EE 287 CIRCUIT THEORY

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What to expect?

R-L Circuits → R-C Circuits → R-L-C Circuits

Introduction to R-L Circuits

- When non-linear elements such as inductors and capacitors are introduced into a circuit, the behaviour is not instantaneous as it would be with resistors.

- A change of state will disrupt the circuit and the non-linear elements require time to respond to the change.

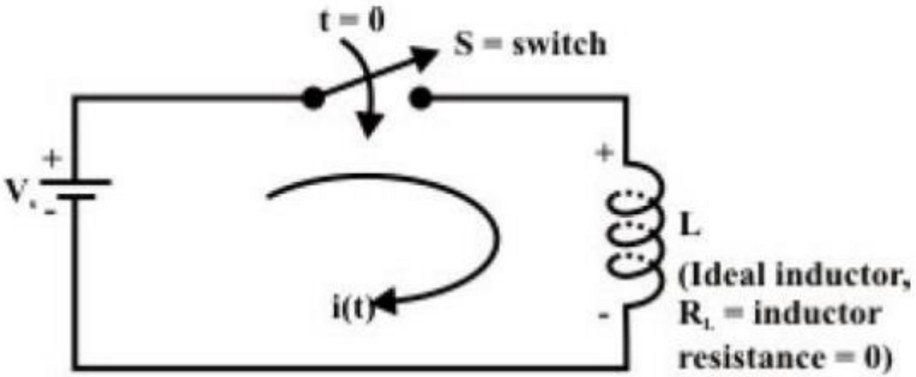
- Some responses can cause jumps in the voltage and current which may be damaging to the circuit.

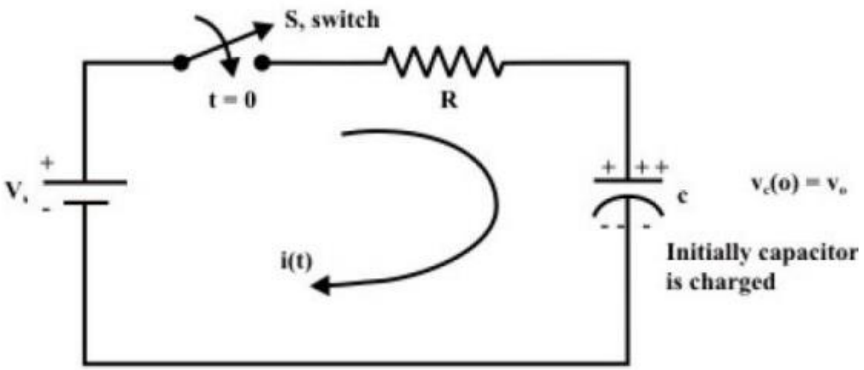
- Accounting for the transient response with circuit design can prevent circuits from acting in an undesirable fashion.

Transient Response

- The response of a circuit (containing resistances, inductances, capacitors and switches) due to sudden application of voltage or current is called **transient response.**

- The most common instance of a transient response in a circuit occurs when a switch is turned on or off in a circuit containing resistance and capacitor/inductor or containing all three passive elements as illustrated below.





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DC Transients in R-L Circuits

Introduction

- An R-L circuit is one that contains a resistor (R) and an inductor (L).

- The figure below shows an R-L circuit where the passive elements are connected through a switch to a DC voltage, V_s

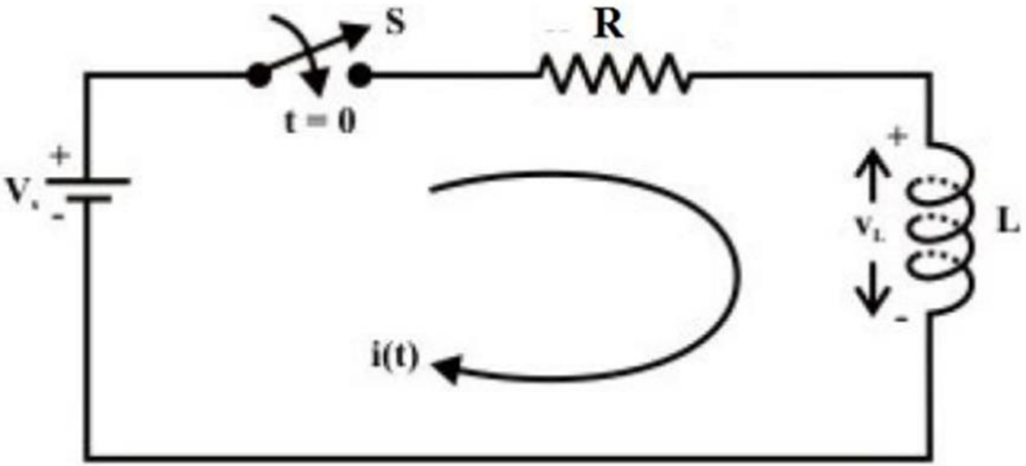


Figure 3.1

Case Problem

Our problem is to study the growth of current in the circuit through two stages:

1. DC Transient response
2. Steady State response

Transient response is the state of the circuit just after the switch is turned on and this elapses some few micro or milliseconds.

Steady state response is the state after the transient response as time goes to infinity.

Case Solution

To find the current expression (response) for the circuit shown in Figure 3.1 above, we can write the KVL equation around the circuit.

$$\mathbf{V_s = Ri(t) + L\frac{di(t)}{dt}} \qquad \mathbf{(3.1)}$$

Where,

V_S is the applied voltage or forcing function.

R is the external resistance

L is the inductance.

Equation (3.1) is a standard first order differential equation and its solution is of the form:

$$\mathbf{i(t) = i_n(t) + i_f(t)} \qquad \mathbf{(3.2)}$$

Where,

- $i_n(t)$ is the complementary solution/natural solution. It is sometimes called the **transient response of the system**.
- $i_f(t)$ is the particular solution or **steady state response of the system** due to the input signal V_S.

Solving the differential equation (3.1) and rearranging the equation into a form that is easier to integrate:

$$\frac{di(t)}{dt} = \frac{R}{L}\left(\frac{V_S}{R} - i(t)\right) \qquad (3.3)$$

Divide by the term in brackets, and integrate.

$$\int \frac{di(t)}{dt} \frac{1}{i(t) - \frac{V_S}{R}}\, dt = -\frac{R}{L}\int dt \qquad (3.4)$$

The integral becomes,

$$\ln\left(\mathrm{i(t)}-\frac{\mathrm{V_S}}{\mathrm{R}}\right)=-\frac{\mathrm{t}}{L/R}+\mathrm{D} \qquad (3.5)$$

Where,

D is the constant of integration

Remove the natural log and solve for the inductor current

$$\mathrm{i(t)}=\frac{\mathrm{V_S}}{\mathrm{R}}+e^{D}e^{-\frac{\mathrm{t}}{L/R}} \qquad (3.6)$$

Let $e^{D}=\mathrm{A}$, a constant

Hence

$$\mathrm{i(t)}=\frac{\mathrm{V_S}}{\mathrm{R}}+\mathrm{A}e^{-\frac{\mathrm{t}}{L/R}} \qquad (3.7)$$

From equation (3.7), the constant A is revealed at time t=0 when the switch is just closed.

$$\mathrm{i}(0) = \mathrm{A} + \frac{\mathrm{V_S}}{\mathrm{R}} \qquad (3.8)$$

Now if the current before the switch is closed $\mathrm{i}(0^-)$ is zero, then $\mathrm{i}(0) = 0$

It means the constant A is,

$$-\frac{\mathrm{V_S}}{\mathrm{R}} = \mathrm{A} \qquad (3.9)$$

As the time goes to infinity, the steady state response from equation (3.7) is found

$$\mathrm{i}(\infty) = \lim_{t \to \infty} i(t) = \frac{\mathrm{V_S}}{\mathrm{R}} \qquad (3.10)$$

The time constant τ is,

$$\tau = \frac{\mathrm{L}}{\mathrm{R}}$$

Therefore the complete response of the current through an inductor connected to a DC voltage is

$$\mathrm{i(t)} = \mathrm{i_n(t)} + \mathrm{i_f(t)} = \frac{\mathrm{V_S}}{\mathrm{R}} - \frac{\mathrm{V_S}}{\mathrm{R}} e^{\frac{-t}{\tau}} = \frac{\mathrm{V_S}}{\mathrm{R}} \left(1 - e^{\frac{-t}{\tau}}\right) \qquad (3.11)$$

How the current i(t) builds up in a R-L circuit

ACTUAL TIME (t) IN SEC	GROWTH OF CURRENT IN INDUCTOR
$t = 0$	$i(0) = 0$
$t = \tau = \frac{L}{R}$	$i(\tau) = 0.632 \times \frac{Vs}{R}$
$t = 2\tau$	$i(2\tau) = 0.865 \times \frac{Vs}{R}$
$t = 3\tau$	$i(3\tau) = 0.950 \times \frac{Vs}{R}$
$t = 4\tau$	$i(4\tau) = 0.982 \times \frac{Vs}{R}$
$t = 5\tau$	$i(5\tau) = 0.993 \times \frac{Vs}{R}$

Note: Theoretically at time t→∞ the current in inductor reaches its steady state value but in practice the inductor reaches 99.3% of its steady state value at time t=5τ (sec)

The expression for voltage across the resistance R is:

$$V_R = i(t)R = \frac{V_s}{R}(1 - e^{\frac{-t}{\tau}})R \qquad (3.12)$$

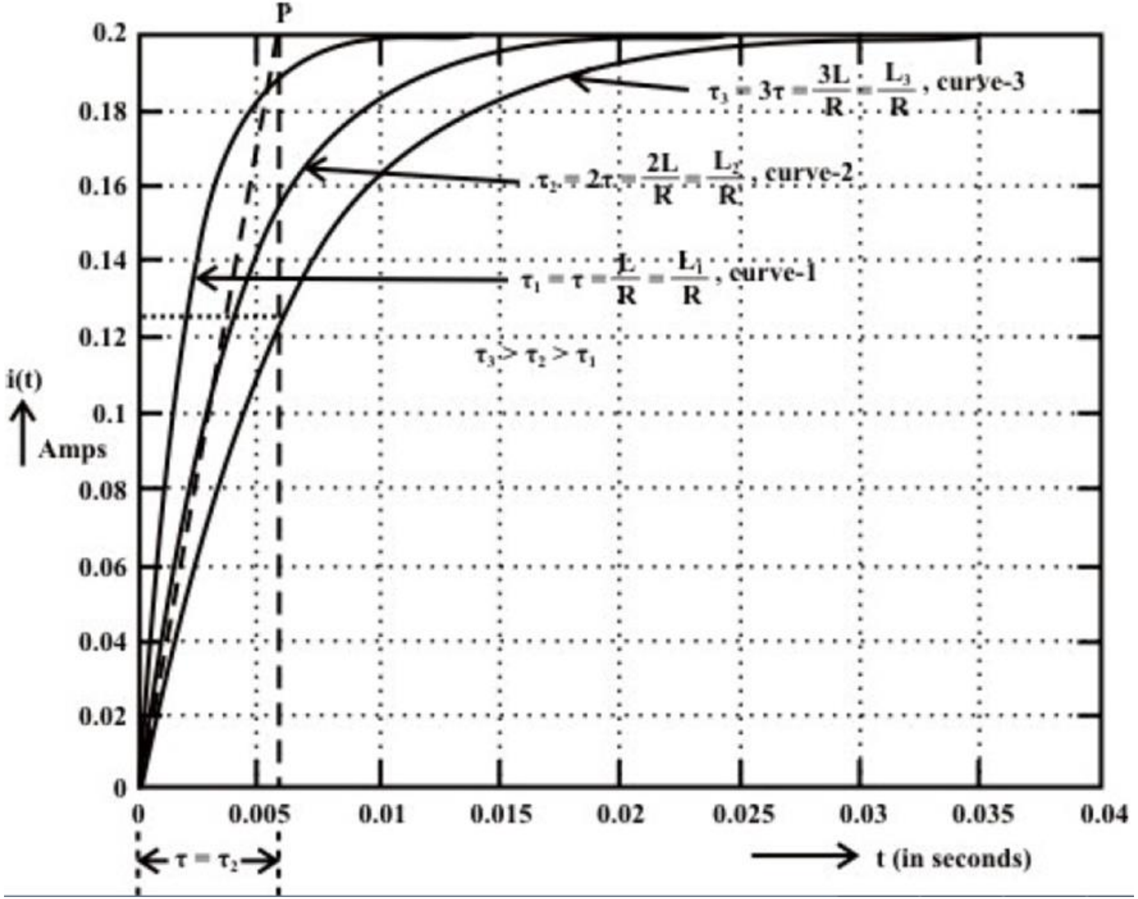
$$V_R = V_s(1 - e^{\frac{-t}{\tau}})$$

The expression for voltage across the inductor is:

