SYSTEM MODELLING

Models are mathematical representations of system dynamics.

- Models allow the dynamics of systems to be simulated and analyzed, without having to build the system.
- Models are never exact but they can be predictive.

Different types of models are used for different purposes.

- Ordinary differential equations for rigid body mechanics.

- Finite-state machines (mathematical model of computation) for manufacturing, internet, information flow, vending machine, elevators, traffic lights.
- Partial differential equations for fluid flow, solid bodies, etc.

ADVANTAGES OF MODELS

Models can be used in ways the actual system can't.

- Certain types of analysis (e.g., parametric variations) can't easily be done on the actual system.
- In many cases, models can be run much more quickly than the actual models.

The model you use depends on the questions you want to answer.

1. A single system may have many models.
2. Time and spatial scale must be chosen to suit the questions you want to answer.
3. Always formulate questions before building a model.

STEPS IN MODELING

1. Understand the physical system and its components.
2. Make appropriate simplifying assumptions.
3. Use basic principles to formulate the mathematical model.
4. Write differential and algebraic equations describing the model.
5. Check the model for validity.

What will the model be used for?

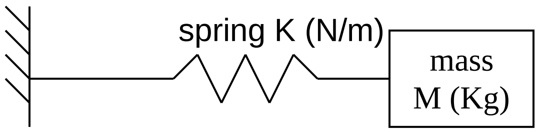
1. Solution of the differential and algebraic equations allows system response and performance to be analyzed and designed.
2. The Laplace transformation will be applied to the model to allow convenient manipulation and dynamic analysis.
3. Input – output relationships for systems and components will be obtained.

CONTROL COMPONENTS

This section presents methods for writing the differential equations for variety of electrical, mechanical, thermal, hydraulic systems, etc. the basic physical laws are given for each system, and the associated parameters are defined. The result is a differential equation or a set of differential equations that describes the system. The equations derived are limited to linear system or to systems that can be represented by linear equations over their useful operating range.

MECHANICAL SYSTEM

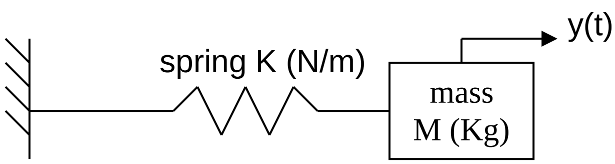
Consider a mass, M, on a frictionless surface connected to a rigid wall by a spring with stiffness, K.



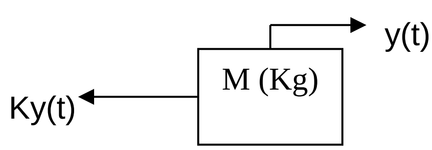
- Need a model for the position of the mass as a function of time.

ANALYSIS OF PHYSICAL SYSTEM

1. Choose a sign convention for the position variable y(t). Note that the sign convention for velocity and acceleration are the same as that for the displacement.



2. Use fundamental physical principles to model the system. (Newton's Law).
3. Draw free – body diagram of the system. In this example, the spring force is the only force acting on the mass



4. Spring exerts a force proportional to and in opposition to movement of the mass. From Newton's Law ∑F=Ma

$$-Ky(t) = M\frac{d^2y}{dt^2}$$

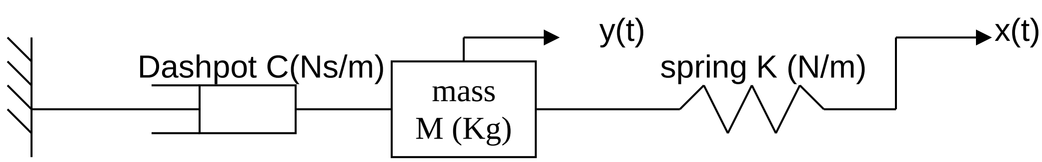
$$M\frac{d^2y}{dt^2}+Ky(t)=0$$

5. Obtain differential equation
$$\frac{M}{K}\frac{d^2y}{dt^2}+y(t)=0$$
$$\frac{1}{w_n^2}\frac{d^2y}{dt^2}+y(t)=0 \quad where\ w_n=\sqrt{\frac{K}{M}}$$

w_n is the undamped natural frequency

6. System characteristics $\frac{1}{w_n^2}\frac{d^2y}{dt^2}+y(t)=0$. The system has no input. No external force acts on the mass. In the differential equation, this is indicated by the zero on the right hand side. The system has no damping. There is no energy dissipation in the system.

