14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

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14.6

Directional Derivatives and the Gradient Vector

In this section, we will learn how to find:

The rate of changes of a function of two or more variables in any direction.

INTRODUCTION

This weather map shows a contour map of the temperature function $T(x, y)$ for:

- The states of California and Nevada at 3:00 PM on a day in October.

INTRODUCTION

The level curves, or isothermals, join locations with the same temperature.

INTRODUCTION

The partial derivative T_x is the rate of change of temperature with respect to distance if we travel east from Reno.

- T_y is the rate of change if we travel north.

INTRODUCTION

However, what if we want to know the rate of change when we travel southeast (toward Las Vegas), or in some other direction?

DIRECTIONAL DERIVATIVE

In this section, we introduce a type of derivative, called a directional derivative, that enables us to find:

- The rate of change of a function of two or more variables in any direction.

Equations 1

Recall that, if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as:

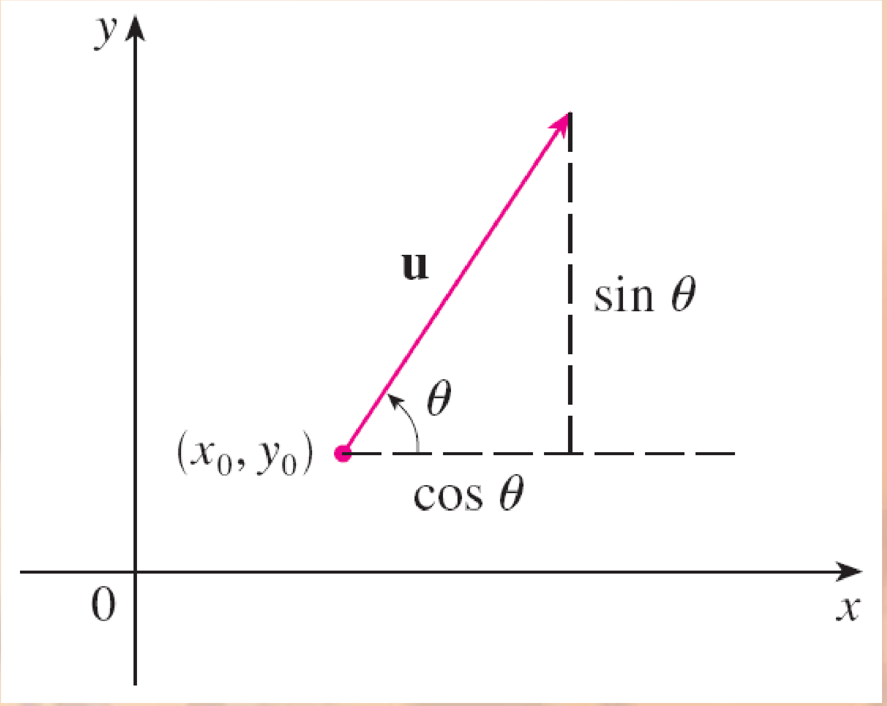
$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Equations 1

They represent the rates of change of z in the x- and y-directions—that is, in the directions of the unit vectors **i** and **j**.

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

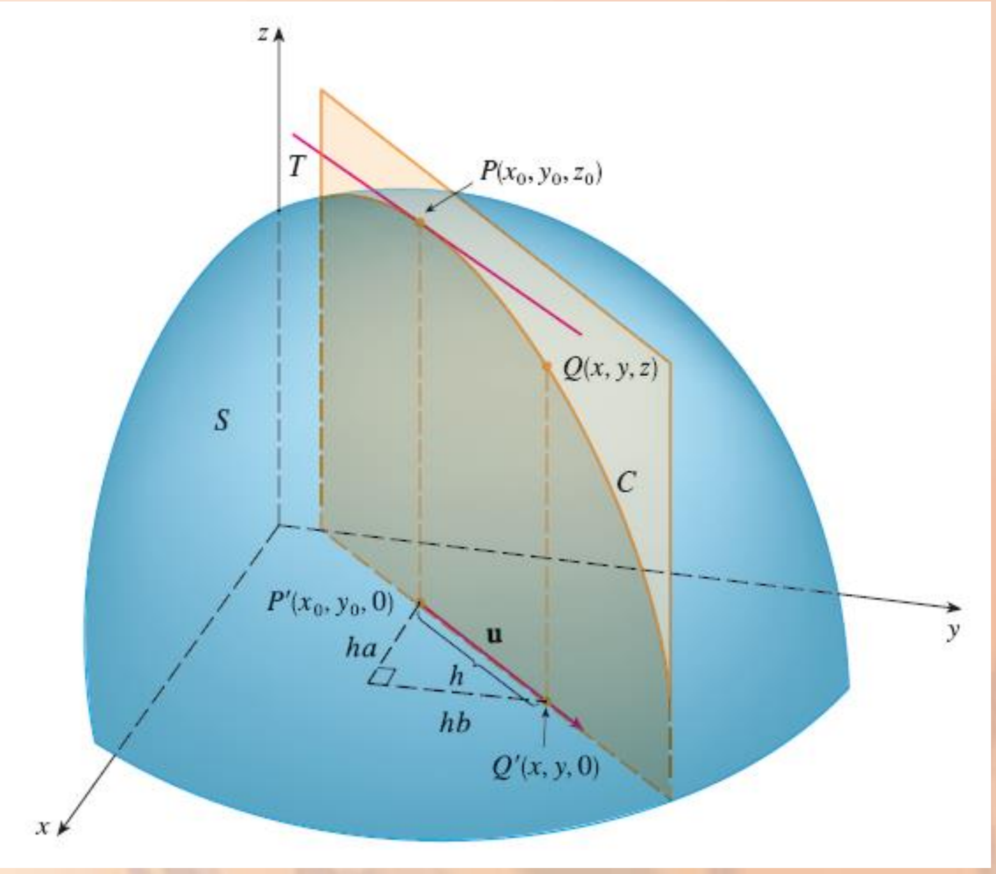


To do this, we consider the surface S with equation $z = f(x, y)$ [the graph of f] and we let $z_0 = f(x_0, y_0)$.

- Then, the point $P(x_0, y_0, z_0)$ lies on S.

