STA201 Assignment 5 Solutions

Random Variables

1. A discrete random variable X has the following probability mass function

$$P(X = x) = \begin{cases} 3kx & x = 1, 3, 5 \\ k(x^2 + 0.5) & x = 7 \\ 0 & otherwise \end{cases}$$

where k is a constant

a. Show that $k = \frac{2}{153}$
b. Find the exact value of $P(3 < x \leq 7)$
c. Find the exact value of $P(3 < x < 5)$
d. What is the expected value of the random variable X?
e. What is the variance of the random variable X?
f. Determine $Var\left(-\frac{1}{3}x + 5\right)$

ANSWER:

a) We know,

$\sum f(x) = 1$

Here,

$\sum f(x) = (3k \times 1) + (9k \times 3) + (15k \times 5) + (k \times 49.5) = 76.5k$

From the above we can write,

76.5*k* = 1

$\therefore k = \frac{2}{153}$ [SHOWED]

x	1	3	5	7
P(X=x)	2/51	2/17	10/51	11/17

b) $P(3 < x \leq 7) = \frac{10}{51} + \frac{11}{17} = \frac{43}{51}$

c) $P(3 < x < 5) = 0$

d) Expected value of *X*,

$E(X) = \sum x \times f(x) = 1 \times \frac{2}{51} + 3 \times \frac{2}{17} + 5 \times \frac{10}{51} + 7 \times \frac{11}{17} = \frac{301}{51} = 5.902$

e) We know,

$Var(X) = E(X^2) - [E(X)]^2$

Here,

$E(X^2) = \sum x^2 \times f(x) = 1^2 \times \frac{2}{51} + 3^2 \times \frac{2}{17} + 5^2 \times \frac{10}{51} + 7^2 \times \frac{11}{17} = 37.706$

Therefore,

$Var(X) = E(X^2) - [E(X)]^2 = 37.706 - 5.902^2 = 2.872$

f) $Var(-\frac{1}{3}x + 5)$

$= \left(-\frac{1}{3}\right)^2 \times Var(x) = 0.319$

2. When travelling from Bangladesh to Vietnam, travellers need to first land at Kuala Lumpur, and then get on a connecting flight to Vietnam. The total time in transit Y in hours can be shown to have the following PDF

$$f(Y=y)=\begin{cases}\frac{1}{20}y & 0<y\le 4\\ \frac{1}{30}(10-y) & 4<y\le 10\\ 0 & otherwise\end{cases}$$

a. What is the probability that total transit time is at most 6 hours?
b. What is the probability that the transit time is either less than 3 hours or more than 7 hours?
c. What is the expected total transit time for travellers going from Bangladesh to Vietnam?
d. Determine the standard deviation in the total transit time.

ASNWER:

a) $f(Y\le 6)=\int_0^4 \frac{1}{20}y\,dy+\int_4^6 \frac{1}{3}-\frac{1}{30}y\,dy$

$=\frac{1}{20}\times\left[\frac{y^2}{2}\right]_0^4+\frac{1}{3}\times[y]_4^6-\frac{1}{30}\times\left[\frac{y^2}{2}\right]_4^6$

$=\frac{11}{15}=0.733$

b) $f(Y<3)+f(Y>7)=\int_0^3 \frac{1}{20}y\,dy+\int_7^{10} \frac{1}{3}-\frac{1}{30}y\,dy$

$=\frac{1}{20}\times\left[\frac{y^2}{2}\right]_0^3+\frac{1}{3}\times[y]_7^{10}-\frac{1}{30}\times\left[\frac{y^2}{2}\right]_7^{10}$

$=\frac{3}{8}=0.375$

c) $E(Y)=\int_0^4 y\times\frac{1}{20}y\,dy+\int_4^{10} y\times\left(\frac{1}{3}-\frac{1}{30}y\right)dy$

$=\frac{1}{20}\times\left[\frac{y^3}{3}\right]_0^4+\frac{1}{3}\times\left[\frac{y^2}{2}\right]_4^{10}-\frac{1}{30}\times\left[\frac{y^3}{3}\right]_4^{10}$