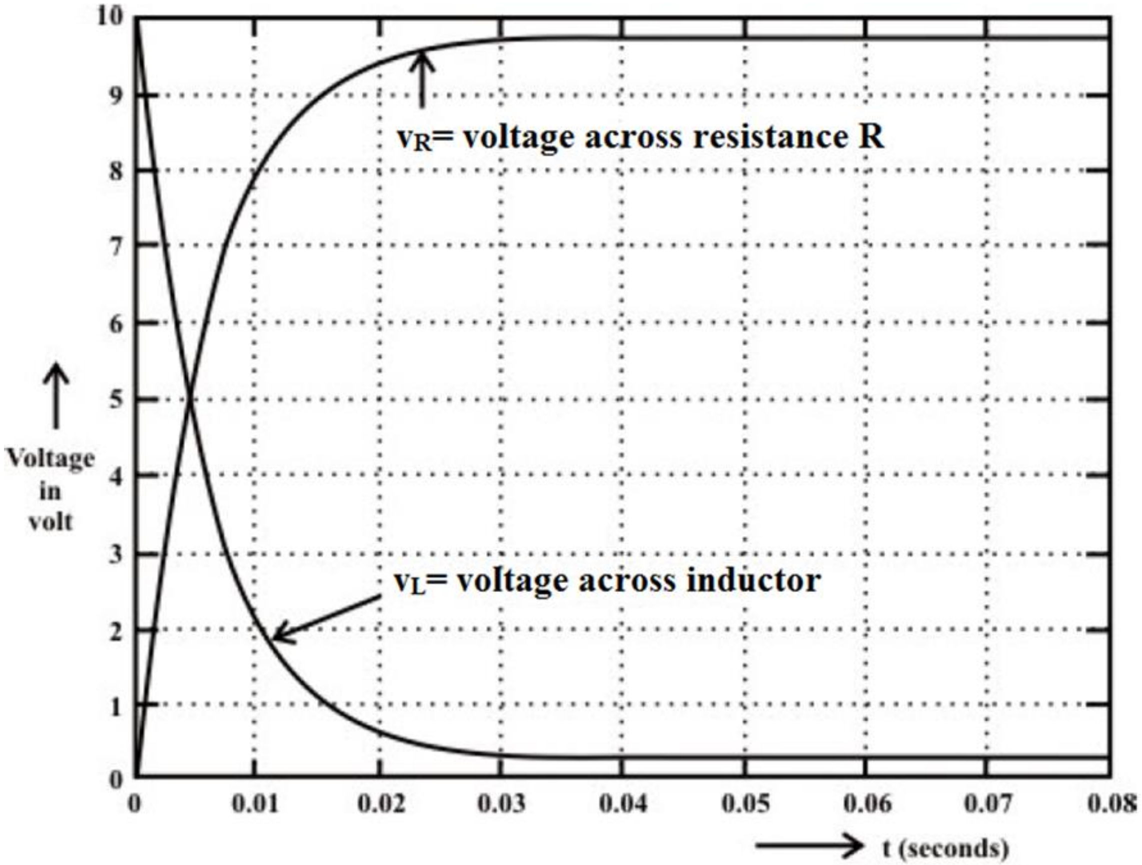
$$V_L(t) = V_s - V_s(1 - e^{\frac{-t}{\tau}}) = V_s\, e^{\frac{-t}{\tau}} \qquad (3.13)$$

Graphical representation of equations (3.11) – (3.13) are shown in the next slides

Growth of current in R-L circuit (assumed initial current through inductor is zero)



Voltage response in different elements of R-L circuit (assumed i_o=0)



Finite Initial Current through an R-L Circuit

Assume current flowing through the inductor just before closing the switch "S" (at $t = 0^{-}$) is $i(0^{-}) = i_o \neq 0$.

Using equation (3.8), we get the value of $A = i(0) - A = i_o - \frac{V_s}{R}$

Using this value in equation (3.7), the expression for current flowing through the circuit is given by:

$$\mathbf{i(t) = \frac{V_s}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \left(i_o e^{\frac{-t}{\tau}}\right)} \qquad (3.14)$$

The second part of the right hand side of the expression (3.14) indicates the current flowing to the circuit due to initial current i_o of inductor and the first part due to the input V_s applied to the circuit.

This means that the complete response of the circuit is the algebraic sum of two outputs due to two inputs; namely

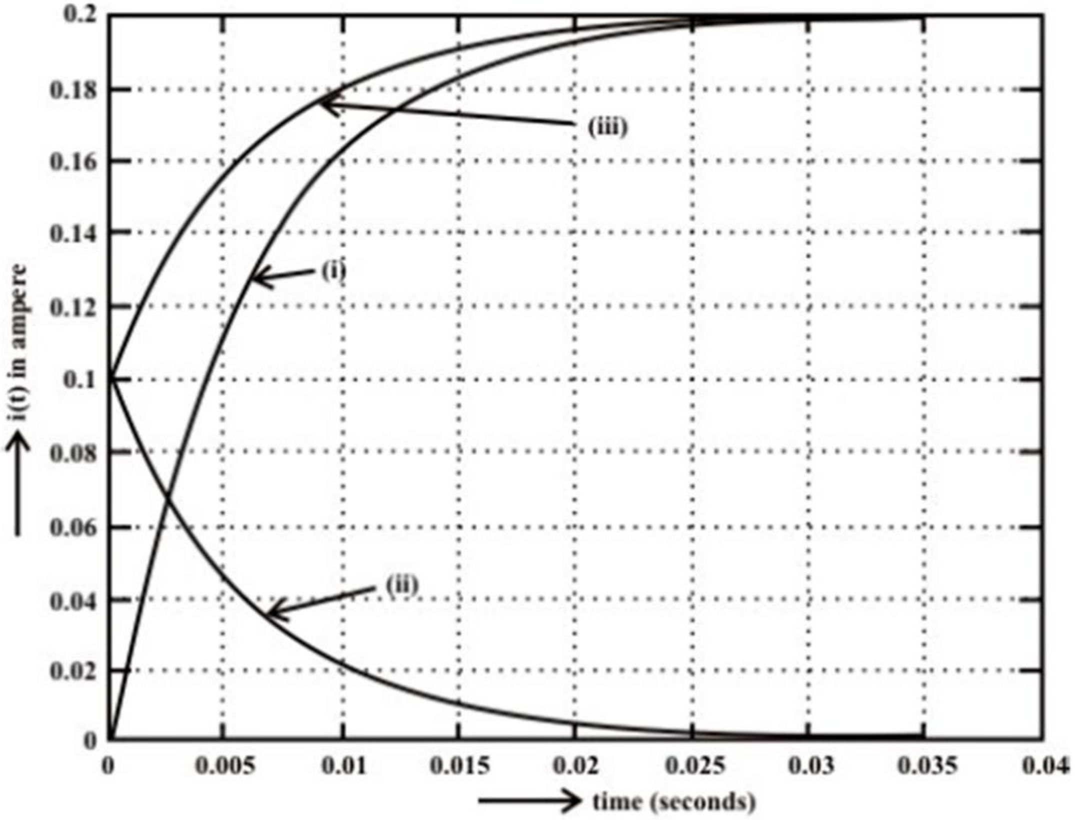
(i) due to input V_S

(ii) due to initial current through the inductor.

This implies that the superposition theorem is also valid for such type of linear circuit.

The response of inductor current when the circuit is excited with a constant voltage source V_S and the initial current through inductor is i_o

Current through inductor due to:
(i) input VS only,
(ii) initial condition io only,
(iii) combined effect of (i) and (ii)



Time Constant (τ) of R-L Circuit

- It is the time required for any variable or signal (in our case either current i(t)) or voltage (VR or VL)) to reach 63.2% of its final value.
- It is possible to write an exact mathematical expression to calculate the time constant (τ) of any first-order differential equation.
- Let 't' be the time required to reach 63.2% of steady-state value of inductor current and the corresponding time 't' expression can be obtained as:

$$i(t) = 0.632 \times \frac{V_s}{R} = \frac{V_s}{R}\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$\text{This implies, } 0.632 = 1 - e^{-\frac{R}{L}t}$$

$$0.368 = e^{-1} = e^{-\frac{R}{L}t}$$

Hence

$$t = \frac{L}{R} = \tau(\text{sec})$$

- The behaviour of all circuit responses (for first-order differential equation) is fixed by a single time constant τ(for R-L circuit) and it provides information about the speed of response or in other words, it indicates how fast or slow the system response reaches its steady state from the instant of switching the circuit.

- Observe the equation (3.11) that the smaller the time constant (τ), the more rapidly the current increases and subsequently it reaches the steady state (or final value) quickly.

- On the other hand, a circuit with a larger time constant (τ) provides a slow response because it takes longer time to reach steady state.

- These facts are illustrated in Fig. 3.4(a).

Fall or Decay of Current in a R-L Circuit

- Let us consider the circuit shown in fig. 3.6(a).
- In this circuit, the switch 'S' is closed sufficiently long duration in position '1'.
- This means that the current through the inductor reaches its steady-state value and it acts, as a short circuit i.e. the voltage across the inductor is nearly equal to zero.
- If the switch 'S' is opened at time 't'=0 and kept in position '2' for $\boldsymbol{t > 0}$ as shown in Fig. 3.6(b), this situation is referred to as source free circuit.

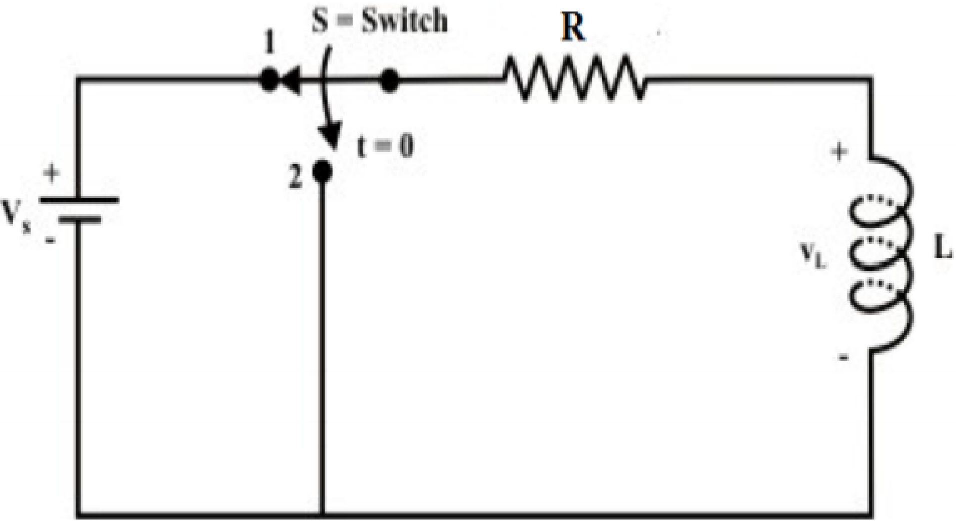


Fig 3.6 (a)

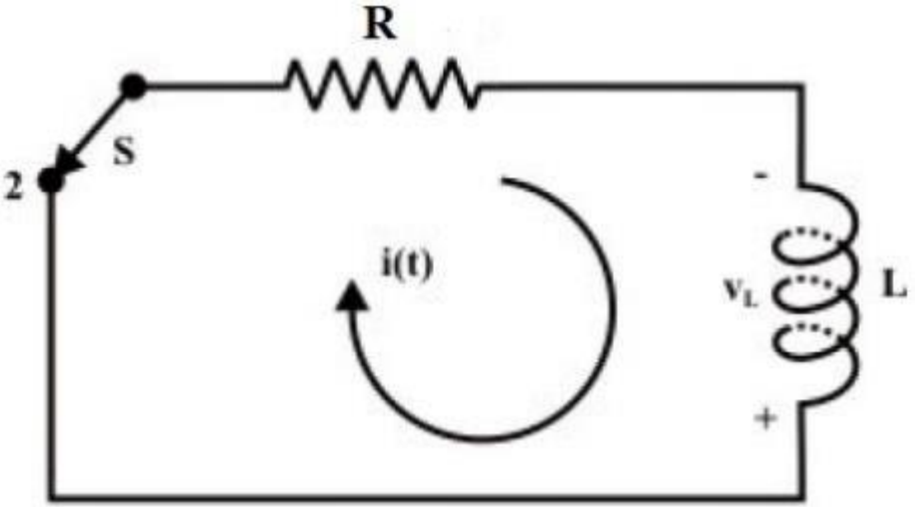


Fig 3.6(b) Decay of current in R-L circuit

- Since the current through an inductor cannot change instantaneously, the current through the inductor just before $i(0^{-})$ and after $i(0^{+})$ opening the switch 'S' must be same.

