# Chapter 2
# Centroids and Moments of Inertia

## 2.1 Centroids and Center of Mass

### *2.1.1 First Moment and Centroid of a Set of Points*

The position vector of a point $P$ relative to a point $O$ is $\mathbf{r}_P$ and a scalar associated with $P$ is $s$, for example, the mass $m$ of a particle situated at $P$. The *first moment* of a point $P$ with respect to a point $O$ is the vector $\mathbf{M} = s\,\mathbf{r}_P$. The scalar $s$ is called the *strength* of $P$. The set of $n$ points $P_i$, $i = 1, 2, \ldots, n$, is $\{S\}$, Fig. 2.1a

$$\{S\} = \{P_1, P_2, \ldots, P_n\} = \{P_i\}_{i=1,2,\ldots,n}.$$

The strengths of the points $P_i$ are $s_i$, $i = 1, 2, \ldots, n$, that is, $n$ scalars, all having the same dimensions, and each associated with one of the points of $\{S\}$.

The *centroid* of the set $\{S\}$ is the point $C$ with respect to which the sum of the first moments of the points of $\{S\}$ is equal to zero. The centroid is the point defining the geometric center of the system or of an object.

The position vector of $C$ relative to an arbitrarily selected reference point $O$ is $\mathbf{r}_C$, Fig. 2.1b. The position vector of $P_i$ relative to $O$ is $\mathbf{r}_i$. The position vector of $P_i$ relative to $C$ is $\mathbf{r}_i - \mathbf{r}_C$. The sum of the first moments of the points $P_i$ with respect to $C$ is $\sum_{i=1}^{n} s_i(\mathbf{r}_i - \mathbf{r}_C)$. If $C$ is to be centroid of $\{S\}$, this sum is equal to zero:

$$\sum_{i=1}^{n} s_i(\mathbf{r}_i - \mathbf{r}_C) = \sum_{i=1}^{n} s_i\mathbf{r}_i - \mathbf{r}_C \sum_{i=1}^{n} s_i = 0.$$

D.B. Marghitu and M. Dupac, *Advanced Dynamics: Analytical and Numerical Calculations with MATLAB*, DOI 10.1007/978-1-4614-3475-7_2,

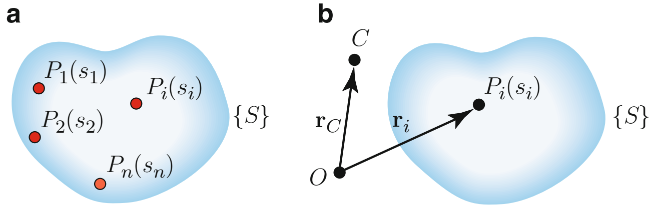



**Fig. 2.1** (**a**) Set of points and (**b**) centroid of a set of points

The position vector $\mathbf{r}_C$ of the centroid $C$, relative to an arbitrarily selected reference point $O$, is given by

$$\mathbf{r}_C = \frac{\sum_{i=1}^{n} s_i \mathbf{r}_i}{\sum_{i=1}^{n} s_i}.$$

If $\sum_{i=1}^{n} s_i = 0$, the centroid is not defined. The centroid $C$ of a set of points of given strength is a unique point, its location being independent of the choice of reference point $O$.

The Cartesian coordinates of the centroid $C(x_C, y_C, z_C)$ of a set of points $P_i$, $i = 1, \ldots, n$, of strengths $s_i$, $i = 1, \ldots, n$, are given by the expressions

$$x_C = \frac{\sum_{i=1}^{n} s_i x_i}{\sum_{i=1}^{n} s_i}, \quad y_C = \frac{\sum_{i=1}^{n} s_i y_i}{\sum_{i=1}^{n} s_i}, \quad z_C = \frac{\sum_{i=1}^{n} s_i z_i}{\sum_{i=1}^{n} s_i}.$$

The *plane of symmetry* of a set is the plane where the centroid of the set lies, the points of the set being arranged in such a way that corresponding to every point on one side of the plane of symmetry there exists a point of equal strength on the other side, the two points being equidistant from the plane.

A set $\{S'\}$ of points is called a *subset* of a set $\{S\}$ if every point of $\{S'\}$ is a point of $\{S\}$. The centroid of a set $\{S\}$ may be located using the *method of decomposition*:

– Divide the system $\{S\}$ into subsets.
– Find the centroid of each subset.
– Assign to each centroid of a subset a strength proportional to the sum of the strengths of the points of the corresponding subset.
– Determine the centroid of this set of centroids.

### 2.1.2 Centroid of a Curve, Surface, or Solid

The position vector of the centroid $C$ of a curve, surface, or solid relative to a point $O$ is

$$\mathbf{r}_C = \frac{\int_\tau \mathbf{r}\,\mathrm{d}\tau}{\int_\tau \mathrm{d}\tau}, \tag{2.1}$$

where $\tau$ is a curve, surface, or solid; $\mathbf{r}$ denotes the position vector of a typical point of $\tau$, relative to $O$; and $\mathrm{d}\tau$ is the length, area, or volume of a differential element of $\tau$. Each of the two limits in this expression is called an "integral over the domain $\tau$ (curve, surface, or solid)." The integral $\int_\tau \mathrm{d}\tau$ gives the total length, area, or volume of $\tau$, that is,

$$\int_\tau \mathrm{d}\tau = \tau.$$

The position vector of the centroid is

$$\mathbf{r}_C = \frac{1}{\tau}\int_\tau \mathbf{r}\,\mathrm{d}\tau.$$

Let $\imath$, $\jmath$, $\mathbf{k}$ be mutually perpendicular unit vectors (Cartesian reference frame) with the origin at $O$. The coordinates of $C$ are $x_C$, $y_C$, $z_C$ and

$$\mathbf{r}_C = x_C\,\imath + y_C\,\jmath + z_C\,\mathbf{k}.$$

It results that

$$x_C = \frac{1}{\tau}\int_\tau x\,\mathrm{d}\tau,\ \ y_C = \frac{1}{\tau}\int_\tau y\,\mathrm{d}\tau,\ \ z_C = \frac{1}{\tau}\int_\tau z\,\mathrm{d}\tau. \tag{2.2}$$

The coordinates for the centroid of a curve $L$, Fig. 2.2, is determined by using three scalar equations

$$x_C = \frac{\int_L x\,\mathrm{d}L}{\int_L \mathrm{d}L},\ \ y_C = \frac{\int_L y\,\mathrm{d}L}{\int_L \mathrm{d}L},\ \ z_C = \frac{\int_L z\,\mathrm{d}L}{\int_L \mathrm{d}L}. \tag{2.3}$$

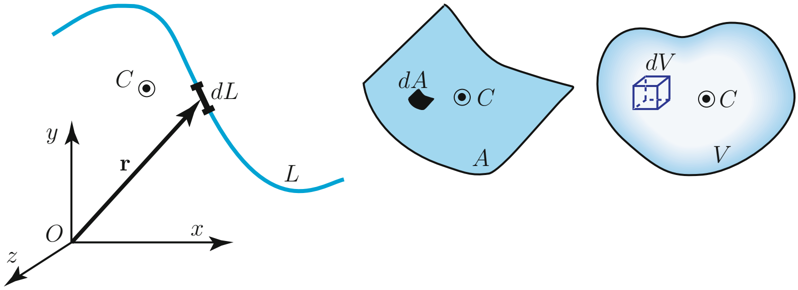



**Fig. 2.2** Centroid of a curve $L$, an area $A$, and a volume $V$

The centroid of an area $A$, Fig. 2.2, is

$$x_C = \frac{\displaystyle\int_A x\,\mathrm{d}A}{\displaystyle\int_A \mathrm{d}A},\quad y_C = \frac{\displaystyle\int_A y\,\mathrm{d}A}{\displaystyle\int_A \mathrm{d}A},\quad z_C = \frac{\displaystyle\int_A z\,\mathrm{d}A}{\displaystyle\int_A \mathrm{d}A}, \tag{2.4}$$

and similarly, the centroid of a volume $V$, Fig. 2.2, is

$$x_C = \frac{\displaystyle\int_V x\,\mathrm{d}V}{\displaystyle\int_V \mathrm{d}V},\quad y_C = \frac{\displaystyle\int_V y\,\mathrm{d}V}{\displaystyle\int_V \mathrm{d}V},\quad z_C = \frac{\displaystyle\int_V z\,\mathrm{d}V}{\displaystyle\int_V \mathrm{d}V}. \tag{2.5}$$

For a curved line in the $xy$ plane, the centroidal position is given by

$$x_C = \frac{\displaystyle\int x\,\mathrm{d}l}{L},\quad y_C = \frac{\displaystyle\int y\,\mathrm{d}l}{L}, \tag{2.6}$$

where $L$ is the length of the line. Note that the centroid $C$ is not generally located along the line. A curve made up of simple curves is considered. For each simple curve, the centroid is known. The line segment, $L_i$, has the centroid $C_i$ with coordinates $x_{C_i}, y_{C_i}$, $i = 1,\ldots,n$. For the entire curve,

$$x_C = \frac{\displaystyle\sum_{i=1}^{n} x_{C_i} L_i}{L},\quad y_C = \frac{\displaystyle\sum_{i=1}^{n} y_{C_i} L_i}{L},\quad \text{where } L = \sum_{i=1}^{n} L_i.$$

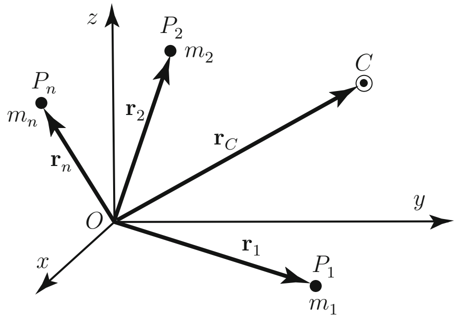



**Fig. 2.3** Mass center position vector

## 2.1.3 Mass Center of a Set of Particles

The *mass center* of a set of particles $\{S\} = \{P_1, P_2, \ldots, P_n\} = \{P_i\}_{i=1,2,\ldots,n}$ is the centroid of the set of points at which the particles are situated with the strength of each point being taken equal to the mass of the corresponding particle, $s_i = m_i$, $i = 1, 2, \ldots, n$. For the system of $n$ particles in Fig. 2.3, one can write

$$\left(\sum_{i=1}^{n} m_i\right) \mathbf{r}_C = \sum_{i=1}^{n} m_i \, \mathbf{r}_i,$$

and the mass center position vector is

$$\mathbf{r}_C = \frac{\sum_{i=1}^{n} m_i \, \mathbf{r}_i}{M}, \tag{2.7}$$

where $M$ is the total mass of the system.

