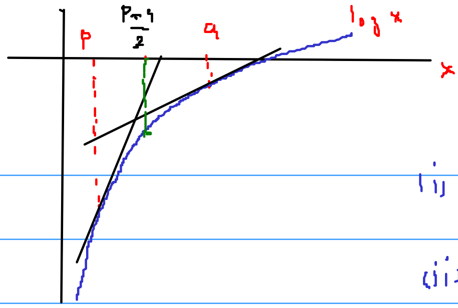


$\log \frac{p+q}{2} \le \log p + \frac{q-p}{2} \cdot \frac{1}{p}$ ← $f'(p)$ (By convexity)

(i) $\times\, p$ $\quad p \log \frac{p+q}{2} \le p \log p + \frac{q-p}{2}$

(ii) $\times\, q$ $\quad q \log \frac{p+q}{2} \le q \log q - \frac{q-p}{2}$

(i) + (ii) $\quad (p+q) \log \frac{p+q}{2} \le p \log p + q \log q$

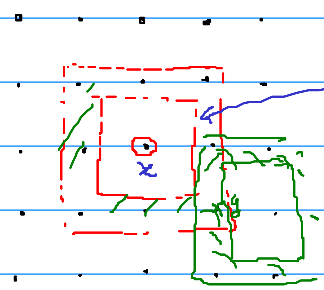
$\left\{ Z = \varepsilon X + (1-\varepsilon) Y \right.$ $\quad 0 < \varepsilon < 1$

$\left\{ Z = \delta X + (1-\delta) Y \right.$, $\quad \delta \in \{0,1\}$

$\sum_i \pi_i \log 1/\pi_i$ $\qquad \frac{p_i + q_i}{2}$ (X → p_i, Y → q_i) $\qquad \varepsilon p_i + (1-\varepsilon) q_i$

Ex-3

$\mathcal{X}$ ↘



N(x)

$|N(x)| = d(x)$

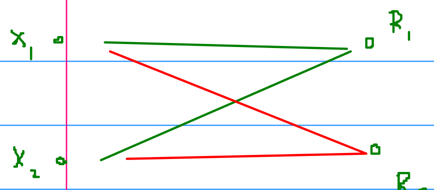
$p_{XY}(x,y) \xrightarrow[\sum_x]{?} p_Y(y)$ Marginalization

$= \sum_x p_{XY}(x,y)$

$p_{Y|X}(y|x)\, p_X(x)$

$e \in \{0, 1\}$

$$\left\{ \begin{aligned} p(y|x) &= \sum_e p(y|x,e) \cdot p(e) \\ &= p(y|x,e=0)\,p(e=0) + p(y|x,e=1)\,p(e=1) \end{aligned} \right.$$



$I(R_1, R_2)$ --→ $\hat{X}_1 = f(R_1, R_2)$, $\hat{X}_2 = g(R_1, R_2)$: $\left\{ \begin{array}{l} I(\hat{X}_1, \hat{X}_2) \simeq 0 \\ f, g \end{array} \right.$

Independent components Analysis

ICA

5/ $|\mathcal{X}| = 26$ $\quad \varepsilon = 0.1$ $\quad d = 4$ → $H(X) \approx 4.6$ bits , $I(X,Y) = 4$ bits → $\{ 0.6$ bits

1/ MAX ENTROPY EST. of PSD (finish)

2/ Markov chains and Hidden Markov Chains. (Rabiner + more)

$\downarrow$ Stochastic Optimization $\qquad\qquad$ $\to$ Viterbi, BCJR, KALMAN

1/ $\underbrace{\{X_1, \dots, X_N\}}$ $\subseteq \{X_1, X_2, \dots\}$ stationary with $c_k \triangleq \mathbb{E}[X_n X_{n \pm k}]$

$$\downarrow \mathcal{F}$$

$$\Gamma(\lambda) = \sum_{k \in \mathbb{Z}} c_k e^{-i2\pi\lambda k}$$

Max Ent. criterium $\longrightarrow$

Fit Gauss-Markov Model to available data

$$\underline{X_n = \alpha_1 X_{n-1} + \dots + \alpha_{\underline{m}} X_{n-m} + u_n} \quad , \quad u_n \text{ Normal, indep}$$

Auto-Regressive order m

$$\varepsilon_n(\underline{\alpha}) = X_n - \sum_{k=1}^{m} \alpha_k X_{n-k} = X_n - \underline{\alpha}^T \cdot X_{n-1:n-m}$$

