Signal Processing.

1. X is a r.v. $X : \Omega \to \mathbb{R}$ with P_X as pdf.

$$\mathbb{E}(x) = \int x\, P(x)\, dx = \bar{X}$$

$\overline{X^n} = \mathbb{E}X^n \Rightarrow$ moment order n

Suppose we have 2 r.v.

$\overline{(X+Y)^2} = \overline{X^2} + \overline{Y^2} + 2\overline{XY} \rightarrow$ sum of expectation.

$\mathrm{Var}(x) = \overline{X^2} - \bar{X}^2 = \mathbb{E}(X-\bar{X})^2$

Suppose $X : \Omega \to \mathbb{R}^N$ i.e. $X = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$

$$E[X \cdot Y^T] = \begin{array}{c c} & m \\ n & \begin{bmatrix} & \vdots & \\ \cdots & X_n Y_m & \end{bmatrix} \end{array}$$

$\rightarrow$ correlation of X_n & Y_m

$\rightarrow$ correlation matrix.

$$\overline{(X+Y)^2} = (\overline{X^2} \cdot \bar{X}^2) + (\overline{Y^2} \cdot \bar{Y}^2) + 2\overline{XY} - 2\bar{X}\bar{Y} + \underbrace{\bar{X}^2 + \bar{Y}^2 + 2\bar{X}\bar{Y}}_{(\bar{X}+\bar{Y})^2}$$

Weak Law of Large Numbers

~~Empirical~~ Sample mean $Z_n = \frac{1}{n}(X_1 + \cdots + X_n)$

$\mathbb{E}Z_n = \frac{1}{n}\sum \mathbb{E}X_i = \frac{1}{n}(n\bar{x}) = \bar{x}$ X_n = independent & identically distributed (iid)

- They have same expectation $\bar{X} = \bar{X}_n$
- Same variance. $\sigma^2 = X_n^2$

Deviation of Z_n from Common mean $\bar{X}$

$$P\{|Z_n - \bar{X}| > \varepsilon\} = P\{|Z_n - \bar{X}|^2 > \varepsilon^2\}$$

where $\varepsilon > 0$

applying chebychev formula. the upper bounded by.

$$P\{|Z_n - \bar{X}|^2 > \varepsilon^2\} \leq \frac{1}{\varepsilon^2} \mathbb{E}[|Z_n - \bar{X}|^2].$$

var of sum is sum of var in iid.

$$\text{But } \mathbb{E}[Z_n - \bar{X}|^2 = Var(Z_n) = \frac{1}{n^2}(n.Var(X_i)) = \frac{n.\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

as $n \to \infty$ hence $\sigma \to 0$

Proof of chebychev. of Markov Inequality

let $X \geq 0$ (positive r.v.) with finite mean $\mathbb{E}X < \infty$

integration by parts. deviation of X

$$\mathbb{E}X = \int_0^\infty x P_x \, dx = \int_0^\infty G_X(x)\, dx \quad G_x(x) = P[X > x].$$

for $\geq \varepsilon G(\varepsilon)$

It is decreasing function.

2. Xtic function $\to$ F.T of pdf

$$\varphi_X(x) \triangleq \mathbb{E} e^{iuX} = \mathbb{E} \sum_n \frac{(iu)^n}{n!} X^n = \sum \frac{(iu)^2}{n!} \bar{X}_n$$

Check $\varphi_X(0) = 1$

$\varphi_x'(u) = i\bar{x}$ - & $\varphi_x''(0) = -\frac{\bar{X^2}}{2}$

$\varphi_x^n(0)$ → derivative at 0 = $\frac{i^n}{n!}\bar{X^n}$

* Suppose we have white noise (iid) Independent $X_1, X_2 \dots X_n$

Common Var → $\text{Var}\, X_n = \bar{X_n^2} = \sigma^2$

Consider
$$Z_n = \frac{1}{\sqrt{n}}(X_1 + \dots + X_n)$$

Central limit theorem (CLT) → sums like this forget individual distribution of collective term X_n converge to a r.v. with gaussian/normal density.

exponential of prod of prod of expo

$$\psi_{Z_n}(v) = \log \varphi_{Z_n}(v), \quad \varphi_{Z_n}(v) = \mathbb{E}\, e^{iu\frac{(X_1 + \dots X_n)}{\sqrt{n}}}$$
$$= \mathbb{E}\left[e^{\frac{iv X_1}{\sqrt{n}}} \times \dots e^{iv\frac{X_n}{\sqrt{n}}}\right].$$

hence
$$\underbrace{\mathbb{E}\left(e^{iv\frac{X_1}{\sqrt{n}}}\right)}_{\varphi_{X_1}'(v)} \cdots \underbrace{\mathbb{E}\left(e^{iv\frac{X_n}{\sqrt{n}}}\right)}_{\varphi_{X_n}'(v)} = \varphi_x^n\left(\frac{v}{\sqrt{n}}\right)$$

→ $\log(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^n (-1)^{n-1}}{n}$

and xtic function → $\psi_{Z_n}(v) = \sum_{K=1}^{n} \log \varphi_{X_K}\frac{(v)}{\sqrt{n}} = n \log\left\{1 - \frac{u^2}{2n}\bar{X^2} + O\left(\frac{1}{n^{3/2}}\right)\right\}$ → order of

$$= -\frac{u^2}{2}\bar{X_2} + O\left(\frac{1}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{} -\frac{u^2\bar{X^2}}{2}.$$

order.

