CHAPTER 44

Multiple Integrals and their Applications

44.1 Evaluate the iterated integral $I=\int_2^3\int_1^5 (x+2y)\,dx\,dy$.

$\int_1^5 (x+2y)\,dx = (\frac{1}{2}x^2+2yx)\big]_1^5 = (\frac{25}{2}+10y)-(\frac{1}{2}+2y) = 12+8y$. Therefore, $I=\int_2^3 (12+8y)\,dy = (12y+4y^2)\big]_2^3 = (36+36)-(24+16)=32$.

44.2 Evaluate the iterated integral $I=\int_0^1\int_{x^2}^{x^3} (x^2+y^2)\,dy\,dx$.

$\int_{x^2}^{x^3}(x^2+y^2)\,dy = (x^2y+\frac{1}{3}y^3)\big]_{x^2}^{x^3} = (x^5+\frac{1}{3}x^9)-(x^4+\frac{1}{3}x^6)$. Therefore, $I=\int_0^1 (x^5+\frac{1}{3}x^9-x^4-\frac{1}{3}x^6)\,dx = \frac{1}{6}x^6+\frac{1}{30}x^{10}-\frac{1}{5}x^5-\frac{1}{21}x^7\big]_0^1 = \frac{1}{6}+\frac{1}{30}-\frac{1}{5}-\frac{1}{21}=-\frac{1}{21}$.

44.3 Evaluate the iterated integral $I=\int_{-\pi}^{\pi}\int_0^2 r\sin\theta\,dr\,d\theta$.

$\int_0^2 r\sin\theta\,dr = \frac{1}{2}r^2\sin\theta\big]_0^2 = 2\sin\theta$. Therefore, $I=\int_{-\pi}^{\pi} 2\sin\theta\,d\theta = 0$ (Problem 20.48).

44.4 Evaluate the iterated integral $I=\int_0^{\pi/2}\int_0^{\cos\theta} \rho^2\sin\theta\,d\rho\,d\theta$.

$\int_0^{\cos\theta}\rho^2\sin\theta\,d\rho = \frac{1}{3}\rho^3\sin\theta\big]_0^{\cos\theta} = \frac{1}{3}\cos^3\theta\sin\theta$. Hence, $I=\int_0^{\pi/2}\frac{1}{3}\cos^3\theta\sin\theta\,d\theta = -\frac{1}{12}\cos^4\theta\big]_0^{\pi/2} = -\frac{1}{12}[\cos^4(\pi/2)-\cos^4 0] = -\frac{1}{12}(0-1)=\frac{1}{12}$.

44.5 Evaluate the iterated integral $I=\int_0^1\int_0^z\int_0^y (x+y+z)\,dx\,dy\,dz$.

$\int_0^y (x+y+z)\,dx = \frac{1}{2}x^2+(y+z)x\big]_0^y = \frac{1}{2}y^2+(y+z)z$. Hence, $\int_0^z\int_0^y (x+y+z)\,dx\,dy = \int_0^z(\frac{1}{2}y^2+yz+z^2)\,dy = \frac{1}{6}y^3+\frac{1}{2}y^2z+z^2y\big]_0^z = \frac{1}{6}z^3+\frac{1}{2}z^3+z^3=\frac{5}{3}z^3$. Therefore, $I=\int_0^1\frac{5}{3}z^3\,dz = \frac{5}{12}z^4\big]_0^1=\frac{5}{12}$.

44.6 Evaluate $I=\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\,dx\,dy$.

In this case the double integral may be replaced by a product: $I=(\int_0^{\ln 3}e^x\,dx)(\int_0^{\ln 4}e^y\,dy)=(3-1)(4-1)=6$. (See Problem 44.71.)

44.7 Evaluate $\int_1^2\int_0^y x\sqrt{y^2-x^2}\,dx\,dy$.

$\int_0^y x\sqrt{y^2-x^2}\,dx = -\frac{1}{2}\cdot\frac{2}{3}(y^2-x^2)^{3/2}\big]_0^y = -\frac{1}{3}(y^2-x^2)^{3/2}\big]_0^y = -\frac{1}{3}[-(y^2)^{3/2}] = \frac{1}{3}y^3$. Therefore, $I=\int_1^2\frac{1}{3}y^3\,dy = \frac{1}{12}y^4\big]_1^2 = \frac{1}{12}(16-1)=\frac{15}{12}=\frac{5}{4}$.

44.8 Evaluate $I=\int_0^1\int_y^1 e^{x^2}\,dx\,dy$.

$\int e^{x^2}\,dx$ cannot be evaluated in terms of standard functions. Therefore, we change the order of integration, using Fig. 44-1. $I=\int_0^1\int_0^x e^{x^2}\,dy\,dx = \int_0^1 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\big]_0^1 = \frac{1}{2}(e-1)$.

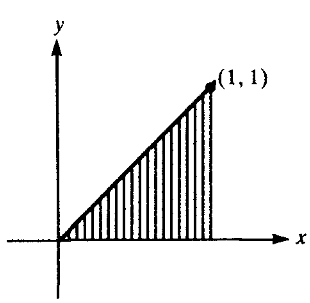

Fig. 44-1

44.9 Evaluate $\int_0^{\pi/2}\int_0^{\cos y} e^x\sin y\,dx\,dy$.

$\int_0^{\cos y} e^x\sin y\,dx = (\sin y)(e^x)\big]_0^{\cos y} = (\sin y)(e^{\cos y}-1)$. Therefore, $I=\int_0^{\pi/2}[(\sin y)e^{\cos y}-\sin y]\,dy = (-e^{\cos y}+\cos y)\big]_0^{\pi/2} = (-e^0+0)-(-e+1)=e-2$.

44.10 Evaluate $I = \int_0^1 \int_{y^4}^{y^2} \sqrt{y/x}\, dx\, dy$.

$\int_{y^4}^{y^2} \sqrt{y/x}\, dx = 2\sqrt{y}\sqrt{x}\,\Big]_{y^4}^{y^2} = 2\sqrt{y}(y - y^2) = 2(y^{3/2} - y^{5/2})$. Therefore, $I = 2\int_0^1 (y^{3/2} - y^{5/2})\, dy = 2(\frac{2}{5}y^{5/2} - \frac{2}{7}y^{7/2})\,\Big]_0^1 = 2(\frac{2}{5} - \frac{2}{7}) = \frac{8}{35}$.

44.11 Evaluate $I = \iint_{\mathscr{R}} x\, dA$, where $\mathscr{R}$ is the region bounded by $y = x$ and $y = x^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$, and, for $0 < x < 1$, $y = x$ is above $y = x^2$ (see Fig. 44-2). $I = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 xy\,\Big]_{x^2}^{x} dx = \int_0^1 (x^2 - x^3)\, dx = (\frac{1}{3}x^3 - \frac{1}{4}x^4)\,\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

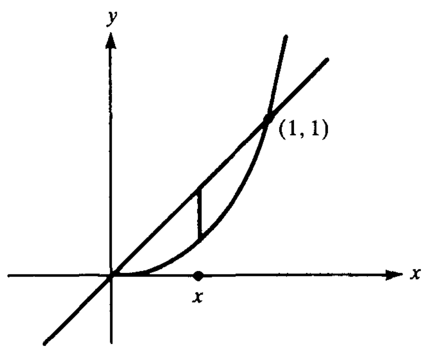

Fig. 44-2

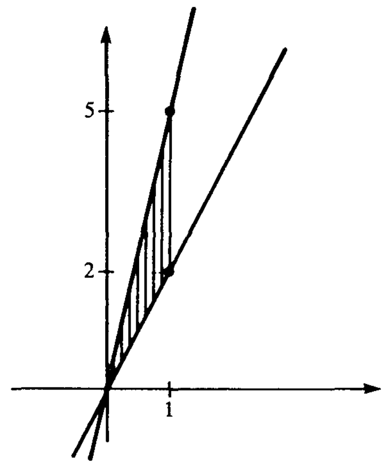

Fig. 44-3

44.12 Evaluate $I = \iint_{\mathscr{R}} y^2\, dA$, where $\mathscr{R}$ is the region bounded by $y = 2x$, $y = 5x$, and $x = 2$.

The lines $y = 2x$ and $y = 5x$ intersect at the origin. For $0 < x \le 1$, the region runs from $y = 2x$ up to $y = 5x$ (Fig. 44-3). Hence, $I = \int_0^1 \int_{2x}^{5x} y^2\, dy\, dx = \int_0^1 \frac{1}{3}y^3\,\Big]_{2x}^{5x} dx = \frac{1}{3}\int_0^1 (125x^3 - 8x^3)\, dx = \frac{1}{3}\int_0^1 117x^3\, dx = \frac{39}{4}x^4\,\Big]_0^1 = \frac{39}{4}$.

44.13 Evaluate $I = \iint_{\mathscr{R}} (x - y)\, dA$, where $\mathscr{R}$ is the region above the x-axis bounded by $y^2 = 3x$ and $y^2 = 4 - x$ (see Fig. 44-4).

It is convenient to evaluate I by means of strips parallel to the x-axis. $I = \int_0^{\sqrt{3}} \int_{y^2/3}^{4-y^2} (x - y)\, dx\, dy = \int_0^{\sqrt{3}} (\frac{1}{2}x^2 - yx)\,\Big]_{y^2/3}^{4-y^2} dy = \int_0^{\sqrt{3}} [\frac{1}{2}(4 - y^2)^2 - y(4 - y^2)] - [\frac{1}{2}(y^2/3)^2 - y^3/3]\, dy = \int_0^{\sqrt{3}} (8 - 4y^2 + \frac{1}{2}y^4 - 4y + y^3 - \frac{1}{18}y^4 - \frac{1}{3}y^3)\, dy = \int_0^{\sqrt{3}} (8 - 4y - 4y^2 + \frac{2}{3}y^3 + \frac{4}{9}y^4)\, dy = 8y - 2y^2 - \frac{4}{3}y^3 + \frac{1}{6}y^4 + \frac{4}{45}y^5\,\Big]_0^{\sqrt{3}} = 8\sqrt{3} - 6 - 4\sqrt{3} + \frac{3}{2} + \frac{4}{5}\sqrt{3} = \frac{24}{5}\sqrt{3} - \frac{9}{2}$.

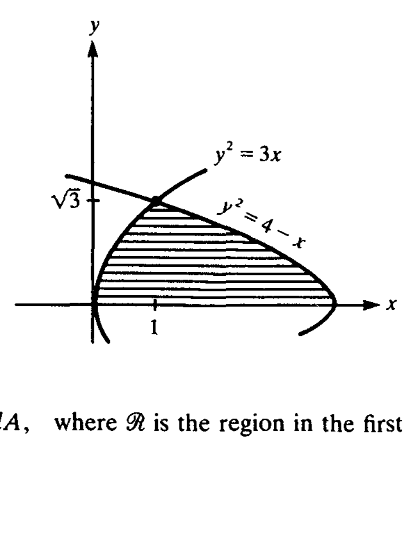

Fig. 44-4

44.14 Evaluate $I = \iint_{\mathscr{R}} \dfrac{1}{\sqrt{2y - y^2}}\, dA$, where $\mathscr{R}$ is the region in the first quadrant bounded by $x^2 = 4 - 2y$.

The curve $x^2 = 4 - 2y$ is a parabola with vertex at $(0,2)$ and passing through the x-axis at $x = 2$ (Fig. 44-5). Hence, $I = \int_0^2 \int_0^{\sqrt{4-2y}} \frac{1}{\sqrt{2y - y^2}}\, dx\, dy = \int_0^2 \frac{x}{\sqrt{2y-y^2}} \Big]_0^{\sqrt{4-2y}} dy = \int_0^2 \frac{\sqrt{4-2y}}{\sqrt{2y-y^2}}\, dy = \int_0^2 \frac{\sqrt{2}}{\sqrt{y}} \frac{\sqrt{2-y}}{\sqrt{2-y}}\, dy = \sqrt{2} \int_0^2 y^{-1/2}\, dy = \sqrt{2} \cdot 2y^{1/2} \Big]_0^2 = 2\sqrt{2}(\sqrt{2} - 0) = 4$. Note that, if we integrate using strips parallel to the y-axis, the integration is difficult.

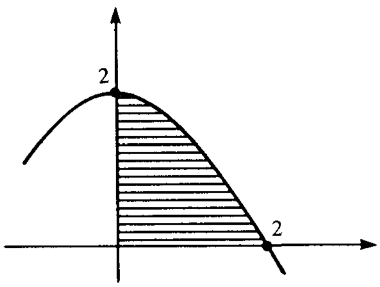

Fig. 44-5

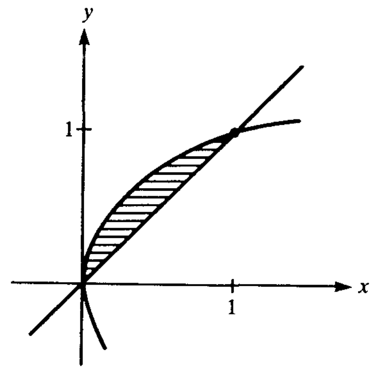

Fig. 44-6

44.15 Let $\mathscr{R}$ be the region bounded by the curve $y = \sqrt{x}$ and the line $y = x$ (Fig. 44-6). Let $f(x, y) = \frac{\sin y}{y}$ if $y \neq 0$ and $f(x, 0) = 1$. Compute $I = \iint_{\mathscr{R}} f(x, y)\, dA$.

$I = \int_0^1 \int_{y^2}^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^1 \frac{\sin y}{y} x \Big]_{y^2}^{y} dy = \int_0^1 (\sin y - y \sin y)\, dy$. Integration by parts yields $\int y \sin y\, dy = \sin y - y \cos y$. Hence, $I = (-\cos y + y \cos y - \sin y)\,\big]_0^1 = (-\sin 1) - (-1) = 1 - \sin 1$.

44.16 Find the volume V under the plane $z = 3x + 4y$ and over the rectangle $\mathscr{R}$: $1 \le x \le 2$, $0 \le y \le 3$.

