CHAPTER 1
THE REAL NUMBER SYSTEM

1.0 Introduction:

The set of **natural numbers** (or the counting numbers) is the set

$N = \{1, 2, 3, \ldots, \}$.

This set is **closed** under + and x , but not closed under $-$ and $\div$.

The inclusion of 0 gives us the set

$W = \{0, 1, 2, 3, \ldots \}$,

commonly called the set of **whole numbers**. The set N may be extended to include all the negative whole numbers to give the set

$$\mathbf{Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}}$$

which is called the set of **integers**. This set Z is closed under +, x and $-$, but not closed under $\div$. A further extension of this set to include all fractions, both proper and improper, gives the set

$Q = \{ \frac{a}{b} : a, b \in Z , b \neq 0 \}$.

This is called the set of **Rational Numbers**. Q is closed under all the arithmetic operations +, x, –, ÷ . In the case of closure under ÷ , division by 0 must be avoided, as the result is either indeterminate (%) or infinity (∞).

Note that

(i) $0 \in Q$, since $\frac{o}{b}$, $b \neq o$ is 0.

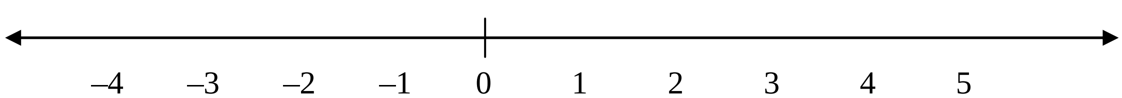
(ii) $Z \subset Q$, since $\frac{a}{1} = a$, for all $a \in Z$.

(iii) It is clear that

$N \subset Z \subset Q$.

Graphical Representation

The above number systems may be represented graphically as follows:



Z

We may identify the subsets of Z as follows:

$$Z^{-} = \{\ldots -3, -2, -1\}$$

called the negative integers; and

$$Z^{+} = \{1, 2, 3, \ldots\}$$

called the positive integers. The set

$$\{0\} \cup Z^{+} = \{0, 1, 2, 3, \ldots\}$$

is called the set of non-negative integers.

Note that

$$Z = Z^{-} \cup \{0\} \cup Z^{+} .$$

The rational numbers may be represented by the straight line

$$\longleftrightarrow$$

$$-3 \quad -2 \quad -\tfrac{3}{2} \quad -1 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{3}{2} \quad 2 \quad \tfrac{5}{2} \quad 3$$

Q

Every fraction whether proper or improper may be represented by a point on this line.

The Set of Irrational Numbers

There are numbers that do not belong to Q, that is, they cannot be expressed in the form $\frac{a}{b}$, where a, b $\in$ Z. There are points on the number line which correspond to these numbers. The set of such numbers is called the set of Irrational Numbers. Examples of Irrational Numbers are:

$\sqrt{2}$, $\sqrt{3}$, π, e, etc.

Example:

Prove that $\sqrt{2}$ is an irrational number.

Let us assume that $\sqrt{2}$ is rational. Then

$\sqrt{2} = \frac{a}{b}$, a, b $\in$ Z, b $\neq$ o, and such that $\frac{a}{b}$ is in its least form.

Then

$$2 = \frac{a^2}{b^2}$$

$$\Rightarrow \quad a^2 = 2b^2$$

$\Rightarrow \quad a^2$ is an even number

$\therefore \quad a$ is even

Let $a = 2c,\ c \in Z$.

Then $(2c)^2 = 2b^2$

i.e. $\quad 4c^2 = 2b^2$

$\Rightarrow \quad b^2 = 2c^2$

$\Rightarrow \quad b^2$ is even $\Rightarrow$ b is even.

Since a is even and b is even, it implies that $\frac{a}{b}$ is not in its least form. This contradiction is because of the false premise that $\sqrt{2}$ is rational. $\therefore \sqrt{2}$ is irrational.

Exercise:

Prove that $\sqrt{3}$ is irrational.

Note

(i) Every rational number may be expressed either as a terminating decimal or as a recurring decimal.

E.g. $\frac{1}{2} = 0.5$,

$\frac{2}{5} = 0.4$

$\frac{1}{3} = 0.333 \ldots$ or $0.\dot{3}$

$\frac{1}{12} = 0.0833 \ldots$ or $0.08\dot{3}$

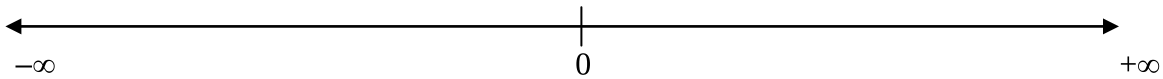
(ii) Every irrational number can only be expressed as a non-terminating and non-recurring decimal number.

E.g. $\sqrt{2} = 1.41423 \ldots..$

$\pi = 3.14159 \ldots..$

The Set of Real Numbers

The union of the set of Rational Numbers and the set of Irrational Numbers is called the set of Real Numbers. Its graphical (or geometrical) representation is the straight line from $-\infty$ to $+\infty$ with an origin 0.



Every real number corresponds to exactly one point on the line and every point on the line corresponds to a real number. Thus this line is called the Real Number Line and represents the **Real Number System**, denoted by R.

1.1 <u>OPERATIONS WITH REAL NUMBERS</u>

Let a, b, c $\in$ R, then

(i) $a + b \in R$ and $ab \in R$ – closure law .

(ii) $a + b = b + a$ – commutative law of addition +.

(iii) $(a + b) + c = a + (b + c)$ – associative law of +.

(iv) $ab = b\,a$ – commutative law of x .

(v) $(ab)c = a\,(bc)$ – associative law of x.

(vi) $a\,(b + c) = ab + ac$ – Distributive law of x over +.

(vii) $a + o = o + a = a$ – existence of an identity under +.

(viii) $a.1 = 1.a = 1$ – existence of an identity under x .

(ix) $a + x = x + a = o$ – existence of an inverse under +

x is denoted by $(-a)$.

(x) $a.a^{-1} = a^{-1}\,a = 1$ – existence of an inverse under x

a^{-1} is denoted by $\frac{1}{a}$.

Any set, such as R, which satisfies the above rules is called a <u>**field**</u>.

1.2 INEQUALITIES

Let a, b $\in$ R. If $a - b$ is a positive number, we say that a is greater than b or b is less than a. We write $a > b$ or $b < a$.

If the possibility exists that a equals b, then we write $a \geq b$ or $b \leq a$.

Theorems on Inequalities:

If a, b, c $\in$ R, then

(i) Either $a > b$, $a = b$ or $a < b$

This is called the law of trichotomy.

(ii) If $a > b$ and $b > c$, then $a > c$

This is called the law of transitivity.

(iii) If $a > b$, then $a + c > b + c$.

(iv) If $a > b$ and $c > o$, then $ac > bc$.

(v) If $a > b$ and $c < o$, then $ac < bc$.

THE ABSOLUTE VALUE OF REAL NUMBERS

Let $x \in$ R. The absolute value of x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} . \qquad (1)$$

E.g.

$$|2| = 2 ,$$

$$|0| = 0$$

$$|-3| = -(-3) = 3$$

The following theorem hold for inequalities.

For x, y $\in$ R,

(i) $|x\,y| = |x|\,|y|$

(ii) $|x + y| \leq |x| + |y|$, called the triangle inequality.

(iii) $|x - y| \geq |x| - |y|$

(iv) $|x| = 0 \Leftrightarrow x = 0.$

Note: The property

(i) may be extended to n real numbers, namely

$$|x_1 . x_2 \ \ldots \ x_n| = |x_1| \ |x_2| \ \ldots \ |x_n| \ ;$$

(ii) may be extended to n real numbers as

$$|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$$

(iii) For the case, $|x - y|$ may be geometrically interpreted as the distance between any two points x and y on the number line. It is clear then that

$$|x - y| = |y - x| \ .$$

Worked Examples

1. Prove that (i) $|x\,y| = |x|\,|y|$

(ii) $|x + y| \leq |x| + |y|$

Solution:

(i) We consider the four cases:

$x = 0, \quad y = 0$

$x > 0, \ y > 0$

$x < 0, \ y > 0$

$x < 0, \ y < 0.$

If $x = 0$ and $y = 0$, $xy = 0.0 = 0$

$\therefore \ |x\,y| = |0| = |0|\,|0| = |x| \ |y|.$

If $x > 0$ and $y > 0$, $xy > 0$, $|x| = x$, $|y| = y$ and $|xy| = xy = |x| \ |y|.$

If $x < 0$ and $y > 0$, then $|x| = -x$ and $|y| = y$ and $x\,y < 0$

$\Rightarrow |xy| = -(xy) = (-x)\,(y) = |x| \ |y|.$

If $x < 0$ and $y < 0$, then $|x| = -x$ and $|y| = -y$ and $xy > 0 \Rightarrow |xy| = xy$

But $|x| \ |y| = (-x) \ (-y) = xy$

$\therefore |xy| = |x| \ |y| \ .$

This completes the proof.

(ii) We again consider the cases

$x = 0$, $y = 0$

$x > 0$, $y > 0$

$x < 0$, $y > 0$

$x < 0$, $y < 0$

If $x = 0$ and $y = 0$, then $|x + y| = |0 + 0| = |0| = 0$.

Also $|x| = |0| = 0$ and $|y| = |0| = 0$

$\therefore |x| + |y| = 0 + 0 = 0$

$\therefore |x + y| = |x| + |y|$.

If $x > 0$ and $y > 0$, then $x + y > 0 \Rightarrow |x + y| = x + y$.

Also $|x| = x$ and $|y| = y$, so $|x| + |y| = x + y$.

$\therefore |x + y| = |x| + |y|$.

If $x < 0$ and $y > 0$; and suppose that $|x| > |y|$,

then $x + y < 0 \Rightarrow |x + y| = -(x + y) = -x + (-y)$.

Also $|x| = -x$ and $|y| = y \Rightarrow |x| + |y| = -x + y$

Clearly, $|x + y| - (|x| + |y|) = -2y < 0$

$\therefore |x + y| < |x| + |y|$.

On the other hand if $|x| < |y|$., then

$x + y > 0 \Rightarrow |x + y| = x + y$

Also $|x| = -x$ and $|y| = y \Rightarrow |x| + |y| = -x + y$.

$\therefore |x + y| - (|x| + |y|) = 2x < 0$

$\Rightarrow |x + y| < |x| + |y|$

Finally, if $|x| = |y|$, then $x + y = 0$.

So that $|x + y| = |0| = 0$.

But

$|x| + |y| = -x + y > 0$

$\therefore |x + y| < |x| + |y|$

Now, if $x < 0$ and $y < 0$, then $x + y < 0$

$\Rightarrow |x + y| = -(x + y) = (-x) + (-y) = |x| + |y|$

Thus for all $x, y \in \Re$,

$$|x + y| \leq |x| + |y| .$$

This completes the proof.

2. Find the values of x for which

$$x^2 - 3x - 2 < 10 - 2x.$$

Solution:

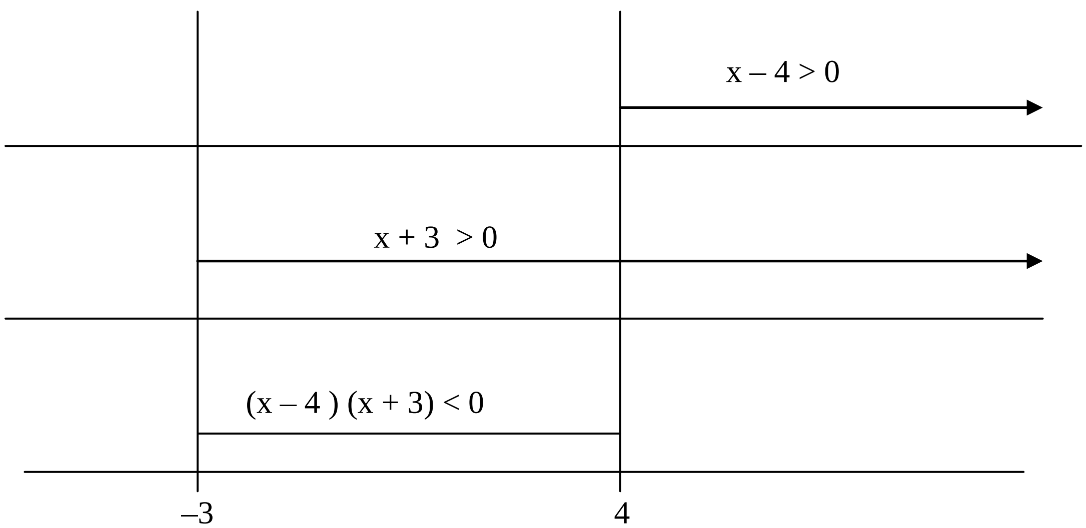
$x^2 - 3x - 2 < 10 - 2x.$

$\Leftrightarrow x^2 - x - 12 < 0.$

$\Leftrightarrow (x - 4)(x + 3) < 0..$

Method I (Geometrical)

The set $x - 4 > 0$ and $x + 3 > 0$ are represented graphically below:



The dotted portion indicate the regions where the expressions $x - 4$ and $x + 3$ are negative.

For the product $(x - 4)(x + 3)$ to be negative, the expressions should be opposite in sign.

Clearly, the solution is the set

$$[x : -3 < x < 4\} .$$

Method II (Analytical)

Note that

$$(x - 4)(x + 3) < 0$$

$$\Rightarrow \quad x - 4 > 0 \text{ and } (x + 3) < 0$$

or

$$x - 4 < 0 \text{ and } (x + 3) > 0$$

$$\Rightarrow \; x > 4 \text{ and } x < -3$$

or

$$x < 4 \text{ and } x > -3$$

Clearly, the first option is impossible. The second option provides the solution, namely

$$x < 4 \text{ and } x > -3$$

or $\{x : -3 < x < 4\}$.

Exercises:

1. Using the triangle inequality (i.e. property ii), or otherwise, prove property (iii), i.e. $|x - y| \geq |x| - |y|$

2. Prove that $-|x| < x < |x|$.

   Hence prove property (ii), i.e.

   $$|x + y| \leq |x| + |y|$$

3. Show that

   $$|x + y + z| \leq |x| + |y| + |z|$$

   and hence the fact that

   $$|x_1 + x_2 + \ldots + x_n| \leq |x| + |x_2| + \ldots + |x_n| .$$

4. Prove that for $x, y \in \Re$, $y \neq 0$, $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$.

CHAPTER 2

POINT SETS

2.1 <u>Definition</u>:

Any set whose members are real numbers is called a **one-dimensional point set**. All the elements of a point set can be located on the real number line.

E.g. The sets $\{0, 1, 2, 3, 4, 5\}$, $\{x: -1 < x < 0\}$ are point sets.

2.2 Intervals

The set of values of x such that $a \le x \le b$, where $a, b \in R$, is called a **closed interval**; it is denoted by $[a, b]$. The set $\{x: a < x < b,\ a, b \in R\}$ is called an **open interval** and is denoted by (a, b). The half-open or half closed intervals are $\{x: a < x \le b\} \equiv (a, b]$.

$\{x: a \le x < b\} \equiv [a, b)$.

The intervals indicated above are said to bounded intervals.

There are unbounded intervals such as

$\{x : -\infty < x < b\} \equiv (-\infty, b)$ or $\{x: x < b\}$,

$\{x: a < x < \infty\} \equiv \{x: x > a\} \equiv (a, \infty)$

$\{x: -\infty < x < \infty\} \equiv \{x: x \in R\} \equiv R$, or $(-\infty, \infty)$.

2.3 Countable Sets

A set is called **countable or denumerable** if its elements can be placed in 1–1 correspondence with the set of natural numbers.

E.g.

The set $P = \{2, 4, 6, \ldots\}$ is a countable set because of the 1 – 1 correspondence

$$\begin{array}{cccccc} 2 & 4 & 6 & 8 & 10 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & \ldots \end{array}$$

A set is called infinite if it can be placed in 1 – 1 correspondence with a subset of itself.

An infinite set which is countable is said to be countably infinite.

E.g. Q is countably infinite.

2.4 NEIGHBOURHOODS AND LIMIT POINTS

The set of all points x such that $|x - a| < \delta$, where $a \in R$ and $\delta > 0$ is called a **δ–neighbourhood of the point *a***

$$|x - a| < \delta \;\Rightarrow\; -\delta < x - a < \delta$$

$$\Rightarrow\; a - \delta < x < a + \delta$$

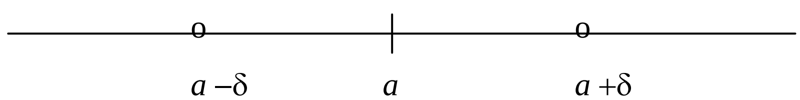
The set of all points x such that

$$0 < |x - a| < \delta$$

is called a **deleted δ–neighbourhood of *a*** .

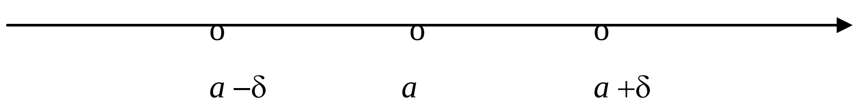
Note

(i) the δ–neighbourhood of a is an interval with centre a and length 2δ .



δ–Neighbourhood of a

(ii) the deleted δ–neighbourhood of a is an interval with centre a and length 2δ , with the point a excluded from the interval.



deleted δ–neighbourhood of a

<u>Examples:</u>

(a) The 1 – neighbourhood of 0 is the open interval $(-1, 1)$. It has length 2 and centre 0.

(b) The deleted 2 – neighbourhood of 0 is the interval $\{x : -2 < x < 2,\ x \neq 0\}$.

2.5 **Definition**

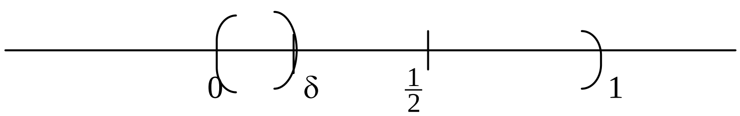
A *limit point*, *point of accumulation*, or *cluster point* of a set A of numbers is a number l such that every deleted δ–neighbourhood of l contains elements of the set A. That is, for every $\delta > 0$, $\exists\, x \in A$ s.t. $0 < |x - l| < \delta$. Since δ may be as small as we please, it is clear that every δ–neighbourhood of 0 contains an infinity of elements of A.