$\varphi_{Z_n}(v) \xrightarrow[n\to\infty]{} e^{-u^2\bar{X^2}/2}$ ⇒ Xtic funct of Gaussian (normal) R.V.

Scanned with CamScanner

Basic manipulation of random quantities. $i = \sqrt{-1}$

Proof

→ In general, if you have r. vector $\underline{X} \sim N(0, R)$ — normal distr., 0 mean, covariance matrix R

$\Leftrightarrow P_X(x) = \frac{1}{(2\pi|R|)^{\frac{1}{2}}} \exp\{-\frac{1}{2} x^T R^{-1} x\}$ — normalization c

x is vector hence T (scalar prod)

$$\varphi_{\underline{X}}(u) = \mathbb{E} e^{iU^T\underline{X}} = \mathbb{E} e^{i(u_1x_1 + \dots u_n x_n)}$$

$$= c\int \exp -\frac{1}{2}\underbrace{\left[x^T R^{-1} x - 2iU^T x\right]}_{(x-iRU)^T R^{-1}(x-iRU) + U^T R U} dx$$

$$= e^{-\frac{U^T R U}{2}} \cdot \underbrace{c\int e^{-\frac{1}{2}(x-iRU)^T R^{-1}(\dots)} dx}_{1}$$

$$= e^{-\frac{U^T R U}{2}}$$

hence proved.

3. No of times a random event is repeated = Poisson.

$X \in \{0, 1, 2 \dots\} = \mathbb{N}$. is poisson if $P(X=n) = e^{-\lambda}\frac{\lambda^n}{n!}$

where $\lambda = \mathbb{E}(X)$ mean value of X (parameter)

<u>Xtic functn of Poisson</u>

discrete sum instead of integral

$$\varphi_X(u) = \mathbb{E} e^{iuX} = \sum e^{iun} \cdot P(X=n).$$

$$= \sum_{n\geq 0} \frac{\lambda^n}{n!} e^{-\lambda + iun} = e^{\lambda(e^{iu}-1)}$$

Sum of 2 r.v.

$Z = X + Y$; X, Y are independent & same poisson distribution

$$\varphi_Z(u) = \varphi_X(u) \cdot \varphi_Y(u) = e^{(\lambda + r)(e^{iu} - 1)}.$$

When you add a poisson, the sum is distributed as poisson also → same for adding 2 gaussians.

Suppose $X_1 \ldots X_n$ → gaussian each $X_k \sim N(0, \sigma_k^2)$ & independent.

$Z = \alpha_1 X_1 + \ldots \alpha_n X_N$ → sum.

char func of sum: $\varphi_Z(u) = \mathbb{E}\, e^{iu(\alpha_1 X_1 + \ldots + \alpha_n X_n)}$ → Exp of prod = prod of expo.

$$= \varphi_{X_1}(\alpha_1 u) \ldots \varphi_{X_n}(\alpha_n u)$$

$$= \cancel{\exp -\tfrac{1}{2}(\alpha_1^2 + \ldots + \alpha_n^2}$$

$$= \exp -\tfrac{1}{2}(\alpha_1^2 \sigma_1^2 + \ldots \alpha_n^2 \sigma_n^2) u^2$$

hence $Z \sim$ Gaussian.

4. X & Y be poisson with λ_1 & λ_2 mean
Suppose $Z = X + Y$ X, Y → Poisson distr. independent with parameters λ_1 & λ_2

$P[X = K \mid Z = n] =$ Suppose X is quantity of interest → signal. & Y is noise.

→ has a Binomial nature.

Any r.v. is distr as Binomial if $X \in \{0, 1, 2 \ldots N\}$.

$P(X = K) = \binom{N}{K} \alpha^K (1 - \alpha)^{N-K}$ $0 \le \alpha$ → probability ≤ 1

$\binom{N}{K}$ → $\frac{N!}{K!(N-K)!}$

$$P[A|B] = \frac{P[A\cap B]}{P[B]}.$$

by reversing

$$\frac{P[Z_n=n|X=k]\ P(X=k)}{P[Z=n]} = \frac{P[Y=n-k]\ P[X=k]}{P[Z=n}$$

Sum of two poisson is again poisson.

$X=\lambda_1$ & $Y=\lambda_2$.

$$= \frac{e^{\lambda_2}\lambda_2^{n-k}\ e^{\lambda_1}\lambda_1^{k}}{(n-k)!\ k!\ (\lambda_1+\lambda_2)^n}\ n!$$

$$= \binom{n}{k} q^k (1-q)^{n-k} \xrightarrow{} \text{Binomial.} \quad \text{where } q = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

When we conditiond $\dashrightarrow$ no longer poison

Conditional expectations. (X, Y) → observable. $P_{XY}(x,y)$

$E[X|Y=y]$ → y is observable. $= \int x\, p_{XY}(x|y)\, dx$

by observing Y you want to say sthg abt X

$$= \int x\ \frac{p_{XY}(x,y)}{P_Y(y)}\, dx = \frac{1}{P_Y(y)} \int x\ p_{XY}(x,y)\, dx.$$

→ Conditional Expectation as a information processing tool.

