Markov Chains for Optimization and Information Processing - I

Exercises

EXERCISE 1. *Let $\{X_n;\ n \in \mathbb{N}\}$ be an $E = \{1, 2, 3, 4, 5\}$-valued Markov chain, with transition matrix*

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

Find the equivalence classes, the transient and recurrent states, and all invariant measures of $\{X_n\}$.

EXERCISE 2. *Show that whenever x is recurrent, $\sum_{n\geq 0}(P^n)_{xy}$ equals $+\infty$ if and only if $x \leftrightarrow y$, and equals 0 if and only if $x \not\to y$.*

EXERCISE 3. *Let $x \in E$. Define $N_x = \{n;\ (P^n)_{xx} > 0\}$.*

1. *Show that whenever N_x contains two consecutive integers, the greatest common divisor of the elements of N_x is 1.*
2. *Show that if $n, n+1 \in N_x$, then $\{n^2, n^2+1, n^2+2, \ldots\} \subset N_x$.*

3. *Show that if the greatest common divisor of the elements of N_x is 1, then there exists $n \in \mathbb{N}$ such that $\{n, n+1\} \subset N_x$.*

4. *Conclude that the two definitions of aperiodicity of a state x are equivalent.*

<u>EXERCISE 4.</u> *Consider an $E = \{1, 2, 3, 4, 5, 6\}$-valued Markov chain $\{X_n;\ n \in \mathbb{N}\}$ with transition matrix P, whose off-diagonal entries are specified by*

$$P = \begin{pmatrix} \cdot & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 7/8 & 0 \\ 1/4 & 1/4 & 0 & \cdot & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 0 & \cdot & 0 \\ 0 & 1/5 & 0 & 1/5 & 1/5 & \cdot \end{pmatrix}.$$

1. *Find the diagonal terms of P.*

2. *Find the equivalence classes of the chain.*

3. *Show that* 4 *and* 6 *are transient states, and that the other states can be grouped into two recurrent classes to be specified. In the sequel, we let* $\mathcal{T} = \{4, 6\}$, $\mathcal{C}$ *be the recurrent class containing* 1, *and* $\mathcal{C}'$ *the other recurrent class. For all* $x, y \in E$, *define* $\rho_x := \mathbb{P}_x(T < \infty)$, *where* $T := \inf\{n \geq 0;\ X_n \in \mathcal{C}\}$.

4. *Show that*

$$\rho_x = \begin{cases} 1, & \text{if } x \in \mathcal{C}, \\ 0, & \text{if } x \in \mathcal{C}', \end{cases}$$

and that $0 < \rho_x < 1$ *if* $x \in \mathcal{T}$.

5. *Using the decomposition* $\{T < \infty\} = \{T = 0\} \cup \{T = 1\} \cup \{2 \leq T < \infty\}$ *and conditioning in the computation of* $\mathbb{P}_x(2 \leq T < \infty)$ *by the value of* X_1, *establish the formula*

$$\rho_x = \sum_{y \in E} P_{xy} \rho_y, \qquad \text{if } x \in \mathcal{T}.$$

6. *Compute* ρ_4 *and* ρ_6.

7. *Deduce (without any serious computation!) the values of* $\mathbb{P}_4(T_{\mathcal{C}'} < \infty)$ *and* $\mathbb{P}_6(T_{\mathcal{C}'} < \infty)$, *where* $T_{\mathcal{C}'} := \inf\{n \geq 0;\ X_n \in \mathcal{C}'\}$.

<u>EXERCISE 5.</u> *Consider an* $E = \{1,2,3,4,5,6\}$*-valued Markov chain* $\{X_n;\, n \in \mathbb{N}\}$ *with transition matrix* P*, whose off-diagonal entries are given by*

$$P = \begin{pmatrix} \cdot & 1/4 & 1/3 & 0 & 0 & 0 \\ 1/4 & \cdot & 0 & 1/4 & 1/3 & 0 \\ 1/2 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 1/2 & 1/3 \\ 0 & 0 & 0 & 1/2 & \cdot & 1/2 \\ 0 & 0 & 0 & 1/3 & 1/4 & \cdot \end{pmatrix}.$$

1. *Find the diagonal entries of* P.

2. *Show that* E *is the union of two equivalence classes to be specified, one* $(\mathcal{R})$ *being recurrent and the other* $(\mathcal{T})$ *transient.*

3. *Define* $T := \inf\{n \geq 0;\ X_n \in \mathcal{R}\}$ *and* $h_x = \mathbb{E}_x(T)$*, for* $x \in E$*. Show that* $h_x = 0$ *for* $x \in \mathcal{R}$*, and that* $1 < h_x < \infty$ *for* $x \in \mathcal{T}$.

