Image : just a 2D signal and nothing else...

Image and signal processing

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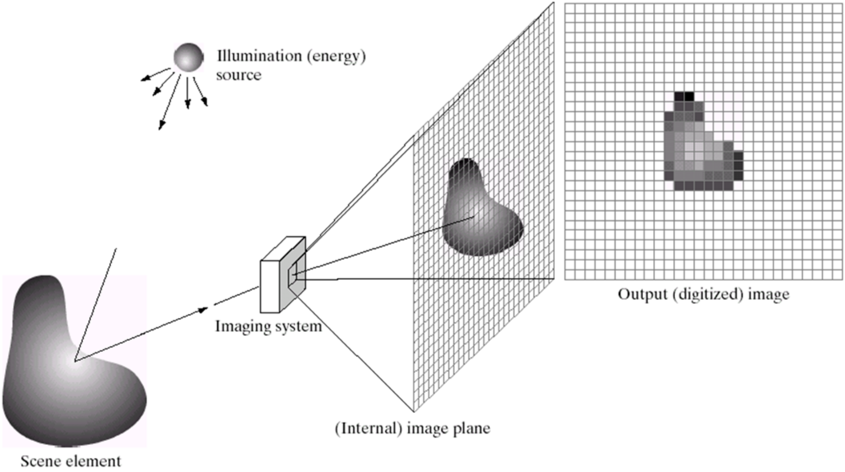
2D Signal

Several sensors to get a 2D or 3D image

Physical phenomenon	What is measured	Sensor
Emission, reflection of light	Reflectance, luminance	CCD, CMOS
Infrared radiation	Heat	Bolometer
Ultrasonic echo	Distances, densities	Ultrasound, Sonar
Magnetic resonance	Presence of a chemical body	MRI, NMR
Electromagnetic echo	Distance, specularity	Radar, SAR
X-ray absorption	density	Radiography, tomography

An image is a set of values called pixels (PICture Elements)



- **4 components :**
  1. Optics
  2. Conversion of light energy into electrical energy
  3. Sampling
  4. Formatting and storage

1+2 : **camera** 3+4 : **numerization card** = sampling + quantification

- Linearity
- Sensitivity
- Spectral Sensitivity
- Integration time
- Resolution

- Definition

$$c = R\{w_1 a + w_2 b\} = w_1 R\{a\} + w_2 R\{b\}$$

- Non-linear sensor

$$c = gain \times a^{\gamma} + offset$$

R : sensor response

a, b : Physical input signals (photons)

c : output signal (voltage)

- The human eye does not perceive gray levels linearly.

- Absolute Sensitivity
  - Minimum number of detectable photons.
- Quantum efficiency
  - Ratio between the number of electrons produced and the number of photons received.
- Relative Sensitivity
  - Number of photons needed to go from one level to the next level.

Example

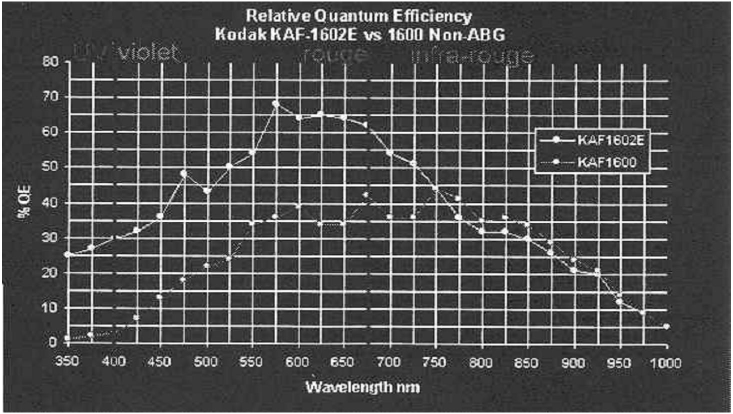
Scientific CCD : 16 photons from one level to the next one.

Example

Standard CCD : 512 photons separate one level from the next.

Acquisition, storage and representation of an image

Spectral Sensitivity



- **Definition**
  - Photon collect time

Example

Video Cameras : 33.37ms (NTSC) and 40.0 ms (PAL, SECAM)

Example

Scientific Cameras : from 500 ns to 1hour !

1. Spatial Resolution = number of raws $\times$ number of columns
2. Dynamic range $[\min, \max]$ of level gray variation
3. Number of bits needed to store a gray level value

	Symbol	Typical values
Raws	N	256,512,525,625,1024,1035
Columns	M	256,512,768,1024,1320
Level	L	2,64,256,1024,4096,16384

- In most of cases,

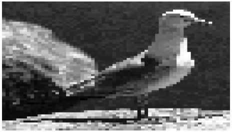
$$M = N = 2^k \text{ with } k = 8, 9, 10, etc.$$
$$L = 2^B \text{ with } B : \text{number of bits}$$

Acquisition, storage and representation of an image

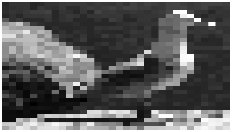
Example when varying spatial resolution

256×256

128×128



64×64



32×32

A too low spatial resolution induce Aliasing.

Acquisition, storage and representation of an image

Example when varying the number of bits

8, 7, 6, 5, 4, 3, 2 et 1 bit to represent a gray level.

Image file

- Header (N bytes) : image format, resolution, number of bits to code a level, comments, pixel organization
- Data : pixel values

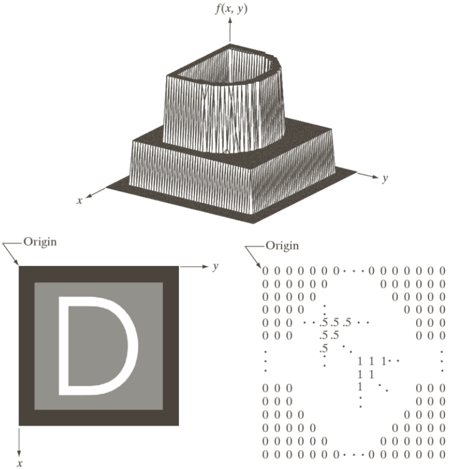
- **Known extensions :** tiff, pcx, gif, pict, png, raw, ppm, bmp, xbm, xwd, sun raster, etc.

