ASSIGNMENT 1

Index N°: 3562718
Course: TE 262
Department: Electrical Engineering

1) (a) Magnitude of |A|

$$\boldsymbol{A} = -a_x + 2a_y - 2a_z$$

$$|A| = \sqrt{(-1)^2 + (2)^2 + (-2)^2}$$

$$|A| = \sqrt{9}$$

$$|A| = 3\ units$$

(b) Unit vector $a_{\boldsymbol{A}} = \frac{\boldsymbol{A}}{|A|}$

$$a_{\boldsymbol{A}} = \frac{-a_x + 2a_y - 2a_z}{3}$$

$$a_{\boldsymbol{A}} = \frac{-1}{3}a_x + \frac{2}{3}a_y + \frac{-2}{3}a_z$$

(c) Angle that vector A makes with the z-axis

Taking the dot product of **A** and a_z, we have

$$\mathbf{A} \cdot a_z = |\mathbf{A}||a_z|cos\theta_{A/a_z}$$

$$\mathbf{A} \cdot a_z = \left(-a_x + 2a_y - 2a_z\right) \cdot \left(0a_x + 0a_y + a_z\right) = -2$$

$$|\mathbf{A}||a_z|cos\theta = 3 * 1 * cos\theta = 3cos\theta$$

So we have

$$-2 = 3cos\theta$$

$$cos\theta = \frac{-2}{3}$$

$$\theta = cos^{-1}\left(\frac{-2}{3}\right) = 131.81°$$

2) (a) $\boldsymbol{A} \cdot \mathbf{B}$

$$\boldsymbol{A} \cdot \mathbf{B} = \left(3a_x - 2a_y + 2a_z\right) \cdot \left(-2a_x + 4a_z\right)$$

$$= 3 * (-2) + (-2) * 0 + 2 * 4$$

$$= -6 + 0 + 8$$

$$\boldsymbol{A} \cdot \mathbf{B} = 2$$

(b) $\boldsymbol{A} \times \mathbf{B}$

$$\boldsymbol{A} \times \mathbf{B} = \begin{vmatrix} a_x & a_y & a_z \\ 3 & -2 & 2 \\ -2 & 0 & 4 \end{vmatrix}$$

$$\boldsymbol{A} \times \mathbf{B} = [(-2 * 4) - (2 * 0)]a_x - [(3 * 4) - (2 * -2)]a_y + [(3 * 0) - (-2 * -2)]a_z$$

$$\boldsymbol{A} \times \mathbf{B} = -8a_x - 16a_y + 4a_z$$

(c) $\boldsymbol{\theta_{AB}}$

Taking the dot product of **A** and **B**, we have

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|cos\theta_{AB}$$

$$\mathbf{A}\cdot\mathbf{B} = \left(3a_x - 2a_y + 2a_z\right)\cdot\left(-2a_x + 0a_y + 4a_z\right) = 2$$

$$|\mathbf{A}| = \sqrt{(3)^2 + (-2)^2 + (2)^2} = \sqrt{17}$$

$$|\mathbf{B}| = \sqrt{(-2)^2 + (0)^2 + (4)^2} = 2\sqrt{5}$$

$$|\mathbf{A}||\mathbf{B}|cos\theta_{AB} = \sqrt{17} * 2\sqrt{5} * cos\theta_{AB} = 2\sqrt{85}cos\theta_{AB}$$

$$2 = 2\sqrt{85}cos\theta_{AB}$$

$$cos\theta_{AB} = \frac{1}{\sqrt{85}}$$

$$\theta = cos^{-1}\left(\frac{1}{\sqrt{85}}\right) = 83.77°$$

3) (a) The magnitude of B

$$\boldsymbol{B} = 2a_x - 4a_y + 2a_z$$

$$|\mathbf{B}| = \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}$$

(b) The expression of $a_{\boldsymbol{B}}$

$$a_{\boldsymbol{B}} = \frac{\boldsymbol{B}}{|B|}$$

$$a_{\boldsymbol{B}} = \frac{2a_x - 4a_y + 2a_z}{2\sqrt{6}}$$

$$a_{\boldsymbol{B}} = \frac{1}{\sqrt{6}}a_x - \frac{2}{\sqrt{6}}a_y + \frac{1}{\sqrt{6}}a_z$$