## 2.1.4 Mass Center of a Curve, Surface, or Solid

To study problems concerned with the motion of matter under the influence of forces, that is, dynamics, it is necessary to locate the mass center. The position vector of the mass center $C$ of a continuous body $\tau$, curve, surface, or solid, relative to a point $O$ is

$$\mathbf{r}_C = \frac{\int_\tau \mathbf{r} \rho \, \mathrm{d}\tau}{\int_\tau \rho \, \mathrm{d}\tau} = \frac{1}{m} \int_\tau \mathbf{r} \rho \, \mathrm{d}\tau, \tag{2.8}$$

or using the orthogonal Cartesian coordinates

$$x_C = \frac{1}{m}\int_\tau x\rho \, \mathrm{d}\tau, \; y_C = \frac{1}{m}\int_\tau y\rho \, \mathrm{d}\tau, \; z_C = \frac{1}{m}\int_\tau z\rho \, \mathrm{d}\tau,$$

where $\rho$ is the mass density of the body: mass per unit of length if $\tau$ is a curve, mass per unit area if $\tau$ is a surface, and mass per unit of volume if $\tau$ is a solid; $\mathbf{r}$ is the position vector of a typical point of $\tau$, relative to $O$; $\mathrm{d}\tau$ is the length, area, or volume of a differential element of $\tau$; $m = \int_\tau \rho \, \mathrm{d}\tau$ is the total mass of the body; and $x_C$, $y_C$, $z_C$ are the coordinates of $C$.

If the mass density $\rho$ of a body is the same at all points of the body, $\rho =$ constant, the density, as well as the body, are said to be *uniform*. The mass center of a uniform body coincides with the centroid of the figure occupied by the body.

The density $\rho$ of a body is its mass per unit volume. The mass of a differential element of volume $\mathrm{d}V$ is $\mathrm{d}m = \rho \, \mathrm{d}V$. If $\rho$ is not constant throughout the body and can be expressed as a function of the coordinates of the body, then

$$x_C = \frac{\int_\tau x\rho \mathrm{d}V}{\int_\tau \rho \mathrm{d}V}, \quad y_C = \frac{\int_\tau y\rho \mathrm{d}V}{\int_\tau \rho \mathrm{d}V}, \quad z_C = \frac{\int_\tau z\rho \mathrm{d}V}{\int_\tau \rho \mathrm{d}V}. \tag{2.9}$$

The centroid of a volume defines the point at which the total moment of volume is zero. Similarly, the center of mass of a body is the point at which the total moment of the body's mass about that point is zero.

The *method of decomposition* can be used to locate the mass center of a continuous body:

– Divide the body into a number of simpler body shapes, which may be particles, curves, surfaces, or solids; holes are considered as pieces with negative size, mass, or weight.
– Locate the coordinates $x_{C_i}, y_{C_i}, z_{C_i}$ of the mass center of each part of the body.
– Determine the mass center using the equations

$$x_C = \frac{\sum_{i=1}^{n}\int_\tau x \mathrm{d}\tau}{\sum_{i=1}^{n}\int_\tau \mathrm{d}\tau}, \quad y_C = \frac{\sum_{i=1}^{n}\int_\tau y \mathrm{d}\tau}{\sum_{i=1}^{n}\int_\tau \mathrm{d}\tau}, \quad z_C = \frac{\sum_{i=1}^{n}\int_\tau z \mathrm{d}\tau}{\sum_{i=1}^{n}\int_\tau \mathrm{d}\tau}, \tag{2.10}$$

where $\tau$ is a curve, area, or volume, depending on the centroid that is required. Equation (2.10) can be simplify as

$$x_C = \frac{\sum_{i=1}^{n} x_{C_i}\tau_i}{\sum_{i=1}^{n}\tau_i}, \quad y_C = \frac{\sum_{i=1}^{n} y_{C_i}\tau_i}{\sum_{i=1}^{n}\tau_i}, \quad z_C = \frac{\sum_{i=1}^{n} z_{C_i}\tau_i}{\sum_{i=1}^{n}\tau_i}, \tag{2.11}$$

where $\tau_i$ is the length, area, or volume of the $i$th object, depending on the type of centroid.

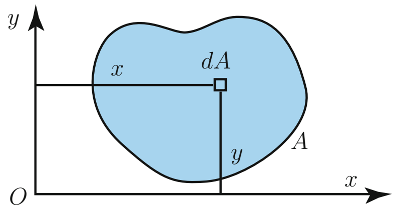



**Fig. 2.4** Planar surface of area $A$

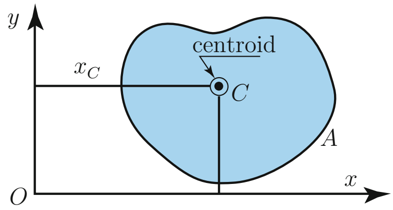



**Fig. 2.5** Centroid and centroidal coordinates for a planar surface

## 2.1.5 *First Moment of an Area*

A planar surface of area $A$ and a reference frame $xOy$ in the plane of the surface are shown in Fig. 2.4. The first moment of area $A$ about the $x$-axis is

$$M_x = \int_A y \, \mathrm{d}A, \tag{2.12}$$

and the first moment about the $y$-axis is

$$M_y = \int_A x \, \mathrm{d}A. \tag{2.13}$$

The first moment of area gives information of the shape, size, and orientation of the area.

The entire area $A$ can be concentrated at a position $C(x_C, y_C)$, the centroid, Fig. 2.5. The coordinates $x_C$ and $y_C$ are the centroidal coordinates. To compute the centroidal coordinates, one can equate the moments of the distributed area with that of the concentrated area about both axes

$$A\, y_C = \int_A y \, \mathrm{d}A, \implies y_C = \frac{\int_A y \, \mathrm{d}A}{A} = \frac{M_x}{A}, \tag{2.14}$$

$$A\, x_C = \int_A x \, \mathrm{d}A, \implies x_C = \frac{\int_A x \, \mathrm{d}A}{A} = \frac{M_y}{A}. \tag{2.15}$$

The location of the centroid of an area is independent of the reference axes employed, that is, the centroid is a property only of the area itself.

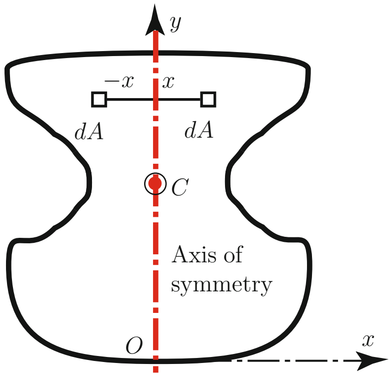



**Fig. 2.6** Plane area with axis of symmetry

If the axes $xy$ have their origin at the centroid, $O \equiv C$, then these axes are called *centroidal axes*. The first moments about centroidal axes are zero. All axes going through the centroid of an area are called centroidal axes for that area, and the first moments of an area about any of its centroidal axes are zero. The perpendicular distance from the centroid to the centroidal axis must be zero.

Finding the centroid of a body is greatly simplified when the body has axis of symmetry. Figure 2.6 shows a plane area with the axis of symmetry collinear with the axis $y$. The area $A$ can be considered as composed of area elements in symmetric pairs such as shown in Fig. 2.6. The first moment of such a pair about the axis of symmetry $y$ is zero. The entire area can be considered as composed of such symmetric pairs and the coordinate $x_C$ is zero

$$x_C = \frac{1}{A}\int_A x \, \mathrm{d}A = 0.$$

Thus, *the centroid of an area with one axis of symmetry must lie along the axis of symmetry*. The axis of symmetry then is a centroidal axis, which is another indication that the first moment of area must be zero about the axis of symmetry. With two orthogonal axes of symmetry, the centroid must lie at the intersection of these axes. For such areas as circles and rectangles, the centroid is easily determined by inspection. If a body has a single plane of symmetry, then the centroid is located somewhere on that plane. If a body has more than one plane of symmetry, then the centroid is located at the intersection of the planes.