- **Two categories**
  1. Formats with compression
     a - Lossy (exploit spatial or temporal redundancy)
     b - Without loss
  2. Formats without compression
- Example : PPM format
  - PBM : P1 or P4
  - PGM : P2 or P5
  - PNM : P3 or P6

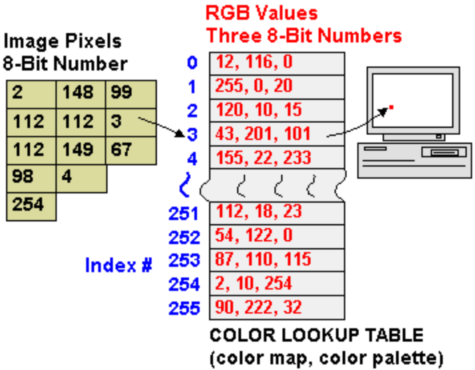
```
P2
CREATOR: The GIMP's PNM Filter Version 1.0
256 256
255
137 136 133 136 138 134 134 132 132 138 129 131....
```

- **Several possible representations**
  1. Matrix $\longrightarrow$ *Linear algebra*
  2. 2D Signal$\longrightarrow$ *filtering in the frequency domain*
  3. Continuous function $f(x, y)$ of two variables x and y $\longrightarrow$ *derivation*
  4. Points clouds$\longrightarrow$ *statistics*
  5. Set of points $\longrightarrow$ *Mathematical morphology*

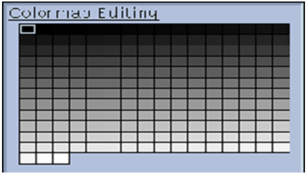
Acquisition, storage and representation of an image
2D image Representation



1. 3D Representation
   - 3 color maps
2. Indexed Representation
   - Image (index) + colormap (3 columns)



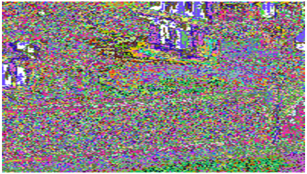
- **Change of colormap**
  - A gray level image with its colormap :



Acquisition, storage and representation of an image

Color image representation

- **Change of colormap**
  - Random colormap



Acquisition, storage and representation of an image

Relations between pixels - Neighborhood

- 4-connectivity

	●	
●	$\underline{\bullet}$	●
	●	

	$(x-1,y)$	
$(x,y-1)$	(x,y)	$(x,y+1)$
	$(x+1,y)$	

- 8-connectivity

●	●	●
●	$\underline{\bullet}$	●
●	●	●

$(x-1,y-1)$	$(x-1,y)$	$(x-1,y+1)$
$(x,y-1)$	(x,y)	$(x,y+1)$
$(x+1,y-1)$	$(x+1,y)$	$(x+1,y+1)$

Let $p(x,y)$, $q(s,t)$ and $z(v,w)$. A function D is a distance metric if :

1. $D(p,q) \geq 0$ ($D(p,q) = 0$ iff $p = q$)
2. $D(p,q) = D(q,p)$
3. $D(p,z) \preceq D(p,q) + D(q,z)$

- Euclidean Distance : $D_e(p,q) = \sqrt{(x-s)^2 + (y-t)^2}$
- D_4 Distance or "City-block" or Manathan :
  $D_4(p,q) = |x-s| + |y-t|$
- D_8 Distance or "Chessboard" : $D_8(p,q) = \max(|x-s|, |y-t|)$

- Example 1 : pixels with $D_4 \preceq 2$ from (x, y)

		2		
	2	1	2	
2	1	0	1	2
	2	1	2	
		2		

- Example 2 : pixels with $D_8 \preceq 2$ from (x, y)

2	2	2	2	2
2	1	1	1	2
2	1	0	1	2
2	1	1	1	2
2	2	2	2	2

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Frequency approach in image processing

A digital image is defined as a two-dimensional finite signal sampled with quantized values in a given color space. So :

- an image has finite dimensions (finite signal) ;
- an image is characterized by two spatial dimensions (two-dimensional signal) ;
- the pixels are generally arranged in a rectangular grid (sampled signal) ;
- the pixel values belong to a discrete space (quantized values).

2D discrete Fourier Transform

Objective : deal with the image as a signal and nothing else

- Two-dimensional discrete Fourier transform
- Fourier transform of basic images
- Fast Fourier transform
- Frequency domain filtering

- **Useful recalls**

$$
\begin{aligned}
&C = R + jI \\
&C = |C| (\cos\theta + j\sin\theta) \text{ with } |C| = \sqrt{R^2 + I^2} \text{ and } \theta = \arctan(I/R) \\
&C^* = R - jI \\
&e^{j\theta} = \cos\theta + j\sin\theta \text{ with } j^2 = -1 \\
&C = |C| e^{j\theta} \\
&\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\
&\sin\theta = \tfrac{-j}{2}\left(e^{j\theta} - e^{-j\theta}\right)
\end{aligned}
$$

Dirac distribution

- **Continuous case :**

$$\delta(x) = \begin{cases} \infty \text{ si } x = 0 \\ 0 \text{ si } x \neq 0 \end{cases} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x)dx = 1$$

If f is a continuous function on $\Re$, we can prove that :

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$$

Similarly :

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

Let $f(x)$ be a continuous function of variable x, defined on $\Re$ and summable (i.e. $\int_{-\infty}^{+\infty} |f(x)|\, dx$ is finite).

- **Direct Fourier Tranform of** $f(x)$:

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{+\infty} f(x) \exp[-j2\pi ux]\, dx$$

- **Inverse Fourier Transform of** $F(u)$:

$$\Im^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{+\infty} F(u) \exp[j2\pi ux]\, du$$

$F(u) = R(u) + jI(u) = |F(u)|\, e^{j\Phi(u)}$
$|F(u)| = \sqrt{R^2(u) + I^2(u)}\,;\ \Phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$

$|F(u)|$: Fourier spectrum of $f(x)$
$\Phi(u)$: Phase and $E(u) = |F(u)|^2$: Energy spectrum

- **Function F is bounded :**

1. $|F(u)| = \left|\int_{-\infty}^{+\infty} f(x) \exp\left[-j2\pi ux\right] dx\right| \leq \int |f(x)| \, dx$ because $|\exp\left[-j2\pi ux\right]| = 1$.
2. F is continuous on $\Re$
3. $\lim_{|u| \to \infty} |F(u)| = 0$ (the spectrum of f has the Ou axis as an asymptote)

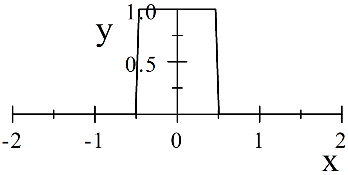
- **Fourier Transform of $f(x,y)$:**

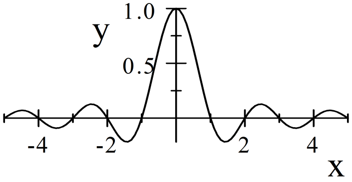
$$\Im\left\{f(x,y)\right\} = F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\exp\left[-j2\pi(ux+vy)\right]dxdy$$

