HEAT AND MASS TRANSFER

Module 1: Introduction (2)

Units, definitions, Basic modes of Heat transfer, Thermal conductivity for various types of materials, convection heat transfer co-efficient, Stefan Boltzman's law of Thermal radiation.

Module 2: One Dimensional Steady State Heat Conduction (7)

Thermal conductivity and other relevant properties, Heat diffusion equation in Cartesian co-ordinates, boundary and initial conditions. One dimensional, steady state heat conduction without and with heat generation through plane slabs, cylinders and spheres, Concept of thermal resistance, Electrical analogy. Heat transfer through composite slabs, cylinders and spheres, contact resistance. Critical thickness of insulation for cylinder and sphere. Steady state heat conduction through fins of uniform cross section, fin effectiveness and fin efficiency.

Module 3: Multi-dimensional Steady State Heat Conduction (4)

Two-dimensional steady state conduction, analytical solution, conduction shape factor, finite difference and finite volume methods

Module 4: Unsteady State Heat Conduction (4)

Transient conduction in solids with negligible internal temperature gradients (lumped parameter), Biot number and Fourier number. One-dimensional transient conduction in slab and radial systems: exact and approximate solutions. Finite difference methods: explicit and implicit formulations.

Module 5: Convection (8)

Flow over a body, velocity and thermal boundary layers, drag-co-efficient and heat transfer coefficient. Flow inside a duct; hydrodynamics and thermal entry lengths; fully developed and developing flow. Use of various correlations in forced convection heat transfer, flow over a flat plate, and flow across a single cylinder and tube bundles. Free convection heat transfer from vertical surface and vertical cylinder, horizontal surface and horizontal cylinders.

Module 6: Heat Exchangers (3)

Heat exchanger types, flow arrangements, overall heat transfer coefficient, fouling factor, LMTD for parallel flow and counter flow heat exchangers. Effectiveness-NTU method, expression for effectiveness of a parallel flows and counter flow heat exchangers. Multi-pass and cross flow heat exchangers.

Module 7: Boiling and Condensation (3)

Different regimes of boiling, mechanism of condensation, Nusselt’s theory of film condensation on a vertical surface, use of correlations in solving film wise condensation on plane surfaces, horizontal tubes and tube banks.

Module 8: Radiation Heat Transfer (4)

Definitions, concept of a black body, Kirchoff's law, Lambert's Cosine Law, Stefan-Boltzman's law, Plank's distribution law, Wein's displacement law, configuration factor. Radiation heat exchange between two parallel plates, radiation shielding, radiation heat exchange in an enclosure.

Module 9: Mass Transfer (2)

Fick's law of diffusion, Mass transfer coefficient, Evaporation of water into air, Schmidt number, Sherwood number.

Lecture Plan

Module	Learning Units	Hours per topic	Total Hours
1. Introduction	1. Modes of heat transfer	1	2
	2. Rate equations: conduction, convection and radiation	1	
2. One Dimensional Steady State Heat Conduction	3. Heat diffusion equation, boundary and initial conditions	1	7
	4. One dimensional steady state conduction	3	
	5. Conduction with thermal energy generation	1	
	6. Extended surface heat transfer	2	
3. Multi-dimensional Steady State Heat Conduction	7. Two-dimensional steady state conduction: analytical solutions.	1	4
	8. Conduction shape factor	1	
	9. Finite difference and finite volume methods	2	
4. Unsteady State Heat Conduction	10. Transient conduction: lumped capacity, Biot and Fourier numbers	1	4
	11. One-dimensional transient conduction in slab and radial systems: exact and approximate solutions.	2	
	12. Finite difference methods: explicit and implicit formulations.	1	
5. Convection	13. Flow over a body, velocity and thermal boundary layers, drag-co-efficient and heat transfer coefficient.	3	8
	14. Use of various correlations in forced convection heat transfer, flow over a flat plate, and flow across a single cylinder and tube bundles.	1	
	15. Flow inside a duct; hydrodynamics and thermal entry lengths; fully developed and developing flow.	2	
	16. Free convection heat transfer from vertical surface and vertical cylinder, horizontal surface and horizontal cylinders.	2	
6. Heat Exchangers	17. Heat exchanger types, flow arrangements, overall heat transfer coefficient, fouling factor, LMTD for parallel flow and counter flow heat exchangers.	2	3
	18. Effectiveness-NTU method, expression for effectiveness of a parallel flows and counter flow heat exchangers. Multi-pass and cross flow heat exchangers.	1	
7. Boiling and Condensation	19. Different regimes of boiling	1	3
	20. Mechanism of condensation, Nusselt's theory of film condensation on a vertical surface, use of correlations in solving film wise condensation on plane surfaces, horizontal tubes and tube banks.	2	

8. Radiation Heat Transfer	21. Definitions, concept of a black body, Kirchoff's law, Lambert's Cosine Law, Stefan-Boltzman's law, Plank's distribution law, Wein's displacement law, configuration factor.	2	4
	22. Radiation heat exchange between two parallel plates, radiation shielding,	1	
	23. Radiation heat exchange in an enclosure	1	
9. Mass Transfer	24. Fick's law of diffusion, Mass transfer coefficient	1	2
	25. Evaporation of water into air, Schmidt number, Sherwood number.	1	

Module 1: Learning objectives

- Overview: Although much of the material of this module will be discussed in greater detail, the objective of this module is to give you a reasonable overview of heat transfer.
- Heat transfer modes: You should be aware of the several modes of transfer modes of transfer and their physical origins.
- Physical insight: Given a physical situation, you should be able to perceive the relevant transport phenomena. The importance of developing this insight must not be underestimated. You will be investing a lot of time to acquire the tools needed to calculate heat transfer phenomena. However, before you can begin to use these tools to solve practical problems, you must have the intuition to determine what is happening physically. In short, you must be able to look at a problem and identify the pertinent transport phenomenon. The example and problems at the end of this module should help you to begin developing this intuition.
- Rate equations and conservation laws: You should also appreciate the significance of the rate equations and feel comfortable in using them to compute transport rates. You must also recognize the importance of the conservation laws and the need to carefully identify control volumes. With the rate equations, the conservation laws may be used to solve numerous heat transfer problems.

MODULE I

BASICS OF HEAT TRANSFER

1.1 Difference between heat and temperature

In describing heat transfer problems, we often make the mistake of interchangeably using the terms heat and temperature. Actually, there is a distinct difference between the two. *Temperature* is a measure of the amount of energy possessed by the molecules of a substance. It is a relative measure of how hot or cold a substance is and can be used to predict the direction of heat transfer. The usual symbol for temperature is *T*. The scales for measuring temperature in SI units are the Celsius and Kelvin temperature scales. On the other hand, *heat* is energy in transit. The transfer of energy as heat occurs at the molecular level as a result of a temperature difference. The usual symbol for heat is Q. Common units for measuring heat are the Joule and calorie in the SI system.

What is Heat Transfer?
"*Energy in transit due to temperature difference.*"

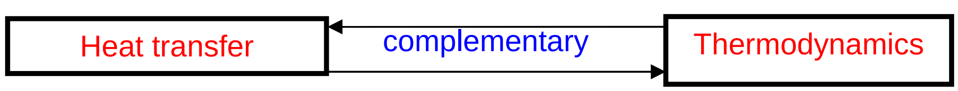
1.2 Difference between thermodynamics and heat transfer

Thermodynamics tells us:

- how much heat is transferred (δQ)
- how much work is done (δW)
- final state of the system

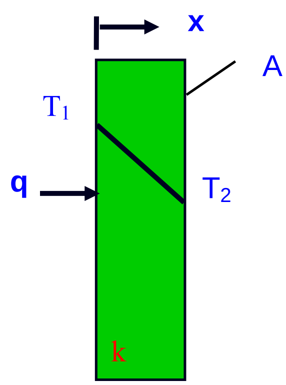
Heat transfer tells us:

- how (with what **modes**) δQ is transferred
- at what **rate** δQ is transferred
- temperature distribution inside the body



1.3 Modes of Heat Transfer

- **Conduction:** An energy transfer across a system boundary due to a temperature difference <u>by the mechanism of inter-molecular interactions</u>. Conduction needs matter and does not require any bulk motion of matter.



Conduction rate equation is described by the Fourier Law:

$$\vec{q} = -kA\nabla T$$

where: q = heat flow vector, (W)
k = thermal conductivity, a thermodynamic property of the material. (W/m K)
A = Cross sectional area in direction of heat flow. (m^2)
∇T = Gradient of temperature (K/m)
= $\partial T/\partial x\ \mathbf{i} + \partial T/\partial y\ \mathbf{j} + \partial T/\partial z\ \mathbf{k}$

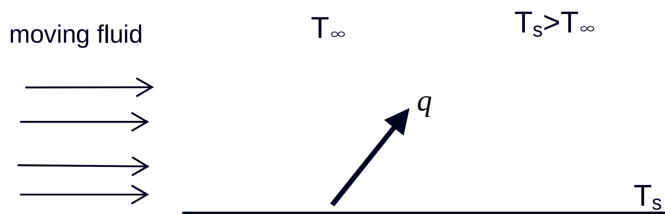
Note: Since this is a vector equation, it is often convenient to work with one component of the vector. For example, in the x direction:

$$q_x = - k\ A_x\ dT/dx$$

In circular coordinates it may convenient to work in the radial direction:

$$q_r = - k\ A_r\ dT/dr$$

- **Convection:** An energy transfer across a system boundary due to a temperature difference by the combined mechanisms of intermolecular interactions and bulk transport. Convection needs fluid matter.



Newton's Law of Cooling:

$$q = h\ A_s\ \Delta T$$

where: q = heat flow from surface, a scalar, (W)
h = heat transfer coefficient (which is not a thermodynamic property of the material, but may depend on geometry of surface, flow characteristics, thermodynamic properties of the fluid, etc. (W/m^2 K)
A_s = Surface area from which convection is occurring. (m^2)
$\Delta T = T_S - T_\infty$ = Temperature Difference between surface and coolant. (K)

Convection → Free or natural convection (induced by buoyancy forces)
Convection → Forced convection (induced by external means)

May occur with phase change (boiling, condensation)

Table 1. Typical values of h (W/m^2K)

Free convection	gases: 2 - 25 liquid: 50 – 100
Forced convection	gases: 25 - 250 liquid: 50 - 20,000
Boiling/Condensation	2500 -100,000

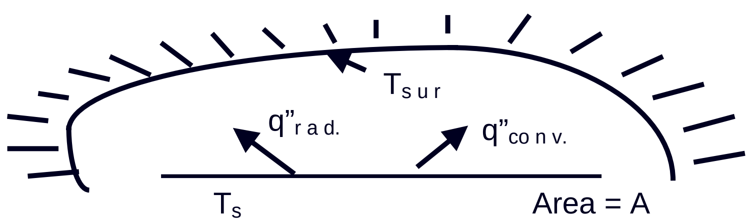
- **Radiation:** Radiation heat transfer involves the transfer of heat by electromagnetic radiation that arises due to the temperature of the body. Radiation does not need matter.

Emissive power of a surface:

$$E=\sigma\varepsilon T_s^4 \ (W/\ m^2)$$

where: ε = emissivity, which is a surface property (ε = 1 is black body)
σ = Steffan Boltzman constant = 5.67×10^{-8} $W/m^2\ K^4$.
T_s = Absolute temperature of the surface (K)

The above equation is derived from Stefan Boltzman law, which describes a gross heat emission rather than heat transfer. The expression for the actual radiation heat transfer rate between surfaces having arbitrary orientations can be quite complex, and will be dealt with in Module 9. However, the rate of radiation heat exchange between a small surface and a large surrounding is given by the following expression:



$$q = \varepsilon \cdot \sigma \cdot A \cdot (T_s^4 - T_{sur}^4)$$

where: ε = Surface Emissivity
A= Surface Area
T_s = Absolute temperature of surface. (K)
T_{sur} = Absolute temperature of surroundings.(K)

1.4 Thermal Conductivity, k

As noted previously, thermal conductivity is a thermodynamic property of a material. From the State Postulate given in thermodynamics, it may be recalled that thermodynamic properties of pure substances are functions of two independent thermodynamic intensive properties, say temperature and pressure. Thermal conductivity of real gases is largely independent of pressure and may be considered a function of temperature alone. For solids and liquids, properties are largely independent of pressure and depend on temperature alone.

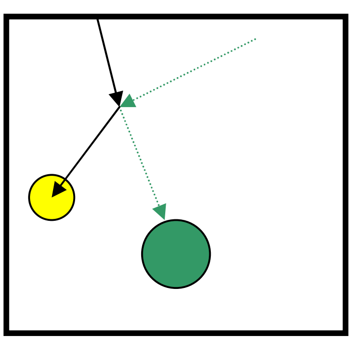
$$k = k\,(T)$$

Table 2 gives the values of thermal conductivity for a variety of materials.

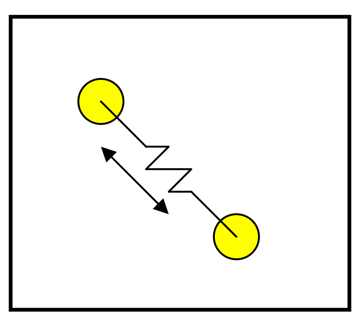
Table 2. Thermal Conductivities of Selected Materials at Room Temperature.

Material	Thermal Conductivity, W/m K
Copper	401
Silver	429
Gold	317
Aluminum	237
Steel	60.5
Limestone	2.15
Bakelite	1.4
Water	0.613
Air	0.0263

It is important that the student gain a basic perspective of the magnitude of thermal conductivity for various materials. The background for this comes from the introductory Chemistry courses. Molecules of various materials gain energy through various mechanisms. Gases exhibit energy through the kinetic energy of the molecule. Energy is gained or lost through collusions of gaseous molecules as they travel through the medium.



Kinetic energy transfer between gaseous molecules.



Lattice vibration may be transferred between molecules as nuclei attract/repel each other.

Solids, being are much more stationary, cannot effectively transfer energy through these same mechanisms. Instead, solids may exhibit energy through vibration or rotation of the nucleus.

Another important mechanism in which materials maintain energy is by shifting electrons into higher orbital rings. In the case of electrical conductors the electrons are weakly bonded to the molecule and can drift from one molecule to another transporting their energy with them. This is an especially effective transport mechanism, so that materials which are excellent electrical conductors are excellent thermal conductors.

HEAT TRANSFER

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What is Heat Transfer?

"*Energy in transit due to temperature difference*."

Thermodynamics tells us:

- how much heat is transferred (δQ)
- how much work is done (δW)
- final state of the system

Heat transfer tells us:

- how (with what **modes**) δQ is transferred
- at what **rate** δQ is transferred
- temperature distribution inside the body

Heat transfer $\rightleftarrows$ Thermodynamics

complementary

MODES:

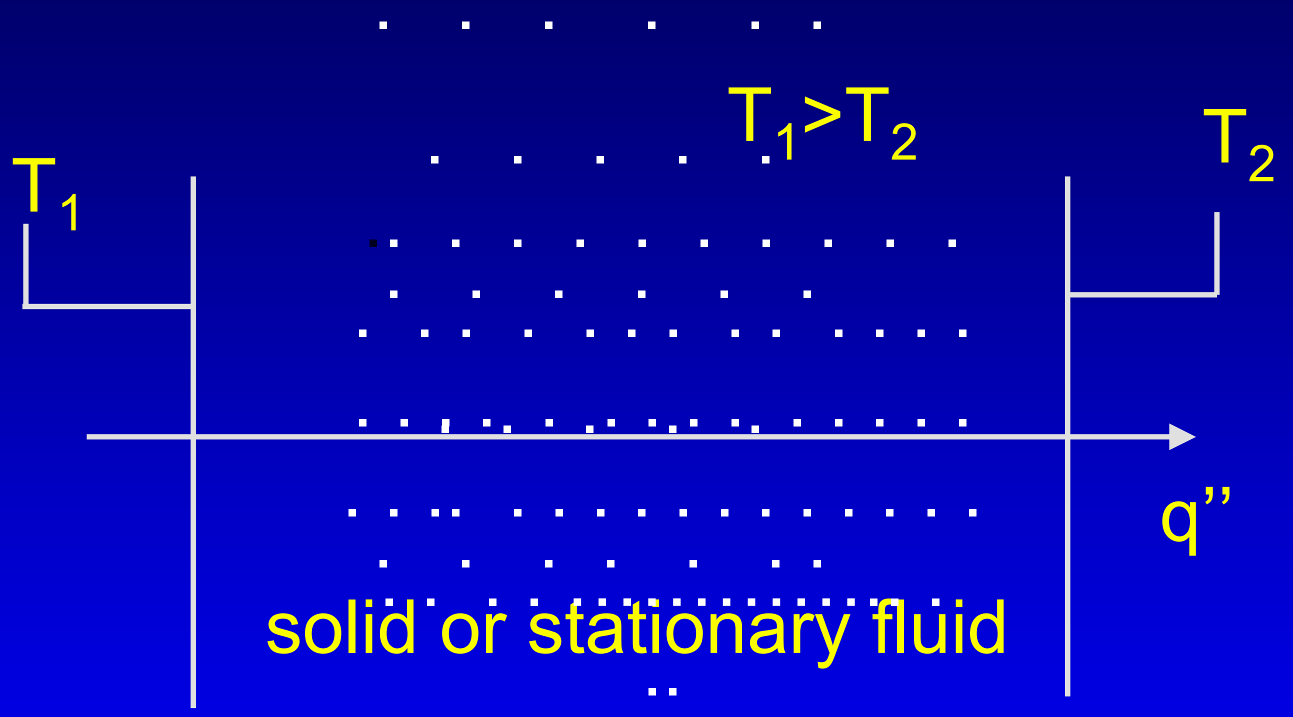
- ✓ **Conduction**
  - **- needs matter**
  - **- molecular phenomenon (diffusion process)**
  - **- without bulk motion of matter**
- ✓ **Convection**
  - **- heat carried away by bulk motion of fluid**
  - **- needs fluid matter**
- ✓ **Radiation**
  - **- does not needs matter**
  - **- transmission of energy by electromagnetic waves**

APPLICATIONS OF HEAT TRANSFER

- ✓ Energy production and conversion
  - steam power plant, solar energy conversion etc.
- ✓ Refrigeration and air-conditioning
- ✓ Domestic applications
  - ovens, stoves, toaster
- ✓Cooling of electronic equipment
- ✓ Manufacturing / materials processing
  - welding,casting, soldering, laser machining
- ✓Automobiles / aircraft design
- ✓ Nature (weather, climate etc)

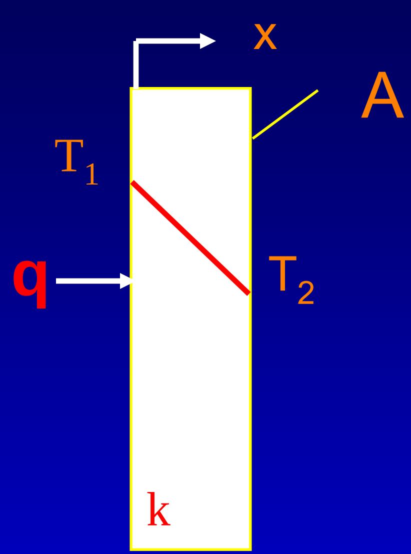
Conduction:

(needs medium, temperature gradient)



RATE:

q(W) or (J/s) (heat flow per unit time)



Rate equations (1D conduction):

- Differential Form

  $q = - k A \, dT/dx$, W

  k = Thermal Conductivity, W/mK

  A = Cross-sectional Area, m^2

  T = Temperature, K or ^oC

  x = Heat flow path, m

- Difference Form

  $q = k A (T_1 - T_2) / (x_1 - x_2)$

Heat flux: $q'' = q / A = - k \, dT/dx$ (W/m^2)

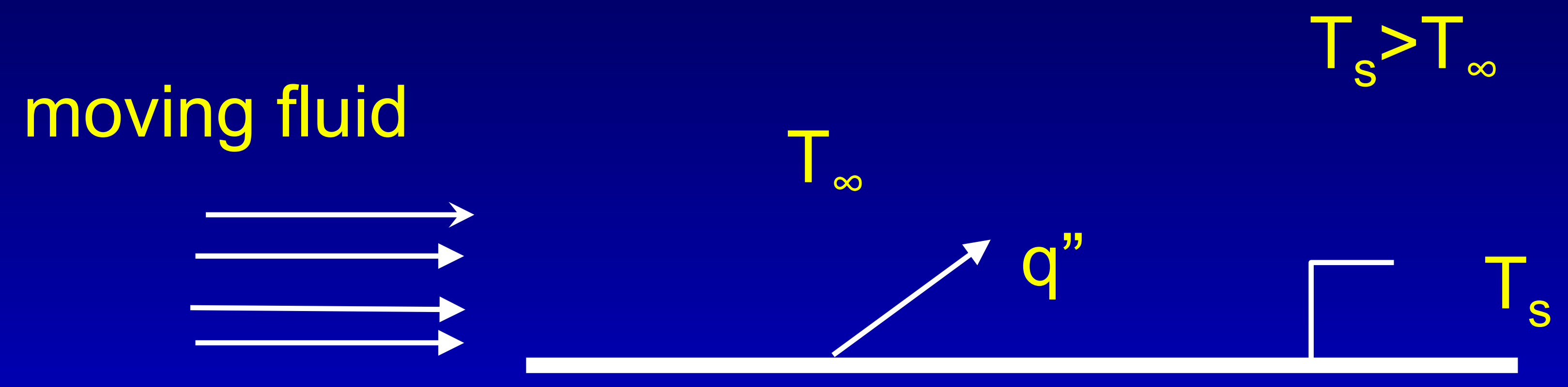
(negative sign denotes heat transfer in the direction of decreasing temperature)

Example:

The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 W/mK. Measurements made during steady state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall which is 0.5 m by 3 m on a side ?

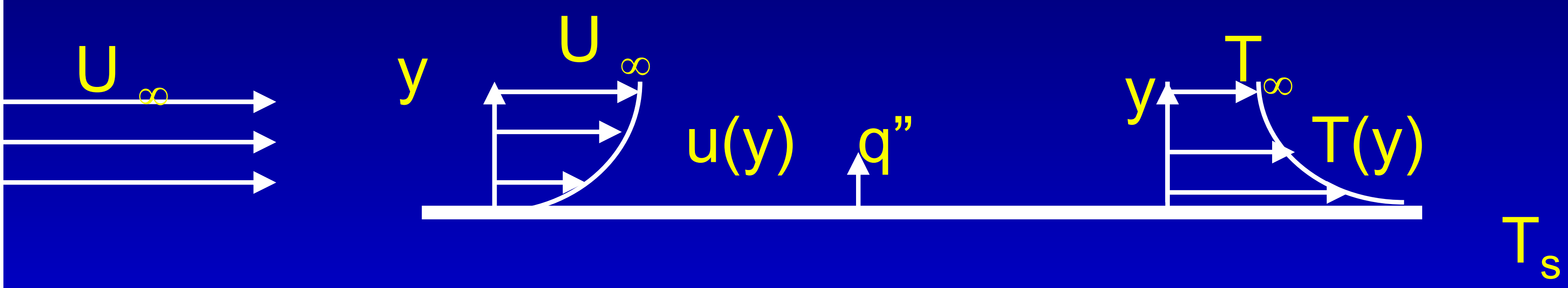
Solution: To be worked out in class

Convection



- Energy transferred by diffusion + bulk motion of fluid)

Rate equation (convection)



Heat transfer rate $q = hA(T_s - T_\infty)$ W

Heat flux $q'' = h(T_s - T_\infty)$ W / m^2

h=heat transfer co-efficient (W /m^2K)

(not a property) depends on geometry ,nature of flow, thermodynamics properties etc.

Convection

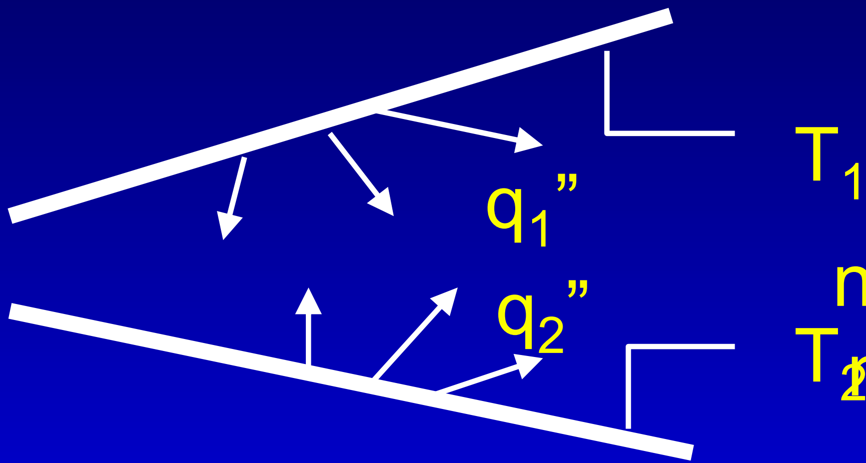
- Free or natural convection (induced by buoyancy forces
- Forced convection (induced by external means

May occur with phase change (boiling, condensation)

Typical values of h (W/m^2K)

Free convection	gases: 2 - 25
	liquid: 50 - 100
Forced convection	gases: 25 - 250
	liquid: 50 - 20,000
Boiling/Condensation	2500 -100,000

Radiation:



no medium necessary

RATE:

q(W) or (J/s) Heat flow per unit time.

Flux : q" (W/m^2)

Rate equation (radiation)

RADIATION:

Heat Transfer by electro-magnetic waves or photons(no medium required.)

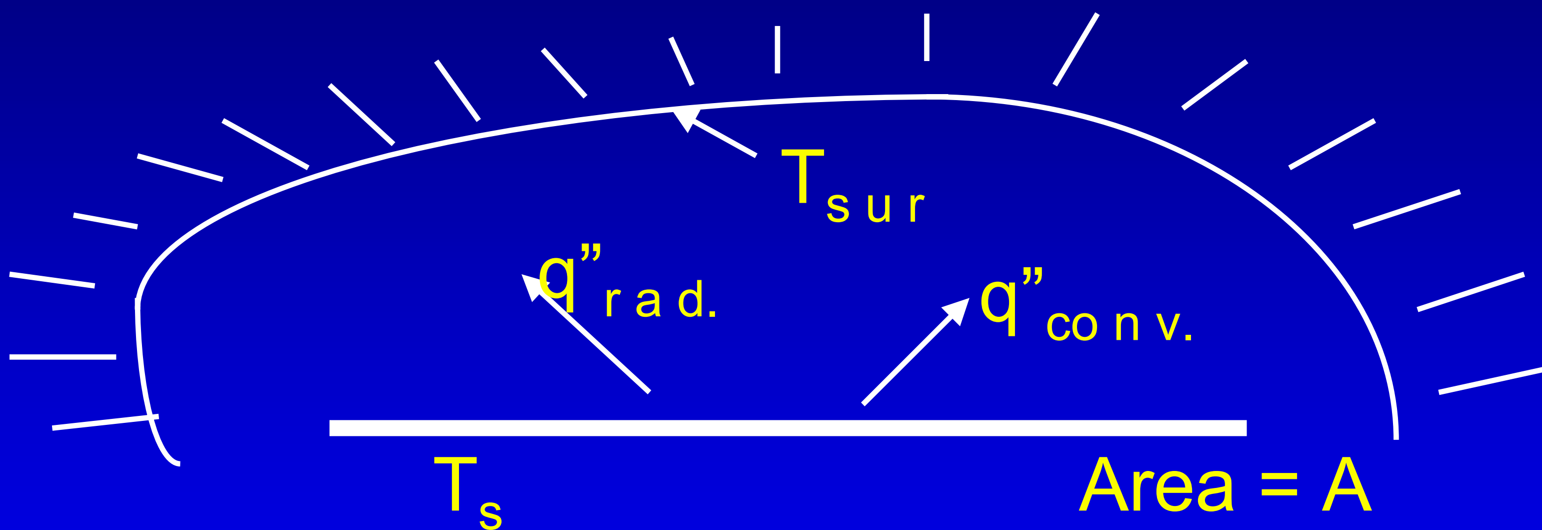
Emissive power of a surface (energy released per unit area):

$$E=\varepsilon\sigma T_s^4 \ (W/\ m^2)$$

ε= emissivity (property)

σ=Stefan-Boltzmann constant

Rate equations(Contd.….)



Radiation exchange between a large surface and surrounding

$$Q''_{rad} = \varepsilon\sigma(T_s^4 - T_{sur}^4)\ W/m^2$$

Example:

An uninsulated steam pipe passes through a room in which the air and walls are at 25 °C. The outside diameter of pipe is 70 mm, and its surface temperature and emissivity are 200 °C and 0.8, respectively. If the coefficient associated with free convection heat transfer from the surface to the air is 5 W/m^2K, what is the rate of heat loss from the surface per unit length of pipe ?

Solution: To be worked out in class

Module 1: Worked out problems

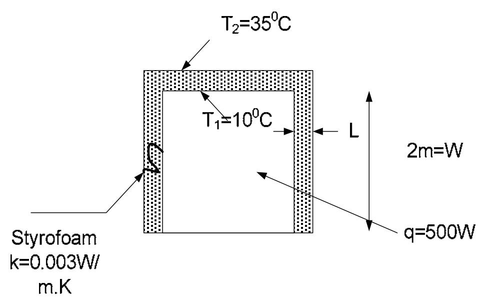
Problem 1:
A freezer compartment consists of a cubical cavity that is 2 m on a side. Assume the bottom to be perfectly insulated. What is the minimum thickness of Styrofoam insulation (k=0.030W/m.K) which must be applied to the top and side walls to ensure a heat load less than 500 W, when the inner and outer surfaces are -10 ºC and 35^{0}C?

Solution:

Known: Dimensions of freezer component, inner and outer surfaces temperatures.

Find: Thickness of Styrofoam insulation needed to maintain heat load below prescribed value.

Schematic:



Assumptions: (1) perfectly insulted bottom, (2) one-dimensional conduction through five walls of areas A=4m^2, (3) steady-state conditions

Analysis: Using Fourier's law, the heat rate is given by

$$q = q^{"}.A = k\frac{\Delta T}{L}A_{total}$$

Solving for L and recognizing that A_{total} =5*W^2

$$L = \frac{5k\Delta TW^2}{q}$$

$$L = \frac{5*0.03W/m.k*45^0C*4m^2}{500W}$$

$$L = 0.054m = 54mm$$

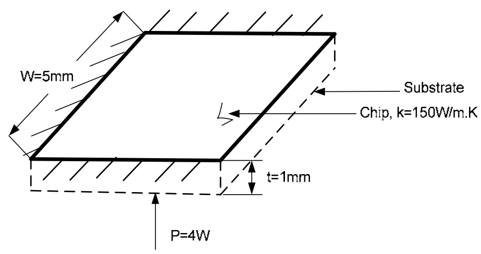
<u>Comments:</u> The corners will cause local departures from one–dimensional conduction and, for a prescribed value of L, a slightly larger heat loss.

Problem 2:
A square silicon chip (k=150W/m.k) is of width W=5mm on a side and of thickness t=1mm. the chip is mounted in a substrate such that its side and back surfaces are insulated, while the front surface is exposed to a coolant. If 4W are being dissipated in circuits mounted to the back surface of the chip, what is the steady-state temperature difference between back and front surfaces?

Known: Dimensions and thermal conductivity of a chip. Power dissipated on one surface.

Find: temperature drop across the chip

Schematic:



Assumptions: (1) steady-state conditions, (2) constant properties, (3) uniform dissipation, (4) negligible heat loss from back and sides, (5) one-dimensional conduction in chip.

Analysis: All of the electrical power dissipated at the back surface of the chip is transferred by conduction through the chip. Hence, Fourier's law,

$$P = q = kA\frac{\Delta T}{t}$$

$$\Delta T = \frac{t.P}{kW^2} = \frac{0.001m * 4W}{150W/m.K(0.005m^2)}$$

$$\Delta T = 1.1^0 C$$

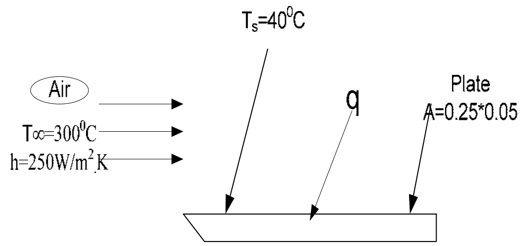
Comments: for fixed P, the temperature drop across the chip decreases with increasing k and W, as well as with decreasing t.

Problem 3:
Air at 300^0C flows over a plate of dimensions 0.50 m, by 0.25 m. if the convection heat transfer coefficient is 250 W/m2.K; determine the heat transfer rate from the air to one side of the plate when the plate is maintained at 40^0C.

Known: air flow over a plate with prescribed air and surface temperature and convection heat transfer coefficient.

Find: heat transfer rate from the air to the plate

Schematic:



Assumptions: (1) temperature is uniform over plate area, (2) heat transfer coefficient is uniform over plate area

Analysis: the heat transfer coefficient rate by convection from the airstreams to the plate can be determined from Newton's law of cooling written in the form,

$$q = q^{"}.A = hA(T_\infty - T_s)$$

where A is the area of the plate. Substituting numerical values,

$$q = 250W/m^2.K * (0.25 * 0.50)m^2 (300 - 40)^0 C$$
$$q = 8125W$$

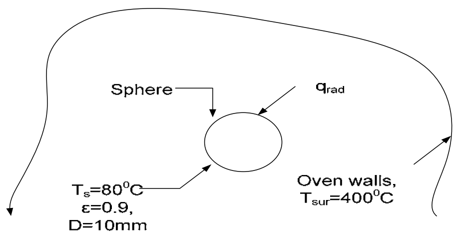
Comments: recognize that Newtown's law of cooling implies a direction for the convection heat transfer rate. Written in the form above, the heat rate is from the air to plate.

Problem 4 :
A water cooled spherical object of diameter 10 mm and emissivity 0.9 is maintained at 4000C. What is the net transfer rate from the oven walls to the object?

Known: spherical object maintained at a prescribed temperature within a oven.

Find: heat transfer rate from the oven walls to the object

Schematic:



Assumptions: (1) oven walls completely surround spherical object, (2) steady-state condition, (3) uniform temperature for areas of sphere and oven walls, (4) oven enclosure is evacuated and large compared to sphere.

Analysis: heat transfer rate will be only due to the radiation mode. The rate equation is

$$q_{rad} = \varepsilon A_s \sigma (T_{sur}^4 - T_s^4)$$

Where As=πD2, the area of the sphere, substituting numerical values,

$$q_{rad} = 0.9 * \pi (10 * 10^{-3})^2 m^2 * 5.67 * 10^{-8} W/m^2.K[(400+273)^4 - (80+273)^4]K^4$$
$$q_{rad} = 3.04W$$

Comments: (1) this rate equation is useful for calculating the net heat exchange between a small object and larger surface completely surrounds the smaller one . this is an essential, restrictive condition.

(2) Recognize that the direction of the net heat exchange depends upon the manner in which T_{sur} and T_s are written.

(3) When performing radiant heat transfer calculations, it is always necessary to have temperatures in Kelvin (K) units.

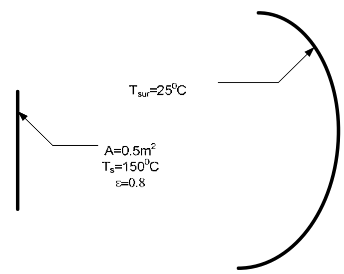
Problem 5:

A surface of area $0.5m^2$, emissivity 0.8 and temperature 150^0C is placed in a large, evacuated chamber whose walls are maintained at $25\,^0C$. What is the rate at which radiation is emitted by the surface? What is the net rate at which radiation is exchanged between the surface and the chamber walls?

Known: Area, emissivity and temperature of a surface placed in a large, evacuated chamber of prescribed temperature.

Find: (a) rate of surface radiation emission, (b) net rate of radiation exchange between the surface and chamber walls.

Schematic:



Assumptions: (1) area of the enclosed surface is much less than that of chamber walls.

Analysis (a) the rate at which radiation is emitted by the surface is emitted

$$q_{emit} = q_{emit.}A = \varepsilon A\sigma Ts^4$$
$$qemit = 0.8(0.5m^2)5.67*10^{-8}\,W/m^2.K^4[(150+273)K]^4$$
$$qemit = 726W$$

(b) The net rate at which radiation is transferred from the surface to the chamber walls is

$$q = \varepsilon A\sigma(Ts4 - Tsurr4)$$
$$q = 0.8(0.5m^2)5.67*10^{-8}\,W/m^2.K^4[(423K)^4 - (298K)^4$$
$$q = 547W$$

Comments: the foregoing result gives the net heat loss from the surface which occurs at the instant the surface is placed in the chamber. The surface would, of course, cool due to this heat loss and its temperature, as well as the heat loss, would decrease with increasing time. Steady-state conditions would eventually be achieved when the temperature of the surface reached that of the surroundings.

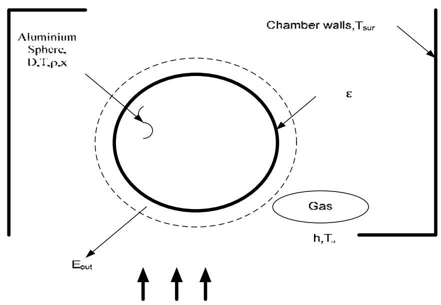
Problem 6:

A solid aluminium sphere of emissivity ε is initially at an elevated temperature and is cooled by placing it in chamber. The walls of the chamber are maintained at a lower temperature and a cold gas is circulated through the chamber. Obtain an equation that could be used to predict the variation of the aluminium temperature with time during the cooling process. Do not attempt to solve.

Known: Initial temperature, diameter and surface emissivity of a solid aluminium sphere placed in a chamber whose walls are maintained at lower temperature. Temperature and convection coefficient associated with gas flow over the sphere.

Find: equation which could be used to determine the aluminium temperature as a function of time during the cooling process.

Schematic:



Assumptions: (1) at any time t, the temperature T of the sphere is uniform, (2) constant properties; (3) chamber walls are large relative to sphere.

Analysis: applying an energy balance at an instant of time to a control volume about the sphere, it follows that

$$\dot{E}_{st} = -\dot{E}_{out}$$

Identifying the heat rates out of the CV due to convection and radiation, the energy balance has the form

$$\frac{d}{dt}(\rho VcT) = -(q_{conv} + q_{rad})$$

$$\frac{dT}{dt} = -\frac{A}{\rho Vc}[h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{surr}^4)]$$

$$\frac{dT}{dt} = \frac{6}{\rho cD}[h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{surr}^4)]$$

Where $A=\pi D^2$, $V=\pi D^3/6$ and $A/V=6/D$ for the sphere.

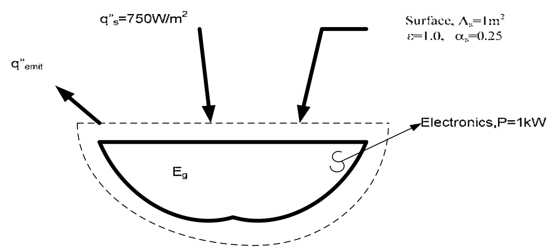
Comments: (1) knowing T=Ti at t =0, the foregoing equation could be solved by numerical integration to obtain T (t). (2) The validity of assuming a uniform sphere temperature depends upon h, D. and the thermal conductivity of the solid (k). The validity of the assumption improves with increasing k and decreasing h and D.

Problem 7: In an orbiting space station, an electronic package is housed in a compartment having surface area $A_s = 1m^2$ which is exposed to space. Under normal operating conditions, the electronics dissipate 1 kW, all of which must be transferred from the exposed surface to space. If the surface emissivity is 1.0 and the surface is not exposed to the sun, what is its steady- state temperature? If the surface is exposed to a solar flux of $750W/m^2$ and its absorptivity to solar radiation is 0.25, what is its steady –state temperature?

Known: surface area of electronic package and power dissipation by the electronics. Surface emissivity and absorptivity to solar radiation. Solar flux.

Find: surface temperature without and with incident solar radiation.

Schematic:



Assumptions: steady state condition

Analysis: applying conservation of energy to a control surface about the compartment, at any instant

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0$$

It follows that, with the solar input,

$$\alpha_S A_s q''_s - A_s q''_{emit} + P = 0$$

$$\alpha_S A_s q''_s - A_s \varepsilon \sigma T_s^4 + P = 0$$

$$T_s = \left(\frac{\alpha_S A_s q''_s + P}{A_s \varepsilon \sigma} \right)^{\frac{1}{4}}$$

In the shade ($q'' = 0$)

$$T_s = \left(\frac{1000W}{1m^2 * 1 * 5.67 * 10^{-8} W/m^2.K^4} \right)^{\frac{1}{4}} = 364K$$

In the sun,

$$T_s = \left(\frac{0.25 * 1m^2 * 750W/m^2 + 1000W}{1m^2 * 1 * 5.67 * 10^{-8} W/m^2.K^4} \right)^{\frac{1}{4}} = 380K$$

Comments: in orbit, the space station would be continuously cycling between shade, and a steady- state condition would not exist.

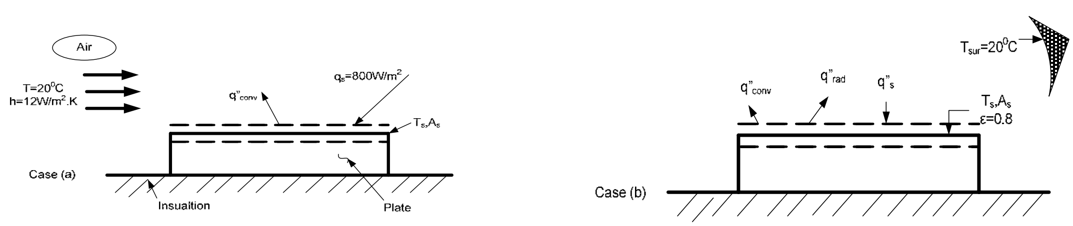
Problem 8: The back side of a metallic plate is perfectly insulated while the front side absorbs a solar radiant flux of 800 W/m^2. The convection coefficient between the plate and the ambient air is 112 w /m^2. K.

(a) Neglecting radiation exchange with the surroundings, calculate the temperature of the plate under steady-state conditions if the ambient air temperature is 20^{0}C.
(b) For the same ambient air temperature, calculate the temperature of the plate if its surface emissivity is 0.8 and the temperature of the surroundings is also20^{0}C.

Known: front surface of insulated plate absorbs solar flux, qs” and experiences for case
(a) Convection process with air at T and for case
(b): the same convection process and radiation exchange with surroundings at T_{sur}

Find: temperature of the plate, T_s, for the two cases.

Schematic:



Assumptions: (1) steady state conditions, (2) no heat loss out backside of plate, (3) surroundings large in comparison plate.

Analysis: (a) apply a surface energy balance, identifying the control surface as shown on the schematic. For an instant of time the conversation requirement is $\dot{E}_{in} - \dot{E}_{out}$ =0. The relevant processes are convection between the plate and the air, q_{conv}, and absorbed solar flux, qs”. Considering the plate to have an area A_s solve for T_s and substitute numerical values to find

$$q''_s.A_s - hA_s(T_s - T_\infty) = 0$$

$$T_s = T_\infty + q''_s / h$$

$$T_s = 20^0C + \frac{800W/m^2}{12W/m^2.K} = 20^0C + 66.7^0C = 87^0C$$

(b) Considering now the radiation exchange between the surface and its surroundings, the surface energy balance has the form $\dot{E}_{in} - \dot{E}_{out}$ =0.

$$q''_s . A_s - qconv - qrad = 0$$

$$qs'' A_s - hA_s (T_s - T_\infty) - \varepsilon A_s (T_s^{\ 4} - T_\infty^4) = 0$$

$$800 \frac{W}{m^2} - 12 \frac{W}{m^2 . K} (T_s - [20 + 273]K) - 0.8 * 5.67 * 10 - 8 \frac{W}{m^2 . K^4} (T_s^4 - [20 + 273]4K^4) = 0$$

$$12T_s + 4.536 * 10 - 8T_s^{\ 4} = 4650.3$$

By trial and error method, find that T_s=338K=65^{0}C.

Comments: note that by considering radiation exchange, Ts decreases as expected. Note the manner in which q_{conv} is formulated using qconv is formulated using Newton's law of cooling: since qconv is shown leaving the control surface, the rate equation must be $h(T_s - T_\infty)$ and not h (T_∞ -T_s).

Module 1: Short questions

1. What is the driving force for (a) heat transfer (b) electric current flow and (c) fluid flow?

2. Which one of the following is not a property of the material ?
   A. thermal conductivity
   B. heat transfer coefficient
   C. emissivity

3. What is the order of magnitude of thermal conductivity for (a) metals (b) solid insulating materials (c) liquids (d) gases?

4. What is the order of magnitude for the convection heat transfer coefficient in free convection? Forced convection? Boiling?

5. When may one expect radiation heat transfer to be important?

6. An ideal gas is heated from 50 ºC to 70 ºC (a) at constant volume and (b) at constant pressure. For which case do you think the energy required will be greater? Why?

7. A person claims that heat cannot be transferred in a vacuum. How do you respond to this claim?

8. Discuss the mechanism of thermal conduction in gases, liquids and solids.

9. Name some good conductors of heat; some poor conductors.

10. Show that heat flow lines must be normal to isotherms in conduction heat transfer. Will it be true for convection heat transfer?

Module 2: Learning objectives

- The primary purpose of this chapter is to improve your understanding of the conduction rate equation (Fourier’s law) and to familiarize you with heat equation. You should know the origin and implication of Fourier’s law, and you should understand the key thermal properties and how they vary for different substances. You should also know the physical meaning of each term appearing in the heat equation.
- The student should understand to what form does the heat equation reduce for simplified conditions, and what kinds of boundary conditions may be used for its solution?
- The student should learn to evaluate the heat flow through a 1-D, SS system with no heat sources for rectangular and cylindrical geometries. Many other geometries exist in nature or in common engineering designs. The student, using a similar development, should be able to develop an appropriate equation to describe systems of arbitrary, simple geometry.
- The student should be comfortable with the use of equivalent thermal circuits and with the expressions for the conduction resistances that pertain to each of the three common geometric.
- Composite thermal resistances for 1-D, Steady state heat transfer with no heat sources placed in parallel or in series may be evaluated in a manner similar to electrical resistances placed in parallel or in series.
- The student should learn to evaluate the heat flow through a 1-D, SS system with no heat sources for rectangular and cylindrical geometries.
- In short, by the end of the module, the student should have a fundamental understanding of the conduction process and its mathematical description.

MODULE 2

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION

2.1 Objectives of conduction analysis:

The primary objective is to determine the temperature field, T(x,y,z,t), in a body (i.e. how temperature varies with position within the body)
T(x,y,z,t) depends on:

- Boundary conditions
- Initial condition
- Material properties (k, cp, ρ)
- Geometry of the body (shape, size)

Why we need T (x, y, z, t)?

- To compute heat flux at any location (using Fourier's eqn.)
- Compute thermal stresses, expansion, deflection due to temp. Etc.
- Design insulation thickness
- Chip temperature calculation
- Heat treatment of metals

2.2 General Conduction Equation

Recognize that heat transfer involves an energy transfer across a system boundary. A logical place to begin studying such process is from Conservation of Energy (1st Law of Thermodynamics) for a closed system:

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} - \dot{W}_{out}$$

The sign convention on work is such that negative work out is positive work in.

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} + \dot{W}_{in}$$

The work in term could describe an electric current flow across the system boundary and through a resistance inside the system. Alternatively it could describe a shaft turning across the system boundary and overcoming friction within the system. The net effect in either case would cause the internal energy of the system to rise. In heat transfer we generalize all such terms as "heat sources".

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} + \dot{Q}_{gen}$$

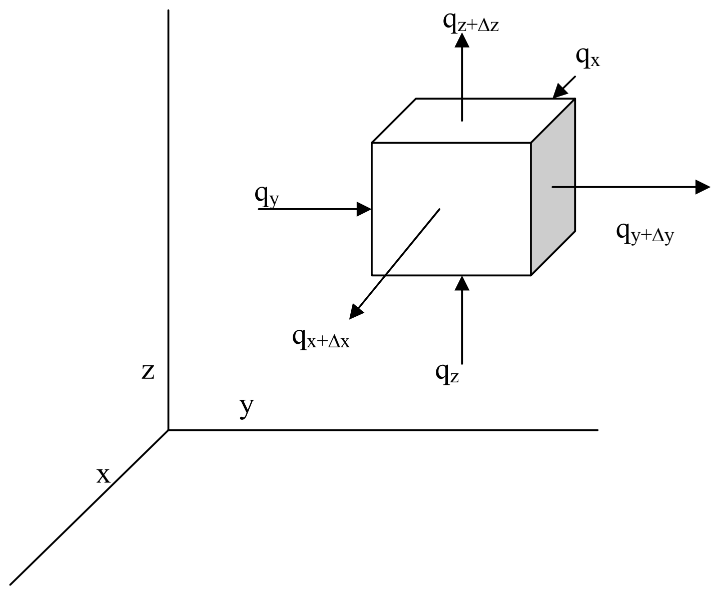
The energy of the system will in general include internal energy, U, potential energy, ½ mgz, or kinetic energy, ½ mΥ^2. In case of heat transfer problems, the latter two terms could often be neglected. In this case,

$$E = U = m \cdot u = m \cdot c_p \cdot \left(T - T_{ref}\right) = \rho \cdot V \cdot c_p \cdot \left(T - T_{ref}\right)$$

where T_{ref} is the reference temperature at which the energy of the system is defined as zero. When we differentiate the above expression with respect to time, the reference temperature, being constant, disappears:

$$\rho \cdot c_p \cdot V \cdot \left.\frac{dT}{dt}\right|_{system} = \dot{Q}_{in} + \dot{Q}_{gen}$$

Consider the differential control element shown below. Heat is assumed to flow through the element in the positive directions as shown by the 6 heat vectors.



In the equation above we substitute the 6 heat inflows/outflows using the appropriate sign:

$$\rho \cdot c_p \cdot \left(\Delta x \cdot \Delta y \cdot \Delta z\right) \cdot \left.\frac{dT}{dt}\right|_{system} = q_x - q_{x+\Delta x} + q_y - q_{y+\Delta y} + q_z - q_{z+\Delta z} + \dot{Q}_{gen}$$

Substitute for each of the conduction terms using the Fourier Law:

$$\begin{aligned}
\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \left.\frac{\partial T}{\partial t}\right|_{system} &= \left\{ -k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} - \left[-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} \right) \cdot \Delta x \right] \right\} \\
&+ \left\{ -k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} - \left[-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} \right) \cdot \Delta y \right] \right\} \\
&+ \left\{ -k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} + \left[-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} \right) \cdot \Delta z \right] \right\} \\
&+ \ddot{q} \cdot \left(\Delta x \cdot \Delta y \cdot \Delta z\right)
\end{aligned}$$

where $\ddot{q}$ is defined as the internal heat generation per unit volume.
The above equation reduces to:

$$\rho \cdot c_p \cdot \left(\Delta x \cdot \Delta y \cdot \Delta z\right) \cdot \left.\frac{dT}{dt}\right|_{system} = \left\{ -\left[\frac{\partial}{\partial x}\left(-k \cdot \left(\Delta y \cdot \Delta z\right) \cdot \frac{\partial T}{\partial x} \right) \right] \cdot \Delta x \right\}$$

$$+\left\{-\left[\frac{\partial}{\partial y}\left(-k\cdot(\Delta x\cdot\Delta z)\cdot\frac{\partial T}{\partial y}\right)\cdot\Delta y\right]\right\}$$

$$+\left\{\left[\frac{\partial}{\partial z}\left(-k\cdot\left(\Delta x\cdot\Delta y\right)\cdot\frac{\partial T}{\partial z}\right)\cdot\Delta z\right]\right\}+\dddot{q}\cdot\left(\Delta x\cdot\Delta y\cdot\Delta z\right)$$

Dividing by the volume (Δx·Δy·Δz),

$$\rho\cdot c_p\cdot\left.\frac{dT}{dt}\right|_{system}=-\frac{\partial}{\partial x}\left(-k\cdot\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(-k\cdot\frac{\partial T}{\partial y}\right)-\frac{\partial}{\partial z}\left(-k\cdot\frac{\partial T}{\partial z}\right)+\dddot{q}$$

which is the **general conduction equation** in three dimensions.

In the case where k is independent of x, y and z then

$$\frac{\rho\cdot c_p}{k}\cdot\left.\frac{dT}{dt}\right|_{system}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k}$$

Define the thermodynamic property, α, the thermal diffusivity:

$$\alpha\equiv\frac{k}{\rho\cdot c_p}$$

Then

$$\frac{1}{\alpha}\cdot\left.\frac{dT}{dt}\right|_{system}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k}$$

or, :

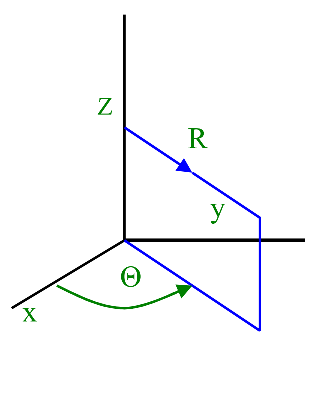
$$\frac{1}{\alpha}\cdot\left.\frac{dT}{dt}\right|_{system}=\nabla^2 T+\frac{\dddot{q}}{k}$$

The vector form of this equation is quite compact and is the most general form. However, we often find it convenient to expand the del-squared term in specific coordinate systems:

Cartesian Coordinates

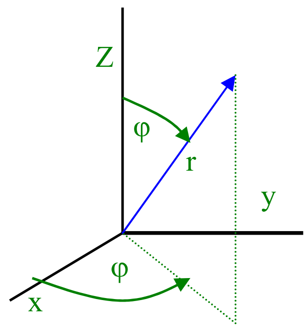
$$\frac{1}{a}\cdot\frac{\partial T}{\partial\tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k}$$

Circular Coordinates



$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k}$$

Spherical Coordinates



$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\cdot\sin^2\theta}\cdot\frac{\partial^2 T}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\cdot\frac{\partial T}{\partial z}\right)+\frac{\dddot{q}}{k}$$

In each equation the dependent variable, T, is a function of 4 independent variables, (x,y,z,τ); (r,θ,z,τ); (r,φ,θ,τ) and is a 2nd order, partial differential equation. The solution of such equations will normally require a numerical solution. For the present, we shall simply look at the simplifications that can be made to the equations to describe specific problems.

- Steady State: Steady state solutions imply that the system conditions are not changing with time. Thus $\partial T/\partial\tau=0$.
- One dimensional: If heat is flowing in only one coordinate direction, then it follows that there is no temperature gradient in the other two directions. Thus the two partials associated with these directions are equal to zero.
- Two dimensional: If heat is flowing in only two coordinate directions, then it follows that there is no temperature gradient in the third direction. Thus the partial derivative associated with this third direction is equal to zero.
- No Sources: If there are no heat sources within the system then the term, $\dddot{q}=0$.

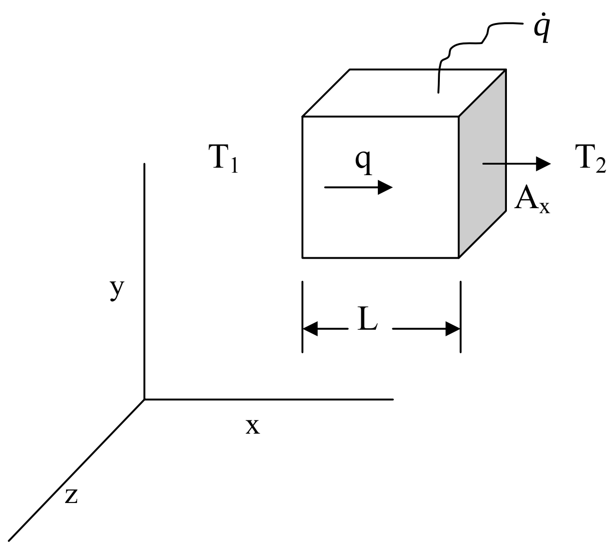
Note that the equation is 2nd order in each coordinate direction so that integration will result in 2 constants of integration. To evaluate these constants two additional equations must be written for each coordinate direction based on the physical conditions of the problem. Such equations are termed “boundary conditions’.

2.3 Boundary and Initial Conditions

- The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

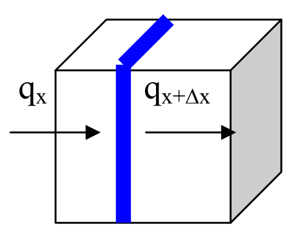
- We have set up a differential equation, with T as the dependent variable. The solution will give us T(x,y,z). Solution depends on boundary conditions (BC) and initial conditions (IC).
- How many BC's and IC's ?
  - Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
    - * 1D problem: 2 BC in x-direction
    - * 2D problem: 2 BC in x-direction, 2 in y-direction
    - * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
  - Heat equation is first order in time. Hence one IC needed.

2.4 Heat Diffusion Equation for a One Dimensional System



Consider the system shown above. The top, bottom, front and back of the cube are insulated, so that heat can be conducted through the cube only in the x direction. The internal heat generation per unit volume is $\dot{q}$ (W/m^3).

Consider the heat flow through an arbitrary differential element of the cube.



From the 1st Law we write for the element:

$$(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st} \tag{2.1}$$

$$q_x - q_{x+\Delta x} + A_x(\Delta x)\dot{q} = \frac{\partial E}{\partial t} \tag{2.2}$$

$$q_x = -kA_x \frac{\partial T}{\partial x} \tag{2.3}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x \tag{2.4}$$

$$-kA\frac{\partial T}{\partial x} + kA\frac{\partial T}{\partial x} + A\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x + A\Delta x\dot{q} = \rho\ Ac\ \Delta x\frac{\partial T}{\partial t} \tag{2.5}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho\, c\,\Delta x\frac{\partial T}{\partial t} \tag{2.6}$$

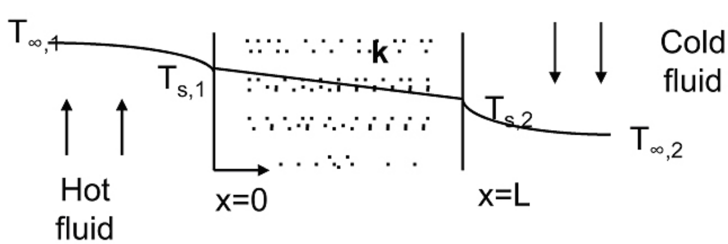
Longitudinal conduction   Internal heat generation   Thermal inertia

If k is a constant, then $$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{\rho\, c}{k}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.7}$$

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

2.5 One Dimensional Steady State Heat Conduction

The plane wall:



The differential equation governing heat diffusion is: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$

With constant k, the above equation may be integrated twice to obtain the general solution:

$$T(x)=C_1x+C_2$$

where C_1 and C_2 are constants of integration. To obtain the constants of integration, we apply the boundary conditions at $x = 0$ and $x = L$, in which case

$$T(0)=T_{s,1} \quad \text{and} \quad T(L)=T_{s,2}$$

Once the constants of integration are substituted into the general equation, the temperature distribution is obtained:

$$T(x)=(T_{s,2}-T_{s,1})\frac{x}{L}+T_{s,1}$$

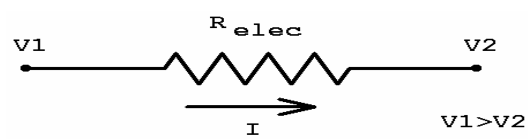
The heat flow rate across the wall is given by:

$$q_x=-kA\frac{dT}{dx}=\frac{kA}{L}\left(T_{s,1}-T_{s,2}\right)=\frac{T_{s,1}-T_{s,2}}{L/kA}$$

Thermal resistance (electrical analogy):

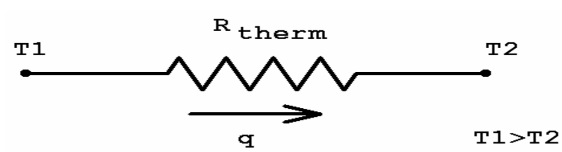
Physical systems are said to be analogous if that obey the same mathematical equation. The above relations can be put into the form of Ohm's law:

$$\mathbf{V=IR_{elec}}$$



Using this terminology it is common to speak of a thermal resistance:

$$\Delta T=qR_{therm}$$



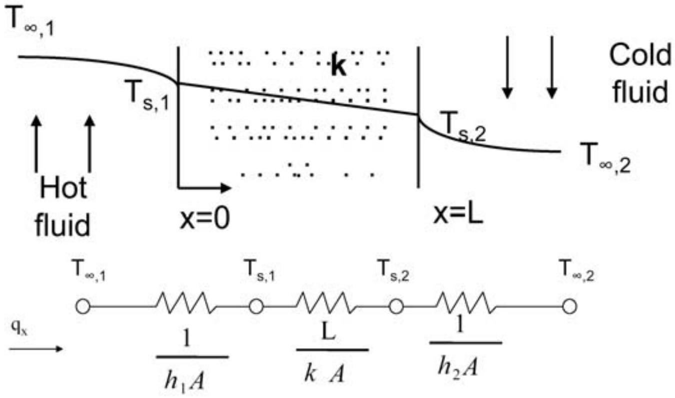
A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$q=hA(T_s-T_\infty)$$

the thermal resistance for convection is then

$$R_{t,conv}=\frac{T_s-T_\infty}{q}=\frac{1}{hA}$$

Applying thermal resistance concept to the plane wall, the equivalent thermal circuit for the plane wall with convection boundary conditions is shown in the figure below



The heat transfer rate may be determined from separate consideration of each element in the network. Since q_x is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

In terms of the overall temperature difference $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

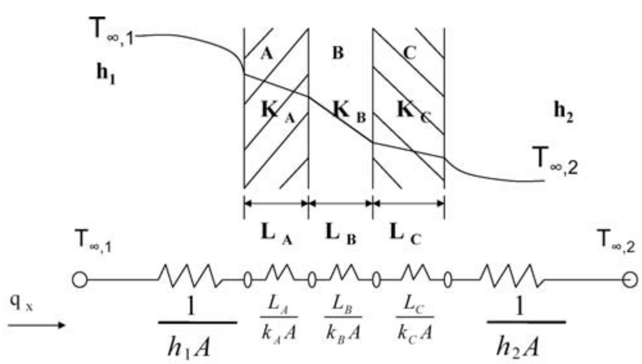
Since the resistance are in series, it follows that

$$R_{tot} = \sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Composite walls:

Thermal Resistances in Series:

Consider three blocks, A, B and C, as shown. They are insulated on top, bottom, front and back. Since the energy will flow first through block A and then through blocks B and C, we say that these blocks are thermally in a series arrangement.



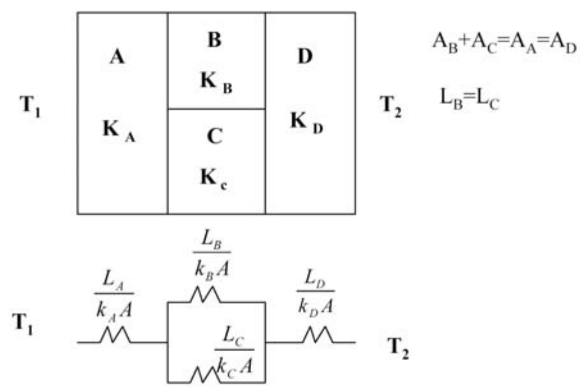
The steady state heat flow rate through the walls is given by:

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA\Delta T$$

where $U = \frac{1}{R_{tot} A}$ is the overall heat transfer coefficient. In the above case, U is expressed as

$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-parallel arrangement:

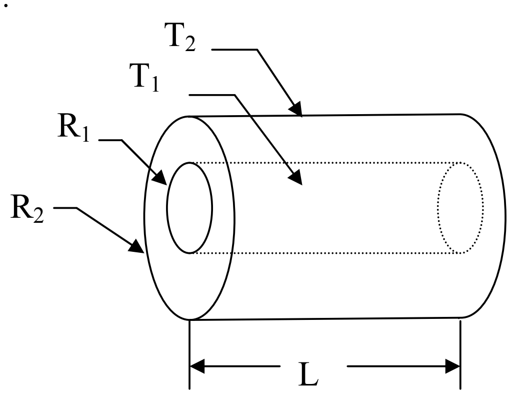


The following assumptions are made with regard to the above thermal resistance model:

1) Face between B and C is insulated.
2) Uniform temperature at any face normal to X.

1-D radial conduction through a cylinder:

One frequently encountered problem is that of heat flow through the walls of a pipe or through the insulation placed around a pipe. Consider the cylinder shown. The pipe is either insulated on the ends or is of sufficient length, L, that heat losses through the ends is negligible. Assume no heat sources within the wall of the tube. If $T_1 > T_2$, heat will flow outward, radially, from the inside radius, R_1, to the outside radius, R_2. The process will be described by the Fourier Law.



The differential equation governing heat diffusion is: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$

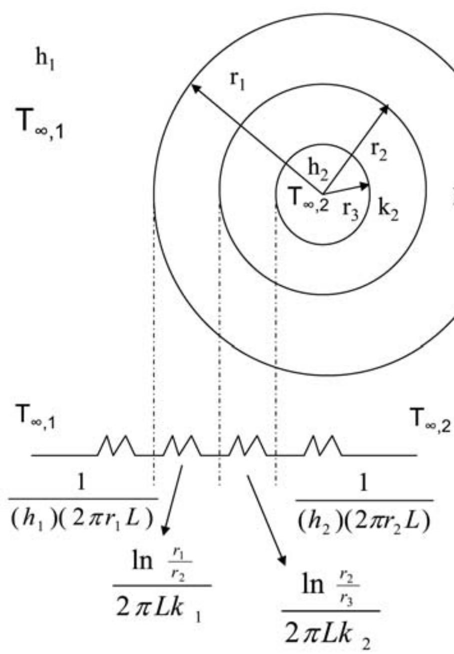
With constant k, the solution is

The heat flow rate across the wall is given by:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1}-T_{s,2}\right) = \frac{T_{s,1}-T_{s,2}}{L/kA}$$

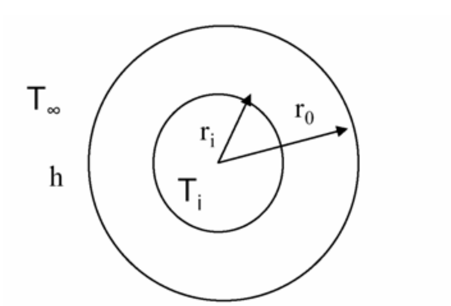
Hence, the thermal resistance in this case can be expressed as: $\frac{\ln\frac{r_1}{r_2}}{2\pi kL}$

Composite cylindrical walls:



$$q_r = \frac{T_{\infty,2}-T_{\infty,1}}{\sum R_t}$$

Critical Insulation Thickness :



$$R_{tot} = \frac{\ln(\frac{r_0}{r_i})}{2\pi kL} + \frac{1}{(2\pi r_0 L)h}$$

Insulation thickness : r_o-r_i

Objective : decrease q , increase R_{tot}

Vary r_o ; as r_o increases, first term increases, second term decreases.

