TD 1 et 2 : Fourier transform of 2D discrete functions, filtering in the frequency space

M1 E3A international track, site Evry

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Exercise 1 : Properties of the Fourier transform

Let $f(x)$ a summable function, its Fourier transform is given by :
$\Im\{f(x)\} = F(u) = \int f(x)\exp(-2\pi jux)dx$

1. Prove the linearity of the Fourier Transform (FT).
2. Computer the FT of the transposed function $f(-x)$.
3. Scale change : consider $a \neq 0$, compute the FT of $f(ax)$.
4. Compute the FT of the translated function $f(x-a)$.
5. Compute the FT of the modulated function $f(x)\exp(2\pi ju_0 x)$.
6. Compute the FT of the the derivative function $f'(x)$.
7. Find the function $g(x)$ with a FT equal to $\frac{\partial F(u)}{\partial u}$.

Exercise 2 : Fourier Transform and convolution

Let $f(x)$ and $g(x)$ two summable functions and their Fourier Transform $F(u)$ and $G(u)$.

1. Show that the FT of $f \star g$ is $F(u)G(u)$.

Exercise 3 : Fourier transform of a Gaussian

1. Determine the Fourier Transform $F(u)$ of $f(x) = \exp(-\pi x^2)$.
2. Using a change of variable, determine the Fourier transform of $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})$
3. Comment the relation between the standard deviation of the Gaussian and its Fourier transform.

Exercise 4 : 2D Fourier transform

Consider : $F(u,v) = \int\int f(x,y)\exp\left[-2\pi i(ux+vy)\right]dxdy$

1. Show that we can reduce this expression to a composition of two monodimensional transformations.

Exercise 5 : Fourier Transform of a Laplacian

Calculate the Fourier of a Laplacian : $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Exercise 6 : Discrete Fourier Transform

$\rightarrow X_K = \sum_{n=0}^{N-1} x_n e^{-2\pi i K\frac{n}{N}}$ $K=0, \dots N-1$

X_0, X_1, X_2, X_3

1. Consider the function f defined for 4 samples :
   $f(0) = 2$; $f(1) = 3$; $f(2) = 4$; $f(3) = 4$
   Compute its discrete Fourier Transform.

2. Consider the filter : $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$.

   Compute its discrete Fourier Transform (DFT).

Exercise 7 : Fast Fourier Transform

The objective is to reveal some properties that could be useful to compute fast the Fourier Transform in few operations.

$F(u) = \sum_{x=0}^{N-1} f(x) \exp(\frac{-2\pi i u x}{N})$

where N is the number of samples.

Let write $w_N = \exp(\frac{-2\pi i}{N})$ and $N = 2M$,

1. Show that the previous expression could be written :

   $F(u) = \frac{1}{2}\left[\frac{1}{M}\sum_{x=0}^{M-1} f(2x).w_M^{ux} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1) w_M^{ux}.w_{2M}^{u}\right]$

   We will use :

   $F_P(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x) w_M^{ux}$ and $F_I(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1) w_M^{ux}$

   Then :

   $F(u) = F_P(u) + F_I(u) w_{2M}^{u}.$

2. Show that : $F(u+M) = F_P(u) - F_I(u) w_{2M}^{u}$
3. Explain why this implementation is faster.
4. Compute the FFT for $N = 4$. Show how the partial results are re-used for upper dimensions.

Exercise 8 : Isotropy✓

1. Show that the Laplacian $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is an isotropic operator (invariant toward rotation).
2. Show that the amplitude of Gradient $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$ is isotropic.
3. Show that the amplitude of Gradient $\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$ is not isotropic.

Exercise 9 : Average Filter

Let's consider the average filter 3×3 excluding from calculation the pixel in the middle.

1. Find the filter $H(u, v)$ that is equivalent in the frequency space.
2. Show that the resulting filter is a low pass filter.

Exercise 10 : Laplacian Filter

Let's condider the Laplacien 3×3 filter : $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ → Basic Laplacian filter in discrete form

1. Find the filter $H(u, v)$ that is equivalent in the frequency space.

2. Show that the resulting filter is a high pass filter.

$\nabla^2 \simeq f - \bar{f}$ → result of average filter (h) → either 8 or 4 with 1 in centre (classic).

