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Part II Numerical Analysis (J6) Lent Term, 2000

Exercise Sheet 1 [1]

1. Prove that the Gauss–Seidel method for the solution of $A\mathbf{x}=\mathbf{b}$ converges whenever the matrix A is symmetric and positive definite. Show, however, by a 3×3 counterexample, that the Jacobi method need not converge.

2. Verify that the $n\times n$ real tridiagonal matrix

$$A=\begin{pmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & \alpha & \beta & \ddots & \vdots \\ 0 & \beta & \alpha & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta \\ 0 & \cdots & 0 & \beta & \alpha \end{pmatrix},$$

has the eigenvalues $\alpha+2\beta\cos(\frac{j\pi}{n+1})$, $j=1,2,\ldots,n$. Hence deduce $\rho(A)=|\alpha|+2|\beta|\cos(\frac{\pi}{n+1})$.
Hint: Show that $\mathbf{v}\in\mathcal{R}^n$ satisfies the eigenvalue equation $A\mathbf{v}=\lambda\mathbf{v}$ for some $\lambda\in\mathcal{R}$ if it has the components $v_k=\sin(k\ell\pi/[n+1])$, $k=1,2,\ldots,n$, where ℓ is any integer.

3. Let the Gauss–Seidel method be applied to the equations $A\mathbf{x}=\mathbf{b}$ when A is the unsymmetric 2×2 matrix $\begin{pmatrix} 10 & -3 \\ 3 & 1 \end{pmatrix}$. Find the spectral radius of the iteration matrix. Then show that the relaxation method, described in Lecture 2, can reduce the spectral radius by a factor of 2.9. Further, show that two iterations of Gauss–Seidel with this relaxation decreases the error $\|\mathbf{x}^{(k)}-\mathbf{x}^{(\infty)}\|$ by more than a factor of ten. Estimate the number of iterations of the original Gauss–Seidel method that would be required to achieve this decrease in the error.

4. Apply the standard form of the conjugate gradient method to the linear system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{x}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

starting as usual with $\mathbf{x}^{(1)}=0$. Verify that the residuals $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ and $\mathbf{r}^{(3)}$ are mutually orthogonal, that the search directions $\mathbf{d}^{(1)}$, $\mathbf{d}^{(2)}$ and $\mathbf{d}^{(3)}$ are mutually conjugate, and that $\mathbf{x}^{(4)}$ satisfies the equations.

5. Let the standard form of the conjugate gradient method be applied when A is positive definite. Express $\mathbf{d}^{(k)}$ in terms of $\mathbf{r}^{(j)}$, $j=1,2,\ldots,k$, and $\beta^{(j)}$, $j=1,2,\ldots,k$. Then deduce in a few lines from the formula $\mathbf{x}^{(k+1)}=\sum_{j=1}^{k}\omega^{(j)}\mathbf{d}^{(j)}$, from $\omega^{(j)}>0$, and from the theorem on the conjugate gradient method in the lecture notes, that the sequence $\{\|\mathbf{x}^{(j)}\| : j=1,2,\ldots,k+1\}$ increases monotonically.

6. The polynomial $p(x)=x^m+\sum_{\ell=0}^{m-1}c_\ell x^\ell$ is the *minimal polynomial* of the $n\times n$ matrix A if it is the polynomial of lowest degree that satisfies $p(A)=0$. Note that $m\le n$ holds because of the Cayley–Hamilton theorem.

Give an example of a 3×3 symmetric matrix with a quadratic minimal polynomial.

Prove that (in exact arithmetic) the conjugate gradient method requires at most m iterations to calculate the exact solution of $A\mathbf{x}=\mathbf{b}$, where m is the degree of the minimal polynomial of A.

[1] Please send any corrections and comments by e-mail to mjdp@cam.ac.uk

7. Let A be the 3×3 matrix

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is real and nonzero. Find an explicit expression for A^k, $k=1,2,3,\ldots$.

