Simulation of a rocket engine

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Brining an object far away from our planet is conveniently done with a rocket. In order to better understand the rocket designee and properties I have through a simple model simulated a rocket engine and estimated features necessary to bring a 1000 kg satellite up to escape velocity from a planet with a radius of 9261 km and a mass of 3.2 times the earth. The planet escape velocity for this satellite is estimated to 16.5 km/s. The focus of the simulation is a number of boxes with hydrogen gas that creates thrust through momentum when escaping a hole at the bottom of the boxes. The pressure, density and temperature of the gas is manipulated. The gas is assumed to be ideal. The model is thoroughly evaluated with and without gravitational force, and with respect to the effect of carried fuel. It is concluded that the model is too simple to give precise quantitative knowledge of real rocked engine.

Introduction

Launching objects from earth to space is important for our everyday life as satellites are used among others in communication, television broadcasting and navigating. Furthermore, in astronomy and astrophysical research gaining access to information of other planets and outer space can in some circumstances be better achieved by moving research equipment out of disturbance of the earth atmosphere. As we continue to explore and understand the universe it might also in the future be of interest to launch object form other planets than our own. Rockets are the preferred spacecraft in launching object from planets. Of importance in rocked design is to estimate the necessary power of the engine and the amount of fuel necessary for a specific mission. Here, I have made a simple model of a rocket engine and explored the model's properties with the goal of understanding the parameters important for the trust and fuel use.

The modeled rocket engine was composed of a fuel storage tank which provided molecular hydrogen to a combustion chamber. The combustion process specifies the specific pressure, density and temperature of the hydrogen, which then is delivered to a number of boxes at the bottom of the rocket. The boxes are cubic and each have a quadratic whole in the bottom with dimension half the length of the individual box floors from where the gas particles are propelled out. The pressure and number of particles in the boxes is kept constant over time causing the particles to escape through the holes with constant momentum. In the model these boxes are the target of simulation and based on that the momentum of the propelled particle the thrust on the rocked is estimated according to newtons laws of motion.

In the process of understanding the model a set of intermediate developmental stage was characterized. First, the necessary number of boxes needed to bring a rocket of mass of 1000 kg to escape velocity from the planet within 20 minutes is estimated. The amount of fuel needed for this achievement is also estimated. Here, the gas was simulated numerically as individual particles. Next, the model is extended to include the mass of fuels and in addition to the force from thrust, the force from gravitation was added. Here, the gas was simulated analytically by use of the ideal gas law. The temperature and density of the gas in addition to the number of boxes was varied with the aim of finding a combination that used least energy to achieve escape velocity of the planet. Theoretically, increasing the temperature will increase the momentum of the molecules, which leads to more thrust per fuel consumption. However, increasing the temperature also involved energy use. Increase in the number of boxes will increase the thrust proportional to the fuel usage as number of particles passing through the engine just increase with time. Increasing the density of the gas can be done by reducing the box size while maintaining the same number of particles. This will increase the pressure and presumably the thrust as and more particles will leave the box per time [1]. Here also, the fuel consumption will increase.

Method

Escape velocity

Escape velocity is reached when the kinetic energy of an object is above the energy of the gravitational field created by the planet and the object. When an object has reached this velocity, it can I principle move to an infinity distance with respect to the planet. Obviously in space there are other gravitational field to consider. The following equation is derived from the kinetic energy and the potential energy of the gravitational field and determines the escape velocity of the planet

$$v_{esc} = \sqrt{\frac{2GM}{R}} \tag{1}$$

[1], where v_{esc} is escape velocity, G is the gravitational constant, M the mass of the planet and R the distance between the planet and the escaping object.

