ME 356 STRENGTH OF MATERIALS II

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Course Outline

1. Deflection of Beams;
2. Struts and Columns;
3. Compressions, Bending and Torsion under Plastic Conditions
4. Bending of Curved Bars
5. Mechanical Springs
6. Thin Shells

CHAPTER 1

DEFLECTION OF BEAMS

Chapter Outline

(A) DIFFENTIAL EQUATION OF THE DEFLECTION CURVE

1. Introduction
2. Relationship between Bending Stress, Deflection and Radius of Curvature
3. Relationship between Bending Moment and Radius of Curvature
4. Differential Equation of the Deflection Curve

(B) METHODS FOR DETERMINING THE DEFLECTION OF A BEAM

1. The Method of Calculus
2. Singularity Function (Macaulay’s Method)
3. Strain Energy Method (Castigliano’s Theorem)

DIFFERENTIAL EQUATION OF THE DEFLECTION CURVE

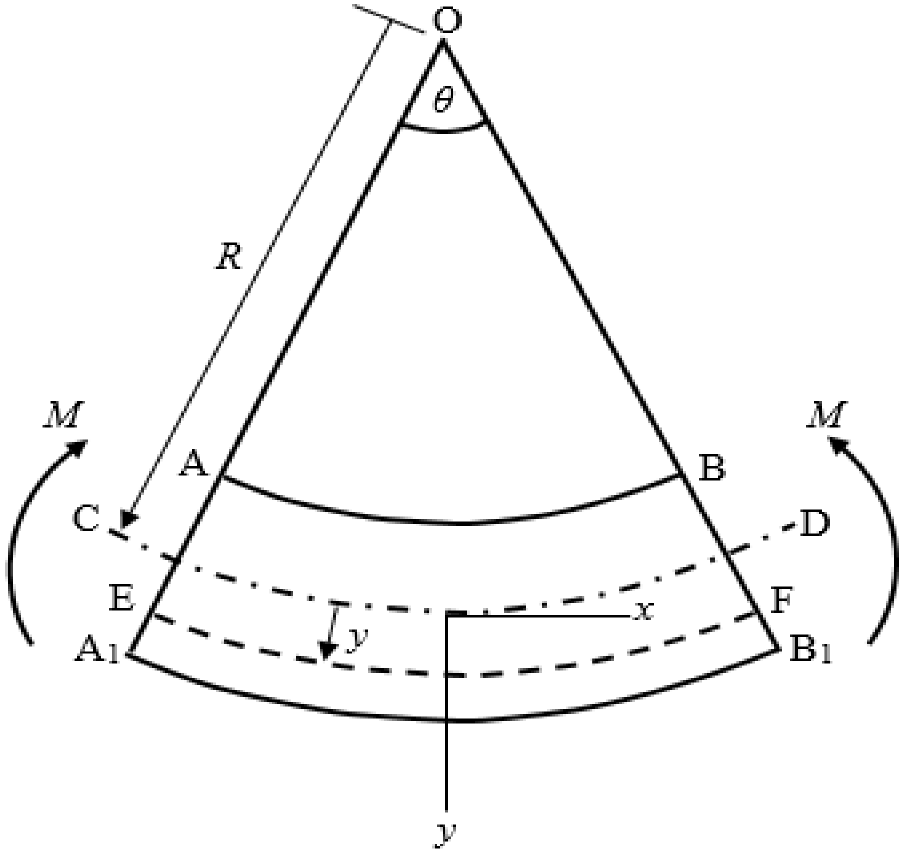
Introduction

- In the design of beams, it is necessary to quantify the deflection to ensure that it does not exceed the maximum allowable deflection

- A knowledge of deflection is also required in the analysis of statically indeterminate beams where there are more unknowns than there are equilibrium equations to be solved

Relationship between Bending Stress, Beam Deflection and Radius of Curvature

Consider an initially straight prismatic beam ABA_1B_1 subjected to a pure bending moment *M*



Under the action of *M* the beam deforms, causing surfaces AB and A_1B_1 to bend into circular arcs.

Surface AB experiences a compressive (-ve) stress
Surface A_1B_1 experiences a tensile (+ve) stress

Therefore, there must be an inner surface such as CD where the bending stress σ is zero. This is called the Neutral Surface

Relationship between Bending Stress, Beam Deflection and Radius of Curvature

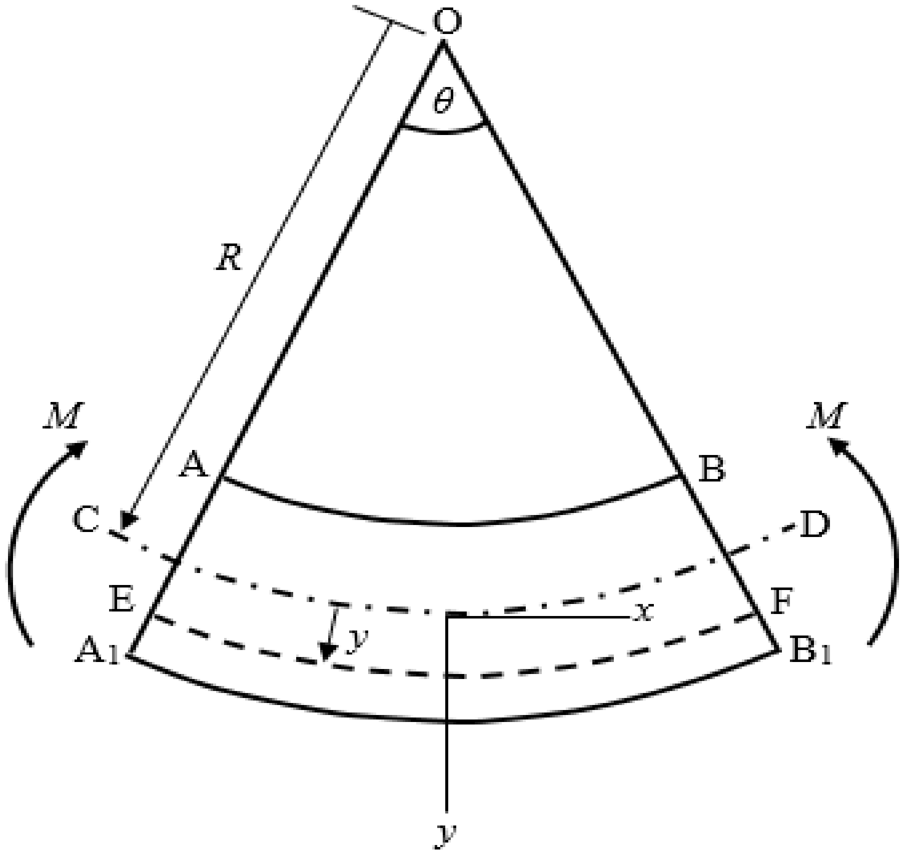
Let θ be the angle at the centre of curvature

R the radius of curvature

EF a filament or fibre of the deformed beam

y the deflection of the filament from the neutral surface

x-y is the coordinate system with the y-axis direction positively downwards



From ME 255/256, the stress in the filament is given by

$$\sigma = E\epsilon$$

where E is the Young's modulus of elasticity and ε is the normal strain

Relationship between Bending Stress, Beam Deflection and Radius of Curvature

The strain ε is

$$\epsilon = \frac{\sigma}{E} = \frac{EF - CD}{CD}$$

For θ in radians, lengths of the arcs CD and EF are given by $CD = R\theta$ and $EF = (R + y)\theta$.