Examples:

1) 0 is a limit point of the set

$A = (0, 1)$,

Since every deleted δ–neighbourhood of l has elements of A.

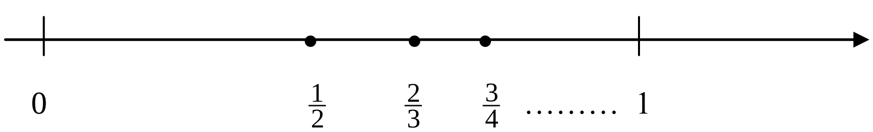


It is clear that $\{x : -\delta < x < \delta$ will always have elements of A.

$, \ x \neq 0 \ \}$

2) 1 is a limit point of the set

$$B = \left\{ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots \right\}$$



Clearly, for every deleted δ–neighbourhood of 1, there will always be elements of the set.

Notes

(i) No finite set has a limit point

E.g. the set $\{1, 2, 3, 4, 5\}$

has no limit points since for any suggested number, a deleted δ–neighbourhood can be constructed with no elements of the set present.

(ii) An infinite set may or may not have a limit point. In particular, the natural numbers N = {1, 2, 3, … } has no cluster points. The set of rational numbers has infinitely many limit points.

(iii) If a set contains all its limit points, it is called a **closed set.**

E.g. the set [0, 1] is a closed set, while the set (0, 1) is not a closed set since 0 and 1 even though are limit points do not belong to the set.

Exercises:

(1) How many limit points does the set (–1, 1) have? Why is 0 a limit point?

(2) Does the set

$$A = \left\{ \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \dots \right\}$$

have any limit points? If yes, find them.

(3) Why does the set

$\{-10,\ -9,\ -8,\ \dots, 0, 1, 2, 3, \dots, 10\}$

not have a limit point. Show why the point $x = \frac{1}{2}$ cannot be a limit point of the set.

(4) An infinite set is defined by

$$A = \{ \frac{1}{2^n} : n = 0, 1, 2, 3, \dots \} .$$

Does it have a limit point? If yes, determine it. Is the set A closed? Why?

2.6 BOUNDS

Let A be a set. If $\exists$ M(real) such that $x \le M$, $\forall x \in A$, then A is said to be **bounded above**. M is called an **upper bound of the set A**. If $\exists$ m(real) such that $x \ge m$, $\forall x \in A$, then A is said to be **bounded below**. m is called a **lower bound** of the set A.

Let $\underline{M}$ be an upper bound of A. Then if for any $\in > 0$ (however small), $\exists$ at least one element x in A such that $x > \underline{M} - \in$, the number $\underline{M}$ is called the **least upper bound** (l.u.b.) of A.

Let $\overline{m}$ be a lower bound of A. then if for any $\in > 0$ (however small), $\exists$ at least one element x in A such that $x < \overline{m} + \in$, the number $\overline{m}$ is called the **greatest lower bound** (g.l.b.).

If $\forall\ x \in A$, $\exists m$, M (real) such that $m \le x \le M$, then the set A is said to be **bounded**.

Examples:

(1) The set [0, 1] is a bounded set since for example, $-2 < x < 3, \forall\ x \in [0, 1]$.

The least upper bound is 1

The greatest lower bound is 0.

(2) The set

$$A = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\right\}$$

is a bounded set, since $-1 < x < z$, $\forall\ x \in A$. The least upper bound is 1 and the greatest lower bound is 0.

2.7 THE BOLZANO -WEIERSTRASS THEOREM

The Bolzano -Weierstrass theorem states that every bounded infinite set has at least one limit point.

Examples:

(1) Show that the set

$$S = \{ \tfrac{1}{n} : n = 1, 2, 3, \ldots \}$$

satisfies the Weierstrass – Bolzano theorem. Determine the limit and prove it.

Solution:

$$S = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\right\}.$$

S is bounded above by 2 and bounded below by -1. $\therefore$ S is bounded.

Also S is infinite set since it can be placed in 1 – 1 correspondence with a subset of itself.

(E.g. $1,\quad \frac{1}{2},\quad \frac{1}{3},\quad \frac{1}{4},\quad \ldots$

$\quad\;\, \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow$

$\quad\;\, \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{6} \quad \frac{1}{8},\quad \ldots$)

$\therefore$ S has at least one limit point, by the Bolzano – Weierstrass Theorem.

The limit point is 0.

A deleted δ–neighbourhood of 0 is

$$0 < \left| \frac{1}{n} - 0 \right| < \delta \quad , \quad n \in N.$$

$$\Rightarrow 0 < \frac{1}{n} < \delta \qquad \Rightarrow n\,\delta > 1$$

$$\Rightarrow n > \frac{1}{\delta}$$

It is clear that for every δ, there will always be an integer $N(\delta)$

$$\left| \frac{1}{n} - 0 \right| < \delta \quad , \quad \forall\, n > N(\delta)\,.$$

(2) Does the set

$$T = \{ \frac{1}{2^n} \; : \; n = 1, 2, 3, \ldots \}$$

satisfy the Bolzano – Weierstrass theorem? Why? Does the set T have a limit point? If yes, find it and prove it.

Solution:

$$T = \left\{ \frac{1}{2},\; \frac{1}{4},\; \frac{1}{8},\; \ldots \right\}$$

Clearly T is bounded. For example, $-1 < x < 1$, $\forall\, x \in T$. T is also infinite and so satisfies the Weierstrass-Bolzano theorem. Therefore, T has at least one limit point. Infact, T has only one limit point 0.

To prove that 0 is a limit point of T, we show that every deleted δ-neighbourhood of 0 has elements of T.

$$0 < \left| \frac{1}{2^n} - 0 \right| < \delta\,,\; n \in N$$

$$\frac{1}{2^n} < \delta$$

$$2^n > \frac{1}{\delta}$$

$$\Rightarrow \quad n \log 2 > \log\left(\frac{1}{\delta}\right)$$

$$n > \frac{\log\left(\frac{1}{\delta}\right)}{\log 2}$$

It is clear that for every δ, there will always be an integer $N(\delta)$ such that $\left|\frac{1}{2^n} - 0\right| < \delta$.

The existence of an integer N(δ) immediately greater than $\frac{\log\left(\frac{1}{\delta}\right)}{\log 2}$ completes the proof.

SEQUENCES

DEFINITION

A set of numbers

$$U_1 , U_2 , U_3 , \ldots$$

in a definite order of arrangement and formed by a definite rule is called a **sequence of numbers .**

Every number in the sequence is called a term of the sequence. If the sequence has a last term, then it is called a **finite sequence**. If the number of terms is infinite, then the sequence is called an **infinite sequence**. The nth term of the sequence will be denoted by U_n, while the sequence itself will be denoted by $\{ U_n \}$.

Example of sequences:

(a)

$$\{1, 2, 3, \ldots, 20\}$$

is a finite sequence of the counting numbers up to 20.

(b) $\left\{ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \ldots \right\}$

This is an infinite sequence with the nth term $U_n = \frac{1}{n}$.

3.2 TYPES OF SEQUENCES

(a) Arithmetic Sequence

A sequence $\{ U_n \}$ of numbers is said to be an arithmetic sequence if

$$U_{n+1} - U_n = d \ , \ \forall n \qquad \text{--------------} \quad (1)$$

where d is a constant. The constant d is called the Common Difference.

The sum of the first n terms of an Arithmetic sequence is given by

$$Sn = \frac{n}{2} \{2a + (n - 1) d\} , \qquad \text{---------------} \quad (2)$$

where a is the first term of the sequence,

E.g.

The sequence

2, 5, 8, 11, …

is an arithmetic sequence, since d = 3.

The sum of the first 10 terms is:

$$S_{10} = \frac{10}{2}\,(2\,(2) + (10 - 1)\,(3))$$

$$= 5\,(4 + 27)$$

$$= 5\,(31)$$

$$= \underline{\underline{155}}$$

(b) **Geometric Sequence**

A sequence $\{U_n\}$ is said to be a geometric sequence if

$$U_{n+1} = r\,U_n, \ \forall n, \qquad \text{------------------(3)}$$

where r is a constant. The number r is called the Common Ratio.

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}, \qquad \text{---------------} \quad (4)$$

where a is the first term.

E.g. The sequence

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$$

is a geometric sequence with $a = 1$ and $r = \frac{1}{2}$.

(c) **Sequences which are neither arithmetic nor geometric**.

There are sequences which are neither arithmetic nor geometric. In what follows general properties of sequences will be discussed.

LIMIT OF A SEQUENCE

An infinite sequence $\{U_n\}_{n=1}^{\infty}$ is said to have a limit l as n approaches ∞ if U_n can be made as close to l as we please by choosing n sufficiently large.

We then write.

$$\lim_{n \to \infty} U_n = l . \qquad \text{------------------} \qquad (5)$$

The above definition may be more rigorously written as follows:

$\lim_{n \to \infty} U_n = l$ if for every $\in > 0$ (however small), $\exists$ a positive integer N (depending on $\in$) s.t. $|U_n - l| < \in$, $\forall$ n > N.

Examples:

(1) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ has limit 0. Thus $U_n = \frac{1}{n}$ and $l = 0$.

The difference between U_n and l is $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right|$ suppose we wish

$\left|\frac{1}{n}\right| < \frac{1}{10}$. Then $\frac{1}{n} < \frac{1}{10} \Rightarrow n > 10$.

Thus any choice of n greater than 10 would satisfy the required condition.

It is clear that no matter how small the difference $\left|\frac{1}{n} - 0\right|$ is made there will always be the number N.

(2) The sequence $\left\{\frac{n+1}{n}\right\}$ has limit 1.

(3) The sequence $\left\{\frac{n^2+1}{n}\right\}$ has no limit.

Note:

If a sequence has a limit then it is called a **convergent sequence**; otherwise it is called a **divergent sequence**.

THEOREMS ON SEQUENCES

Theorem 1:

If a sequence $\{U_n\}$ has a limit, then the limit is unique.

(In other words, a convergent sequence has only one limit).

Proof:

Suppose that $\{U_n\}$ has two limits l_1 and l_2. Then $\forall \in > 0$, $\exists$ N (*a* + ve integer) such that

$|U_n - l_1| < \in, \quad \forall\, n > N$, and

$|U_n - l_2| < \in, \quad \forall\, n > N$.

Then

$$\begin{aligned} |l_1 - l_2| &= |l_1 - U_n + U_n - l_2| \\ &\leq |l_1 - U_n| + |U_n - l_2| \\ &< \in + \in \end{aligned}$$

i.e.

$$|l_1 - l_2| < 2 \in$$

Since $\in > 0$ (however small), it follows that $|l_1 - l_2|$ is less than every positive integer.

$\therefore |l_1 - l_2| = 0 \Rightarrow l_1 - l_2 = 0 \Rightarrow l_1 = l_2$.

This completes the proof.

Theorem 2:

Let $\{U_n\}$ and $\{V_n\}$ be two sequences such that $\lim U_n = A$ and $\lim V_n = B$.

Then the following results hold

(1) $\lim\limits_{n\to\infty} (U_n + V_n) = \lim\limits_{n\to\infty} U_n + \lim\limits_{n\to\infty} V_n$

$= A + B.$

(2) $\lim\limits_{n\to\infty} (U_n - V_n) = \lim\limits_{n\to\infty} U_n - \lim\limits_{n\to\infty} V_n$

$= A - B$

(3) $\lim\limits_{n\to\infty} (U_n \cdot V_n) = \lim\limits_{n\to\infty} U_n \cdot \lim\limits_{n\to\infty} V_n$

$= A \cdot B$

(4) $\lim\limits_{n\to\infty} \left(\dfrac{U_n}{V_n}\right) = \dfrac{\lim\limits_{n\to\infty} U_n}{\lim\limits_{n\to\infty} V_n} = \dfrac{A}{B}$ provided that $\lim\limits_{n\to\infty} V_n = B \neq 0.$

(a) If $B = 0$ and $A \neq 0$, then $\lim\limits_{n\to\infty} \dfrac{U_n}{V_n}$ does not exist.

(b) If $B = 0$ and $A = 0$, then $\lim\limits_{n\to\infty} \dfrac{U_n}{V_n}$ may or may not exist.

(5) $\lim\limits_{n\to\infty} U_n^p = \left(\lim\limits_{n\to\infty} U_n\right)^p = A^p$

(6) $\lim\limits_{n\to\infty} p^{u_n} = p^{\lim\limits_{n\to\infty} u_n} = p^A.$

For the conditions (5) and (6), p must be real and A^p or p^A must exist.

BOUNDED, MONOTONIC SEQUENCES

A sequence $\{U_n\}$ is said to be bounded above if $\exists$ M (a constant) such that $U_n \leq M, \forall n$. M is called an upper bound. If $\exists$ m a constant such that $U_n \geq m$, $\forall$ n , then the sequence $\{U_n\}$ is said to be bounded below and m is called a lower bound. If $\exists$ m, M such that $m \leq U_n \leq M, \forall n$, then the sequence $\{U_n\}$is said to be a bounded sequence.

An upper bound $\underline{M}$ is called the least upper bound (l. u. b.) of the sequence if for every $\in > 0$, there will always exist at least one term of the sequence greater than $M - \in$.

A lower bound $\overline{m}$ is called the greatest lower bound (g. l. b.) of the sequence if for every

$\in > 0$, $\exists$ at least one term of the sequence is less than $\overline{m} + \in$.

Example:

The sequence $\left\{\frac{2n+1}{n}\right\} \equiv \left\{2 + \frac{1}{n}\right\}$ is bounded below by 1 and bounded above by 4. Thus $\left\{\frac{2n+1}{n}\right\}$ is a bounded sequence.

The least upper bound is 3 and the greatest lower bound is 2.

A sequence $\{U_n\}$ is said to be a **monotonic increasing sequence or a non-decreasing sequence** if $Un \leq U_{n+1}$, $\forall$ n . If $U_{n+1} \leq U_n$, $\forall$ n, then the sequence is called a **monotonic decreasing sequence or a non-increasing sequence**.

If $U_n < Un_{+1,}$ $\forall$ n, the sequence is said to be strictly increasing. If $Un_{+1} < Un_{,}$ $\forall$ n, the sequence is strictly decreasing.

If a sequence is either monotonic increasing or monotonic decreasing, it is said to be a **monotonic sequence.**

Example:

Show that the sequence $\left\{\dfrac{n+1}{n}\right\}$ is:

(i) a monotonic decreasing sequence

(ii) a bounded sequence.

Solution:

(i) $U_n = \dfrac{n+1}{n}$

$$U_{n+1} - U_n = \frac{n+2}{n+1} - \frac{n+1}{n}$$

$$= \frac{n(n+2) - (n+1)(n+1)}{n(n+1)}$$

$$= \frac{n^2 + 2n - n^2 - 2n - 1}{n(n+1)}$$

$$= \frac{-1}{n(n+1)} \leq 0$$

$\Rightarrow \quad U_{n+1} - U_n \leq 0$

$\Rightarrow \quad U_{n+1} \leq U_n \;, \quad \forall\, n.$

$\therefore \quad \{U_n\}$ is a monotonic decreasing sequence.

(ii) $\left\{\dfrac{n+1}{n}\right\} \equiv \left\{\dfrac{2}{1}, \dfrac{3}{2}, \dfrac{4}{3}, \dfrac{5}{4}, \ldots\right\}$

The sequence is bounded above by 3 (say) and bounded below by 0(say).

$\therefore$ The sequence is bounded.

Theorem: Every bounded monotonic sequence has a limit.

Example:

Use the above theorem to establish that the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ has a limit.

Prove that $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right) = 1.$

Solution:

The sequence $\left\{\dfrac{n}{n+1}\right\} \equiv \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\right\}$ is bounded above by 2(say) and bounded below by 0(say). $\therefore$ the sequence is bounded. The sequence is monotonic increasing since

$$U_{n+1} - U_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)} \geq 0, \ \forall\, n$$

$\Rightarrow \quad U_{n+1} - U_n \geq 0$

$\Rightarrow \ U_{n+1} \geq U_n, \ \forall\, n.$

Since the sequence is bounded and monotonic, it has a limit.

To prove that $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right) = 1$, we wish to show that for every $\delta > 0$, $\exists$ a positive integer N, such that $\left|\dfrac{n}{n+1} - 1\right| < \delta \quad \forall\, n > N.$

Now $\left|\dfrac{n}{n+1} - 1\right| = \left|\dfrac{n-(n+1)}{n+1}\right| = \left|\dfrac{-1}{n+1}\right| = \dfrac{1}{n+1}$ **.**

For any $\delta > 0$

$$\frac{1}{n+1} < \delta$$

$$\Rightarrow (n+1)\,\delta > 1$$

$$\Rightarrow n > \frac{1}{\delta} - 1 .$$

Thus choosing N a positive integer immediately larger than $\left(\frac{1}{\delta} - 1\right)$, the result follows.

Example:

Prove that:

(a) $\lim\limits_{n \to \infty} \left(\frac{4 - 2n}{3n + 2} \right) = -\frac{2}{3}$ (b) $\lim\limits_{n \to \infty} \left(\frac{3}{4} \right)^n = 0.$

Solution:

(a) For every $\delta > 0$

$$\left| \frac{4 - 2n}{3n + 2} - \left(\frac{-2}{3} \right) \right| < \delta$$

$$\Leftrightarrow \left| \frac{4 - 2n}{3n + 2} + \frac{2}{3} \right| < \delta$$

$$\Leftrightarrow \left| \frac{3(4 - 2n) + 2(3n + 2)}{3(3n + 2)} \right| < \delta$$

$$\Leftrightarrow \left| \frac{16}{3(3n + 2)} \right| < \delta \Leftrightarrow \frac{16}{3(3n + 2)} < \delta$$

$$\Leftrightarrow (3n + 2)\,\delta > \frac{16}{3} \Leftrightarrow 3n + 2 > \frac{16}{3\delta}$$

$$\therefore n > \frac{1}{3} \left(\frac{16}{3\delta} - 2 \right).$$

Hence choosing N immediately larger than $\frac{1}{3} \left(\frac{16}{3\delta} - 2 \right)$, the result follows.

