Numeric simulation of planetary orbits in a solar system

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I have here simulated the orbit of eight planet in a one-star solar system by numerical simulations. I assume a two-body problem between each planet and the star ignoring forces between the planets. I have evaluated the numerical methods forward Euler, Euler-Cromer and Runge-Kutta four with respect to trajectory of the planets and energy stability. In addition, I have simulated a solar system with two stars and one planet including forces between all three objects. I found that RK4 is superior to forward Euler and Euler-Cromer and Euler-Cromer to be far superior to forward Euler. The eight planets of the on star solar system showed stable elliptical trajectories. The planet of the two-star solar system on the other hand shoved a peculiar trajectory and the planet appeared to be thrown out of the solar system with time.

Introduction

Of fundamental importance in understanding the astrophysical world is the description of the position and movement of celestial objects in relation to each other. Central to this is the orbiting behavior. Here I have examined the orbits of planets within two modeled solar systems using numerical approaches. One solar system is composed of one star and eight planets, while the other contains two stars and one planet. To characterize the orbits of the eight planets in a one-star solar system I only included gravitational forces from the star on the planets, omitting all forces from the planets. This reduces the problem to a classical two-body problem [1]. I used numerical simulation to solve the problem. However, the two body problem have an analytical solutions which describes the trajectory of the orbits as elliptical [1]. In the second solar system I include all forces between the stars and the planet. The system is not much more complex than the above described system, but nevertheless with no analytical solution [1]. Interestingly, about half of the know stars are binary stars [1].

In order to describe motions of astronomical objects, numerical approaches are of high relevance as the complexity of such systems is most often out of scop for an analytically approach. There are numerous numerical methods that are suitable for computer-based algorithms. I have evaluated here evaluated the forward Euler, Euler-Cromer, Runge-Cutta four methods as they are extensively in use and fast and easy to implement. Conservation of energy and deviation between trajectories of successive orbits is used to compare the and evaluate the stabilities of the various method.

The mass of the star was 1.99 sun masses. The masses of the planets is presented in table 1 together with their initial position and velocity.

Precise knowledge of orbits to nearby planets is necessary information for potential space voyages. Furthermore, planetary orbits will likely influence the environment of the planet and thus is of interest to one of the greatest questions to humankind: do their exits biological life on other planets. Surely, emphasizing the relevance of these types of studies.

Table 1. Overview of the masses and initial position and velocities of the planet in the solar system. Astronomical units (AU) is the distance between the earth and the sun.

Planet	Mass (earth masses)	x-position (AU)	y-position (AU)	x-velocity (m/year)	y-velocity (m/year)
0	3.18	1.943	0.0	0.0	30053
1	0.620	0.260	2.649	-25799	3605
2	814	-16.89	5.594	-3667	-9465
3	0.287	-5.025	-3.487	9871	-13523
4	0.00179	-12.518	1.465	-1576	-11721
5	0.136	-7.246	1.671	-3542	-15219
6	0.0306	2.786	-2.665	14460	15299
7	0.0171	1.194	0.499	-15467	33936

Method

Simulation of the one-star solar system

In the one star-system I have modeled the planets trajectory based on the forces between the respective planet and the star. I have neglected interplanetary forces and external forces of the solar system. To simplify the problem further I placed the star in the origo and only characterized the orbit of the planets. Since the mass of the star is in order 1000 times the largest planets the movement of the star caused by gravitational pull from the planets is of minor importance to describe the orbits of the planets. To simulate the force on the planets, I used classical mechanics which for this system follows Newton's law of universal gravitation given by

$$\boldsymbol{F} = -\frac{G\,M\,m}{r^2}\boldsymbol{e}_r,$$

where $\boldsymbol{F}$ is the force, *G* is the gravitational constant, *M* is the mass of the star, *m* is the mass of the planet, *r* is the distance between the star and the planet and $\boldsymbol{e}_r$ is the unit vector from the star to the planet. As the forces of interest here is the attractive forces on the planet by the star while the unit vector points from the star to the planet a minus sign is included in the equation.

Numerical integration methods

To translate force on the planets into description of motion I have used Newton's second law of motion. Newton second law can be written as $F = ma$, which leads to $a = F/m$. *a* is the acceleration, which is the derivative of the velocity *v*. *v* is the derivative of the position *r*. Thus, this leads to a second ordered differential equation. To implement a numerical solution of the equation it is first written as two coupled first-order differential equations

$$\frac{\mathrm{d}v}{\mathrm{dt}} = \mathrm{a(t)}, \qquad \frac{\mathrm{d}r}{\mathrm{dt}} = \mathrm{v(t)}$$