SPRING – MASS DAMPER

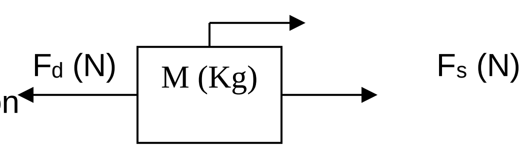


- Mass is perturbed by manipulating end position x(t) of the spring with stiffness K.
- Dashpot with damping co-efficient C resists motion in proportion to velocity

$$F_s=K(x(t)-y(t))$$
$$F_d=C\frac{dy}{dt}$$

FREE – BODY DIAGRAM

y(t)

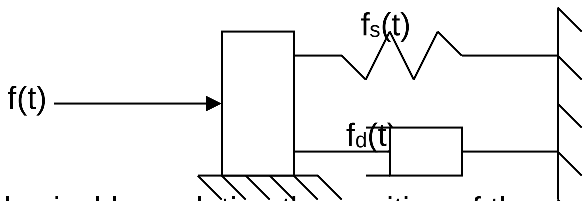


Expression equation

$$M\frac{d^2y}{dt^2}=F_s-F_d$$
$$M\frac{d^2y}{dt^2}=K(x(t)-y(t))-C\frac{dy}{dt}$$
$$\frac{1}{w_n^2}\frac{d^2y}{dt^2}+\frac{2\xi}{w_n}\frac{dy}{dt}+y(t)=x(t) \quad where\ w_n=\sqrt{\frac{K}{M}}\ rad/s \quad \xi=\frac{C}{2\sqrt{KM}}$$

MASS – SPRING – DAMPER

A frequently occurring physical arrangement that can be represented by a force acting on a mass which is restrained by spring and viscous damper is shown diagrammatically in fig.1. It is assumed that the mass is constrained to move in the direction of the applied force by friction-free guiding surface.



The basic physical law relating the position of the mass to the force acting upon it is Newton's second law of motion which states as the sum of the applied forces is equal to the rate of change of the momentum of the system.

$f(t) - f_s(t) - f_d(t) = M\frac{d^2x(t)}{dt^2}$ But spring force, $f_s(t) = Kx(t)$ and the damper force $f_d(t) = Cdx(t)/dt$

Therefore
$$f(t) = M\frac{d^2x(t)}{dt^2} + f_s(t) + f_d(t)$$
$$= M\frac{d^2x(t)}{dt^2} + C\frac{dx(t)}{dt} + Kx(t)$$

For zero initial conditions, the Laplace transform is $F(s) = Ms^2X(s) + CsX(s) + KX(s)$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + K}$$

The most general form of this equation, using dot notation for differentiation is

$\ddot{x}(t) + 2\xi w_n \dot{x}(t) + w_n^2 x(t) = w_n^2 u(t)$ Where u (t) is the forcing term and $w_n = \sqrt{\frac{K}{M}}$ is the system natural frequency and the symbol ξ is known as the damping factor. How such mass-spring-damper system react to a force, or respond to inputs of an electrical or other system were shown in the time response.

ELECTRICAL CIRCUITS AND COMPONENTS

The equations for an electric circuit obey Kirchhoff's Law, which state the following:

1. The algebraic sum of the potential differences around a closed path equals zero.
2. The algebraic sum of the currents entering or leaving a node is equal to zero. In other words, the sum of currents entering the junction equals the sum of the currents leaving the junction.

The voltage sources are usually alternating (AC) or direct current (DC) generators. The usual dc voltage source is a battery. The voltage drops appear across the three basic electrical elements: Resistors, Inductors, and Capacitors. These elements have constant components values.