(b) For every $\delta > 0$,

$$\left| \left(\frac{3}{4}\right)^n - 0 \right| < \delta$$

$$\Rightarrow \quad \left(\frac{3}{4}\right)^n < \delta$$

$$\Rightarrow \quad n \log_e \left(\frac{3}{4}\right) < \log_e \delta$$

$$\Rightarrow \quad n > \frac{\log_e \delta}{\log_e \frac{3}{4}} \quad (\text{since } \log_e \frac{3}{4} < 0).$$

Hence choosing N, a positive integer immediately larger than $\dfrac{\log_e \delta}{\log_e \frac{3}{4}}$, the result follows.

3.6 EVALUATING LIMITS OF SEQUENCES

The limit l of a sequence $\{U_n\}$, if it exists, may be evaluated using the theorems on limits.
Consider the following examples.

Examples:
Evaluate the limits of the following sequences:

(a) $\left\{ \dfrac{4 - 2n - 3n^2}{4n^2 + 5n} \right\}$ (b) $\left\{ \sqrt{\dfrac{n^2(n-1)}{1 + \frac{1}{2}n^3}} \right\}$

(c) $\left\{ \sqrt{n^2 + n} - n \right\}$ (d) $\left\{ 2^{\frac{n}{n^2+1}} \right\}$.

Solutions:

(a) Using the theorems on limits we have

$$\lim_{n\to\infty} \frac{4-2n-3n^2}{4n^2+5n} = \lim_{n\to\infty} \frac{\frac{4}{n^2} - \frac{2}{n} - 3}{4 + \frac{5}{n}}$$

$$= \frac{\lim_{n\to\infty}\left(\frac{4}{n^2} - \frac{2}{n} - 3\right)}{\lim_{n\to\infty}\left(4 + \frac{5}{n}\right)} = \frac{3}{4}.$$

(b) $$\lim_{n\to\infty} \sqrt{\frac{n^2(n-1)}{1+\frac{1}{2}n^3}} = \sqrt{\lim_{n\to\infty}\left(\frac{n^2(n-1)}{1+\frac{1}{2}n^3}\right)}$$

$$= \sqrt{\lim_{n\to\infty}\left(\frac{1-\frac{1}{n}}{\frac{1}{n^3}+\frac{1}{2}}\right)} = \sqrt{\frac{1}{\frac{1}{2}}}$$

$$= \underline{\underline{\sqrt{2}}}$$

(c) $$\lim_{n\to\infty}\left\{\sqrt{n^2+n} - n\right\} = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty}\left(\frac{\left(n^2+n\right) - n^2}{\sqrt{n^2+n} + n}\right)$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{2}$$

(d) $$\lim_{n\to\infty}\left(2^{\frac{n}{n^2+1}}\right) = 2^{\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right)}$$

$$= 2^{\circ} = 1.$$

3.7 **EXERCISES:**

(1) Let $U_n = \dfrac{n+3}{2n+1}$

(a) Prove that the sequence $\{ U_n \}$ is (i) bounded (ii) monotonic decreasing

(b) Guess the limit of $\{ U_n \}$ and prove it.

(2) Prove that $\lim\limits_{n \to \infty} \dfrac{n+1}{3n - 2} = \dfrac{1}{3}$, using the definition of a limit.

(3) Prove that $\lim\limits_{n \to \infty} 2^{\frac{1}{n}} = 1$, using he definition of a limit.

(4) Evaluate $\lim\limits_{n \to \infty} U_n$, where U_n is given by each of the following:

(a) $\dfrac{5n^2 + 3n + 3}{2 - n + 7n^2}$ (b) $\left(\dfrac{1}{n^2} \left(\dfrac{n^2 - 1}{n^2} \right) \right)$

(c) $\dfrac{\sqrt{3n^2 - 4n + 5}}{n + 1}$ (d) $5^{\sqrt{\left(1 + \frac{1}{n^2}\right)}}$

(5) Show that
$\lim\limits_{n \to \infty} \left(\sqrt{n + 1} - \sqrt{n} \right) = 0$.

(6) If $U_{n+1} = \sqrt{U_n + 1}$, $U_1 = 1$, prove that $\lim\limits_{n \to \infty} U_n = \dfrac{1}{2} \left(1 + \sqrt{5} \right)$.

(7) A sequence $\{ U_n \}$ is defined inductively by $U_1 = 1$, $U_{n+1} = \sqrt{2U_n}$.

(a) Prove that $\lim\limits_{n \to \infty} U_n$ exists.

(b) Determine $\lim\limits_{n \to \infty} U_n$.

(8) Prove that $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n} \right)^n = e$.

(Hint : Expand $\left(1 + \dfrac{1}{n} \right)^n$ binomially).

CHAPTER FOUR

FUNCTIONS

4.1 **Definition**:

A real function f is a relation which takes elements from a point set D (called the domain) to elements of another point set (called the co-domain) such that each element in the domain has exactly one image in the co-domain.

Any number $x \in D$ is called an independent variable. The function may be represented in the form

$$f : x \rightarrow f(x) \text{ or } f : x \rightarrow y$$

The domain of the function is also defined as the set of all real number for which the function f is defined. When we write $y = f(x)$ to represent a function, then y is called the dependent variable.

4.2 **The Graph of a function:**

The graph of the function $y = f(x)$ is a pictorial representation of the function. This representation may be on a rectangular co-ordinate system, the points being defined by the pairs (x, y) or (x, f(x)).

4.3 **BOUNDED FUNCTIONS:**

Let $x \in [a, b]$. If $\exists$ M (real) such that $f(x) \le M$, $\forall x \in [a, b]$, then f(x) is said to be bounded above in $[a, b]$. M is called an upper bound of f.

If $\exists$ m (real) such that $f(x) \ge m$, $\forall x \in [a, b]$ we say that f(x) is bounded below in (a, b) and m is called a lower bound of f.

If $\exists$ m, M such that $m \le f(x) \le M$, $\forall x \in [a, b)$ then the function f is said to be bounded.

If f(x) has an upper bound, it has a least upper bound; if it has a lower bound it has a greatest lower bound.

Examples:

Consider the interval $[-1, 1]$.

1. The function $f(x) = 4 - x$ is bounded above by 5 (or any number greater than 5) and bounded below by 3 (or any number less than 3).

4.4 MOTONONIC FUNCTIONS

A function f(x) is called a monotonic increasing function in an interval $[a, b]$ if for $\forall x_1, x_2 \in [a, b]$ such that $x_1, < x_2$, $f(x_1) \le f(x_2)$.

f(x) is called monotonic decreasing if for $x_1 < x_2$, $f(x_1) \ge f(x_2)$.

If $x_1 < x_2$ and $f(x_1) < f(x_2)$, then f(x) is strictly increasing; if $f(x_1) > f(x_2)$, then f(x) is strictly decreasing.

Examples:

(1) The function f(x) = 2x + 1 is monotonic increasing in the interval [0, 1].

(2) The function f(x) = $\frac{1}{x}$ is monotonic decreasing in the interval [1, 3].

(3) The function

$$f(x) = \frac{x + 1}{x}$$

is monotonic decreasing in the interval [1, 5].

Exercises:

(1) Consider the function

$$f(x) = \frac{x}{x+1}$$

in the interval [0, 1]. Investigate the following properties of the function f.

(a) Boundedness.

(b) Monotonicity.

(2) Define a function by

$$f(x) = x + \frac{1}{x}$$

in the interval [1, 3]. Describe the following properties of the function:

(a) Boundedness

(b) Monotonicity

(3) Repeat (2) for the interval (0, 1].

MAXIMA AND MINIMA:

Let $x_0 \in [a, b]$. If a function f(x) is such that $f(x) \leq f(x_0)$, $\forall\, x \in [a, b]$, then f(x) is said to have an **absolute maximum** in the interval $[a, b]$ at $x = x_0$. The maximum value is $f(x_0)$. If $f(x) \geq f(x_0)$, $\forall\, x \in [a, b]$, then f(x) is said to have **an absolute minimum** in the interval $[a, b]$ at $x = x_0$. The minimum value is $f(x_0)$.

If the above statements are true only for values of x in some deleted δ - neighbourhood of x_0 then f(x) is said to have a **relative maximum** or a **relative minimum** at x_0.

4.6 TYPES OF FUNCTIONS

There are three major types of functions: Polynomial functions, algebraic functions and transcendental functions.

1. **Polynomial Functions**

These are functions of the form

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \quad a_n \neq 0. \qquad \text{------------------} \qquad (1)$$

This is called a polynomial in x of degree n. The numbers $a_0, a_1, \dots, a_n$ are constants and n is a positive integer.

Theorem

Every polynomial equation

$$f(x) = 0$$

has at least one root.

If the degree of the polynomial is n, then the equation $P_n(x) = 0$ has exactly n roots.

2. **Algebraic Function**

Any function $y = f(x)$ satisfying an equation of the form

$$P_0(x)\, y^n + P_1(x)\, y^{n-1} + \dots + P_{n-1}(x) y + P_n(x) = 0, \qquad \text{----------------} \quad (2)$$

where $P_0(x), \dots, P_n(x)$ are polynomials in x, is called an algebraic function.

If an algebraic function may be expressed as the quotient of two polynomials, i.e. $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, then the function is called a rational algebraic function. Otherwise it is an irrational algebraic function.

3. **Transcendental Functions**

Any function which is not algebraic is called a Transcendental Function. A transcendental function is expressible as an infinite sum of polynomials.

Examples:

(a) $2x^4 + x3 - 2x^2 + x + 9$ – a polynomial of degree 4.

(b) $y = \dfrac{1}{x^3}$ is an algebraic function since

$$x^3 \cdot \frac{1}{x^3} - 1 = 0 \quad (x^3, \ -1 \text{ are polynomials })$$

(c) $y = \dfrac{x}{x+1}$ is algebraic, since

$$x \cdot \frac{x}{x+1} + \frac{x}{x+1} - x = 0$$

$$\Rightarrow (x + 1)\frac{x}{x+1} - x = 0$$

4.7 ELEMENTARY TRANSCENDENTAL FUNCTIONS

1. **Exponential Functions**.
   These are functions of the form

   $$f(x) = a^x,\ a \neq 0, 1$$

2. **Logarithmic Functions**
   These have the form

   $$f(x) = \log_a x,\ a \neq 0, 1$$

   If a = 2.71828 … = e , we write

   $$f(x) = \log_e x \equiv \ln x$$

   usually called logarithm to the natural base.

3. **Trigonometric Functions**
   These are the circular functions
   Sinx, Cos x and tanx.
   There are also their reciprocals

   $$\sec x = \frac{1}{\cos x},\quad \operatorname{cosec} x = \frac{1}{\sin x} \quad \text{and} \quad \cot x = \frac{1}{\tan x}.$$

   Furthermore, $\tan x = \dfrac{\sin x}{\cos x}$ and so $\cot x = \dfrac{\cos x}{\sin x}$.

   The following results hold:
   (a) $\sin^2 x + \cos^2 x = 1$ – Pythagoras' theorem
   (b) $1 + \tan^2 x = \sec^2 x$
   (c) $1 + \cot^2 x = \operatorname{cosec}^2 x$
   (d) $\sin(x + y) = \sin x \cos y + \cos x \sin y$
   (e) $\cos(x + y) = \cos x \cos y - \sin x \sin y$
   (f) $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

4. **Inverse Trigonometric Functions**
   These are the functions
   $f(x) = \sin^{-1} x$ (also called arc sine)
   $f(x) = \cos^{-1} x$ (arc cosine).
   $f(x) = \tan^{-1} x$ (arc tangent)

$f(x) = \operatorname{cosec}^{-1} x$ (arc cosech nt)
$f(x) = \sec^{-1} x$ (arc secc nt)
$f(x) = \cot^{-1} x$ (arc cotan gent)

5. **Hyperbolic Functions**

These are functions defined in terms of the exponential functions as follows:

(a) Sinh (x) or sinh x $= \frac{1}{2}(e^x - e^{-x})$.

(b) Coshx $= \frac{1}{2}(e^x + e^{-x})$.

(c) tanh x $= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(d) cosech x $= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

(e) sech x $= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

(f) coth x $= \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(See special notes on Hyperbolic functions and their graphs and properties on page ?)

6. **Inverse Hyperbolic Functions**

These include the following:

(a) $\text{Sinh}^{-1} x$ (b) $\text{Cosh}^{-1} x$ (c) $\tanh^{-1} x$
(d) $\text{Cosech}^{-1} x$ (e) $\text{Sech}^{-1} x$ (f) $\coth^{-1} x$.

(See notes on hyperbolic functions).

GRAPHS OF FUNCTIONS

Let us now consider the graphs of certain special functions.

(1) **Exponential Functions**:

$f(x) = a^x, \quad a \neq 0, 1 \;, \; x \in \mathbb{R}$.

For $a > 1$,

When $x = 0, \; a^0 = 1$

As $x \to -\infty$, **$a^x \to 0^+$**

As $x \to +\infty$, $a^x \to \infty$

$f(x) = a^{-x}$, $a > 1$, $x \in R$.

When $x = 0$, $a^0 = 1$

As $x \to -\infty$, **$a^x \to +\infty$**

As $x \to +\infty$, $a^x \to 0^+$

The two graphs are as shown below on the same axes.

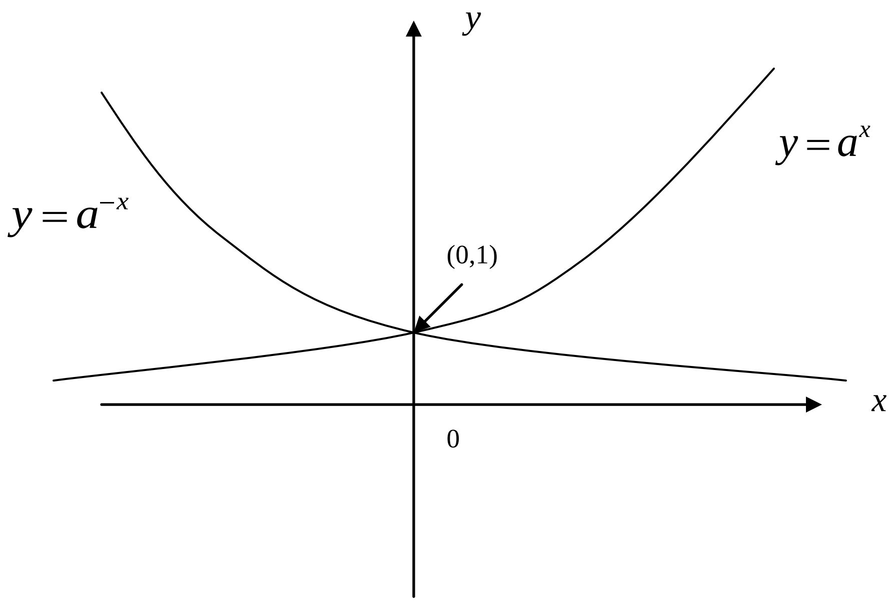


Figure 4.7.1

The exponential function defined by $f(x) = e^x$, where $e \cong 2.72$ is a special exponential function. The number e is called the natural base. How the number e was arrived at will be discussed later.

(2) **Logarithmic Functions**

$f(x) = \log_a x$, $a > 1$, $x \in R$,

When $x = a$, $f(a) = 1$

When $x = 1$, $f(1) = 0$.

As $x \to 0^+$, $f(x) \to -\infty$.

x cannot assume – ve real numbers.

$\text{Log}_a\ x$ is also the inverse function of the exponential function y = ax.
The graph is shown below

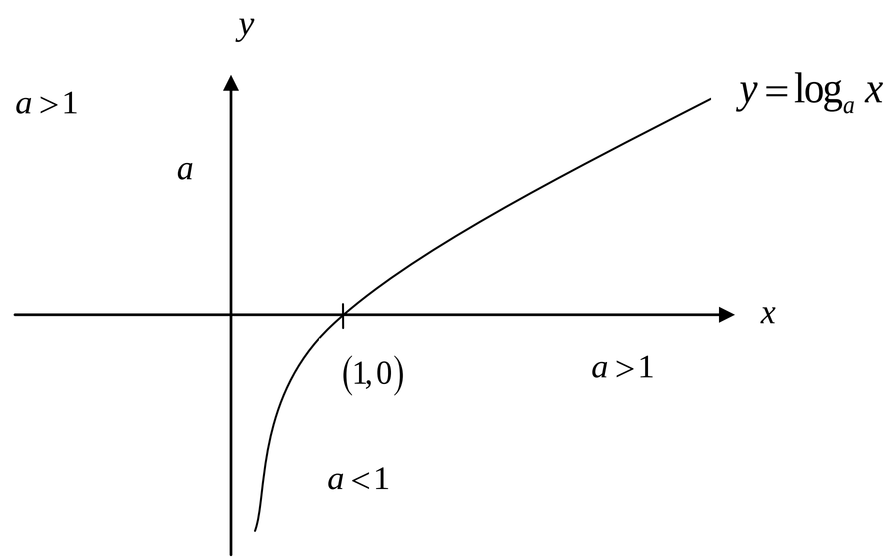


Figure 4.7.2

When a = 10, then $\log_{10} x$ is called the **common logarithm**.

Relationship with the exponential function $y = a^x$, $a > 1$.

The functions y = ax and $y = \log_a x$ are mutually inverse functions. The two graphs are represented on the same graph.

