TD 1 et 2 : Fourier transform of 2D discrete functions, filtering in the frequency space

M1 E3A international track, site Evry

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Answer of exercise 1

1. We have to show that : $\Im\{af_1(x) + bf_2(x)\} = aF_1(u) + bF_2(u)$
   with $F_1(u) = \Im\{f_1(x)\}$ et $F_2(u) = \Im\{f_2(x)\}$.
   We have : $\Im\{af_1(x) + bf_2(x)\}$
   $= \int (af_1(x) + bf_2(x)) \exp(-2\pi jux)dx$
   $= a\int f_1(x) \exp(-2\pi jux)dx + \int bf_2(x) \exp(-2\pi jux)dx$
   $= aF_1(u) + bF_2(u)$
2. Let $y = -x$ then : $dy = -dx$
   When x tends to $+\infty$, y tends to $-\infty$ and vice-versa.
   $\Im\{f(y)\} = \int_{+\infty}^{-\infty} f(y) \exp(-2\pi ju(-y))d(-y)$
   $\Im\{f(y)\} = -(-\int_{-\infty}^{+\infty} f(y) \exp(-2\pi j(-u)y)dy = F(-u)$
3. Let $y = ax$ then : $dy = adx$
   If a>0 then :
   When x tends to $+\infty$, y tends to $+\infty$ and idem for $-\infty$
   $\Im\{f(y)\} = \int_{-\infty}^{+\infty} f(y) \exp(-2\pi ju(\frac{y}{a}))d(\frac{y}{a})$
   $\Im\{f(y)\} = \frac{1}{a}\int_{-\infty}^{+\infty} f(y) \exp(-2\pi ju(\frac{y}{a}))dy$

   If a<0 then :
   When x tends to $+\infty$, y tends to $-\infty$ and vice-versa.
   $\Im\{f(y)\} = (-\int_{-\infty}^{+\infty} f(y) \exp(-2\pi ju(\frac{y}{a}))d\frac{y}{a}$
   $\Im\{f(y)\} = -\frac{1}{a}\int_{-\infty}^{+\infty} f(y) \exp(-2\pi ju(\frac{y}{a}))dy$

   If we group the two cases : $\Im\{f(ax)\} = \frac{1}{|a|}F(\frac{u}{a})$
4. Let $y = x - a$ then $x = y + a$ et $dx = dy$.
   When x tends to $+\infty$, y tends to $+\infty$ and idem for $-\infty$.
   $\Im\{f(x - a)\} = \Im\{f(y)\} = \int f(y) \exp(-2\pi ju(y + a))dy$
   $= \int f(y) \exp(-2\pi juy) \exp(-2\pi jua))dy = \exp(-2\pi jua))F(u)$
5. $\Im\{f(x) \exp(2\pi ju_0x)\} = \int f(x) \exp(2\pi ju_0x) \exp(-2\pi jux)dx$
   $= \int f(x) \exp(2\pi ju_0x - 2\pi jux)dx$
   $= \int f(x) \exp(-2\pi jx(u - u_0))dx = F(u - u_0)$

6. $\Im\{f'(x)\} = \int f'(x) \exp(-2\pi jux)dx$
   Integration by parts. Recall : $\int uv' = [uv]_a^b - \int u'v$
   $\Im\{f'(x)\} = [f(x) \exp(-2\pi jux)]_{-\infty}^{+\infty} - \int [\exp(-2\pi jux)]' f(x)dx$
   Now $[f(x) \exp(-2\pi jux)]_{-\infty}^{+\infty}$ is null because f has a null limit when x tends $\pm\infty$
   $\Im\{f'(x)\} = -\int [\exp(-2\pi jux)]' f(x)dx = 2\pi juF(u)$

7. $\frac{\partial F(u)}{\partial u} = \frac{\partial}{\partial u}\int f(x) \exp(-2\pi jux)dx$
   We derive under the sum symbol :