In many problems, the area of interest can be considered formed by the addition or subtraction of simple areas. For simple areas, the centroids are known by inspection. The areas made up of such simple areas are *composite* areas. For composite areas

$$x_C = \frac{\sum_i A_i x_{Ci}}{A} \quad \text{and} \quad y_C = \frac{\sum_i A_i y_{Ci}}{A}, \tag{2.16}$$

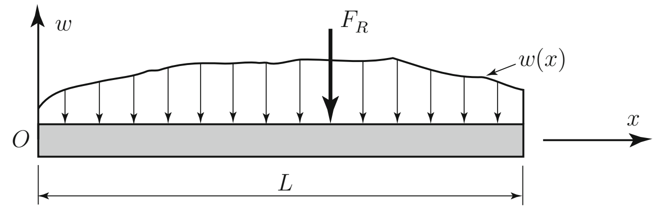



**Fig. 2.7** Distributed load

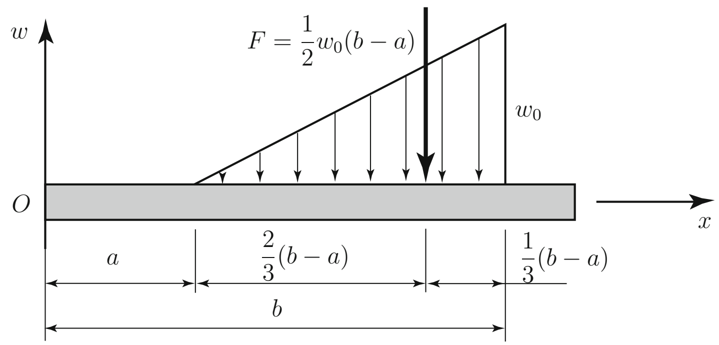



**Fig. 2.8** Triangular distributed load

where $x_{Ci}$ and $y_{Ci}$ are the centroidal coordinates to simple area $A_i$ (with proper signs) and $A$ is the total area.

The centroid concept can be used to determine the simplest resultant of a distributed loading. In Fig. 2.7, the distributed load $w(x)$ is considered. The resultant force $F_R$ of the distributed load $w(x)$ loading is given as

$$F_R = \int_0^L w(x)\,\mathrm{d}x. \tag{2.17}$$

From the equation above, the *resultant force equals the area under the loading curve*. The position, $x_C$, of the *simplest* resultant load can be calculated from the relation

$$F_R x_C = \int_0^L x w(x)\,\mathrm{d}x \implies x_C = \frac{\int_0^L x w(x)\,\mathrm{d}x}{F_R}. \tag{2.18}$$

The position $x_C$ is actually the centroidal coordinate of the loading curve area. Thus, the *simplest resultant force of a distributed load acts at the centroid of the area under the loading curve*. For the triangular distributed load shown in Fig. 2.8, one can replace the distributed loading by a force $F$ equal to $\left(\frac{1}{2}\right)(w_0)(b-a)$ at a position $\left(\frac{1}{3}\right)(b-a)$ from the right end of the distributed loading.

### 2.1.6 Center of Gravity

The *center of gravity* is a point which locates the resultant weight of a system of particles or body. The sum of moments due to individual particle weight about any point is the same as the moment due to the resultant weight located at the center of gravity. The sum of moments due to the individual particles weights about center of gravity is equal to zero. Similarly, the center of mass is a point which locates the resultant mass of a system of particles or body. The center of gravity of a body is the point at which the total moment of the force of gravity is zero. The coordinates for the center of gravity of a body can be determined with

$$x_C = \frac{\int_V x\rho\, g\,\mathrm{d}V}{\int_V \rho\, g\,\mathrm{d}V},\ \ y_C = \frac{\int_V y\rho\, g\,\mathrm{d}V}{\int_V \rho\, g\,\mathrm{d}V},\ \ z_C = \frac{\int_V z\rho\, g\,\mathrm{d}V}{\int_V \rho\, g\,\mathrm{d}V}. \tag{2.19}$$

The acceleration of gravity is $g$, $g = 9.81$ m/s$^2$ or $g = 32.2$ ft/s$^2$. If $g$ is constant throughout the body, then the location of the center of gravity is the same as that of the center of mass.

### 2.1.7 Theorems of Guldinus–Pappus

The theorems of Guldinus–Pappus are concerned with the relation of a surface of revolution to its generating curve, and the relation of a volume of revolution to its generating area.

*Theorem 2.1. Consider a coplanar generating curve and an axis of revolution in the plane of this curve in Fig. 2.9. The surface of revolution A developed by rotating the generating curve about the axis of revolution equals the product of the length of the generating L curve times the circumference of the circle formed by the centroid of the generating curve $y_C$ in the process of generating a surface of revolution*

$$A = 2\pi y_C L. \tag{2.20}$$

*The generating curve can touch but must not cross the axis of revolution.*

*Proof.* An element d$l$ of the generating curve is considered in Fig. 2.9. For a single revolution of the generating curve about the $x$-axis, the line segment d$l$ traces an area

$$\mathrm{d}A = 2\pi y\,\mathrm{d}l.$$

For the entire curve, this area, d$A$, becomes the surface of revolution, $A$, given as

$$A = 2\pi \int y\,\mathrm{d}l = 2\pi y_C L,$$

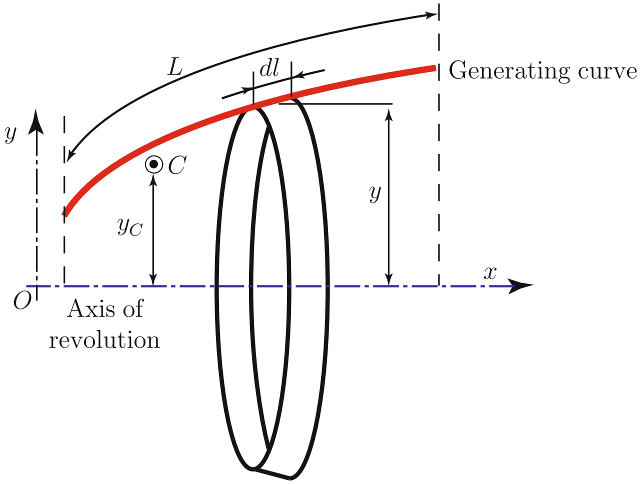



**Fig. 2.9** Surface of revolution developed by rotating the generating curve about the axis of revolution

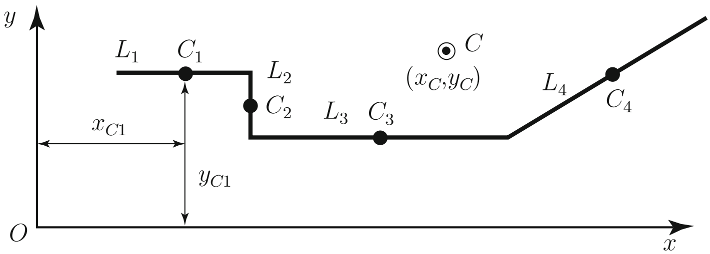



**Fig. 2.10** Composed generating curve

where $L$ is the length of the curve and $y_C$ is the centroidal coordinate of the curve. The circumferential length of the circle formed by having the centroid of the curve rotate about the $x$-axis is $2\pi y_C$, q.e.d.

The surface of revolution $A$ is equal to $2\pi$ times the first moment of the generating curve about the axis of revolution.

If the generating curve is composed of simple curves, $L_i$, whose centroids are known, Fig. 2.10, the surface of revolution developed by revolving the composed generating curve about the axis of revolution $x$ is

$$A = 2\pi \left( \sum_{i=1}^{4} L_i y_{Ci} \right), \tag{2.21}$$

where $y_{Ci}$ is the centroidal coordinate to the $i$th line segment $L_i$. □

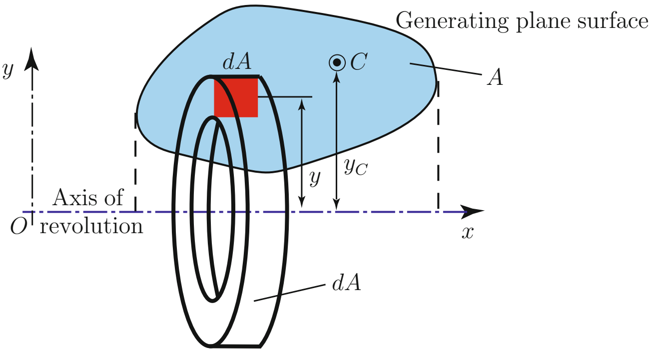



**Fig. 2.11** Volume of revolution developed by rotating the generating plane surface about the axis of revolution

*Theorem 2.2. Consider a generating plane surface A and an axis of revolution coplanar with the surface Fig. 2.11. The volume of revolution V developed by rotating the generating plane surface about the axis of revolution equals the product of the area of the surface times the circumference of the circle formed by the centroid of the surface $y_C$ in the process of generating the body of revolution*

$$V = 2\pi y_C A. \tag{2.22}$$

*The axis of revolution can intersect the generating plane surface only as a tangent at the boundary or have no intersection at all.*

*Proof.* The plane surface $A$ is shown in Fig. 2.11. The volume generated by rotating an element d$A$ of this surface about the $x$-axis is

$$\mathrm{d}V = 2\pi y \mathrm{d}A.$$

The volume of the body of revolution formed from $A$ is then

$$V = 2\pi \int_A y \mathrm{d}A = 2\pi y_C A.$$

Thus, the volume $V$ equals the area of the generating surface $A$ times the circumferential length of the circle of radius $y_C$, q.e.d.

The volume $V$ equals $2\pi$ times the first moment of the generating area $A$ about the axis of revolution. □

## 2.2 Moments of Inertia

### 2.2.1 Introduction

A system of $n$ particle $P_i$, $i = 1, 2, \ldots, n$ is considered. The mass of the particle $P_i$ is $m_i$ as shown in Fig. 2.12.