$X, Y \to 2$ r. vectors. & have common distr.

$\min_h \mathbb{E} \| X - \underbrace{h(y)}^{\text{funct of } y} \|^2$. → we want to minimize over all possible functn h of observed signal s.t. error of estimation of X w.r.t to h(Y) is minimized. the result is

$h(Y) = \mathbb{E}[X|Y]$.

$$\mathbb{E}\left\{ \mathbb{E}\left[\underbrace{\| X - h(y) \|^2}_{X^2 + h(Y)^2 - X h(Y)} \,\Big|\, Y \right] \right\}.$$

$$\mathbb{E}\left\{ \mathbb{E}[X^2 | Y] \mp \mathbb{E}(X - h(Y) | Y)\, h(Y) \right\}.$$

min for $h(Y) = \mathbb{E}(X|Y)$ → best value is to observe X knowing Y

Exercise: $\underline{X}, \underline{Y}$ → gaussian Random vector.

Show that $\mathbb{E}[\underline{X} | \underline{Y}] = L \underline{Y}$ L → linear matrix

$L \to \overbrace{\mathbb{E}[X \cdot Y^T]}^{R_{XY}} \cdot \overbrace{\mathbb{E}[Y \cdot Y^T]}^{R_Y})^{-1}$ → show this

(X, Y) . gaussian.

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad R_Z = \left[\begin{array}{c|c} R_X & R_{XY} \\ \hline R_{YX} & R_Y \end{array}\right] \quad \& \quad R_{YX} = R_{XY}^T$$

→ Using $\mathbb{E}[X|Y] = \int x \cdot \frac{P_{XY}(x,y)}{P_Y(y)}\, dx$.

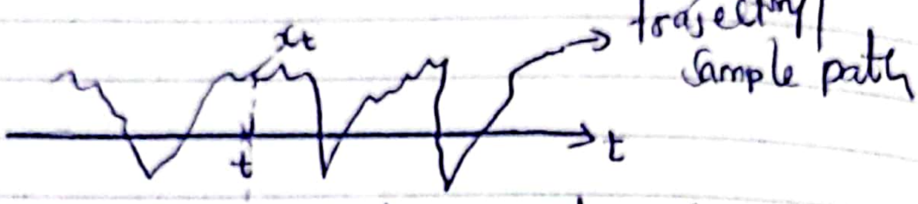
Q1 → 9-11 am

- Brownian motion
- Markov process (random dynamic system)

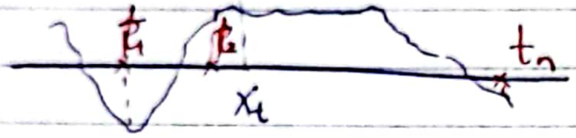
Ex 10.

→ A random process is sequence of random variables

Random process: $\{X_t, t \in R\}$



at each time t we have value X_t



X_t will change.

Independent increament Process:

for any choice of n

$\forall n, t_1 \ldots t_n$: → these variations are independent.

$(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}) \ldots (X_{t_n} - X_{t_{n-1}})$

are independent :

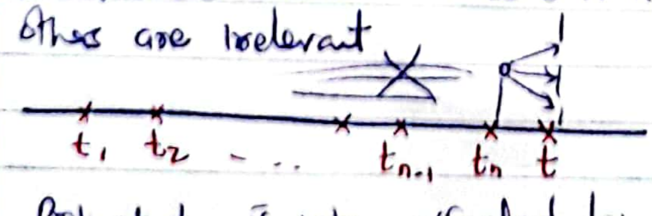
- Example: Brownian process

Show that Independent increment process has markovian property.

Markov process : $P(X_t = x \mid X_{t_1} = x_1 \ldots X_{t_n} = x_n)$ ← previous

$= P(X_t = x \mid X_{t_n} = x_n) \qquad t_1 < \ldots < t_n$

in a markov process if you have observed process at t_n its possible values at time t is affected by only values at t_n others are irrelevant



Prob at t is only affected by values at time t_n

Journal

Suppose given X_t with indep. increments.

$$P[X_t = x \mid X_{t_1} = x_1, \dots X_{t_n} = x_n] = \cancel{P(X_t - X_{t_n})} =$$

$$= P[X_t - X_{t_n} = x - x_n \mid X_{t_n} = x_n, \dots X_{t_i} - X_{t_{i-1}} = x_i - x_{i-1},\ i = 2:n].$$

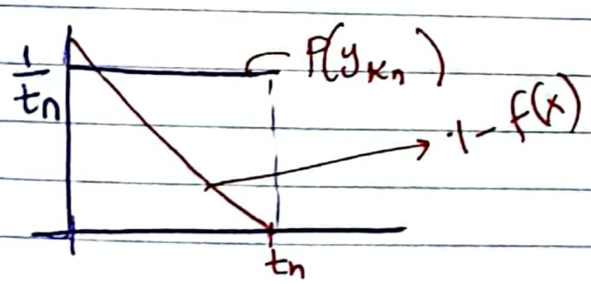
$$= P[X_t - X_{t_n} = x - x_n \mid X_{t_n} = x_n]$$

$$= P[X_t = x \mid X_{t_n} = x_n].$$

$\Rightarrow \{X_t\}$ is Markovian.

