MULTIVARIABLE CALCULUS

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Multiple Integrals

Volume and Double Integrals

Let f be a function of two variables defined over a rectangle $R = [a, b] \times [c, d]$. We would like to define the double integral of f over R as the (algebraic) volume of the solid under the graph of $z = f(x, y)$ over R.

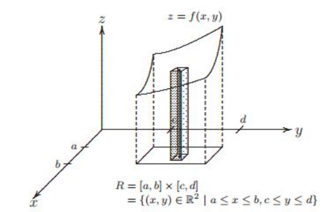


Figure: 1

Volume and Double Integrals

Volume and Double Integrals

The sum of the volume of all small rectangular solids approximate the volume of the solid under the graph of $z = f(x, y)$ over R.

Volume and Double Integrals

This sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A$$

is called a Riemann sum of f. we define the double integral of f over R as the limit of the Riemann sum as m and n tend to infinity.

Multiple Integrals

Double Integral

The double integral of f over R is

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if the limit exists.

Double Integral

Theorem

If $f(x, y)$ is continuous on R,then $\int\int_R f(x, y)dA$ always exists. If $f(x, y) \geqslant 0$,then the volume V of the solid lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \int\int_R f(x, y)dA$$

.

Iterated Integrals

Let $f(x,)$ be a function defined on $R = [a, b] \times [c, d]$. We write $\int_c^d f(x, y)dy$ to mean that x is regarded as a constant and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$.
Therefore, $\int_c^d f(x, y)dy$ is a function of x and we can integrate it with respect to x from $x = a$ to $x = b$.The resulting integral

$$\int_a^b \int_c^d f(x, y)dydx$$

is called an iterated integral.

Iterated Integrals

Examples

Evaluate the iterated integrals (a). $\int_0^3 \int_1^2 x^2 y dy dx$,
(b). $\int_1^2 \int_0^3 x^2 y dy dx$,

solution.

(a) $\int_0^3 \int_1^2 x^2 y dy dx = \int_0^3 [\frac{x^2y^2}{2}]_{y=1}^{y=2} dx = \int_0^3 \frac{3x^2}{2} dx = [\frac{x^3}{2}]_{x=0}^{x=3} = \frac{27}{2}$

(b). $\int_1^2 \int_0^3 x^2 y dy dx$,

solution.

(a) $\int_1^2 \int_0^3 x^2 y dy dx = \int_0^3 [\frac{x^3 y}{3}]_{x=0}^{x=3} dy = \int_1^2 9y dy = [\frac{9y^2}{2}]_{y=1}^{y=2} = \frac{27}{2}$

Iterated Integrals

Consider a positive function $f(x, y)$ defined on a rectangle $R = [a, b] \times [c, d]$. Let V be the volume of the solid under the graph of f over R. We may compute V by means of either one of the iterated integrals.

$$\int_a^b \int_c^d f(x, y)dydx, \quad or$$

$$\int_c^d \int_a^b f(x, y)dxdy,$$

Iterated Integrals

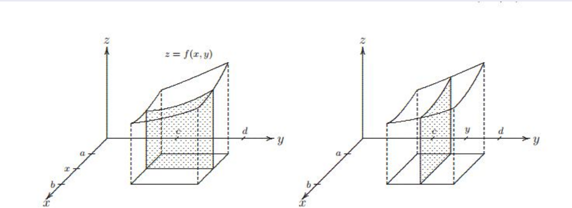


Figure: 2

Fubini's Theorem

Theorem

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$,then
$\int_a^b \int_c^d f(x, y) dydx = \int_c^d \int_a^b f(x, y) dxdy$
More generally, this is true if f is bounded on R,f is discontinuous only at a finite number of smooth curves,and the iterated integrals exist.Furthermore, the theorem is valid for a general closed and bounded region as discussed in the subsequent sections.

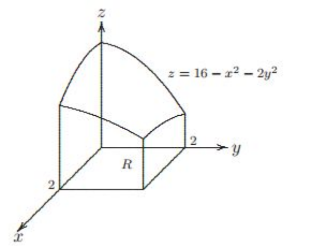
Fubini's Theorem

Examples

Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$,the planes $x = 2, y = 2$, and the 3 coordinate planes.See figure 3

solution

Volume= $\int\int_R 16 - x^2 - y^2 dA = \int_0^2 \int_0^2 16 - x^2 - y^2 dxdy = 48$



Examples

Let $R=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$.Evaluate $\int\int_R \sin x \cos y dA$.

solution

$\int\int_R \sin x \cos y dA = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin x \cos y dxdy = \int_0^{\frac{\pi}{2}} \sin x dx \int_0^{\frac{\pi}{2}} \cos y dy$

