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Linear Algebra I exam 2004 solutions.
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This is a core course. All questions are bookwork; only the details are intended to be unseen.

SECTION A

1. (a) $AB = \begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix}$

2 marks

(b) $AB = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

2 marks

(c) $A^2 =$

$$\begin{pmatrix} x & x^2 \\ x^3 & x^4 \end{pmatrix}\begin{pmatrix} x & x^2 \\ x^3 & x^4 \end{pmatrix} = \begin{pmatrix} x^2 + x^5, & x^3 + x^6 \\ x^4 + x^7, & x^5 + x^8 \end{pmatrix}$$

3 marks

2. (a) $\det A = 3^3 \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 27 \times ((0+6+0) - (8+1+0))$

$= -27 \times 3 = -81$

3 marks

(b) $A^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \times \frac{-1}{3} \times \begin{pmatrix} 2 & 5 & -4 \\ -1 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}^T$

$= -\frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ 5 & -4 & 2 \\ -4 & 2 & -1 \end{pmatrix}$

4 marks

Check: $-\frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ 5 & -4 & 2 \\ -4 & 2 & -1 \end{pmatrix} \times 3 \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ optional

3. The <u>rank</u> of a matrix is the maximum number of linearly independent rows or columns.

3 marks

(a) rank $A = 2$ because $R_3 = R_1 + R_2$
and $aR_1 + bR_2 = 0 \Rightarrow 0a + 1b = 0,\ 1a + 1b = 0$
$\Rightarrow a = b = 0$, so R_1 & R_2 are linearly indep.

2 marks

(b) rank $B = 2$ because $R_2 = R_1 + R_3$
(also $R_1 = R_2 + R_3$ and $R_3 = R_1 + R_2$ over $\mathbb{F}_2$)
and R_1 & R_2 are linearly indep. as in (a).

2 marks

4. A set of vectors $\{v_i\}$ is <u>linearly independent</u> if $\sum k_i v_i = 0$ for scalars $k_i \Rightarrow k_i = 0\ \forall i$.

3 marks

By inspection $(3,2,3) = 3(1,1,1) + (0,-1,0)$
and $(1,0,1) = (1,1,1) + (0,-1,0)$
so take $T = \{(1,1,1),\ (0,-1,0)\}$ **2 marks**
and $U = \{(1,1,1),\ (0,-1,0),\ (1,0,0)\}$
since $a(1,1,1) + b(0,-1,0) + c(1,0,0) = 0$
$\Rightarrow a + c = 0,\ a - b = 0,\ a = 0 \Rightarrow a = b = c = 0$.

2 marks

5. The *span* of a set of vectors $\{v_i\}$ is the set of all vectors of the form $v = \sum k_i v_i$ for any scalars k_i. 3 marks

$v \in \langle v_i \rangle$ if we can solve $v = \sum k_i v_i$ for the scalars k_i. Consider
$(1,2,3,4) = a(1,0,0,0) + b(1,1,0,0) + c(0,1,1,1)$
$\Rightarrow \quad 1 = a + b, \quad 2 = b + c, \quad 3 = c, \quad 4 = c.$
This has no solutions, so $(1,2,3,4) \notin \text{span } S'$.
4 marks

6. An ordered basis for a vector space is an ordered list of linearly independent vectors that span the space. 3 marks

Let $(1,2,3) = x.(0,1,1) + y(1,0,1) + z(1,1,0)$
$\Rightarrow 1 = y + z, \quad 2 = x + z, \quad 3 = x + y$
i.e. $\qquad 1 = x - y$
Hence $\quad 4 = 2x \Rightarrow x = 2, \quad 2 = 2y \Rightarrow y = 1$
so $\quad z = 1 - y = 0$
i.e. $(1,2,3) = 2(0,1,1) + 1(1,0,1) + 0(1,1,0)$
and its coordinate vector is $(2,1,0)$.
4 marks

7. A map $\alpha : U \to V$ is linear if
$\alpha(au + bv) = a\alpha(u) + b\alpha(v)$
$\forall\ u, v \in U$ and all scalars a, b.