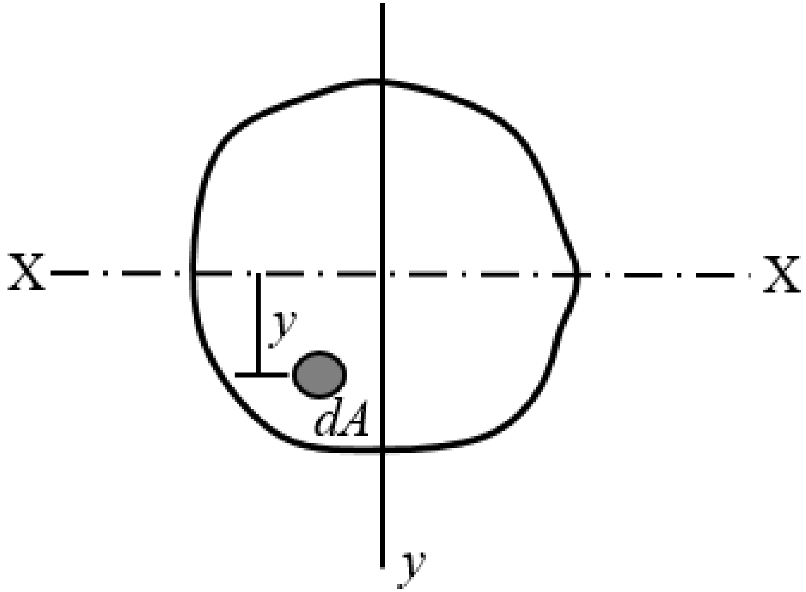
$$\implies \quad \frac{\sigma}{E} = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\implies \quad \sigma = \frac{Ey}{R}$$

Thus, the bending stress is proportional to the deflection, y

Relationship between Bending Moment and Radius of Curvature

transverse section



Consider a transverse section of the beam as shown in the diagram

The neutral surface will intersect the transverse in a straight line XX called the Neutral Axis of the transverse section

Let *dA* be an element of area of the transverse section at a distance *y* from the neutral axis

The differential bending moment *dM* on the area element is given by

$$dM = \sigma dA \cdot y$$

Relationship between Bending Moment and Radius of Curvature

$$\therefore M = \int \sigma y \, dA$$

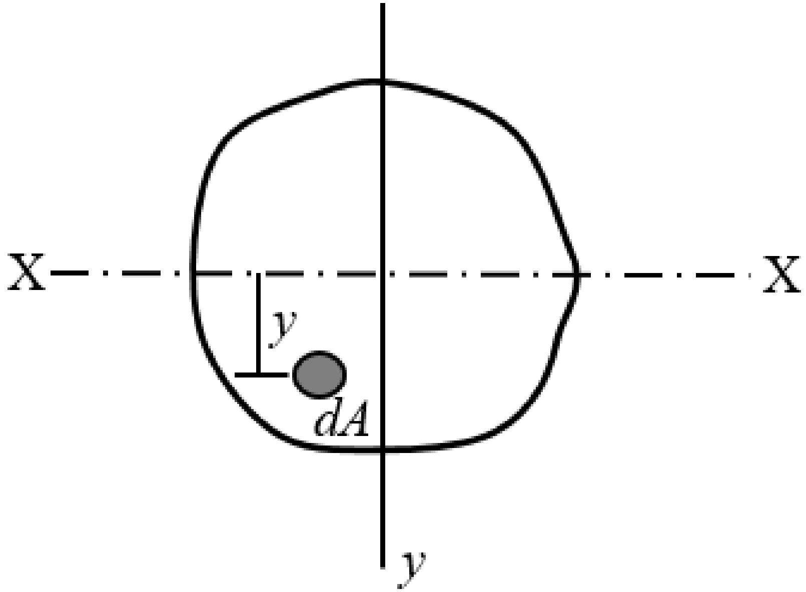
By substituting $\sigma = Ey/R$, we have

$$M = \int Ey^2/R \, dA = (E/R) \int y^2 dA$$

But $\int y^2 \, dA$ is the area moment of inertia or the second moment of area, I, of the beam cross section. Thus, we have

$$M = \frac{EI}{R}$$

transverse section



EI → Flexural Rigidity

Some Important Results

Recall,

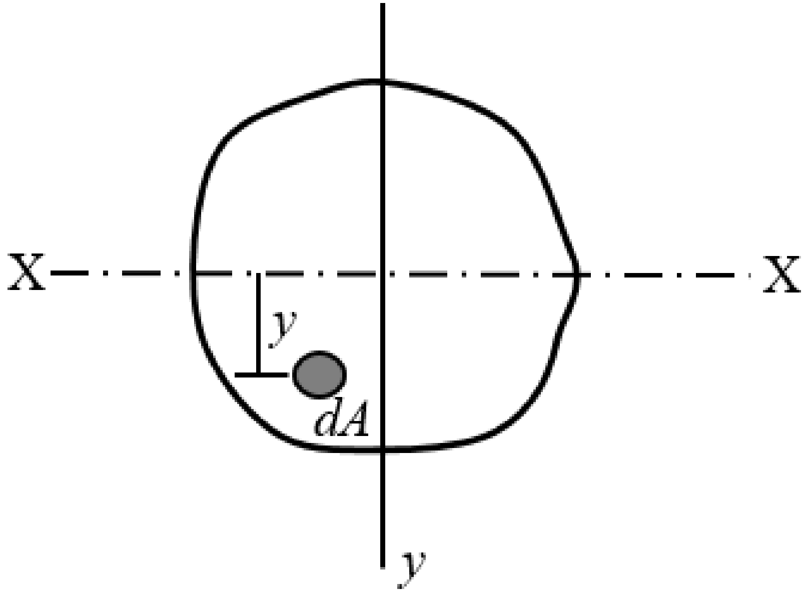
$\sigma = Ey/R$, and $M = EI/R$

Then $\dfrac{\sigma}{y} = \dfrac{M}{I} = \dfrac{E}{R}$

If y_{max} is the maximum deflection of the beam, then the maximum bending stress $\sigma_{\max}$ is given by

$$\sigma_{max} = \frac{My_{max}}{I} = \frac{M}{I/y_{max}}$$

transverse section



The quantity I/y_{max} is called the section modulus, Z.

- **At maximum stress the bending moment that can be supported is called the moment of resistance.**

Differential Equation of the Deflection Curve

From previous analyses we recall, $\frac{M}{I} = \frac{E}{R}$

$$\Rightarrow \quad \frac{M}{EI} = \frac{1}{R}$$

For the beam, the quantity *1/R*, known as the curvature of the beam is given by

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\boxed{M(x) = EI\,\frac{d^2y}{dx^2}}$$

Differential Equation of the Deflection Curve

$$M(x) = EI\,\frac{d^2y}{dx^2} \qquad \text{or} \qquad EI\,\frac{d^2y}{dx^2} = M(x)$$

Either of the above equations is known as the differential equation of the deflection curve. It is a second order differential equation which can be integrated to give the slope *dy/dx* and the deflection *y* of the beam.

Self-assessment:

What is deflection of beams and state the relation between deflection and bending moment.