- **Inverse Fourier Transform of $F(u,v)$:**

$$\Im^{-1}\left\{F(u,v)\right\} = f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)\exp\left[j2\pi(ux+vy)\right]dudv$$

- **Rectangular pulse** : $f(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2} \end{cases}$





$$F(u) = \int_{-\infty}^{+\infty} f(x) \exp\left[-j2\pi ux\right] dx$$
$$= \int_{-1/2}^{+1/2} \exp\left[-j2\pi ux\right] dx$$
$$= -\frac{1}{j2\pi u} \left[\exp(-j2\pi ux)\right]_{-1/2}^{1/2}$$
$$= -\frac{1}{j2\pi u} \left[\exp(-\pi ju) - \exp(\pi ju)\right]$$

Since : $\frac{\exp(-\pi uj)-\exp(\pi uj)}{2j} = \sin(\pi u) \Rightarrow F(u) = \frac{1}{\pi u}\left[\sin(\pi u)\right] = sinc(\pi u)$

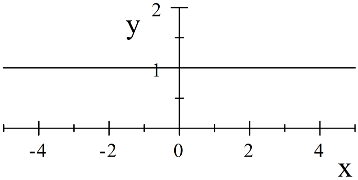
FT of basic functions

Fourier Transform of a centred impulse function

- **Impulse at origin**

$$F(u) = \int_{-\infty}^{+\infty} \delta(x) \exp\left[-j2\pi ux\right] dx$$

$F(u) = \exp\left[-j2\pi u0\right] = 1 \Longrightarrow$ constant
$F(u) = 1$



- $\Im(\delta(x - x_0)) = \exp\left(-2\pi jux_0\right)$

- **Sin and Cos functions**

$\Im\left(\cos(2\pi u_0 x)\right) = \frac{1}{2}\left[\delta\left(u - u_0\right) + \delta\left(u + u_0\right)\right]$
$\Im(\sin(2\pi u_0 x) = \frac{j}{2}\left[\delta\left(u - u_0\right) - \delta\left(u + u_0\right)\right]$
car $\Im(\exp(2\pi j u_0 x) = \delta\left(u - u_0\right)$

- **Gaussian**
  $f(x) = \exp(-\pi x^2)$
  $F(u) = \int_{-\infty}^{+\infty} \exp(-\pi x^2) \exp(-j2\pi u x) dx$
  $= \exp(-\pi u^2) \exp(\pi u^2) \int_{-\infty}^{+\infty} \exp(-\pi x^2) \exp(-j2\pi u x) dx$
  $= \exp(-\pi u^2) \int_{-\infty}^{+\infty} \exp(-\pi (x + ju)^2) dx$
  $= \exp(-\pi u^2) \int_{-\infty}^{+\infty} \exp(-\pi w^2) dw$ with $w = x + ju$
  Since $\int_{-\infty}^{+\infty} \exp(-\pi w^2) dw = 1$

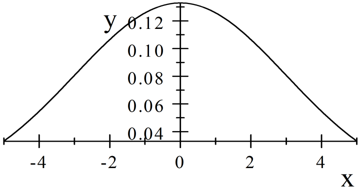
$$F(u) = \exp(-\pi u^2)$$

$$\Im \left\{ \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2}) \right\} = \sigma\sqrt{2\pi} \exp(-2\pi^2\sigma^2 u^2)$$

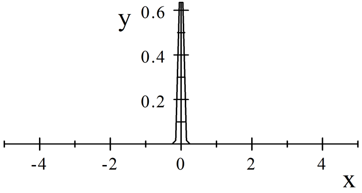
- **Gaussian**

$\sigma = 3$

$f(x) = \left[\frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})\right]_{\sigma=3} = \frac{1}{6}\frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{1}{18}x^2}$



$F(u) = \left[\exp(-2\pi^2\sigma^2u^2)\right]_{\sigma=3} = e^{-18\pi^2u^2}$



- **Laplacian**

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\Im\left\{\nabla^2 f(x,y)\right\} \leftrightsquigarrow (2\pi)^2(u^2+v^2)F(u,v)$$

- **1D Convolution**

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

$$f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$$

- Important property of convolution :

$$\Im\{f(x) * g(x)\} = F(u) \cdot G(u)$$

$$\Im\{f(x) \cdot g(x)\} = F(u) * G(u)$$

Recall about the Dirac distribution

- **Dirac Distribution (discrete case)**

$$\delta(x) = \begin{cases} 1 \text{ si } x = 0 \\ 0 \text{ si } x \neq 0 \end{cases} \quad \text{with} \quad \sum_{-\infty}^{+\infty} \delta(x) = 1$$

If f is continuous on $\Re$,, we can prove that :

$$\sum_{-\infty}^{+\infty} f(x)\delta(x) = f(0)$$

In the same way : :

$$\sum_{-\infty}^{+\infty} f(x)\delta(x - x_0) = f(x_0)$$

- **Impulse train**

Let x be a discrete variable, we define :

$$\text{Ш}(x) = \sum_{n=-\infty}^{n=+\infty} \delta(x-n)$$

In the case of an impulse train of period T :

$$\Im\left(\sum_{n=-\infty}^{n=+\infty} \delta(x-nT)\right) = \frac{1}{T}\sum_{n=-\infty}^{n=+\infty} \delta(u-\frac{n}{T})$$

- **Sampled Function**

$f_e(t) = f(t) \amalg_{\Delta T} (t) = \sum_{n=-\infty}^{n=+\infty} f(t)\delta(t - n\Delta T)$

- **Fourier Transform**

$\Im\{f_e(t)\} = F_e(u) = \Im\{f(t) \amalg_{\Delta T} (t)\} = F(u) * \Im\{\amalg_{\Delta T}(t)\}$. Let $\Im\{\amalg_{\Delta T}(t)\} = S(u)$, then :

$$F_e(u) = \int F(\tau)S(u-\tau)d\tau = \frac{1}{\Delta T}\int F(\tau)\sum_{n=-\infty}^{n=+\infty}\delta(u-\tau-\frac{n}{\Delta T})d\tau$$

$$= \tfrac{1}{\Delta T}\textstyle\sum_{n=-\infty}^{n=+\infty}\displaystyle\int F(\tau)\delta(u-\tau-\frac{n}{\Delta T})d\tau = \frac{1}{\Delta T}\sum_{n=-\infty}^{n=+\infty}F\left(u-\frac{n}{\Delta T}\right)$$

$$\Im\{f_e(t)\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{n=+\infty}F\left(u-\frac{n}{\Delta T}\right)$$

