LECTURE NOTES SEVEN

Lagrangian Interpolation

After reading this lecture, you should be able to:

1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation, and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this lecture.

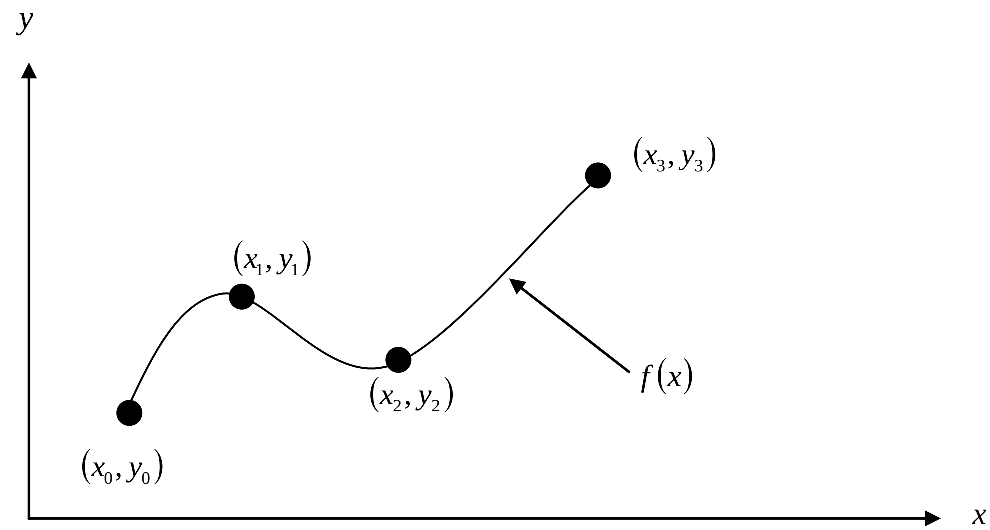

Figure 1 Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

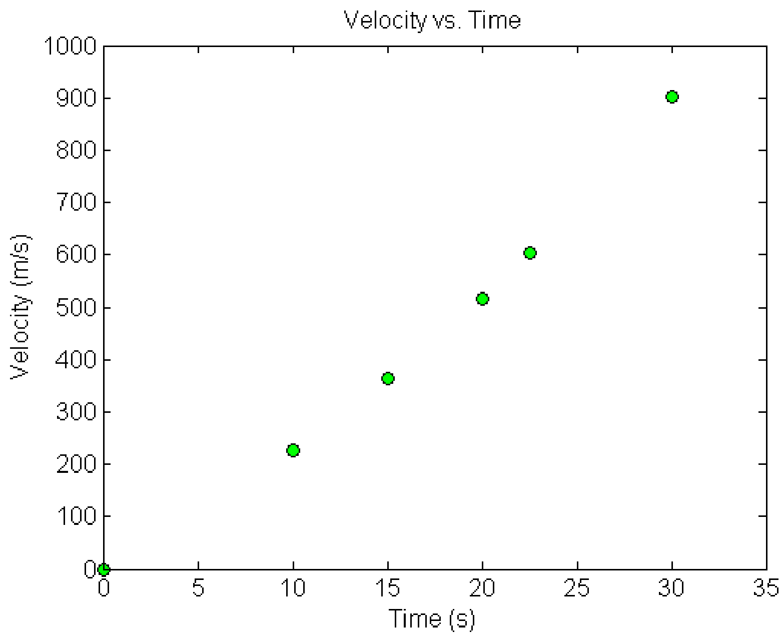

Figure 2 Graph of velocity vs. time data for the rocket example.

