Part II Numerical Analysis (J6) Lent Term, 2000

Exercise Sheet 3 [1]

25. Find the intervals of absolute stability on the real line of the following methods:

(1) $y_{n+1} = y_n + h\,f_n$ (2) $y_{n+1} = y_n + \frac{1}{2}h\,(f_n + f_{n+1})$

(3) $y_{n+2} = y_n + 2h\,f_{n+1}$ (4) $y_{n+2} = y_{n+1} + \frac{1}{2}h\,(3f_{n+1} - f_n)$

(5) The RK method $k_1 = f(t_n, y_n)$, $k_2 = f(t_n{+}h,\, y_n{+}hk_1)$, $y_{n+1} = y_n + \frac{1}{2}h\,(k_1{+}k_2)$.

26. Show that, if z is a nonzero complex number that is on the boundary of the absolute stability region of the two step BDF method

$$y_{n+2} - \tfrac{4}{3}y_{n+1} + \tfrac{1}{3}y_n = \tfrac{2}{3}hf_{n+2},$$

then the real part of z is positive. Thus deduce that this method is A-stable.

27. The stiff differential equation

$$y'(t) = -10^4\,(y{-}t^{-1}) - t^{-2}, \qquad t\geq 1, \qquad y(1)=1,$$

has the analytic solution $y(t)=t^{-1}$, $t\geq 1$. Let it be solved numerically by Euler's method $y_{n+1} = y_n + h_n f_n$ and the backward Euler method $y_{n+1} = y_n + h_n f_{n+1}$, where $h_n = t_{n+1} - t_n$ is allowed to depend on n and to be different in the two cases. Suppose that, for any $t_n \geq 1$, we have $|y_n - y(t_n)| \leq 10^{-6}$, and that we require $|y_{n+1} - y(t_{n+1})| \leq 10^{-6}$. Show that Euler's method can fail if $h_n = 2\times 10^{-4}$, but that the backward Euler method always succeeds if $h_n \leq 10^{-2} t_n t_{n+1}^2$.
Hint: Find relations between $y_{n+1} - y(t_{n+1})$ and $y_n - y(t_n)$ for general y_n and t_n.

28. This question concerns the predictor-corrector pair

$$\left.\begin{array}{rcl} y_{n+3}^{(p)} & = & -\frac{1}{2}\,y_n + 3\,y_{n+1} - \frac{3}{2}\,y_{n+2} + 3h\,f_{n+2} \\ y_{n+3}^{(c)} & = & \frac{1}{11}\,(2\,y_n - 9\,y_{n+1} + 18\,y_{n+2} + 6h\,f_{n+3}) \end{array}\right\}.$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value $\frac{6}{17}\,|y_{n+3}^{(p)} - y_{n+3}^{(c)}|$.

29. Let p be the cubic polynomial that is defined by $p(t_j) = y_j$, $j = n, n+1, n+2$, and by $p'(t_{n+2}) = f_{n+2}$. Show that the predictor formula of the previous exercise is $y_{n+3}^{(p)} = p(t_{n+2}{+}h)$. Further, show that the corrector formula is equivalent to the equation

$$y_{n+3} = p(t_{n+2}) + \tfrac{5}{11}\,h\,p'(t_{n+2}) - \tfrac{1}{22}\,h^2 p''(t_{n+2}) - \tfrac{7}{66}\,h^3 p'''(t_{n+2}) + \tfrac{6}{11}\,h\,f(t_{n+2}{+}h, y_{n+3}).$$

The point of these remarks is that p can be derived from available data, and then the above forms of the predictor and corrector can be applied for any choice of $h = t_{n+3} - t_{n+2}$.