EX 11

- $Y_{1n}, Y_{2n} \dots Y_{nn}$ triangular array of R.V.
- Suppose they are iid, uniformly distributed on interval $(0, t_n)$ $t_n \to$ some no



Consider

$Z_n = \min(Y_{1n}, \dots Y_{nn})$ → min is random dro.

a) $P(Z_n > x) = P(Y_{1n} > x, \dots Y_{nn} > x)$

$$= P(Y_{1n} > x) \dots P(Y_{nn} > x) \to$$

since same & identical distr.

$$= [1 - F(x)]^n \quad \text{where } F(x) = P[Y_{kn} \le x]$$

$$= \exp n \log[1 - F(x)]$$

Poisson Process

b) $\log P(Z_n > x) = n \log\left(1 - \frac{x}{t_n}\right)$

$$= n\left\{-\frac{x}{t_n} + o\left(\frac{x}{t_n}\right)\right\} = -x\left(\frac{n}{t_n} + o\left(\frac{n}{t_n}\right)\right)$$

$\frac{n}{t_n} \to \lambda$ limit

$$P[Z_n > x] \xrightarrow[n\to\infty]{} e^{-\lambda x}$$

* Exercise on Correlation Functions

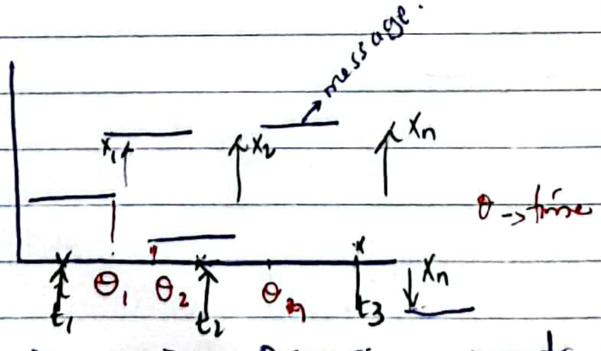
A random Process:

is defined by all distribution functions $P(x_{t_1}, \dots x_{t_n})$ $\forall n, \forall t_1 < t_2 < \dots t_n$.

Cor function of process X is $\gamma_X(t_1, t_2)$ is fnctn of any 2 instance fixed times.

$$\gamma_X(t_1,t_2) = E(X_{t_1} \cdot X_{t_2}) \Rightarrow \gamma_X(t_2 - t_1)$$

between time lapse in stationary case



$\{\theta_n\}$ is poison process $\to$ events occuring at random time (θ) such that

i) for any times $t - \tau, t$, $\tau \geq 0$

$N_{(t, t-\tau)} =$ no of random pulses t_i occuring in $]t-\tau, t]$. is a Poisson R.V. with parameter $\lambda\tau$ → mean of r.v.

$\lambda\tau \to$ proportional to time of event.

$$P[N_{t,t-\tau}=n] = e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}$$

ii) For any times $t_1, t_2 \ldots t_n$

$N_{t_i, t_{i-1}}$, are independent $\to (t_1\ t_2\ t_3)$
pulses diff at t_1, t_2, t_3 e.g. a car crossing bridge at any time. Crossing times are random & no. b/w two times is poisson distributed.

iii) $\{x_n\}$ independent, 0, mean, gaussian with var σ^2

$\to$ cross correlation of X_t & $X_{t-\tau}$

Exercise: Compute $\gamma_Y(t, t-\tau) = \mathbb{E}[X_t \cdot X_{t-\tau}]$, $\tau \geq 0$

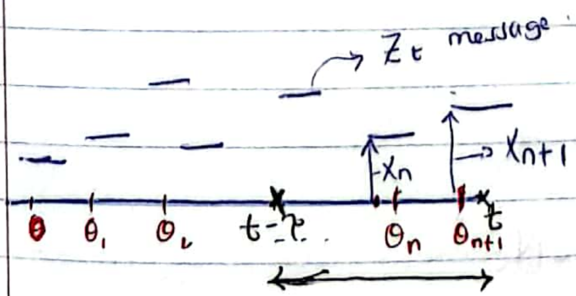
show that $\to \sim e^{-\sigma_i^2 \tau}$

- use conditional expectation.

