \dots, X_{T+m} = x_m\} | X_T = x, T < \infty)$$
$$= \mathbb{P}(A | X_T = x,\ T < \infty) \times \mathbb{P}_x(X_1 = x_1, \dots, X_m = x_m).$$

Proof: It suffices to show that for all integer *n*,

$$\mathbb{P}(A \cap \{T = n\} \cap \{X_{T+1} = x_1, \ldots, X_{T+m} = x_m\} | X_T = x)$$
$$= \mathbb{P}(A \cap \{T = n\} | X_T = x) \mathbb{P}_x(X_1 = x_1, \ldots, X_m = x_m),$$

which follows from Theorem 3.1, and then to sum over all possible values of n.

In the sequel we will mainly focuse on the long terme behavior of homogeneous Markov chains.

4. RECURRENT AND TRANSIENT STATES

Define $T_x = \inf\{n \geq 1; X_n = x\}$ as in Example 3.3.

Definition: *$x \in E$ is said to be recurrent if $\mathbb{P}_x(T_x < \infty) = 1$, and transient otherwise (i.e. if $\mathbb{P}_x(T_x < \infty) < 1$).*

We define the number of returns to the state x:

$$N_x = \sum_{n \geq 1} \mathbf{1}_{\{X_n = x\}}.$$

Proposition: *(a) if x is recurrent, then*

$$\mathbb{P}_x(N_x = +\infty) = 1.$$

(b) If x is transient, then

$$\mathbb{P}_x(N_x = k) = (1 - \Pi_x)\Pi_x^k, \; k \geq 0,$$

where $\Pi_x = \mathbb{P}_x(T_x < \infty)$ *(in particular,* $N_x < \infty$*,* $\mathbb{P}_x$ *almost surely).*

Proof: Let

$$\begin{aligned} T_x^2 &= \inf\{n > T_x; X_n = x\} \\ &= T_x + \inf\{n \geq 1; X_{T_x+n} = x\}. \end{aligned}$$

It is not hard to show that T_x^2 is a stopping time:

$$\begin{aligned} \mathbb{P}_x(T_x^2 < \infty) &= \mathbb{P}_x(T_x^2 < \infty | T_x < \infty)\mathbb{P}_x(T_x < \infty) \\ &= \sum_{n=1}^{\infty} \mathbb{P}_x(T_x^2 = T_x + n | T_x < \infty)\mathbb{P}_x(T_x < \infty). \end{aligned}$$

From a theorem of §3 we deduce that:

$$
\begin{aligned}
\mathbb{P}_x(T_x^2 &= T_x + n | T_x < \infty) \\
&= \mathbb{P}_x(X_{T_x+1} \neq x, \ldots, X_{T_x+n-1} \neq x, X_{T_x+n} = x | T_x < \infty) \\
&= \mathbb{P}_x(X_1 \neq x, \ldots, X_{n-1} \neq x, X_n = x) \\
&= \mathbb{P}_x(T_x = n).
\end{aligned}
$$

Finally,

$$\mathbb{P}_x(T_x^2 < \infty) = (\mathbb{P}_x(T_x < \infty))^2$$

or

$$\mathbb{P}_x(N_x \geq 2) = (\mathbb{P}_x(T_x < \infty))^2$$

and, iterating the same argument, we deduce that

$$\mathbb{P}_x(N_x \geq k) = (\mathbb{P}_x(T_x < \infty))^k, \quad k \in \mathbb{N}.$$

Both statements of the proposition follow easily from this identity.

Corollary: *x is recurrent if and only if*

$$\sum_{n=0}^{\infty} (P^n)_{xx} = +\infty.$$

Proof:

$$\mathbb{E}_x(N_x) = \sum_{n \geq 1} \mathbb{P}_x(X_n = x)$$

$$= \sum_{n \geq 1} (P^n)_{xx}.$$

It follows from the last proposition that this quantity is infinite whenever is recurrent. On the other hand, if x is transient, then

$$\mathbb{E}_x(N_x) = \sum_{k=1}^{\infty} k(1 - \Pi_x)\Pi_x^k$$

$$= \frac{\Pi_x}{1 - \Pi_x} < \infty.$$

Definition: *we say that the state x is accessible from x* (denoted by $x \to y$) *whenever there exists $n \geq 0$ such that $\mathbb{P}_x(X_n = y) > 0$. We say that x and y communicate (written $\leftrightarrow$) whenever both $x \to y$ and $y \to x$.*

The relation $x \leftrightarrow y$ is an equivalence relation, and we can partition E into equivalence classes modulo the relation $\leftrightarrow$.