The second step can be understood through the definition of the derivative. Her presented for *a*

$$a = \lim_{dt \to 0} \frac{v(t + dt) - v(t)}{dt}$$

This lim is converted to a numeric relation by assuming linear acceleration when the time steps are very small. Based on this assumption the infinitesimal *dt* is substituted for a real value Δt

$$a \approx \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

Solving for $v(t + \Delta t)$ gives the equation

$$v(t + \Delta t) = v(t) + a\Delta t.$$

Numerical implementation of change in position over time can be derived equivalently as for velocity. However, by changing *a* for *v* and *v* for *r* in the derivation above. This leads to

$$r(t + \Delta t) = r(t) + v\Delta t.$$

Thus, for each time step firstly the velocity is updated based on the acceleration that is derived from the forces on the object. Thereafter the velocity is used to update the position. This method is caller forward Euler [1]. An extension of this method is the Euler-Cromer which use the velocity at the end of the step $v(t + \Delta t)$ instead of $v(t)$ to update the position. The Euler-Cromer is shown to perform better in energy conserved systems, which is relevant for the simulations here [2]. The following set of equations describes the total algorithm of one timestep for Euler-Cromer,

$$\boldsymbol{v}(t + 1) = \boldsymbol{v}(t) - \frac{G\ M\ m}{r^2}\boldsymbol{e}_r\Delta t,$$

$$\boldsymbol{r}(t + 1) = \boldsymbol{r}(t) + \boldsymbol{v}(t + 1)\Delta t.$$

The forward Euler use the slope at the beginning of the given step to calculate the next value, while the Euler-Cromer used the velocity slope at the end of the step. An even more refined method uses a combination of the beginning step, the final step, but also two intermediate steps to estimate the slope. This method is called the fourth order Runge-Kutta and it is regard as significantly more accurate than the two other methods [2]. In principle the forward Euler is used four times and a slope value k is calculated at four locations. First a half step with the slope k1 calculated at the beginning of the step. Second another half step from the beginning, but now with the slope calculated at the end of the previous half step k2. The slope is also calculated at the end of this half step giving k3. Finally, a full step from the beginning with slope k3. The last slope k4 is calculated at the end of this final step. The actuall slope used to move on step uses the forward Euler with the weighted average of these four slopes

$$k_{average} = (k1 + 2k2 + 2k3 + k4)/6,$$

where *k* is either *a* of *v* depending of whether it is the next velocity or position that is determined, respectably. It can be regarded as a combination of forward Euler, Euler-Cromer and evaluations at the midpoint of the timestep and is among the more popular numerical methods of differential equations [2]

Numerical integrations of the two-star system

I have approached the binary-star-one-planet system as a Newtonian three-body problem, which means I have used all the gravitational interaction between the three objects. I have used the same numerical approach as described above besides the force acting on each object is now from the two other objects instead of one.

Energy calculations

As the orbit of a planet is an energy conserved system, I calculated the sum of the kinetic energy KE and potential energy U of the simulations at each time step with

$$KE = \frac{1}{2}m\boldsymbol{v}^2,$$

$$U = GmM.$$

Results

Orbits of the planets using various numerical integration methods

To compare the three numeric methods, I ran the different algorithms with the same timesteps of 526 hours which is just enough time for the outermost planet to orbit one time (fig. 1). The orbits, as known from the analytical solution of the two-body problem, are not supposed to deviate over time. From the plot of the trajectories it is clear that this is not the case for the forward Euler. The Euler-Cromer is substantially better, however a closer inspection revealed multiple trajectories of the innermost planets. The Runge-Kutta 4 showed an improvement over Euler-Cromer, but the innermost planet showed multiple trajectories. Notably, the distance between the orbits of the planets is changing. This is in accordance with the analytical solution of the Keppler two body problem, which is elliptical and eccentric [1].