$V = \iint_{\mathscr{R}} (3x + 4y)\, dA = \int_0^3 \int_1^2 (3x + 4y)\, dx\, dy = \int_0^3 (\tfrac{3}{2}x^2 + 4yx)\,\big]_1^2\, dy = \int_0^3 [(6 + 8y) - (\tfrac{3}{2} + 4y)]\, dy$

$= \int_0^3 (\tfrac{9}{2} + 4y)\, dy = (\tfrac{9}{2}y + 2y^2)\,\big]_0^3 = \tfrac{27}{2} + 18 = \tfrac{63}{2}$.

44.17 Find the volume V in the first octant bounded by $z = y^2$, $x = 2$, and $y = 4$.

$V = \iint_{\mathscr{R}} y^2\, dA = \int_0^2 \int_0^4 y^2\, dy\, dx = \int_0^2 \tfrac{1}{3}y^3\,\big]_0^4\, dx = \int_0^2 \tfrac{64}{3}\, dx = \tfrac{64}{3} \cdot 2 = \tfrac{128}{3}$.

44.18 Find the volume V of the solid in the first octant bounded by $y = 0$, $z = 0$, $y = 3$, $z = x$, and $z + x = 4$ (Fig. 44-7).

For given x and y, the z-value in the solid varies from $z = x$ to $z = -x + 4$. So $V = \int_0^3 \int_0^2 [(-x + 4) - x]\, dx\, dy = \int_0^3 \int_0^2 (4 - 2x)\, dx\, dy = \int_0^3 (4x - x^2)\,\big]_0^2\, dy = \int_0^3 (8 - 4)\, dy = 4 \int_0^3 dy = 4 \cdot 3 = 12$.

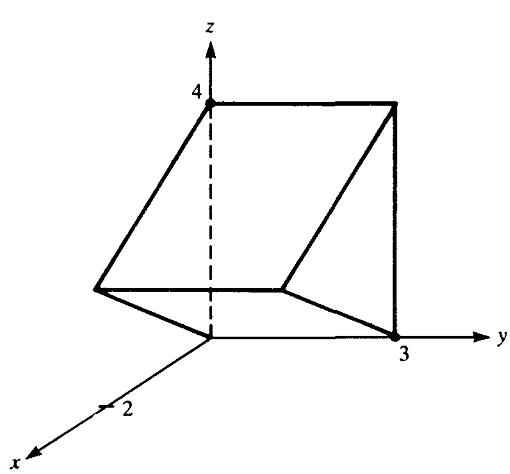

Fig. 44-7

44.19 Find the volume V of the tetrahedron bounded by the coordinate planes and the plane $z = 6 - 2x + 3y$.

As shown in Fig. 44-8, the solid lies above the triangle in the xy-plane bounded by $2x + 3y = 6$ and the x and y axes. $V = \int_0^3 \int_0^{2-2x/3} (6 - 2x - 3y)\,dy\,dx = \int_0^3 6y - 2xy - \frac{3}{2}y^2 \Big]_0^{2-2x/3} dx = \int_0^3 (2 - \frac{2}{3}x)(6 - 2x - 3 + x)\,dx = \int_0^3 \frac{2}{3}(3 - x)(3 - x)\,dx = \frac{2}{3}\int_0^3 (3 - x)^2\,dx = \frac{2}{3}(-\frac{1}{3})(3 - x)^3 \Big]_0^3 = -\frac{2}{9}(-3^3) = 6$. (Check against the formula $V = \frac{1}{6}abc$.)

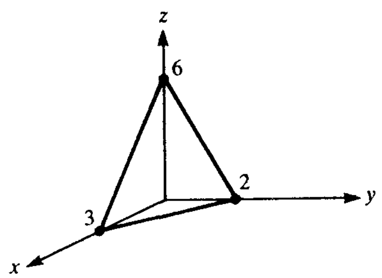

Fig. 44-8

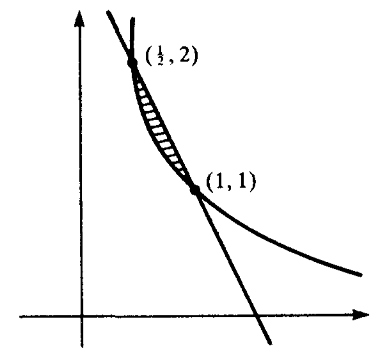

Fig. 44-9

44.20 Use a double integral to find the area of the region $\mathscr{R}$ bounded by $xy = 1$ and $2x + y = 3$.

Figure 44-9 shows the region $\mathscr{R}$. $A = \iint_{\mathscr{R}} 1\,dA = \int_{1/2}^1 \int_{1/x}^{3-2x} 1\,dy\,dx = \int_{1/2}^1 (3 - 2x - 1/x)\,dx = 3x - x^2 - \ln x \Big]_{1/2}^1 = (3 - 1 - 0) - (\frac{3}{2} - \frac{1}{4} - \ln\frac{1}{2}) = 2 - (\frac{5}{4} + \ln 2) = \frac{3}{4} - \ln 2$.

44.21 Find the volume V of the solid bounded by the right circular cylinder $x^2 + y^2 = 1$, the xy-plane, and the plane $x + z = 1$.

As seen in Fig. 44-10, the base is the circle $x^2 + y^2 = 1$ in the xy-plane, the top is the plane $x + z = 1$. $V = \iint_{\mathscr{R}} (1 - x)\,dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x)\,dy\,dx = \int_{-1}^1 (1 - x)y \Big]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^1 2(1 - x)\sqrt{1 - x^2}\,dx = 2\int_{-1}^1 \sqrt{1 - x^2}\,dx - 2\int_{-1}^1 x\sqrt{1 - x^2}\,dx = 2(\pi/2) + \frac{2}{3}(1 - x^2)^{3/2} \Big]_{-1}^1 = \pi + \frac{2}{3}(0) = \pi$. (Note: We know that $\int_{-1}^1 \sqrt{1 - x^2}\,dx = \pi/2$, since the integral is the area of the unit semicircle.)

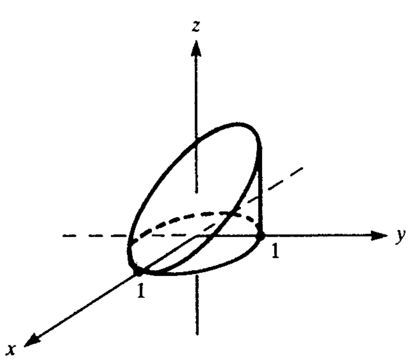

Fig. 44-10

44.22 Find the volume V of the solid bounded above by the plane $z = 3x + y + 6$, below by the xy-plane, and on the sides by $y = 0$ and $y = 4 - x^2$.

Since $-2 \le x \le 2$ and $y \ge 0$, we have $z = 3x + y + 6 \ge 0$. Then $V = \int_{-2}^2 \int_0^{4-x^2} (3x + y + 6)\,dy\,dx = \int_{-2}^2 (3xy + \frac{1}{2}y^2 + 6y) \Big]_0^{4-x^2} dx = \int_{-2}^2 [3x(4 - x^2) + \frac{1}{2}(4 - x^2)^2 + 6(4 - x^2)]\,dx = \int_{-2}^2 (32 + 12x - 10x^2 - 3x^3 + \frac{1}{2}x^4)\,dx = (32x + 6x^2 - \frac{10}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{10}x^5) \Big]_{-2}^2 = (64 + 24 - \frac{80}{3} - 12 + \frac{32}{10}) - (-64 + 24 + \frac{80}{3} - 12 - \frac{32}{10}) = \frac{1216}{15}$.

44.23 Find the volume of the wedge cut from the elliptical cylinder $9x^2 + 4y^2 = 36$ by the planes $z = 0$ and $z = y + 3$.

On $9x^2 + 4y^2 = 36$, $-3 \le y \le 3$. Hence, $z = y + 3 \ge 0$. So the plane $z = y + 3$ will be above the plane $z = 0$ (see Fig. 44-11). Since the solid is symmetric with respect to the yz-plane, $V = 2\int_{-3}^3 \int_0^{2\sqrt{9-y^2}/3} (y + 3)\,dx\,dy = 2\int_{-3}^3 (y + 3)x \Big]_0^{2\sqrt{9-y^2}/3} dy = 2\int_{-3}^3 (y + 3)\cdot\frac{2}{3}\sqrt{9 - y^2}\,dy = \frac{4}{3}\int_{-3}^3 y\sqrt{9 - y^2}\,dy + 4\int_{-3}^3 \sqrt{9 - y^2}\,dy = 0 + 4\cdot\frac{1}{2}(9\pi) = 18\pi$. [The integral $\int_{-3}^3 \sqrt{9 - y^2}\,dy$ represents the area of the upper semicircle of the circle $x^2 + y^2 = 9$. Hence, it is equal to $\frac{1}{2}\cdot\pi(3)^2 = \frac{1}{2}\cdot 9\pi$.]

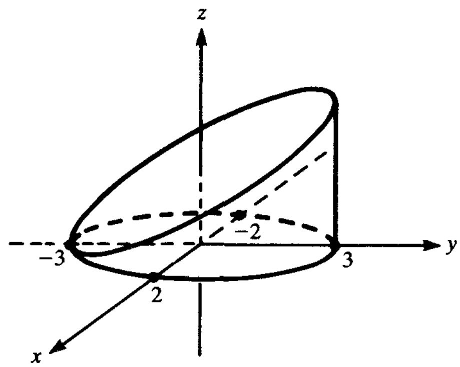

Fig. 44-11

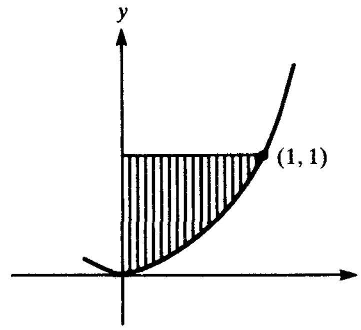

Fig. 44-12

44.24 Express the integral $I = \int_0^1 \int_0^{\sqrt{y}} f(x, y)\, dx\, dy$ as an integral with the order of integration reversed.

In the region of integration, the x-values for $0 \le y \le 1$ range from 0 to $\sqrt{y}$. Hence, the bounding curve is $x = \sqrt{y}$, or $y = x^2$. Thus (see Fig. 44-12), $I = \int_0^1 \int_{x^2}^1 f(x, y)\, dy\, dx$.

44.25 Express the integral $I = \int_0^4 \int_{x/2}^2 f(x, y)\, dy\, dx$ as an integral with the order of integration reversed.

For $0 \le x \le 4$, the region of integration runs from $x/2$ to 2. Hence, the region of integration is the triangle indicated in Fig. 44-13. So, if we use strips parallel to the x-axis, $I = \int_0^2 \int_0^{2y} f(x, y)\, dx\, dy$.

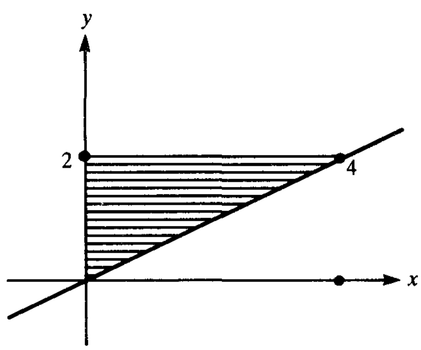

Fig. 44-13

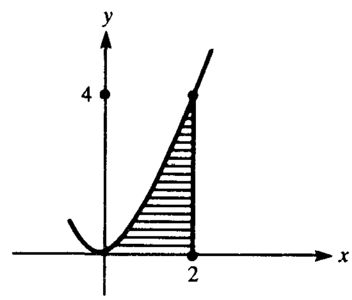

Fig. 44-14

44.26 Express $I = \int_0^2 \int_0^{x^2} f(x, y)\, dy\, dx$ as a double integral with the order of integration reversed.

The region of integration is bounded by $y = 0$, $x = 2$, and $y = x^2$ (Fig. 44-14). $I = \int_0^4 \int_{\sqrt{y}}^2 f(x, y)\, dx\, dy$.

44.27 Express $I = \int_0^{\pi/2} \int_0^{\cos x} x^2\, dy\, dx$ as double integral with the order of integration reversed and compute its value.

The region of integration is bounded by $y = \cos x$, $y = 0$, and $x = 0$ (Fig. 44-15). So $I = \int_0^1 \int_0^{\cos^{-1} y} x^2\, dx\, dy$. The original form is easier to calculate. $I = \int_0^{\pi/2} x^2 y \,\Big]_0^{\cos x} dx = \int_0^{\pi/2} x^2 \cos x\, dx$. Two integrations by parts yields $\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x$. Hence, $I = (x^2 \sin x + 2x \cos x - 2 \sin x)\,\Big]_0^{\pi/2} = \pi^2/4 - 2$.

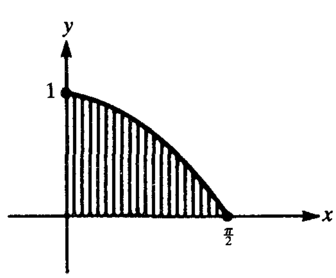

Fig. 44-15

44.28 Find $I = \iint_{\mathscr{R}} e^{x^3}\, dA$, where $\mathscr{R}$ is the region bounded by $y = x^2$, $x = 3$, and $y = 0$.

Use strips parallel to the y-axis (see Fig. 44-16). $I = \int_0^3 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^3 e^{x^3} y \,\Big]_0^{x^2} dx = \int_0^3 e^{x^3} x^2\, dx = \frac{1}{3} e^{x^3} \,\Big]_0^3 = \frac{1}{3}(e^{27} - e^0) = \frac{1}{3}(e^{27} - 1)$. Note that the integral with the variables in reverse order would have been impossible to calculate.