METHODS FOR DETERMINING THE SLOPE AND DEFLECTION OF THE BEAM

Methods for Slope and Deflection at a Section

1. Calculus (Double Integration Method)

2. Singularity Function (Macaulay’s Method)

3. Strain Energy Method (Castigliano’s Theorem)

Double Integration Method

Bending moment at a point is given by

$$M = EI \frac{d^2 y}{dx^2}$$

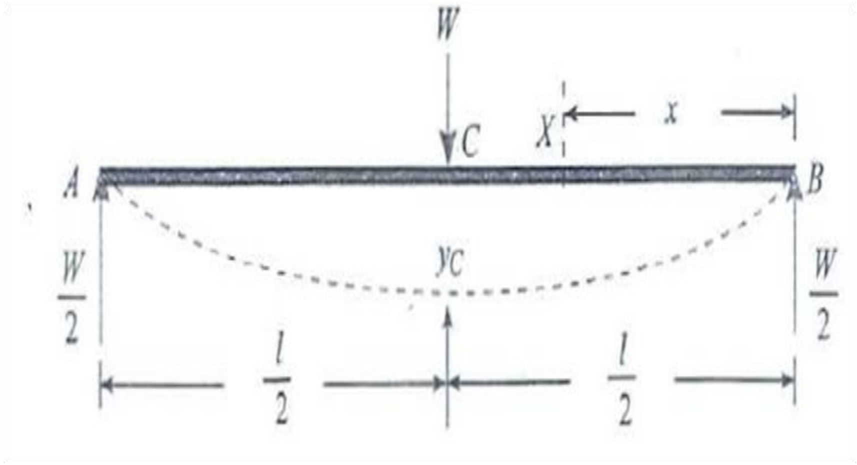
The value of slope at any point

$$EI \frac{dy}{dx} = \int M$$

The value of deflection at any point,

$$EI.y = \iint M$$

Case 1: Simple Supported Beam with a Central Point Load



Reactions at A and B are $$R_A = R_B = \frac{W}{2}$$

The bending moment at this section

$$M_x = R_B x = \frac{Wx}{2}$$

Therefore $$EI\frac{d^2y}{dx^2} = \frac{Wx}{2}$$

Integrating $$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1$$

Using Boundary Condition

$$x = \frac{l}{2} \qquad \frac{dy}{dx} = 0 \qquad C_1 = -\frac{Wl^2}{16}$$

Hence

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}$$

Case 1: Simple Supported Beam with a Central Point Load

Integrating the above equation

$$EI.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16} + C_2$$

Using Boundary Condition, $x = 0$, $y = 0$, $C_2 = 0$

Hence,

$$EI.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16}$$

Example 1-1

A simply supported beam of span 3 m is subjected to a central load of 10 kN. Find the maximum slope and deflection of the beam. Take I= 12 X 10^6 mm^4 and E = 200 GPa

Solution

Given: Span (*l*) = 3 m = 3 x 10^3 mm; Central load *(W)* = 10 kN = 10 x 10^3 N; Moment of inertia (*I*)= 12 x 10^6 mm^4 and Modulus of elasticity *(E)*= 200 GPa = 200 x 10^3 N/mm^2,

Maximum slope of the beam, @ x= 0

$$EI\,\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}$$

$$EI\left(\frac{dy}{dx}\right)_{\max} = \frac{W(0)^2}{4} - \frac{Wl^2}{16}$$

$$\left(\frac{dy}{dx}\right)_{\max} = -\frac{Wl^2}{16EI}$$

Example 1-1 (continued)

$$\left(\frac{dy}{dx}\right)_{\max} = -\frac{Wl^2}{16EI} = -\frac{\left(10x10^3\right)\left(3x10^3\right)^2}{16\left(200x10^3\right)\left(12x10^6\right)} = -0.0023\,rads$$

Maximum deflection of the beam, ***@ x= l/2***

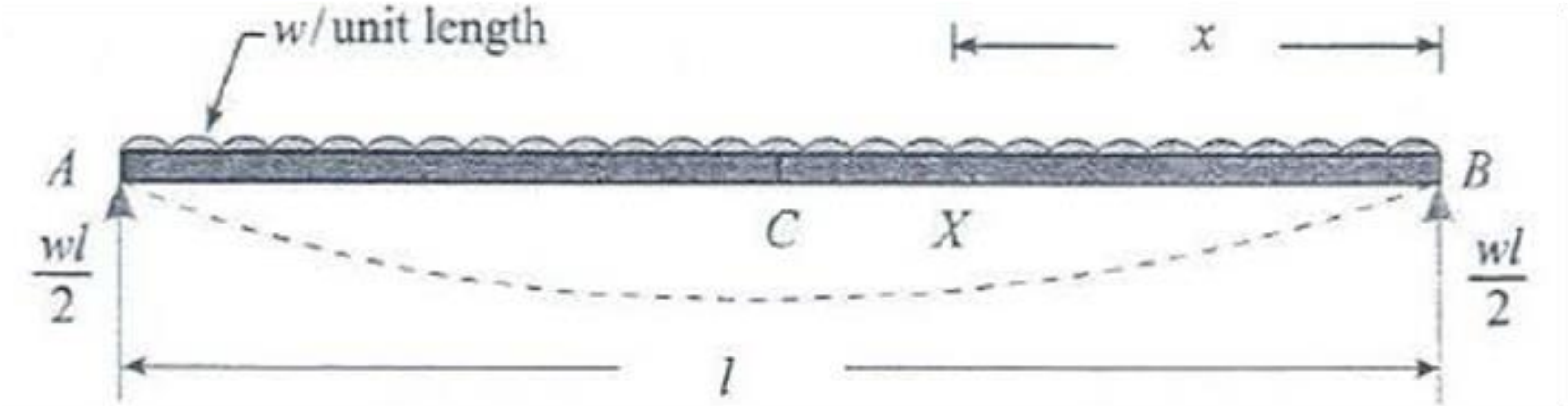
$$EI.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16}$$

$$EI.y_{\max} = \frac{W(l/2)^3}{12} - \frac{Wl^2(l/2)}{16}$$

$$y_{\max} = \frac{Wl^3}{48EI}$$

$$y_{\max} = -\frac{Wl^3}{48EI} = \frac{\left(10000\right)\left(3000\right)^3}{48\left(200x10^3\right)\left(12x10^6\right)} = 2.34\,mm$$

Case 2: Simple Supported Beam with a Uniformly Distributed Load



Reactions at *A and B,* $$R_A = R_B = \frac{wl}{2}$$

The bending moment at this section

$$M_x = R_B x = \frac{wlx}{2} - \frac{wx^2}{2}$$

Therefore $$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}$$

Case 2: Simple Supported Beam with a Uniformly Distributed Load

Integrating the above equation,

$$EI\,\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1$$

Using Boundary Condition

$x = \frac{l}{2}$ $\quad \frac{dy}{dx} = 0 \quad C_1 = -\frac{wl^3}{24}$

Hence,

$$EI\,\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}$$

Integrating the above equation,

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2$$

Using Boundary Condition, $x = 0$, $y = 0$, $C_2 = 0$

Hence,

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}$$

Example 1-2

A simply supported beam of span 4 m is carrying a uniformly distributed load of 2 kN/m over the entire span. Find the maximum slope and deflection of the beam. Take EI for the beam as 80 x 10^9 N-mm^2

Solution

Given: Span *(l)* = 4 m = 4 *x* 10^3 mm; Uniformly distributed load *(w)* = 2 kN/m = *2Nlmm* and flexural rigidity (*EI*)= 80 *x* 10^9 N-mm

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}$$

Maximum slope of the beam, @ x= 0

$$EIi_{\max} = \frac{wl(0)^2}{4} - \frac{w(0)^3}{6} - \frac{wl^3}{24}$$

$$i_{\max} = -\frac{wl^3}{24EI}$$

$$i_{\max} = -\frac{wl^3}{24EI} = -\frac{2(4000)^3}{24(80x10^9)} = 0.067\ rad$$

Example 1-2 (continued)

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}$$

Maximum deflection of the beam, @ x=l/2

$$EI.y_{\max} = \frac{wl(l/2)^3}{12} - \frac{w(l/2)^4}{24} - \frac{wl^3(l/2)}{24}$$

$$y_{\max} = \frac{wl^4}{384EI}$$

$$y_C = \frac{5wl^4}{384EI} = \frac{5(2)(4000)^4}{384(80x10^9)} = 83.3\,mm$$

Example 1-3

A simply supported beam of span 6 m is subjected to a uniformly distributed load over the entire span. If the deflection at the centre of the beam is not to exceed 4 mm, find the value of the load. Take E= 200 GPa and I = 300 x 10^6 mm^4.