Note that $x \to y \Leftrightarrow \exists\, n \geq 0$ such that $(P^n)_{xy} > 0$, since $\mathbb{P}_x(X_n = y) = (P^n)_{xy}$

Theorem: *Let $C \subset E$ be an equivalence class for the relation $\leftrightarrow$. Then all states in C either are recurrent, or else they all are transient.*

Proof: Let $x, y \in C$. It suffices to show that x transient $\Rightarrow$ y transient (since then y recurrent $\Rightarrow$ x recurrent). Since $x \leftrightarrow y$, there exist $n, m > 0$ such that $(P^n)_{xy} > 0$ and $(P^m)_{yx} > 0$. But for all $r \geq 0$,

$$(P^{n+r+m})_{xx} \geq (P^n)_{xy}(P^r)_{yy}(P^m)_{yx})$$

and

$$\sum_{r=0}^{\infty}(P^r)_{yy} \leq \frac{1}{(P^n)_{xy}(P^m)_{yx}}\sum_{n=0}^{\infty}(P^{n+r+m})_{xx} < \infty.$$

Definition: *A (μ, P) Markov chain is said to be irreducible whenever E consists of a single equivalence class. It is said to be irreducible and recurrent if it is irreducible and all states are recurrent.*

Proposition: *Any irreducible Markov chain on a finite state space E is recurrent.*

Proof Whenever E is finite, at least one state must be visited infinitely many times with positive probability, hence almost surely by Proposition 4.2, and that state (as well as all states) is (are) recurrent. □

5. THE IRREDUCIBLE AND RECURRENT CASE

In this section, we assume that the chain is both irreducible and recurrent. We start by studying the *excursions* of the chain between two successive returns to state x:

$$\mathcal{E}_k = (X_{T_x^k}, X_{T_x^k+1}, \dots, X_{T_x^{k+1}}), \quad k \geq 0.$$

These excursions are random sequences whose length is random, at least 2 and finite, composed of elements of $E \backslash \{x\}$, except for the first and the last, which are equal to x. Denote by U the set of sequences

$$u = (x, x_1, \dots, x_n, x),$$

with $n \geq 0$, $x_\ell \neq x$, $1 \leq \ell \leq n$. U is countable, and it is the set of all possible excursions $\mathcal{E}_0, \mathcal{E}_1, \dots$. Hence these random variables take their values in a countable set, and their probability law is characterized by the quantities

$$\mathbb{P}(\mathcal{E}_k = u), \quad u \in U.$$

Proposition: *Under $\mathbb{P}_x$, the sequence $(\mathcal{E}_0, \mathcal{E}_1, \dots)$ of excursions is i.i.d.; in other words, there exists a probability $\{p_u;\ u \in U\}$ on U such that, for all $k > 0$, $u_0, \dots, u_k \in U$,*

$$\mathbb{P}_x(\mathcal{E}_0 = u_0, \mathcal{E}_1 = u_1, \dots, \mathcal{E}_k = u_k) = \prod_{\ell=0}^{k} p_{u_\ell}.$$

<u>Proof:</u> This is a consequence of the strong Markov property. Indeed, $\{\mathcal{E}_0 = u_0\} \in \mathcal{F}_{T_x}$, and the event

$$\{\mathcal{E}_1 = u_1, \dots, \mathcal{E}_k = u_k\}$$

is of the form

$$\{X_{T_x+1} = x_1, \dots, X_{T_x+p} = x_p\},$$

for some $p > 0$, $x_1, \dots, x_p \in E$. Consequently,

$$\begin{aligned}
&\mathbb{P}_x(\mathcal{E}_0 = u_0, \mathcal{E}_1 = u_1, \dots, \mathcal{E}_k = u_k) \\
&= \mathbb{P}_x(\{\mathcal{E}_0 = u_0\} \cap \{X_{T_x+1} = x_1, \dots, X_{T_x+p} = x_p\} | T_x < \infty) \\
&= \mathbb{P}_x(\mathcal{E}_0 = u_0)\mathbb{P}_x(X_1 = x_1, \dots, X_p = x_p) \\
&= \mathbb{P}_x(\mathcal{E}_0 = u_0)\mathbb{P}_x(\mathcal{E}_0 = u_1, \dots, \mathcal{E}_{k-1} = u_k) \\
&= \mathbb{P}_x(\mathcal{E}_0 = u_0)\mathbb{P}_x(\mathcal{E}_0 = u_1) \times \dots \times \mathbb{P}_x(\mathcal{E}_0 = u_k) \\
&= p_{u_0} p_{u_1} \cdots p_{u_k},
\end{aligned}$$

where $\{p_u; u \in U\}$ is the law of $\mathcal{E}_0$ under $\mathbb{P}_x$.

A measure on the set E is a 'row vector' $\{\gamma_x;\ x \in E\}$ such that $0 \le \gamma_x < \infty$, for all x. Whenever the measure is finite, $\sum_{x\in E} \gamma_x < \infty$, we can normalize it, to make it a probability on E, $\left(\gamma_x / \sum_z \gamma_z,\ x \in E\right)$. A measure γ is said to be invariant (with respect to the transition matrix P) whenever

$$\gamma P = \gamma,$$

that is,

$$\sum_{y \in E} \gamma_y P_{yx} = \gamma_x,\ x \in E.$$

A measure γ is said to be strictly positive if $\gamma_x > 0$, for all $x \in E$.

A probability measure γ is invariant if and only if the chain (γ, P) has the property that γ is the law of X_n, for all $n \in \mathbb{N}$, hence for all n, $\{X_{n+m};\ m \in \mathbb{N}\}$ is a (γ, P) Markov chain.