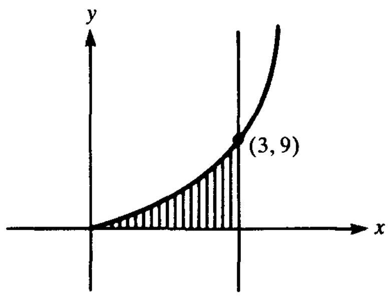

Fig. 44-16

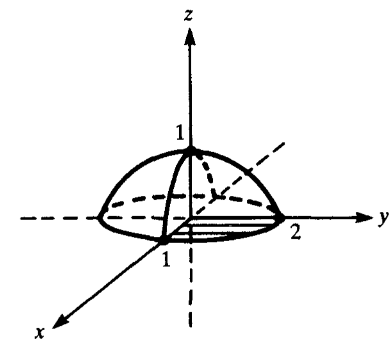

Fig. 44-17

44.29 Find the volume cut from $4x^2 + y^2 + 4z = 4$ by the plane $z = 0$.

The elliptical paraboloid $4x^2 + y^2 + 4z = 4$ has its vertex at $(0, 0, 1)$ and opens downward. It cuts the xy-plane in an ellipse, $4x^2 + y^2 = 4$, which is the boundary of the base $\mathscr{R}$ of the solid whose volume is to be computed (see Fig. 44-17). Because of symmetry, we need to integrate only over the first-quadrant portion of $\mathscr{R}$ and then multiply by 4. $V = 4 \cdot \frac{1}{4} \int_0^1 \int_0^{\sqrt{4-4x^2}} (4 - 4x^2 - y^2)\, dy\, dx = \int_0^1 [(4 - 4x^2)y - \frac{1}{3}y^3] \,\Big]_0^{\sqrt{4-4x^2}} dx = \int_0^1 \sqrt{4 - 4x^2}[4 - 4x^2 - \frac{1}{3}(4 - 4x^2)]\, dx = \int_0^1 \frac{2}{3}(4 - 4x^2)^{3/2}\, dx = \frac{2}{3}\int_0^1 8(1 - x^2)^{3/2}\, dx = \frac{16}{3}\int_0^1 (1 - x^2)^{3/2}\, dx$. Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$. Then $V = \frac{16}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \dfrac{16}{3}\displaystyle\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta = \frac{4}{3}\int_0^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta = \dfrac{4}{3}\displaystyle\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) d\theta = \frac{4}{3}(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta) \,\Big]_0^{\pi/2} = \frac{4}{3}(3\pi/4) = \pi$.

44.30 Find the volume in the first octant bounded by $x^2 + z = 64$, $3x + 4y = 24$, $x = 0$, $y = 0$, and $z = 0$.

See Fig. 44-18. The roof of the solid is given by $z = 64 - x^2$. The base $\mathscr{R}$ is the triangle in the first quadrant of the xy-plane bounded by the line $3x + 4y = 24$ and the coordinate axes. Hence, $V = \iint_{\mathscr{R}} (64 - x^2)\, dA = \int_0^8 \int_0^{(24-3x)/4} (64 - x^2)\, dy\, dx = \int_0^8 (64 - x^2) \,\Big]_0^{(24-3x)/4} dx = \int_0^8 (64 - x^2) \cdot \frac{3}{4}(8 - x)\, dx = \frac{3}{4}\int_0^8 (512 - 64x - 8x^2 + x^3)\, dx = \frac{3}{4}(512x - 32x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4) \,\Big]_0^8 = \frac{3}{4}(2^{12} - 2^{11} - \frac{1}{3} \cdot 2^{12} + \frac{1}{4} \cdot 2^{12}) = 3 \cdot 2^{10}(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}) = 3 \cdot 2^{10} \cdot \frac{5}{12} = 2^8 \cdot 5 = 1280$.

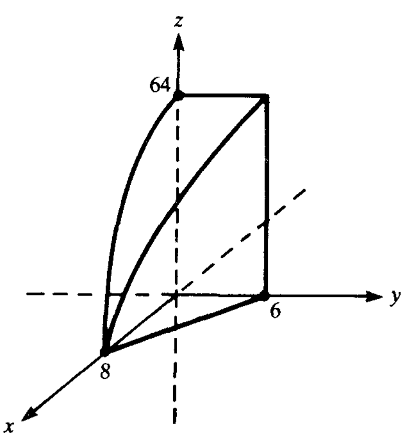

Fig. 44-18

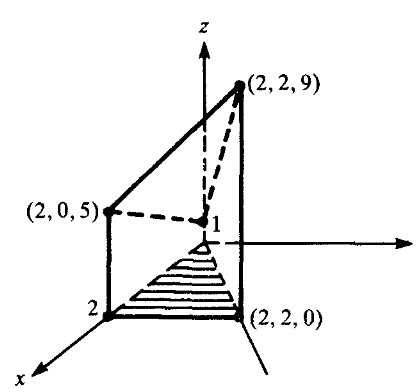

Fig. 44-19

44.31 Find the volume in the first octant bounded by $2x + 2y - z + 1 = 0$, $y = x$, and $x = 2$.

See Fig. 44-19. $V = \iint_{\mathscr{R}} (2x + 2y + 1)\, dA = \int_0^2 \int_0^x (2x + 2y + 1)\, dy\, dx = \int_0^2 (2xy + y^2 + y) \,\Big]_0^x dx = \int_0^2 (3x^2 + x)\, dx = (x^3 + \frac{1}{2}x^2) \,\Big]_0^2 = 8 + 2 = 10$.

44.32 Find the volume of a wedge cut from the cylinder $4x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = my$.

The base is the semidisk $\mathcal{R}$ bounded by the ellipse $4x^2 + y^2 = a^2$, $y \ge 0$. Because of symmetry, we need only double the first-octant volume. Thus, $V = 2\int_0^{a/2}\int_0^{\sqrt{a^2-4x^2}} my\,dy\,dx = m\int_0^{a/2} y^2\Big]_0^{\sqrt{a^2-4x^2}} dx = m\int_0^{a/2}(a^2 - 4x^2)\,dx = m(a^2x - \frac{4}{3}x^3)\Big]_0^{a/2} = m\cdot(a/2)[a^2 - \frac{4}{3}(a^2/4)] = (ma/2)(\frac{2}{3}a^2) = (m/3)a^3$.

44.33 Find $I = \iint_{\mathcal{R}} \sin\theta\,dA$, where $\mathcal{R}$ is the region outside the circle $r = 1$ and inside the cardioid $r = 1 + \cos\theta$ (see Fig. 44-20).

For polar coordinates, recall that the factor r is introduced into the integrand via $dA = r\,dr\,d\theta$. By symmetry, we can restrict the integration to the first quadrant and double the result. $I = 2\int_0^{\pi/2}\int_1^{1+\cos\theta}(\sin\theta)\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{2}(\sin\theta)r^2\Big]_1^{1+\cos\theta} d\theta = \int_0^{\pi/2}[(1+\cos\theta)^2\sin\theta - \sin\theta]\,d\theta = [-(1+\cos\theta)^3/3 + \cos\theta]\Big]_0^{\pi/2} = -\frac{1}{3} - (-\frac{8}{3} + 1) = \frac{4}{3}$.

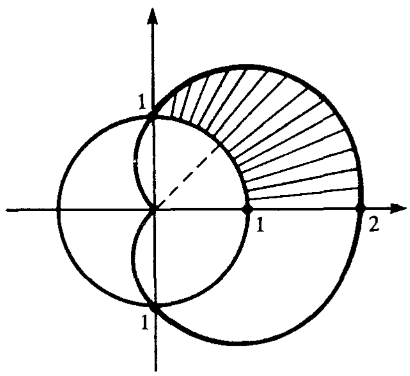

Fig. 44-20

44.34 Use cylindrical coordinates to calculate the volume of a sphere of radius a.

In cylindrical coordinates, the sphere with center $(0,0,0)$ is $r^2 + z^2 = a^2$. Calculate the volume in the first octant and multiply it by 8. The base is the quarter disk $0 \le r \le a$, $0 \le \theta \le \pi/2$. $V = 8\int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\,r\,dr\,d\theta = 8\int_0^{\pi/2} -\frac{1}{2}\cdot\frac{2}{3}(a^2 - r^2)^{3/2}\Big]_0^a d\theta = 8\int_0^{\pi/2}\frac{1}{3}a^3\,d\theta = (8a^3/3)\int_0^{\pi/2} d\theta = (8a^3/3)(\pi/2) = \frac{4}{3}\pi a^3$, the standard formula.

44.35 Use polar coordinates to evaluate $I = \int_0^1\int_0^{\sqrt{1-x^2}}(x^2 + y^2)^{5/2}\,dy\,dx$.

The region of integration is the part of the unit disk in the first quadrant: $0 \le \theta \le \pi/2$, $0 \le r \le 1$. Hence, $I = \int_0^{\pi/2}\int_0^1 r^5\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^6\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{7}r^7\Big]_0^1 d\theta = \frac{1}{7}\int_0^{\pi/2} d\theta = \frac{1}{7}(\pi/2) = \pi/14$.

44.36 Find the area of the region enclosed by the cardioid $r = 1 + \cos\theta$.

$A = \iint_{\mathcal{R}} 1\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}r^2\Big]_0^{1+\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta =$
$$\frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big]_0^{2\pi} = \frac{1}{2}\left(\frac{3}{2}\cdot 2\pi\right) = \frac{3\pi}{2}.$$

44.37 Use polar coordinates to find the area of the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = \frac{3}{2}$.

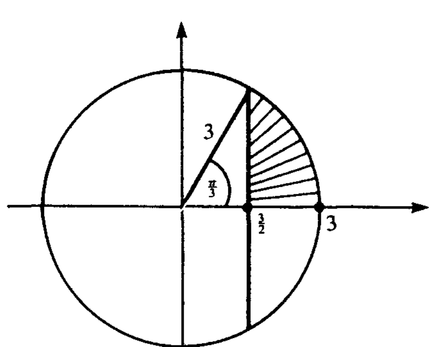

Fig. 44-21

It suffices to double the area in the first quadrant. $x^2+y^2=9$ becomes $r=3$ in polar coordinates, and $x=\frac{3}{2}$ is equivalent to $r\cos\theta=\frac{3}{2}$. From Fig. 44-21, $0\le\theta\le\pi/3$. So the area is $2\int_0^{\pi/3}\int_{3/(2\cos\theta)}^{3} r\,dr\,d\theta = \int_0^{\pi/3} r^2\big]_{(3\sec\theta)/2}^{3}\,d\theta = \int_0^{\pi/3}(9-\frac{9}{4}\sec^2\theta)\,d\theta = (9\theta-\frac{9}{4}\tan\theta)\big]_0^{\pi/3} = 3\pi-(9\sqrt{3}/8)$.

44.38 Describe the planar region $\mathscr{R}$ whose area is given by the iterated integral $I=\int_\pi^{2\pi}\int_1^{1-\sin\theta} r\,dr\,d\theta$.

$r=1-\sin\theta$ is a cardioid, and $r=1$ is the unit circle. Between $\theta=\pi$ and $\theta=2\pi$, $1<1-\sin\theta$ and the cardioid is outside the circle. Therefore, $\mathscr{R}$ is the region outside the unit circle and inside the cardioid (Fig. 44-22).

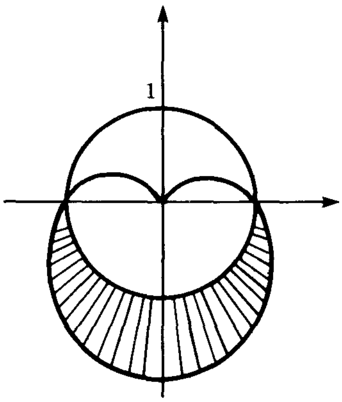

Fig. 44-22

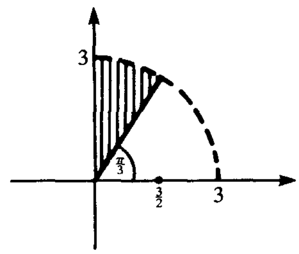

Fig. 44-23

44.39 Evaluate the integral $I=\int_0^{3/2}\int_{\sqrt{3}x}^{\sqrt{9-x^2}} 2xy\,dy\,dx$ using (*a*) rectangular coordinates and (*b*) polar coordinates.

(*a*) $I=\int_0^{3/2} xy^2\big]_{\sqrt{3}x}^{\sqrt{9-x^2}}\,dx = \int_0^{3/2}[x(9-x^2)-3x^3]\,dx = \int_0^{3/2}(9x-4x^3)\,dx = (\frac{9}{2}x^2-x^4)\big]_0^{3/2} = \frac{9}{4}(\frac{9}{2}-\frac{9}{4}) = (\frac{9}{4})^2 = \frac{81}{16}$. (*b*) As indicated in Fig. 44-23, the region of integration lies under the semicircle $y=\sqrt{9-x^2}$ (or $r=3$) and above the line $y=\sqrt{3}x$ (or $\theta=\pi/3$). Hence, $I=2\int_{\pi/3}^{\pi/2}\int_0^3 r\cos\theta\cdot r\cos\theta\cdot r\,dr\,d\theta = 2\int_{\pi/3}^{\pi/2}\int_0^3 r^3\cos\theta\sin\theta\,dr\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi/2} r^4\cos\theta\sin\theta\,\big]_0^3\,d\theta = \frac{81}{2}\int_{\pi/3}^{\pi/2}\cos\theta\sin\theta\,d\theta = \frac{81}{2}\cdot\frac{1}{2}\sin^2\theta\,\big]_{\pi/3}^{\pi/2} = \frac{81}{4}(1-\frac{3}{4}) = \frac{81}{16}$.

44.40 Find the volume of the solid cut out from the sphere $x^2+y^2+z^2\le 4$ by the cylinder $x^2+y^2=1$ (see Fig. 44-24).

It suffices to multiply by 8 the volume of the solid in the first octant. Use cylindrical coordinates. The sphere is $r^2+z^2=4$ and the cylinder is $r=1$. Thus, we have $V=8\int_0^{\pi/2}\int_0^1\sqrt{4-r^2}\,r\,dr\,d\theta = 8\int_0^{\pi/2}[-\frac{1}{2}\cdot\frac{2}{3}(4-r^2)^{3/2}]\,]_0^1\,d\theta = -\frac{8}{3}\int_0^{\pi/2}[(3)^{3/2}-8]\,d\theta = \frac{8}{3}(8-3\sqrt{3})\int_0^{\pi/2}d\theta = \frac{4\pi}{3}(8-3\sqrt{3})$.