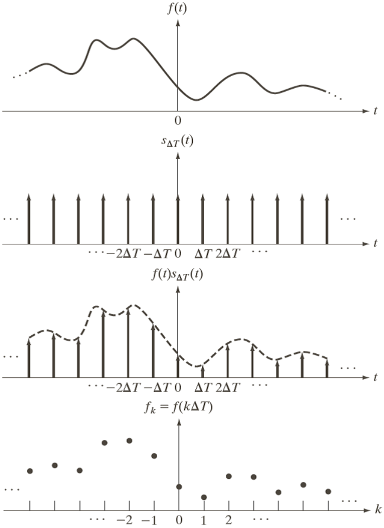
Fourier Transform of a sampled function

$$\Im\left\{f_e(t)\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{n=+\infty} F\left(u - \frac{n}{\Delta T}\right)$$

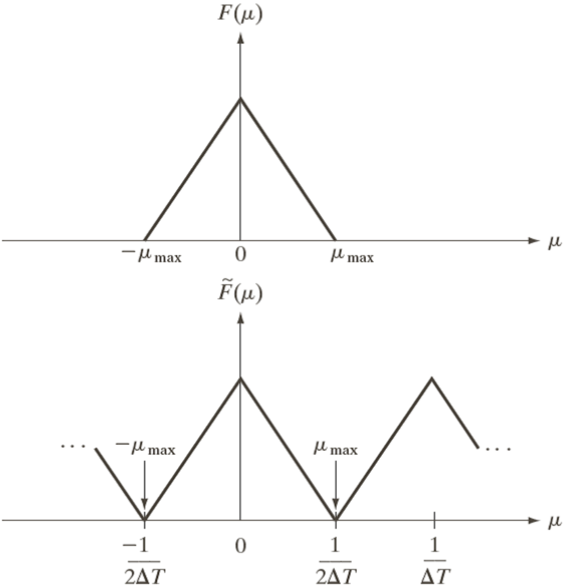
Let $[-\mu_{\max}, \mu_{\max}]$ the support of $F(u)$ ($F(u) = 0$ if $u \notin [-\mu_{\max}, \mu_{\max}]$).
The graphic of $F_e(u)$ is then composed by translations of $\frac{n}{\Delta T}$ from the graphic of $F(u)$.
The sub-graphics are disjoints if :

$$\frac{1}{\Delta T} \geq 2\mu_{\max}$$

Fourier Transform of a sampled function



Fourier Transform of a sampled function



Discrete Fourier Transform

Let $f(x)$ with N samples :
$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), ..., f(x_0+[N-1]\Delta x)\}$.
Let $f(x) = f(x_0 + x\Delta x)$.

- **Discrete Fourier Transform (DFT) :**

$$\Im\{f(x)\} = F(u) = \sum_{x=0}^{N-1} f(x) \exp\left[\frac{-j2\pi ux}{N}\right]_{\text{pour } u=0,1,2,...,N-1}$$

- **Inverse Discrete Fourier Transform (IDFT) :**

$$\Im^{-1}\{F(u)\} = f(x) = \frac{1}{N}\sum_{u=0}^{N-1} F(u) \exp\left[\frac{j2\pi ux}{N}\right]_{\text{pour } x=0,1,2,...,N-1}$$

2D Discrete Fourier Transform

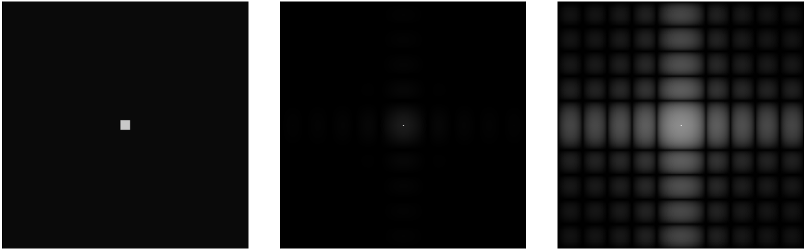
- **2D DFT :**

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp\left[-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right]_{\substack{u=0,1,2,\ldots,M-1\\ v=0,1,2,\ldots,N-1}}$$

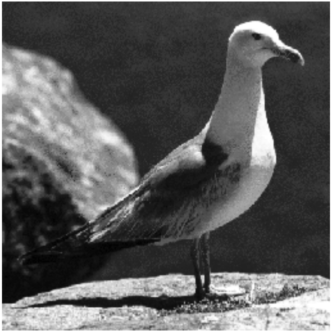
- **2D IDFT :**

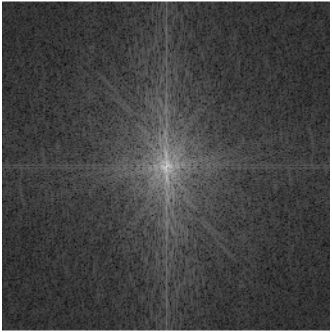
$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp\left[j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right]_{\substack{x=0,1,2,\ldots,M-1\\ y=0,1,2,\ldots,N-1}}$$

- **Visualization :** we display $|F(u,v)|$. High frequencies have lower values than low frequencies. We rather display : $k \log(1 + |F(u,v)|)$. k allows to normalize between 0 and 255.