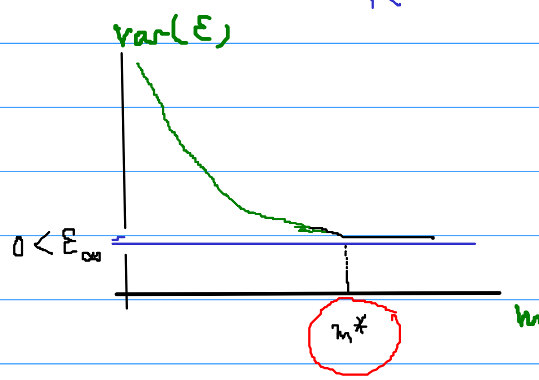
$$\alpha^T \Sigma V_n \cdot V_n^T \cdot \alpha$$

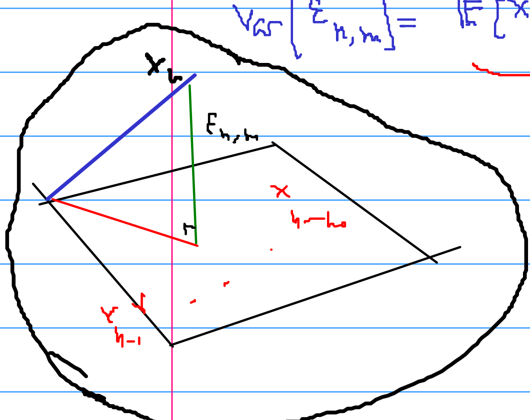
$$\mathbb{E}\,|\varepsilon_n(\underline{\alpha})|^2 \simeq \frac{1}{N-m-1}\sum_{n=m+1}^{N} \left\{ X_n^2 - 2\underline{\alpha}^T \cdot X_n \cdot X_{n-1:n-m} + (\alpha^T \cdot X_{n-1:n-m})^2 \right\}$$

$$= \underbrace{\frac{1}{N-m-1}\sum x_n^2}_{\simeq\, \mathbb{E}[X_n^2]} - 2\alpha^T \underbrace{\overbrace{\frac{1}{N-m-1}\sum x_n \cdot X_{n-1:n-m}}^{\hat{\varphi}}}_{\underbrace{\mathbb{E}[X_n \cdot X_{n-1:n-m}]}_{\varphi}} + \alpha^T \underbrace{\overbrace{\frac{1}{N-m-1}\sum X_{n-1:n-m} \cdot X^T_{n-1:n-m}}^{\hat{R}}}_{\underbrace{\mathbb{E}[X_{n-1:n-m} \cdot X^T_{n-1:n-m}]}_{R}} \alpha$$

$$\Rightarrow \quad \overline{\hat{\underline{\alpha}}_m = \hat{R}_m^{-1}\hat{\varphi}_m}$$

$$\mathrm{Var}[\varepsilon_{n,m}] = \underbrace{\mathbb{E}[X_n^2] - \varphi_m^T R_m^{-1} \varphi_m}_{\hat{\sigma}_m^2}$$



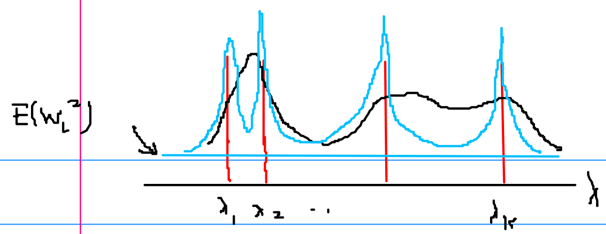


$$\varepsilon_{n,m} \perp\!\!\!\perp X_{n-1}, \dots, X_{n-m}.$$

$$\varepsilon_{n,m^*} \perp\!\!\!\perp \{X_{n-1}, X_{n-2}, \dots\}$$

$$X_n = \alpha_1 X_{n-1} + \dots + \alpha_m X_{n-m} + u_n$$

$$\hat{\Gamma}(\lambda) = \frac{\hat{\sigma}_m^2}{\left|1 - \sum_{k=1}^{m} \hat{\alpha}_k e^{-i2\pi k\lambda}\right|^2}$$



$x_n = \sum \text{Sinusoids} + w_n$

$X_n = a_1 X_{n-1} + \cdots + a_m X_{n-m} + U_n$

U_n: iid $\sim N(0, \sigma^2)$

$x_1, \ldots, x_N$

$y_n = f(x_n, w_n)$

$\downarrow$

$\hat{x}_n \simeq x_n$

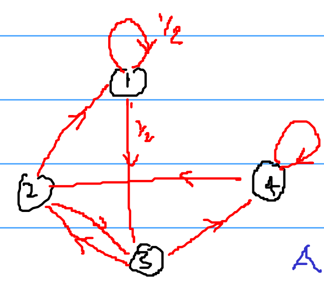
- $X_n \in \mathbb{R}$ — Gaussian → Kalman Filter
- $x_n \in \mathcal{S}$ finite → Viterbi, BCJR ... EM Alg. - Baum-Welsh.