Fig. 44-24

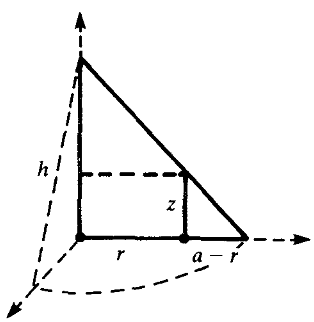

Fig. 44-25

44.41 Use integration in cylindrical coordinates to find the volume of a right circular cone of radius a and height h.

The base is the disk of radius a, given by $r\le a$. For given r, the corresponding value of z on the cone is determined by $z/(a-r)=h/a$ (obtained by similar triangles; see Fig. 44-25.) Then $V=\int_0^{2\pi}\int_0^a (h/a)(a-r)r\,dr\,d\theta = \int_0^{2\pi}[\frac{1}{2}hr^2-(h/3a)r^3]_0^a\,d\theta = \int_0^{2\pi}(\frac{1}{2}ha^2-\frac{1}{3}ha^2)\,d\theta = \frac{1}{6}ha^2\int_0^{2\pi}d\theta = \frac{1}{6}ha^2(2\pi) = \frac{1}{3}\pi a^2h$, the standard formula.

44.42 Find the average distance from points in the unit disk to a fixed point on the boundary.

For the unit circle $r = 2\sin\theta$, with the pole as the fixed point (Fig. 44-26), the distance of an interior point to the pole is r. Thus, $\bar{r} = \frac{1}{\text{area of } \mathscr{R}} \cdot \iint_{\mathscr{R}} r\, dA = \frac{1}{\pi}\int_0^\pi \int_0^{2\sin\theta} r \cdot r\, dr\, d\theta = \frac{1}{\pi}\int_0^\pi \frac{1}{3} r^3 \Big]_0^{2\sin\theta} d\theta = \frac{8}{3\pi}\int_0^\pi \sin^3\theta\, d\theta = \frac{8}{3\pi}\int_0^\pi (\sin\theta - \cos^2\theta\sin\theta)\, d\theta = \frac{8}{3\pi}(-\cos\theta + \frac{1}{3}\cos^3\theta)\Big]_0^\pi = \frac{8}{3\pi}[(1-\frac{1}{3}) - (-1+\frac{1}{3})] = \frac{8}{3\pi}\left(\frac{4}{3}\right) = \frac{32}{9\pi}$.

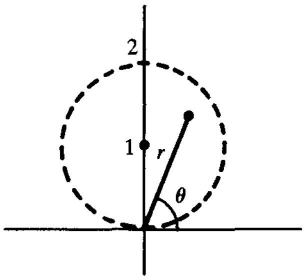

Fig. 44-26

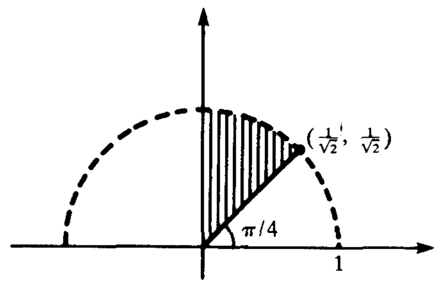

Fig. 44-27

44.43 Use polar coordinates to evaluate $I = \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

The region of integration is indicated in Fig. 44-27. $I = \int_{\pi/4}^{\pi/2} \int_0^1 r \cdot r\, dr\, d\theta = (\int_0^1 r^2\, dr)(\int_{\pi/4}^{\pi/2} d\theta) = \left(\frac{1}{3}\right)\left(\frac{\pi}{4}\right) = \frac{\pi}{12}$.

44.44 Use polar coordinates to evaluate $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} x\, dy\, dx$.

$y = \sqrt{2x - x^2}$ is equivalent to $y^2 = 2x - x^2$, $y \ge 0$, or the upper half of the circle $(x-1)^2 + y^2 = 1$ (see Fig. 44-28). In polar coordinates, $x^2 + y^2 = 2x$ is equivalent to $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus, $I = \int_0^{\pi/2} \int_0^{2\cos\theta} (r\cos\theta) r\, dr\, d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \cos\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{3} r^3 \cos\theta\, \Big]_0^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{\pi}{2}$ (by Problem 44.29).

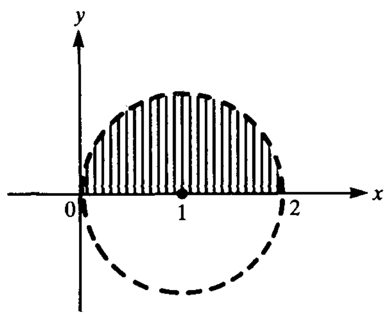

Fig. 44-28

44.45 If the depth of water provided by a water sprinkler in a given unit of time is 2^{-r} feet at a distance of r feet from the sprinkler, find the total volume of water within a distance of a feet from the sprinkler after one unit of time.

$$V = \int_0^{2\pi} d\theta \int_0^a e^{-r\ln 2}\, r\, dr = 2\pi\left[\frac{e^{-r\ln 2}}{(\ln 2)^2}(-r\ln 2 - 1)\right]_0^a = \frac{2\pi}{(\ln 2)^2}[1 - e^{-a\ln 2}(1 + a\ln 2)]$$
$$= \frac{2\pi}{(\ln 2)^2}\left(1 - \frac{1 + a\ln 2}{2^a}\right).$$

44.46 Evaluate $I = \int_{\sqrt{2}/2}^1 \int_{\sqrt{1-x^2}}^x \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$.

The region of integration (Fig. 44-29) consists of all points in the first quadrant above the circle $x^2 + y^2 = 1$ and under the line $y = x$. Transform to polar coordinates, noting that $x = 1$ is equivalent to $r = \sec\theta$. $I = \int_0^{\pi/4} \int_1^{\sec\theta} \frac{1}{r} \cdot r\, dr\, d\theta = \int_0^{\pi/4} \int_1^{\sec\theta} dr\, d\theta = \int_0^{\pi/4} (\sec\theta - 1)\, d\theta = (\ln|\sec\theta + \tan\theta| - \theta)\,\Big]_0^{\pi/4} = \ln(\sqrt{2} + 1) - \pi/4$.

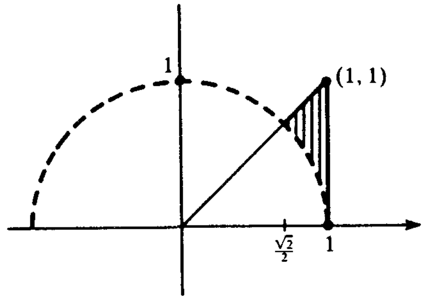

Fig. 44-29

44.47 Show that $\int_0^\infty e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$.

Let $c = \int_0^\infty e^{-x^2}\,dx$. Then $c = \int_0^\infty e^{-y^2}\,dy$. Hence, $c^2 = \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy = \int_0^\infty e^{-y^2}(\int_0^\infty e^{-x^2}\,dx)\,dy = \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}\,dx\,dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$. The region of integration is the entire first quadrant. Change to polar coordinates. $c^2 = \int_0^\infty \int_0^{\pi/2} e^{-r^2}\,r\,d\theta\,dr = \int_0^\infty e^{-r^2}\cdot r\theta\,\big]_0^{\pi/2}\,dr = \dfrac{\pi}{2}\displaystyle\int_0^\infty e^{-r^2}\,r\,dr = (\pi/2)(-\tfrac{1}{2})e^{-r^2}\,\big]_0^\infty = -(\pi/4)(\lim_{r\to+\infty} e^{-r^2} - e^0) = -(\pi/4)(0-1) = \pi/4$. Since $c^2 = \pi/4$, $c = \sqrt{\pi}/2$. (The rather loose reasoning in this computation can be made rigorous.)

44.48 Evaluate $I = \displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$.

Let $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$. Then $I = \displaystyle\int_0^\infty \frac{e^{-u^2}}{u}\cdot 2u\,du = 2\int_0^\infty e^{-u^2}\,du = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$, by Problem 44.47.

44.49 Evaluate $I = \int_0^\infty x^2 e^{-x^2}\,dx$.

Consider $I_w = \int_0^w x^2 e^{-x^2}\,dx$. Use integration by parts. Let $u = x$, $dv = xe^{-x^2}\,dx$, $du = dx$, $v = -\tfrac{1}{2}e^{-x^2}$. Then $I_w = -\tfrac{1}{2}xe^{-x^2}\,\big]_0^w + \tfrac{1}{2}\int_0^w e^{-x^2}\,dx$. Hence, $I = \lim_{w\to+\infty} I_w = \lim_{w\to+\infty}(-\tfrac{1}{2}we^{-w^2}) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = 0 + \tfrac{1}{2}\sqrt{\pi}/2 = \sqrt{\pi}/4$, by Problem 44.47.

44.50 Use spherical coordinates to find the volume of a sphere of radius a.

In spherical coordinates a sphere of radius a is characterized by $0 \le \rho \le a$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. Recall that the volume element is given by $dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$. $V = \int_0^\pi \int_0^{2\pi} \int_0^a \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \int_0^{2\pi} \tfrac{1}{3}\rho^3 \sin\phi\,\big]_0^a\,d\theta\,d\phi = \int_0^\pi \int_0^{2\pi} \tfrac{1}{3}a^3 \sin\phi\,d\theta\,d\phi = \int_0^\pi \tfrac{1}{3}a^3 \sin\phi\cdot\theta\,\big]_0^{2\pi}\,d\phi = \int_0^\pi \tfrac{1}{3}a^3 \sin\phi\cdot 2\pi\,d\phi = \tfrac{2}{3}\pi a^3 \int_0^\pi \sin\phi\,d\phi = -\tfrac{2}{3}\pi a^3 \cos\phi\,\big]_0^\pi = -\tfrac{2}{3}\pi a^3(-1-1) = \tfrac{4}{3}\pi a^3$.

44.51 Use spherical coordinates to find the volume of a right circular cone of height h and radius of base b.

For the orientation shown in Fig. 44-30, the points of the cone satisfy $0 \le \theta \le 2\pi$, $0 \le \phi \le \tan^{-1}(b/h)$, $0 \le \rho \le h\sec\phi$. Thus, $V = \int_0^{2\pi} \int_0^{\tan^{-1}(b/h)} \int_0^{h\sec\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_0^{\tan^{-1}(b/h)} \tfrac{1}{3}\rho^3 \sin\phi\,\big]_0^{h\sec\phi}\,d\phi\,d\theta = \displaystyle\int_0^{2\pi}\int_0^{\tan^{-1}(b/h)} \frac{1}{3}\,h^3\,\frac{\sin\phi}{\cos^3\phi}\,d\phi\,d\theta = \frac{1}{3}\,h^3 \int_0^{2\pi} \frac{1}{2\cos^2\phi}\,\Big]_0^{\tan^{-1}(b/h)}\,d\theta = \frac{1}{3}\,h^3 \int_0^{2\pi} \frac{1}{2}\left(\frac{s^2}{h^2} - 1\right)d\theta = \frac{h^3}{6h^2}\int_0^{2\pi}(s^2 - h^2)\,d\theta = \frac{hb^2}{6}\int_0^{2\pi} d\theta = \frac{hb^2}{6}\cdot 2\pi = \frac{1}{3}\,\pi b^2 h$.

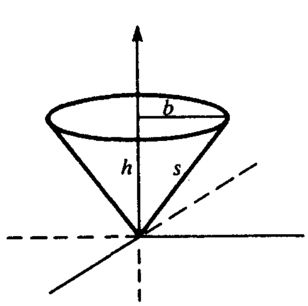

Fig. 44-30

44.52 Find the average distance $\bar{\rho}$ from the center of a ball $\mathcal{B}$ of radius a to all other points of the ball.

$\bar{\rho} = \frac{1}{\text{volume } V} \cdot \iiint_{\mathcal{B}} \rho\, dV = \frac{1}{\frac{4}{3}\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi} \frac{1}{4} \rho^4 \sin\phi \Big]_0^a d\phi\, d\theta = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi} \frac{1}{4} a^4 \sin\phi\, d\phi\, d\theta = \frac{3a}{16\pi} \int_0^{2\pi} -\cos\phi \Big]_0^{\pi} d\theta = \frac{3a}{16\pi} \int_0^{2\pi} -(-1-1)\, d\theta = \frac{3a}{8\pi} \int_0^{2\pi} d\theta = \frac{3a}{8\pi} \cdot 2\pi = \frac{3}{4} a.$

44.53 Find the area S of the part of the plane $x + 2y + z = 4$ which lies inside the cylinder $x^2 + y^2 = 1$.

Recall (Fig. 44-31) the relation $dA = dS \cos\theta = dS \dfrac{1}{\sqrt{1 + z_x^2 + z_y^2}}$ between an element of area dS of a surface $z = f(x, y)$ and its projection dA in the xy-plane. Thus, the formula for S is $\iint_{\mathcal{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$, where, here, $\mathcal{R}$ is the disk $x^2 + y^2 \le 1$. $1 + \frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial x} = -1$. $2 + \frac{\partial z}{\partial y} = 0$, $\frac{\partial z}{\partial y} = -2$. Hence, $S = \iint_{\mathcal{R}} \sqrt{6}\, dA = \sqrt{6}(\text{area } \mathcal{R}) = \sqrt{6}\,\pi$.

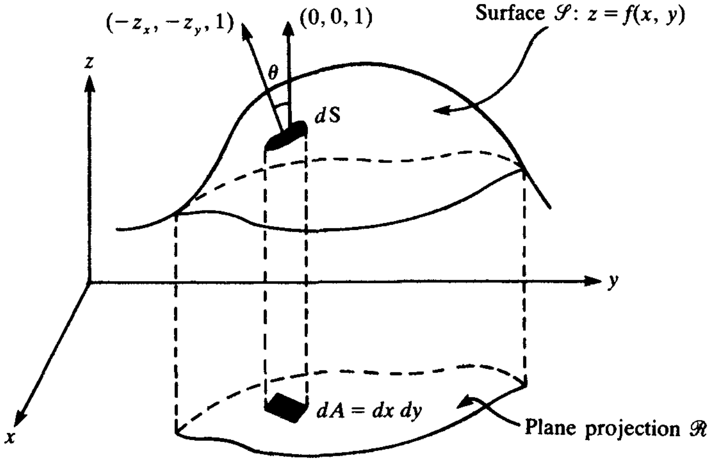

Fig. 44-31

44.54 Find the surface area S of the part of the sphere $x^2 + y^2 + z^2 = 36$ inside the cylinder $x^2 + y^2 = 6y$ and above the xy-plane.

