## Exercises on second order statistics

Note: all random vectors are assumed of finite variance and zero mean.

1. For any two random vectors $X, Y$ compute the minimum error of the linear estimation of $X$ from $Y$:

$$\sigma^2_{min} = \underset{G}{min} E \parallel X - GY \parallel^2$$

2. Suppose $Y = HX + W$ with $(W, X)$ independent. Compute

$$\hat{G} = \underset{G}{argmin} E \parallel X - GY \parallel^2$$

3. Take $X = X_{1:N}$ , $W = W_{1:N}$ where the sequence $(X_n)$ is binary $\pm 1$ i.i.d., $(W_n)$ is gaussian with variance $\beta$. Give the expression of $\hat{G}$ in this case .

4. Suppose $(X, Y) \sim N(0, \Gamma)$ with

$$\Gamma = \begin{bmatrix} R_X & R_{XY} \\ R_{YX} & R_Y \end{bmatrix}$$

Show that $X|Y \sim N(\hat{X}, R_e)$ with $\hat{X} = R_{XY} R_Y^{-1} Y$ and $R_e = R_X - R_{XY} R_Y^{-1} R_X$
.

5. Now $(X_n)$ is a Markov sequence such that $X_n = -X_{n-1}$ with probability $\varepsilon$ independently from $X_{1:n-1}$. Compute the auto-correlation function $\gamma_X(k) = E(X_n X_{n-k})$. Explicit the structure of the correlation matrix $R_X$. Compute the spectrum density function of $(X_n)$.

6. Denote $L_{X|Y} = E(XY^T)E(YY^T)^{-1}$. Prove the following properties and interpret in terms of estimation:
(a) $L_{AX|Y} = AL_{AX|Y}$
(b) $L_{X|AY} = L_{X|Y}$
(c) If $Y = (Y_1, Y_2)$ with $(Y_1, Y_2)$ uncorrelated, $L_{X|Y} = L_{X|Y_1} + L_{X|Y_2}$.

7. The sequence $(X_n)$ is AR-m:

$$X_n = a_1 X_{n-1} + ... + a_m X_{n-m} + U_n$$

where $(U_n)$ is i.i.d . Compute the spectrum density of $(X_n)$.