4. *Show that, for all* $x \in \mathcal{T}$,

$$h_x = 1 + \sum_{y \in E} P_{xy} h_y.$$

Deduce the values of h_x, $x \in \mathcal{T}$.

<u>EXERCISE 7.</u> *Given* $0 < p < 1$, *we consider an* $E = \{1, 2, 3, 4\}$*-valued Markov chain* $\{X_n;\ n \in \mathbb{N}\}$ *with transition matrix* P *given by*

$$P = \begin{pmatrix} p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p \\ p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p \end{pmatrix}.$$

1. *Show that the chain* $\{X_n\}$ *is irreducible and recurrent.*

2. *Compute its unique invariant probability* π.

3. *Show that the chain is aperiodic. Deduce that* P^n *tends, as* $n \to \infty$, *towards the matrix*
$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}.$$

4. *Compute* P^2. *Show that this transition matrix coincides with the above limit. Determine the law of* X_2, *as well as that of* X_n, $n \geq 2$.

5. *Define* $T_4 = \inf\{n \geq 1;\ X_n = 4\}$. *Compute* $\mathbb{E}_4(T_4)$.

EXERCISE 8. *Consider an* $E = \{0, 1, 2, 3, 4\}$*-valued Markov chain* $\{X_n;\ n \in \mathbb{N}\}$ *with transition matrix*

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ p & 0 & \frac{1-p}{2} & 0 & \frac{1-p}{2} \\ p & \frac{1-p}{2} & 0 & \frac{1-p}{2} & 0 \\ p & 0 & \frac{1-p}{2} & 0 & \frac{1-p}{2} \\ p & \frac{1-p}{2} & 0 & \frac{1-p}{2} & 0 \end{pmatrix},$$

where $0 < p < 1$. *Let* $T := \inf\{n \geq 1;\ X_n = 0\}$.

1. *Show that the chain* $\{X_n\}$ *is irreducible and recurrent. Denote its invariant probability by* π.

2. *Show that under* $\mathbb{P}_0$, *the law of* T *is a geometric law to be specified. Show that* $\mathbb{E}_0(T) = (p+1)/p$.

3. *Let*

$$N_n = \sum_{k=1}^{n} \mathbf{1}_{\{X_k = 0\}}, \quad M_n = \sum_{k=1}^{n} \mathbf{1}_{\{X_k \neq 0\}}.$$

Compute the limits as $n \to \infty$ *of* $n^{-1} N_n$ *and* $n^{-1} M_n$.

4. *Give an intuitive argument to support the identity*

$$\pi_1 = \pi_2 = \pi_3 = \pi_4.$$

Deduce the probability π, *exploiting this identity.*

5. *Show the following general result. If there exists a one-to-one mapping* τ *from* E *into itself, such that*

$$P_{\tau x, \tau y} = P_{xy}, \quad \forall x, y \in E,$$

then the invariant probability π *has the property* $\pi_{\tau x} = \pi_x$, $x \in E$. *Deduce a rigorous argument for the result in part 4.*

EXERCISE 9. (Random Walk) Let

$$X_n = X_0 + Y_1 + \ldots + Y_n,$$

where the X_n *take their values in* $\mathbb{Z}^d$,*the* Y_n *being i.i.d., globally independent of* X_0, *and their law specified by*

$$\mathbb{P}(Y_n = \pm e_i) = (2d)^{-1}, \quad 1 \le i \le d,$$

where $\{e_1, \ldots, e_d\}$ *is the canonical basis of* $\mathbb{Z}^d$.

1. *Show that the common characteristic function of the Y_n is given by*

$$\phi(t) = d^{-1} \sum_{j=1}^{d} \cos(t_j),$$

and that

$$(P^n)_{00} = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \phi^n(t) dt.$$

2. *Deduce that, for all* $0 < r < 1$,

$$\sum_{n \geq 0} r^n (P^n)_{00} = (2\pi)^{-d} \int_{[-\pi,\pi]^d} (1 - r\phi(t))^{-1} dt.$$

3. *Show that, for all* $\alpha > 0$, *the mapping*

$$(r, t) \to (1 - r\phi(t))^{-1}$$

is bounded on $]0, 1] \times ([-\pi, \pi]^d \backslash C_\alpha)$, *where* $C_\alpha = \{t \in \mathbb{R}^d; \|t\| \leq \alpha\}$, *and that whenever* $\|t\|$ *is sufficiently small,* $r \to (1 - r\phi(t))^{-1}$ *is positive and increasing.*

4. *Deduce from the fact that* $1 - \phi(t) \simeq \|t\|^2/2$, *as* $t \to 0$, *that* $\{X_n\}$ *is an irreducible* $\mathbb{Z}^d$*-valued Markov chain, which is null recurrent if* $d = 1, 2$, *and transient if* $d \geq 3$.