(c) The angles that **B** makes with the x, y, and z axis

- **x-axis**

Taking the dot product of **B** and a_x, we have

$$\mathbf{B} \cdot a_x = |\,\mathbf{B}||a_x|cos\theta_{Bx}$$

$$\mathbf{B} \cdot a_x = \left(2a_x - 4a_y + 2a_z\right) \cdot \left(a_x + 0a_y + 0a_z\right) = 2$$

$$|\mathbf{B}||a_x|cos\theta_{Bx} = 2\sqrt{6} * 1 * cos\theta_{Bx} = 2\sqrt{6}cos\theta_{Bx}$$

$$2 = 2\sqrt{6}cos\theta_{Bx}$$

$$cos\theta_{Bx} = \frac{1}{\sqrt{6}}$$

$$\theta_{Bx} = cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9°$$

- **y-axis**

Taking the dot product of **B** and a_y, we have

$$\mathbf{B} \cdot a_y = |\,\mathbf{B}||a_y|cos\theta_{By}$$

$$\mathbf{B} \cdot a_y = \left(2a_x - 4a_y + 2a_z\right) \cdot \left(0a_x + a_y + 0a_z\right) = -4$$

$$|\mathbf{B}||a_y|cos\theta_{By} = 2\sqrt{6} * 1 * cos\theta_{By} = 2\sqrt{6}cos\theta_{By}$$

$$-4 = 2\sqrt{6}cos\theta_{By}$$

$$cos\theta_{By} = \frac{-2}{\sqrt{6}}$$

$$\theta_{By} = cos^{-1}\left(\frac{-2}{\sqrt{6}}\right) = 144.73°$$

- **z-axis**

Taking the dot product of **B** and a_z, we have

$$\mathbf{B} \cdot a_z = |\,\mathbf{B}||a_z|cos\theta_{By}$$

$$\mathbf{B} \cdot a_z = \left(2a_x - 4a_y + 2a_z\right) \cdot \left(0a_x + a_y + 0a_z\right) = 2$$

$$|\mathbf{B}||a_z|cos\theta_{Bz} = 2\sqrt{6} * 1 * cos\theta_{Bz} = 2\sqrt{6}cos\theta_{Bz}$$

$$2 = 2\sqrt{6}cos\theta_{Bz}$$

$$cos\theta_{Bz} = \frac{1}{\sqrt{6}}$$

$$\theta_{Bz} = cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9°$$

4) The total charges contained in the region

Let ρ be the charge density. We have

$$\rho = \frac{3 \times 10^{-4}}{R^2} cos^2\emptyset \quad (C/m^3)$$

The number of charges Q is given by Q $= \int \rho v$, with v being the volume. We'll need to perform a triple integral, and use the spherical coordinates.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{0.01}^{0.03} \frac{3\times10^{-4}}{R^2}\, dv$$

In this case, $dv \;=\; R^2 sin\theta dR d\theta d\phi$, so we have:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{0.01}^{0.03} \frac{3\times10^{-4}}{R^2}\, R^2 sin\theta dR d\theta d\phi$$

$$= 3\times10^{-4}\int_0^{2\pi}\int_0^{\pi} cos^2\emptyset sin\theta \int_{0.01}^{0.03}\frac{1}{R^2}\, R^2 dR d\theta d\phi$$

$$= 3\times10^{-4}\int_0^{2\pi}\int_0^{\pi} cos^2\emptyset sin\theta \left[\int_{0.01}^{0.03} 1 dR\right] d\theta d\phi$$

$$= 3\times10^{-4}\int_0^{2\pi}\int_0^{\pi} cos^2\emptyset sin\theta [0.03 - 0.01]\, d\theta d\phi$$

$$= 6\times10^{-6}\int_0^{2\pi}\int_0^{\pi} cos^2\emptyset sin\theta \; d\theta d\phi$$

Using integration by parts to integrate $cos^2\emptyset sin\theta$, we have

$$cos^2\emptyset sin\theta = \frac{-cos^3(\theta)}{3}$$

$$\int_0^{\pi} cos^2\emptyset sin\theta \; d\theta = \left[\frac{-cos^3(\theta)}{3}\right]_0^{\pi} = \frac{-cos^3(\pi) + cos^3(0)}{3} = \frac{2}{3}$$

$$Q = \frac{2}{3} 6\times10^{-6}\int_0^{2\pi} d\phi$$

$$= 4\times10^{-6}\int_0^{2\pi} d\phi$$

$$= 4\times10^{-6}[\phi]_0^{2\pi}$$

$$Q = 4 \times 10^{-6}(2\pi - 0)$$

$$Q = 8\pi \times 10^{-6}\ C$$

5) (a) Total outward flux

To obtain the total outward flux, we should use the divergence theorem.

$$Total\ outward\ flux = \iiint (\nabla \cdot A) dv$$

Where $dv = rdrd\theta dz$ and the limits of integration are given by the dimensions of the cylinder.

$$Total\ outward\ flux = \int_0^5 \int_0^{2\pi} \int_0^1 \frac{1}{r}\left(\frac{\delta r}{\delta r}(r) + r\frac{\delta}{\delta r}(z)\right) rdrd\theta dz$$