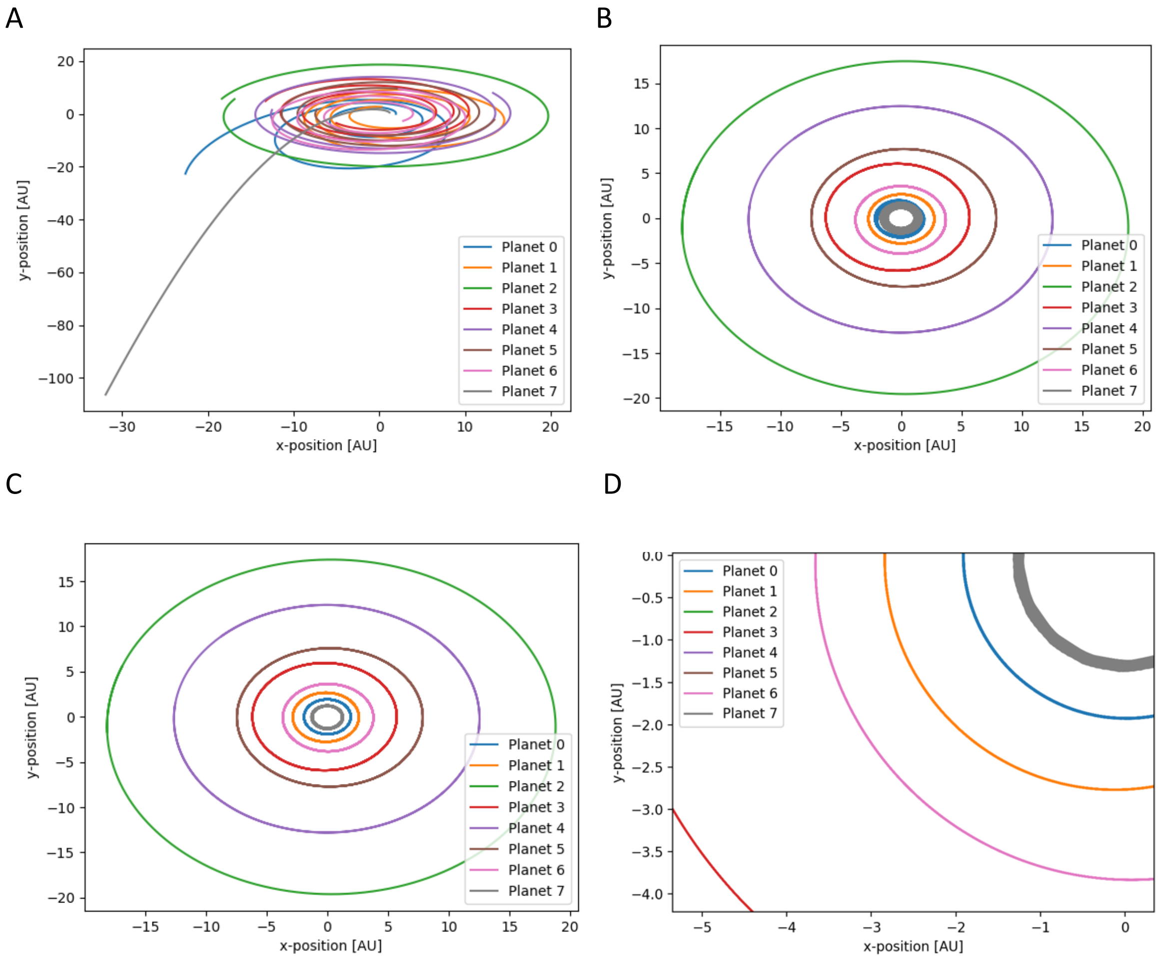

Figure 1. Trajectories of the eight planets for 60 days with three different numerical methods with same time step. A) Forward Euler gives an unstable system. B) Euler-Cromer is significantly more stable, but deviation between the different orbits particularly or the innermost planets. C) Runge-Cutta 4 performs best, however close inspection of the innermost orbit reveals some instability here, as seen in panel D.

Energy conservations of the different numerical methods

As the planet orbit conserves energy, I compared the change in the sum of potential energy and kinetic energy as given in equation 1 and 2. (fig. 2). Forward Euler was substantially worse than the two other methods. Euler-Cromer and Runge-Kutta four showed for some planet the same variation in energy, but for two planets the Runge-Kutta four performed a little better.