* Asymptotic Behavior of Markov Chains: ($\leftrightarrow$ Ergodicity)

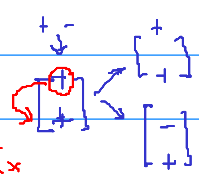
$u_n \in \{-1, +1\} \to \boxed{h_0, h_1} \to z_n = h_0 u_n + h_1 u_{n-1} \xrightarrow{w_n} y_n = z_n + w_n \qquad \pm 1 \quad y_2$

$z_{n+1} = h_0 u_{n+1} + h_1 u_n$

$$X_n = \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} \in \left\{ \underset{(1)}{\begin{bmatrix} + \\ + \end{bmatrix}}, \underset{(2)}{\begin{bmatrix} + \\ - \end{bmatrix}}, \underset{(3)}{\begin{bmatrix} - \\ + \end{bmatrix}}, \underset{(4)}{\begin{bmatrix} - \\ - \end{bmatrix}} \right\} = \mathcal{S}$$



$P(X_n = j \,/\, X_{n-1} = i) \qquad i, j \in \mathcal{S}$



$= A_{ij}$ Transition Matrix

$$A = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

$\mathbb{P}[X_n = i_n \,/\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}]$ MARKOV CHAIN

$\mathbb{P}[X_0]$ $\qquad p_i = \mathbb{P}[X_0 = i]$

$\mathbb{P}[X_n = j] = \sum_i \mathbb{P}[X_n = j, X_{n-1} = i]$
$j \in \mathcal{S}$

$= \sum_{i \in \mathcal{S}} \mathbb{P}[X_n = j \,/\, X_{n-1} = i]\, \mathbb{P}[X_{n-1} = i]$

$\mathbb{P}[X_n = j] = \underline{p_n(j)} = \sum_{i \in \mathcal{S}} p_{n-1}(i) A_{ij}$

$j \in \mathcal{S}$ $\qquad \boxed{\underline{p}_n} = \underline{p}_{n-1} \cdot A = \cdots = \boxed{\underline{p}_0 \cdot A^n}$

$0 \leq A_{ij} \leq 1$

$\sum_j A_{ij} = 1$

Ergodic Case: $\forall \underline{p}_0, \quad \underline{p}_0 A^n \longrightarrow \underline{p}_\infty$

$Z_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$

$\begin{bmatrix} x_n = f(x_{n-1}, x_{n-2}) \\ x_{n-1} = f(x_{n-2}, x_{n-3}) \end{bmatrix}$

$$\underbrace{X_n}= \alpha_1 X_{n-1} + \dots + \alpha_m X_{n-m} + U_n .$$

$$Z_n = \begin{bmatrix} X_{n-m+1} \\ \vdots \\ X_n \end{bmatrix}$$

with order

mark.

$$Z_n = \begin{bmatrix} 0 & 1 & 0 & - \\ & \ddots & \ddots & 1 \\ \alpha_m & \cdots & & \alpha_1 \end{bmatrix} \underbrace{\begin{bmatrix} X_{n-m} \\ \\ X_{n-1} \end{bmatrix}}_{Z_{n-1}} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ U_n \end{bmatrix}}_{q_n}$$

$$\forall \pi_0 \ : \ \pi_0 A^n \xrightarrow{n\infty} \pi_\infty$$

$$\pi_\infty A = \pi_\infty$$

$\rightarrow \pi_\infty$ invariant distribution

$$P_n(i) = \mathbb{P}[k_n = i] \quad i = 1:r$$

$$\pi_n = \pi_{n-1} \cdot A = \pi_0 A^n$$

Def

A is { ergodic / irreducible } if $\exists s \quad A^{(s)}_{ij} > 0$

π is stationary or invariant if $\pi P = \pi$

Ergodic theorem

If A is Ergodic, an invariant probability π exists

$$\lim_{n\infty} A^{(n)}_{ij} = \pi_j$$