$$= \int_0^5 \int_0^{2\pi} \int_0^1 \frac{1}{r}(2r + r)\, rdrd\theta dz$$

$$= \int_0^5 \int_0^{2\pi} \int_0^1 \frac{1}{r}(3r)\, rdrd\theta dz$$

$$= 3\int_0^5 \int_0^{2\pi} \left[\int_0^1 rdr\right] d\theta dz$$

$$= 3\int_0^5 \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta dz$$

$$= 3\int_0^5 \int_0^{2\pi} \frac{1}{2} d\theta dz$$

$$= \frac{3}{2}\int_0^5 \left[\int_0^{2\pi} d\theta\right] dz$$

$$= \frac{3}{2}\int_0^5 [\theta]_0^{2\pi}\, dz$$

$$= \frac{3}{2}\int_0^5 (2\pi - 0)\, dz$$

$$= 2\pi \frac{3}{2}\int_0^5 dz$$

$$= 3\pi[z]_0^5$$

$$= 3\pi(5 - 0)$$

$$Total\ outward\ flux = 15\pi$$

(b) The divergence of A

$$\nabla \cdot A = \frac{1}{r}\left(\frac{\delta r}{\delta r}(r) + r\frac{\delta}{\delta r}(z)\right)$$

$$= \frac{1}{r}\left(\frac{\delta}{\delta r}(r^2) + r\frac{\delta}{\delta r}(z)\right)$$

$$= \frac{1}{r}(2r + r(1))$$

$$= \frac{1}{r}(3r)$$

$$\nabla \cdot A = 3$$

(c) Verification of the divergence theorem

The divergence theorem states that

$$\int_v \nabla \cdot A \cdot dv = \oint_S A \cdot ds\ and$$

$$\oint_S A \cdot ds = \int_{\substack{front\\surface}} A \cdot ds + \int_{\substack{bottom\\surface}} A \cdot ds + \int_{\substack{top\\surface}} A \cdot ds$$

- Front surface, we have $z = 5 \; and \; ds = rdrd\theta$

$$\int_{\substack{front \\ surface}} A \cdot ds = \int\int_{\substack{front \\ surface}} 5rdrd\theta \; = \; 5\int_0^1 rdr \int_0^{2\pi} d\theta$$

$$= 5\left(\frac{1}{2}\right)(2\pi) = \; 5\pi$$

- Bottom surface, we have $z = 0 \; and \; ds = rdrd\theta$

$$\int_{\substack{bottom \\ surface}} A \cdot ds = \int\int_{\substack{bottom \\ surface}} 0 \cdot rdrd\theta \; = \; 0$$

- Top surface, we have r$= 5 \; and \; ds = rdzd\theta$

$$\int_{\substack{top \\ surface}} A \cdot ds = \int\int_{\substack{top \\ surface}} rdrd\theta \; = \; \int_0^5 dz \int_0^{2\pi} d\theta$$

$$= 5\,(1)(2\pi) = \; 10\pi$$

$$\oint_S A \cdot ds = 5\pi + 0 + 10\pi = \; 15\pi$$

The divergence theorem is verified.

6) Prove the two null identities mathematically

The two null identities involves repeated del operation and are important in the study of electromagnetism especially when dealing with potential function.

- **Identity 1**$= \boldsymbol{\nabla \times (\nabla V) = 0}$

This implies the curve of the gradient of any scalar field is indistinguishably zero. Character 1 can be demonstrated in Cartesian directions. Generally if the surface integral of $(\nabla * \nabla V)$ over any surface, the outcome is equivalent to the line necessary of ∇V around the path bounded limited by the strokes theorem.

$$\int_s [\nabla * (\nabla V)]\,.\,ds = \oint_c (\nabla V)\, dt$$

Therefore,

$$\oint_c (\nabla V).\, dt = \oint_c dV = 0$$

The combination of these two conditions shows that the surface vital of ∇*∇V is equivalent to zero. If the vector field is curve free, at that point it tends to be expressed as the gradient of a scalar field. In the event that the vector field is E. At that point if ∇*E=0. We can characterize the scalar field V with the end goal that E=-∇V. Where the negative is inconsequential as long as character 1 is concerned.

Realizing that a curl free vector field is a conservative field, hence an irrotational vector field can be expressed as the gradient of the scalar field.

- **Identity 2** $= \boldsymbol{\nabla \cdot (\nabla \times V) = 0}$

The divergence of the curl of any vector field is identically zero, it can be proved taking the volume integral of $\nabla \cdot (\nabla \times V)$. We have

$$\int_V \nabla \cdot (\nabla \times V) dv = \oint_s (\nabla \times V) ds \quad Eq.1$$

Taking a volume V with a surface S and assuming the surface is in reality to surfaces with a common boundary. We apply Stokes' theorem to those two surfaces. We then have:

$$\oint_s (\nabla \times V) ds = \oint_{s1} (\nabla \times V) \cdot a_{n1} ds + \oint_{s2} (\nabla \times V) \cdot a_{n2} ds$$

$$= \oint_{C1} V \cdot dl + \oint_{C2} V \cdot dl$$

a_{n1} *and* a_{n2} are the vectors normal to surfaces S1 and S2 respectively. Since the two boundary contours of S1 and S2 are actually the same but in opposite direction, their sum is zero and the volume integral of $(\nabla \cdot (\nabla \times V))$ on the left side of Eq.1 disappears. Therefore the identity is verified since this is true for any arbitrary volume: the integrand must zero.