<u>Definition:</u> *An invariant probability is a probability π which satisfies $\pi P = \pi$, or equivalently, for all $x \in E$,*

$$\sum_{y \neq x} \pi_y P_{yx} = \pi_x (1 - P_{xx}),$$

that is,

$$\mathbb{P}(X_n \neq x, X_{n+1} = x) = \mathbb{P}(X_n = x, X_{n+1} \neq x),$$

This means that at equilibrium, the mean number of departures from state *x* between time *n* and time *n+1* equals the mean number of arrivals at state *x* between *n* and *n+1*.

Theorem: *Let $\{X_n; n \in \mathbb{N}\}$ be a Markov chain with transition matrix P, which we assume to be irreducible and recurrent. Then there exists a strictly positive invariant measure γ, which is unique up to a multiplicative constant.*

Proof: To prove existence, let γ_y^x denote the mean number of visits to state y during the excursion $\mathcal{E}_0$ starting at x, that is,

$$\begin{aligned}
\gamma_y^x &= \mathbb{E}_x \sum_{n=1}^{T_x} \mathbf{1}_{\{X_n=y\}} \\
&= \sum_{n=1}^{\infty} \mathbb{P}_x(X_n = y, n \le T_x) \\
&= \sum_{z\in E} \sum_{n=1}^{\infty} \mathbb{P}_x(\{X_{n-1} = z, n-1 < T_x\} \cap \{X_n = y\}) \\
&= \sum_{z\in E} \left(\sum_{n=2}^{\infty} \mathbb{P}_x(X_{n-1} = z, n-1 \le T_x) \right) P_{zy} \\
&= (\gamma^x P)_y.
\end{aligned}$$

Note that we have used recurrence to obtain the penultimate equality. We now exploit the irreducibility of the chain. There exist n, m such that $(P^n)_{xy} > 0$, $(P^m)_{yx} > 0$. Hence, since $\gamma_x^x = 1$,

$$0 < (P^n)_{xy} = \gamma_x^x (P^n)_{xy} \le (\gamma^x P^n)_y = \gamma_y^x,$$

$$\gamma_y^x (P^m)_{yx} \le (\gamma^x P^m)_x = \gamma_x^x = 1.$$

Consequently, γ^x is a strictly positive measure, which satisfies $\gamma_x^x = 1$.

Turning to uniqueness, let λ denote an invariant measure such that $\lambda_x = 1$. We shall first prove that $\lambda \ge \gamma^x$, then that $\lambda = \gamma^x$. Note that this part of the proof of the theorem exploits only irreducibility (and not recurrence). We have

$$\begin{aligned}
\lambda_y &= P_{xy} + \sum_{z_1 \ne x} \lambda_{z_1} P_{z_1 y} \\
&= P_{xy} + \sum_{z_1 \ne x} P_{x z_1} P_{z_1 y} + \sum_{z_1, z_2 \ne x} \lambda_{z_2} P_{z_2 z_1} P_{z_1 y} \\
&\ge \sum_{n=0}^{\infty} \sum_{z_1, \dots, z_n \ne x} P_{x z_n} P_{z_n z_{n-1}} \cdots P_{z_1 y} \\
&= \sum_{n=0}^{\infty} \mathbb{P}_x(X_{n+1} = y, T_x \ge n+1) \\
&= \gamma_y^x.
\end{aligned}$$

Hence $\mu = \lambda - \gamma^x$ is also an invariant measure, and $\mu_x = 0$. Let $y \in E$, and n be such that $(P^n)_{yx} > 0$. Then

$$0 = \mu_x = \sum_{z \in E} \mu_z (P^n)_{zx} \geq \mu_y (P^n)_{yx}.$$

Hence $\mu_y = 0$, and this holds for all $y \in E$. $\square$

We have seen that a state x is recurrent whenever

$$\mathbb{P}_x(T_x < \infty) = 1.$$

Let $m_x = \mathbb{E}_x(T_x)$. If this quantity is finite, then x is called *positive recurrent*, and otherwise it is called *null recurrent*.

Theorem: *Assume again that the chain is irreducible. A state x is positive recurrent if and only if all the states are positive recurrent, if and only if there exists an invariant probability $\pi = (\pi_x = m_x^{-1},\ x \in E)$.*

Proof: Note that

$$m_x = \sum_{y \in E} \gamma_y^x .$$

Hence if x is positive recurrent, then the probability $\pi = (\pi_y = \gamma_y^x / m_x,\ y \in E)$ is invariant.

Conversely, if π is an invariant probability, from the irreducibility (see the end of the proof of existence in Theorem 5.3), π is strictly positive, hence if x is an arbitrary state, $\lambda = \left(\lambda_y = \pi_y/\pi_x,\ y \in E\right)$ is an invariant measure which satisfies $\lambda_x = 1$. From the irreducibility and the proof of uniqueness in Theorem 5.3,

$$m_x = \sum_{y \in E} \gamma_y^x = \sum_{y \in E} \frac{\pi_y}{\pi_x} = \frac{1}{\pi_x} < \infty.$$

Hence x is positive recurrent, as are all the states. □

The following dichotomy follows from the two preceding theorems: in the *irreducible and recurrent* case, the chain is *positive recurrent* whenever there exists an *invariant probability*, *null recurrent* if *one (hence all) invariant measure(s)* has *infinite* total mass ($\sum_i \pi_i = +\infty$). In particular, if $|E| < \infty$, there do not exist null recurrent states, rather, any recurrent state is positive recurrent.