Model parameters of the cubic boxes

One cubic box with side lengths 10^{-6} m was initially filled with 10^5 particles using a uniform probability distribution for all position coordinates to evenly spread out the particles in the box. As the gas is assumed to be an ideal gas the velocity follows the Maxwell-Boltzmann distribution, which is a normal distribution,

$$P(v) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{v^2}{2\sigma^2}}$$

where the standard deviation $\sigma = \sqrt{kT/m}$ [1]. T is temperature which was set to 10^4 K. k is Boltzmann's constant and m is the mass of the particles which for hydrogen molecules is $3.32 \cdot 10^{-27}$. As the three-dimensional coordinates of velocity is independent of each other the distribution can be written

$$P(v) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{v_x^2}{2\sigma^2}} \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{v_y^2}{2\sigma^2}} \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{v_z^2}{2\sigma^2}}. \tag{2}$$

Thus, the velocity of the particles in each dimension can be initiated from the individual normal distributions.

Numerical expressions for kinetic energy and absolute velocity
The mean velocity $\langle v \rangle$ and mean kinetid energy $\langle KE \rangle$ of the particles in an ideal gas can be calculated using [1]

$$\langle v \rangle = \sum_{i=1}^{N} |v_i| \quad \text{and} \tag{3}$$

$$\langle KE \rangle = \sum_{i=1}^{N} \frac{1}{2} m |v_i|^2. \tag{4}$$

Analytical expressions for kinetic energy and absolute velocity
The mean kinetid energy $\langle KE \rangle$ of the particles in an ideal gas can be calculated with

$$\langle KE \rangle = \frac{3}{2} kT \tag{5}$$

[1]. The mean absolute velocity in an ideal gas can be calculated using:

$$\langle v \rangle = \sqrt{\frac{8}{\pi m} kT} \tag{6}$$

[2].

Numeric pressure
As the gas particles are assumed to behave as in an ideal gas, they only change velocity upon colliding with a wall of the box. The collision is elastic and thus the velocity change orientation, but the absolute velocity stays the same (fig. 1). Between the collisions the position follows the algorithm $r_{n+1} = r_n + \Delta t * v$, where r is the position of the particles. This can be called a forward Euler, which is more thoroughly described below, however here it also an equation of motion as the particle does not change direction. To estimate the pressure, I follow the particles for 10^{-9} s in timesteps of 10^{-12} s and summate the momentum of the particles that come in contact with the bottom wall of the box (negative z-direction). To rationalize the algorithm, I only update the collision change of the particles in the z directions ignoring changes in x and y direction. Thus, in practice I don't only summate particles that hit the bottom wall of the box, but all particles that pass an extended bottom floor (particles with position bellow the x-y plane). From the total momentum $p = 2vm$, I calculate the pressure using the definition of pressure

$$P = \frac{\Delta p / \Delta t}{A}, \tag{7}$$

Where $\Delta p / \Delta t$ is the force and A is the area of the box floor. Here Δp is estimated to be 2 times the moment of the particles (p) that collide with the wall as $\Delta p = p_{before} - p_{after} = 2p$.

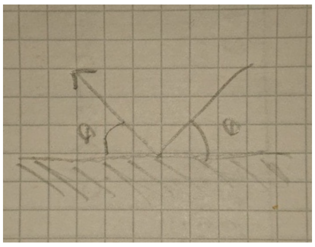

Fig 1. Schematic drawing of an elastic collision between the particle and the wall of the box.

Analytical pressure

From the ideal gas law, the pressure in a box can be calculated by

$$P = \frac{NkT}{V}, \tag{8}$$

where N is the number of particles in the box and V is the volume of the box.

Simulation of the rocket engine

In order to transform the boxes into part of an operational rocket engine a quadratic hole with side length half the box width is placed at the center of the bottom floor of the box. To numerically estimate the momentum over time I summed up the downward momentum of particles that escape from the hole. To calculate the fuel loss, I summated the number of particles escaping from the hole. To refill the box with fuel I assumed that the particles escaping the box was duplicated at the moment of escape with one particle returning into the box in as it experienced an elastic collision with the wall. The simulation ran over 10^{-9} s in timesteps of 10^{-12} s.