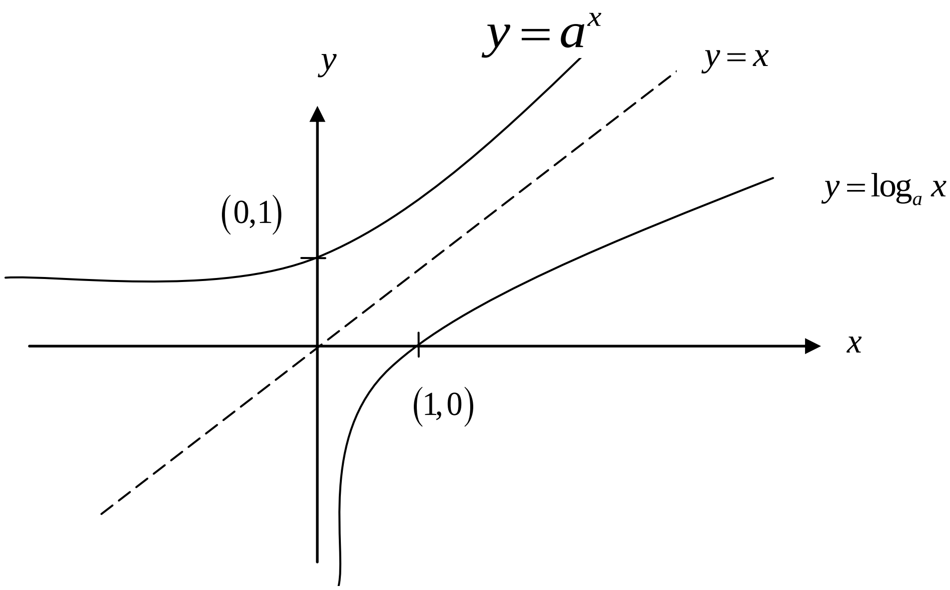


Figure 4.7.3

4.8 EVEN, ODD AND PERIODIC FUNCTIONS

A function f(x) is said to be an <u>even</u> function if

$$f(-x) = f(x), \; x \in \Re \qquad \text{-------------------} \quad (1)$$

A function is called <u>odd</u> if

$$f(-x) = -f(x), \; x \in \Re \qquad \text{-------------------} \quad (2)$$

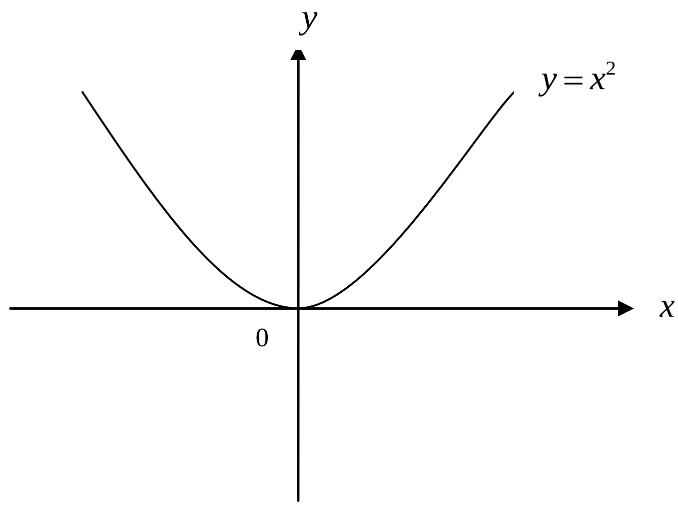
<u>A function f(x) is said to be a periodic function if $\exists$ a real number T such that</u>

$$f(x + T) = f(x), \; x \in \Re \qquad \text{-------------------} \quad (3)$$

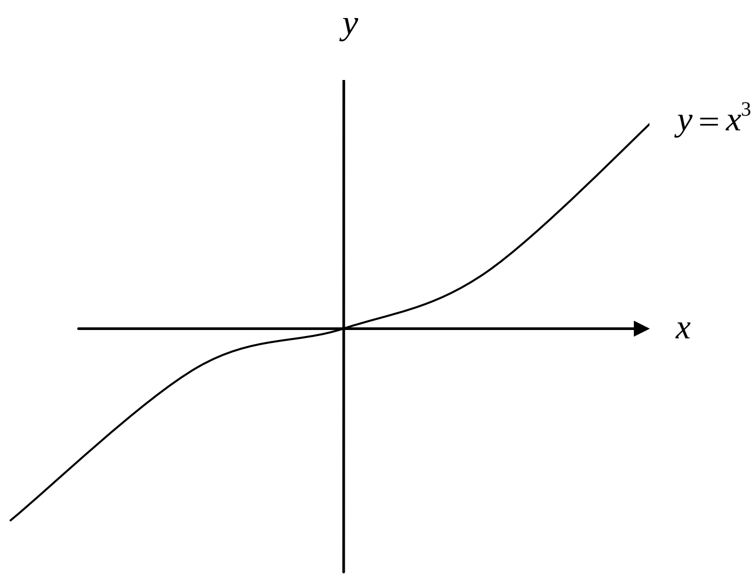
Symmetry
An even function is symmetrical about the y-axis.
An odd function has a rotational symmetry of order 2 about the origin.
A periodic function repeats its value after every value x = T.

<u>Examples:</u>

(1) $f(x) = x^2$ is an even function
$f(-x) = (-x)^2 = x^2 = f(x)$



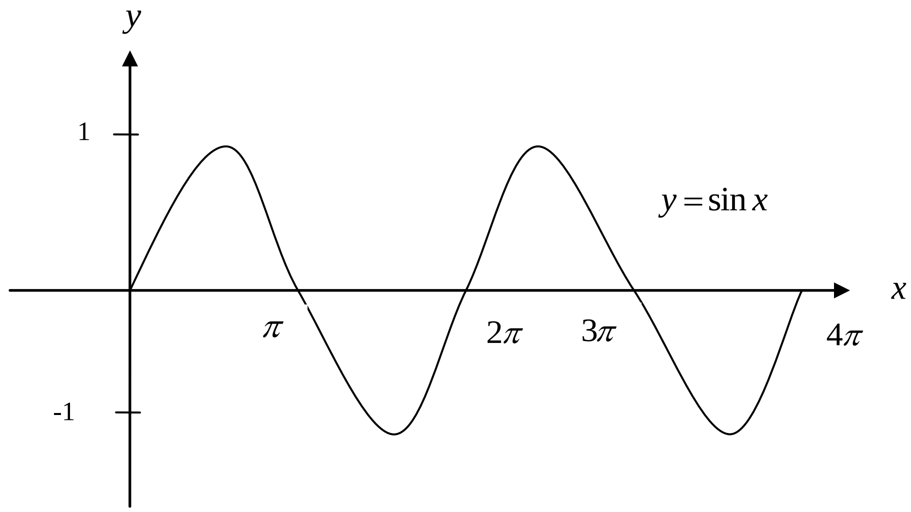
(2) $f(x) = x^3$ is an odd function:
$f(-x) = (-x)^3 = (-1)^3 x^3 = -x^3 = -f(x)$



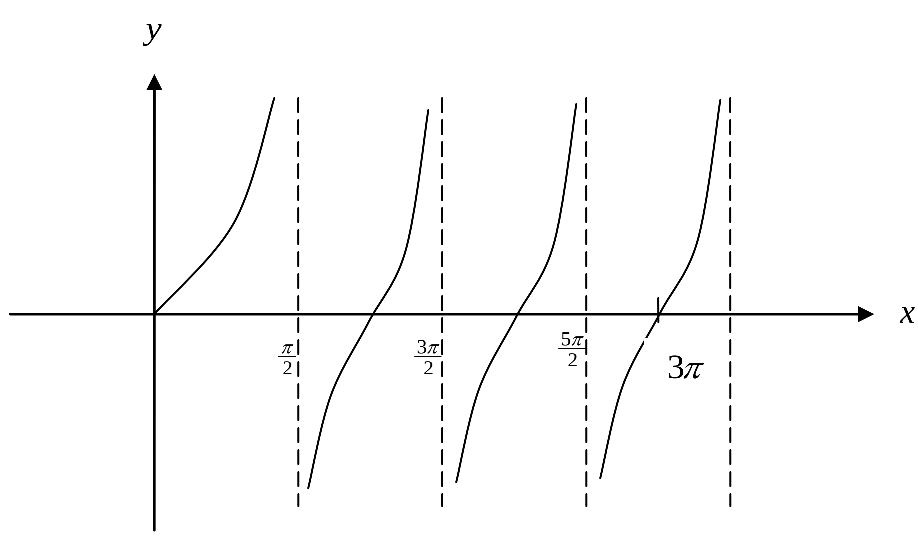
(3) (a) $f(x) = \sin x$ is periodic with period 2π
$f(x + 2\pi) = \sin(x + 2\pi) = \sin x = f(x)$

(b) $f(x) = \tan x$ is periodic with period π

$f(x + \pi) = \tan(x + \pi) = \tan x = f(x)$.



(b)



Remarks:

1) There are functions which are neither even nor odd, and many are not periodic.

E.g. $f(x) = x^2 + x^3$

2) Every function f(x) may be expressed as the sum of an even and an odd function, since

$$f(x) = \underbrace{\frac{1}{2}\{f(x) + f(-x)\}}_{\text{even}} + \underbrace{\frac{1}{2}\{f(x) - f(-x)\}}_{\text{odd.}}$$

E.g. For $f(x) = (x + 1)^2$.
$f(-x) = (-x + 1)^2$
$\therefore f(x) + f(-x) = (x + 1)^2 + (1 - x)^2$ is an even function;
while
$f(x) - f(-x) = (x + 1)^2 - (1 - x)^2$ is an odd function.

3) **Special Even and odd functions**.

(a) $f(x) = \sin x$ is an odd function since $\sin(-x) = -\sin x$.
(b) $f(x) = \cos x$ is an even function since, $\cos(-x) = \cos x$.

Exercises:

1. Show that
   (a) $f(x) = x \sin x$ is an even function
   (b) $f(x) = x \cos x$ is an odd function.

2. From the function
   $f(x) = (x - 1)^2$,
   construct
   (a) an even function
   (b) an odd function
   Prove each of the cases (a) and (b).

3. Write down any function which is neither even nor odd. Construct an even and an odd function from it.

CHAPTER FIVE

CO-ORDINATE GEOMETRY

5.0 **Introduction:** Students are already familiar with topics in co-ordinate geometry such as Equations of straight lines and circles.

In this chapter we shall concentrate on loci which define plane shapes we generally call CONIC SECTIONS. Treatment will be carried out in the Cartesian rectangular co-ordinates. The last section of this chapter will deal with polar curves – curves whose equations are presented in the polar co-ordinate system.

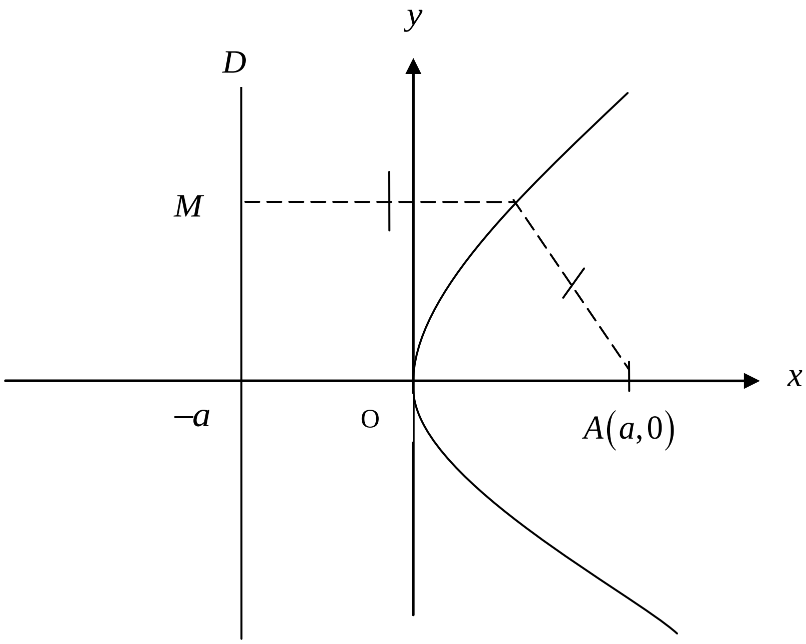
5.1 CONIC SECTIONS

There are three basic conic sections – the Parabola, the Ellipse and the Hyperbola.

A. THE PARABOLA

Let P(x, y) be a variable point in the x – y plane. If the locus of the point P(x, y) is such that it is the same distance from a fixed point (called the Focus) and from a fixed line (called the Directrix), then the locus of the point P defines a curve called a PARABOLA.

Consider a point A (a, o) on the x-axis and a line D with equation $x = -a$ (parallel to the y-axis). A (a, 0) is the focus while D($x = -a$) is the directrix.



Let M be a point on D such that PM is parallel to the x-axis. Then by the definition of a parabola

$$|PM| = |PA| .$$

That is,

$$x + a = \sqrt{(x-a)^2 + (y-o)^2} ,$$

$\Rightarrow$

$$y^2 + (x-a)^2 = (x + a)^2$$

$\Rightarrow$ $y^2 + x^2 - 2ax + a^2 = x^2 + 2ax + a^2$

$\Rightarrow$ $y^2 = 4ax.$

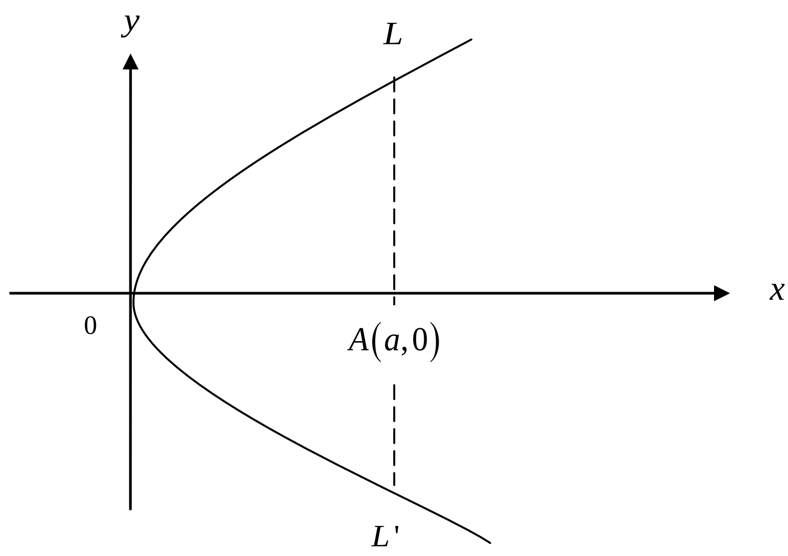
Thus the general equation of a parabola is

$$\boxed{y^2 = 4ax} \qquad \text{------------------(1)}$$

This general equation is in its simplest form.

<u>The Graph:</u>

When $x = 0$, $y^2 = 0 \Rightarrow y = 0$ (repeated) thus the curve touches the y-axis at 0; when $y = 0$, $x = 0$. Curve meets the axes only at the origin 0.



The curve is symmetrical about the x-axis. The line L L' is called the Latus Rectum of the parabola.

Parametric Equations
The equations

$$x = at^2,\ \ y = 2at, \qquad \text{--------------------------} \qquad (2)$$

where t is a parameter, are called the parametric equations of the parabola. They satisfy the equation of the curve since

$$y^2 = (2at)^2 = 4a^2t^2 = 4a(at^2) = 4ax\ . \qquad \text{-----------------------------} \qquad (3)$$

Any point $(at^2, 2at)$ lies on the curve.

Equations of Tangents and Normals

$$y^2 = 4ax$$

$\Rightarrow$

$$2y\frac{dy}{dx} = 4a \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{2a}{y} \qquad \text{---------} \qquad (4)$$

Thus at any point $(at^2, 2at)$, on the curve the tangent has gradient

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}\ . \qquad \text{--------} \qquad (5)$$

The equation of the tangent at any point 't' is then

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$\Rightarrow \qquad y - 2at = \frac{x}{t} - at$$

$$\Rightarrow \quad \boxed{ty - x - at^2 = 0} \qquad \text{---------} \qquad (6)$$

The gradient of the normal at $(at^2, 2at)$ is $-t$; the equation is then

$$y - 2at = -t\,(x - at^2).$$

$$\Rightarrow \quad \boxed{y + tx - 2at - at^3 = 0\ .} \qquad \text{-----------} \qquad (7)$$

<u>Length of the Latus Rectum</u>

When $x = a$

$$y^2 = 4a(a) = 4a^2$$

$$\therefore\ y = \pm 2a.$$

Hence the Latus Rectum has length

$$|L\,L'| = 4a. \qquad \text{---------} \qquad (8)$$

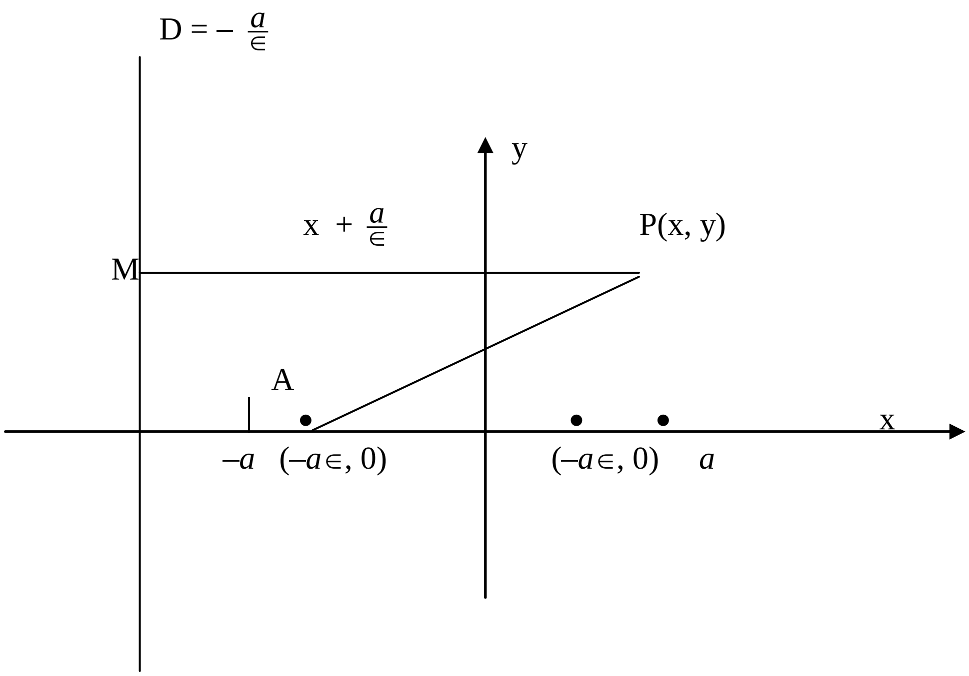
<u>Exercise:</u>

(1) A straight line passing through the focus of the parabola $y^2 = 4ax$ has gradient 1. Determine the points of intersection of the line with the parabola.

(2) The normal to the parabola $y^2 = 4ax$ at the point t $(t \neq 0)$ meets the curve again at the point t'. Find t' in terms of t. Determine a point on the x-axis where the tangent at t' meets the x-axis.

B. **<u>THE ELLIPSE</u>**

The locus of points $P(x, y)$ in a plane such that its distance from a fixed point (the focus) is always less than its distance from a fixed line (the directrix).