- Because there is no source to sustain the current flow in inductor, the magnetic field in inductor starts to collapse and this, in turn, will induce a voltage across the inductor.

- The polarity of this induced voltage across the inductor is just in reverse direction compared to the situation that occurred during the growth of current in inductor (i.e. when the switch ‘S’ is kept in position ‘1’).

- This is illustrated in fig. 3.6(b), where the voltage induced in inductor is positive at the bottom of the inductor terminal and negative at the top.

- This implies that the current through inductor will still flow in the same direction, but with a magnitude decaying toward zero.

- Applying KVL around the closed circuit in Fig. 3.6(b), we obtain:

$$\mathrm{R}i(\mathrm{t}) + \mathrm{L}\frac{\mathrm{d}i(\mathrm{t})}{\mathrm{dt}} = 0 \qquad (3.15)$$

The solution of the homogenous (input free) first-order differential equation above is given by:

$$i(t) = i_n(t) = Ae^{\alpha t}$$

Solving the differential equation (3.15) and rearranging the equation into a form that is easier to integrate:

$$\frac{di(t)}{dt} = -\frac{R}{L}i(t)$$

Divide by the i(t) and integrate.

$$\int \frac{di(t)}{dt}\ \frac{1}{i(t)}\ dt = -\frac{R}{L}\int dt$$

The integral becomes,

$$\ln\big(i(t)\big) = -\frac{t}{L/R} + D$$

Where D is the constant of integration

Remove the natural log and solve for the inductor current

$$\mathrm{i(t)} = e^{D} e^{-\frac{\mathrm{t}}{L/R}}$$

Let $e^{D} = \mathrm{A}$, a constant

Hence

$$\mathrm{i(t)} = \mathrm{A}e^{-\frac{\mathrm{t}}{L/R}} \qquad (3.20)$$

The constant A is revealed at time t=0 when the switch 'S' is just closed in position 2. From equation (3.20)

$\mathrm{i(0)} = \mathrm{A}$

Now current before the switch is closed in position 2 $\mathrm{i}(0^{-})$ is $\frac{\mathrm{V_s}}{\mathrm{R}}$, hence $\mathrm{i(0)} = \frac{\mathrm{V_s}}{\mathrm{R}}$

It means the constant A is,

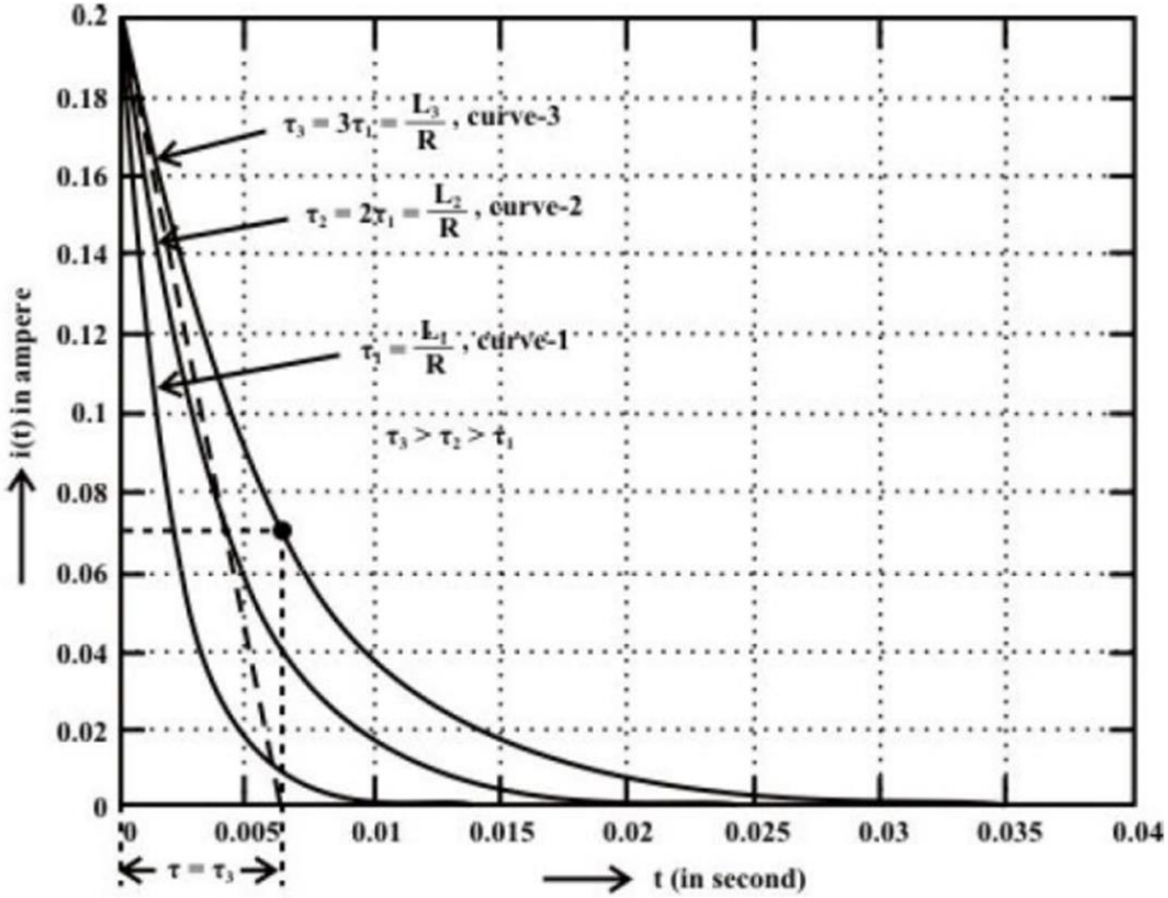
$$\frac{V_s}{R} = A$$

Therefore the expression of the current in a source-free R-L circuit is

$$i(t) = \frac{V_s}{R} e^{\frac{-t}{\tau}} \quad \text{where } \tau = \frac{L}{R}$$

A sketch of i(t) for $t \geq 0$ is shown in fig.3.7. Here, transient has ended and steady state has been reached when both current in inductor i(t) and voltage across the inductor including its internal resistance are zero.

Fall of current in R-L circuit (assumed initial current through inductor is I)



Time Constant (τ) for exponential decay response

- For the source free circuit, it is the time τ by which the current falls to 36.8% of its initial value.

- The initial condition in this case (see fig. 3.6(b)) is considered to be the value of inductor's current at the moment the switch is opened and kept in position '2'.

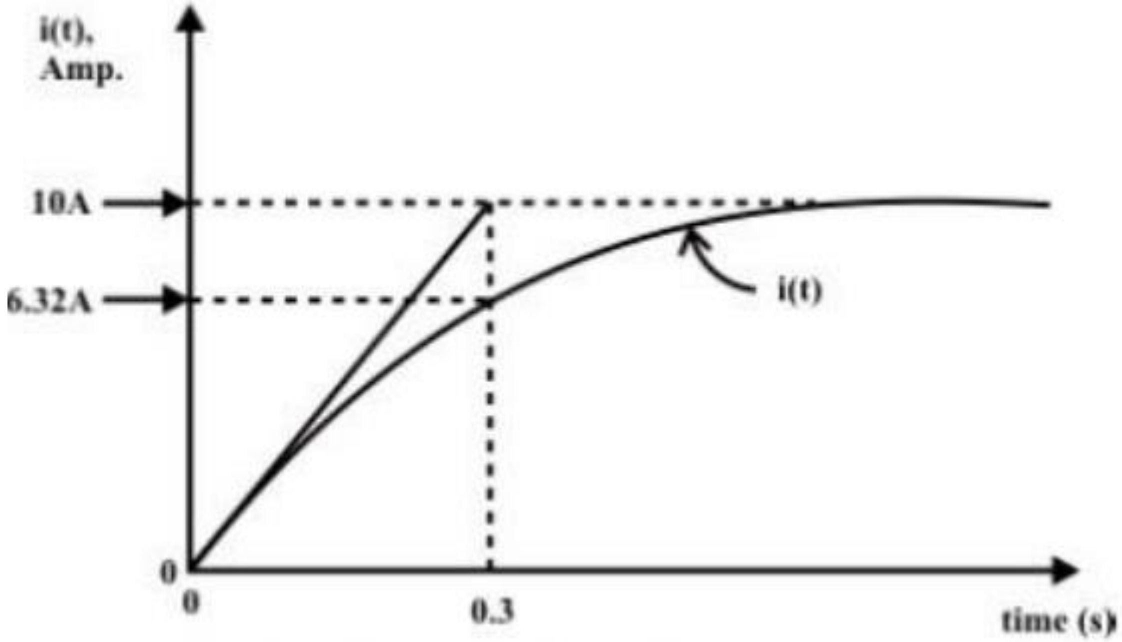
- Mathematically, τ is computed as

$$0.368 \times \frac{V_s}{R} = \frac{V_s}{R} e^{-\frac{R}{L}t}$$

$$t = \tau = \frac{L}{R}$$

Example 1

The figure below shows the plot of current i(t) through a series R-L circuit when a constant forcing function of magnitude $VS = 50\ V$ is applied to it. Calculate the values of resistance R and inductance L.



Current-time characteristics

Solution

From the diagram, one can easily see that the steady state current flowing through the circuit is 10A and the time constant of the circuit $\tau = 0.3$ sec.

The following relationships can be written as:

$$i_{\text{steady state}} = \frac{V_s}{R}, \quad \Rightarrow \quad 10 = \frac{50}{R}$$

$$R = 5\ \Omega$$

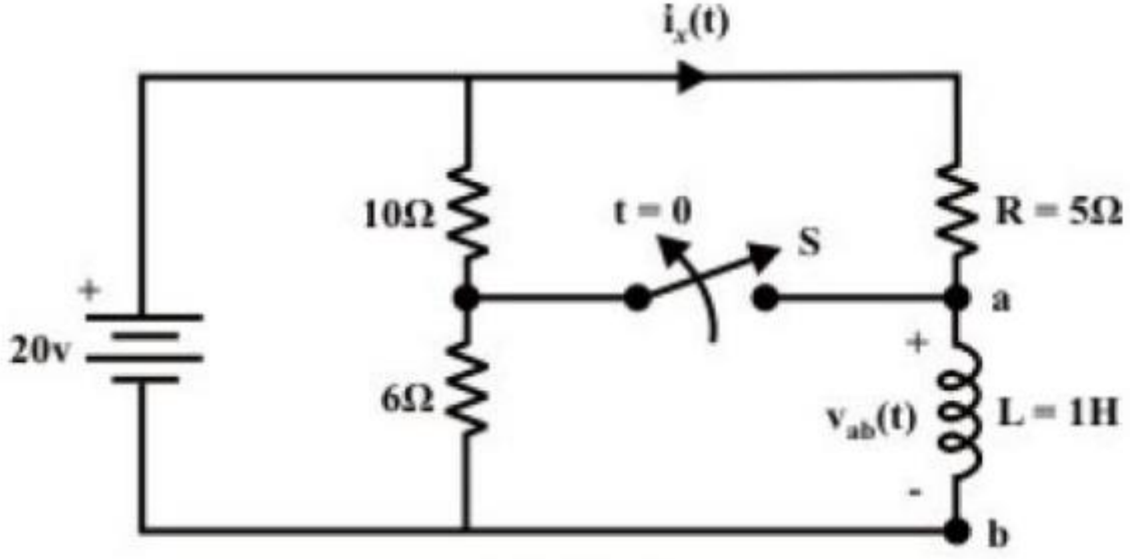
And

$$\tau = \frac{L}{R}, \quad \Rightarrow \ 0.3 = \frac{L}{5}$$

$$L = 1.5\ H$$

Example 2

For the circuit shown below the switch ‘S’ has been closed for a long time and then opens at t=0.