- **Interpretation of the frequency domain contain : case of real images**

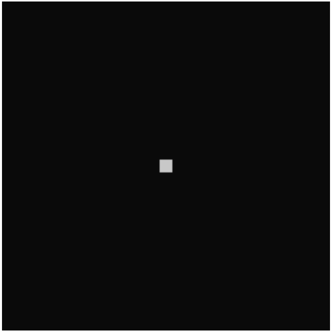


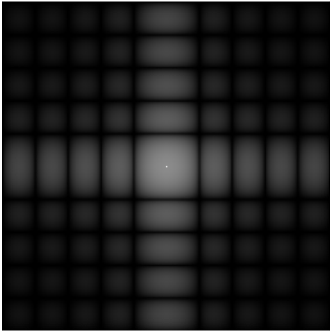


DFT of basic images

Rectangular impulse

- **DFT of a rectangle function**





DFT of basic images

Rectangle function

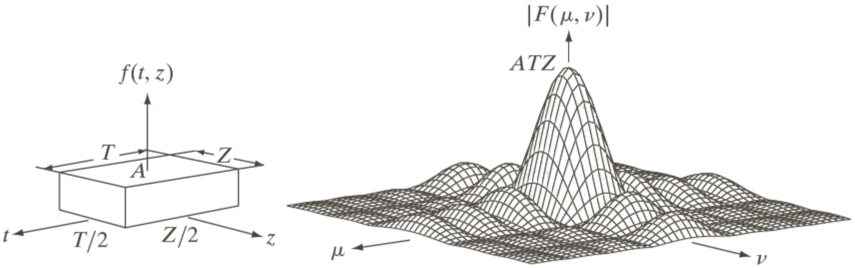
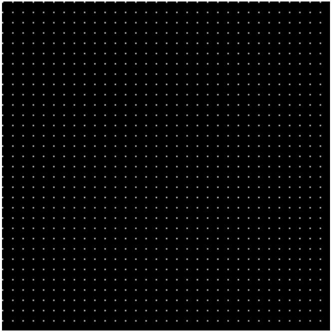
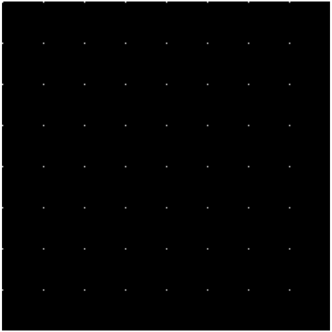


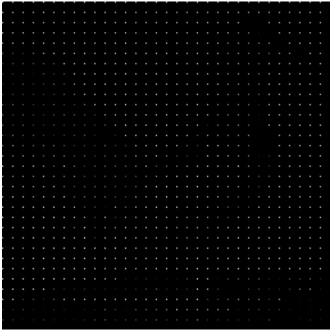
FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

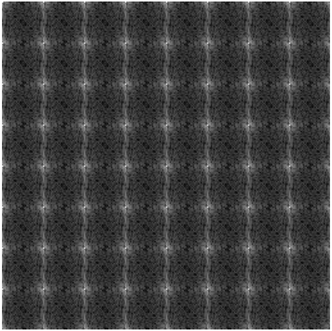
- **DFT of an impulse train**



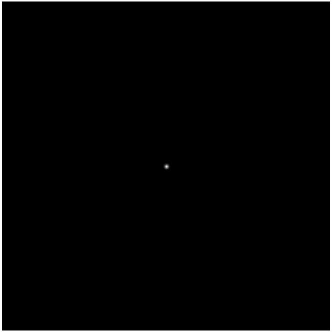


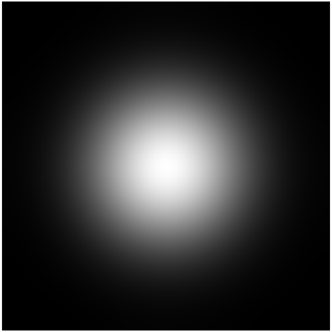
- **DFT of a sampled image**





- **DFT of a 2D Gaussian**

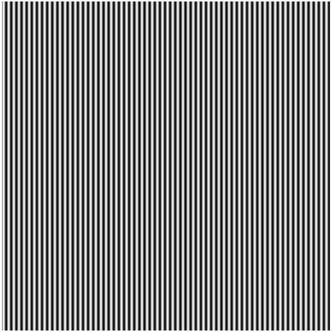




- **DFT of a 2D sinusoidal function**

$sin(2\pi u_0 x + 2\pi v_0 y) \rightsquigarrow$

$$F(u) = \frac{A}{2}\left[\delta(u + Mu_0, v + Nv_0) + \delta(u + Mu_0, v + Nv_0)\right]$$



Fourier Transform properties

- **Separability**

$$F(u,v) = \sum_{x=0}^{M-1} \exp\left[-j2\pi\left(\frac{ux}{M}\right)\right] \sum_{y=0}^{N-1} f(x,y) \exp\left[-j2\pi\left(\frac{vy}{N}\right)\right]$$

$$\begin{cases} F(u,v) = \sum_{x=0}^{M-1} F(x,v) \exp\left[-j2\pi\left(\frac{ux}{M}\right)\right] \\ F(x,v) = \sum_{y=0}^{N-1} f(x,y) \exp\left[-j2\pi\left(\frac{vy}{N}\right)\right] \end{cases}$$

$\Longrightarrow$ *1D DFT of raws then 1D DFT for the columns of the intermediate result.*

- **Translation**

$$f(x,y)\exp\left[-j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)\right] \leftrightsquigarrow F(u-u_0,v-v_0)$$

$$f(x-x_0,y-y_0) \leftrightsquigarrow F(u,v)\exp\left[-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)\right]$$

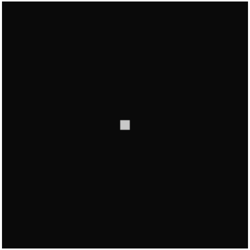
- Case where : $u_0 = v_0 = \frac{N}{2}$ and $N = M$
  $\exp\left[-j2\pi\left(\frac{u_0x+v_0y}{N}\right)\right] = \exp[j\pi(x+y)] = (-1)^{x+y}$

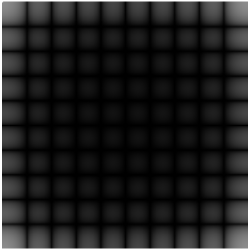
$$f(x,y)(-1)^{x+y} \leftrightsquigarrow F(u-\frac{N}{2},v-\frac{N}{2})$$

Remark : *Translation of* $f(x,y)$ *do not change the amplitude.*

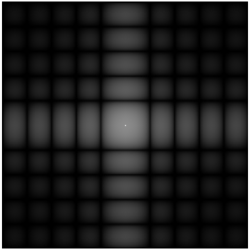
$$\left|F(u,v)\exp\left[-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)\right]\right| = |F(u,v)|$$

- **Example showing the translation property**





Fourier of Transform of $f(x, y)$



Fourier of Transform of $f(x, y)(-1)^{x+y}$

- **Example showing the translation property**

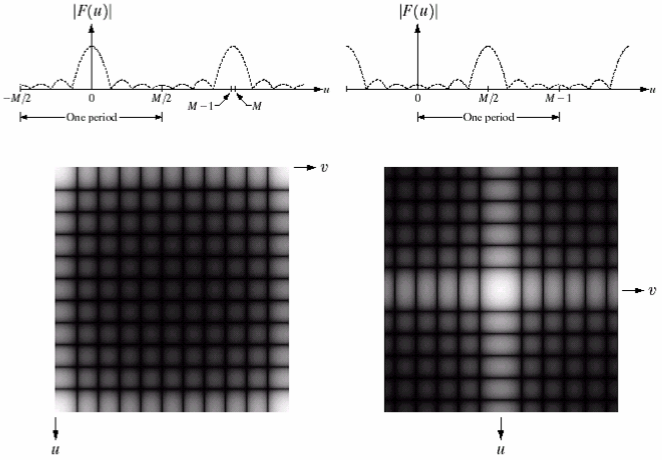


FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.