Corollary: *Let $\{X_n\}$ be an irreducible Markov chain which is positive recurrent. With any $x \in E$ we associate $T_x = \inf\{n > 0;\ X_n = x\}$. Then for all $y \in E$,*

$$\mathbb{E}_y(T_x) < \infty.$$

Proof: Note that

$$T_x \geq T_x \mathbf{1}_{\{T_y < T_x\}},$$

whence, taking the expectation under $\mathbb{P}_x$,

$$m_x \geq \mathbb{E}_x(T_x | T_y < T_x)\mathbb{P}_x(T_y < T_x).$$

But it follows from the strong Markov property that $\mathbb{E}_x(T_x | T_y < T_x) > \mathbb{E}_y(T_x)$, and from the irreducibility that $\mathbb{P}_x(T_y < T_x) > 0$, and the proof is complete. □

Remark: (The non irreductible case) *For simplicity we consider here only the case $|E| < \infty$. There exists at least one recurrent class (which is positive recurrent), hence there exists a least one invariant probability. Any invariant probability charges only recurrent states. If there is only one recurrent class, then the chain possesses one and only one invariant probability. Otherwise, we can associate with each recurrent class a unique invariant probability whose support is that class, and all invariant measures are convex linear combinations of these, which are the extremal ones. Hence, if there are at least two different recurrent classes, there are an uncountable number of invariant probabilities.*

We restrct ourself again to the irreducible case. We can now establish the ergodic theorem which is a generalization of the law of large numbers.

Theorem: *Suppose that the chain is irreducible and positive recurrent. Let $\pi = (\pi_x,\ x \in E)$ denote its unique invariant probability. If $f : E \to \mathbb{R}$ is bounded, then $\mathbb{P}$ almost surely, as $n \to \infty$,*

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) \to \sum_{x\in E} \pi_x f(x).$$

Proof: By assumption, there exists c such that $|f(x)| \le c$, for all $x \in E$.

Let

$$N_x(n) = \sum_{1\le k\le n} \mathbf{1}_{\{X_k=x\}}$$

denote the number of returns to state x before time n. We wish to study the limit as $n \to \infty$ of

$$\frac{N_x(n)}{n}.$$

Let $S_x^0, S_x^1, \ldots, S_x^k, \ldots$ denote the lengths of the excursions $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_k, \ldots$ starting at x. We have

$$S_x^0 + \ldots + S_x^{N_x(n)-1} \leq n < S_x^0 + \ldots + S_x^{N_x(n)}.$$

Hence

$$\frac{S_x^0 + \ldots + S_x^{N_x(n)-1}}{N_x(n)} \leq \frac{n}{N_x(n)} \leq \frac{S_x^0 + \ldots + S_x^{N_x(n)}}{N_x(n)}.$$

But since the random variables $\mathcal{E}_k$ are i.i.d. (hence the same is true for the S_x^k), as $n \to \infty$,

$$\frac{S_x^0 + \ldots + S_x^{N_x(n)}}{N_x(n)} \to \mathbb{E}_x(T_x) = m_x \;\; \mathbb{P}_x \text{ a.s.},$$

since $N_x(n) \to +\infty \;\; \mathbb{P}_x$ almost surely. Again from the law of large numbers,

$$\frac{n}{N_x(n)} \to m_x \;\; \mathbb{P}_x \text{ a.s.},$$

that is

$$\frac{N_x(n)}{n} \to \frac{1}{m_x} \ \mathbb{P}_x \text{ a.s.}$$

This convergence is also true $\mathbb{P}_\mu$ almost surely, for any initial law μ , since the limit of $N_x(n)/n$ is the same for the chains $\{X_n; n \geq 0\}$ and $\{X_{T_x+n}; n \geq 0\}$.

$$\begin{aligned}
\left| \frac{1}{n} \sum_{k=1}^{n} f(X_k) - \bar{f} \right| &= \left| \sum_{x \in E} \left(\frac{N_x(n)}{n} - \pi_x \right) f(x) \right| \\
&\leq c \sum_{x \in F} \left| \frac{N_x(n)}{n} - \pi_x \right| + c \sum_{x \notin F} \left(\frac{N_x(n)}{n} + \pi_x \right) \\
&= c \sum_{x \in F} \left| \frac{N_x(n)}{n} - \pi_x \right| + c \sum_{x \in F} \left(\pi_x - \frac{N_x(n)}{n} \right) + 2c \sum_{x \notin F} \pi_x \\
&\leq 2c \sum_{x \in F} \left| \frac{N_x(n)}{n} - \pi_x \right| + 2c \sum_{x \notin F} \pi_x .
\end{aligned}$$