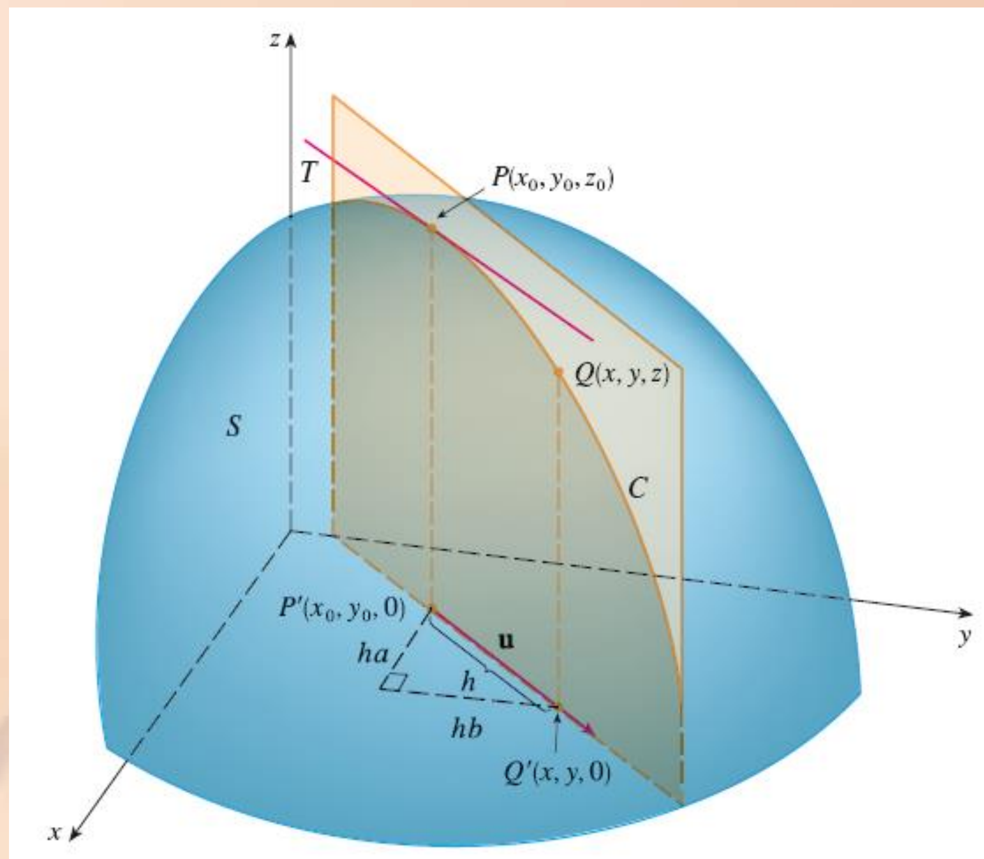
DIRECTIONAL DERIVATIVES

The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*.



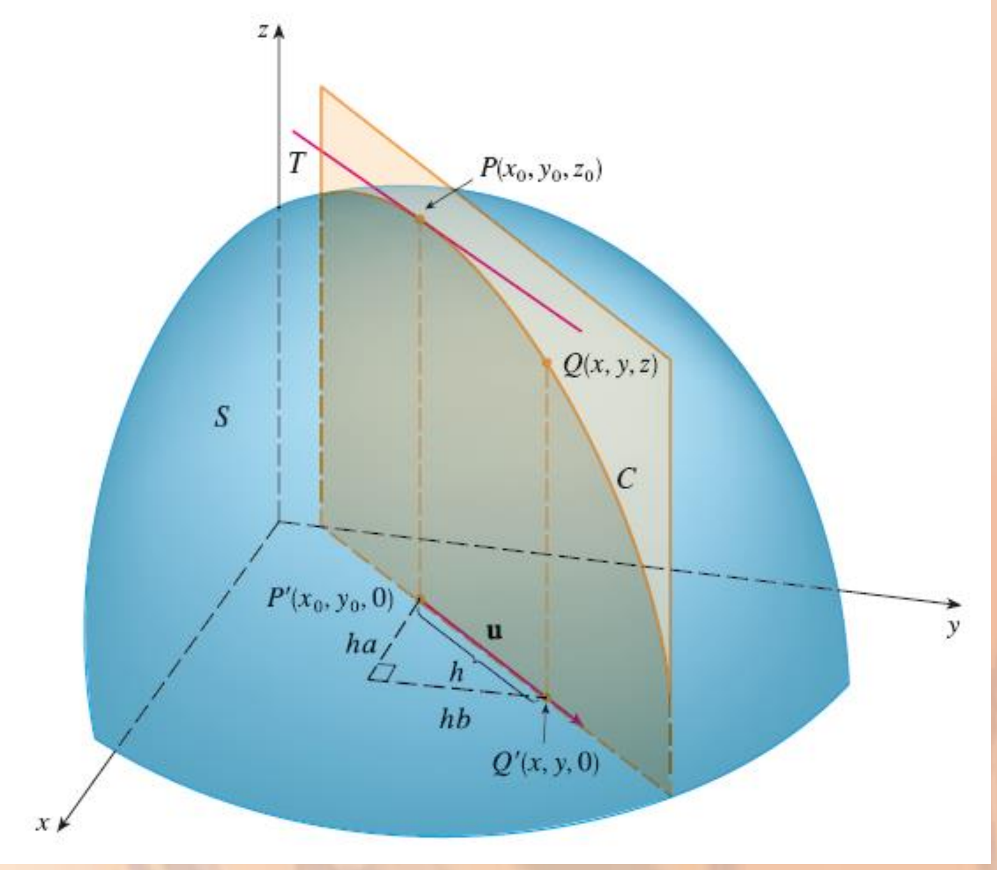
DIRECTIONAL DERIVATIVES

The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.



Now, let:

- $Q(x, y, z)$ be another point on C.
- P', Q' be the projections of P, Q on the xy-plane.



DIRECTIONAL DERIVATIVES

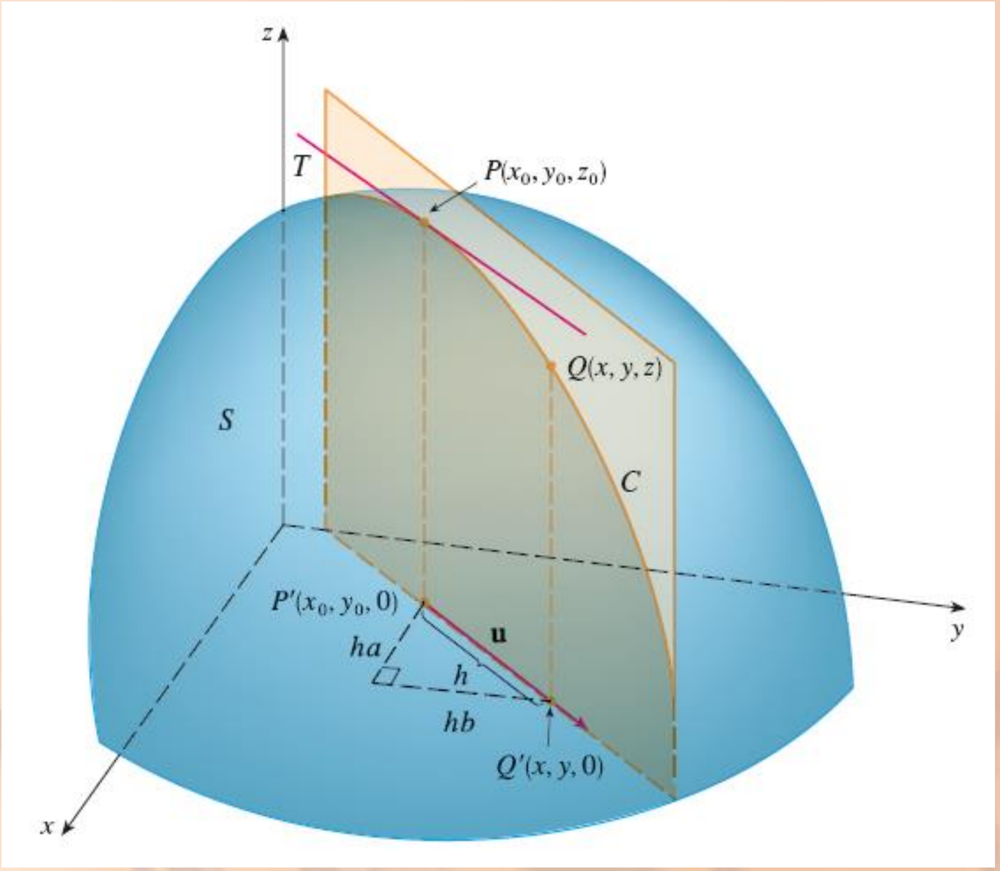
Then, the vector $\overrightarrow{P'Q'}$ is parallel to **u**.