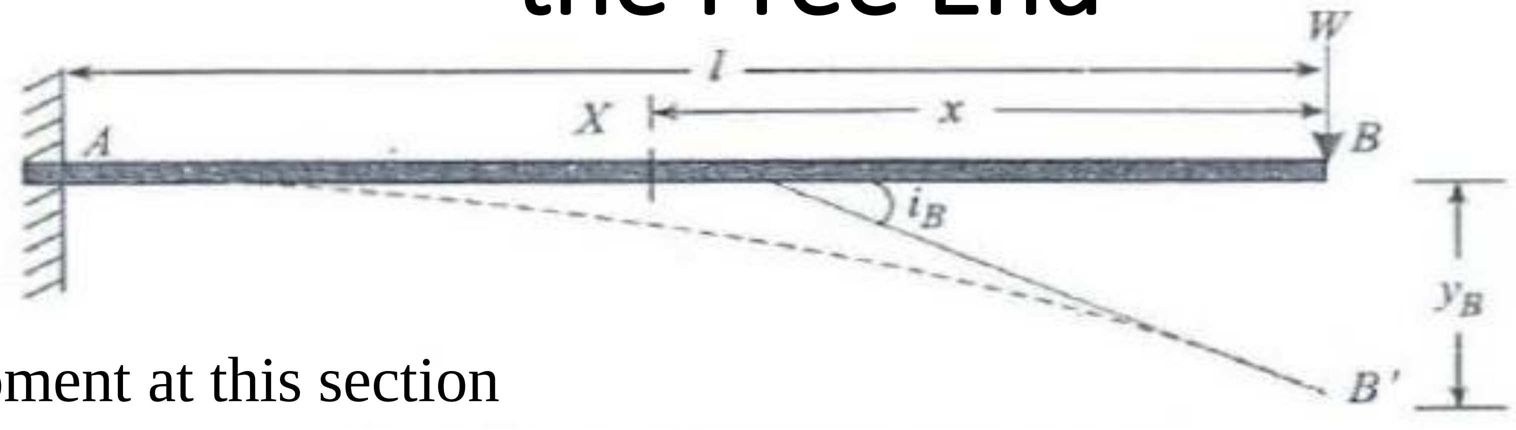
Solution

Given: Span (*l*)= -6 m = 6 x 10^3 mm ; Deflection at the centre *(Y_c)* = 4 mm ; modulus of elasticity(*E*)= 200 GPa = 200 x 10^3 N/mm^2 and moment of inertia(*I*)= 300 x 10^6 mm^4

Let *w* = Value of uniformly distributed load in N/mm or kN/m.

$$4 = \frac{5wl^4}{384EI} = \frac{5w(6000)^4}{384(200x10^3)(300x10^6)} = 0.281w \Rightarrow w = \frac{4}{0.281} = 14.2\ N/mm$$

Case 3: Cantilever Beam with a Point Load at the Free End



The bending moment at this section

$$M_x = -Wx$$

Therefore

$$EI\frac{d^2y}{dx^2} = -Wx$$

Integrating the above equation

$$EI\frac{dy}{dx} = -\frac{Wx}{2} + C_1$$

Case 3: Cantilever Beam with a Point Load at the Free End

Using Boundary Condition

$$x = l \qquad \frac{dy}{dx} = 0$$

$$C_1 = \frac{Wl^2}{2}$$

Hence,

$$EI\,\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{Wl^2}{2}$$

Integrating the above equation

$$EI.y = -\frac{Wx^3}{6} + \frac{Wl^2 x}{2} + C_2$$

Using Boundary Condition

$x = l$ and y = 0

$$C_2 = -\frac{Wl^3}{3}$$

Hence,

$$EI.y = -\frac{Wx^3}{6} + \frac{Wl^2 x}{2} - \frac{Wl^3}{3}$$

Example 1-4

A cantilever beam 120 nun wide and 150 mm deep is 1.8 m long. Determine the slope and deflection ac the free end of the beam, when it carries a point load of20 kN at its free end. Take E for the cantilever beam as 200 GPa.

Solution

Given: Width *(b)*= 120 mm; Depth (d)= 150 mm; Span(*l*)= 1.8 m = 1.8 x 10^3 mm; Point load *(W)* = 20 kN = 20 x 10^3 N and modulus of elasticity (*E*) = 200 GPa = 200 x 10^3 N/mm^2

$$I = \frac{bd^3}{12} = \frac{(120)(150)^3}{12} = 33.75x10^6 \ mm^4$$

Example 1-4 (continued)

Slope at the free end

$$EIi_B = -\frac{W(0)^2}{2} + \frac{Wl^2}{2}$$

$$i_B = \frac{Wl^2}{2EI} = \frac{(20000)(1800)^2}{2(200x10^3)(33.75x10^6)} = 0.0048\ rad$$

Deflection at the free end

$$EI.y_B = -\frac{W(0)^3}{6} + \frac{Wl^2(0)}{2} - \frac{Wl^3}{3}$$

$$y_B = -\frac{Wl^3}{3EI} = -\frac{(20000)(18000)^3}{3(200x10^3)(33.75x10^6)} = -5.76\ mm$$

Example 1-5

A cantilever beam of 160 mm width and 240 mm depth is 1.75 m long. What load can be placed at the free end of the cantilever if its deflection under the load is not to exceed 4.5 mm? Take E for the beam material as 180 GPa