We choose a finite F such that $\sum_{x \notin F} \pi_x \leq \varepsilon/4c$, and then $N(\omega)$ such that, for all $n \geq N(\omega)$,

$$\sum_{x \in F} \left| \frac{N_x(n)}{n} - \pi_x \right| \leq \frac{\varepsilon}{4c},$$

which proves the result. □

6. THE APERIODIC CASE

We have just shown that in the irreducible, positive recurrent case,

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{X_k = y\}} \to \pi_y \text{ a.s.,}$$

as $n \to \infty$. Taking the expectation under $\mathbb{P}_x$, we deduce that

$$\frac{1}{n} \sum_{k=1}^{n} (P^k)_{xy} \to \pi_y, \quad \forall x, y \in E.$$

We see that the Cesaro means of the $(P^k)_{xy}$ converge. This raises the natural question whether it is true under the above assumptions that, as $n \to \infty$,

$$(P^n)_{xy} \to \pi_y, \quad \forall x, y \in E.$$

It is easily seen this is not the case.

Consider a random walk on $E = \mathbb{Z}/N$, where N is an even integer (we identify 0 and N),

$$X_n = X_0 + Y_1 + \ldots + Y_n,$$

with the Y_n i.i.d. with values in $\{-1, 1\}$; in other words,

$$X_n = (X_0 + Y_1 + \ldots + Y_n) \bmod N.$$

This chain is irreducible, and positive recurrent since E is finite. But $(P^{2k+1})_{xx} = 0$, for all $x \in E$. In the particular case $N = 2$, we have $P^{2k} = I$ and $P^{2k+1} = P$.

<u>In order for the desired convergence to be true an additional assumption is needed</u>:

<u>Definition:</u> *A state $x \in E$ is said to be aperiodic if there exists N such that*

$$(P^n)_{xx} > 0, \quad \forall n \geq N.$$

<u>Lemma:</u> *If P is irreducible and there exists an aperiodic state x, then for all $y, z \in E$, there exists M such that $(P^n)_{yz} > 0$, for all $n \geq M$. In particular, all states are aperiodic.*

Proof: From the irreducibility, there exist $r, s \in \mathbb{N}$ such that $(P^r)_{yx} > 0$, $(P^s)_{xz} > 0$. Moreover,

$$\left(P^{r+n+s}\right)_{yz} \geq (P^r)_{yx}(P^n)_{xx}(P^s)_{xz} > 0$$

if $n \geq N$. Hence we have the desired property with $M = N + r + s$.

Remark: *Suppose we are in the irreducible, positive recurrent case. Let π be the invariant probability, so that $\pi_y > 0$, for all $y \in E$. Hence the fact that there exists N such that, for all $n \geq N$, $(P^n)_{xy} > 0$ is a necessary condition for the convergence $(P^n)_{xy} \to \pi_y$ to hold. We shall now see that it is a sufficient condition.*

Theorem: *Suppose that P is irreducible, positive recurrent and aperiodic. Let π denote the unique invariant probability. If $\{X_n; n \in \mathbb{N}\}$ is a (μ, P) Markov chain, for all $y \in E$,*

$$\mathbb{P}(X_n = y) \to \pi_y, \quad n \to \infty;$$

in other words,

$$(\mu P^n)_y \to \pi_y,$$

for any initial law μ. In particular, for all $x, y \in E$,

$$(P^n)_{xy} \to \pi_y.$$

Proof: We shall use a coupling argument. Let $\{Y_n; n \in \mathbb{N}\}$ be a (π, P) Markov chain, independent of $\{X_n; n \in \mathbb{N}\}$, and $x \in E$ be arbitrary. Let

$$T = \inf\{n \geq 0;\ \ X_n = Y_n = x\}.$$

Step 1. We show that $\mathbb{P}(T < \infty) = 1$. $\{W_n = (X_n, Y_n); n \in \mathbb{N}\}$ is an $(E \times E)$-valued Markov chain, with initial law λ (where $\lambda_{(x,u)} = \mu_x \pi_u$) and transition matrix $\tilde{P}_{(x,u)(y,v)} = P_{xy} P_{uv}$. Since P is aperiodic, for all x, u, y, v, for all n large enough,

$$(\tilde{P}^n)_{(x,u)(y,v)} = (P^n)_{xy}(P^n)_{uv} > 0.$$

Step 2. Define

$$Z_n = \begin{cases} X_n, & n \leq T; \\ Y_n, & n > T. \end{cases}$$

By the strong Markov property, both processes $\{X_{T+n}; n \geq 0\}$ and $\{Y_{T+n}; n \geq 0\}$ are (δ_x, P) Markov chains, independent of $(X_0, \ldots, X_T)$. Consequently, $\{Z_n; n \in \mathbb{N}\}$ is, like $\{X_n\}$, a (μ, P) Markov chain.

Step 3. We now conclude. We have the three identities

$$\mathbb{P}(Z_n = y) = \mathbb{P}(X_n = y),$$

$$\mathbb{P}(Y_n = y) = \pi_y,$$

$$\mathbb{P}(Z_n = y) = \mathbb{P}(X_n = y, n \leq T) + \mathbb{P}(Y_n = y, n > T).$$

Hence

$$|\mathbb{P}(X_n = y) - \pi_y| = |\mathbb{P}(Z_n = y) - \mathbb{P}(Y_n = y)| \leq \mathbb{P}(n < T) \to 0,$$

as $n \to \infty$.