Speed gain of rocket

In understanding my rocket model, I first estimated the speed gain over a 10^{-9} s time period by numerically sum the amount of momentum leaving the hole in direction normal to the planet surface. From this I have the relatively more amount of momentum reaching the roof of the box compared to the floor of the box. Thus, this is the momentum responsible for the thrust accelerating the rocket. By conservation of momentum and Newton's third law, the rocket's momentum changes by this same amount in opposite direction [1]. Thus, the speed gain of a rocket composed of a single box is

$$\Delta v_{rocket} = -\frac{\Delta v_{hydrogen}\, m_{hydrogen}}{m_{rocket}}. \tag{9}$$

Escape velocity after 20 min with constant weight and zero gravitational pull

From the speed gain of one box driving the satellite I calculated the number of boxes needed to reach escape velocity within 20 min by first estimating the speed gain achieved by one box in 20 minutes; and then dividing the escape velocity with that value

$$boxes = -\frac{v_{escape}}{\Delta v_{rocket}\; time_{20\ minutes}/10^{-9}}\,. \tag{10}$$

Fuel consumes

The fuel loss of one box from the simulation per timestep of 1e-9 s was multiplied by the mass of hydrogen and number of boxes and corrected for time with the formula

$$fuel = m_{hydrogen} * boxes * particles * time_{20\ minutes}/10^{-9}. \tag{11}$$

Analytical acceleration

From the equation of state (equation 8), the pressure in each box can be calculated from gas density and temperature,

$$P = N\,k\,T/V.$$

Further, thrust on the rocket, and thus acceleration, can be estimated from pressure. As noticed above the numerical estimation of pressure was a result of two times the momentum of each particle colliding with the wall and the thrust was only a result of one time the momentum leaving the boxes. In similar manner I have dived the pressure by two to estimate the analytical trust. As I understand it this is because the upward force of the rocket is the momentum against the roof minus the momentum leaving the rocket at the bottom per time. The momentum leaving the rocket is half the size of the momentum against the roof. Thus, the force of the thrust is

$$F = P\,A\,n,$$

where A is the area of the hole and n is the number of boxes. In this part of the simulation I also include change in fuel mass of the rocket over time and the gravitational pull of the planet as described by the Newtons law of gravitation

$$F = -G\frac{Mm}{r^2},$$

where F is the force pulling the rocket against the earth. The minus sign is necessary because the positive direction om movements is away from the earth. G is the gravitational constant, M is the mass of the earth and m is the mass of the rocket. r is the distance between the earth and the rocket. Newton second law, $F = ma$ is used find the acceleration from the force of gravitation. This leads to a second ordered differential equation which is solved numerically [1, 3]. Briefly, I have used the numerical integration algorithm to update velocity and position called Euler-Cromer. This method is an improvement of the forward Euler and it is proven to be more stable [1, 3]. Forward Euler can be derived from the definition of acceleration in the following way;

$$a = \lim_{dt \to 0} \frac{v(t + dt) - v(t)}{dt}$$

$$\lim_{dt \to 0} v(t + dt) = v(t) + a\,dt$$

$$v(t + \Delta t) \approx v(t) + a\,\Delta t.$$

Here, the infinitesimal *dt* is changed for a real value Δt. The numeric equation for determining position over time is then derived from the velocity in the same manner. However, in the Euler-Cromer the velocity of the next time step is used

$$r(t + \Delta t) = r(t) + v(t + 1)\Delta t).$$

Results

Escape velocity of planet
The escape velocity of an object from the planet was estimated to 16539 m/s using equation 1.

Initial properties of the particles in a single box
A graphical representation of the of the initial positions of the particles in a box confirms a uniform distribution, and a visual examination of a histogram over the velocity distribution in a box clearly suggests normally distributed velocities (fig. 2). The velocities were derived from equation 2.