EXERCISE 10. (Queue) *Consider a discrete time queue such that, at each time $n \in \mathbb{N}$, one customer arrives with probability p ($0 < p < 1$) and no customer arrives with probability $1 - p$. During each unit time interval when at least one customer is present, one customer is served and leaves the queue with probability q, $0 < q < 1$, and no customer leaves the queue with probability $1 - q$ (a customer who arrives at time n leaves at the earliest at time $n + 1$). All the above events are mutually independent. Denote by X_n the number of customers in the queue at time n.*

1. *Show that $\{X_n;\ n \in \mathbb{N}\}$ is an irreducible $E = \mathbb{N}$-valued Markov chain. Determine its transition matrix $P_{xy},\ x, y \in \mathbb{N}$.*

2. *Give a necessary and sufficient condition on p and q for the chain $\{X_n\}$ to possess an invariant probability. We assume below that this condition is satisfied. Specify the unique invariant probability $\{\pi_x;\ x \in \mathbb{N}\}$ of the chain $\{X_n\}$.*

3. *Compute $\mathbb{E}_\pi(X_n)$.*

4. *Customers are served according to the order in which they arrive. Denote by T the sojourn time in the queue of a customer who arrives at an arbitrary fixed time. Assuming that the queue is initialized with its invariant probability, what is the expectation of T?*

EXERCISE 11. (Queue) *Let X denote the random number of individuals in a given population, and $\phi(u) = \mathbb{E}[u^X]$, $0 \leq u \leq 1$, its generating function. Each individual is selected with probability q $(0 < q < 1)$, independently of the others. Let Y denote the number of individuals selected in the initial population of X individuals.*

1. *Show that the generating function ψ of Y (defined as $\psi(u) = \mathbb{E}[u^Y]$) is given by*

$$\psi(u) = \phi(1 - q + qu).$$

Consider a service system (equipped with an infinite number of servers), and denote by X_n $(n = 0, 1, 2, \dots)$ the number of customers present in the system at time n. Assume that, at time $n + \frac{1}{3}$, each of the X_n customers leaves the system with probability $1 - p$, and stays with probability p (independently of the others, and of all the other events), denote by X'_n the number of remaining customers, and assume that, at time $n + \frac{2}{3}$, Y_{n+1} new customers join the queue. Assume that the random variables $X_0, Y_1, Y_2, \dots$ are mutually independent, and globally independent of the service times, and that the joint law of the Y_n is the Poisson distribution with parameter $\lambda > 0$ (i.e. $\mathbb{P}(Y = k) = e^{-\lambda}\lambda^k/k!$ and $\mathbb{E}[u^{Y_n}] = \exp[\lambda(u - 1)]$).

2. *Show that $\{X_n;\ n \geq 0\}$ is an irreducible $E = \mathbb{N}$-valued Markov chain.*

3. *Compute $\mathbb{E}[u^{X_{n+1}} | X_n = x]$ in terms of u, p, λ and x.*

4. *Denote by* $\phi_n(u) = \mathbb{E}[u^{X_n}]$ *the generating function of* X_n. *Compute* ϕ_{n+1} *in terms of* ϕ_n, *and show that*

$$\phi_n(u) = \exp\left[\lambda(u-1)\sum_0^{n-1} p^k\right]\phi_0(1 - p^n + p^n u).$$

5. *Show that* $\rho(u) = \lim_{n\to\infty}\phi_n(u)$ *exists and does not depend on* ϕ_0, *and that* ρ *is the generating function of a Poisson distribution whose parameter is to be specified in terms of* λ *and* p.