This is a maximum – minimum problem. The point of extrema can be found by setting

$$\frac{dR_{tot}}{dr_0} = 0$$

or, $$\frac{1}{2\pi k r_0 L} - \frac{1}{2\pi h L r_o^2} = 0$$

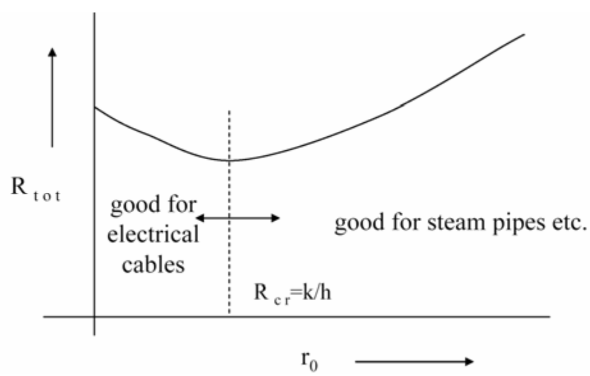
or, $$r_0 = \frac{k}{h}$$

In order to determine if it is a maxima or a minima, we make the second derivative zero:

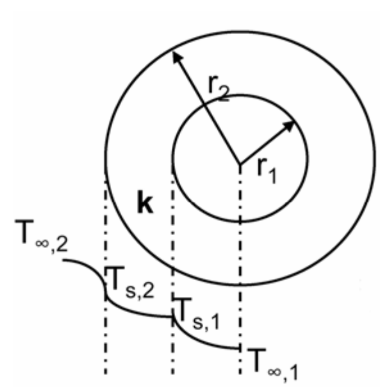
$$\frac{d^2 R_{tot}}{dr_o^2} = 0 \quad \text{at} \quad r_0 = \frac{k}{h}$$

$$\frac{d^2 R_{tot}}{dr_o^2} = \frac{-1}{2\pi k r_o^2 L} + \left.\frac{1}{\pi r_o^2 h L}\right|_{r_0 = \frac{k}{h}} = \frac{h^2}{2\pi L k^3} \rangle 0$$

Minimum q at r_o =(k/h) = r_{cr} (critical radius)



1-D radial conduction in a sphere:



$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)=0$$

$$\rightarrow T(r)=T_{s,1}-\{T_{s,1}-T_{s,2}\}\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$$

$$\rightarrow q_r=-kA\frac{dT}{dr}=\frac{4\pi k(T_{s,1}-T_{s,2})}{(1/r_1-1/r_2)}$$

$$\rightarrow R_{t,cond}=\frac{1/r_1-1/r_2}{4\pi k}$$

2.6 Summary of Electrical Analogy

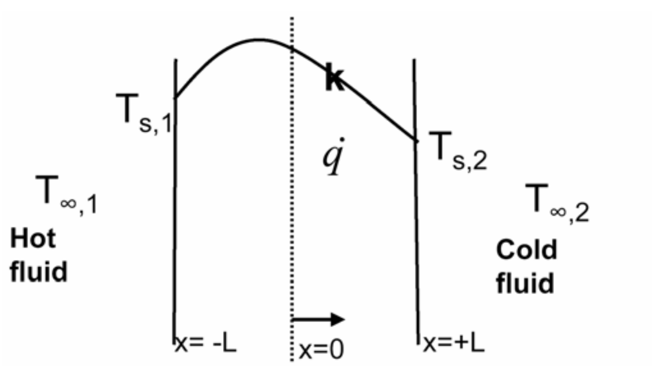
System	Current	Resistance	Potential Difference
Electrical	I	R	ΔV
Cartesian Conduction	q	$\frac{L}{kA}$	ΔT
Cylindrical Conduction	q	$\frac{\ln r_2/r_1}{2\pi kL}$	ΔT
Conduction through sphere	q	$\frac{1/r_1-1/r_2}{4\pi k}$	ΔT
Convection	q	$\frac{1}{h\cdot A_s}$	ΔT

2.7 One-Dimensional Steady State Conduction with Internal Heat Generation

Applications: current carrying conductor, chemically reacting systems, nuclear reactors.

Energy generated per unit volume is given by $\dot{q}=\frac{\dot{E}}{V}$

<u>Plane wall with heat source:</u> Assumptions: 1D, steady state, constant k, uniform $\dot{q}$



$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0$$

Boundary cond.: $x=-L, \quad T=T_{s,1}$

$x=+L, \quad T=T_{s,2}$

Solution: $T=-\frac{\dot{q}}{2k}x^2+C_1x+C_2$

Use boundary conditions to find C_1 and C_2

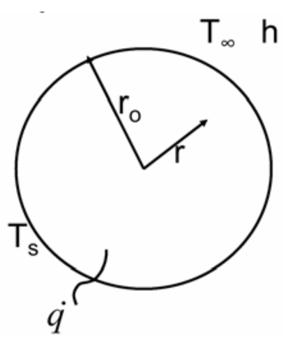
Final solution: $T=\frac{\dot{q}L^2}{2k}\left(1-\frac{x^2}{L^2}\right)+\frac{T_{s,2}-T_{s,1}}{2}\frac{x}{L}+\frac{T_{s,2}+T_{s,1}}{2}$

Heat flux: $\quad q''_x=-k\frac{dT}{dx}$

Note: From the above expressions, it may be observed that the solution for temperature is no longer linear. As an exercise, show that the expression for heat flux is no longer independent of *x. Hence thermal resistance concept is not correct to use when there is internal heat generation.*

Cylinder with heat source: Assumptions: 1D, steady state, constant k, uniform $\dot{q}$

Start with 1D heat equation in cylindrical co-ordinates



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

Boundary cond.: $r=r_0, \quad T=T_s$

$r=0, \quad \frac{dT}{dr}=0$

Solution: $T(r)=\frac{\dot{q}}{4k}r_0^{\,2}\left(1-\frac{r^2}{r_0^{\,2}}\right)+T_s$

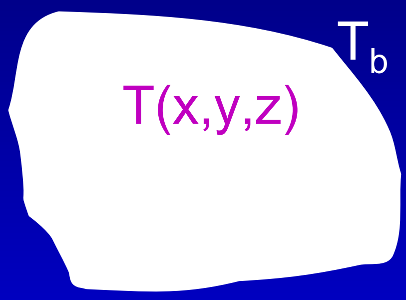
Exercise: T_s may not be known. Instead, T_∞ and h may be specified. Eliminate T_s, using T_∞ and h.

Objectives of conduction analysis:

- to determine the temperature field, $T(x,y,z,t)$, in a body

(i.e. how temperature varies with position within the body)

$T(x,y,z,t)$ depends on:

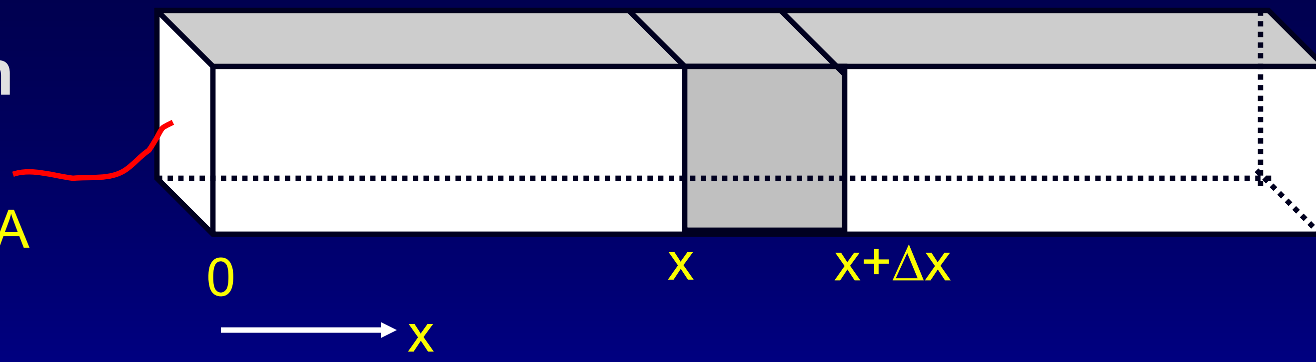


- boundary conditions
- initial condition
- material properties (k, c_p, ρ …)
- geometry of the body (shape, size)

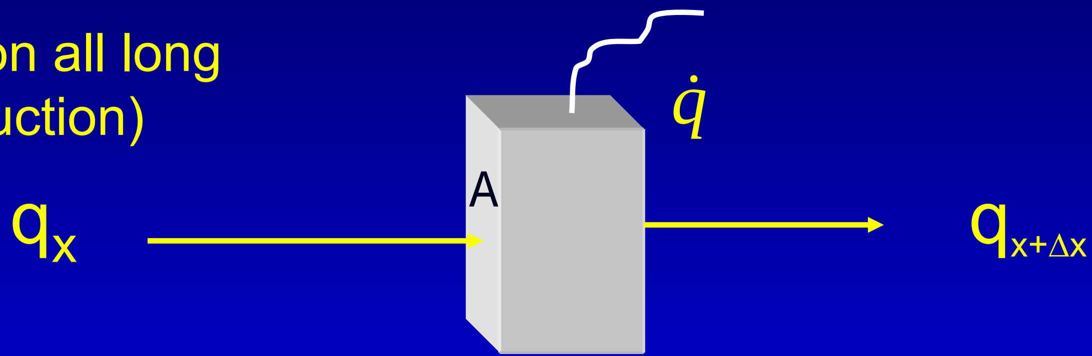
Why we need $T(x,y,z,t)$?

- to compute heat flux at any location (using Fourier's eqn.)
- compute thermal stresses, expansion, deflection due to temp. etc.
- design insulation thickness
- chip temperature calculation
- heat treatment of metals

Unidirectional heat conduction (1D)



Solid bar, insulated on all long sides (1D heat conduction)



$\dot{q}$ = internal heat generation per unit vol. (W/m^3)

First Law (energy balance) $(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st}$

$$q_x - q_{x+\Delta x} + A(\Delta x)\dot{q} = \frac{\partial E}{\partial t}$$

$$E = (\rho A \Delta x) u$$

$$\frac{\partial E}{\partial t} = \rho A \Delta x \frac{\partial u}{\partial t} = \rho A \Delta x c \frac{\partial T}{\partial t}$$

$$q_x = -kA \frac{\partial T}{\partial x}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x} \Delta x$$

$$-kA\frac{\partial T}{\partial x}+kA\frac{\partial T}{\partial x}+A\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x+A\Delta x\dot{q}=\rho Ac\Delta x\frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}=\rho c\Delta x\frac{\partial T}{\partial t}$$

Longitudinal conduction

Internal heat generation

Thermal inertia

If k is a constant

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}}{k}=\frac{\rho c}{k}\frac{\partial T}{\partial t}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

Boundary and initial conditions:

• The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

• We have set up a differential equation, with T as the dependent variable. The solution will give us T(x,y,z). Solution depends on boundary conditions (BC) and initial conditions (IC).

• How many BC's and IC's ?

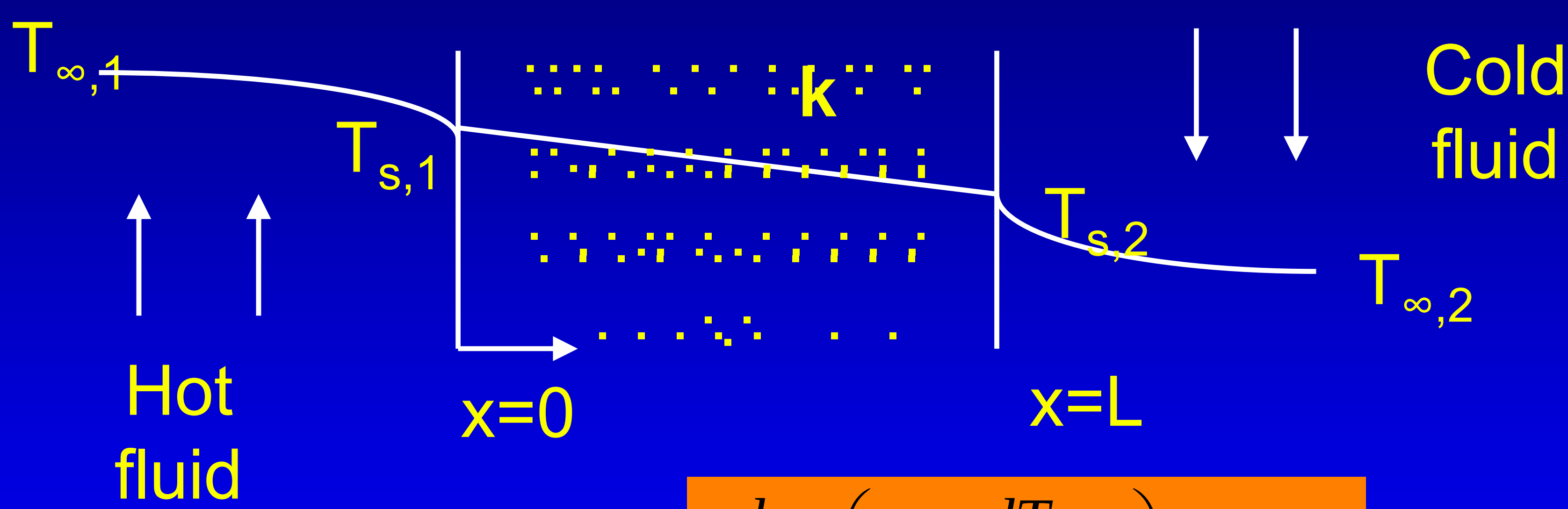
- Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.

  * 1D problem: 2 BC in x-direction
  * 2D problem: 2 BC in x-direction, 2 in y-direction
  * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.

- Heat equation is first order in time. Hence one IC needed.

1- Dimensional Heat Conduction

The Plane Wall :



$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

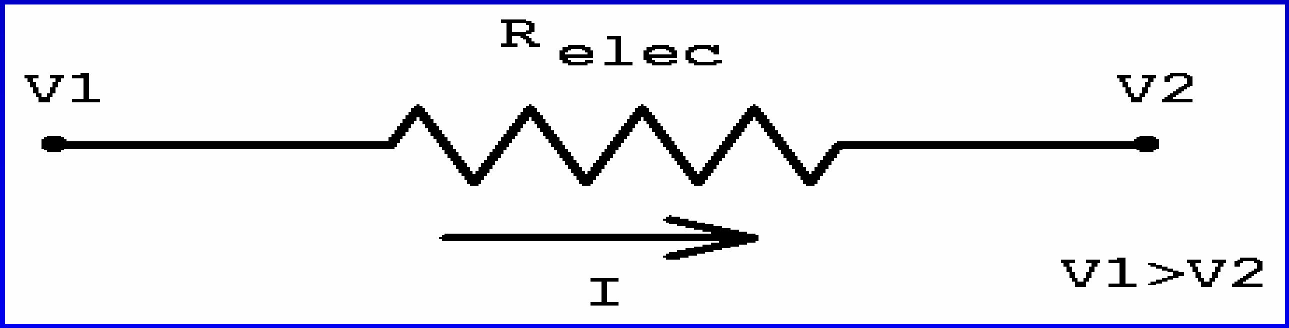
Const. K; solution is:

$$q_x=-kA\frac{dT}{dx}=\frac{kA}{L}\left(T_{s,1}-T_{s,2}\right)=\frac{T_{s,1}-T_{s,2}}{L/kA}$$

Thermal resistance (electrical analogy)

OHM's LAW :Flow of Electricity

$$V = IR_{elect}$$

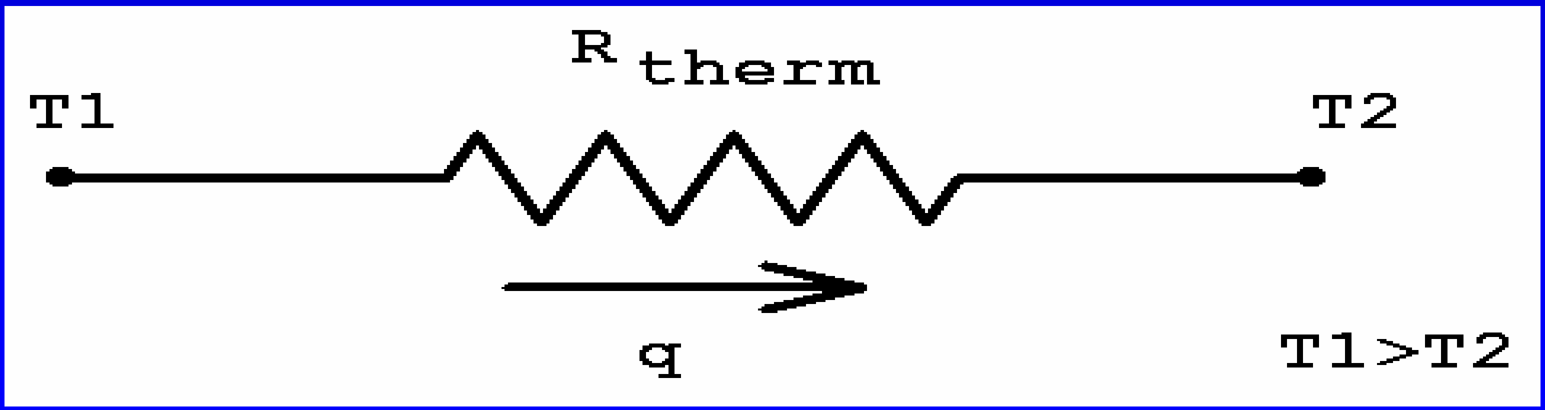


Voltage Drop = Current flow × Resistance

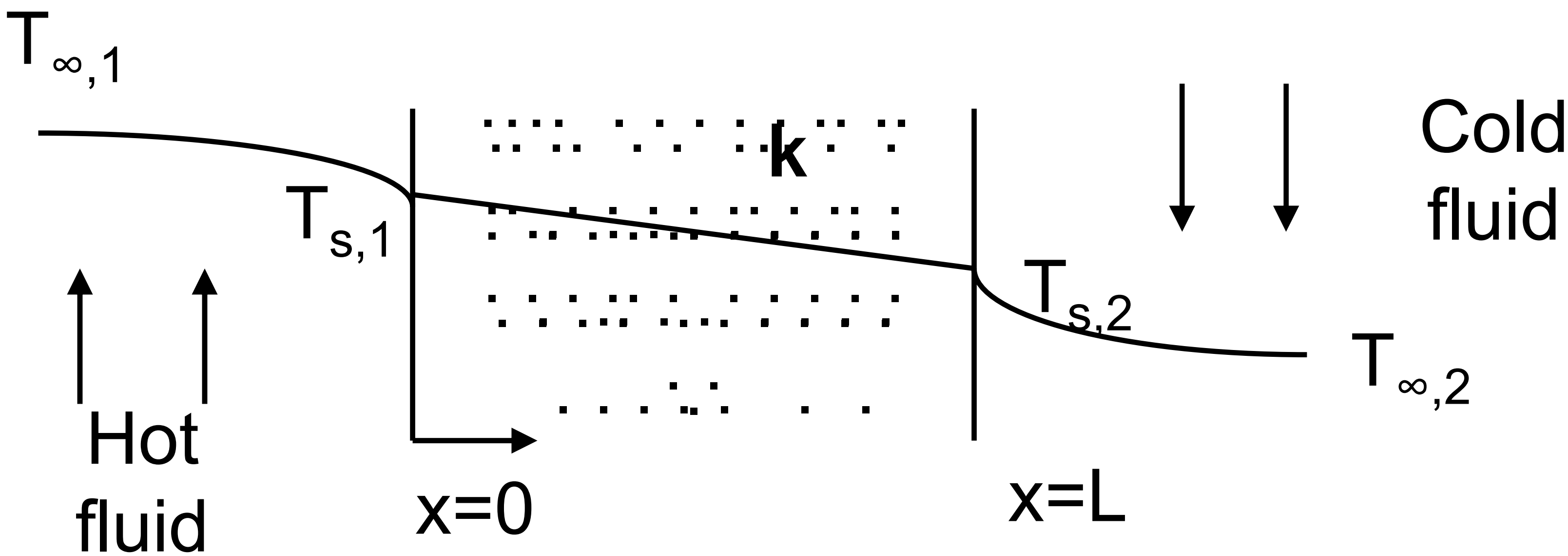
Thermal Analogy to Ohm's Law :

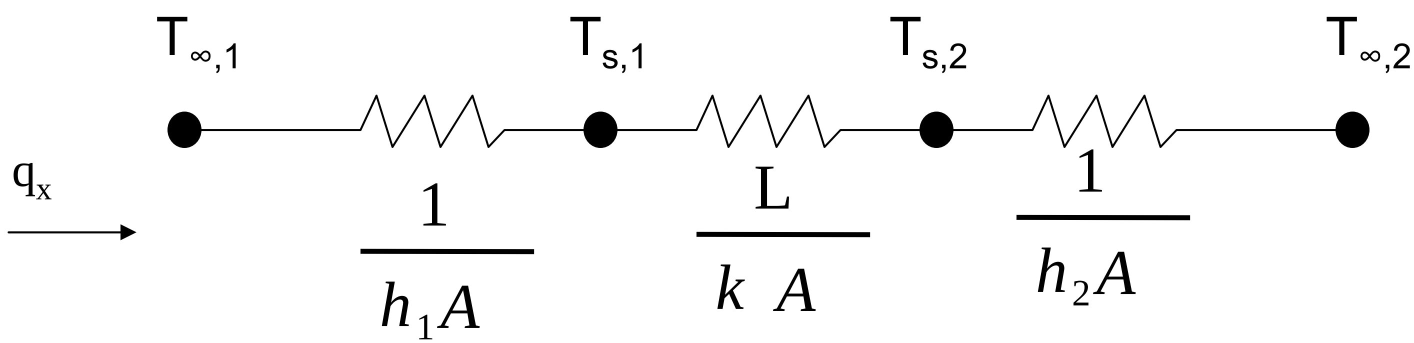
$$\Delta T = qR_{therm}$$

Temp Drop=Heat Flow×Resistance



1 D Heat Conduction through a Plane Wall





$$\sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad \text{(Thermal Resistance)}$$

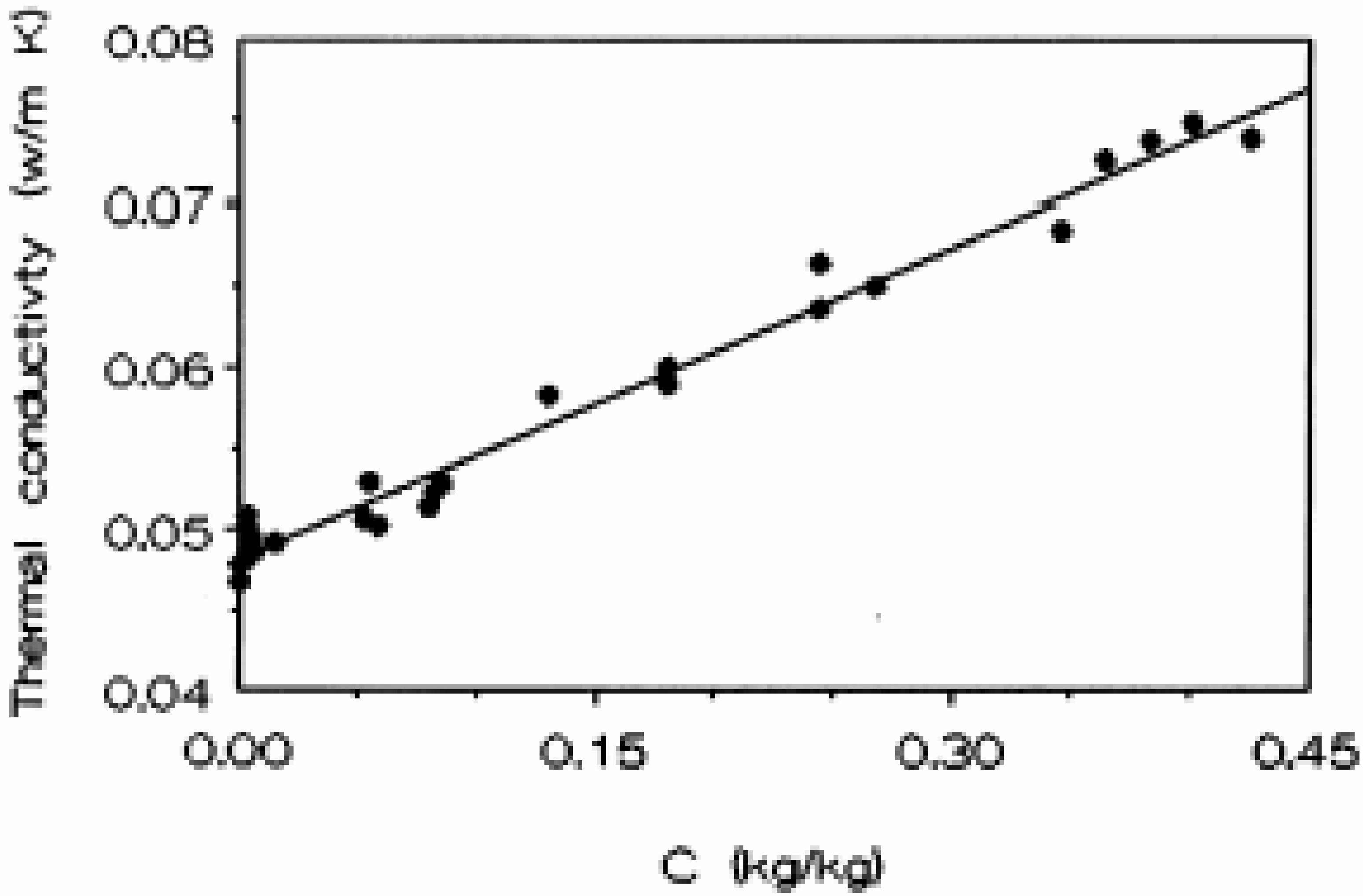


Fig. 2. Variation of thermal conductivity of moisture laden cork slab with the mass concentration of moisture.

Resistance expressions

THERMAL RESISTANCES

- Conduction

$$R_{cond} = \Delta x/kA$$

- Convection

$$R_{conv} = (hA)^{-1}$$

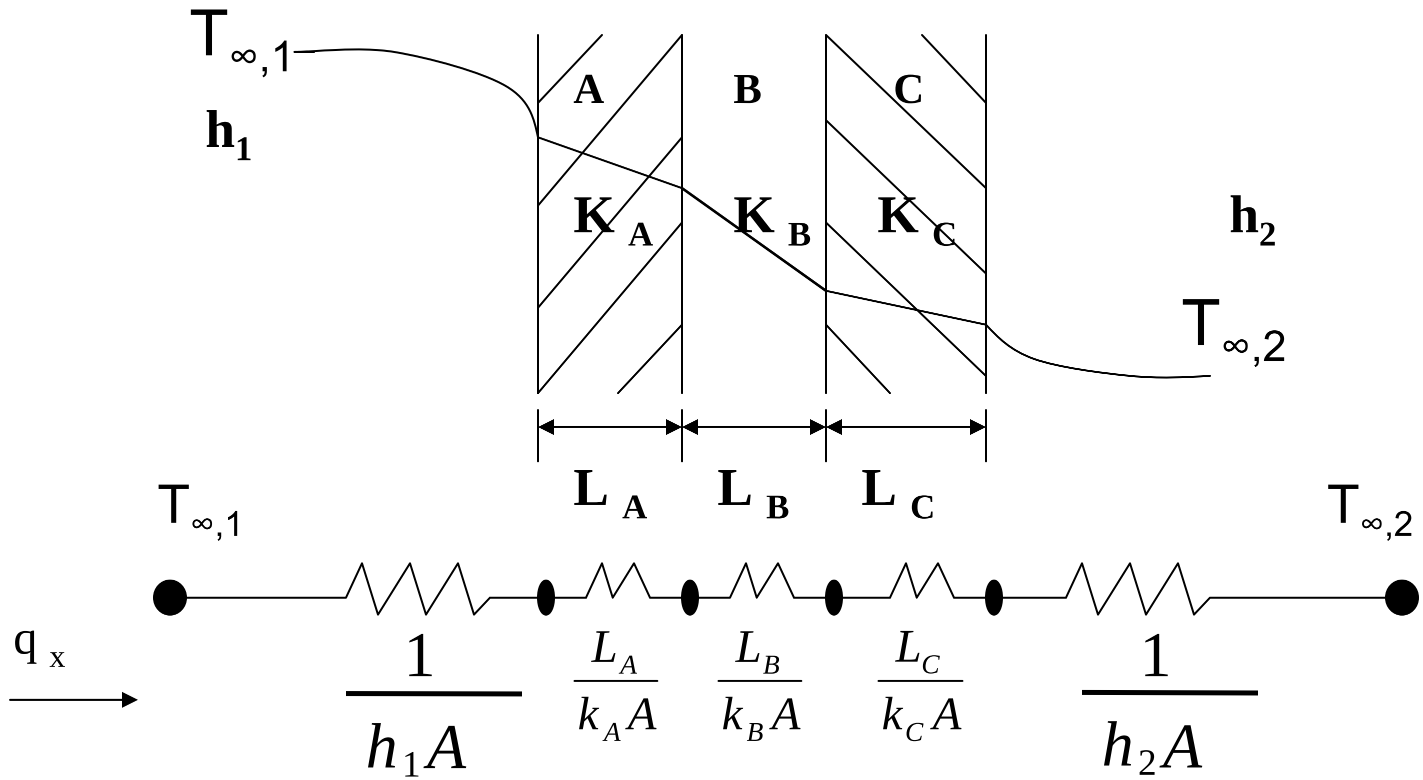
- Fins

$$R_{fin} = (h\eta A)^{-1}$$

- Radiation*(aprox)*

$$R_{rad} = [4A\sigma F(T_1T_2)^{1.5}]^{-1}$$

Composite Walls :



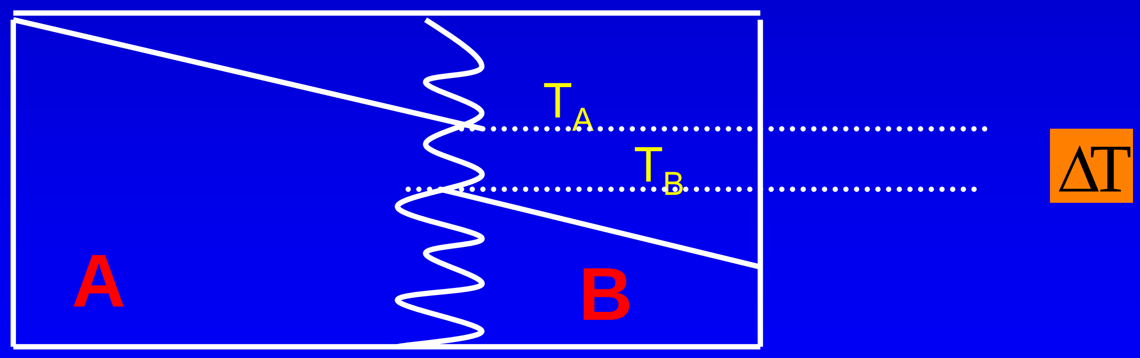
$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA\ \Delta T$$

$$where, U = \frac{1}{R_{tot} A} = \text{ Overall heat transfer coefficient}$$

Overall Heat transfer Coefficient

$$U = \frac{1}{R_{total} A} = \frac{1}{\frac{1}{h_1} + \Sigma \frac{L}{k} + \frac{1}{h_2}}$$

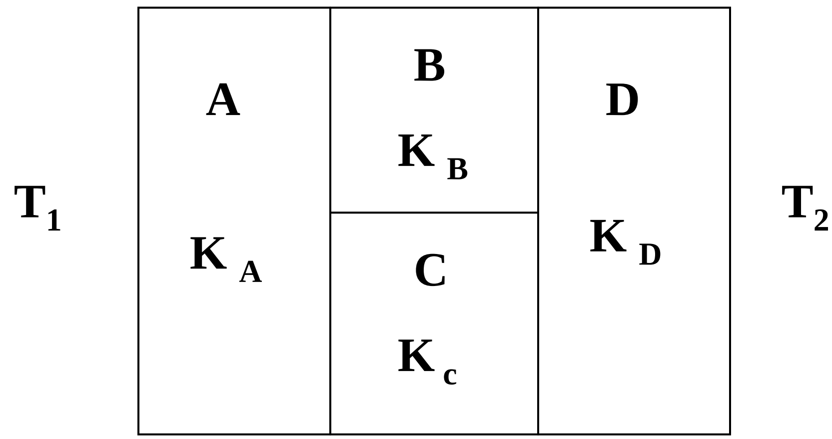
Contact Resistance :



$$R_{t,c} = \frac{\Delta T}{q_x}$$

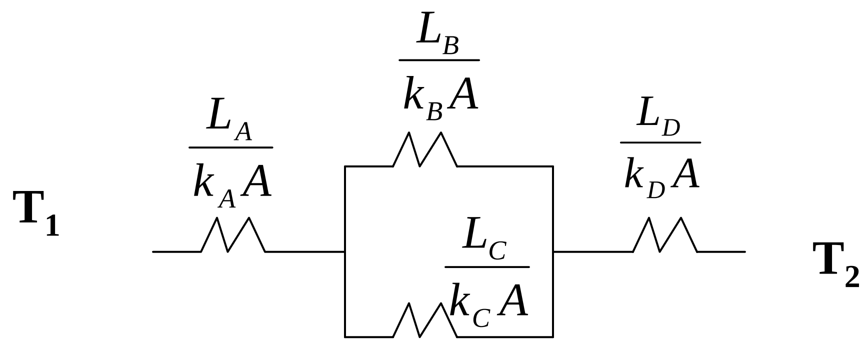
$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-Parallel :



$$A_B + A_C = A_A = A_D$$

$$L_B = L_C$$



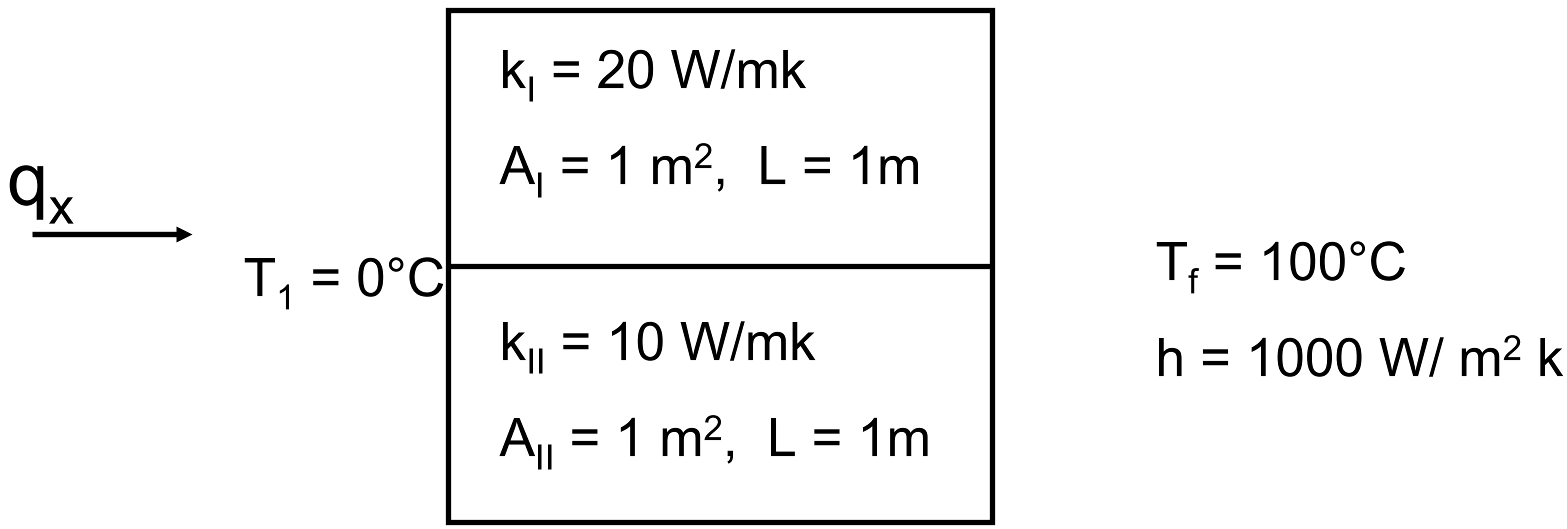
Assumptions :

(1) Face between B and C is insulated.

(2) Uniform temperature at any face normal to X.

Example:

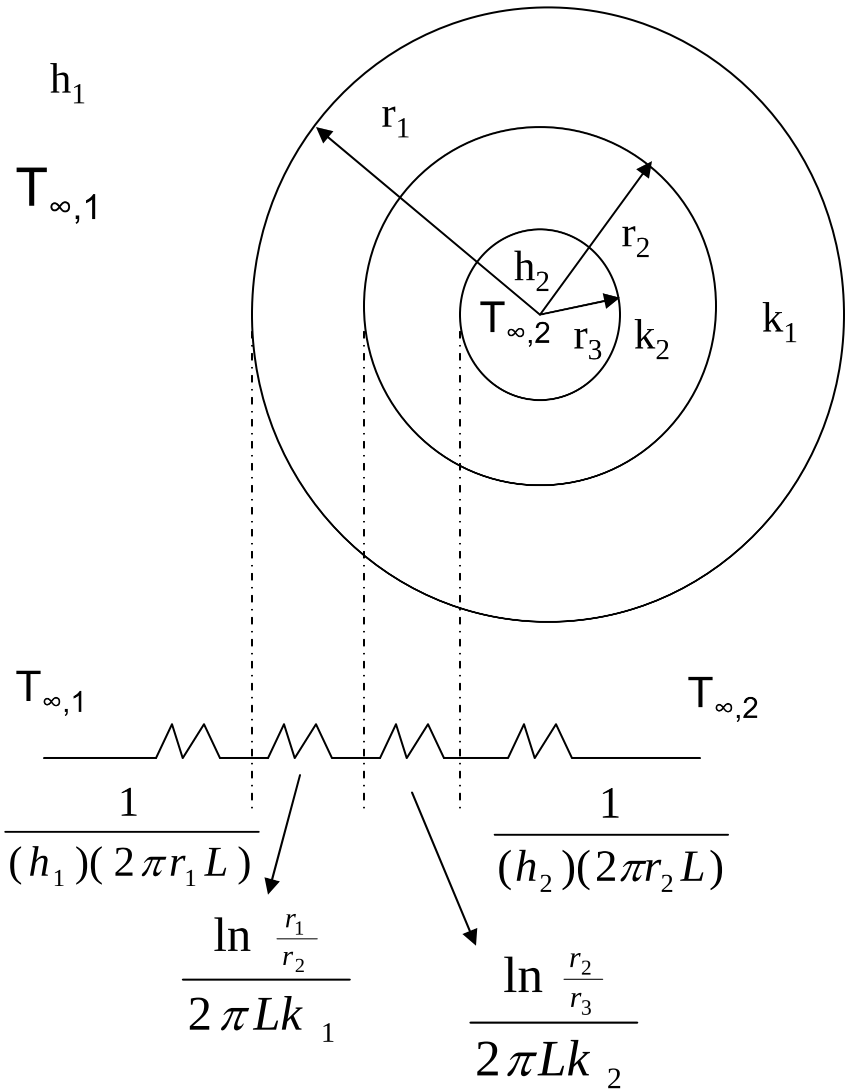
Consider a composite plane wall as shown:



Develop an approximate solution for th rate of heat transfer through the wall.

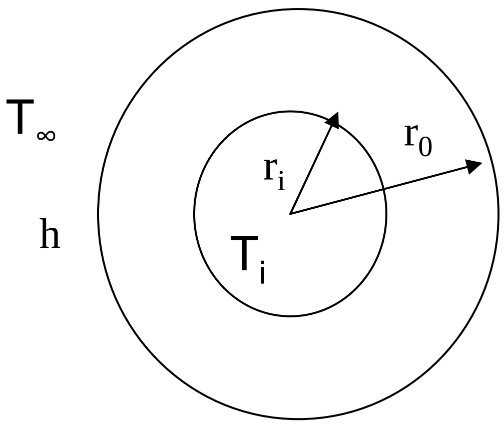
Solution: to be worked out in class

1 D Conduction(Radial conduction in a composite cylinder)



$$q_r = \frac{T_{\infty,2} - T_{\infty,1}}{\sum R_t}$$

Critical Insulation Thickness :



Insulation Thickness : $r_o - r_i$

$$R_{tot} = \frac{\ln\left(\frac{r_0}{r_i}\right)}{2\pi k L} + \frac{1}{(2\pi r_0 L)h}$$

Objective : decrease q , increases R_{tot}

Vary r_0 ; as r_0 increases ,first term increases, second term decreases.

Maximum – minimum problem

Set $\dfrac{dR_{tot}}{dr_0} = 0$

$$\frac{1}{2\pi k r_0 L} - \frac{1}{2\pi h L r^2{}_0} = 0$$

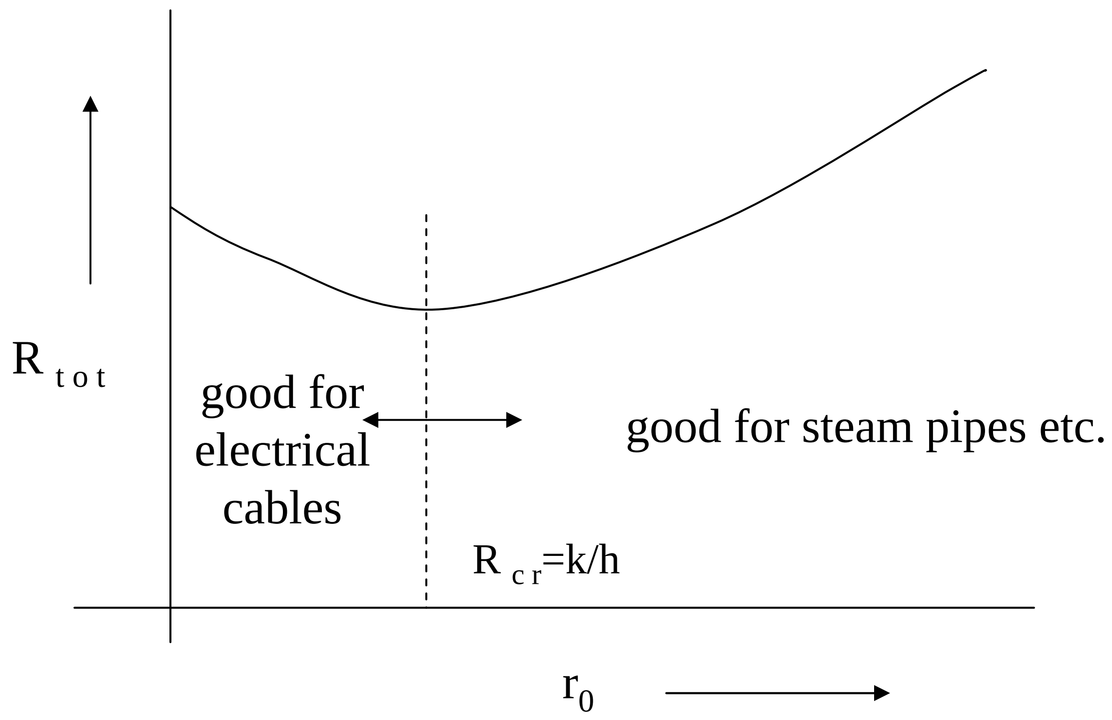
$$r_0 = \frac{k}{h}$$

Max or Min. ? Take : $\dfrac{d^2 R_{tot}}{dr^2{}_0} = 0$ at $r_0 = \dfrac{k}{h}$

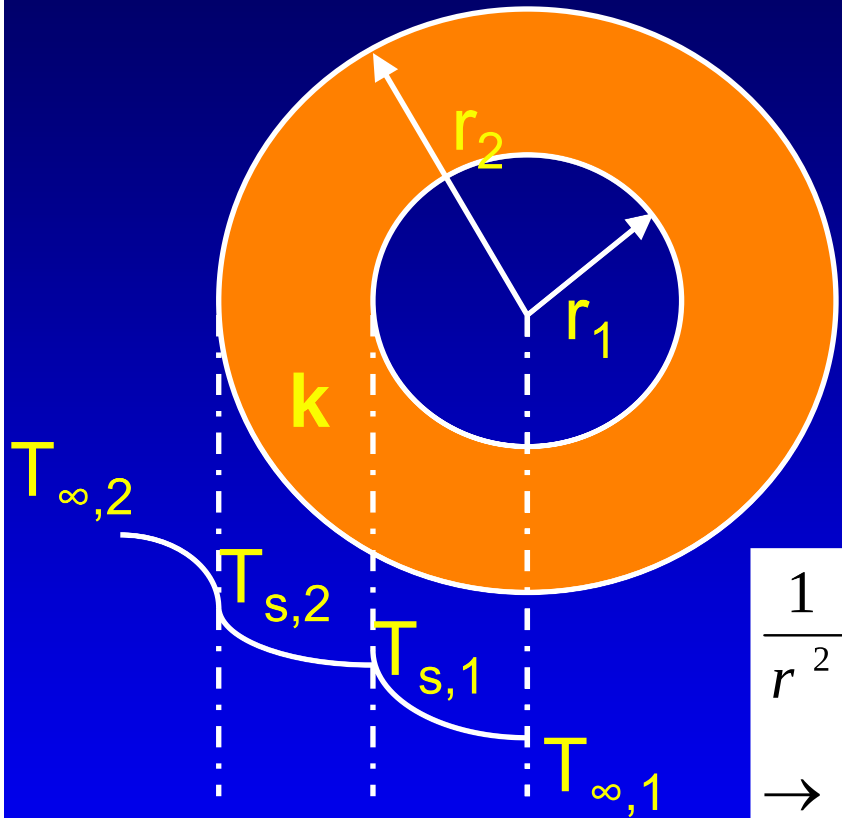
$$\frac{d^2 R_{tot}}{dr^2{}_0} = \frac{-1}{2\pi k r^2{}_0 L} + \frac{1}{\pi r^2{}_0 h L}\Bigg|_{r_0 = \frac{k}{h}}$$

$$= \frac{h^2}{2\pi L k^3} \Big\rangle 0$$

Minimum q at r_0 =(k/h)=r_{cr} (critical radius)



1 D Conduction in Sphere



Inside Solid:

$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$$

$$\rightarrow T(r) = T_{s,1} - \{T_{s,1} - T_{s,2}\}\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$$

$$\rightarrow q_r = -kA\frac{dT}{dr} = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1 - 1/r_2)}$$

$$\rightarrow R_{t,cond} = \frac{1/r_1 - 1/r_2}{4\pi k}$$

Conduction with Thermal Energy Generation

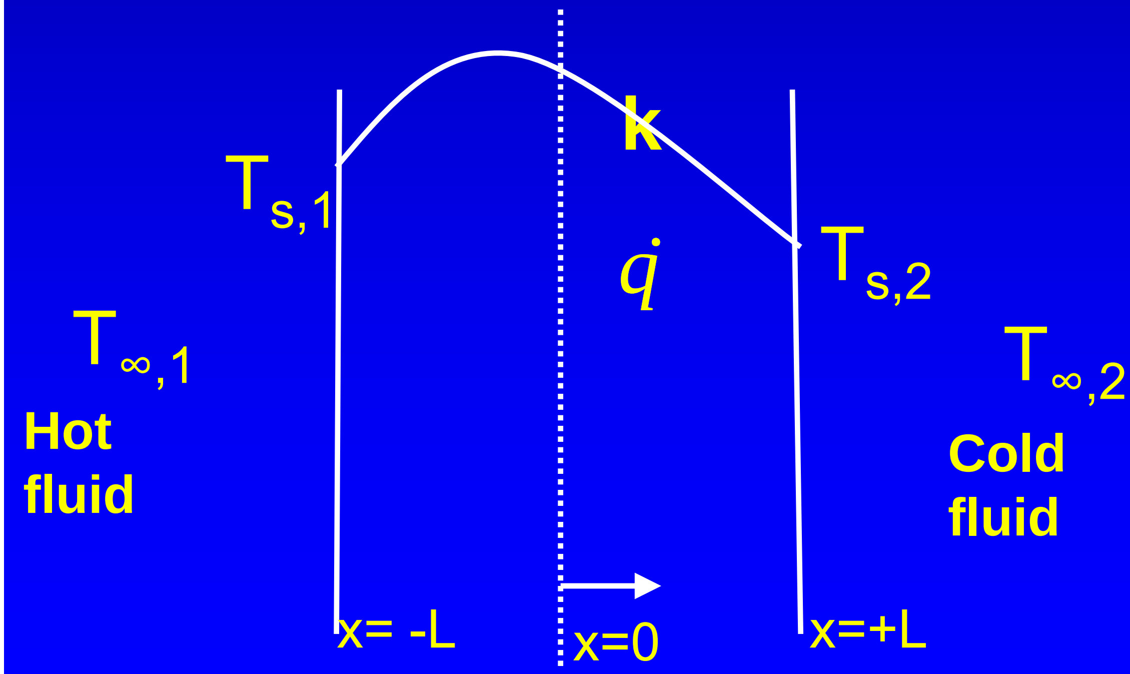
$\dot{q} = \frac{\dot{E}}{V}$ = Energy generation per unit volume

Applications: * current carrying conductors

* chemically reacting systems

* nuclear reactors

The Plane Wall :



Assumptions:

1D, steady state, constant k,

uniform $\dot{q}$

Conduction with thermal energy generation (cont..)

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$Boundary\ \ cond.: \quad x = -L, \quad T = T_{s,1}$$

$$x = +L, \quad T = T_{s,2}$$

$$Solution: \quad T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

$$Use\ boundary\ conditions\ to\ \ find\ C_1\ and\ C_2$$

$$Final\ solution: T = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

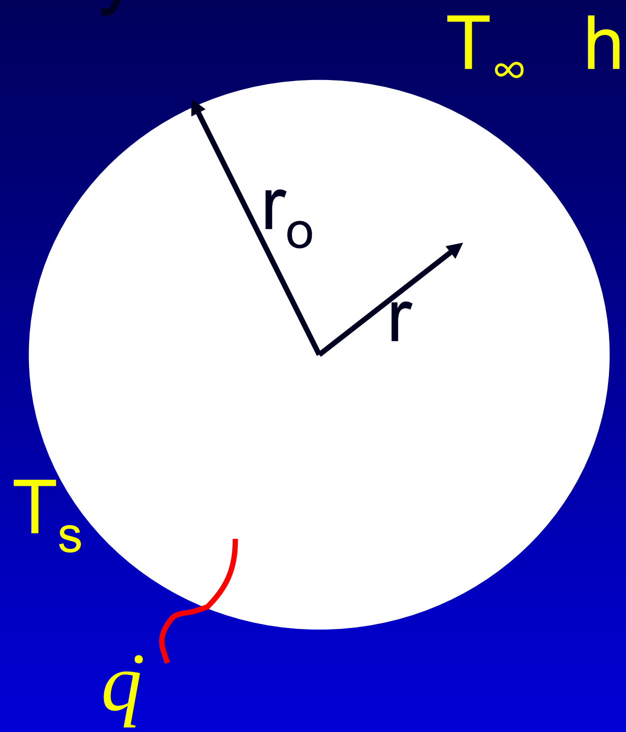
No more linear

$$Heat\ \ flux: \qquad q''_x = -k\frac{dT}{dx}$$

Derive the expression and show that it is no more independent of x

Hence thermal resistance concept is not correct to use when there is internal heat generation

Cylinder with heat source



Assumptions:

1D, steady state, constant k, uniform $\dot{q}$

Start with 1D heat equation in cylindrical co-ordinates:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

$$Boundary\ \ cond.: \quad r=r_0, \quad T=T_s$$

$$r=0, \quad \frac{dT}{dr}=0$$

$$Solution: \quad T(r)=\frac{\dot{q}}{4k}r_0^{\,2}\left(1-\frac{r^2}{r_0^{\,2}}\right)+T_s$$

T_s may not be known. Instead, $T_\propto$ and h may be specified.

Exercise: Eliminate T_s, using $T_\propto$ and h.

Cylinder with heat source

Example:

A current of 200A is passed through a stainless steel wire having a thermal conductivity k=19W/mK, diameter 3mm, and electrical resistivity R = 0.99 Ω. The length of the wire is 1m. The wire is submerged in a liquid at 110°C, and the heat transfer coefficient is 4W/m^{2}K. Calculate the centre temperature of the wire at steady state condition.

Solution: to be worked out in class

Problem 1:

The steady-state temperature distribution in a one–dimensional wall of thermal conductivity 50W/m.K and thickness 50 mm is observed to be T(^{0}C)= a+bx^2, where a=200^{0}C, B=-2000^{0}c/ m^2, and x in meters.

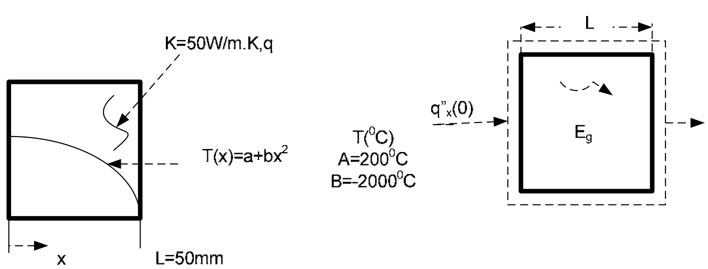
(a) What is the heat generation rate in the wall?

(b) Determine the heat fluxes at the two wall faces. In what manner are these heat fluxes related to the heat generation rate?

Known: Temperature distribution in a one dimensional wall with prescribed thickness and thermal conductivity.

Find: (a) the heat generation rate, q in the wall, (b) heat fluxes at the wall faces and relation to q.

Schematic:



Assumptions: (1) steady-state conditions, (2) one –dimensional heat flow, (3) constant properties.

Analysis: (a) the appropriate form of heat equation for steady state, one dimensional condition with constant properties is

$$\dot{q} = -K\frac{d}{dx}\left[\frac{dT}{dx}\right]$$

$$\dot{q} = -k\frac{d}{dx}\left[\frac{d}{dx(a+bx^2)}\right] = -k\frac{d}{dx}[2bx] = -2bk$$

$$\dot{q} = -2(-2000^0\,CC/m^2)\times 50W/m.K = 2.0\times 10^5\,W/m^3$$

(b) The heat fluxes at the wall faces can be evaluated from Fourier's law,

$$q''_x(x) = -k\left.\frac{dT}{dx}\right|_x$$

Using the temperature distribution T(x) to evaluate the gradient, find

$$q_x''(x) = -k\frac{d}{dx}[a+bx^2] = -2kbx.$$

The flux at the face, is then x=0

$$q''_x(0) = 0$$

$$\text{at}X = L, q''_x(l) = -2kbL = -2\times 50W/m.K(-2000^0 C/m^2)\times 0.050m$$

$$q''_x(L) = 10{,}000 W/m^2$$

Comments: from an overall energy balance on the wall, it follows that

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0 \qquad q''_x(0) - q''_x(L) + \dot{q}L = 0$$

$$\dot{q} = \frac{q''_x(L) - q''_x(0)}{L} = \frac{10{,}00w/m^2 - 0}{0.050m} = 2.0\times 10^5 W/m^3$$

Problem 2:

A salt gradient solar pond is a shallow body of water that consists of three distinct fluid layers and is used to collect solar energy. The upper- and lower most layers are well mixed and serve to maintain the upper and lower surfaces of the central layer at uniform temperature T_1 and T_2, where $T_1>T_2$. Although there is bulk fluid motion in the mixed layers, there is no such motion in the central layer. Consider conditions for which solar radiation absorption in the central layer provides non uniform heat generation of the form $q=Ae^{-ax}$, and the temperature distribution in the central layer is

$$T(x) = -\frac{A}{ka^2}e^{-ax} + bx + c$$

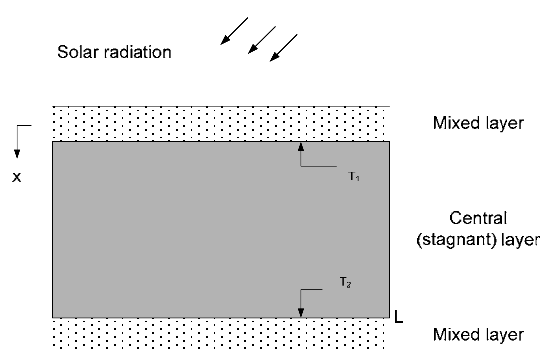
The quantities A (W/m3), a (1/m), B (K/m) and C(K) are known constants having the prescribed units, and k is the thermal conductivity, which is also constant.

(a) Obtain expressions for the rate at which heat is transferred per unit area from the lower mixed layer to the central layer and from central layer to the upper mixed layer.
(b) Determine whether conditions are steady or transient.
(c) Obtain an expression for the rate at which thermal energy is generated in the entire central layer, per unit surface area.

Known: Temperature distribution and distribution of heat generation in central layer of a solar pond.

Find: (a) heat fluxes at lower and upper surfaces of the central layer, (b) whether conditions are steady or transient (c) rate of thermal energy generation for the entire central layer.

Schematic:



Assumptions: (1) central layer is stagnant, (2) one-dimensional conduction, (3)constant properties.

Analysis (1) the desired fluxes correspond to conduction fluxes in the central layer at the lower and upper surfaces. A general form for the conduction flux is

$$q''_{cond} = \left[-k\frac{A}{ka}e^{-ax+}B\right]$$

Hence

$$q''_1 = q''_{cond(x=L0} = \left[-k\frac{A}{ka}e^{-al}+B\right] q''_u = q''_{cond(x=0)} = -k\left[\frac{A}{ka}+B\right]$$

(b) Conditions are steady if $\partial T/\partial t=0$. Applying the heat equation,

$$\frac{\partial^2 T}{\partial^2 t}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad -\frac{A}{k}e^{-ax}+\frac{A}{k}e^{-ax}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Hence conditions are steady since

$$\frac{\partial T}{\partial t}=0 \quad \text{(for all 0<=x<=L)}$$

For the central layer, the energy generation is

$$\dot{E}''_g = \int_0^L q dx = A\int_0^L e - axdx$$

$$\dot{E}_g = -\frac{A}{a}e^{-ax}\Big|_0^L = -\frac{A}{a}(e^{-aL}-1) = \frac{A}{a}(1-e^{-aL})$$

Alternatively, from an overall energy balance,

$$q''_2 - q''_1 + \dot{E}_g = 0 \qquad \dot{E}''_g = q''_1 - q''_2 = (-q''_{cond(x=0)})-(q''_{cond(x=L)})$$

$$\dot{E}_g = k\frac{A}{ka}+B-K\frac{A}{ka}e^{-aL}+B = \frac{A}{a}(1-e^{-aL})$$

Comments: Conduction is the negative x-direction, necessitating use of minus signs in the above energy balance.

Problem 3:

The steady state temperatures distribution in a one-dimensional wall of thermal conductivity and thickness L is of the form $T=ax^3+bx^2+cx+d$. derive expressions for the heat generation rate per unit volume in the wall and heat fluxes at the two wall faces(x=0, L).

Known: steady-state temperature distribution in one-dimensional wall of thermal conductivity, $T(x)=Ax^3+Bx^2+CX+d$.

Find: expressions for the heat generation rate in the wall and the heat fluxes at the two wall faces(x=0, L).

Assumptions: (1) steady state conditions, (2) one-dimensional heat flow, (3) homogeneous medium.

Analysis: the appropriate form of the heat diffusion equation for these conditions is

$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0 \qquad \text{Or} \qquad \dot{q}=-k\frac{d^2T}{dx^2}$$

Hence, the generation rate is

$$\dot{q}=-\frac{d}{dx}\left[\frac{dT}{dx}\right]=-k\frac{d}{dx}[3Ax^2+2Bx+C+0]$$

$$\dot{q}=-k[6Ax+2B]$$

which is linear with the coordinate x. The heat fluxes at the wall faces can be evaluated from Fourier's law,

$$q''_x=-k\frac{dT}{dx}=-k[3Ax^2+2Bx+C]$$

Using the expression for the temperature gradient derived above. Hence, the heat fluxes are:

Surface x=0; $\qquad q''_x(0)=-kC$

Surface x=L;

$q''_x(L) = -K\ [3AL^2+2BL+C]$

COMMENTS: (1) from an over all energy balance on the wall, find

$$\dot{E}_{in}-\dot{E}_{out}+\dot{E}_g=0$$

$$q''_x(0)-q''_x(L)=(-kC)-(-K)[3AL^2+2BL+C]+\dot{E}_g=0$$

$$\dot{E}''_g=-3AkL^2-2BkL$$

From integration of the volumetric heat rate, we can also find

$$\dot{E}''_g=\int_0^L \dot{q}(x)dx=\int_0^L -k[6Ax+2B]dx=-k[3Ax^2+2Bx]_0^L$$

$$\dot{E}''_g=-3AkL^2-2BkL$$

Problem 4:

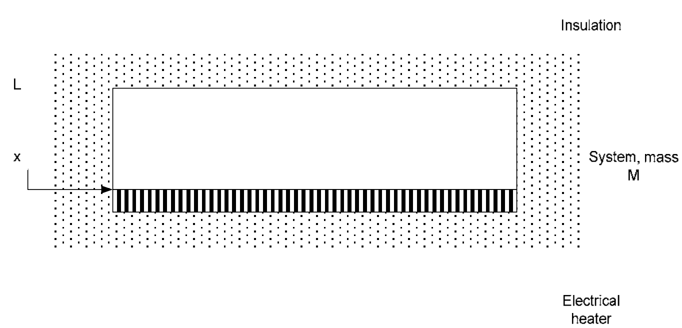
The one dimensional system of mass M with constant properties and no internal heat generation shown in fig are initially at a uniform temperature T_i. The electrical heater is suddenly energized providing a uniform heat flux q''_o at the surface x=0. the boundaries at x=L and else where are perfectly insulated.

(a) Write the differential equation and identify the boundary and initial conditions that could be used to determine the temperature as a function of position and time in the system.
(b) On T-x coordinates, sketch the temperature distributions for the initial condition (t<=0) and for several times after the heater is energized. Will a steady-state temperature distribution ever be reached?
(c) On q''_x-t coordinates, sketch the heat flux $q''_x(x,t)$ at the planes x=0, x=L/2, and x=L as a function of time.
(d) After a period of time t_e has elapsed, the heater power is switched off. Assuming that the insulation is perfect, the system will eventually reach final uniform temperature T_f. Derive an expression that can be used to determine T_f a function of the parameters q''_o,t_e,T_i, and the system characteristics M,c_p, and A(the heater surface area).

Known: one dimensional system, initially at a uniform temperature Ti, is suddenly exposed to a uniform heat flux at one boundary while the other boundary is insulated.

Find: (a) proper form of heat diffusion equation; identify boundary and initial conditions, (b) sketch temperature distributions for following conditions: initial condition (t<=0), several times after heater is energized ;will a steady-state condition be reached?, (c) sketch heat flux for x=0, L/2, L as a function of time, (d) expression for uniform temperature, Tf, reached after heater has been switched off the following an elapsed time , te, with the heater on.]

Schematic:



Assumptions: (1) one dimensional conduction, (2) no internal heat generation, (3) constant properties.

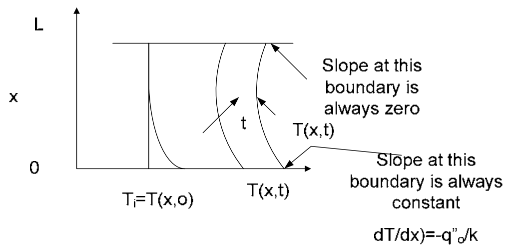
Analysis: (a) the appropriate form of the heat equation follows. Also the appropriate boundary and initial conditions are:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ Initial condition: $T(x, 0) = T_i$ uniform temperature

Boundary conditions: x=0 $q''_o = -k\partial T/\partial x)_0 \; t > 0$

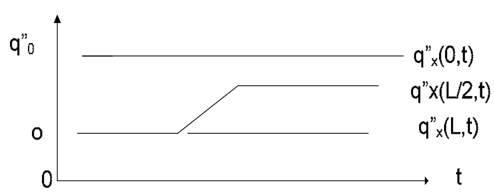
x=L $\partial T/\partial x)_L = 0$ Insulated

(b) The temperature distributions are as follows:



No steady-state condition will be reached since $\dot{E}_{in} - \dot{E}_{out}$ and $\dot{E}_{in}$ is constant.

(c) The heat flux as a function of time for positions x=0, L/2 and L appears as:



(d) If the heater is energized until t=t_o and then switched off, the system will eventually reach a uniform temperature , T_f. Perform an energy balance on the system, for an interval of time Δt=t_e,

$$\dot{E}_{in} = \dot{E}_{st} \qquad E_{in} = Q_{in} = \int_0^{t_e} q''_0 A_s dt = q''_o A_s t_e \qquad E_{st} = Mc(T_f - T_i)$$

It follows that $q''_o A_s t_e = Mc(T_f - T_i)$ OR $T_f = T_i + \dfrac{q''_o A_s t_e}{Mc}$

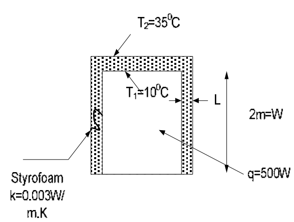
Problem 5:

A 1–m-long steel plate (k=50W/m.K) is well insulated on its sides, while the top surface is at 100^0C and the bottom surface is convectively cooled by a fluid at 20^0C. Under steady state conditions with no generation, a thermocouple at the midpoint of the plate reveals a temperature of 85^0C. What is the value of the convection heat transfer coefficient at the bottom surface?

Known: length, surfacethermal conditions, and thermal conductivity of a Plate. Plate midpoint temperature.

Find: surface convection coefficient

Schematic:



Assumptions: (1) one-dimensional, steady conduction with no generation, (2) Constant properties

Analysis: for prescribed conditions, is constant. Hence,

$$q''_{cond} = \frac{T_{1-}T_2}{L/2} = \frac{15^0\,C}{0.5m/50W/m.k} = 1500W/m^2$$

$$q'' = \frac{T_1 - T_\infty}{(L/k)+(1/h)} = \frac{30^0\,C}{(0.02+1/h)m^2.K/W} = 1500W/m^2$$

$$h = 30W/m^2.K$$

Comments: The contributions of conduction and convection to the thermal resistance are

$$R''_{t,cond} = \frac{L}{K} = 0.02m^2.K/W$$

$$R''_{t,cond} = \frac{1}{h} = 0.033m^2.K/W$$

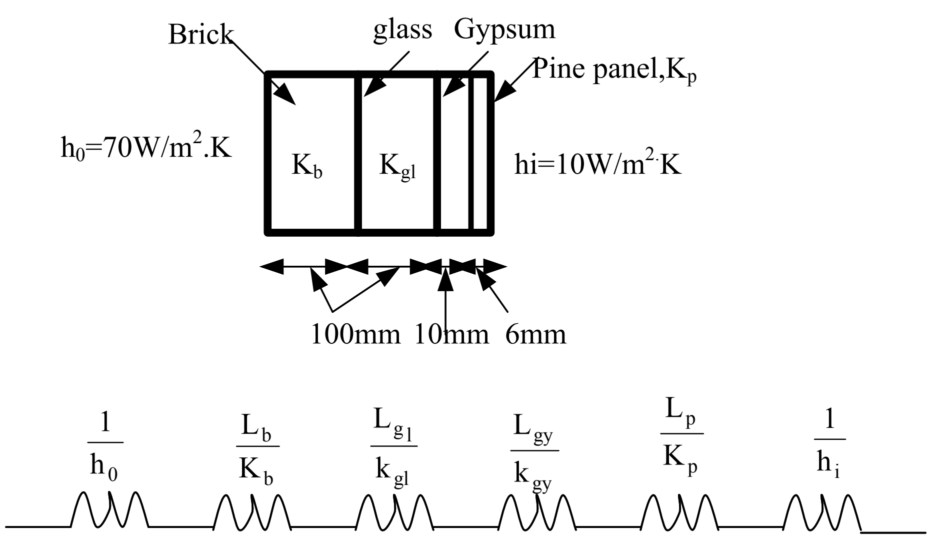
Problem 6:

The wall of a building is a composite consisting of a 100-mm layer of common brick, a 100-mm layer of glass fiber(paper faced. 28kg/m^2), a 10-mm layer of gypsum plaster (vermiculite), and a 6-mm layer of pine panel. If the inside convection coefficient is 10W/m^2.K and the outside convection coefficient is 70W/m^2.K, what are the total resistance and the overall coefficient for heat transfer?

Known: Material thickness in a composite wall consisting of brick, glass fiber, and vermiculite and pine panel. Inner and outer convection coefficients.

Find: Total thermal resistance and overall heat transfer coefficient.

Schematic:



Assumptions: (1) one dimensional conduction, (2) constant properties, (3) negligible contact resistance.

Properties: Table A-S, T= 300K: Brick, k_b=1.3 W/m.K: Glass fiber (28kg/m^3), k_{gl}= 0.038W/m.K: gypsum, k_{gy}=0.17W/m.K: pine panel, k_p=0.12W/m.K.