Find:

(i) $v_{ab}(0^-)$ (ii) $i_x(0^-)$, $i_L(0^-)$ (iii) $i_x(0^+)$

(iv) $v_{ab}(0^+)$ (v) $i_x(t \to \infty)$ (vi) $v_{ab}(t \to \infty)$

Solution

When the switch S was in closed position for a long time, the circuit reached in steady state condition i.e. the current through inductor is constant and hence, the voltage across the inductor terminals a and b is zero or in other words, inductor acts as short circuit i.e.,

(i) $v_{ab}(0^{-}) = 0\text{ V}$.

It can be seen that no current is flowing through resistor 6Ω. The following are the currents through different branches just before the switch 'S' is opened i.e., at $t = 0^{-}$

(ii) $i_x(0^{-}) = \frac{20}{5} = 4\text{A}$

The current through the 10Ω resistor, $i_{10\Omega}(0^{-}) = \frac{20}{10} = 2\text{A}$

The algebraic sum of these two currents is flowing through the inductor i.e.,

$i_L(0^-) = 2 + 4 = 6A$

(iii) When the switch 'S' is in open position, the current through inductor at time $t = 0^+$ is same as that of current $i_L(0^-)$, since inductor cannot change its current instantaneously. Therefore the current $i_x(0^+)$ is given by

$$i_x(0^+) = i_L(0^+) = 6A$$

(iv)Applying KVL around the closed loop at $t = 0^+$ we get,

$20 - i_x(0^+) \times R = v_{ab}(0^+) \qquad \Rightarrow 20 - 6 \times 5 = v_{ab}(0^+)$

Hence $v_{ab}(0^+) = -10V$

The negative sign indicates that inductor terminal ‘b’ as +ve terminal and it acts as a source of energy.

(v and vi)At steady state condition ($t \to \infty$) the current through the inductor is constant and hence inductor acts as a short circuit.

This establishes the following relations:

$$\mathrm{v_{ba}}(t = \infty) = 0V \qquad \text{and} \qquad \mathrm{i_x}(t = \infty) = \frac{20}{5} = 4\mathrm{A}$$

2

DC Transients in R-C Circuits

Case Problem

Consider a simple series R-C circuit connected through a switch ‘S’ to a constant voltage source V_S as shown in the figure below.

The switch ‘S’ is closed at time ‘t=0’.

Let $V_C(t)$ be the capacitor voltage and i(t) be the current flowing through the circuit at any instant of time ‘t’ after closing the switch.

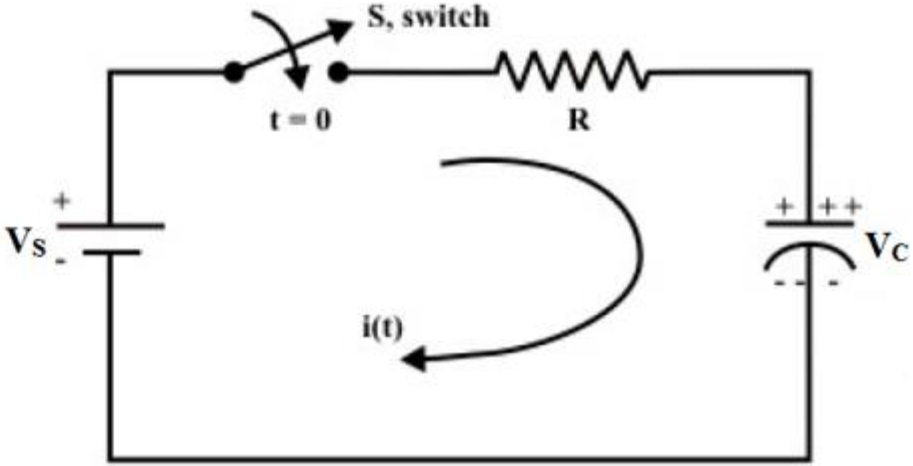


Fig 3.10 Charging of a RC Circuit

Case Solution

The KVL equation around the loop can be written as

$$V_s = Ri(t) + V_c(t) \qquad \Rightarrow \qquad V_s = RC\frac{dV_c(t)}{dt} + V_c(t) \quad (3.23)$$

The solution of the above first-order differential equation due to forcing function V_s is given by:

$$V_c(t) = V_{cn}(t) + V_{cf}(t) = A_1 e^{\alpha t} + A$$

The equation (3.23) simplifies into the differential equation,

$$\frac{dV_c(t)}{dt} + \frac{V_c(t)}{RC} = \frac{V_s}{RC}$$

Move the second term to the right hand side and then divide by the numerator

$$\frac{\mathrm{dV_c(t)}}{\mathrm{dt}} \frac{1}{\mathrm{V_{c(t)} - Vs}} \mathrm{dt} = -\frac{1}{\mathrm{RC}} \mathrm{dt}$$

The indefinite integral resolves to the following form

$$\ln(\mathrm{V_c(t) - V_s}) = -\frac{\mathrm{t}}{\mathrm{RC}} + \mathrm{D}$$

D is a constant of integration. Removing the natural log and solving for $\mathrm{V_c(t)}$

$$\mathrm{V_c(t) = V_s} + e^{D} e^{-\left(\frac{\mathrm{t}}{\mathrm{RC}}\right)} \qquad (3.28)$$

The constant e^D, represented by A, can be found at time $\mathrm{t = 0}$

$$e^D = \mathrm{A = V_c(0) - V_s}$$

Assuming the voltage across the capacitor before closing the circuit is zero, it means

$V_c(0) = 0$

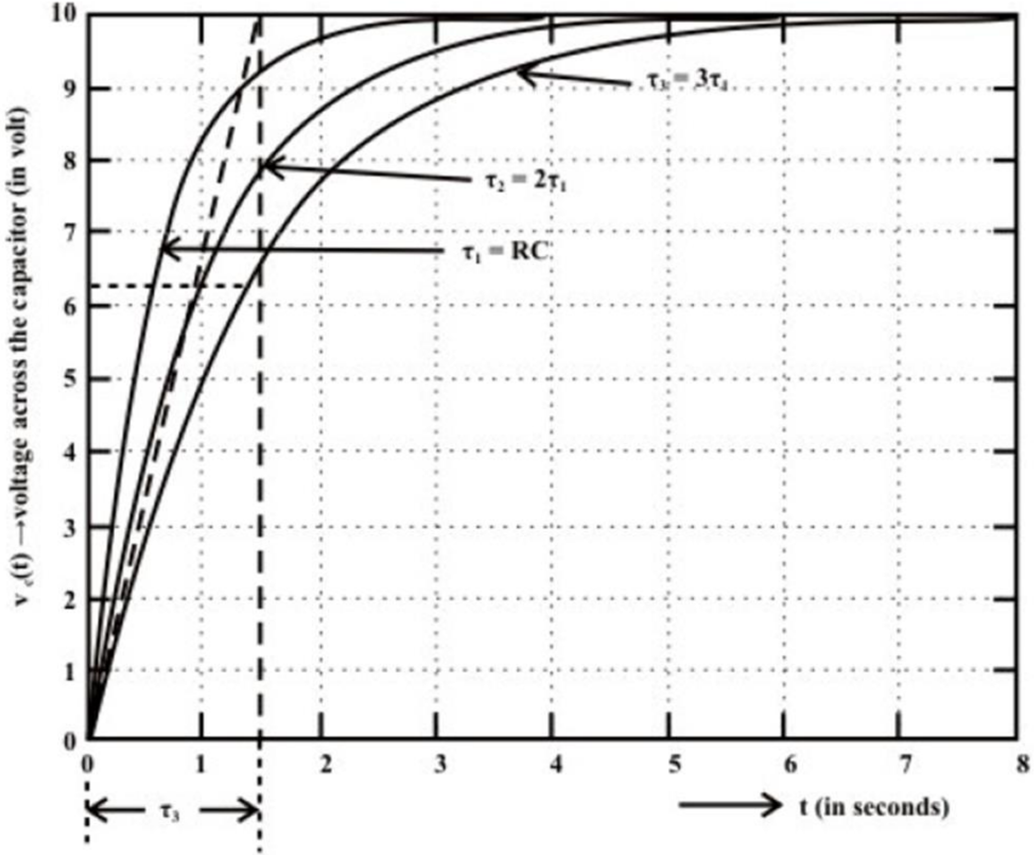
Hence, $A = -V_s$

Equation (3.28) becomes

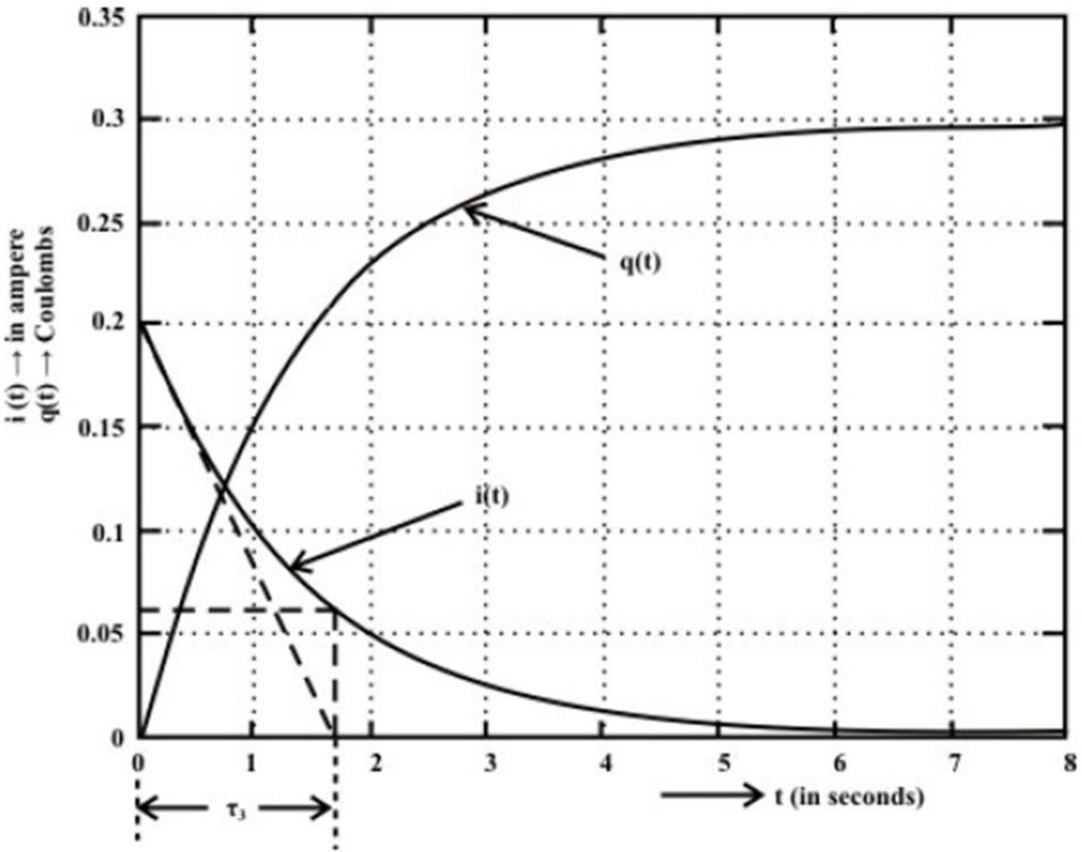
$V_c(t) = V_s(1 - e^{\frac{-t}{\tau}})$

Where the time constant $\tau = RC$

Growth of capacitor voltage (assumed initial capacitor voltage is zero)



System response due to the forcing function V_S (assumed capacitor initial voltage V_o = 0



