Exercises on Spectral Analysis

1/ (Wiener-Khinchin formula)

Given a 2nd order signal $(X(n), n \in \mathbb{Z})$ and a sequence of weight functions $W_N(n), N = 1, 2, ...$ consider the sequence of Fourier transformations

$$\tilde{X}_N(\lambda) = \frac{1}{\sqrt{N}} \sum_n X(n) W_N(n) \exp(-i2\pi\lambda n)$$

and its mean square limit $\tilde{X}(\lambda) = \lim_{N\to\infty} \tilde{X}_N(\lambda)$.

1.1 Under which conditions $\tilde{X}(\lambda)$ and $\tilde{X}(\lambda')$ are uncorrelated for $\lambda \neq \lambda'$?

1.2 Show that $E|\tilde{X}(\lambda)|^2 = \mathit{\Gamma}(\lambda)$ where $\Gamma(.)$ is the spectral density of $X(.)$.

2/ (Interferences formula)

Let $X(n)$ and $Y(n)$ be resp. the input and output sequences of a stationary linear system with the transfer function $H(\lambda)$. Show that their spectral densities $\mathit{\Gamma}_X$ and $\mathit{\Gamma}_Y$ verify the relation

$$\mathit{\Gamma}_Y(\lambda) = \mathit{\Gamma}_X(\lambda)|H(\lambda)|^2.$$

3/ (Phase shifts)

During its transmission, a continuous-time stationary signal $X(t)$ is subject to randomly occuring phase shifts at Poisson times $t_1, t_2, ...$ of a Poisson Process $N(t)$ such that the received signal is

$$Y(t) = X(t + \Phi + \varphi(t))$$

with

$$\varphi(t) = \sum_{\{t_k<t\}} \varphi_k$$

where $\varphi_k$are i.i.d. with known characteristic function $g(u) = E(\exp iu\varphi_k)$ and $\Phi$is a constant r.v. equidistributed on $(0, 2\pi)$ independent of the φ_k . The underlying Poisson process has intensity μ.

3.1 Consider first a single harmonic component $X(t) = \exp(i2\pi\lambda_0 t)$. Calculate $E(Y(t))$ and the correlation function $\gamma_Y(t, t-\tau) = E[Y(t)Y^*(t-\tau)]$.

3.2. Compute the spectral density $\mathit{\Gamma}_Y(\lambda)$ and analyze the effect of μ on the line shift and widening.

3.3 Supposing that the statistics of X, N, Φ, φ_k are independent calculate the above functions with the random signal $X(t)$ sum of its harmonics.