Solutions to Final Practice Problems

Math 22

March 15, 2012

1. Change the Cartesian integral into an equivalent polar integral and evaluate:
$$I = \int_{-5}^{0} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \left(x^2 + y^2\right)\, dydx.$$

   Solution
   The domain of integration for this integral is
   $$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid -5 \leq x \leq 0,\ -\sqrt{25 - x^2} \leq y \leq \sqrt{25 - x^2} \right\}.$$

   Geometrically $\mathcal{D}$ is the region in the second and third quadrants bounded by the circle of radius 5 centered at the origin and the line $x = 0$ (draw a picture!). In polar coordinates
   $$\mathcal{D} = \left\{ (r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 5,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\}.$$

   Changing variables:
   $$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ dydx &= rdrd\theta, \end{aligned}$$

we obtain

$$
\begin{aligned}
I &= \int_{\pi/2}^{3\pi/2} \int_0^5 r\left((r\cos\theta)^2 + (r\sin\theta)^2\right) dr d\theta \\
&= \int_{\pi/2}^{3\pi/2} \int_0^5 r^3 \, dr d\theta \\
&= \left(\frac{3\pi}{2} - \frac{\pi}{2}\right)\left(\frac{5^4}{4}\right) \\
&= \frac{625\pi}{4}.
\end{aligned}
$$

2. Evaluate the double integral:

$$
I = \int\int_{\mathcal{D}} \sin\left(x^2 + y^2\right) dA
$$

where $\mathcal{D}$ is the part of the unit circle that lies in the first quadrant.

Solution

In polar coordinates we describe $\mathcal{D}$ by:

$$
\mathcal{D} = \left\{(r,\theta) \in \mathbb{R}^2 \mid 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\right\}.
$$

Changing to polar coordinates we find (note that $dA = rdrd\theta$ and $x^2 + y^2 = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2$):

$$
\begin{aligned}
I &= \int_0^{\pi/2} \int_0^1 r\sin\left(r^2\right) dr d\theta \\
&= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^1 r\sin\left(r^2\right) dr\right) \\
&= \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\int_0^1 \sin u \, du\right) \\
&= \frac{\pi}{4}\left(-\cos(1) + \cos(0)\right) = \frac{\pi}{4}\left(1 - \cos(1)\right)
\end{aligned}
$$

(in the penultimate line we made the u-subs $u = r^2$).

3. Set up, but do not evaluate, the integral to find the volume of the solid bounded by the planes $2x + y + z = 4$, $z = -6$, $y - x = 4$, and $y = 0$.

Solution
We find the intersection of the plane $2x + y + z = 4$ with the plane $z = -6$. Substituting the later into the former gives

$$y = 10 - 2x.$$

To find the volume of the solid we therefore integrate the function

$$z = (4 - y - 2x) - (-6) = 10 - y - 2x$$

over the region $\mathcal{D}$ in the xy plane bounded by the lines $y = 0$, $y = 4 + x$ and $y = 10 - 2x$ (sketch this region!). Your sketch should show that $\mathcal{D}$ is the interior of the triangle with vertices at $(-4, 0)$, $(5, 0)$ and $(2, 6)$. Expressing $\mathcal{D}$ as a 'type two' region gives

$$\mathcal{D} = \left\{(x, y) \in \mathbb{R}^2 \mid y - 4 \leq x \leq 5 - y/2,\ 0 \leq y \leq 6\right\}.$$

Hence the volume of the solid is described by the iterated integral

$$V = \int_0^6 \int_{y-4}^{5-y/2} (10 - y - 2x)\ dxdy.$$

Alternative Solution
An alternative answer to this question comes from expressing $\mathcal{D}$ as the union of two 'type one' regions. From the sketch,

$$\mathcal{D} = \left\{(x, y) \in \mathbb{R}^2 \mid -4 \leq x \leq 2,\ 0 \leq y \leq x + 4\right\} \bigcup \left\{(x, y) \in \mathbb{R}^2 \mid 2 \leq x \leq 5,\ 0 \leq y \leq 10 - 2x\right\}.$$

This leads to

$$V = \int_{-4}^2 \int_0^{x+4} (10 - y - 2x)\ dydx + \int_2^5 \int_0^{10-2x} (10 - y - 2x)\ dydx.$$