The position vector of the particle $P_i$ is

$$\mathbf{r}_i = x_i \mathbf{\imath} + y_i \mathbf{\jmath} + z_i \mathbf{k}.$$

The *moments of inertia* of the system about the planes $xOy$, $yOz$, and $zOx$ are

$$I_{xOy} = \sum_i m_i z_i^2, \quad I_{yOz} = \sum_i m_i x_i^2, \quad I_{zOx} = \sum_i m_i y_i^2. \tag{2.23}$$

The *moments of inertia* of the system about $x$, $y$, and $z$ axes are

$$\begin{aligned} I_{xx} &= A = \sum_i m_i \left(y_i^2 + z_i^2\right), \\ I_{yy} &= B = \sum_i m_i \left(z_i^2 + x_i^2\right), \\ I_{zz} &= C = \sum_i m_i \left(x_i^2 + y_i^2\right). \end{aligned} \tag{2.24}$$

The *moment of inertia* of the system about the origin $O$ is

$$I_O = \sum_i m_i \left(x_i^2 + y_i^2 + z_i^2\right). \tag{2.25}$$

The *products of inertia* of the system about the axes $xy$, $yz$, and $zx$ are

$$I_{yz} = D = \sum_i m_i y_i z_i, \quad I_{zx} = E = \sum_i m_i z_i x_i, \quad I_{xy} = F = \sum_i m_i x_i y_i. \tag{2.26}$$

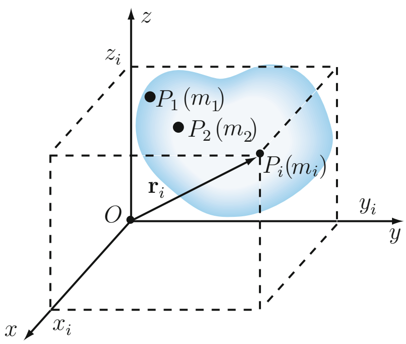



**Fig. 2.12** Particle $P_i$ with the mass $m_i$

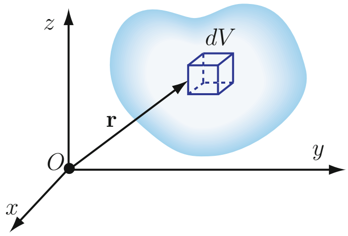



**Fig. 2.13** Rigid body in space with mass $m$ and differential volume d$V$

Between the different moments of inertia, one can write the relations

$$I_O = I_{xOy} + I_{yOz} + I_{zOx} = \frac{1}{2}\left(I_{xx} + I_{yy} + I_{zz}\right),$$

and

$$I_{xx} = I_{yOz} + I_{zOx}.$$

For a continuous domain $D$, the previous relations become

$$\begin{aligned}
I_{xOy} &= \int_D z^2 \mathrm{d}m, \;\; I_{yOz} = \int_D x^2 \mathrm{d}m, \;\; I_{zOx} = \int_D y^2 \mathrm{d}m, \\
I_{xx} &= \int_D \left(y^2 + z^2\right) \mathrm{d}m, \;\; I_{yy} = \int_D \left(x^2 + z^2\right) \mathrm{d}m, \;\; I_{zz} = \int_D \left(x^2 + y^2\right) \mathrm{d}m, \\
I_O &= \int_D \left(x^2 + y^2 + z^2\right) \mathrm{d}m, \\
I_{xy} &= \int_D xy\, \mathrm{d}m, \;\; I_{xz} = \int_D xz\, \mathrm{d}m, \;\; I_{yz} = \int_D yz\, \mathrm{d}m.
\end{aligned} \tag{2.27}$$

The infinitesimal mass element d$m$ can have the values

$$\mathrm{d}m = \rho_v\, \mathrm{d}V, \quad \mathrm{d}m = \rho_A\, \mathrm{d}A, \quad \mathrm{d}m = \rho_l\, \mathrm{d}l,$$

where $\rho_v$, $\rho_A$, and $\rho_l$ are the volume density, area density, and length density.

*Moment of Inertia about an Arbitrary Axis*
For a rigid body with mass $m$, density $\rho$, and volume $V$, as shown in Fig. 2.13, the moments of inertia are defined as follows:

$$I_{xx} = \int_V \rho \left(y^2 + z^2\right) \mathrm{d}V, \;\; I_{yy} = \int_V \rho \left(z^2 + x^2\right) \mathrm{d}V, \;\; I_{zz} = \int_V \rho \left(x^2 + y^2\right) \mathrm{d}V, \tag{2.28}$$

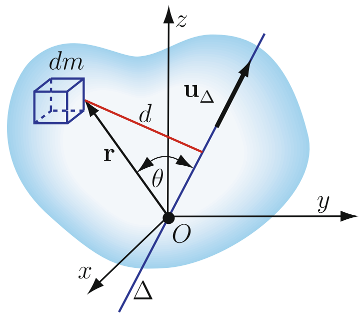



**Fig. 2.14** Rigid body and an arbitrary axis $\Delta$ of unit vector $\mathbf{u}_\Delta$

and the products of inertia

$$I_{xy} = I_{yx} = \int_V \rho\, x\, y\, \mathrm{d}V, \quad I_{xz} = I_{zx} = \int_V \rho\, x\, z\, \mathrm{d}V, \quad I_{yz} = I_{zy} = \int_V \rho\, y\, z\, \mathrm{d}V. \quad (2.29)$$

The moment of inertia given in (2.28) is just the *second moment* of the mass distribution with respect to a Cartesian axis. For example, $I_{xx}$ is the integral of summation of the infinitesimal mass elements $\rho\, \mathrm{d}V$, each multiplied by the square of its distance from the $x$-axis.

The effective value of this distance for a certain body is known as its *radius of gyration* with respect to the given axis. The *radius of gyration* corresponding to $I_{jj}$ is defined as

$$k_j = \sqrt{\frac{I_{jj}}{m}},$$

where $m$ is the total mass of the rigid body and where the symbol $j$ can be replaced by $x$, $y$ or $z$. The *inertia matrix* of a rigid body is represented by the matrix

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}.$$

*Moment of Inertia about an Arbitrary Axis*

Consider the rigid body shown in Fig. 2.14. The reference frame $x$, $y$, $z$ has the origin at $O$. The direction of an arbitrary axis $\Delta$ through $O$ is defined by the unit vector $\mathbf{u}_\Delta$

$$\mathbf{u}_\Delta = \cos\alpha\,\imath + \cos\beta\,\mathrm{J} + \cos\gamma\,\mathbf{k},$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines. The moment of inertia about the $\Delta$ axis, for a differential mass element d$m$ of the body, is by definition

$$I_\Delta = \int_D d^2\, \mathrm{d}m,$$

where $d$ is the perpendicular distance from d$m$ to $\Delta$. The position of the mass element d$m$ is located using the position vector $\mathbf{r}$ and then $d = r\sin\theta$, which represents the magnitude of the cross product $\mathbf{u}_\Delta \times \mathbf{r}$. The moment of inertia can be expressed as

$$I_\Delta = \int_D |\mathbf{u}_\Delta \times \mathbf{r}|^2 \,\mathrm{d}m = \int_D (\mathbf{u}_\Delta \times \mathbf{r}) \cdot (\mathbf{u}_\Delta \times \mathbf{r})\,\mathrm{d}m.$$

If the position vector is $\mathbf{r} = x\boldsymbol{\imath} + y\mathrm{J} + z\mathbf{k}$, then

$$\mathbf{u}_\Delta \times \mathbf{r} = (z\cos\beta - y\cos\gamma)\,\boldsymbol{\imath} + (x\cos\gamma - z\cos\alpha)\,\mathrm{J} + (y\cos\alpha - x\cos\beta)\,\mathbf{k}.$$

After substituting and performing the dot-product operation, one can write the moment of inertia as

$$\begin{aligned} I_\Delta &= \int_D \left[(z\cos\beta - y\cos\gamma)^2 + (x\cos\gamma - z\cos\alpha)^2 + (y\cos\alpha - x\cos\beta)^2\right] \mathrm{d}m \\ &= \cos^2\alpha \int_D \left(y^2 + z^2\right) \mathrm{d}m + \cos^2\beta \int_D \left(z^2 + x^2\right) \mathrm{d}m + \cos^2\gamma \int_D \left(x^2 + y^2\right) \mathrm{d}m \\ &\quad -2\cos\alpha\cos\beta \int_D xy\,\mathrm{d}m - 2\cos\beta\cos\gamma \int_D yz\,\mathrm{d}m - 2\cos\gamma\cos\alpha \int_D zx\,\mathrm{d}m. \end{aligned}$$

The moment of inertia with respect to the $\Delta$ axis is

$$\begin{aligned} I_\Delta &= I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\gamma \\ &\quad -2I_{xy}\cos\alpha\cos\beta - 2I_{yz}\cos\beta\cos\gamma - 2I_{zx}\cos\gamma\cos\alpha. \end{aligned} \tag{2.30}$$