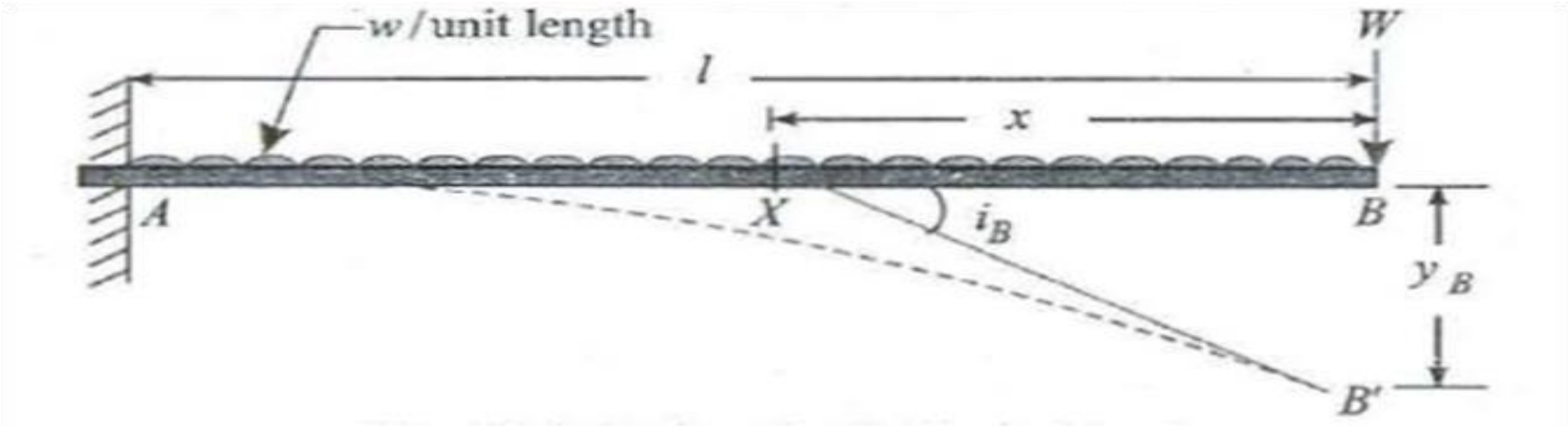
Solution

Given: Width *(b)*= 160 mm: Depth (d)= 240 mm; Span(*l*) = 1.75 m = 1.75 x 10^3 mm ; Deflection under the load (y_B) = 4.5 mm and modulus of elasticity *(E)* = 180 GPa = 180 x 10^3 N/mm^2

$$I = \frac{bd^3}{12} = \frac{(160)(240)^3}{12} = 184.32x10^6 \ mm^4$$

$$4.5 = \frac{Wl^3}{3EI} = \frac{W(18000)^3}{3(801x10^3)(1824.32x10^6)} \Rightarrow W = 83.57\,kN$$

Case 4: Cantilever Beam with a Uniformly Distributed Load



The bending moment at this section

$$M_x = -\frac{wx^2}{2}$$

Therefore

$$EI\frac{d^2y}{dx^2} = -\frac{w.x^2}{2}$$

Integrating the above equation

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$$

Case 4: Cantilever Beam with a Uniformly Distributed Load

Using Boundary Condition

$$x = l \qquad \frac{dy}{dx} = 0 \qquad C_1 = \frac{wl^3}{6}$$

Hence,

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6}$$

Integrating the above equation

$$EI.y = -\frac{wx^4}{24} + \frac{wl^3x}{6} + C_2$$

Using Boundary Condition

$x = l$ and y = 0 $\qquad C_2 = -\frac{wl^4}{8}$

Hence,

$$EI.y = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8}$$

Example 1-6

A cantilever beam 2 m long is subjected to a uniformly distributed load of 5 kN/m over its entire length. Find the slope and deflection of the cantilever beam at its free end. Take EI= 2.5 x 10^{12} mm^2

Solution

Given: Span (*l*) = 2 m = 2 x 10^3 mm ; Uniformly distributed load (w) = *5* kN/m =*5N/mm* and flexural rigidity (*EI*)= 2.5 x 10^{12} N-mm^2

Slope of the cantilever beam at its free end

$$EIi_B = -\frac{w(0)^3}{6} + \frac{wl^3}{6}$$

$$i_B = -\frac{wl^3}{6EI} = \frac{(5)(2000)^3}{6(2.5x10^{12})} = 0.0027\ rad$$

Deflection of the cantilever beam at its free end

$$EI.y_B = -\frac{w(0)^4}{24} + \frac{wl^3(0)}{6} - \frac{wl^4}{8}$$

$$y_B = \frac{wl^4}{8EI} = \frac{(5)(2000)^4}{8(2.5x10^{12})} = 4\ mm$$

Further Examples

7. A cantilever beam 100 mm wide and 180 mm deep is projecting 2 m from a wall. Calculate the uniformly distributed load, which the beam should carry, if the deflection of the free end should not exceed 3.5 mm. Take E as 200 Gpa

8. A cantilever beam of length 3 m is carrying a uniformly distributed load of w kN/m. Assuming rectangular cross-section with depth (d) equal to twice the width (b), determine the dimensions of the beam, so that vertical deflection at the free end does not exceed 8 mm. Take maximum bending stress = 100 MPa and E = 200 Gpa

Singularity Function (Macaulay's Method)

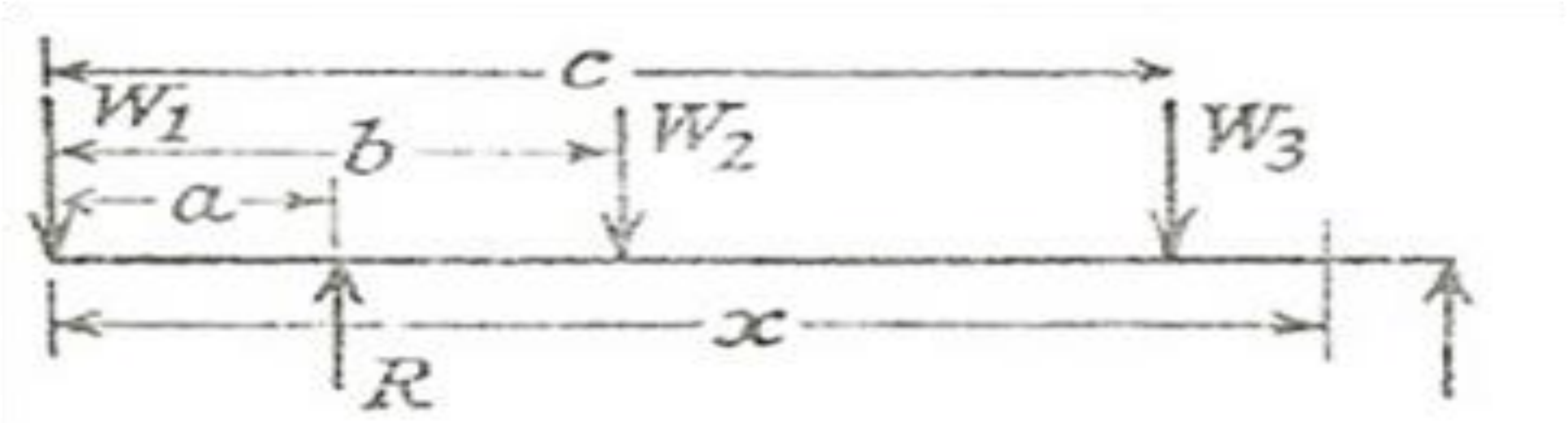
Macaulay's method enables

one continuous expression for bending moment to be obtained, and

the same constants of integration for all sections of the beam.

For the purpose of illustration, it is advisable to deal with the different types of loading separately.

