Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT

MAS212 Linear Algebra I

Monday 8 May 2006, 2:30 pm – 4:30 pm

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators are **not** *permitted in this examination. The unauthorised use of a calculator constitutes an examination offence. Show your working.*

If not specified then assume that the field of scalars is the field of rational numbers $\mathbb{Q}$.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A

This section carries 56 marks and each question carries 7 marks. You should attempt ALL 8 questions. ***Do* not *begin each answer in this section on a fresh page. Write the number of the question in the* left *margin.***

A1. (a) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, compute A^2, B^2, AB and BA.

(b) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, compute $AB - BA$.

A2. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Compute M^n for $n = 2, 3, 4$. Find a function $c(n)$ such that $M^n = c(n)M$ for all $n \in \mathbb{Z}, n \geq 1$. Also deduce the value of M^n for $n \in \mathbb{Z}, n \geq 2$ if M is regarded as a matrix over the Boolean field $\mathbb{F}_2$. [You are not required to prove any of your results.]

A3. Consider the following determinants:

$$D_1 = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 4 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 3 & 2 & 5 & -1 \end{vmatrix}.$$

Evaluate D_1 by direct expansion. Evaluate D_2 by relating it to D_1. Evaluate D_3 by using standard properties of determinants. In each case, explain your method briefly.

A4. Let V be a vector space over a field $\mathbb{K}$ and let $v_1, v_2, \ldots, v_n$ be vectors in V that span U, a vector subspace of V. Write down a formula for the general vector $u \in U$.

Let $(1,0,1), (0,1,0) \in \mathbb{R}^3$ span a vector subspace $U \subseteq \mathbb{R}^3$.

(a) Write down any vector u, other than $(1,0,1)$ or $(0,1,0)$, that is in U and *prove* that $u \in U$.

(b) Write down any vector $v \in \mathbb{R}^3$ that is *not* in U and *prove* that $v \notin U$.

A5. Define the terms *linear independence*, *spanning set*, *basis* and *dimension* for a finite-dimensional vector space V over a field $\mathbb{K}$.

One of the following statements is false; decide which one, and give a counterexample to it:

(a) the basis for V is unique,

(b) the dimension of V is unique.

A6. Define the term *linear map* between two vector spaces.

State whether or not each of the maps $\alpha, \beta : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and *prove your assertion* when

$$\alpha(x, y, z) = (yz, zx, xy),$$
$$\beta(x, y, z) = (y + z, z + x, x + y).$$

A7. Define the terms *eigenvalue* and *eigenvector* for an $n \times n$ matrix A over $\mathbb{C}$.

Define the term *similar matrices.*

State a condition that ensures that A is similar to a diagonal matrix (assuming that A itself is not diagonal).

Explain briefly how to construct the appropriate similarity transformation when the condition you stated holds.

A8. Define the terms *symmetric* and *antisymmetric* applied to a *real* matrix.

If M is any real square matrix, find expressions for a symmetric matrix S and an antisymmetric matrix A such that $M = S + A$. Hence, write down the "symmetric part" of the matrix $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Prove that $x^T A x = 0$ for all column vectors x if A is any (conformable) real *antisymmetric* matrix.

SECTION B

This section carries 44 marks and each question carries 22 marks. You may attempt all 4 questions but, except for the award of a bare pass, only marks for the best 2 questions will be counted. ***Begin each answer in this section on a fresh page. Write the number of the question at the top of each page.***

B1. A vector space V over a field $\mathbb{K}$ is a nonempty set of vectors $v \in V$ together with an addition operation $(v, w) \mapsto v + w$ for $v, w \in V$ and a scalar multiplication operation $(k, v) \mapsto kv$ for $k \in \mathbb{K}$ and $v \in V$.

(a) [5 marks] State the axioms relating to addition that the vector space V must satisfy.

(b) [5 marks] Define the vector space $\mathbb{K}^n$ (including the operations of addition and scalar multiplication).

(c) [3 marks] Let $U \subseteq V$ inherit the operations defined on V. Give a minimal set of explicitly testable conditions that ensures that U is also a vector space.

(d) [9 marks] Using your set of conditions, state whether U is a vector space and *prove your assertion* when

(i) $U = \{(w, x, y, z) \mid w = y, x = z\} \subseteq \mathbb{R}^4$,

(ii) $U = \{(w, x, y, z) \mid w = y^2, x = z^2\} \subseteq \mathbb{R}^4$.

B2. Let $\alpha : U \to V$ be a linear map between vector spaces U and V over the same field $\mathbb{K}$.

(a) [2 marks] Define the *kernel*, $\ker(\alpha)$, and *image*, $\operatorname{im}(\alpha)$, of α.

(b) [8 marks] Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\alpha(x, y, z) = (x + y, 2x + 5y + z, 3y + z).$$

Construct a basis for $\ker(\alpha)$ and *prove that it is a basis.* Then extend it, as necessary, on the left to give an *ordered* basis $\mathcal{B}$ for the domain of α.

Use the *image* of this ordered basis to construct a corresponding ordered basis for $\operatorname{im}(\alpha)$ and *prove that it is a basis.* Then extend it, as necessary, on the right to give an *ordered* basis $\mathcal{C}$ for the codomain of α.

(c) [2 marks] Construct the matrix A' of α with respect to the ordered bases $\mathcal{B}, \mathcal{C}$ and the matrix A of α with respect to the standard ordered bases.

(d) [10 marks] Construct matrices P, Q such that $A' = PAQ$, stating clearly how P, Q are defined.

B3. (a) [6 marks] Define the *elementary row operations* on a matrix and state the effect of each elementary row operation on the determinant of a square matrix.

(b) [3 marks] Explain, with justification, how to use elementary row operations to compute the inverse, if it exists, of a square matrix A over a field.

(c) [11 marks] Use elementary row operations to compute the inverse of the matrix

$$A = \begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & -2 \\ 4 & 2 & -1 \end{pmatrix}$$

over $\mathbb{Q}$. You may combine elementary row operations if you wish. State clearly what each row operation you use is, using the notation R_i to denote the i^{th} row of the matrix being operated on.

(d) [2 marks] Use the effect of each row operation in the previous computation of A^{-1} to deduce the value of $\det(A)$.

B4. (a) [4 marks] Define the terms *orthogonal* and *orthonormal* applied to a set of vectors in a vector space on which an inner product is defined.

(b) [4 marks] State the relationship between an *orthogonal matrix* and its transpose. Prove that the set of columns of an orthogonal matrix forms an orthonormal set of vectors.

(c) [14 marks]

(i) Show that $(1, 0, 1, 0)$ and $(1, 0, -1, 0)$ are eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

and find the corresponding eigenvalues.

(ii) Find the two other eigenvalues and corresponding eigenvectors of A.

(iii) Find matrices P, Q, Λ such that $PQ = I$ and $PAQ = \Lambda$ is diagonal.

 [End of examination paper]