Cont'd

This implies, $\int\int_R \sin x \cos y dA = 1\times 1 = 1$

Note

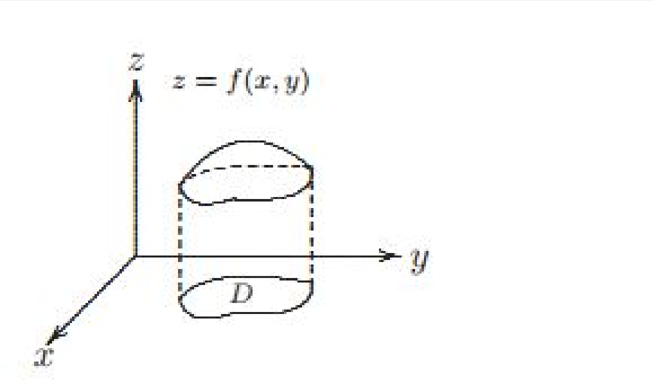
In general, if $f(x,y)=g(x)h(y)$, then

$\int\int_R g(x)h(y)dA = (\int_a^b g(x)dx)(\int_c^d h(y)dy)$, where

$R=[a,b]\times[c,d]$.

Double Integral over General Regions

Let $f(x, y)$ be a continuous function defined on a closed and bounded region D in $\Re^2$. The double integral $\int\int_R f(x, y)dA$ can be defined similarly as the limit of a Riemann sum.



Double Integral(Type 1 region)

If D is the region bounded by two curves $y = g_1(x)$ and $y = g_2(x)$ from $x = a$ to $x = b$, where $g_2(x) \geqslant g_1(x)$ for all $x \in [a, b]$, we called it a type 1 region. In this case, the double integral of f over D can be expressed as an iterated integral as given below;

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

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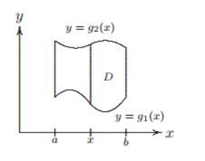
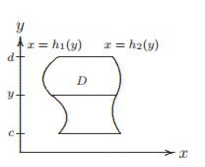


Double Integral(Type 2 region)

Similarly, If D is the region bounded by two curves $x = h_1(x)$ and $x = h_2(x)$ from $y = c$ to $y = d$, where $h_2(y) \geqslant h_1(y)$ for all $y \in [c, d]$, we called it a type 2 region. In this case, the double integral of f over D can be expressed as an iterated integral as given below;

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$



Double Integrals over a General region

Examples

Evaluate $\int\int_D (x+2y)dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

solution

The region D is a type 1 region as shown below;

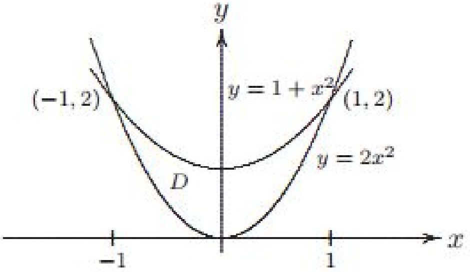


Figure: 7

$$\int\int_D (x+2y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx = \int_{-1}^{1}[xy+y]_{y=2x^2}^{y=1+x^2}dx$$

$$= \int_{-1}^{1}(-3x^4 - x^3 + 2x^2 + x + 1)dx = \frac{32}{15}$$

Exercise

Evaluate $\int\int_D xydA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

[Answer:36]

Properties of Double Integrals

1. $\int\int_D (f(x,y)+g(x,y))dA = \int\int_D f(x,y)dA + \int\int_D g(x,y)dA$
2. $\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA$,where c is a constant.
3. If $f(x,y) \geqslant g(x,y)$ for all $(x,y) \in D$,then $\int\int_D f(x,y)dA \geqslant \int\int_D g(x,y)dA$
4. $\int\int_D f(x,y)dA = \int\int_{D_1} f(x,y)dA + \int\int_{D_2} f(x,y)dA$, where $D = D_1 \cup D_2$ and D_1, D_2 do not overlap except perhaps on their boundary.

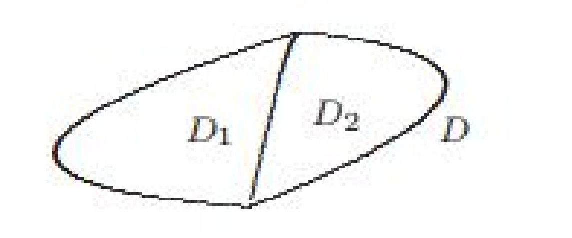


Figure: 8

Properties of Double Integrals

5. $\int\int_D dA = (\int\int_D 1 dA) = A(D)$, the area of D.
6. If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$,then
   $mA(D) \leq \int\int_D f(x,y)dA \leq MA(D)$.