Case 1: Concentrated Load

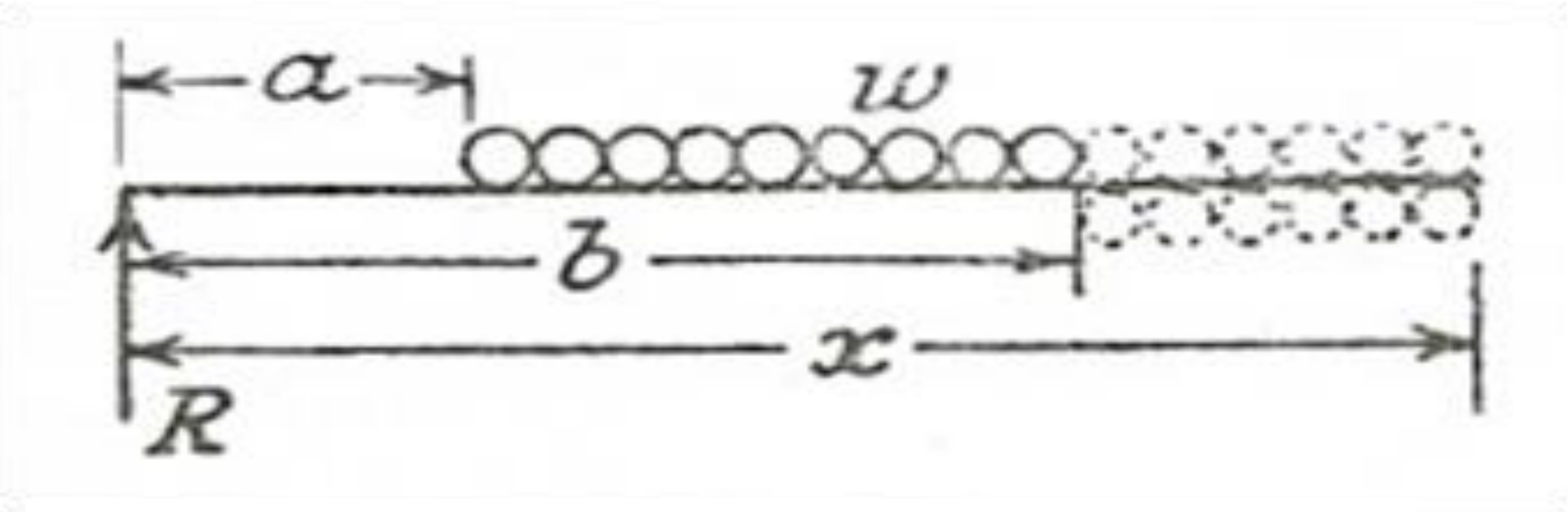


$$EI\, {d^2 y}\Big/{dx^2} = M = \left(-W_1 x + R[x-a] - W_2[x-b] - W_3[x-c]\right)$$

$$EI\, {dy}\Big/{dx} = \frac{1}{2}\left(-W_1 x^2 + R[x-a]^2 - W_2[x-b]^2 - W_3[x-c]^2\right) + A$$

$$EIy = \frac{1}{6}\left(-W_1 x^3 + R[x-a]^3 - W_2[x-b]^3 - W_3[x-c]^3\right) + Ax + B$$

Case 2: Uniformly Distributed Load

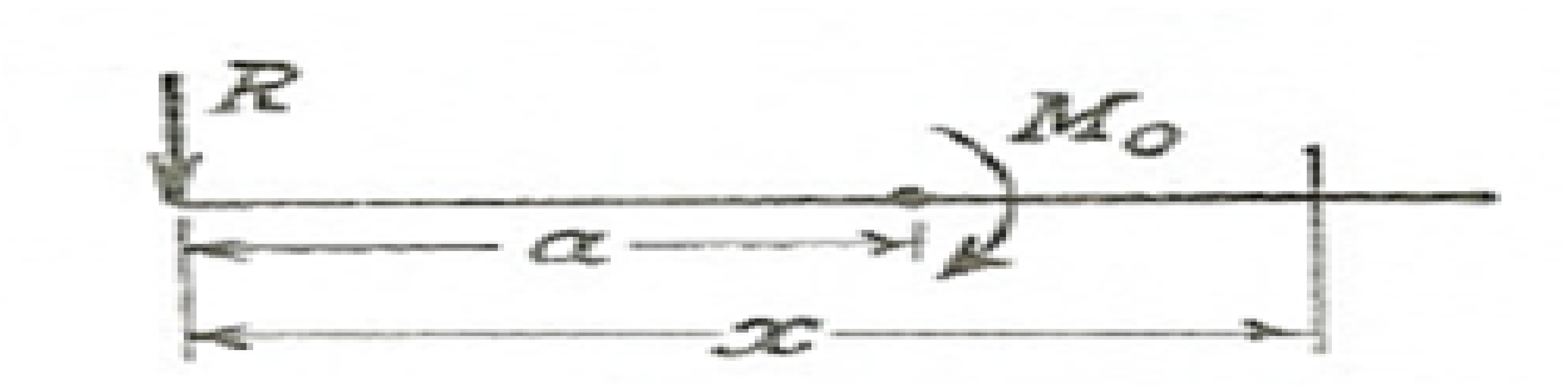


$$EI\, {d^2y}\Big/{dx^2} = M = Rx - w[x-a]^2 + w[x-b]^2$$

$$EI\, {dy}\Big/{dx} = M = \frac{R}{2}x^2 - \frac{w}{3}[x-a]^3 + \frac{w}{3}[x-b]^3 + A$$

$$EI.y = M = \frac{R}{6}x^3 - \frac{w}{12}[x-a]^4 + \frac{w}{12}[x-b]^4 + Ax + B$$

Case 3: Concentrated Moment



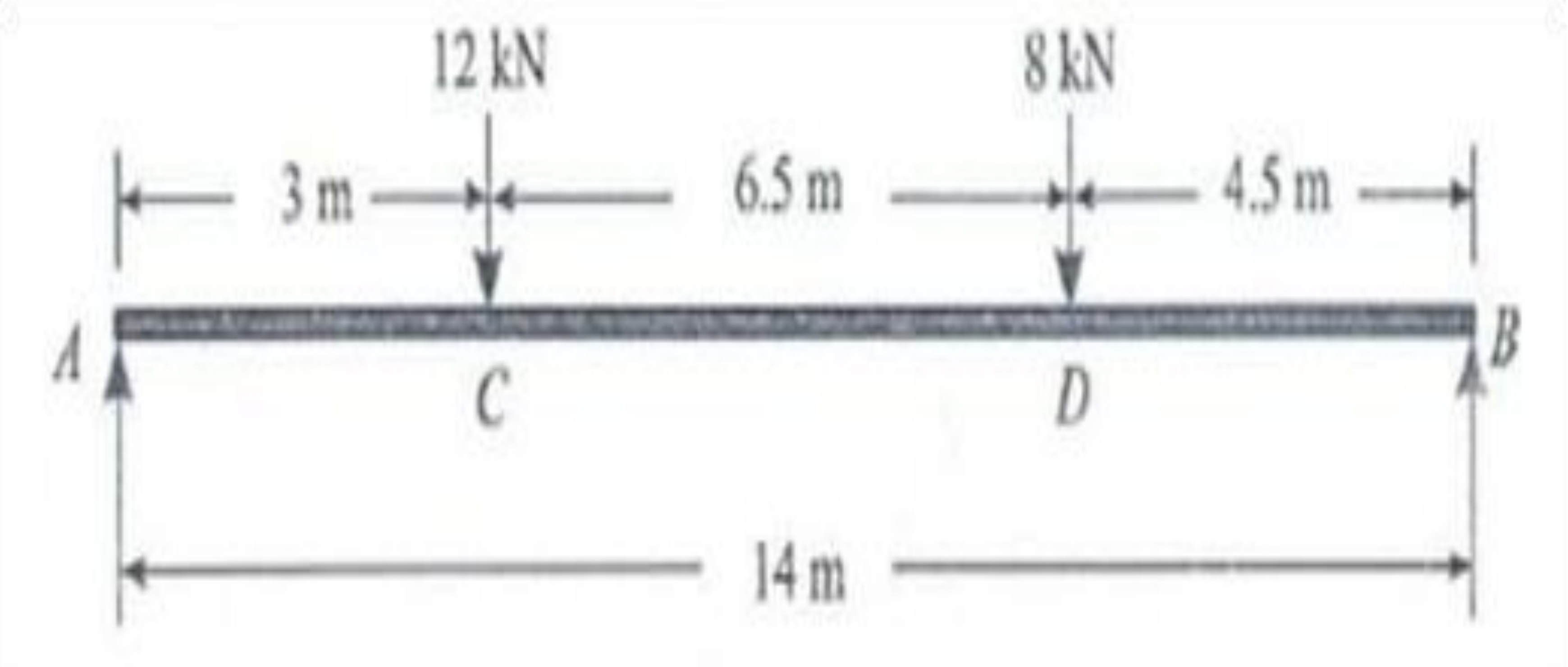
$$EI \, {d^2 y}/{dx^2} = M = \left(Rx - M_0 [x-a]^0\right)$$