$x^2 + y^2 = 6y$ is equivalent to $x^2 + (y-3)^2 = 9$. So the cylinder has axis $x = 0$, $y = 3$, and radius 3. The base $\mathcal{R}$ is the circle $x^2 + (y-3)^2 = 9$. (See Fig. 44-32.) $S = \iint_{\mathcal{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$. $2x + 2z\frac{\partial z}{\partial x} = 0$. Hence, $\frac{\partial z}{\partial x} = -\frac{x}{z}$. Similarly, $\frac{\partial z}{\partial y} = -\frac{y}{z}$. Thus, $1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2 + z^2}{z^2} = \frac{36}{z^2}$. Therefore, $S = \iint_{\mathcal{R}} \frac{6}{z}\, dA$. Now use polar coordinates. The circle $x^2 + y^2 = 6y$ is equivalent to $r = 6\sin\theta$. $x^2 + y^2 + z^2 = 36$ is equivalent to $r^2 + z^2 = 36$, or $z^2 = 36 - r^2$. Hence, $S = \int_0^{\pi} \int_0^{6\sin\theta} \frac{6}{\sqrt{36 - r^2}}\, r\, dr\, d\theta = \int_0^{\pi} -6\sqrt{36 - r^2}\,\Big]_0^{6\sin\theta} d\theta = -6\int_0^{\pi} (6|\cos\theta| - 6)\, d\theta = -12\int_0^{\pi/2} (6\cos\theta - 6)\, d\theta$, since $|\cos(\pi - \theta)| = |\cos\theta|$. Hence, $S = -72(\sin\theta - \theta)\,\Big]_0^{\pi/2} = 72(\pi/2 - 1)$.

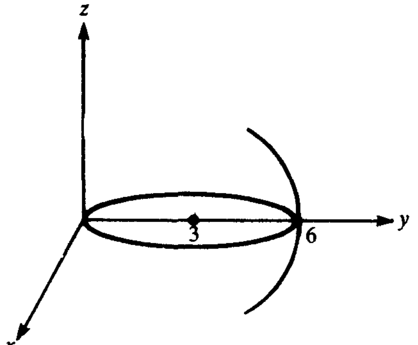

Fig. 44-32

44.55 Find the surface area S of the part of the sphere $x^2 + y^2 + z^2 = 4z$ inside the paraboloid $z = x^2 + y^2$.

▌ The region $\mathscr{R}$ under the spherical cap (Fig. 44-33) is obtained by finding the intersection of $x^2 + y^2 + z^2 = 4z$ and $z = x^2 + y^2$. This gives $z(z-3) = 0$. Hence, the paraboloid cuts the sphere when $z = 3$, and $\mathscr{R}$ is the disk $x^2 + y^2 \le 3$. $S = \iint_{\mathscr{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$. $2x + 2z \frac{\partial z}{\partial x} = 4\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial x} = -\frac{x}{z-2}$. Similarly, $\frac{\partial z}{\partial y} = -\frac{y}{z-2}$. Hence,

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{(z-2)^2} + \frac{y^2}{(z-2)^2} = \frac{(z-2)^2 + x^2 + y^2}{(z-2)^2} = \frac{(x^2+y^2+z^2) - 4z + 4}{(z-2)^2} = \frac{4}{(z-2)^2}$$

Therefore,

$$S = \iint_{\mathscr{R}} \frac{2}{z-2}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}\, r\, dr\, d\theta = -\int_0^{2\pi} 2\sqrt{4-r^2}\,\Big]_0^{\sqrt{3}}\, d\theta = -2\int_0^{2\pi}(1-2)\, d\theta = 2\cdot 2\pi = 4\pi$$

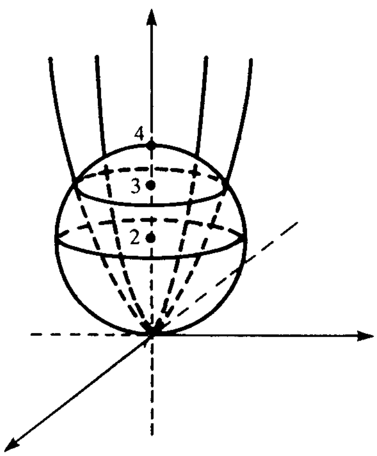

Fig. 44-33

44.56 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 25$ between the planes $z = 2$ and $z = 4$.

▌ The surface lies above the ring-shaped region $\mathscr{R}$: $3 \le r \le \sqrt{21}$. $S = \iint_{\mathscr{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_{\mathscr{R}} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dA = \iint_{\mathscr{R}} \sqrt{\frac{25}{z^2}}\, dA = \iint_{\mathscr{R}} \frac{5}{z}\, dA = \int_0^{2\pi}\int_3^{\sqrt{21}} \frac{5}{\sqrt{25-r^2}}\, r\, dr\, d\theta$, since $r^2 + z^2 = 25$. Hence, $S = \int_0^{2\pi} -5\sqrt{25-r^2}\,]_3^{\sqrt{21}}\, d\theta = -5\int_0^{2\pi}(2-4)\, d\theta = 10\int_0^{2\pi} d\theta = 20\pi$.

44.57 Find the surface area of a sphere of radius a.

▌ Consider the upper hemisphere of the sphere $x^2 + y^2 + z^2 = a^2$. It projects down onto the disk $\mathscr{R}$ of radius a whose center is at the origin. Hence, the surface area of the entire sphere is $2\iint_{\mathscr{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 2\iint_{\mathscr{R}} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dA = 2\iint_{\mathscr{R}} \sqrt{\frac{x^2+y^2+z^2}{z^2}}\, dA = 2\iint_{\mathscr{R}} \frac{a}{z}\, dA = 2\int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta = 2a\int_0^{2\pi}(-\sqrt{a^2-r^2})\,]_0^a\, d\theta = -2a\int_0^{2\pi}(-a)\, d\theta = 2a^2\int_0^{2\pi} d\theta = 2a^2\cdot 2\pi = 4\pi a^2$.

44.58 Find the surface area of a cone of height h and radius of base b.

▌ Consider the cone $z = \frac{h}{b}\sqrt{x^2+y^2}$, or $b^2z^2 = h^2x^2 + h^2y^2$ (see Fig. 44-30). The portion of the cone under $z = h$ projects onto the interior $\mathscr{R}$ of the circle $r = b$ in the xy-plane. Then

$$S = \iint_{\mathscr{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_{\mathscr{R}} \sqrt{1 + \left(\frac{h^2 x}{b^2 z}\right)^2 + \left(\frac{h^2 y}{b^2 z}\right)^2}\, dA = \iint_{\mathscr{R}} \sqrt{\frac{b^4 z^2 + h^4 x^2 + h^4 y^2}{b^4 z^2}}\, dA$$

$$= \iint_{\mathscr{R}} \sqrt{\frac{h^2(b^2 + h^2)(x^2 + y^2)}{b^4 z^2}}\, dA = \iint_{\mathscr{R}} \frac{h}{b^2} \sqrt{b^2 + h^2}\, \frac{\sqrt{x^2 + y^2}}{z}\, dA = \iint_{\mathscr{R}} \frac{h}{b^2} \sqrt{b^2 + h^2} \cdot \frac{b}{h}\, dA$$

$$= \frac{1}{b} \sqrt{b^2 + h^2} \iint_{\mathscr{R}} dA = \frac{\sqrt{b^2 + h^2}}{b} (\pi b^2) = \pi b \sqrt{b^2 + h^2} = \pi b s$$

where $s = \sqrt{b^2 + h^2}$ is the slant height of the cone.

44.59 Use a triple integral to find the volume V inside $x^2 + y^2 = 9$, above $z = 0$, and below $x + z = 4$.

▮ $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{4-x} 1\, dz\, dy\, dx$. The bounds on z correspond to the requirement that the solid is above $z = 0$ and below $x + z = 4$. The bounds on y come from the equation $x^2 + y^2 = 9$. $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\,\Big]_0^{4-x} dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)\, dy\, dx = \int_{-3}^{3} (4 - x) y\,\Big]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = \int_{-3}^{\sqrt{3}} 2(4 - x)\sqrt{9 - x^2}\, dx = 8 \int_{-3}^{3} \sqrt{9 - x^2}\, dx + \int_{-3}^{3} (-2x)\sqrt{9 - x^2}\, dx$. Now, $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$ gives the area of the upper half of the disk $x^2 + y^2 \le 9$ of radius 3, namely, $\frac{1}{2}(9\pi) = \frac{9}{2}\pi$. Also, $\int_{-3}^{3} (-2x)\sqrt{9 - x^2}\, dx = 0$ (odd function). Therefore, $V = 8(\frac{9}{2}\pi) = 36\pi$.

44.60 Use a triple integral to find the volume V inside $x^2 + y^2 = 4x$, above $z = 0$, and below $x^2 + y^2 = 4z$.

▮ $x^2 + y^2 = 4x$ is equivalent to $(x - 2)^2 + y^2 = 4$, the cylinder of radius 2 with axis $x = 2$, $y = 0$. Hence, $V = \int_0^4 \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} \int_0^{(x^2+y^2)/4} 1\, dz\, dy\, dx$. This is difficult to compute; so let us switch to cylindrical coordinates. The circle $x^2 + y^2 = 4x$ becomes $r = 4\cos\theta$, and we get $V = \int_0^{\pi} \int_0^{4\cos\theta} \theta \int_0^{r^2/4} r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^{4\cos\theta} rz\,\Big]_0^{r^2/4} dr\, d\theta = \int_0^{\pi} \int_0^{4\cos\theta} \frac{1}{4} r^3\, dr\, d\theta = \int_0^{\pi} \frac{1}{16} r^4 \Big]_0^{4\cos\theta} d\theta = \int_0^{\pi} 16\cos^4\theta\, d\theta = 32 \int_0^{\pi/2} \cos^4\theta\, d\theta = 6\pi$ (Problem 44.29).

44.61 Use a triple integral to find the volume inside the cylinder $r = 4$, above $z = 0$, and below $2z = y$.

▮ The solid is wedge-shaped. The base is the half-disk $0 \le \theta \le \pi$, $0 \le r \le 4$. The height is $\frac{y}{2} = \frac{r\sin\theta}{2}$. Then

$$V = \int_0^{\pi} \int_0^4 \int_0^{(r\sin\theta)/2} r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^4 rz\,\Big]_0^{(r\sin\theta)/2} dr\, d\theta = \int_0^{\pi} \int_0^4 \frac{1}{2} r^2 \sin\, dr\, d\theta = \frac{1}{6} \int_0^{\pi} r^3 \sin\theta\,\Big]_0^4 d\theta$$

$$= \frac{1}{6} \int_0^{\pi} 64\sin\theta\, d\theta = \frac{32}{3} (-\cos\theta)\Big]_0^{\pi} = -\frac{32}{3}(-1 - 1) = \frac{64}{3}$$

44.62 Use a triple integral to find the volume cut from the cone $\phi = \pi/4$ by the sphere $\rho = 2a\cos\phi$.

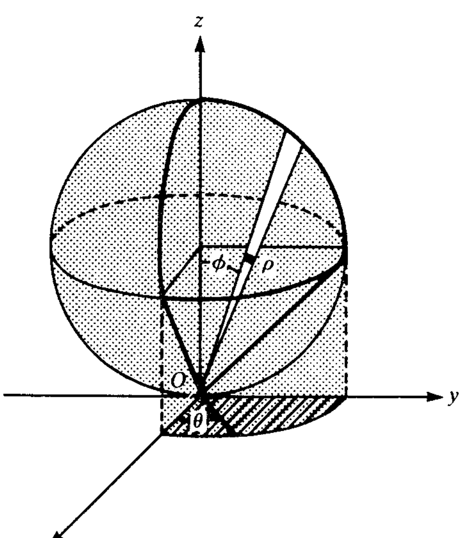

Fig. 44-34

Refer to Fig. 44-34. Note that $\rho = 2a\cos\phi$ has the equivalent forms $\rho^2 = 2a\rho\cos\phi$, $x^2+y^2+z^2 = 2az$, $x^2+y^2+(z-a)^2 = a^2$. Thus, it is the sphere with center at $(0,0,a)$ and radius a. Then, $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\rho^3 \sin\phi \Big]_0^{2a\cos\phi} d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{8a^3}{3}\sin\phi\cos^3\phi\, d\phi\, d\theta = \frac{8a^3}{3}\int_0^{2\pi}\left(-\frac{\cos^4\phi}{4}\right)\Big]_0^{\pi/4} d\theta = -\frac{2a^3}{3}\int_0^{2\pi}\left(\frac{1}{4}-1\right)d\theta = \frac{1}{2}a^3\int_0^{2\pi} d\theta = \frac{1}{2}a^3\cdot 2\pi = \pi a^3$.

44.63 Find the volume within the cylinder $r = 4\cos\theta$ bounded above by the sphere $r^2+z^2 = 16$ and below by the plane $z = 0$.

Integrate first with respect to z from $z=0$ to $z = \sqrt{16-r^2}$, then with respect to r from $r=0$ to $r = 4\cos\theta$, and then with respect to θ from $\theta = 0$ to $\theta = \pi$. (See Fig. 44-35.) $V = \int_0^{\pi}\int_0^{4\cos\theta}\int_0^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta = \int_0^{\pi}\int_0^{4\cos\theta} rz\Big]_0^{\sqrt{16-r^2}} dr\, d\theta = \int_0^{\pi}\int_0^{4\cos\theta} r\sqrt{16-r^2}\, dr\, d\theta = \int_0^{\pi}[-\frac{1}{2}\cdot\frac{2}{3}(16-r^2)^{3/2}]\Big]_0^{4\cos\theta} d\theta = -\frac{1}{3}\int_0^{\pi}(64\sin^3\theta - 64)\,d\theta = -\frac{64}{3}\int_0^{\pi}(\sin\theta - \cos^2\theta\sin\theta - 1)\,d\theta = -\frac{64}{3}(-\cos\theta + \frac{1}{3}\cos^3\theta - \theta)\Big]_0^{\pi} = -\frac{64}{3}[(1-\frac{1}{3}-\pi)-(-1+\frac{1}{3})] = -\frac{64}{3}(\frac{4}{3}-\pi) = \frac{64}{9}(3\pi - 4)$.

