

Chapter 13

Quadric Surfaces

The quadrics are all surfaces that can be expressed as a second degree polynomial in x , y and z . They include important principle shapes such as those shown in Figure 13.1.

The general implicit form for a 3D quadric surface can be written in homogeneous x, y, z, w coordinates as:

$$ax^2 + 2bxy + 2cxz + 2dxw + ey^2 + 2fyz + 2gyw + hz^2 + 2izw + jw^2 = 0. \quad (13.1)$$

Setting $w = 1$, this provides the ability to position the quadrics in space. This equation can also be written in the matrix form:

$$\mathbf{x}^t Q \mathbf{x} = 0, \quad (13.2)$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, $\mathbf{x}^t = [x \ y \ z \ w]$, and $Q = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$

(multiply this out and you will see that you get the original polynomial form. You will also see why we put the 2s in the original equation).

Now, if we wish to raytrace a quadric, we insert the ray equation $\mathbf{x} = \mathbf{p} + t\mathbf{u}$ into Equation 13.2, giving

$$(\mathbf{p} + t\mathbf{u})^t Q (\mathbf{p} + t\mathbf{u}) = 0,$$

or

$$\mathbf{p}^t Q \mathbf{p} + t\mathbf{p}^t Q \mathbf{u} + t\mathbf{u}^t Q \mathbf{p} + t^2 \mathbf{u}^t Q \mathbf{u} = 0.$$

Since Q is symmetric, the two middle terms of this equation are identical, so we can rearrange and write

$$t^2 \mathbf{u}^t Q \mathbf{u} + 2t\mathbf{u}^t Q \mathbf{p} + \mathbf{p}^t Q \mathbf{p} = 0. \quad (13.3)$$

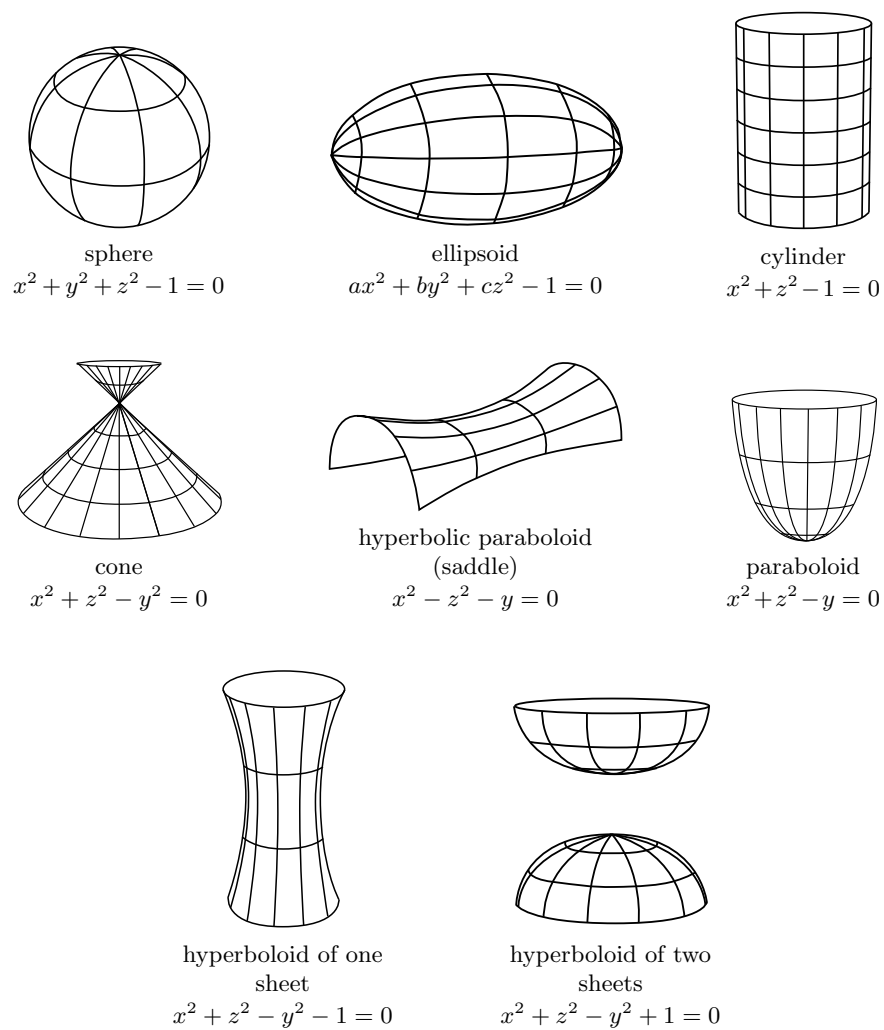


Figure 13.1: The important quadric surfaces.