$$EI \, {dy}/{dx} = M = \left(Rx^2/2 - M_0 [x-a]\right) + A$$

$$EIy = M = \left(Rx^3/6 - M_0 [x-a]^2\right) + Ax + B$$

Example 1-9

A horizontal steel girder having uniform cross-section is 14m long and is simply supported at its ends. It carries two concentrated loads as shown in Fig. 17. Calculate the deflections of the beam under the loads C and D. Take E = 200 GPa and 1 = $160x10^6$ mm^4



Example 1-9 (continued)

Solution

Given: Span (l)= 14m= 14 x 10^3 mm; Load at C (W_1) = 12 kN = 12 x 10^3 N; Load at D (W_2) = 8 kN = 8 x 10^3 N; Modulus of elasticity (E)= 200 GPa = 200 x I 0^3 N/mm^2 and moment of inertia(I)= 160 x 10^6 mm^4

Taking moments about A and equating the same

$$R_A = 12 + 8 - 8 = 12000\,N$$

$$14R_B = 12(3) + 8(9.5) = 112 \Rightarrow R_B = 8000\,N$$

Now taking A as the origin and using Macaulay's method, the bending moment at any section X at a distance x from A,

$$EI\frac{d^2y}{dx^2} = 12000x - 12000[x - 3000] - 8000[x - 9500]$$

Example 1-9 (continued)

Integrating the above equation

$$EI\frac{dy}{dx} = 6000x^2 - 6000[x-3000]^2 - 4000[x-9500]^2 + C_1$$

Integrating the above equation once again

$$EI.y = 2000x^3 - 2000[x-3000]^3 - 1333[x-9500]^3 + C_1x + C_2$$

Using Boundary Condition

x= 0 and $y = 0$, then $C_2 = 0$

x= 14000 mm and y = 0, then $C_1 = 193.2x10^9$

Hence

$$EI.y = 2000x^3 - 2000[x-3000]^3 - 1333[x-9500]^3 - 193.2x10^6\,x..(i)$$

Example 1-9 (continued)

For 12 kN load; *x* = 3 m (or 3 x 10³ mm)

$$EIy_C = 2000(3000)^3 - 193.2x10^9(3000) = -525.6x10^{12}$$

$$\Rightarrow y_C = \frac{-525.6x10^{12}}{(200x10^3)(160x10^6)} = 16.4\ mm$$

For 12 kN load; *x* = 9.5 m (or 9.5 x 10³ mm)

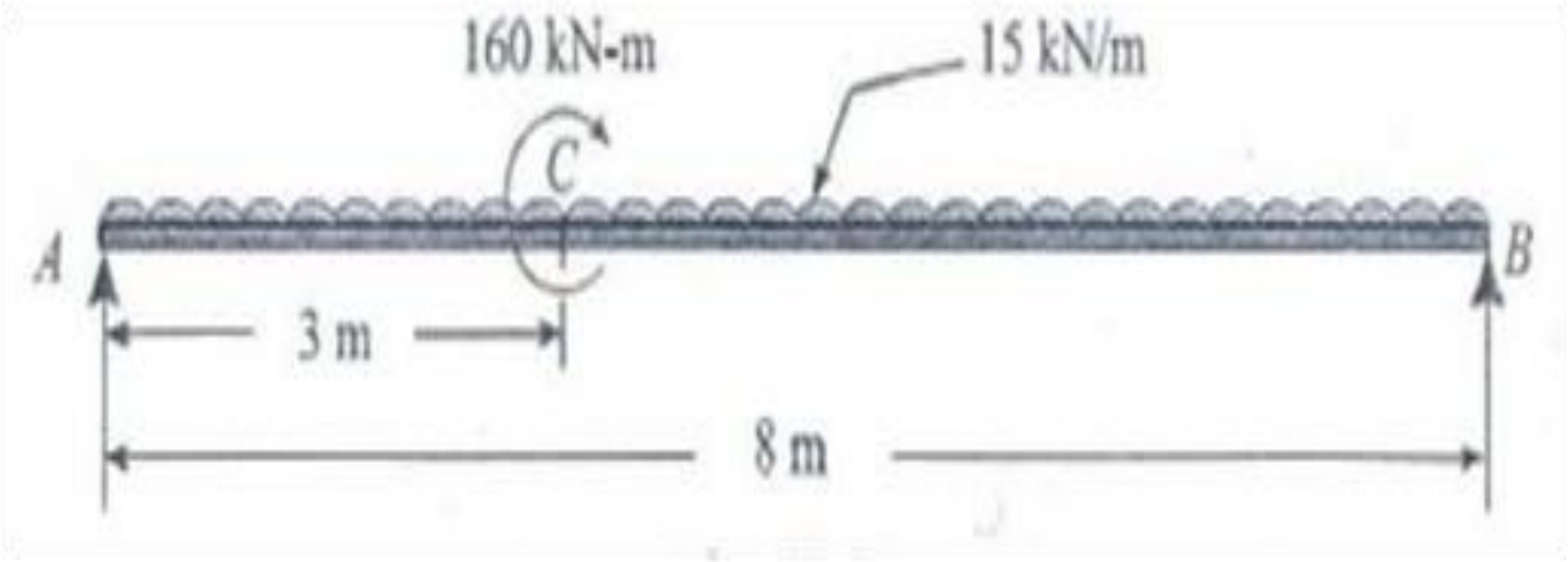
$$EIy_D = 2000(9500)^3 - 193.2x10^9(9500) - 2000[6500]^3 = -669.9x10^{12}$$