Hint: $\mathbb{E}[X_t \cdot X_{t-\tau} \mid N_{t,t-\tau} = n]$. 0 or >0 pulses b/wn $t, t-\tau$

- sum over all possible values of n = of pulses

$$\mathbb{E}(z) = E\{\mathbb{E}[z \mid u]\} = \sum_n E[z/u=n] \cdot P[u=n]$$



* Correlation of X_n (message) & poisson time θ_n.
$\{\theta_n\}$, $\{X_n\}$.

$$\gamma_z(t, t-\tau) = \mathbb{E}[z_t \cdot z_{t-\tau}]$$

$\tau \geq 0$

$$= \sum_{n=0}^{\infty} \mathbb{E}[z_t z_{t-\tau} \mid N(t-\tau, t) = n] \, P[N_z = n]$$

$\to$ condition correlation on each value of n

$N(t-\tau,t)$: No. of poisson events in $[t-\tau, t]$

$P[N_z = n]$: no. of θ (times)

$\mathbb{P}[N_z] \to$ proba. distribution.

correlation n discrete $e^{-2\pi i j}$ $2\pi f j$ $\omega j \zeta$

stationarity $\rightarrow \mathbb{E}[X_m X_{m+n}] = \gamma_X(n)$ → function of n (Wide sense) stationarity

$$= \sum_{n=0}^{\infty} \mathbb{E}[Z_t Z_{t-\tau} \mid N(t-\tau, t) = n] \, P[N_\tau = n].$$

$$= \sum_{n\geq 0} \mathbb{E}[X_0 X_n] \, e^{-\lambda\tau} \left(\frac{(\lambda\tau)^n}{n!}\right) = e^{-\lambda\tau} \sum_{n\geq 0} \gamma_X(n) \frac{(\lambda\tau)^n}{n!}$$

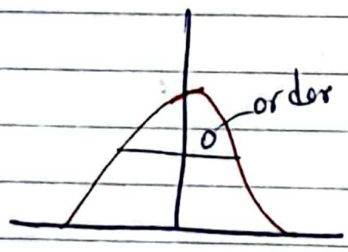
(mean)

* Suppose $\{X_n\}$ iid, $\mathbb{E}X_n = 0 \rightarrow \gamma_X(n) = \sigma^2 \delta(n) = \begin{cases} \sigma^2 & n=0 \\ 0 & n \text{ otherwise} \end{cases}$

$$\gamma_Z(t, t-\tau) = \gamma_Z(\tau) = \sigma^2 e^{-\lambda\tau}$$

* Spectrum of Z (PSD) $= \int_{-\infty}^{\infty} \gamma_Z(\tau) e^{-i2\pi\gamma\tau} \, d\tau =$ $\quad \gamma \rightarrow$ frequency

$$= \sigma^2 \left\{ \int_{-\infty}^{0} e^{-i2\pi\gamma\tau + \lambda\tau} \, d\tau + \int_0^{\infty} e^{-i2\pi\gamma\tau - \lambda\tau} \, d\tau \right\}$$



$$\frac{\sigma^2}{\lambda^2 + 4\pi^2\gamma^2} \sim \Gamma_Z(\gamma)$$

* $P_{(t-\tau, t)}(Z_{t-\tau} = \zeta_1, Z_t = \zeta_2) = P_\tau(\zeta_1, \zeta_2)$ → joint probability

$\{X_n\}$ gaussian iid $N(0, \sigma^2)$ → Normal distribution.

$$= \sum_{n\geq 0} P_\tau(\zeta_1, \zeta_2 \mid N_\tau = n) \, P[N_\tau = n].$$

$$= \underbrace{P_\tau[\zeta_1, \zeta_2 \mid N_\tau > 0] \, P[N_\tau > 0]}_{(i)} + \underbrace{P_\tau(\zeta_1, \zeta_2 \mid N_\tau = 0) \, P[N_\tau = 0]}_{(ii)}$$

$$(i) = \underbrace{P[Z_t = \zeta_2 \mid Z_{t-\tau} = \zeta_1, N_\tau > 0]}_{P_X(X_0 = \zeta_2).} \; \underbrace{P[Z_{t-\tau} = \zeta_1 \mid N_\tau > 0]}_{\underbrace{P[Z_{t-\tau} = \zeta_1]}_{P_X(X_0 = \zeta_1)}.} \; \underbrace{P[N_\tau > 0]}_{(1 - e^{-\lambda\tau})}$$

→ depends on past only

$$= \left(1 - e^{-\lambda\tau}\right) \frac{1}{2\pi \sigma_X^2} \exp - \frac{(\zeta_1^2 + \zeta_2^2)}{2\sigma^2}$$

$t-\tau$ N_τ t

→ No switching (change b/w $t-\tau$ & τ)

(ii) $\underbrace{P(Z_t = \zeta_2 \mid Z_{t-\tau} = \zeta_1, N_\tau = 0)}_{\delta(\zeta_2 - \zeta_1)} \underbrace{P(Z_{t-\tau} = \zeta_1 \mid N_\tau = 0)}_{P_{X_0}(\zeta_1)} \underbrace{P(N_\tau = 0)}_{e^{-\lambda\tau}}$.