Analysis: considering a unit surface Area, the total thermal resistance

$$R''_{tot} = \frac{1}{h_0} + \frac{L_B}{K_B} + \frac{L_{g1}}{k_{g1}} + \frac{L_{gy}}{k_{gy}} + \frac{L_p}{K_p} + \frac{1}{h_i}$$

$$R''_{tot} = \left[\frac{1}{70} + \frac{0.1}{1.3} + \frac{0.1}{0.038} + \frac{0.01}{0.17} + \frac{0.006}{0.12} + \frac{1}{10}\right]\frac{m^2.K}{W}$$

$$R''_{tot} = (0.0143 + 0.0769 + 2.6316 + 0.0588 + 0.0500 + 0.1)m^2.K/W$$

$$R''_{tot} = 2.93m^2..K/W$$

The overall heat transfer coefficient is

$$U = \frac{1}{R_{tot}A} = \frac{1}{R''_{tot}} = (2.93m^2.K/W)^{-1}$$

$$U = 0.341W/m^2.K.$$

Comments: an anticipated, the dominant contribution to the total resistance is made by the insulation.

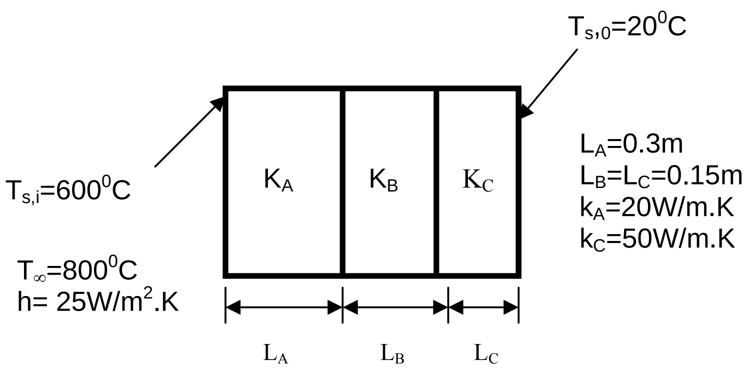
Problem 7:

The composite wall of an oven consists of three materials, two of which are known thermal conductivity, k_A=20W/m.K and k_C=50W/m.K, and known thickness, L_A=0.30m and L_C=0.15m. The third material, B, which is sandwiched between materials A and C, is of known thickness, L_B=0.15m, but unknown thermal conductivity k_B. Under steady-state operating conditions, measurements reveal an outer surface temperature of $T_{s,0}$=200C, an inner surface temperature of $T_{s,i}$=600^{0}C and an oven air temperature of T =800^{0}C. The inside convection coefficient h is known to be 25W/m^2.K. What is the value of k_B?

Known: Thickness of three material which form a composite wall and thermal conductivities of two of the materials. Inner and outer surface temperatures of the composites; also, temperature and convection coefficient associated with adjoining gas.

Find: value of unknown thermal conductivity, k_B.

Schematic:



Assumptions: (1) steady state conditions, (2) one-dimensional conduction, (3) constant properties, (4) negligible contact resistance, (5) negligible radiation effects.

Analysis: Referring to the thermal circuit, the heat flux may be expressed as

$$q'' = \frac{T_{s,i} - T_{s,0}}{\frac{L_A}{K_A} + \frac{L_B}{K_B} + \frac{L_C}{K_C}} = \frac{(600-20)^0 C}{\frac{0.3m}{0.018} + \frac{0.15m}{K_B} + \frac{0.15m}{50W/m.K}}$$

$$= \frac{580}{0.018 + 0.15/K_B} W/m^2$$

The heat flux can be obtained from

$$q'' = h(T_\infty - T_{s,i}) = 25W/m^2.K(800-600)^0 C$$
$$q'' = 5000W/m^2$$

Substituting for heat flux,

$$\frac{0.15}{K_B} = \frac{580}{q''} - 0.018 = \frac{580}{5000} - 0.018 = 0.098$$
$$K_B = 1.53W/m.K.$$

Comments: radiation effects are likely to have a significant influence on the net heat flux at the inner surface of the oven.

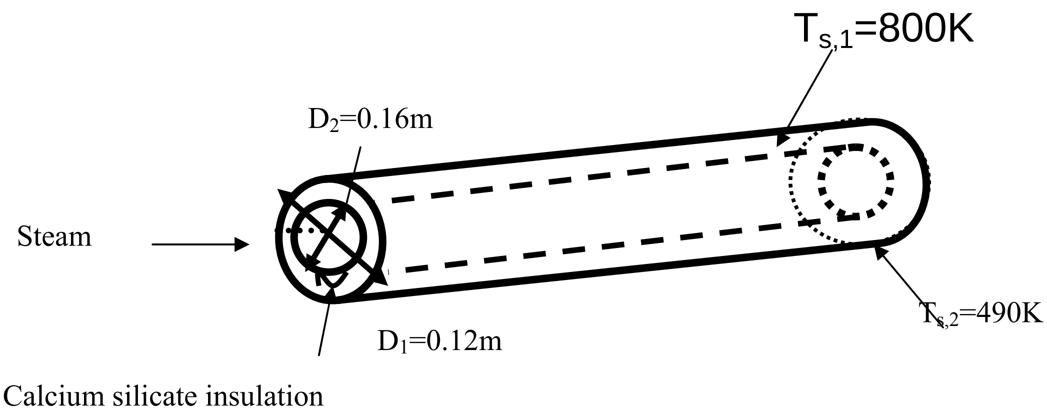
Problem 8:

A steam pipe of 0.12 m outside diameter is insulated with a 20-mm-thick layer of calcium silicate. If the inner and outer surfaces of the insulation are at temperatures of $T_{s,1}$=800K and $T_{s,2}$=490 K, respectively, what is the heat loss per unit length of the pipe?

Known: Thickness and surface temperature of calcium silicate insulation on a steam pipe.

Find: heat loss per unit pipe length.

Schematic:



Assumptions: (steady state conditions, (2) one-dimensional conduction, (3) constant properties.

Properties: Table, A-3, calcium silicate (T=645K): k=0.089W/m.K

Analysis: The heat per unit length is

$$q_r' = \frac{q_r}{q_L} = \frac{2\pi K(T_{s,1} - T_{s,2})}{\ln(D_2 / D_1)}$$

$$q_r' = \frac{2\pi(0.089\,W/m.K)(800-490)K}{\ln(0.16m/0.12m)}$$

$$q_r' = 603W/m$$

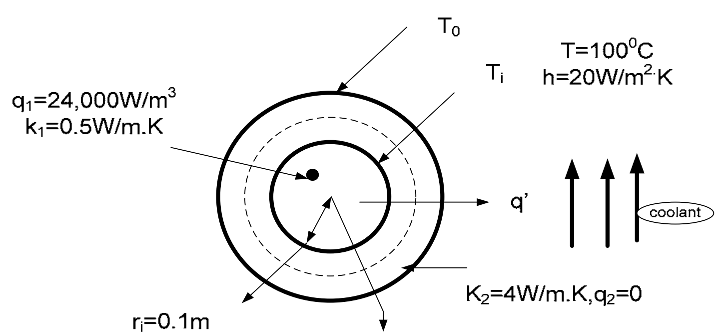
Comments: heat transferred to the outer surface is dissipated to the surroundings by convection and radiation.

Problem 9:

A long cylindrical rod of 10 cm consists of a nuclear reacting material (k=0.0W/m.K) generating 24,000W/m^3 uniformly throughout its volume. This rod is encapsulated within another cylinder having an outer radius of 20 cm and a thermal conductivity of 4W/m.K. the outer surface is surrounded by a fluid at 100^{0}C, and the convection coefficient between the surface and the fluid is 20W/m^2.K. Find the temperatures at the interface between the two cylinders and at the outer surface.

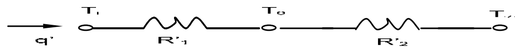
Known: A cylindrical rod with heat generation is cladded with another cylinder whose outer surface is subjected to a convection process.
Find: the temperature at the inner surfaces, $T_{1,}$ and at the outer surface, T_c.

Schematic:



Assumptions: (1) steady-state conditions, (2) one-dimensional radial conduction, (3), negligible contact resistance between the cylinders.

Analysis: The thermal circuit for the outer cylinder subjected to the convection process is



$$R_1' = \frac{\ln r_o / r_1}{2\pi k_2}$$

$$R_2' = \frac{1}{h 2\pi r_o}$$

Using the energy conservation requirement, on the inner cylinder,

$$\dot{E}_{out} = \dot{E}_g$$

Find that

$$q' = \dot{q}_1 \times \pi r_1^2$$

The heat rate equation has the form $\dot{q} = \Delta T / R'$, hence

$$T_i - T_\infty = q' \times (R_1' \times R_2') \text{and} q' = \Delta T / R'$$

Numerical values:
$$R_1' = \ln 0.2/0.1/2\pi \times 4\,\mathrm{W/m.K} = 0.0276\,\mathrm{K.m/W}$$
$$R_2' = 1/20\,\mathrm{W/m^2.K} \times 2\pi \times 0.20\,\mathrm{m} = 0.0398\,\mathrm{K.m/W}$$
$$q' = 24{,}000\,\mathrm{W/m^3} \times \pi \times (0.1)^2\,\mathrm{m^2} = 754.0\,\mathrm{W/m}$$

Hence

$$T_i = 100^0\,\mathrm{C} + 754.0\,\mathrm{W/m} \times (0.0276 + \mathrm{cccc})\,\mathrm{K.m/W} = 100 + 50.8 = 150.8^0\,\mathrm{C}$$
$$T_C = 100^0\,\mathrm{C} + 754.0\,\mathrm{W/m} \times 0.0398\,\mathrm{K.m/W} = 100 + 30 = 130^0\,\mathrm{C}$$

Comments: knowledge of inner cylinder thermal conductivity is not needed.

Problem 10:

An electrical current of 700 A flows through a stainless steel cable having a diameter of 5mm and an electrical resistance of $6*10^{-4}$ /m (i.e. perimeter of cable length). The cable is in an environment having temperature of 300C, and the total coefficient associated with convection and radiation between the cable and the environment is approximately $25W/m^2.K$.

(a) If the cable is bar, what is its surface temperature?

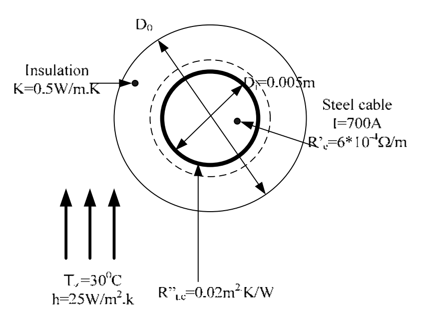
(b) If a very thin coating of electrical insulation is applied to the cable, with a contact resistance of $0.02m^2K/W$, what are the insulation and cable surface temperatures?

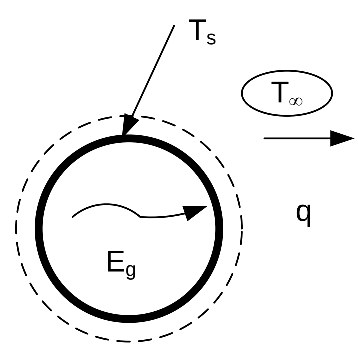
(c) There is some concern about the ability of the insulation to withstand elevated temperatures. What thickness of this insulation (k=0.5W/m.K) will yields the lowest value of the maximum insulation temperature? What is the value of the maximum temperature when the thickness is used?

Known: electric current flow, resistance, diameter and environmental conditions associated with a cable.

Find: (a) surface temperature of bare cable, (b) cable surface and insulation temperatures for a thin coating of insulation, (c) insulation thickness which provides the lowest value of the maximum insulation temperature. Corresponding value of this temperature.

Schematic:





Assumptions: (1) steady-state conditions, (2) one-dimensional conduction in r, (3) constant properties.

Analysis: (a) the rate at which heat is transferred to the surroundings is fixed by the rate of heat generation in the cable. Performing an energy balance for a control surface about the cable, it follows that $\dot{E}_g = q$ or, for the bare cable, $I^2R'_eL = h(\pi D_iL)(T_s - T_\infty)$.with$q' = I^2R'_e = (700A)^2(6\times10^{-4}\Omega/m) = 294W/m.$

It follows that

$$T_s = T_\infty + \frac{q'}{h\pi D_i} = 30^0C + \frac{294W/m}{(25W/m^2.K)\pi(0.005m)}$$

$$T_s = 778.7^0C$$

(b) With thin coating of insulation, there exists contact and convection resistances to heat transfer from the cable. The heat transfer rate is determined by heating within the cable, however, and therefore remains the same,

$$q = \frac{T_s - T_\infty}{R_{t,c} + \frac{1}{h\pi D_iL}} = \frac{T_s - T\infty}{\frac{R_{t,c}}{\pi D_iL} + \frac{1}{h\pi D_iL}}$$

$$q' = \frac{\pi D_i(T_s - T_\infty)}{R_{t,c} + \frac{1}{h}}$$

And solving for the surface temperature, find

$$T_s = \frac{q'}{\pi D_i}\left(R_{t,c} + \frac{1}{h}\right) + T_\infty = \frac{294W/m}{\pi(0.005m)}\left(0.02\frac{m^2.K}{W} + 0.04\frac{m^2.K}{W}\right) + 30^0C$$

$$T_s = 1153^0C$$

The insulation temperature is then obtained from

$$q = \frac{T_s - T_\infty}{R_{t,e}}$$

Or

$$T_i = T_s - qR_{t,c} = 1153^0C - q\frac{R''_{t,c}}{\pi D_i L} = 1153^0C - \frac{294\frac{W}{m}\times 0.02\frac{m^2.K}{W}}{\pi(0.005m)}$$

$$T_i = 778.7^0C$$

(c) The maximum insulation temperature could be reduced by reducing the resistance to heat transfer from the outer surface of the insulation. Such a reduction is possible $D_i<D_{cr.}$

$$r_{cr} = \frac{k}{h} = \frac{0.5W/m.K}{25W/m^2.K} = 0.02m$$

Hence, $D_{cr} = 0.04m > D_i = 0.005m$. To minimize the maximum temperature, which exists at the inner surface of the insulation, add insulation in the amount.

$$t = \frac{D_0 - D_i}{2} = \frac{D_{cr} - D_i}{2} = \frac{(0.04-0.005)m}{2}$$
$$t = 0.0175m$$

The cable surface temperature may then be obtained from

$$q'' = \frac{T_s - T_\infty}{\frac{R''_{t,c}}{\pi D_i} + \frac{\ln(D_{c,r}/D_i)}{2\pi\pi} + \frac{1}{h\pi\pi_{c,r}}} = \frac{T_s - 30^0C}{\frac{0.02m^2.K/W}{\pi(0.005m)} + \frac{\ln(0.04/0.005)}{2\pi\pi(0.5W/.)} + \frac{1}{25\frac{W}{m^2.K}\pi(0.04m)}}$$

hence,

$$294\frac{W}{m} = \frac{T_s - 30^0C}{(1.27+0.66+0.32)m.K/W} = \frac{T_s - 30^0C}{2.25m.K/W}$$

$$T_s = 692.5^0C$$

recognizing that, $q = (T_s - T_i)/R_{t,c,}$

$$T_i = T_s - qR_{t,c,} = T_s - q\frac{R''_{t,c}}{\pi D_i L} = 692.5^0C - \frac{294\frac{W}{m}\times 0.02\frac{m^2.K}{W}}{\pi(0.005m)}$$

$$T_i = 318.2^0C$$

Comments: use of the critical insulation in lieu of a thin coating has the effect of reducing the maximum insulation temperature from 778.7^0C to 318.2^0C. Use of the critical insulation thickness also reduces the cable surface temperatures to 692.5^0C from 778.7^0C with no insulation or fro 1153^0C with a thin coating.

Problem 11:

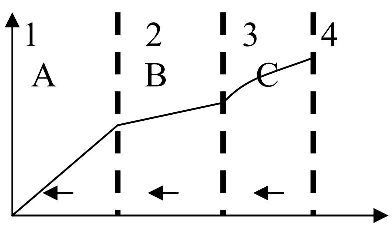
The steady state temperature distribution in a complete plane wall of three different methods, each of constant thermal conductivity, is shown below.

(a) On the relative magnitudes of $q_2^{''} and q_3^{''}$ and of $q_3^{''} and q_4^{''}$.
(b) Comment on the relative magnitudes of k_A and k_B and ok k_B and k_C.
(c) Plot the heat flux as a function of x.

Known: Temperature distribution in a composite wall.

Find: (a) relative magnitudes of interfacial heat fluxes, (b) relative magnitudes of thermal conductivities, and (c) heat fluxes as a function of distance x.

Schematic:



Assumptions: (1) steady-state conditions, (2) one-dimensional conduction, (3) constant properties.

Analysis: (a) for the prescribed conditions (one-dimensional, steady state, constant k), the parabolic temperature distribution in C implies the existence of heat generation. Hence, since dT/dx increases with decreasing x, the heat flux in C increases with decreasing. Hence,

$$q_3^{''} > q_4^{''}$$

However, the linear temperatures distributions in A and B indicate no generation, in which case

$$q_2^{''} = q_3^{''}$$

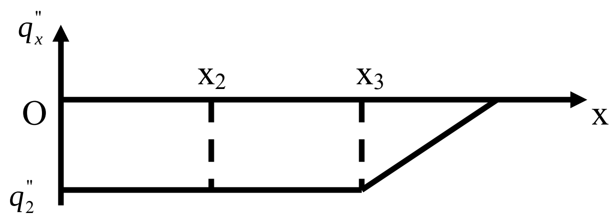
(b) Since conservation of energy requires that $q''_{3,B} = q''_{3,C}$ and dT/dx/$_B$ <dT/dx)$_C$, it follows from Fourier's law that

$K_A > K_C$

Similarly, since $q''_{2,A} = q''_{2,B}$ and dT/dx)$_A$ > dT/dx)$_B$, it follows that

$K_A <$ K$_B$.

(d) It follows that the flux distribution appears as shown below.



Comments: Note that, with dT/dx)$_{4,C}$=0, the interface at 4 is adiabatic.

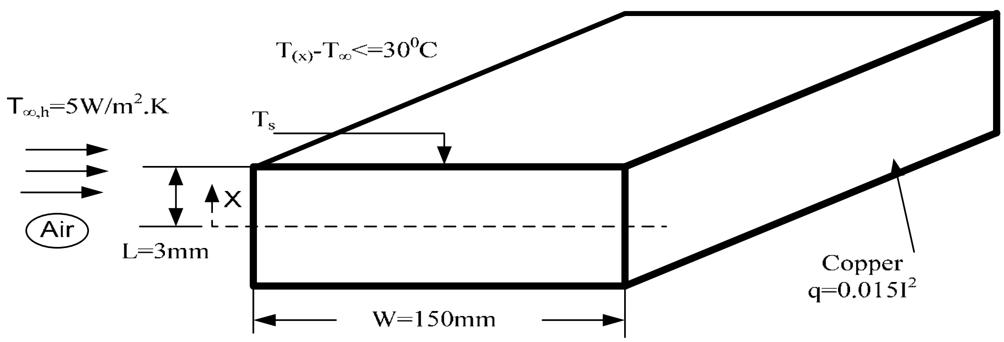
Problem 12:

When passing an electrical current I, a copper bus bar of rectangular cross section (6mm*150mm) experiences uniform heat generation at a rate $\dot{q} = aI^2, \text{where} \alpha = 0.015 W/m^2.A^2$. If the bar is in ambient air with h=5W/m^2. K and its maximum temperature must not exceed that of the air by more than 300C, what is the allowable current capacity for the bar?

Known: Energy generation, q (I), in a rectangular bus bar.

Find: maximum permissible current.

Schematic:



Assumptions: (1) one-dimensional conduction in x (W>>L), (2) steady-state conditions, (3) constant properties, (4) negligible radiation effects.

Properties: table A-I, copper: k 400W/m. K

Analysis: the maximum mid plane temperature is

$$T_0 = \frac{\dot{q}L^2}{2K} + T_s$$

Or substituting the energy balance results,

$$T_o - T_\infty = \dot{q}L\left(\frac{L}{2k} + \frac{1}{h}\right) = 0.015\ I^2 L\left(\frac{L}{2k} + \frac{1}{h}\right).$$

hence ,

$$T_s = T_\infty + \dot{q}L/h,$$

$$I = \left(\frac{T_o - T_\infty}{0.015\ L(L/2k + 1/h)}\right)^{\frac{1}{2}}$$

$$I_{\max} = \left(\frac{30\ ^0C}{0.015\ (W/m^3.A^2)0.003\ m\ \frac{0.003\ m}{800\ W/m.K} + \frac{1}{5W/m^2.K}}\right)^{\frac{1}{2}}$$

$$I_{\max} = 1826\ A$$

Module 2: Short questions

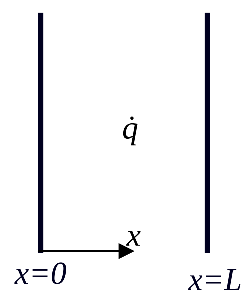
1. How does transient heat transfer differ from steady state heat transfer?

2. What is meant by the term "one-dimensional" when applied to conduction heat transfer?

3. What is meant by thermal resistance? Under what assumptions can the concept of thermal resistance be applied in a straightforward manner?

4. For heat transfer through a single cylindrical shell with convection on the outside, there is a value for the shell radius for a nonzero shell thickness at which the heat flux is maximized. This value is
   (A) k/h
   (B) h/k
   (C) h/r
   (D) r/h

5. The steady temperature profile in a one-dimensional heat transfer across a plane slab of thickness L and with uniform heat generation, $\dot{q}$, has one maximum. If the slab is cooled by convection at x = 0 and insulated at x = L, the maximum occurs at a value of x given by



   (A) 0
   (B) $\frac{L}{2}$
   (C) $\frac{\dot{q}}{k}$
   (D) L

6. Consider a cold canned drink left on a dinner table. Would you model the heat transfer to the drink as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyse this heat transfer problem, and where would you place the origin?

7. Consider a round potato being baked in an oven? Would you model the heat transfer to the potato as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyse this heat transfer problem, and where would you place the origin?

8. Consider an egg being cooked in boiling water in a pan? Would you model the heat transfer to the egg as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyse this heat transfer problem, and where would you place the origin?

Learning Objectives:

- Students should recognize the fin equation.
- Students should know the 2 general solutions to the fin equation.
- Students should be able to write boundary conditions for (a) very long fins, (b) insulated tip fins, (c) convective tip fins and (d) fins with a specified tip temperature.
- Students should be able to apply the boundary conditions to the fin equation and obtain a temperature profile.
- Students should be able to apply the temperature profile to the Fourier Law to obtain a heat flow through the fin.
- Students should be able to apply the concept of fin efficiency to define an equivalent thermal resistance for a fin.
- Students should be able to incorporate fins into an overall electrical network to solve 1-D, SS problems with no sources.

MODULE 3

Extended Surface Heat Transfer

3.1 Introduction:

Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s\text{-}T_\infty)$, where T_s is the surface temperature and T_∞ is the fluid temperature. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference $(T_s\text{-}T_\infty)$ between the surface and the fluid.
- Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the *h*. Example: a cooling fan.
- Increase the contact surface area A. Example: a heat sink with fins.

Many times, when the first option is not in our control and the second option (i.e. increasing *h*) is already stretched to its limit, we are left with the only alternative of increasing the effective surface area by using fins or extended surfaces. Fins are protrusions from the base surface into the cooling fluid, so that the extra surface of the protrusions is also in contact with the fluid. Most of you have encountered cooling fins on air-cooled engines (motorcycles, portable generators, etc.), electronic equipment (CPUs), automobile radiators, air conditioning equipment (condensers) and elsewhere.

3.2 Extended surface analysis:

In this module, consideration will be limited to steady state analysis of rectangular or pin fins of constant cross sectional area. Annular fins or fins involving a tapered cross section may be analyzed by similar methods, but will involve solution of more complicated equations which result. Numerical methods of integration or computer programs can be used to advantage in such cases.

We start with the General Conduction Equation:

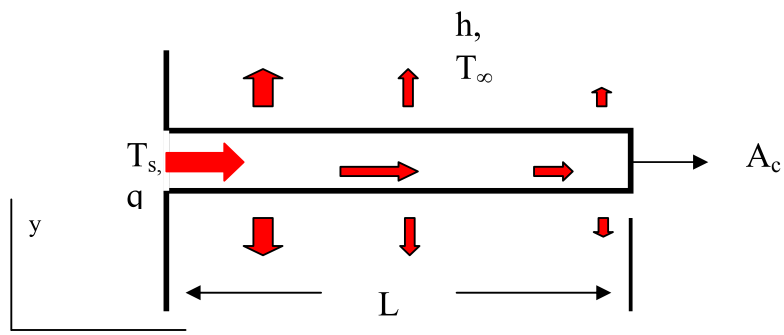
$$\frac{1}{\alpha}\cdot\left.\frac{dT}{d\tau}\right|_{system} = \nabla^2 T + \frac{\ddot{q}}{k} \tag{1}$$

After making the assumptions of Steady State, One-Dimensional Conduction, this equation reduces to the form:

$$\frac{d^2T}{dx^2} + \frac{\ddot{q}}{k} = 0 \tag{2}$$

This is a second order, ordinary differential equation and will require 2 boundary conditions to evaluate the two constants of integration that will arise.

Consider the cooling fin shown below:

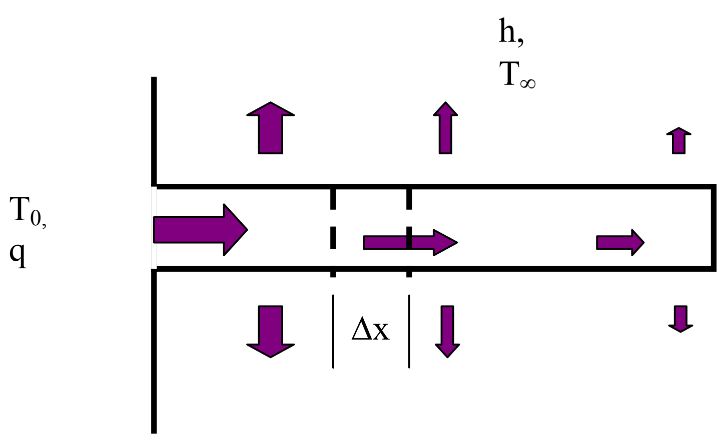


The fin is situated on the surface of a hot surface at T_s and surrounded by a coolant at temperature T_∞, which cools with convective coefficient, h. The fin has a cross sectional area, A_c, (This is the area through with heat is conducted.) and an overall length, L.

Note that as energy is conducted down the length of the fin, some portion is lost, by convection, from the sides. Thus the heat flow varies along the length of the fin.

We further note that the arrows indicating the direction of heat flow point in both the x and y directions. This is an indication that this is truly a two- or three-dimensional heat flow, depending on the geometry of the fin. However, quite often, it is convenient to analyse a fin by examining an equivalent one–dimensional system. The equivalent system will involve the introduction of heat sinks (negative heat sources), which remove an amount of energy equivalent to what would be lost through the sides by convection.

Consider a differential length of the fin.



Across this segment the heat loss will be $h\cdot(P\cdot\Delta x)\cdot(T-T_\infty)$, where P is the perimeter around the fin. The equivalent heat sink would be $\ddot{q}\cdot\left(A_c\cdot\Delta x\right)$.

Equating the heat source to the convective loss:

$$\ddot{q} = \frac{-h \cdot P \cdot \left(T - T_\infty\right)}{A_c} \tag{3}$$

Substitute this value into the General Conduction Equation as simplified for One-Dimension, Steady State Conduction with Sources:

$$\frac{d^2T}{dx^2} - \frac{h \cdot P}{k \cdot A_c} \cdot \left(T - T_\infty\right) = 0 \tag{4}$$

which is the equation for a fin with a constant cross sectional area. This is the Second Order Differential Equation that we will solve for each fin analysis. Prior to solving, a couple of simplifications should be noted. First, we see that h, P, k and A_c are all independent of x in the defined system (They may not be constant if a more general analysis is desired.). We replace this ratio with a constant. Let

$$m^2 = \frac{h \cdot P}{k \cdot A_c} \tag{5}$$

then:

$$\frac{d^2T}{dx^2} - m^2 \cdot \left(T - T_\infty\right) = 0 \tag{6}$$

Next we notice that the equation is non-homogeneous (due to the T_∞ term). Recall that non-homogeneous differential equations require both a general and a particular solution. We can make this equation homogeneous by introducing the temperature relative to the surroundings:

$$\theta \equiv T - T_\infty \tag{7}$$

Differentiating this equation we find:

$$\frac{d\theta}{dx} = \frac{dT}{dx} + 0 \tag{8}$$

Differentiate a second time:

$$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2} \tag{9}$$

Substitute into the Fin Equation:

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \tag{10}$$

This equation is a Second Order, Homogeneous Differential Equation.

3.3 Solution of the Fin Equation

We apply a standard technique for solving a second order homogeneous linear differential equation.

Try $\theta = e^{\alpha \cdot x}$. Differentiate this expression twice:

$$\frac{d\theta}{dx} = \alpha \cdot e^{\alpha \cdot x} \tag{11}$$

$$\frac{d^2\theta}{dx^2} = \alpha^2 \cdot e^{\alpha \cdot x} \tag{12}$$

Substitute this trial solution into the differential equation:

$$\alpha^2 \cdot e^{\alpha \cdot x} - m^2 \cdot e^{\alpha \cdot x} = 0 \tag{13}$$

Equation (13) provides the following relation:

$$\alpha = \pm\ m \tag{14}$$

We now have two solutions to the equation. The general solution to the above differential equation will be a linear combination of each of the independent solutions.

Then:

$$\theta = A \cdot e^{m \cdot x} + B \cdot e^{-m \cdot x} \tag{15}$$

where A and B are arbitrary constants which need to be determined from the boundary conditions. Note that it is a 2^{nd} order differential equation, and hence we need two boundary conditions to determine the two constants of integration.

An alternative solution can be obtained as follows: Note that the hyperbolic sin, sinh, the hyperbolic cosine, cosh, are defined as:

$$\sinh(m \cdot x) = \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} \qquad \cosh(m \cdot x) = \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} \tag{16}$$

We may write:

$$C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) = C \cdot \frac{e^{m \cdot x} + e^{-m \div x}}{2} + D \cdot \frac{e^{m \cdot x} - e^{-m \div x}}{2} = \frac{C + D}{2} \cdot e^{m \cdot x} + \frac{C - D}{2} \cdot e^{-m \cdot x} \tag{17}$$

We see that if (C+D)/2 replaces A and (C-D)/2 replaces B then the two solutions are equivalent.

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \tag{18}$$

Generally the exponential solution is used for very long fins, the hyperbolic solutions for other cases.

Boundary Conditions:

Since the solution results in 2 constants of integration we require 2 boundary conditions. The first one is obvious, as one end of the fin will be attached to a hot surface and will come into thermal equilibrium with that surface. Hence, at the fin base,

$$\theta(0) = T_0 - T_\infty \equiv \theta_0 \qquad (19)$$

The second boundary condition depends on the condition imposed at the other end of the fin. There are various possibilities, as described below.

Very long fins:
For very long fins, the end located a long distance from the heat source will approach the temperature of the surroundings. Hence,

$$\theta(\infty) = 0 \qquad (20)$$

Substitute the second condition into the exponential solution of the fin equation:

$$\theta(\infty) = 0 = A\cdot \underbrace{e^{m\cdot\infty}}_{\infty} + B\cdot \underbrace{e^{-m\cdot\infty}}_{0} \qquad (21)$$

The first exponential term is infinite and the second is equal to zero. The only way that this equation can be valid is if A = 0. Now apply the second boundary condition.

$$\theta(0) = \theta_0 = B\cdot e^{-m\cdot 0} \Rightarrow B = \theta_0 \qquad (22)$$

The general temperature profile for a very long fin is then:

$$\theta(x) = \theta_0 \cdot e^{-m\cdot x} \qquad (23)$$

If we wish to find the heat flow through the fin, we may apply Fourier Law:

$$q = -k\cdot A_c \cdot \frac{dT}{dx} = -k\cdot A_c \cdot \frac{d\theta}{dx} \qquad (24)$$

Differentiate the temperature profile:

$$\frac{d\theta}{dx} = -\theta_o \cdot m\cdot e^{-m\cdot x} \qquad (25)$$

So that:

$$q = k\cdot A_c\cdot\theta_0\cdot\left[\frac{h\cdot P}{k\cdot A_c}\right]^{1/2}\cdot e^{-m\cdot x} = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot e^{-m\cdot x}\cdot\theta_0 = M\theta_0 e^{-mx} \qquad (26)$$

where $M = \sqrt{hPkA_c}$.

Often we wish to know the total heat flow through the fin, i.e. the heat flow entering at the base (x=0).

$$q = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_0 = M\theta_0 \tag{27}$$

The insulated tip fin

Assume that the tip is insulated and hence there is no heat transfer:

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0 \tag{28}$$

The solution to the fin equation is known to be:

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \tag{29}$$

Differentiate this expression.

$$\frac{d\theta}{dx} = C \cdot m \cdot \sinh(m \cdot x) + D \cdot m \cdot \cosh(m \cdot x) \tag{30}$$

Apply the first boundary condition at the base:

$$\theta(0) = \theta_0 = C \overset{0}{\sinh(m \cdot 0)} + D \overset{1}{\cosh(m \cdot 0)} \tag{31}$$

So that D = θ_0. Now apply the second boundary condition at the tip to find the value of C:

$$\frac{d\theta}{dx}(L) = 0 = Cm \sinh(m \cdot L) + \theta_0 m \cosh(m \cdot L) \tag{32}$$

which requires that

$$C = -\theta_0 \frac{\cosh(mL)}{\sinh(mL)} \tag{33}$$

This leads to the general temperature profile:

$$\theta(x) = \theta_0 \frac{\cosh m(L-x)}{\cosh(mL)} \tag{34}$$

We may find the heat flow at any value of x by differentiating the temperature profile and substituting it into the Fourier Law:

$$q = -k \cdot A_c \cdot \frac{dT}{dx} = -k \cdot A_c \cdot \frac{d\theta}{dx} \tag{35}$$

So that the energy flowing through the base of the fin is:

$$q = \sqrt{hPkA_c}\,\theta_0 \tanh(mL) = M\theta_0 \tanh(mL) \tag{36}$$

If we compare this result with that for the very long fin, we see that the primary difference in form is in the hyperbolic tangent term. That term, which always results in a number equal to or less than one, represents the reduced heat loss due to the shortening of the fin.

Other tip conditions:

We have already seen two tip conditions, one being the long fin and the other being the insulated tip. Two other possibilities are usually considered for fin analysis: (i) a tip subjected to convective heat transfer, and (ii) a tip with a prescribed temperature. The expressions for temperature distribution and fin heat transfer for all the four cases are summarized in the table below.

Table 3.1

Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	$M\theta_o \dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L}=0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M\theta_0 \tanh mL$
C	Given temperature: $\theta(L)=\theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh m(L-x) + \sinh m(L-x)}{\sinh mL}$	$M\theta_0 \dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L)=0$	e^{-mx}	$M\ \theta_0$

3.4 Fin Effectiveness

How effective a fin can enhance heat transfer is characterized by the fin effectiveness, ε_f, which is as the ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{\sqrt{hPkA_C}\tanh(mL)}{hA_C} = \sqrt{\frac{kP}{hA_C}}\tanh(mL) \tag{37}$$

If the fin is long enough, mL>2, tanh(mL)→1, and hence it can be considered as infinite fin (case D in Table 3.1). Hence, for long fins,

$$\varepsilon_f \to \sqrt{\frac{kP}{hA_C}} = \sqrt{\left(\frac{k}{h}\right)\frac{P}{A_C}} \tag{38}$$

In order to enhance heat transfer, ε_f should be greater than 1 (In case ε_f <1, the fin would have no purpose as it would serve as an insulator instead). However $\varepsilon_f \geq 2$ is considered unjustifiable because of diminishing returns as fin length increases.

To increase ε_f, the fin's material should have higher thermal conductivity, k. It seems to be counterintuitive that the lower convection coefficient, h, the higher ε_f. Well, if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low.

Observations:

- If fins are to be used on surfaces separating gas and liquid, fins are usually placed on the gas side. (Why?)
- P/A_C should be as high as possible. Use a square fin with a dimension of W by W as an example: P=4W, A_C=W^2, P/A_C=(4/W). The smaller the W, the higher is the P/A_C, and the higher the ε_f .Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

  The effectiveness of a fin can also be characterized by

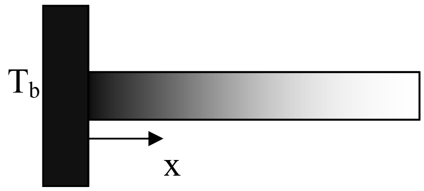
$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}} \tag{39}$$

  It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than the resistance due only to convection.

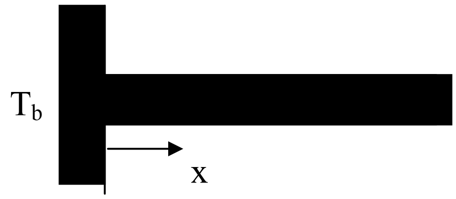
3.5 Fin Efficiency

The fin efficiency is defined as the ratio of the energy transferred through a real fin to that transferred through an ideal fin. An ideal fin is thought to be one made of a perfect or infinite conductor material. A perfect conductor has an infinite thermal conductivity so that the entire fin is at the base material temperature.

$$\eta = \frac{q_{real}}{q_{ideal}} = \frac{\sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_L \cdot \tanh(m \cdot L)}{h \cdot (P \cdot L) \cdot \theta_L} \tag{40}$$



Real situation



Ideal situation

Simplifying equation (40):

$$\eta = \sqrt{\frac{k \cdot A_c}{h \cdot P}} \frac{\theta_L \cdot \tanh(m \cdot L)}{L \cdot \theta_L} = \frac{\tanh(m \cdot L)}{m \cdot L} \tag{41}$$

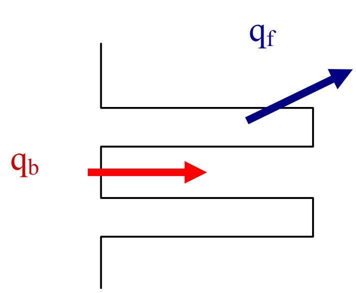
The heat transfer through any fin can now be written as:

$$q.\left[\frac{1}{\eta.h.A_f}\right] = (T - T_\infty \tag{42}$$

The above equation provides us with the concept of fin thermal resistance (using electrical analogy) as

$$R_{t,f} = \frac{1}{\eta.h.A_f} \tag{43}$$

Overall Fin Efficiency:
Overall fin efficiency for an array of fins



Define terms: A_b: base area exposed to coolant

A_f: surface area of a single fin

A_t: total area including base area and total finned surface, $A_t = A_b + NA_f$

N: total number of fins

Heat Transfer from a Fin Array:

$$q_t = q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty)$$
$$= h[(A_t - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_t - NA_f(1-\eta_f)](T_b - T_\infty)$$
$$= hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)](T_b - T_\infty) = \eta_O hA_t(T_b - T_\infty)$$

Define overall fin efficiency: $\eta_O = 1 - \frac{NA_f}{A_t}(1-\eta_f)$

$$q_t = hA_t\eta_O(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,O}} \text{ where } R_{t,O} = \frac{1}{hA_t\eta_O}$$

Compare to heat transfer without fins

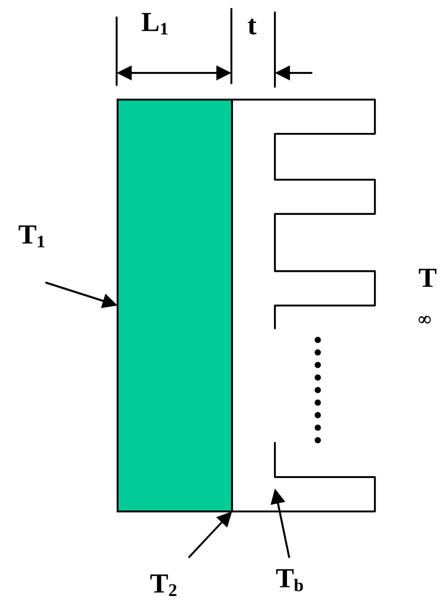
$$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$$

where $A_{b,f}$ is the base area (unexposed) for the fin

To enhance heat transfer $A_t\eta_O >> A$

That is, to increase the effective area $\eta_O A_t$.

Thermal Resistance Concept:



A=Ab+NAb,f

Rb= t/(kbA)

R_1=L_1/(k_1A)

$R_{t,O} = 1/(hA_t\eta_O)$

$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,O}}$$

Multidimensional Heat Transfer

Heat Diffusion Equation

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q} = k\nabla^2 T + \dot{q}$$

- This equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation.
- For steady state $\partial / \partial t = 0$
- No generation $\dot{q} = 0$
- To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior.

Two-D, Steady State Case

For a 2 - D, steady state situation, the heat equation is simplified to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$ it needs two boundary conditions in each direction.

There are three approaches to solve this equation:

- **Numerical Method**: Finite difference or finite element schemes, usually will be solved using computers.
- **Graphical Method**: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see Table 4.1 for selected configurations)
- **Analytical Method**: The mathematical equation can be solved using techniques like the method of separation of variables. (refer to handout)

Conduction Shape Factor

This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T_1) to the other surface (at T_2) can be expressed as: $q=Sk(T_1-T_2)$ where k is the thermal conductivity of the solid and S is the conduction shape factor.

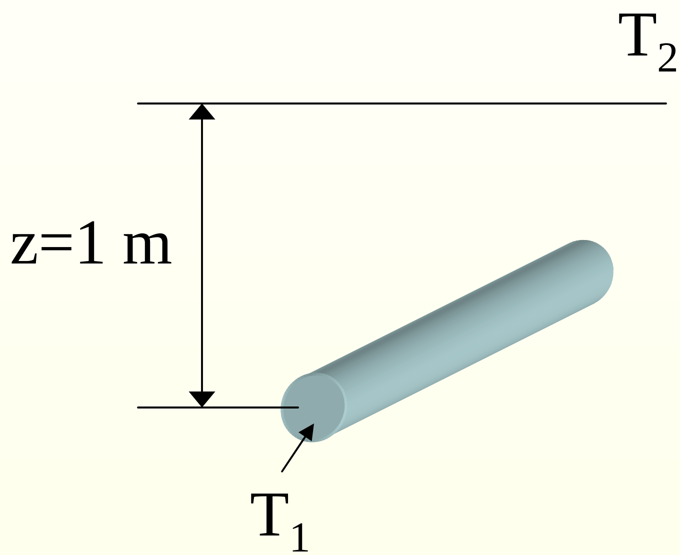
• The shape factor can be related to the thermal resistance:
$q=Sk(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$
where $R_t = 1/(kS)$

• 1-D heat transfer can use shape factor also. Ex: heat transfer inside a plane wall of thickness L is $q=kA(\Delta T/L)$, $S=A/L$

• Common shape factors for selected configurations can be found in Table 4.1

Example

An Alaska oil pipe line is buried in the earth at a depth of 1 m. The horizontal pipe is a thin-walled of outside diameter of 50 cm. The pipe is very long and the averaged temperature of the oil is 100°C and the ground soil temperature is at -20 °C (k_{soil}=0.5W/m.K), estimate the heat loss per unit length of pipe.



From Table 8.7, case 1.
L>>D, z>3D/2

$$S = \frac{2\pi L}{\ln(4z/D)} = \frac{2\pi(1)}{\ln(4/0.5)} = 3.02$$

$$q = kS(T_1 - T_2) = (0.5)(3.02)(100+20)$$

$$= 181.2(W) \text{ heat loss for every meter of pipe}$$

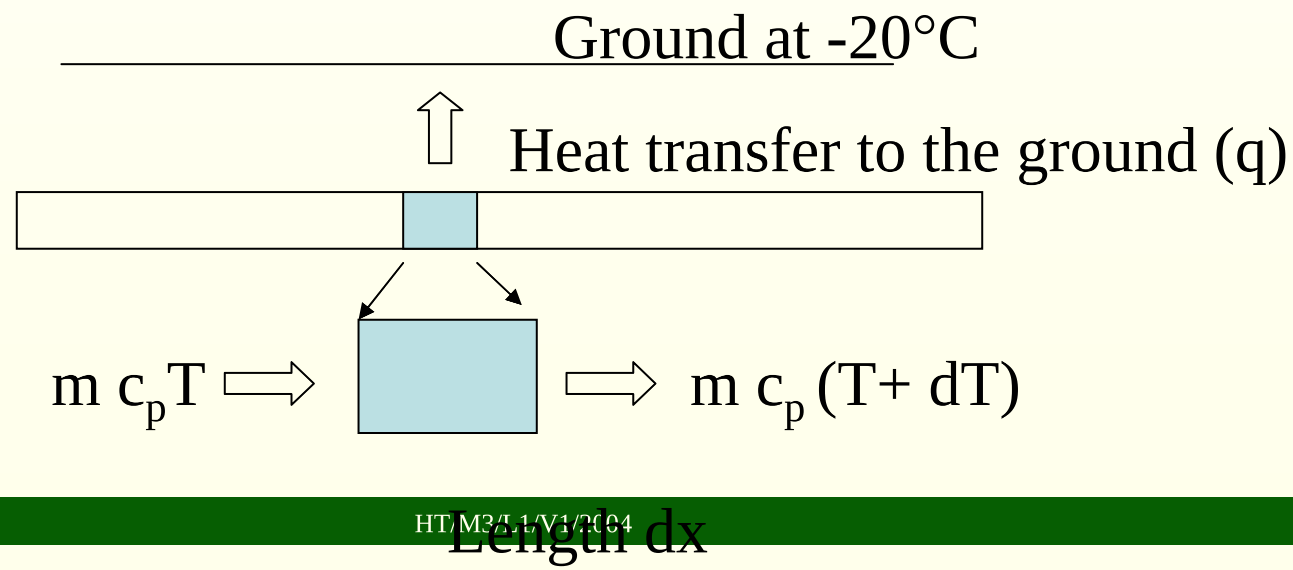
Example (cont.)

If the mass flow rate of the oil is 2 kg/s and the specific heat of the oil is 2 kJ/kg.K, determine the temperature change in 1 m of pipe length.

$$q = \dot{m}C_P\Delta T,\ \Delta T = \frac{q}{\dot{m}C_P} = \frac{181.2}{2000 * 2} = 0.045(^\circ C)$$

Therefore, the total temperature variation can be significant if the pipe is very long. For example, 45°C for every 1 km of pipe length.

- Heating might be needed to prevent the oil from freezing up.
- The heat transfer can not be considered constant for a long pipe



Example (cont.)

Heat Transfer at section with a temperature T(x)

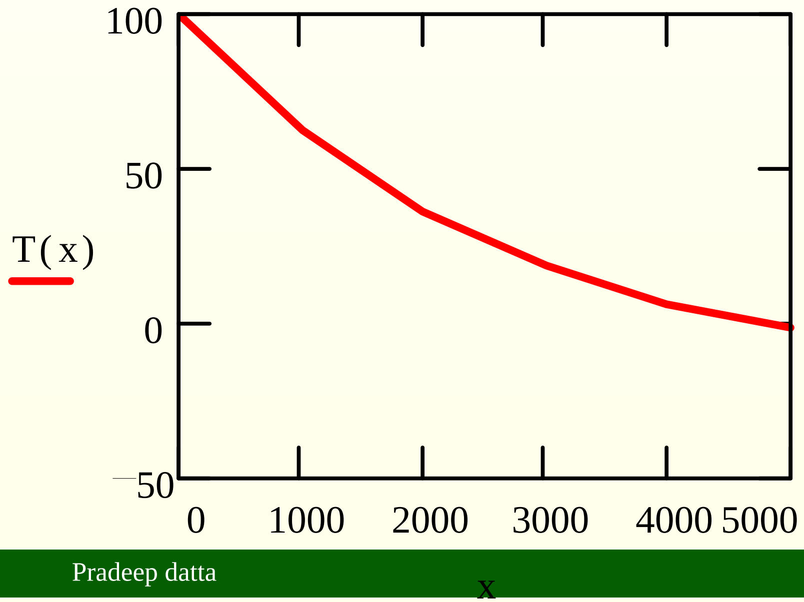
$$q = \frac{2\pi k(dx)}{\ln(4z/D)}(T+20) = 1.51(T+20)(dx)$$

Energy balance: $\dot{m}C_P T - q = \dot{m}C_P(T+dT)$

$$\dot{m}C_P\frac{dT}{dx} + 1.51(T+20) = 0,\ \frac{dT}{T+20} = -0.000378dx, \text{integrate}$$

$$T(x) = -20 + Ce^{-0.000378x},\ \text{at inlet x = 0, T(0) = 100° C, C = 120}$$

$$\text{T(x) = -20} + 120e^{-0.000378x}$$



• Temperature drops exponentially from the initial temp. of 100°C

• It reaches 0°C at x=4740 m, therefore, reheating is required every 4.7 km.

Numerical Methods

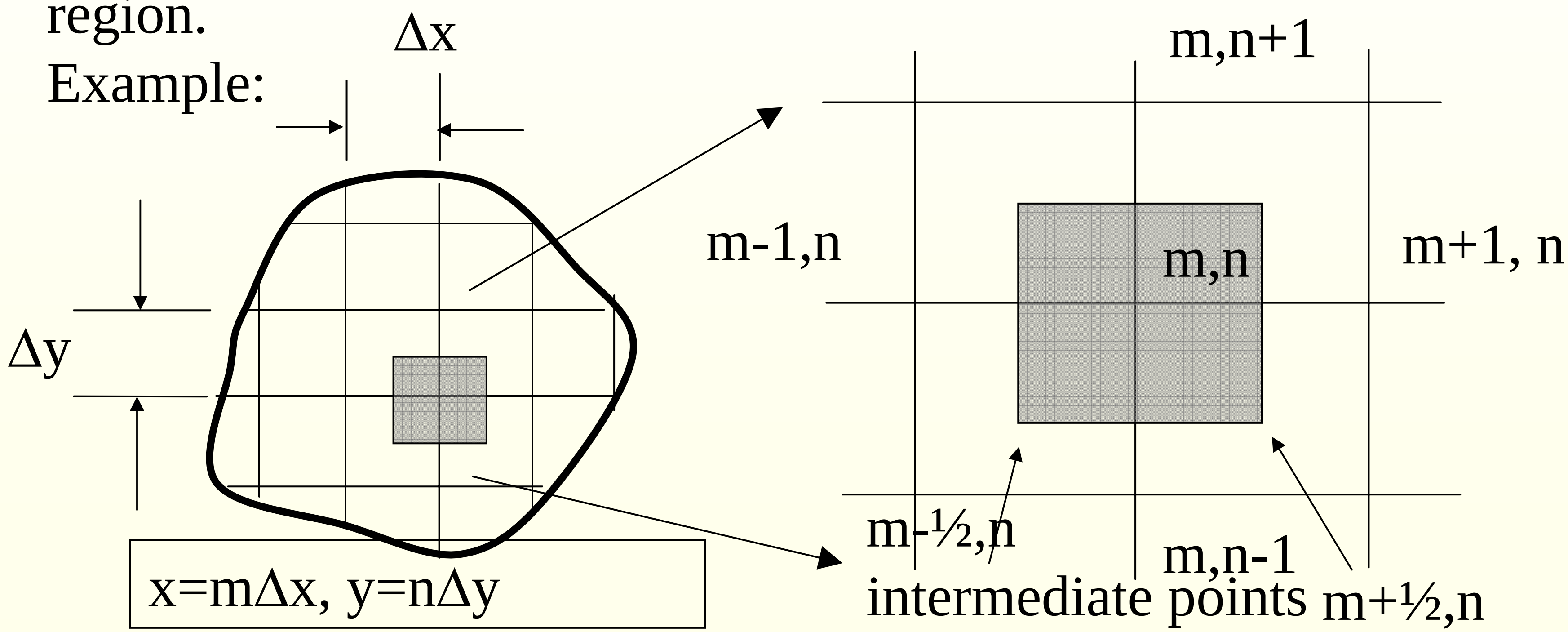
Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, two different approaches are commonly used: the finite difference and the finite element methods. In heat transfer problems, the finite difference method is used more often and will be discussed here. The finite difference method involves:

- Establish nodal networks
- Derive finite difference approximations for the governing equation at both interior and exterior nodal points
- Develop a system of simultaneous algebraic nodal equations
- Solve the system of equations using numerical schemes

The Nodal Networks

The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Dx and Dy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.

Example:



Finite Difference Approximation

Heat Diffusion Equation: $\nabla^2 T + \dfrac{\dot{q}}{k} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$,

where $\alpha = \dfrac{\mathrm{k}}{\rho \mathrm{C_P} V}$ is the thermal diffusivity

No generation and steady state: $\dot{\mathrm{q}}=0$ and $\dfrac{\partial}{\partial \mathrm{t}} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation

at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Finite Difference Approximation (cont.)

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\partial T/\partial x\big|_{m+1/2,n} - \partial T/\partial x\big|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Finite Difference Approximation (cont.)

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify $\Delta x=\Delta y$

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\dfrac{\Delta T}{\Delta x}=\dfrac{\partial T}{\partial x}, \lim(\Delta y \to 0)\dfrac{\Delta T}{\Delta y}=\dfrac{\partial T}{\partial y}$

A System of Algebraic Equations

• The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.
For example: for nodal point m=3, n=4, the equation is
$T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$
$T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$

• Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks (Table 4.2 of our textbook).

• Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

$$
\begin{array}{l}
a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N = C_1 \\
a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N = C_2 \\
\vdots \qquad \vdots \qquad \vdots \quad \vdots \qquad \vdots \\
a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N = C_N
\end{array}
$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**]

$$
where\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}
$$

Numerical Solutions

Matrix form: [**A**][**T**]=[**C**].
From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$
where $[\mathbf{A}]^{-1}$ is the inverse of matrix [**A**]. [**T**] is the solution vector.

- Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)

- Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.

- For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij}T_j + a_{ii}T_i + \sum_{j=i+1}^{N} a_{ij}T_j = C_i,$$

$$(\text{Example}: a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{1N}T_N = C_1,\ i = 3)$$

Rewrite the equation of the form:

Replace (k) by (k-1) for the Jacobi iteration

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{\boxed{(k)}} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated
- The iteration will be stopped when $\max\left| T_i^{(k)} - T_i^{(k-1)} \right| \leq \varepsilon$, where ε specifies a predetermined value of acceptable error

Example

Solve the following system of equations using (a) the Jacobi methos, (b) the Gauss Seidel iteration method.

$$\begin{bmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 16 \end{bmatrix}$$

$$4X + 2Y + Z = 11,$$
$$-X + 2Y + 0 * Z = 3,$$
$$2X + Y + 4Z = 16$$

Reorganize into new form:

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z$$
$$Y = \frac{3}{2} + \frac{1}{2}X + 0 * Z$$
$$Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

(a) Jacobi method: use initial guess $X^0=Y^0=Z^0=1$,

stop when max$\left| X^k-X^{k-1}, Y^k-Y^{k-1}, Z^k-Z^{k-1} \right| \le 0.1$

First iteration:

$X^1= (11/4) - (1/2)Y^0 - (1/4)Z^0 = 2$

$Y^1= (3/2) + (1/2)X^0 = 2$

$Z^1= 4 - (1/2) X^0 - (1/4)Y^0 = 13/4$

Example (cont.)

Second iteration: use the iterated values $X^1=2$, $Y^1=2$, $Z^1=13/4$

$X^2 = (11/4) - (1/2)Y^1 - (1/4)Z^1 = 15/16$

$Y^2 = (3/2) + (1/2)X^1 = 5/2$

$Z^2 = 4 - (1/2) X^1 - (1/4)Y^1 = 5/2$

Converging Process:

$$[1,1,1], \left[2,2,\frac{13}{4}\right], \left[\frac{15}{16},\frac{5}{2},\frac{5}{2}\right], \left[\frac{7}{8},\frac{63}{32},\frac{93}{32}\right], \left[\frac{133}{128},\frac{31}{16},\frac{393}{128}\right]$$

$\left[\frac{519}{512},\frac{517}{256},\frac{767}{256}\right]$. Stop the iteration when

$$\max \left| X^5 - X^4, Y^5 - Y^4, Z^5 - Z^4 \right| \leq 0.1$$

Final solution [1.014, 2.02, 2.996]

Exact solution [1, 2, 3]

Example (cont.)

(b) Gauss-Seidel iteration: Substitute the iterated values into the iterative process immediately after they are computed.

Use initial guess $X^0 = Y^0 = Z^0 = 1$

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z,\ Y = \frac{3}{2} + \frac{1}{2}X,\ Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

First iteration: $X^1 = \frac{11}{4} - \frac{1}{2}(Y^0) - \frac{1}{4}(Z^0) = 2$

Immediate substitution

$$Y^1 = \frac{3}{2} + \frac{1}{2}X^1 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$$

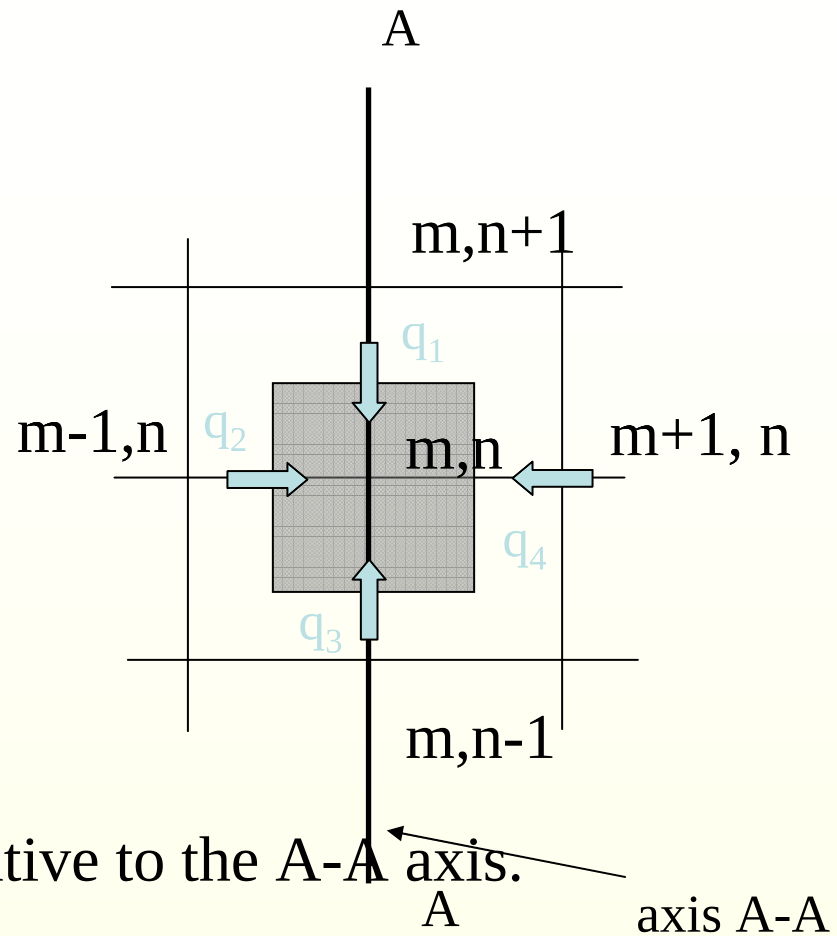
$$Z^1 = 4 - \frac{1}{2}X^1 - \frac{1}{4}Y^1 = 4 - \frac{1}{2}(2) - \frac{1}{4}\left(\frac{5}{2}\right) = \frac{19}{8}$$

Converging process: [1,1,1], $\left[2, \frac{5}{2}, \frac{19}{8}\right], \left[\frac{29}{32}, \frac{125}{64}, \frac{783}{256}\right], \left[\frac{1033}{1024}, \frac{4095}{2048}, \frac{24541}{8192}\right]$

The iterated solution [1.009, 1.9995, 2.996] and it converges faster

Numerical Method (Special Cases)

For all the special cases discussed in the following, the derivation will be based on the standard nodal point configuration as shown to the right.



☐ Symmetric case: symmetrical relative to the A-A axis.

In this case, $T_{m-1,n}=T_{m+1,n}$

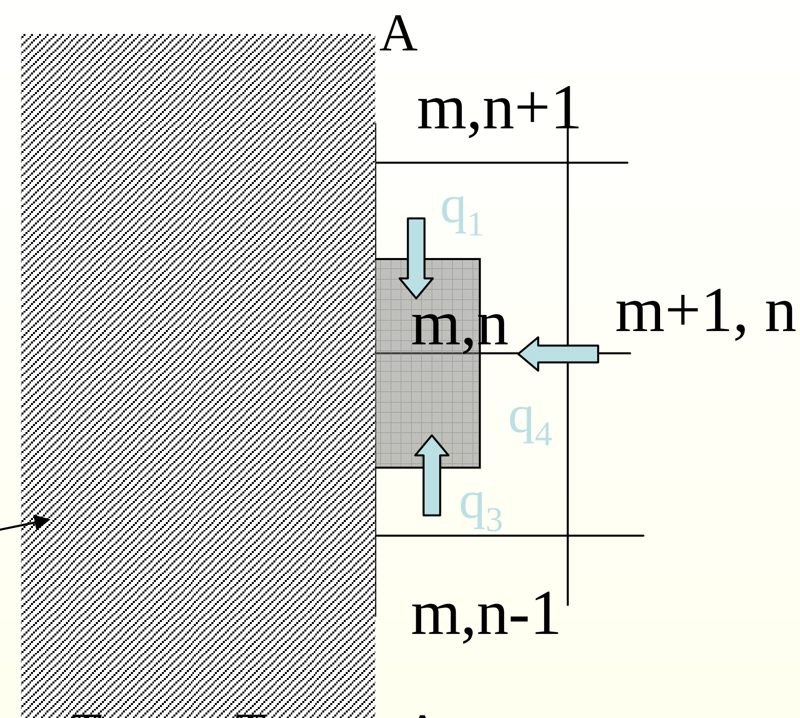
Therefore the standard nodal equation can be written as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}$$

$$=2T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

Special cases (cont.)

- Insulated surface case: If the axis A-A is an insulated wall, therefore there is no heat transfer across A-A. Also, the surface area for q_1 and q_3 is only half of their original value. Write the energy balance equation (q_2=0):



$$q_1 + q_3 + q_4 = 0$$

$$k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k\Delta y\frac{T_{m+1,n} - T_{m,n}}{\Delta x} = 0$$

$$2T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

This equation is identical to the symmetrical case discussed previously.

Special cases (cont.)

- With internal generation G=gV where g is the power generated per unit volume (W/m^3). Based on the energy balance concept:

$$q_1 + q_2 + q_3 + q_4 + G$$

$$q_1 + q_2 + q_3 + q_4 + g(\Delta x)(\Delta y)(1) = 0$$

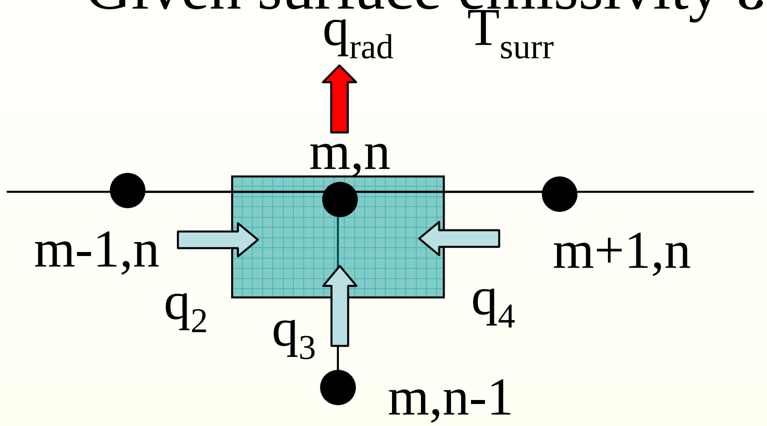
Use 1 to represent the dimension along the z-direction.

$$k(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n}) + g\left(\Delta x\right)^2 = 0$$

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{g(\Delta x)^2}{k} = 0$$

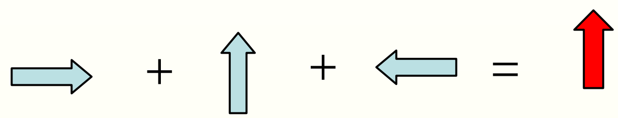
Special cases (cont.)

☐ Radiation heat exchange with respect to the surrounding (assume no convection, no generation to simplify the derivation). Given surface emissivity ε, surrounding temperature T_{surr}.



From energy balance concept:

$q_2+q_3+q_4=q_{rad}$



$$k\left(\frac{\Delta y}{2}\right)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}+k\left(\Delta x\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}+k\left(\frac{\Delta y}{2}\right)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}=\varepsilon\sigma\left(\Delta x\right)\left({T_{m,n}}^4-{T_{surr}}^4\right)$$

$$k\left(T_{m-1,n}+T_{m+1,n}+2T_{m,n-1}-4T_{m,n}\right)=2\varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}^4-T_{surr}^4\right)$$

$$T_{m-1,n}+T_{m+1,n}+2T_{m,n-1}-4T_{m,n}\boxed{-\frac{2\varepsilon\sigma\left(\Delta x\right)}{k}T_{m,n}^4}=-2\frac{\varepsilon\sigma\left(\Delta x\right)}{k}T_{surr}^4$$

Non-linear term, can solve using the iteration method

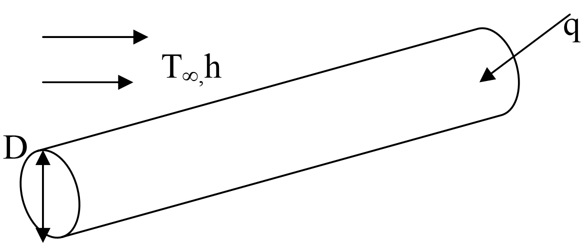
Problem 1:

A long, circular aluminium rod attached at one end to the heated wall and transfers heat through convection to a cold fluid.
(a) If the diameter of the rod is triples, by how much would the rate of heat removal change?
(b) If a copper rod of the same diameter is used in place of aluminium, by how much would the rate of heat removal change?

Known: long, aluminum cylinder acts as an extended surface.

Find: (a) increase in heat transfer if diameter is tripled and (b) increase in heat transfer if copper is used in place of aluminum.

Schematic:



Assumptions: (1) steady-state conditions, (2) one-dimensional conduction, (3) constant properties, (4) uniform convection coefficient, (5)rod is infinitely long.

Properties: Table A-1, aluminum (pure): k=240W/m. K;
Table A-1, copper (pure): k=400W/m. K

Analysis: (a) for an infinitely long fin, the fin rate is

$$qr = M = (hpkA_c)^{\frac{1}{2}}\theta_b$$

$$q_f = (h\pi D k \pi D^2/4)^{\frac{1}{2}}\theta_b = \frac{\pi}{2}(hk)^{\frac{1}{2}} D^{\frac{3}{2}}\theta_b$$

Where P=πD and $A_c=\pi D^2/4$ for the circular cross-section. Note that $q_f \approx D^3/^2$. Hence, if the diameter is tripled,

$$\frac{q_{f(3D)}}{q_{f(D)}} = 3^{\frac{3}{2}} = 5.2$$

And there is a 520 % increase in heat transfer.

(b) in changing from aluminum to copper, since $q_f \approx k^{1/2}$, it follows that

$$\frac{q_f(Cu)}{q_f(Al)} = \left(\frac{k_{Cu}}{k_{Al}}\right)^{\frac{1}{2}} = \left(\frac{400}{240}\right)^{\frac{1}{2}} = 1.29$$

And there is a 29 % increase in the heat transfer rate.

Comments: (1) because fin effectiveness is enhanced by maximum $P/A_c = 4/D$. the use of a larger number of small diameter fins is preferred to a single large diameter fin.

(2) From the standpoint of cost and weight, aluminum is preferred over copper.