We now make precise the rate of convergence in the preceding theorem, under an additional assumption, called Doeblin's condition : there exist $n_0 \in \mathbb{N}$, $\beta > 0$ and a probability ν on E such that

$$(D) \quad (P^{n_0})_{xy} \geq \beta \nu_y, \ \forall x, y \in E.$$

Remark: *Condition (D) is equivalent to the condition*

$$\exists x \in E, n_0 \geq 1 \text{ such that } \inf_{y \in E} (P^{n_0})_{yx} > 0.$$

This implies that this state x is aperiodic. But it does not imply irreducibility. In fact, this condition implies existence of a unique recurrence class and of unique invariant probability.

Lemma: *If P is irreducible and aperiodic, and E is finite, then condition (D) Is satisfied.*

Proof: Choose $x \in E$. For all $y \in E$, there exists n_y such that $n \geq n_y \Rightarrow (P^n)_{yx} > 0$. Let

$$\bar{n} = \sup_{y \in E} n_y, \quad \alpha = \inf_y (P^{\bar{n}})_{yx}.$$

Then $\alpha > 0$, and for all $y \in E$,

$$(P^{\bar{n}})_{yx} \geq \alpha.$$

Hence condition (D) is satisfied with $n_0 = \bar{n}$, $\beta = \alpha$, $\nu = \delta_x$.

We state now without proof the important theorem:

Theorem: *Suppose that P is irreducible and satisfies Doeblin's condition (D). Then P is aperiodic, positive recurrent, and if π denotes its invariant probability,*

$$\sum_{y \in E} |(P^n)_{xy} - \pi_y| \leq 2\,(1-\beta)^{[n/n_0]}, \quad \forall x \in E, \; n \in \mathbb{N},$$

where $[n/n_0]$ stands for the integer part of n/n_0.

We now state a central limit theorem for irreducible, positive recurrent and aperiodic Markov chains. Such a chain, if it also satisfies

$$\sum_{y \in E} |(P^n)_{xy} - \pi_y| \le M t^n, \quad x \in E, n \in \mathbb{N}$$

with $M \in \mathbb{R}$ and $0 < t < 1$, is said to be *uniformly ergodic*. We have just shown that Doeblin's condition implies uniform ergodicity. That property implies the central limit theorem.

Theorem: *Let $\{X_n; n \in \mathbb{N}\}$ be an E-valued Markov chain, with an irreducible transition matrix P, which is moreover uniformly ergodic and aperiodic. Let π denote the unique invariant probability of the chain, and $f : E \to \mathbb{R}$ be such that*

$$\sum_{x \in E} \pi_x f^2(x) < \infty \quad \text{and} \quad \sum_{x \in E} \pi_x f(x) = 0.$$

Then as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{1}^{n} f(X_k) \text{ converges in law to } \sigma_f Z,$$

where $Z \simeq N(0, 1)$ and

$$\sigma_f^2 = \sum_{x \in E} \pi_x (Qf)_x^2 - \sum_x \pi_x (PQf)_x^2$$
$$= 2 \sum_x \pi_x (Qf)_x f_x - \sum_x \pi_x f_x^2,$$

with

$$(Qf)_x = \sum_{n=0}^{\infty} \mathbb{E}_x[f(X_n)], \quad x \in E.$$

7. REVERSIBLE MARKOV CHAINS

Consider the irreducible, positive recurrent case. The Markov property – that the past and future are conditionally independent given the present – tells us that whenever $\{X_n; n \in \mathbb{N}\}$ is a Markov chain, it follows that, for all N, $\{\hat{X}_n^N = X_{N-n}; 0 \leq n \leq N\}$ is also a Markov chain. In general, the time-reversed chain is not homogeneous, unless $\{X_n\}$ is initialized with its invariant probability π.

Proposition: *Let $\{X_n; n \in \mathbb{N}\}$ be a (π, P) Markov chain whose transition matrix P is supposed to be irreducible, and π be its invariant probability. Then the time-reversed chain $\{\hat{X}_n^N; 0 \leq n \leq N\}$ is a $(\pi, \hat{P})$ Markov chain, with*

$$\pi_y \hat{P}_{yx} = \pi_x P_{xy}, \quad \forall\, x, y \in E.$$

Proof:

$$
\begin{aligned}
&\mathbb{P}(\hat{X}_{p+1} = x | \hat{X}_p = y) \\
&= \mathbb{P}(X_n = x | X_{n+1} = y) \\
&= \mathbb{P}(X_{n+1} = y | X_n = x) \times \frac{\mathbb{P}(X_n = x)}{\mathbb{P}(X_{n+1} = y)}.
\end{aligned}
$$

We say that the chain $\{X_n; n \in \mathbb{N}\}$ is *reversible* if $\hat{P} = P$, which holds if and only if the following *detailed balance equation* is satisfied:

$$\pi_x P_{xy} = \pi_y P_{yx}, \ \forall x, y \in E,$$

where π denotes the invariant probability. It is easily checked that whenever a probability π satisfies this relation, then it is P-invariant. The converse need not be true.

