Assignment 3

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Section : 09

Answer

Given, Nodes : $[0, \frac{\pi}{2}, \pi]$

$$f(x) = \sin(x)$$

So, for, $x_0 = 0$, $f(x_0) = \sin(0) = 0$

$x_1 = \frac{\pi}{2}$, $f(x_1) = \sin(\frac{\pi}{2}) = 1$

$x_2 = \pi$, $f(x_2) = \sin(\pi) = 0$

From Newtons divided difference method we get,

$x_0 = 0$, $f[x_0] = 0$

$$f[x_0, x_1] = \frac{1-0}{\frac{\pi}{2} - 0} = \frac{2}{\pi}$$

$x_1 = \frac{\pi}{2}$, $f[x_1] = 1$

$$f(x_0, x_1, x_2) =$$

$$f[x_1, x_2] = \frac{0-1}{\pi - \frac{\pi}{2}} = -\frac{2}{\pi}$$

$x_2 = \pi$, $f(x_2) = 0$

$$= \frac{-\frac{2}{\pi} - \frac{2}{\pi}}{\pi - 0}$$

$$= -\frac{4}{\pi^2}$$

1

as we know, $a_0 = f[x_0]$, $a_1 = f[x_0, x_1]$, $a_2 = f[x_0, x_1, x_2]$

$$\therefore a_0 = 0$$

$$a_1 = \frac{2}{\pi}$$

$$a_2 = -\frac{4}{\pi^2} \quad \text{(Ans)}$$

2

we know the equation for interpolating polynomial is,

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots a_n(x - x_0)\ldots(x - x_n)$$

So for our term,

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$= 0 + \frac{2}{\pi}(x - 0) + \left(-\frac{4}{\pi^2}\right)(x - 0)\left(x - \frac{\pi}{2}\right) \quad \boxed{\text{from } 1}$$

$$= \frac{2x}{\pi} - \frac{4x^2}{\pi^2} + \frac{2x}{\pi}$$

$$= \frac{4x}{\pi} - \frac{4x^2}{\pi^2} \quad \text{(Ans)}$$

$x_0 = 0 : f[x_0] = 0$

$x_1 = \frac{\pi}{2} : f(x_1) = 1$

$x_2 = \pi : f[x_2] = 0$

$x_3 = \frac{3\pi}{2} : f[x_3] = -1$

$f(x_0, x_1) = \frac{2}{\pi}$

$f(x_1, x_2) = -\frac{2}{\pi}$

$f(x_2, x_3) = \frac{-1-0}{\frac{3\pi}{2} - \pi} = \frac{-2}{\pi}$

$f(x_0, x_1, x_2) = -\frac{4}{\pi^2}$

$$f(x_1, x_2, x_3) = \frac{-\frac{2}{\pi} + \frac{2}{\pi}}{\frac{3\pi}{2} - \frac{\pi}{2}} = 0$$

$$\therefore f(x_0, x_1, x_2, x_3) = \frac{0 - \left(-\frac{4}{\pi^2}\right)}{\frac{3\pi}{2} - 0} = \frac{8}{3\pi^3}$$

$$\therefore a_3 = \frac{8}{3\pi^3}$$

$\therefore$ ~~Now~~ $\cancel{P_3(x) = a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)(x_1 - x_1) + a_3(x_0 - x_1)(x_0 - x_2)(x_2 - x_3)}$

$$\text{Now } P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= \frac{4x}{\pi} - \frac{4x^2}{\pi^2} + \frac{8}{3\pi^3}(x - 0)\left(x - \frac{\pi}{2}\right)(x - \pi)$$

$$= \frac{4x}{\pi} - \frac{4x^2}{\pi^2} + \frac{8}{3\pi^3}\left(x^3 - \frac{3\pi}{2}x^2 + \frac{\pi^2}{2}x\right)$$

$$= \frac{4x}{\pi} - \frac{4x^2}{\pi^2} + \frac{8x^3}{3\pi^3} - \frac{8 \times 3\pi x^2}{3\pi^3 \times 2} + \frac{8\pi^2 x}{3.\cancel{2}\pi^3}$$

$$= \frac{4x}{\pi} - \frac{4x^2}{\pi^2} + \frac{8x^3}{3\pi^3} - \frac{4x^2}{\pi^2} + \frac{4x}{3\pi}$$

$$= \frac{8x^3}{3\pi^3} - \frac{8x^2}{\pi^2} + \frac{16x}{3\pi} \quad \text{(Ans)}$$

4

we know

Interpolation Error $= |f(x) - P_n(x)| \Rightarrow \frac{f^{n+1}(\xi)}{(n+1)!} w(x)$