## 2.2.2 *Translation of Coordinate Axes*

The defining equations for the moments and products of inertia, as given by (2.28) and (2.29), do not require that the origin of the Cartesian coordinate system be taken at the mass center. Next, one can calculate the moments and products of inertia for a given body with respect to a set of parallel axes that do not pass through the mass center. Consider the body shown in Fig. 2.15. The mass center is located at the origin $O' \equiv C$ of the primed system $x'y'z'$. The coordinate of $O'$ with respect to the unprimed system $xyz$ is $(x_c, y_c, z_c)$. An infinitesimal volume element d$V$ is located at $(x, y, z)$ in the unprimed system and at $(x', y', z')$ in the primed system. These coordinates are related by the equations

$$x = x' + x_c, \quad y = y' + y_c, \quad z = z' + z_c. \tag{2.31}$$

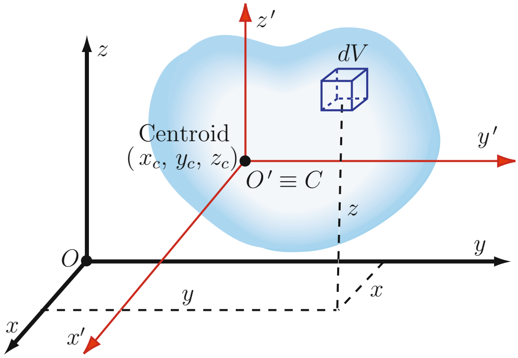



**Fig. 2.15** Rigid body and centroidal axes $x'y'z'$: $x = x' + x_c$, $y = y' + y_c$, $z = z' + z_c$

The moment of inertia about the $x$-axis can be written in terms of primed coordinates by using (2.28) and (2.31)

$$
\begin{aligned}
I_{xx} &= \int_V \rho \left[ \left(y' + y_c\right)^2 + \left(z' + z_c\right)^2 \right] \mathrm{d}V \\
&= I_{Cx'x'} + 2y_c \int_V \rho y' \mathrm{d}V + 2z_c \int_V \rho z' \mathrm{d}V + m\left(y_c^2 + z_c^2\right),
\end{aligned}
\tag{2.32}
$$

where $m$ is the total mass of the rigid body, and the origin of the primed coordinate system was chosen at the mass center. One can write

$$
\int_V \rho x' \mathrm{d}V = \int_V \rho y' \mathrm{d}V = \int_V \rho z' \mathrm{d}V = 0, \tag{2.33}
$$

and therefore, the two integrals on the right-hand side of (2.32) are zero. In a similar way, one can obtain $I_{yy}$ and $I$. The results are summarized as follows:

$$
\begin{aligned}
I_{xx} &= I_{Cx'x'} + m\left(y_c^2 + z_c^2\right), \\
I_{yy} &= I_{Cy'y'} + m\left(x_c^2 + z_c^2\right), \\
I_{zz} &= I_{Cz'z'} + m\left(x_c^2 + y_c^2\right),
\end{aligned}
\tag{2.34}
$$

or, in general,

$$
I_{kk} = I_{Ck'k'} + md^2, \tag{2.35}
$$

where $d$ is the distance between a given unprimed axis and a parallel primed axis passing through the mass center $C$. Equation (2.35) represents the *parallel − axes*

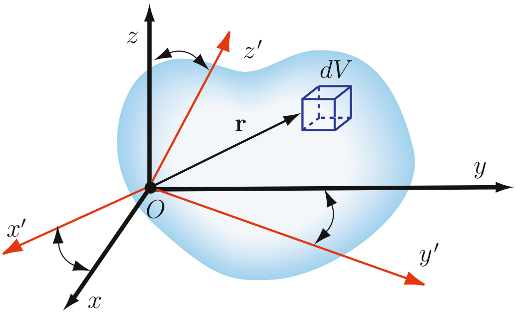



**Fig. 2.16** Rotation of coordinate axes

*Theorem*. The products of inertia are obtained in a similar manner, using (2.29) and (2.31)

$$I_{xy} = \int_V \rho \left(x' + x_c\right)\left(y' + y_c\right) \mathrm{d}V$$

$$= I_{Cx'y'} + x_c \int_V \rho y' \mathrm{d}V + y_c \int_V \rho x' \mathrm{d}V + m x_c y_c.$$

The two integrals on the previous equation are zero. The other products of inertia can be calculated in a similar manner, and the results can be written as follows:

$$\begin{aligned} I_{xy} &= I_{Cx'y'} + m x_c y_c, \\ I_{xz} &= I_{Cx'z'} + m x_c z_c, \\ I_{yz} &= I_{Cy'z'} + m y_c z_c. \end{aligned} \tag{2.36}$$

Equations (2.34) and (2.36) shows that a translation of axes away from the mass center results in an increase in the moments of inertia. The products of inertia may increase or decrease, depending upon the particular case.

### *2.2.3 Principal Axes*

Next, the changes in the moments and product of inertia of a rigid body due to a rotation of coordinate axes are considered, as shown in Fig. 2.16. The origin of the coordinate axes is located at the fixed point $O$. In general, the origin $O$ is not the mass center $C$ of the rigid body. From the definitions of the moments of inertia given in (2.28), it results that the moments of inertia cannot be negative. Furthermore,

$$I_{xx} + I_{yy} + I_{zz} = 2 \int_V \rho r^2 \mathrm{d}V, \tag{2.37}$$

where $r$ is the square of the distance from the origin $O$,

$$r^2 = x^2 + y^2 + z^2.$$

The distance $r$ corresponding to any mass element $\rho\mathrm{d}V$ of the rigid body does not change with a rotation of axes from $xyz$ to $x'y'z'$ (Fig. 2.16). Therefore, the sum of the moments of inertia is invariant with respect to a coordinate system rotation. In terms of matrix notation, the sum of the moments of inertia is just the sum of the elements on the principal diagonal of the inertia matrix and is known as the trace of that matrix. So the trace of the inertia matrix is unchanged by a coordinate rotation because the trace of any square matrix is invariant under an orthogonal transformation.

Next, the products of inertia are considered. A coordinate rotation of axes can result in a change in the signs of the products of inertia. A 180° rotation about the $x$-axis, for example, reverses the signs of $I_{xy}$ and $I_{xz}$, while the sign of $I_{yz}$ is unchanged. This occurs because the directions of the positive $y$ and $z$ axes are reversed. On the other hand, a 90° rotation about the $x$-axis reverses the sign of $I_{yz}$. It can be seen that the moments and products of inertia vary smoothly with changes in the orientation of the coordinate system because the direction cosines vary smoothly. Therefore, an orientation can always be found for which a given product of inertia is zero. It is always possible to find an orientation of the coordinate system relative to a given rigid body such that all products of inertia are zero simultaneously, that is, the inertia matrix is *diagonal*. The three mutually orthogonal coordinate axes are known as *principal axes* in this case, and the corresponding moments of inertia are the *principal moments of inertia.* The three planes formed by the principal axes are called *principal planes.*

If $I$ is a principal moment of inertia, then $I$ satisfies the cubic characteristic equation

$$\begin{vmatrix} I_{xx} - I & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} - I & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} - I \end{vmatrix} = 0. \tag{2.38}$$

Equation (2.38) is used to determine the associated principal moments of inertia.

Suppose that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are mutually perpendicular unit vectors each parallel to a principal axis of the rigid body relative to $O$. The principal moments of inertia associated to $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ for the rigid body relative to $O$ are $I_1$, $I_2$, and $I_3$. The inertia matrix, in this case, is

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

When the point $O$ under consideration is the mass center of the rigid body, one speaks of *central principal moments of inertia*.

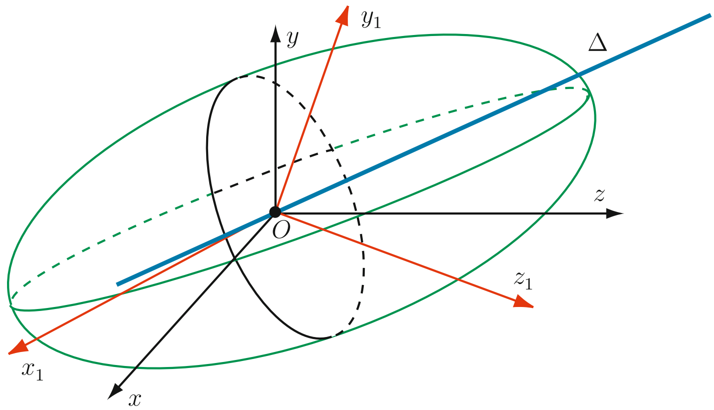



**Fig. 2.17** Ellipsoid of inertia

### *2.2.4 Ellipsoid of Inertia*

The ellipsoid of inertia for a given body and reference point is a plot of the moment of inertia of the body for all possible axis orientations through the reference point. This graph in space has the form of an ellipsoid surface. Consider a rigid body in rotational motion about an axis $\Delta$. The *ellipsoid of inertia* with respect to an arbitrary point $O$ is the geometrical locus of the points $Q$, where $Q$ is the extremity of the vector $\overrightarrow{OQ}$ with the module $\left|\overrightarrow{OQ}\right| = \dfrac{1}{\sqrt{I_\Delta}}$ and where $I_\Delta$ is the moment of inertia about the instantaneous axis of rotation $\Delta$, as shown in Fig. 2.17. The segment $OQ$ is calculated with

$$|\overrightarrow{OQ}| = \frac{1}{\sqrt{I_\Delta}} = \frac{1}{k_0\, m},$$

where $k_0$ is the radius of gyration of the body about the given axis and $m$ is the total mass. For a Cartesian system of axes, the equation of the ellipsoid surface centered at $O$ is

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{xz}xz + 2I_{yz}yz = 1. \tag{2.39}$$

The radius of gyration of a given rigid body depends upon the location of the axis relative to the body and is not depended upon the position of the body in space. The ellipsoid of inertia is fixed in the body and rotates with it. The $x_1$, $y_1$, and $z_1$ axes are assumed to be the principal axes of the ellipsoid, as shown in Fig. 2.17. For the principal axes, the equation of the inertia ellipsoid, (2.39), takes the following simple form

$$I_1 x_1^2 + I_2 y_1^2 + I_3 z_3^2 = 1. \tag{2.40}$$

The previous equation is of the same form as (2.39) for the case where the $x_1$, $y_1$, and $z_1$ axes are the principal axes of the rigid body, and all products of inertia vanish.