Double Integrals In Polar Coordinates

Consider a point (r, θ) on the plane in polar coordinates as in the figure below;An increment dr in r and $d\theta$ in θ give rise to an area

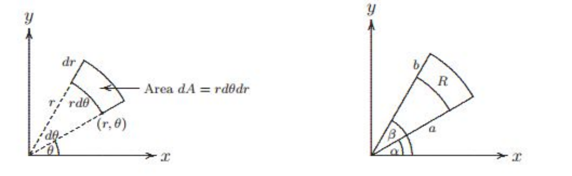


Figure: 9

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Let f be a continuous function defined on a polar rectangle.

$$R = (r, \theta)|0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$$

, where $0 \leq \beta - \alpha \leq 2\pi$. The double integral of f over R can be expressed in polar coordinates as follow:

$$\int\int_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

.

Double Integrals In Polar Coordinates

Example 1

Evaluate $\int\int_R (3x + 4y^2)dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

[Answer: $\frac{15\pi}{2}$**]**

and see figure 10;

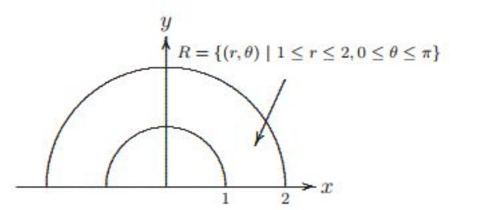

Figure: 10

Double Integrals In Polar Coordinates

Note

In general,if f is continuous on a polar region of the form

$$D = ((r, \theta)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)),$$

then

$$\int\int_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

.

A general Polar Region

Figure 11

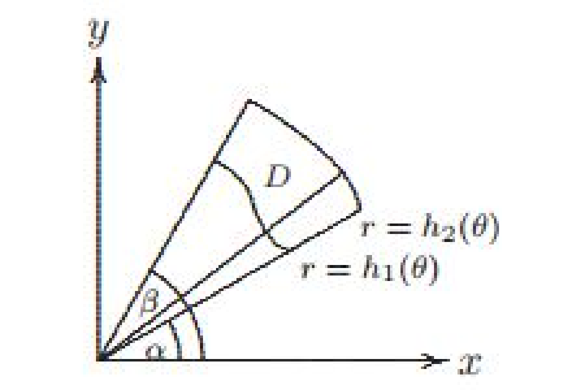

Figure: 11

Double Integrals In Polar Coordinates

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the $xy-$plane, and inside the cylinder $x^2 + y^2 = 2x$.

solution

The cylinder $x^2 + y^2 = 2x$ lies over the circular disk D which can be described in polar as

$$D = ((r, \theta)| -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2\cos\theta).$$

cont'd

Figure 12

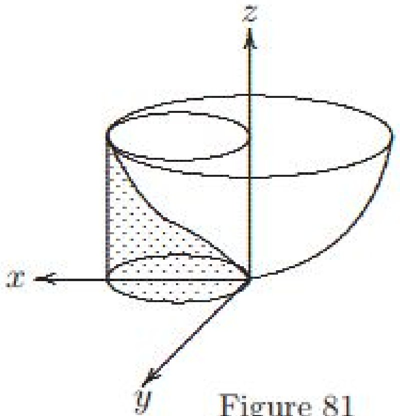

Figure 81

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The height of the solid is the $z - value$ of the paraboloid. Hence the volume V of the solid is

$$V = \int\int_D (x^2 + y^2) dA = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 r dr d\theta = \frac{3\pi}{2}.$$

Surface Area

Let f be a differentiable function of 2 variables defined on a D.We wish to find the surface area of the graph of f over D.It is simply equal to $\int\int_D dS$. Therefore we need to express the differential of the surface area dS in terms of the differential dA of the domain. To do so, take any point $P'(x, y)$ in D and let P be the corresponding point of the graph of f.Consider an increment dx along the x-direction and an increment dy along the y-direction at the point P'. Thus $dA = |dxdy|$. These increments sweep out an increment of surface area on the surface area on the surface at P. The differential dS of this area at P is given by the corresponding area on the tangent plane to the surface at P.