$=\frac{14}{3}=4.667$

d) We Know,

$E(Y^2)=\int_0^4 y^2\times\frac{1}{20}y\,dy+\int_4^{10} y^2\times\left(\frac{1}{3}-\frac{1}{30}y\right)dy$

$=\frac{1}{20}\times\left[\frac{y^4}{4}\right]_0^4+\frac{1}{3}\times\left[\frac{y^3}{3}\right]_4^{10}-\frac{1}{30}\times\left[\frac{y^4}{4}\right]_4^{10}$

$=26$

$Var(Y)=E(Y^2)-[E(Y)]^2$

$=26-\frac{14^2}{3}=\frac{38}{9}$

$\therefore SD(Y)=\sqrt{Var(Y)}=\sqrt{\frac{38}{9}}=2.055$

3. There are two food carts serving food at a local park. At any given time, Let A denote the number of customers in line at Food Cart A, and let B denote the number of customers in line at Food Cart B. The joint PMF of A and B is as given in the following table.

A \ B	0	1	2	3
0	0.09	0.05	0.03	0
1	0.01	0.01	0.05	0.04
2	0.08	0.06	0.1	0.07
3	0	0.03	0.01	0.1
4	0.01	0.15	0.05	0.06

a. What is $P(A = B)$, that is, the probability that the numbers of customers in the two lines are identical?
b. What is the probability that the total number of customers in the two lines is exactly four? At least four?
c. Determine the marginal PMF of A and B and then calculate the expected number of customers in line at Food Cart B.
d. If at a given time there are 3 customers in line at Food cart A, what is the probability of 2 customers being in line at Food Cart B?
e. Are A and B independent random variables? Explain.

ANSWER:

a) $P(A = B) = P(0,0) + P(1,1) + P(2,2) + P(3,3) = 0.09 + 0.01 + 0.1 + 0.1 = 0.3$

b) $P(A + B = 4) = P(1,3) + P(3,1) + P(2,2) + P(4,0) = 0.04 + 0.03 + 0.1 + 0.01 = 0.18$
Therefore,
$P(A + B \geq 4) = P(A + B = 4) + P(A + B > 4)$
$= 0.17 + P(4,1) + P(2,3) + P(4,2) + P(3,2) + P(3,3) + P(4,3) = 0.62$

c) The marginal PMF of A and B is given below:

A \ B	0	1	2	3	$P_A(A)$
0	0.09	0.05	0.03	0	0.17
1	0.01	0.01	0.05	0.04	0.11
2	0.08	0.06	0.1	0.07	0.31
3	0	0.03	0.01	0.1	0.14
4	0.01	0.15	0.05	0.06	0.27
$P_B(B)$	0.19	0.3	0.24	0.27	1

The expected number of customers at Food Cart B,
$E(B) = \sum B.P(B) = \sum(0 \times .19) + (1 \times .3) + (2 \times .24) + (3 \times .27) = 1.59$

d) $P(B = 2 \mid A = 3) = \frac{P(A=3,B=2)}{P(A=3)} = \frac{0.01}{0.14} = \frac{1}{14} = 0.0714$

e) We know, a random variable is independent if and only if,
$P(A, B) = P_A(A) \times P_B(B)$ for all values of A and B
Here,
$P(A, B) = P(0,0) = 0.09$
$P_A(A) \times P_B(B) = P_A(0) \times P_B(0) = 0.17 \times 0.19 = 0.0323$
$\therefore P(A, B) \neq P_A(A) \times P_B(B)$
Therefore, A and B are not independent.

Discrete Probability Distributions

4. Only 3 hearts, 7 clubs and 2 spades were found undamaged in a moth-eaten deck of ancient playing cards. Suppose on every single turn, you randomly select a card from the set of these 12 cards, see it and put it back in the deck. You keep doing this repeatedly.
   a. What is the probability that you get the first heart on the 5th turn?
   b. How many turns are expected to get one non-spade card?
   c. What is the variance of the number of turns required to get one club?