$w(x) = (x-x_0)(x-x_1)(x-x_2) \cdots (x-x_n)$

In our term,

$$|f(x) - P_3(x)| = \frac{f^4(\xi)}{4!} \underbrace{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}_{w(x)}$$

$$\therefore w(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3) = (x-0)\left(x-\frac{\pi}{2}\right)(x-\pi)\left(x-\frac{3\pi}{2}\right)$$

$$= x\left(x-\frac{\pi}{2}\right)(x-\pi)\left(x-\frac{3\pi}{2}\right) \quad \text{(Ans)}$$

from 4,

we get $\left| f(x) - P_3(x) \right| = \dfrac{f^4(x)}{4!} \, w(x)$

$f(x) = \sin x$, $f'(x) = +\cos x$, $f''(x) = -\sin x$, $f''' = -\cos x$

$f''''(x) = \sin x$

$\therefore f^4(x) = \sin x$

Now for $I = \left[0, \frac{3\pi}{2}\right]$ we know $\sin\left(\frac{\pi}{2}\right)$ contains the highest value among them.

$\therefore f^4(x) = \sin\left(\frac{\pi}{2}\right) = 1$

$\therefore \dfrac{f^4(x)}{4!} = \dfrac{1}{24}$

Now,

$w(x) = x\left(x - \frac{\pi}{2}\right)(x - \pi)\left(x - \frac{3\pi}{2}\right)$

$= \left(x^2 - \frac{\pi}{2}x\right)\left(x^2 - \frac{3\pi x}{2} - x\pi + \frac{3\pi^2}{2}\right)$

$= \left(x^2 - \frac{\pi}{2}x\right)\left(x^2 - \frac{5\pi x}{2} + \frac{3\pi^2}{2}\right)$

$= x^4 - \dfrac{5\pi x^3}{2} + \dfrac{3\pi^2 x^2}{2} - \dfrac{x^3\pi}{2} + \dfrac{5\pi^2 x^2}{4} - \dfrac{3\pi^3 x}{4}$

$\therefore w(x) = x^4 - 3\pi x^3 + \frac{11}{4}\pi^2 x^2 - \frac{3}{4}\pi^3 x$

$\therefore w'(x) = 4x^3 - 9\pi x^2 + \frac{11}{2}\pi^2 (x) - \frac{3}{4}\pi^3$

hence we know,

$$w'(x) = 0$$

$$\therefore 4x^3 - 9\pi x^2 + \frac{11}{2}\pi^2 (x) - \frac{3}{4}\pi^3 = 0$$

$$\Rightarrow 4x^3 - 9 \cdot 3.1416 \times x^2 + \frac{11}{2}(3.1416)^2 x - \frac{3}{4}(3.1416)^3 = 0$$

$$= 4x^3 - 28.274 x^2 + 54.2828 x - 23.254 = 0$$

using cubic formula we get,

$$x = 0.5999 \approx 0.6, \quad 2.356, \quad 4.1123$$

Now lets find $w(x)$ for these x values,

for $x = 0.6 \Rightarrow (0.6)^4 - 3\pi(0.6)^3 + \frac{11}{4}\pi^2(0.6)^2 - \frac{3}{4}\pi^3(0.6)$

$$= -6.08$$

for $x = 4.112 \Rightarrow (4.112)^4 - 3\pi(4.112)^3 + \frac{11}{4}\pi^2(4.112)^2 - \frac{3}{4}\pi^3(4.112)$

$$= -6.08$$

and

$$x = 2.356 \Rightarrow (2.356)^4 - 3\pi(2.356)^3 + \frac{11}{4}\times\pi^2(2.356)^2 - \frac{3}{4}\pi^3(2.356)$$

$$= 3.424$$

for $x = 0 \Rightarrow 0^4 - 3\pi 0^3 + \frac{11}{4}\pi^2 0^2 - \frac{3}{4}\pi^3 0$

$$= 0$$

for $x = \frac{3\pi}{2} = 4.71 = (4.71)^4 - 3\pi(4.71)^3 + \frac{11}{4}\pi^2(4.71)^2 - \frac{3}{4}\pi^3(4.71)$

$$= -0.0554$$

$\therefore$ $x = 2.356$ giving us the maximum value which is $3.424 = w(x)$

$\therefore$ Interpolation Error: $|f(x) - P_3(x)|$

$$= \frac{f^4(\xi)}{4!}\, w(x)$$

$$= \frac{1}{24} \times 3.424$$

$$= 0.1426 \quad \text{(Answer)}$$