Remark: *If π is the invariant probability of an irreducible (and hence also positive recurrent) Markov chain, the chain need not be reversible. Suppose that card$(E) \geq 3$. Then there may exist $x \neq y$ such that $P_{xy} = 0 \neq P_{yx}$. Consequently, $\pi_x P_{xy} = 0 \neq \pi_y P_{yx}$. The transitions from y to x of the original chain correspond to the transitions from x to y of the time-reversed chain, hence $P_{yx} \neq 0 \Rightarrow \hat{P}_{xy} \neq 0$, whence $\hat{P} \neq P$.*

Remark: *Given the transition matrix P of an irreducible positive recurrent Markov chain, one might wish to compute its invariant probability. This problem is not always solvable.*

Another problem, which will appear in the next chapter, is to determine an irreducible transition matrix P whose associated Markov chain admits a given probability π as its invariant probability.

The second problem is rather easy to solve. In fact there are always many solutions. The simplest way to solve it is to look for P such that the associated chain is reversible with respect to π. In other words, it suffices to find an irreducible transition matrix P such that the quantity $\pi_x P_{xy}$ is symmetric in x, y.

In order to solve the first problem, one can try to find π such that

$$\pi_x P_{xy} = \pi_y P_{yx}, \quad \forall x,\ y \in E,$$

which, unlike solving $\pi P = \pi$, implies no summation with respect to x. But that equation has a solution only if the chain is reversible with respect to its unique invariant probability measure, which need not be the case.

Suppose now that we are given a pair (P, π), and that we wish to check whether or not π is the invariant probability of the chain with the irreducible transition matrix P. If the quantity $\pi_x P_{xy}$ is symmetric in x, y, then the answer is yes, and we have an additional property, namely the reversibility. If this is not the case, one needs to check whether or not $\pi P = \pi$. One way to carry out that verification is given by the next proposition, whose elementary proof is left to the reader.

Proposition: *Let P be an irreducible transition matrix, and π a strictly positive probability on E. For each pair $x, y \in E$, we define*

$$\hat{P}_{xy} = \begin{cases} \frac{\pi_y}{\pi_x} P_{yx}, & \text{if } x \neq y, \\ P_{xx}, & \text{if } x = y. \end{cases}$$

π is the invariant probability of the chain having the transition matrix P, and $\hat{P}$ is the transition matrix of the time-reversed chain if and only if, for all $x \in E$,

$$\sum_{y \in E} \hat{P}_{xy} = 1.$$

8. RATE OF CONVERGENCE TO EQUILIBRIUM

Suppose we are in the irreducible, positive recurrent and aperiodic case. We then know that for all $x, y \in E$, $(P^n)_{x,y} \to \pi_y$ as $n \to \infty$, where π denotes the unique invariant probability measure. More generally, we expect that for a large class of functions $f : E \to \mathbb{R}$, $(P^n f)_x \to \langle f, \pi \rangle$ as $n \to \infty$ for all $x \in E$, where, here and below,

$$\langle f, \pi \rangle = \sum_{x \in E} f(x)\pi_x.$$

In this section, we discuss the rate at which the above convergence holds.

The reversible finite case:

Let us first consider the simplest case, in which we assume that E is finite (we write $d = |E|$) and that the process is reversible. We first note that we can identify $L^2(\pi)$ with $\mathbb{R}^d$, equipped with the scalar product

$$\langle f, g\rangle_\pi = \sum_{x\in E} f(x)g(x)\pi_x.$$

Next the reversibility of P is equivalent to the fact that P, as an element of $\mathcal{L}(L^2(\pi))$, is a self-adjoint operator, in the sense that

$$\begin{aligned}\langle Pf, g\rangle_\pi &= \sum_{x,y\in E} P_{x,y} f(y)g(x)\pi_x \\ &= \sum_{x,y\in E} P_{y,x} f(y)g(x)\pi_y \\ &= \langle f, Pg\rangle_\pi,\end{aligned}$$

where we have used the detailed balance equation for the second identity. We now check that the operator norm of P, as an element of $\mathcal{L}(L^2(\pi))$, is at most 1. Indeed, if $\|\cdot\|_\pi$ denotes the usual norm in $L^2(\pi)$,

$$
\begin{aligned}
\|Pf\|_\pi^2 &= \sum_{x\in E} \left[(Pf)_x\right]^2 \pi_x \\
&= \sum_{x\in E} (\mathbb{E}[f(X_t)|X_0 = x])^2 \pi_x \\
&\le \mathbb{E}[f^2(X_t)|X_0 = x]\pi_x \\
&= \sum_{x\in E} f^2(x)\pi_x,
\end{aligned}
$$

where we have used Schwarz's (or equivalently Jensen's) inequality for the inequality, and the invariance of π for the last identity.