Voltage across the resistance of Fig 3.10 is

$$V_R(t) = V_s - V_c(t) = V_s\, e^{\frac{-t}{\tau}}$$

Charging current through the capacitor

$$i(t) = \frac{V_R}{R} = \frac{V_s}{R}\, e^{\frac{-t}{\tau}}$$

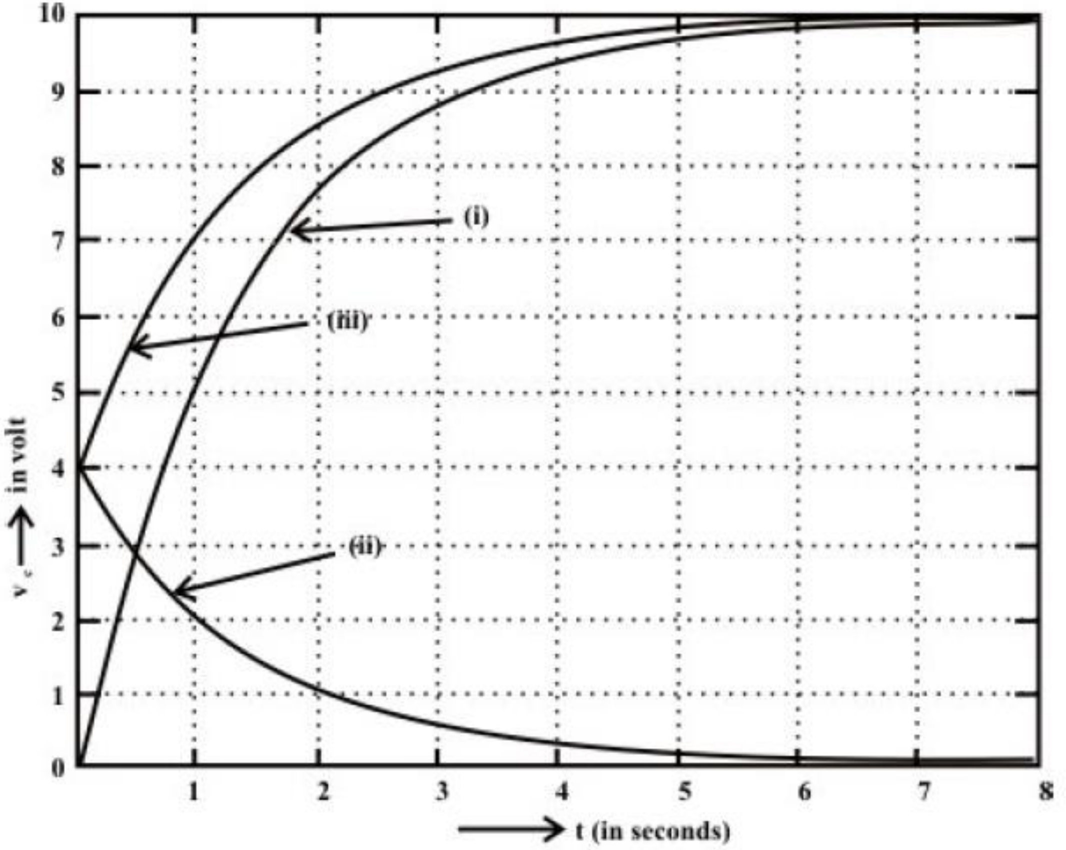
Now if the capacitor has some initial charge and hence initial voltage, the capacitor voltage at the time of switching, $V_c(0)$ will be finite and not zero.

i.e $V_c(0) \neq 0$

Assume $V_c(0) = V_o$

$$V_c(t) = V_s(1 - e^{\frac{-t}{\tau}}) + V_o\, e^{\frac{-t}{\tau}}$$

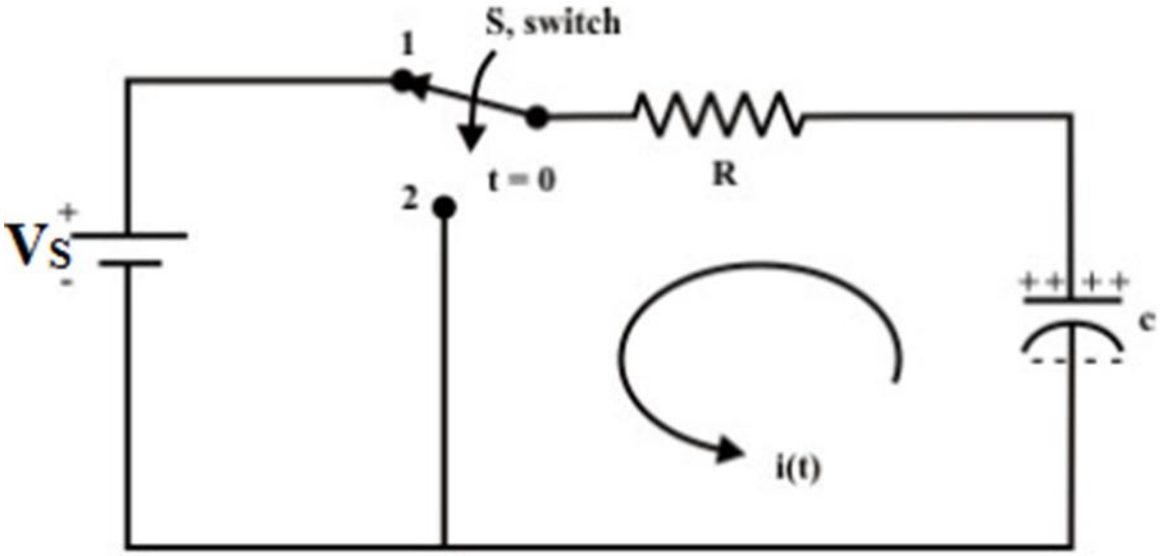
Voltage across the capacitor due to (i) the forcing function V_S acting alone (ii) discharge of capacitor initial voltage V_0 (iii) Combined effect of (i) and (ii)



Discharging of a Capacitor or Fall of a Capacitor Voltage in DC Circuits

The figure below shows that the switch ‘S’ is closed at position ‘1’ for sufficiently long time and the circuit has reached its steady-state condition.

At ‘t=0’ the switch ’S’ is opened and kept in position ‘2’ and remains there.



Discharging of Capacitor voltage Circuit

Discharging of a Capacitor or Fall of a Capacitor Voltage in DC Circuits

Our job is to find the expressions for:

(i) voltage across the capacitor V_c

(ii) voltage across the resistance V_R

(iii) current through the capacitor(discharging current)

Discharging of a Capacitor or Fall of a Capacitor Voltage in DC Circuits

For $t<0$, the switch 'S' in position 1.

The capacitor acts like an open circuit to dc, but the voltage across the capacitor is same as the supply voltage V_S.

Since, the capacitor voltage cannot change instantaneously, this implies that

$V_c(0^-) = V_c(0^+) = V_s$

When the switch is closed in position '2', the current will flow through the circuit until capacitor is completely discharged through the resistance R.

In other words, the discharging cycle will start at $t = 0$.

Discharging of a Capacitor or Fall of a Capacitor Voltage in DC Circuits

Now applying KVL around the loop, we get

$$RC\frac{dV_c(t)}{dt} + V_c(t) = 0 \qquad (3.34)$$

The solution of input free differential equation (3.34) is given by:

$$V_c(t) = Ae^{\alpha t} \qquad (3.35)$$

Where $\alpha = -\frac{1}{RC}$

The constant A is obtained using the initial condition of the circuit in equation (3.35)

Note, at ‘$t = 0$’ (when the switch is just closed in position ‘2’) the voltage across the capacitor $V_c(0) = V_s$

Discharging of a Capacitor or Fall of a Capacitor Voltage in DC Circuits

Using this condition in equation 3.35, we get

$$V_c(0) = V_s = A \qquad \Rightarrow \qquad A = V_s$$

Hence the discharging voltage across the capacitor is:

$$V_c(t) = V_s e^{-\left(\frac{t}{RC}\right)} \qquad (3.36)$$

Voltage across the resistance is

$$V_R(t) = -V_c(t) = -V_s e^{-\left(\frac{t}{RC}\right)} \qquad (3.37)$$

Discharging current through the capacitor is

$$i(t) = \frac{V_R}{R} = -\frac{V_s}{R} e^{-\left(\frac{t}{RC}\right)} \qquad (3.38)$$

Discharging of a Capacitor or Fall of a Capacitor Voltage in DC Circuits

An inspection of the above exponential terms of equations from (3.36) to (3.38) reveals that the time constant of circuit is given by:

$\tau = \text{RC (sec)}$

This means that at time $t = \tau$, the capacitor's voltage drops to 36.8% of its initial value (see the fig 3.15).

For all practical purposes, the dc transient is considered to end after a time span of 5τ.

At such time steady state condition is said to be reached.

Plots of above equations as a function of time are depicted in fig. 3.15(a) and fig. 3.15(b) respectively.

Discharge of capacitor voltage with time

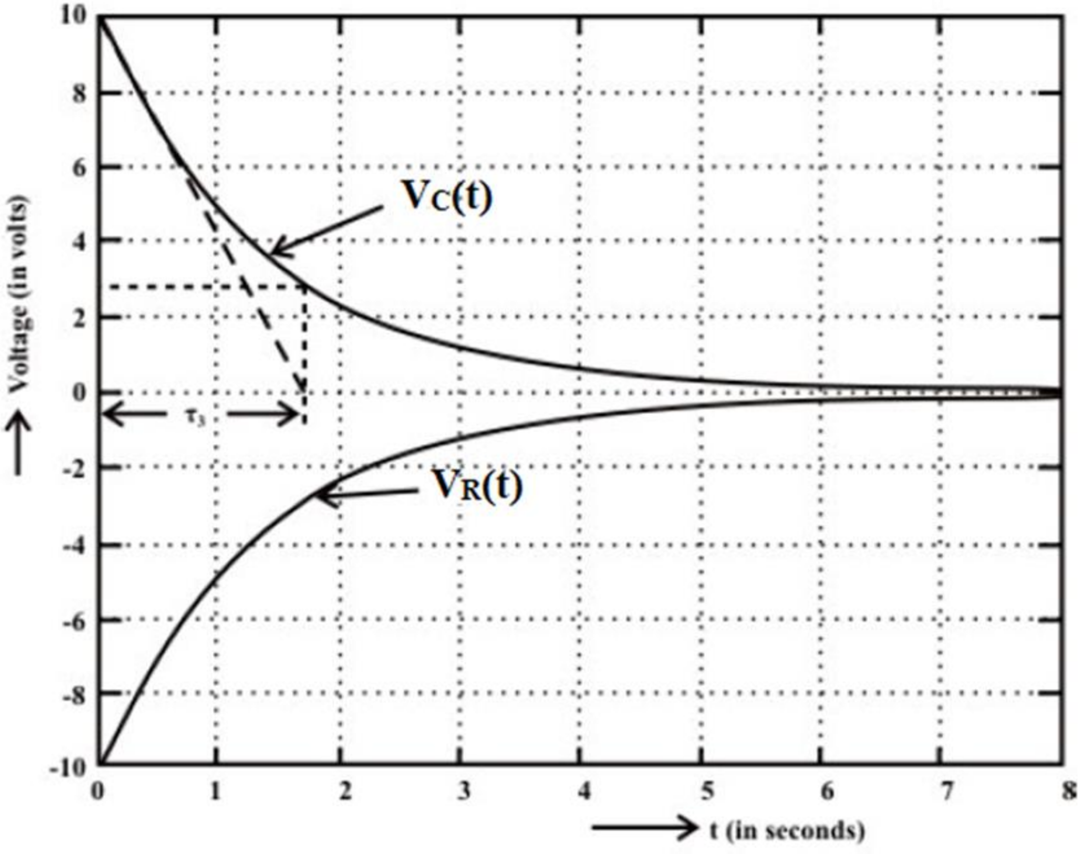


Figure 3.15(a)