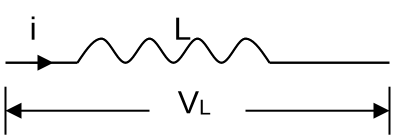
- The voltage drop across a Resistor is given by Ohm's Law, which states that the voltage drop across a resistor is equal to the product of the current through the resistor and its resistance. Resistors absorb energy from the system. Symbolically, this voltage is written as $V_R = Ri$
- The voltage drop across an inductor is given by Henry's Law, which is written as V_L= Ldi/dt = LDi.. This equation states that the voltage drop across an inductor is equal to the product of the inductance and the time rate of change of the current. A positive valued derivative Di, implies an increasing current, and thus a negative voltage.
- The voltage drop across a capacitor is defined by Faraday's Law. The positively directed voltage drop across a capacitor is defined as the ratio of the magnitude of the positive electric charge on its positive plate to the value of its capacitance. Its direction is from the positive plate to the negative plate. The charge on a capacitor plate is equal to the time integral of the current entering the plate from the initial instant to the arbitrary time, t, plus the initial value of the charge. The capacitor voltage is written in the form $$V_c = \frac{q}{C} = \frac{1}{C}\int_0^t i\,dt + \frac{Q_0}{C} = \frac{i}{CD}$$

Units for these electrical quantities in the practical system are given in the table below.

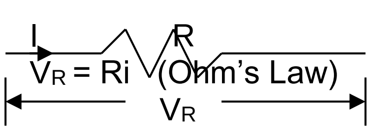
SYMBOL	QUANTITY	UNIT
e or V	Voltage	Volts
I	Current	Amperes
L	Inductance	Henrys
C	Capacitance	Farads
R	resistance	Ohms

Table 1: Electrical Symbols and Units

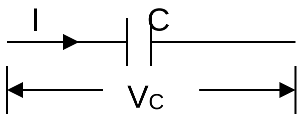
SUMMARY



V_L= Ldi/dt = LDi.. (Henry's Law)



$V_R = Ri$ (Ohm's Law)



$$V_c = \frac{q}{C} = \frac{1}{C}\int_0^t i\,dt + \frac{Q_0}{C} = \frac{i}{CD}$$ (Faraday's Law)

The ratio V/i = impedance, it is denoted by Z

- Z_L= LS   where S = d/dt. (Inductance)
- Z_R = R (Resistance)
- Z_c = 1/CS (Capacitance)

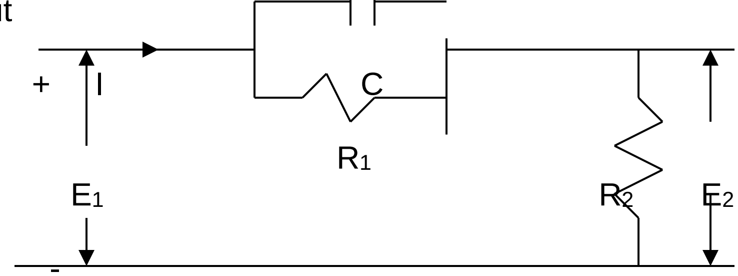
- For series impedance

$Z_s = Z_1 + Z_2 + Z_3 + \ldots\ldots..$

- For parallel impedance

$(1/Z_p)= (1/ Z_1 +1/ Z_2 + 1/Z_3 + \ldots\ldots..)$

Example: for the circuit below, determine the equation relating the output to the input



Solution

The parallel combination of R_1 and C_1 is in series with R_2, so that the total impedance Z is $Z = Z_1+Z_2$

E = ZI which implies Z = E/I

$$Z = \frac{1}{1/R_1 + CD_1} + R_2 = \frac{R_1}{1+R_1CD_1} + R_2$$

The voltages are given as

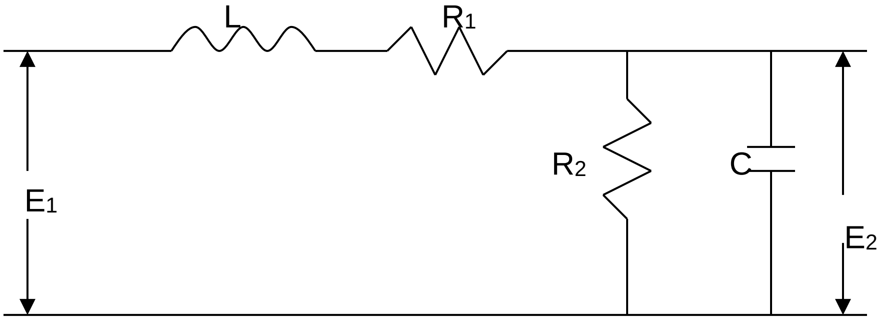
$$E_1 = ZI = \frac{R_1 + R_2 + R_1R_2CD_1}{1+R_1CD_1} I$$