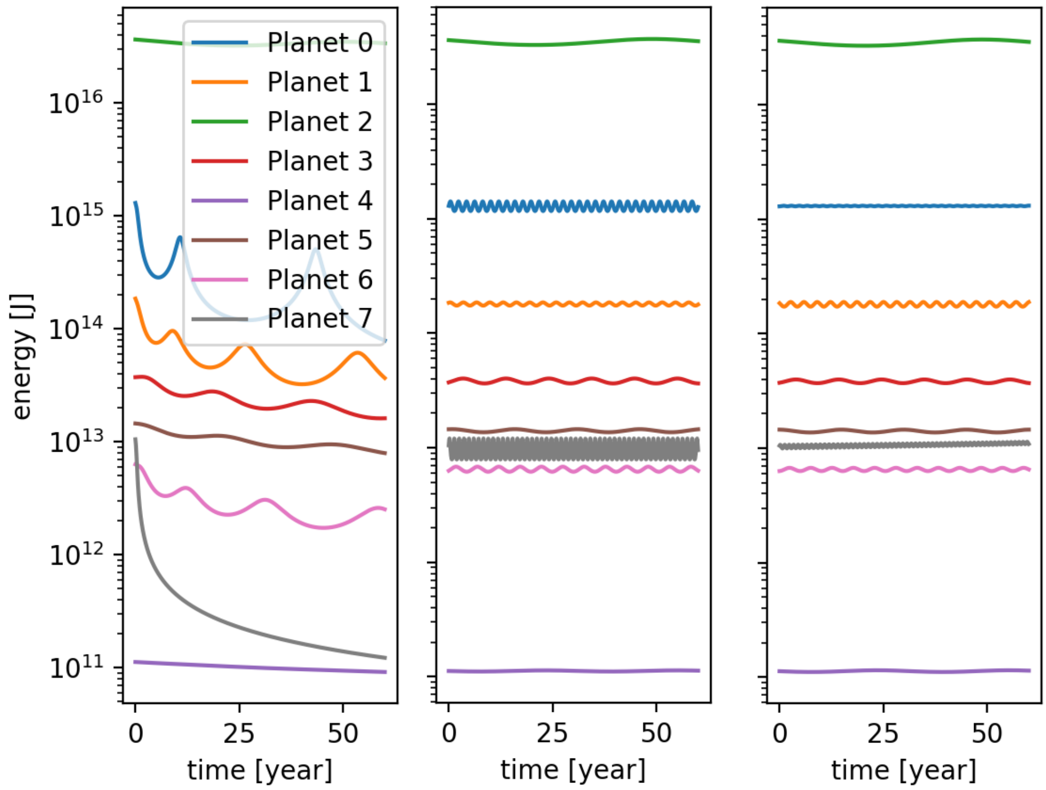

Figure 2. Total mechanical energy over orbit time for three numerical methods. From left to right forward Euler, Euler-Cromer and Runge-Cutta 4.

The two-star one-planet solar system

The solar system of the three objects show circular orbit of the two stars, however the planet follows a completely crazy trajectory where it with time is thrown out of the solar system (fig. 3). Runge-Kutta was used with time step of 400 000 s.

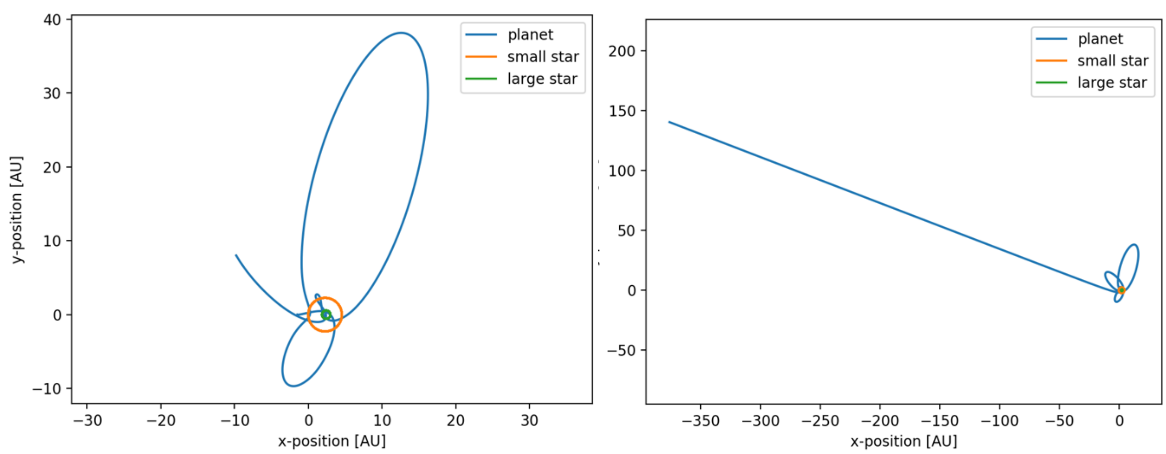

Figure 3. The orbits of the two-star one-planet solar system. The stars have circular orbits. The planets have something else.

Discussion

The numerical solution led planet orbiting the sun with a slight elliptic trajectory. This in accordance with Kepler's third law of planetary motions [1]. The Euler-Cromer method was substantially better than forward Euler. Since they have the same computational time Euler Cromer is clearly preferred. Runge-Kutta four was better than Euler-Cromer, however the difference was less. Runge-Kutta four contains 2.5 times more computational steps, which have to be taken into account when comparing the methods. I was little surprised by this as Runge-Kutta four is often presented as a far superior method to the other two [2]. None of the methods worked perfectly as seen for the orbit of the innermost planets and the energy analysis, however this could have been substantially improved by decreasing the length of the timesteps.

With respect to environmental condition for life of the different planet this will be particularly challenge for the planet in the two-star system. The temperature differences will be enormous making biochemical reactions as we know them impossible. In the one-star solar system the planet follows orbits similar to our solar systems. As I don't know the geological and chemical composition of this planets I can only speculate with respect to temperature. The innermost planet is approximately twice the distance away for the star compare to the earth and sun, However the star is twice the since of the sun, which makes it warmer. Therefore, I believe there could potentially be life on this planet.

Reference

[1] Hansen F. K. (2020) AST2000 Lecture Notes; part 1B
[2] Vistnes A. I. (2018) Book: Physics of oscillations and waves