→ last X recorded

$$= \frac{e^{-\lambda\tau}}{\sqrt{2\pi\sigma^2}} e^{\frac{-\zeta_1^2}{2\sigma^2}} \delta(\zeta_2 - \zeta_1)$$

② $r_Z(\tau) = e^{-\lambda\tau} \sum_{k=0}^{\infty} r_X(k) \frac{(\lambda\tau)^n}{n!}$

Suppose $X_n = \alpha X_{n-1} + u_n$ $\{u_n\}$ iid $\mathbb{E}[u_n] = 0$, $\mathbb{E}[u_n^2] = \beta$

Correlation: stationary, depend on n only not K

$$r_X(n) = \mathbb{E}[X_K \cdot X_{K-n}].$$

$$r_X(0) = \mathbb{E}[X_n^2] = \alpha^2 \underbrace{\mathbb{E}[X_{n-1}^2]}_{r_X(0)} + \underbrace{\mathbb{E}[u_n^2]}_{\beta} + \underbrace{2\alpha\mathbb{E}[X_{n-1} \cdot u_n]}_{0 \to \text{non-correlated}}.$$

$$r_X(0) = \frac{\beta}{1-\alpha^2} \quad \text{--- (i)}$$

$$r_X(k) = \underbrace{\mathbb{E}[X_n \cdot X_{n-K}]}_{r_X(K)} = \alpha \underbrace{\mathbb{E}[X_{n-1} X_{n-K}]}_{r_X(K-1)} + \underbrace{\mathbb{E}[u_n X_{n-K}]}_{0}.$$

$$r_X(k) = \alpha r_X(k-1) \quad \text{--- (ii)}$$

Combining (i) & (ii)

$$r_X(k) = \frac{\beta}{1-\alpha^2} \cdot \alpha^K, \quad |\alpha| < 1$$

$$r_Z(\tau) = e^{-\lambda\tau} \frac{\beta}{1-\alpha^2} \cdot \sum_{K \geq 0} \frac{(\lambda\tau)^K}{K!} \alpha^K = \sigma_X^2 e^{-\lambda(1-\alpha)\tau}$$

→ let σ_X^2

* Conditioning on gaussian vectors.
* linear Mean square estimation of parameters.
* Application: source Antenna processing.
: spectral estimation.

$Y, X \to$ subvectors

Gaussian Conditioning

Consider random vector $Z = \begin{bmatrix} X \\ Y \end{bmatrix} \begin{matrix} \updownarrow m \\ \updownarrow n \end{matrix}$ $\quad Z \in \mathbb{R}^{n+m}$.

$Y = X + \text{noise}$

$\mathbb{E}[Z] = 0, \quad R_Z = \mathbb{E}[Z \cdot Z^T].$

$$R_Z = \begin{bmatrix} \mathbb{E}(XX^T) & \mathbb{E}(XY^T) \\ \mathbb{E}(YX^T) & \mathbb{E}[YY^T] \end{bmatrix} = \begin{bmatrix} R_X & R_{XY} \\ R_{YX} & R_Y \end{bmatrix} \quad R_{YX} = R_{XY}^T$$

$Z \sim N(0, R_Z)$ $\to$ Covariance matrix

$$P_Z(z) = \frac{1}{|2\pi R_Z|^{1/2}} \exp -\frac{1}{2} z^T R_Z^{-1} z$$

$\to$ determinant

Mean sqr. estimation: $\to$ best estimator of X knowing Y

$\hat{X}(Y) = \mathbb{E}[X|Y]$ = Best mean square estimator of X observing Y

$$= \int x \, P_{X|Y}(x|y)\, dx = \int x \frac{P_Z(x,y)}{P_Z(y)}\, dx$$

$\to$ subtract exponents to get maximum of gaussian

Calculate R_Z^{-1}

let $M = R_Z^{-1} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} R_X & R_{XY} \\ R_{YX} & R_Y \end{bmatrix} \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$

$$= \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$$

$R_X A + R_{XY} B = I \rightarrow$ Identity

~~$R_{XY} B^T$~~ extracting:

$$R_{YX} A + R_Y B = 0 \Rightarrow B = -R_Y^{-1} R_{YX} A \rightarrow B^T = -A R_{XY} R_Y^{-1}$$

$$* \; Z^T R_Z Z - Y^T R_Y^{-1} Y = [X^T Y^T] \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} - Y^T R_Y^{-1} Y$$

$$= X^T A X + X^T B^T Y + Y^T B X + Y^T C Y - Y^T R_Y^{-1} Y$$

multiply both sides by A^{-1}

$$= (X + A^{-1} B^T Y)^T A \; (X + A^{-1} B^T Y) - Y^T [B A^{-1} B^T Y + R_Y^{-1}] Y$$

$\rightarrow$ maximum X

Extraction at $X^* = -A^{-1} B^T Y = \underbrace{R_{XY} R_Y^{-1}}_{L_{X|Y}} Y = \hat{X}(Y)$ gives best estimate of X $= \mathbb{E}[X|Y]$

$\rightarrow$ Conditional expectation

$Cov(X|Y) = Cov(\underbrace{X - \hat{X}(Y)}_{error})$ $\rightarrow$ how much error is done when you estimate X by (Y)

$$Cov(X|Y) = A^{-1}$$

(*) $$A^{-1} = R_X + R_{XY} B A^{-1}$$

$$= R_X - R_{XY} R_Y^{-1} R_{YX} = R_X - L_{X|Y} R_Y L_{X|Y}^T$$

Now minimize directly

$$\min_L \mathbb{E}\{\|\underbrace{X - LY}_{\varepsilon}\|^2\}$$

$\rightarrow$ euclidean norm.