$$E_2 = Z_2 I = R_2 I$$

$$\Rightarrow \frac{E_2}{E_1} = \frac{R_2(1+R_1CD_1)}{R_1 + R_2 + R_1R_2CD_1}$$

Example 2

Find the ratio of the output to the input for the circuit below



$$Z_p = \frac{1}{1/R_2 + CS} = \frac{R_2}{1 + R_2CS}$$

$$Z_s = R_1 + LS\,\frac{R_2}{1 + R_2CS}$$

$$E_2 = Z_pI = \frac{R_2}{1 + R_2CS}I$$

$$E_1 = Z_sI = (R_1 + LS\,\frac{R_2}{1 + R_2CS})I$$

$$\frac{E_2}{E_1} = \frac{R_2}{(LR_2C)S^2 + (CR_1R_2 + L)S + (R_1 + R_2)}$$

THEMAL SYSTEMS

- Liquid in glass thermometer: For a simple thermometer (or equivalent temperature measuring device) there is relationship between the indicated temperature and the temperature being measured. Let $\theta_i(t)$ be the temperature of the fluid around the bulb, or the input, and $\theta_0(t)$ the temperature of the fluid in the thermometer be the system output. The thermometer fluid volume will vary in proportion to its temperature, and the stem is graduated accordingly. The temperatures are both time varying and hence functions of time, t.

The rate of heat flow, q (t), into the thermometer fluid is proportional to the temperature difference across the walls.

$q(t) = \dfrac{\theta_i(t) - \theta_0(t)}{K_1}$ Where K_1 is the thermal resistance and is determined by the coefficients of heat transfer from fluid to glass, through the glass and from glass to inner fluid. Also the rate of heat flow, q(t), is proportional to the rate of temperature rise of the thermometer fluid.

$q(t) = Cm\dfrac{d\theta_0(t)}{dt}$ Where C is the specific heat and m is the mass of the thermometer fluid. $\theta_i(t) - \theta_0(t) = K_2 \dfrac{d\theta_0(t)}{dt}$ *where* $K_2 = K_1 Cm$

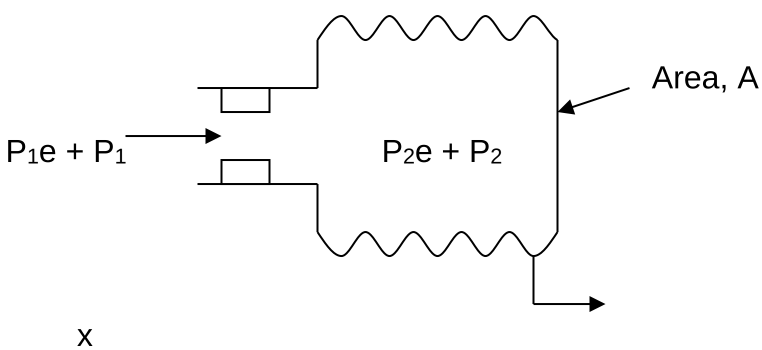
$$\theta_i(s) - \theta_0(s) = K_2 s \theta_0(s)$$

Taking Laplace transform of both sides $\dfrac{\theta_0(s)}{\theta_i(s)} = \dfrac{1}{1 + K_2 s}$

NB:

1. The thermal capacity of the glass has been assumed negligible and the overall coefficient of heat transfer assumed to be constant.
2. Also, the parameters have been considered to be lumped, which means that the thermometer fluid temperature has been assumed to be uniform in a spatial sense, as has the temperature of the fluid being measured. If the temperatures were to be considered as functions of both time and position, it would be necessary to describe the system by partial differential equations, and it would be termed as distributed parameter system.