$$\Rightarrow y_D = \frac{-669.6x10^{12}}{(200x10^3)(160x10^6)} = -20.9\ mm$$

Further Examples

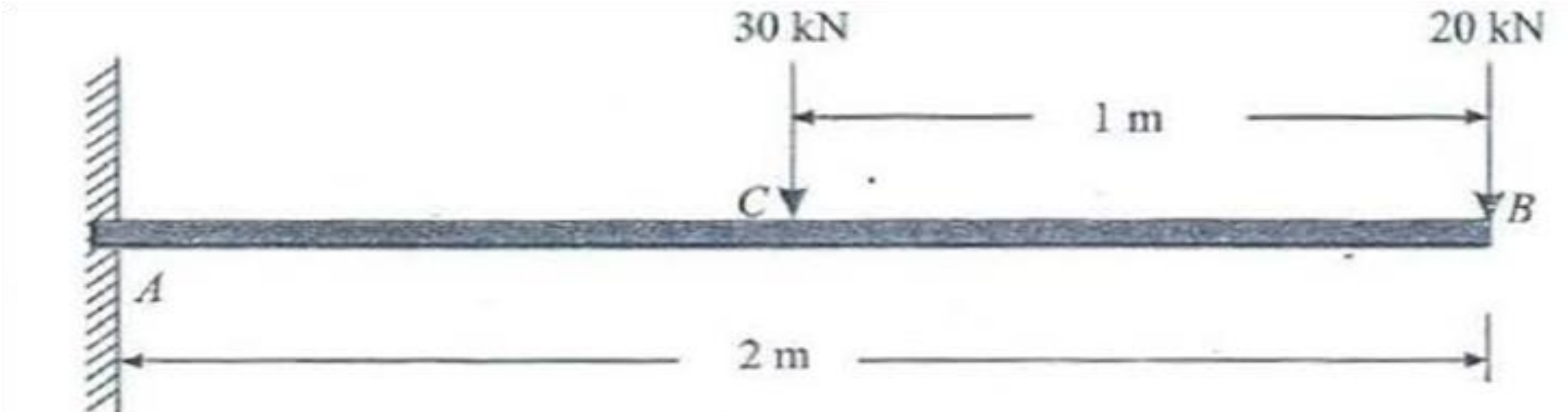
Example 10

A horizontal beam AB is freely supported at A and B 8 m apart and carries a uniformly distributed load of 15 kN/m run (including its own weight). A clockwise moment of 160 kN-m is applied to the beam at a point C, 3m from the left hand support A. Calculate the slope of the beam at C, if EI = 40 x 10^3 kN-m^2.



Example 11

A cantilever AB 2 m long is carrying a load of 20 kN at free end and 30 kN at a distance 1 m from the free end. Find the slope and deflection at the free end. Take E = 200 GPa and I = 150 x 10^6 mm^4



Strain Energy Method

- Consider a short length of beam δc ,under the action of a bending moment *M*.
- The strain energy of the length δx is given by

$$\delta U = \int \left(\sigma^2 \big/ 2E \right) dV$$

- But $dV = dA.\delta xx$ and $\sigma^2 = \left(My \big/ I \right)^2$

$$\Rightarrow \delta U = \left(M^2 \delta x \big/ 2EI^2 \right) \int y^2 dA$$

Strain Energy Method

But

$$\int y^2 dA = I$$

hence

$$\delta U = \left(M^2 \big/ 2EI \right) \delta x$$

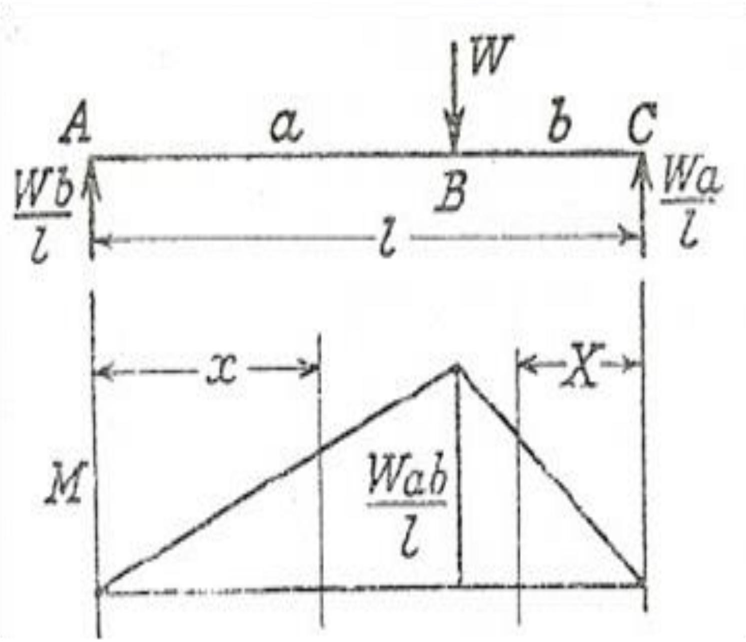
For the whole beam:

$$U = \int \left(M^2 \big/ 2EI \right) dx$$

The product ***EI*** is called the ***Flexural Rigidity*** of the beam

Example 12

A simply supported beam of length I carries a concentrated load W at distances of a and b from the two ends. Find expressions for the total strain energy of the beam and the deflection under the load.



Solution

The integration for strain energy can only be applied over a length of beam for which a continuous expression for *M* can be obtained.

This usually implies a separate integration for each section between two concentrated loads or reactions.

Referring to Figure 3, for the section AB,

$$M = (Wb/l)x$$

Example 1-12 (continued)

Taking a variable *x* measured from A

$$U_a = \int \left(M^2 / 2EI \right) dx = \int \frac{W^2 b^2}{2l^2 EI} x^2 dx = \frac{W^2 b^2}{2l^2 EI} \left[\frac{x^3}{3} \right]_0^a = W^2 a^3 b^2 / 6EIl^2$$

Similarly, by taking a variable *X* measured from C

$$U_b = \int \left(M^2 / 2EI \right) dx = \int \frac{W^2 a^2}{2l^2 EI} X^2 dX = \frac{W^2 a^2}{2l^2 EI} \left[\frac{X^3}{3} \right]_0^b = W^2 a^2 b^3 / 6EIl^2$$

The total strain energy is

$$U = U_a + U_b$$

$$\therefore U = \left(\frac{W^2 a^2 b^2}{6EIl^2} \right) (a+b) = \frac{W^2 a^2 b^2}{6EIl}$$

But, if δ is the deflection under the load, the strain energy must equal the work done by the load (gradually applied),

i.e. $$\frac{1}{2} W\delta = \frac{W^2 a^2 b^2}{6EIl} \Rightarrow \delta = \frac{Wa^2 b^2}{3EIl}$$

For a central load, *a =b =1/2,* and

$$\delta = \frac{W \left(l/2 \right)^2 \left(l/2 \right)^2}{3EIl} = \frac{Wl^3}{48EI}$$

Example 1-13

Compare the strain energy of a beam, simply supported at its ends and loaded with a uniformly distributed load, with that of the same beam centrally loaded and having the same value of maximum bending stress.

Solution

If l is the span and EI the flexural rigidity, then for a uniformly distributed load w, the end reactions are $wl/2$, and at a distance x from one end

$$M = \left({wl}/{2}\right)x - {wx^2}/{2} = \left({wx}/{2}\right)(l - x)$$

$$U_1 = \int \left({M^2}/{2EI}\right)dx = \int_0^l \frac{w^2 x^2 (l-x)^2}{8EI}\,dx = \frac{w^2}{8EI}\int_0^l \left(l^2 x^2 - 2lx^3 + x^4\right)dx = \frac{w^2 l^5}{240EI}$$

Example 1-13 (continued)

For central load of *W*, (Example 1-1), $$U_2 = \tfrac{1}{2}W\delta = \frac{1}{2}W\left(\frac{Wl^3}{48EI}\right) = \frac{W^2l^3}{96EI}$$

Maximum bending stress=*M/Z*.

Since the section modulus is the same then for the two bending stresses to be the same, the maximum bending moment must be the same.

Hence equating maximum bending moments, we have $$wl^2/8 = Wl/4 \Rightarrow W = \tfrac{1}{2}wl$$

Hence $$U_2 = \frac{W^2l^3}{96EI} = \frac{\left(\tfrac{1}{2}wl\right)^2 l^3}{96EI} = \frac{wl^5}{384EI}$$

Ratio $$\frac{U_1}{U_2} = \left(\frac{wl^5}{240EI}\right)\left(\frac{384EI}{wl^5}\right) = 1.6$$