Surface Area

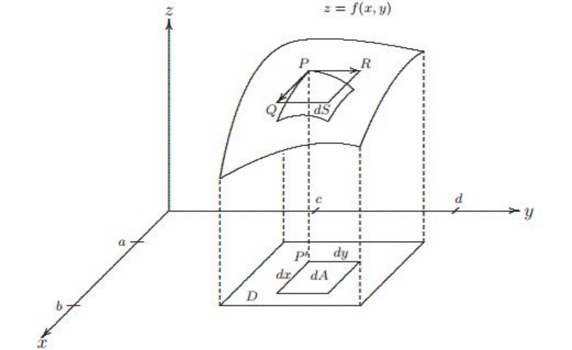

Figure: 13

Surface Area

Let $\vec{PQ}$ be the vector on the tangent plane at P with x-component dx, and $\vec{PR}$ the vector with y-component dy. Thus, $\vec{PQ} = \langle dx, 0, f_x(x,y)dx \rangle$ and $\vec{PR} = \langle 0, dy, f_y(x,y)dy \rangle$.The area of the parallelogram spanned by $\vec{PQ}$ and $\vec{PR}$ is the magnitude of the cross product $\vec{PQ} \times \vec{PR}$.

Therefore, $dS = |\langle -f_x, -f_y, 1 \rangle dxdy| = \sqrt{f_x^2 + f_y^2 + 1} dA$.

Consequently,

$$Surface \quad area = \int \int_D \sqrt{f_x^2 + f_y^2 + 1} dA$$

Surface Area

Examples

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

solution

The paraboloid lies above the circular disk D;

$$D = ((r, \theta)|0 \leq \theta \leq 2\pi, 0 \leq r \leq 3).$$

$$Surface \quad area = \int\int_D \sqrt{f_x^2 + f_y^2 + 1} dA$$

cont'd

$$\begin{aligned} Surface \quad area &= \int\int_D \sqrt{1+4(x^2+y^2)}dA &(1)\\ &= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}rdrd\theta &(2)\\ &= \frac{\pi}{6}(37\sqrt{37}-1). &(3)\end{aligned}$$

Triple Integrals

Let $f : B \subseteq R^3 \to R$ be a continuous function ,where $B = [a, b] \times [c, d] \times [r, s]$ is a rectangular solid. Divide $[a, b], [c, d]$ and $[r, s]$ into l, m and n equal subintervals,respectively.Thus B is divided into $l \times m \times n$ small rectangular solids.Label each small rectangular solid by C_{ijk}, where $1 \leq i \leq l, 1 \leq j \leq m, and 1 \leq k \leq m$. Inside each such C_{ijk},pick a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$.Denote the volume of C_{ijk} by $\triangle V$.Then we form the Riemann sum:

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f((x^*_{ijk}, y^*_{ijk}, z^*_{ijk}))\triangle V$$

Triple Integrals

The triple integral of f over B is defined to

$$\int\int\int_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f((x_{ijk}^*, y_{ijk}^*, z_{ijk}^*))\triangle V$$

Triple Integrals

Figure 14

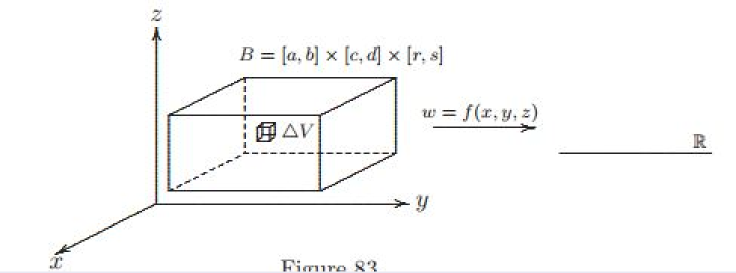

Figure: 14

The limits exists if f is continuous.The triple integral of a continuous function defined on a more general closed and bounded solid in $\Re^3$ can be defined in a similar way.

Fubini's Theorem for triple integrals

If $f(x, y, z)$ is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then

$$\begin{aligned}\int\int\int_B f(x, y, z)dV &= \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz \\ &= \int_r^s \int_c^d \int_a^b f(x, y, z)dydxdz = etc.\end{aligned}$$

Triple Integrals

Examples

Evaluate $\int\int\int_B xyz^2 dV$, where $B = [0,1] \times [-1,2] \times [0,3]$.

Solution

$$\int\int\int_B xyz^2 dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz = \frac{27}{4}$$

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Triple Integrals over a General Bounded Region

Type 1 solid region

For each of the following three types of solid regions,we may write down the triple integral as an iterated integral of a double integral and a simple integral.