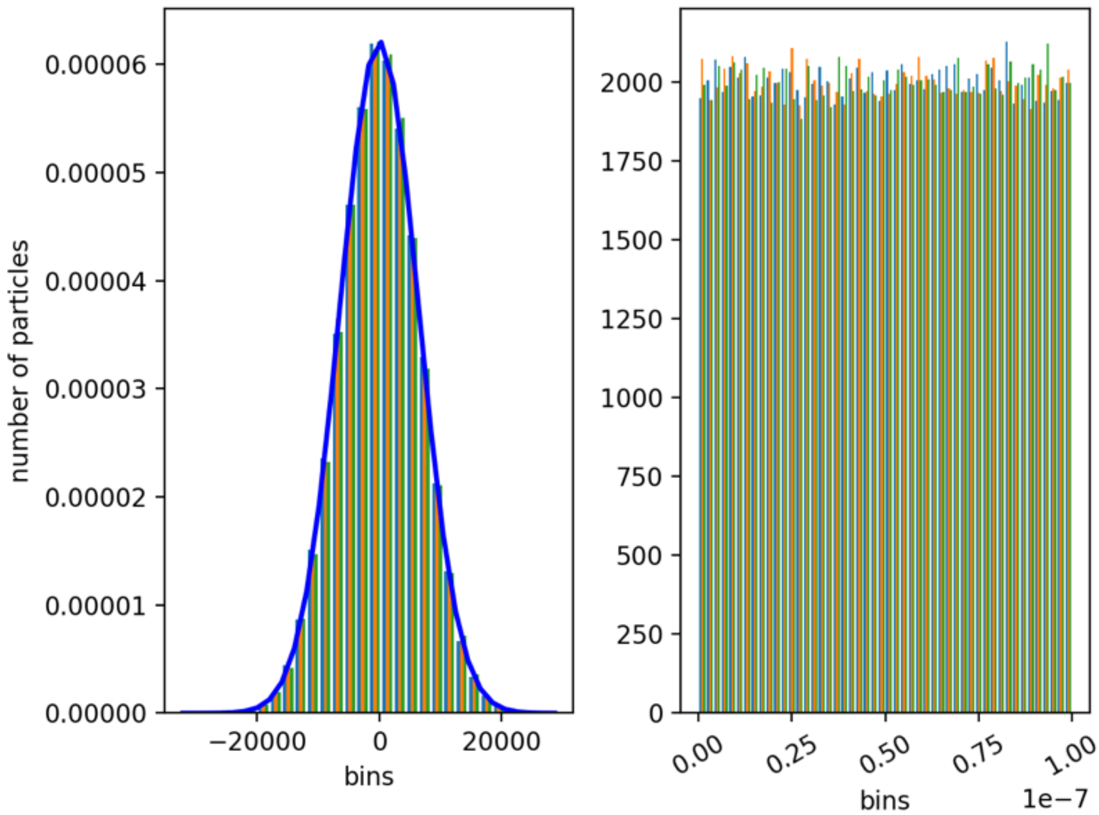

Fig. 2. Histograms of the initial velocity (left panel) and positions of the particles in a box (right panel). The three different colors represent the three orientations. The velocity plot has included a normal distribution of same standard deviation as used to derive the velocity to confirm normality of data. The of positions of the particles show clear uniform distribution.

To confirm that the properties of the particle in the initial configuration is in agreement with expected values, the mean velocity and mean kinetic energy of the particles in a box is compared to the theoretical mean velocity and mean kinetic energy of particles in an ideal gas with same temperature and mass using the equation 3-6. The differences were 0.04% m/s for the absolute velocity and 0.01% J for the kinetic energy (table 1). Furthermore, the numerical calculation of the pressure is compared to the analytical calculation by equation 7 and 8, respectably (table 1).

Table 1. Comparison of velocity and kinetic energy of the numerically created particles in a box compared to the theoretical values.

	Numerical	Analytical	Difference
Mean absolute velocity [m/s]	10 252	10 248	0.04%
Mean kinetic energy [J]	$2.072 \cdot 10^{-19}$	$2.071 \cdot 10^{-19}$	0.01%
Pressure [Pa]	13729	13806	0.5%

Speed gain and fuel use of a rocket

To understand the function of the rocket model, I first estimated the speed gain over a 10^{-9} s time period. The numerical calculation suggested the amount of momentum leaving the hole in a single box to be $1.734 \cdot 10^{18}$ kg m/s. When using conservation of momentum (equation 9) this leads to an estimate of the speed gain of a rocket composed of a single box with total weight of 1000 kg to be $1.734 \cdot 10^{-21}$ m/s. To further explore the properties of the rocked I estimated the number of boxes needed to bring the 1000 kg rocked to escape velocity within 20 minutes using equation 10. In my simulations suggested number of boxes were $7.95 \cdot 10^{12}$. As each box releases $2.52 \cdot 10^{6}$ particles during 10^{-9} s of the simulations. The total fuel loss within these 20 minutes became 80501 kg (equation 11).