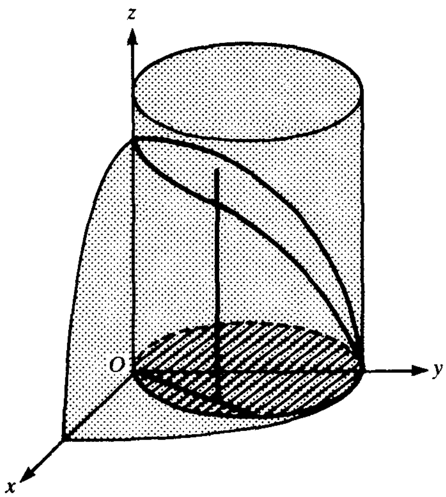

Fig. 44-35

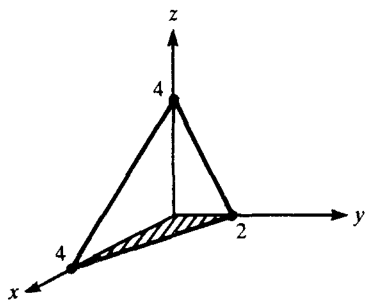

Fig. 44-36

44.64 Evaluate $I = \iiint\limits_{\mathscr{R}} x\, dV$, where $\mathscr{R}$ is the tetrahedron bounded by the coordinate planes and the plane $x + 2y + z = 4$ (see Fig. 44-36).

y can be integrated from 0 to 2. In the base triangle, for a given y, x runs from 0 to $4-2y$. For given x and y, z varies from 0 to $4-x-2y$. Hence, $I = \int_0^2\int_0^{4-2y}\int_0^{4-x-2y} x\, dz\, dx\, dy = \int_0^2\int_0^{4-2y} xz\Big]_0^{4-x-2y} dx\, dy = \int_0^2\int_0^{4-2y} x(4-x-2y)\,dx\,dy = \int_0^2\int_0^{4-2y}(4x - x^2 - 2xy)\,dx\,dy = \int_0^2 (2x^2 - \frac{1}{3}x^3 - yx^2)\Big]_0^{4-2y} dy = \int_0^2 (4-2y)^2[2 - \frac{1}{3}(4-2y) - y]\,dy = \int_0^2 (4-2y)^2(\frac{2}{3} - \frac{1}{3}y)\,dy = \frac{4}{3}\int_0^2 (2-y)^3\,dy = \frac{4}{3}\cdot(-\frac{1}{4})(2-y)^4\Big]_0^2 = -\frac{1}{3}(-16) = \frac{16}{3}$.

44.65 Evaluate $I = \iiint\limits_{\mathscr{R}} (x^2+y^2+z^2)\,dV$, where $\mathscr{R}$ is the ball of radius a with center at the origin.

Use spherical coordinates. $I = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{1}{5}\rho^5\sin\phi\Big]_0^a d\phi\, d\theta = \frac{1}{5}a^5\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = \frac{1}{5}a^5\int_0^{2\pi}(-\cos\phi)\Big]_0^{\pi} d\theta = \frac{1}{5}a^5\int_0^{2\pi}[1-(-1)]\,d\theta = \frac{2}{5}a^5\int_0^{2\pi}d\theta = \frac{2}{5}a^5\cdot 2\pi = \frac{4}{5}\pi a^5$.

44.66 Use a triple integral to find the volume V of the solid inside the cylinder $x^2+y^2 = 25$ and between the planes $z = 2$ and $x + z = 8$.

The projection on the xy-plane is the circle $x^2+y^2 = 25$. Use cylindrical coordinates.

$$V = \int_0^{2\pi}\int_0^5\int_2^{8-(r\cos\theta)} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^5 rz\Big]_2^{8-(r\cos\theta)} dr\, d\theta = \int_0^{2\pi}\int_0^5 r(6 - r\cos\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}(3r^2 - \tfrac{1}{3}r^3\cos\theta)\Big]_0^5 d\theta = \int_0^{2\pi} 25(3 - \tfrac{5}{3}\cos\theta)\,d\theta = 25(3\theta - \tfrac{5}{3}\sin\theta)\Big]_0^{2\pi} = 25(6\pi) = 150\pi.$$

44.67 Find the volume of the solid enclosed by the paraboloids $z = x^2+y^2$ (upward-opening) and $z = 36 - 3x^2 - 8y^2$ (downward-opening).

The projection of the intersection of the surfaces is the ellipse $4x^2+9y^2=36$. By symmetry, we can calculate the integral with respect to x and y in the first quadrant and then multiply it by 4.

$$V=4\int_0^3\int_0^{2\sqrt{9-x^2}/3}\int_{x^2+y^2}^{36-3x^2-8y^2} dz\,dy\,dx = 4\int_0^3\int_0^{2\sqrt{9-x^2}/3}[(36-3x^2-8y^2)-(x^2+y^2)]\,dy\,dx$$

$$=4\int_0^3\int_0^{2\sqrt{9-x^2}/3}(36-4x^2-9y^2)\,dy\,dx = 4\int_0^3 (36y-4x^2y-3y^3)\Big]_0^{2\sqrt{9-x^2}/3}dx$$

$$=\frac{8}{3}\int_0^3 \sqrt{9-x^2}\,[36-4x^2-4(9-x^2)]\,dx=\frac{32}{3}\int_0^3\sqrt{9-x^2}\left[\frac{2}{3}(9-x^2)\right]dx=\frac{64}{9}\int_0^3(9-x^2)^{3/2}\,dx$$

Let $x=3\sin\theta$, $dx=3\cos\theta\,d\theta$. Then $V=\dfrac{64}{9}\displaystyle\int_0^{\pi/2}27\cos^3\theta\cdot 3\cos\theta\,d\theta=576\int_0^{\pi/2}\cos^4\theta\,d\theta=108\pi$ (from Problem 44.29).

44.68 Describe the solid whose volume is given by the integral $\int_0^2\int_{2x}^4\int_0^1 dz\,dy\,dx$.

See Fig. 44-37. The solid lies under the plane $z=1$ and above the region in the first quadrant of the xy-plane bounded by the lines $y=2x$ and $y=4$.

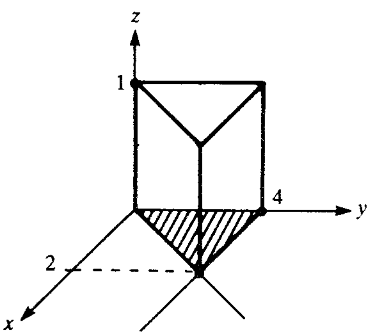

Fig. 44-37

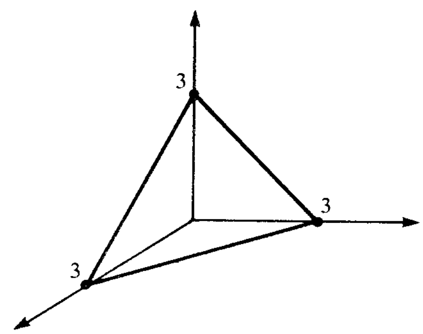

Fig. 44-38

44.69 Describe the solid whose volume is given by the integral $\int_0^3\int_0^{3-x}\int_0^{3-x-y} dz\,dy\,dx$.

See Fig. 44-38. The solid lies under the plane $z=3-x-y$ and above the triangle in the xy-plane bounded by the coordinate axes and the line $x+y=3$. It is a tetrahedron, of volume $\frac{1}{6}(3)^3=\frac{9}{2}$.

44.70 Describe the solid whose volume is given by the integral $\int_0^5\int_0^{\sqrt{25-x^2}}\int_0^3 dz\,dy\,dx$, and compute the volume.

The solid is the part of the solid right circular cylinder $x^2+y^2\le 25$ lying in the first octant between $z=0$ and $z=3$ (see Fig. 44-39). Its volume is $\frac{1}{4}(\pi r^2h)=\frac{1}{4}[\pi(5)^2\cdot 3]=\frac{75}{4}\pi$.

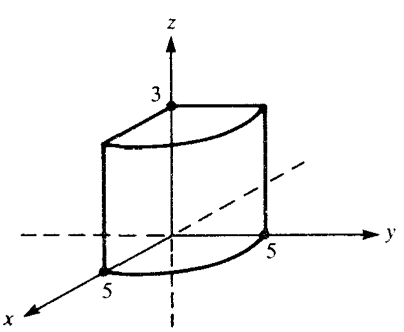

Fig. 44-39

44.71 If $\mathcal{R}$ is a rectangular box $x_1\le x\le x_2$, $y_1\le y\le y_2$, $z_1\le z\le z_2$, show that $I=\iiint_{\mathcal{R}} f(x)g(u)h(z)\,dV = \underbrace{\left(\int_{x_1}^{x_2}f(x)\,dx\right)}_{a}\underbrace{\left(\int_{y_1}^{y_2}g(y)\,dy\right)}_{b}\underbrace{\left(\int_{z_1}^{z_2}h(z)\,dz\right)}_{c}$.

$I=\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}f(x)g(y)h(z)\,dx\,dy\,dz = \int_{z_1}^{z_2}\int_{y_1}^{y_2}g(y)\,h(z)[\int_{x_1}^{x_2}f(x)\,dx]\,dy\,dz = a\int_{z_1}^{z_2}h(z)[\int_{y_1}^{y_2}g(y)\,dy]\,dz = ab\int_{z_1}^{z_2}h(z)\,dz=abc$.

44.72 Evaluate $I=\iiint_{\mathcal{R}} x^3ye^z\,dV$, where $\mathcal{R}$ is the rectangular box: $1\le x\le 2$, $0\le y\le 1$, $0\le z\le \ln 2$.

By Problem 44.71, $I=\int_1^2 x^3\,dx\cdot\int_0^1 y\,dy\cdot\int_0^{\ln 2}e^z\,dz=\frac{1}{4}x^4\Big]_1^2\cdot\frac{1}{2}y^2\Big]_0^1\cdot e^z\Big]_0^{\ln 2}=(4-\frac{1}{4})\cdot\frac{1}{2}\cdot(2-1)=\frac{15}{8}$.

44.73 Evaluate $J = \iiint_{\mathcal{R}} x^2\,dV$ for the ball $\mathcal{R}$ of Problem 44.65.

By spherical symmetry, $I \equiv \iiint_{\mathcal{R}} x^2\,dV + \iiint_{\mathcal{R}} y^2\,dV + \iiint_{\mathcal{R}} z^2\,dV = J + J + J$; so $J = \frac{1}{3}I = \dfrac{4\pi a^5}{15}$.

44.74 Find the mass of a plate in the form of a right triangle $\mathcal{R}$ with legs a and b, if the density (mass per unit area) is numerically equal to the sum of the distances from the legs.

Let the right angle be at the origin and the legs a and b be along the positive x and y axes, respectively (Fig. 44-40). The density $\delta(x, y) = x + y$. Hence, the mass $M = \iint (x+y)\,dA = \int_0^a \int_0^{b-(b/a)x} (x+y)\,dy\,dx = \int_0^a (xy + \frac{1}{2}y^2)\Big]_0^{b-(b/a)x} dx = \dfrac{b}{a}\displaystyle\int_0^a (a-x)[x + \tfrac{1}{2}(b/a)(a-x)]\,dx = \dfrac{b}{a}\int_0^a [ax - x^2 + \tfrac{1}{2}(b/a)(a-x)^2]\,dx = (b/a)[\frac{1}{2}ax^2 - \frac{1}{3}x^3 - \frac{1}{2}(b/a)\cdot\frac{1}{3}(a-x)^3]\Big]_0^a = (b/a)\{(\frac{1}{2}a^3 - \frac{1}{3}a^3) - [-\frac{1}{2}(b/a)\cdot\frac{1}{3}a^3]\} = (b/a)(\frac{1}{6}a^3 + \frac{1}{6}ba^2) = \frac{1}{6}ba(a+b)$.

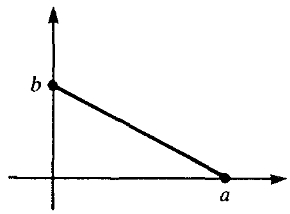

Fig. 44-40

44.75 Find the mass of a circular plate $\mathcal{R}$ of radius a whose density is numerically equal to the distance from the center.

Let the circle be $r = a$. Then $M = \iint_{\mathcal{R}} r\,dA = \int_0^{2\pi}\int_0^a r\cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{3}r^3\Big]_0^a d\theta = \int_0^{2\pi} \frac{1}{3}a^3\,d\theta = \frac{1}{3}a^3\cdot 2\pi = \frac{2}{3}\pi a^3$.

44.76 Find the mass of a solid right circular cylinder $\mathcal{R}$ of height h and radius of base b, if the density (mass per unit volume) is numerically equal to the square of the distance from the axis of the cylinder.

$M = \iiint_{\mathcal{R}} r^2\,dV = \int_0^{2\pi}\int_0^b\int_0^h r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^b r^3h\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}hr^4\Big]_0^b d\theta = \int_0^{2\pi} \frac{1}{4}hb^4\,d\theta = \frac{1}{4}hb^4\cdot 2\pi = \frac{1}{2}\pi hb^4$.

44.77 Find the mass of a ball $\mathcal{B}$ of radius a whose density is numerically equal to the distance from a fixed diametral plane.