$\rightarrow$ best estimation of X knowing Y.

$\varepsilon \rightarrow$ variance of error of estimation.

For any (non necessarily joint gaussian) random vectors X, Y

~~Var(e)~~

Reminder: For any M square matrix M, trace of M

$$\mathrm{tr}(M) = \underbrace{\sum_{k=1}^{n} M_{kk}}_{\text{sum of diagonals}} = \sum \text{eigen values} = \sum_{k=1}^{n} \lambda_k \text{ if } M \text{ is diagonalizable}$$

Suppose you have two vectors $\underline{X}$, $\underline{Y}$

$$\underline{X}^T \cdot Y = \mathrm{tr}(Y \cdot X^T)$$

hence;

$$\mathrm{Var}(\underline{\varepsilon}) = \mathbb{E}\{\|E\|^2\} = \mathbb{E}(\varepsilon^T \varepsilon) = \mathbb{E}[\mathrm{tr}(\varepsilon \cdot \varepsilon^T)] = \mathrm{tr}\, \mathbb{E}[\varepsilon \cdot \varepsilon^T]$$

Suppose $E[W] = \bar{W}$

$$= \mathrm{tr}\{\underbrace{E[X \cdot X^T]}_{R_X} - L\underbrace{YX^T}_{R_{YX}} - \underbrace{XY^T}_{R_{XY}}L^T + L\underbrace{YY^T}_{R_Y}L^T\}.$$

Unknown here is L

$$\mathrm{Var}(\underline{\varepsilon}) = \mathrm{tr}\{(L - R_{XY}R_Y^{-1})R_Y(L - R_{XY}R_Y^{-1})^T\} + \mathrm{tr}\{R_X - R_{XY}R_Y^{-1}R_{YX}\}$$

We want minimum of L

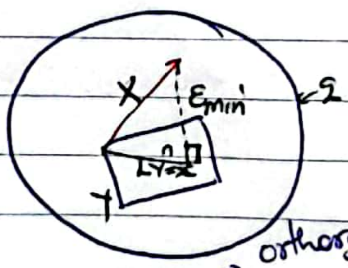
Since R_Y is positive definite ; $\forall W$; $W^{-1,T}R_Y W \geq 0$

$\mathrm{Var}(\varepsilon)$ is minimised by

$$L = R_{XY} \cdot R_Y^{-1} \rightarrow \text{to be } 0 \text{ (minimum)}.$$

$$\min_L \mathrm{var}(\varepsilon) = \mathrm{tr}(R_X - \underbrace{R_{XY}R_Y^{-1}R_{YX}}_{LR_YL^T}\} \quad \text{------ } 1$$

Geometrical Interpretation.



minimum error (ε) is obtained by projecting X on Y.

orthogonal thro' Y.

$$\mathbb{E}[\varepsilon^T \cdot Y] = 0$$

$$\mathrm{tr}\,\mathbb{E}[(X - LY)Y^T] = 0$$

Van trees (2 vol) → LibGen

→ Estimation, Detection & Modulation.

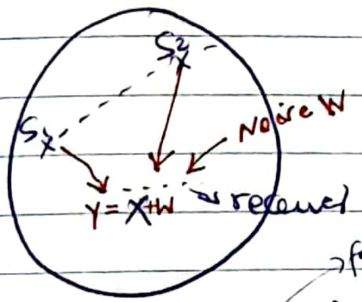
- random processes.

$$\underbrace{E[X\cdot Y^T]}_{R_{xy}} - L\,\underbrace{E[YY^T]}_{R_Y} = 0$$ → Normal equation

$cov(e) = Cov(X) - Cov(LY)$ by Pythagoras theorem and hence same as ①

Application to Antenna Processing (MUSIC Algorithm) multi signal

Source-S



→ function Random scalar parameters.

→ deterministic signals

$\vec{X} = S\cdot\theta$ ← $\vec{X} = \sum_{i=1} \theta_i \vec{S}_i$ → Signal received by antenna.

where $S = [S_1, \dots S_m]$, $\theta = \begin{bmatrix}\theta_1\\ \vdots \\ \theta_m\end{bmatrix}$

$\vec{Y} = \vec{X} + \vec{W}$ → noise (0 mean $\bar{W}=0$, $\overline{WW^T} = R_W$ → Covariance)

Suppose X & W are Uncorrelated

So that $R_{XW} = E[XW^T] = 0$ → Correlation: Covariance $= 0$

Optimal Receiver:

$$L = R_{xy}R_y^{-1} = \overline{X(X+W)^T}\cdot\{\overline{(X+W)(X+W)^T}\}^{-1}$$

$$= R_X[\underbrace{R_X+R_W}_{R_Y}]^{-1} = S\,\underbrace{\overline{\theta\theta^T}}_{C_\theta}\cdot S[S C_\theta S^T + R_W]^{-1}$$

C_θ ← cov(θ)

Applied to received signal Y to obtain the best estimate.

diagonal.