Determine the value of the velocity at $t = 16$ seconds using a first order Lagrange polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), the velocity is given by

$$\begin{aligned} v(t) &= \sum_{i=0}^{1} L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) \end{aligned}$$

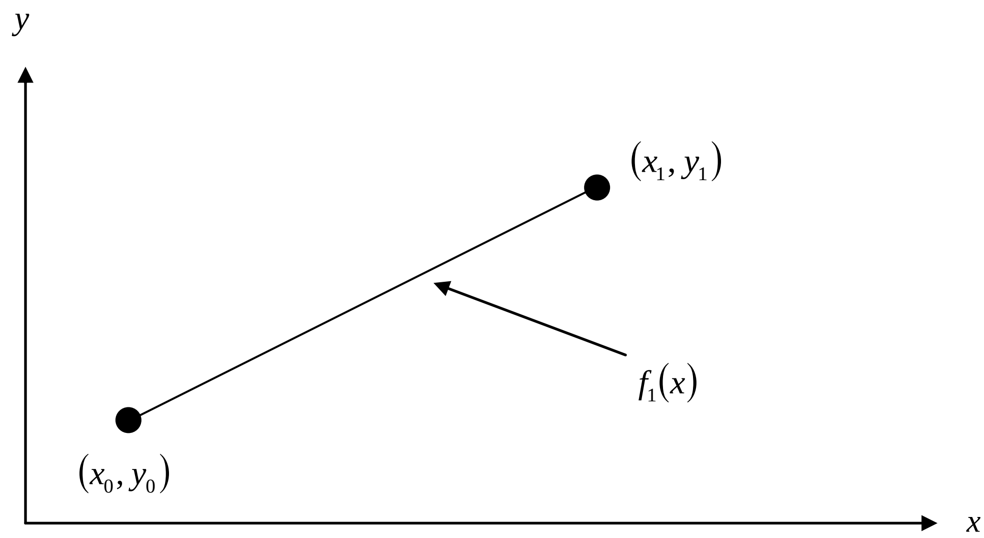

Figure 3 Linear interpolation.

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$
$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0\\ j\neq 0}}^{1} \frac{t - t_j}{t_0 - t_j}$$
$$= \frac{t - t_1}{t_0 - t_1}$$
$$L_1(t) = \prod_{\substack{j=0\\ j\neq 1}}^{1} \frac{t - t_j}{t_1 - t_j}$$
$$= \frac{t - t_0}{t_1 - t_0}$$

Hence

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1)$$
$$= \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35), \quad 15 \le t \le 20$$
$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$
$$= 0.8(362.78) + 0.2(517.35)$$

$= 393.69 \text{ m/s}$

You can see that $L_0(t) = 0.8$ and $L_1(t) = 0.2$ are like weightages given to the velocities at $t = 15$ and $t = 20$ to calculate the velocity at $t = 16$.

Quadratic Interpolation

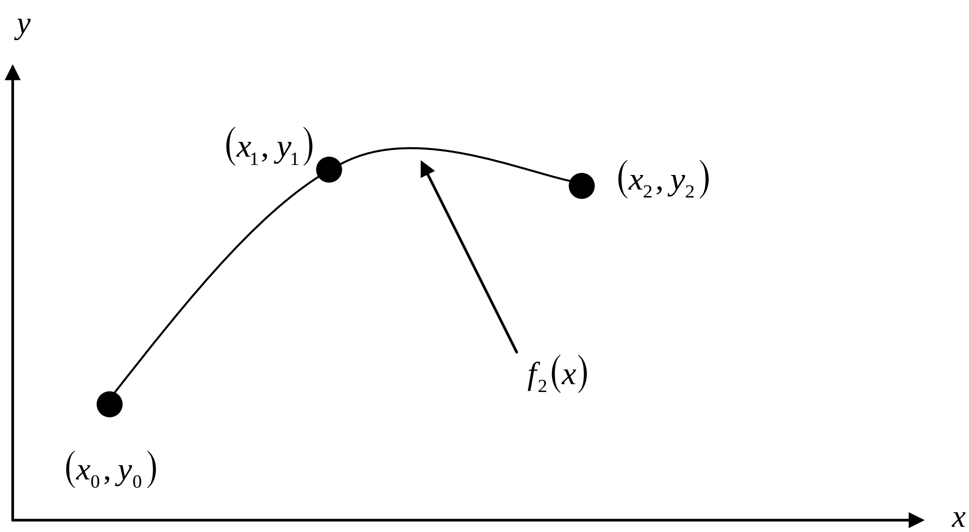

Figure 4 Quadratic interpolation.