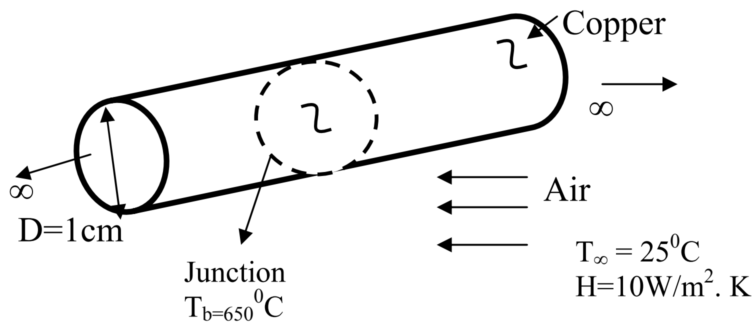
Problem 2:

Two long copper rods of diameter D=1 cm are soldered together end to end, with solder having melting point of 650^0C.The rods are in the air at 250C with a convection coefficient of $10W/m^2$. K. What is the minimum power input needed to effect the soldering?

Known: Melting point of solder used to join two long copper rods.

Find: Minimum power needed to solder the rods.

Schematic:



Assumptions: (1) steady-state conditions, (2) one-dimensional conduction along the rods, (3) constant properties, (4) no internal heat generation, (5) negligible radiation exchange with surroundings, (6) uniform, h and (7) infinitely long rods.

Properties: table A-1: copper $\bar{T} = (650+25)^0C \approx 600K$: k=379 W/m.K

Analysis: the junction must be maintained at 650^0C while energy is transferred by conduction from the junction (along both rods). The minimum power is twice the fin heat rate for an infinitely long fin,

$$q_{min} = 2q_f = 2(hpkA_c)^{\frac{1}{2}}(T_b - T_\infty)$$

substituting numerical values,

$$q_{min} = 2\left(10\frac{W}{m^2.K}(\pi \times 0.01m)\left(379\frac{W}{m.K}\right)\frac{\pi}{4}(0.01m)^2\right)^{\frac{1}{2}}(650 - 25)^0C.$$

therefore,

$$q_{min} = 120.9W$$

Comments: radiation losses from the rod are significant, particularly near the junction, thereby requiring a larger power input to maintain the junction at 650^0C.

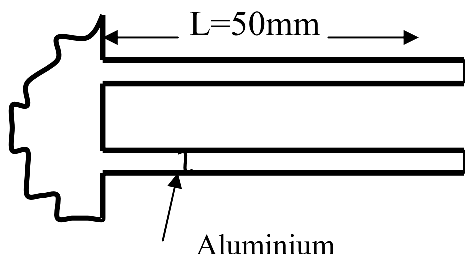
Problem 3:

Determine the percentage increase in heat transfer associated with attaching aluminium fins of rectangular profile to a plane wall. The fins are 50mm long, 0.5mm thick, and are equally spaced at a distance of 4mm (250fins/m). The convection coefficient affected associated with the bare wall is $40W/m^2$. K, while that resulting from attachment of the fins is $30W/m^2$. K.

Known: Dimensions and number of rectangular aluminum fins. Convection coefficient with and without fins.

Find: percentage increase in heat transfer resulting from use of fins.

Schematic:



$N=250m^{-1}$
W=width
$h_w = 30W/m^2.K$(with fins)
$h_{wo}=40W/m^2.K$(without fins)

Assumptions: (1) steady-state conditions, (2) one-dimensional conduction, (3) constant properties, (4) negligible fin contact resistance, (6) uniform convection coefficient.

Properties: table A-1, Aluminum, pure: k≈240W/m. K

Analysis: evaluate the fin parameters

$$L_c = L + \frac{t}{2} = 0.0505m$$

$$A_p = L_c t = 0.0505m \times 0.5 \times 10^{-3} m = 25.25 \times 10^{-6} m^2$$

$$L_c^{3/2}(h_w / KA_P)^{1/2} (0.0505m)^{3/2} \left(\frac{30W/m^2.K}{240W/m.K \times 25.25 \times 10^{-6} m^2} \right)^{1/2}$$

$$L_c^{3/2}(h_w / KA_P)^{1/2} = 0.80$$

it follows that, $\eta_f = 0.72$. hence

$$q_f = \eta_f q_{max} = 0.72 h_w 2wL\theta_b$$

$$q_f = 0.72 \times 30W/m^2.K \times 2 \times 0.05m \times (w\theta_b) = 2.16W/m/K(w\theta_b)$$

With the fins, the heat transfer from the walls is

$$q_w = Nq_f + (1 - Nt)wh_w\theta_b$$

$$q_w = 250 \times 2.16\frac{W}{m.K}(w\theta_b) + (1m - 250 \times 5 \times 10^{-4} m) \times 30W/m^2.K(w\theta_b)$$

$$q_w = (540 + 26.63)\frac{W}{m.K}(w\theta_b) = 566w\theta_b$$

Without the fins, $q_{wo} = h_{wo}1m \times w\theta_b = 40w\theta_b$.

Hence the percentage increases in heat transfer is

$$\frac{q_w}{q_{wo}} = \frac{566w\theta_b}{40w\theta_b} = 14.16 = 1416\%$$

Comments: If the finite fin approximation is made, it follows that q_f = $(hPkA_{c)}$ $^{1/2}\theta_b$ = $[h_w 2wkwt]^{1/2}\theta_b$ = $(30*2*240*5*10^{-4})^{1/2}$ $w\theta_b$ =2.68$w\theta_b$. Hence q_f is overestimated.

Module 3: Short questions

1. Is the heat flow through a fin truly one-dimensional? Justify the one-dimensional assumption made in the analysis of fins.

2. Insulated tip condition is often used in fin analysis because (choose one answer)
   A. Fins are usually deliberately insulated at their tips
   B. There is no heat loss from the tip
   C. The heat loss from the tip is usually insignificant compared to the rest of the fin, and the insulated tip condition makes the problem mathematically simple

3. What is the difference between fin effectiveness and fin efficiency?

4. The fins attached to a surface are determined to have an effectiveness of 0.9. Do you think the rate of heat transfer from the surface has increased or decreased as a result of addition of fins?

5. Fins are normally meant to enhance heat transfer. Under what circumstances the addition of fins may actually decrease heat transfer?

6. Hot water is to be cooled as it flows through the tubes exposed to atmospheric air. Fins are to be attached in order to enhance heat transfer. Would you recommend adding fins to the inside or outside the tubes? Why?

7. Hot air is to be cooled as it is forced through the tubes exposed to atmospheric air. Fins are to be attached in order to enhance heat transfer. Would you recommend adding fins to the inside or outside the tubes? Why? When would you recommend adding fins both inside and outside the tubes?

8. Consider two finned surface which are identical except that the fins on the fist surface are formed by casting or extrusion, whereas they are attached to the second surface afterwards by welding or tight fitting. For which case do you think the fins will provide greater enhancement in heat transfer? Explain.

9. Does the (a) efficiency and (b) effectiveness of a fin increase or decrease as the fin length is increased?

10. Two pin fins are identical, except that the diameter of one of them is twice the diameter of the other? For which fin will the a) efficiency and (b) effectiveness be higher? Explain.

11. Two plate fins of constant rectangular cross section are identical, except that the thickness of one of them is twice the thickness of the other? For which fin will the a) efficiency and (b) effectiveness be higher? Explain.

12. Two finned surface are identical, except that the convection heat transfer coefficient of one of them is twice that of the other? For which fin will the a) efficiency and (b) effectiveness be higher? Explain.

Module 4: Learning objectives

- The primary objective of this module is to develop an appreciation for the nature of two- or multi-dimensional conduction problems and the methods that are available for its solutions.
- For a multi-dimensional problem, the student should be able to determine whether an exact solution is known. This may be done by examining one or more of the many excellent references in which exact solutions to the heat equation are obtained.
- The student should understand what a conduction shape factor is, and link it to the concept of thermal resistances in 2D problems. The student should be able to determine whether the shape factor is known for the system, and if available, use to solve the heat transfer problem.
- However, if conditions are such that the use of a shape factor or an exact solution is not possible, the student should be able to use a numerical solution, such as the finite difference method..
- The student should appreciate the inherent nature of the *discretization process*, and know how to formulate the finite difference equations for the discrete points of a nodal network. Although one may find it convenient to solve these equations using hand calculations for a coarse mesh, one should be able to treat fine meshes using standard computer algorithms involving direct or iterative techniques.

MODULE 4

MULTI-DIMENSIONAL STEADY STATE HEAT CONDUCTION

4.1 Introduction

We have, to this point, considered only One Dimensional, Steady State problems. The reason for this is that such problems lead to ordinary differential equations and can be solved with relatively ordinary mathematical techniques.

In general the properties of any physical system may depend on both location (x, y, z) and time (τ). The inclusion of two or more independent variables results in a partial differential equation. The multidimensional heat diffusion equation in a Cartesian coordinate system can be written as:

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k} \tag{1}$$

The above equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation. To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior. For 2D, steady state ($\partial/\partial t = 0$) and without heat generation, the above equation reduces to:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{2}$$

Equation (2) needs 2 boundary conditions in each direction. There are three approaches to solve this equation:

- **Analytical Method**: The mathematical equation can be solved using techniques like the method of separation of variables.
- **Graphical Method**: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see Table 4.1 for selected configurations)
- **Numerical Method**: Finite difference or finite volume schemes, usually will be solved using computers.

Analytical solutions are possible only for a limited number of cases (such as linear problems with simple geometry). Standard analytical techniques such as separation of variables can be found in basic textbooks on engineering mathematics, and will not be reproduced here. The student is encouraged to refer to textbooks on basic mathematics for an overview of the analytical solutions to heat diffusion problems. In the present lecture material, we will cover the graphical and numerical techniques, which are used quite conveniently by engineers for solving multi-dimensional heat conduction problems.

4.2 Graphical Method: Conduction Shape Factor

This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T1) to the other surface (at T2) can be expressed as: q=Sk(T1-T2) where k is the thermal conductivity of the solid and S is the conduction shape factor.

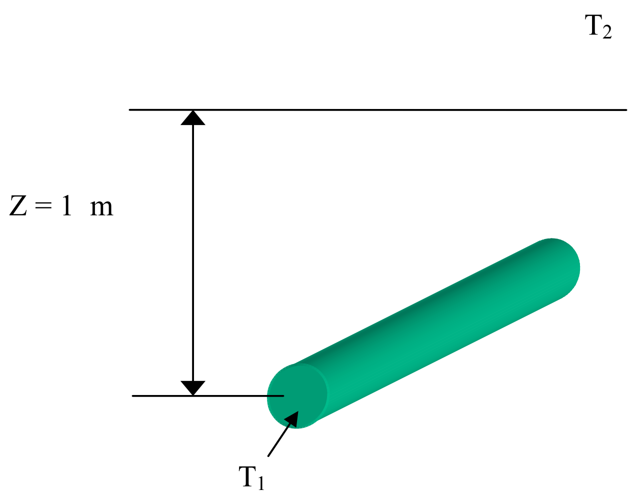
The shape factor can be related to the thermal resistance:

$$q=S.k.(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$$

where R_t = 1/(kS) is the thermal resistance in 2D. Note that 1-D heat transfer can also use the concept of shape factor. For example, heat transfer inside a plane wall of thickness L is q=kA(ΔT/L), where the shape factor S=A/L. Common shape factors for selected configurations can be found in Table 4.1

<u>Example:</u> A 10 cm OD uninsulated pipe carries steam from the power plant across campus. Find the heat loss if the pipe is buried 1 m in the ground is the ground surface temperature is 50 ºC. Assume a thermal conductivity of the sandy soil as k = 0.52 w/m K.

<u>Solution:</u>



The shape factor for long cylinders is found in Table 4.1 as Case 2, with L >> D:

$$S = 2\cdot\pi\cdot L/\ln(4\cdot z/D)$$

Where z = depth at which pipe is buried.

$$S = 2\cdot\pi\cdot 1\cdot m/\ln(40) = 1.7 \text{ m}$$

Then

$$q' = (1.7\cdot m)(0.52 \text{ W/m}\cdot K)(100\,^oC - 50\,^oC)$$

$$q' = 44.2 \text{ W}$$

Table 4.1
Conduction shape factors for selected two-dimensional systems [$q = Sk(T_1 - T_2)$]

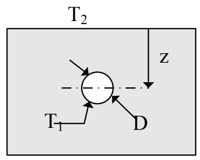
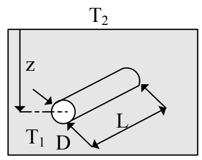
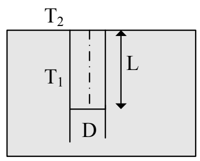
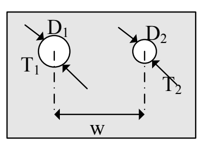
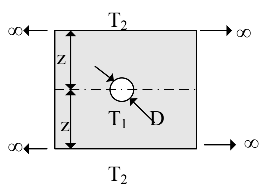
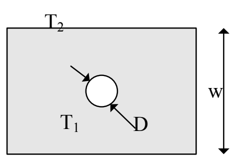
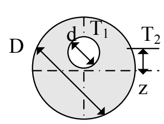
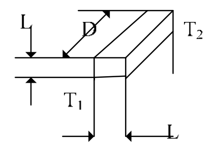
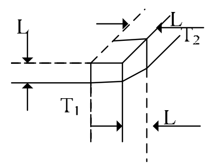
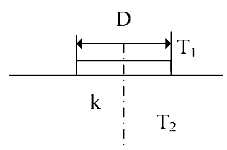
System	Schematic	Restrictions	Shape Factor
Isothermal sphere buried in as finite medium		z>D/2	$\frac{2\pi D}{1 - D/4z}$
Horizontal isothermal cylinder of length L buried in a semi finite medium		L>>D L>>D z>3D/2	$\frac{2\pi L}{\cosh^{-1}(2z/D)}$ $\frac{2\pi L}{\ln(4z/D)}$
Vertical cylinder in a semi finite medium		L>>D	$\frac{2\pi L}{\ln(4L/D)}$
Conduction between two cylinders of length L in infinite medium		$L \gg D_1, D_2$ L>>w	$\frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$
Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		z>>D/2 L>>2	$\frac{2\pi L}{\ln(8z/\pi D)}$
Circular cylinder of length L centered in a square solid of equal length		W>D L>>w	$\frac{2\pi L}{\ln(1.08w/D)}$
Eccentric circular cylinder of length L in a cylinder of equal length		D>d L>>D	$\frac{2\pi L}{\cosh^{-1}\left(\frac{D^2 + d^2 - 4z^2}{2Dd}\right)}$

Table 4.1	Continued		
System	Schematic	Restrictions	Shape Factor
Conduction through the edge of adjoining walls		D>L/5	0.54D
Conduction through corners of three walls with a temperature difference of ΔT_{1-2} across the walls		L<<length and width of wall	0.15L
Disk of diameter D and T1 on a semi finite medium of thermal conductivity k and T_2		None	2D

4.3 Numerical Methods

Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, three different approaches are commonly used: the finite difference, the finite volume and the finite element methods. Brief descriptions of the three methods are as follows:

The Finite Difference Method (FDM)

This is the oldest method for numerical solution of PDEs, introduced by Euler in the 18th century. It's also the easiest method to use for simple geometries. The starting point is the conservation equation in differential form. The solution domain is covered by grid. At each grid point, the differential equation is approximated by replacing the partial derivatives by approximations in terms of the nodal values of the functions. The result is one algebraic equation per grid node, in which the variable value at that and a certain number of neighbor nodes appear as unknowns.

In principle, the FD method can be applied to any grid type. However, in all applications of the FD method known, it has been applied to structured grids. Taylor series expansion or polynomial fitting is used to obtain approximations to the first and second derivatives of the variables with respect to the coordinates. When necessary, these methods are also used to obtain variable values at locations other than grid nodes (interpolation).

On structured grids, the FD method is very simple and effective. It is especially easy to obtain higher-order schemes on regular grids. The disadvantage of FD methods is that the conservation is not enforced unless special care is taken. Also, the restriction to simple geometries is a significant disadvantage.

Finite Volume Method (FVM)

In this dissertation finite volume method is used. The FV method uses the integral form of the conservation equations as its starting point. The solution domain is subdivided into a finite number of contiguous control volumes (CVs), and the conservation equations are applied to each CV. At the centroid of each CV lies a computational node at which the variable values are to be calculated. Interpolation is used to express variable values at the CV surface in terms of the nodal (CV-center) values. As a result, one obtains an algebraic equation for each CV, in which a number of neighbor nodal values appear. The FVM method can accommodate any type of grid when compared to FDM, which is applied to only structured grids. The FVM approach is perhaps the simplest to understand and to program. All terms that need be approximated have physical meaning, which is why it is popular.

The disadvantage of FV methods compared to FD schemes is that methods of order higher than second are more difficult to develop in 3D. This is due to the fact that the FV approach requires two levels of approximation: interpolation and integration.

Finite Element Method (FEM)

The FE method is similar to the FV method in many ways. The domain is broken into a set of discrete volumes or finite elements that are generally unstructured; in 2D, they are usually triangles or quadrilaterals, while in 3D tetrahedra or hexahedra are most often used. The distinguishing feature of FE methods is that the equations are multiplied by a weight function before they are integrated over the entire domain. In the simplest FE methods, the solution is approximated by a linear shape function within each element in a way that guarantees continuity of the solution across element boundaries. Such a function can be constructed from its values at the corners of the elements. The weight function is usually of the same form.

This approximation is then substituted into the weighted integral of the conservation law and the equations to be solved are derived by requiring the derivative of the integral with respect to each nodal value to be zero; this corresponds to selecting the best solution within the set of allowed functions (the one with minimum residual). The result is a set of non-linear algebraic equations.

An important advantage of finite element methods is the ability to deal with arbitrary geometries. Finite element methods are relatively easy to analyze mathematically and can be shown to have optimality properties for certain types of equations. The principal drawback, which is shared by any method that uses unstructured grids, is that the matrices of the linearized equations are not as well structured as those for regular grids making it more difficult to find efficient solution methods.

4.4 The Finite Difference Method Applied to Heat Transfer Problems:

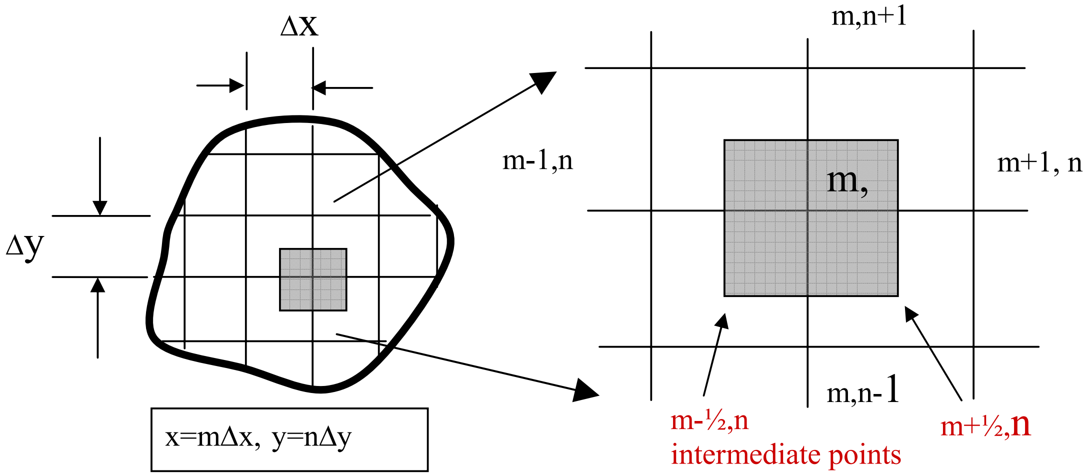
In heat transfer problems, the finite difference method is used more often and will be discussed here in more detail. The finite difference method involves:

- Establish nodal networks
- Derive finite difference approximations for the governing equation at both interior and exterior nodal points
- Develop a system of simultaneous algebraic nodal equations
- Solve the system of equations using numerical schemes

<u>The Nodal Networks:</u>

The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Δx and Δy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.

Example:



Finite Difference Approximation:

Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{\mathrm{k}}{\rho \mathrm{C_P} V}$ is the thermal diffusivity

No generation and steady state: $\dot{\mathrm{q}}=0$ and $\frac{\partial}{\partial \mathrm{t}} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation

at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\partial T/\partial x|_{m+1/2,n} - \partial T/\partial x|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to

the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify Δx=Δy

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\frac{\Delta T}{\Delta x}=\frac{\partial T}{\partial x}, \lim(\Delta y \to 0)\frac{\Delta T}{\Delta y}=\frac{\partial T}{\partial y}$

Table 4.2 provides a list of nodal finite difference equation for various configurations.

A System of Algebraic Equations

- The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.

  For example: for nodal point m=3, n=4, the equation is

  $T_{2,4}+T_{4,4}+T_{3,3}+T_{3,5}-4T_{3,4}=0$

  $T_{3,4}=(1/4)(T_{2,4}+T_{4,4}+T_{3,3}+T_{3,5})$
- Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks.
- Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

$$\begin{aligned}
a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N &= C_1 \\
a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N &= C_2 \\
\vdots \quad\quad \vdots \quad\quad \vdots \quad \vdots \quad\quad & \quad \vdots \\
a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N &= C_N
\end{aligned}$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**] .

$$where\ A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T=\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C=\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

Table 4.2 Summary of nodal finite-difference methods

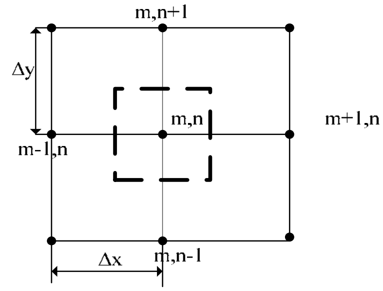
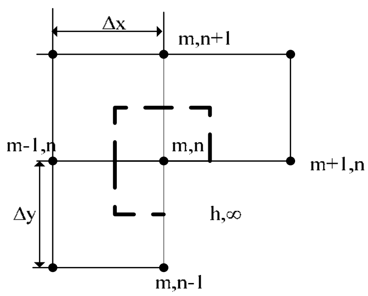
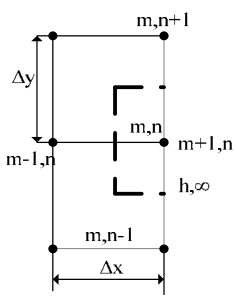
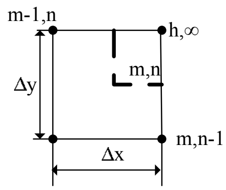
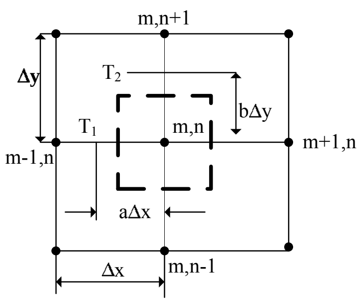
Configuration	Finite-Difference equations for Δx=Δy
	$T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}-4T_{m,n}=0$ Case 1. Interior node
	$2(T_{m-1,n}+T_{m,n+1})+(T_{m+1,n}+T_{m,n-1})+2\frac{h\Delta x}{k}T_{\infty}-2\left(3+\frac{h\Delta x}{k}\right)T_{m,n}=0$ Case 2. Node at an internal corner with convection
	$2(T_{m-1,n}+T_{m,n+1}+T_{m,n-1})+2\frac{h\Delta x}{k}T_{\infty}-2\left(\frac{h\Delta x}{k}+2\right)T_{m,n}=0$ Case 3. Node at a plane surface with convection

Table 4.2 Summary of nodal finite-difference methods



$$2(T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + 2\frac{h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{m,n} = 0$$

Case 4. Node at an external corner with convection



$$\frac{2}{a+1}T_{m+1,n} + \frac{2}{b+1}T_{m,n-1} + \frac{2}{a(a+1)}T_1 + \frac{2}{b)b+1)}T_2 - \left(\frac{2}{a} + \frac{2}{b}\right)T_{m,n} = 0$$

Case 5. Node near a curved surface maintained at a non uniform temperature

<u>Numerical Solutions</u>

Matrix form: [**A**][**T**]=[**C**].

From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$

where [**A**]-1 is the inverse of matrix [**A**]. [**T**] is the solution vector.

- Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)
- Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.
- For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

<u>Iteration</u>

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij}T_j + a_{ii}T_i + \sum_{j=i+1}^{N} a_{ij}T_j = C_i,$$

$$(\text{Example}: a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{1N}T_N = C_1,\ i = 3)$$

Rewrite the equation of the form:

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}T_j^{(k)} - \sum_{j=i+1}^{N}\frac{a_{ij}}{a_{ii}}T_j^{(k-1)}$$

Replace (k) by (k-1)
for the Jacobi iteration

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- • By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated The iteration will be stopped when max$| T_i^{(k)}-T_i^{(k-1)} |$ε≤ , where ε specifies a predetermined value of acceptable error

UNSTEADY HEAT TRANSFER

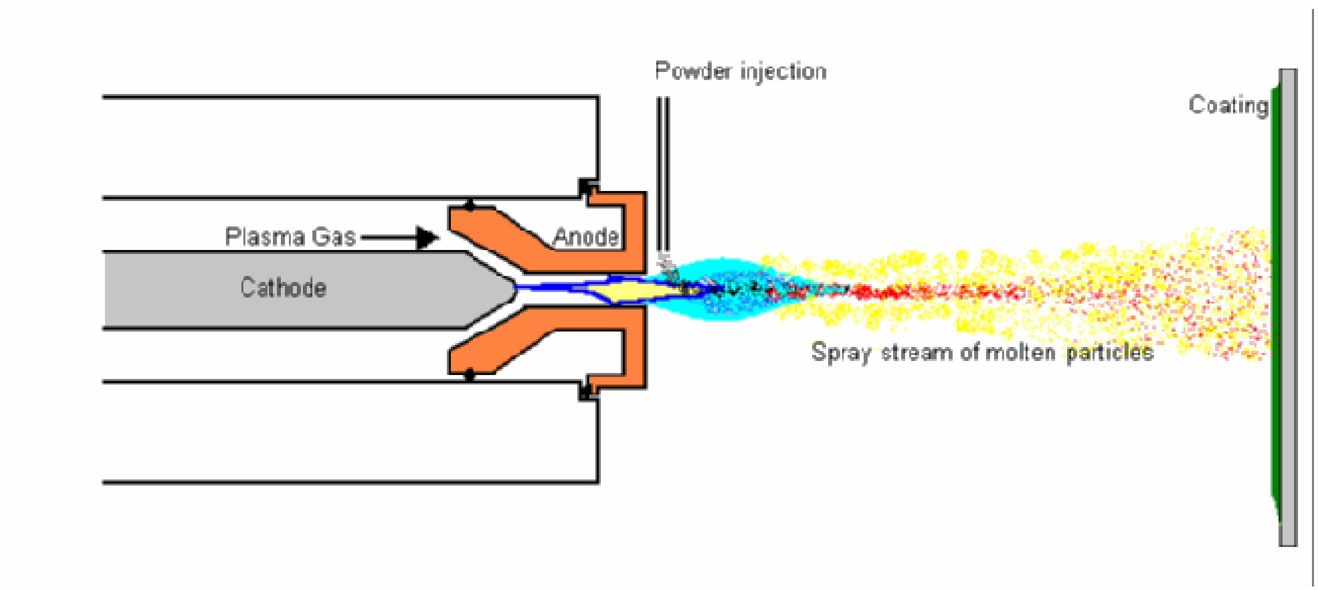
Many heat transfer problems require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T, \text{ or } \frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T$$

where $\alpha = \dfrac{\mathrm{k}}{\rho \mathrm{c}}$ is the thermal diffusivity

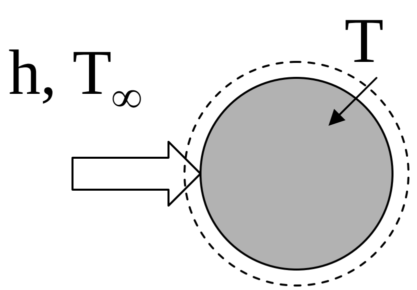
Lumped Capacitance Method (LCM)

The simplest situation in an unsteady heat transfer process is to neglect the temperature distribution inside the solid and only deals with the heat transfer between the solid and the ambient fluids. In other words, we are going to assume the temperature inside the solid is constant and equals to the surface temperature. Let us look at a practical example about a plasma spray process involving the injection of tiny particles into a plasma jet at a very high temperature. These particles will eventually melt and impinge on the processing surface and solidify to form a layer of protecting coating.



Example: (Plasma Spray)

Assume spherical alumina particles are used in the plasma jet. (diameter D=50 mm, density ρ=3970 kg/m3, thermal conductivity k=10.5 W/m.K and specific heat c_p=1560 J/kg, and an initial temperature of 300 K) The plasma is maintained at a temperature of 10,000 K and has a convection coefficient of h=30,000 W/m^2.K. The melting temperature of the particle is 2318 K and the latent heat of fusion is 3577 kJ/kg. (a) Determine the time required to heat a particle to its melting point, (b) determine the time for the particle to melt completely after it reaches the melting temperature. (Special notes: why the particles follow the plasma jet? Does the particles travel at the same velocity as the local jet velocity? Does the jet has a uniform velocity?



Energy balance: energy in = energy storage in solid

$$hA_S(T_\infty - T) = \frac{dE}{dt} = \frac{d(mc_P T)}{dt}$$

$$hA_S(T_\infty - T) = \rho c_P V \frac{dT}{dt}$$

wher A_S is the surface area of the sphere

Example (cont.)

Define a new variable θ=T-T_∞

$$\rho V c_P \frac{d\theta}{dt} = -hA_S\theta, \ \frac{d\theta}{\theta} = -\frac{hA_S}{\rho V c_P} dt$$

Integrate with respect to time and apply the initial condition

T(t=0)=T_i=300(K) or θ(t=0)=θ_i=T_i-T_∞=300-10000=-9700(K)

$$\ln\theta = \frac{-hA_S}{\rho V c_P} t + C_1, \ \theta(t) = C_2 \exp\left[-\left(\frac{hA_S}{\rho V c_P}\right)t\right]$$

Substitute the initial condition: θ(0)=$\theta_i = C_2$

The general solution for lumped capacitance method

$$\frac{\theta(\mathrm{t})}{\theta_\mathrm{i}} = \frac{T(t)-T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_S}{\rho V c_P}\right)t\right]$$

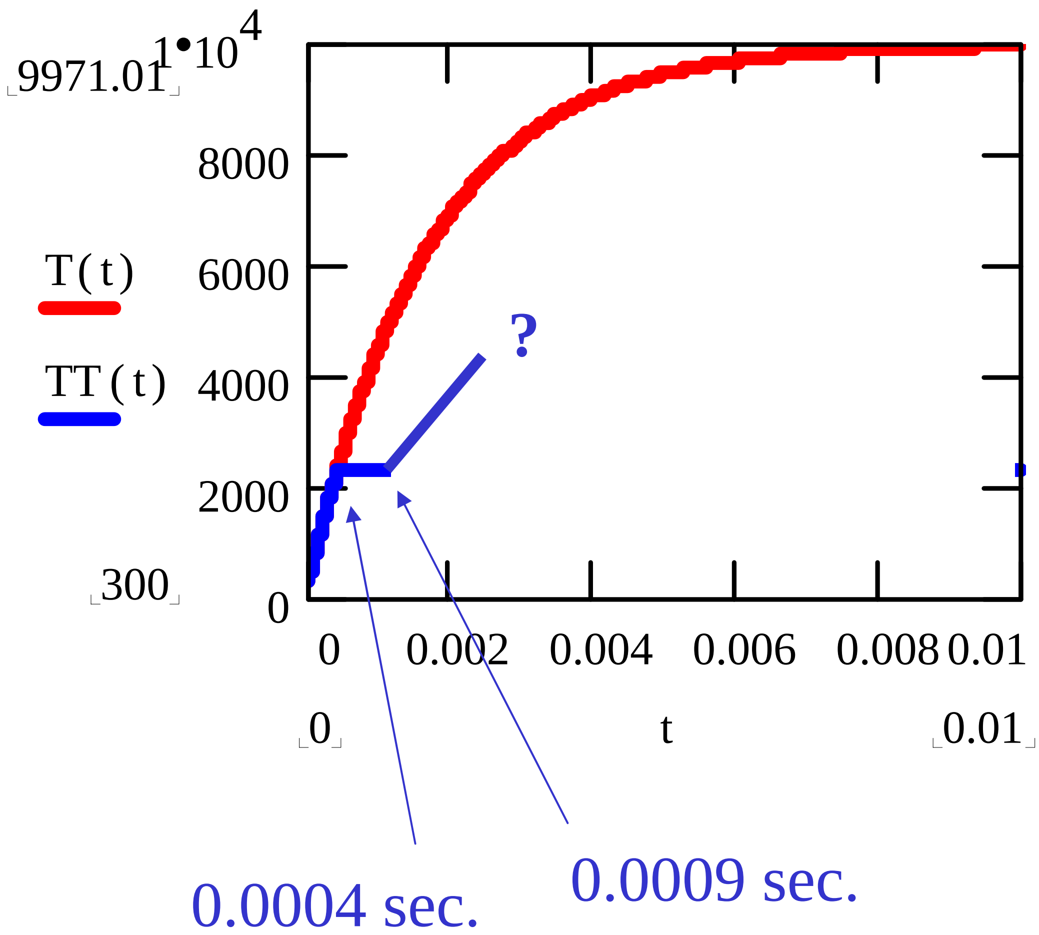
$$\frac{T(t)-10000}{-9700} = \exp\left[-\left(\frac{30000*\pi D^2}{3970*(1/6)\pi D^3 * 1560}\right)t\right] = \exp(-581.3t)$$

$$T(t) = 10000 - 9700\exp(-581.3t)$$

Example (cont.)

Temperature distribution as a function of time:

$$T(t) = 10000 - 9700\exp(-581.3t)$$



- Temperature of the particle increases exponentially from 300 K to 10000K in a very short time (<0.01 sec.)
- It only takes 0.0004 sec. To reach the melting temperature
- Therefore, the true temperature variation is described by the blue curve. (why?)

Example (cont.)

After the particle reaches its melting temperature, the heat input will not increase the temperature of the particle anymore. Rather, the heat will be absorbed by the solid particle as latent heat of melting in order for it to melt.

$$\int_{t_1}^{t_2} q_{conv} dt == \Delta E_{sf} = m h_{sf}$$

$$h A_S (T_\infty - T_{melt})(t_2 - t_1) = \rho V h_{sf}$$

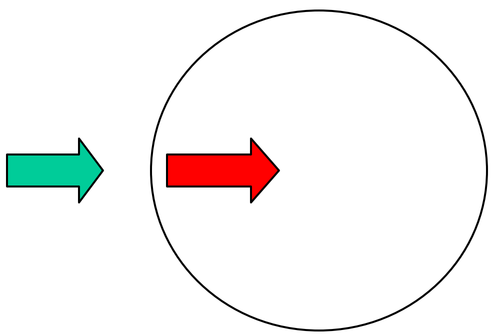
$$t_2 - t_1 = \frac{\rho V h_{sf}}{h A_S (T_\infty - T_{melt})} = \frac{3970(50\times 10^{-6})(3577000)}{6(30000)(10000-2318)} = 5\times 10^{-4}(s)$$

It will take an additional 5×10^{-4} s. to melt the particle

The total time to completely melt the particle will be 9×10^{-4} s.

SPATIAL EFFECTS

QUESTION: When can we neglect the temperature variation inside a solid? We can do that if the heat transfer inside the solid is much more effective than that occurs between the solid and the ambient fluid. If can be demonstrated as the following:



- Heat transfer from the fluid to the solid through convection: $q=hA(T_S-T_\infty)$ and the thermal resistance of this process is $R_{conv}=1/(hA)$.
- Heat transfer from the exterior of the solid to the interior of the solid is through the conductive heat transfer:

$$q=\frac{4\pi k(T_{S,1}-T_{S,2})}{(1/r_1)-(1/r_2)}$$

The thermal resistance is

$$R_{sphere}=\frac{1}{4\pi k}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Biot Number

In order to be able to use the lumped capacitance assumption, the temperature variation inside the solid should be much smaller than the temperature difference between the surface and the fluid: $T_S - T_\infty << T_{S,1} - T_{S,2}$.
In other words, the thermal resistance between the solid and the fluid should be much greater compared to the thermal resistance inside the solid.
$R_{conv} >> R_{solid}$.

$$\frac{1}{hA} = \frac{1}{h(4\pi r_2^2)} (\text{of the order } \frac{1}{\mathrm{hL_C}^2}) >> \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) (\text{of the order } \frac{1}{\mathrm{kL_C}})$$

$$1 >> \frac{hr_2^2}{k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \text{ (of the order } \frac{\mathrm{hL_C}}{\mathrm{k}} = \mathbf{BI}, \text{ defined as the } \textbf{Biot} \text{ number)}$$

where L_C is a characteristic length scale: relate to the size of the solid involved in the problem.

For example, $L_C = r_O/2$ (half-radius) when the solid is a cylinder.

$L_C = r_O / 3$ (one-third radius) when the solid is a sphere.

$L_C = L$ (half thickness) when the solid is a plane wall with a 2L thickness.

Biot and Fourier Numbers

Define Biot number $Bi = hL_C/k$

In general, $Bi < 0.1$ for the lumped capacitance assumption to be valid.

Use temperature variation of the Alumina particles in a plasma jet process as an example:

$$\frac{T(t)-T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_S}{\rho V c_P}\right)t\right] = \exp\left[-\left(\frac{hA_S}{Vk}\right)\left(\frac{kt}{\rho c_P}\right)\right]$$

For a sphere: $\dfrac{A_S}{V} = \dfrac{4\pi r_O^{\ 2}}{\frac{4}{3}\pi r_O^{\ 3}} = \dfrac{3}{r_O} = \dfrac{1}{L_C}$

$$\frac{T(t)-T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{h}{kL_C}\right)\left(\frac{kt}{\rho c_P}\right)\right] = \exp\left[-\left(\frac{hL_C}{k}\right)\left(\frac{kt}{\rho c_P L_C^2}\right)\right]$$

$$= \exp\left[-\left(Bi\right)\left(\frac{\alpha t}{L_C^2}\right)\right] = \exp(-Bi \cdot \tau)$$

Define Fourier number (Fo): $\tau = \dfrac{\alpha t}{L_C^2}$ as dimensionless time

Spatial Effects

Re-examine the plasma jet example: h=30,000 W/m^2.K, k=10.5 W/m.K, and a diameter of 50 μm. Bi=(h/k)(r_O/3)=0.0238<0.1. Therefore, the use of the LCM is valid in the previous example. However, if the diameter of the particle is increased to 1 mm, then we have to consider the spatial effect since Bi=0.476>0.1. The surface temperature can not be considered as the same as the temperature inside the particle. To determine the temperature distribution inside the spherical particle, we need to solve the unsteady heat equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

with initial condition of $T(r,t=0)=T_i$ and boundary conditions of the form:

$\frac{\partial \mathrm{T}}{\partial \mathrm{r}}=0$ at r = 0: symmetric, temperature is either a max. or a min. at the center.

$-\mathrm{k}\frac{\partial \mathrm{T}}{\partial \mathrm{r}}=h(T(r_O,t)-T_\infty)$ at r = r_O: conduction = convection at the surface.

Example

The solution can be determined using the method of separation of variables. Equations 9-10,11,12 show the one-term approximate solutions of the exact solutions for a plane wall, a cylinder and a sphere, respectively. This approximation can also be represented in a graphical form as shown in Appendix D. (Sometimes they are called the Heisler charts)

Terminology:

$$\theta_0^* = \frac{\theta_0}{\theta_i} = \frac{T_0 - T_\infty}{T_i - T_\infty}$$ where T_0 is the centre temperature,

T_∞ is the ambient temperature, T_i is the initial temperature.

$\dfrac{Q}{Q_{\max}}$ where Q is the amount of heat transfer into (or out of) the solid

Q_{max} is the maximum amount of energy transfer when $t \to \infty$

$Q_{\max} = \rho c_p V \left(T_i - T_\infty \right)$ or the energy to increase (or decrease) the overall temperature of the solid from Ti to T_∞ .

Example (cont.)

Recalculate the plasma jet example by assuming the diameter of the particles is 1 mm. (a) Determine the time for the center temperature to reach the melting point, (b) Total time to melt the entire particle. First, use the lumped capacitance method and then compare the results to that determined using Heisler charts. ($h=30000\ W/m^2$, $k=10.5\ W/m.K$, $\rho=3970\ kg/m^3$, $r_O=0.5$ mm, $c_P=1560\ J/kg.K$, $T_\infty=10000$ K, $T_i=300$ K, $T_{melt}=2318$, $h_{sf}=3577$ kJ/kg, $\alpha=k/(\rho c_P)=1.695\times10^{-6}(m^2/s)$.

First, $Bi=(h/k)(r_O/3)=0.476>0.1$ should not use LCM. However, we will use it anyway to compare the difference.

$$\frac{T(t)-T_\infty}{T_i-T_\infty}=\exp(-Bi\cdot\tau),\ \text{or}\ \frac{\text{T-10000}}{\text{300-10000}}=\exp(-0.476\cdot\tau)$$

To reach T=2318 K, $\tau=\dfrac{\alpha t}{L_C^2}=0.49$, or $t=0.008(s)$

since $L_C=\dfrac{r_O}{3}$ for a spherical particle.

Example (cont.)

After the particle reaches the melting point, the heat transfer will be used to supply to the solid for phase transition

$$\int_{t_1}^{t_2} q_{conv} dt == \Delta E_{sf} = mh_{sf}, \; hA_S(T_\infty - T_{melt})(t_2 - t_1) = \rho V h_{sf}$$

$$t_2 - t_1 = \frac{\rho V h_{sf}}{hA_S(T_\infty - T_{melt})} = \frac{3970(0.001)(3577000)}{6(30000)(10000 - 2318)} = 0.01(s)$$

It will take an additional 0.01 s. to melt the particle

The total time to completely melt the particle will be 0.018 *s*.

To consider the spatial effects, we need to use the Heisler chart:

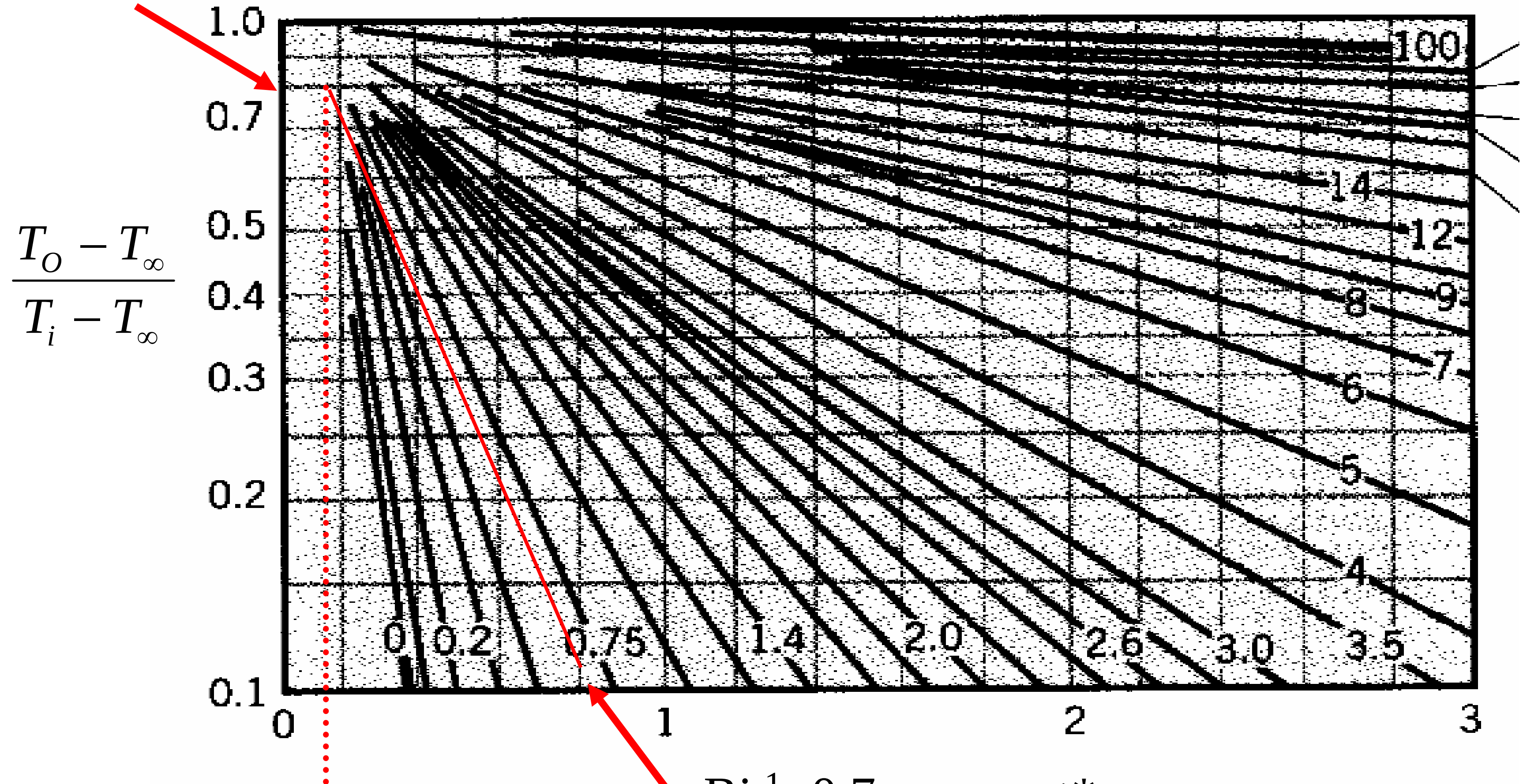
First, $\mathrm{Bi}^{-1} = \dfrac{k}{hr_O} = 0.7$ (Note : this definition is different from the previous Biot number used to validate the lumped capacitance method)

$$\theta_O^* = \frac{T_O - T_\infty}{T_i - T_\infty} = \frac{2318 - 10000}{300 - 10000} = 0.792.$$ Now look it up in figure 9-15

Heisler Chart

Note: this is the magnified upper left corner of Fig. D-1

$\theta_O^*=0.792$



$Bi^{-1}=0.7$

$t^*=\tau$

$t^*=0.16=\alpha t/r_O^2$

$t=(r_O^2/\alpha)(0.16)=0.023$(s) much longer than the value calculated using LCM ($t=0.008$ s)

Approximate Solution

We can also use the one-term approximation solution in equation 9-12 to calculate.

Determine the constants in Table 9.1: for Bi=1.43 through interpolation, $\mathrm{A}_1 = 1.362, \lambda_1 = 1.768.$

$$\theta^* = A_1 \exp(-\lambda_1^2 \tau) \frac{1}{\lambda_1 r^*} \sin(\lambda_1 r^*), r^* = (r / r_O)$$

$\theta_O^* = A_1 \exp(-\lambda_1^2 \tau)$ at the center

$$\tau = \frac{-1}{\lambda_1^2} \ln\left(\frac{\theta_O^*}{A_1}\right) = \frac{-1}{(1.768)^2} \ln\left(\frac{0.792}{1.362}\right) = 0.173$$

It is close to the value (0.16) estimated from using the chart

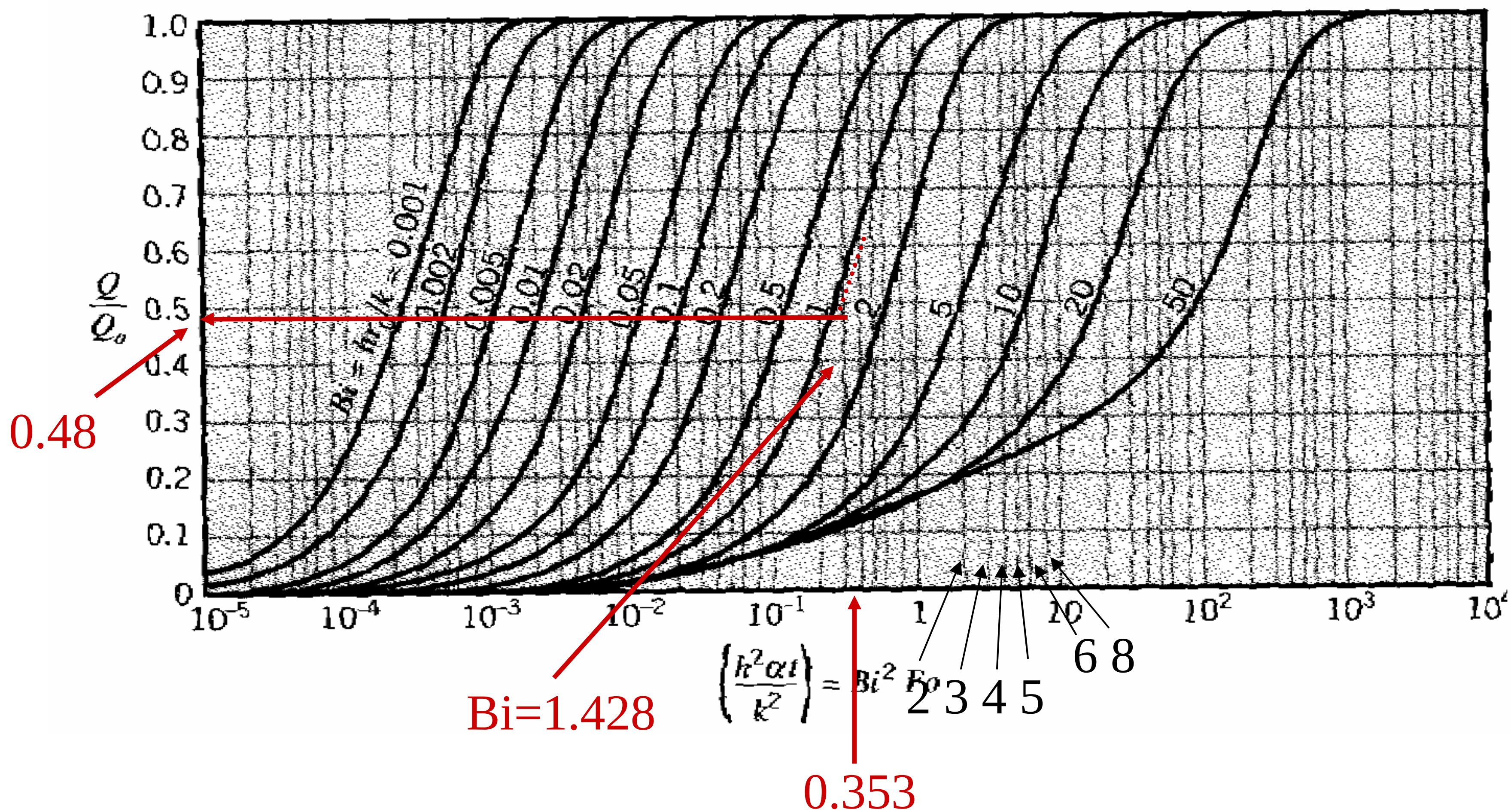
Heat Transfer

How much heat has been transfered into the particle during this period of time?

Determine $Q_{max} = \rho c_P V(T_i - T_\infty)$

$= (3970)(1560)(1/6)\pi(0.001)^2(300-10000) = 31,438(J)$

$Bi = 1.428,\ Bi^2\tau = (1.428)^2(0.173) = 0.353$



Heat Transfer (cont.)

Total heat transfer during this period:
$Q=Q_{max}(Q/Q_{max})=31438(0.48)=15090(J)$

- The actual process is more complicated than our analysis. First, the temperature will not increase once the solid reaches the melting temperature unless the solid melts into liquid form. Therefore, the actual heat transfer process will probably slower than the estimation.

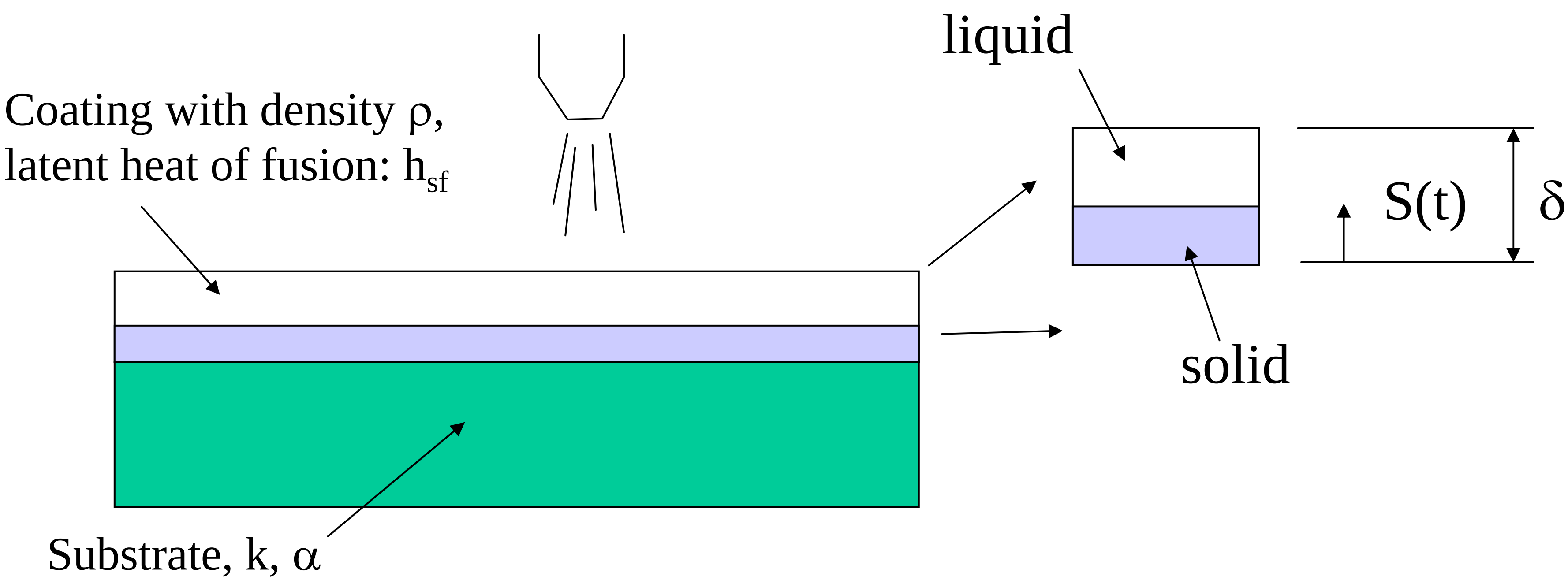
- If the outer layer melts then we have double convective conditions: convection from the plasma gas to the liquid alumina and then from there to the inner solid.

- To make matter even more complicated, the interface between the melt alumina and the solid is continuously moving inward.

- No analytical solutions or numerical analysis is necessary.

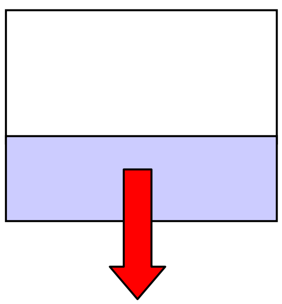
Unsteady Heat Transfer in Semi-infinite Solids

Solidification process of the coating layer during a thermal spray operation is an unsteady heat transfer problem. As we discuss earlier, thermal spray process deposits thin layer of coating materials on surface for protection and thermal resistant purposes, as shown. The heated, molten materials will attach to the substrate and cool down rapidly. The cooling process is important to prevent the accumulation of residual thermal stresses in the coating layer.



Example

As described in the previous slide, the cooling process can now be modeled as heat loss through a semi-infinite solid. (Since the substrate is significantly thicker than the coating layer) The molten material is at the fusion temperature T_f and the substrate is maintained at a constant temperature T_i. Derive an expression for the total time that is required to solidify the coating layer of thickness δ.



Heat transfer from the molten material to the substrate ($q=q''A$)

- Assume the molten layer stays at a constant temperature T_f throughout the process. The heat loss to the substrate is solely supplied by the release of the latent heat of fusion.

From energy balance:

$$h_{sf}\Delta m \text{(solidified mass during } \Delta t) = \Delta Q = q'' A \Delta t \text{ (energy input)}$$

$$h_{sf}\frac{dm}{dt} = q'' A, \text{ where } m = \rho V = \rho A S,$$

where S is solidified thickness

$$\rho \frac{dS}{dt} = q''$$

Example (cont.)

Identify that the previous situation corresponds to the case discussed in chapter 9-3 in the text as a semi-infinite transient heat transfer problem with a constant surface temperature boundary condition (note: the case in the textbook corresponds to an external convection case. However, it can be modelled as constant surface temperature case by setting h=∞, therefore, $T_s=T_\infty$).

If the surface temperature is T_s and the initial temperature of the bolck is T_i, the analytical solution of the problem can be found:

The temperature distribution and the heat transfer into the block are:

$$\frac{T(x,t)-T_s}{T_i - T_s} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right),$$ where erf( ) is the Gaussian error function.

It is defined as $\mathrm{erf}(w)=\frac{2}{\sqrt{\pi}}\int_0^w e^{-v^2}dv$

$$q_s''(t)=\frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Example (cont.)

From the previous equation

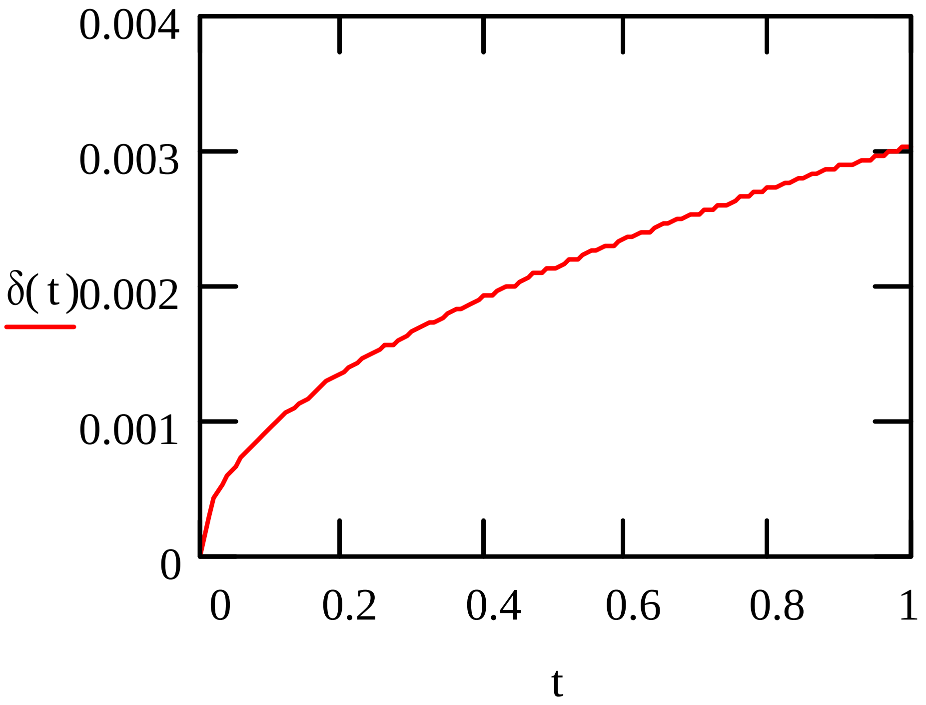
$$\rho h_{sf}\frac{dS}{dt} = q'' = \frac{k(T_f - T_i)}{\sqrt{\pi\alpha t}}, \quad \text{and} \int_0^{\delta} dS = \frac{k(T_f - T_i)}{\rho h_{sf}\sqrt{\pi\alpha}}\int_0^t \frac{dt}{\sqrt{t}}$$

$$\delta(t) = \frac{2k(T_f - T_i)}{\rho h_{sf}\sqrt{\pi\alpha}}\sqrt{t}, \text{ therefore, } \delta \propto \sqrt{t}. \text{ Cooling time } t = \frac{\pi\alpha}{4k^2}\left(\frac{\delta\rho h_{sf}}{T_f - T_i}\right)^2$$

Use the following values to calculate: $k=120$ W/m.K, $\alpha=4\times10^{-5}$ m^2/s, $\rho=3970$ kg/m^3, and $h_{sf}=3.577\times10^6$ J/kg, $T_f=2318$ K, $T_i=300$K, and $\delta=2$ mm

Example (cont.)

$$\delta(t) = \frac{2k(T_f - T_i)}{\rho h_{sf}\sqrt{\pi\alpha}}\sqrt{t} = 0.00304\sqrt{t}$$



- $\delta(t) \propto t^{1/2}$
- Therefore, the layer solidifies very fast initially and then slows down as shown in the figure
- Note: we neglect contact resistance between the coating and the substrate and assume temperature of the coating material stays the same even after it solidifies.

- To solidify 2 mm thickness, it takes 0.43 seconds.

Example (cont.)

What will be the substrate temperature as it varies in time? The temperature distribution is:

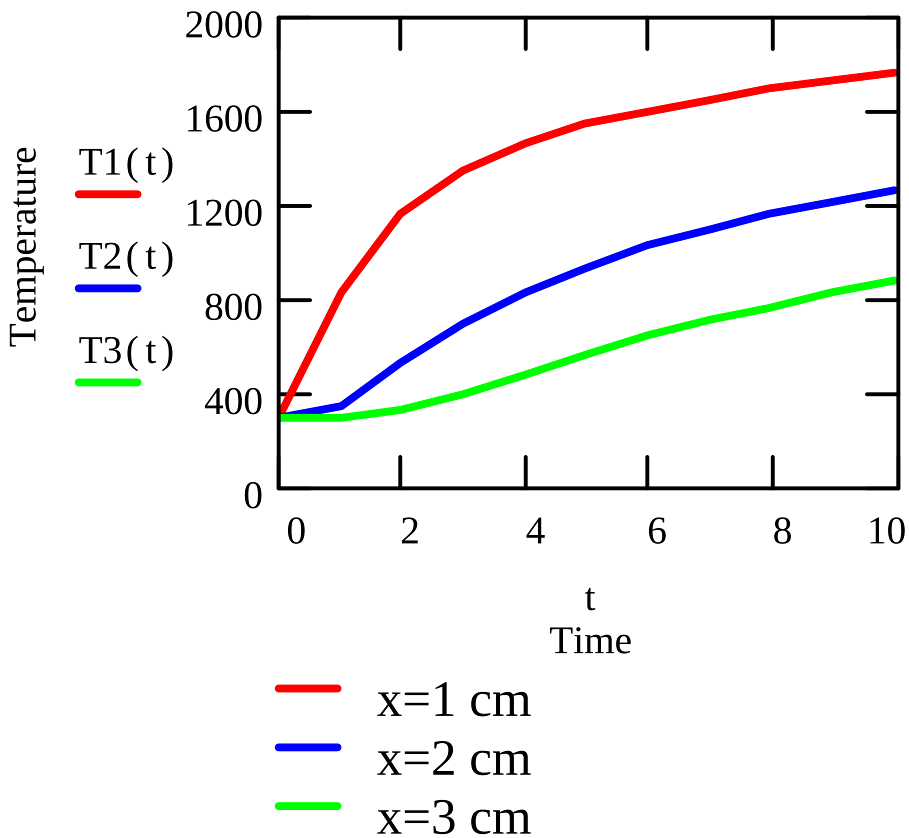
$$\frac{T(x,t)-T_S}{T_i - T_S} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

$$T(x,t) = 2318 + (300-2318)erf\left(\frac{x}{2\sqrt{\alpha t}}\right) = 2318 - 2018erf\left(79.06\frac{x}{\sqrt{t}}\right)$$

Example (cont.)

• For a fixed distance away from the surface, we can examine the variation of the temperature as a function of time. Example, 1 cm deep into the substrate the temperature should behave as:

$$T(x=0.01,t) = 2318 - 2018\,erf\left(79.06\frac{x}{\sqrt{t}}\right) = 2318 - 2018\,erf\left(\frac{0.79}{\sqrt{t}}\right)$$



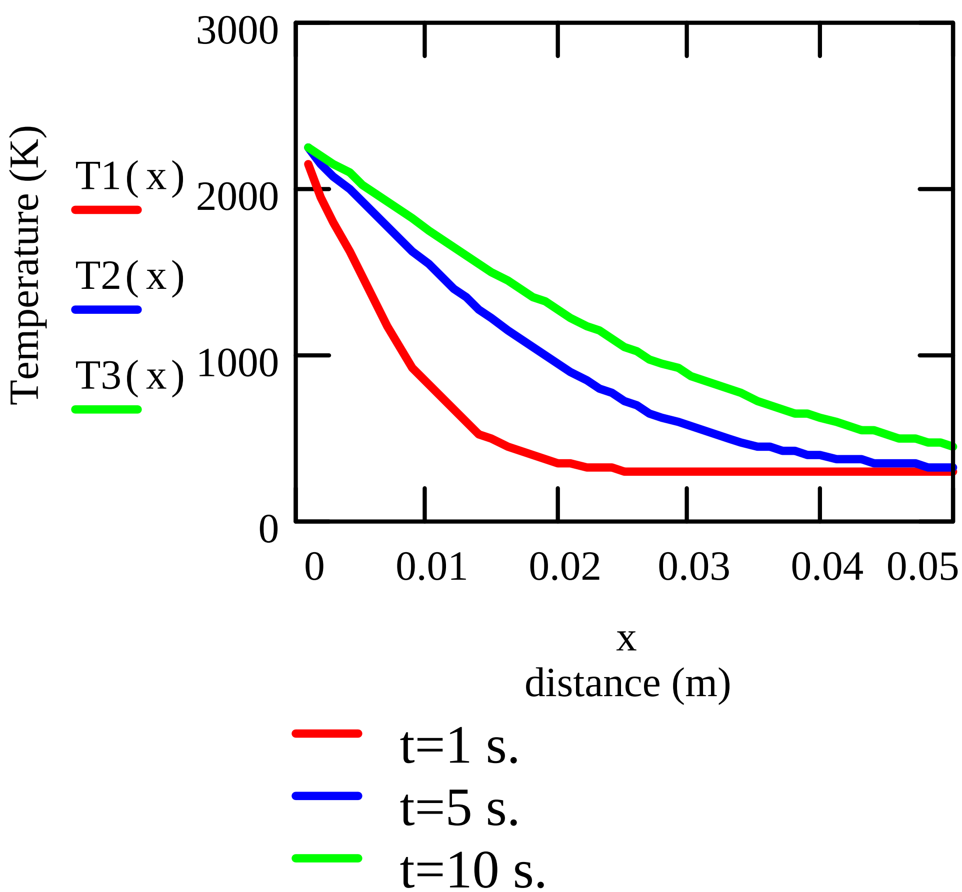
• At x=1 cm, the temperature rises almost instantaneously at a very fast rate. A short time later, the rate of temp. increase slows down significantly since the energy has to distribute to a very large mass.

• At deeper depth (x=2 & 3 cm), the temperature will not respond to the surface condition until much later.

Example (cont.)

We can also examine the spatial temperature distribution at any given time, say at t=1 second.

$$T(x, t=1) = 2318 - 2018\,erf\;\; 79.06\frac{x}{\sqrt{t}} \;\; = 2318 - 2018\,erf\;\; 79.06x$$



- Heat penetrates into the substrate as shown for different time instants.
- It takes more than 5 seconds for the energy to transfer to a depth of 5 cm into the substrate
- The slopes of the temperature profiles indicate the amount of conduction heat transfer at that instant.

Numerical Methods for Unsteady Heat Transfer

Unsteady heat transfer equation, no generation, constant k, two-dimensional in Cartesian coordinate:

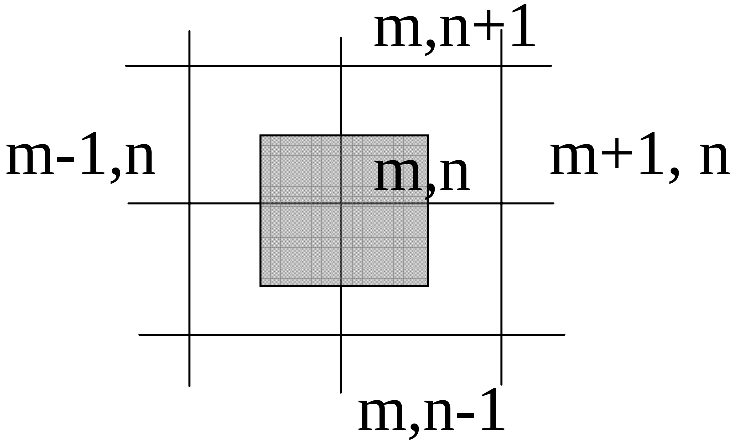
$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

We have learned how to discretize the Laplacian operator into system of finite difference equations using nodal network. For the unsteady problem, the temperature variation with time needs to be discretized too. To be consistent with the notation from the book, we choose to analyze the time variation in small time increment Δt, such that the real time t=pΔt. The time differentiation can be approximated as:

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{P+1} - T_{m,n}^{P}}{\Delta t},$$ while m & n correspond to nodal location

such that x=mΔx, and y=nΔy as introduced earlier.