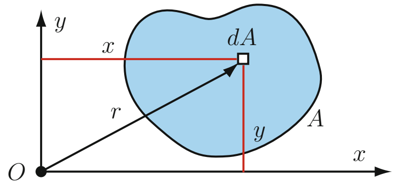



**Fig. 2.18** Moments of inertia for area $A$ about $x$ and $y$ axes

Therefore, $I_1$, $I_2$, and $I_3$ are the principal moments of inertia of the rigid body, and furthermore, the principal axes of the body coincide with those of the ellipsoid of inertia. From (2.40), the lengths of the principal semiaxes of the ellipsoid of inertia are

$$l_1 = \frac{1}{\sqrt{I_1}},$$

$$l_2 = \frac{1}{\sqrt{I_2}},$$

$$l_3 = \frac{1}{\sqrt{I_3}}.$$

From the parallel-axis theorem, (2.35), one can remark that the minimum moment of inertia about the mass center is also the smallest possible moment of inertia for the given body with respect to any reference point.

If the point $O$ is the same as the mass center $(O \equiv C)$, the ellipsoid is named *principal ellipsoid of inertia.*

## *2.2.5 Moments of Inertia for Areas*

The moment of inertia (*second moment*) of the area $A$ about $x$ and $y$ axes, see Fig. 2.18, denoted as $I_{xx}$ and $I_{yy}$, respectively, are

$$I_{xx} = \int_A y^2 \, \mathrm{d}A, \tag{2.41}$$

$$I_{yy} = \int_A x^2 \, \mathrm{d}A. \tag{2.42}$$

The second moment of area cannot be negative.

The entire area may be concentrated at a single point $(k_x, k_y)$ to give the same second moment of area for a given reference. The distances $k_x$ and $k_y$ are called the radii of gyration. Thus,

$$A\,k_x^2 = I_{xx} = \int_A y^2 \, \mathrm{d}A \quad \Longrightarrow \quad k_x^2 = \frac{\int_A y^2 \, \mathrm{d}A}{A} = \frac{I_{xx}}{A},$$

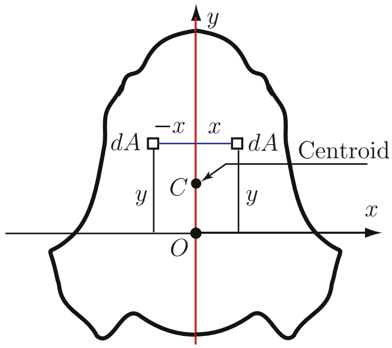



**Fig. 2.19** Area, $A$, with an axis of symmetry $Oy$

$$A\,k_y^2 = I_{yy} = \int_A x^2\,\mathrm{d}A \quad\Longrightarrow\quad k_y^2 = \frac{\int_A x^2\,\mathrm{d}A}{A} = \frac{I_{yy}}{A}\,. \tag{2.43}$$

This point $(k_x, k_y)$ depends on the shape of the area and on the position of the reference. The centroid location is independent of the reference position.

The *product of inertia* for an area $A$ is defined as

$$I_{xy} = \int_A xy\,\mathrm{d}A. \tag{2.44}$$

This quantity may be positive or negative and relates an area directly to a set of axes.

If the area under consideration has an axis of symmetry, the product of area for this axis is zero. Consider the area in Fig. 2.19, which is symmetrical about the vertical axis $y$. The planar Cartesian frame is $xOy$. The centroid is located somewhere along the symmetrical axis $y$. Two differential element of areas that are positioned as mirror images about the $y$-axis are shown in Fig. 2.19. The contribution to the product of area of each elemental area is $x\,y$ d$A$, but with opposite signs, and so the result is zero. The entire area is composed of such elemental area pairs, and the product of area is zero. The product of inertia for an area $I_{xy}$ is zero ($I_{xy} = 0$) if either the $x$- or $y$-axis is an axis of symmetry for the area.

*Transfer Theorem or Parallel-Axis Theorem*

The $x$-axis in Fig. 2.20 is parallel to an axis $x'$, and it is at a distance $b$ from the axis $x'$. The axis $x'$ is going through the centroid $C$ of the $A$ area, and it is a centroidal axis. The second moment of area about the $x$-axis is

$$I_{xx} = \int_A y^2\,\mathrm{d}A = \int_A (y' + b)^2\,\mathrm{d}A,$$

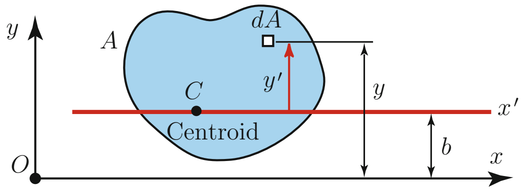



**Fig. 2.20** Area and centroidal axis $Cx'x'||xx$

where the distance $y = y' + b$. Carrying out the operations

$$I_{xx} = \int_A y'^2 \, dA + 2b \int_A y' \, dA + A\,b^2.$$

The first term of the right-hand side is by definition $I_{x'x'}$

$$I_{Cx'x'} = \int_A y'^2 \, dA.$$

The second term involves the first moment of area about the $x'$ axis, and it is zero because the $x'$ axis is a centroidal axis

$$\int_A y' \, dA = 0.$$

The second moment of the area $A$ about any axis $I_{xx}$ is equal to the second moment of the area $A$ about a parallel axis at centroid $I_{Cx'x'}$ plus $A\,b^2$, where $b$ is the perpendicular distance between the axis for which the second moment is being computed and the parallel centroidal axis

$$I_{xx} = I_{Cx'x'} + A\,b^2.$$

With the transfer theorem, the second moments or products of area about any axis can be computed in terms of the second moments or products of area about a parallel set of axes going through the centroid of the area in question.

In handbooks, the areas and second moments about various centroidal axes are listed for many of the practical configurations, and using the parallel-axis theorem (or Huygens–Steiner theorem), second moments can be calculated for axes not at the centroid.

In Fig. 2.21 are shown two references, one $x'y'$ at the centroid $C$ and the other $xy$ arbitrary but positioned parallel relative to $x'y'$. The coordinates of the centroid $C(x_C, y_C)$ of area $A$ measured from the reference $x, y$ are $c$ and $b$, $x_C = c$, $y_C = b$. The centroid coordinates must have the proper signs. The product of area about the noncentroidal axes $xy$ is

$$I_{xy} = \int_A x\,y \, dA = \int_A (x' + c)(y' + b) \, dA,$$

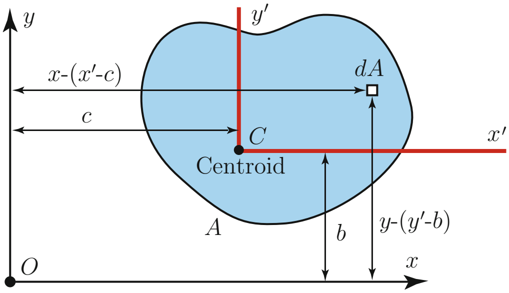



**Fig. 2.21** Centroidal axes $x'y'$ parallel to reference axes $xy$: $Cx'x'||xx$ and $Cy'y'||yy$

or

$$I_{xy} = \int_A x'\,y'\;\mathrm{d}A + c\int_A y'\;\mathrm{d}A + b\int_A x'\;\mathrm{d}A + A\,b\,c.$$

The first term of the right-hand side is by definition $I_{x'y'}$

$$I_{x'y'} = \int_A x'\,y'\;\mathrm{d}A.$$

The next two terms of the right-hand side are zero since $x'$ and $y'$ are centroidal axes

$$\int_A y'\;\mathrm{d}A = 0 \;\text{ and }\; \int_A x'\;\mathrm{d}A = 0.$$

Thus, the parallel-axis theorem for products of area is as follows.

The product of area for any set of axes $I_{xy}$ is equal to the product of area for a parallel set of axes at centroid $I_{Cx'y'}$ plus $A\,c\,b$, where $c$ and $b$ are the coordinates of the centroid of area $A$,

$$I_{xy} = I_{Cx'y'} + A\,c\,b.$$

With the transfer theorem, the second moments or products of area can be found about any axis in terms of second moments or products of area about a parallel set of axes going through the centroid of the area.

*Polar Moment of Area*

In Fig. 2.18, there is a reference $xy$ associated with the origin $O$. Summing $I_{xx}$ and $I_{yy}$,

$$\begin{aligned} I_{xx} + I_{yy} &= \int_A y^2\;\mathrm{d}A + \int_A x^2\;\mathrm{d}A \\ &= \int_A (x^2+y^2)\;\mathrm{d}A = \int_A r^2\;\mathrm{d}A, \end{aligned}$$

where $r^2 = x^2 + y^2$. The distance $r^2$ is independent of the orientation of the reference, and the sum $I_{xx} + I_{yy}$ is independent of the orientation of the coordinate system.

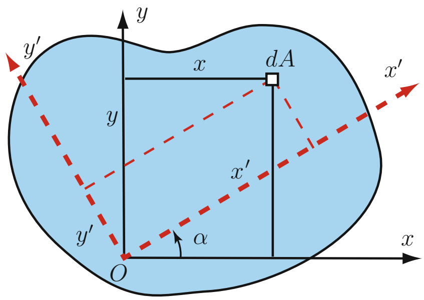



**Fig. 2.22** Reference $xy$ and reference $x'y'$ rotated with an angle $\alpha$

Therefore, the sum of second moments of area about orthogonal axes is a function only of the position of the origin $O$ for the axes.