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with second order polynomial interpolation using Lagrangian polynomial interpolation.
b) Find the absolute relative approximate error for the second order polynomial approximation.

Solution

a) For second order polynomial interpolation (also called quadratic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$
$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$

Since we want to find the velocity at $t = 16$, and we are using a second order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10,\ t_1 = 15,$ and $t_2 = 20$.

Then

$$t_0 = 10,\ v(t_0) = 227.04$$
$$t_1 = 15,\ v(t_1) = 362.78$$
$$t_2 = 20,\ v(t_2) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j}$$
$$= \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$
$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j}$$
$$= \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$
$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j}$$
$$= \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

Hence

$$v(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)v(t_0) + \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)v(t_1) + \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)v(t_2),\ t_0 \le t \le t_2$$

$$v(16) = \frac{(16-15)(16-20)}{(10-15)(10-20)}(227.04) + \frac{(16-10)(16-20)}{(15-10)(15-20)}(362.78)$$
$$+ \frac{(16-10)(16-15)}{(20-10)(20-15)}(517.35)$$
$$= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(517.35)$$
$$= 392.19 \text{ m/s}$$

b) The absolute relative approximate error $|\in_a|$ for the second order polynomial is calculated by considering the result of the first order polynomial (Example 1) as the previous approximation.

$$\left|\in_a\right| = \left|\frac{392.19-393.69}{392.19}\right| \times 100$$
$$= 0.38410\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t=16$ seconds using third order Lagrangian polynomial interpolation.
b) Find the absolute relative approximate error for the third order polynomial approximation.
c) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t=11\text{s}$ to $t=16\text{s}$.
d) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t=16\text{s}$.

Solution

a) For third order polynomial interpolation (also called cubic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{3} L_i(t)v(t_i)$$
$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$

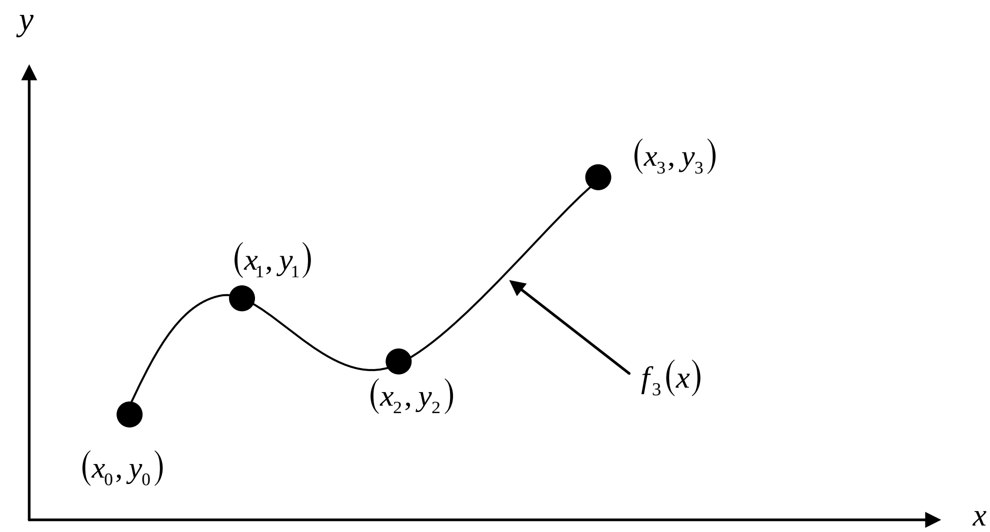

Figure 5 Cubic interpolation.

Since we want to find the velocity at $t=16$, and we are using a third order polynomial, we need to choose the four data points closest to $t=16$ that also bracket $t=16$ to evaluate it. The four points are $t_0=10,\ t_1=15,\ t_2=20$ and $t_3=22.5$.