Exercise 11 : Approximation of a Laplacian

Show that $f(x,y) - \nabla^2 f(x,y)$ could be approximated by $f(x,y) - \bar{f}(x,y)$ where $\bar{f}$ is the average f in a given neighborhood.

Recall :

- If $y = -x$ then $dy = -dx$ and $\int_{-\infty}^{+\infty} dx = -\int_{+\infty}^{-\infty} dy = \int_{-\infty}^{+\infty} dy$
- $\exp(x+y) = \exp(x)\exp(y)$
- $\exp(x)^a = \exp(ax)$
- $\frac{1}{\exp(x)} = \exp(-x)$
- Gauss Integral : $\int \exp(-\alpha x^2)dx = \sqrt{\frac{\pi}{\alpha}}$
- $\int \exp(ax)dx = \frac{1}{a}\exp(ax)$
- $\exp(i\theta) = \cos(\theta) + j\sin(\theta)$
- $\cos(\theta) = \cos(-\theta)$ and $\sin(\theta) = -\sin(-\theta)$
- $\cos(\theta) = \frac{1}{2}(\exp(j\theta) + \exp(-j\theta))$
- $\sin(\theta) = -\frac{j}{2}(\exp(j\theta) - \exp(-j\theta))$

TD2: Filtering in the spatial domain

M1 E3A international track, Evry site

UE "Image and signal processing", Upsay / UEVE

Exercise 1 : Spatial Convolution

Calculate and name the following filters:

1. $\begin{bmatrix} 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 \end{bmatrix} * \begin{bmatrix} -1 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 & -1 \end{bmatrix}$
4. $\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$
5. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
6. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
7. $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Exercise 2 : Filter Decomposition

Give the usefulness of the filters below by studying their answer on simple images. For each of them, break down the filter into several filters if possible. Otherwise express the filter continuously using the classical differential operators ($\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, etc.$). Give images of your choice allowing you to highlight the responses of the filters.

1. $\begin{bmatrix} -1 & 0 & 2 & 0 & -1 \end{bmatrix}$
2. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$
3. $\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

Exercise 3 : Filter synthesis

1. Give the filter h of size 3×3 that correspond to the identity filter for the convolution with an image f ($f * h = f$)
2. Compute the sum of the resulting filter h with a Laplacian of size 3×3.
3. Explain the utility of this kind of resulting filter.

Exercise 4 : Binomial filters

1. Give two ways to generate the binomial filter of size 1×7.

2. Deduce the binomial filter of size 7×7.
3. Give its normalized form.
4. In the general case, what is the point of normalizing the filters?
5. What is the use of binomial filtering?

Exercise 5 : Gaussian Differences

1. Calculate the Gaussian filter of size 1×5 with a variance equal to 4.
2. Calculate the Gaussian filter of size 1×5 with a variance equal to 1.
3. What is the use of Gaussian filtering?
4. Calculate the difference between the two filters of question 1 and 2.
5. Display the intensity profile of the difference. What do you deduce?
6. Give the utility and use of the new filter thus generated.

Exercise 6 : Median Filter

We consider a small neighborhood around a pixel of size 3×3:

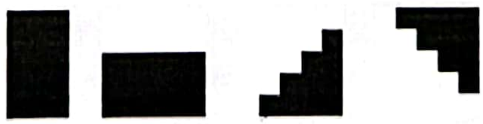
1	2	5
3	3	4
5	4	5

1. Calculate the result of applying a median filter on this neighborhood.
2. Calculate the result of applying a majority filter on this same neighborhood.
3. Is the median value kept when the image is multiplied by a scalar?
4. Give the density of the noise in the neighborhood of size $n \times n$ to from which the median filter is no longer effective.

Exercise 7 : Derivative Filters

Let's consider the following filters: $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

1. Calculate the modulus and direction of the gradient for small binary images of sizes 5×5 that correspond to ideal edges (vertical, horizontal, left diagonal, right diagonal).



2. Determine for each of the 4 considered ideal edges the zero crossing of the Laplacian (convolve with $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$).