ANSWER:

a) $p = probability\ of\ success = P(drawing\ a\ Heart) = \frac{3}{12} = \frac{1}{4}$
$Let, X = number\ of\ tirals\ until\ first\ success$
$P(X = 5) = (1-p)^4 \times p = \left(1 - \frac{1}{4}\right)^4 \times \frac{1}{4} = \frac{81}{1024} = 0.0791$

b) No. of non-spade cards = 10
$p\ =\ probability\ of\ success\ =\ P(drawing\ a\ non-spade\ card)\ = \frac{10}{12} = \frac{5}{6}$
$Let, X = number\ of\ tirals\ until\ first\ success$
Therefore, expected number of turns to get one non-spade card:
$E(X) = \frac{1}{p} = \frac{6}{5} = 1.2$

c) $p = probability\ of\ success = P(drawing\ a\ club) = \frac{7}{12}$
$Let, X = number\ of\ tirals\ until\ first\ success$
$\therefore V(X) = \frac{1-p}{p^2} = \frac{1-\frac{7}{12}}{\frac{7}{12}^2} = 1.2245$

5. Only 4 hearts, 6 clubs and 2 spades were found undamaged in a moth-eaten deck of ancient playing cards. Suppose on every single turn, you randomly select a card from the set of these 12 cards, see it and put it back in the deck. Let's say, you do this 6 times.
   a. What is the probability that you get exactly 4 clubs after 6 turns?
   b. What is the probability that you pick more than 3 clubs after 6 turns?
   c. What is the mean number of hearts picked after 60 turns?
   d. What is the standard deviation of the number of spades picked after 36 turns?

ANSWER:

a) $p = probability\ of\ success = P(drawing\ a\ club) = \frac{6}{12} = \frac{1}{2}$
Total number of trials, $n = 6$
$Let, X = number\ of\ clubs\ drawn$
$\therefore P(X = 4) = {}^6C_4 \times \frac{1}{2}^4 \times \left(1 - \frac{1}{2}\right)^{6-4} = 0.2344$

b) Total number of trials, $n = 6$
$Let, X = number\ of\ clubs\ drawn$
$\therefore P(X > 3) = P(X = 4) + P(x = 5) + P(x = 6)$
$= {}^6C_4 \times \frac{1}{2}^4 \times \left(1 - \frac{1}{2}\right)^{6-4} + {}^6C_5 \times \frac{1}{2}^5 \times \left(1 - \frac{1}{2}\right)^{6-5} + {}^6C_6 \times \frac{1}{2}^6 \times \left(1 - \frac{1}{2}\right)^{6-6} = 0.3438$

c) Probability of drawing a heart, $p = \frac{4}{12} = \frac{1}{3}$; Total number of turns, $n = 60$
Therefore, mean number of hearts drawn, $\mu = E(X) = np = 60 \times \frac{1}{3} = 20$

d) Probability of drawing a spade, $p = \frac{2}{12} = \frac{1}{6}$; and Total number of turns, $n = 36$
Therefore, standard deviation of number of spades drawn,
$\sigma = \sqrt{np(1-p)} = \sqrt{36 \times \frac{1}{6} \times \frac{5}{6}} = \sqrt{5}$

6. Suppose on average, Mymensingh registers 14 fatalities per week from the novel coronavirus.
    a. What is the mean number of fatalities from the novel coronavirus in Mymensingh in a month?
    b. What is the probability that 40 fatalities from COVID-19 will be registered in Mymensingh in the next two weeks ?
    c. What is the probability of at most 9 deaths to be registered from COVID-19 in Mymensingh in a week?