- **Periodicity and conjugate symmetry**

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

$$F(u,v) = F^*(-u,-v) \text{ et } |F(u,v)| = |F(-u,-v)|$$

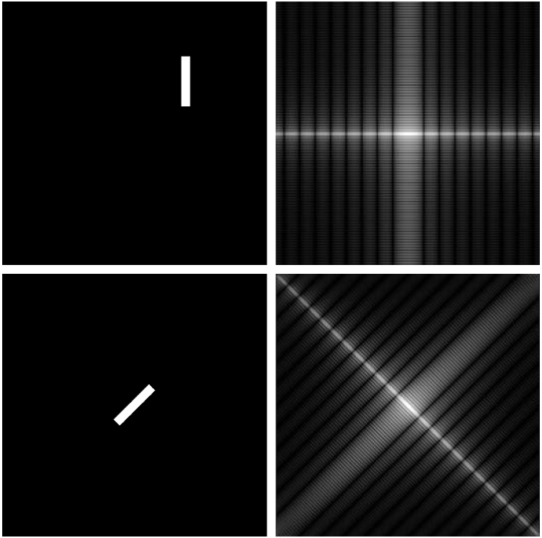
Fourier Transform properties

- **Rotation**
  In polar coordinates : $x = r\cos\theta$, $y = r\sin\theta$, $u = \omega\cos\varphi$, $v = \omega\sin\varphi$

$$\begin{aligned} f(x,y) &\rightarrow f(r,\theta) \\ F(u,v) &\rightarrow F(\omega,\varphi) \\ f(r,\theta+\theta_0) &\leftrightsquigarrow F(\omega,\varphi+\theta_0) \end{aligned}$$

Fourier Transform properties

- **Example : rotation and translation**



Fourier Transform properties

- **Distributivity and scale**

$$\Im\{f_1(x,y) + f_2(x,y)\} = \Im\{f_1(x,y)\} + \Im\{f_2(x,y)\}$$

$$\Im\{f_1(x,y) \cdot f_2(x,y)\} \neq \Im\{f_1(x,y)\} \cdot \Im\{f_2(x,y)\}$$

$$af(x,y) \leftrightsquigarrow aF(u,v)$$

$$f(ax,by) \leftrightsquigarrow \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

- **Mean value**

$$\overline{f} = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

Or : $F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = MN\overline{f}$

2D discrete Convolution

- **2D Convolution**

$$f(x,y) * g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)g(x-\alpha,y-\beta)d\alpha d\beta$$

$$f_e(x,y) * g_e(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e(m,n)g_e(x-m,y-n)$$

- Important property for convolution :

$$\Im\{f(x,y) * g(x,y)\} = F(u,v)\cdot G(u,v)$$

$$\Im\{f(x,y)\cdot g(x,y)\} = F(u,v) * G(u,v)$$

Example of 1D discrete convolution

- The objective is to compute : $f * g(x) = \sum_t f(t)g(x-t)$
- Here t vary from -2 to 2 if we consider that f is centered (frame origin in the middle). $f * g(x) = \sum_{t=-2}^{t=2} f(t)g(x-t) =$ $f(-2)g(x+2)+f(-1)g(x+1)+f(0)g(x)+f(1)g(x-1)+f(2)g(x-2)$
- Let's compute the resulting value in $x = -1$ for example : $f * g(-1) = \sum_{t=-2}^{t=2} f(t)g(-1-t) =$ $f(-2)g(1) + f(-1)g(0) + f(0)g(-1) = 1 \times 3 + 2 \times 2 + 3 \times 1 = 10$

$f \, [\; 1 \;\; 2 \;\; \mathbf{3} \;\; 4 \;\; 5 \;] \quad * \quad g \, [\; 1 \;\; \mathbf{2} \;\; 3 \;]$

1	2	3	4	5
3	2	1		
3	4	3		
	10			

1	2	3	4	5
	3	2	1	
	6	6	4	
	10	16	22	

2D discrete Convolution

- **Convolution in the borders**
  Many possibilities depending on the application :

1. Add zeros (zero padding)
2. Reflexion
3. Defaut

- **Example of 2D discrete convolution**

$$\begin{array}{|c|c|c|c|c|}\hline 10 & 5 & 20 & 20 & 20 \\ \hline 10 & 5 & 20 & 20 & 20 \\ \hline 10 & 5 & 20 & 20 & 20 \\ \hline 10 & 5 & 20 & 20 & 20 \\ \hline 10 & 5 & 20 & 20 & 20 \\ \hline \end{array} * \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|}\hline -10 & 5 & -15 & 0 & 0 \\ \hline -10 & 15 & -10 & 20 & 20 \\ \hline -10 & 15 & -10 & 20 & 20 \\ \hline -10 & 15 & -10 & 20 & 20 \\ \hline -10 & 15 & -10 & 20 & 20 \\ \hline \end{array}$$

We have to consider the symmetric of $\begin{array}{|c|c|}\hline -1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & -1 \\ \hline \end{array}$

- **Correlation**

$$f(x) \circ g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x+\alpha) d\alpha$$

$$f_e(x) \circ g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x+m)$$

$$f(x,y) \circ g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha,\beta) g(x+\alpha, y+\beta) d\alpha d\beta$$

$$f_e(x,y) \circ g_e(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) g_e(x+m, y+n)$$

$$\Im \{ f(x,y) \circ g(x,y) \} = F(u,v) \cdot G^*(u,v)$$

$$\Im \{ f(x,y) \cdot g^*(x,y) \} = F(u,v) \circ G(u,v)$$

- **FFT**

$F(u) = \sum_{x=0}^{N-1} f(x) \exp\left[\frac{-j2\pi ux}{N}\right] \Longrightarrow O(N^2)$ operations.