So,

$$\overrightarrow{P'Q'} = h\mathbf{u}$$

$$= \langle ha, hb \rangle$$

for some scalar *h*.



Therefore,

$$x - x_0 = ha$$

$$y - y_0 = hb$$

DIRECTIONAL DERIVATIVES

So,

$$x = x_0 + ha$$

$$y = y_0 + hb$$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h}$$

$$= \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

DIRECTIONAL DERIVATIVE

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u.**

- This is called the directional derivative of f in the direction of **u**.

DIRECTIONAL DERIVATIVE

Definition 2

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is:

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Comparing Definition 2 with Equations 1, we see that:

- If **u** = **i** = $\langle 1, 0 \rangle$, then $D_i f = f_x$.
- If **u** = **j** = $\langle 0, 1 \rangle$, then $D_j f = f_y$.

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

Use this weather map to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

The unit vector directed toward the southeast is:

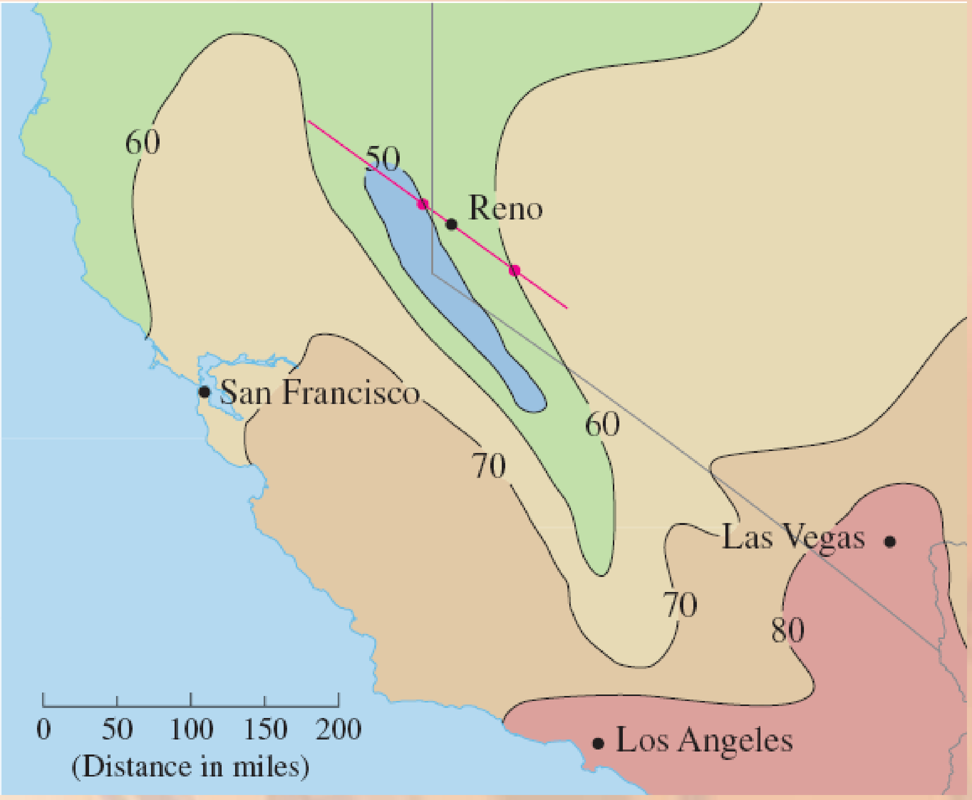
$$\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$$

- However, we won't need to use this expression.

We start by drawing a line through Reno toward the southeast.

We approximate the directional derivative $D_{\mathbf{u}}T$ by:

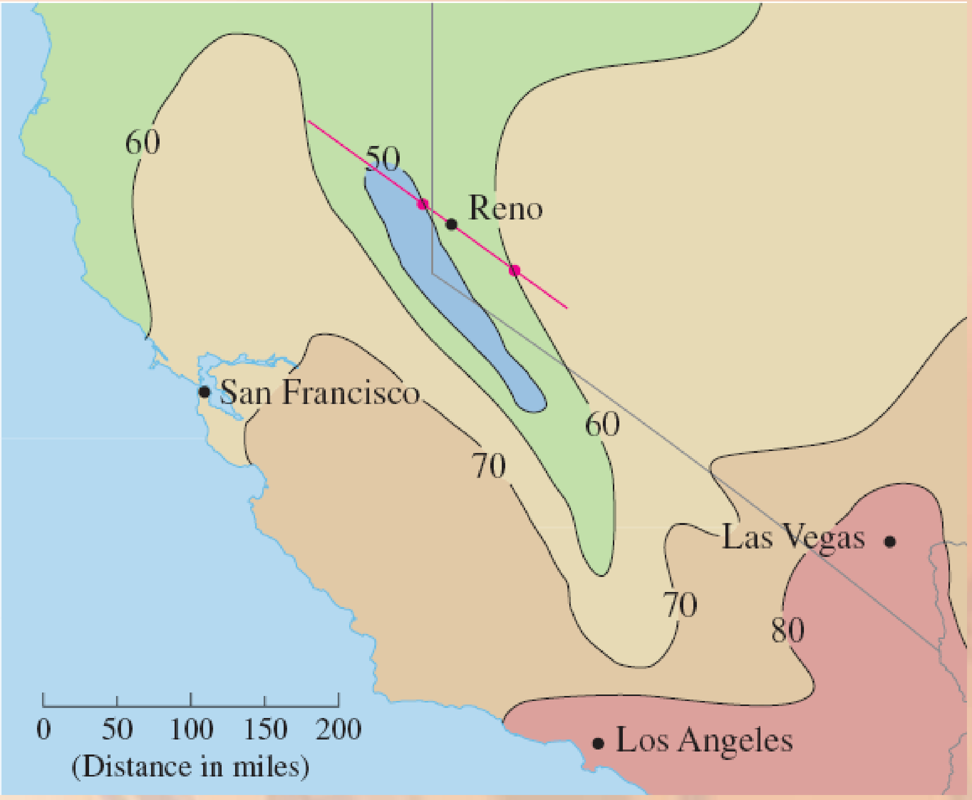
- The average rate of change of the temperature between the points where this line intersects the isothermals T = 50 and T = 60.



The temperature at the point southeast of Reno is T = 60°F.

The temperature at the point northwest of Reno is T = 50°F.

The distance between these points looks to be about 75 miles.



So, the rate of change of the temperature in the southeasterly direction is:

$$\begin{aligned} D_{\mathbf{u}}T &\approx \frac{60-50}{75} \\ &= \frac{10}{75} \\ &\approx 0.13^{\circ}\ \mathrm{F/mi} \end{aligned}$$

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

Theorem 3

If *f* is a differentiable function of *x* and *y*, then *f* has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

If we define a function *g* of the single variable *h* by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have the following equation.