4. Evaluate the integral
$$\int_0^2 \int_0^3 e^{x-y}\ dydx.$$

Solution

$$
\begin{aligned}
\int_0^2 \int_0^3 e^{x-y}\, dydx &= \int_0^2 \int_0^3 e^x e^{-y}\, dydx \\
&= \int_0^2 e^x\, dx \int_0^3 e^{-y}\, dy \\
&= \left(e^2 - 1\right)\left(1 - e^{-3}\right) \\
&= e^2 - e^{-1} - 1 + e^{-3}.
\end{aligned}
$$

Alternative Solution

$$
\begin{aligned}
\int_0^2 \int_0^3 e^{x-y}\, dydx &= \int_0^2 \left[-e^{x-y}\right]_{y=0}^{y=3} dx \\
&= \int_0^2 e^x - e^{x-3}\, dx \\
&= \left[e^x - e^{x-3}\right]_{x=0}^{x=2} \\
&= e^2 - e^{-1} - (1 - e^{-3}) \\
&= e^2 - e^{-1} + e^{-3} - 1.
\end{aligned}
$$

5. Sketch the region over which we are integrating and evaluate the integral

$$I = \int_0^1 \int_{3y}^3 e^{x^2}\, dxdy.$$

Solution
The domain of integration is

$$\mathcal{D} = \left\{(x, y) \in \mathbb{R}^2 \mid 3y \leq x \leq 3,\ 0 \leq y \leq 1\right\}.$$

Geometrically this region represents the interior of a triangle with vertices at $(0,0)$, $(3,0)$ and $(3,1)$ (this should be your sketch). Since the integral is difficult to evaluate as expressed in the question, we change the order of integration. As a 'type one' region, $\mathcal{D}$ may be expressed

$$\mathcal{D} = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3,\ 0 \leq y \leq \frac{x}{3}\right\}.$$

Hence

$$\begin{aligned}
I &= \int_0^3 \int_0^{x/3} e^{x^2}\, dydx \\
&= \int_0^3 \left[ye^{x^2} \right]_{y=0}^{y=x/3} dx \\
&= \frac{1}{3} \int_0^3 xe^{x^2}\, dx \\
&= \frac{1}{3} \int_0^9 e^u \frac{du}{2} \\
&= \frac{1}{6} \left(e^9 - 1 \right)
\end{aligned}$$

(in the penultimate line we made the u-substitution $u = x^2$).

6. Calculate the iterated integral

$$\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dydx$$

and completely simplify your answer.

Solution

$$\begin{aligned}
\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dydx &= \int_1^4 \left[x \ln|y| + \frac{y^2}{2x} \right]_{y=1}^{y=2} dx \\
&= \int_1^4 \left(\ln(2)x + \frac{3}{2x} \right) dx \\
&= \left[\frac{\ln(2)x^2}{2} + \frac{3\ln|x|}{2} \right]_1^4 \\
&= \frac{1}{2} \left(16\ln(2) + 3\ln(4) - \ln(2) \right) \\
&= \frac{1}{2} \left(16\ln(2) + 6\ln(2) - \ln(2) \right) \\
&= \frac{21\ln(2)}{2}.
\end{aligned}$$

7. Find the volume of the solid lying under the elliptic paraboloid

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

and above the rectangle $[-1, 1] \times [-2, 2]$.

Solution

The volume in question corresponds to the iterated integral

$$\begin{aligned} V &= \int_{-2}^{2} \int_{-1}^{1} \left(\frac{x^2}{4} + \frac{y^2}{9} \right) dxdy \\ &= \int_{-2}^{2} \left[\frac{x^3}{12} + \frac{xy^2}{9} \right]_{x=-1}^{x=1} dy \\ &= \int_{-2}^{2} \left(\frac{1}{6} + \frac{2y^2}{9} \right) dy \\ &= \left[\frac{y}{6} + \frac{2y^3}{27} \right]_{-2}^{2} \\ &= \frac{4}{6} + \frac{32}{27} = \frac{50}{27}. \end{aligned}$$

8. Sketch the region of integration and change the order of integration,

$$\int_{1}^{2} \int_{0}^{\ln x} f(x, y)\, dydx.$$

Solution

Reading off the limits of integration we see that the domain of integration may be expressed (as a type one region):

