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Mech 3

Question 1

a) When a linear system is symmetric positive definite
b) Let's find LU factorization of the coefficient matrix using Cholesky reduction. Hence find the solution

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11} \quad L_{21} & L_{11} \quad L_{31} \\ L_{21} \quad L_{11} & L_{21}^2 \quad L_{22}^2 & L_{21} \quad L_{31} \quad + \quad L_{22} \quad L_{32} \\ L_{31} \quad L_{11} & L_{31} \quad L_{21} \quad + \quad L_{32} \quad L_{22} & L_{31}^2 \quad + \quad L_{32}^2 \quad + \quad L_{33}^2 \end{bmatrix}$$

By comparison

•$L_{11}=\sqrt{2}$ •$L_{31}=\frac{3\sqrt{2}}{2}$ •$L_{32}=\frac{\sqrt{6}}{6}$

•$L_{21}=\frac{-\sqrt{2}}{2}$ •$L_{22}=\frac{\sqrt{6}}{2}$ •$(L_{33})^2=\frac{-8}{3}$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}$$

Ax=b $LL^Tx=b$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{after solving we have} \quad \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 1.633 \\ \frac{-0.6667}{L_{33}} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 1.633 \\ \frac{-0.6667}{L_{33}} \end{bmatrix} \quad \text{after solving we have} \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.25 \\ 0.25 \end{bmatrix}$$

c) Let's find the inverse of the coefficient matrix from b

$AA^{-1}=I$ but $Ax=b$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$; with $A^{-1}=\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$

Solving this special system

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}\begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} Y_{11} \\ Y_{21} \\ Y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ after solving we have $\begin{bmatrix} Y_{11} \\ Y_{21} \\ Y_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\sqrt{6}}{6} \\ \frac{5}{3L_{33}} \end{bmatrix}$

$\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}\begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\sqrt{6}}{6} \\ \frac{5}{3L_{33}} \end{bmatrix}$ after solving we have $\begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{bmatrix} = \begin{bmatrix} -0.375 \\ 0.125 \\ 0.625 \end{bmatrix}$

We repeat the same process for X_{12}, X_{22} and X_{32}

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}\begin{bmatrix} X_{12} \\ X_{22} \\ X_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix}\begin{bmatrix} Y_{12} \\ Y_{22} \\ Y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ after solving we have $\begin{bmatrix} Y_{12} \\ Y_{22} \\ Y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{6}}{3} \\ \frac{-1}{3L_{33}} \end{bmatrix}$

$\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix}\begin{bmatrix} X_{12} \\ X_{22} \\ X_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{6}}{3} \\ \frac{-1}{3L_{33}} \end{bmatrix}$ after solving we have $\begin{bmatrix} X_{12} \\ X_{22} \\ X_{32} \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.625 \\ 0.125 \end{bmatrix}$

we repeat the same process for X_{13}, X_{23} and X_{33}

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix} \begin{bmatrix} X_{13} \\ X_{23} \\ X_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & 0 \\ \frac{3}{\sqrt{2}} & \frac{\sqrt{6}}{6} & L_{33} \end{bmatrix} \begin{bmatrix} Y_{13} \\ Y_{23} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ after solving we have $\begin{bmatrix} Y_{13} \\ Y_{23} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_{33}} \end{bmatrix}$

$\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & L_{33} \end{bmatrix} \begin{bmatrix} X_{13} \\ X_{23} \\ X_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_{33}} \end{bmatrix}$ after solving we have $\begin{bmatrix} X_{13} \\ X_{23} \\ X_{33} \end{bmatrix} = \begin{bmatrix} 0.625 \\ 0.125 \\ -0.375 \end{bmatrix}$

A^{-1}= $\begin{bmatrix} -0.375 & 0.125 & 0.625 \\ 0.125 & 0.625 & 0.125 \\ 0.625 & 0.125 & -0.375 \end{bmatrix}$ This matrix is also symmetric.

Question 2 :

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 4 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

1. For a matrix to be symmetric, A=A^T

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 4 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 4 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

- Since A=A^T, the matrix is said to be symmetric

2. For a matrix to be positively defined, all the leading principal minors should be greater than zero

$$A_1 = A$$

$$A_2 = \begin{bmatrix} 6 & 10 \\ 10 & 20 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{bmatrix}$$

$$A_4 = 20$$

Det of A_1=1 >0

Det of A_2=20>0

Det of A_3=4>0

Det of A_4=20>0

- Since all the leading principal minors should be greater than zero, the matrix A is said to be positively defined

Question 3 :

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 30 \\ -24 \end{bmatrix}$$

First step:

X_1^1= 1.25/4 [24-3X_2^0]-0.25X_1^0

X_1^1=1.25/4[24-3]-0.25=6.3125

X_2^1=1.25/4[30-3(6.3125) + 1]- 0.25=3.5195

X_3^1=1.25/4(-24+3.5195)-0.25= -6.650

X^1=$[\mathbf{6.3125}, \mathbf{3.5195}, -\mathbf{6.650}]^T$

Second step:

X_1^2=1.25/4[24-3(3.5195)]-0.25(6.3125)=2.6223

X_2^2=1.25/4[30-3(2.6223)-6.650]-0.25(3.5194) =3.9586

X_3^2=1.25/4[-24+3.9586]-0.25(-6.650) =-4.600

X^2=$[\mathbf{2.6223}, \mathbf{3.9586}, -\mathbf{4.600}]^T$

Question 4:

$$\begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

Let X_1=X, X_2=Y, X_3=Z

The equation becomes

12X+7Y+3Z=2 [1]

X+5Y+Z=-5 [2]

2X+7Y-11Z=6 [3]

Making X, Y and Z subject of the formula in equation 1, 2 and 3 respectively

X=1/12(2-7y-3z)

Y=1/5(-5-x-z)

Z=1/11(2x+7y-6)

Using $[1; 3; 5]^T$ as the initial values and solving for the first 3 iterations on calculator gives

<u>1st Iteration:</u>

X=-17/6 =-2.83

Y= -43/30 =-1.43

Z= -217/110 = -1.97

<u>2nd Iteration:</u>

X= 1481/990 =1.49

Y= -2239/2475 = -0.90

Z= -7706/9075 = -0.85

<u>3rd Iteration:</u>

X=0.91

Y=-1.01

Z=-1.02

<u>QUESTION 5:</u>

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 11 \\ 20 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & 1 & 2 \\ 2 & 4 & 1 & 0 & 3 \\ 1 & 1 & 4 & -1 & 11 \\ -1 & 0 & -1 & 3 & 20 \end{array}\right] \quad \begin{array}{c} R_2 - \frac{1}{3}R_1 \\ R_3 - \frac{1}{6}R_1 \\ R_4 - -\frac{1}{6}R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & 1 & 2 \\ 0 & 10/3 & 2/3 & -1/3 & 7/3 \\ 0 & 2/3 & 23/6 & -7/6 & 32/3 \\ 0 & 1/3 & -5/6 & 19/6 & 61/3 \end{array}\right] \quad \begin{array}{c} R_3 - \frac{1}{5}R_2 \\ R_4 - \frac{1}{10}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & 1 & 2 \\ 0 & 10/3 & 2/3 & -1/3 & 7/3 \\ 0 & 0 & 37/10 & -11/10 & 51/5 \\ 0 & 0 & -9/10 & 16/5 & 201/10 \end{array}\right] \quad R_4 - -\frac{9}{37}R_3$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & 1 & 2 \\ 0 & 10/3 & 2/3 & -1/3 & 7/3 \\ 0 & 0 & 37/10 & -11/10 & 51/5 \\ 0 & 0 & 0 & 217/74 & 22.58 \end{array}\right]$$

As seen above;

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.33 & 1 & 0 & 0 \\ 0.17 & 0.2 & 1 & 0 \\ -0.17 & 0.1 & -0.24 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & 3.33 & 0.67 & -0.33 \\ 0 & 0 & 3.7 & -1.1 \\ 0 & 0 & 0 & 2.93 \end{bmatrix}$$

The LU decomposition of the original matrix becomes;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.33 & 1 & 0 & 0 \\ 0.17 & 0.2 & 1 & 0 \\ -0.17 & 0.1 & -0.24 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & 3.33 & 0.67 & -0.33 \\ 0 & 0 & 3.7 & -1.1 \\ 0 & 0 & 0 & 2.93 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 11 \\ 20 \end{bmatrix}$$

The above equation can be reduced to

$$\textit{Equation 1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.33 & 1 & 0 & 0 \\ 0.17 & 0.2 & 1 & 0 \\ -0.17 & 0.1 & -0.24 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 11 \\ 20 \end{bmatrix}$$

$$\textit{Equation 2} \begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & 3.33 & 0.67 & -0.33 \\ 0 & 0 & 3.7 & -1.1 \\ 0 & 0 & 0 & 2.93 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

Solving for the values of Y from equation 1

$Y_1=2$ [1]

$0.33Y_1+Y_2=3$ [2]

$0.17Y_1+0.2Y_2+Y_3=11$ [3]

$-0.17Y_1+0.1Y_2-0.24Y_3+Y_4=20$ [4]

Solving subequation1,2,3 and 4 simultaneously gives

$Y_1=2$

$Y_2=2.34$

$Y_3=10.192$

$Y_4=22.55$

Substituting the values of Y in equation 2 gives;

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & 3.33 & 0.67 & -0.33 \\ 0 & 0 & 3.7 & -1.1 \\ 0 & 0 & 0 & 2.93 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.34 \\ 10.192 \\ 22.55 \end{bmatrix}$$

Solving for the values of X from equation 2

$6X_1+2X_2+X_3+X_4=2$ [1]

$3.33X_2+0.67X_3-0.33X_4=2.34$ [2]

$3.7X_3-1.1X_4=10.192$ [3]

$2.93X_4=22.55$ [4]

Solving subequation1,2,3 and 4 simultaneously gives

$X_1=-1.98$

$X_2=0.56$

$X_3=5.04$

$X_4=7.7$