Finite Difference Equations



From the nodal network to the left, the heat equation can be written in finite difference form:

$$\frac{1}{\alpha}\frac{T_{m,n}^{P+1}-T_{m,n}^{P}}{\Delta t}=\frac{T_{m+1,n}^{P}+T_{m-1,n}^{P}-2T_{m,n}^{P}}{(\Delta x)^2}+\frac{T_{m,n+1}^{P}+T_{m,n-1}^{P}-2T_{m,n}^{P}}{(\Delta y)^2}$$

Assume $\Delta x=\Delta y$ and the discretized Fourier number Fo= $\dfrac{\alpha\Delta \mathrm{t}}{\left(\Delta \mathrm{x}\right)^2}$

$$T_{m,n}^{P+1}=Fo\left(T_{m+1,n}^{P}+T_{m-1,n}^{P}+T_{m,n+1}^{P}+T_{m,n-1}^{P}\right)+(1-4Fo)T_{m,n}^{P}$$

This is the **explicit**, finite difference equation for a 2-D, unsteady heat transfer equation.

The temperature at time p+1 is explicitly expressed as a function of neighboring temperatures at an earlier time p

Nodal Equations

Some common nodal configurations are listed in table for your reference. On the third column of the table, there is a stability criterion for each nodal configuration. This criterion has to be satisfied for the finite difference solution to be stable. Otherwise, the solution may be diverging and never reach the final solution.

For example, $Fo \leq 1/4$. That is, $\alpha\Delta t/(\Delta x)^2 \leq 1/4$ and $\Delta t \leq (1/4\alpha)(\Delta x)^2$. Therefore, the time increment has to be small enough in order to maintain stability of the solution.

This criterion can also be interpreted as that we should require the coefficient for $T^p_{m,n}$ in the finite difference equation be greater than or equal to zero.

Question: Why this can be a problem? Can we just make time increment as small as possible to avoid it?

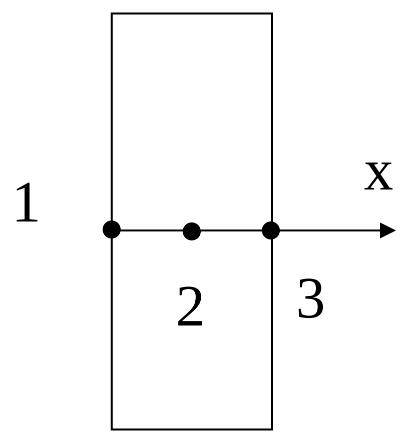
Finite Difference Solution

Question: How do we solve the finite difference equation derived?

- First, by specifying initial conditions for all points inside the nodal network. That is to specify values for all temperature at time level p=0.
- Important: check stability criterion for each points.
- From the explicit equation, we can determine all temperature at the next time level p+1=0+1=1. The following transient response can then be determined by marching out in time p+2, p+3, and so on.

Example

Example: A flat plate at an initial temperature of 100 deg. is suddenly immersed into a cold temperature bath of 0 deg. Use the unsteady finite difference equation to determine the transient response of the temperature of the plate.



L(thickness)=0.02 m, k=10 W/m.K, $\alpha=10\times10^{-6}$ m^2/s,
h=1000 $W/m^2.K$, T_i=100°C, T_∞=0°C, Δx=0.01 m
Bi=$(h\Delta x)/k$=1, Fo=$(\alpha\Delta t)/(\Delta x)^2$=0.1

There are three nodal points: 1 interior and two exterior points: For node 2, it satisfies the case 1 configuration in table.

$$T_2^{P+1} = Fo(T_1^P + T_3^P + T_2^P + T_2^P) + (1-4Fo)T_2^P = Fo(T_1^P + T_3^P) + (1-2Fo)T_2^P$$
$$= 0.1(T_1^P + T_3^P) + 0.8T_2^P$$

Stability criterion: 1-2Fo ≥ 0 or Fo=0.1 $\leq \dfrac{1}{2}$, it is satisfied

Example

For nodes 1 & 3, they are consistent with the case 3 in table.

Node 1: $T_1^{P+1} = Fo(2T_2^P + T_1^P + T_1^P + 2BiT_\infty) + (1 - 4Fo - 2BiFo)T_1^P$

$= Fo(2T_2^P + 2BiT_\infty) + (1 - 2Fo - 2BiFo)T_1^P = 0.2T_2^P + 0.6T_1^P$

Node 3: $T_3^{P+1} = 0.2T_2^P + 0.6T_3^P$

Stability criterion: (1-2Fo-2BiFo) ≥ 0, $\frac{1}{2} \geq Fo(1 + Bi) = 0.2$ and it is satisfied

System of equations

$$T_1^{P+1} = 0.2T_2^P + 0.6T_1^P$$

$$T_2^{P+1} = 0.1(T_1^P + T_3^P) + 0.8T_2^P$$

$$T_3^{P+1} = 0.2T_2^P + 0.6T_3^P$$

Use initial condition, $\mathrm{T}_1^0 = \mathrm{T}_2^0 = \mathrm{T}_3^0 = 100$,

$$T_1^1 = 0.2T_2^0 + 0.6T_1^0 = 80$$

$$T_2^1 = 0.1(T_1^0 + T_3^0) + 0.8T_2^0 = 100$$

$$T_3^1 = 0.2T_2^0 + 0.6T_3^0 = 80$$

Marching in time, $\mathrm{T}_1^1 = \mathrm{T}_3^1 = 80$, $\mathrm{T}_2^1 = 100$

$$T_1^2 = 0.2T_2^1 + 0.6T_1^1 = 0.2(100) + 0.6(80) = 68$$

$$T_2^2 = 0.1(T_1^1 + T_3^1) + 0.8T_2^1 = 0.1(80 + 80) + 0.8(100) = 96$$

$T_3^2 = 0.2T_2^1 + 0.6T_3^1 = 0.2(100) + 0.6(80) = 68$, and so on

Module 4: Worked out problems

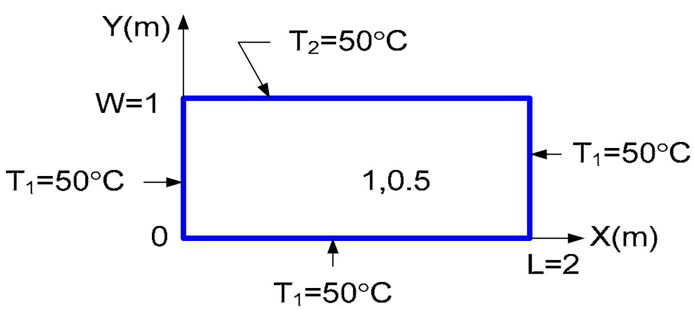
Problem 1:

A two dimensional rectangular plate is subjected to the uniform temperature boundary conditions shown. Using the results of the analytical solution for the heat equation, calculate the temperature at the midpoint (1, 0.5) by considering the first five nonzero terms of the infinite series that must be evaluated. Assess the error from using only the first three terms of the infinite series.

Known: Two-dimensional rectangular plate to prescribed uniform temperature boundary conditions.

Find: temperature at the midpoint using the exact solution considering the first five nonzero terms: Assess the error from using only the first three terms.

Schematic:



Assumptions: (1) Two-dimensional, steady-state conduction, (2) constant properties.

Analysis: From analytical solution, the temperature distribution is

$$\theta(x,y) = \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi\, x}{L}\cdot\frac{\sinh(n\pi\, y/L)}{\sinh(n\pi\, W/L)}$$

Considering now the point (x, y) = (1.0, 0.5) and recognizing x/L =1/2, y/L=1/4 and W/L=1/2, the distribution has the form

$$\theta(1{,}0.5) = \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi}{L}\cdot\frac{\sinh(n\pi/L)}{\sinh(n\pi/L)}$$

When n is even (2,4,6…..), the corresponding term is zero; hence we need only consider n=1,3,5,7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi}\left\{\begin{array}{r} 2\sin\left(\frac{\pi}{2}\right)\frac{\sinh\frac{\pi}{4}}{\sinh\frac{\pi}{2}} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh\frac{3\pi}{4}}{\sinh\frac{3\pi}{2}} + \frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh\frac{5\pi}{4}}{\sinh\frac{5\pi}{2}} + \\ \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh\frac{7\pi}{4}}{\sinh\frac{7\pi}{2}} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh\frac{9\pi}{4}}{\sinh\frac{9\pi}{2}} \end{array}\right\}$$

$$\theta(1,0.5) = \frac{2}{\pi}\left[0.755 + 0.063 + 0.008 - 0.001 + 0.000\right] = 0.445$$

Since $\theta = (T-T_1)/(T_2-T_1)$, it follows that

$T(1, 0.5) = \theta(1, 0.5)(T_2-T_1) + T_1 = 0.445(150-50) + 50 = 94.5°C$

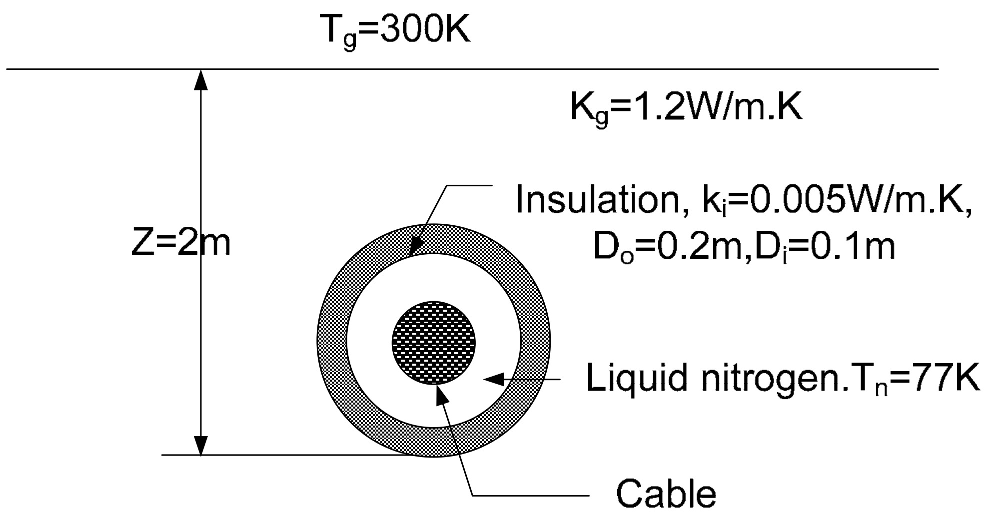
If only the first term of the series, Eq (2) is considered, the result will be $\theta(1, 0.5) = 0.446$ that is, there is less than a 0.2% effect.

Problem 2:
A long power transmission cable is buried at a depth (ground to cable centerline distance) of 2m. The cable is encased in a thin walled pipe of 0.1 m diameter, and to render the cable superconducting (essentially zero power dissipation), the space between the cable and pipe is filled with liquid nitrogen at 77 K. If the pipe is covered a super insulator(k_i=0.005W/m.K) of 0.05-m thickness and the surface of the earth (k_g=1.2W/m.K) is at 300K, what is the cooling load in W/m which must be maintained by a cryogenic refrigerator per unit pipe length.

Known: Operating conditions of a buried superconducting cable.

Find: required cooling load.

Schematic:



Assumptions: (1) steady-state conditions, (2) constant properties, (3) two-dimensional conduction in soil, (4) one-dimensional conduction in insulation.

Analysis: The heat rate per unit length is

$$q' = \frac{T_g - T_n}{R'_g + R'_I}$$

$$q' = \frac{T_g - T_n}{[k_g(2\pi / \ln(4z/D_o))]^{-1} + \ln(D_o/D_i)/2\pi\, k_i}$$

where table 4.1 have been used to evaluate the ground resistance. Hence,

$$q' = \frac{(300-77)K}{[(1.2W/m.K)\,(2\pi/\ln(8/0.2))]^{-1} + \ln(2)/2\pi\times 0.005\,W/m.K}$$

$$q' = \frac{223K}{(0.489+22.064)m.K/W}$$

$$q' = 9.9W/m$$

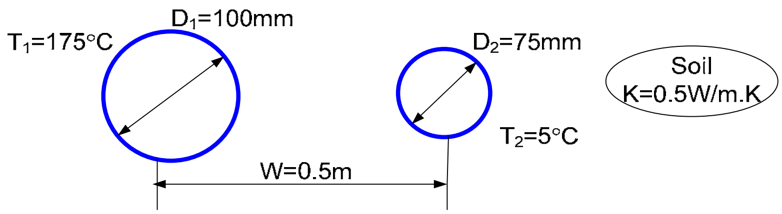
Comments: the heat gain is small and the dominant contribution to the thermal resistance is made by the insulation.

Problem 3:
Two parallel pipelines spaced 0.5 m apart are buried in soil having a thermal conductivity of 0.5W/m.K. the pipes have outer-diameters of 100 and 75 mm with surface temperatures of 175°C and 5°C, respectively. Estimate the heat transfer rate per unit length between the two pipe lines.

Known: Surfaces temperatures of two parallel pipe lines buried in soil.

Find: heat transfer per unit length between the pipe lines.

Schematic:



Assumptions: (1) steady state conditions, (2) two-dimensional conduction, (3) constant properties, (4) pipe lines are buried very deeply approximating burial in an infinite medium, (5) pipe length>> D_1 or D_2 and w>D_1 or D_2

Analysis: the heat transfer rate per length from the hot pipe to the cool pipe is

$$q^{'} = \frac{q}{L} = \frac{S}{L} k(T_1 - T_2)$$

The shape factor S for this configuration is given in table 4.1 as

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$$

Substituting numerical values

$$\frac{S}{L} = 2\pi / \cosh^{-1} \frac{4\times(0.5m)^2 - (0.1m)^2 - (0.075m)^2}{2\times 0.1m\times 0.075m} = 2\pi / \cosh^{-1}(65.63)$$

$$\frac{S}{l} = 2\pi / 4.88 = 1.29.$$

hence, the heat rate per unit length is

$$\text{q}^{'} = 1.29\times 0.5W / m.K(175-5)°C = 110W / m$$

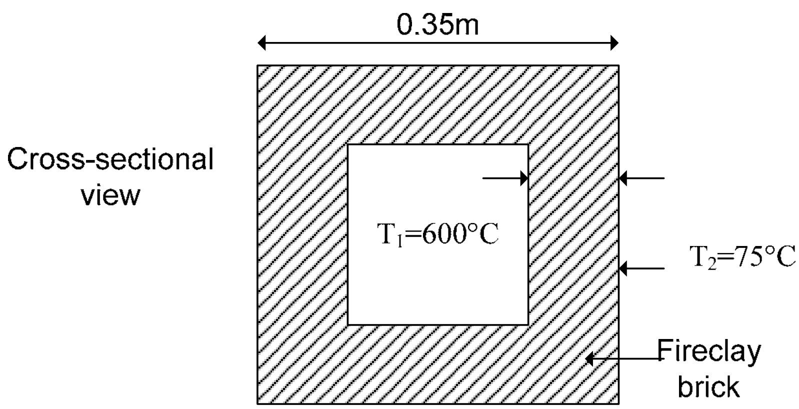
Comments: The heat gain to the cooler pipe line will be larger than 110W/m if the soil temperature is greater than 5°C. How would you estimate the heat gain if the soil were at 25°C?

Problem 4:
A furnace of cubical shape, with external dimensions of 0.35m, is constructed from a refractory brick (fireclay). If the wall thickness is 50mm, the inner surface temperature is 600°C, and the outer surface temperature is 75°C, calculate the heat loss from the furnace.

Known: Cubical furnace, 350mm external dimensions, with 50mm thick walls.

Find: The heat loss, q(W).

Schematic:



Assumptions: (1) steady-state conditions, (2) two-dimensional conduction, (3) constant properties.

Properties: From table of properties, fireclay brick ($\bar{T} = (T_1 + T_2)/2 = 610K) : k \approx 1.1W/m.K$

Analysis: using relations for the shape factor from table 4.1,

Plane walls (6) $$S_W = \frac{A}{L} = \frac{0.25 \times 0.25m^2}{0.05m} = 1.25m$$

Edges (12) $$S_E = 0.54D = 0.52 \times 0.25m = 0.14m$$

Corners (8) $$S_C = 0.15L = 0.15 \times 0.05m = 0.008m$$

The heat rate in terms of the shape factor is

$$q = kS(T_1 - T_2) = k(6S_W + 12S_E + 8S_C)(T_1 - T_2)$$

$$q = 1.1\frac{W}{m.K}(6\times 1.25m + 12\times 0.14m + 0.15\times 0.008m)(600-75)°C$$

$$q = 5.30kW$$

Comments: Be sure to note that the restriction for S_E and S_C has been met.

Problem 5:

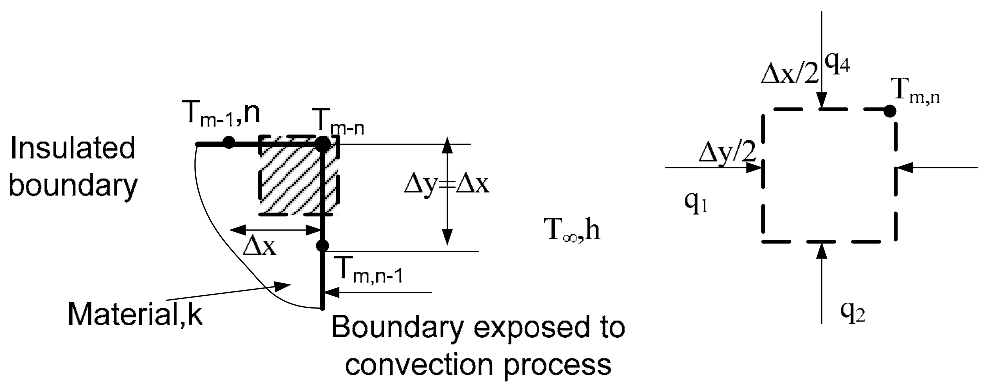
Consider nodal configuration 4 of table 4.2. Derive the finite-difference equation under steady-state conditions for the following situations.

(a) The upper boundary of the external corner is perfectly insulated and the side boundary is subjected to the convection process (T_∞, h)
(b) Both boundaries of external corner are perfectly insulated.

Known: External corner of a two-dimensional system whose boundaries are subjected to prescribed conditions.

Find: finite-difference equations for these situations: (a) upper boundary is perfectly insulated and side boundary is subjected to a convection process, (b) both boundaries are perfectly insulated.

Schematic:



Assumptions: (1) steady-state conditions, (2) two-dimensional conduction, (3) constant properties, (4) no internal generation.

Analysis: Consider the nodal point configuration shown in schematic and also as case 4, table 4.2. the control volume about the nodes shaded area above of unit thickness normal to the page has dimensions, (Δx/2)(Δy/2).1. The heat transfer processes at the surface of the CV are identified as q_1, q2….perform an energy balance wherein the processes are expressed using the appropriate rate equations.

With the upper boundary insulated and the side boundary and the side boundary subjected to a convection process, the energy balance has the form

$$\dot{E}_{in} - E_{out} = 0 \qquad q_1 + q_2 + q_3 + q_4 = 0$$

$$k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\left(\frac{\Delta y}{2}.1\right)(T_\infty - T_{m,n}) + 0 = 0$$

Letting Δx=Δy, and regrouping, find

$$T_{m,n-1} + T_{m-1,n} + \frac{h\Delta x}{k} T_{\infty} - 2\left(\frac{1}{2}\frac{h\Delta x}{k} + 1\right) T_{m,n} = 0$$

with both boundaries insulated, the energy balance of Eq(2) would have q3=q4=0. the same result would be obtained by letting h=0 in the finite difference equation, Eq(3). The result is

$$T_{m,n-1} + T_{m-1,n} - 2T_{m,n} = 0$$

Comments: Note the convenience resulting formulating the energy balance by assuming that all the heat flow is into the node.

Module 4: Short questions

1. For what kind of problems in multidimensional heat transfer are analytical solutions possible? Name some common analytical methods in steady state multidimensional heat transfer?

2. What are the limitations of analytical methods? In spite of their limitations, why are analytical solutions useful?

3. What is meant by "shape factor" in two-dimensional heat transfer analysis? What is the advantage of using such a method? Is there a shape factor in 1D heat transfer?

4. How do numerical solution methods differ from analytical methods? What are the advantages and disadvantages of numerical and analytical methods?

5. What is the basis of energy balance method in numerical analysis? How does it differ from the formal finite difference method using Taylor series approximation? For a specified nodal network, will these two methods result in the same or a different set of equations?

6. Consider a medium in which the finite difference formulation of a general interior node is given in its simplest form as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0$$

   (a) Is heat transfer in this medium steady or transient?
   (b) Is heat transfer one-, two-, or three-dimensional?
   (c) Is there heat generation in the medium?
   (d) Is the nodal spacing constant or variable?
   (e) Is thermal conductivity of the medium constant or variable?

7. Consider a medium in which the finite difference formulation of a general interior node is given in its simplest form as

$$T_{left} + T_{top} + T_{right} + T_{bottom} - 4T_{node} + \frac{\dot{g}_m l^2}{k} = 0$$

   (a) Is heat transfer in this medium steady or transient?
   (b) Is heat transfer one-, two-, or three-dimensional?
   (c) Is there heat generation in the medium?
   (d) Is the nodal spacing constant or variable?
   (e) Is thermal conductivity of the medium constant or variable?

8. What is an irregular boundary? What is a practical way of handling irregular boundary surfaces with the finite difference method?

9. When performing numerical calculations of heat diffusion on a structured Cartesian grid in two dimensions, a simplified form of the equations states that the

temperature at a node is simply the average of its four adjacent neighbours. What assumption is NOT required to allow this simplified form

a) must have no heat generation
b) must not be at a domain boundary
c) must have uniform cell dimensions in both directions
d) must be a solid medium

Module 5: Learning objectives

- The primary objective of this chapter is to learn various methods of treating transient conduction which occurs in numerous engineering applications. The student should learn to recognize several classes of problems in unsteady conduction, and should be able to apply appropriate simplicity conditions before attempting to solve the problems.
- The simplest case is the lumped capacity condition, and the student should understand under what conditions one can apply this assumption. The first thing a student should do is calculate the Biot number. If this number is much less than unity, he/she may use the lumped capacitance method to obtain accurate results with minimum computational requirements.
- However, if the Biot number is not much less than unity, spatial effects must be considered and some other method must be used. Analytical results are available in convenient graphical and equation form for the plane wall, the finite cylinder, the sphere, and the semi-infinite solid. The student should know when and how to use these results.
- If geometrical complexities and /or the form of the boundary conditions preclude the use of analytical solutions, recourse must be made to an approximate numerical technique, such as the finite difference method or the finite volume method.

MODULE 5

UNSTEADY STATE HEAT CONDUCTION

5.1 Introduction

To this point, we have considered conductive heat transfer problems in which the temperatures are independent of time. In many applications, however, the temperatures are varying with time, and we require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k} \tag{5.1}$$

where $\alpha=\frac{k}{\rho c}$ is the thermal diffusivity. Without any heat generation and considering spatial variation of temperature only in x-direction, the above equation reduces to:

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial x^2} \tag{5.2}$$

For the solution of equation (5.2), we need two boundary conditions in x-direction and one initial condition. Boundary conditions, as the name implies, are frequently specified along the physical boundary of an object; they can, however, also be internal – e.g. a known temperature gradient at an internal line of symmetry.

5.2 Biot and Fourier numbers

In some transient problems, the internal temperature gradients in the body may be quite small and insignificant. Yet the temperature at a given location, or the average temperature of the object, may be changing quite rapidly with time. From eq. (5.1) we can note that such could be the case for large thermal diffusivity α .

A more meaningful approach is to consider the general problem of transient cooling of an object, such as the hollow cylinder shown in figure 5.1.

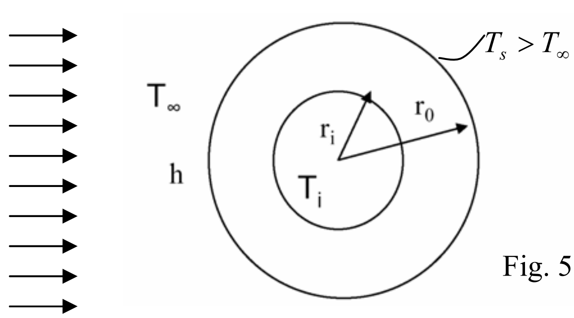


Fig. 5.1

For very large r_i, the heat transfer rate by conduction through the cylinder wall is approximately

$$q \approx -k(2\pi r_o l)\left(\frac{T_s - T_i}{r_o - r_i}\right) = k(2\pi r_o l)\left(\frac{T_i - T_s}{L}\right) \quad (5.3)$$

where l is the length of the cylinder and L is the material thickness. The rate of heat transfer away from the outer surface by convection is

$$q = \bar{h}(2\pi r_o l)(T_s - T_\infty) \quad (5.4)$$

where $\bar{h}$ is the average heat transfer coefficient for convection from the entire surface. Equating (5.3) and (5.4) gives

$$\frac{T_i - T_s}{T_s - T_\infty} = \frac{\bar{h}L}{k} = \text{Biot number} \quad (5.5)$$

The Biot number is dimensionless, and it can be thought of as the ratio

$$\mathbf{Bi} = \frac{\textbf{resistance to internal heat flow}}{\textbf{resistance to external heat flow}}$$

Whenever the Biot number is small, the internal temperature gradients are also small and a transient problem can be treated by the "lumped thermal capacity" approach. The lumped capacity assumption implies that the object for analysis is considered to have a single mass-averaged temperature.

In the derivation shown above, the significant object dimension was the conduction path length, $L = r_o - r_i$. In general, a characteristic length scale may be obtained by dividing the volume of the solid by its surface area:

$$L = \frac{V}{A_s} \quad (5.6)$$

Using this method to determine the characteristic length scale, the corresponding Biot number may be evaluated for objects of any shape, for example a plate, a cylinder, or a sphere. As a thumb rule, if the Biot number turns out to be less than 0.1, lumped capacity assumption is applied.

In this context, *a dimensionless time*, known as the **Fourier number**, can be obtained by multiplying the dimensional time by the thermal diffusivity and dividing by the square of the characteristic length:

$$\textbf{dimensionless time} = \frac{\alpha \mathbf{t}}{L^2} = \mathbf{Fo} \quad (5.7)$$

5.3 Lumped thermal capacity analysis

The simplest situation in an unsteady heat transfer process is to use the lumped capacity assumption, wherein we neglect the temperature distribution inside the solid and only deal with the heat transfer between the solid and the ambient fluids. In other words, we are assuming that the temperature inside the solid is constant and is equal to the surface temperature.

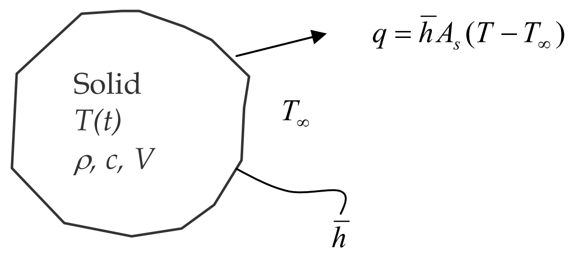


Fig. 5.2

The solid object shown in figure 5.2 is a metal piece which is being cooled in air after hot forming. Thermal energy is leaving the object from all elements of the surface, and this is shown for simplicity by a single arrow. The first law of thermodynamics applied to this problem is

$$\begin{pmatrix} \text{heat out of object} \\ \text{during time } dt \end{pmatrix} = \begin{pmatrix} \text{decrease of internal thermal} \\ \text{energy of object during time } dt \end{pmatrix}$$

Now, if Biot number is small and temperature of the object can be considered to be uniform, this equation can be written as

$$\bar{h} A_s \left[T(t) - T_\infty \right] dt = -\rho c V dT \qquad (5.8)$$

or,

$$\frac{dT}{\left(T - T_\infty\right)} = -\frac{\bar{h} A_s}{\rho c V} dt \qquad (5.9)$$

Integrating and applying the initial condition $T(0) = T_i$,

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{\bar{h} A_s}{\rho c V} t \qquad (5.10)$$

Taking the exponents of both sides and rearranging,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad (5.11)$$

where

$$b = \frac{\bar{h} A_s}{\rho c V} \qquad (1/\text{s}) \qquad (5.12)$$

Note: In eq. 5.12, b is a positive quantity having dimension $(\text{time})^{-1}$. The reciprocal of b is usually called *time constant*, which has the dimension of time.
Question: What is the significance of b?
Answer: According to eq. 5.11, the temperature of a body approaches the ambient temperature T_∞ exponentially. In other words, the temperature changes rapidly in the beginning, and then slowly. A larger value of b indicates that the body will approach the surrounding temperature in a shorter time. You can visualize this if you note the variables in the numerator and denominator of the expression for b. As an exercise, plot T vs. t for various values of b and note the behaviour.

Rate of convection heat transfer at any given time t:

$$\dot{Q}(t) = hA_s\left[T(t) - T_\infty\right]$$

Total amount of heat transfer between the body and the surrounding from t=0 to t:

$$Q = mc\,\left[T(t) - T_i\right]$$

Maximum heat transfer (limit reached when body temperature equals that of the surrounding):

$$Q = mc\,\left[T_\infty - T_i\right]$$

5.4 Spatial Effects and the Role of Analytical Solutions

If the lumped capacitance approximation can not be made, consideration must be given to spatial, as well as temporal, variations in temperature during the transient process.

The Plane Wall: Solution to the Heat Equation for a Plane Wall withSymmetrical Convection Conditions

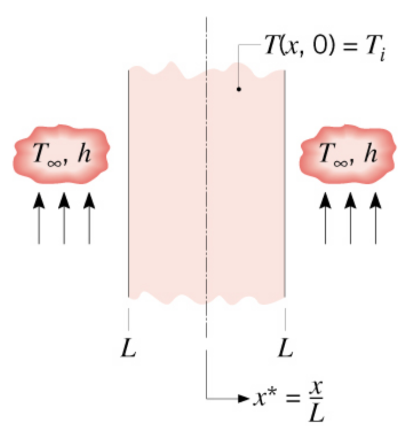
- For a plane wall with symmetrical convection conditions and constant properties, the heat equation and initial/boundary conditions are:

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=l} = h\left[T(L,t) - T_\infty\right]$$



Note: Once spatial variability of temperature is included, there is existence of seven different independent variables.

$$T = T(x, t, T_i, T_\infty, h, k, \alpha)$$

How may the functional dependence be simplified?

- The answer is **Non-dimensionalisation.** We first need to understand the physics behind the phenomenon, identify parameters governing the process, and group them into meaningful non-dimensional numbers.

Non-dimensionalisation of Heat Equation and Initial/Boundary Conditions:

The following dimensionless quantities are defined.

Dimensionless temperature difference: $\theta^* = \dfrac{\theta}{\theta_i} = \dfrac{T - T_\infty}{T_i - T_\infty}$

Dimensionless coordinate: $x^* = \dfrac{x}{L}$

Dimensionless time: $t^* = \dfrac{\alpha t}{L^2} = Fo$

The Biot Number: $Bi = \dfrac{hL}{k_{solid}}$

The solution for temperature will now be a function of the other non-dimensional quantities

$$\theta^* = f(x^*, Fo, Bi)$$

Exact Solution:

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right)$$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin(2\zeta_n)} \qquad \zeta_n \tan\zeta_n = Bi$$

The roots (eigenvalues) of the equation can be obtained from tables given in standard textbooks.

The One-Term Approximation $Fo > 0.2$

Variation of mid-plane $(x^* = 0)$ temperature with time (Fo)

$$\theta_0^* = \frac{T - T_\infty}{T_i - T_\infty} \approx C_1 \exp\left(-\zeta_1^2 Fo\right)$$

From tables given in standard textbooks, one can obtain C_1 and ζ_1 as a function of *Bi*.

Variation of temperature with location (x^*) and time (Fo):

$$\theta^* = \theta_0^* = \cos\left(\zeta_1 x^*\right)$$

Change in thermal energy storage with time:

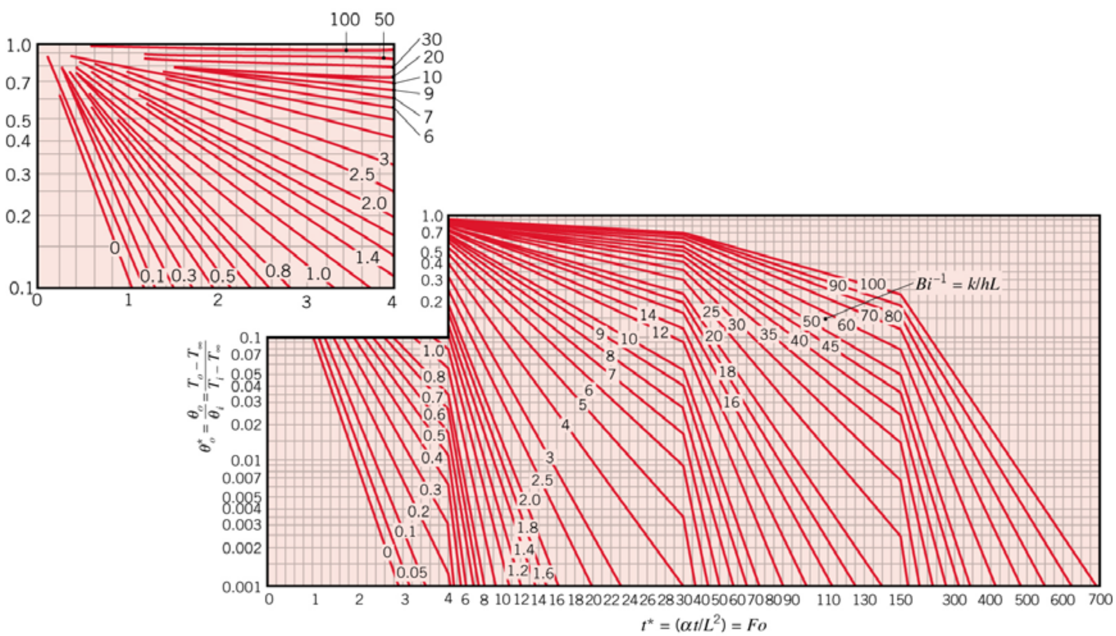
$$\Delta E_{st} = -Q$$

$$Q = Q_0\left(1 - \frac{\sin \zeta_1}{\zeta_1}\right)\theta_0^*$$

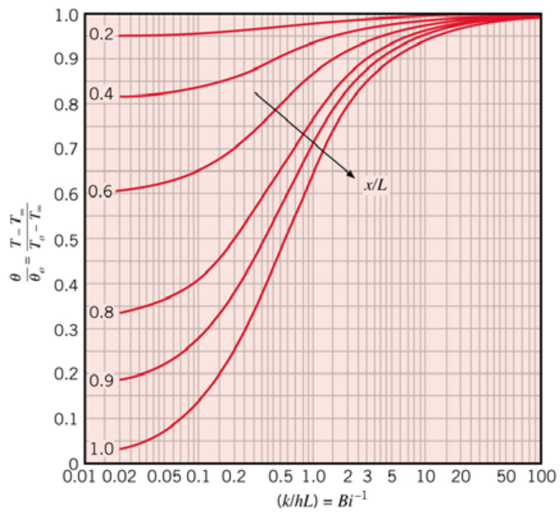
$$Q_0 = \rho c V\left(T_i - T_\infty\right)$$

Can the foregoing results be used for a plane wall that is well insulated on one side and convectively heated or cooled on the other?
Can the foregoing results be used if an isothermal condition $\left(T_s \neq T_i\right)$ is instantaneously imposed on both surfaces of a plane wall or on one surface of a wall whose other surface is well insulated?

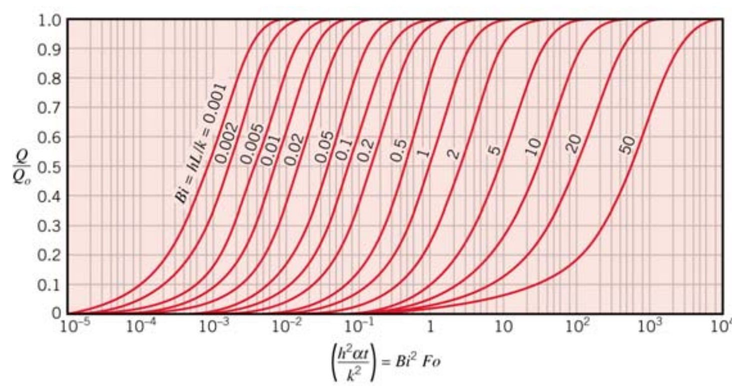
Graphical Representation of the One-Term Approximation:
The Heisler Charts
Midplane Temperature:



Temperature Distribution



Change in Thermal Energy Storage



§ Assumptions in using Heisler charts:
  l Constant Ti and thermal properties over the body
  l Constant boundary fluid T_∞ by step change
  l Simple geometry: slab, cylinder or sphere

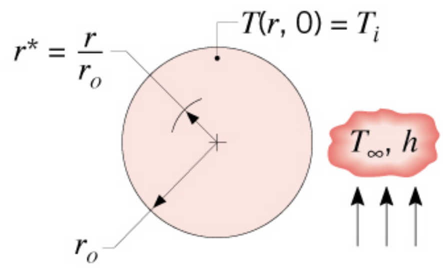
§ Limitations:
  l Far from edges
  l No heat generation (Q=0)
  l Relatively long after initial times (Fo >0.2)

Radial Systems

Long Rods or Spheres Heated or Cooled by Convection

$Bi = hr_0 / k$

$Fo = \alpha t / r_0^2$



Similar Heisler charts are available for radial systems in standard text books.

Important tips: Pay attention to the length scale used in those charts, and calculate your Biot number accordingly.

5.5 Numerical methods in transient heat transfer: The Finite Volume Method

Considering the steady convection-diffusion equation:

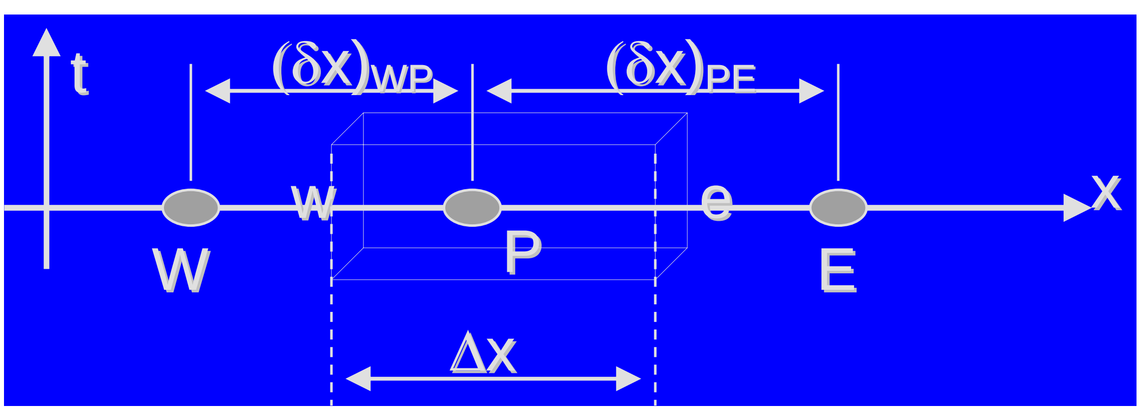
$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\Gamma\, grad\ \phi) + S_\phi$$

- The time and control volume integrations give:

$$\int_{CV}\left(\int_t^{t+\Delta t}\frac{\partial}{\partial t}(\rho\phi)dt\right)dV + \int_t^{t+\Delta t}\left(\int_A \mathbf{n}\cdot(\rho\phi\,\mathbf{u})dA\right)dt = \int_t^{t+\Delta t}\left(\int_A \mathbf{n}\cdot(\Gamma\, grad\ \phi)dA\right)dt + \int_t^{t+\Delta t}\left(\int_{CV} S_\phi dV\right)dt$$

- Unsteady one-dimensional heat conduction:

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + S$$



- Consider the one-dimensional control volume. Integration over the control volume and over a time interval gives:
$$\int_t^{t+\Delta t}\left(\int_{CV}\left(\rho c\frac{\partial T}{\partial t}\right)dV\right)dt = \int_t^{t+\Delta t}\left(\int_{cv}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dV\right)dt + \int_t^{t+\Delta t}\left(\int_{CV}SdV\right)dt$$
- Re-written
$$\int_w^e\left(\int_t^{t+\Delta t}\rho c\frac{\partial T}{\partial t}dt\right)dV = \int_t^{t+\Delta t}\left(\left(kA\frac{\partial T}{\partial x}\right)_e - \left(kA\frac{\partial T}{\partial x}\right)_w\right)dt + \int_t^{t+\Delta t}\left(\bar{S}\Delta V\right)dt$$
- If the temperature at a node is assumed to prevail over the whole control volume, applying the central differencing scheme, we have:
$$\rho c\left(T_P - T_P^0\right)\Delta V = \int_t^{t+\Delta t}\left(\left(k_e A\frac{T_E - T_P}{\delta x_{PE}}\right) - \left(k_w A\frac{T_P - T_W}{\delta x_{WP}}\right)\right)dt + \int_t^{t+\Delta t}\left(\bar{S}\Delta V\right)dt$$
- An assumption about the variation of T_P, T_E and T_w with time. By generalizing the approach by means of a weighting parameter θ between 0 and 1:
$$I_T = \int_t^{t+\Delta t}T_P dt = \left[\theta T_P + (1-\theta)T_P^0\right]\Delta t$$
- Therefore,
$$\rho c\left(\frac{T_P - T_P^0}{\Delta t}\right)\Delta x = \theta\left[\left(k_e\frac{T_E - T_P}{\delta x_{PE}}\right) - \left(k_w\frac{T_P - T_W}{\delta x_{WP}}\right)\right]$$
$$+ (1-\theta)\left[\left(k_e\frac{T_E^0 - T_P^0}{\delta x_{PE}}\right) - \left(k_w\frac{T_P^0 - T_W^0}{\delta x_{WP}}\right)\right] + \bar{S}\Delta x$$
- Re-arranging:
$$\left[\rho c\frac{\Delta x}{\Delta t} + \theta\left(\frac{k_e}{\delta x_{PE}} + \frac{k_w}{\delta x_{WP}}\right)\right]T_P = \frac{k_e}{\delta x_{PE}}\left[\theta T_E + (1-\theta)T_E^0\right] + \frac{k_w}{\delta x_{WP}}\left[\theta T_W + (1-\theta)T_W^0\right]$$
$$+\left[\rho c\frac{\Delta x}{\Delta t} - (1-\theta)\frac{k_e}{\delta x_{PE}} - (1-\theta)\frac{k_w}{\delta x_{WP}}\right]T_P^0 + \bar{S}\Delta x$$
- Compared with standard form:
$a_P T_P = a_W\left[\theta T_W + (1-\theta)T_W^0\right] + a_E\left[\theta T_E + (1-\theta)T_E^0\right] + \left[a_P^0 - (1-\theta)a_W - (1-\theta)a_E\right]T_P^0 + b$ where
$$a_P = \theta\left(a_W + a_E\right) + a_P^0$$
$$a_P^0 = \rho c\frac{\Delta x}{\Delta t}$$
$$a_W = \frac{k_w}{\delta x_{WP}}$$
$$a_E = \frac{k_e}{\delta x_{PE}}$$
$$b = \bar{S}\Delta x$$
  - ✓ When $\theta = 0$, the resulting scheme is "explicit".
  - ✓ When $0 < \theta \leq 1$, the resulting scheme is "implicit".
  - ✓ When $\theta = 1$, the resulting scheme is "fully implicit".

- ✓ When $\theta = 1/2$, the resulting scheme is "the Crank-Nicolson".

- Explicit scheme

  $$a_P T_P = a_W\left[\theta T_W + (1-\theta)T_W^0\right] + a_E\left[\theta T_E + (1-\theta)T_E^0\right] + \left[a_P^0 - (1-\theta)a_W - (1-\theta)a_E\right]T_P^0 + b$$

  ✓ The source term is linearised as $b = S_u + S_p T_p^0$ and set θ = 0

  ✓ The explicit discretisation:

  $$a_P T_P = a_W T_W^0 + a_E T_E^0 + \left[a_P^0 - (a_W + a_E)\right]T_P^0 + S_u$$

  where

  $$a_P = a_P^0$$

  $$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

  $$a_W = \frac{k_w}{\delta x_{WP}}$$

  $$a_E = \frac{k_e}{\delta x_{PE}}$$

  ✓ The scheme is based on backward differencing and its Taylor series trunction error accuracy is first-order with respect to times.

  ✓ All coefficient must be positive in the discretised equation:

  $$a_P^0 - (a_W + a_E - S_P) > 0$$

  or

  $$\rho c \frac{\Delta x}{\Delta t} - \left(\frac{k_w}{\delta x_{WP}} + \frac{k_e}{\delta x_{PE}}\right) > 0$$

  or

  $$\rho c \frac{\Delta x}{\Delta t} > \frac{2k}{\Delta x}$$

  or

  $$\Delta t < \rho c \frac{(\Delta x)^2}{2k}$$

  ✓ It becomes very expensive to improve spatial accuracy. This method is not recommended for general transient problems.

  ✓ Nevertheless, provided that the time step size is chosen with care, the explicit scheme described above is efficient for simple conduction calculations.

- Crank-Nicolson scheme

$$a_P T_P = a_W\left[\theta T_W + (1-\theta)T_W^0\right] + a_E\left[\theta T_E + (1-\theta)T_E^0\right] + \left[a_P^0 - (1-\theta)a_W - (1-\theta)a_E\right]T_P^0 + b$$

✓ Set θ = 1/2

$$a_P T_P = a_E\left(\frac{T_E + T_E^0}{2}\right) + a_W\left(\frac{T_W + T_W^0}{2}\right) + \left[a_P^0 - \frac{a_E}{2} - \frac{a_W}{2}\right]T_P^0 + b$$

where

$$a_P = \frac{1}{2}(a_E + a_W) + a_P^0 - \frac{1}{2}S_P$$

$$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

$$a_W = \frac{k_w}{\delta x_{WP}}$$

$$a_E = \frac{k_e}{\delta x_{PE}}$$

$$b = S_u + \frac{1}{2}S_p T_p^0$$

✓ The method is implicit and simultaneous equations for all node points need to be solved at each time step.

✓ All coefficient must be positive in the discretised equation:

$$a_P^0 > \frac{a_E + a_W}{2}$$

or

$$\Delta t < \rho c \frac{(\Delta x)^2}{k}$$

✓ This is only slightly less restrictive than the explicit method. The Crank-Nicolson method is based on central differencing and hence it is second-order accurate in time. So, it is normally used in conjunction with spatial central differencing.

- The fully implicit scheme

$$a_P T_P = a_W\left[\theta T_W + (1-\theta)T_W^0\right] + a_E\left[\theta T_E + (1-\theta)T_E^0\right] + \left[a_P^0 - (1-\theta)a_W - (1-\theta)a_E\right]T_P^0 + b$$

  - ✓ Set θ = 1

    $$a_P T_P = a_E T_E + a_W T_W + a_P^0 T_P^0$$

    where

    $$a_P = a_P^0 + a_E + a_W - S_P$$

    $$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

    $$a_W = \frac{k_w}{\delta x_{WP}}$$

    $$a_E = \frac{k_e}{\delta x_{PE}}$$

  - ✓ A system of algebraic equations must be solved at each time level. The accuracy of the scheme is first-order in time.
  - ✓ The time marching procedure starts with a given initial field of temperature T^0. The system is solved after selecting time step Δt.
  - ✓ All coefficients are positive, which makes the implicit scheme unconditionally stable for any size of time step.
  - ✓ The implicit method is recommended for general purpose transient calculations because of its robustness and unconditional stability.

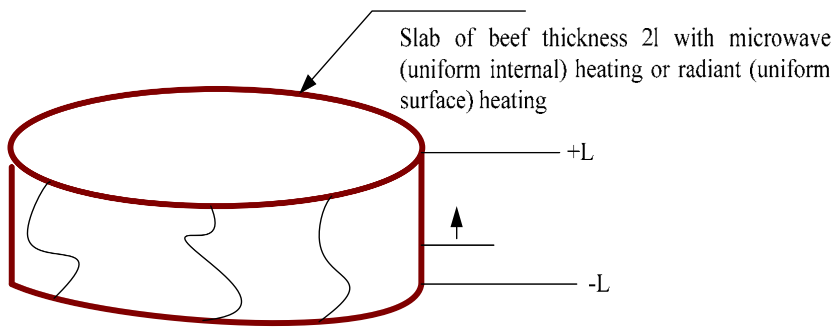
Module 5: Worked out problems

Problem 1:
A microwave oven operates on the principle that application of a high frequency field causes electrically polarized molecules in food to oscillate. The net effect is a uniform generation of thermal energy within the food, which enables it to be heated from refrigeration temperatures to 90º in as short a time as 30 s.

Known: Microwave and radiant heating conditions for a slab of beef.

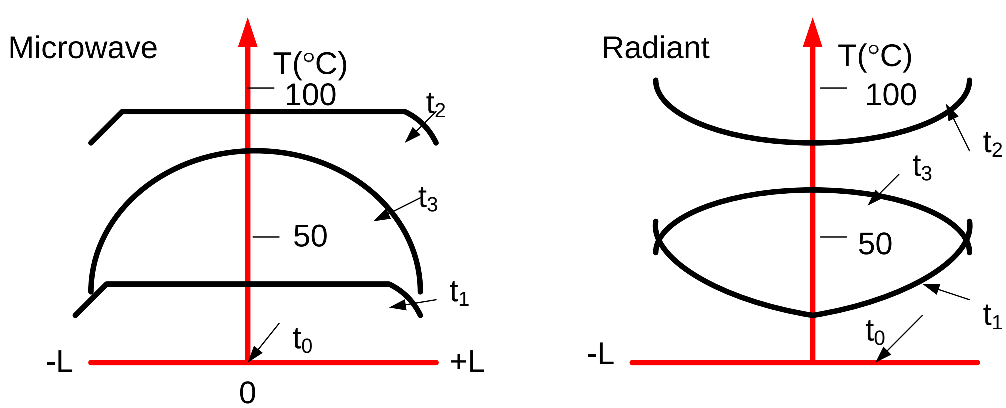
Find: Sketch temperature distributions at specific times during heating and cooling.

Schematic:



Assumptions: (1) one-dimensional conduction in x, (2) uniform internal heat generation for microwave, (3) uniform surface heating for radiant oven, (4) heat loss from surface of meat to surroundings is negligible during the heat process, (5) symmetry about mid plane.

Analysis:



Comments:

(1) With uniform generation and negligible surface heat loss, the temperature distribution remains nearly uniform during microwave heating. During the subsequent surface cooling, the maximum temperature is at the mid plane.

(2) The interior of the meat is heated by conduction from the hotter surfaces during radiant heating, and the lowest temperature is at the mid plane. The situation is reversed shortly after cooling begins, and the maximum temperature is at the mid plane.

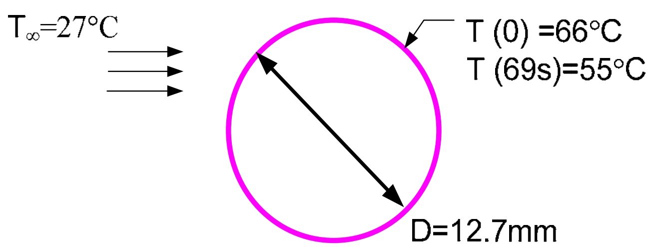
Problem 2:

The heat transfer coefficient for air flowing over a sphere is to be determined by observing the temperature- time history of a sphere fabricated from pure copper. The sphere which is 12.7 mm in diameter is at 66º C before it is inserted into an air stream having a temperature of 27ºC. A thermocouple on the outer surface of the sphere indicates 55ºC, 69 s after the sphere is inserted into an air stream. Assume, and then justify, that the sphere behaves as a space-wise isothermal object and calculate the heat transfer coefficient.

Known: The temperature-time history of a pure copper sphere in air stream.

Find: The heat transfer coefficient between and the air stream

Schematic:



Assumptions: (1) temperature of sphere is spatially uniform, (2) negligible radiation exchange, (3) constant properties.

Properties: From table of properties, pure copper (333K): =8933 kg/m^3, c_p=389 J/kg.K, k=389W/m.K

Analysis: the time temperature history is given by

$$\frac{\theta(t)}{\theta_i} = \exp\left(-\frac{t}{R_t C_t}\right)$$

Where $R_t = \frac{1}{hA_s}$, $C_t = \rho V c_p$, $\theta = T - T_\infty$, $A_s = \pi D^2$, $V = \frac{\pi D^3}{6}$

Recognize that when t = 69 s

$$\frac{\theta(t)}{\theta_i} = \frac{(55-27)^\circ C}{(66-27)^\circ C} = 0.718 = \exp\left(-\frac{t}{\tau_t}\right) = \exp\left(-\frac{69s}{\tau_t}\right)$$

And noting that $\tau_t = R_t C_t$ find

$\tau_t = 208s$

Hence,

$$h = \frac{\rho V c_p}{A_s \tau_t} = \frac{8933 kg/m^3 (\pi 0.0127^3 m^3 / 6) 389 J/kg.K}{\pi 0.0127^2 m^2 \times 208 s}$$

$$h = 35.3\, W/m^2.K$$

Comments: Note that with $L_c = D_0 / 6$

$$Bi = \frac{hL_c}{k} = 35.3 W/m^2.K \times \frac{0.0127}{6} m / 398\, W/m.K = 1.88 \times 10^{-4}$$

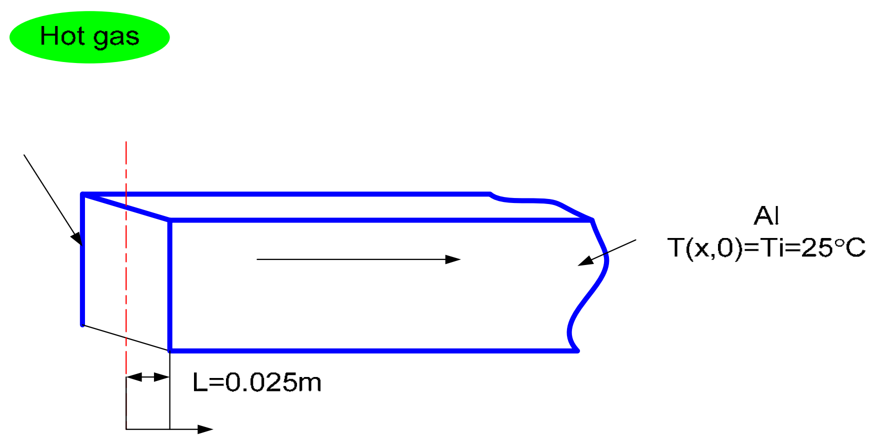
Hence Bi<0.1 and the spatially isothermal assumption is reasonable.

Problem 3:
A thermal energy storage unit consists of a large rectangular channel, which is well insulated on its outer surface and enclosed alternating layers of the storage material and the flow passage. Each layer of the storage material is aluminium slab of width=0.05m which is at an initial temperatures of 25ºC. consider the conditions for which the storage unit is charged by passing a hot gas through the passages, with the gas temperature and convection coefficient assumed to have constant values of T=600ºC and h=100W/m^2.K throughout the channel how long will it take to achieve 75% of the maximum possible energy storage? What is the temperature of the aluminium at this time?

Known: Configuration, initial temperature and charging conditions of a thermal energy storage unit.

Find: Time required achieving 75% of maximum possible energy storage. Temperature of storage medium at this time.

Schematic:



Assumptions: (1) one-dimensional conduction, (2) constant properties, (3) negligible heat exchange with surroundings.

Properties: From any table of properties: Aluminum, pure (T 600K=327C): k=231W/m.K, c= 1033 J/kg.K, =2702kg/m3.

Analysis: recognizing the characteristic length is the half thickness, find

$$Bi = \frac{hL}{k} = \frac{100\,W/m^2.K \times 0.025m}{231\,W/m.K} = 0.011$$

Hence, the lumped capacitance method may be used.

$$Q = (\rho Vc)\theta_i[1 - \exp(-t/\tau_i)] = -\Delta E_{st}$$
$$-\Delta E_{st,\max} = (\rho Vc)\theta_i$$

Dividing eq. (1) and (2), the condition sought is for

$$\Delta E_{st} / \Delta E_{st,\max} = 1 - \exp(-t / \tau_{th}) = 0.75$$

Solving for τ_{th} and substituting numerical values, find

$$\tau_{th} = \frac{\rho V c}{h A_s} = \frac{\rho L c}{h} = \frac{2702 kg / m^3 \times 0.025 m \times 1033 J / kg.K}{100 W / m^2.K} = 698 s$$

Hence, the time required is

-exp (-t/698s) =-0.25 or t=968s.

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-t / \tau_{th})$$

$$T = T_\infty + (T_i - T_\infty) \exp(-t / \tau_{th}) = 600°C - (575°C) \exp(-968 / 698)$$

T=456°C

Comments: for the prescribed temperatures, the property temperatures dependence is significant and some error is incurred by assuming constant properties. However, selecting at 600K was reasonable for this estimate.

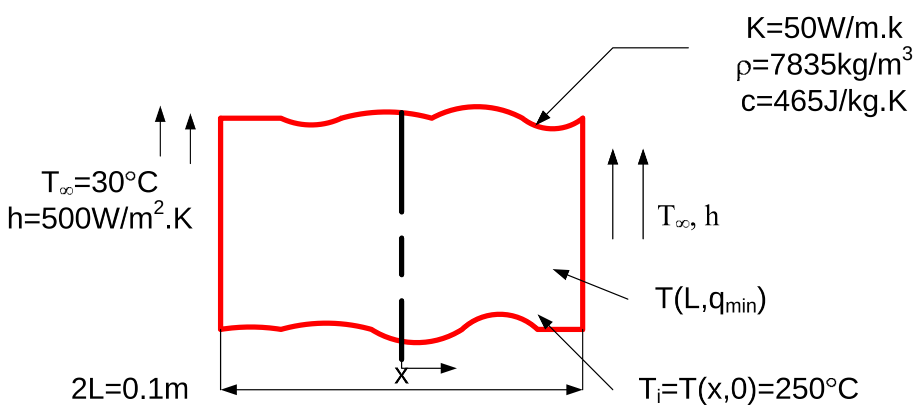
Problem 4:

A one-dimensional plane wall with a thickness of 0.1 m initially at a uniform temperature of 250C is suddenly immersed in an oil bath at 30°C. assuming the convection heat transfer coefficient for the wall in the bath is 500 $W/m^2.K$. Calculate the surface temperature of the wall 9 min after immersion. The properties of the wall are k=50 W/m.K, ρ=7835 kg/m^3, and c=465 J/kg.K.

Known: plane wall, initially at a uniform temperature, is suddenly immersed in an oil bath and subjected to a convection cooling process.

Find: Surface temperature of the wall nine minutes after immersion, T (L, 9 min).

Schematic:



Assumptions: The Biot number for the plane wall is

$$Bi = \frac{hL_c}{k} = \frac{500\, W/m^2.K \times 0.05\, m}{50\, W/m.K} = 0.50$$

Since Bi>0.1, lumped capacitance analysis is not appropriate.

$$Fo = \frac{\alpha t}{L^2} = \frac{(k/\rho c)_t}{L^2} = \frac{50W/m.K/7835\,kg/m^3 \times 465\,J/kg.K \times (9\times 60)s}{(0.05m)^2} = 2.96$$

And Bi^{-1}=1/0.50 = 2, find

$$\frac{\theta_0}{\theta_i} = \frac{T(0,t)-T_\infty}{T_i - T_\infty} \approx 0.3$$

We know that Bi^{-1}=1/0.50 = 2 and for X/L=1, find

$$\frac{\theta(1,t)}{\theta_0} \approx 0.8$$

By combining equation, $\theta(1,t)$=0.8(θ_0) = 0.8(0.3 θ_i) =0.24 θ_i

Recalling that $\theta = T(L,t) - T_\infty$ and $\theta_i = T_i - T_\infty$, it follows that

$$T(L,t) = T_\infty + 0.24\,(T_i - T_\infty) = 30°C + 0.24\,(250 - 30)°C = 83°C$$

Comments: (1) note that figure provides a relationship between the temperature at any x/L and the centerline temperature as a function of only the Biot number. Fig applies to the centerline temperature which is a function of the Biot number and the Fourier number. The centerline temperature at t=9min follows from equation with

$$T(0,t) - T_\infty = 0.3(T_i - T_\infty) = 0.3(250 - 30)°C = 66°C$$

(2) Since F_0>=0.2, the approximate analytical solution for θ^* is valid. From table with Bi=0.50, and ζ_1 =0.6533 rad and C_1=1.0701. Substituting numerical values into equations

θ^*=0.303 and $\theta^*(1, F_O)$ =0.240

From this value, find T (L, 9 min) =83°C which is identical to graphical result.

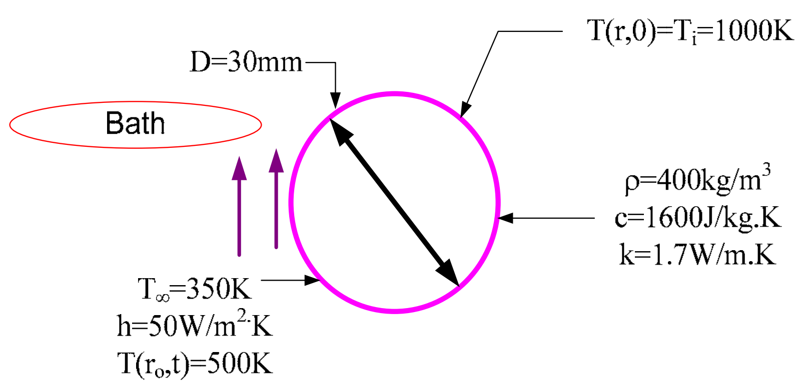
Problem 5:

A long cylinder of 30mm diameter, initially at a uniform temperature of 1000K, is suddenly quenched in a large, constant-temperature oil bath at 350K. The cylinder properties are k=1.7W/m.K, c=1600 J/kg.K, and ρ=400 kg/m^3, while the convection coefficient is 50W/m^2.K. Calculate the time required for the surface cylinder to reach 500K.

Known: A long cylinder, initially at a uniform temperature, is suddenly quenched in large oil bath.

Find: time required for the surface to reach 500K.

Schematic:



Assumptions: (1) one dimensional radial conduction, (2) constant properties

Analysis: check whether lumped capacitance methods are applicable.

$$BI_c = \frac{hL_c}{k} = \frac{h(r_0/2)}{k} = \frac{50W/m^2.K(0.015m/2)}{1.7W/m.K} = 0.221$$

Since BI_c>0.1, method is not suited. Using the approximate series solutions for the infinite cylinder,

$$\theta^*(r^*, Fo) = C_1 \exp(-\varsigma_1^2 Fo) \times J_o(\varsigma_1^2 r^*)$$

Solving for F_o and letting =1, find

$$F_o = -\frac{1}{\varsigma_1^2}\ln\left[\frac{\theta^*}{C_1 J_o(\varsigma_1^2)}\right]$$

where $\theta^*(1, F_0) = \frac{T(r_o, t_{o)} - T\infty}{T_i - T_\infty} = \frac{(500-350)K}{(1000-350)K} = 0.231$

From table, Bi=0.441, find ς_1 =0.8882 rad and C_1=1.1019. From table find J_o (ς_1^2) =0.8121. Substituting numerical values into equation,

$$F_o = -\frac{1}{(0.8882)^2}\ln[0.231/1.1019\times 0.8121] = 1.72$$

From the definition of the Fourier number, $F_o = \frac{\alpha t}{r_o{}^2} = F_o . r_o{}^2 \frac{\rho c}{k}$

$$t = 1.72\ (0.015m)^2 \times 400\ kg/m^3 \times 1600\ J/kg.K/1.7W/m.K = 145\ s$$

Comments: (1) Note that F_o>=0.2, so approximate series solution is appropriate.

(2) Using the Heisler chart, find F_o as follows. With Bi^{-1}=2.27, find from fir r/r_o=1 that

$\frac{\theta(r_o,t)}{\theta_o} = \frac{T(r_o,t)-T_\infty}{T(0,t)0-T_\infty} \approx 0.8$ or $T(0,t) = T_\infty + \frac{1}{0.8}[T(r_o,t)-T_\infty] = 537K$

hence $\frac{\theta_o}{\theta_i} = \frac{(537-350)K}{(1000-350)K} = 0.29$

From fig, with $\frac{\theta_o}{\theta_i}$=0.29 and Bi^{-1}=2.27, find Fo $\approx$ 1.7 and eventually obtain t $\approx$144s.

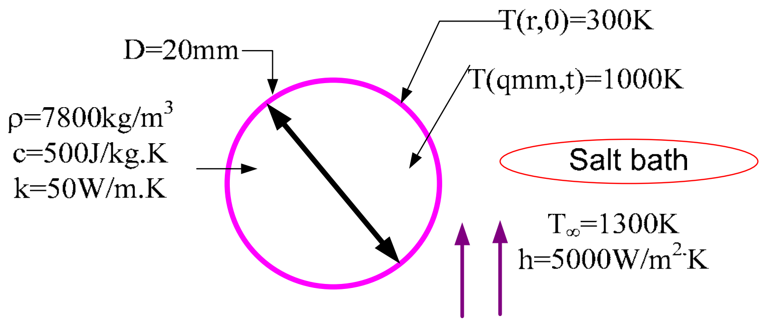
Problem 6:

In heat treating to harden steel ball bearings (c=500 J/kg.K, ρ=7800 kg/m^3, k=50 W/m.K) it is desirable to increase the surface temperature for a short time without significantly warming the interior of the ball. This type of heating can be accomplished by sudden immersion of the ball in a molten salt bath with T_∞=1300 K and h= 5000 W/m^2.K. Assume that any location within the ball whose temperature exceeds 1000 K will be hardened. Estimate the time required to harden the outer millimeter of a ball of diameter 20 mm if its initial temperature is 300 K.

Known: A ball bearing is suddenly immersed in a molten salt bath; heat treatment to harden occurs at locations with T>1000K.

Find: time required to harden outer layer of 1mm.

Schematic:



Assumptions: (1) one-dimensional radial conduction, (2) constant properties, (3) Fo≥0.2.

Analysis: since any location within the ball whose temperature exceeds 1000K will be hardened, the problem is to find the time when the location r=9mm reaches 1000K. Then a 1mm outer layer is hardened. Using the approximate series solution, begin by finding the Biot number.

$$Bi = \frac{hr_o}{k} = \frac{5000\,W/m^2.K(0.020m/2)}{50\,W/m.K} = 1.00$$

Using the appropriate solution form for a sphere solved for F_o , find

$$F_o = -\frac{1}{\varsigma_1^2}\ln\left[\theta^* / C_1 \frac{1}{\varsigma_1 r^*}\sin(\varsigma_1 r^*)\right]$$

From table, with Bi=1.00, for the sphere find ς_1=1.5708 rad and C_1 =1.2732. with r^* =r/r_o= (9mm/10mm)=0.9, substitute numerical values.

$$F_o = -\frac{1}{(1.5708)^2}\ln\left[\frac{(1000-1300)K}{(300-1300)K}/1.2732\frac{1}{1.5708\times 0.9}\sin(1.5708\times 0.9rad)\right] = 0.441$$

From the definition of the Fourier number with α=k/ρc,

$$t = F_o\frac{r^2{}_o}{\alpha} = F_o.r^2\frac{\rho c}{k} = 0.441\times\left(\frac{0.020}{2}\right)^2 7800\frac{kg}{m^3}\times 500\frac{J}{kg.K}/50W/m.K = 3.4s$$

Comments: (1) note the very short time required to harden the ball. At this time it can be easily shown the center temperature is T(0,3.4s)=871K.