The sequence $\mathbf{x}^{(k+1)}$, $k=1,2,3,\ldots$, is generated by the power method $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}/\|A\mathbf{x}^{(k)}\|$, where $\mathbf{x}^{(1)}$ is any nonzero vector in $\mathcal{R}^3$. Deduce from your expression for A^k that the second and third components of $\mathbf{x}^{(k+1)}$ tend to zero as $k\to\infty$. Further, show that this remark implies $A\mathbf{x}^{(k+1)} - \lambda\mathbf{x}^{(k+1)} \to 0$, so the power method tends to provide a solution to the eigenvalue equation.

8. Let A be an $n\times n$ matrix with the real and distinct eigenvalues $\lambda_1 > \lambda_2 = 1 > \lambda_3 > \cdots > \lambda_n = 0$. Three versions of the power method are used to estimate the eigenvector of A whose eigenvalue is λ_1. Specifically, the k-th iteration has the form $\mathbf{x}^{(k+1)} = (A - s^{(k)} I)\mathbf{x}^{(k)} / \|A\mathbf{x}^{(k)}\|$, $k = 1,2,3,\ldots$, where each $s^{(k)}$ is a real shift. In the first version every $s^{(k)}$ is zero, in the second version every $s^{(k)}$ is one half, and in the third version shifts of $\frac{1}{4}(2-\sqrt{2})$ and $\frac{1}{4}(2+\sqrt{2})$ are used alternately. Compare these shift strategies when $\lambda_1 = 1+\varepsilon$ where ε is a tiny positive number, assuming $\mathbf{x}^{(1)}$ contains substantial components of all the eigenvectors. You should find that the use of the single shift (version 2) instead of no shift (version 1) approximately halves the number of iterations that are needed to achieve a prescribed accuracy, while the use of a double shift (version 3) instead of a single shift (version 2) approximately halves the number of iterations again.

9. Let A be a symmetric 2×2 matrix with distinct eigenvalues and normalized eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Given $\mathbf{x}^{(1)} \in \mathcal{R}^2$, the sequence $\mathbf{x}^{(k+1)}$, $k = 1,2,3,\ldots$, is generated in the following way. The Rayleigh quotient $\lambda_k = \mathbf{x}^{(k)T} A\mathbf{x}^{(k)} / \|\mathbf{x}^{(k)}\|^2$ is taken as an estimate of an eigenvalue of A, the vector norm being Euclidean. Then inverse iteration gives

$$\mathbf{y} = (A - \lambda_k I)^{-1}\mathbf{x}^{(k)}, \quad \text{and we set} \quad \mathbf{x}^{(k+1)} = \mathbf{y} \,/\, \|\mathbf{y}\|.$$

Show that, if $\mathbf{x}^{(k)} = (\mathbf{v}_1 + \varepsilon_k\mathbf{v}_2)/(1+\varepsilon_k^2)^{1/2}$, where $|\varepsilon_k|$ is small, then $|\varepsilon_{k+1}|$ is of magnitude $|\varepsilon_k|^3$. In other words, the method enjoys a "third order" rate of convergence.

10. The symmetric matrix

$$A = \begin{pmatrix} 9 & -8 & 2 \\ -8 & 9 & -2 \\ 2 & -2 & 10 \end{pmatrix} \quad \text{has the eigenvector} \quad \mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Calculate an orthogonal matrix Ω by a Householder transformation such that $\Omega\mathbf{v}$ is a multiple of the first coordinate vector. Then form the product $\Omega^T A\,\Omega$. You should find that this matrix is suitable for deflation. Hence identify all the eigenvalues and eigenvectors of A.

11. Show that the vectors $\mathbf{x}$, $A\mathbf{x}$ and $A^2\mathbf{x}$ are linearly dependent in the case

$$A = \begin{pmatrix} 4 & 5 & 2 & 0 \\ -26 & -14 & 1 & 4 \\ -2 & 2 & 3 & 1 \\ -43 & -8 & 13 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}.$$

Hence calculate two of the eigenvalues of A. Obtain by deflation a 2×2 matrix whose eigenvalues are the remaining eigenvalues of A. Then find the other two eigenvalues of A.

12. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix

$$A = \begin{pmatrix} 9 & -1 & 2 & 2 \\ -1 & 3 & 4 & 2 \\ 2 & 4 & 14 & -3 \\ 2 & 2 & -3 & 4 \end{pmatrix}.$$

Your final matrix should be symmetric and should have the same trace as A.