$\frac{\partial F(u)}{\partial u} = \int f(x) \frac{\partial}{\partial u} \exp(-2\pi jux)dx$
$\frac{\partial F(u)}{\partial u} = \int -2\pi jxf(x) \exp(-2\pi jux)dx$
The resulting function is : $g(x) = -2\pi jxf(x)$

Answer of exercise 2

$\Im\{f \star g_{(x)}\} = \Im\{\int f(t)g(x-t)dt\} = \int\int f(t)g(x-t)dt \exp(-2\pi jux)dx$
Variable change : $x - t = y$ then $dx = dy$
$\Im\{f \star g_{(x)}\} = \int\int f(t)g(y) \exp(-2\pi ju(t+y))dtdy$
$= \int\int f(t) \exp(-2\pi jut)g(y) \exp(-2\pi juy)dtdy$
$= \int f(t) \exp(-2\pi jut)dt \int g(y) \exp(-2\pi juy)dy$
$= F(u)G(u)$

Answer of exercise 3

1. $f(x) = \exp(-\pi x^2)$
   $F(u) = \int_{-\infty}^{+\infty} \exp(-\pi x^2) \exp(-j2\pi ux)dx$
   $= \exp(-\pi u^2) \exp(\pi u^2) \int_{-\infty}^{+\infty} \exp(-\pi x^2) \exp(-j2\pi ux)dx$
   $= \exp(-\pi u^2) \int_{-\infty}^{+\infty} \exp(-\pi(x+ju)^2)dx$
   $= \exp(-\pi u^2) \int_{-\infty}^{+\infty} \exp(-\pi w^2)dw$ with $w = x + ju$
   Knowning that $\int_{-\infty}^{+\infty} \exp(-\pi w^2)dw = 1$ (sea Gauss integral)

   $F(u) = \exp(-\pi u^2)$

   Other method (using a differential equation) :
   $f'(x) = -2\pi x \exp(-\pi x^2) = -2\pi xf(x)$
   Or $\Im\{f'(x)\} = 2\pi juF(u)$ et $\Im\{-2\pi jxf(x)\} = \frac{\partial F(u)}{\partial u}$

   $\Im\{f'(x)\} = \Im\{-2\pi xf(x)\} = 2\pi juF(u)$
   $j\Im\{-2\pi xf(x)\} = 2\pi j^2uF(u)$
   $\Im\{-2\pi jxf(x)\} = 2\pi j^2uF(u) = -2\pi uF(u)$
   Then $-2\pi uF(u) = \frac{\partial F(u)}{\partial u}$
   Just solve the differential equation up to a constant$F(u) = exp(-\pi u^2)$
2. Let write $a = \frac{1}{\pi 2\sigma^2}$ and then we use : $\Im\{f(ax)\} = \frac{1}{|a|}F(\frac{u}{a})$. We write $b = \frac{1}{\sigma\sqrt{2\pi}}$ and the we use : $\Im\{bf(x)\} = bF(u)$.

   $\Im\{f(ax)\} = 2\pi\sigma^2 exp(-2\pi^2\sigma^2u^2)$

   $\Im\{bf(x)\} = \frac{1}{\sigma\sqrt{2\pi}} 2\pi\sigma^2 exp(-2\pi^2\sigma^2u^2) = \sigma\sqrt{2\pi} exp(-2\pi^2\sigma^2u^2)$
3. The higher the standard deviation of the initial Gaussian, the lower the standard deviation of the resulting Gaussian (Fourier transform).

Answer of exercise 4

$F(u,v) = \int\int f(x,y) \exp[-2\pi j(ux+vy)]dxdy = \int f(x,y) \exp(-2\pi jux)dx \int \exp(-2\pi jvy)dy$
Let write (x fixed and y variable) : $F(x,v) = \int f(x,y) \exp(-2\pi jvy)dy$
Then : $F(u,v) = \int F(x,v) \exp(-2\pi jux)dx$
The two-dimensional Fourier transform can therefore be composed by two one-dimensional transforms.