System response due to capacitor discharge

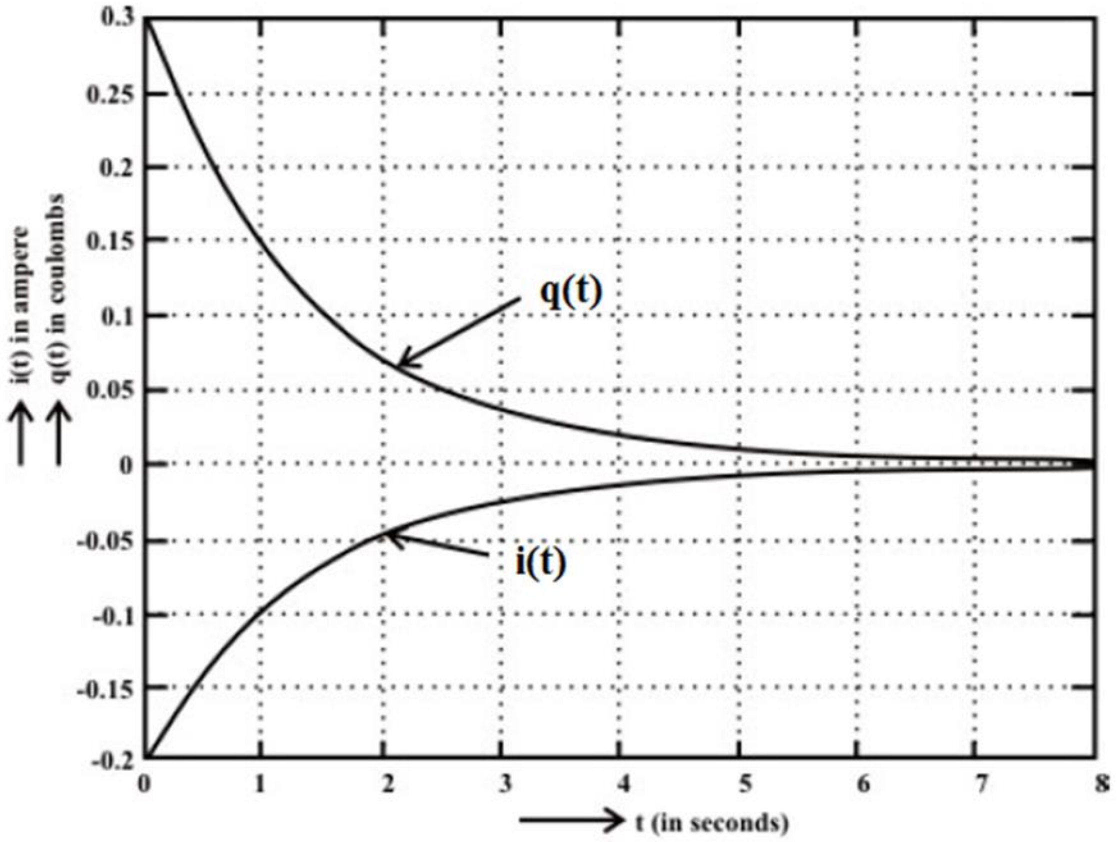
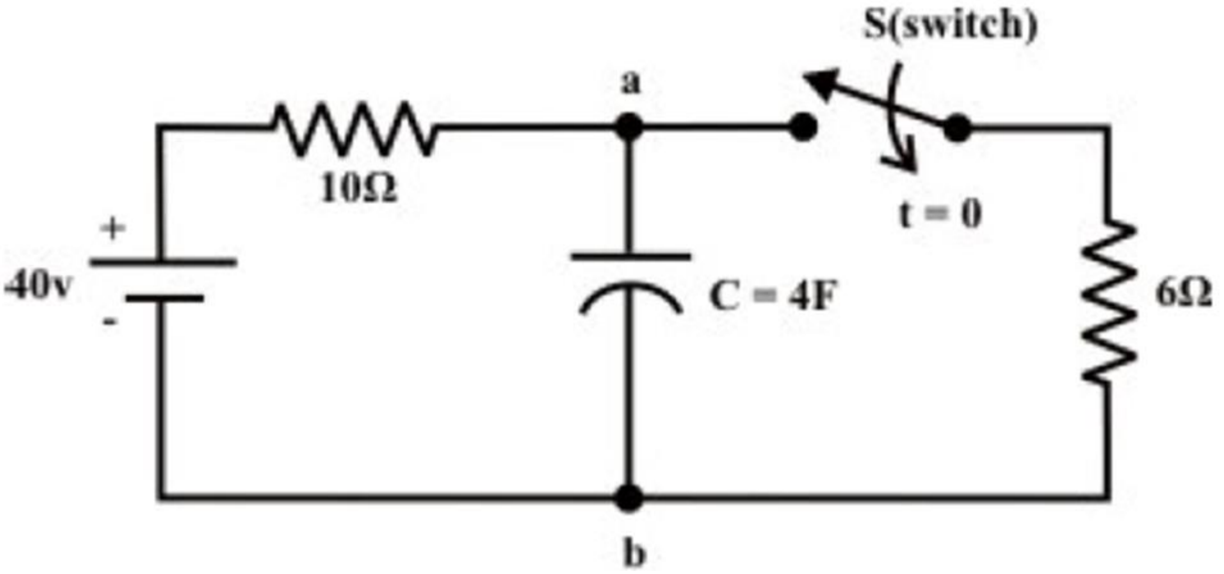


Figure 3.15(b)

Example 1

The switch 'S' shown in Fig. 3.16 is kept open for a long time and then it is closed at time 't=0'.



Find

(i) $V_c(0^-)$ (ii) $V_c(0^+)$ (iii) $i_c(0^-)$

(iv) $i_c(0^+)$ (v) find the time constants of the circuit before and after the switch is closed

Solution

(i) $V_c(0^-) = 40\text{V}$

(ii) $V_c(0^+) = 40\text{V}$

As we know the voltage across the capacitor cannot change instantaneously. Therefore, the voltage across the capacitor just before the switch is closed is same as voltage across the capacitor just after the switch is closed (note the terminal ‘a’ is positively charged).

(iii) $\mathrm{i_c}(0^-) = 0$

(iv) $\mathrm{i_c}(0^+) = \frac{\mathrm{Vc(0)}}{6} = \frac{40}{6} = 6.66\ \mathrm{A}$

(v) Time constant of the circuit before the switch was closed $= \tau = \mathrm{RC} = 10 \times 4 = 40\mathrm{sec}$

Time constant of the circuit after the switch is closed $\tau = \mathrm{R_{th}C} = \dfrac{10 \times 6}{10 + 6} \times 4 = 15$ sec

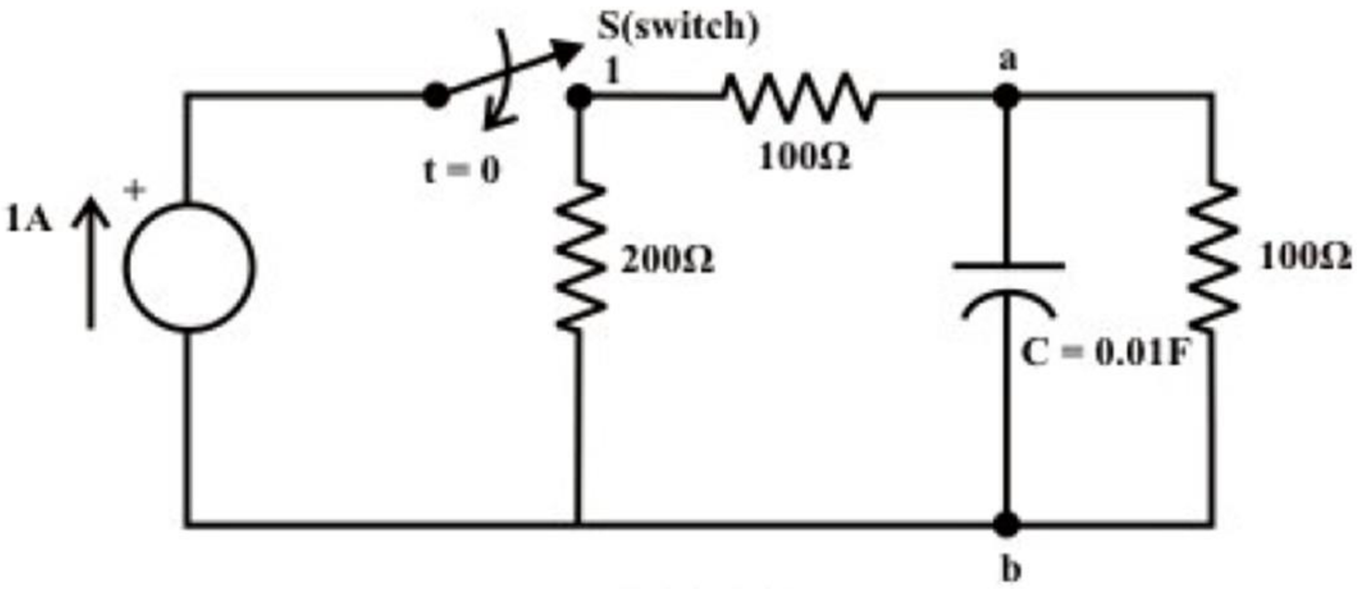
(Replace the part of the circuit than contains only independent sources and resistive elements by an equivalent Thevenin's circuit.

Example 2

The circuit shown below is switched on at time t=0.

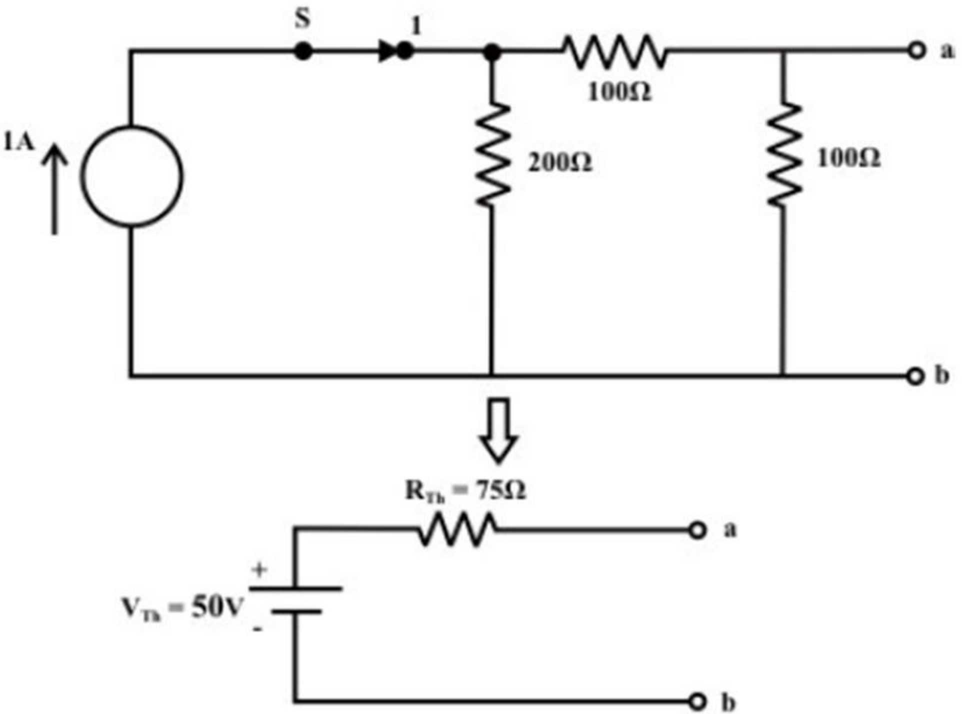
(i) How long does it take for the capacitor to attain 70 % of its final voltage? Assume the capacitor is not charged initially

(ii) Find also the time constant (τ) of the circuit after the switch is closed

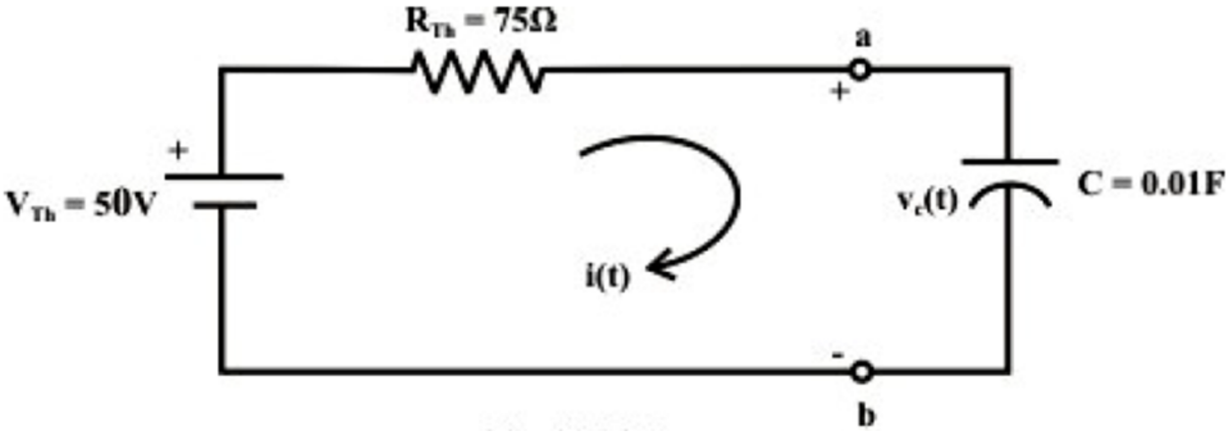


Solution

The circuit containing only resistive elements and independent current source (i.e., non-transient part of the circuit) is converted to an equivalent voltage source which is shown in the figure below.