30. Let $u(x)$, $0\leq x\leq 1$, be a six times differentiable function that satisfies $u''(x) = f(x)$, $0\leq x\leq 1$, $u(0)$ and $u(1)$ being given. Further, we let $x_m = mh = m/M$, $m = 0, 1, \ldots, M$, for some positive integer M, and we calculate the estimates $U_m \approx u(x_m)$, $m = 1, 2, \ldots, M{-}1$, by solving the difference equation

$$U_{m-1} - 2\,U_m + U_{m+1} = h^2 f(x_m) + \alpha\,h^2\,[f(x_{m-1}) - 2\,f(x_m) + f(x_{m+1})]\,, \quad m = 1, 2, \ldots, M{-}1,$$

where $U_0 = u(0)$, $U_M = u(1)$, and α is a positive parameter. Show that there exists a choice of α such that the local truncation error of the difference equation is $\mathcal{O}(h^6)$. In this case, deduce that the Euclidean norm of the vector of errors $u(x_m) - U_m$, $m = 0, 1, \ldots, M$, is bounded above by a constant multiple of $\|u^{(6)}\|_\infty\, h^{7/2}$, and provide an upper bound on this constant.

[1] Please send any corrections and comments by e-mail to mjdp@cam.ac.uk

31. Let f be a smooth function from $\mathcal{R}$ to $\mathcal{R}$, and let $f^{(k)}$ denote its k-th derivative. Further, let Δ_0 be the "central difference" operator $\Delta_0 f(mh) = f(mh+\frac{1}{2}h) - f(mh-\frac{1}{2}h)$ and Υ be the "averaging" operator $\Upsilon f(mh) = \frac{1}{2}\,[f(mh-\frac{1}{2}h) + f(mh+\frac{1}{2}h)]$. Deduce that the approximation

$$f^{(2q+1)}(mh) \approx h^{-2q-1}\,\Upsilon\,[\,\Delta_0^{2q+1} - \tfrac{1}{12}\,(q+2)\,\Delta_0^{2q+3}\,]\,f(mh)$$

has the form $f^{(2q+1)}(mh) \approx \sum_{j=-q-2}^{q+2} c_j f(mh+jh)$, where q is a nonnegative integer. We set $q=1$ for the rest of the question. In this case, find the values of the coefficients c_j, $j=-3,-2,\ldots,3$ (which are multiples of h^{-3}). Then show that the order of the approximation to $f'''(mh)$ is 3.

32. The Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is approximated by the nine point formula

$$\begin{aligned} h^2\,\nabla^2 u(mh,nh) \;\; &\approx \;\; -\tfrac{10}{3}\,U_{mn} + \tfrac{2}{3}\,(U_{m+1\,n} + U_{m-1\,n} + U_{m\,n+1} + U_{m\,n-1}) \\ &\qquad + \tfrac{1}{6}\,(U_{m+1\,n+1} + U_{m+1\,n-1} + U_{m-1\,n+1} + U_{m-1\,n-1}), \end{aligned}$$

where $U_{mn} \approx u(mh,nh)$. Find the order of this approximation when u is any infinitely differentiable function. Show that the order is higher if u happens to satisfy Laplace's equation $\nabla^2 u = 0$.

33. Let $M \geq 2$ and $N \geq 2$ be integers and let $U \in \mathcal{R}^{(M-1)(N-1)}$ have the components U_{mn}, $1 \leq m \leq M-1$, $1 \leq n \leq N-1$, where two subscripts occur because we associate the components with the interior points of a rectangular grid. Further, let U_{mn} be zero on the boundary of the grid, which means $U_{mn}=0$ if $0 \leq m \leq M$ and $0 \leq n \leq N$ and at least one of these four inequalities holds as an equation. Thus, for any real constants α, β and γ, we can define a linear operator A from $\mathcal{R}^{(M-1)(N-1)}$ to $\mathcal{R}^{(M-1)(N-1)}$ by the equations

$$\begin{aligned} (A\,U)_{mn} \;\; &= \;\; \alpha\,U_{mn} + \beta\,(U_{m-1\,n} + U_{m+1\,n} + U_{m\,n-1} + U_{m\,n+1}) \\ &\qquad + \gamma\,(U_{m-1\,n-1} + U_{m+1\,n-1} + U_{m-1\,n+1} + U_{m+1\,n+1}), \quad 1 \leq m \leq M-1, \;\; 1 \leq n \leq N-1. \end{aligned}$$