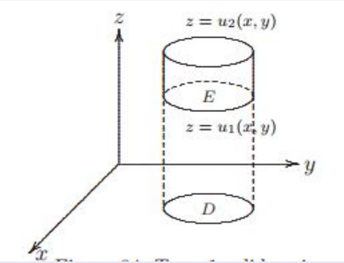


Figure: 15

Triple Integrals over a General Bounded Region

Type 1 solid region

$$\int\int\int_E f(x,y,z)dV = \int\int_D[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dA$$

Type 2 solid region

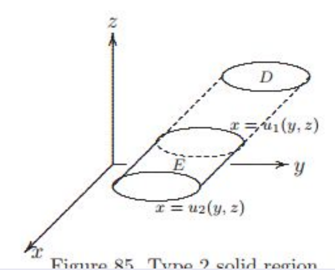


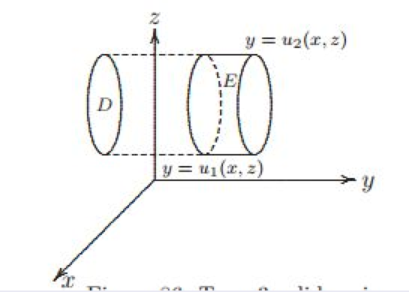
Figure: 16

Triple Integrals over a General Bounded Region

Type 2 solid region

$$\int\int\int_E f(x,y,z)dV = \int\int_D [\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx]dA$$

Type 3 solid region



Triple Integrals over a General Bounded Region

Type 3 solid region

$$\int\int\int_E f(x,y,z)dV = \int\int_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dA$$

Note

Note that if $f(x,y,z) = 1$ for all $(x,y,z) \in E$, then $\int\int\int_E 1dV$ is just the volume of E.

Triple Integrals over a General Bounded Region

Examples

Evaluate $\int\int\int_E \sqrt{x^2+z^2}dV$,where E is the solid region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. See the below figure with the corresponding solution.

Solution

E is a type 3 solid region whose projection onto the xz-plane is

$$D = ((x,z)|x^2+z^2 \leq 4) = ((r,\theta)|0 \leq \theta \leq 2\pi, 0 \leq r \leq 2)$$

cont'd

$$\begin{aligned}
\int\int\int_E \sqrt{x^2+z^2}\,dV &= \int\int_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\,dA \\
&= \int\int_D [y\sqrt{x^2+z^2}]_{x^2+z^2}^{4}\,dA \\
&= \int\int_D \sqrt{x^2+z^2}(4-x^2-z^2)\,dA \\
&= \int_0^{2\pi}\int_0^{2} r(4-r^2)\,r\,dr\,d\theta \\
&= \frac{128\pi}{5}.
\end{aligned}$$

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A Type 3 Solid

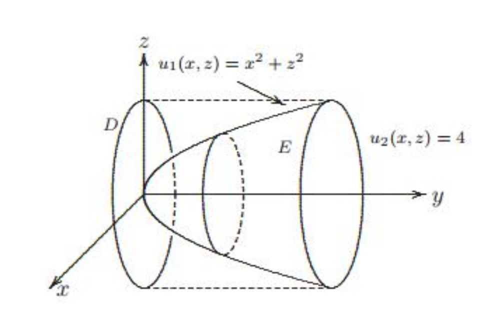

Figure: 18

Triple Coordinates in Polar Coordinates

Exercises

1. Evaluate $\int\int\int_E z dV$, where E is the solid tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.
2. Find the volume of the solid tetrahedron bounded by the planes $x = 2y, x = 0, z = 0$ and $x + 2y + z = 2$.

Triple Integrals in Cylindrical Coordinates

Consider a rectangle in cylindrical coordinates as in the figure below;

$$E = ((r,\theta,z)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(r,\theta) \leq z \leq u_2(r,\theta))$$

The triple integral of $f(x,y,z)$ over E can be expressed as:

$$\int\int\int_E f(x,y,z)dV = \int\int_D [\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)dz]dA$$

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r dr d\theta$$

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A cylindrical rectangle

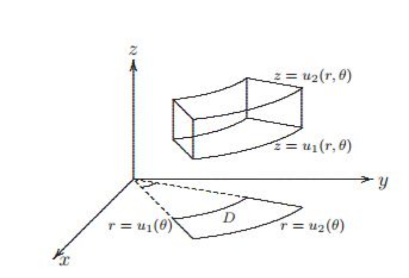


Figure: 19

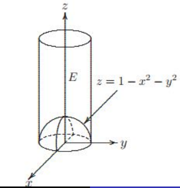
Triple Integrals in Cylindrical Coordinates

Examples

Let E be the solid within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$ and the above paraboloid $z = 1 - x^2 - y^2$.Evaluate $\int\int\int_E \sqrt{x^2 + y^2} dV$.

Solution The solid can be described in cylindrical coordinates as:

$$E = ((r, \theta, z)|0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4)$$



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Thus,

$$\int\int\int_E \sqrt{x^2+y^2}dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 rrdzdrd\theta = \frac{12\pi}{5}.$$

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx$.

