* GAUSSIAN CONDITIONING

$$\underline{Z} = \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow n \end{matrix} \qquad z \in \mathbb{R}^{n+m} \qquad E[\underline{Z}] = \underline{0} \qquad R_z = E[\underline{z}\,\underline{z}^T]$$

$$R_z = \begin{bmatrix} E\,\underline{x}\,\underline{x}^T & E(xY^T) \\ E(Yx^T) & E[YY^T] \end{bmatrix} = \begin{bmatrix} R_x & R_{xY} \\ R_{Yx} & R_Y \end{bmatrix} \qquad R_{Yx} = R_{xY}^T$$

$$Z \sim N(0, R_z) \rightarrow p_z(\underline{z}) = \frac{1}{|2\pi R_z|^{1/2}} \exp -\frac{1}{2}\left(\underline{z}^T R_z^{-1} \underline{z}\right) \leftarrow$$

$$\hat{X}(Y) = E[X \mid Y] = \text{Best m.s. estim of } X \text{ observing } Y$$

$$= \int x\, p_{x|y}(x \mid y)\, dx = \frac{\int x\, p_z(x, y)\, dx}{p_y(y)}$$

→ Calculate R_z^{-1}

let $M = R_z^{-1} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$; $\begin{bmatrix} R_x & R_{xY} \\ R_{Yx} & R_Y \end{bmatrix} \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$

$$R_{Yx} A + R_Y B = 0 \Rightarrow \underline{B = -R_Y^{-1} R_{Yx} A} \rightarrow \underline{B^T = -A R_{xY} R_Y^{-1}}$$

$$R_x A + R_{xy} B = I$$

$$z^T R_z^{-1} z - y^T R_y^{-1} y = [x^T\ y^T] \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - y^T R_Y^{-1} Y = x^T A x + \underbrace{x^T B^T y} + \underbrace{y^T B x} + y^T C y - y^T R_Y^{-1} y$$

$$= (x + A^{-1} B^T y)^T A (x + A^{-1} B^T y) - y^T \left[B A^{-1} B^T + R_Y^{-1} \right] Y$$

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Extremum at $\left\{ x^* = -A^{-1} B^T y = \underbrace{R_{xy} R_y^{-1}}_{L_{x|y}} y = \hat{X}(y) = \underline{E[x \mid Y]} \right.$

$$\operatorname{cov}(x \mid y) = \operatorname{cov}\big(\underbrace{x - \hat{x}(y)}_{\text{error}}\big) = A^{-1}$$

$$A^{-1} = R_x + R_{xY} B A^{-1} = \swarrow$$

$$= R_x - R_{xY} R_Y^{-1} R_{Yx} = \underline{R_x - L_{x|y} R_Y L_{x|Y}^T}$$