We now let the components of U have the special form $U_{mn} = \sin(mk\pi/M)\sin(n\ell\pi/N)$, $1 \leq m \leq M-1$, $1 \leq n \leq N-1$, where k and ℓ are integers. Prove that U is an eigenvector of A and find its eigenvalue. Hence deduce that, if α, β and γ provide the nine point formula of Exercise 32, and if M and N are large, then the least modulus of an eigenvalue is approximately $4\sin^2(\frac{\pi}{2M}) + 4\sin^2(\frac{\pi}{2N})$.

34. The function $u(x) = x(x-1)$, $0 \leq x \leq 1$, is defined by the equations $u''(x) = 2$, $0 \leq x \leq 1$, and $u(0) = u(1) = 0$. A difference approximation to the differential equation provides the estimates $U_m \approx u(mh)$, $m = 1,2,\ldots,M-1$, through the system of equations $U_{m-1} - 2U_m + U_{m+1} = 2h^2$, $m = 1,2,\ldots,M-1$, where $U_0 = U_M = 0$, $h = 1/M$, and M is a large positive integer. Show that the exact solution of the system is just $U_m = u(mh)$, $m = 1,2,\ldots,M-1$.

We employ the notation $U_m^{\infty} = u(mh)$, because we let the system be solved by the Jacobi iteration of Methods 1.6, using the starting values $U_m^{(1)} = 0$, $m = 1,2,\ldots,M-1$. Prove that the iteration matrix of Revision 1.5 has the spectral radius $\rho(H) = \cos(\pi/M)$. Further, by regarding the initial error vector $\mathbf{U}^{(1)} - \mathbf{U}^{(\infty)}$ as a linear combination of the eigenvectors of H, show that the largest component of $\mathbf{U}^{(k+1)} - \mathbf{U}^{(\infty)}$ for large k is approximately $(8/\pi^3)\cos^k(\pi/M)$. Hence deduce that the Jacobi method requires about $2.5M^2$ iterations to achieve $\|\mathbf{U}^{(k+1)} - \mathbf{U}^{(\infty)}\|_\infty \leq 10^{-6}$.

35. The function $u(x,y) = 18\,x(1-x)y(1-y)$, $0 \leq x, y \leq 1$, is the solution of Poisson's equation $u_{xx} + u_{yy} = 36\,(x^2 + y^2 - x - y) = f(x,y)$, say, subject to u being zero on the boundary of the unit square. We pick $h = 1/6$ and we seek the solution of the five point difference equation

$$U_{m-1\,n} + U_{m+1\,n} + U_{m\,n-1} + U_{m\,n+1} - 4\,U_{mn} = h^2 f(mh,nh), \qquad 1 \leq m \leq 5, \;\; 1 \leq n \leq 5,$$

where U_{mn} is zero if (mh,nh) is on the boundary of the square. Let the multigrid method be applied, using only this fine grid and a coarse grid of mesh size $1/3$, and let every U_{mn} be zero initially. Calculate the 25 residuals of the starting vector on the fine grid. Then, following the "restriction" procedure in the hand-outs, find the residuals for the initial calculation on the coarse grid. Further, show that if the equations on the coarse grid are solved exactly, then the resultant estimates of u at the four interior points of the coarse grid all have the value $5/6$. By applying the "prolongation operator" to these estimates, find the 25 starting values of U_{mn} for the subsequent iterations of Gauss–Seidel or Jacobi on the fine grid. Further, show that if one Jacobi iteration is performed, then $U_{33} = 23/24$ occurs, which is the estimate of $u(\frac{1}{2},\frac{1}{2}) = 9/8$.