PNEUMATIC BELLOWS



For equilibrium position $P_1e = P_2e$

Let $P_1(t)$ and $P_2(t)$ and $x(t)$ denote changes from equilibrium position. The mass flow rate is given by

$$\dot{m}_1 = \frac{P_1 - P_2}{R} \quad \textit{where R is the pneumatic resistance}$$

$$\dot{m}_2 = 0$$

$$\dot{m}_1 = \frac{cdP_2}{dt} = \frac{P_1 - P_2}{R}$$

$$AP_2 = Kx$$

$$RC\frac{P_2}{dt} + P_2 = P_1 \quad \textit{taking Laplace transfrom of both sides}$$

$$(RCS+1)P_2(s) = P_1(s)$$

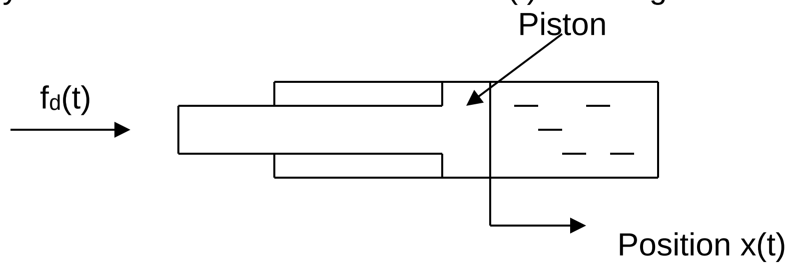
$$(RCS+1)\frac{K}{A}X(s) = P_1(s)$$

$$\tau = RC \qquad Kg = \frac{A}{K}$$

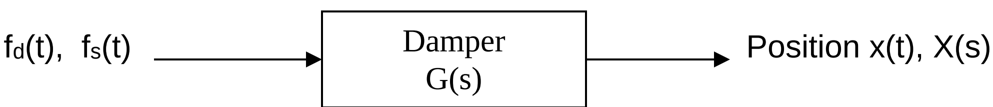
$$\Rightarrow (\tau S+1)X(s) = KgP_1(s)$$

IDEAL HYDRAULIC DAMPER

Consider a piston of negligible mass sliding with some clearance in an oil-filled cylinder under the action of a force $f_d(t)$ in the figure below.



Assumptions: there is a free flow through the piston so that no pressure difference can arise across the piston.

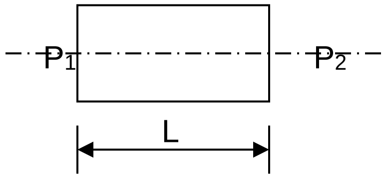


There will be a relationship between the position of the position x(t) and the applied force. Using basic physical considerations that the viscous drag on the piston is proportional to the velocity of the piston in the cylinder, $f_d(t) = Cdx(t)/dt$, where C is the viscous damping coefficient with units of force per unit velocity.

Taking Laplace transform gives, for zero initial conditions $F_d(s) = CsX(s)$. Hence the transfer function is $G(s) = \frac{X(s)}{F_d(s)} = \frac{1}{Cs}$

HYDRAULIC INERTANCE

Inertance force required to accelerate a column of fluid in pipe is $A(P_1 - P_2) = mdv/dt$



$$If\ \ m = \rho AL \qquad then\ \ A(P_1 - P_2) = \rho AL\frac{dv}{dt}$$

$$\dot{q} = A\frac{dv}{dt} = A\dot{v} \quad \Rightarrow (P_1 - P_2) = \frac{\rho L}{A}\frac{dq}{dt}$$

taking laplace of the above equation, we have

$$P_1 - P_2) = P(s) = IsQ(s)\ where\ I\ is\ the\ hydraulic\ iner\tan ce\ given\,by\ I = \frac{\rho L}{A}$$

CONTROLLER DESIGN REQUIREMENTS

1. closed loop stability
2. decreased sensitivity to plant variations
3. steady state error
4. dynamical response
5. disturbance rejection
6. robust stability