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Numerical Analysis – Lecture 9

6.6 The Newton interpolation formula

Recalling that $f[x_i] = f(x_i)$, the recursive formula allows for fast evaluation of the *divided difference table*

$$\begin{array}{ccccccccc} f[x_0] & \to & f[x_0,x_1] & \to & f[x_0,x_1,x_2] & \to & f[x_0,x_1,x_2,x_3] & \to & \cdots \\ & \nearrow & & \nearrow & & \nearrow & & & \\ f[x_1] & \to & f[x_1,x_2] & \to & f[x_1,x_2,x_3] & \to & & \cdots & \\ \vdots & & & & & & & & \\ f[x_n] & & & & & & & & \end{array}$$

This can be done in $\mathcal{O}(n^2)$ operations.
We now provide an alternative representation of the interpolating polynomial.. Again, $f(x_i)$, $i = 0,1,\ldots,k$, are given and we seek $p \in \mathbb{P}_k[x]$ such that $p(x_i) = f(x_i)$, $i = 0,\ldots,k$.

Theorem Suppose that $x_0, x_1, \ldots, x_k$ are pairwise distinct. The polynomial

$$p_k(x) := f[x_0] + f[x_0,x_1](x-x_0) + \cdots + f[x_0,x_1,\ldots,x_k]\prod_{i=0}^{k-1}(x-x_i) \in \mathbb{P}_k[x]$$

obeys $p_k(x_i) = f(x_i)$, $i = 0,1,\ldots,k$.

Proof. By induction on k. The statement is obvious for $k = 0$ and we suppose that it is true for k. We now prove that $p_{k+1}(x) - p_k(x) = f[x_0,x_1,\ldots,x_{k+1}]\prod_{i=0}^{k}(x-x_i)$. Clearly, $p_{k+1} - p_k \in \mathbb{P}_{k+1}[x]$ and the coefficient of x^{k+1} therein is, by definition, $f[x_0,\ldots,x_{k+1}]$. Moreover, $p_{k+1}(x_i) - p_k(x_i) = 0$, $i = 0,1,\ldots,k$, hence it is a multiple of $\prod_{i=0}^{k}(x-x_i)$, and this proves the asserted form of $p_{k+1} - p_k$. The explicit form of p_{k+1} follows. □

We obtain the *Newton interpolation formula,* which requires only the top row of the divided difference table. It has several advantages over Lagrange's. In particular, its evaluation at a given point x (provided that divided differences are known) requires just $\mathcal{O}(k)$ operations, as long as we do it by the *Horner scheme*

$$\begin{aligned} p_k(x) &= \{\{\{f[x_0,\ldots,x_k](x-x_{k-1}) + f[x_1,\ldots,x_{k-1}]\} \times (x-x_{k-2}) + f[x_0,\ldots,x_{k-2}]\} \\ &\qquad \times (x-x_3) + \cdots\} + f[x_0]. \end{aligned}$$

7 Orthogonal polynomials

7.1 Orthogonality amongst functions

We have already seen the scalar product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i=1}^n x_i y_i$, acting on $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Likewise, given $w_1, w_2, \ldots, w_n > 0$, we may define $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i=1}^n w_i x_i y_i$. In general, a *scalar* (or *inner*) *product* is any function $\mathbb{V}\times\mathbb{V} \to \mathbb{R}$, where $\mathbb{V}$ is a vector space over the reals, subject to the following three axioms: symmetry: $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{y}, \boldsymbol{x}\rangle \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$; nonnegativity: $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \ge 0 \ \forall \boldsymbol{x} \in \mathbb{V}$ and $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$; and linearity: $\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}\rangle = a\langle \boldsymbol{x}, \boldsymbol{z}\rangle + b\langle \boldsymbol{y}, \boldsymbol{z}\rangle \ \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}$, $a, b \in \mathbb{R}$.
Any scalar product defines *orthogonality:* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$ are orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.