3 marks

$$\alpha(1,0,0) = (1,0) = 1(1,0) + 0(0,1)$$
$$\alpha(0,1,0) = (2,0) = 2(1,0) + 0(0,1)$$
$$\alpha(0,0,1) = (0,3) = 0(1,0) + 3(0,1)$$

Hence the matrix of α is $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

4 marks

8. Let $\langle u, v \rangle$ denote the inner product of two vectors u and v.
Two vectors u, v are orthogonal if $\langle u, v \rangle = 0$
They are orthonormal if $\langle u, v \rangle = 0$, and
$\langle u, u \rangle = \langle v, v \rangle = 1$.

3 marks

$\langle u, v \rangle = 2 - 2 + 0 = 0$.
Therefore u and v are orthogonal.

$$\hat{u} = \frac{u}{\|u\|} = \frac{(1,2,3)}{\sqrt{1+4+9}} = \frac{(1,2,3)}{\sqrt{14}}$$

Let $w = \hat{v} = \dfrac{(2,-1,0)}{\sqrt{4+1}} = \dfrac{(2,-1,0)}{\sqrt{5}}$

Then $\hat{u}$ and w are orthonormal.

4 marks

SECTION B

1. (a) U is a vector subspace of V if $0 \in U$ and $au + bv \in U$ $\forall$ $u, v \in U$ and all scalars a, b.

3 marks

(i) U is a vector subspace of V because $(0,0) \in U$ since $0 + 2.0 = 0$

Let $(x_1, y_1), (x_2, y_2) \in U$

$\Rightarrow x_1 + 2y_1 = 0, \quad x_2 + 2y_2 = 0$

Then $a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2)$ $\in U$ since $a(x_1 + 2y_1) + b(x_2 + 2y_2) = 0$

$\Rightarrow (ax_1 + bx_2) + 2(ay_1 + by_2) = 0.$

5 marks

(ii) Let $u = (1,1) \in U$, $v = (-1,1) \in U$

Then $u + v = (0, 2) \notin U$ since $|0| \neq 2$.

Hence U is not a vector subspace of V.

3 marks

(b)(i) $S + T$ is the set of all sums of a vector in S and a vector in T.

2 marks

(ii) $S + T = T$ since $S \subseteq T$

2 marks

(iii) Let $S = \langle (1,0) \rangle$, $T = \langle (0,1) \rangle$.

Then $S \cup T = \{(x, y) \mid xy = 0\}$ is not a vector subspace because $(1,0), (0,1) \in S \cup T$ but $(1,0) + (0,1) = (1,1) \notin S \cup T$ since $xy \neq 0$.

3 marks

(iv) A basis is a linearly independent set by defn.

2 marks

(v) $\{(1,0), (0,1), (1,1)\}$ is a spanning set for $\mathbb{R}^2$ that is not a basis (since it is not lin. ind.).

2 marks

2. (a) • α is "onto" if $\forall\, v \in V\ \exists\, u \in U$ st $\alpha(u) = v$.
• α is "one-to-one" if $\forall\, u, v \in U$, $\alpha(u) = \alpha(v) \Rightarrow u = v$.
• α is an isomorphism if it is onto and one-to-one

3 marks

(b) (i) Let $u \in U$. Then $\alpha(0) = \alpha(u-u) = \alpha(u) - \alpha(u) = 0$. Hence $0 \in \ker(\alpha)$.
Let $u, v \in \ker(\alpha) \Rightarrow \alpha(u) = \alpha(v) = 0$.
Then $\alpha(au + bv) = a\alpha(u) + b\alpha(v) = 0$
$\Rightarrow au + bv \in \ker(\alpha)$ $\forall$ scalars a, b.
Hence $\ker(\alpha)$ is a vector subspace of U.

3 marks

(ii) $\alpha(u) = \alpha(v)$, $u, v \in U$, $\Leftrightarrow \alpha(u-v) = 0$
$\Leftrightarrow u - v \in \ker(\alpha)$
If $\ker(\alpha) = \{0\}$ then $u - v = 0\ \forall\, u, v$
If $u - v = 0\ \forall\, u, v$ then $\ker(\alpha) = \{0\}$.
Hence, α one-to-one iff $\ker(\alpha) = \{0\}$.