Answer of exercise 5

Definition of a Laplacian in 2D : $\triangle = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

$\Im\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\} = \Im\{\frac{\partial^2 f}{\partial x^2}\} + \Im\{\frac{\partial^2 f}{\partial y^2}\}$

Let compute for the 1D case, the result could be transposed to the 2D case :

$\Im\{\frac{\partial^2 f}{\partial x^2}\} = \int\int \frac{\partial^2 f(x,y)}{\partial x^2} \exp(-2\pi j(ux + vy))dxdy$

Let $g = \frac{\partial f}{\partial x}$ and $\Im\{\frac{\partial g}{\partial x}\} = 2\pi ju\Im\{\ g\} = 2\pi ju\Im\{\ \frac{\partial f}{\partial x}\} = (2\pi ju) \times (2\pi ju) = -4\pi^2u^2F(u,v)$

We can do the same for $\Im\{\frac{\partial^2 f}{\partial y^2}\}$ then we will get $-4\pi^2v^2F(u,v)$

Finally : $\Im\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\} = -4\pi^2(u^2 + v^2)F(u,v)$

Answer of exercise 6

1. We apply : $F(u) = \sum f(x)\exp(-2\pi jux/N)$ with $N = 4$ in this example. We find : $F(0) = 13$, $F(1) = -2 + j$, $F(2) = -1$, $F(3) = -2 - j$
2. It is possible to use one of the two methods : a) Apply : $F(u,v) = \sum\sum f(x,y)\exp(-2\pi(ux/M + vy/N)$. b) Use the separability property by applying a 1D transform raw by raw then 1D transform of the resulting columns. Be careful : you have to center the filter un the middle. (x,y) vary then as in the following table :
$\begin{bmatrix} (-1,-1) & (-1,0) & (-1,1) \\ (0,-1) & (0,0) & (0,1) \\ (1,-1) & (1,0) & (1,1) \end{bmatrix}$

Answer of exercise 7

Sea lecture

Answer of exercise 8

1. After transformation, we set : $x = x'\cos(\theta) - y'\sin(\theta)$ and $y = x'\sin(\theta) + y'\cos(\theta)$

   $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ and after rotation : $\nabla^2 = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$

   Knowing that : $\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)$

   To compute : $\frac{\partial^2 f}{\partial x'^2}$, then : $g(x) = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)$

   We must compute : $\frac{\partial^2 f}{\partial x'^2} = \frac{\partial g}{\partial x'} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial g}{\partial x}\cos(\theta) + \frac{\partial g}{\partial y}\sin(\theta)$

   $= \frac{\partial^2 f}{\partial x^2}\cos^2(\theta) + \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin(\theta)\cos(\theta) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\cos(\theta)\sin(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$

   The same process is done for y :
   $\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} = -\frac{\partial f}{\partial x}\sin(\theta) + \frac{\partial f}{\partial y}\cos(\theta)$

   To compute : $\frac{\partial^2 f}{\partial y'^2}$, then : $g(x) = -\frac{\partial f}{\partial x}\sin(\theta) + \frac{\partial f}{\partial y}\cos(\theta)$

   We must compute : $\frac{\partial^2 f}{\partial y'^2} = \frac{\partial g}{\partial y'} = -\frac{\partial g}{\partial x}\sin(\theta) + \frac{\partial g}{\partial y}\cos(\theta)$

   $= \frac{\partial^2 f}{\partial x^2}\sin^2(\theta) - \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\cos(\theta)\sin(\theta) - \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\cos^2(\theta)$

   Adding the two resulting expressions for x and for y, we get :

   $\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
   The Laplacien is then isotropic.
2. Same method, same conclusion.
3. Same method, not the same conclusion.