Let $\mathbb{V} = C[a,b]$, $w \in \mathbb{V}$ be a fixed *positive* function and define $\langle f, g \rangle := \int_a^b w(x)f(x)g(x)\,\mathrm{d}x$ for all $f, g \in \mathbb{V}$. It is easy to verify all three axioms of the scalar product.

7.2 Orthogonal polynomials – definition, existence, uniqueness

We say that $p_n \in \mathbb{P}_n[x]$ is the nth *orthogonal polynomial* if $\langle p_n, p \rangle = 0$ for all $p \in \mathbb{P}_{n-1}[x]$. *[Note: different inner products lead to different orthogonal polynomials.]* A polynomial in $\mathbb{P}_n[x]$ is *monic* if the coefficient of x^n therein is one.

Theorem For every $n \in \mathbb{Z}^+$ there exists a unique monic orthogonal polynomial of degree n. Moreover, any $p \in \mathbb{P}_n[x]$ can be expanded as a linear combination of $p_0, p_1, \ldots, p_n$,

Proof. We let $p_0(x) \equiv 1$ and prove the theorem by induction on n. Thus, suppose that $p_0, p_1, \ldots, p_n$ have been already derived consistently with both assertions of the theorem and let $q(x) := x^{n+1} \in \mathbb{P}_{n+1}[x]$. Guided by the *Gram–Schmidt algorithm,* we choose

$$p_{n+1}(x) = q(x) - \sum_{k=0}^{n} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} p_k(x), \qquad x \in \mathbb{R}. \tag{7.1}$$

Clearly, $p_{n+1} \in \mathbb{P}_{n+1}[x]$ and it is monic (since all the terms in the sum are of lower degree).
Let $m \in \{0, 1, \ldots, n\}$. It follows from (7.1) and the induction hypothesis that

$$\langle p_{n+1}, p_m \rangle = \langle q, p_m \rangle - \sum_{k=0}^{n} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} \langle p_k, p_m \rangle = \langle q, p_m \rangle - \frac{\langle q, p_m \rangle}{\langle p_m, p_m \rangle} \langle p_m, p_m \rangle = 0.$$

Hence, p_{n+1} is orthogonal to $p_0, \ldots, p_n$. Consequently, according to the second inductive assertion, it is orthogonal to all $p \in \mathbb{P}_n[x]$.
To prove uniqueness, we suppose the existence of two monic orthogonal polynomials $p_{n+1}, \tilde{p}_{n+1} \in \mathbb{P}_{n+1}[x]$. Let $p := p_{n+1} - \tilde{p}_{n+1} \in \mathbb{P}_n[x]$, hence $\langle p_{n+1}, p \rangle = \langle \tilde{p}_{n+1}, p \rangle = 0$, and this implies

$$0 = \langle p_{n+1}, p \rangle - \langle \tilde{p}_{n+1},, p \rangle = \langle p_{n+1} - \tilde{p}_{n+1}, p \rangle = \langle p, p \rangle,$$

and we deduce $p \equiv 0$.
Finally, in order to prove that each $p \in \mathbb{P}_{n+1}[x]$ is a linear combination of $p_0, \ldots, p_{n+1}$, we note that we can always write it in the form $p = cp_{n+1} + q$, where c is the coefficient of x^{n+1} in p and where $q \in \mathbb{P}_n[x]$. The theorem follows by induction. $\square$

Well known examples of orthogonal polynomials include

Legendre polynomials P_n: $[a,b] = [-1,1]$, $w(x) \equiv 1$;
Chebyshev polynomials T_n: $[a,b] = [-1,1]$, $w(x) = (1-x^2)^{-\frac{1}{2}}$;
Laguerre polynomials L_n: $[a,b) = [0,\infty)$, $w(x) = \mathrm{e}^{-x}$;
Hermite polynomials H_n: $(a,b) = (-\infty,\infty)$, $w(x) = \mathrm{e}^{-x^2}$.