(2) The Heisler charts can also be used. From fig, with Bi-1=1.0 and r/r0=0.9, read θ/θ_o =0.69(±0.03). since

$$\theta = T - T_\infty = 1000 - 1300 = -300K \qquad \theta_i = T_i - T_\infty = -1000K$$

It follows that

$$\frac{\theta}{\theta_i} = 0.30 \qquad \text{since}\ \frac{\theta}{\theta_i} = \frac{\theta}{\theta_o}.\frac{\theta_o}{\theta_i} \qquad \text{then}\ \frac{\theta}{\theta_i} = 0.69\frac{\theta_o}{\theta_i},$$

And then $\frac{\theta_o}{\theta_i} = \frac{0.30}{0.69} = 0.43(\pm 0.02)$

From fig at $\frac{\theta_o}{\theta_i}$=0.43, Bi^{-1}=1.0, read F_O=0.45(± 0.3) and t=3.5 (± 0.2) s.

Note the use of tolerances assigned as acceptable numbers dependent upon reading the charts to ±5%.

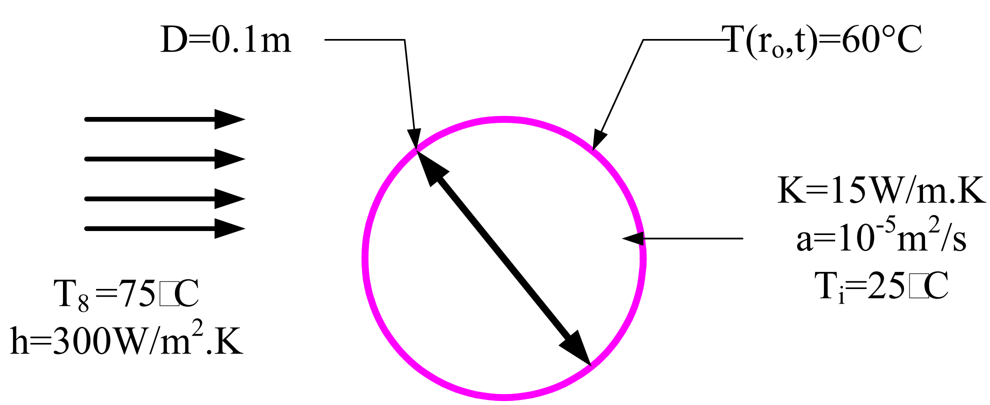
Problem 7:

The convection coefficient for flow over a solid sphere may be determined by submerging the sphere, which is initially at 25°C, into the flow, which is at 75°C and measuring its surface temperature at some time during the transient heating process. The sphere has a diameter of 0.1m, and its thermal conductivity and thermal diffusivity are 15 W/m.K and 10^{-5} m^2/s, respectively. If the convection coefficient is 300W/m^2.K, at what time will a surface temperature of 60°C be recorded?

Known: Initial temperatures and properties of solid sphere. Surface temperatures after immersion in a fluid of prescribed temperatures and convection coefficient.

Find: The process time

Schematic:



Assumptions: (1) one-dimensional, radial conduction, (2) constant properties.

Analysis: the Biot number is

$$Bi = \frac{h(r_0/3)}{k} = \frac{300W/m^2.K(0.05m/3)}{15W/m.K} = 0.333$$

Hence the lumped capacitance methods should be used. From equation

$$\frac{T - T_\infty}{T_i - T_\infty} = C_1 \exp(-\varsigma_1^2 F_o)\frac{\sin(\varsigma_1 r^*)}{\varsigma_1 r^*}$$

At the surface, r^*=1. from table , for Bi=1.0, ς_1=1.5708 rad and C_1=1.2732. hence,

$$\frac{60-75}{25-75} = 0.30 = 1.2732\exp(-1.5708^2 F_o)\frac{\sin 90°}{1.5708}$$ +Exp (-2.467F_0) =0.370

$$F_o = \frac{\alpha t}{r_0^2} = 0.403 \frac{(0.05m)^2}{10^{-5} m^2 / s}$$

t=100s

Comments:

Use of this technique to determine h from measurement of T (r_o) at a prescribed t requires an iterative solution of the governing equations.

Module 5: Short questions

1. What is lumped capacity analysis? When is it applicable? What is the physical significance of the Biot number?

2. The Biot number is used when considering a solid body subject to convection in a surrounding fluid. It is a comparison of
   a) Convection to conduction in the surrounding fluid
   b) Conduction in the surrounding fluid to conduction in the solid
   c) Convection at the solid surface to conduction within the solid
   d) The thermal diffusivity in the solid to the kinematic viscosity in the fluid
   e) None of the above

3. Consider heat transfer between two identical hot solid bodies and the surrounding air. The first solid is cooled by a fan, while the second one is cooled by natural convection in air. For which case is the lumped capacity assumption more applicable?

4. Consider a hot boiled potato kept on a plate and cooled by natural convection in air. During the first minute, the temperature drops by 10ºC. During the second minute, will the temperature drop be more than, less than or same as that during the first minute?

5. In what medium will the lumped capacity assumption more likely to be valid: in air or in water?

6. Consider a sphere and a cylinder of equal volume and made of copper. Both are heated to the same temperature and then kept in air for cooling. Which one is likely to cool faster?

7. A block of metal is cooled in a water bath. Its unsteady temperature is considered uniform and is thus modelled using a lumped capacitance method. The product of the block’s resistance to convection and its lumped thermal capacitance is
   a) Bi
   b) Nu
   c) Fo
   d) τ
   e) None of the above

8. In transient heat transfer analysis, when is it proper to treat an actual cylinder as an infinitely long one, and when is it not?

9. Why are the transient temperature charts prepared using non-dimensionalised quantities such as the Biot and the Fourier numbers and not the actual variables such as thermal conductivity and time?

10. What is the physical significance of the Fourier number? Will the Fourier number of a specific transient heat transfer problem double if the time is doubled?

11. What is a semi-infinite medium? Give examples of solid bodies that can be treated as semi-infinite mediums for the purpose of transient heat transfer studies? Under what conditions can a plane wall be treated as a semi-infinite medium?

12. When modelling the unsteady conduction in a semi-infinite slab with convection at the surface, there is no geometric length scale with which to construct a Biot number. The appropriate length scale for this is therefore
    (a) $(\alpha t)^{0.5}$
    (b) αt
    (c) $(ht)^{0.5}$
    (d) $\rho c t$
    (e) none of the above

Introduction to Heat Exchangers

Contents:

What are exchangers for?

Main heat exchanger types

Heat exchanger analysis

- LMTD method
- Effectiveness method

What are heat exchangers for?

Heat exchangers are practical devices used to transfer energy from one fluid to another

To get fluid streams to the right temperature for the next process

reactions often require feeds at high temp.

To condense vapours

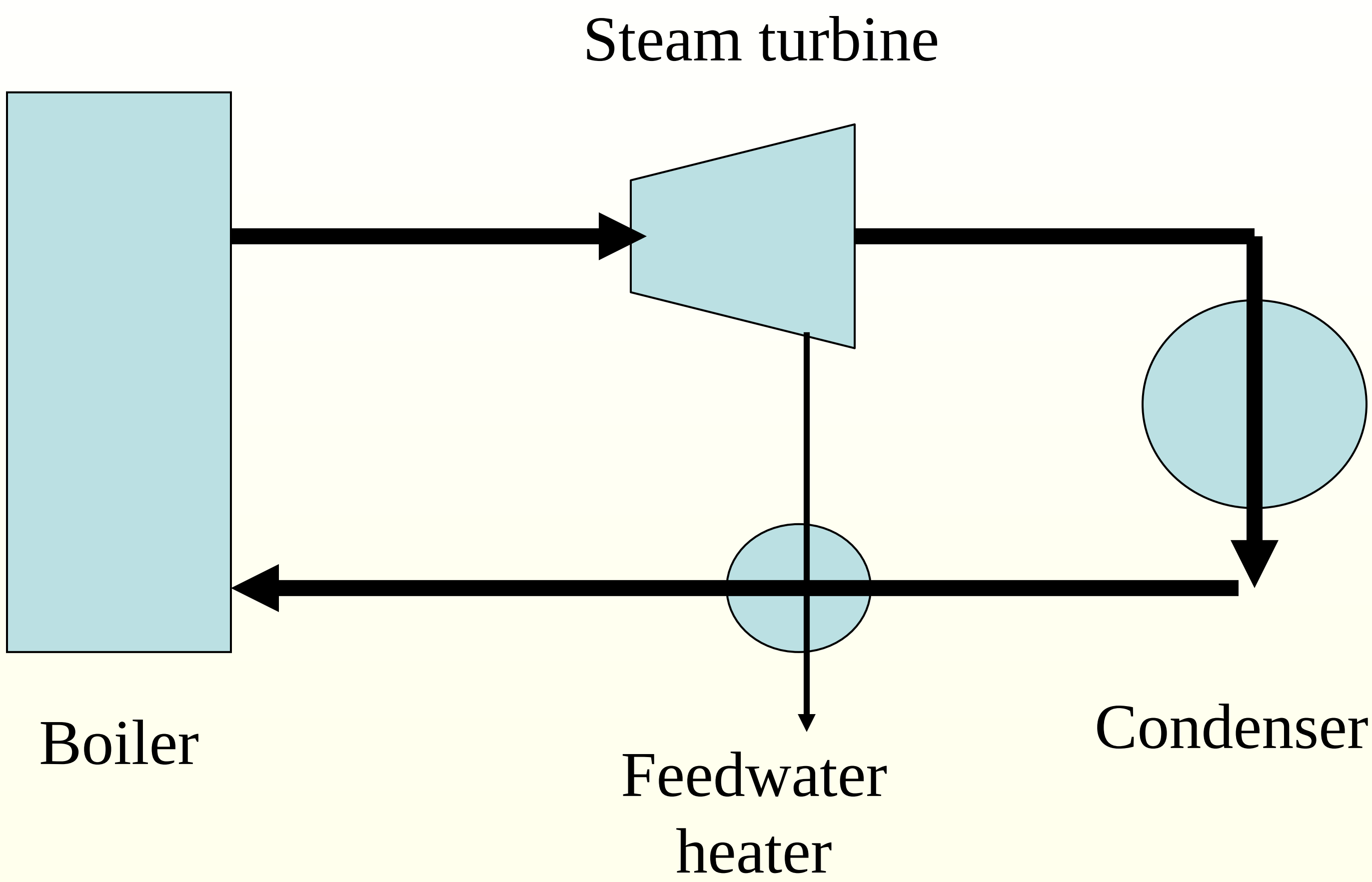
To evaporate liquids

To recover heat to use elsewhere

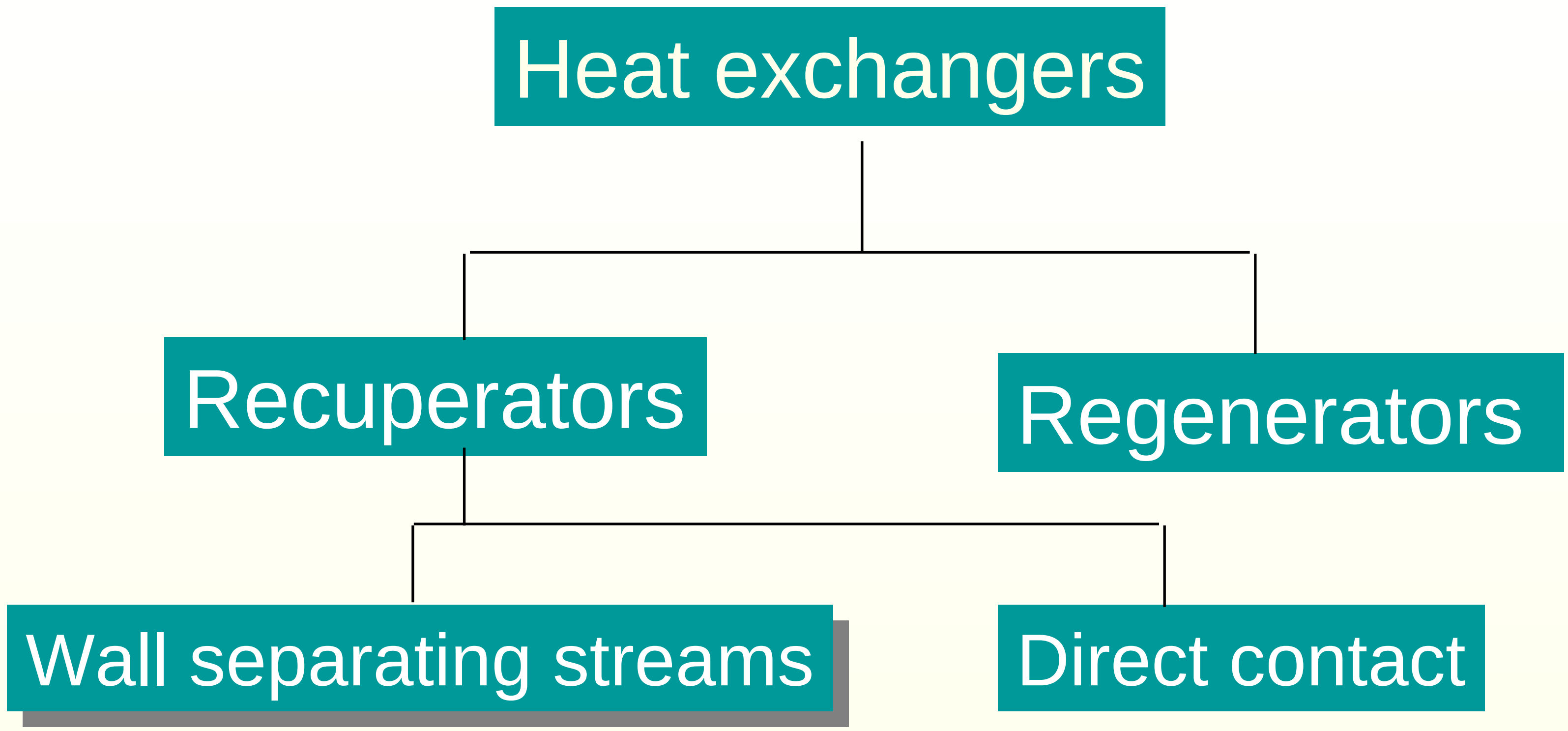
To reject low-grade heat

To drive a power cycle

Application: Power cycle



Main categories of exchanger

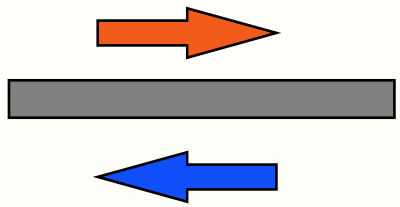


Most heat exchangers have two streams, *hot* and *cold*, but some have more than two

Recuperators/Regenerators

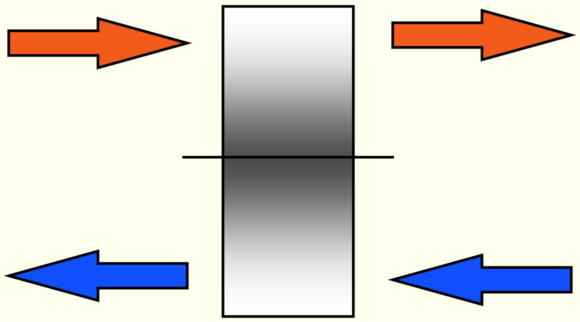
Recuperative

Has separate flow paths for each fluid which flow simultaneously through the exchanger transferring heat between the streams



Regenerative

Has a single flow path which the hot and cold fluids alternately pass through.



Compactness

Can be measured by the heat-transfer area per unit volume **or** by channel size

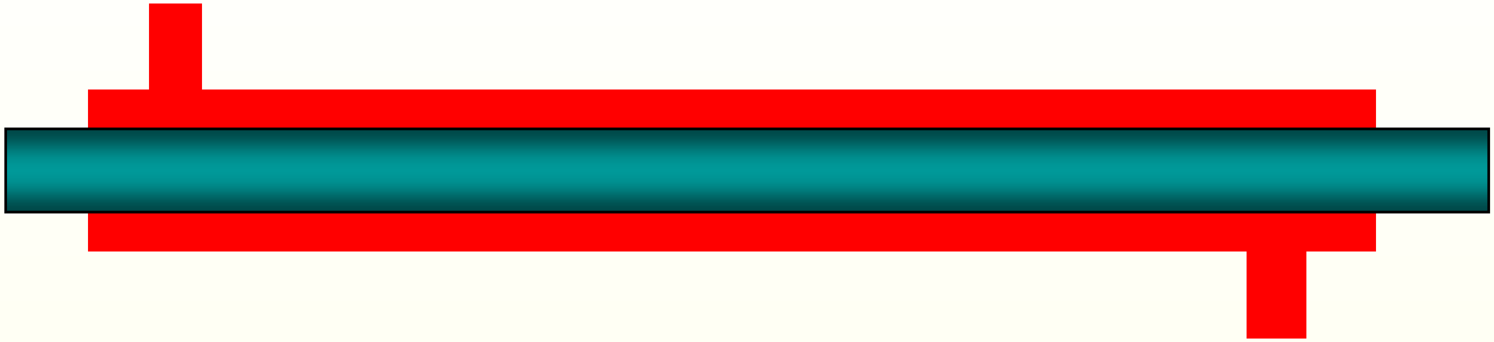
Conventional exchangers (shell and tube) have channel size of 10 to 30 mm giving about $100 m^2/m^3$

Plate-type exchangers have typically 5mm channel size with more than $200 m^2/m^3$

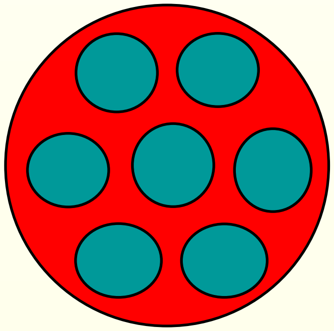
More compact types available

Double Pipe

Simplest type has one tube inside another - inner tube may have longitudinal fins on the outside

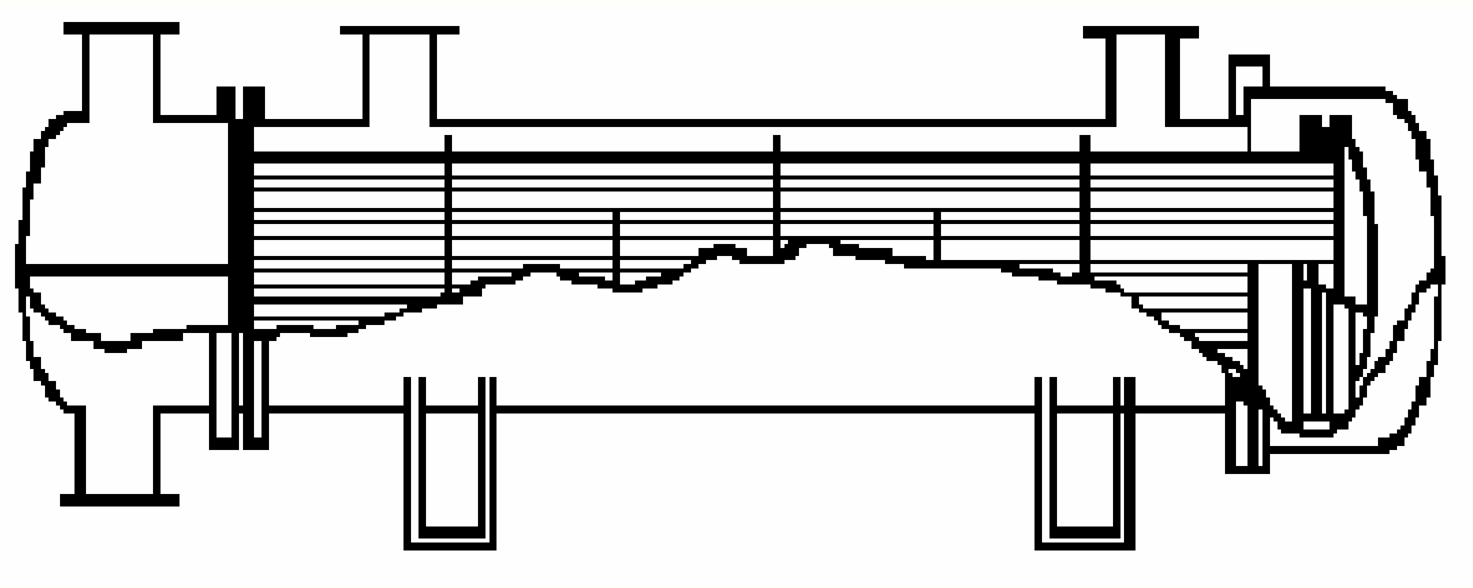


However, most have a number of tubes in the outer tube - can have very many tubes thus becoming a shell-and-tube



Shell and Tube

Typical shell and tube exchanger as used in the process industry



Shell side fl

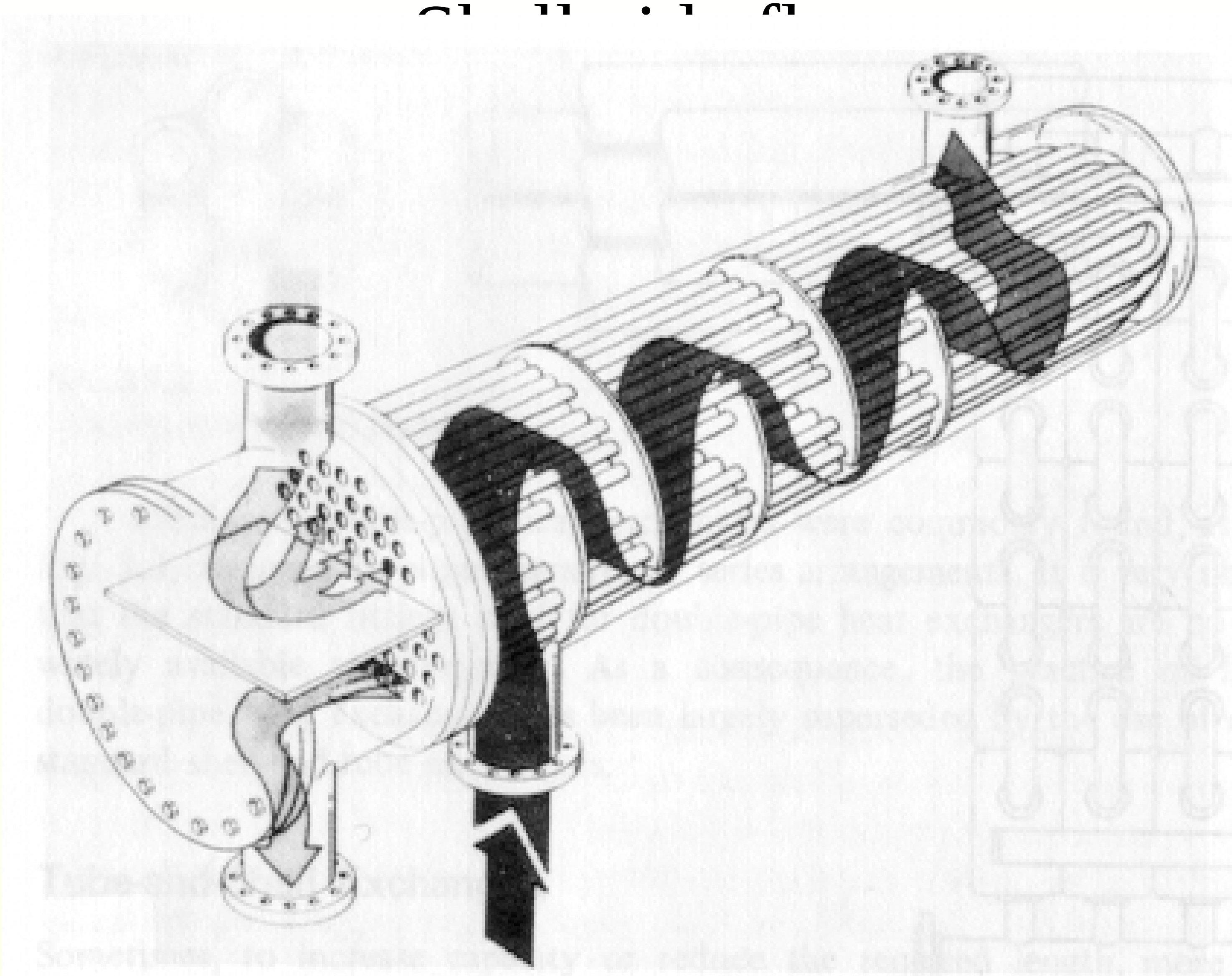
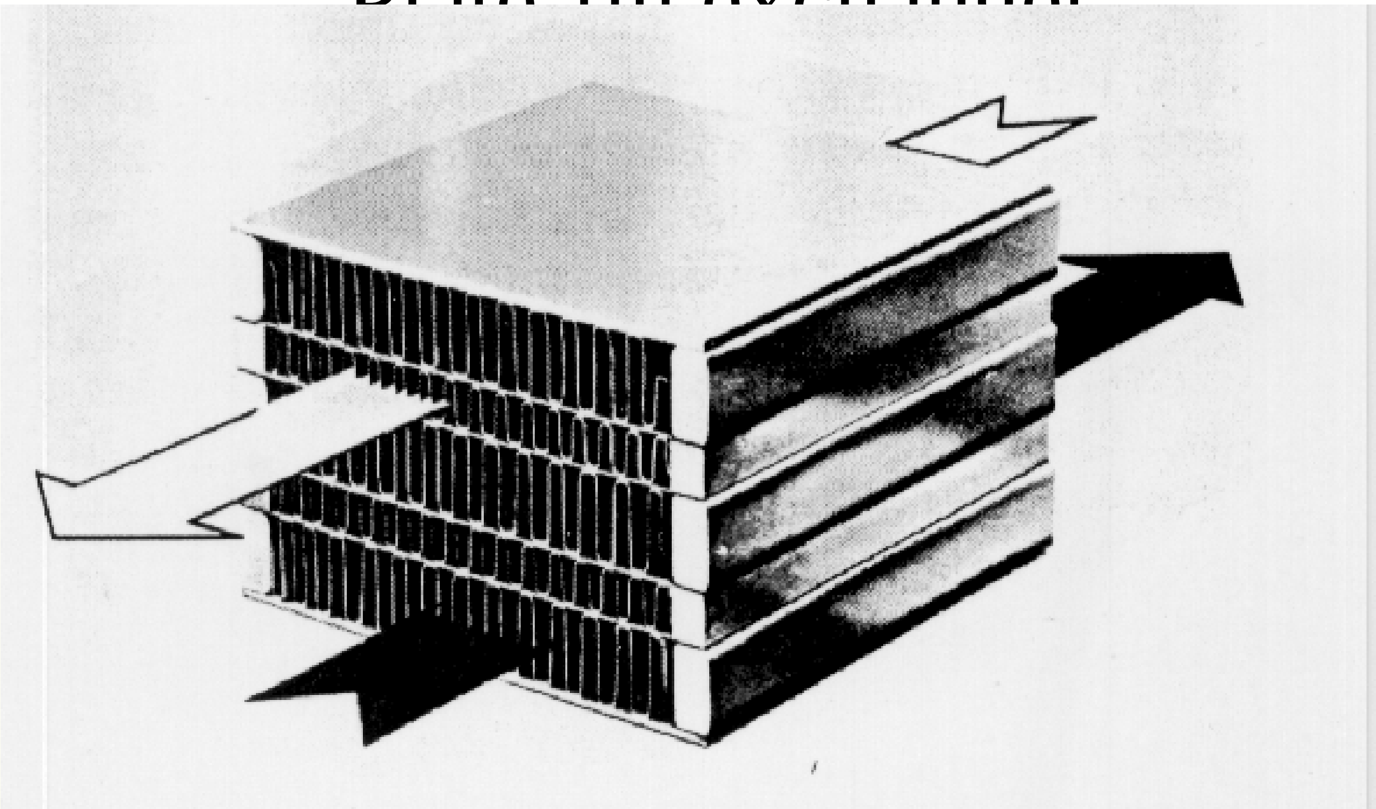


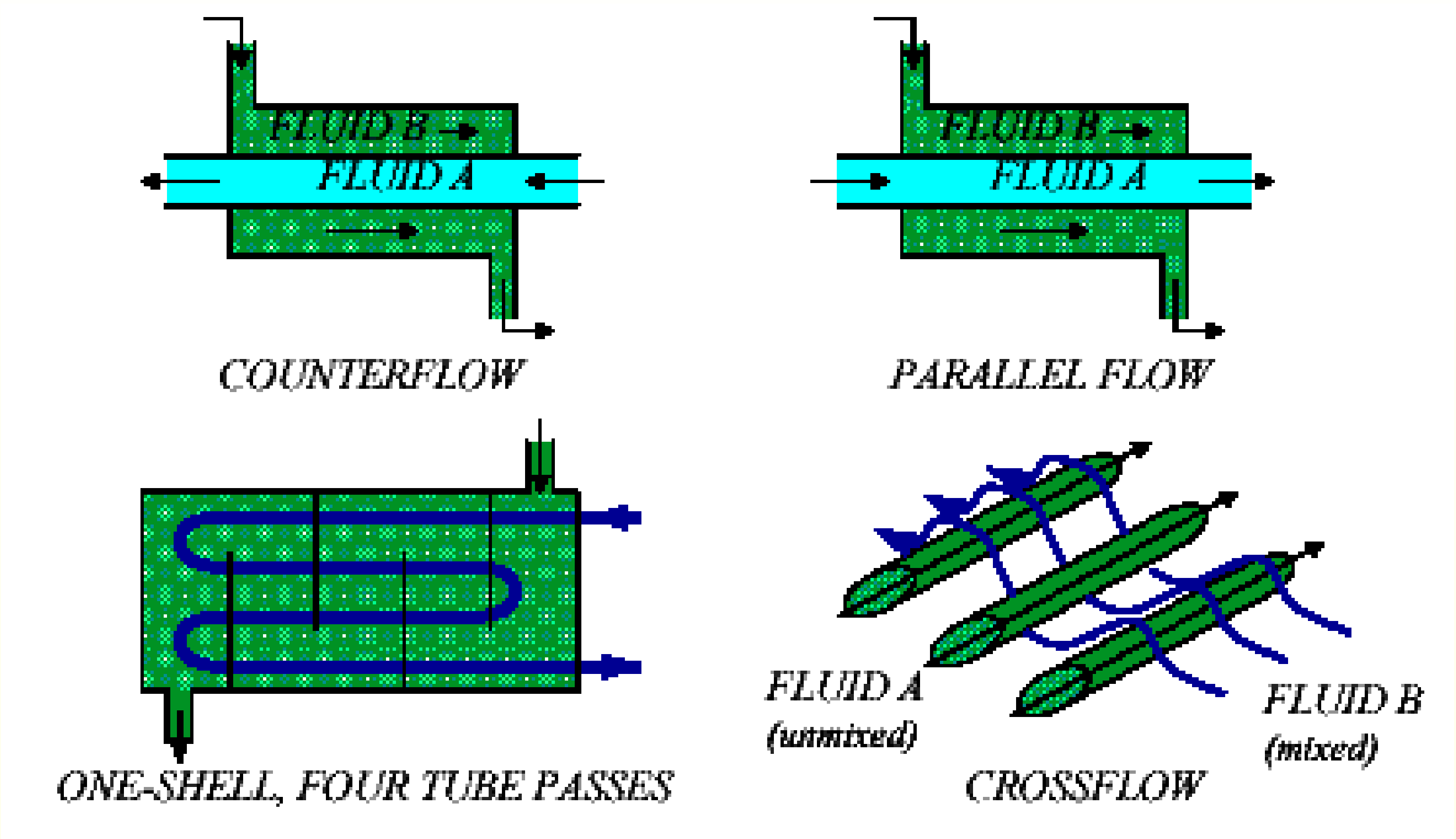
Plate-fin exchanger



Made up of flat plates (parting sheets) and corrugated sheets which form fins

Brazed by heating in vacuum furnace

Configurations



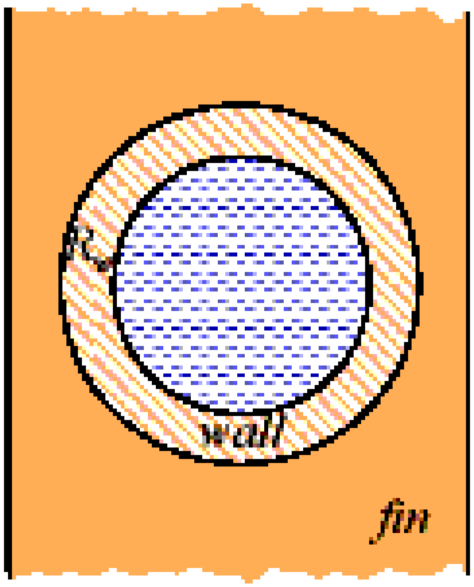
Heat Transfer considerations:

- Overall heat transfer coefficient

Internal and external thermal resistances in series

$$\frac{1}{UA} = \frac{1}{(UA)_c} = \frac{1}{(UA)_h}$$

$$\frac{1}{UA} = \frac{1}{(h\eta_o A)_c} + \frac{R''_{f,c}}{(\eta_o A)_c} + R_w + \frac{1}{(h\eta_o A)_h} + \frac{R''_{f,h}}{(\eta_o A)_h}$$



- A is wall total surface area on hot or cold side
- R''_f is fouling factor (m^2K/W)
- η_o is overall surface efficiency (if finned)

Heat Transfer considerations:

- Fouling factor

Material deposits on the surfaces of the heat exchanger tube may add further resistance to heat transfer in addition to those listed above. Such deposits are termed fouling and may significantly affect heat exchanger performance.

- *Scaling* is the most common form of fouling and is associated with inverse solubility salts. Examples of such salts are $CaCO_3$, $CaSO_4$, $Ca_3(PO_4)_2$, $CaSiO_3$, $Ca(OH)_2$, $Mg(OH)_2$, $MgSiO_3$, Na_2SO_4, $LiSO_4$, and Li_2CO_3.

- *Corrosion fouling* is classified as a chemical reaction which involves the heat exchanger tubes. Many metals, copper and aluminum being specific examples, form adherent oxide coatings which serve to passivate the surface and prevent further corrosion.

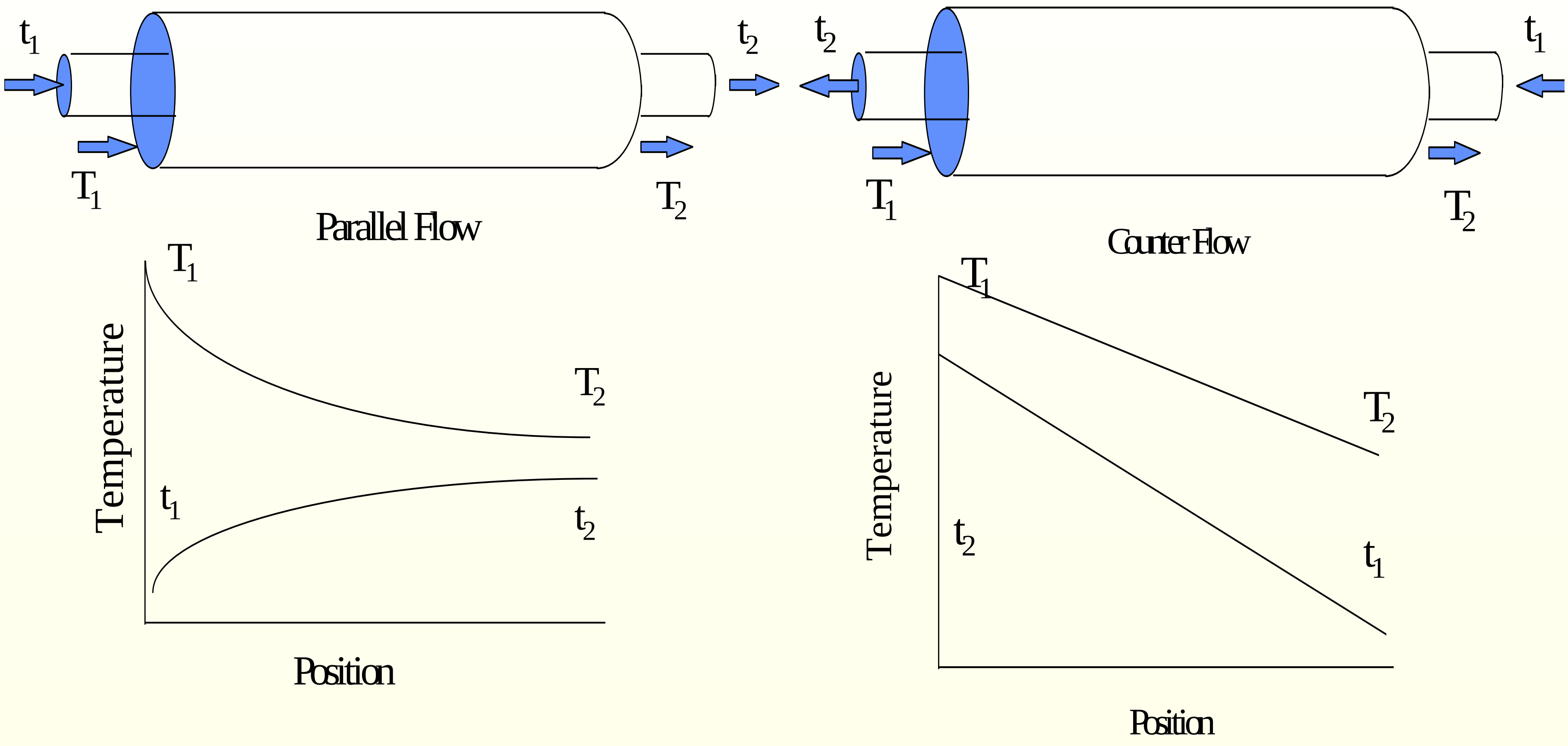
• *Chemical reaction fouling* involves chemical reactions in the process stream which results in deposition of material on the heat exchanger tubes. When food products are involved this may be termed scorching but a wide range of organic materials are subject to similar problems.

• *Freezing fouling* is said to occur when a portion of the hot stream is cooled to near the freezing point for one of its components. This is most notable in refineries where paraffin frequently solidifies from petroleum products at various stages in the refining process, obstructing both flow and heat transfer.

• *Biological fouling* is common where untreated water is used as a coolant stream. Problems range from algae or other microbes to barnacles.

Fluid	R”, $m^2K/watt$
Seawater and treated boiler feedwater (below 50^oC)	0.0001
Seawater and treated boiler feedwater (above 50^oC)	0.0002
River water (below 50^oC)	0.0002-0.001
Fuel Oil	0.0009
Regrigerating liquids	0.0002
Steam (non-oil bearing)	0.0001

Basic flow arrangement in tube in tube flow

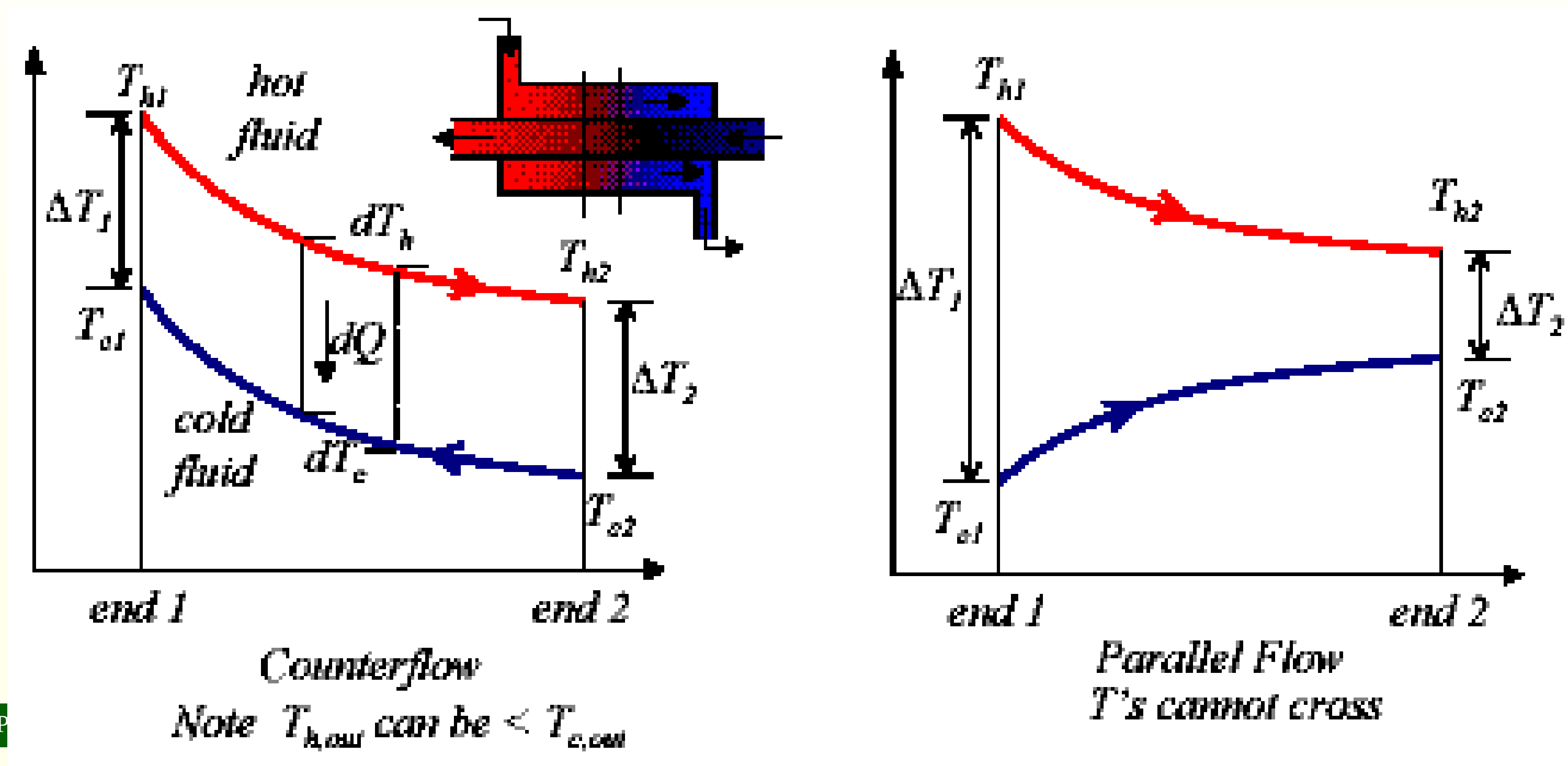


Heat Exchanger Analysis

Log mean temperature difference (LMTD) method

Want a relation $\dot{Q} = UA\Delta T_m$

where ΔT_m is some mean ΔT between hot and cold fluid

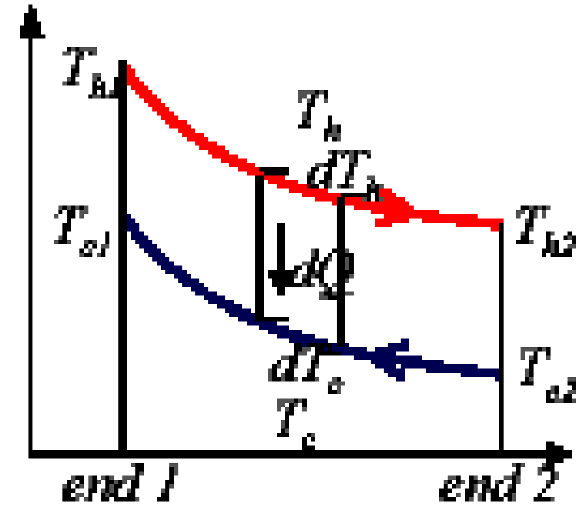


Energy Balance (counterflow)

on element shown.

$$d\dot{Q} = -\dot{m}_h c_h dT_h = -\dot{m}_c c_c dT_c \qquad (1)$$

$\dot{m}$ *mass flow rate of fluid*
c *specific heat*



(dT in direction $1 \rightarrow 2$)

Rate Equation

$$d\dot{Q} = UdA(T_h - T_c) \qquad (2)$$

Now from (1) $\quad dT_h = \dfrac{-d\dot{Q}}{\dot{m}_h c_h} \qquad dT_c = \dfrac{-d\dot{Q}}{\dot{m}_c c_c}$

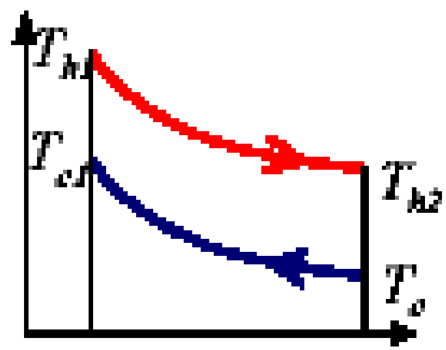
$$\therefore d(T_h - T_c) = d\dot{Q}\left(\frac{1}{\dot{m}_c c_c} - \frac{1}{\dot{m}_h c_h}\right)$$

Sub. $d\dot{Q}$ from (2),

$$\frac{d\left(T_h - T_c\right)}{\left(T_h - T_c\right)} = U\left(\frac{1}{\dot{m}_c c_c} - \frac{1}{\dot{m}_h c_h}\right) dA$$

Integrate $1 \rightarrow 2$

$$\ln\left(\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right) = UA\left(\frac{1}{\dot{m}_c c_c} - \frac{1}{\dot{m}_h c_h}\right)$$



Total heat transfer rate

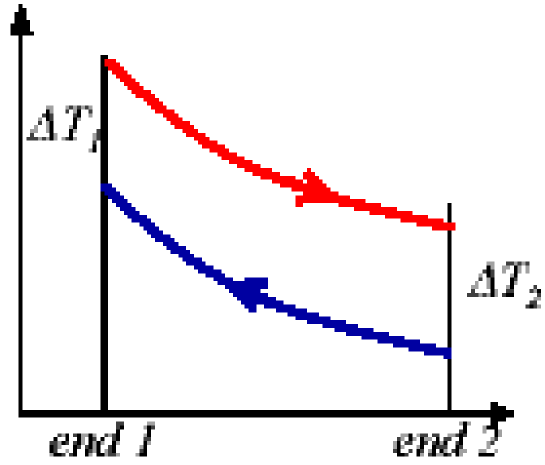
$$\dot{Q} = \dot{m}_h c_h\left(T_{h1} - T_{h2}\right) \qquad and \qquad \dot{Q} = \dot{m}_c c_c\left(T_{c1} - T_{c2}\right)$$

Sub for $\dot{m}c$ and put

$$\Delta T_1 = T_{h1} - T_{c1} \qquad END\ 1$$

$$\Delta T_2 = T_{h2} - T_{c2} \qquad END\ 2$$

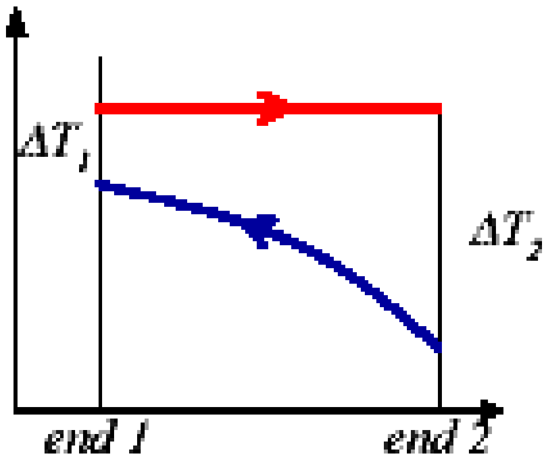
$$\dot{Q} = UA\left[\frac{\Delta T_2 - \Delta T_1}{\ln\left(\Delta T_2 / \Delta T_1\right)}\right]$$



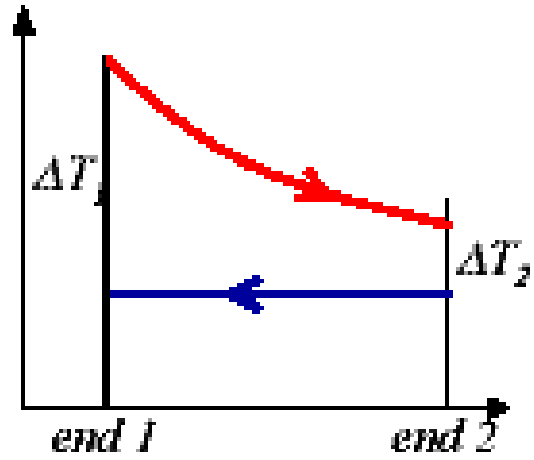
$$\dot{Q} = UA(\mathrm{LMTD})$$

LOG MEAN TEMPERATURE DIFFERENCE

- Remember - 1 and 2 are ends, not fluids
- Same formula for parallel flow (but ΔT's are different)
- Counterflow has highest LMTD, for given T's therefore smallest area for Q.



Condenser



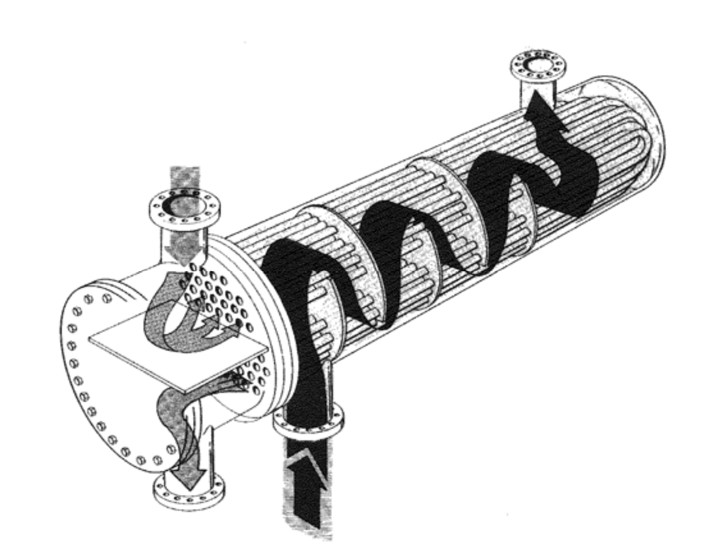
Evaporator

Example:A counterflow, concentric tube heat exchanger is used to cool the lubricating oil for a large industrial gas turbine engine. The flow rate of cooling water through the inner tube (D_i = 25 mm) is 0.2 kg/s, while the flow rate of oil through the outer annulus (D_o = 45 mm) is 0.1 kg/s. The oil and water enter at temperatures of 100 and 30 °C, respectively. What should be the length of the tube if if the outlet temperature of the oil is to be 60 °C ?

Soultion: To be worked out in class.

Multipass HX Flow Arrangements

- In order to increase the surface area for convection relative to the fluid volume, it is common to design for multiple tubes within a single heat exchanger.
- With multiple tubes it is possible to arrange to flow so that one region will be in parallel and another portion in counter flow.



1-2 pass heat exchanger, indicating that the shell side fluid passes through the unit once, the tube side twice. By convention the number of shell side passes is always listed first.

The LMTD formulas developed earlier are no longer adequate for multipass heat exchangers. Normal practice is to calculate the LMTD for counter flow, $LMTD_{cf}$, and to apply a correction factor, F_T, such that

$$\Delta\theta_{eff} = F_T \cdot LMTD_{CF}$$

The correction factors, F_T, can be found theoretically and presented in analytical form. The equation given below has been shown to be accurate for any arrangement having 2, 4, 6, ...,2n tube passes per shell pass to within 2%.

$$F_T = \frac{\sqrt{R^2+1}\ln\left[\frac{1-P}{1-R\cdot P}\right]}{(R-1)\ln\left[\frac{2-P\left(R+1-\sqrt{R^2+1}\right)}{2-P\left(R+1+\sqrt{R^2+1}\right)}\right]}$$

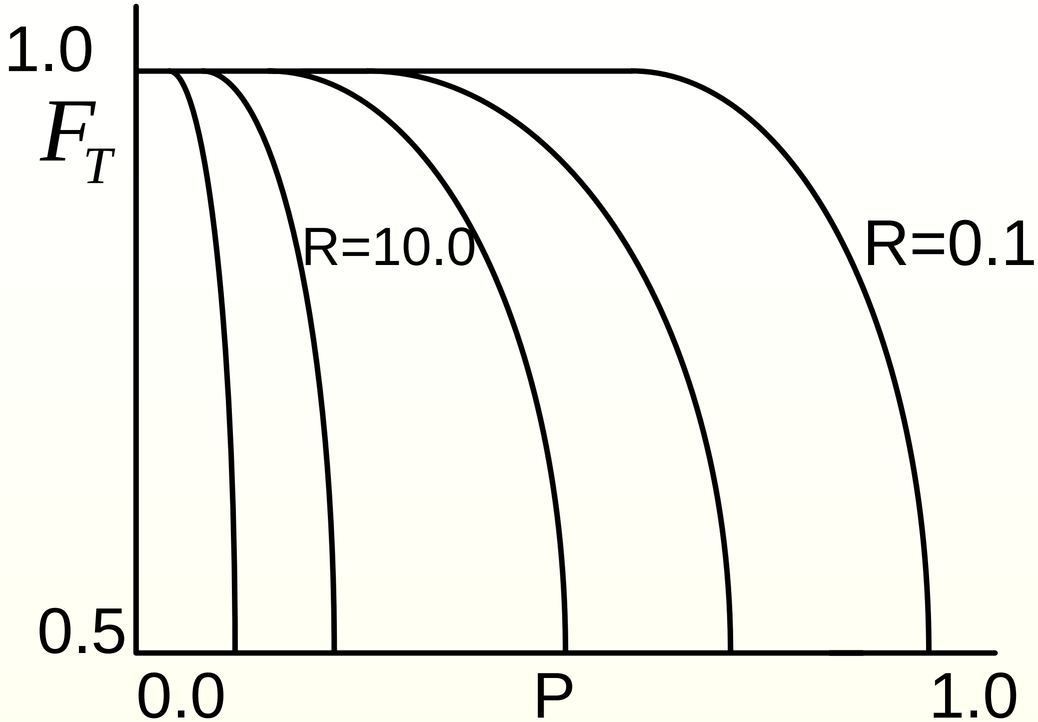
$$\text{Effectiveness}: \ P = \frac{1-X^{1/N_{shell}}}{R-X^{1/N_{shell}}}, \quad \text{for } R \neq 1$$

$$P = \frac{P_o}{N_{shell} - P_o \cdot (N_{shell}-1)}, \quad \text{for } R = 1$$

$$P_o = \frac{t_2 - t_1}{T_1 - t_1} \qquad X = \frac{P_o \cdot R - 1}{P_o - 1}$$

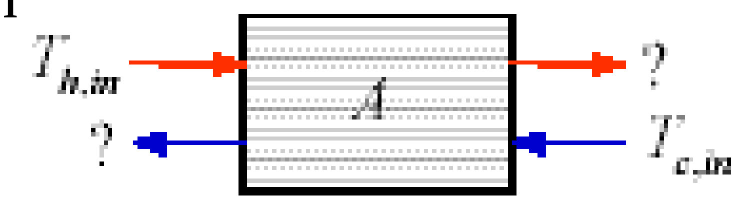
$$\text{Capacity ratio } R = \frac{T_1 - T_2}{t_2 - t_1}$$

T,t = Shell / tube side; 1, 2 = inlet / outlet


1.0
F_T
R=10.0
R=0.1
0.5
0.0
P
1.0

Effectiveness-NTU method

How will existing H.Ex. perform for given inlet conditions ?



Define effectiveness : $\varepsilon = \dfrac{\dot{Q}_{actual}}{\dot{Q}_{\max}}$

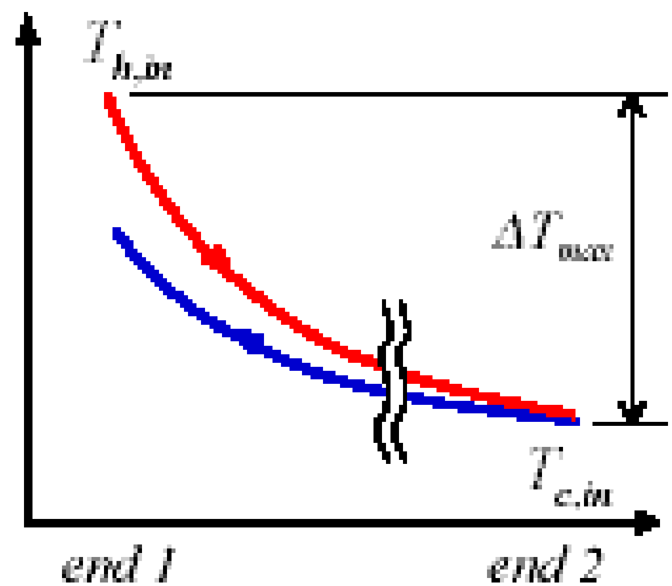
where $\dot{Q}_{\max}$ is for an infinitely long H.Ex.

One fluid $\Delta T \rightarrow \Delta T_{\max} = T_{h,in} - T_{c,in}$

and since $\dot{Q} = (\dot{m}c_A)\Delta T_A = (\dot{m}c_B)\Delta T_B$

$$= C_A \Delta T_A = C_B \Delta T_B$$

then only the fluid with lesser of C_A, C_B heat capacity rate can have $\Delta T_{\max}$



i.e. $\dot{Q}_{\max} = C_{\min} \Delta T_{\max}$ and $\varepsilon = \dfrac{\dot{Q}}{C_{\min}\left(T_{h.in} - T_{c.in}\right)}$

$$\text{or,} \quad \dot{Q} = \varepsilon C_{\min}\left(T_{h.in} - T_{c.in}\right)$$

Want expression for ε which does not contain outlet T's

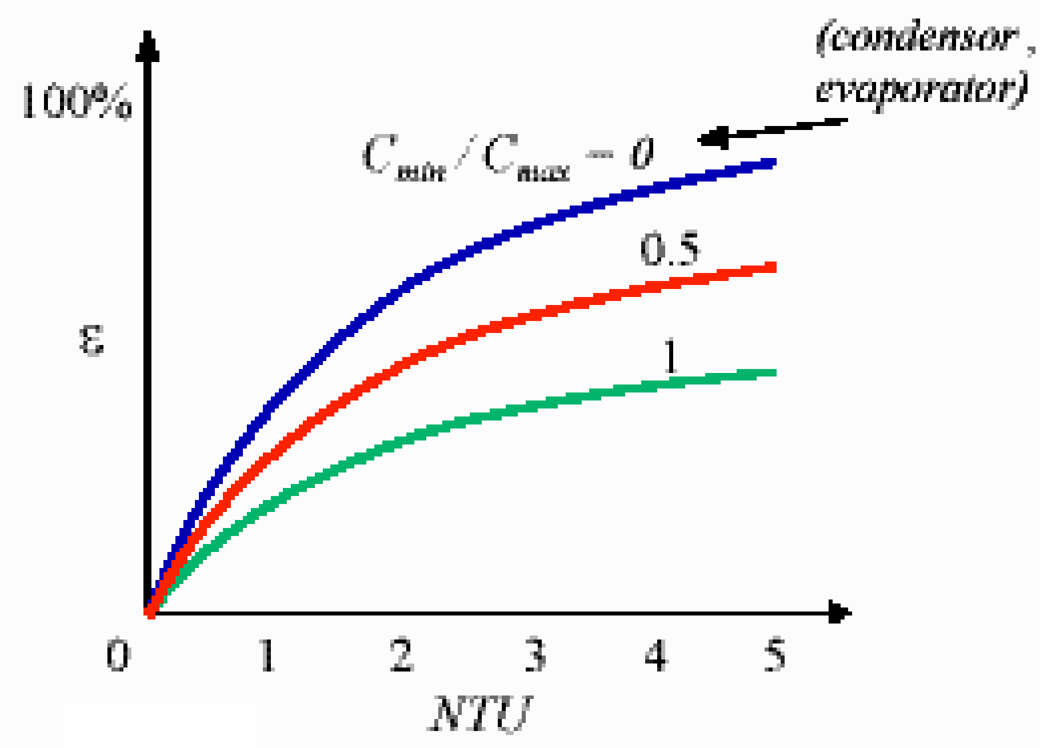
Substitute back into $\dot{Q} = UA(LMTD)$ ....

$$\varepsilon = \frac{1 - \exp\left[\dfrac{-\mathrm{UA}}{C_{\min}}\left(1 - \dfrac{C_{\min}}{C_{\max}}\right)\right]}{1 - \dfrac{C_{\min}}{C_{\max}}\exp\left[\dfrac{-\mathrm{UA}}{C_{\min}}\left(1 - \dfrac{C_{\min}}{C_{\max}}\right)\right]}$$

$$\therefore \varepsilon = \varepsilon\left(NTU, \frac{C_{\min}}{C_{\max}}\right)$$

and No. of transfer units (size of HEx.) $\quad NTU = \dfrac{UA}{C_{\min}}$

CHARTS FOR EACH CONFIGURATION …



Procedure:

Determine C_{max}, C_{min}/C_{max}

$$\dot{Q} = \varepsilon C_{\min}\left(T_{h.in} - T_{c.in}\right)$$

Get $UA/C_{min,} \rightarrow$ ε from chart

Example:

Hot exhaust gases are used in a finned-tube cross-flow HEX to heat 2.5 kg/s of water from 35°C to 85°C. The gases (c_{pg}=1.09 kJ/kg-°C) enter at 200°C and leave at 93°C. The overall heat-transfer coefficient is 180 W/m^2°C. Determine area of the HEX using the effectiveness-NTU method.

Solution: (to be worked out in class)

$$NTU_{\max} = \frac{AU}{C_{\min}} \Longrightarrow A = \frac{NTU_{\max} C_{\min}}{U}$$

- NTU_{max} can be obtained from Fig. [illegible] using ε and C_{min}/C_{max}. First, however, we must determine which fluid has C_{min}
- For the type of the HEX used in this problem, [illegible], yields

$$\dot{m}_g c_{pg}(T_1 - T_2) = \dot{m}_w c_w (t_2 - t_1) \Longrightarrow \dot{m}_g c_{pg} = \dot{m}_w c_w \frac{t_2 - t_1}{T_1 - T_2}$$

- Examination of the last equation, subject to values given, indicates that the gas will have C_{min}

$$C_{\min} = \dot{m}_g c_{pg} = \dot{m}_w c_w \frac{t_2 - t_1}{T_1 - T_2} = \left(2.5\frac{kg}{s}\right)\left(4179\frac{J}{kg\cdot°C}\right)\frac{85-35}{200-93} = 4{,}882\frac{W}{°C}$$

$$C_{\max} = \dot{m}_w c_w = \left(2.5\frac{kg}{s}\right)\left(4179\frac{J}{kg\cdot°C}\right) = 10{,}448\frac{W}{°C}$$

$$\frac{C_{\min}}{C_{\max}} = \frac{4{,}882}{10{,}448} = 0.467$$

- Effectiveness can be calculated using

$$\varepsilon = \frac{T_1 - T_2}{T_1 - t_1} = \frac{200-93}{200-35} = 0.649$$

$$\left.\begin{array}{l}\dfrac{C_{\min}}{C_{\max}} = 0.467 \\ \varepsilon = 0.649\end{array}\right\} \Rightarrow \qquad \Rightarrow NTU_{\max} = 1.4$$

$$A = \frac{NTU_{\max} C_{\min}}{U} = \frac{1.4\left(4{,}882\dfrac{W}{°C}\right)}{180\dfrac{W}{m^2°C}} = 38.0\ m^2$$

Condensation and Boiling

- Until now, we have been considering convection heat transfer in *homogeneous single-phase* (*HSP*) systems
- *Boiling and condensation*, however, provide much higher heat transfer rates than those possible with the *HSP* systems

Condensation

- Condensation occurs when the temperature of a vapour is reduced below its saturation temperature

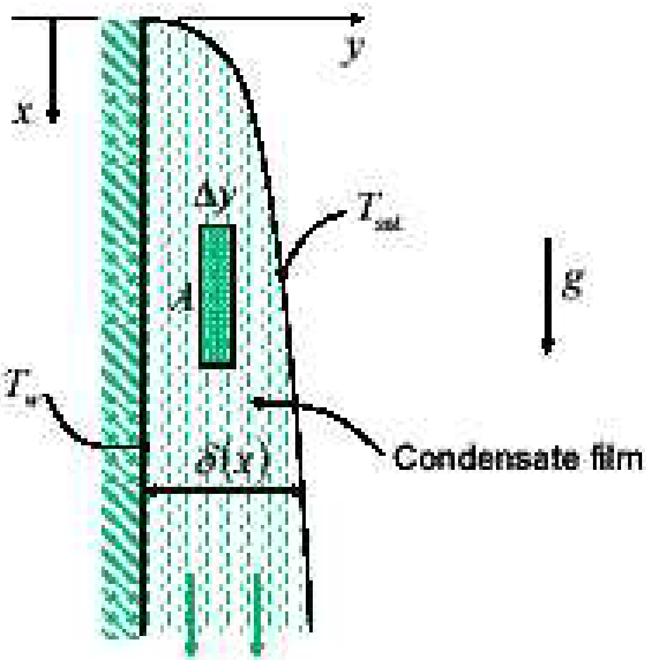
- Condensation heat transfer

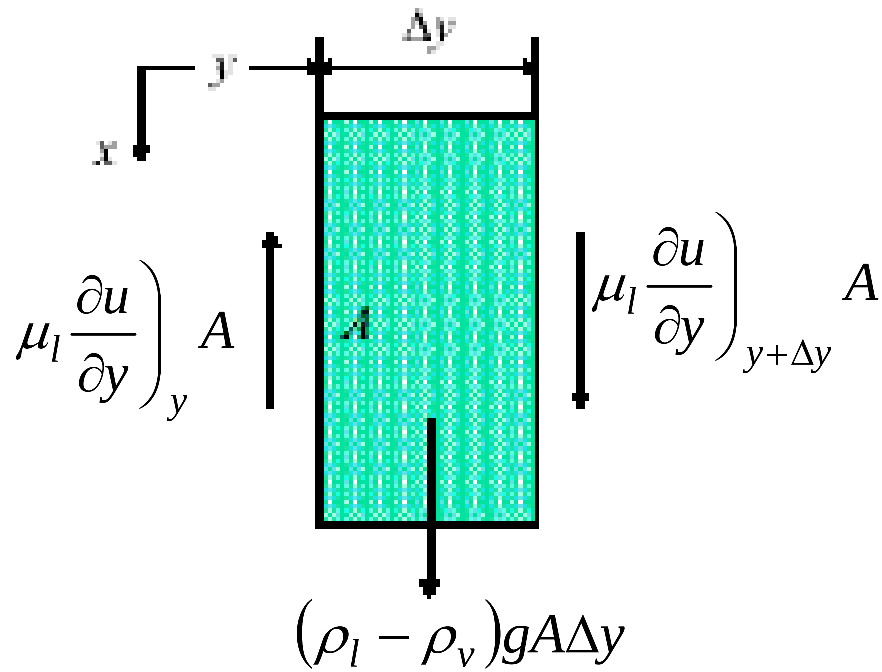
  Film condensation

  Dropwise condensation

- Heat transfer rates in dropwise condensation *may be as much as 10 times higher* than in film condensation

Laminar film condensation on a vertical wall (VW)





Laminar film condensation on a vertical wall (cont..)

$$\delta(x) = \left[\frac{4xk_l(T_{sat} - T_w)\nu_l}{h_{fg}\, g(\rho_l - \rho_v)} \right]^{1/4}$$

$$h(x) = \left[\frac{h_{fg}\, g(\rho_l - \rho_v)k_l^3}{4x(T_{sat} - T_w)\nu_l} \right]^{1/4}$$

Average coeff. $\quad \bar{h}_L = 0.943 \left[\frac{h_{fg}\, g(\rho_l - \rho_v)k_l^3}{L(T_{sat} - T_w)\nu_l} \right]^{1/4}$

where L is the plate length.