Proof—Equation 4

$$
\begin{aligned}
&g'(0) \\
&= \lim_{h\to 0} \frac{g(h)-g(0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb)-f(x_0, y_0)}{h} \\
&= D_{\mathbf{u}} f(x_0, y_0)
\end{aligned}
$$

On the other hand, we can write:

$$g(h) = f(x, y)$$

where:

- $x = x_0 + ha$
- $y = y_0 + hb$

Hence, the Chain Rule (Theorem 2 in Section 14.5) gives:

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$= f_x(x, y)a + f_y(x, y)b$$

If we now put $h = 0$,

then $$x = x_0$$

$$y = y_0$$

and

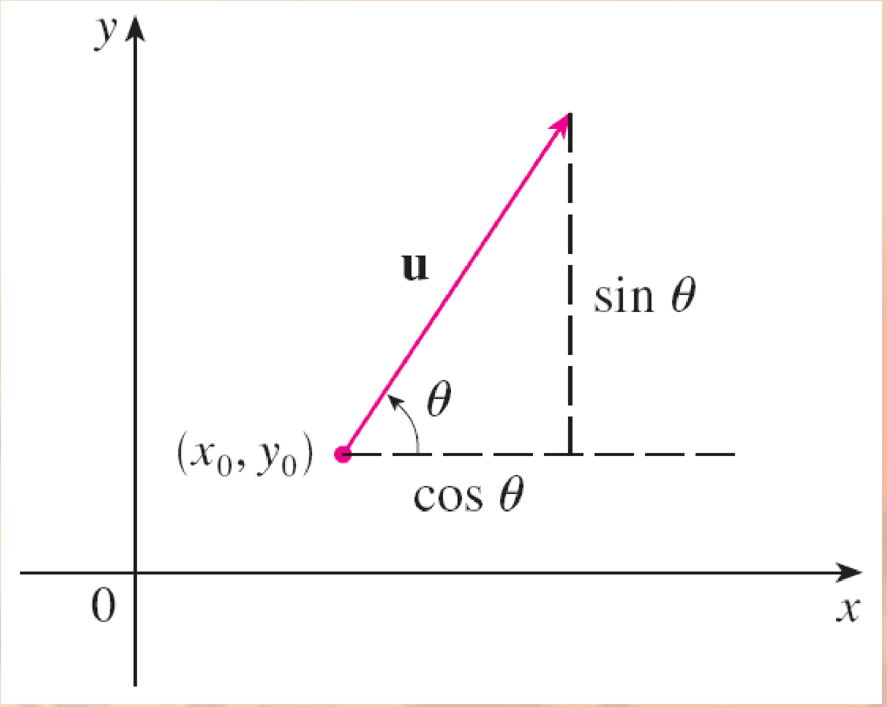
$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Comparing Equations 4 and 5,

we see that:

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Suppose the unit vector **u** makes an angle *θ* with the positive *x*-axis, as shown.



 Equation 6

Then, we can write

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$$

and the formula in Theorem 3 becomes:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if:

- $f(x, y) = x^3 - 3xy + 4y^2$
- **u** is the unit vector given by angle $\theta = \pi/6$

What is $D_{\mathbf{u}}f(1, 2)$?

Formula 6 gives:

$$
\begin{aligned}
D_{\mathbf{u}} f(x, y) &= f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6} \\
&= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2} \\
&= \tfrac{1}{2}\left[3\sqrt{3}x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]
\end{aligned}
$$

Therefore,

$$D_{\mathbf{u}} f(1,2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right]$$

$$= \frac{13 - 3\sqrt{3}}{2}$$

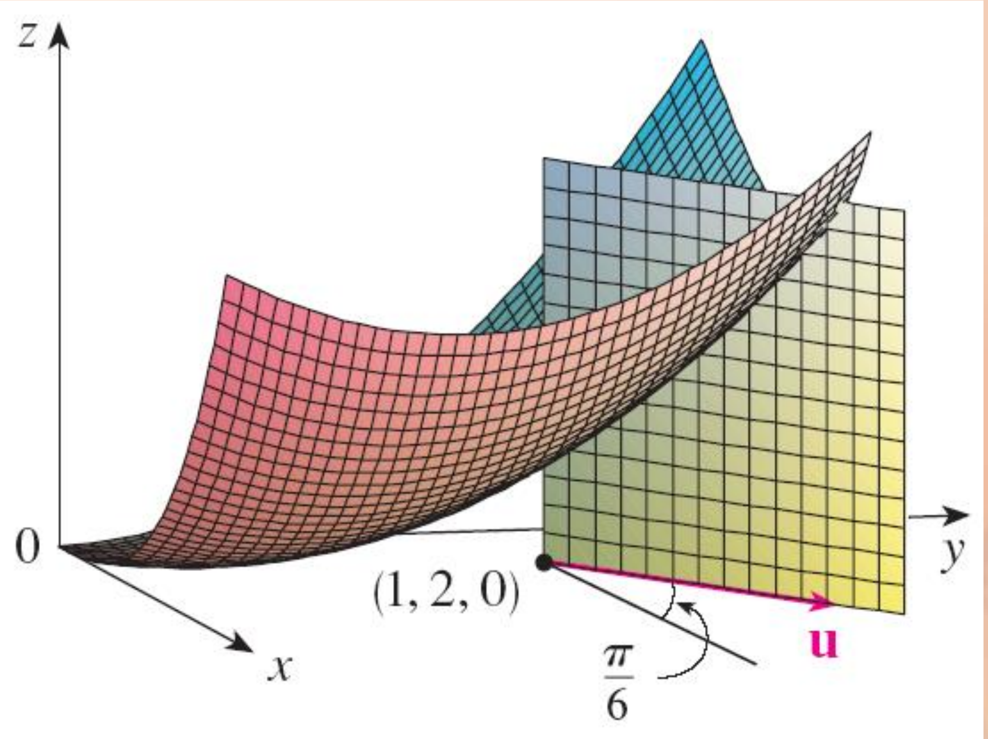
DIRECTIONAL DERIVATIVES

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of **u**.

This is the slope of the tangent line to the curve of intersection of the surface

$$z = x^3 - 3xy + 4y^2$$

and the vertical plane through (1, 2, 0) in the direction of **u** shown here.



Expression 7

Notice from Theorem 3 that the directional derivative can be written as the dot product of two vectors:

$$
\begin{aligned}
D_{\mathbf{u}} f(x, y) &= f_x(x, y)\,a + f_y(x, y)\,b \\
&= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\
&= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}
\end{aligned}
$$

THE GRADIENT VECTOR

The first vector in that dot product occurs not only in computing directional derivatives but in many other contexts as well.

THE GRADIENT VECTOR

So, we give it a special name:

- The gradient of f

We give it a special notation too:

- grad f or ∇f , which is read "del f "

THE GRADIENT VECTOR

Definition 8

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j}$$

Example 3

If $f(x, y) = \sin x + e^{xy}$,

then

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

$$\nabla f(0,1) = \langle 2, 0 \rangle$$

THE GRADIENT VECTOR

Equation 9

With this notation for the gradient vector, we can rewrite Expression 7 for the directional derivative as:

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

- This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

Example 4

Find the directional derivative of the function

$$f(x, y) = x^2y^3 - 4y$$

at the point (2, –1) in the direction of the vector **v** = 2 **i** + 5 **j**.

We first compute the gradient vector at (2, –1):

$$\nabla f(x, y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\,\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\,\mathbf{j}$$

Note that **v** is not a unit vector.

