Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT

MAS212 Linear Algebra I

Monday 26 April 2004, 10:00 am – 12:00 noon

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators are **not** *permitted in this examination. The unauthorised use of a calculator constitutes an examination offence. Show your working.*

SECTION A

This section carries 56 marks and each question carries 7 marks. You should attempt ALL 8 questions. ***Do* not *begin each answer in this section on a fresh page. Write the number of the question in the* left *margin.***

1. (a) Compute AB, where $A = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = 2\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ are matrices over the field $\mathbb{Q}$ of rational numbers.

(b) Compute AB, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are matrices over the Boolean field $\mathbb{F}_2$.

(c) Compute A^2, where $A = \begin{pmatrix} x & x^2 \\ x^3 & x^4 \end{pmatrix}$ is a matrix over the field $\mathbb{Q}(x)$ of rational functions of x.

2. Let A be the square matrix over the field $\mathbb{Q}$ defined by $A = 3\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$. Compute (a) $\det A$ and (b) A^{-1} (if it exists).

3. Define the term *rank of a matrix.*

Calculate the rank of (a) the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ over the field $\mathbb{Q}$ and (b) the matrix $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ over the field $\mathbb{F}_2$.

4. Define the term *linearly independent set of vectors.*

Let S be a set of row vectors in $\mathbb{R}^3$ defined by $S = \{(1,1,1),(0,-1,0),(3,2,3),(1,0,1)\}$. Find a maximal linearly independent subset $T \subseteq S$. Extend T to a set $U \supseteq T$ containing 3 linearly independent vectors in $\mathbb{R}^3$. Briefly explain your method.

5. Define the term *span of a set of vectors.*

Let S be a set of row vectors in $\mathbb{R}^4$ defined by $S = \{(1,0,0,0),(1,1,0,0),(0,1,1,1)\}$. Determine whether the vector $(1,2,3,4) \in \mathbb{R}^4$ is in the vector subspace of $\mathbb{R}^4$ spanned by S. Briefly explain your method.

6. Define the term *ordered basis for a vector space.*

Let $\mathcal{B} = (0,1,1),(1,0,1),(1,1,0)$ be an ordered basis for $\mathbb{R}^3$. Find the coordinate vector of $(1,2,3) \in \mathbb{R}^3$ with respect to $\mathcal{B}$.

7. Define the term *linear map.*

Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by $\alpha(x,y,z) = (x+2y, 3z)$. Find the matrix of α with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

8. Define the terms *orthogonal* and *orthonormal* applied to a pair of vectors.

Let $u = (1,2,3)$ and $v = (2,-1,0)$ be vectors in $\mathbb{R}^3$. Calculate the standard inner product (or dot product) of u and v, and state whether u and v are orthogonal. Calculate $\hat{u}$, the vector obtained by normalizing u, and find a vector w such that $\hat{u}$ and w are orthonormal.

SECTION B

This section carries 44 marks and each question carries 22 marks. You may attempt all 4 questions but, except for the award of a bare pass, only marks for the best 2 questions will be counted. ***Begin each answer in this section on a fresh page. Write the number of the question at the top of each page.***

1. Let V be a vector space over a field $\mathbb{K}$.

(a) [11 marks] Let U be a subset of V.
Define what it means for U to be a *vector subspace* of V.
In the following, either prove that U is a vector subspace of V or give a reason why it is not:

(i) $V = \mathbb{R}^2$, $U = \{(x, y) \mid x, y \in \mathbb{R},\ x + 2y = 0\}$;

(ii) $V = \mathbb{R}^3$, $U = \{(x, y, z) \mid x, y, z \in \mathbb{R},\ |x| = y\}$.

(b) [11 marks]

(i) Define $S + T$, where S and T are vector subspaces of the vector space V.

(ii) If $V = \mathbb{R}^3$, $S = \langle(2, 3, 0)\rangle$ and $T = \langle(1, 0, 0), (0, 1, 0)\rangle$, find $S + T$.

(iii) Give an example to show that if S and T are vector subspaces of the vector space $V = \mathbb{R}^2$ then $S \cup T$ is not necessarily a vector subspace.

(iv) Either give an example of a basis for $\mathbb{R}^2$ that is *not a linearly independent set* or explain why it is not possible to have such a basis.

(v) Either give an example of a spanning set for $\mathbb{R}^2$ that is *not a basis* or explain why it is not possible to have such a spanning set.

2. Let U and V be vector spaces over a field $\mathbb{K}$ and let $\alpha : U \to V$ be a linear map.

(a) [3 marks] Define what is meant by the statements "α is *onto*", "α is *one-to-one*" and "α is an *isomorphism*".

(b) [9 marks] Prove that

(i) $\ker(\alpha)$ is a vector subspace of U;

(ii) α is one-to-one if and only if $\ker(\alpha) = \{0_U\}$;

(iii) $\mathrm{im}(\alpha)$ is a vector subspace of V.

(c) [6 marks] For the case that α is an isomorphism, define the *inverse map* α^{-1} and prove that it is linear.

(d) [4 marks] For the map $\alpha : \mathbb{F}_2^3 \to \mathbb{F}_2^3$ defined by $\alpha(x, y, z) = (x + z, x + y, y + z)$, determine $\ker(\alpha)$ and $\mathrm{im}(\alpha)$ by finding bases for each of them.

[This question continues overleaf . . .]

3. (a) [8 marks] Let U and V be finite-dimensional vector spaces over a field $\mathbb{K}$ with ordered bases $\mathcal{B} = u_1, u_2, \ldots, u_m$ and $\mathcal{C} = v_1, v_2, \ldots, v_n$, respectively, and let $\alpha : U \to V$ be a linear map.

Give the definition of the matrix representation $A = (\alpha, \mathcal{B}, \mathcal{C})$ of α with respect to $\mathcal{B}$ and $\mathcal{C}$.

Write down the matrix A, with respect to the standard bases for the domain and codomain, of the linear map $\alpha : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\alpha(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3, x_2 + x_3 + x_4).$$

(b) [14 marks] Find bases $\mathcal{B}'$ for $\mathbb{R}^4$ and $\mathcal{C}'$ for $\mathbb{R}^2$ with respect to which the map α defined above has the matrix representation

$$A' = (\alpha, \mathcal{B}', \mathcal{C}') = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and each O is either a zero matrix of appropriate size or absent. What is (i) the value of r and (ii) the precise form of the matrix A'?

4. (a) [12 marks] Let A be an $n \times n$ matrix over a field $\mathbb{K}$. Define the terms *eigenvalue* and *eigenvector* of A. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

(i) Write down the characteristic equation of A, and obtain the eigenvalues and eigenvectors of A.

(ii) Write down an invertible matrix X and a diagonal matrix Λ that satisfy $X^{-1}AX = \Lambda$.

(b) [10 marks] Prove that the eigenvectors of a general square matrix corresponding to distinct eigenvalues are linearly independent.

[End of examination paper]