From the figure above, with $0 < \in < 1$, and a constant a, the directrix has equation

$$x = -\frac{a}{\in} \quad . \qquad \text{-------------} \qquad (9)$$

Then

$$\frac{|PA|}{|PM|} = \in \qquad \text{------------} \qquad (10)$$

$$\Rightarrow \sqrt{(x + a\in)^2 + y^2} = \in\sqrt{\left(x + \frac{a}{\in}\right)^2} \quad .$$

$$\Rightarrow \quad (x + a\in)^2 + y^2 = \in^2\left(x + \frac{a}{\in}\right)^2 \quad .$$

$$\Rightarrow \quad x^2 + 2a\in x + a^2\in^2 + y^2 = \in^2\left(x^2 + 2\frac{a}{\in}x + \frac{a^2}{\in^2}\right)$$

$$\Rightarrow \quad x^2(1 - \in^2) + y^2 = a^2(1 - \in^2).$$

Dividing through by $a^2(1 - \in^2)$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - \in^2)} = 1. \qquad \text{---------------} \qquad (11)$$

Setting $b^2 = a^2(1 - \in^2)$, we have the equation

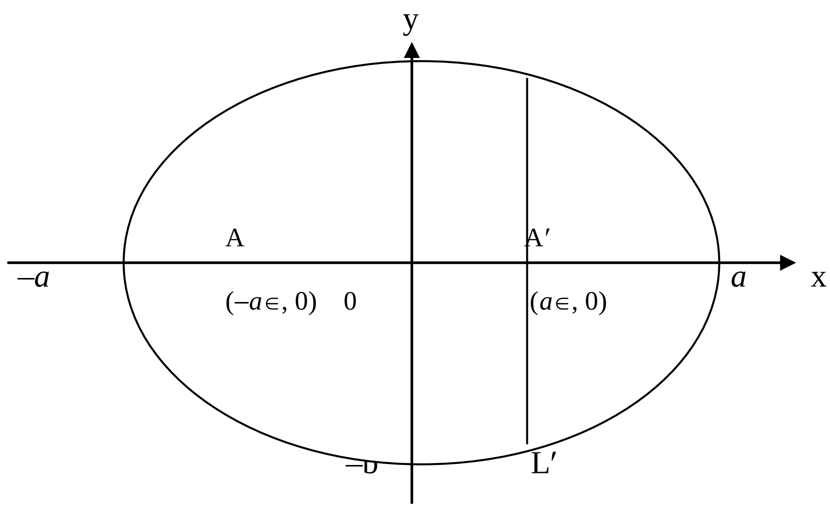
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad \text{---------------} \quad (12)$$

This is the equation of the ellipse.

Notes:

1) Since $0 < \in < 1$, $a > b$.

The constant a is called the **semi-major axis** of the ellipse, while b is called its **semi-minor axis**. The ellipse has the shape



2) The number $\in$ is called the **Eccentricity** of the ellipse.
If $\in = 0$, the ellipse reduces to a circle of radius a.
If $\in = 1$, the locus is a parabola.

3) The ellipse is symmetrical in both the x and the y-axis; these are called the **axes** of the ellipse.

4). When $x = a \in$, we have

$$\frac{a^2 \in^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \in^2$$

$$\Rightarrow \quad y^2 = b^2 (1 - \in^2).$$

$$\Rightarrow \quad y = \pm\, b \sqrt{1 - \in^2}\ . \qquad \text{---------------------} \quad (13)$$

Thus the Latus Rectum of the ellipse has length

$LL' = 2b\ (1 - \in^2)^{½}$.

Equations of Tangents and Normals

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \qquad \text{---------------------} \qquad (14)$$

At any point (x_1, y_1) on the ellipse, the tangent there has gradient

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\left(\frac{x_1}{y_1}\right) .$$

$\therefore$ The equation of the tangent is

$$y - y_1 = -\frac{b^2}{a^2}\left(\frac{x_1}{y_1}\right)(x - x_1) .$$

$$\Rightarrow \quad a^2 y_1 (y - y_1) = - b^2 x_1 (x - x_1) .$$

$$\Rightarrow \quad a^2 y_1 y - a^2 y_1^2 = - b^2 x_1 x + b^2 x_1^2 .$$

Dividing through by $a^2 b^2$ and re-arranging, we have

$$\frac{y_1 y}{b^2} + \frac{x_1 x}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \qquad \text{----------------------} \qquad (15)$$

The rhs equals 1, since (x_1, y_1) lies on the ellipse.
Thus the tangent has equation

$$\boxed{\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.} \qquad \text{------------------------} \qquad (16)$$

The gradient of the normal at (x_1, y_1) is $\dfrac{a^2 y_1}{b^2 x_1}$.

$\therefore$ The normal at (x_1, y_1) has equation

$$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1) .$$

$$\Rightarrow \quad b^2 x_1 (y - y_1) = a^2 y_1 (x - x_1) .$$

$$\Rightarrow \quad b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 y_1 x_1$$

$$\frac{x_1 y}{a^2} - \frac{y_1 x}{b^2} = \frac{x_1 y_1}{a^2} - \frac{x_1 y_1}{b^2}$$

$$\Rightarrow \boxed{\frac{x_1 y}{a^2} - \frac{y_1 x}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) x_1 y_1} \qquad \text{---------------------} \qquad (17)$$

Parametric Equations of the Ellipse.

The equations are

$$x = a\cos t \;, \quad y = b\sin t \;, \qquad \text{-------------------} \qquad (18)$$

where t is a parameter. The point (a cost, b sint) lies on the ellipse, since

$$\frac{(a\cos t)^2}{a^2} + \frac{(b\sin t)^2}{b^2} = \cos^2 t + \sin^2 t = 1.$$

The equation of the tangent to the ellipse at any point 't' (ie. (*a* cost, b sint)) will be

$$\frac{a\cos t}{a^2}\, x + \frac{b\sin t}{b^2}\, y = 1 \,,$$

i.e. using equation (5.1.16)

$$\boxed{\left(\frac{\cos t}{a}\right) x + \left(\frac{\sin t}{b}\right) y = 1.} \qquad \text{---------------------} \qquad (19)$$

By a similar procedure, the equation of the normal at any point 't' on the curve can be obtained.

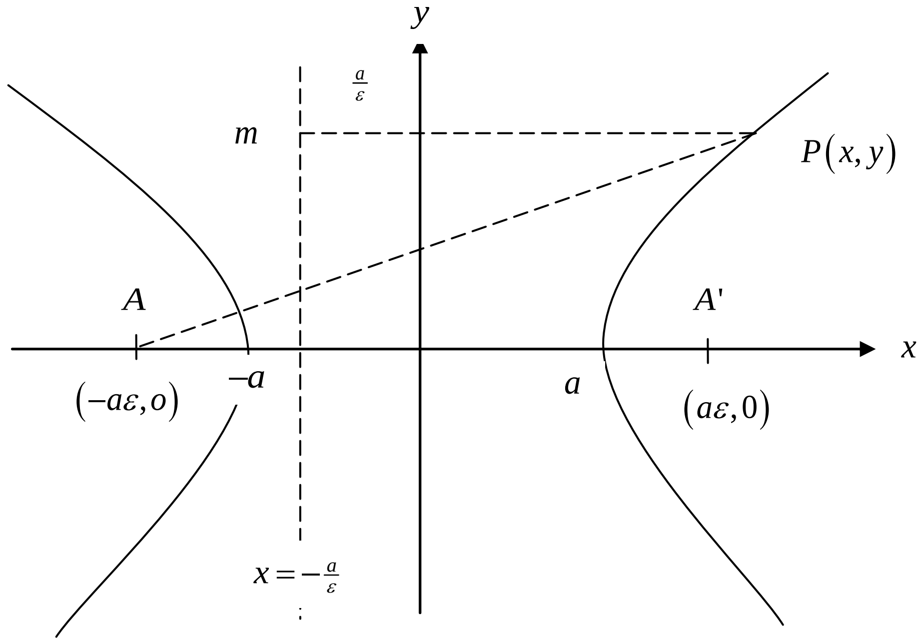
<u>Exercises:</u>

(1) The normal at the point 't' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the ellipse again at t_1.

Find t_1 in terms of t .

(2) Find the equations of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the Latus rectum meets the curve. Show that the two tangents meet on the x-axis.

C THE HYPERBOLA



In the case of the hyperbola, the eccentricity $\in > 1$. Thus by definition

$$\frac{|PA|}{|PM|} > 1 \qquad \text{-----------------------} \qquad (20)$$

i.e.

$$\frac{|PA|}{|PM|} = \in \text{ , where } \in > 1.$$

$$\Rightarrow \quad (x + a\in)^2 + y^2 = \in^2 \left(x + \frac{a}{\in}\right)^2 .$$

$$\Rightarrow \quad x^2 + 2a\in x + a^2\in^2 + y^2 = \in^2 \left(x^2 + 2\frac{a}{\in}x + \frac{a^2}{\in^2}\right) .$$

$$= \in^2 x^2 + 2a\in x + a^2 .$$

Hence

$$x^2 (1 - \in^2) + y^2 = a^2 (1 - \in^2).$$

$\Rightarrow$

$$x^2 (\in^2 - 1) - y^2 = a^2 (\in^2 - 1) \qquad \text{-------------------} \qquad (21)$$

which leads to the equation

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \qquad \text{--------------------} \qquad (22)$$

where $b^2 = a^2(\in^2 - 1)$.

Parametric Equations

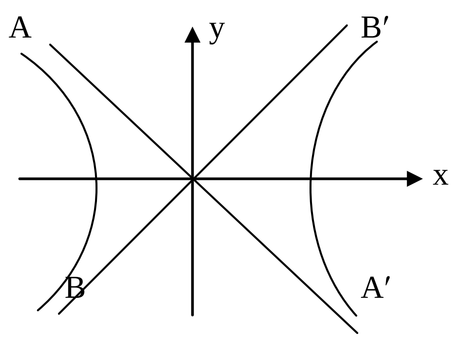
These are

$$x = a \sec t \quad , \quad y = b \tan t \qquad \text{----------------} \quad (23)$$

or

$$x = a \cosh t \quad , \quad y = b \sin h\, t. \qquad \text{---------------} \quad (24)$$

Asymptotes



The tangents to the hyperbola at $x = \pm\infty$ are called the asymptotes of the hyperbola. They have equations

$$y = \pm \frac{b}{a} x \qquad \text{-----------------------} \qquad (25)$$

In the figure, they are shown by the lines AA′ and BB′ .

When AA′ and BB′ are perpendicular, then the hyperbola is called a **rectangular hyperbola**. In this case it may be shown that the equation of the hyuperbola reduces to

$$\boxed{xy = c^2,} \qquad \text{----------------------} \qquad (26)$$

where c is a real constant.

Parametric Equations of the Rectangular Hyperbola.

These are

$$x = ct, \; y = \frac{c}{t} \qquad \text{-----------------------} \qquad (27)$$

Thus, the point ‘t’ has co-ordinates $(ct, \frac{c}{t})$.

Exercises:

(1) Show that the equation of the tangent to the hyperbola at any point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(2) Find the equations of the tangents and normals to the rectangular hyperbola at any point 't' on it.

(3) The normal to the rectangular hyperbola at the point $(ct, \frac{c}{t})$ meets the hyperbola again at the point 't_1'. Find t_1 in terms of t.

CHAPTER SIX

D I F F E R E N T I A T I O N

In this chapter topics to be treated will enable the student gain a more complete understanding of the process of differentiation.

6.1 LIMITS OF FUNCTIONS

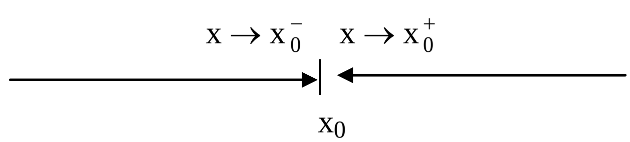
6.1.1 **Definition**:
Let f(x) be a function defined in a deleted δ - neighbourhood of the point $x = x_0$. The function f(x) is said to have the limit l as x approaches x_0 if f(x) can be made as close to l as we please by choosing x sufficiently close to x_0.

We write

$$\lim_{x \to x_0} f(x) = l . \qquad \text{-------------------} \qquad (1)$$

In a rigorous language we say that $\lim_{x \to x_0} f(x) = l$ if for every $\in > 0$ (however small, $\exists\ \delta > 0$ ($\delta = \delta(\in)$) such that $|f(x) - l| < \in$, whenever $o < |x - x_0| < \delta$.

Right Hand /Left Hand Limits
The lim f(x) as $x \to x_0$ usually depends on how x approaches x_0. Since x_0 is a real number, x can approach x_0 in only two possible directions. (See the figure).



Different results could be obtained for each of the cases. Let us consider the following examples:

1) $$\lim_{x \to 0^-} (2x + 1) = 1$$

$$\lim_{x \to 0^+} (2x + 1) = 1$$

2) $$\lim_{x \to 0^-} \left(\frac{1}{x}\right) = -\infty$$

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = +\infty$$

3) $\lim_{x \to 0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$

$\lim_{x \to 0^+} \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

4) Let f(x) be defined by

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

(See graph)

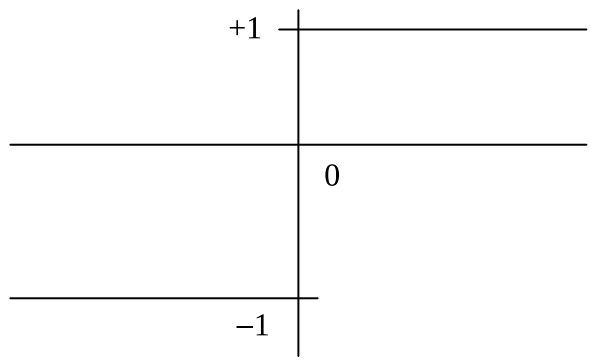


Fig. 1

$\lim_{x \to 0^-} f(x) = -1$

$\lim_{x \to 0^+} f(x) = +1$

The above examples illustrate the fact that $\lim_{x \to x_0} f(x) \neq \lim_{x \to x_0^+} f(x)$ (in general).

Let $\lim_{x \to x_0^-} f(x) = l_1$, and

$\lim_{x \to x_0^+} f(x) = l_2$.

Then l_1 is called the left hand limit, while l_2 is called the right hand limit of f(x) as x approaches x_0.

In general $l_1 \neq l_2$. However, if $l_1 = l_2 = l$ then

$$\lim_{x \to x_0} f(x) = l. \qquad \text{-------------------} \qquad (2)$$

Remarks:

1) The $\in - \delta$ definition for limit of $f(x)$ is used only to prove that $f(x)$ has a limit at $x = x_0$.
2) A simple way of showing that a function $f(x)$ has no limit at $x = x_0$ is by determining l_1 and l_2 .
3) It is important to note that $\lim_{x \to x_0} f(x)$ is not the same as $f(x_0)$ since it is only the deleted δ – neighbourhood of x_0 that is being considered when finding $\lim_{x \to x_0} f(x)$.

Worked Examples:

(1) **Prove that $\lim_{x \to 0} (2x + 1) = 1$.**

We wish to show that given any $\in > 0$, $\exists\ \delta > 0\ (\delta(\in))$ such that $|(2x+1) - 1| < \in$ whenever $0 < |x - 0| < \delta$.

Now $|(2x + 1) - 1| = |2x| = 2|x|$.

Since $|x| < \delta$ $(x \neq 0)$, it follows that $|(2x + 1) - 1| < 2\delta$.

$\therefore$ Choosing $\delta = \dfrac{\in}{2}$, we obtain the result that

$|(2x + 1) - 1| < \in$ whenever $0 < |x| < \delta$. This completes the proof.

(2) **Prove that $\lim_{x \to 1} x^2 = 1$.**

We show that given $\in > 0$, $\exists\ \delta > 0\ (\delta(\in))$ s.t. $|x^2 - 1| < \in$ whenever $0 < |x - 1| < \delta$.

Now, $|x^2 - 1| = |(x + 1)(x - 1)| = |x + 1|\ |x - 1|$

Since $0 < |x - 1| < \delta$, it follows that $|x^2 - 1| < |x + 1|\ \delta$

Suppose that $\delta = 1$, then $|x - 1| < 1 \Rightarrow -| < x - 1 < | \Leftrightarrow 0 < x < 2,\ x \neq 1$

$\Rightarrow$

$|x^2 - 1| < 3\delta$

Setting $\delta = \dfrac{\in}{3}$ or 1, whichever is smaller, will give us the result that $|x^2 - 1| < \in$ whenever $|x - 1| < \delta$.

This completes the proof.

(3) **Define**

$$f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

(i) Draw the graph , and (ii) show that $\lim\limits_{x \to 1} f(x)$ does not exist.

(i) **Graph** : for x < 1 , $|x - 1| = -(x - 1)$

$\Rightarrow f(x) = -\dfrac{(x-1)}{x-1} = -1$

for x > 1, $|x - 1| = x - 1$

$\Rightarrow f(x) = \dfrac{x-1}{x-1} = 1$

f(1) = 0.

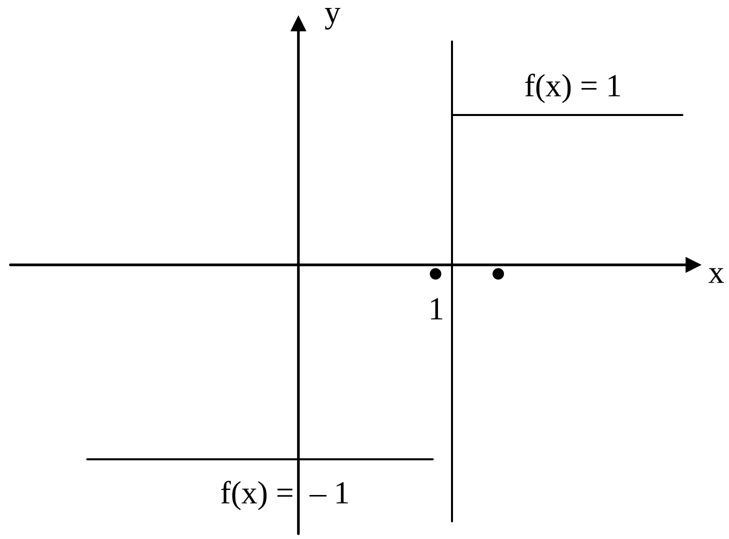


Fig. 2

(ii) $$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -\frac{(x-1)}{x-1} = \lim_{x \to 1^-} (-1) = -1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x-1}{x-1} = \lim_{x \to 1^+} (+1) = 1$$

$\therefore \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist.