$F(u) = \sum_{x=0}^{N-1} f(x) W_N^{ux}$ with $W_N^{ux} = \exp\left[\frac{-j2\pi ux}{N}\right]$

We suppose that $N = 2^n$, where : $N = 2M$

$$F(u) = \sum_{x=0}^{2M-1} f(x) W_{2M}^{ux}$$

$$F(u) = \left(\underbrace{\sum_{x=0}^{M-1} f(2x) W_{2M}^{u(2x)}}_{pair} + \underbrace{\sum_{x=0}^{M-1} f(2x+1) W_{2M}^{u(2x+1)}}_{impair} \right)$$

- **FFT**

Hence : $W_{2M}^{2ux} = W_M^{ux}$

$$F(u) = \left(\sum_{x=0}^{M-1} f(2x) W_M^{ux} + \sum_{x=0}^{M-1} f(2x+1) W_M^{ux} W_{2M}^{u} \right)$$

Let $F_{pair}(u) = \sum_{x=0}^{M-1} f(2x) W_M^{ux}$
and $F_{impair}(u) = \sum_{x=0}^{M-1} f(2x+1) W_M^{ux}$
Then : $F(u) = [F_{pair}(u) + F_{impair}(u) W_{2M}^{u}]$
Hence : $W_M^{u+M} = W_M^{u}$ et $W_{2M}^{u+M} = -W_{2M}^{u}$ then :

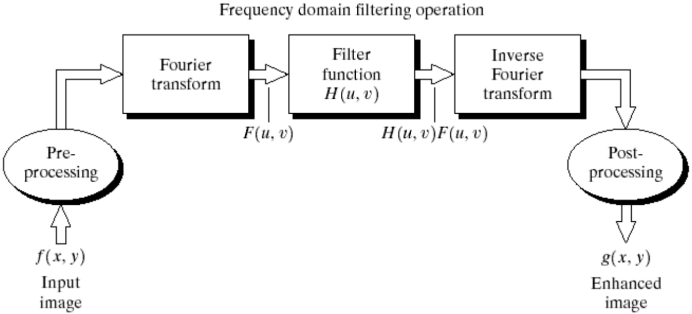
$$F(u+M) = [F_{pair}(u) - F_{impair}(u) W_{2M}^{u}]$$

- **FFT**
  - Remark : $F_{pair}(u)$ and $F_{impair}(u)$ are also Fourier Transforms but using a reduced number of samples.
  - Example : let compute the FFT of : $f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)$
    - * The FFT for 8 samples is based on twio FFTs of 4 samples and so on.
    - * The algorithm complexity is : $N/\log_2 N$

- **FFT**

N	N^2	$N\log_2 N$	$N/\log_2 N$
2	4	2	2.00
4	16	8	2.00
8	64	24	2.67
16	256	64	4.00
32	1024	160	6.40
64	4096	384	10.67
128	16384	896	18.29
256	65536	2048	32.00
512	262144	4608	56.89
1024	1048576	10240	102.40
2048	4194304	22528	186.18
4096	16777216	49152	341.33
8192	67108864	106496	630.15

- **General principle**

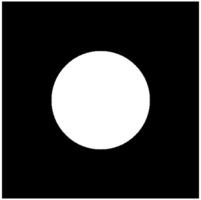


$$f(x,y) * h(x,y) \blacktriangleleft\!\blacktriangleright F(u,v)H(u,v)$$
$$f(x,y) * h(x,y) = \Im^{-1}\{F(u,v)H(u,v)\}$$
h : spatial filter
H : transfer function of the filter, impulse response.

- **Ideal Low Pass Filter : definition**

$$H(u,v) = \begin{cases} 1 \text{ Si } D(u,v) \leq D_0 \\ 0 \text{ Si } D(u,v) > D_0 \end{cases}$$

D_0 : cut-off filter frequency
$D(u,v) = \sqrt{u^2 + v^2}$



Ideal 2D low pass filter, diameter=256, image 512x512.

Filtering in the frequency domain

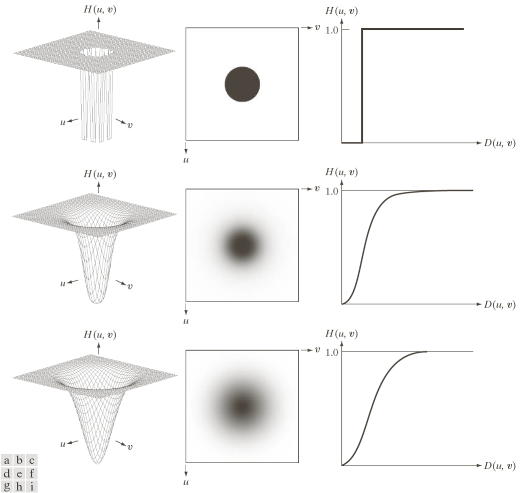
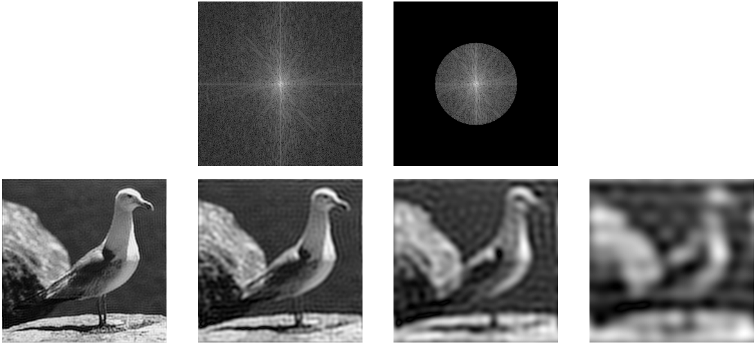


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Filtering in the frequency domain
Ideal Low Pass Filter

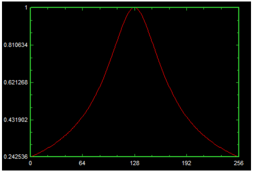
- **Ideal Low Pass Filter : examples when varying the cut-off frequency**



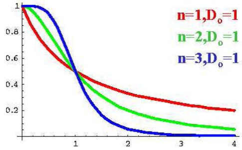
- **ButterWorth Filter :** $H(u,v) = \frac{1}{1+(D(u,v)/D_0)^{2n}}$

D_0 : cut-off frequency
n : filter order
$D(u,v) = \sqrt{u^2+v^2}$

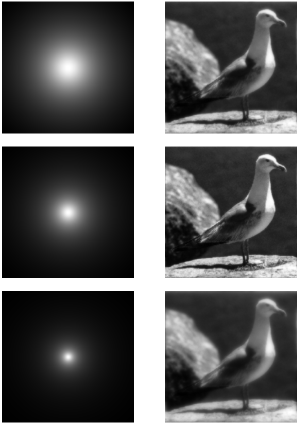
Frequency de cut-off = 0.25



2D Profile 2D



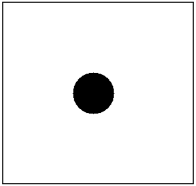
- **ButterWorth Filter :** cut-off =0.25, 0125, 0.0625



- **Ideal High Pass Filter : definition**

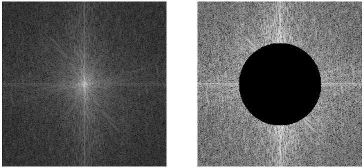
$$H(u,v) = \begin{cases} 1 \text{ Si } D(u,v) \geq D_0 \\ 0 \text{ Si } D(u,v) < D_0 \end{cases}$$

D_0 : cut-off filter frequency $D(u,v) = \sqrt{u^2 + v^2}$



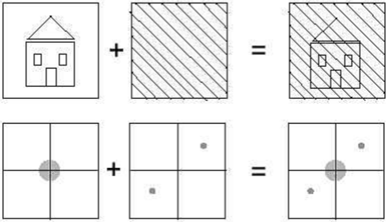
Ideal High pass Filter. Cut-off $= 0.25$.