Simulation with fuel and gravitational field

To expand the model into more realistic sceneries fuel was included in the total mass and the force from the gravitational field of the planet was added in addition to the thrust (fig. 3). The fuel mass was updated at each time step based on calculation of fuel use as described above. First, I tested the parameters from the simulation above where the rocket of 1000 kg reached escape velocity. The rocked had a negative velocity during the whole time because the gravitational force overcame the thrust. I adjusted the number of boxes just slightly because the total volume already was extremally small. In my calculation 10^{-6} m^3. I adjusted to 10^{14} boxes giving a total volume of 10^{-4} m^3 with side length of 46.4 mm. The rocked now reached escape velocity within 1012 s. I noticed that the rocked still had small negative speed in the beginning. As this is not feasible, I changed the simulation so that negative speed is not possible (fig. 3C).

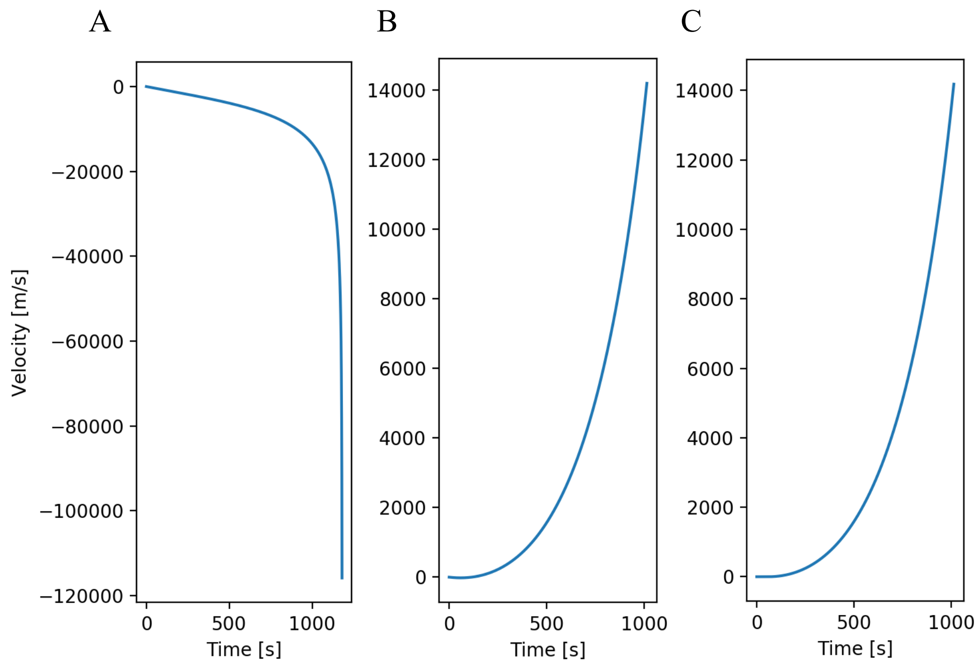

Figure 3. Simulation of rocket engine with fuel and gravitational field. A) Same parameter as used in the engine with gravitational force of the earth and on-board fuel. The gravitational force drags the rocked down. B) Increasing the number of boxes by 10 times gave the rocket enough velocity to escape the gravitational field. Hover in the beginning of the mission the rocket is still dragged into the ground. C) Implementing an algorithm that prevent dragging the rocket into the ground.