Let the ball be the inside of the sphere $x^2 + y^2 + z^2 = a^2$, and let the fixed diametral plane be $z = 0$. Then $M = \iiint |z|\,dV$. Use the upper hemisphere and double the result. In spherical coordinates,

$$M = 2\int_0^{2\pi}\int_0^{2\pi}\int_0^a z\cdot\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= 2\int_0^{2\pi}\int_0^{\pi/2} \tfrac{1}{4}\rho^4\cos\phi\sin\phi\Big]_0^a d\phi\,d\theta = \tfrac{1}{2}a^4\int_0^{2\pi} \tfrac{1}{2}\sin^2\phi\Big]_0^{\pi/2} d\theta = \tfrac{1}{4}a^4\int_0^{2\pi} d\theta = \tfrac{1}{4}a^4\cdot 2\pi = \tfrac{1}{2}\pi a^4$$

44.78 Find the mass of a solid right circular cone $\mathscr{C}$ of height h and radius of base b whose density is numerically equal to the distance from its axis.

Let $\alpha = \tan^{-1}(b/h)$. In spherical coordinates, the lateral surface is $\phi = \alpha$. $M = \iiint_{\mathscr{C}} r\,dV = \int_0^{2\pi}\int_0^{\alpha}\int_0^{h\sec\phi} \rho\sin\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\alpha} \frac{1}{4}\rho^4\sin^2\phi\Big]_0^{h\sec\phi} d\phi\,d\theta = \frac{1}{4}h^4\int_0^{2\pi}\int_0^{\alpha}\sec^4\phi\sin^2\phi\,d\phi\,d\theta = \frac{1}{4}h^4\int_0^{2\pi}\int_0^{\alpha}\sec^2\phi\tan^2\phi\,d\phi\,d\theta = \frac{1}{4}h^4\int_0^{2\pi}\frac{1}{3}\tan^3\phi\Big]_0^{\alpha} d\theta = \frac{1}{4}h^4\cdot\frac{1}{3}\tan^3\alpha\int_0^{2\pi} d\theta = \frac{1}{12}h^4(b/h)^3\cdot 2\pi = \frac{1}{6}\pi hb^3$.

44.79 Find the mass of a spherical surface $\mathscr{S}$ whose density is equal to the distance from a fixed diametral plane.

Let $\mathscr{S}$ be $x^2 + y^2 + z^2 = a^2$, and let the fixed diametral plane be the xy-plane. Then $M = \iint_{\mathscr{C}} |z|\dfrac{dS}{dA}\,dA$, where $\mathscr{C}$ is the disk $x^2 + y^2 = a^2$. Then, if we double the mass of the upper hemisphere, $M = 2\displaystyle\iint_{\mathscr{C}} z\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$. Then $1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = 1 + \dfrac{x^2}{z^2} + \dfrac{y^2}{z^2} = \dfrac{x^2+y^2+z^2}{z^2} = \dfrac{a^2}{z^2}$. Hence, $M = 2\iint_{\mathscr{C}} a\,dA = 2a\iint_{\mathscr{C}} dA = 2a(\text{area of a circle of radius } a) = 2a(\pi a^2) = 2\pi a^3$.

44.80 Find the center of mass $(\bar{x}, \bar{y})$ of the plate cut from the parabola $y^2 = 8x$ by its latus rectum $x = 2$ if the density is numerically equal to the distance from the latus rectum.

▌ See Fig. 44-41. By symmetry, $\bar{y} = 0$. The mass $M = \iint (2-x)\,dA = \int_{-4}^{4}\int_{y^2/8}^{2} (2-x)\,dx\,dy = \int_{-4}^{4}\left(2x - \frac{1}{2}x^2\right)\Big]_{y^2/8}^{2} dy = \int_{-4}^{4}\left[2 - \left(\frac{y^2}{4} - \frac{y^4}{128}\right)\right] dy = \left(2y - \frac{1}{12}y^3 + \frac{1}{128}\frac{y^5}{5}\right)\Big]_{-4}^{4} = 8\left(2 - \frac{16}{12} + \frac{1}{128}\cdot\frac{256}{5}\right) = \frac{128}{15}$. The moment about the y-axis is given by $M_y = \int_{-4}^{4}\int_{y^2/8}^{2} x(2-x)\,dx\,dy = \int_{-4}^{4}\left(x^2 - \frac{1}{3}x^3\right)\Big]_{y^2/8}^{2} dy = \int_{-4}^{4}\left[\frac{4}{3} - \left(\frac{y^4}{64} - \frac{y^6}{3\cdot 512}\right)\right] dy = \left(\frac{4y}{3} - \frac{1}{64}\frac{y^5}{5} + \frac{1}{3\cdot 512}\frac{y^7}{7}\right)\Big]_{-4}^{4} = 8\left(\frac{4}{3} - \frac{4}{5} + \frac{8}{21}\right) = \frac{256}{35}$. Hence, $\bar{x} = \frac{M_y}{M} = \frac{\frac{256}{35}}{\frac{128}{15}} = \frac{6}{7}$. Thus, the center of mass is $\left(\frac{6}{7}, 0\right)$.

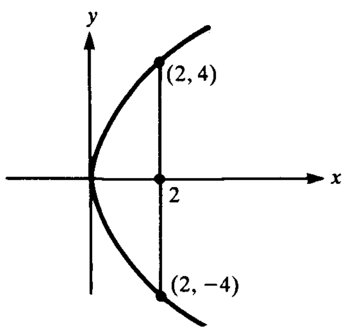

Fig. 44-41

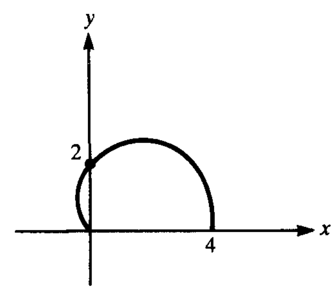

Fig. 44-42

44.81 Find the center of mass of a plate in the form of the upper half of the cardioid $r = 2(1+\cos\theta)$ if the density is numerically equal to the distance from the pole.

▌ See Fig. 44-42. The mass $M = \iint_{\mathcal{R}} r\,dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} r\cdot r\,dr\,d\theta = \int_0^{\pi}\frac{1}{3}r^3\Big]_0^{2(1+\cos\theta)} d\theta = \frac{8}{3}\int_0^{\pi}(1+\cos\theta)^3\,d\theta = \frac{8}{3}\int_0^{\pi}\left[1 + 3\cos\theta + 3\left(\frac{1+\cos 2\theta}{2}\right) + (\cos\theta - \sin^2\theta\cos\theta)\right] d\theta = \frac{8}{3}\left(\frac{5}{2}\theta + 4\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{3}\sin^3\theta\right)\Big]_0^{\pi} = \frac{8}{3}\left(\frac{5}{2}\cdot\pi\right) = \frac{20\pi}{3}$. The moment about the x-axis is $M_x = \int_0^{\pi}\int_0^{2(1+\cos\theta)} yr^2\,dr\,d\theta = \int_0^{\pi}\int_0^{2(1+\cos\theta)} r^3\sin\theta\,dr\,d\theta = \int_0^{\pi}\frac{1}{4}r^4\sin\theta\Big]_0^{2(1+\cos\theta)} d\theta = 4\int_0^{\pi}(1+\cos\theta)^4\sin\theta\,d\theta = -4\,\frac{(1+\cos\theta)^5}{5}\Big]_0^{\pi} = -\frac{4}{5}(-32) = \frac{128}{5}$. Hence, $\bar{y} = \frac{M_x}{M} = \frac{128/5}{20\pi/3} = \frac{96}{25\pi}$. The moment about the y-axis is $M_y = \int_0^{\pi}\int_0^{2(1+\cos\theta)} xr^2\,dr\,d\theta = \int_0^{\pi}\int_0^{2(1+\cos\theta)} r^3\cos\theta\,dr\,d\theta = \int_0^{\pi}\frac{1}{4}r^4\cos\theta\Big]_0^{2(1+\cos\theta)} d\theta = 4\int_0^{\pi}(1+\cos\theta)^4\cos\theta\,d\theta = 4\int_0^{\pi}\left[\cos\theta + 2(1+\cos 2\theta) + 6(\cos\theta - \sin^2\theta\cos\theta) + \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) + (\cos\theta - 2\sin^2\theta\cos\theta + \sin^4\theta\cos\theta)\right] d\theta = 4(\frac{7}{2}\theta + 8\sin\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta - \frac{8}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta)\,]_0^{\pi} = 4(\frac{7}{2}\pi) = 14\pi$. Hence, $\bar{x} = \frac{M_y}{M} = \frac{14\pi}{20\pi/3} = \frac{21}{10}$. So the center of mass is $\left(\frac{21}{10}, \frac{96}{25\pi}\right)$.

44.82 Find the center of mass of the first-quadrant part of the disk of radius a with center at the origin, if the density function is y.

▌ The mass $M = \int_0^a\int_0^{\sqrt{a^2-y^2}} y\,dx\,dy = \int_0^a y\sqrt{a^2-y^2}\,dy = -\frac{1}{2}\cdot\frac{2}{3}(a^2-y^2)^{3/2}\,]_0^a = -\frac{1}{3}(-a^3) = \frac{1}{3}a^3$. The moment about the x-axis is

$$M_x = \iint_{\mathcal{R}} y\cdot y\,dA = \int_0^{\pi/2}\int_0^a y^2\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_0^a r^3\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}a^4\sin^2\theta\,d\theta$$

$$= \frac{1}{4}a^4\int_0^{\pi/2}\frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{8}a^4\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/2} = \frac{a^4}{8}\left(\frac{\pi}{2}\right) = \frac{\pi a^4}{16}$$

Hence, $\bar{y} = \frac{M_x}{M} = \frac{\pi a^4/16}{a^3/3} = \frac{3\pi a}{16}$. The moment about the y-axis is $M_y = \int_0^a\int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy = \int_0^a \frac{1}{2}yx^2\Big]_0^{\sqrt{a^2-y^2}} dy = \frac{1}{2}\int_0^a y(a^2-y^2)\,dy = \frac{1}{2}\left(\frac{a^2}{2}y^2 - \frac{y^4}{4}\right)\Big]_0^a = \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{a^4}{8}$. Therefore, $\bar{x} = \frac{M_y}{M} = \frac{a^4/8}{a^3/3} = \frac{3a}{8}$. Hence, the center of mass is $\left(\frac{3a}{8}, \frac{3\pi a}{16}\right)$.

44.83 Find the center of mass of the cube of edge a with three faces on the coordinate planes, if the density is numerically equal to the sum of the distances from the coordinate planes.

The mass $M = \int_0^a \int_0^a \int_0^a (x+y+z)\,dx\,dy\,dz = 3(\int_0^a x\,dx)(\int_0^a dy)(\int_0^a dz) = \frac{3}{2}a^4$. The moment about the xz-plane is $M_{xz} = \int_0^a \int_0^a \int_0^a y(x+y+z)\,dx\,dy\,dz = \int_0^a \int_0^a y\left[\frac{1}{2}x^2 + (y+z)x\right]\Big]_0^a dy\,dz = \int_0^a \int_0^a y\left[\frac{a^2}{2} + a(y+z)\right] dy\,dz = \int_0^a \left[\frac{a^2}{4}y^2 + a\left(\frac{1}{3}y^3 + \frac{1}{2}zy^2\right)\right]\Big]_0^a dz = \int_0^a \left(\frac{a^4}{4} + \frac{a^4}{3} + \frac{a^3}{2}z\right) dz = \frac{7}{12}a^4 z + \frac{a^3}{4}z^2\Big]_0^a = \frac{7}{12}a^5 + \frac{1}{4}a^5 = \frac{5}{6}a^5$. So $\bar{y} = \frac{M_{xz}}{M} = \frac{\frac{5}{6}a^5}{\frac{3}{2}a^4} = \frac{5}{9}a$. By symmetry, $\bar{x} = \frac{5}{9}a$, $\bar{z} = \frac{5}{9}a$.

44.84 Find the center of mass of the first octant of the ball of radius a, $x^2+y^2+z^2 \le a^2$, if the density is numerically equal to z.

The mass is one-eighth of that of the ball in Problem 44.77, or $\pi a^4/16$. The moment about the xz-plane is

$$M_{xz} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \underbrace{\rho\sin\phi\sin\theta}_{y}\cdot\underbrace{\rho\cos\phi}_{z}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \rho^5\sin^2\phi\cos\phi\sin\theta\Big]_0^a d\phi\,d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2} \frac{1}{3}\sin^3\phi\sin\theta\Big]_0^{\pi/2} d\theta = \frac{a^5}{5}\int_0^{\pi/2}\frac{1}{3}\sin\theta\,d\theta = \frac{a^5}{15}(-\cos\theta)\Big]_0^{\pi/2} = \frac{a^5}{15}.$$

Hence, $\bar{y} = \frac{M_{xz}}{M} = \frac{a^5/15}{\pi a^4/16} = \frac{16a}{15\pi}$. By symmetry, $\bar{x} = \frac{16a}{15\pi}$. The moment about the xy-plane is $M_{xy} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2\cos^2\theta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{5}a^5\cos^2\phi\sin\theta\,d\phi\,d\theta = \frac{a^5}{5}\int_0^{\pi/2}\left(-\frac{\cos^3\phi}{3}\right)\Big]_0^{\pi/2} d\theta = \frac{a^5}{5}\int_0^{\pi/2}\frac{1}{3}\,d\theta = \frac{a^5}{15}\cdot\frac{\pi}{2} = \frac{\pi a^5}{30}$. Hence $\bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^5/30}{\pi a^4/16} = \frac{8a}{15}$. Thus, the center of mass is $\left(\frac{16a}{15\pi}, \frac{16a}{15\pi}, \frac{8a}{15}\right)$.

44.85 Find the center of mass of a solid right circular cone $\mathscr{C}$ of height h and radius of base b, if the density is equal to the distance from the base.