The polar moment of area about the origin $O$ is

$$I_O = I_{xx} + I_{yy}. \tag{2.45}$$

The polar moment of area is an *invariant* of the system. The group of terms $I_{xx}\, I_{yy} - I_{xy}^2$ is also invariant under a rotation of axes. The polar radius of gyration is

$$k_O = \sqrt{\frac{I_O}{A}}. \tag{2.46}$$

*Principal Axes*

In Fig. 2.22, an area $A$ is shown with a reference $xy$ having its origin at $O$. Another reference $x'y'$ with the same origin $O$ is rotated with an angle $\alpha$ from $xy$ (counterclockwise as positive). The relations between the coordinates of the area elements d$A$ for the two references are

$$x' = x\cos\alpha + y\sin\alpha, \quad y' = -x\sin\alpha + y\cos\alpha.$$

The second moment $I_{x'x'}$ can be expressed as

$$\begin{aligned} I_{x'x'} &= \int_A (y')^2 \,\mathrm{d}A = \int_A (-x\sin\alpha + y\cos\alpha)^2 \,\mathrm{d}A \\ &= \sin^2\alpha \int_A x^2 \,\mathrm{d}A - 2\sin\alpha\cos\alpha \int_A xy \,\mathrm{d}A + \cos^2\alpha \int_A y^2 \,\mathrm{d}A \\ &= I_{yy}\sin^2\alpha + I_{xx}\cos^2\alpha - 2I_{xy}\sin\alpha\cos\alpha. \end{aligned} \tag{2.47}$$

Using the trigonometric identities

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \quad \sin 2\alpha = 2\sin\alpha\cos\alpha,$$

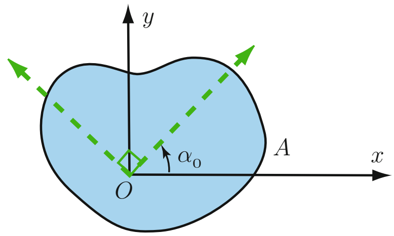



**Fig. 2.23** Principal axis of area

(2.47) becomes

$$I_{x'x'} = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha - I_{xy}\sin 2\alpha. \tag{2.48}$$

Replacing $\alpha$ with $\alpha+\pi/2$ in (2.48) and using the trigonometric relations

$$\cos(2\alpha+\pi) = -\cos 2\alpha,\ \ \sin(2\alpha+\pi) = -\sin 2\alpha,$$

the second moment $I_{y'y'}$ is

$$I_{y'y'} = \frac{I_{xx}+I_{yy}}{2} - \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha + I_{xy}\sin 2\alpha. \tag{2.49}$$

The product of area $I_{x'y'}$ is computed in a similar manner

$$I_{x'y'} = \int_A x'y'\,\mathrm{d}A = \frac{I_{xx}-I_{yy}}{2}\sin 2\alpha + I_{xy}\cos 2\alpha. \tag{2.50}$$

If $I_{xx}$, $I_{yy}$, and $I_{xy}$ are known for a reference $xy$ with an origin $O$, then the second moments and products of area for every set of axes at $O$ can be computed. Next, it is assumed that $I_{xx}$, $I_{yy}$, and $I_{xy}$ are known for a reference $xy$. The sum of the second moments of area is constant for any reference with origin at $O$. The *minimum* second moment of area corresponds to an axis at *right angles* to the axis having the *maximum* second moment, as shown in Fig. 2.23. This particular set of axes if called *principal axis* of area and the corresponding moments of inertia with respect to these axes are called *principal moments of inertia*.

The second moments of area can be expressed as functions of the angle variable $\alpha$. The maximum second moment may be determined by setting the partial derivative of $I_{x'x'}$ with respect to $\alpha$ equal to zero. Thus,

$$\frac{\partial I_{x'x'}}{\partial \alpha} = (I_{xx}-I_{yy})(-\sin 2\alpha) - 2I_{xy}\cos 2\alpha = 0, \tag{2.51}$$

or

$$(I_{yy}-I_{xx})\sin 2\alpha_0 - 2I_{xy}\cos 2\alpha_0 = 0,$$

where $\alpha_0$ is the value of $\alpha$ which defines the orientation of principal axes. Hence,

$$\tan 2\alpha_0 = \frac{2I_{xy}}{I_{yy} - I_{xx}}. \tag{2.52}$$

The angle $\alpha_0$ corresponds to an extreme value of $I_{x'x'}$ (i.e., to a maximum or minimum value). There are two roots for $2\alpha_0$, which are $\pi$ radians apart, that will satisfy the previous equation. Thus,

$$2\alpha_{0_1} = \tan^{-1}\frac{2I_{xy}}{I_{yy} - I_{xx}} \implies \alpha_{0_1} = \frac{1}{2}\tan^{-1}\frac{2I_{xy}}{I_{yy} - I_{xx}},$$

and

$$2\alpha_{0_2} = \tan^{-1}\frac{2I_{xy}}{I_{yy} - I_{xx}} + \pi \implies \alpha_{0_2} = \frac{1}{2}\tan^{-1}\frac{2I_{xy}}{I_{yy} - I_{xx}} + \frac{\pi}{2}.$$

This means that there are two axes orthogonal to each other having extreme values for the second moment of area at $O$. One of the axes is the maximum second moment of area, and the minimum second moment of area is on the other axis. These axes are the principal axes.

With $\alpha = \alpha_0$, the product of area $I_{x'y'}$ becomes

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\alpha_0 + I_{xy}\cos 2\alpha_0. \tag{2.53}$$

For $\alpha_0 = \alpha_{0_1}$, the sine and cosine expressions are

$$\sin 2\alpha_{0_1} = \frac{2I_{xy}}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}, \quad \cos 2\alpha_{0_1} = \frac{-(I_{xx} - I_{yy})}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}.$$

For $\alpha_0 = \alpha_{0_2}$, the sine and cosine expressions are

$$\sin 2\alpha_{0_2} = \frac{-2I_{xy}}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}, \quad \cos 2\alpha_{0_2} = \frac{I_{xx} - I_{yy}}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}.$$

Equation (2.53) and $\alpha_0 = \alpha_{0_1}$ give

$$I_{x'y'} = -(I_{yy} - I_{xx})\frac{I_{xy}}{\left[(I_{yy} - I_{xx})^2 + 4I_{xy}^2\right]^{1/2}} + I_{xy}\frac{I_{yy} - I_{xx}}{\left[(I_{yy} - I_{xx})^2 + 4I_{xy}^2\right]^{1/2}} = 0.$$

In a similar way, (2.53) and $\alpha_0 = \alpha_{0_2}$ give $I_{x'y'} = 0$. The product of area corresponding to the principal axes is zero.

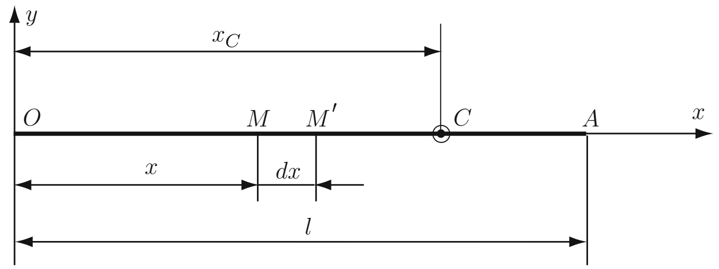



**Fig. 2.24** Example 2.1

The maximum or minimum moment of inertia for the area are

$$I_{1,2} = I_{\max,\min} = \frac{I_{xx}+I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}. \tag{2.54}$$

If $I$ is a principal moment of inertia, then $I$ satisfies the quadratic characteristic equation

$$\begin{vmatrix} I_{xx}-I & I_{xy} \\ I_{yx} & I_{yy}-I \end{vmatrix} = 0. \tag{2.55}$$

## 2.3 Examples

*Example 2.1.* Find the position of the mass center for a nonhomogeneous straight rod, with the length $OA = l$ (Fig. 2.24). The linear density $\rho$ of the rod is a linear function with $\rho = \rho_0$ at $O$ and $\rho = \rho_1$ at $A$.

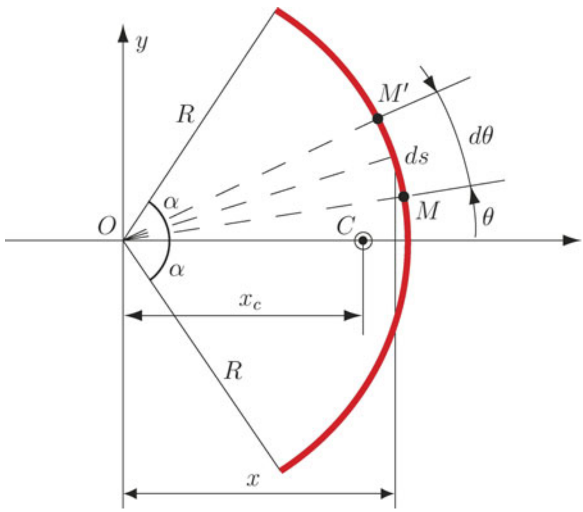
*Solution*
A reference frame $xOy$ is selected with the origin at $O$ and the $x$-axis along the rectilinear rod (Fig. 2.24). Let $M(x,0)$ be an arbitrarily given point on the rod, and let $MM'$ be an element of the rod with the length $\mathrm{d}x$ and the mass $\mathrm{d}m = \rho\,\mathrm{d}x$. The density $\rho$ is a linear function of $x$ given by

$$\rho = \rho(x) = \rho_0 + \frac{\rho_1-\rho_0}{l}x.$$

The center mass of the rod, $C$, has $x_c$ as abscissa. The mass center $x_c$, with respect to point $O$, is

$$x_c = \frac{\displaystyle\int_L x\,\mathrm{d}m}{\displaystyle\int_L \mathrm{d}m} = \frac{\displaystyle\int_0^l x\rho\,\mathrm{d}x}{\displaystyle\int_0^l \rho\,\mathrm{d}x} = \frac{\displaystyle\int_0^l x\left(\rho_0 + \frac{\rho_1-\rho_0}{l}x\right)\mathrm{d}x}{\displaystyle\int_0^l \left(\rho_0 + \frac{\rho_1-\rho_0}{l}x\right)\mathrm{d}x} = \frac{\dfrac{\rho_0+2\rho_1}{6}l^2}{\dfrac{\rho_0+\rho_1}{2}l}.$$

**Fig. 2.25** Example 2.2




The mass center $x_c$ is

$$x_c = \frac{\rho_0 + 2\rho_1}{3\,(\rho_0 + \rho_1)} l.$$

In the special case of a homogeneous rod, the density and the position of the mass center are given by

$$\rho_0 = \rho_1 \quad \text{and} \quad x_c = \frac{l}{2}.$$

The MATLAB program is

```
syms rho0 rhoL L x
rho = rho0 + (rhoL - rho0)*x/L;
m = int(rho, x, 0, L);
My = simplify(int(x*rho, x, 0, L));
xC = simplify(My/m);
fprintf('m = %s \n', char(m))
fprintf('My = %s \n', char(My))
fprintf('xC = My/m = %s \n', char(xC))
```

The MATLAB statement `int(f, x, a, b)` is the definite integral of `f` with respect to its symbolic variable `x` from `a` to `b`.