Triple Integrals in Spherical Coordinates

Consider the volume element in spherical coordinates.To do so,take any point $P(r, \theta, \phi)$. Make an increment in each of the coodinates. See the figure below; Let's calculate the volume of the solid arising from these increments. The projection of OP onto the xy-plane has length $\rho \sin \phi$. Thus the thickness of this volume element is $\rho \sin \phi d\theta$.It opens up a sector of width of the $\rho d\phi$. Thus, the volume of is

$$dV = \rho^2 \sin \phi d\rho d\theta d\phi$$

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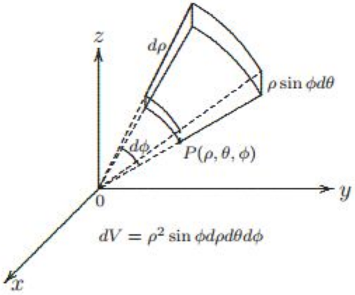
Triple Integrals in Spherical Coordinates

The volume element in spherical coordinate



'e 90 The volume element in spherical coordinate

Figure: 21

Triple Integrals in Spherical Coordinates

Now consider a spherical rectangle

$$E = ((\rho, \theta, \phi)|a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d),$$

where $a \geqslant 0, \beta - \alpha \leq 2\pi, d - c \leq \pi$.The triple integral of f over E can be expressed as follow:

$$\int\int\int_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2$$

Triple Integrals in Spherical Coordinates

Examples

Evaluate $\int\int\int e^{(}x^2+y^2+z^2)^{\frac{3}{2}}dV$,where B; is the unit ball

$$((x,y,z)|x^2+y^2+z^2 \leq 1).$$

Solution

Using spherical coordinates,we have

$$\int\int\int_B e^{(}x^2+y^2+z^2)^{\frac{3}{2}}dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(}\rho^2)^{\frac{3}{2}}\rho^2 \sin\phi d\rho d\theta d\phi$$

cont'd

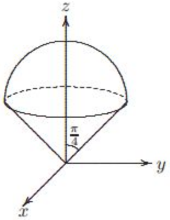
$$\int\int\int_B e^(x^2 + y^2 + z^2)^{\frac{3}{2}} dV = \frac{4}{3}\pi(e - 1)$$

Exercise

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.see the figure below;

cont'd

Figure



[Answer: $\pi/8$]

Figure: 22

Change of Variables in Multiple Integrals

Let T be a transformation from the uv-plane to the xy-plane. That is $(x, y) = T(u, v)$ or $x = x(u, v), y = y(u, v)$.We assume that T is a C^1-transformation,i.e. both $x(u, v)$ and $y(u, v)$ have continuous partial derivatives with respect to u and v.We also assume T is an injective function so that its inverse T^{-1} exists(from the range of T back to the domain of T).Thus T maps a region S in the uv-plane bijectively onto a region R in the xy-plane.

Change of Variables in Multiples Integrals

Figure

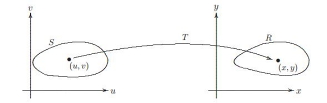


Figure: 23

For example if T is the transformation to polar coordinates $T(r,\theta) = (r\cos\theta, r\sin\theta)$,then T maps rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$ in the $r\theta$-plane to a polar rectangle in the xy-plane .

Change of Variables in Multiple Integrals

Polar coordinates

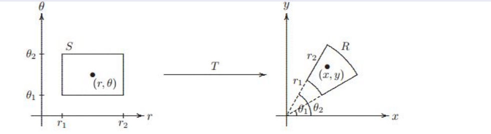


Figure: 24

Examples

Solution.First let's find out the boundary of the image.Label the edges of the square S by S_1, S_2, S_3 and S_4 as shown in figure 24;

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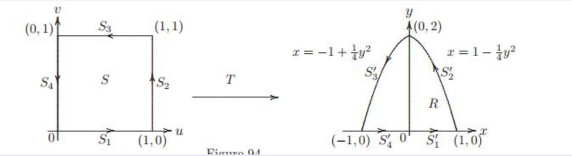

Figure: 25

S_1 is described by $v = 0, 0 \leq u \leq 1$.Thus the image S_1' in the xy-plane is given by $x = u^2 - 0^2 = u^2, y = 2u(0) = 0$.That is $x = u^2$ for $0 \leq u \leq 1$ and $y = 0$. Therefore,S_1' is described by $y = 0,0 \leq x \leq 1$,which is just the line segment on the x-axis from (0,0) to (1,0).

cont'd

Next S_2' is described by $u = 1, 0 \leq v \leq 1$.Thus the image S_2' in the xy-plane is given by $x = 1 - v^2, y = 2v$.Eliminating v,we obtain $x = 1 - \frac{1}{4}(y^2)$.As $0 \leq v \leq 1$,we have $0 \leq y \leq 2$.Therefore S_2' is described by $x = 1 - \frac{1}{4}(y^2)$ for $0 \leq y \leq 2$.
Similarly, we found that S_3' as $x = -1 + \frac{1}{4}(y^2)$ for y from 2 to 0 and S_4' as $y = 0$ for x from -1 to 0.