ANSWER:

a) Average number of number of fatalities per week, $\lambda_1 = 14$
So, number of number of fatalities per month (let 1 month = 4 weeks), $\lambda_4 = 14 \times 4 = 56$

b) $\lambda_2 = 14 \times 2 = 28$
$Let, X = Number\ of\ fatalities$
$\therefore P(X = 40) = e^{-\lambda_2} \cdot \frac{\lambda_2^x}{x!} = e^{-28} \cdot \frac{28^{40}}{40!}$ =0.0065

c) $\lambda_1 = 14$
$Let, X = Number\ of\ fatalaties$
$\therefore P(X \leq 9) = \left(\sum_{x=0}^{9} e^{-14} \cdot \frac{14^x}{x!}\right) = 0.1094$

Continuous Probability Distributions

7. The mode and variance of the daily income of 2000 workers are Tk. 100 and Tk^2 36 respectively. The income of workers are distributed normally.
    a. Find the expected number of workers whose daily income are
        i. between Tk. 100 and Tk. 105
        ii. greater than Tk. 110
        iii. less than Tk. 90
        iv. less than Tk. 90 or greater than Tk. 107
    b. Determine the 85th percentile value for the daily income of the 2000 workers.

ANSWER:

Mode=Mean: μ = 100; SD: σ = 6, X~N(100, 36), $Z = \frac{X-\mu}{\sigma}$

a) Expected number of workers:

i. $P(100 < x < 105) = P(0 < z < 0.83) = 0.2967$
Therefore, expected number of workers = 0.2967 X 2000 = 593.4

ii. $P(x > 110) = P(z > 1.67) = 1 - P(z \leq 1.67) = 1 - 0.9525 = 0.0475$
Therefore, expected number of workers = 0.0475 X 2000 = 95

iii. $P(x < 90) = P(z < -1.67) = 0.0475$
Therefore, expected number of workers = 0.0475 X 2000 = 95

iv. $P(x < 90 \cup x > 107) = P(z < -1.67) + P(z > 1.17)$
$= 0.0475 + 0.1210 = 0.1685$
Therefore, expected number of workers = 0.1685 X 2000 = 337

b) Let, A = 85th percentile value for the daily income of the 2000 workers
$P(x < A) = 0.85$
$\therefore P\left(z < \frac{A-100}{6}\right) = 0.85$
$\therefore \frac{A-100}{6} = 1.035$
$\therefore A = (1.035 \times 6) + 100 = 106.21$

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8. Suppose that an average of 30 customers per hour arrive at Shwapno and the time between arrivals is exponentially distributed.
   a. On average, how many minutes elapse between two successive arrivals?
   b. After a customer arrives, find the probability that it takes more than 8 minutes for the next customer to arrive.
   c. After a customer arrives, find the probability that it takes less than 2 minutes for the next customer to arrive.
   d. 70% of the customers arrive within how many minutes of the previous customer?
   e. Which is larger, the mean or the median?

ANSWER:

Let X = Time between arrival of customers (in minutes)

$\lambda = 30\ customers\ per\ hour = \frac{1}{2}\ customers\ per\ minute$

a) $E(X) = \frac{1}{\lambda} = \frac{1}{0.5} = 2\ minutes$

b) $P(x > 8) = e^{-\lambda x} = e^{-\frac{1}{2}\times 8} = 0.0183$

c) $P(x < 2) = 1 - e^{-\lambda x} = 1 - e^{-\frac{1}{2}\times 2} = 0.6321$

d) $P(x < a) = 70\% = 0.70$

$\therefore 1 - e^{-\lambda x} = 1 - e^{-\frac{1}{2}\times a} = 0.70$

$\therefore e^{-\frac{1}{2}\times a} = 0.3$

$\therefore a = 2.408\ minutes$

Therefore, 70% of the customers arrive within 2.408 minutes of the previous customer

e) Let the median be: M

$P(x < M) = 50\% = 0.50$

$\therefore 1 - e^{-\lambda x} = 1 - e^{-\frac{1}{2}\times M} = 0.50$

$\therefore e^{-\frac{1}{2}\times M} = 0.5$

$\therefore a = 1.386\ minutes$

Therefore, the mean is larger