However, since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of **v** is:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore, by Equation 9,

we have:

$$\begin{aligned} D_{\mathbf{u}} f(2,-1) &= \nabla f(2,-1)\cdot \mathbf{u} \\ &= (-4\mathbf{i}+8\mathbf{j})\cdot\left(\frac{2}{\sqrt{29}}\mathbf{i}+\frac{5}{\sqrt{29}}\mathbf{j}\right) \\ &= \frac{-4\cdot 2+8\cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}} \end{aligned}$$

FUNCTIONS OF THREE VARIABLES

For functions of three variables, we can define directional derivatives in a similar manner.

- Again, $D_{\mathbf{u}} f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is:

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

THREE-VARIABLE FUNCTIONS

If we use vector notation, then we can write both Definitions 2 and 10 of the directional derivative in a compact form, as follows.

THREE-VARIABLE FUNCTIONS

Equation 11

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where:

- $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$
- $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$

This is reasonable.

- The vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{u}$ (Equation 1 in Section 12.5).

- Thus, $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

THREE-VARIABLE FUNCTIONS **Formula 12**

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that:

$$D_{\mathbf{u}} f(x, y, z)$$
$$= f_x(x, y, z)\,a + f_y(x, y, z)\,b + f_z(x, y, z)\,c$$

For a function *f* of three variables, the gradient vector, denoted by ∇f or grad *f*, is:

$$\nabla f(x, y, z)$$
$$= \langle f_x(x, y, z), f_y(x, y, z,), f_z(x, y, z) \rangle$$

For short,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

THREE-VARIABLE FUNCTIONS Equation 14

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as:

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

If $f(x, y, z) = x \sin yz$, find:

a. The gradient of f

b. The directional derivative of f at (1, 3, 0) in the direction of $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

The gradient of *f* is:

$$
\begin{aligned}
&\nabla f(x,y,z) \\
&= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \\
&= \langle \sin yz, xz\cos yz, xy\cos yz\rangle
\end{aligned}
$$

At (1, 3, 0), we have:

$$\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$$

The unit vector in the direction of **v** = **i** + 2 **j** – **k** is:

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Hence, Equation 14 gives:

$$
\begin{aligned}
D_{\mathbf{u}} f(1,3,0) &= \nabla f(1,3,0) \cdot \mathbf{u} \\
&= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\
&= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}}
\end{aligned}
$$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point.

- These give the rates of change of f in all possible directions.

We can then ask the questions:

- In which of these directions does *f* change fastest?
- What is the maximum rate of change?

MAXIMIZING THE DIRECTIONAL DERIVATIVE

The answers are provided by the following theorem.

MAXIMIZING DIRECTIONAL DERIV. Theorem 15

Suppose f is a differentiable function of two or three variables.

The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is: $|\nabla f(\mathbf{x})|$

- It occurs when **u** has the same direction as the gradient vector $\nabla f(\mathbf{x})$

From Equation 9 or 14, we have:

$$
\begin{aligned}
D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} &= |\nabla f||\mathbf{u}| \cos\theta \\
&= |\nabla f| \cos\theta
\end{aligned}
$$

where θ is the angle between ∇f and **u**.

MAXIMIZING DIRECTIONAL DERIV. Proof

The maximum value of cos θ is 1.

This occurs when $\theta = 0$.

- So, the maximum value of $D_{\mathbf{u}} f$ is: $|\nabla f|$
- It occurs when $\theta = 0$, that is, when **u** has the same direction as ∇f.

a. If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

b. In what direction does *f* have the maximum rate of change?

What is this maximum rate of change?

MAXIMIZING DIRECTIONAL DERIV. **Example 6 a**

We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \langle e^y, xe^y \rangle$$

$$\nabla f(2,0) = \langle 1, 2 \rangle$$

MAXIMIZING DIRECTIONAL DERIV. Example 6 a

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$

is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$.

So, the rate of change of *f* in the direction from *P* to *Q* is:

$$\begin{aligned} D_{\mathbf{u}} f(2,0) &= \nabla f(2,0) \cdot u \\ &= \langle 1, 2 \rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5} \rangle \\ &= 1(-\tfrac{3}{5}) + 2(\tfrac{4}{5}) = 1 \end{aligned}$$

MAXIMIZING DIRECTIONAL DERIV. Example 6 b

According to Theorem 15, *f* increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1, 2 \rangle$.

So, the maximum rate of change is:

$$\left| \nabla f(2,0) \right| = \left| \langle 1, 2 \rangle \right| = \sqrt{5}$$

Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$$

where:

- T is measured in degrees Celsius.
- x, y, z is measured in meters.

MAXIMIZING DIRECTIONAL DERIV. Example 7

In which direction does the temperature increase fastest at the point (1, 1, –2)?

What is the maximum rate of increase?

The gradient of *T* is:

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

$$= -\frac{160x}{(1+x^2+2y^2+3z^2)^2}\mathbf{i} - \frac{320y}{(1+x^2+2y^2+3z^2)^2}\mathbf{j}$$

$$- \frac{480z}{(1+x^2+2y^2+3z^2)^2}\mathbf{k}$$

$$= \frac{160}{(1+x^2+2y^2+3z^2)^2}(-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k})$$

At the point (1, 1, –2), the gradient vector is:

$$\nabla T(1,1,-2) = \tfrac{160}{256}(-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k})$$
$$= \tfrac{5}{8}(-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k})$$

MAXIMIZING DIRECTIONAL DERIV. Example 7

By Theorem 15, the temperature increases fastest in the direction of the gradient vector

$$\nabla T(1,1,-2) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

- Equivalently, it does so in the direction of $-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k})/\sqrt{41}$

MAXIMIZING DIRECTIONAL DERIV. **Example 7**

The maximum rate of increase is the length of the gradient vector:

$$\left|\nabla T(1,1,-2)\right| = \tfrac{5}{8}\left|-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k}\right|$$

$$= \tfrac{5}{8}\sqrt{41}$$

- Thus, the maximum rate of increase of temperature is: $\tfrac{5}{8}\sqrt{41} \approx 4^\circ \mathrm{C/m}$

TANGENT PLANES TO LEVEL SURFACES

Suppose S is a surface with equation

$$F(x, y, z)$$

- That is, it is a level surface of a function F of three variables.

Then, let

$$P(x_0, y_0, z_0)$$

be a point on *S*.

TANGENT PLANES TO LEVEL SURFACES

Then, let C be any curve that lies on the surface S and passes through the point P.

- Recall from Section 13.1 that the curve C is described by a continuous vector function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

TANGENT PLANES TO LEVEL SURFACES

Let t_0 be the parameter value corresponding to *P*.

- That is,

$$\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$$

Equation 16

Since *C* lies on *S*, any point $(x(t), y(t), z(t))$ must satisfy the equation of *S*.