cont'd

The boundary of the image of S encloses a region R.We are going to show that T maps S bijectively onto R.We leave it the reader to verify that T is a bijective function $u, v \geqslant 0$.As we traverse the boundary of S in the counterclockwise direction,the above calculation shows that the image of the boundary of S also traverses in the counterclockwise direction.This means that T preserves orientation.In other words,points on the left hand side of the boundary of S go under T to points on the left hand side of the boundary of R.Therefore,T maps S onto R. Another easy way to confirm this is to a pick a point P,say$(1/2, 1/2)$ inside S and check that $T(P)$ is inside R.Then the region S must be mapped by T into R.

cont'd

Before we derive the formula for change of variables in a multiple integral, let's review the formula for functions of 1 variable.Let the continuous function $f(x)$ be integrated over the interval [a,b].Suppose we make a substitution $x = g(u)$ so that $a = g(c)$ and $b = g(d)$.
Thus we obtain:

$$\int_a^b f(x)dx = \int_c f(g(u))g'(u)du$$

.
Here the formula is valid provided g is differentiable and $g'(u) \neq 0$,except possibly at a finite number of points. The function f is also required to be bijective so that g^{-1} exists. Observe that c may not be less than d.

cont'd

More precisely, If $g'(u) > 0$ for all u between c and d,then g and g^{-1} is increasing.Thus g^{-1} preserves orientation or ordering.This means that $c < d$ and $[c, d]$ is an interval. On the other hand,if $g'(u) < 0$ for all u between c and d,then g and g^{-1} is decreasing. Thus g^{-1} reverses orientation or ordering.This means that $c > d$ and it does not make sense to write $[c, d]$ though we could still integrate c to d. In this case,the formula can be rewritten as:

$$\int_a^b f(x)dx = \int_d^c f(g(u))(-g'(u))du,$$

so as to keep the lower limit of integration smaller than the upper limit.

cont'd

Therefore,if the interval $[c,d]$,$(c<d)$ is mapped onto the interval [a,b] under $x=g(u)$,then the formula for change of variables can be stated as:

$$\int_{[a,b]} f(x)dx = \int_{[c,d]} f(g(u))|-g'(u)|du$$

.It is this formula we are going to generalize.
How does a change of variables affect a double integral? Let T be a transformation mapping a point (u_0, v_0) along the u and v directions respectively.These increments generates a rectangle of area $dudv$ whose image under T is a curved parallelogram in the xy-plane.

cont'd

The area of this curved parallelogram up to the first order approximation is given by the area of the parallelogram generated by the two tangent vectors $\mathbf{a}du$ and $\mathbf{b}dv$ at (x_0, y_0), where $\mathbf{a}$ is the derivative of the curve $T(u, v_0)$ at (u_0, v_0) and $\mathbf{b}$ is the derivative of curve $T(u_0, v)$ at (u_0, v_0). That is

$$\mathbf{a} = \left.\frac{dT(u, v_0)}{du}\right|_{u=u_0} = \langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0)\rangle,$$

$$\mathbf{b} = \left.\frac{dT(u_0, v)}{dv}\right|_{v=v_0} = \langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0)\rangle.$$

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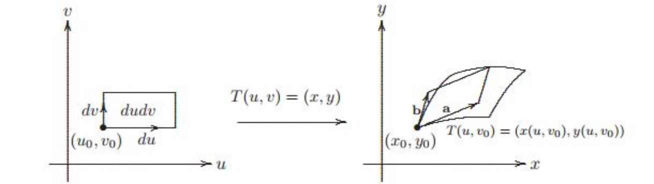


Figure: 26

Therefore, the area element dA in the xy-plane is $dudv$ times the magnitude of

$$\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \rangle \times \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \rangle = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{k}$$

That is $dA = |\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}|dudv$.

Jacobian

The jacobian of the transformation T is given by $x = x(u, v), y = y(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Therefore,

$$dA = |\frac{\partial(x, y)}{\partial(u, v)}|dudv$$

Jacobian

Theorem

Let $T(u,u)$ be a bijective C^{-1}-transformation whose Jacobian is nonzero except possibly at a finite number of points.Suppose T maps a region S in the uv-plane onto a region R of the xy-plane.Suppose f is continuous on R.Then

$$\int\int_R f(x,y)dA = \int\int_S f(x(u,v),y(u,v))|\frac{\partial(x,y)}{\partial(u,v)}|dudv.$$

Jacobian

Examples

Find the Jacobian of the transformation from polar coordinates to Cartesian coordinates.