3 marks

(iii) Let $\alpha(u_1) = v_1$, $\alpha(u_2) = v_2$. Then $v_1, v_2 \in \text{im}(\alpha)$. Also $\alpha(au_1 + bu_2) = av_1 + bv_2 \in \text{im}(\alpha)$. Hence $av_1 + bv_2 \in \text{im}(\alpha)$ $\forall\, v_1, v_2 \in \text{im}(\alpha)$ and scalars a, b.
Moreover $\alpha(0) = 0$ from (i) $\Rightarrow 0 \in \text{im}(\alpha)$.
Hence $\text{im}(\alpha)$ is a vector subspace of V.

3 marks

(c) The inverse map α^{-1} is a map such that $\alpha^{-1}(\alpha(u)) = u$ $\forall\, u \in U$ and $\alpha(\alpha^{-1}(v)) = v$ $\forall\, v \in V$.

2 marks

Let $u, v \in U$, a, b scalars.
Then $\alpha(au + bv) = a\alpha(u) + b\alpha(v)$ $\quad (*)$
Let $u' = \alpha(u)$, $v' = \alpha(v)$
Then $u = \alpha^{-1}(u')$, $v = \alpha^{-1}(v')$
Substituting in $(*)$ gives
$\alpha(a\alpha^{-1}(u') + b\alpha^{-1}(v')) = au' + bv'$
and applying α^{-1} gives
$a\alpha^{-1}(u') + b\alpha^{-1}(v') = \alpha^{-1}(au' + bv')$
This holds $\forall\, u', v' \in V$ since α is onto.
Hence α^{-1} is linear.

4 marks

(d) $\ker(\alpha)$: $\left.\begin{array}{l} x + z = 0 \\ x + y = 0 \\ y + z = 0 \end{array}\right\}$

$x = 0 \Rightarrow z = 0$ & $y = 0$.
$x = 1 \Rightarrow z = 1$ & $y = 1$.
Hence $\ker(\alpha) = \langle (1,1,1) \rangle$.

2 marks

$\operatorname{im}(\alpha)$: $\alpha(1,0,0) = (1,1,0)$
$\alpha(0,1,0) = (0,1,1)$
$\alpha(0,0,1) = (1,0,1)$
But $(1,1,0) + (0,1,1) = (1\,0\,1)$ in $\mathbb{F}_2$
Hence $\operatorname{im}(\alpha) = \langle (1,1,0), (0,1,1) \rangle$.

2 marks

(This satisfies the dimension theorem!)

3. (a) Let $\alpha(u_i) = \sum_{j=1}^{n} a_{ji} v_j$, $i = 1..m$

Then $A = (\alpha, \mathcal{B}, \mathcal{C})$ is the matrix whose elements are a_{ji}, $i = 1..m$, $j = 1..n$.

<u>2 marks</u>

$$\begin{aligned}
\alpha(1,0,0,0) &= (1,0) = 1(1,0) + 0(0,1)\\
\alpha(0,1,0,0) &= (1,1) = 1(1,0) + 1(0,1)\\
\alpha(0,0,1,0) &= (1,1) = 1(1,0) + 1(0,1)\\
\alpha(0,0,0,1) &= (0,1) = 0(1,0) + 1(0,1)
\end{aligned}$$ <u>4 marks</u>

Hence $A = (\alpha, \mathcal{B}, \mathcal{C}) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ <u>2 marks</u>

(b) $\mathcal{B}'$ & $\mathcal{C}'$ must respect the kernel & image respectively of α.

$\ker \alpha$: $\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}\right\}$

$\Rightarrow x_1 = x_4 = -(x_2 + x_3)$. Let $x_2 = a$, $x_3 = b$.

The general vector in $\ker \alpha$ is

$(-a-b, a, b, -a-b) = a(-1,1,0,-1) + b(-1,0,1,-1)$

Hence $\ker \alpha = \langle (-1,1,0,-1), (-1,0,1,-1) \rangle$ which is a basis.

<u>4 marks</u>

Extend this basis (forwards) to a basis for $\mathbb{R}^4$ as

$\mathcal{B}' = (1,0,0,0), (0,0,0,1), (-1,1,0,-1), (-1,0,1,-1)$

(Adding the first two vectors to the last two gives the standard bases, hence $\mathcal{B}'$ is a basis.)