Next, because the temperature of the gas is to my knowledge higher than any material can handle, I lowered the temperature to 2000 K (fig. 4). The rocked reached escape velocity but it was standing on the ground burning fuel before it was light enough to take off. It could have started with less fuel as the fuel on board was based on a 1200 s operation time of the engine. However, the goal is to use around 1200 s to reach escape velocity. Just by increasing the number of boxes to 10^{15} the rocked reached 1190 s, but the take-off problem was still prominent (fig. 4B), and the rocked used 52 times its weight in fuel. I could to some extend overcome the problem by increasing the gas density in each box by reducing the side walls down to $5 \cdot 10^{-9}$ m and reducing the box number to $1.5 \cdot 10^{12}$ m (fig. 4C). The rocket now reached escape velocity in 1171 s and used 9 times its weight in fuel. The pressure is $1.2 \cdot 10^{7}$ kPa and the size of the boxes is very small.

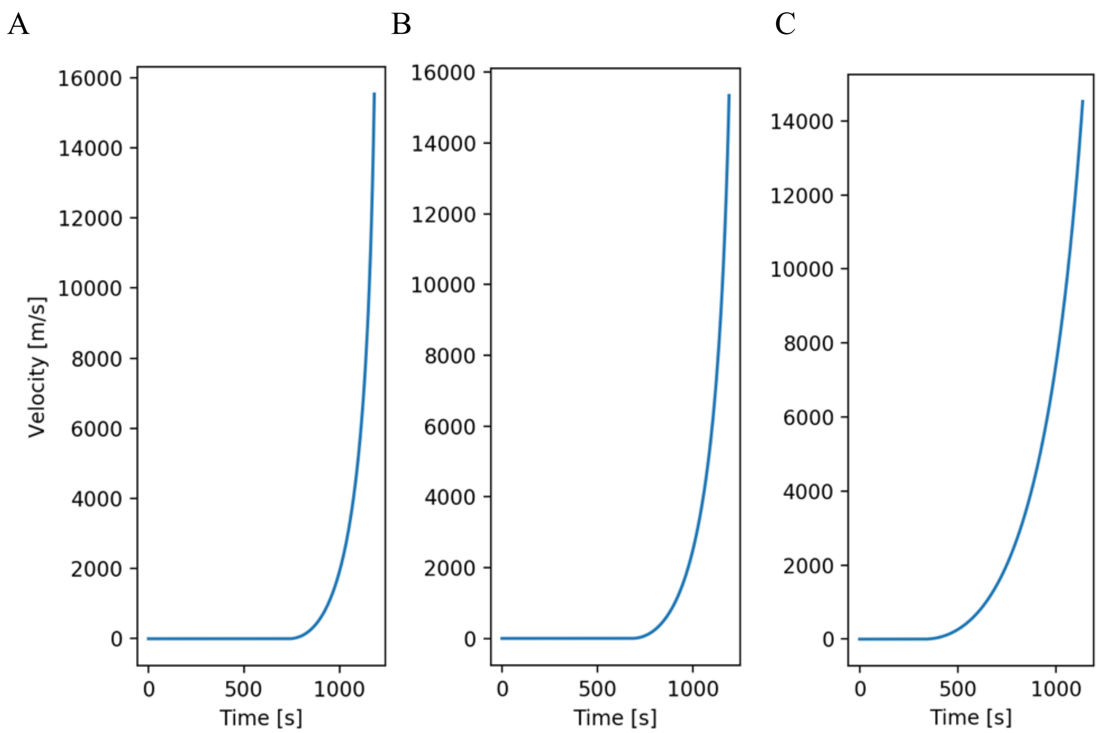

Figure 4. A) Gas temperature set to 2000K. B) increasing the number of boxes to 10^{15}.
C) Lowered box number with increased gas density.

Finally, I went back to a temperature of 10 000 K and focused on using as little fuel as possible. Box length of $5 \cdot 10^{-7}$ m with $2.1 \cdot 10^{13}$ boxes reached escape velocity in 1191s using 6.2 times the rocket weight in fuel (fig. 5C). Reducing the density with box length of $2 \cdot 10^{-6}$ m I needed $3.8 \cdot 10^{13}$ boxes to reach escape velocity at 1187s with fuel consume of 5.7 times rocked weight (fig. 5B). In the end the lowest fuel use become with box length of 10^{-6} with $3.8 \cdot 10^{13}$ m boxes reaching escape velocity in 1187s using 5.2 times the weight in fuel (fig. 5C).