Total heat transfer rate : $\quad q = \bar{h}_L A(T_{sat} - T_w)$

Condensation rate : $\quad \dot{m} = \frac{q}{h_{fg}} = \frac{\bar{h}_L A(T_{sat} - T_w)}{h_{fg}}$

Example

Laminar film condensation of steam

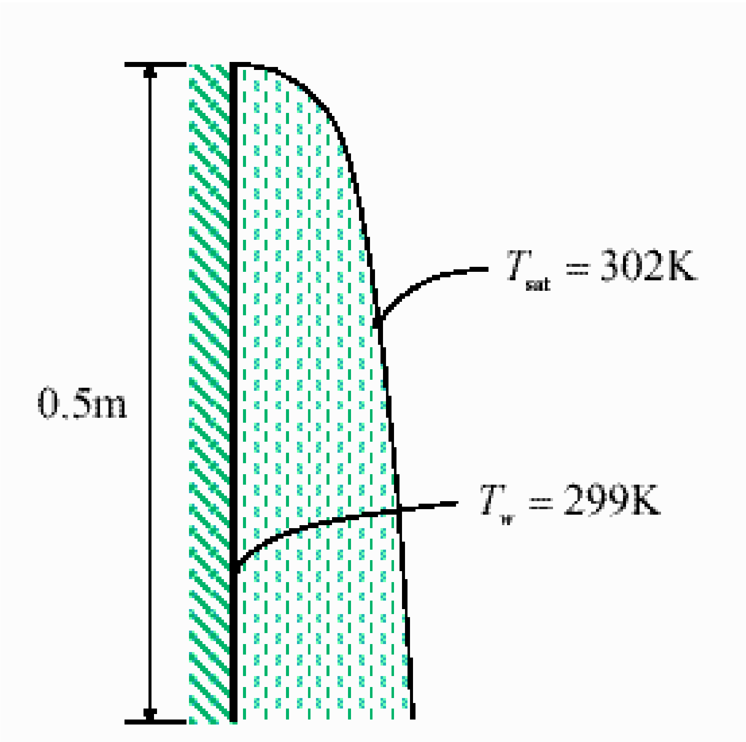
Saturated steam condenses on the outside of a 5 cm-diameter vertical tube, 50 cm high. If the saturation temperature of the steam is 302 K, and cooling water maintains the wall temperature at 299 K, determine: (i) the average heat transfer coefficient, (ii) the total condensation rate, and (iii) the film thickness at the bottom of the tube.

Given: Film condensation of saturated steam

Required: (i) Average heat transfer coefficient, (ii) total condensation rate, (iii) and film thickness

1. Effect of tube curvature negligible
2. Effect of liquid subcooling negligible
3. Laminar

Solution:



Example (cont.)

The average heat transfer coefficient is given by:

$$\bar{h} = 0.943\left[\frac{h'_{fg}\,g(\rho_l - \rho_v)k_l^{\,3}}{L(T_{sat} - T_w)\nu_l}\right]^{1/4}$$

evaluate h_{fg} at the saturation temperature of 302 K.

From table of water properties:

$$h_{fg} = 2.432 \times 10^6\ \text{J/kg} \qquad \rho_v = 0.03\text{kg/m}^3$$

Example (cont.)

Also, for water:

$$k_l = 0.611 \text{ W/mK}$$

$$\rho_l = 996 \text{ kg/m}^3$$

$$v_l = 0.87 \times 10^{-6} \text{m}^2\text{/s}$$

$$\bar{h} = 0.943 \left[\frac{h_{fg} g(\rho_l - \rho_v) k_l^{\,3}}{L(T_{sat} - T_w) v_l} \right]^{1/4}$$

$$= 0.943 \left[\frac{(2.432 \times 10^6)(9.81)(996 - 0.03)(0.611)^3}{(0.5)(3)(0.87 \times 10^{-6})} \right]^{1/4} = 7570 \text{W/m}^2\text{K}$$

ii) The total condensation rate is:

$$\dot{m} = \frac{\dot{Q}}{h_{fg}} = \frac{\bar{h}\Delta T A}{h_{fg}} = \frac{(7570)(3)(\pi)(0.05)(0.5)}{(2.432 \times 10^6)} = 7.33 \times 10^{-4} \text{kg/s}$$

Example (cont.)

(iii) The film thickness is obtained to be

$$\delta = \left(\frac{3 v_l \Gamma}{\rho_l g}\right)^{1/3} \qquad \rho_v << \rho_l$$

The mass flow rate per unit width of film Γ is:

$$\Gamma = \frac{\dot{m}}{\pi D} = \frac{\left(7.33\times 10^{-4}\right)}{(\pi)(0.05)} = 4.67\times 10^{-3}\,\mathrm{kg/m\cdot s}$$

Hence,

$$\delta = \left[\frac{3\left(0.87\times 10^{-6}\right)\left(4.67\times 10^{-3}\right)}{(996)(9.81)}\right]^{1/3} = 1.08\times 10^{-4}\,\mathrm{m} \qquad (0.108\mathrm{mm})$$

Boiling

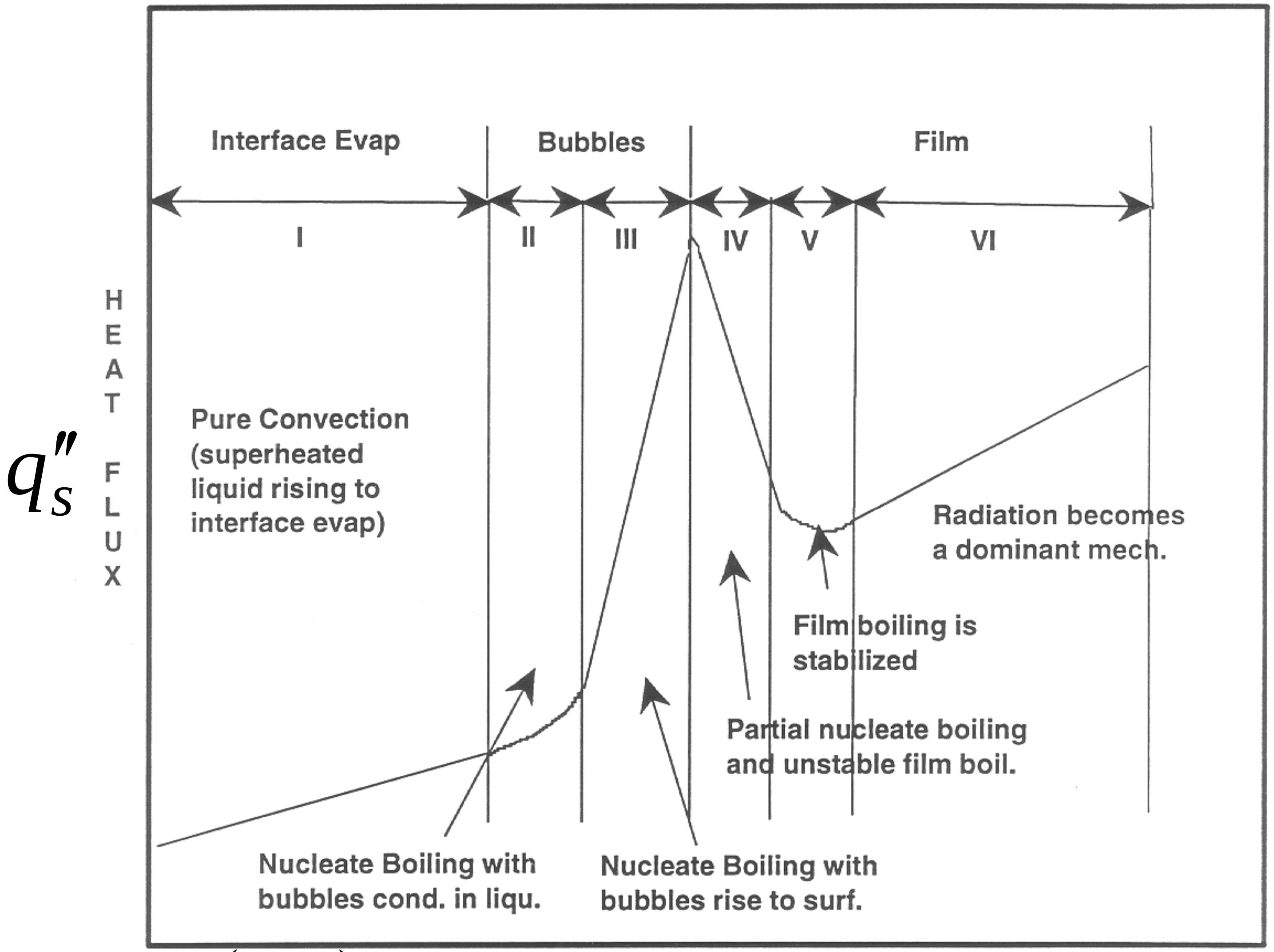
- Boiling occurs when the surface temperature T_w exceeds the saturation temperature T_{sat} corresponding to the liquid pressure

$$\text{Heat transfer rate}: q_s'' = h(T_w - T_{sat}) = h\Delta T_e$$

$$\text{where} \quad \Delta T_e = T_w - T_{sat} \quad \text{(excess temperature)}$$

- Boiling process is characterised by formation of vapour bubbles, which grow and subsequently detach from the surface
- Bubble growth and dynamics depend on several factors such as excess temp., nature of surface, thermophysical properties of fluid (e.g. surface tension, liquid density, vapour density, etc.). Hence, heat transfer coefficient also depends on those factors.

Pool Boiling Curve



$(\Delta T_e) =$ SURFACE TEMP. -- LIQUID SATURATION TEMP.

Modes of Pool Boiling

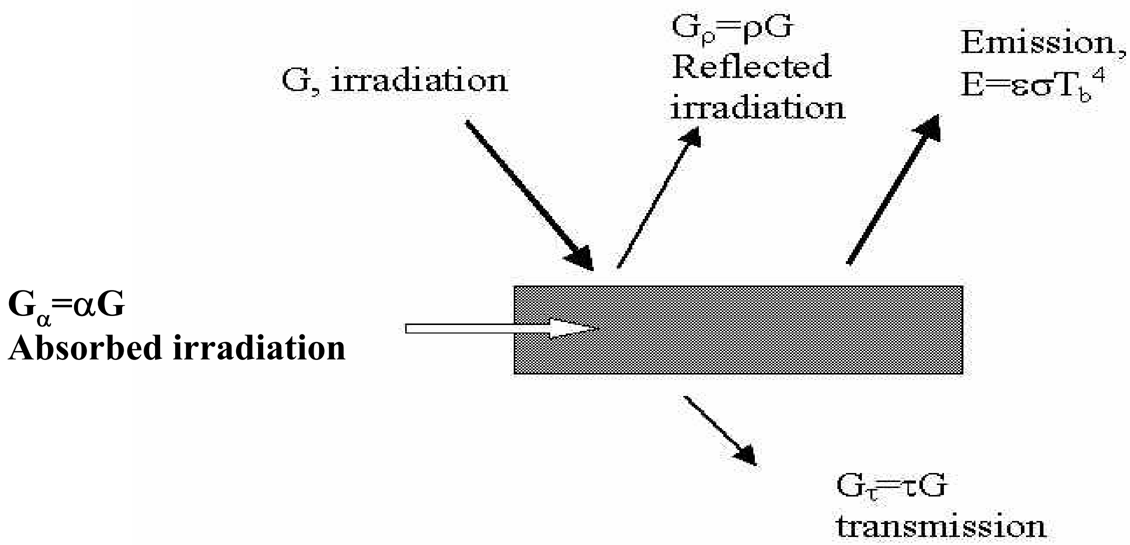
- Free convection boiling $\Delta T_e \approx 5^\circ C$
- Nucleate boiling $5^\circ C \leq \Delta T_e \leq 30^\circ C$
- Transition boiling $30^\circ C \leq \Delta T_e \leq 120^\circ C$
- Film boiling $\Delta T_e \geq 120^\circ C$

Radiation Heat Transfer

Thermal energy emitted by matter as a result of vibrational and rotational movements of molecules, atoms and electrons. The energy is transported by electromagnetic waves (or photons). Radiation requires no medium for its propagation, therefore, can take place also in vacuum. All matters emit radiation as long as they have a finite (greater than absolute zero) temperature. The rate at which radiation energy is emitted is usually quantified by the modified Stefan-Bolzmann law:

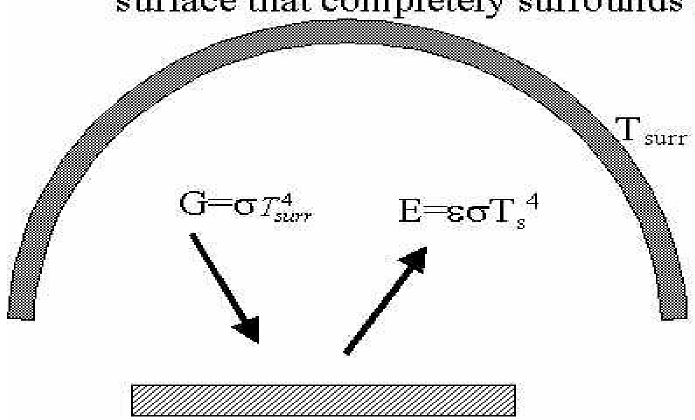
$$EA = q = \frac{dQ}{dt} = \varepsilon \sigma A T_b^4$$

where the emissivity, ε , is a property of the surface characterizing how effectively the surface radiates compared to a "blackbody" ($0<\varepsilon<1$). $E=q/A$ (W/m^2) is the surface emissive power. σ is the Stefan-Boltzmann constant ($\sigma=5.67\times10^{-8}$ $W/(m^2K^4)$). T_b is the absolute temperature of the surface (in Kelvin).



Net radiation energy exchange:
$q''_{net} = q/A = \alpha G - E = \alpha G - \varepsilon\sigma T_b^4$

Special case: a small surface inside a much large isothermal surface that completely surrounds the small one



$q''_{net} = \varepsilon\sigma T_s^4 - \alpha G$
if $\varepsilon = \alpha$ (a gray surface)
$= \varepsilon\sigma(T_s^4 - T_{surr}^4)$

Electromagnetic radiation spectrum

Thermal radiation spectrum range: 0.1 to 100 mm

It includes some ultraviolet (UV) radiation and all visible (0.4-0.76 mm) and infrared radiation (IR).

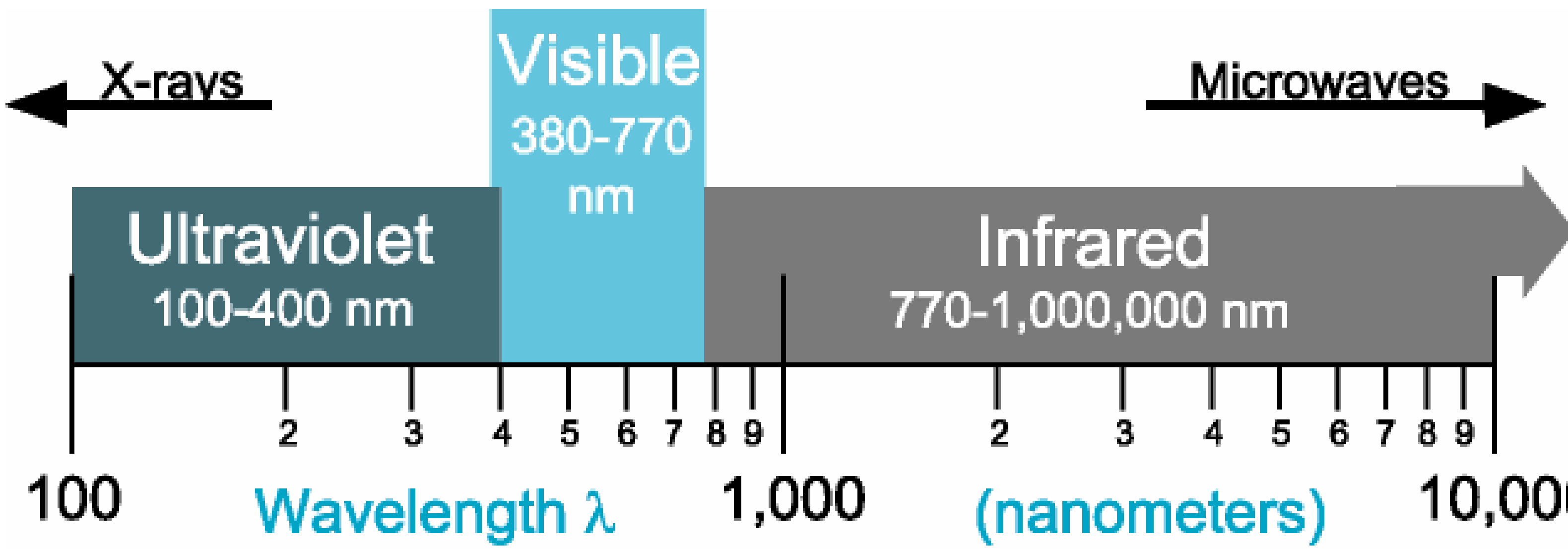
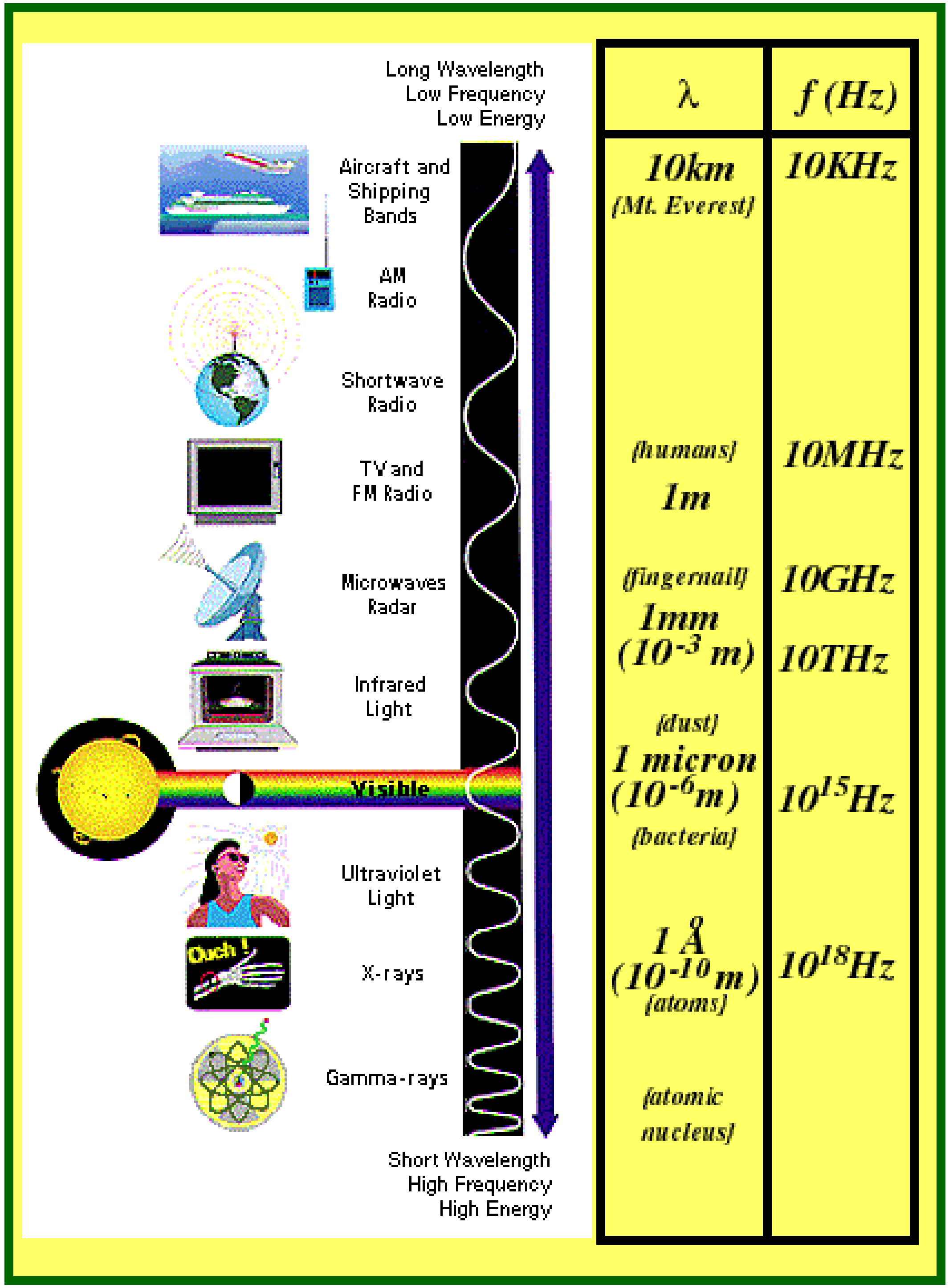


Fig. 1.1 The optical portion of the electromagnetic spectrum


Long Wavelength
Low Frequency
Low Energy
Aircraft and Shipping Bands
AM Radio
Shortwave Radio
TV and FM Radio
Microwaves Radar
Infrared Light
Visible
Ultraviolet Light
Ouch!
X-rays
Gamma-rays
Short Wavelength
High Frequency
High Energy
λ
f (Hz)
10km
[Mt. Everest]
10KHz
[humans]
1m
10MHz
[fingernail]
1mm
$(10^{-3}$ m$)$
10GHz
10THz
[dust]
1 micron
$(10^{-6}$m$)$
[bacteria]
10^{15}Hz
1 Å
$(10^{-10}$ m$)$
[atoms]
10^{18}Hz
[atomic nucleus]

The Planck Distribution

The Planck law describes theoretical spectral distribution for the emissive power of a black body. It can be written as

$$E_{\lambda,b} = \frac{C_1}{\lambda^5\left[\exp(C_2/\lambda T)-1\right]}$$

where C_1=3.742x10^8 (W.μm^4/m^2) and C_2=1.439x10^4 (μm.K) are two constants. The Planck distribution is shown in the following figure as a function of wavelength for different body temperatures.

Spectral blackbody emissive power

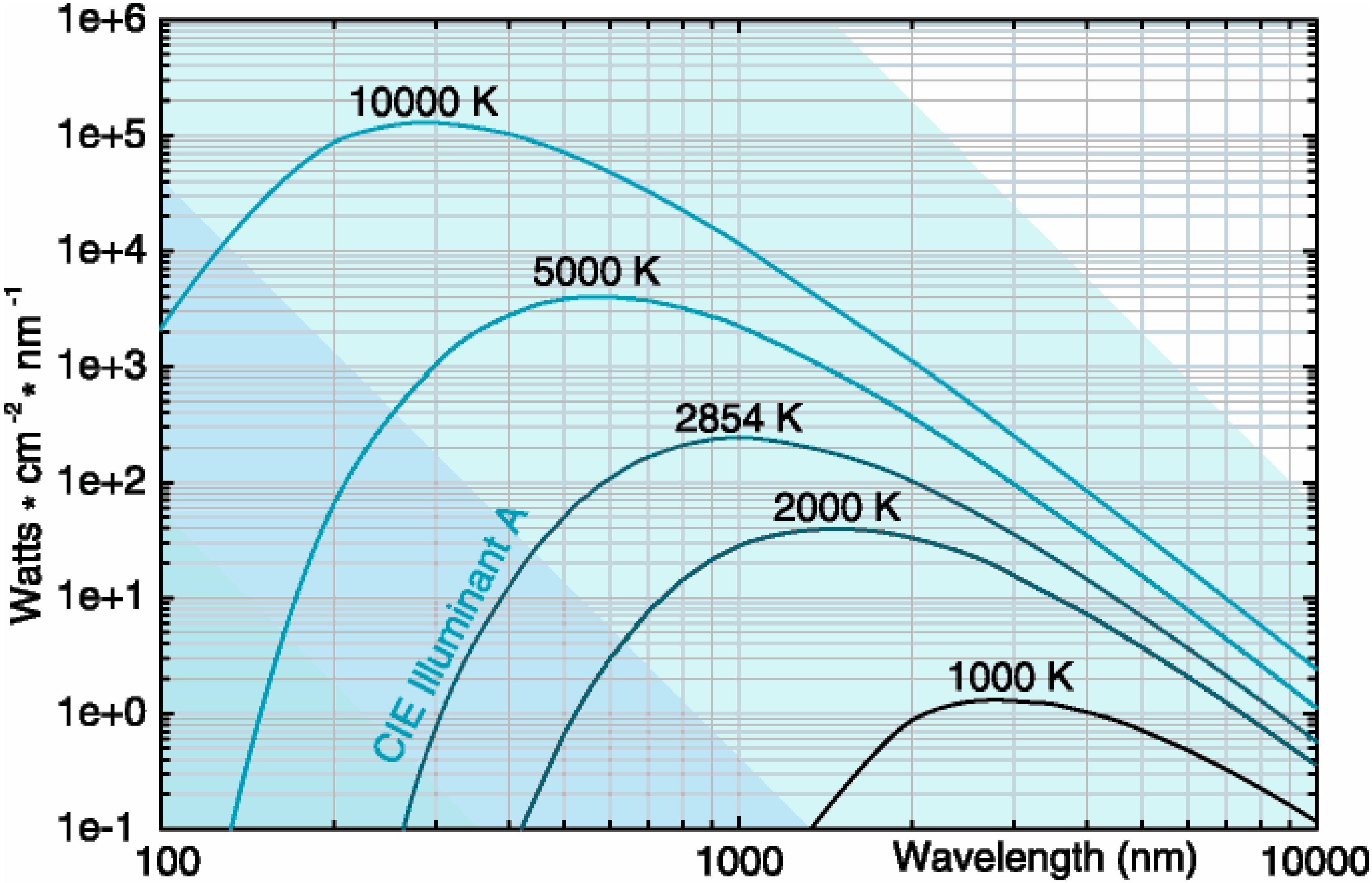


Fig. 5.1 Blackbody radiation at several color temperatures.

Planck Distribution

- At given wavelength, the emissive power increases with increasing temperature

- As the temperature increases,more emissive energy appear at shorter wavelengths

- For low temperature (< 800 K), all radiant energy falls in the infrared region and is not visible to the human eyes. That is why only very high temperature objects, such as molten iron, can glow.

- Sun can be approximated as a blackbody at 5800 K

Solar Irradiation

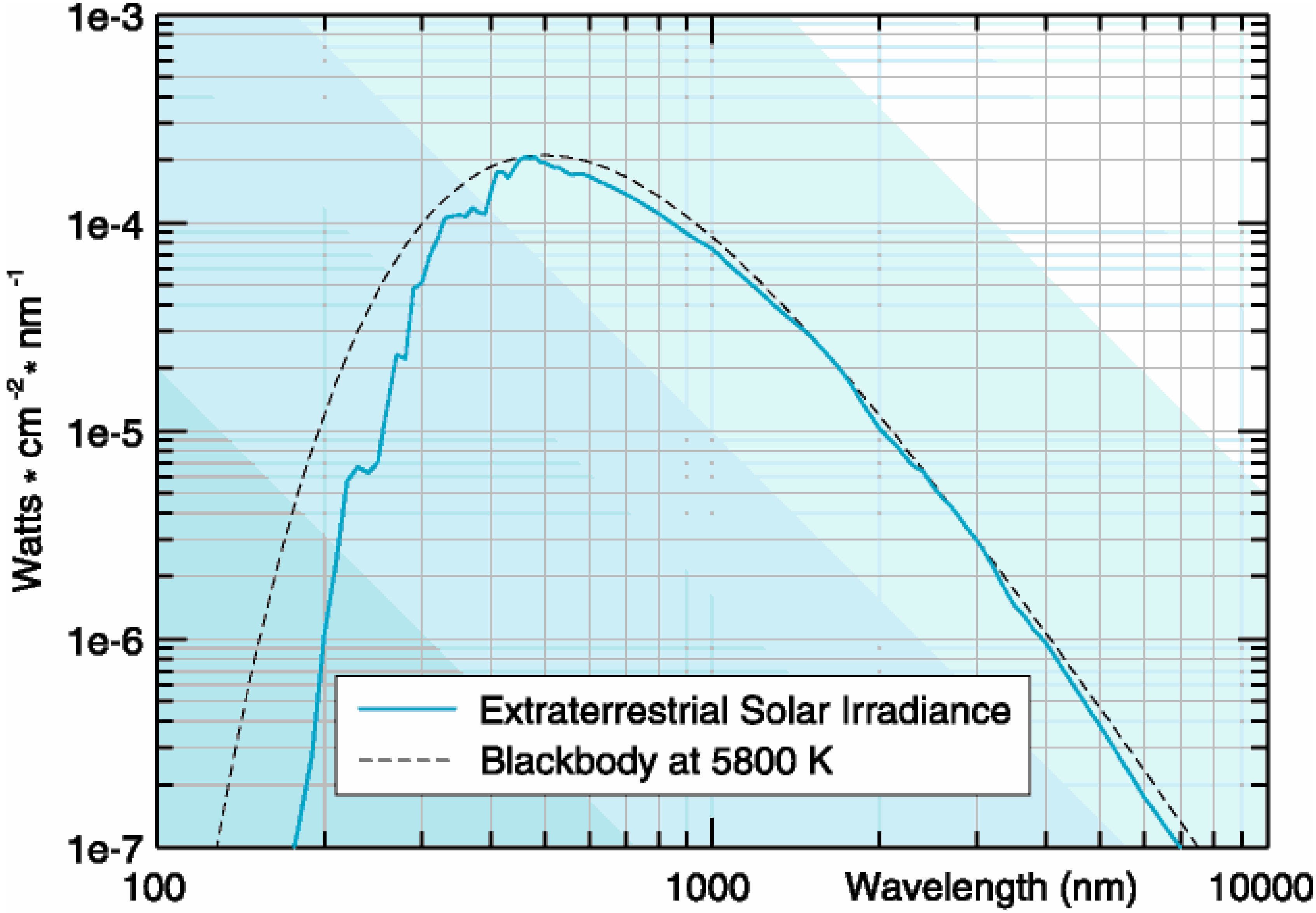
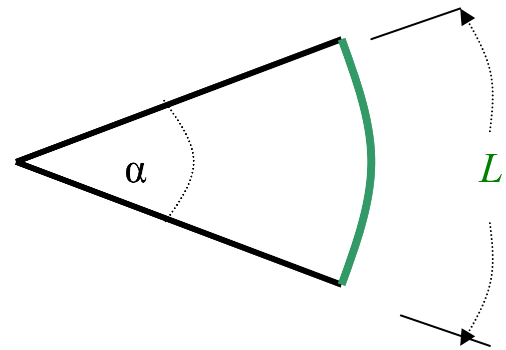


Fig. 5.7 Extraterrestrial solar irradiance compared to a blackbody.

Angles and Arc Length

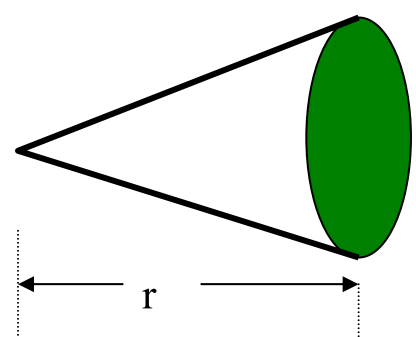
We are well accustomed to thinking of an angle as a two dimensional object. It may be used to find an arc length:



$L = r \cdot \alpha$

Solid Angle

We generalize the idea of an angle and an arc length to three dimensions and define a solid angle, Ω, which like the standard angle has no dimensions. The solid angle, when multiplied by the radius squared will have dimensions of length squared, or area, and will have the magnitude of the encompassed area.

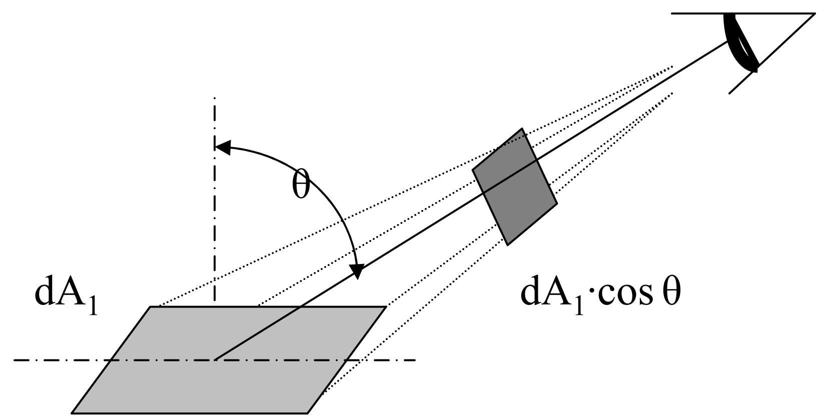


$A = r^2 \cdot d\Omega$

Projected Area

The area, dA_1, as seen from the prospective of a viewer, situated at an angle θ from the normal to the surface, will appear somewhat smaller, as $\cos\theta \cdot dA_1$. This smaller area is termed the projected area.

$A_{projected} = \cos\theta \cdot A_{normal}$



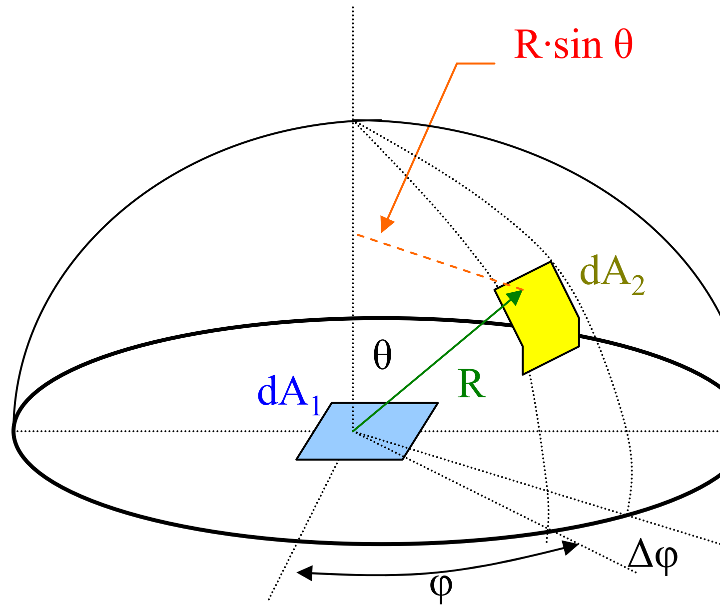
Intensity

The ideal intensity, I_b, may now be defined as the energy emitted from an ideal body, per unit projected area, per unit time, per unit solid angle.

$$I = \frac{dq}{\cos\theta \cdot dA_1 \cdot d\Omega}$$

Spherical Geometry

Since any surface will emit radiation outward in all directions above the surface, the spherical coordinate system provides a convenient tool for analysis. The three basic coordinates shown are R, φ, and θ, representing the radial, azimuthal and zenith directions.



In general dA_1 will correspond to the emitting surface or the source. The surface dA_2 will correspond to the receiving surface or the target. Note that the area proscribed on the hemisphere, dA_2, may be written as:

$$dA_2 = [(R \cdot \sin\theta) \cdot d\varphi] \cdot [R \cdot d\theta]$$

$$dA_2 = R^2 \cdot \sin\theta \cdot d\varphi \cdot d\theta]$$

Recalling the definition of the solid angle, $dA = R^2 \cdot d\Omega$
we find that: $d\Omega = R^2 \cdot \sin\theta \cdot d\theta \cdot d\varphi$

Real Surfaces

Thus far we have spoken of ideal surfaces, i.e. those that emit energy according to the Stefan-Boltzman law: $E_b = \sigma \cdot T_{abs}^4$

Real surfaces have emissive powers, E, which are somewhat less than that obtained theoretically by Boltzman. To account for this reduction, we introduce the emissivity, ε.

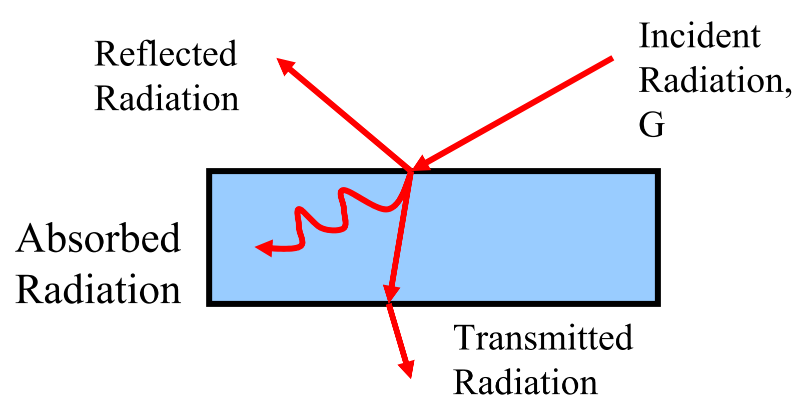
$$\varepsilon \equiv \frac{E}{E_b}$$

Emissive power from any real surface is given by: $E = \varepsilon \cdot \sigma \cdot T_{abs}^4$

Receiving Properties

Targets receive radiation in one of three ways; they absorption, reflection or transmission.

- •Absorptivity, α, the fraction of incident radiation absorbed.
- •Reflectivity, ρ, the fraction of incident radiation reflected.
- • Transmissivity, τ, the fraction of incident radiation transmitted.



We see, from Conservation of Energy, that:

$$\alpha + \rho + \tau = 1$$

In this course, we will deal with only opaque surfaces, $\tau = 0$, so that:

$\alpha + \rho = 1$

Relationship Between Absorptivity,α, and Emissivity,ε

Consider two flat, infinite planes, surface A and surface B, both emitting radiation toward one another. Surface B is assumed to be an ideal emitter, i.e. $\varepsilon_B = 1.0$. Surface A will emit radiation according to the Stefan-Boltzman law as:

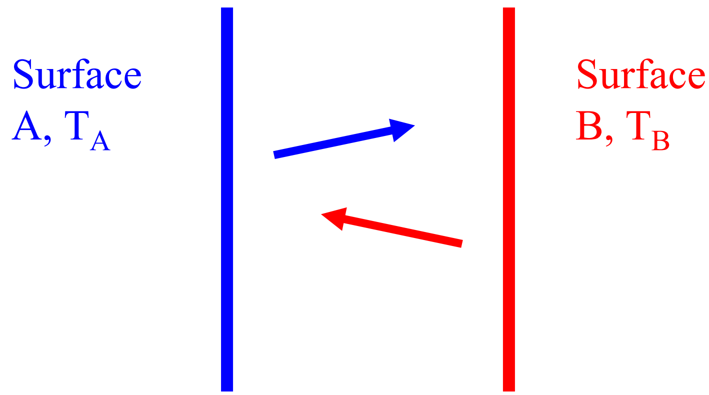
$$E_A = \varepsilon_A \cdot \sigma \cdot T_A^4$$

and will receive radiation as:

$$G_A = \alpha_A \cdot \sigma \cdot T_B^4$$

The net heat flow from surface A will be:

$q'' = \varepsilon_A \cdot \sigma \cdot T_A^4 - \alpha_A \cdot \sigma \cdot T_B^4$



Now suppose that the two surfaces are at exactly the same temperature. The heat flow must be zero according to the 2nd law. If follows then that: $\alpha_A = \varepsilon_A$

Thermodynamic properties of the material, α and ε may depend on temperature. In general, this will be the case as radiative properties will depend on wavelength, λ. The wave length of radiation will, in turn, depend on the temperature of the source of radiation.

The emissivity, ε, of surface A will depend on the material of which surface A is composed, i.e. aluminum, brass, steel, etc. and on the temperature of surface A.

The absorptivity, α, of surface A will depend on the material of which surface A is composed, i.e. aluminum, brass, steel, etc. and on the temperature of surface B.

Black Surfaces

Within the visual band of radiation, any material, which absorbs all visible light, appears as black. Extending this concept to the much broader thermal band, we speak of surfaces with $\alpha = 1$ as also being "black" or "thermally black". It follows that for such a surface, $\varepsilon = 1$ and the surface will behave as an ideal emitter. The terms ideal surface and black surface are used interchangeably.

Diffuse Surface: Refers to directional independence of the intensity associated with emitted,reflected ,or incident radiation.

Grey Surface: A surface for which the spectral absorptivity and the emissivity are independent of wavelength over the spectral regions of surface irradiation and emission.

Relationship Between Emissive Power and Intensity

By definition of the two terms, emissive power for an ideal surface, E_b, and intensity for an ideal surface, I_b.

$$E_b = \int_{hemisphere} I_b \cdot \cos\theta \cdot d\Omega$$

Replacing the solid angle by its equivalent in spherical angles:

$$E_b = \int_0^{2\cdot\pi} \int_0^{\pi/2} I_b \cdot \cos\theta \cdot \sin\theta \cdot d\theta \cdot d\varphi$$

Integrate once, holding I_b constant: $E_b = 2\cdot\pi\cdot I_b \cdot \int_0^{\pi/2} \cos\theta \cdot \sin\theta \cdot d\theta$

Integrate a second time. (Note that the derivative of sin θ is cos θ·dθ.)

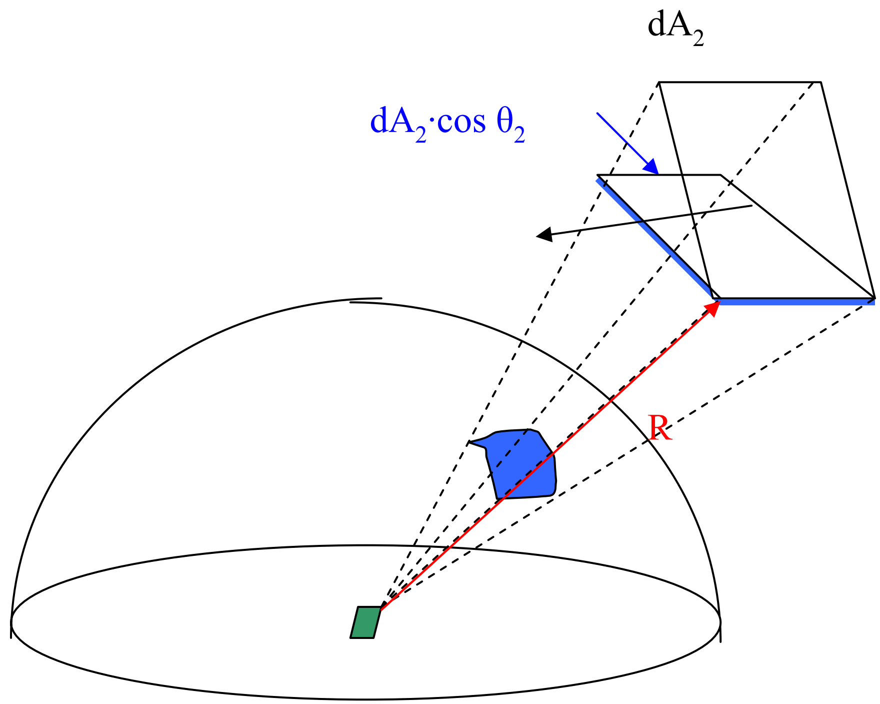
$$E_b = 2\cdot\pi\cdot I_b \cdot \left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2} = \pi \cdot I_b$$

$$\boxed{E_b = \pi\cdot I_b}$$

$$I = \frac{dq}{\cos\theta \cdot dA_1 \cdot d\Omega}$$

$$dq = I \cdot \cos\theta \cdot dA_1 \cdot d\Omega$$

Next we will project the receiving surface onto the hemisphere surrounding the source. First find the projected area of surface dA_2, $dA_2 \cdot \cos\theta_2$. (θ_2 is the angle between the normal to surface 2 and the position vector, R.) Then find the solid angle, Ω, which encompasses this area.



To obtain the entire heat transferred from a finite area, dA_1, to a finite area, dA_2, we integrate over both surfaces:

$$q_{1\to 2} = \int_{A_2}\int_{A_1} \frac{I \cdot \cos\theta_1 \cdot dA_1 \cdot \cos\theta_2 dA_2}{R^2}$$

Total energy emitted from surface 1: $q_{emitted} = E_1 \cdot A_1 = \pi \cdot I_1 \cdot A_1$

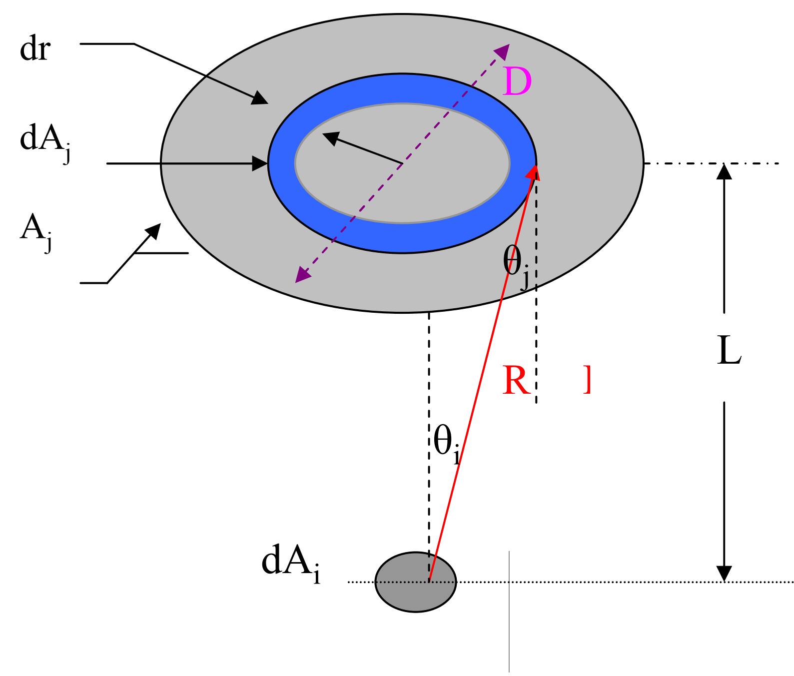
View Factors-Integral Method

Define the view factor, $F_{1\text{-}2}$, as the fraction of energy emitted from surface 1, which directly strikes surface 2.

$$F_{1\to 2} = \frac{q_{1\to 2}}{q_{emitted}} = \frac{\int_{A_2}\int_{A_1} \frac{I \cdot \cos\theta_1 \cdot dA_1 \cdot \cos\theta_2 dA_2}{R^2}}{\pi \cdot I \cdot A_1}$$

$$F_{1\to 2} = \frac{1}{A_1} \cdot \int_{A_2}\int_{A_1} \frac{\cos\theta_1 \cdot \cos\theta_2 \cdot dA_1 \cdot dA_2}{\pi \cdot R^2}$$

Example Consider a diffuse circular disk of diameter D and area A_j and a plane diffuse surface of area $A_i \ll A_j$. The surfaces are parallel, and A_i is located at a distance L from the center of A_j. Obtain an expression for the view factor F_{ij}.



$$F_{1\to 2} = \frac{1}{A_1}\cdot\int_{A_2}\int_{A_1}\frac{\cos\theta_1\cdot\cos\theta_2\cdot dA_1\cdot dA_2}{\pi\cdot R^2}$$

Since dA_1 is a differential area $F_{1\to 2} = \int_{A_2}\frac{\cos\theta_1\cdot\cos\theta_2\cdot dA_2}{\pi\cdot R^2}$

$$F_{1\to 2} = \int_{A_2}\frac{\left(L/R\right)^2\cdot 2\pi\cdot r\cdot dr}{\pi\cdot R^2} \qquad F_{1\to 2} = \int_{A_2}\frac{L^2\cdot 2\cdot r\cdot dr}{R^4}$$

Let $\rho^2 \equiv L^2 + r^2 = R^2$. Then $2\cdot\rho\cdot d\rho = 2\cdot r\cdot dr$. $F_{1\to 2} = \int_{A_2}\frac{L^2\cdot 2\cdot\rho\cdot d\rho}{\rho^4}$

$$F_{1\to 2} = -2\cdot L^2\cdot\left.\frac{\rho^{-2}}{2}\right|_{A_2} = -L^2\cdot\left[\frac{1}{L^2+\rho^2}\right]_0^{D/2}$$

$$F_{1\to 2} = -L^2\cdot\left[\frac{4}{4\cdot L^2+D^2}-\frac{1}{L^2}\right]_0^{D/2} = \frac{D^2}{4\cdot L^2+D^2}$$

Enclosures

In order that we might apply conservation of energy to the radiation process, we must account for all energy leaving a surface. We imagine that the surrounding surfaces act as an enclosure about the heat source which receive all emitted energy. For an N surfaced enclosure, we can then see that:

$$\sum_{j=1}^{N} F_{i,j} = 1$$

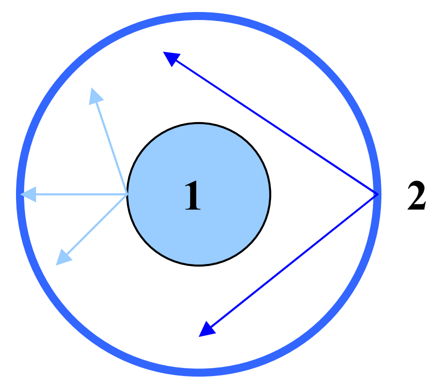
This relationship is known as the Conservation Rule".

Reciprocity

$$A_i \cdot F_{i \to j} = A_j \cdot F_{j \to i}$$

This relationship is known as "Reciprocity".

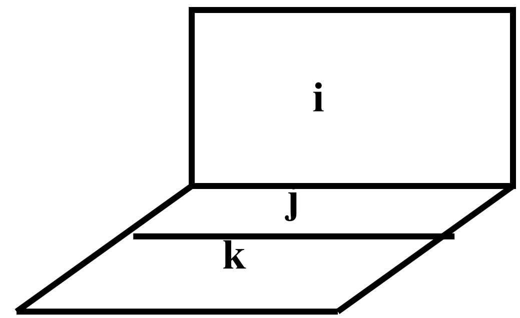
Example: Consider two concentric spheres shown to the right. All radiation leaving the outside of surface 1 will strike surface 2. Part of the radiant energy leaving the inside surface of object 2 will strike surface 1, part will return to surface 2. Find $F_{2,1}$. Apply reciprocity.



$$A_2 \cdot F_{2,1} = A_1 \cdot F_{1,2} \Rightarrow F_{2,1} = \frac{A_1}{A_2} \cdot F_{1,2} = \frac{A_1}{A_2} = \left[\frac{D_1}{D_2}\right]^2$$

Associative Rule

Consider the set of surfaces shown to the right: Clearly, from conservation of energy, the fraction of energy leaving surface i and striking the combined surface j+k will equal the fraction of energy emitted from i and striking j plus the fraction leaving surface i and striking k.



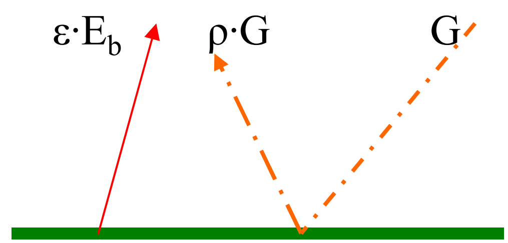
$$F_{i\Rightarrow(j+k)} = F_{i\Rightarrow j} + F_{i\Rightarrow k}$$

This relationship is known as the "Associative Rule".

Radiosity

Radiosity, J, is defined as the total energy leaving a surface per unit area and per unit time.

$$J \equiv \varepsilon \cdot E_b + \rho \cdot G$$



Net Exchange Between Surfaces

Consider the two surfaces shown. Radiation will travel from surface i to surface j and will also travel from j to i.

$$q_{i \to j} = J_i \cdot A_i \cdot F_{i \to j}$$

likewise,

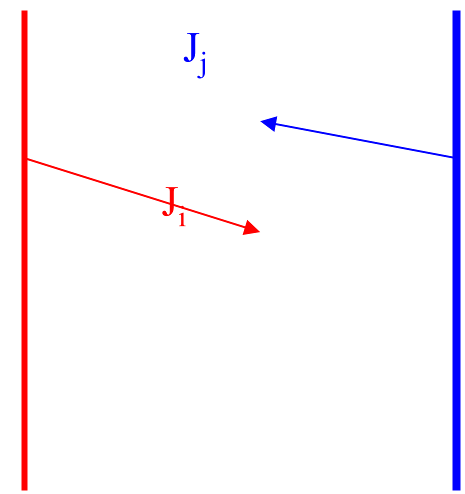
$$q_{j \to i} = J_j \cdot A_j \cdot F_{j \to j}$$

The net heat transfer is then:

$$q_{j \to i}\ (\text{net}) = J_i \cdot A_i \cdot F_{i \to j} - J_j \cdot A_j \cdot F_{j \to j}$$

From reciprocity we note that $F_{1 \to 2} \cdot A_1 = F_{2 \to 1} \cdot A_2$ so that

$$q_{j \to i}\ (\text{net}) = J_i \cdot A_i \cdot F_{i \to j} - J_j \cdot A_i \cdot F_{i \to j} = A_i \cdot F_{i \to j} \cdot (J_i - J_j)$$

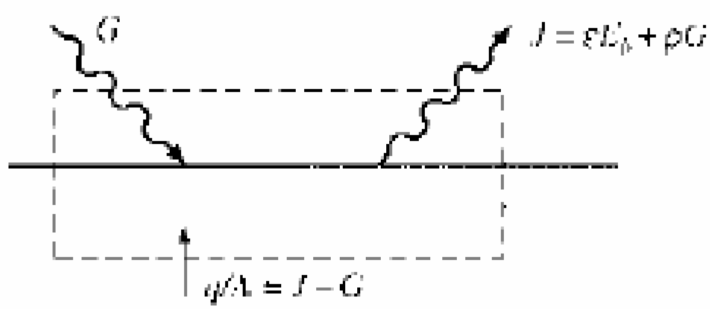


Heat exchange between nonblackbodies

- *All energy striking a surface of a blackbody is absorbed*
- To calculate radiation heat transfer between blackbodies we need to *determine shape factors*
- Energy striking a surface of a *nonblackbody* is
    - partially *absorbed*, and
    - partially *reflected*
- The *reflected energy* may be directed to
    - another surface, or
    - out of the system
- The energy reflected to another surface may be *reflected back and forth* between the heat transfer surfaces a number of times between
- The analysis of the radiation heat transfer must take into considerations these *multiple reflections* if accurate results are to be obtained

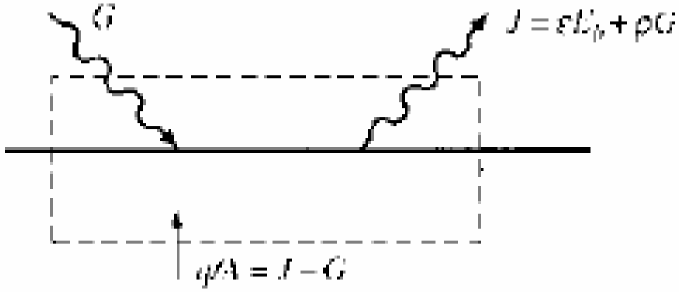
Heat exchange between nonblackbodies, cont'd

- All surfaces considered in the analysis are assumed to be
  - *diffuse*
  - *uniform* in temperature
  - reflective and emissive properties are *spatially constant*
- Total radiation incident upon a surface per unit time and per unit area is called *irradiation* and is represented by G
- Total radiation leaving a surface per unit time and per unit area is called *radiosity* and is represented by J
- Usually, analysis assumes that *irradiation and radiosity are spatially constant*
- The *radiosity is the sum of the energy emitted and the energy reflected*



$$J = \varepsilon E_b + \rho G$$

Heat exchange between nonblackbodies, cont'd



$$J = \varepsilon E_b + \rho G$$

$$\rho + \alpha + \tau = 1 \qquad \text{for } \tau = 0 \qquad \rho + \alpha = 1$$

using Kirchhoff's identity

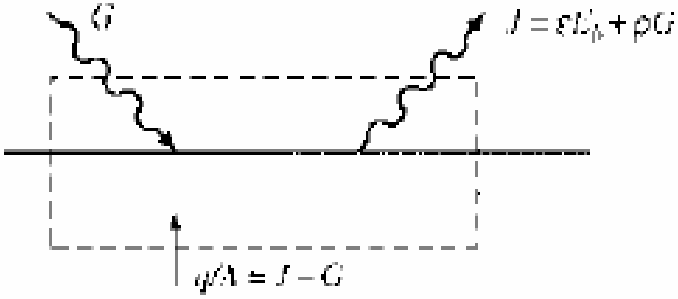
$$\rho = 1 - \varepsilon \;\Longrightarrow\; J = \varepsilon E_b + (1-\varepsilon)G \;\Longrightarrow\; G = \frac{J - \varepsilon E_b}{1-\varepsilon}$$

The *net energy* q_r leaving the surface is

$$q_r = \frac{Q_r}{A_r} = J - G = \varepsilon E_b + (1-\varepsilon)G - G = \varepsilon E_b - \varepsilon G$$

$$\frac{Q_r}{A_r} = \varepsilon E_b + \varepsilon\left(\frac{J - \varepsilon E_b}{1-\varepsilon}\right) = \left(\frac{\varepsilon}{1-\varepsilon}\right)(E_b - J) \;\Longrightarrow\; q_r = \frac{E_b - J}{\left(\dfrac{1-\varepsilon}{\varepsilon A_r}\right)}$$

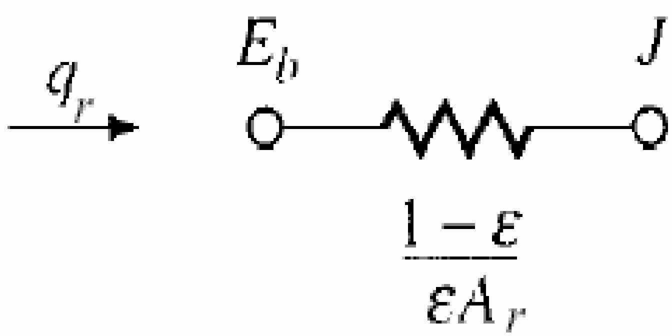
Heat exchange between nonblackbodies, cont'd



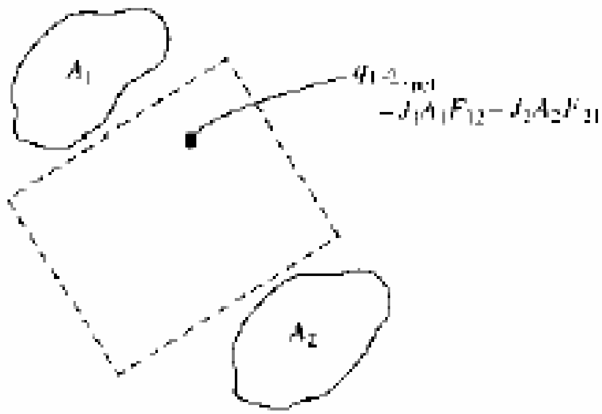
$$q_r = \frac{E_b - J}{\left(\dfrac{1-\varepsilon}{\varepsilon A_r}\right)} \quad \Longrightarrow \quad \boxed{q_r = \frac{E_b - J}{R_{surf}}}$$

where we have defined the *surface resistance to radiation heat transfer* as

$$\boxed{R_{surf} = \frac{1-\varepsilon}{\varepsilon A_r}}$$



Heat exchange between nonblackbodies, cont'd



The net energy interchange between the two surfaces is

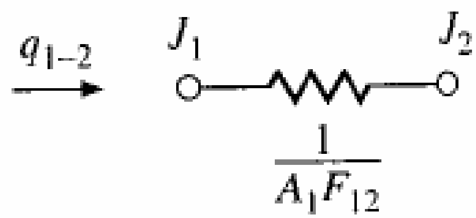
$$Q_{r_{1-2}} = J_1A_1F_{12} - J_2A_2F_{21}$$

Reciprocity

$$A_1F_{12} = A_2F_{21}$$

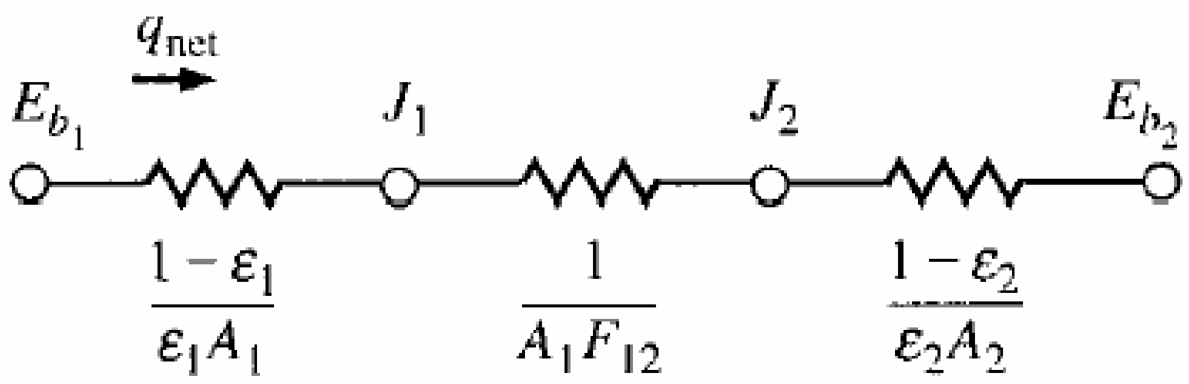
$$Q_{r_{1-2}} = (J_1 - J_2)A_1F_{12} = (J_2 - J_1)A_2F_{21} \Rightarrow Q_{r_{1-2}} = \frac{J_1 - J_2}{\left(\dfrac{1}{A_1F_{12}}\right)}$$

$$Q_{r_{1-2}} = \frac{J_1 - J_2}{R_{space}} \Rightarrow R_{space} = \frac{1}{A_1F_{12}} \Rightarrow \boxed{\textit{space resistance}}$$



Heat exchange between nonblackbodies, cont'd

Therefore, two surfaces that exchange heat with each other and nothing else can be represented by the following "*radiation network*"



$$Q_{r_{1-2}} = \frac{E_{b1} - E_{b2}}{\left(\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1}\right) + \left(\dfrac{1}{A_1 F_{12}}\right) + \left(\dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}\right)} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\left(\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1}\right) + \left(\dfrac{1}{A_1 F_{12}}\right) + \left(\dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}\right)}$$

Net Energy Leaving a Surface

The net energy leaving a surface will be the difference between the energy leaving a surface and the energy received by a surface:

$$q1\rightarrow = [\varepsilon \cdot Eb - \alpha \cdot G] \cdot A1$$

Combine this relationship with the definition of Radiosity to eliminate G.

$$J \equiv \varepsilon \cdot Eb + \rho \cdot G \Rightarrow G = [J - \varepsilon \cdot Eb]/\rho$$

$$q1\rightarrow = \{\varepsilon \cdot Eb - \alpha \cdot [J - \varepsilon \cdot Eb]/\rho\} \cdot A1$$

Assume opaque surfaces so that $\alpha + \rho = 1 \Rightarrow \rho = 1 - \alpha$, and substitute for ρ.

$$q1\rightarrow = \{\varepsilon \cdot Eb - \alpha \cdot [J - \varepsilon \cdot Eb]/(1 - \alpha)\} \cdot A1$$

Put the equation over a common denominator:

$$q_{1\rightarrow} = \left[\frac{(1-\alpha)\cdot \varepsilon \cdot E_b - \alpha \cdot J + \alpha \cdot \varepsilon \cdot E_b}{1-\alpha}\right] \cdot A_1 = \left[\frac{\varepsilon \cdot E_b - \alpha \cdot J}{1-\alpha}\right] \cdot A_1$$

assume that $\alpha = \varepsilon$

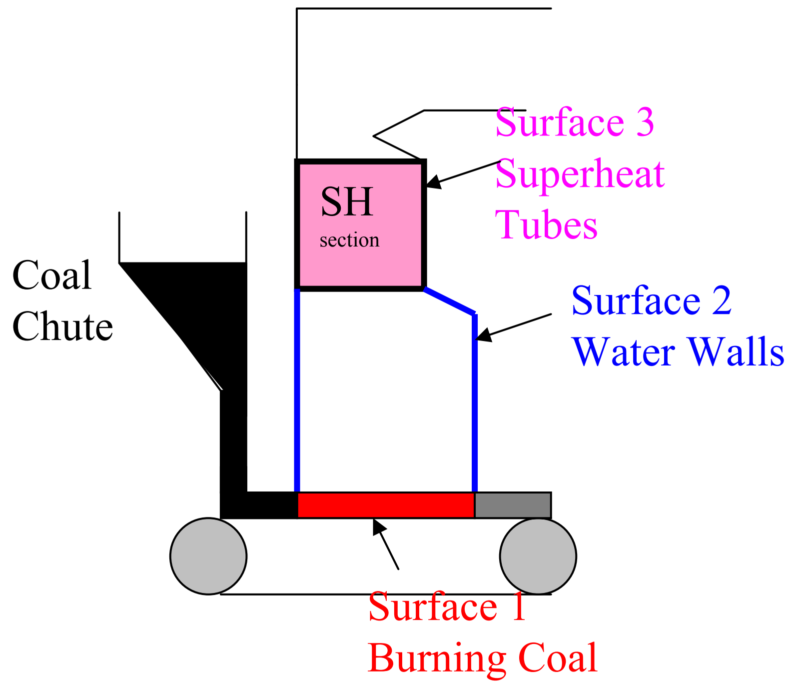
$$q_{1\rightarrow} = \left[\frac{\varepsilon \cdot E_b - \varepsilon \cdot J}{1-\varepsilon}\right] \cdot A_1 = \left[\frac{\varepsilon \cdot A_1}{1-\varepsilon}\right] \cdot (E_b - J)$$

Electrical Analogy for Radiation

We may develop an electrical analogy for radiation, similar to that produced for conduction. The two analogies should not be mixed: they have different dimensions on the potential differences, resistance and current flows.

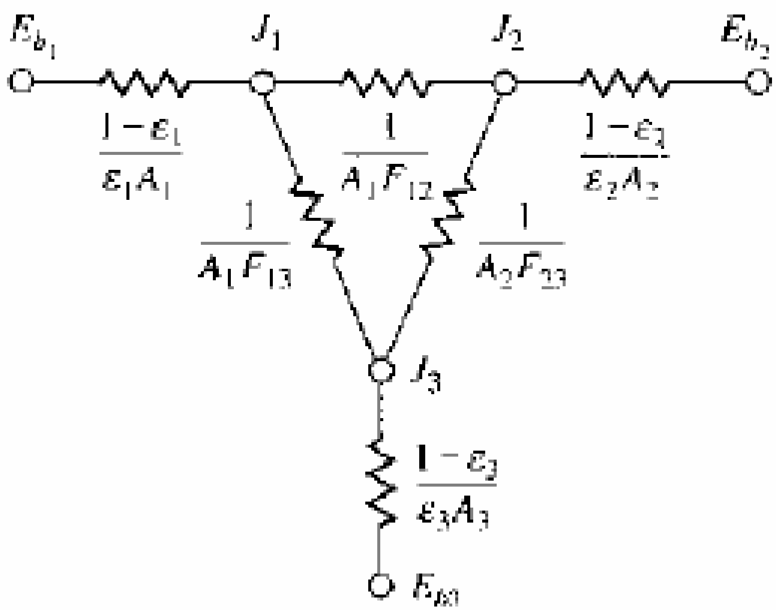
	Equivalent Current	Equivalent Resistance	Potential Difference
Ohms Law	I	R	ΔV
Net Energy Leaving Surface	$q_{1?}$	$\left[\dfrac{1-\varepsilon}{\varepsilon \cdot A}\right]$	E_b - J
Net Exchange Between Surfaces	$q_{i?j}$	$\dfrac{1}{A_1 \cdot F_{1\to 2}}$	$J_1 - J_2$

Example: Consider a grate fed boiler. Coal is fed at the bottom, moves across the grate as it burns and radiates to the walls and top of the furnace. The walls are cooled by flowing water through tubes placed inside of the walls. Saturated water is introduced at the bottom of the walls and leaves at the top at a quality of about 70%. After the vapour is separated from the water, it is circulated through the superheat tubes at the top of the boiler. Since the steam is undergoing a sensible heat addition, its temperature will rise. It is common practice to subdivide the super-heater tubes into sections, each having nearly uniform temperature. In our case we will use only one superheat section using an average temperature for the entire region.