Answer of exercise 9

1. The considered filter is a 4-connectivity average filter where the pixel in the middle is set to zero. In a 3×3 neighborhood, one can write analytically the result of a convolution between the filter H and the function f : $g(x,y) = \frac{1}{4}(f(x,y+1)+f(x+1,y)+f(x-1,y)+f(x,y-1))$

   We use the translation property of the Fourier transform : $\Im\{f(x-x_0)\} = F(u)\exp(-2\pi jux_0/N)$

   In 2D : $\Im\{f(x-x_0, y-y_0)\} = F(u)\exp(-2\pi j(ux_0/M + vy_0/N))$

   Then : $G(u,v) = \frac{1}{4}(\exp(2j\pi v/N)+\exp(2j\pi u/M)+\exp(-2j\pi u/M)+\exp(-2j\pi v/N))F(u,v)$

   $G(u,v) = H(u,v)F(u,v)$ with :

   $H(u,v) = \frac{1}{2}(cos(2\pi u/M) + cos(2\pi v/N))$

   This function admits a maximum in $u = 0$ and $v = 0$ (it is equal to 1. In $(u,v) = (-1,-1)$ and in $(u,v) = (1.0)$, it is equal to $-1/2$ and $1/4$ resp. The other values are deduced by symmetry, so the filter has the typical shape of a low pass.

Answer of exercise 10

The considered filter is a 4-connectivity Laplacian filter. On a 3×3 neighborhood , we can write the result of the convolution between the filter H and the function f : $g(x,y) = -f(x,y+1) - f(x+1,y) - f(x-1,y) - f(x,y-1) + 4f(x,y)$

We use the translation property of the Fourier transform : $\Im\{f(x-x_0)\} = F(u)\exp(-2\pi jux_0/N)$

In 2D : $\Im\{f(x-x_0, y-y_0)\} = F(u)\exp(-2\pi j(ux_0/M + vy_0/N))$

Donc : $G(u,v) = (\exp(2j\pi u/M)+\exp(-2j\pi u/M)+\exp(2j\pi v/N)+\exp(-2j\pi v/N)-4)F(u,v)$

$G(u,v) = H(u,v)F(u,v)$ avec :

$H(u,v) = 2(cos(2\pi u/M) + cos(2\pi v/N) - 2)$

This function admits a zero in $u = 0$ and $v = 0$ (it is equal to 0. In $(u,v) = (-1,-1)$ and in $(u,v) = (1.0)$ This is typical of a high pass, so the filter has the typical shape of a high pass.

Answer of exercise 11

Let us consider the 4-connectivity Laplacian :
$f(x,y) - \nabla^2 f(x,y) = f(x,y) - [-f(x+1,y) - f(x-1,y) - f(x,y+1) - f(x,y-1) + 4f(x,y)]$
$f(x,y) - \nabla^2 f(x,y) = -3f(x,y) + f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)$

$f(x,y) - \nabla^2 f(x,y) = -4f(x,y) + f(x,y) + f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)$

$f(x,y) - \nabla^2 f(x,y) = -4f(x,y) + 5\bar{f}(x,y)$
where : $\bar{f}(x,y) = \frac{f(x,y)+f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)}{5}$

$f(x,y) - \nabla^2 f(x,y) = -5(\frac{4}{5}f - \bar{f}) \approx f - \bar{f}$

Recall :

- If $y = -x$ then $dy = -dx$ and $\int_{-\infty}^{+\infty} dx = -\int_{+\infty}^{-\infty} dy = \int_{-\infty}^{+\infty} dy$
- $\exp(x + y) = \exp(x)\exp(y)$
- $\exp(x)^a = \exp(ax)$
- $\frac{1}{\exp(x)} = \exp(-x)$
- Gauss Integral : $\int \exp(-\alpha x^2)dx = \sqrt{\frac{\pi}{\alpha}}$
- $\int \exp(ax)dx = \frac{1}{a}\exp(ax)$
- $\exp(i\theta) = \cos(\theta) + j\sin(\theta)$
- $\cos(\theta) = \cos(-\theta)$ and $\sin(\theta) = -\sin(-\theta)$
- $\cos(\theta) = \frac{1}{2}(\exp(j\theta) + \exp(-j\theta))$
- $\sin(\theta) = -\frac{j}{2}(\exp(j\theta) - \exp(-j\theta))$