Eigen decomposition of Symmetrical matrix

Hypothesis :

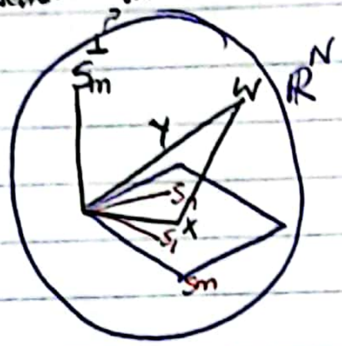
→ Noise is 'white' and isotropic → $R_W = \beta I$ → Direct diagonal matrix

$$L = R_X R_Y^{-1} = (R_Y - R_W) R_Y^{-1} = I - R_W R_Y^{-1} = I - \beta R_Y^{-1}$$

– By observing signal Y by WLLN we can make L.

$R_Y = R_X + \beta I$ → i) Eigen vectors of R_X are eigen vectors of R_Y

Geometrical Interpretation



S_m → subspace.

$S_m^\perp$ → orthogonal complement to S_m

ii) S_m & $S_m^\perp$ → both are eigen spaces of R_Y

$$R_Y = \beta I + \underset{\sim}{S} C_\theta \underset{\sim}{S}^T$$

$$R_Y S_i = \beta S_i + \underbrace{\underset{\sim}{S} C_\theta \underset{\sim}{S}^T \underset{\sim}{S_i}}_{\in S \to \text{in } S} \in S$$ → are

such that $R_Y S \subseteq \underset{=}{S}$

If $\vec{V} = S^\perp$; $R_Y \vec{V} = \beta \vec{V} + \underset{\sim}{S} C_\theta \underbrace{\underset{\sim}{S}^T \vec{V}}_{0 \text{ – because of orthogonality}} = \beta \vec{V}$

any vector in $S^\perp$ is an eigen vector in R_X with eigen value β

Let $R_X = E XX^T = \sum_{i=1}^{m} \lambda_i u_i u_i^T$: $\{u_i\}$ are orthogonal for $R_X = R_Y^T$

$\underset{\sim}{U}$ → matrix
$\underline{U}$ → vector

× eigen vectors of symmetrical vectors are orthogonal, and $\lambda_i \geq 0$ (R_X is positive definite).

$u_i \perp u_j$, $i \neq j$

$\{u_i\}$ span S

$$R = \underset{\sim}{U} \Lambda \underset{\sim}{U}^{-1} = \underset{\sim}{U} \cdot \Lambda \underset{\sim}{U}^T = [\underset{\sim}{u_1} \cdots \underset{\sim}{u_N}] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_m \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_N^T \end{bmatrix}$$

Scanned with CamScanner

* let $u_{m+1}, \dots, u_N$ span $S^{\perp}$ and $u_i \perp u_j = 0$, $i \neq j$

$i, j = m+1 : N$

$$I = \sum_{i=1}^{N} u_i u_i^T \Rightarrow R_W = \beta I = \beta \underbrace{\sum_{i=1}^{N} u_i u_i^T}_{\text{Projector on } S} + \beta \underbrace{\sum_{i=m+1}^{N} u_i \cdot u_i^T}_{\text{projector on } S^{\perp}}$$

N → dim of subspace S

projector in direction u_i

$$R_Y = \sum_{i=1}^{M} \mu_i u_i u_i^T + \beta \sum_{i=m+1} u_i u_i^T$$

$$R_Y = \sum \underbrace{(\lambda_i + \beta)}_{\mu_i} u_i u_i^T + \beta \sum_{i=m+1}^{N} u_i u_i^T$$

$\lambda_i \geq 0 \Rightarrow \beta$ = least eigen value of R_Y

$\mu_i \Rightarrow$ Dominant eigen values of R_Y

$$R_Y^{-1} = \sum \mu_i^{-1} u_i u_i^T + \frac{1}{\beta} \sum u_i u_i^T$$

$\lambda_i \to$ eigen values of R_X

$$L = I - \beta R_Y^{-1} = \sum_{i=1}^{m} \left(1 - \frac{\beta}{\mu_i}\right) u_i u_i^T$$

$$1 - \frac{\beta}{\mu_i} = 1 - \frac{\beta}{\lambda_i + \beta} = \frac{\lambda_i}{\lambda_i + \beta} = \frac{\lambda_i / \beta}{1 + \frac{\lambda_i}{\beta}}$$

if $\beta / \mu_i \downarrow 0$: $L \longrightarrow \text{Proj } S = \sum_{i=1}^{M} u_i u_i^T$.

⊛ Exercise.

Show that:

$$Var(\varepsilon_{min}) = tr\{R_X - R_{XY} R_Y^{-1} R_{YX}\}$$

is

$$= \beta \sum_{i=1}^{m} \frac{\lambda_i}{\beta + \lambda_i} \leq m\beta$$

$\beta \to$ variance of noise.

$Var(w) = N\beta$

$$\frac{Var(\varepsilon_{min})}{Var(w)} \leq \frac{m}{N}$$

If N is high (many sensors) → best performance → No of sources M is less than No of sensors.