<u>3 marks</u>

im α: Mapping the standard basis gives
$$\text{im } \alpha = \langle (1,0), (1,1), (0,1) \rangle$$
$$= \langle (1,0), (0,1) \rangle \ (= \mathbb{R}^2),$$
which is a basis both for im α and for $\mathbb{R}^2$.
Hence $\mathcal{C}' = (1,0), (0,1)$.

3 marks

The rows of A are clearly linearly independent, therefore rank A = 2.
But r = rank A, so $r = 2$.

2 marks

Therefore $A' = (\alpha, \mathcal{B}', \mathcal{C}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

2 marks

Check (optional):

$$\left.\begin{aligned} \alpha(1,0,0,0) &= (1,0) \\ \alpha(0,0,0,1) &= (0,1) \\ \alpha(-1,1,0,-1) &= (0,0) \\ \alpha(-1,0,1,-1) &= (0,0) \end{aligned}\right\} \Rightarrow A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

4. (a) If $\exists$ a nonzero vector $x \in \mathbb{K}^n$ and a scalar $\lambda \in \mathbb{K}$ such that $Ax = \lambda x$ then x is an <u>eigenvector</u> of A and λ is the corresponding <u>eigenvalue</u>.

<u>2 marks</u>

(i) The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 3 & -\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

i.e. $(2-\lambda)(-\lambda)(-3-\lambda) - 3(-3-\lambda) = 0$

$\Rightarrow (\lambda+3)(\lambda^2 - 2\lambda - 3) = 0$

$\Rightarrow (\lambda+3)(\lambda+1)(\lambda-3) = 0$

Hence the eigenvalues are $\lambda = \pm 3, -1$. <u>2 marks</u>

$\underline{\lambda = +3}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow x_1 = x_2,\ x_3 = 0$$

i.e. $\underset{\sim}{x} = (1,1,0)$.

<u>2 marks</u>

$\underline{\lambda = -3}$

$$\begin{pmatrix} 5 & 1 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow \left.\begin{array}{l} 5x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{array}\right\} \Rightarrow x_1 = x_2 = 0$$

$\underset{\sim}{x} = (0,0,1)$

<u>2 marks</u>

$\underline{\lambda = -1}$

$$\begin{pmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow 3x_1 + x_2 = 0 \Rightarrow x_2 = -3x_1$$

$x_3 = 0$

$\underset{\sim}{x} = (1, -3, 0)$

<u>2 marks</u>

(ii) Hence $X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 3 & & 0 \\ & -3 & \\ 0 & & -1 \end{pmatrix}$.

(b) (i) Let x_i, $i = 1..t+1$, be eigenvectors of a matrix A with corresponding <u>distinct</u> eigenvalues λ_i, and suppose that x_i, $i = 1..t$, are linearly independent. [1]

Consider $\sum_{i=1}^{t+1} k_i x_i = 0$ (*) [1]

Premultiply (*) by A to give $\sum_{i=1}^{t+1} k_i \lambda_i x_i = 0$ [1]

and multiply (*) by λ_{t+1} to give $\sum_{i=1}^{t+1} k_i \lambda_{t+1} x_i = 0$ [1]

Subtract to give $\sum_{i=1}^{t} k_i (\lambda_i - \lambda_{t+1}) x_i = 0$ [1]

Linear independence of x_i, $i = 1..t$, implies that $k_i (\lambda_i - \lambda_{t+1}) = 0$, $i = 1..t$, and hence $k_i = 0$, $i = 1..t$, since the λ_i are all distinct.
Substituting in (*) gives $k_{t+1} x_{t+1} = 0$ and hence $k_{t+1} = 0$ since x_{t+1} is an eigenvector. [2]

Therefore $k_i = 0$, $i = 1..t+1$, in (*), so x_i, $i = 1..t+1$, are linearly independent. [1]

A single eigenvector is necessarily linearly independent. Therefore, by induction on the number of linearly independent eigenvectors, all the eigenvectors with distinct eigenvalues are linearly independent. [2]