That is,

$$F(x(t), y(t), z(t)) = k$$

Equation 17

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16:

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial x}\frac{dz}{dt}=0$$

However, as $\nabla F = \langle F_x, F_y, F_z \rangle$

and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Equation 17 can be written in terms of a dot product as:

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Equation 18

In particular, when $t = t_0$,

we have:

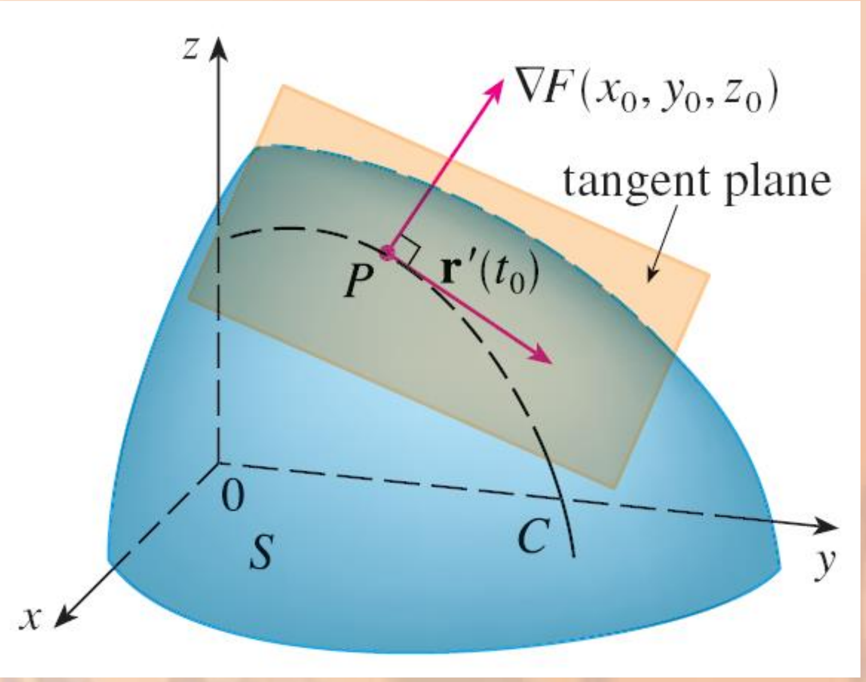
$$\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$$

So,

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says:

- The gradient vector at *P*, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve *C* on *S* that passes through *P*.



TANGENT PLANES

If $\nabla F(x_0, y_0, z_0) \neq 0$, it is thus natural to define the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as:

- The plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$

Equation 19

Using the standard equation of a plane (Equation 7 in Section 12.5), we can write the equation of this tangent plane as:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

NORMAL LINE

The normal line to S at P is the line:

- Passing through P
- Perpendicular to the tangent plane

Thus, the direction of the normal line is given by the gradient vector

$$\nabla F(x_0, y_0, z_0)$$

So, by Equation 3 in Section 12.5, its symmetric equations are:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

TANGENT PLANES

Consider the special case in which the equation of a surface S is of the form

$$z = f(x, y)$$

- That is, S is the graph of a function f of two variables.

Then, we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F.

Then,

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$

$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$

$$F_z(x_0, y_0, z_0) = -1$$

So, Equation 19 becomes:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

- This is equivalent to Equation 2 in Section 14.4

TANGENT PLANES

Thus, our new, more general, definition of a tangent plane is consistent with the definition that was given for the special case of Section 14.4

Example 8

Find the equations of the tangent plane and normal line at the point (–2, 1, –3) to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Example 8

The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

So, we have:

$$F_x(x,y,z)=\frac{x}{2}$$

$$F_y(x,y,z)=2y$$

$$F_z(x,y,z)=\frac{2z}{9}$$

$$F_x(-2,1,-3)=-1$$

$$F_y(-2,1,-3)=2$$

$$F_z(-2,1,-3)=-\tfrac{2}{3}$$

Then, Equation 19 gives the equation of the tangent plane at (–2, 1, –3) as:

$$-1(x+2)+2(y-1)-\tfrac{2}{3}(z+3)=0$$

- This simplifies to:

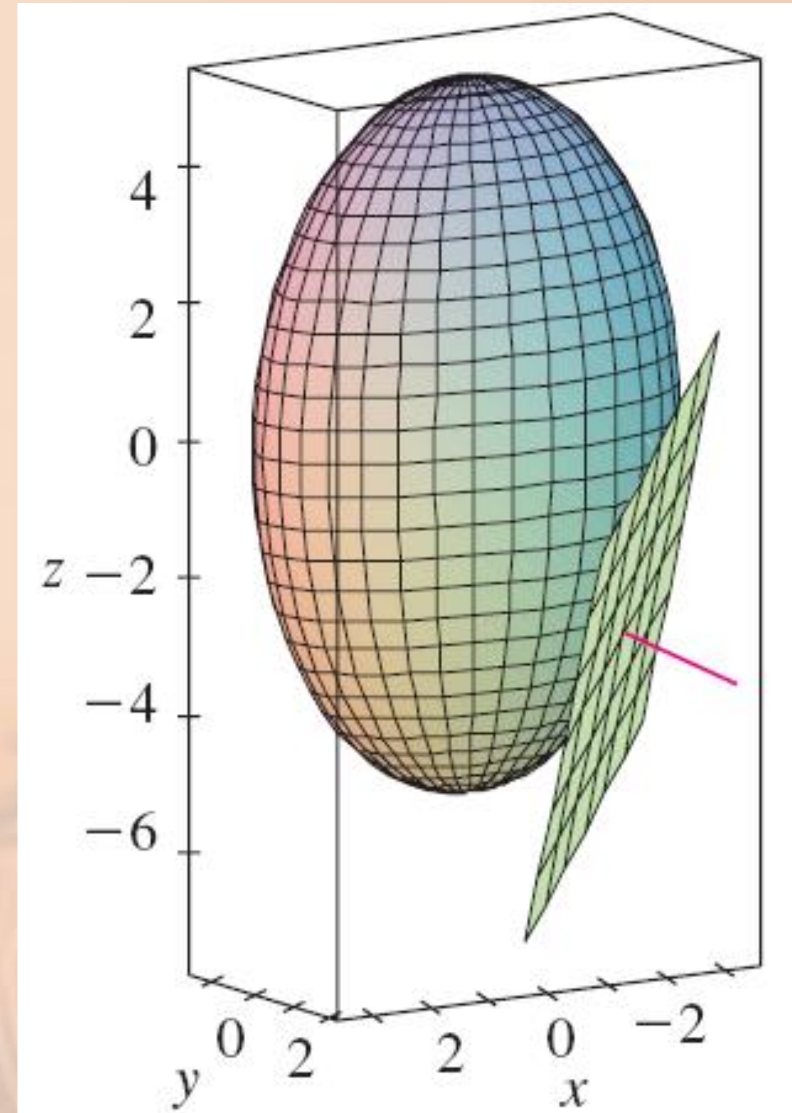
$$3x - 6y + 2z + 18 = 0$$

By Equation 20, symmetric equations of the normal line are:

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$$

Example 8

The figure shows the ellipsoid, tangent plane, and normal line in Example 8.



SIGNIFICANCE OF GRADIENT VECTOR

We now summarize the ways in which the gradient vector is significant.