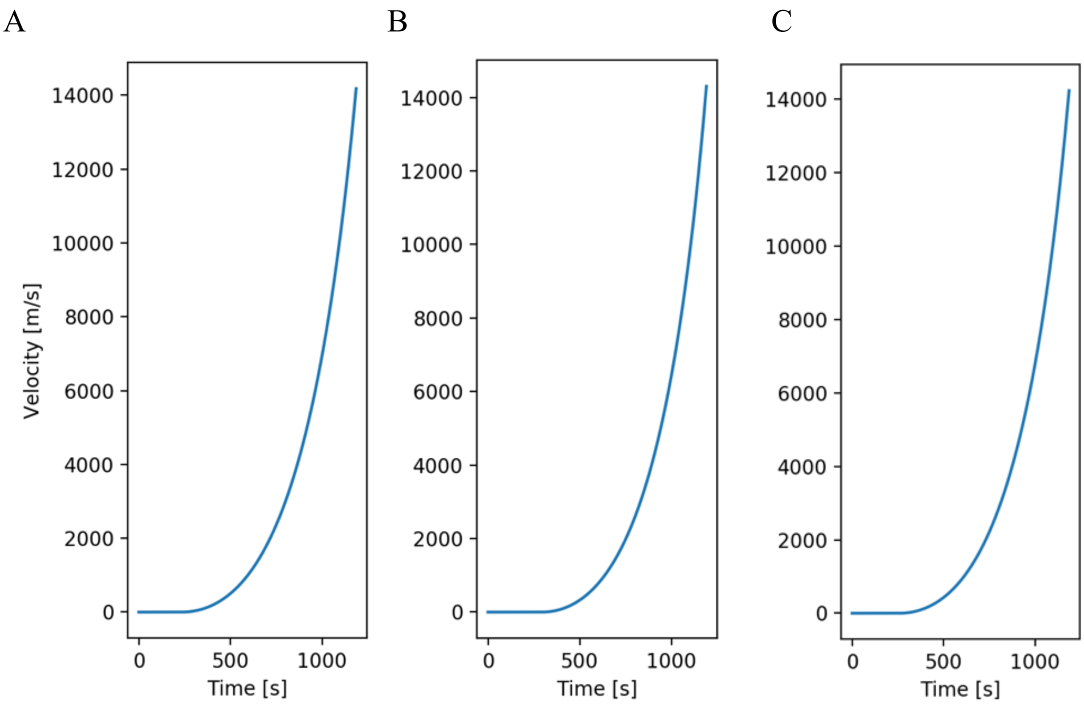

Figure 5. Fine adjustment of box number and gas density in order to use as little fuel as necessary to reach escape velocity at a time point around 20 minutes of launch.

Discussion

My model of a rocket engine assumed the hydrogen gas to follow the Ideal gas law. At the extremely high temperature and gas density this might lead to a flaw in the model. I have also omitted the friction from air which is of relevance at these high velocities. One goal was the determine the most energy saving parameter setting. This was not fully possible as I could not estimate the energy consumed by heating the gas. However, neglecting the temperature, I found parameters that caused a temperature use of 5.2 times the weight of the rocket. This is lower but in the same order as real rockets, which is in the range 8-20. The model used hydrogen molecules to give a driving moment and thrust, while a normal hydrogen rocket burns the hydrogen by oxygenation in the combustion chamber producing water which is propelled out. The temperatures of the gas were likely higher than normal material can handle. Highest laboratory made material, to may knowledge, can manage up to 4000 K. In conclusion, I believe this model is far from useful to quantitively describe a real rocked engine, however in includes very relevant parameters and can gain some qualitative understand of the operation of the rocket engine.

Reference
[1] Hansen F. K. (2020) AST2000 Lecture Notes; part 1A
[2] Hansen F. K. (2020) AST2000 Solution to Exercise 1A.5
[3] Kielland A. (2020) AST2000 Solution to Exercise 1B.7