The Thevenin Equivalent circuit is shown below:



The parameters of Thevenin's equivalent circuit are:

$$V_{th} = \frac{200}{200 + 100 + 100} \times 1 \times 100 = 50V$$

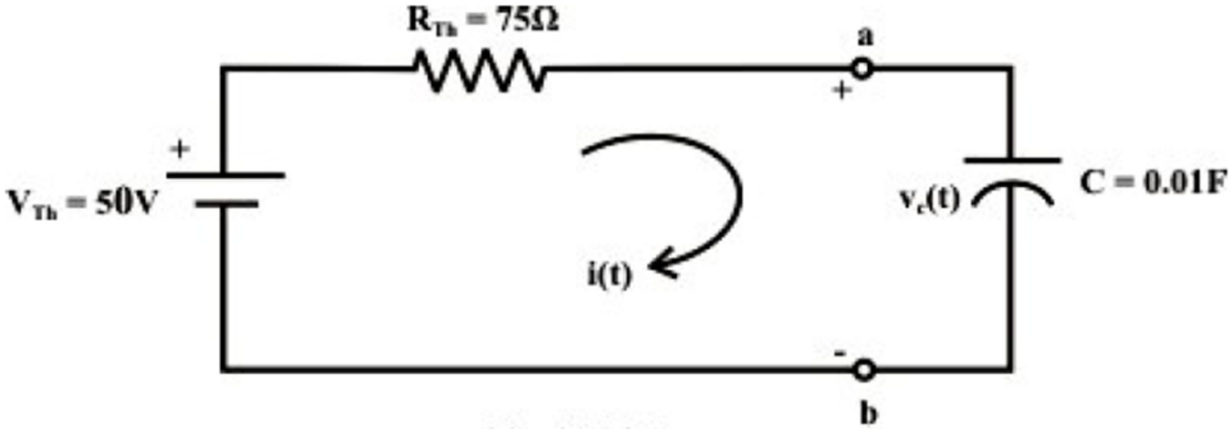
$$R_{th} = \frac{100 \times 300}{100 + 300} = 75\ \Omega$$

The time constant of the circuit is:

$\tau = \mathrm{RC} = 75 \times 0.01 = 0.75\ \mathrm{sec}$

The capacitor voltage expression for the circuit is

$V_c(t) = V_s(1 - e^{\frac{-t}{\tau}}) = 50(1 - e^{-\frac{t}{0.75}})$



Thevenin Equivalent

Let ‘t’ be the time required for the capacitor voltage to reach 70% of its final voltage.

Hence

$$50 \times 0.7 = 35 = 50(1 - e^{-1.33t})$$

$$\therefore t = 0.91 \text{ sec}$$

3
DC Transients RLC Circuits

Case Problem

Consider a series R-L-C circuit excited with a DC voltage source V_S

Applying KVL around the closed path for $t > 0$.

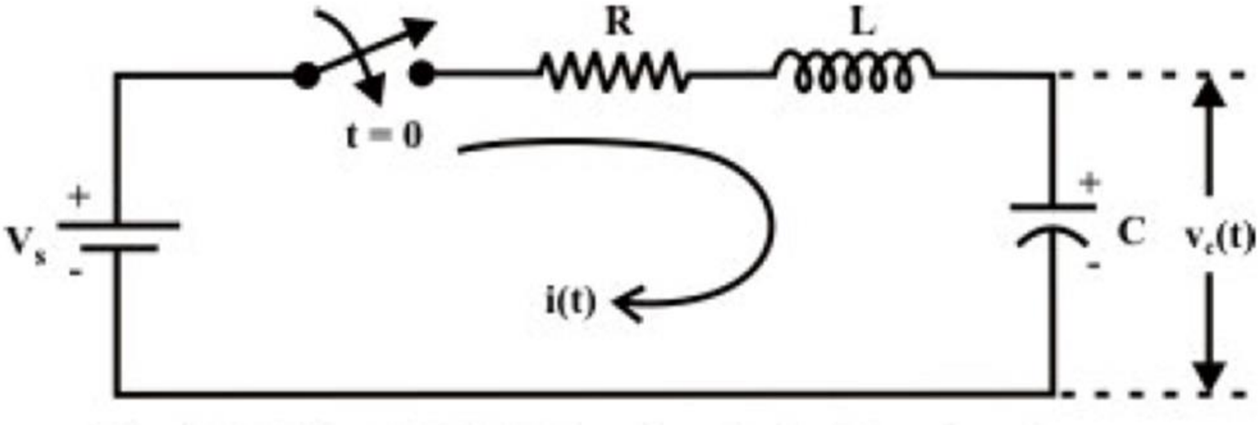
$$L\frac{di(t)}{dt} + Ri(t) + V_c(t) = V_s$$

The current through the capacitor can be written as:

$$i(t) = C\,\frac{dVc(t)}{dt}$$

Substituting the current 'i(t)' expression in equation (3.39) and rearranging the terms,

$$LC\,\frac{d^2V_c(t)}{dt^2} + RC\,\frac{dV_c(t)}{dt} + V_c(t) = V_s \qquad (3.40)$$



A Simple R-L-C circuit excited with a DC voltage source

Equation (3.40) is a 2nd-order linear differential equation

The complete solution of the above differential equation has two components:

- The transient response
- The steady state response.

Mathematically, one can write the complete solution as:

$$V_c(t) = V_{cn}(t) + V_{cf}(t) = (A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}) + A \qquad (3.41)$$

The nature of the steady state response is same as that of the forcing function (input voltage) and it is given by a constant value A.

The natural or transient response of second order differential equation can be obtained from the homogenous equation (i.e. from force free system) that is expressed by:

$$LC\frac{d^2V_c(t)}{dt^2} + RC\frac{dV_c(t)}{dt} + V_c(t) = 0$$

$$\frac{d^2V_c(t)}{dt^2} + \frac{R}{L}\frac{dV_c(t)}{dt} + \frac{1}{LC}V_c(t) = 0$$

$$a\frac{d^2V_c(t)}{dt^2} + b\frac{dV_c(t)}{dt} + cV_c(t) = 0 \ \ (\text{where } a = 1,\ b = \frac{R}{L} \text{ and } c = \frac{1}{LC})$$

The characteristic equation of the above homogenous differential equation (using the operator $\alpha = \frac{d}{dt}$, $\alpha^2 = \frac{d^2}{dt^2}$ and $V_c(t) \neq 0$) is given by

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$$

$$a\alpha^2 + b\alpha + c = 0 \qquad (3.43)$$

The solution of the above equation has a solution α_1 and α_2, which are associated with the exponential terms associated with the transient part of the complete solution (eqn 3.41) and they are given below:

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)$$

$$\alpha_2 = \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)$$

The roots of the characteristic equation (3.43) determine the kind of response depending upon the values of the parameters R, L and C of the circuit.

Case A: Overdamped Response I

When $\left(\frac{R}{2L}\right)^2 - \frac{1}{\mathrm{LC}} > 0$, this implies that the roots are distinct with negative real parts.

Under this situation, the natural or transient part of the complete solution is written as:

$$\mathrm{V_{cn}(t)} = (\mathrm{A_1}e^{\alpha_1 t} + \mathrm{A_2}e^{\alpha_2 t})$$

Each term of the above expression decays exponentially and ultimately reduces to zero as $t \rightarrow \infty$ and it is termed as overdamped response of input free system.

A system that is overdamped responds slowly to any change in excitation.

Case A: Overdamped Response II

It may be noted that the exponential term $A_1 e^{\alpha_1 t}$ takes longer time to decay its value to zero than the term $A_2 e^{\alpha_2 t}$.

One can introduce a factor ξ that provides an information about the speed of system response and it is defined by damping ratio:

$$\xi = \frac{\text{Actual damping}}{\text{Critical damping}} = \frac{R/L}{2/\sqrt{LC}} > 1$$

Case B: Critically damped Response

When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$, this implies that the roots of eq.(3.43) are same with negative real parts.

Under this situation, the form of the natural or transient part of the complete solution is written as:

$$\mathrm{V_{cn}(t) = (A_1t + A_2)}e^{\alpha t} \qquad \text{(where } \alpha = -\frac{R}{2L}\text{)}$$

Where the natural or transient response is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term.

The response is the speediest response possible without any overshoot .

The response of such a second order system is defined as a critically damped system’s response.

In this case damping ratio:

$$\xi = \frac{\text{Actual damping}}{\text{Critical damping}} = \frac{\text{R/L}}{2/\sqrt{\text{LC}}} = 1$$

Case C: Underdamped Response

When $\left(\frac{R}{2L}\right)^2 - \frac{1}{\mathrm{LC}} < 0$, this implies that the roots of eq.(3.43) are complex conjugates and they are expressed as:

$$\alpha_1 = \left(-\frac{R}{2L} + \mathrm{j}\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \beta + \mathrm{j}\gamma$$

$$\alpha_2 = \left(-\frac{R}{2L} - \mathrm{j}\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \beta - \mathrm{j}\gamma$$

Under this situation, the form of the natural or transient part of the complete solution is written as:

$$\mathrm{V_{cn}(t)} = \mathrm{A_1}e^{\alpha_1 t} + \mathrm{A_2}e^{\alpha_2 t} = \mathrm{A_1}e^{(\beta + \mathrm{j}\gamma)} + \mathrm{A_2}e^{(\beta - \mathrm{j}\gamma)}$$

$$V_{cn}(t) = e^{\beta t}[(A_1 + A_2)\cos(\gamma t) + j(A_1 - A_2)\sin(\gamma t)]$$

$$V_{cn}(t) = e^{\beta t}[B1\cos(\gamma t) + B2\sin(\gamma t)] \qquad \text{where } B_1 = A_1 + A_2; \quad B_2 = j(A_1 - A_2) \quad (3.48)$$

The equation (3.48) further can be simplified as:

$$e^{\beta t}K\sin(\gamma t + \theta)$$

Where β = real part of the root, γ = complex part of the root, $K = \sqrt{B_1{}^2 + B_2{}^2}$ and

$$\theta = \tan^{-1}\left(\frac{B_1}{B_2}\right)$$

The values of K and θ can be calculated using the initial conditions of the circuit.

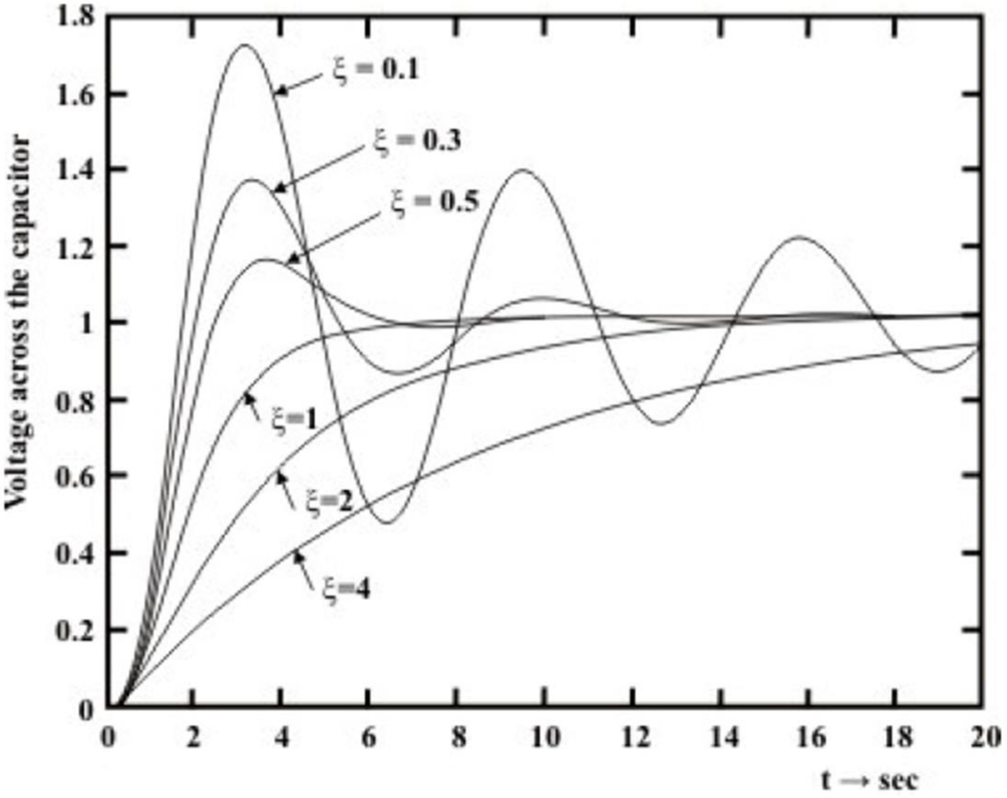
The system response exhibits oscillation around the steady state value when the roots of characteristic equation are complex and results in an under-damped system's response.