SIGNIFICANCE OF GRADIENT VECTOR

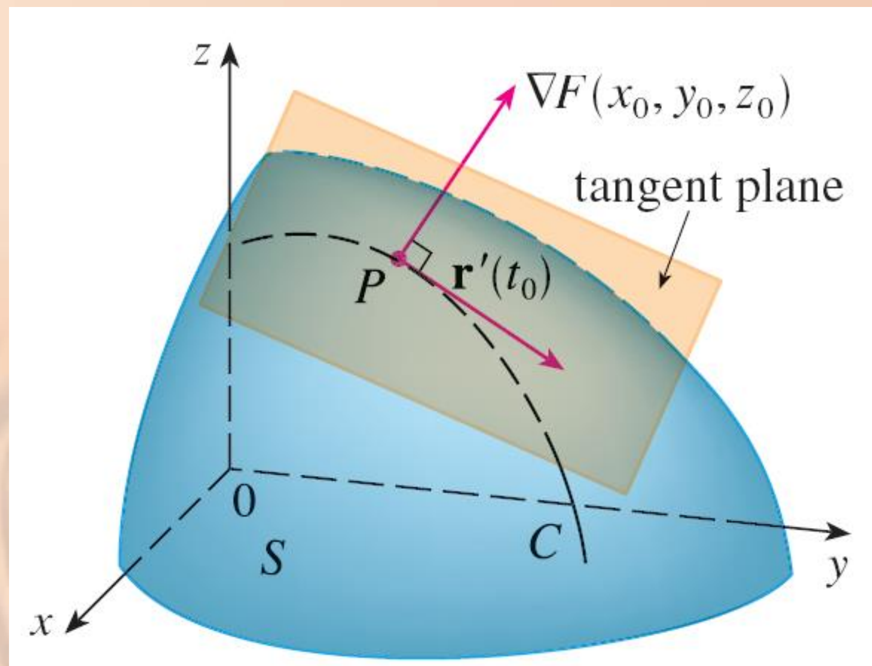
We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain.

SIGNIFICANCE OF GRADIENT VECTOR

On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of *f*.

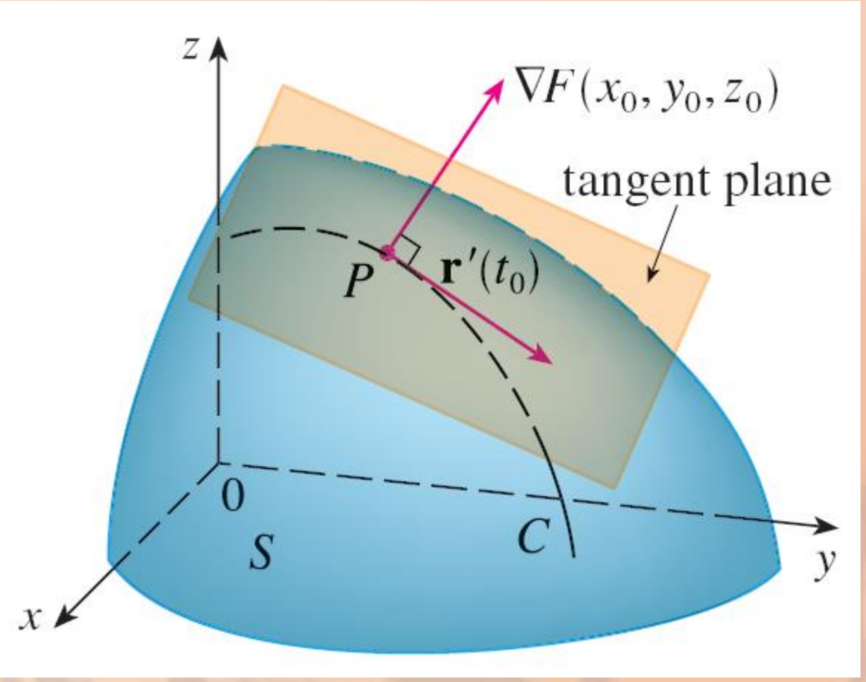
SIGNIFICANCE OF GRADIENT VECTOR

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface *S* of *f* through *P*.



These two properties are quite compatible intuitively.

- As we move away from *P* on the level surface *S*, the value of *f* does not change at all.



SIGNIFICANCE OF GRADIENT VECTOR

So, it seems reasonable that, if we move in the perpendicular direction, we get the maximum increase.

SIGNIFICANCE OF GRADIENT VECTOR

In like manner, we consider a function *f* of two variables and a point $P(x_0, y_0)$ in its domain.

SIGNIFICANCE OF GRADIENT VECTOR

Again, the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of *f*.

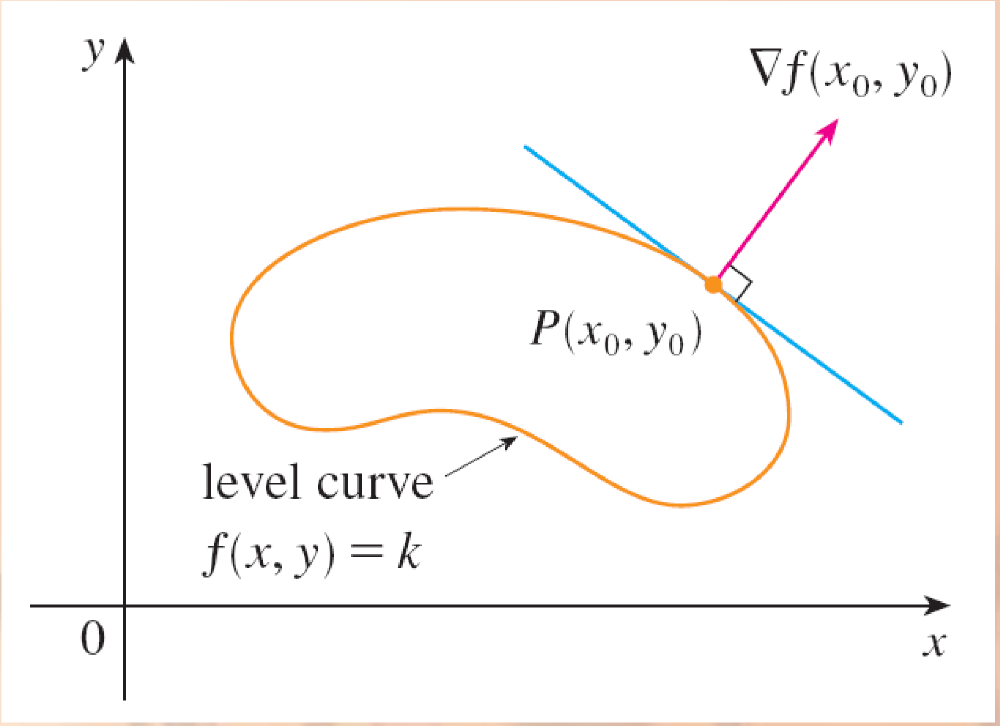
SIGNIFICANCE OF GRADIENT VECTOR

Also, by considerations similar to our discussion of tangent planes, it can be shown that:

- $\nabla f(x_0, y_0)$ is perpendicular to the level curve *f*(*x*, *y*) = *k* that passes through *P*.

Again, this is intuitively plausible.

- The values of f remain constant as we move along the curve.