Note that Equation 13.3 is quadratic in t , so by the quadratic formula it has solutions

$$t = \frac{-\mathbf{u}^t Q \mathbf{p} \pm \sqrt{(\mathbf{u}^t Q \mathbf{p})^2 - (\mathbf{u}^t Q \mathbf{u})(\mathbf{p}^t Q \mathbf{p})}}{\mathbf{u}^t Q \mathbf{u}}. \quad (13.4)$$

This gives a general form for all of the quadrics that is completely and uniquely specified by the matrix Q . This representation is especially nice, since we can transform the quadric into any desired coordinate system via the transformation

$$Q' = M^{-1} Q (M^{-1})^t,$$

where M is the transform from the object coordinates in which the quadric is defined to the desired scene coordinates.

Because of this, we need only consider the important canonical principle quadric shapes, like those in Figure 13.1, and then apply the appropriate transform to create the shape desired, before transforming to world coordinates. For example, a sphere of radius 1, centered at the origin, is given by

$$Q_{sphere} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

To make a sphere of radius 3, we simply scale via

$$M = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ with } M^{-1} = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that

$$\begin{aligned} M^{-1} Q_{sphere} (M^{-1})^t &= \\ \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= \\ \begin{bmatrix} 1/9 & 0 & 0 & 0 \\ 0 & 1/9 & 0 & 0 \\ 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \end{aligned}$$

which gives a sphere of radius $3 = \sqrt{9}$.

We can go further. For example, we can make an ellipsoid with desired radius

in the x , y and z directions by a non-uniform scale

$$M = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, we can translate, rotate or shear these shapes to make a large variety of shapes from this one representation.

Raytracing is as simple as solving Equation 13.4. *Well – not quite! Except for the sphere and ellipsoid, all of the quadrics need caps to be useful, since they are open, infinite surfaces.*

For example, the cylinder $x^2 + z^2 - 1 = 0$ is infinite in the y direction. We can apply caps at $y = 0$ and $y = 1$ (for example) by intersecting the cylinder with the planes $y = 0$ and $y = 1$.

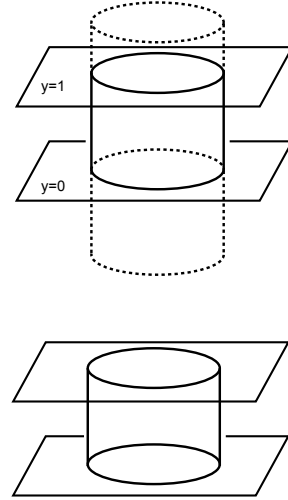
When we shoot a ray, we look for all intersections between the ray and the cylinder and the two planes, and sort these in order by distance along the ray. From this we can tell if the ray hits the truncated cylinder (between $y = 0$ and $y = 1$), and if it hits it, if the hit is on one of the caps or on the body.

Let c_1, c_2 be the cap plane hits, and let b_1, b_2 be hits on the cylinder body. The value of each is the ray parameter at the hit, and assume (without loss of generality) that $c_1 < c_2$ and $b_1 < b_2$. We have several non degenerate cases and some special cases. The general cases are:

1. c_1, c_2, b_1, b_2 : miss
2. c_1, b_1, c_2, b_2 : hit body, exit cap
3. c_1, b_1, b_2, c_2 : hit body, exit body
4. b_1, c_1, b_2, c_2 : hit cap, exit body
5. b_1, c_1, c_2, b_2 : hit cap, exit cap
6. b_1, b_2, c_1, c_2 : miss

The special cases are when one of the equations has only one solution, for example if the ray is tangent to the cylinder, or no solutions, like when the ray misses the cylinder, or is parallel to the caps.

Note that it is also possible to find a parameterization for each of the quadrics.



In the last chapter, we already discussed how to do this for a sphere using spherical coordinates. Parameterizing a cylinder or a cone is straightforward using cylindrical coordinates. In this case, the angle parameter measures around the body of the cylinder or cone, and the second parameter measures height. If a uniform spacing is desired in the vertical direction all of the other quadrics except saddle can also be parameterized using cylindrical coordinates.