Heat exchange between nonblackbodies, cont'd

Thermal radiation exchange between three bodies

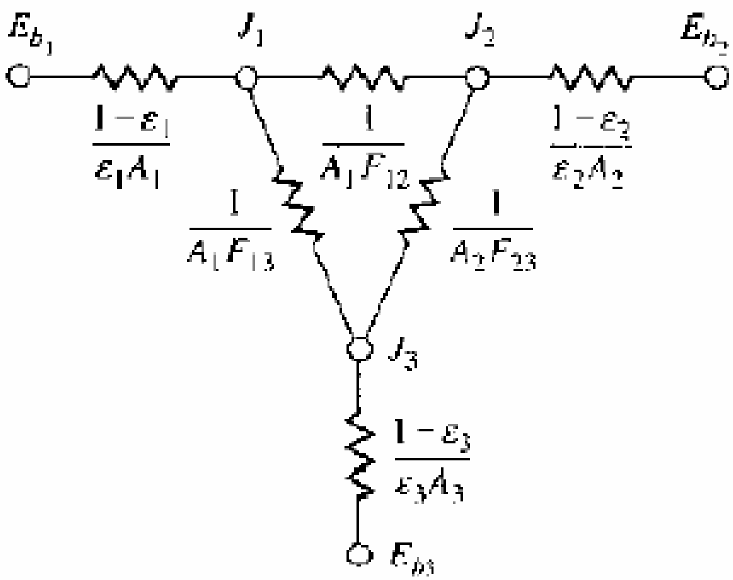


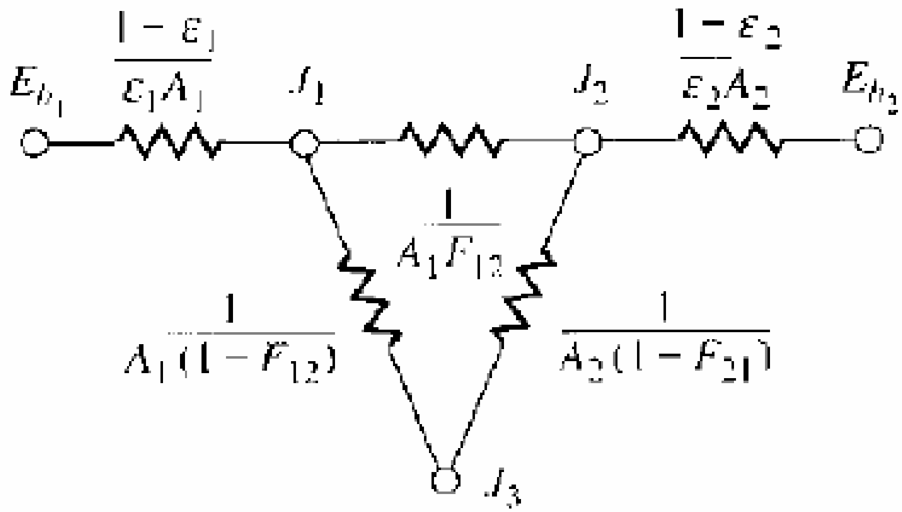
$$Q_{r_{1-2}} = \frac{J_1 - J_2}{\left(\dfrac{1}{A_1 F_{12}}\right)}$$

$$Q_{r_{1-3}} = \frac{J_1 - J_3}{\left(\dfrac{1}{A_1 F_{13}}\right)}$$

Heat exchange between nonblackbodies, cont'd

Insulated surface

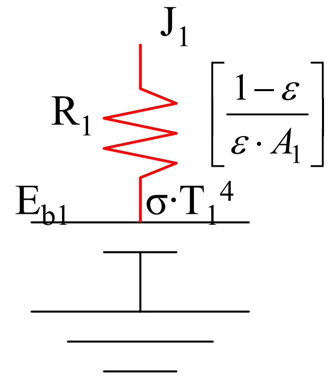




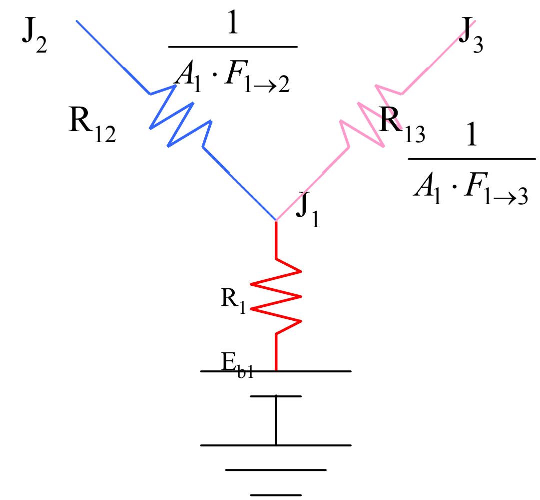
$$F_{12} + F_{13} = 1 \qquad F_{21} + F_{23} = 1$$

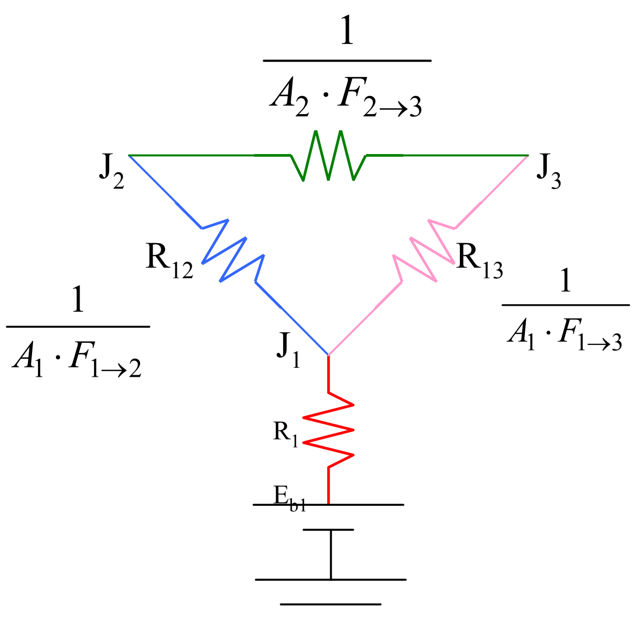
$$\boxed{F_{13} = 1 - F_{12}} \qquad \boxed{F_{23} = 1 - F_{21}}$$

Energy will leave the coal bed, Surface 1, as described by the equation for the net energy leaving a surface. We draw the equivalent electrical network as seen to the right:



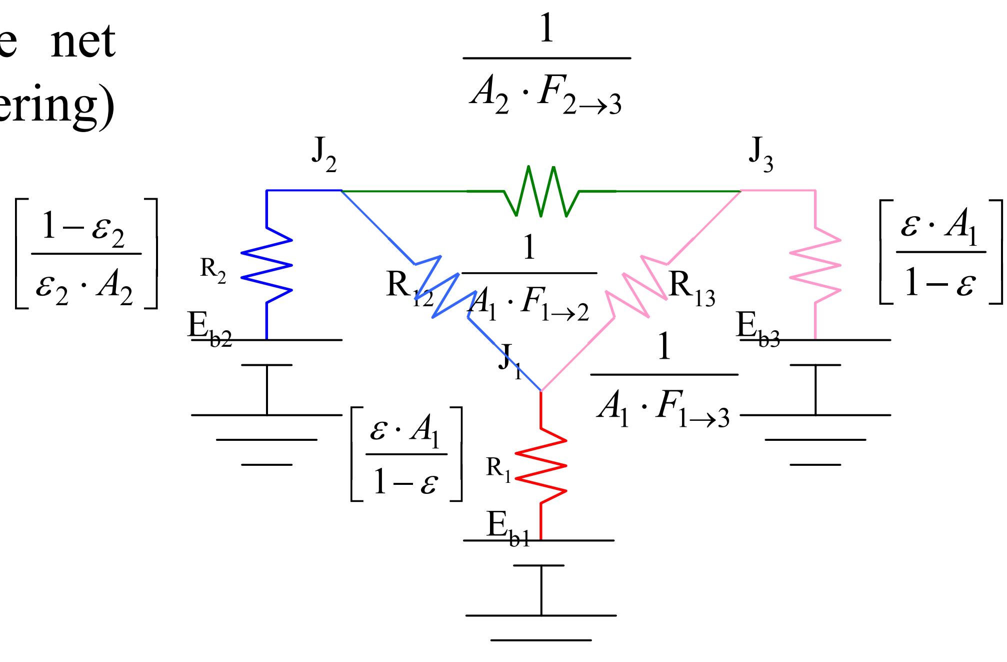
The heat leaving from the surface of the coal may proceed to either the water walls or to the super-heater section. That part of the circuit is represented by a potential difference between Radiosity:





It should be noted that surfaces 2 and 3 will also radiate to one another.

It remains to evaluate the net heat flow leaving (entering) nodes 2 and 3.



- **Insulated surfaces.** In steady state heat transfer, a surface cannot receive net energy if it is insulated. Because the energy cannot be stored by a surface in steady state, all energy must be re-radiated back into the enclosure.

Insulated surfaces are often termed as re-radiating surfaces.

- **Black surfaces:** A black, or ideal surface, will have no surface resistance:

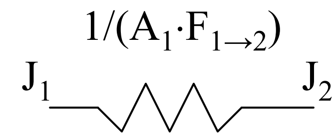
$$\left[\frac{1-\varepsilon}{\varepsilon \cdot A}\right] = \left[\frac{1-1}{1 \cdot A}\right] = 0$$

In this case the nodal Radiosity and emissive power will be equal.

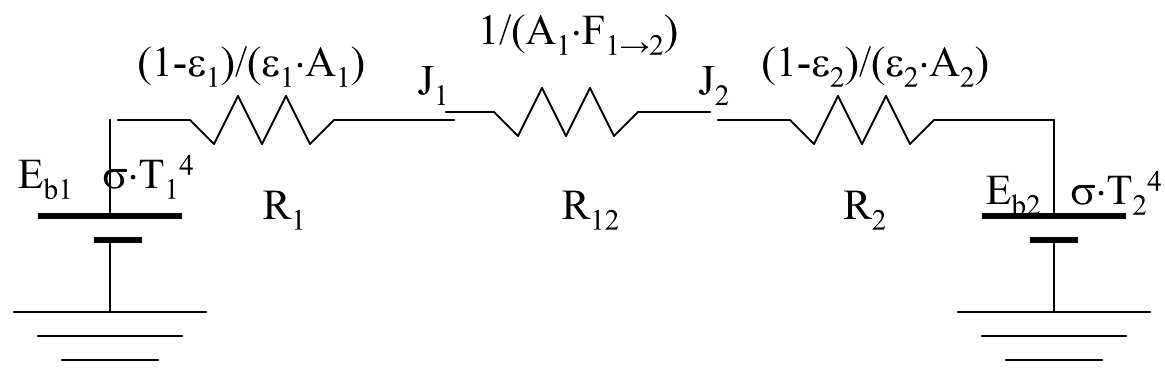
- Large surfaces: Surfaces having a large surface area will behave as black surfaces, irrespective of the actual surface properties:

$$\left[\frac{1-\varepsilon}{\varepsilon \cdot A}\right] = \left[\frac{1-\varepsilon}{\varepsilon \cdot \infty}\right] = 0$$

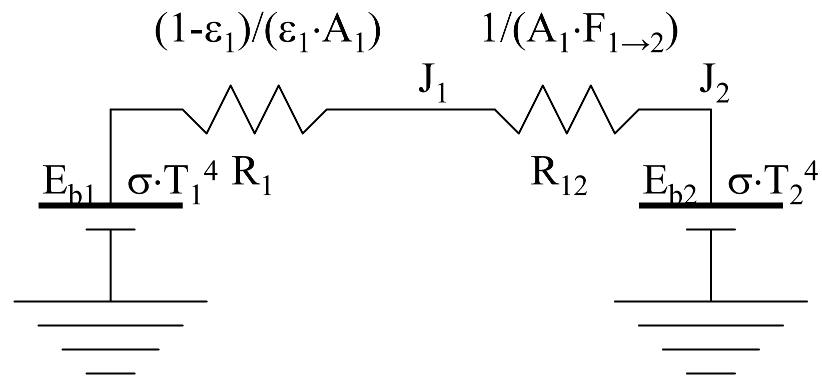
Consider the case of an object, 1, placed inside a large enclosure, 2. The system will consist of two objects, so we proceed to construct a circuit with two radiosity nodes.



Now we ground both Radiosity nodes through a surface resistance.



Since A_2 is large, $R_2 = 0$. The view factor, $F_{1\rightarrow2} = 1$



Sum the series resistances:

$$R_{Series} = (1-\varepsilon_1)/(\varepsilon_1 \cdot A_1) + 1/A_1 = 1/(\varepsilon_1 \cdot A_1)$$

Ohm's law:

$$i = \Delta V/R$$

or by analogy:

$$q = \Delta E_b/R_{Series} = \varepsilon_1 \cdot A_1 \cdot \sigma \cdot (T_1^4 - T_2^4)$$

Returning for a moment to the coal grate furnace, let us assume that we know (a) the total heat being produced by the coal bed, (b) the temperatures of the water walls and (c) the temperature of the super heater sections.

Apply Kirchoff's law about node 1, for the coal bed:

$$q_1 + q_{2\to 1} + q_{3\to 1} = q_1 + \frac{J_2 - J_1}{R_{12}} + \frac{J_3 - J_1}{R_{13}} = 0$$

Similarly, for node 2:

$$q_2 + q_{1\to 2} + q_{3\to 2} = \frac{E_{b2} - J_2}{R_2} + \frac{J_1 - J_2}{R_{12}} + \frac{J_3 - J_2}{R_{23}} = 0$$

And for node 3:

$$q_3 + q_{1\to 3} + q_{2\to 3} = \frac{E_{b3} - J_3}{R_3} + \frac{J_1 - J_3}{R_{13}} + \frac{J_2 - J_3}{R_{23}} = 0$$

The three equations must be solved simultaneously. Since they are each linear in J, matrix methods may be used:

$$\begin{bmatrix} -\frac{1}{R_{12}} - \frac{1}{R_{13}} & \frac{1}{R_{12}} & \frac{1}{R_{13}} \\ \frac{1}{R_{12}} & -\frac{1}{R_2} - \frac{1}{R_{12}} - \frac{1}{R_{13}} & \frac{1}{R_{23}} \\ \frac{1}{R_{13}} & \frac{1}{R_{23}} & -\frac{1}{R_3} - \frac{1}{R_{13}} - \frac{1}{R_{23}} \end{bmatrix} \cdot \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} -q_1 \\ -\frac{E_{b2}}{R_2} \\ -\frac{E_{b3}}{R_3} \end{bmatrix}$$

Surface 1: Find the coal bed temperature, given the heat flow:

$$q_1 = \frac{E_{b1} - J_1}{R_1} = \frac{\sigma \cdot T_1^4 - J_1}{R_1} \Rightarrow T_1 = \left[\frac{q_1 \cdot R_1 + J_1}{\sigma} \right]^{0.25}$$

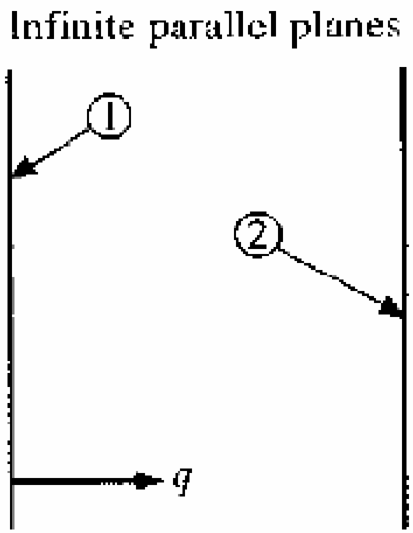
Surface 2: Find the water wall heat input, given the water wall temperature:

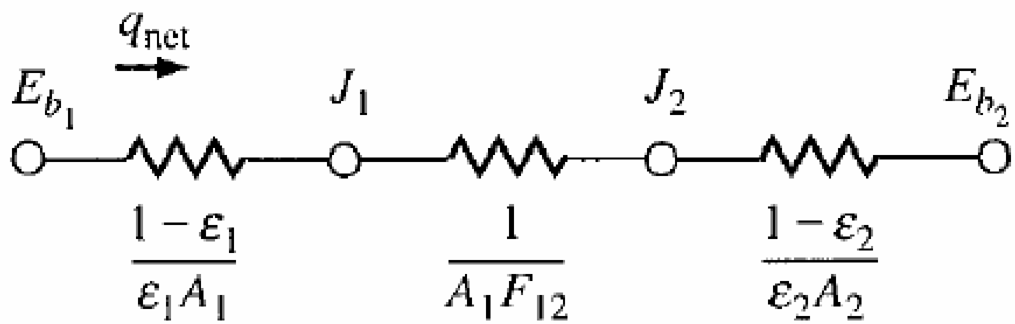
$$q_2 = \frac{E_{b2} - J_2}{R_2} = \frac{\sigma \cdot T_2^4 - J_2}{R_2}$$

Surface 3: (Similar to surface 2) Find the water wall heat input, given the water wall temperature:

$$q_3 = \frac{E_{b3} - J_3}{R_3} = \frac{\sigma \cdot T_3^4 - J_3}{R_3}$$

Infinite parallel surfaces



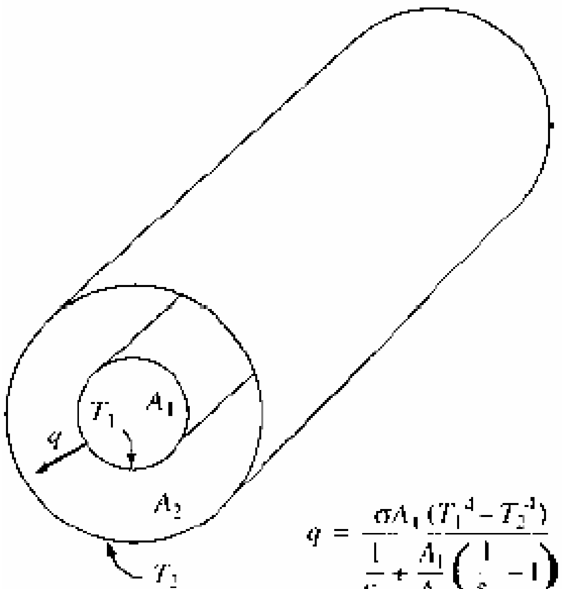


$$A_1 = A_2$$

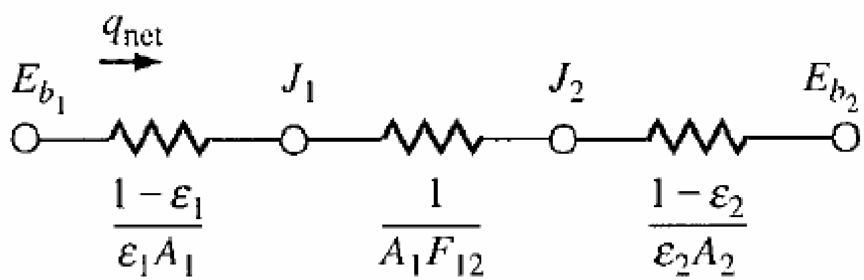
$$F_{12} = 1$$

$$q_{r_{1-2}} = \frac{Q_{r_{1-2}}}{A_r} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\varepsilon_1}\right) + \left(\frac{1}{\varepsilon_2}\right) - 1}$$

Infinite concentric cylinders



$$q = \frac{\sigma A_1 (T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)}$$

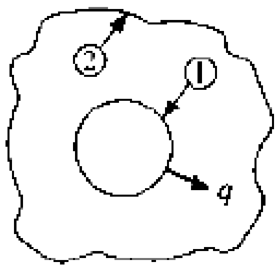


$$A_1 \neq A_2$$

$$F_{12} = 1$$

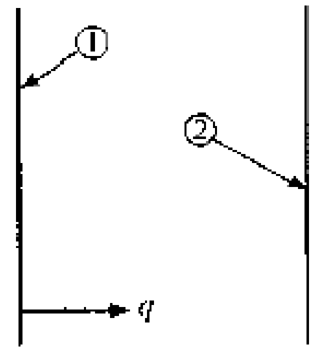
$$Q_{r_{1-2}} = \frac{\sigma A_1\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\varepsilon_1}\right) + \left(\frac{A_1}{A_2}\right)\left(\frac{1}{\varepsilon_2} - 1\right)} = \frac{\sigma A_1\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\varepsilon_1}\right) + \left(\frac{D_1}{D_2}\right)\left(\frac{1}{\varepsilon_2} - 1\right)}$$

Small convex object in large enclosure



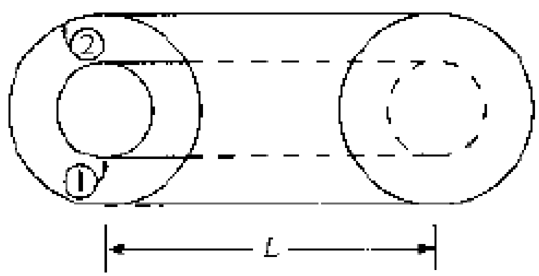
$$q = A_1 \epsilon_1 \sigma \left(T_1^4 - T_2^4 \right)$$
for $A_1/A_2 \to 0$

Infinite parallel planes



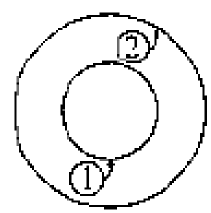
$$(q/A) = \frac{\sigma \left(T_1^4 - T_2^4 \right)}{1/\epsilon_1 + 1/\epsilon_2 - 1}$$
with $A_1 = A_2$

Infinite concentric cylinders



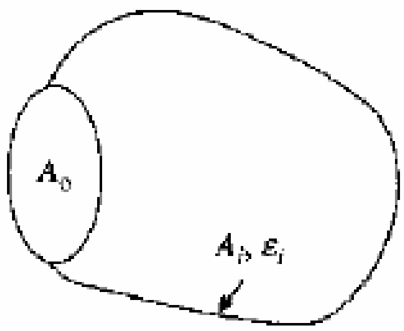
$$q = \frac{\sigma A_1 \left(T_1^4 - T_2^4 \right)}{1/\epsilon_1 + (1/\epsilon_2 - 1)(r_1/r_2)}$$
with $A_1/A_2 = r_1/r_2$; $\quad r_1/L \to 0$

Concentric spheres



$$q = \frac{\sigma A_1 \left(T_1^4 - T_2^4 \right)}{1/\epsilon_1 + (1/\epsilon_2 - 1)(r_1/r_2)^2}$$
for $A_1/A_2 = (r_1/r_2)^2$

Apparent emissivity of a cavity



$$F_{oi} = 1$$

$$A_o F_{oi} = A_i F_{io}$$

$$F_{io} = F_{is}$$

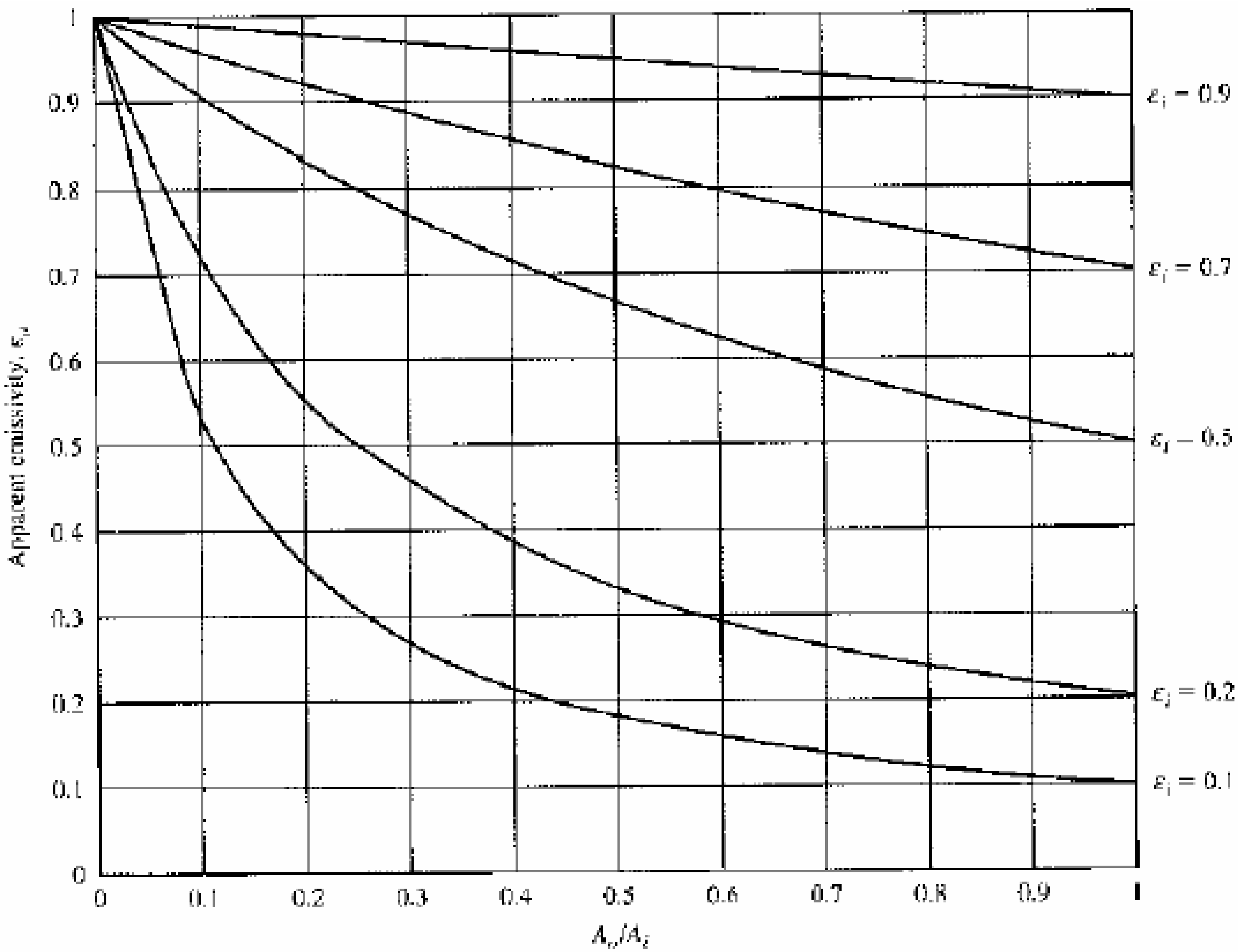
$$A_i F_{is} = A_o$$

$$Q_{is} = \frac{E_{bi} - E_{bs}}{\left(\frac{1-\varepsilon_i}{\varepsilon_i A_i}\right) + \left(\frac{1}{A_i F_{is}}\right)}$$

$$Q_r = \sigma A_1 \varepsilon_1 \left(T_1^4 - T_2^4\right)$$

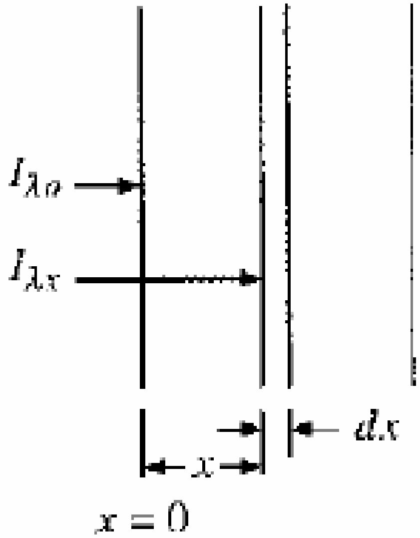
$$Q_{o-s} = A_o \varepsilon_a \left(E_{bi} - E_{bs}\right)$$

$$\varepsilon_a = \frac{\varepsilon_i A_i}{A_o + \varepsilon_i \left(A_i - A_o\right)}$$



Apparent emissivity, ε_a
A_o/A_i
$\varepsilon_i = 0.9$
$\varepsilon_i = 0.7$
$\varepsilon_i = 0.5$
$\varepsilon_i = 0.2$
$\varepsilon_i = 0.1$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Gas radiation



$$dI_{\lambda} = -\alpha_{\lambda} I_{\lambda} dx$$

monochromatic absorption coefficient

$$\int_{I_{\lambda_0}}^{I_{\lambda x}} \frac{dI_{\lambda}}{I_{\lambda}} = \int_{0}^{x} \frac{dI_{\lambda}}{I_{\lambda}}$$

$$\frac{I_{\lambda_x}}{I_{\lambda_0}} = e^{-\alpha_{\lambda} x}$$

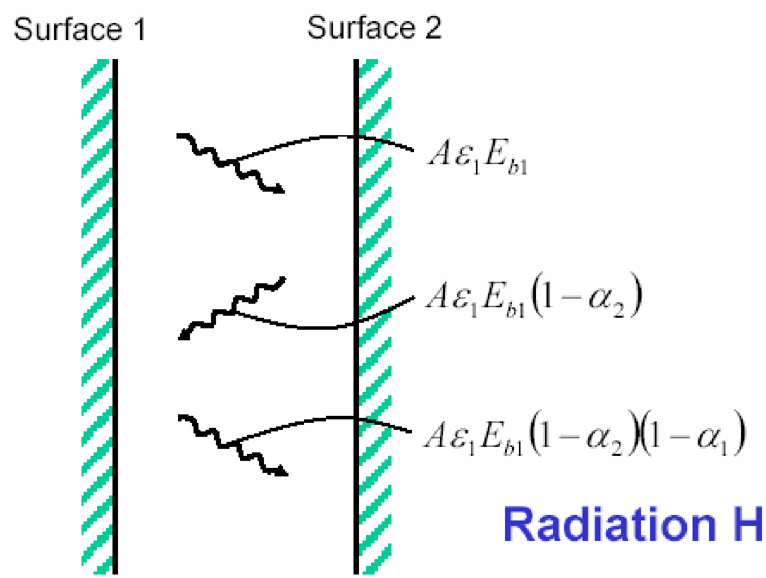
Beer law

Gases frequently absorb only in a narrow bandwidth

Calculations of *gas-radiation* properties is very complicated

Radiation heat transfer by multiple reflections

The ray tracing method



Radiation HT by multiple reflections, cont'd

Surface 1 emits	$A\varepsilon_1 E_{b1}$
Surface 2 absorbs	$A\varepsilon_1 E_{b1}\alpha_2$
Surface 2 reflects	$A\varepsilon_1 E_{b1}(1-\alpha_2)$
Surface 1 absorbs	$A\varepsilon_1 E_{b1}(1-\alpha_2)\alpha_1$
Surface 1 reflects	$A\varepsilon_1 E_{b1}(1-\alpha_2)(1-\alpha_1)$
Surface 2 absorbs	$A\varepsilon_1 E_{b1}(1-\alpha_2)(1-\alpha_1)\alpha_2$
Surface 2 reflects	$A\varepsilon_1 E_{b1}(1-\alpha_2)(1-\alpha_1)(1-\alpha_2)$
Surface 1 absorbs	$A\varepsilon_1 E_{b1}(1-\alpha_2)(1-\alpha_1)(1-\alpha_2)\alpha_1$

Extended Surfaces / Fins

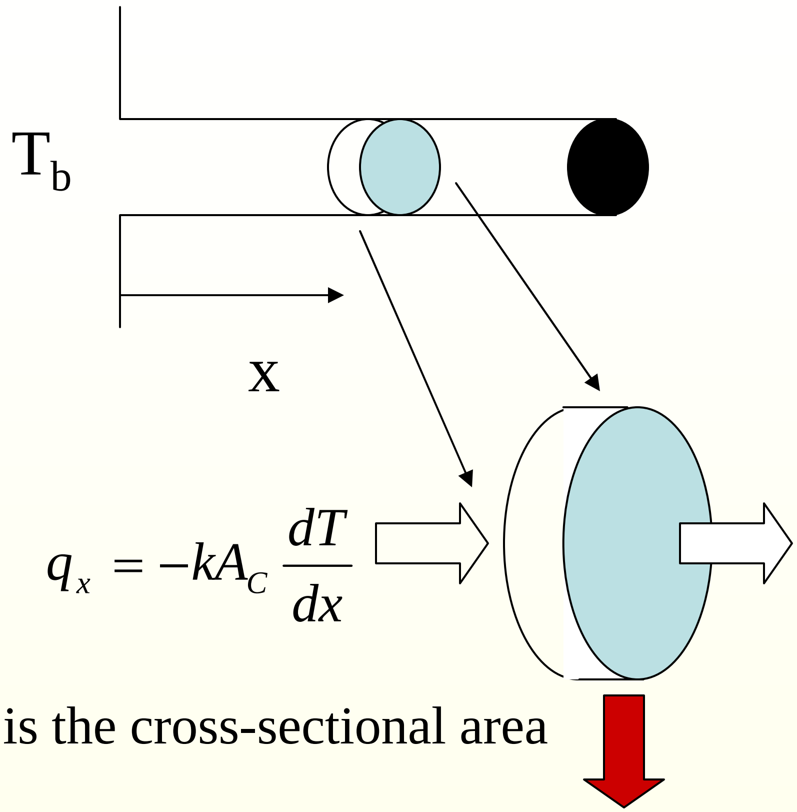
Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference $(T_s - T_\infty)$ between the surface and the fluid.

- Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h. Example: a cooling fan.

- Increase the contact surface area A. Example: a heat sink with fins.

Extended Surface Analysis



P: the fin perimeter

A_c: the fin cross-sectional area

$$q_x = -kA_C \frac{dT}{dx}$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

A_C is the cross-sectional area

$dq_{conv} = h(dA_S)(T - T_\infty)$, where dA_S is the surface area of the element

Energy Balance: $q_x = q_{x+dx} + dq_{conv} = q_x + \dfrac{dq_x}{dx} dx + hdA_S(T - T_\infty)$

$-kA_C \dfrac{d^2T}{dx^2} dx + hP(T - T_\infty)dx = 0$, if k, A_C are all constants.

Extended Surface Analysis (cont.)

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_C}(T - T_\infty) = 0, \text{ A second-order, ordinary differential equation}$$

Define a new variable $\theta(x) = T(x) - T_\infty$, so that

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \text{ where } m^2 = \frac{hP}{kA_C}, \ (D^2 - m^2)\theta = 0$$

Characteristics equation with two real roots: $+m$ & $-m$

The general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

To evaluate the two constants C_1 and C_2, we need to specify two boundary conditions:

The first one is obvious: the base temperature is known as $T(0) = T_b$

The second condition will depend on the end condition of the tip

Extended Surface Analysis (cont.)

For example: assume the tip is insulated and no heat transfer $d\theta/dx(x=L)=0$

The temperature distribution is given by

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

The fin heat transfer rate is

$$q_f = -kA_C \frac{dT}{dx}(x=0) = \sqrt{hPkA_C}\tanh mL = M\tanh mL$$

These results and other solutions using different end conditions are tabulated in the following fins table

Temperature distribution for fins of different configurations

Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$	$\dfrac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}$	$M\dfrac{\sinh mL+(h/mk)\cosh mL}{\cosh mL+(h/mk)\sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L}=0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M\tanh mL$
C	Given temperature: $\theta(L)=\theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh m(L-x)+\sinh m(L-x)}{\sinh mL}$	$M\dfrac{(\cosh mL-\theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L)=0$	e^{-mx}	M

$$\theta \equiv T - T_\infty, \qquad m^2 \equiv \frac{hP}{kA_C}$$

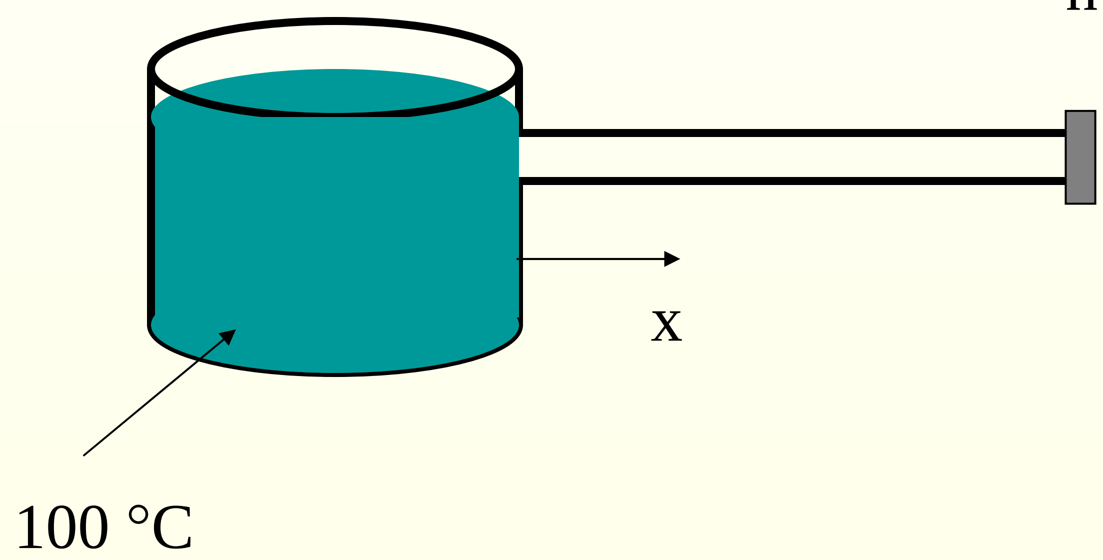
$$\theta_b = \theta(0) = T_b - T_\infty, \qquad M = \sqrt{hPkA_C}\,\theta_b$$

Note: This table is adopted from Fundamentals of Heat and Mass Transfer by Frank Incropera and David DeWitt

Example

An Aluminum pot is used to boil water as shown below. The handle of the pot is 20-cm long, 3-cm wide, and 0.5-cm thick. The pot is exposed to room air at 25°C, and the convection coefficient is 5 W/m^2 °C. Question: can you touch the handle when the water is boiling? (k for aluminum is 237 W/m °C)

T_∞ = 25 °C
h = 5 W/ m^2 °C



Example (cont.)

We can model the pot handle as an extended surface. Assume that there is no heat transfer at the free end of the handle. The condition matches that specified in the fins Table, case B.

h=5 W/ m^2 °C, P=2W+2t=2(0.03+0.005)=0.07(m), k=237 W/m °C, A_C=Wt=0.00015(m^2), L=0.2(m)

Therefore, $m=(hP/kA_C)^{1/2}=3.138$,

$M=\sqrt{(hPkA_C)}(T_b-T_\infty)=0.111\theta_b=0.111(100-25)=8.325(W)$

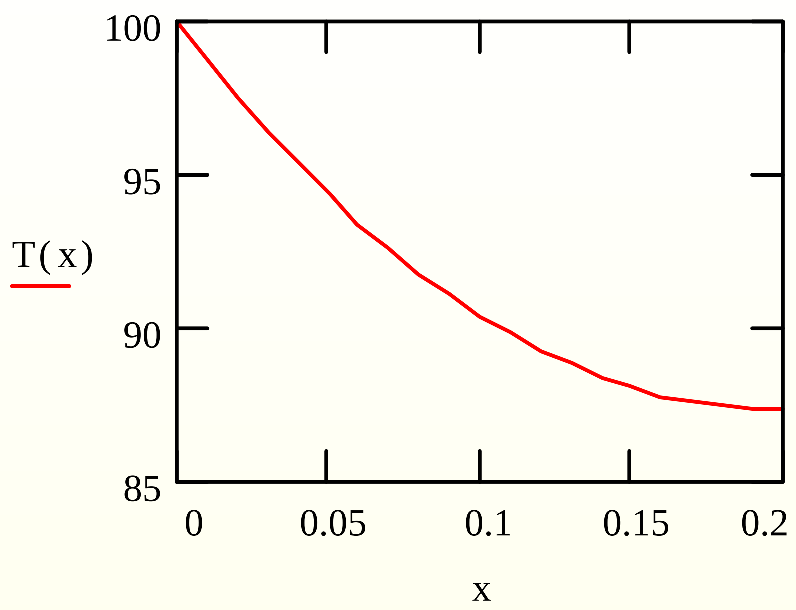
$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T-25}{100-25}=\frac{\cosh[3.138(0.2-x)]}{\cosh(3.138*0.2)},$$

$$T(x)=25+62.32*\cosh[3.138(0.2-x)]$$

Example (cont.)

Plot the temperature distribution along the pot handle



As shown, temperature drops off very quickly. At the midpoint T(0.1)=90.4°C. At the end T(0.2)=87.3°C.
Therefore, it should not be safe to touch the end of the handle

Example (cont.)

The total heat transfer through the handle can be calculated also. q_f=Mtanh(mL)=8.325*tanh(3.138*0.2)=4.632(W) Very small amount: latent heat of evaporation for water: 2257 kJ/kg. Therefore, the amount of heat loss is just enough to vaporize 0.007 kg of water in one hour.

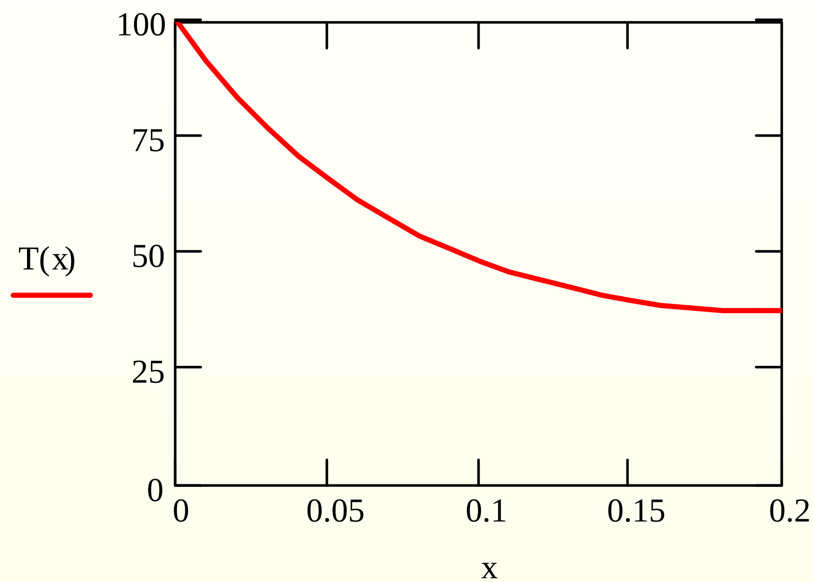
If a stainless steel handle is used instead, what will happen: For a stainless steel, the thermal conductivity k=15 W/m°C. Use the same parameter as before:

$$m = \left(\frac{hP}{kA_C}\right)^{1/2} = 12.47, \quad M = \sqrt{hPkA_C} = 0.0281$$

Example (cont.)

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$T(x)=25+12.3\cosh[12.47(L-x)]$$



Temperature at the handle (x=0.2 m) is only 37.3 °C, not hot at all. This example illustrates the important role played by the thermal conductivity of the material in terms of conductive heat transfer.

Fins-2

If the pot from previous lecture is made of other materials other than the aluminum, what will be the temperature distribution? Try stainless steel (k=15 W/m.K) and copper (385 W/m.K).

Recall: h=5 W/ m^2 °C, P=2W+2t=2(0.03+0.005)=0.07(m)

A_C=Wt=0.00015(m^2), L=0.2(m)

Therefore, $m_{ss}=(hP/kA_C)^{1/2}=12.47$, $m_{cu}=2.46$

$M_{ss}=\sqrt{(hPk_{ss}A_C)}\,(T_b-T_\infty)=0.028(100-25)=2.1(W)$

$M_{cu}=\sqrt{(hPk_{ss}A_C)}\,\theta_b=0.142(100-25)=10.66(W)$

For stainless steel, $\dfrac{T_{ss}(x)-T_\infty}{T_b-T_\infty}=\dfrac{\theta}{\theta_b}=\dfrac{\cosh m(L-x)}{\cosh mL}$

$$\frac{T_{ss}-25}{100-25}=\frac{\cosh[12.47(0.2-x)]}{\cosh(12.47*0.2)},$$

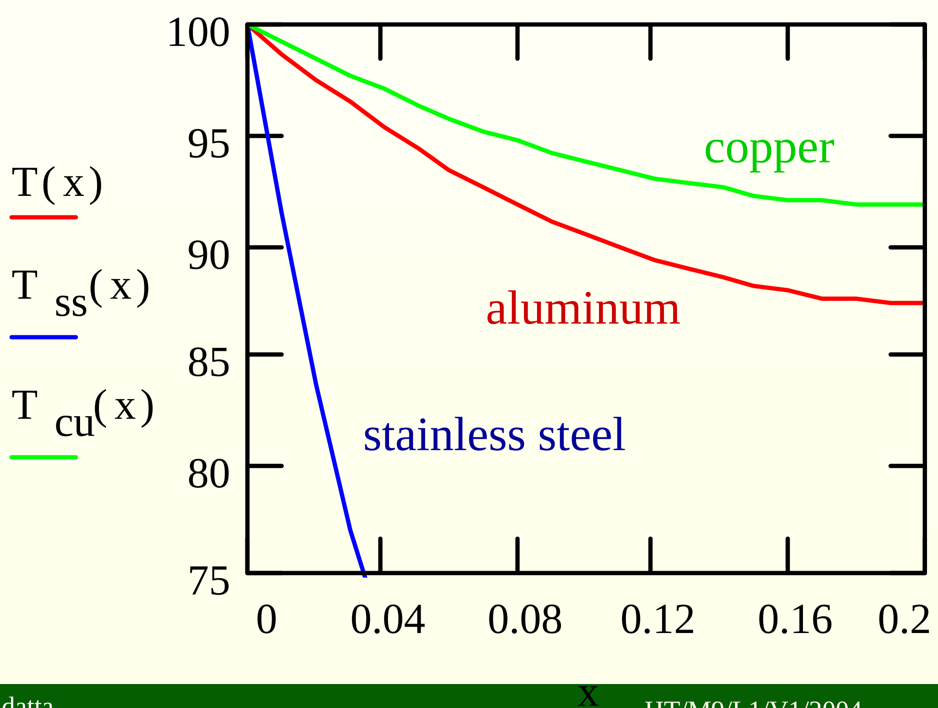
$$T_{ss}(x)=25+12.3*\cosh[12.47(0.2-x)]$$

Fins-2 (cont.)

For copper, $$\frac{T_{cu}(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T_{cu} - 25}{100 - 25} = \frac{\cosh[2.46(0.2 - x)]}{\cosh(2.46 * 0.2)},$$

$$T_{cu}(x) = 25 + 66.76 * \cosh[2.46(0.2 - x)]$$



Fins-2 (cont.)

• Inside the handle of the stainless steel pot, temperature drops quickly. Temperature at the end of the handle is 37.3°C. This is because the stainless steel has low thermal conductivity and heat can not penetrate easily into the handle.

• Copper has the highest k and, correspondingly, the temperature inside the copper handle distributes more uniformly. Heat easily transfers into the copper handle.

• Question? Which material is most suitable to be used in a heat sink?

Fins-2 (cont.)

How do we know the adiabatic tip assumption is good? Try using the convection heat transfer condition at the tip (case A in fins table) We will use the aluminum pot as the example.
h=5 W/m^2.K, k=237 W/m.K, *m*=3.138, M=8.325W

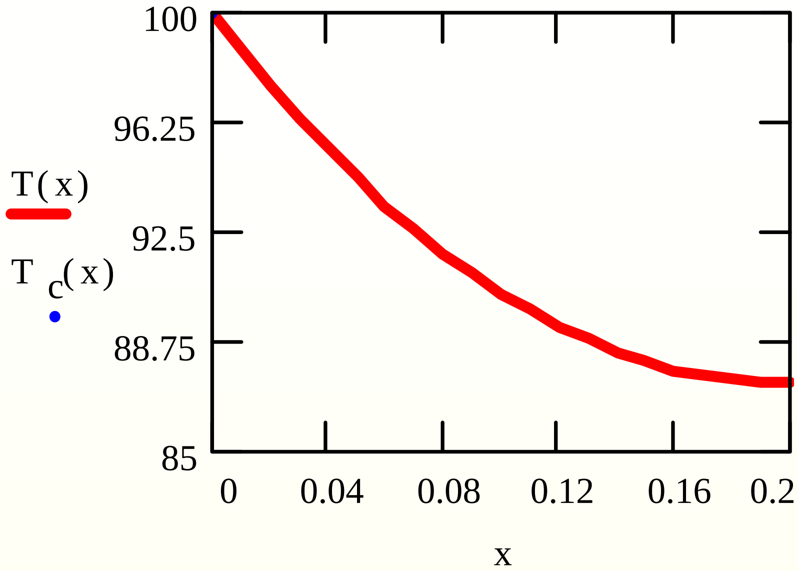
Long equation

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh[m(L-x)] + (h/mk)\sinh[m(L-x)]}{\cosh mL + (h/mk)\sinh mL}$$

$$= \frac{\cosh[3.138(0.2-x)] + 0.00672\sinh[3.138(0.2-x)]}{\cosh(0.6276) + 0.00672\sinh(0.6276)}$$

$$T(x) = 25 + 62.09\{\cosh(0.6276 - 3.138x) + 0.00672\sinh(0.6276 - 3.138x)\}$$

Fins-2 (cont.)



T: adiabatic tip

T_c: convective tip

T(0.2)=87.32 °C

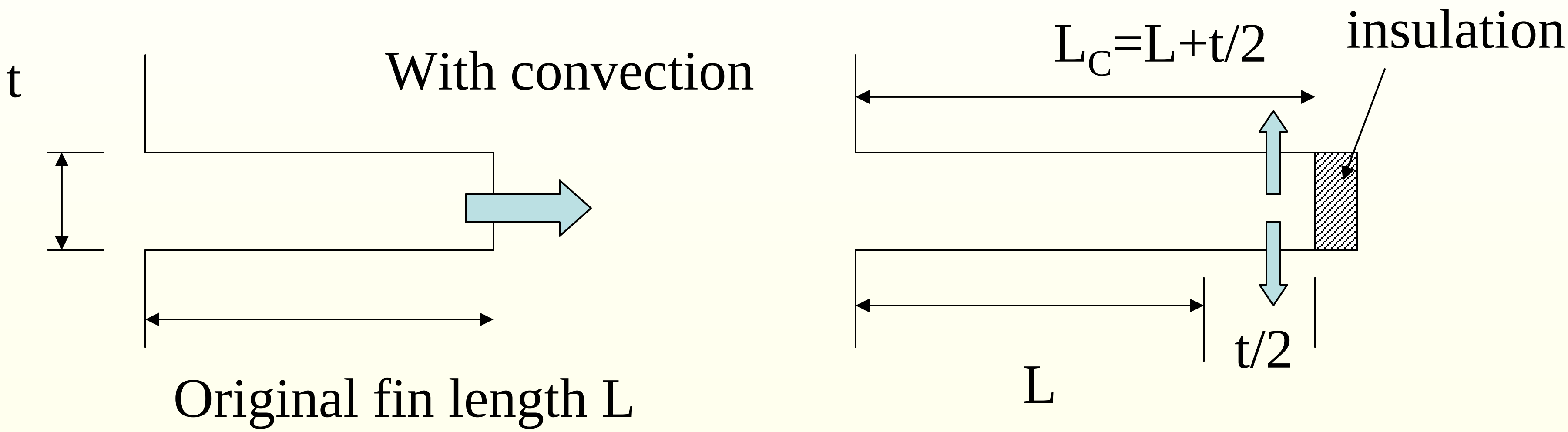
T_c(0.2)=87.09 °C

Note 1: Convective tip case has a slightly lower tip temperature as expected since there is additional heat transfer at the tip.

Note 2: there is no significant difference between these two solutions, therefore, correct choice of boundary condition is not that important here. However, sometimes correction might be needed to compensate the effect of convective heat transfer at the end. (especially for thick fins)

Fins-2 (cont.)

In some situations, it might be necessary to include the convective heat transfer at the tip. However, one would like to avoid using the long equation as described in case A, fins table. The alternative is to use case B instead and accounts for the convective heat transfer at the tip by extending the fin length L to L_C=L+(t/2).



Then apply the adiabatic condition at the tip of the extended fin as shown above.

Fins-2 (cont.)

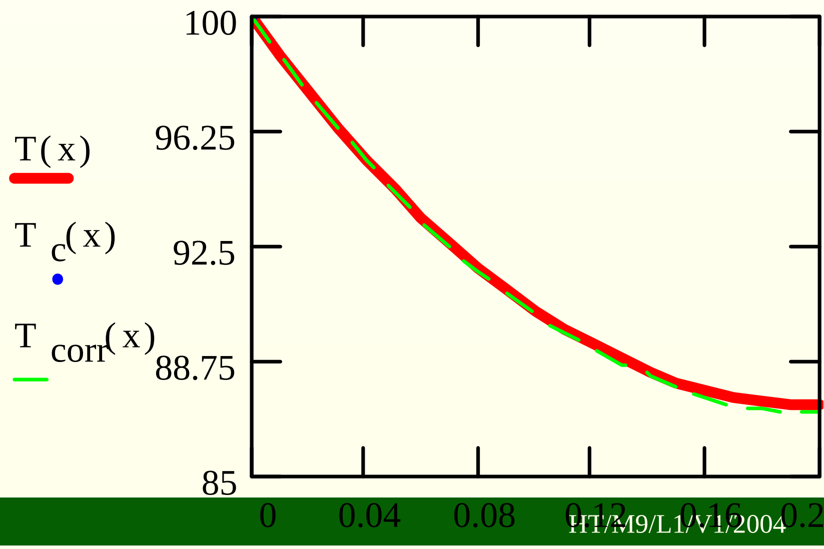
Use the same example: aluminum pot handle, *m*=3.138, the length will need to be corrected to

L_C=l+(t/2)=0.2+0.0025=0.2025(m)

$$\frac{T_{corr}(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L_c - x)}{\cosh mL_c}$$

$$\frac{T_{corr} - 25}{100 - 25} = \frac{\cosh[3.138(0.2025 - x)]}{\cosh(3.138 * 0.2025)},$$

$$T_{corr}(x) = 25 + 62.05 * \cosh[3.138(0.2025 - x)]$$



T(0.2)=87.32 °C

T_c(0.2)=87.09 °C

T_{corr}(0.2025)=87.05 °C

slight improvement over the uncorrected solution

Correction Length

The correction length can be determined by using the formula: $L_c=L+(A_c/P)$, where A_c is the cross-sectional area and P is the perimeter of the fin at the tip.

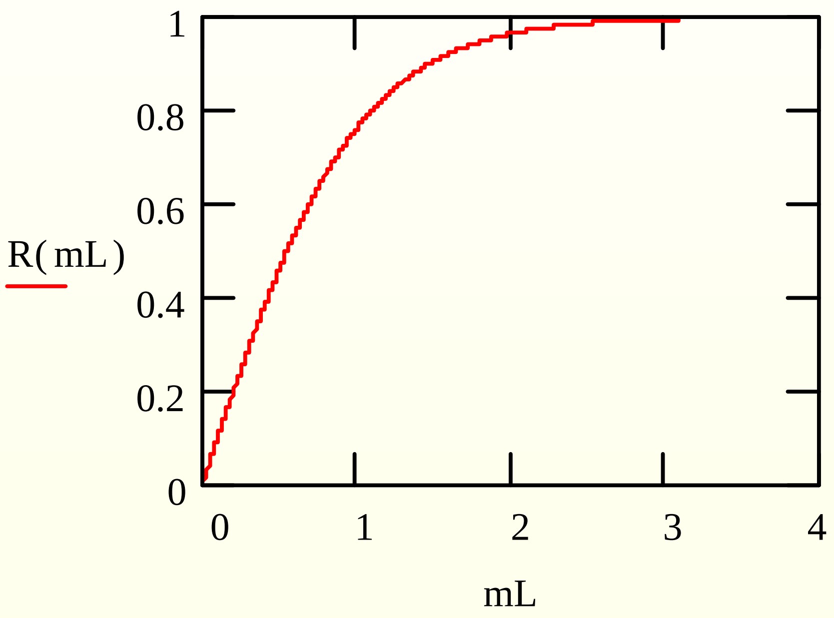
- Thin rectangular fin: $A_c=Wt$, $P=2(W+t)\approx 2W$, since $t << W$
$L_c=L+(A_c/P)=L+(Wt/2W)=L+(t/2)$

- Cylindrical fin: $A_c=(\pi/4)D^2$, $P= \pi D$, $L_c=L+(A_c/P)=L+(D/4)$

- Square fin: $A_c=W^2$, $P=4W$,
$L_c=L+(A_c/P)=L+(W^2/4W)=L+(W/4)$

Optimal Length of a fin

In general, the longer the fin, the higher the heat transfer. However, a long fin means more material and increased size and cost. Question: how do we determine the optimal fin length? Use the rectangular fin as an example:



$q_f = M \tanh mL,$ for an adiabatic tip fin

$(q_f)_\infty = M,$ for an infinitely long fin

Their ratio: $R(mL) = \dfrac{q_f}{(q_f)_\infty} = \tanh mL$

Note: heat transfer increases with mL as expected. Initially the rate of change is large and slows down drastically when mL> 2.

R(1)=0.762, means any increase beyond mL=1 will increase no more than 23.8% of the fin heat transfer.

Temperature Distribution

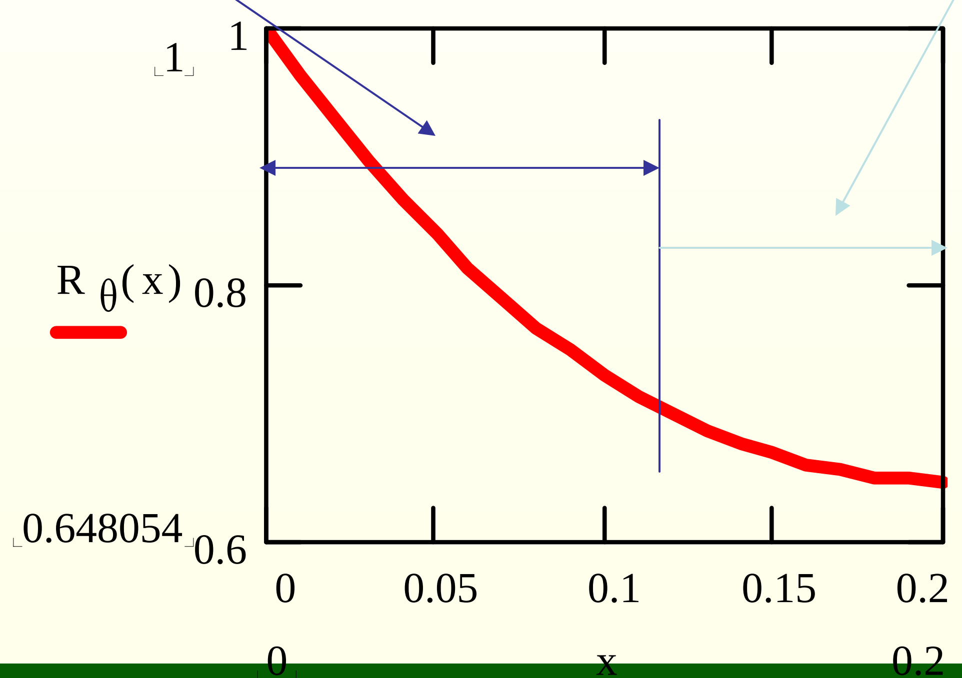
For an adiabatic tip fin case:

$$R_\theta = \frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

- Use m=5, and L=0.2 as an example:

Low ΔT, poor fin heat transfer

High ΔT, good fin heat transfer



Correction Length for a Fin with a Non-adiabatic Tip

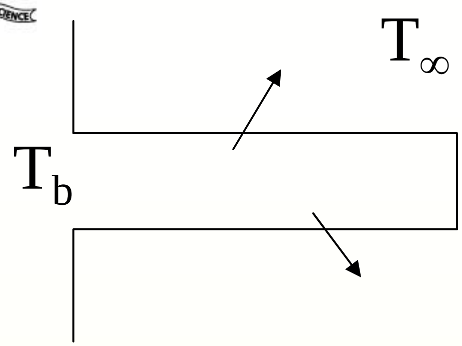
The correction length can be determined by using the formula: $L_c=L+(A_c/P)$, where A_c is the cross-sectional area and P is the perimeter of the fin at the tip.

• Thin rectangular fin: $A_c=Wt$, $P=2(W+t)\approx 2W$, since $t \ll W$
$L_c=L+(A_c/P)=L+(Wt/2W)=L+(t/2)$

• Cylindrical fin: $A_c=(\pi/4)D^2$, $P= \pi D$, $L_c=L+(A_c/P)=L+(D/4)$

• Square fin: $A_c=W^2$, $P=4W$,
$L_c=L+(A_c/P)=L+(W^2/4W)=L+(W/4)$

Fin Design



Total heat loss: q_f=Mtanh(mL) for an adiabatic fin, or q_f=Mtanh(mL_C) if there is convective heat transfer at the tip

where $m=\sqrt{\frac{hP}{kA_c}}$, and $M=\sqrt{hPkA_C}\,\theta_b = \sqrt{hPkA_C}\,(T_b - T_\infty)$

Use the thermal resistance concept:

$$q_f = \sqrt{hPkA_C}\,\tanh(mL)(T_b - T_\infty) = \frac{(T_b - T_\infty)}{R_{t,f}}$$

where $R_{t,f}$ is the thermal resistance of the fin.

For a fin with an adiabatic tip, the fin resistance can be expressed as

$$R_{t,f} = \frac{(T_b - T_\infty)}{q_f} = \frac{1}{\sqrt{hPkA_C}\,[\tanh(mL)]}$$

Fin Effectiveness

How effective a fin can enhance heat transfer is characterized by the fin effectiveness ε_f: Ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{\sqrt{\mathrm{hPkA_C}}\tanh(mL)}{hA_C} = \sqrt{\frac{kP}{hA_C}}\tanh(mL)$$

If the fin is long enough, mL>2, tanh(mL) $\rightarrow 1$,

it can be considered an infinite fi)

$$\varepsilon_f \rightarrow \sqrt{\frac{kP}{hA_C}} = \sqrt{\frac{k}{h}\left(\frac{P}{A_C}\right)}$$

In order to enhance heat transfer, $\varepsilon_f > 1$.

However, $\varepsilon_f \geq 2$ will be considered justifiable

If ε_f <1 then we have an insulator instead of a heat fin

Fin Effectiveness (cont.)

$$\varepsilon_f \to \sqrt{\frac{kP}{hA_C}} = \sqrt{\frac{k}{h}\left(\frac{P}{A_C}\right)}$$

• To increase ε_f, the fin's material should have higher thermal conductivity, k.

• It seems to be counterintuitive that the lower convection coefficient, h, the higher ε_f. But it is not because if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low. Observation: If fins are to be used on surfaces separating gas and liquid. Fins are usually placed on the gas side. (Why?)

• P/A_C should be as high as possible. Use a square fin with a dimension of W by W as an example: $P=4W$, $A_C=W^2$, $P/A_C=(4/W)$. The smaller W, the higher the P/A_C, and the higher ε_f.

•Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

Fin Effectiveness (cont.)

The effectiveness of a fin can also be characterized as

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}}$$

It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than that of the resistance due only to convection.

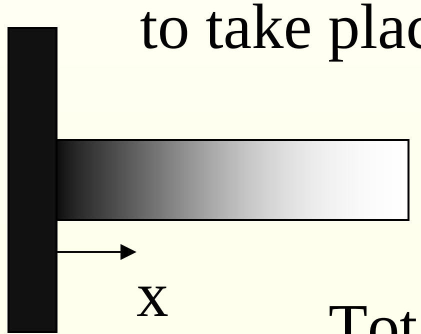
Fin Efficiency

Define Fin efficiency: $\eta_f = \dfrac{q_f}{q_{max}}$

where q_{max} represents an idealized situation such that the fin is made up of material with infinite thermal conductivity. Therefore, the fin should be at the same temperature as the temperature of the base.

$$q_{max} = hA_f(T_b - T_\infty)$$

$T(x) < T_b$ for heat transfer to take place



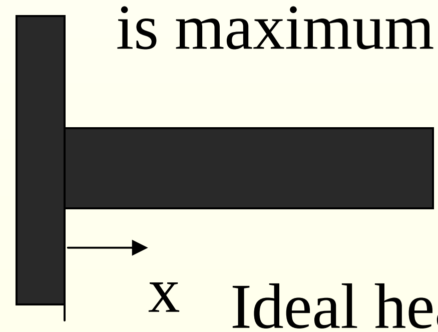
T_b

x

Total fin heat transfer q_f

Real situation

For infinite k
$T(x) = T_b$, the heat transfer is maximum



x

Ideal heat transfer q_{max}

Ideal situation

Fin Efficiency (cont.)

Use an adiabatic rectangular fin as an example:

$$\eta_f = \frac{q_f}{q_{\max}} = \frac{M \tanh mL}{hA_f(T_b - T_\infty)} = \frac{\sqrt{hPkA_c}(T_b - T_\infty)\tanh mL}{hPL(T_b - T_\infty)}$$

$$= \frac{\tanh mL}{\sqrt{\frac{hP}{kA_c}}L} = \frac{\tanh mL}{mL}$$ (see Table 3.5 for η_f of common fins)

The fin heat transfer: $q_f = \eta_f q_{\max} = \eta_f hA_f(T_b - T_\infty)$

$$q_f = \frac{T_b - T_\infty}{1/(\eta_f hA_f)} = \frac{T_b - T_\infty}{R_{t,f}}, \text{ where } R_{t,f} = \frac{1}{\eta_f hA_f}$$

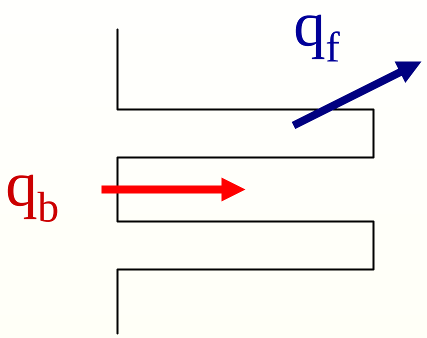
Thermal resistance for a single fin.

As compared to convective heat transfer: $R_{t,b} = \frac{1}{hA_b}$

In order to have a lower resistance as that is required to enhance heat transfer: $R_{t,b} > R_{t,f}$ or $A_b < \eta_f A_f$

Overall Fin Efficiency

Overall fin efficiency for an array of fins:



Define terms: A_b: base area exposed to coolant

A_f: surface area of a single fin

A_t: total area including base area and total finned surface, $A_t = A_b + NA_f$

N: total number of fins

$$q_t = q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty)$$

$$= h[(A_t - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_t - NA_f(1-\eta_f)](T_b - T_\infty)$$

$$= hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)](T_b - T_\infty) = \eta_O hA_t(T_b - T_\infty)$$

Define overall fin efficiency: $\eta_O = 1 - \frac{NA_f}{A_t}(1-\eta_f)$

Heat Transfer from a Fin Array

$$q_t = hA_t\eta_O(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,O}} \text{ where } R_{t,O} = \frac{1}{hA_t\eta_O}$$

Compare to heat transfer without fins

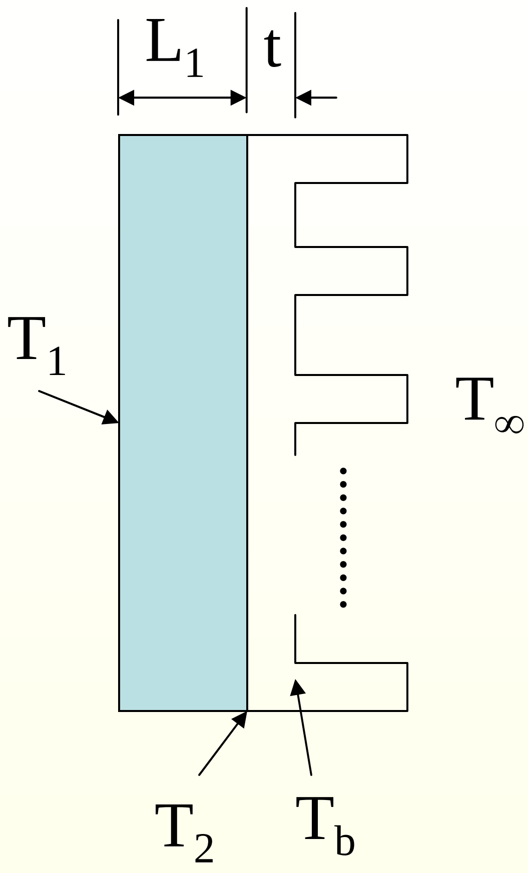
$$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$$

where $A_{b,f}$ is the base area (unexposed) for the fin

To enhance heat transfer $A_t\eta_O >> A = A_b + NA_{b,f}$

That is, to increase the effective area $\eta_O A_t$.

Thermal Resistance Concept



$A = A_b + NA_{b,f}$

$R_b = t/(k_b A)$

$R_1 = L_1/(k_1 A)$

$$R_{t,O} = 1/(hA_t\eta_O)$$

$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,O}}$$