Solution. $x = r\cos\theta$ and $y = r\sin\theta$. Thus,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Therefore,

$$\int\int_R f(x,y)dA = \int\int_S f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Jacobian

Examples

Use the change of variables $x = u^2 - v^2, y = 2uv$ to evaluate the integral where R is the region by the parabolas $y = 4 - 4x$ and $y^2 = 4 + 4x$,and the x-axis.

Solution. First,let's compute the Jacobian of T.

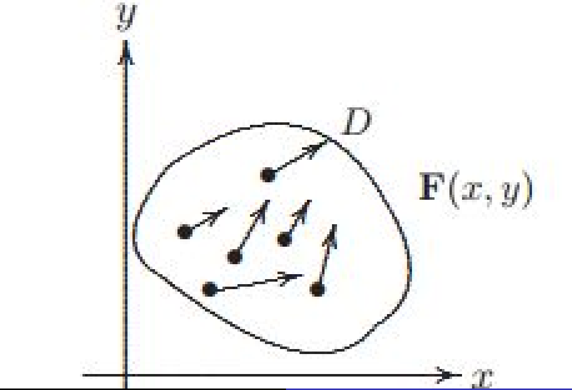
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2.$$

Therefore,

$$\int\int_R ydA = \int\int_S (2uv)|\frac{\partial(x, y)}{\partial(u, v)} dudv = \int_0^1 \int_0^1 (2uv)4(u^2+v^2)dudv = 2$$

Vector Fields

Let $D \subseteq \Re^2$. A vector field on D is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two dimensional vector $\mathbf{F}(x, y)$.



Vector Fields

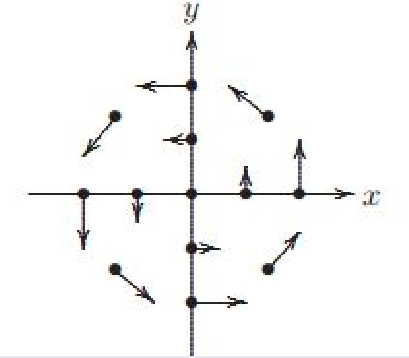
We may write $\mathbf{F}(x, y)$ in terms of its component functions.That is; $F(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$,or simply $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

Definition

Let $E \subseteq \Re^3$.A vector field on E is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three dimensional vector $\mathbf{F}(x, y, z)$.That is $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$

Examples

A vector field on $\Re^2$ is defined by $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ Show that $\mathbf{F}(x, y)$ is always perpendicular to the position vector of the point (x, y).



The above figure shows the vector field **F**.Note that $\langle x, y\rangle.\langle -y, x\rangle = 0$. Also $|\mathbf{F}(x, y)| = \sqrt{x^2 + y^2}$.The vector assigned by **F** to the origin is the zero vector.

Gradient Fields

If $f : \Re^2 \to \Re$ is a differentiable function,then ∇f is a vector field on $\Re^3$ and it is called the gradient vector field of f.

Examples

Find the gradient vector field of $f(x, y) = x^2y - y^3$.

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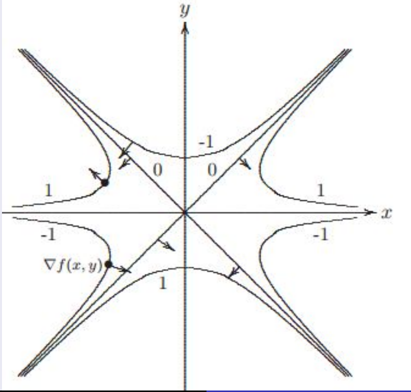
Solution. $\nabla f(x, y) = 2xy\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. The gradient field and the contours of f are drawn on the diagram in figure 24 ;



Conservative Vector Field

A vector field **F** is called a conservative vector field if it is the gradient of some scalar function, that is there exists a differentiable function f such that $\mathbf{F} = \nabla f$.In this situation,f is called a potential function for **F**.
For example,$F(x, y) = 2xy\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative since it has a potential function $f(x, y) = x^2y - y^3$.
Not all vector fields are conservative , but such fields do arise frequently in physics.For instance,the gravitational field given by

$$\mathbf{F} = \frac{-mMGx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\mathbf{k}$$

cont'd

is conservative because it is the gradient of the gravitational potential function.

$$f(x,y,z) = \frac{-mMG}{\sqrt{(x^2+y^2+z^2)}}$$

,where G is gravitational constant, M and m are the masses of two objects.Think of the mass M at the origin that creates the field and f is the potential energy attained by the mass m situated at (x,y,y).In later sections, we will derive conditions when a vector fields is conservative.