This oscillation will die down with time if the roots are with negative real parts.

In this case the damping ratio

$$\xi = \frac{\text{Actual damping}}{\text{Critical damping}} = \frac{R/L}{2/\sqrt{LC}} < 1$$

Finally, the response of a second order system when excited with a dc voltage source is presented in the figure below for different cases, i.e., (i) under-damped (ii) over-damped (iii) critically damped system response.



System response for series R-L-C circuit:
(a) underdamped
(b) critically damped
(c) overdamped system

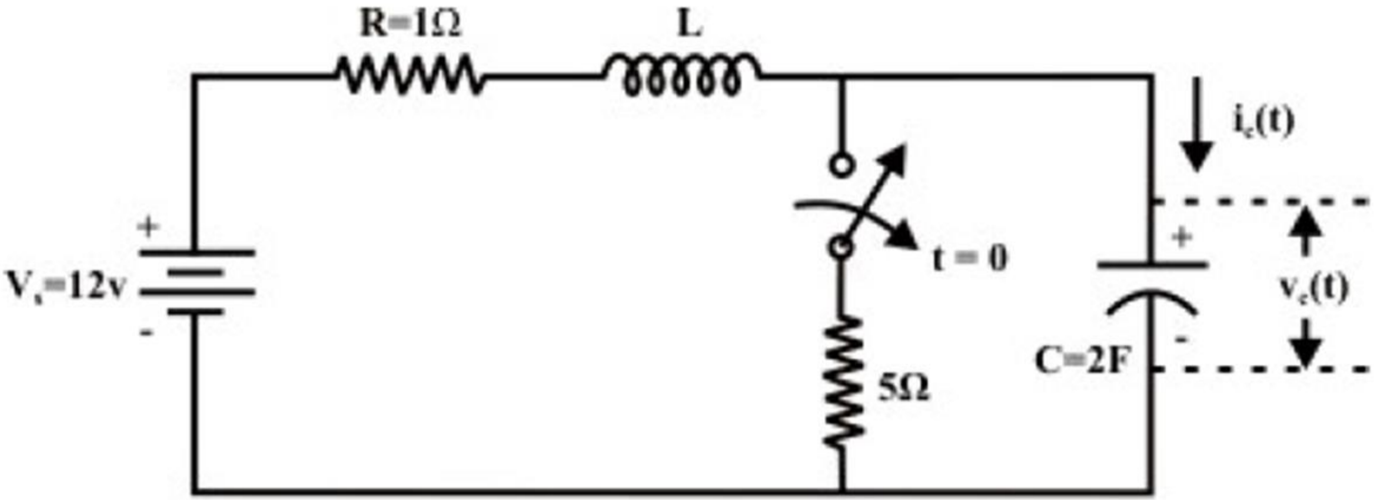
Example 1

The switch S_1 has been closed for a sufficiently long time and then it is opened at $t = 0$. Find the expression for (a) $v_c(t)$ (b) $i_c(t)$, $t > 0$ for inductor values of

(i) L=0.5 H

(ii) L=0.2 H

(iii) L=0.1 H and plot $v_c(t) - v_s - t$ and $i(t) - v_s - t$ for each case.



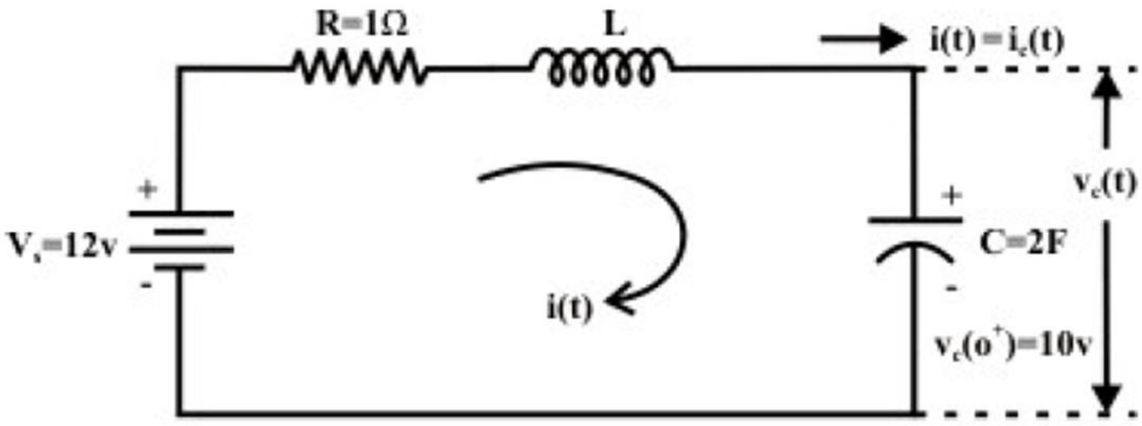
Solution

At $t = 0^-$ (before the switch is opened) the capacitor acts as an open circuit or block the current through it but the inductor acts as short circuit.

Using the properties of inductor and capacitor, one can find the current in inductor at time $t = 0^+$ as $i_L(0^+) = i_L(0^-) = \frac{12}{1+5} = 2$ A (note inductor acts as a short circuit) and voltage across the 5 Ω resistor $= 2 \times 5 = 10$ volts.

The capacitor is fully charged with the voltage across the 5 Ω resistor and the capacitor voltage at $t = 0^+$ is given by $v_c(0^+) = v_c(0^-) = 10$ V

The circuit is opened at time t=0 and the corresponding circuit diagram is shown in Fig. 3.21



Case 1: L = 0.5 H, R = 1Ω and C = 2F

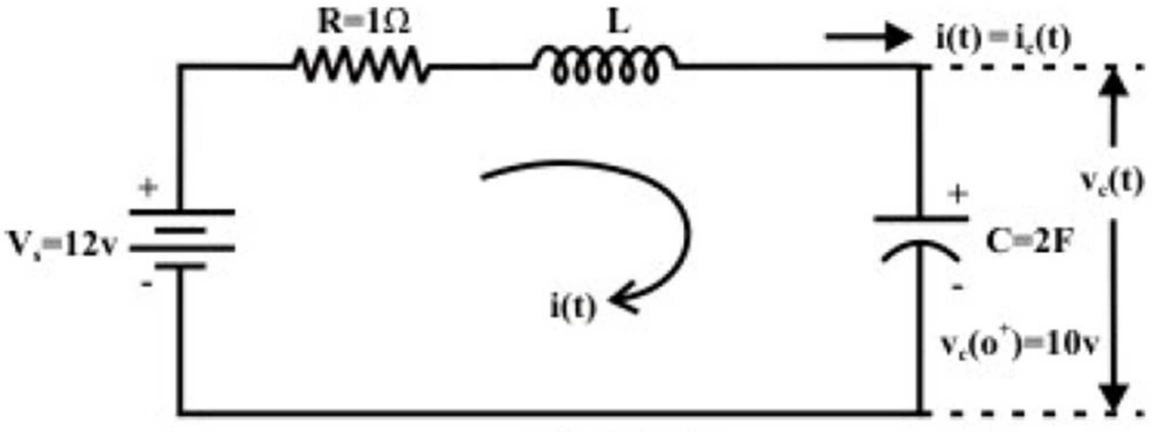
Let us assume the current flowing through the circuit is i(t) and apply KVL equation around the closed path:

$$L\frac{di(t)}{dt} + Ri(t) + V_c(t) = V_s$$

$$LC\frac{d^2V_c(t)}{dt^2} + RC\frac{dV_c(t)}{dt} + V_c(t) = V_s$$

The solution of the above is given by

$$V_c(t) = V_{cn}(t) + V_{cf}(t)$$



The solution of natural or transient response $V_{cn}(t)$ is obtained from the force free equation or homogenous equation which is:

$$\frac{d^2V_c(t)}{dt^2} + \frac{R}{L}\frac{dV_c(t)}{dt} + \frac{1}{LC}V_c(t) = 0$$

The characteristic equation of the above homogenous equation is written as:

$$\mathrm{a}\alpha^2 + \mathrm{b}\alpha + \mathrm{c} = 0$$

The roots of the characteristic equation are given as:

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = -1$$

$$\alpha_2 = \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = -1$$

The roots are equal with negative real sign. The expression for the natural response is given by:

$$\mathrm{V_{cn}(t)} = (\mathrm{A_1 t} + \mathrm{A_2})e^{\alpha t} \quad (\text{where } \alpha = \alpha_1 = \alpha_2 = -1)$$

The forced or the steady state response $v_{cf}(t)$ is the form of the applied input voltage and it is constant 'A'.

Now the final expression for $v_c(t)$ is

$$v_c(t) = (A_1 t + A_2)e^{\alpha t} + A = (A_1 t + A_2)e^{-t} + A$$

The initial and final conditions needed to evaluate the constants are based on

$v_c(0^+) = v_c(0^-) = 10\text{ V}; \qquad i_L(0^+) = i_L(0^-) = 2\text{ A}$

At $t = 0^+$

$$v_c(t)|_{t\,=\,0^+} = A_2 e^{-1\times 0} + A = A_2 + A \;\Rightarrow\; A_2 + A = 10 \quad (1)$$

Also,

$$\frac{dv_c(t)}{dt} = -(A_1 t + A_2)e^{-t} + A_1 e^{-t}$$

$$\left.\frac{dv_c(t)}{dt}\right|_{t\ =\ 0^+} = A_1 - A_2 \qquad (2)$$

$$V_c(\infty) = A \quad \Rightarrow \; A = 12$$

Using the value of A in equation (1) and then solving (1) and (2) we get, $A_1 = -1; A_2 = -2$

The total solution is:

$$V_c(t) = -(t+2)\ e^{-t} + 12 = 12 - (t+2)\ e^{-t}$$

$$i(t) = C\,\frac{dV_c(t)}{dt} = 2 \times [(t+2)e^{-t} - e^{-t}] = 2 \times (t+1)e^{-t}$$

Case 2: L=0.2 H, R= 1Ω and C=2F

It can be noted that the initial and final conditions of the circuit are all same as in case-1 but the transient or natural response will differ.

In this case the roots of characteristic equation are computed and the values of roots are:

$\alpha_1 = -0.563;$ $\alpha_2 = -4.436$

$$V_c(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + A = A_1 e^{-4.436t} + A_2 e^{-0.563t} + A \qquad (1)$$

$$\frac{dV_c(t)}{dt} = \alpha_1 A_1 e^{-\alpha_1 t} + \alpha_2 A_2 e^{-\alpha_2 t} = -4.436 A_1 e^{-4.436t} - 0.563 A_2 e^{-0.563t} \qquad (2)$$

Using the initial conditions ($v_c(0^+) = 10, \frac{dv_c(0^+)}{dt} = 1\frac{volt}{sec}.$) and $A = 12$ obtained in case-1 in the equations (1) and (2) above, we have

$$A_1 = 0.032;\ A_2 = -2.032$$

The total response is

$$V_c(t) = 0.032e^{-4.436t} - 2.032e^{-0.563t} + 12$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2[1.14e^{-0.563t} - 0.14e^{-4.436t}]$$

Case 3

Again the initial and final conditions will remain same and the natural response of the circuit will be decided by the roots of the characteristic equation and they are obtained as

$$\alpha_1 = \beta + j\gamma = -0.063 + j0.243;$$

$$\alpha_2 = \beta - j\gamma = -0.063 - j0.242$$

The expression for the total response is

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = e^{\beta t} K \sin(\gamma t + \Theta) + A \qquad (1)$$

$$\frac{dv_c(t)}{dt} = Ke^{\beta t}[\beta \sin(\gamma t + \Theta) + \gamma \cos(\gamma t + \Theta)] \qquad (2)$$

Again the initial conditions that were obtained in case-1 are used in above equations with

A=12 (final steady state condition) and simultaneous solution gives

$$K = 4.13; \;\; \Theta = -28.98^{o}$$

The total response is

$$v_c(t) = e^{\beta t} K \sin(\gamma t + \Theta) + 12 = e^{-0.063t} 4.13 \sin(0.242t - 28.99^{o}) + 12$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2e^{-0.063t}[0.999 * \cos(0.242t - 28.99^{o}) - 0.26 \sin(0.242t - 28.99^{o})]$$

Thanks!

Any questions?