Then

$$t_0=10,\ v(t_0)=227.04$$
$$t_1=15,\ v(t_1)=362.78$$
$$t_2=20,\ v(t_2)=517.35$$
$$t_3=22.5,\ v(t_3)=602.97$$

gives

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{3}\frac{t-t_j}{t_0-t_j}$$
$$=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$
$$L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{3}\frac{t-t_j}{t_1-t_j}$$
$$=\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)$$
$$L_2(t)=\prod_{\substack{j=0\\ j\neq 2}}^{3}\frac{t-t_j}{t_2-t_j}$$
$$=\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)$$

$$L_3(t) = \prod_{\substack{j=0 \\ j\neq 3}}^{3} \frac{t-t_j}{t_3-t_j}$$
$$= \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)$$

Hence

$$v(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_1)$$
$$+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2) + \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3),\ t_0 \le t \le t_3$$

$$v(16) = \frac{(16-15)(16-20)(16-22.5)}{(10-15)(10-20)(10-22.5)}(227.04) + \frac{(16-10)(16-20)(16-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+\frac{(16-10)(16-15)(16-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+\frac{(16-10)(16-15)(16-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97)$$
$$= (-0.0416)(227.04) + (0.832)(362.78) + (0.312)(517.35) + (-0.1024)(602.97)$$
$$= 392.06 \text{ m/s}$$

b) The absolute percentage relative approximate error, $|\in_a|$ for the value obtained for $v(16)$ can be obtained by comparing the result with that obtained using the second order polynomial (Example 2)

$$|\in_a| = \left|\frac{392.06-392.19}{392.06}\right| \times 100$$
$$= 0.033269\%$$

c) The distance covered by the rocket between $t = 11\text{s}$ to $t = 16\text{s}$ can be calculated from the interpolating polynomial as

$$v(t) = \frac{(t-15)(t-20)(t-22.5)}{(10-15)(10-20)(10-22.5)}(227.04) + \frac{(t-10)(t-20)(t-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+\frac{(t-10)(t-15)(t-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+\frac{(t-10)(t-15)(t-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97), 10 \le t \le 22.5$$
$$= \frac{(t^2-35t+300)(t-22.5)}{(-5)(-10)(-12.5)}(227.04) + \frac{(t^2-30t+200)(t-22.5)}{(5)(-5)(-7.5)}(362.78)$$
$$+\frac{(t^2-25t+150)(t-22.5)}{(10)(5)(-2.5)}(517.35) + \frac{(t^2-25t+150)(t-20)}{(12.5)(7.5)(2.5)}(602.97)$$

$$= (t^3 - 57.5t^2 + 1087.5t - 6750)(-0.36326) + (t^3 - 52.5t^2 + 875t - 4500)(1.9348)$$
$$+ (t^3 - 47.5t^2 + 712.5t - 3375)(-4.1388) + (t^3 - 45t^2 + 650t - 3000)(2.5727)$$
$$= -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3,\ 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$= \int_{11}^{16} (-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3)dt$$
$$= \left[-4.245t + 21.265\frac{t^2}{2} + 0.13195\frac{t^3}{3} + 0.00544\frac{t^4}{4} \right]_{11}^{16}$$
$$= 1605 \text{ m}$$

d) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

Given that

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3,\ 10 \le t \le 22.5$$
$$a(t) = \frac{d}{dt} v(t)$$
$$= \frac{d}{dt}\left(-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3\right)$$
$$= 21.265 + 0.26390t + 0.01632t^2,\ 10 \le t \le 22.5$$
$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$
$$= 29.665 \text{ m/s}^2$$

Note: There is no need to get the simplified third order polynomial expression to conduct the differentiation. An expression of the form

$$L_0(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)\left(\frac{t - t_3}{t_0 - t_3}\right)$$

gives the derivative without expansion as

$$\frac{d}{dt}\left(L_0(t)\right) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right) + \left(\frac{t - t_2}{t_0 - t_2}\right)\left(\frac{t - t_3}{t_0 - t_3}\right) + \left(\frac{t - t_3}{t_0 - t_3}\right)\left(\frac{t - t_1}{t_0 - t_1}\right)$$