In order to be able to work in $\mathbb{R}^d$ equipped with the Euclidean norm, let us introduce the new $d \times d$ matrix

$$
\tilde{P}_{x,y} := \sqrt{\frac{\pi_x}{\pi_y}} P_{x,y}.
$$

In matrix notation, $\tilde{P} = \Pi^{1/2} P \Pi^{-1/2}$, where $\Pi_{x,y} = \delta_{x,y}\pi_x$ is a diagonal matrix. Moreover, if we denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$, for any $f : E \to \mathbb{R}$ (i.e. f is a collection of real numbers indexed by the d elements of the set E, in other words an element of $\mathbb{R}^d$), denoting $g = \Pi^{-1/2} f$, we have

$$\|\tilde{P} f\|^2 = \sum_{x \in E} (P \Pi^{-1/2} f)_x^2 = \|Pg\|_\pi^2 \le \|g\|_\pi^2 = \|f\|^2.$$

First, note that f is an eigenvector of $\tilde{P}$ if and only if $g = \Pi^{-1/2} f$ is a right eigenvector of P, and $g' = \Pi^{1/2} f$ is a left eigenvector of P associated with the same eigenvalue. We have that $\tilde{P}$ is a symmetric $d \times d$ matrix, whose norm is bounded by 1. Hence, from elementary results in linear algebra, $\tilde{P}$ admits the eigenvalues $-1 \le \lambda_d \le \lambda_{d-1} \le \lambda_2 \le \lambda_1 \le 1$.

We have then the following proposition:

Proposition:

$$\|P^n f - \langle f, \pi \rangle\|_\pi \le (1-\beta)^n \|f - \langle f, \pi \rangle\|_\pi,$$

where $\beta := (1 - \lambda_2) \wedge (1 + \lambda_d)$ *is the spectral gap.*

The general case:

More generally, the same is true with

$$\beta := 1 - \sup_{f \in L^2(\pi),\ \|f\|_\pi = 1} \|Pf - \langle f, \pi \rangle\|_\pi.$$

Indeed, with this β, considering only the case $f \neq 0$, since all inequalities below are clearly true for $f = 0$, we have

$$\begin{aligned}\|Pf - \langle f, \pi\rangle\|_\pi &= \left\| P\left(\frac{f}{\|f\|_\pi}\right) - \left\langle \frac{f}{\|f\|_\pi}, \pi\right\rangle \right\|_\pi \times \|f\|_\pi \\ &\leq (1-\beta)\|f\|_\pi.\end{aligned}$$

Finally, we check that Proposition 8.2 still holds in the general case, with β defined above. Note that

$$\begin{aligned}\|P^{n+1}f - \langle f, \pi\rangle\|_\pi &= \|P[P^n f - \langle f, \pi\rangle]\|_\pi \\ &\leq (1-\beta)\|P^n f - \langle f, \pi\rangle\|_\pi.\end{aligned}$$

The result follows by induction.

9. STATISTICS OF MARKOV CHAINS

The aim of this section is to introduce the basic notions for the estimation of the parameters of a Markov chain.

We have seen that, for all $n > 0$, the law of the random vector $(X_0, X_1, \ldots, X_n)$ depends only on the initial law μ and on the transition matrix P. We are interested in the conditions under which one can estimate the pair (μ, P), given the observation of $(X_0, X_1, \ldots, X_n)$, in such a way that the error tends to zero, as $n \to \infty$.

Let us first discuss the estimation of the invariant probability μ. For any $x \in E$,

$$\hat{\mu}_x^n = \frac{1}{n+1} \sum_{\ell=0}^{n} \mathbf{1}_{\{X_\ell = x\}}$$

is a consistent estimator of μ_x, since the following is an immediate consequence of the ergodic theorem:

Proposition: *For any $x \in E$, $\hat{\mu}_x^n \to \mu_x$ almost surely, as $n \to \infty$.*

Let us now discuss the estimation of the P_{xy}, $x, y \in E$. We choose the estimator

$$\hat{P}_{xy}^n = \frac{\sum_{\ell=0}^{n-1} \mathbf{1}_{\{X_\ell = x, X_{\ell+1} = y\}}}{\sum_{\ell=0}^{n-1} \mathbf{1}_{\{X_\ell = x\}}}.$$

We have the following proposition.

Proposition: *For any $x, y \in E$, $\hat{P}_{xy}^n \to P_{xy}$ almost surely as $n \to \infty$.*

Proof: We clearly have:

$$\hat{P}^n_{xy} = \left(\frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{1}_{\{X_\ell = x\}} \right)^{-1} \frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{1}_{\{X_\ell = x, X_{\ell+1} = y\}}.$$

We know that

$$\frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{1}_{\{X_\ell = x\}} \to \mu_x.$$

For $n \geq 0$, define $\widetilde{X}_n = (X_n, X_{n+1})$. It is not very hard to check that $\{\widetilde{X}_n;\ n \geq 0\}$ is an irreducible and positive recurrent $\widetilde{E} = \{(x, y) \in E \times E;\, P_{xy} > 0\}$-valued Markov chain, with transition matrix $\widetilde{P}_{(x,y)(u,v)} = \delta_{yu} P_{uv}$. It admits the invariant probability $\widetilde{\mu}_{(x,y)} = \mu_x P_{xy}$. The ergodic theorem applied to the chain $\{\widetilde{X}_n\}$ implies that almost surely, as $n \to \infty$,

$$\frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{1}_{\{X_\ell = x, X_{\ell+1} = y\}} \to \mu_x P_{xy}.$$

$\square$

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