6.2 **THEOREMS ON LIMITS**

Let

$$\lim_{x \to x_0} f(x) = A, \qquad \lim_{x \to x_0} g(x) = B, \text{ where } A \text{ and } B \text{ are finite constants.}$$

Then

(a) $$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$= A + B$$

(b) $$\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x)$$

$$= A - B$$

(c) $$\lim_{x \to x_0} (f(x) \,.\, g(x)) = \lim_{x \to x_0} f(x) \,.\, \lim_{x \to x_0} g(x)$$

$$= A \,.\, B$$

(d) $$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{A}{B}.$$

provided that $B \neq o$.

(i) If $A \neq o$ and $B = o$, $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right)$ does not exist.

(ii) If $A = o$ and $B = o$, the form $\frac{0}{0}$ is indeterminate and the limit may or may not exist.

6.3 **Evaluation of Limits**

The following rules will assist us when evaluating limits of functions

(1) If $f(x) = P_n(x)$, a polynomial in x, then

$$\lim_{x \to x_0} f(x) = P_n(x_0) = f(x_0).$$

E.g. $\lim_{x \to 1} (x^2 + 2x - 5) = 1^2 + 2(1) - 5 = -2.$

(2) If $f(x)$ is an elementary function, then

$$\lim_{x \to x_0} f(x) = f(x_0).$$

E.g. $\lim_{x \to 0} \sin x = \sin o = 0.$

(3) If $f(x) = \dfrac{(x-x_0)^{\alpha}\ g(x)}{(x-x_0)^{\alpha}\ h(x)}$, $x \neq x_0$ and α is real, then

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \frac{g(x)}{h(x)} .$$

E.g. For $f(x) = \dfrac{x^3 - 8}{x - 2}$, $x \neq 2$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x\to 2} (x^2 + 2x + 4)$$

$$= 2^2 + 2(2) + 4 = \underline{\underline{12}} .$$

6.3 **Special Limits**

(1) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ ------------------- (3)

The proof is shown below:
Consider the arc AB of the circle centre 0 and radius r.

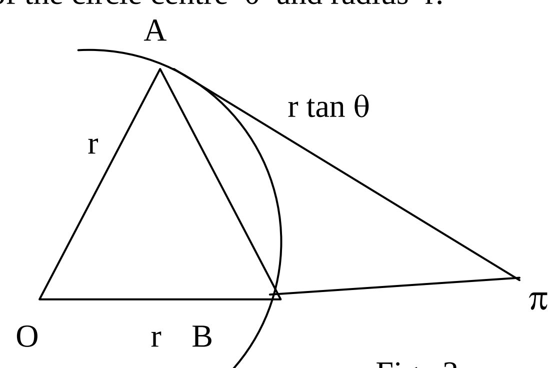


Fig. 3

Clearly,
Area of Δ AOB < Area of Sector < Area of Δ AOT
A O B

$\Rightarrow$ $\dfrac{1}{2} r^2 \sin\theta < \dfrac{1}{2} r^2 \theta < \dfrac{1}{2} r^2 \tan\theta$

$\Rightarrow$

$\sin\theta < \theta < \tan\theta$

$\Rightarrow$ $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

As θ $\to$ 0, $\frac{1}{\cos\theta} \sim 1$

$\Rightarrow \quad \frac{\theta}{\sin\theta} \sim 1$ for small θ

$\therefore \quad \lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$ or $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$

(2) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(3) $\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}} = e$

Exercises:

(1) Prove, using the $\in - \delta$ definition that

(a) $\lim_{x \to 0} (1 - x) = 1$ (b) $\lim_{x \to 2} (3x + 1) = 7$

(2) A function f(x) is defined as

$f(x) = \frac{x^3 + 1}{x + 1}, \quad x \neq -1.$

Evaluate $\lim_{x \to -1} f(x)$.

(3) Draw the graph of the function

$f(x) = |x - 2|$

By determining $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ show that $\lim_{x \to 2} f(x)$ exists.

(4) Evaluate the following limits of functions:

(a) $\lim_{x \to -2} \frac{(x + 2)(x^2 + 3)}{x^2 - 4}$

(b) $\lim_{x \to 0} e^x \sin 2x$

(c) $\lim_{x \to 1} \frac{1}{x-1}\left(\frac{1}{x+3} - \frac{2}{3x+5}\right)$

(5) Prove, using the $\in - \delta$ definition that

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

(6) Prove that if $\lim_{x \to x_0} f(x)$ exists, then it must be unique.

(7) A function f(x) is defined by

$$f(x) = \begin{cases} -1, & x < -1 \\ x & -1 < x < 1 \\ 1, & x > 1 \end{cases}$$

(i) Draw the graph of f(x).
(ii) Show that $\lim_{x \to -1} f(x)$ and $\lim_{x \to 1} f(x)$ both exist.

6.4 <u>CONTINUITY OF A FUNCTION</u>

<u>Definition</u>: A function f(x) is said to be continuous at a point $x = x_0$ if the following conditions hold:

1) $\lim_{x \to x_0} f(x)$ exists,

2) $f(x_0)$ is defined,

3) $\lim_{x \to x_0} f(x) = f(x_0)$.

Equivalently we could say that f(x) is continuous at the point $x = x_0$ if $f(x_0)$ is defined and $\lim_{x \to x_0} f(x) = f(x_0)$.

Using the $\in - \delta$ definition we say that f(x) is continuous at $x = x_0$ if given $\in > o$ (however small) $\exists\, \delta > o$ such that $|f(x) - f(x_0)| < \in$, whenever $|x - x_0| < \delta$.

Right Hand/Left Hand Continuity

If only $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$, where $f(x_0)$ is defined, then $f(x)$ is continuous on the left; if only $\lim\limits_{x \to x_0^+} f(x) = f(x)$, then f(x) is continuous on the right.

Continuity in an Interval

A function $f(x)$ is said to be continuous in an interval (a, b) if $f(x)$ is defined for every $x_0 \in (a, b)$ and $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

For the closed interval $[a, b]$, f(x) is continuous in $[a, b]$ if $f(x)$ is continuous in (a, b) and in addition

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b) ,$$

where $f(a)$ and $f(b)$ are defined.

THEOREMS ON CONTINUITY

1. If $f(x)$ and $g(x)$ are both continuous at a point $x = x_0$, then $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\dfrac{f(x)}{g(x)}$ are all continuous at $x = x_0$ (provided that $g(x_0) \neq 0$ in the last case).

2. Every polynomial in x, $P_n(x)$ is continuous for all real x.

Proof: (of 2)

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$.
Consider $f(x) = x^m$, $m < n$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^m \text{ for some } x_0 \text{ real.}$$
$$= x_o^m = f(x_0).$$

$\therefore$ $f(x) = x^m$ is continuous at $x = x_0$.
Since $P_n(x)$ is a sum of such functions it is continuous at x_0 (using 1). Since x_0 is arbitrary it follows that $P_n(x)$ is continuous for all x real.

3. If $y = f(x)$ is continuous at $x = x_0$ and $z = g(y)$ is continuous at $y = y_0$ and if $y_0 = f(x_0)$, then the function $z = g\{f(x)\}$, called the composite or function of a function, is continuous at $x = x_0$. That is, a continuous function of a continuous function is continuous.

4. If $f(x)$ is continuous in a closed interval, then it is bounded in the interval.

5. For further theorems on continuity, see M. Spiegel (Adv. Calculus) pp. 25 – 26.

6.5 <u>SECTIONAL CONTINUITY</u>

A function f(x) is said to be <u>sectionally</u> <u>continuous</u> or <u>piecewise</u> <u>continuous</u> in an interval [*a*, b] if the interval can be subdivided into a finite number of subintervals in each of which f(x) is continuous and has finite right and left hand limits. Such a function has only a finite number of discontinuities. (See figure).

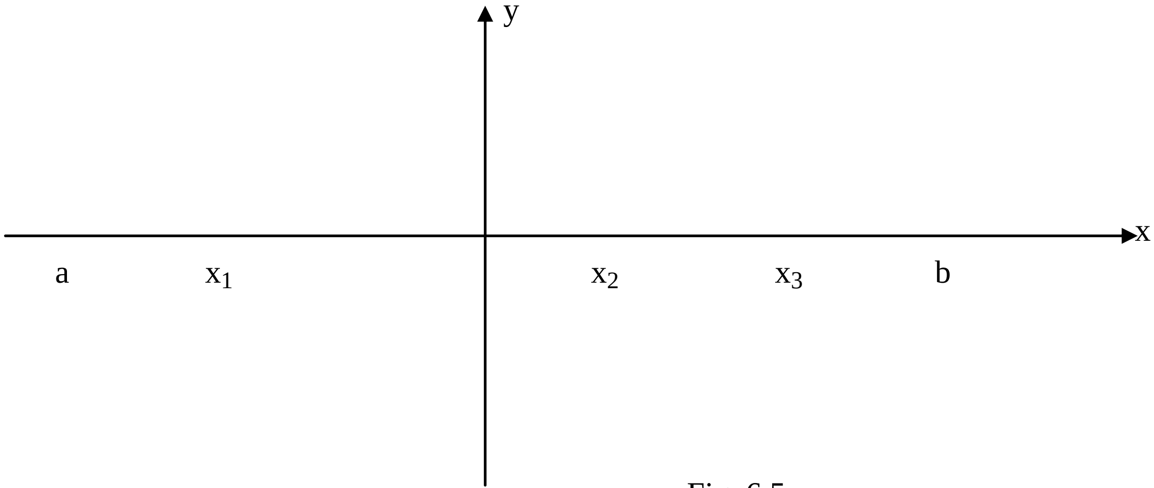


Fig. 6.5

The discontinuities are x_1, x_2 and x_3.

<u>Exercise:</u>
Draw the graph of the function

$$f(x) = \begin{cases} x, & -3 \le x \le -1 \\ x^2 - 1, & -1 < x \le 1 \\ 2, & 1 < x \le 3 \\ 1 - x, & 3 < x \le 5 \end{cases}$$

This is an example of a sectionally continuous or a piecewise continuous function in the interval [–3, 5]. It is continuous in each of the subintervals [– 3, – 1], (– 1, 1], (1, 3], (3, 5] with finite discontinuities at x = – 1, 1 and 3. (Show that f(x) is not continuous at these points!).

6.6 <u>UNIFORM CONTINUITY</u>

Definite:
Let f(x) be continuous in [a, b]. Then at each point x_0, and for every $\in > o, \exists \delta > o$ ($\delta = \delta(\in, x_0)$) s.t. $|f(x) - f(x_0)| < \in$, whenever $|x - x_0| < \delta$.

Now if δ depends only on $\in$ and not and x_0, then f(x) is said to be uniformly continuous in [a, b].

Example:

Prove that $f(x) = x^2$ is uniformly continuous in the interval [0, 2] .

We show that for each x_0 in [0, 2] and for every $\in > o$, $\exists\, \delta = \delta(\in)$ (depending on $\in$ only), s.t. $|f(x) - f(x_0)| < \in$, whenever $|x - x_0| < \delta$.

Now, for any $x_0 \in [0, 2]$

$|x^2 - x_0^2| = |x + x_0|\ |x - x_0|$.

Since $|x - x_0| < \delta$. , we have

$|x^2 - x_0^2| < |x + x_0|\ \delta < 4\,\delta$, since $0 \le x \le 2$.

Hence choosing $\delta = \dfrac{\in}{4}$, the result follows. It is clear that δ does not depend on x_0.

6.7 DIFFERENTIABILITY

Let f(x) be defined at some point x_0 in the interval (a, b). Then, the derivative of f(x) at x_0 denoted by $f'(x_0)$ is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ , if the rhs limit exists.}$$

Setting $h = x - x_0$, the above definition becomes

$$f'(x_0) = \lim_{h \to o} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{-------------------} \qquad (4)$$

A function f(x) is said to be **differentiable** at a point x_0 if $f'(x_0)$ exists.

Right Hand/Left Hand Differentiability

A function f(x) is said to be **left hand differentiable** at $x = x_0$ if

$$f'_{-}(x_0) = \lim_{h \to o^-} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{-------------------} \qquad (5)$$

exists.

It is **right hand differentiable** at $x = x_0$ if

$$f'_{+}(x_0) = \lim_{h \to o^+} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{-------------------} \qquad (6)$$

exists.

The function f(x) is said to be differentiable at $x = x_0$ if and only if $f'_{-}(x_0) = f'_{+}(x_0)$.

Differentiability in an Interval

A function f(x) is differentiable in an interval (a, b) if $f'(x_0)$ exists $\forall\, x_0$ in (a, b). In particular, for the closed interval $[a, b]$ $f'(x_0)$ must exist $\forall\, x_0 \in (a, b)$ and $f'_{+}(a)$ and $f'_{-}(b)$ must exist.

Examples:

(1) Show that the function $f(x) = x^2$ is differentiable at $x = 2$.

Solution:

$$f'(2) = \lim_{h \to 2} \frac{f(2+h) - f(2)}{h}$$

i.e.

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h}$$

$$= \lim_{h \to 0} (4 + h)$$

$$= 4.$$

$\therefore$ $f'(2)$ exists and so f(x) is differentiable at $x = 2$.

(2) Let

$$f(x) = \begin{cases} x^2, & x > 1 \\ 1, & x = 1 \\ x, & x < 1 \end{cases}$$

Find (a) $f'_{-}(1)$

(b) $f'_{+}(1)$

Is f(x) differentiable at x = 1?

Solution:

$$f'_{-}(1) = \lim_{h\to o^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to o^-} \frac{(1+h) - 1}{h}$$

$$= \lim_{h\to o^-} \frac{h}{h} = \lim_{h\to o^-} 1 = 1$$

$$f'_{+}(1) = \lim_{h\to o^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to o^+} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h\to o^+} \frac{2h + h^2}{h}$$

$$= \lim_{h\to o^+} (2 + h) = 2$$

Clearly then, $f'_{-}(1) \neq f'_{+}(1)$ $\therefore$ $f'(1)$ does not exist, and so f(x) is not differentiable at x = 1.

THEOREM:

If a function f(x) is differentiable at a point $x = x_0$, then it is continuous there. The converse, however, is not true.

Proof:

$$f(x_0 + h) - f(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} \cdot h$$

$\Rightarrow$

$$\lim_{h\to 0} \{f(x_0 + h) - f(x_0)\} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot \lim_{h\to 0} h$$

$$= f'(x_0) \cdot \lim_{h\to 0} h$$

$$= 0 \text{, since } f'(x_0) \text{ exists.}$$

$$\therefore \lim_{h\to 0} f(x_0 + h) = f(x_0)$$

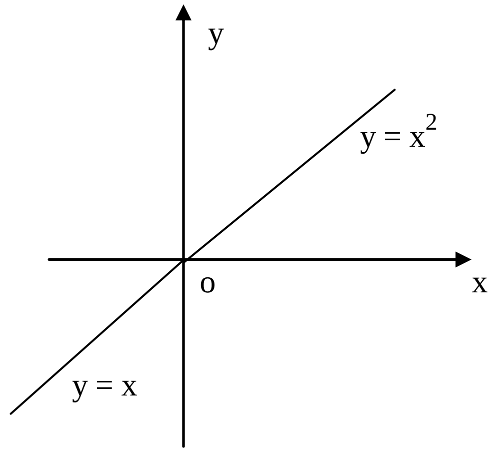
Setting $h = x - x_0$, we have

$$\lim_{x\to x_0} f(x) = f(x_0)$$

which completes the proof.

The converse of this theorem is false. Consider the counter example

$$f(x) = \begin{cases} x^2 & , \quad x > 0 \\ x & , \quad x < 0 \\ 0 & , \quad x = 0 \end{cases}$$



$$\lim_{x \to 0^-} f(x) \;=\; \lim_{x \to 0^-} x \;=\; 0$$

$$\lim_{x \to 0^+} f(x) \;=\; \lim_{x \to 0^+} x^2 \;=\; 0$$

$$f(o) \;=\; 0$$

$\therefore$ f(x) is continuous at x = 0.

Graph of y = f(x).

Now

$$f'_{-}(o) \;=\; \lim_{h \to 0^-} \frac{f(o+h) - f(o)}{h} \;=\; \lim_{h \to 0^-} \frac{f(h) - f(o)}{h}$$

$$= \lim_{h \to 0^-} \frac{h - o}{h}$$

$$= 1$$

$$f'_{+}(o) \;=\; \lim_{h \to 0^+} \frac{f(h) - f(o)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 - o}{h} \;=\; \lim_{n \to 0^+} h \;=\; 0$$

$$\therefore f'_{-}(o) \neq f'_{+}(o)$$

$\therefore$ f(x) is not differentiable at x = 0.

Exercises:

(1) Define

$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

Is f(x) continuous at x = 2? If not, how can f(x) be redefined so that it is continuous at x = 2?

(2) (a) Prove that $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$.

(b) Using the result of (a) show that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at x = 0.

(3) Show that the function

$$f(x) = |x|$$

is continuous at x = 0. Demonstrate analytically, however, that f(x) is not differentiable at x = 0.

(4) Prove that the function $f(x) = x^3$ is uniformly continuous in the interval (0, 2).

(5) Construct a function f(x) which is

(a) Continuous at x = 0,

(b) Sectionally continuous in [– 2, 2]

(c) Differentiable at x = 1,

(d) Continuous at x = 1 but not differentiable there.

(Prove each of the cases (a) to (d)).

6.8 DIFFERENTIATION

The derivative of a function f(x) at any point x is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

The function $f'(x)$ is also called the Derived Function of f(x).
The process of finding the derivative is called **differentiation**. The derived function $f'(x)$ enables one to obtain the derivative of f(x) at any point $x = x_0$.
The following cases will be considered.