Let the cone have the base $x^2+y^2 \le b$ in the xy-plane and vertex at $(0,0,h)$; its equation is then $z = h-(h/b)r$. By symmetry, $\bar{x}=\bar{y}=0$. The mass $M = \iiint_{\mathscr{C}} z\,dV$. Use cylindrical coordinates. Then $M = \int_0^{2\pi}\int_0^b\int_0^{h-(h/b)r} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^b \frac{1}{2}rz^2\Big]_0^{h-(h/b)r} dr\,d\theta = \frac{h^2}{2b^2}\int_0^{2\pi}\int_0^b r(b-r)^2\,dr\,d\theta = \frac{h^2}{2b^2}\int_0^{2\pi}\left(\frac{1}{2}b^2r^2 - \frac{2b}{3}r^3 + \frac{1}{4}r^4\right)\Big]_0^b d\theta = \frac{h^2}{2b^2}\int_0^{2\pi}\left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4\right)d\theta = \frac{h^2b^2}{24}\cdot 2\pi = \frac{\pi h^2b^2}{12}$. The moment about the xy-plane is $M_{xy} = \int_0^{2\pi}\int_0^b\int_0^{h-(h/b)r} z^2r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^b \frac{1}{3}rz^3\Big]_0^{h-(h/b)r} dr\,d\theta = \frac{h^3}{3b^3}\int_0^{2\pi}\int_0^b r(b-r)^3\,dr\,d\theta = \frac{h^3}{3b^3}\int_0^{2\pi}\left(\frac{b^3}{2}r^2 - b^2r^3 + \frac{3}{4}br^4 - \frac{1}{5}r^5\right)\Big]_0^b d\theta = \frac{h^3}{3b^3}\int_0^{2\pi}\left(\frac{b^5}{2} - b^5 + \frac{3}{4}b^5 - \frac{1}{5}b^5\right)d\theta = \frac{h^3b^2}{60}2\pi = \frac{\pi h^3b^2}{30}$. Hence, $\bar{z} = \frac{M_{xy}}{M} = \frac{\pi h^3b^2/30}{\pi h^2b^2/12} = \frac{2}{5}h$. Thus, the center of mass is $\left(0, 0, \frac{2}{5}h\right)$.

44.86 Find the moments of inertia of the triangle bounded by $3x+4y=24$, $x=0$, and $y=0$, and having density 1.

The moment of inertia with respect to the x-axis is $I_x = \int_0^8\int_0^{6-(3/4)x} y^2\,dy\,dx = \int_0^8 \frac{1}{3}y^3\Big]_0^{6-(3/4)x} dx = \frac{9}{64}\int_0^8 (8-x)^3\,dx = \frac{9}{64}(-1)\frac{(8-x)^4}{4}\Big]_0^8 = \frac{9}{64}\cdot\frac{8^4}{4} = 144$. The moment of inertia with respect to the y-axis is $I_y = \int_0^8\int_0^{6-(3/4)x} x^2\,dy\,dx = \int_0^8 x^2\cdot\frac{3}{4}(8-x)\,dx = \frac{3}{4}\left(\frac{8}{3}x^3 - \frac{x^4}{4}\right)\Big]_0^8 = \frac{3}{4}\cdot\frac{1}{12}(8)^4 = 256$.

44.87 Find the moment of inertia of a square plate of side a with respect to a side, if the density is numerically equal to the distance from an extremity of that side.

Let the square be $0 \le x \le a$, $0 \le y \le a$, and let the density at (x, y) be the distance $\sqrt{x^2+y^2}$ from the origin. We want to find the moment of inertia I_x about the x-axis: $I_x = \int_0^a\int_0^a y^2\sqrt{x^2+y^2}\,dy\,dx$. Now, by the symmetry of the situation, the moment of inertia about the y-axis, $I_y = \int_0^a\int_0^a x^2\sqrt{x^2+y^2}\,dy\,dx$, must be equal to I_x. This allows us to write

$$I_x = \frac{1}{2}(I_x+I_y) = \frac{1}{2}\int_0^a\int_0^a (x^2+y^2)^{3/2}\,dy\,dx = \int_0^a\int_0^x (x^2+y^2)^{3/2}\,dy\,dx$$

where symmetry was again invoked in the last step. Change to polar coordinates and use Problem 29.9:

$$I_x = \int_0^{\pi/4}\int_0^{a\sec\theta} r^4\,dr\,d\theta = \frac{a^5}{5}\int_0^{\pi/4}\sec^5\theta\,d\theta = \frac{a^5}{5}\left[\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\cdot\right]_0^{\pi/4}$$

$$= \frac{a^5}{5}\left[\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{8} + \frac{3}{8}\ln(1+\sqrt{2}) - 0 - 0 - 0\right] = \frac{a^5}{40}\left[7\sqrt{2} + \ln(7+5\sqrt{2})\right]$$

44.88 Find the moment of inertia of a cube of edge a with respect to an edge if the density is numerically equal to the square of the distance from one extremity of that edge.

Consider the cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$. Let the density be $x^2+y^2+z^2$, the square of the distance from the origin. Let us calculate the moment of inertia around the x-axis. $I_x = \int_0^a\int_0^a\int_0^a (y^2+z^2)(x^2+y^2+z^2)\,dx\,dy\,dz$. [The distance from (x, y, z) to the x-axis is $\sqrt{y^2+z^2}$.] Then $I_x = \int_0^a\int_0^a (y^2+z^2)[\frac{1}{3}x^3 + (y^2+z^2)x]\,]_0^a\,dy\,dz = \int_0^a\int_0^a (y^2+z^2)[\frac{1}{3}a^3 + (y^2+z^2)a]\,dy\,dz = \int_0^a[\frac{1}{3}a^3(\frac{1}{3}y^3+z^2y) + a(\frac{1}{5}y^5 + \frac{2}{3}y^3z^2 + z^4y)]\,]_0^a\,dz = \int_0^a[\frac{1}{3}a^3(\frac{1}{3}a^3+az^2) + a(\frac{1}{5}a^5 + \frac{2}{3}a^3z^2 + az^4)]\,dz = a^2\int_0^a[\frac{1}{3}a^2(\frac{1}{3}a^2+z^2) + (\frac{1}{4}a^4 + \frac{2}{3}a^2z^2 + z^4)]\,dz = a^2[\frac{1}{3}a^2(\frac{1}{3}a^2z+\frac{1}{3}z^3) + (\frac{1}{4}a^4z + \frac{2}{9}a^2z^3 + \frac{1}{5}z^5)]\,]_0^a = a^2[\frac{1}{3}a^2(\frac{2}{3}a^3) + (\frac{1}{4}a^5 + \frac{2}{9}a^5 + \frac{1}{5}a^5)] = a^7(\frac{2}{9}+\frac{1}{4}+\frac{2}{9}+\frac{1}{5}) = \frac{161}{180}a^7$.

44.89 Find the centroid of the region outside the circle $r=1$ and inside the cardioid $r = 1+\cos\theta$.

Refer to Fig. 44-20. Clearly, $\bar{y}=0$, and $\bar{x}$ is the same for the given region as for the half lying above the polar axis. For the latter, the area $A = \int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}r^2\Big]_1^{1+\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi/2}\left(2\cos\theta + \frac{1+\cos\theta}{2}\right)d\theta = \frac{1}{2}\left(2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big]_0^{\pi/2} = \frac{1}{2}\left(2+\frac{\pi}{4}\right) = \frac{\pi+8}{8}$. The moment about the y-axis is $M_y = \int_0^{\pi/2}\int_1^{1+\cos\theta} xr\,dr\,d\theta = \int_0^{\pi/2}\int_1^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{3}r^3\cos\theta\Big]_1^{1+\cos\theta} d\theta = \frac{1}{3}\int_0^{\pi/2}[(1+\cos\theta)^3 - 1]\cos\theta\,d\theta = \frac{1}{3}\int_0^{\pi/2}(3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\,d\theta = \frac{1}{3}\int_0^{\pi/2}\left[\frac{3}{2}(1+\cos 2\theta) + 3(\cos\theta - \sin^2\theta\cos\theta) + \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\right]\Big]_0^{\pi/2} = \frac{1}{3}\left(\frac{15}{8}\theta + 3\sin\theta + \sin 2\theta + \frac{\sin 4\theta}{32} - \sin^3\theta\right)\Big]_0^{\pi/2} = \frac{1}{3}\left(\frac{15\pi}{16} + 3 - 1\right) = \frac{5\pi}{16} + \frac{2}{3} = \frac{15\pi+32}{48}$. Hence, $\bar{x} = \frac{M_y}{A} = \frac{(15\pi+32)/48}{(\pi+8)/8} = \frac{15\pi+32}{6(\pi+8)}$.

44.90 Find the centroid $(\bar{x}, \bar{y})$ of the region in the first quadrant bounded by $y^2 = 6x$, $y=0$, and $x=6$ (Fig. 44-43).

The area $A = \int_0^6\int_{y^2/6}^6 dx\,dy = \int_0^6\left(6 - \frac{y^2}{6}\right)dy = \left(6y - \frac{1}{18}y^3\right)\Big]_0^6 = 6(6-2) = 24$. The moment about the x-axis is $M_x = \int_0^6\int_{y^2/6}^6 y\,dx\,dy = \int_0^6 y\left(6-\frac{y^2}{6}\right)dy = \left(3y^2 - \frac{1}{24}y^4\right)\Big]_0^6 = 36\left(3-\frac{3}{2}\right) = 54$. Hence, $\bar{y} = \frac{M_x}{A} = \frac{54}{24} = \frac{9}{4}$. The moment about the y-axis is $M_y = \int_0^6\int_{y^2/6}^6 x\,dx\,dy = \int_0^6 \frac{1}{2}x^2\Big]_{y^2/6}^6 dy = \frac{1}{2}\int_0^6\left(36 - \frac{y^4}{36}\right)dy = \frac{1}{2}\left(36y - \frac{1}{180}y^5\right)\Big]_0^6 = 3\left(36 - \frac{36}{5}\right) = \frac{12\cdot 36}{5}$. Hence, $\bar{x} = \frac{M_y}{A} = \frac{12(36)/5}{24} = \frac{18}{5}$. So the centroid is $\left(\frac{18}{5}, \frac{9}{4}\right)$.

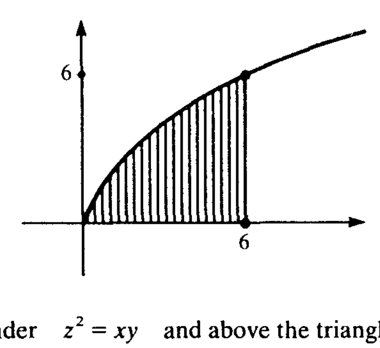

Fig. 44-43

44.91 Find the centroid of the solid under $z^2 = xy$ and above the triangle bounded by $y=x$, $y=0$, and $x=4$.

The volume $V = \int_0^4\int_0^x\int_0^{\sqrt{xy}} dz\,dy\,dx = \int_0^4\int_0^x \sqrt{xy}\,dy\,dx = \int_0^4 \frac{2}{3}x^{1/2}y^{3/2}\Big]_0^x dx = \int_0^4 \frac{2}{3}x^2\,dx = \frac{2}{9}x^3\Big]_0^4 = \frac{128}{9}$. The moment about the yz-plane is $M_{yz} = \int_0^4\int_0^x\int_0^{\sqrt{xy}} x\,dz\,dy\,dx = \int_0^4\int_0^x xz\,]_0^{\sqrt{xy}}\,dy\,dx = \int_0^4\int_0^x x^{3/2}y^{1/2}\,dy\,dx = \int_0^4 \frac{2}{3}x^{3/2}y^{3/2}\Big]_0^x dx = \int_0^4 \frac{2}{3}x^3\,dx = \frac{1}{6}x^4\Big]_0^4 = \frac{128}{3}$. Hence, $\bar{x} = \frac{M_{yz}}{V} = \frac{\frac{128}{3}}{\frac{128}{9}} = 3$. The moment about the

xz-plane is $M_{xz} = \int_0^4 \int_0^x \int_0^{\sqrt{xy}} y\,dz\,dy\,dx = \int_0^4 \int_0^x x^{1/2}y^{3/2}\,dy\,dx = \int_0^4 \frac{2}{5} x^{1/2}y^{5/2}\Big]_0^x dx = \int_0^4 \frac{2}{5} x^3\,dx = \frac{1}{10} x^4 \Big]_0^4 = \frac{128}{5}$. Hence, $\bar{y} = \frac{M_{xz}}{V} = \frac{\frac{128}{5}}{\frac{128}{9}} = \frac{9}{5}$. The moment about the xy-plane is $M_{xy} = \int_0^4 \int_0^x \int_0^{\sqrt{xy}} z\,dz\,dy\,dx = \int_0^4 \int_0^x \frac{1}{2} z^2 \Big]_0^{\sqrt{xy}} dy\,dx = \int_0^4 \int_0^x \frac{1}{2} xy\,dy\,dx = \int_0^4 \frac{1}{4} xy^2 \Big]_0^x dx = \int_0^4 \frac{1}{4} x^3\,dx = \frac{1}{16} x^4 \Big]_0^4 = 16$. Hence, $\bar{z} = \frac{M_{xy}}{V} = \frac{16}{\frac{128}{9}} = \frac{9}{8}$. Thus, the centroid is $\left(3, \frac{9}{5}, \frac{9}{8}\right)$.

44.92 Find the centroid of the upper half $\mathscr{H}$ of the solid ball of radius a with center at the origin.

We know $V = \frac{1}{2}(\frac{4}{3}\pi a^3) = \frac{2}{3}\pi a^3$. By symmetry, $\bar{x} = \bar{y} = 0$. The moment about the xy-plane is $M_{xy} = \iiint_{\mathscr{H}} z\,dV$. Use spherical coordinates.

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \cos\phi \cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_0^{\pi/2} \tfrac{1}{4}\rho^4 \cos\phi \sin\phi \Big]_0^a d\phi\,d\theta$$

$$= \frac{a^4}{4} \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi\,d\phi\,d\theta = \frac{a^4}{4} \int_0^{2\pi} \frac{1}{2} \sin^2\phi \Big]_0^{\pi/2} d\theta = \frac{a^4}{8} \int_0^{2\pi} d\theta = \frac{a^4}{8} \cdot 2\pi = \frac{\pi a^4}{4}$$

Hence, $\bar{z} = \frac{M_{xy}}{V} = \frac{\pi a^4/4}{2\pi a^3/3} = \frac{3}{8} a$.