Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT

MAS212 Linear Algebra I

Thursday 12 May 2005, 2:30 pm – 4:30 pm

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators are **not** *permitted in this examination. The unauthorised use of a calculator constitutes an examination offence. Show your working.*

If not specified then assume that the field of scalars is the field of rational numbers $\mathbb{Q}$*.*

SECTION A

This section carries 56 marks and each question carries 7 marks. You should attempt ALL 8 questions. ***Do*** **not** ***begin each answer in this section on a fresh page. Write the number of the question in the*** **left** ***margin.***

A1. (a) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$, compute $5A$, A^2 and AX.

(b) If $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, compute AB and BA.

(c) If $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$, compute AA^T and A^TA.

A2. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, compute $\det A$ and A^{-1} (if it exists).

A3. Reduce the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

to echelon form by performing elementary row operations and hence determine the rank and determinant of A.

A4. Let $S = \{v_1, v_2, \ldots, v_r\}$ be a set of $r \geq 1$ vectors in a vector space V. Define the terms *linear combination of vectors in S* and *vector space spanned by S*.

Prove whether or not the vector $(1, 2, 3, 4)$ is a linear combination of the set of vectors

$$S = \{(1, 0, 1, 0), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1)\}$$

and whether or not S spans the vector space $\mathbb{R}^4$.

A5. Define the terms *basis* and *dimension* for a finite dimensional vector space V.

If $\{v_1, v_2, \ldots, v_n\}$ with $n \geq 1$ is a fixed basis for V, prove that for any $v \in V$ the representation

$$v = \sum_{i=1}^{n} c_i v_i$$

is unique, i.e. that the scalars c_i are unique.

A6. Define the term *linear map.*

Prove that the map $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\alpha(x, y, z) = (x + y, y + z, z + x)$$

is linear, show how it maps the standard basis, and write down the matrix A of α with respect to the standard basis.

A7. Let V be a vector space over a field $\mathbb{K}$.

Define the terms *eigenvalue* and *eigenvector* for a linear map $\alpha : V \to V$.

Prove that if x is an eigenvector of α then the equation that defines x is also satisfied by kx for any $k \in \mathbb{K}$.

Prove that if x is an eigenvector of α then it is also an eigenvector of $\alpha^{(r)}$, the composition of r copies of the map α, and find the corresponding eigenvalue.

A8. Define the terms *symmetric matrix* and *antisymmetric matrix.*

Express $(AB)^T$ in terms of A^T and B^T.

Prove that if x is any n-dimensional column vector then xx^T is a symmetric $n \times n$ matrix, and give a simple example.

SECTION B

This section carries 44 marks and each question carries 22 marks. You may attempt all 4 questions but, except for the award of a bare pass, only marks for the best 2 questions will be counted. ***Begin each answer in this section on a fresh page. Write the number of the question at the top of each page.***

B1. Let V be a vector space over a field $\mathbb{K}$ and let $U \subseteq V$.

(a) [4 marks] Define what it means for U to be a *vector subspace* of V and give conditions that U must satisfy.

(b) [9 marks] Prove that

$$U = \{(x, y, z) \mid x + 2y = 0, 2y - 3z = 0\} \subseteq \mathbb{R}^3$$

is a vector subspace of $\mathbb{R}^3$. Construct a basis for U and hence determine the dimension of U.

(c) [9 marks] Let S and T be vector subspaces of V. Define the *sum* $S + T$ and state a sufficient condition for the sum to be a *direct sum* $S \oplus T$.

For U defined as in (b) above, write down a basis for a vector subspace $W \subseteq \mathbb{R}^3$ such that $U \oplus W = \mathbb{R}^3$ and prove that $U + W$ is a direct sum.

B2. Let $\alpha : U \to V$ be a linear map between finite dimensional vector spaces U and V.

(a) [8 marks] Define the *kernel* $\ker(\alpha)$ and *image* $\mathrm{im}(\alpha)$ of α. Prove that $\ker(\alpha)$ is a vector subspace of U and that $\mathrm{im}(\alpha)$ is a vector subspace of V.

(b) [3 marks] Prove that the image under α of a spanning set for U is a spanning set for $\mathrm{im}(\alpha)$.

(c) [5 marks] State and prove a general formula relating the dimensions of U, $\ker(\alpha)$ and $\mathrm{im}(\alpha)$.

(d) [6 marks] If $\alpha : \mathbb{R}^3 \to \mathbb{R}^2$ is the map defined by

$$\alpha(x, y, z) = (x + y, y + z),$$

find bases for $\ker(\alpha)$ and $\mathrm{im}(\alpha)$, and hence verify explicitly that the formula referred to in (c) above holds.

B3. (a) [9 marks] Define the matrix P that effects a change from an ordered basis $\mathcal{B}$ to an ordered basis $\mathcal{C}$ in a vector space V. [You may use any reasonable definition provided you use it consistently.] Find P and P^{-1} when $V = \mathbb{R}^3$, $\mathcal{B}$ is the standard basis and

$$\mathcal{C} = (0, 1, 1), (1, 0, 1), (1, 1, 0).$$

(b) [5 marks] Let the vector $v \in V$ have coordinate vector $(1, 2, 3)$ with respect to the standard basis and let v' be the coordinate vector of v with respect to the ordered basis $\mathcal{C}$ defined above. Show how the matrix P can be used to compute v' from v. Check that the result satisfies the definition of v' as a coordinate vector.

(c) [8 marks] If the linear map $\alpha : V \to V$ is represented by the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

with respect to the standard basis, use P to transform A to the matrix A' of α with respect to the ordered basis $\mathcal{C}$. Use A' and v' (computed in part (b)) to compute the coordinate vector $\alpha(v)$ with respect to the ordered basis $\mathcal{C}$. Compare this with the result obtained by transforming the vector Av from the standard basis to the ordered basis $\mathcal{C}$.

B4. (a) [6 marks] Define the term *similarity transformation*. Prove that an $n \times n$ matrix that has n linearly independent eigenvectors can be diagonalized by a similarity transformation.

(b) [8 marks] Define the terms *orthogonal* and *orthonormal* applied to a set of vectors. Prove that the eigenvalues of a real symmetric matrix are real and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

(c) [4 marks] State the relationship between an *orthogonal matrix* and its transpose. Prove that the set of columns of an orthogonal matrix is an orthonormal set of vectors.

(d) [4 marks] Prove that an $n \times n$ real symmetric matrix that has n distinct eigenvalues can be diagonalized by a real orthogonal similarity transformation.

[End of examination paper]