1. **f(x) = x^n , n rational.**

(i) for f(x) = x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \quad \text{-------------} \quad (7)$$

(ii) f(x) = x^2

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} .$$

$$= \lim_{h \to 0} (2x + h) = 2x \quad \text{-----} \quad (8)$$

(iii) f(x) = x^3

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \quad \text{---------------}(9)$$

(iv) f(x) = x^n , n integer.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \ldots + \binom{n}{n-1} x h^{n-1} + h^n - x^n}{h}$$

$$= \lim_{h\to 0} \left(\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \ldots + \binom{n}{n-1} x h^{n-2} + h^{n-1} \right)$$

$$= \binom{n}{1} x^{n-1} = \frac{n!}{(n-1)!\,1!} x^{n-1} = n\, x^{n-1} \qquad \text{-------------} \qquad (10)$$

(v) f(x) = x^{-1} = $\frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h x (x+h)} = \lim_{h\to 0} \frac{-h}{h(x^2 + xh)}$$

$$= \lim_{h\to 0} - \frac{1}{x^2 + xh} = -\frac{1}{x^2} \text{ . ----- (11)}$$

The same result as for (iv) holds for – ve integers.
In fact for all rational numbers n

f(x) = x^n

$\Rightarrow$ f ′(x) = $n\,x^{n-1}$

2. **Trigonometric Functions.**

(i) f(x) = sin x.

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

Using the fact that sin A – Sin B = 2 Cos $\frac{A+B}{2}$. Sin $\frac{A-B}{2}$ we have

$$f'(x) = \lim_{h \to 0} \frac{2\cos\frac{(x+h+x)}{2}\ \sin\frac{(x+h-x)}{2}}{h}.$$

$$= \lim_{h \to 0} \frac{2\cos\frac{(2x+h)}{2}\ \sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \cos\left(x+\frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x.\ 1$$

$$= \cos\ x. \qquad \text{-------------} \qquad (12)$$

(ii) $f(x) = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin\frac{(x+h+x)}{2}\ \sin\frac{(x+h-x)}{2}}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(x+\frac{h}{2}\right)\ \sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} -\sin\left(x+\frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\sin x\ .\ 1 = \underline{\underline{-\sin x}}\ . \qquad \text{------------------} \qquad (13)$$

3. **Exponential Function**

Differentiation of $y = a^x$, $a > 1$

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right) a^x = k.a^x, \qquad \text{--------------} \quad (14)$$

where

$$k = \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right) \qquad \text{----------------} \quad (15)$$

Note that when $x = 0$, for $a \neq 0$, $\frac{dy}{dx} = k$. This is the gradient of the tangent to the curve $y = a^x$ at $x = 0$. In order to determine the value of k, let us attempt to find $\frac{dy}{dx}$ by an alternative approach:

If $y = a^x$

Taking log of both sides we have

$\log_e y = x \log_e a$

$$\Rightarrow \quad \frac{1}{y} \frac{dy}{dx} = \log_e a$$

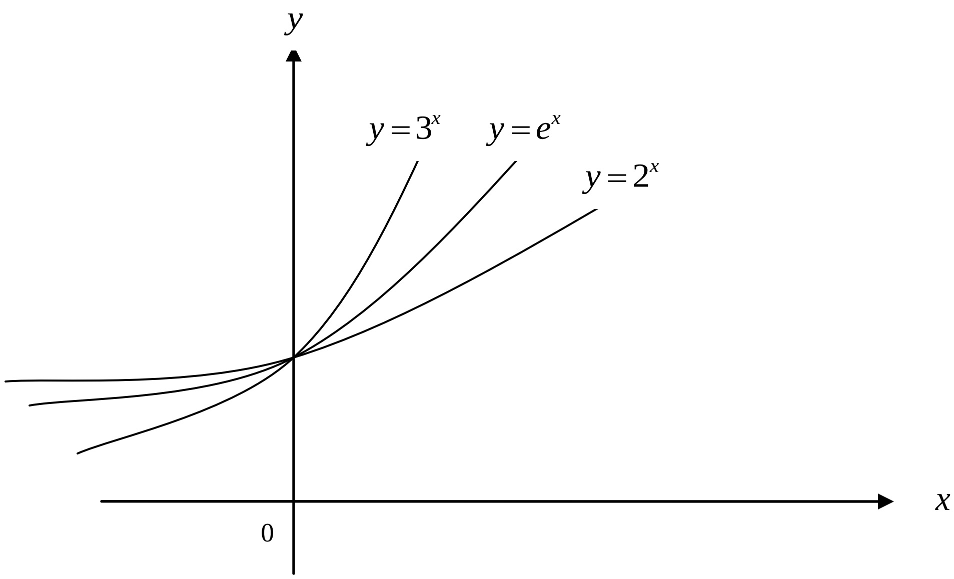
$$\Rightarrow \quad \frac{dy}{dx} = y \log_e a = a^x . \log_e a \qquad \text{----------------} \quad (16)$$

Comparing this result with 14 above, we have $\lim_{h \to 0} \frac{a^h - 1}{h} = \log_e a$.

Thus $y = a^x$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = a^x \log_e a} \qquad \text{-----------------} \ (17)$$

Graphs of 2^x, e^x, 3^x



For $a > 1$, $\frac{dy}{dx}$ at $x = o$ increases. The evidence from the graph shows that $T_2 < T_3$, where T_2, T_3 are the tangents of the curves for 2^x and 3^x respectively.

Gradient of $T_2 < 1$ while grad of $T_3 > 1$.

Let $y = e^x$, $2 < e < 3$ s.t. gradient of $T_e = 1$.

Hence for $a = e$, $k = 1$,

$y = e^x$,

and $\frac{dy}{dx} = e^x$.

We have obtained an important result that

$$\boxed{y = e^x \Rightarrow \frac{dy}{dx} = e^x} \qquad \text{------------} \quad (18)$$

4. **Logarithmic Function**

Let $y = \log_e x$. Then

$$x = e^y$$

$$\therefore \; 1 = e^y \frac{dy}{dx}$$

$$\therefore \; \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

i.e.

$$\boxed{y = \log_e x \Rightarrow \frac{dy}{dx} = \frac{1}{x}} \qquad \text{-----------} \quad (19)$$

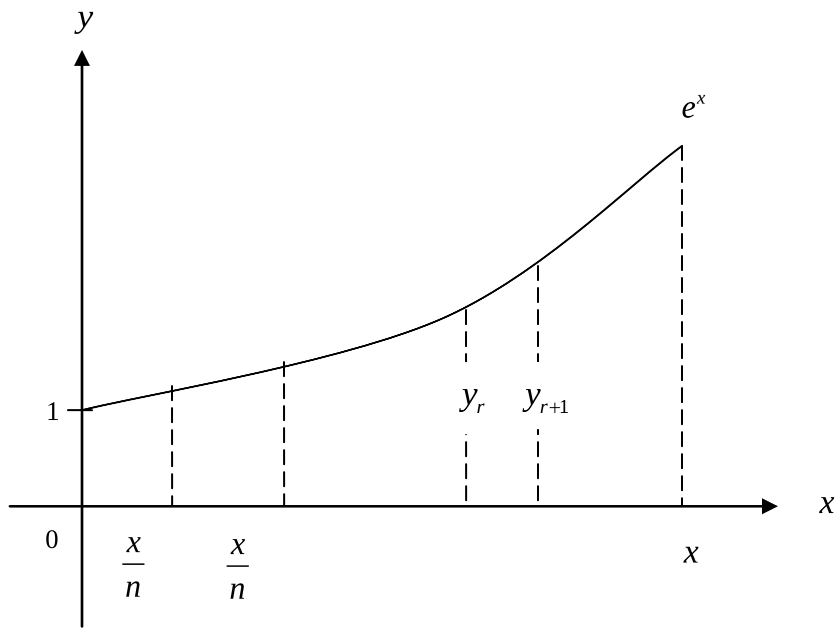
The Value of e :

This can never be done exactly but only as a limiting process.

Consider the interval (o, x). Subdivide it into a large number n of equal intervals. Then the width of each subinterval is $\frac{x}{n}$.

Let

$$\delta x = \frac{x}{n} \qquad \text{----------------} \quad (20)$$



Since $\dfrac{dy}{dx} = y$

$\delta y \sim y \; \delta x$ ----------------- (21)

Let y_r , y_{r+1} be two successive ordinates. Then

$$y_{r+1} = y_r + \delta y$$

$$= y_r + y_r \, \delta x = y_r (1 + \delta x)$$

$$= y_r (1 + \frac{x}{n}) \; .$$

The sequence $\{y_r\}_{r=1}^{n}$ forms a G.P with common ratio $(1 + \frac{x}{n})$.

Hence $y_n = y \simeq (1 + \frac{x}{n})^n$, so that

$$y = \lim_{n \to \infty} (1 + \frac{x}{n})^n$$

$$\therefore \quad e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n \; . \qquad \text{-------------- (22)}$$

Setting x = 1, we have $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ ------------- (23)

6.9 RULES FOR DIFFERENTIATION

The following rules are valid and can be proved from the definition of the derivative.

Let f(x) and g(x) be functions. Then the following rules hold:

(i) $h(x) = f(x) \pm g(x)$

$\Rightarrow h'(x) = f'(x) \pm g'(x)$ – Sum and Difference

(ii) $h(x) = f(x) . g(x)$

$\Rightarrow h'(x) = f'(x) . g(x) + f(x) g'(x)$. – Product Rule

(iii) $h(x) = \frac{f(x)}{g(x)} \Rightarrow$

$$h'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2} .$$ – Quotient Rule

<u>The Chain Rule:</u>

Let

$$h(x) = f[u(x)],$$

then

$$h'(x) = \frac{df}{du} \cdot \frac{du}{dx}$$

<u>Examples:</u>

Differentiate the following functions:

(a) $f(x) = (x^2 + 1)^3$

(b) $f(x) = \tan x$

(c) $f(x) = x^2 \sin x$

(d) $f(x) = \cos (2x + 1)^2$

(e) $f(x) = e^{x^2 + x}$.

CHAPTER SEVEN

ROLLE'S THEOREM AND THE MEAN-VALUE THEOREMS

In this chapter, Rolle's theorem will be studied, proved and used to discuss what is commonly known as the mean-value theorems.

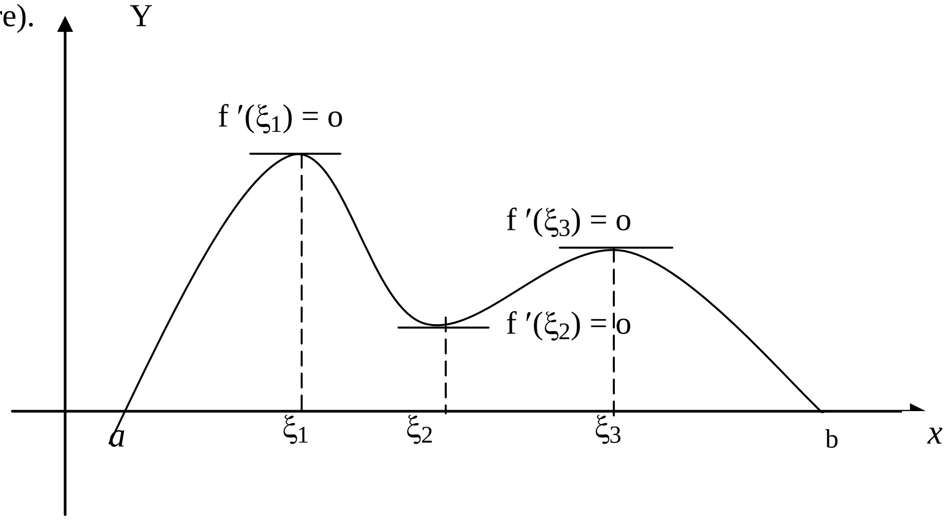
7.1 ROLLE'S THEOREM

If a function f(x) is continuous in a closed interval $[a, b]$ and differentiable in an open interval (a, b), and if $f(a) = f(b) = o$, then $\exists$ at least one value of $x = \xi \in (a, b)$ such that $f'(\xi) = o$.

Geometrical Interpretation

If the curve $y = f(x)$ is continuous in $[a, b]$ and has a unique tangent at every point in $[a, b]$ except possibly at $x = a$ and $x = b$, then there will always be at least one tangent to the curve whose gradient is zero (i.e. parallel to the x-axis).

(see the figure).



Proof:

If $f(x) = 0$, $\forall$ $x \in [a, b]$, then $f'(x) = o$ $\forall$ x in (a, b) and the theorem is trivially true.

Suppose now that $f(x) \neq o$ for some x in $[a, b]$.

Since f(x) is continuous in $[a, b]$, $\exists$ m, M s.t.

M = minimum f(x), and M = maximum f(x).

Since $f(x) \neq o$, M and m are not simultaneously zero. Suppose $M \neq 0$ and that

$f(\xi) = M$ for some $\xi \in (a, b)$.

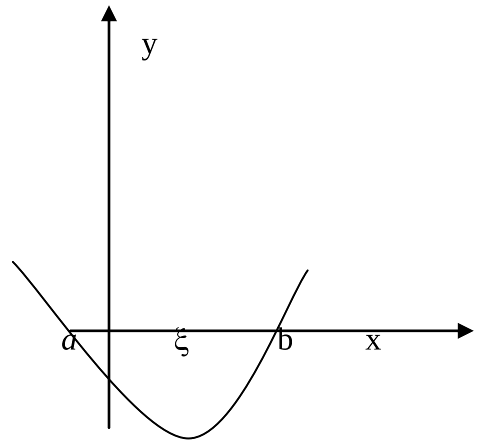
Then for any h,

$$f(\xi + h) \le f(\xi).$$

If h > 0, then

$$\frac{f(\xi+h) - f(\xi)}{h} \le 0$$

$$\Rightarrow \quad \lim_{h \to 0^+} \frac{f(\xi+h) - f(\xi)}{h} \le 0$$



If h < 0, then

$$\frac{f(\xi+h) - f(\xi)}{h} \ge 0$$

$$\Rightarrow \quad \lim_{h \to 0^-} \frac{f(\xi+h) - f(\xi)}{h} \ge 0$$

Since $f'(x)$ exists $\forall\, x \in (a, b)$, in particular $f'(\xi)$ exists

$$\therefore \; f'_-(\xi) = f'_+(\xi)$$

$$\Rightarrow \quad \lim_{h \to 0^-} \frac{f(\xi+h) - f(\xi)}{h} = \lim_{h \to 0^+} \frac{f(\xi+h) - f(\xi)}{h} = 0.$$

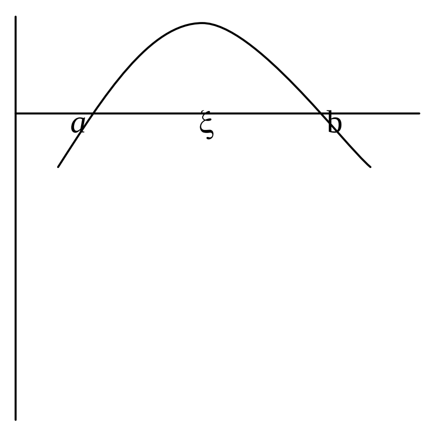
$$\Rightarrow \; f'(\xi) = 0.$$

On the other hand if $m \ne o$, and that $f(\xi) = m$ for some $\xi \in (a, b)$.

Then for any h,

$$f(\xi + h) \ge f(\xi).$$

If h > o, then



$$\frac{f(\xi+h) - f(\xi)}{h} \ge 0$$

$$\Rightarrow \quad \lim_{h \to 0^+} \frac{f(\xi+h) - f(\xi)}{h} \ge 0$$

If $h < 0$, $\dfrac{f(\xi+h) - f(\xi)}{h} \le 0$

$$\Rightarrow \quad \lim_{h \to 0^-} \frac{f(\xi + h) - f(\xi)}{h} \leq 0$$

Since $f'(\xi)$ exists, we have $f'_-(\xi) = f'_+(\xi) = 0$

$\therefore f'(\xi) = 0.$

This completes the proof of Rolles Theorem.

Example:

Verify Rolle's Theorem for the case where

$$f(x) = x(2x - 1), \qquad [0, \tfrac{1}{2}].$$

Solution:

$f(x)$ is continuous $\forall$ x real and so for $[0, \frac{1}{2}]$.

$f'(x) = 4x - 1$; and so $f'(x)$ exists in $(0, \frac{1}{2})$.

$f(0) = f(\frac{1}{2}) = 0.$

$\therefore$ All the conditions of Rolle's Theorem are satisfied

$\therefore \exists\, x = \xi \in (0, \frac{1}{2})$ s.t. $f'(\xi) = 4\xi - 1 = 0$

$\Rightarrow \xi = \frac{1}{4} \in (0, \frac{1}{2}).$

Remarks:

The condition $f(a) = f(b) = 0$ in Rolle's Theorem may be replaced with the condition $f(a) = f(b)$.

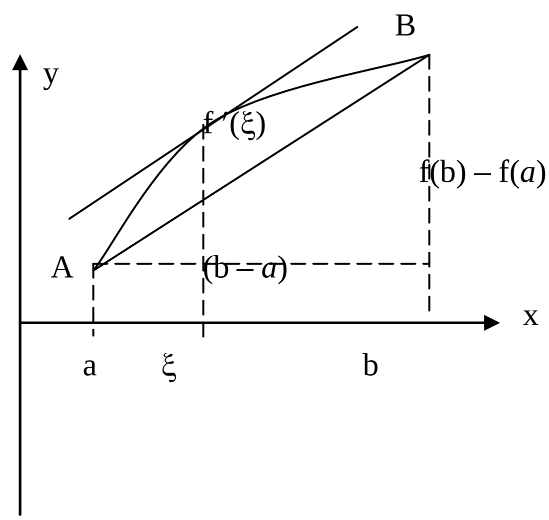
7.2 THE FIRST MEAN-VALUE THEOREM

It states that if f(x) is continuous in the interval [a, b] and differentiable in (a, b) then $\exists$ at least one value of $x = \xi \in (a, b)$ s.t.

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} .$$

Geometrical Interpretation

There is a tangent parallel to the chord AB. (See figure)



Proof

Construct the function

$$G(x) = f(x) - y$$

where y is the equation of the chord AB joining two points A and B on the curve $y = f(x)$ from $x = a$ to $x = b$.

Equation of chord AB is

$$y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a)$$

$$\Rightarrow \qquad y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

$$\therefore \qquad G(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a).$$

It is clear that G(x) is continuous and differentiable in $[a, b]$, since f(x) is.

