A linear system is described by the following equations

$$x_1 - x_2 + x_3 = 1$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$3x_1 + 5x_2 + 3x_3 = 4$$

Answer the following questions (1-4):

Questions-1: [1 Mark] Does this system has any unique solution? Explain.

A = $\begin{pmatrix} 1 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$, det(A) = 40 ≠ 0. So, A is non-singular. This system has unique solution.

Question-2: [3 Marks] Solve the above linear system by Gaussian elimination method.

Matrix form = $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right)$ = $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 7 & -5 & 2 \\ 0 & 8 & 0 & 1 \end{array}\right)$ [R2 - 4 R1 → R2; R3 - 3 R1 → R3]

= $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 7 & -5 & 2 \\ 0 & 1 & 5 & -1 \end{array}\right)$ [R3-R1 → R3]

= $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 7 & -5 & 2 \\ 0 & 0 & 40 & -9 \end{array}\right)$ [7R3 – R2 → R3]

So, $40x_3 = -9$ => $x_3 = -9/40$

$7x_2 - 5x_3 = 2$ => $x_2 = 1/8$

$x_1 - x_2 + x_3 = 1$ => $x_1 = 27/20$

Now solve the same linear system above by the *LU*-decomposition method:

Question-3: [3 Marks] Find the lower triangular matrix *L* and the upper triangular matrix *U*.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}, \quad F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 5 & 1 \end{pmatrix} \quad [m_{21}=4, m_{31}=3]$$

$$A^{(2)} = F^{(1)}A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 7 & -5 \\ 0 & 8 & 0 \end{pmatrix}$$

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -8/7 & 1 \end{pmatrix} \qquad [m_{32}=8/7]$$

$$U = A^{(3)} = F^{(2)}A^{(2)} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 40/7 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 8/7 & 1 \end{pmatrix}$$

Question-4: [3 Marks] Now find the solution of the linear system again using the matrix L and U found in the previous question.

$$Ly=b \qquad \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 8/7 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix} \quad \Rightarrow y_1 = 1, y_2 = 2, y_3 = -9/7$$

$$Ux=y \qquad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 40/7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -9/7 \end{pmatrix} \quad \Rightarrow x_3 = -9/40, x_2 = 1/8, x_1 = 27/20$$