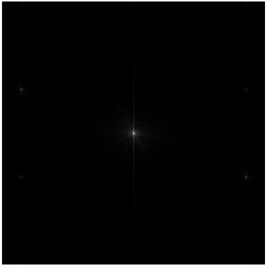
- **Ideal High Pass Filter : example**



- **Other filters :**
  - Band Pass Filter (input : width of the band)
  - Gaussian filter
  - Local Filter (radius around a given frequency + local frequency coordinates)
  - etc.

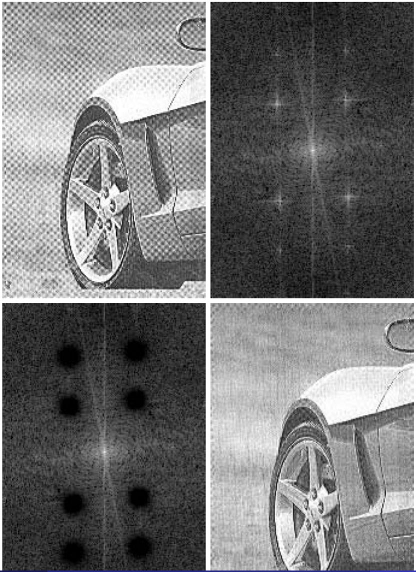
- **Principle :**



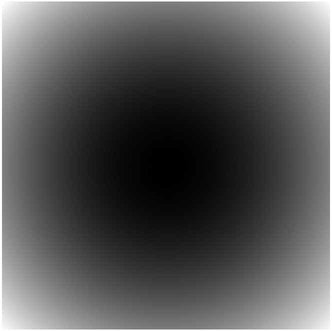


Filtering in the frequency domain

Example of local filter

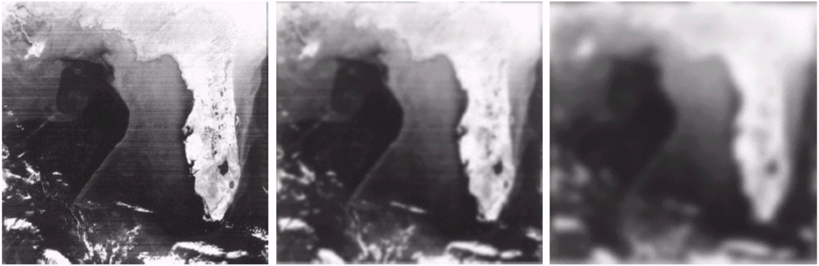


- The Laplacian in the frequency domain



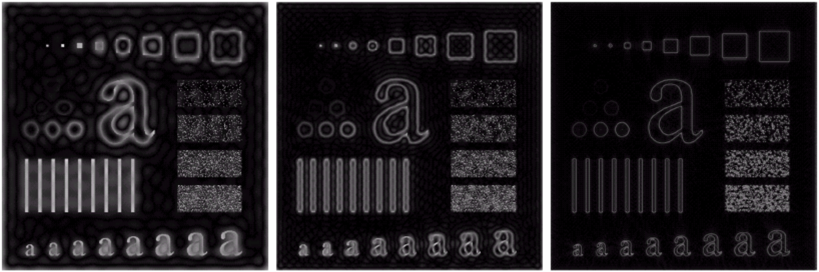
$$H(u,v) = 2\pi(u^2 + v^2)$$

Filtering in the frequency domain
Example



a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

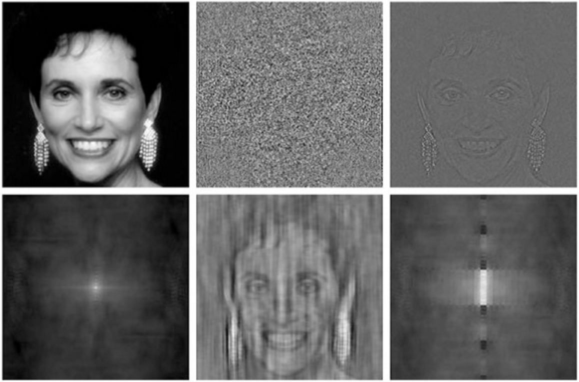
a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Filtering in the frequency domain

Example

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.



a	b	c
d	e	f

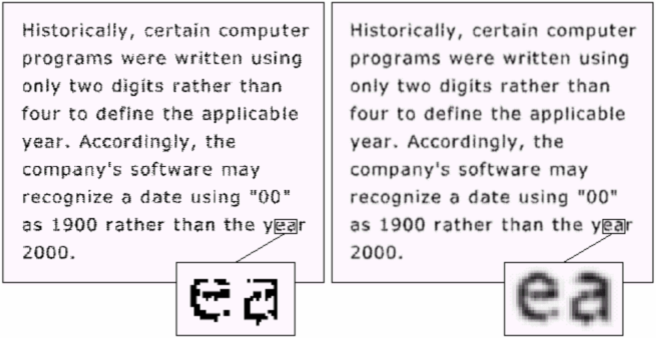
FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Filtering in the frequency domain

Example

a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view).
(b) Result of filtering with a GLPF (broken character segments were joined).

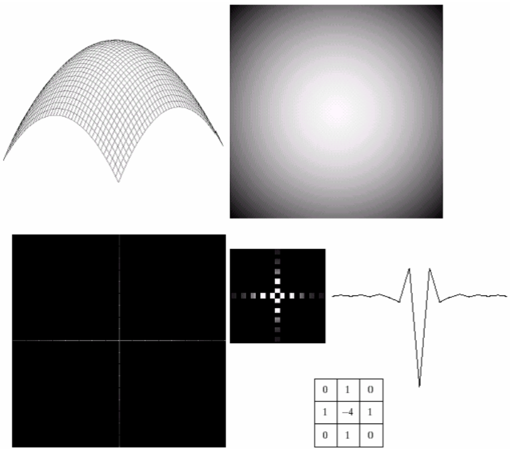


Filtering in the frequency domain

Example

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Filtering in the frequency domain
Discrete Spatial Laplacien



a b
c d e
f

FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.