Furthermore,

$$G(a) = f(a) - 0 - f(a) = 0,$$

$$G(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a), \quad b \neq a$$

$$= f(b) - f(b) + f(a) - f(a) = 0.$$

Thus G(x) satisfies all the conditions of Rolle's theorem. Therefore $\exists\, x = \xi \in (a, b)$ s.t. $G'(\xi) = 0$.

Now $G'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$

$$\therefore\ G'(\xi) = 0 \ \Rightarrow\ f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0$$

$$\Rightarrow\ f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Example:

Verify the first mean-value theorem for the function

$$f(x) = (x^2 - 3x + 1),\ [0, 1].$$

Solution:

f(x) is both continuous and differentiable in [0, 1], since f(x) is a polynomial.

$f'(\xi) = 2x - 3.$

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1^2 - 3(1) + 1 - 1}{1} = -2$$

$$\therefore\ f'(\xi) = 2\xi - 3 = -2$$

$$\Rightarrow\ 2\xi = 1$$

$$\Rightarrow\ \xi = \frac{1}{2} \in (0, 1).$$

which verifies the theorem.

Remarks

If f(b) = f(a), the theorem reduces to Rolle's theorem. Thus Rolles theorem is a special case of the first mean-value theorem.

7.3 THE SECOND MEAN-VALUE THEOREM

(OR CAUCHY'S GENERALIZED MEAN-VALUE THEOREM)

This theorem states that if f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), then there exists at least one value of $x = \xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)},$$ $g(a) \neq g(b)$ and where $f'(x)$ and $g'(x)$ are not simultaneously zero.

Proof:

Construct the function

$H(x) = f(x) - f(a) - \alpha\{g(x) - g(a)\}$, where α is a constant.

H(x) is continuous in [a, b], since f(x) and g(x) are also continuous in [a, b].

H(x) is differentiable in (a, b) since f(x) and g(x) are also differentiable in (a, b).

Now,

$$H(a) = f(a) - f(a) - \alpha\{g(a) - g(a)\} = 0$$

$$H(b) = f(b) - f(a) - \alpha(g(b) - g(a)\}$$

$$= 0, \text{ provided that } \alpha = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

If we suppose that α takes on this value, then there exists at least one value of $\xi \in (a, b)$ s.t. $H'(\xi) = 0$.

$$H'(x) = f'(x) - \alpha g'(x)$$

$$H'(\xi) = o \Rightarrow f'(\xi) - \alpha g'(\xi) = 0$$

$$\Rightarrow \alpha = f'(\xi) / g'(\xi)$$

$$\therefore \qquad \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)} .$$

This completes the proof.

Remarks:

Note that the 2^{nd} Mean-Value theorem reduces to the 1^{st} Mean-Value theorem if g(x) =x.

7.4 TAYLOR'S THEOREM OF THE MEAN

If the n^{th} derivative $f^{(n)}(x)$ of f(x) is continuous in [a, b] and differentiable in (a, b) then there exists at least one value of $x = \xi \in (a, b)$ such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots$$

$$+ \frac{f^{(n)}(a)}{n!}(b-a)^n + Rn ,$$

where Rn, called the remainder, may be written in one of the forms

$$Rn = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1} \qquad \dots\dots\dots \qquad \text{Lagrange's form}$$

$$Rn = \frac{f^{(n+1)}(\xi)(b-\xi)^n(b-a)}{n!} \qquad \dots\dots\dots\dots \qquad \text{Cauchy's form.}$$

Remarks:

1) Lagrange's form of the remainder will be generally used.
2) Consider the sub-interval [x_0, x] in (a, b), then Taylor's theorem may be written as

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

$$+ \frac{f^{n}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} .$$

This last form of Taylor's theorem is also known as Taylor's series for f(x) with a remainder.

3) Taylor's theorem is used to approximate f(x) by a polynomial, and Rn is called the error term.

4) If $\lim_{n\to\infty}$ Rn = 0, the infinite series

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots$$

is called a Taylor's series for f(x) about $x = x_0$. In the case $x_0 = 0$, the Taylor series obtained is called a Maclaurin Series.

CHAPTER EIGHT

LEIBNITZ RULE FOR FINDING THE nth DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS

8.1 Introduction

Transcendental functions can be differentiated any number of times, without the result becoming zero. This is unlike polynomials in the form $y = x^n$, which yield the result zero when differentiated (n + 1) times.

The following examples will clarify what has been said.

Example 1:

Let y = sin x.

Then $\frac{dy}{dx}$ = Cos x = Sin (x + $\frac{\pi}{2}$)

$\frac{d^2y}{dx^2}$ = – Sin x = Sin (x + π)

$\frac{d^3y}{dx^3}$ = – Cos x = Sin (x + $\frac{3\pi}{2}$).

- -

$\frac{d^ny}{dx^n}$ = Sin (x + $\frac{n\pi}{2}$)

i.e. $$\boxed{\frac{d^n}{dx^n}(\text{Sin } x) = \text{Sin}\left(x + \frac{n\pi}{2}\right)}$$

This provides a formula for finding the n^{th} derivative of sin x.

E.g. $\frac{d^5}{dx^5}$ (Sin x) = Sin (x + $\frac{5\pi}{2}$)

i.e. $$\frac{d^5}{dx^5} (\text{Sin } x) = \text{Sin } (x + \frac{5\pi}{2})$$

$$= \text{Sin } (x + 2\pi + \frac{\pi}{2})$$

$$= \text{Sin } (x + \frac{\pi}{2})$$

$$= \text{Cos } x$$

Example 2:

Let y = ln x

Then $$\frac{dy}{dx} = \frac{1}{x} ,$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

$$\frac{d^3y}{dx^3} = \frac{2}{x^3}$$

$$\frac{d^4y}{dx^4} = -\frac{(2)(3)}{x^4} = -\frac{6}{x^4}$$

$$\frac{d^5}{dx^5} = \frac{(-6)(-4)}{x^5} = \frac{24}{x^5}$$

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$$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Thus $$\boxed{\frac{d^n}{dx^n} (\ln x) = \frac{(-1)^{n-1}(n-1)!}{x^n}}$$

This provides a formula for finding the nth derivative of y = ln x.

Example 3:

Let y = e^{ax} , where a is real and non-zero.

Then $\frac{dy}{dx} = ae^{ax}$

$$\frac{d^2y}{dx^2} = a^2 e^{ax}$$

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$$\frac{d^ny}{dx^n} = a^n e^{ax}$$

Thus $\boxed{\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}}$,

which also provides a formula for finding the nth derivative of y = e^{ax}.

8.2 The Product of Two functions

Let y = uv , where u = u(x) and v = v(x).

Denote by $D \equiv \frac{d}{dx}$. This means that

$\frac{d}{dx}\left(\frac{d}{dx}\right) = D(D) = D^2 = \frac{d^2}{dx^2}$ and $D^3 = \frac{d^3}{dx^3}$, and so on.

Now $\underline{\frac{d}{dx}}$ **(uv) ≡ D(uv) = v** $\underline{\frac{du}{dx}}$ **+ u** $\underline{\frac{dv}{dx}}$

= v Du + u Dv .

Further ,

$$\begin{aligned} D(D\,uv) &= D^2(uv) \\ &= D(v\,Du + u\,Dv) \\ &= vD^2u + DvDu + DuDv + u\,D^2v \\ &= vD^2u + 2\,DvDu + uD^2v \end{aligned}$$

Also

$$\begin{aligned} D^3(uv) &= vD^3u + DvD^2u + 2D^2v\,Du + 2\,DvD^2u + DuD^2v + uD^3v \\ &= vD^3u + 3DvD2u + 3DuD2v + uD3v. \end{aligned}$$

It is easily shown that,

$$D^4(uv) = vD^4u + 4DvD^3u + 6D^2vD^2u + 4D^3vDu + uD^4v$$

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$$D^n(uv) = vD^nu + \binom{n}{1} DvD^{n-1} u + \ldots + \binom{n}{r} D^r vD^{n-r} u + \ldots + uD^nv$$

8.3 LEIBNITZ THEOREM

For a function $y = u\,v$, where $u = u(x)$ and $v = v(x)$ each of which can be differentiated n times with respect to x, the nth derivative of y is given by

$$D^ny \equiv D^n(uv) = \sum_{r=0}^{n} \binom{n}{r} D^ru\, D^{n-r}\, v,$$

where $\binom{n}{r} = \dfrac{n\,!}{(n - r)\,!\; r\,!}$, and $D^0 = 1$.

Proof:

Let $n = 1$. Then

$$Dy = D(uv) = u\,Dv + v\,Du$$

$$= \binom{1}{0} D^0u\,Dv + \binom{1}{1} D^1\,u\,D^{1-1}\,v$$

Hence the result holds for $n = 1$.

Suppose that the result holds for $n = k$ (some positive integer). Then

$$D^ky = D^k(uv) = \sum_{r=0}^{k} \binom{k}{r} D^ru\,D^{k-r}\,v\,.$$

Consider $n = k + 1$; then

$$D^{k+1}y = D^{k+1}(uv) = D\sum_{r=0}^{k} \binom{k}{r} D^ru\,D^{k-r}\,v\,.$$

$$= \sum_{r=0}^{k} D\binom{k}{r} D^ru\,D^{k-r}\,v$$

$$= \sum_{r=0}^{k} \binom{k}{r} [\,D^{r+1}\,u\,D^{k-r}\,v + D^ru\,D^{k+1-r}\,v\,]$$

i.e.

$$D^{k+1}(uv) = \sum_{r=0}^{k} \binom{k}{r} D^{r+1} u\, D^{k-r} v + \sum_{r=0}^{k} \binom{k}{r} D^{r} u\, D^{k+1-r} v$$

Setting $r + 1 = r'$ in the first summation, we have

$$r = r' - 1 .$$

$$D^{k+1}(uv) = \sum_{r'=1}^{k+1} \binom{k}{r'-1} D^{r'} u\, D^{k+1-r'} + \sum_{r=0}^{k} \binom{k}{r} D^{r} u\, D^{k+1-r} v$$

$$= \binom{k}{k} D^{k+1} u\, D^{o} v + \sum_{r=1}^{k} \binom{k}{r-1} D^{r} u\, D^{k+1-r}$$

$$+ \sum_{r=1}^{k} \binom{k}{r} D^{r} u\, D^{k+1-r} v + \binom{k}{o} u\, D^{k+1} v$$

$$= v\, D^{k+1} u + \sum_{r=1}^{k} \left(\binom{k}{r-1} + \binom{k}{r} \right) D^{r} u\, D^{k+1-r} v + u\, D^{k+1} v$$

Now,

$$\binom{k}{r-1} + \binom{k}{r} = \frac{k\,!}{(k-r+1)!\,(r-1)!} + \frac{k\,!}{(k-r)!\; r\,!}$$

$$= \frac{k\,!\,r + k!(k+1-r)}{(k+1-r)!\,r!}$$

$$= \frac{k\,!\,(r + k + 1 - r)}{(k+1-r)!\,r!}$$

$$= \frac{k\,!\,(k+1)}{(k+1-r)!\,r!} = \frac{(k+1)!}{(k+1-r)!\,r\,!} = \binom{k+1}{r}$$

$$\therefore \quad D^{k+1} uv = v\, D^{k+1} u + \sum_{r=1}^{k} \binom{k+1}{r} D^{r} u\, D^{k+1-r} v + u\, D^{k+1} v$$

$$= \binom{k+1}{k+1} D^{o} v\, D^{k+1} u + \sum_{r=1}^{k} \binom{k+1}{r} D^{r} u\, D^{k+1-r} v$$

$$+ \binom{k+1}{o} D^{o} u\, D^{k+1} v$$

$$= \sum_{r=0}^{k+1} \binom{k+1}{r} D^{r} u\, D^{k+1-r} v .$$

Thus the result holds for n = k + 1. Since it holds for n = 1, it holds for n = 2, 3, … and for all positive integers by induction.

Worked Examples

(i) Find the n^{th} derivative of $y = x^2 \operatorname{Sin} x$. Hence obtain $\frac{d^{10}}{dx^{10}}(x^2 \operatorname{Sin} x)$.

Solution:

From Leibnitz rule

Let $u = x^2$ and $v = \operatorname{Sin} x$.

Then

$$\frac{d^n}{dx^n}(x^2 \operatorname{Sin} x) = x^2 D^n \operatorname{Sin} x + \binom{n}{1} . 2x D^{n-1} \operatorname{Sin} x + \binom{n}{2} . 2 D^{n-2} \operatorname{Sin} x + 0$$

$$= x^2 \operatorname{Sin}\left(x + \frac{n\pi}{2}\right) + n.2x \operatorname{Sin}\left(x + \frac{(n-1)\pi}{2}\right)$$

$$+ \frac{n(n-1)}{2!} . 2 \operatorname{Sin}\left(x + \frac{(n-2)\pi}{2}\right)$$

$$= x^2 \operatorname{Sin}\left(x + \frac{n\pi}{2}\right) + 2nx(-1) \operatorname{Cos}\left(x + \frac{n\pi}{2}\right)$$

$$+ n(n-1)(-1) \operatorname{Sin}\left(x + \frac{n\pi}{2}\right).$$

$$\frac{d^n}{dx^n}(x^2 \operatorname{Sin} x) = (x^2 - n^2 + n) \operatorname{Sin}\left(x + \frac{n\pi}{2}\right) - 2nx \operatorname{Cos}\left(x + \frac{n\pi}{2}\right).$$

Hence, for n = 10, we have

$$\frac{d^{10}}{dx^{10}}(x^2 \operatorname{Sin} x) = (x^2 - 100 + 10) \operatorname{Sin}(x + 5\pi) - 20x \operatorname{Cos}(x + 5\pi).$$

$$= (x^2 - 90) \operatorname{Sin}(x + \pi) - 20x \operatorname{Cos}(x + \pi)$$

$$= \underline{(90 - x^2) \operatorname{Sin} x + 20x \operatorname{Cos} x .}$$

Exercise:

Find the nth derivative of $y = x^3 \ln x$, and hence obtain $\frac{d^5}{dx^5}(x^3 \ln x)$.

8.4 INDETERMINATE FORMS AND L'HOSPITAL'S RULES

In finding he limit of the quotient of two functions f(x) and g(x) at some point $x = x_0$, it is required that $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$ should exist. If $\lim\limits_{x \to x_0}$ g(x) = 0, and $\lim\limits_{x \to x_0}$ f(x) ≠ 0, then $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$ does not exist. If, however, $\lim\limits_{x \to x_0}$ f(x) = 0 and $\lim\limits_{x \to x_0}$ g(x) = 0, then $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$, and an indeterminate situation arises.

The form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ is called an indeterminate form. Other indeterminate forms include $0.\infty$, ∞^0, 1^∞.

THEOREM (L'HOSPITAL'S RULE)

Let f(x) and g(x) be continuous for all $x \in [a, b]$ except possibly at $x = x_0$ and differentiable in (a, b) except possibly at $x = x_0$. If $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$, provided the right hand side limit exists.

Remarks: We shall not prove the above theorem but will consider some worked examples to demonstrate the use of the theorem. However, a proof of the theorem may be found in 'Calculus' by D.D. Berkey, 2nd Edition, Saunden College Publishing, New York p. 488 – or in Advanced Calculus, Schaum Outline Series by M. Spiegel.

Examples:

1. Using L'Hôspital's rules, determine the following limits:

(a) $\lim\limits_{x \to 1} \frac{(x-1)^2}{x^2 - 1}$ (b) $\lim\limits_{x \to 0} \frac{Sin x}{1 - Cos x}$

Solutions:

(a) $\lim_{x \to 1} \frac{(x-1)^2}{x^2-1} = \frac{0}{0}$

$\therefore \lim_{x \to 1} \frac{(x-1)^2}{x^2-1} = \lim_{x \to 1} \frac{2(x-1)}{2x} = \frac{0}{2} = 0.$

(b) $\lim_{x \to 0} \frac{Sinx}{1-Cos x} = \frac{0}{0}$

$\therefore \lim_{x \to 0} \frac{Sinx}{1-Cos x} = \lim_{x \to 0} \frac{Cos x}{Sinx} = \frac{1}{0} = \infty$

$\therefore \lim_{x \to 0} \frac{Sinx}{1-Cos x}$ does not exist.

2. Find the following limits:

(a) $\lim_{x \to 0} \frac{xe^{3x}}{Sinx}$ (b) $\lim_{x \to 0} \frac{Sinx - xCos x}{\tan x}$

Solution:

(a) $\lim_{x \to 0} \frac{xe^{3x}}{Sinx} = \frac{0}{0}$

$\therefore \lim_{x \to 0} \frac{xe^{3x}}{Sinx} = \lim_{x \to 0} \frac{3xe^{3x} + e^{3x}}{Cos x} = \frac{0 + e^0}{Coso} = \frac{1}{1} = 1$

(b) $\lim_{x \to 0} \frac{Sinx - xCos x}{\tan x} = \frac{0}{0}$

$\therefore \lim_{x \to 0} \frac{Sinx - xCos x}{\tan x} = \lim_{x \to 0} \frac{Cos x - Cos x + x Sinx}{Sec^2 x}$

$= \frac{0}{1} = 0.$

3. Determine the following limits

(a) $\lim_{x \to \infty} \frac{x^3 + x^2 + x + 1}{(x+3)^3}$ (b) $\lim_{x \to \infty} xe^{-2x}$

Solution:

(a) $\lim\limits_{x\to\infty} \dfrac{x^3 + x^2 + x + 1}{2(x+3)^3} = \dfrac{\infty}{\infty}$

$$\therefore \lim_{x\to\infty} \frac{x^3 + x^2 + x + 1}{2(x+3)^3} = \lim_{x\to\infty} \frac{3x^2 + 2x + 1}{6(x+3)^2} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{6x + 2}{12(x+3)} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{6}{12} = \frac{1}{2} .$$

Note: When the indeterminate $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ re-occurs after application of L'Hospital's rules, there should be a repeated application of the rule.

(b) $\lim\limits_{x\to\infty} \mathrm{xe}^{-2x} = \infty \, . \, 0$ an indeterminate form.

However $\lim\limits_{x\to\infty} \mathrm{xe}^{-2x} = \lim\limits_{x\to\infty} \dfrac{x}{e^{2x}} = \dfrac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{1}{2e^{2x}} = 0 .$$