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid 1 \le x \le 2,\ 0 \le y \le \ln x \right\}.$$

A sketch of the region shows that $\mathcal{D}$ is the region bounded by the curve $y = \ln x$ (which intersects the x-axis at $x = 1$), and the lines $y = 0$, $x = 2$. To reverse the order of integration we need to express $\mathcal{D}$ as a type two region. Our sketch gives:

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid e^y \le x \le 2,\ 0 \le y \le \ln(2) \right\}.$$

Hence

$$\int_{1}^{2} \int_{0}^{\ln x} f(x, y)\, dydx = \int_{0}^{\ln(2)} \int_{e^y}^{2} f(x, y)\, dxdy.$$

9. Evaluate the integral in polar coordinates:

$$\int\int_{\mathcal{D}} \cos\left(x^2+y^2\right)\, dA,$$

where $\mathcal{D}$ is the region that lies to the left of the y-axis within the circle $x^2+y^2=9$.

Solution
In polar coordinates the domain of integration may be expressed

$$\mathcal{D} = \left\{ (r,\theta) \in \mathbb{R}^2 \mid 0 \le r \le 3, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \right\}.$$

In polar coordinates we have $x^2+y^2=r^2$ and $dA = r dr d\theta$. Therefore

$$\begin{aligned}
\int\int_{\mathcal{D}} \cos\left(x^2+y^2\right)\, dA &= \int_{\pi/2}^{3\pi/2} \int_0^3 r\cos\left(r^2\right)\, dr d\theta \\
&= \left(\int_{\pi/2}^{3\pi/2} d\theta\right)\left(\int_0^3 r\cos\left(r^2\right)\, dr\right) \\
&= (\pi)\left(\int_0^9 \cos(u)\, \frac{du}{2}\right) \\
&= \frac{\pi}{2}\sin(9)
\end{aligned}$$

(on the penultimate line we made the u-subs $u=r^2$).

10. Evaluate the iterated integral,

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy dx dz.$$

Solution

$$
\begin{aligned}
\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dydxdz &= \int_0^1 \int_0^z \left[6xzy\right]_{y=0}^{y=x+z} dxdz \\
&= \int_0^1 \int_0^z \left(6x^2z + 6xz^2\right) dxdz \\
&= \int_0^1 \left[2x^3z + 3x^2z^2\right]_{x=0}^{x=z} dz \\
&= \int_0^1 \left(5z^4\right) dz \\
&= \left[z^5\right]_0^1 \\
&= 1.
\end{aligned}
$$

11. Evaluate the iterated integral

$$\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\, dxdzdy.$$

Solution

$$
\begin{aligned}
\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\, dxdzdy &= \int_0^3 \int_0^1 ze^y \sqrt{1-z^2}\, dzdy \\
&= \left(\int_0^3 e^y\, dy\right) \left(\int_0^1 z\sqrt{1-z^2}\, dz\right) \\
&= \left(e^3 - 1\right) \left(-\int_1^0 \sqrt{u}\, \frac{du}{2}\right) \\
&= \frac{1}{3}\left(e^3 - 1\right)
\end{aligned}
$$

(we made the u-substitution $u = 1 - z^2$ in the second to last line).

12. If the density is $\rho = 1$ then the center of mass of the thin plate in the xy-plane bounded by $y = \cos x$ and $y = 0$ lies on the y-axis. Find $\overline{y}$, the y coordinate of the center of mass.

Solution

By symmetry it suffices to consider the section of the plate in the region

$-\pi/2 \leq x \leq \pi/2$. This region may be expressed

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq \cos x \right\}.$$

Therefore we find

$$\begin{aligned}
M_y &= \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} \rho(x, y) y \, dy dx \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y \, dy dx \\
&= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} \cos^2 x \right) \, dx \\
&= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \cos(2x) + 1 \, dx \\
&= \frac{1}{4} \left[\frac{\sin(2x)}{2} + x \right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4} \pi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
M &= \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} \rho(x, y) \, dy dx \\
&= \int_{-\pi/2}^{\pi/2} \cos x \, dx \\
&= 2.
\end{aligned}$$

Hence

$$\overline{y} = \frac{M_y}{M} = \frac{\pi}{8}.$$