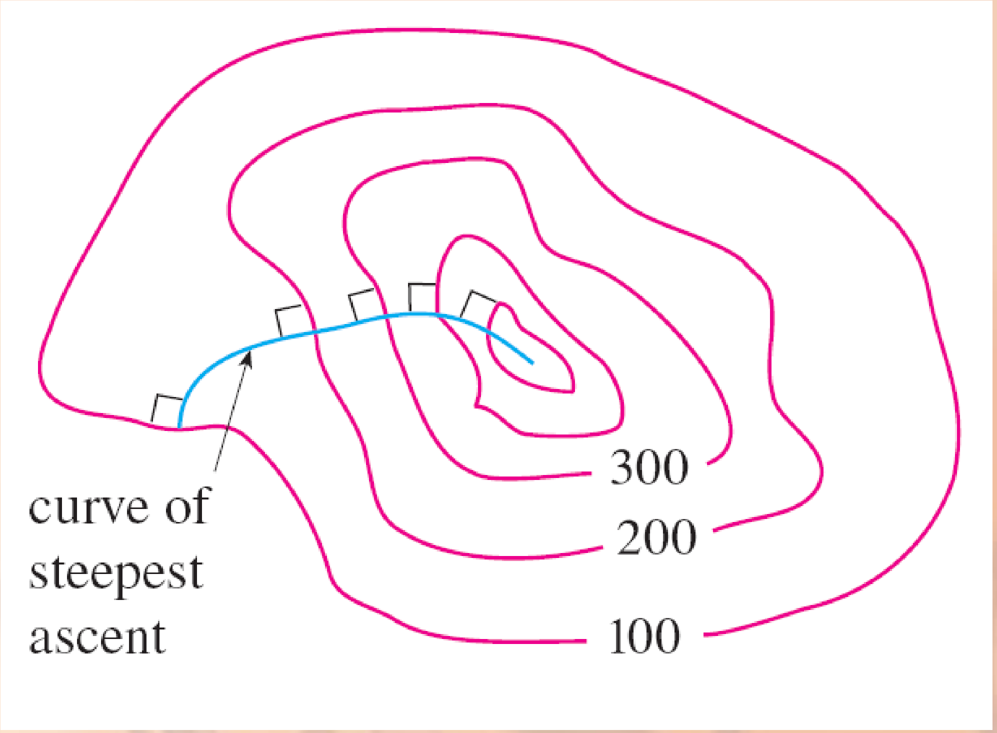
SIGNIFICANCE OF GRADIENT VECTOR

Now, we consider a topographical map of a hill.

Let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y).

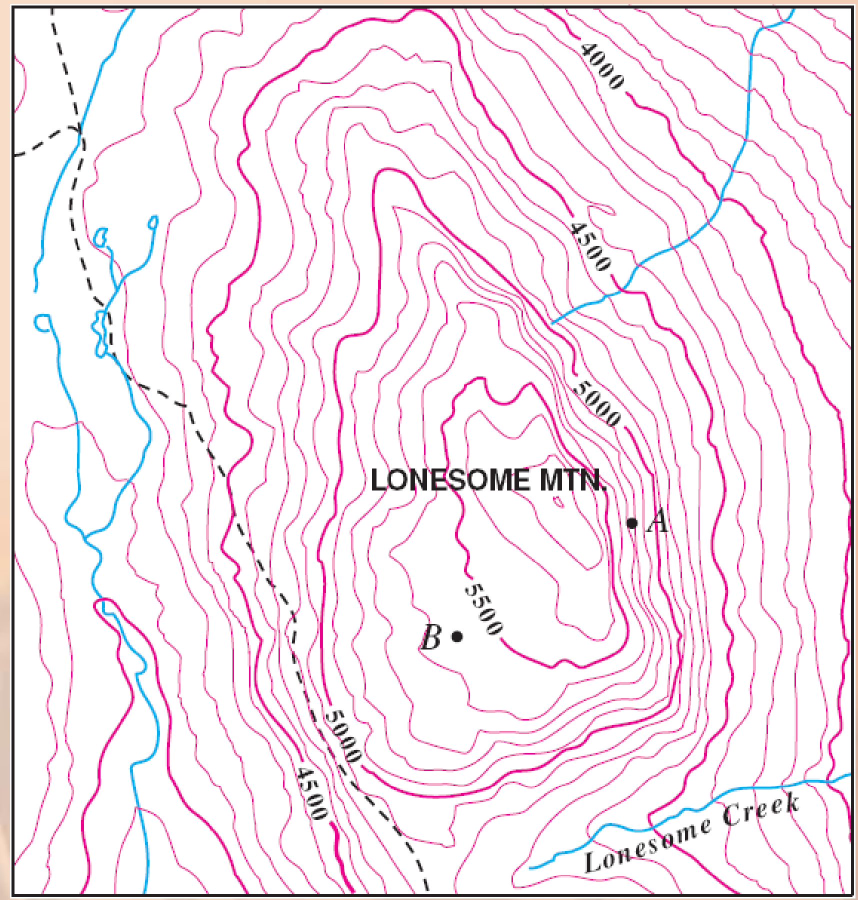
SIGNIFICANCE OF GRADIENT VECTOR

Then, a curve of steepest ascent can be drawn by making it perpendicular to all of the contour lines.



SIGNIFICANCE OF GRADIENT VECTOR

This phenomenon can also be noticed in this figure in Section 14.1, where Lonesome Creek follows a curve of steepest descent.



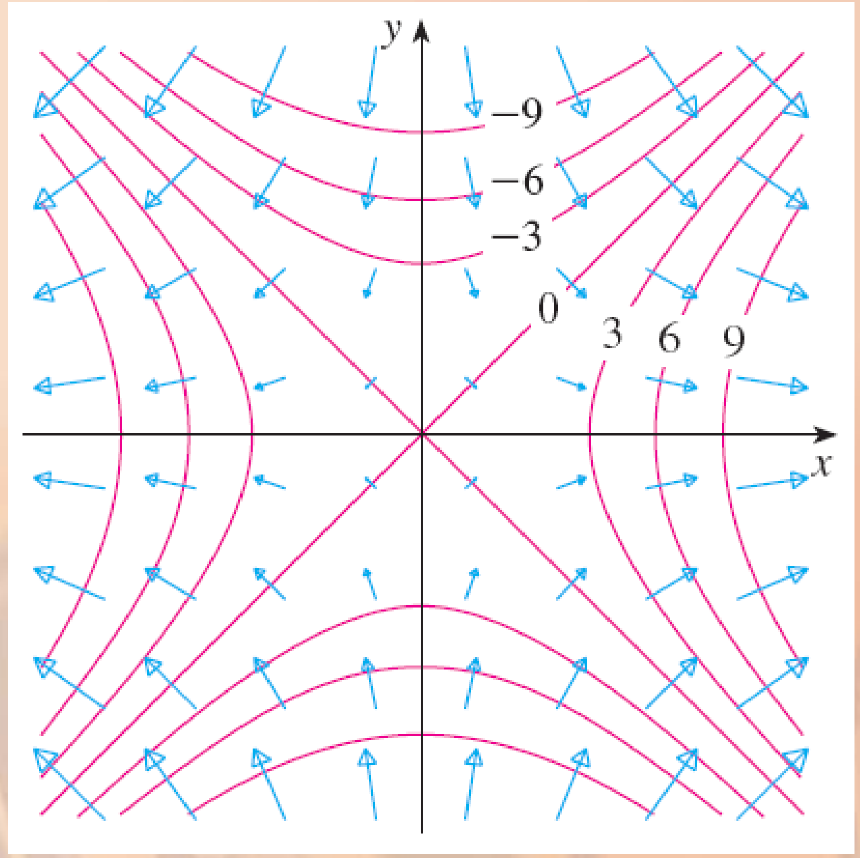
SIGNIFICANCE OF GRADIENT VECTOR

Computer algebra systems have commands that plot sample gradient vectors.

Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b).

GRADIENT VECTOR FIELD

The figure shows such a plot—called a gradient vector field—for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f.



SIGNIFICANCE OF GRADIENT VECTOR

As expected,

the gradient vectors:

- Point “uphill”
- Are perpendicular to the level curves