6. *Show that* $\{X_n;\ n \geq 0\}$ *is positive recurrent and specify its invariant probability.*

EXERCISE 12. *Let* $X_0, A_0, D_0, A_1, D_1, \ldots$ *be* $\mathbb{N}$*-valued mutually independent random variables. The* D_n *are Bernoulli random variables with parameter* q, *that is,* $\mathbb{P}(D_n = 1) = 1 - \mathbb{P}(D_n = 0) = q$, $0 < q < 1$. *The* A_n *all have the same law defined by* $\mathbb{P}(A_n = k) = r_k$, $k \in \mathbb{N}$, *where* $0 \leq r_k < 1$, $0 < r_0 < 1$ *and* $\sum_{k=0}^{\infty} r_k = 1$. *Assume that* $p = \sum_k k r_k < \infty$. *Consider the sequence of random variables* $\{X_n;\ n \in \mathbb{N}\}$ *defined by*

$$X_{n+1} = (X_n + A_n - D_n)^+, \quad n \geq 0,$$

with the usual notation $x^+ = \sup(x, 0)$.

Assume from now on that $X_0 = 0$. *Let* $T = \inf\{n > 0;\ X_n = 0\}$. *Define* $S_n = \sum_{k=0}^{n-1}(A_k - D_k)$.

2. *Show that* $X_n \geq S_n$, *and that* $X_{n+1} = S_{n+1}$ *on the event* $\{T > n\}$.

3. *Show that* $S_n/n \to p - q$ *almost surely, as* $n \to \infty$.

4. *Show that whenever* $p < q$, $T < \infty$ *almost surely.*

5. *Assume that* $p > q$. *Show that* $\{X_n;\ n \in \mathbb{N}\}$ *visits* 0 *at most a finite number of times.*

6. *In the case* $p \neq q$, *specify when the chain is recurrent, and when it is transient.*

Assume from now on that $\mathbb{P}(A_n = 1) = 1 - \mathbb{P}(A_n = 0) = p$, *where* $0 < p < 1$ *(*p *is again the expectation of* A_n*).*

7. *Specify the transition matrix* P *in this case.*

8. *Show that if* $p = q$, *the chain is null recurrent. (Hint: use the result of Exercise 10.11, part 3, in order to show the recurrence, and then look for an invariant measure.)*

9. *Assume that* $p < q$. *Show that the chain has a unique invariant probability* π *on* $\mathbb{N}$, *and that* $\pi_k = (1-a)a^k$, *with* $a = p(1-q)/q(1-p)$. *(Hint: first establish a recurrence relation for the sequence* $\Delta_k = \pi_k - \pi_{k+1}$.*) Show that the chain is positive recurrent.*

EXERCISE 13. (Database) *Suppose that a computer memory contains n items $1, 2, \ldots, n$. The memory receives successive requests, each consisting of one of the items. The closer the item is to the top of the list, the faster the access is. Assume that the successive requests are i.i.d. random variables. If the common law of those random variables were known, the best choice would be to order the data in decreasing order of their associated probability of being requested. But this probability $(p_1, p_2, \ldots, p_n)$ is either unknown or slowly varying. Assume that $p_k > 0$, for all $k \in \{1, 2, \ldots, n\}$.*

We need to choose a method of replacement of the data after they are requested, in such a way that in the long run the time taken to get the requested data will be as small as possible.

We will compare two such methods. The first involves systematically replacing any item which has been requested at the top of the list. The second involves moving each item which has been requested one step ahead. In both cases, we have an irreducible Markov chain with values in the set E of all permutations of the set $\{1, 2, \ldots, n\}$. Denote by Q (P) the transition matrix of the first (second) chain, and by π (μ) the associated invariant measure. We associate with the Markov chain with transition matrix Q, the quantity

$$J_Q \stackrel{def}{=} \sum_{k=1}^{n} \pi(\textit{position of } k)\, p_k,$$

where π(position of k) is the expectation under π of the position of the element k. We associate with the Markov chain with transition matrix P, the quantity

$$J_P \stackrel{def}{=} \sum_{k=1}^{n} \mu(\text{position of } k)\, p_k.$$

1. *Show that the chain with transition matrix Q is not reversible.*

2. *Show that any irreducible and positive recurrent Markov chain which satisfies the following conditions is reversible:*

   (i) $P_{k\ell} > 0 \Leftrightarrow P_{\ell k} > 0$;

   (ii) *for any excursion $k, k_1, k_2, \ldots, k_m, k$,*

   $$P_{kk_1} \prod_{i=2}^{m} P_{k_{i-1}k_i} P_{k_m k} = P_{kk_m} \prod_{i=m-1}^{1} P_{k_{i+1}k_i} P_{k_1 k}.$$

   This is known as the 'Kolmogorov cycle condition'.

3. *Show that P satisfies (i) and (ii).*

4. *Show that the second procedure is preferable, in the sense that $J_P < J_Q$.*