*Example 2.2.* Find the position of the centroid for a homogeneous circular arc. The radius of the arc is $R$, and the center angle is $2\alpha$ radians as shown in Fig. 2.25.

*Solution*
The axis of symmetry is selected as the $x$-axis ($y_C = 0$). Let $MM'$ be a differential element of arc with the length $\mathrm{d}s = R\,\mathrm{d}\theta$. The mass center of the differential element of arc $MM'$ has the abscissa

$$x = R\cos\left(\theta + \frac{d\theta}{2}\right) \approx R\cos(\theta).$$

The abscissa $x_C$ of the centroid for a homogeneous circular arc is calculated from (2.8)

$$x_c \int_{-\alpha}^{\alpha} R d\theta = \int_{-\alpha}^{\alpha} xR d\theta. \tag{2.56}$$

Because $x = R\cos(\theta)$ and with (2.56), one can write

$$x_c \int_{-\alpha}^{\alpha} R d\theta = \int_{-\alpha}^{\alpha} R^2 \cos(\theta) d\theta. \tag{2.57}$$

From (5.19), after integration, it results

$$2x_c \alpha R = 2R^2 \sin(\alpha),$$

or

$$x_c = \frac{R\sin(\alpha)}{\alpha}.$$

For a semicircular arc when $\alpha = \frac{\pi}{2}$, the position of the centroid is

$$x_c = \frac{2R}{\pi},$$

and for the quarter-circular $\alpha = \frac{\pi}{4}$,

$$x_c = \frac{2\sqrt{2}}{\pi} R.$$

*Example 2.3.* Find the position of the mass center for the area of a circular sector. The center angle is $2\alpha$ radians, and the radius is $R$ as shown in Fig. 2.26.

*Solution*
The origin $O$ is the vertex of the circular sector. The $x$-axis is chosen as the axis of symmetry and $y_C = 0$. Let $MNPQ$ be a surface differential element with the area $dA = \rho d\rho d\theta$. The mass center of the surface differential element has the abscissa

$$x = \left(\rho + \frac{d\rho}{2}\right)\cos\left(\theta + \frac{d\theta}{2}\right) \approx \rho\cos(\theta). \tag{2.58}$$

Using the first moment of area formula with respect to $Oy$, (2.15), the mass center abscissa $x_C$ is calculated as

$$x_C \int\int \rho d\rho d\theta = \int\int x\rho d\rho d\theta. \tag{2.59}$$

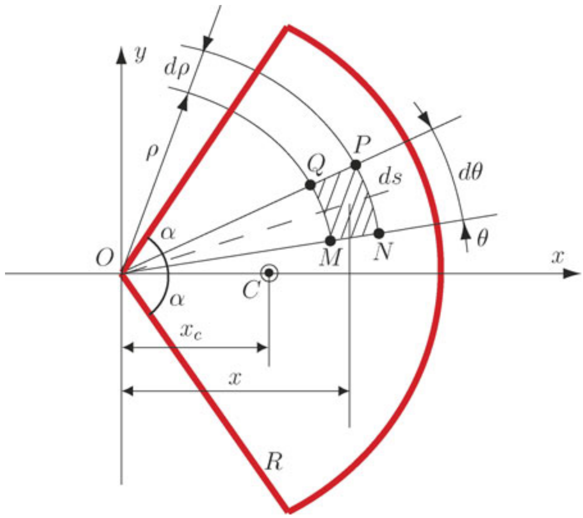



**Fig. 2.26** Example 2.3

Equations (2.58) and (2.59) give

$$x_C \int\int \rho \mathrm{d}\rho \mathrm{d}\theta = \int\int \rho^2 \cos(\theta) \mathrm{d}\rho \mathrm{d}\theta,$$

or

$$x_C \int_0^R \rho \mathrm{d}\rho \int_{-\alpha}^{\alpha} \mathrm{d}\theta = \int_0^R \rho^2 \mathrm{d}\rho \int_{-\alpha}^{\alpha} \cos(\theta) \mathrm{d}\theta. \tag{2.60}$$

From (2.60), after integration, it results

$$x_C \left(\frac{1}{2}R^2\right)(2\alpha) = \left(\frac{1}{3}R^3\right)(2\sin(\alpha)),$$

or

$$x_C = \frac{2R\sin(\alpha)}{3\alpha}. \tag{2.61}$$

For a semicircular area, $\alpha = \dfrac{\pi}{2}$, the $x$-coordinate to the centroid is

$$x_C = \frac{4R}{3\pi}.$$

For the quarter-circular area, $\alpha = \dfrac{\pi}{2}$, the $x$-coordinate to the centroid is

$$x_C = \frac{4R\sqrt{2}}{3\pi}.$$

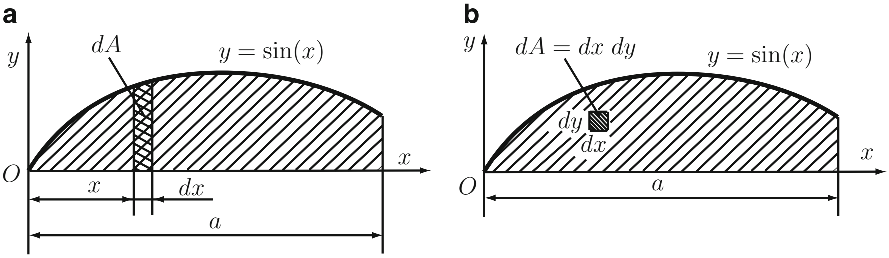



**Fig. 2.27** Example 2.4

*Example 2.4.* Find the coordinates of the mass center for a homogeneous planar plate located under the curve of equation $y = \sin x$ from $x = 0$ to $x = a$.

*Solution*
A vertical differential element of area $\mathrm{d}A = y\,\mathrm{d}x = (\sin x)(\mathrm{d}x)$ is chosen as shown in Fig. 2.27a. The $x$-coordinate of the mass center is calculated from (2.15):

$$x_c \int_0^a (\sin x)\,\mathrm{d}x = \int_0^a x(\sin x)\,\mathrm{d}x \quad \text{or} \quad x_c \{-\cos x\}_0^a = \int_0^a x(\sin x)\,\mathrm{d}x, \quad \text{or}$$
$$x_c(1 - \cos a) = \int_0^a x(\sin x)\,\mathrm{d}x. \tag{2.62}$$

The integral $\int_0^a x(\sin x)\,\mathrm{d}x$ is calculated with

$$\int_0^a x(\sin x)\,\mathrm{d}x = \{x(-\cos x)\}_0^a - \int_0^a (-\cos x)\mathrm{d}x = \{x(-\cos x)\}_0^a + \{\sin x\}_0^a$$
$$= \sin a - a\cos a. \tag{2.63}$$

Using (2.62) and (2.63) after integration, it results

$$x_c = \frac{\sin a - a\cos a}{1 - \cos a}.$$

The $x$-coordinate of the mass center, $x_C$, can be calculated using the differential element of area $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, as shown in Fig. 2.27b. The area of the figure is

$$A = \int_A \mathrm{d}x\,\mathrm{d}y = \int_0^a \int_0^{\sin x} \mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^{\sin x} \mathrm{d}y$$
$$= \int_0^a \mathrm{d}x\,\{y\}_0^{\sin x} = \int_0^a (\sin x)\mathrm{d}x = \{-\cos x\}_0^a = 1 - \cos a.$$

The first moment of the area $A$ about the $y$-axis is

$$M_y = \int_A x \mathrm{d}A = \int_0^a \int_0^{\sin x} x \mathrm{d}x \mathrm{d}y = \int_0^a x \mathrm{d}x \int_0^{\sin x} \mathrm{d}y$$
$$= \int_0^a x \mathrm{d}x \, \{y\}_0^{\sin x} = \int_0^a x (\sin x) \mathrm{d}x = \sin a - a \cos a.$$

The $x$-coordinate of the mass center is $x_C = M_y/A$. The $y$-coordinate of the mass center is $y_C = M_x/A$, where the first moment of the area $A$ about the $x$-axis is

$$M_x = \int_A y \mathrm{d}A = \int_0^a \int_0^{\sin x} y \mathrm{d}x \mathrm{d}y = \int_0^a \mathrm{d}x \int_0^{\sin x} y \mathrm{d}y$$
$$= \int_0^a \mathrm{d}x \left\{ \frac{y^2}{2} \right\}_0^{\sin x} = \int_0^a \frac{\sin^2 x}{2} \mathrm{d}x = \frac{1}{2} \int_0^a \sin^2 \mathrm{d}x.$$

The integral $\int_0^a \sin^2 x \mathrm{d}x$ is calculated with

$$\int_0^a \sin^2 x \mathrm{d}x = \int_0^a \sin x d(-\cos x) = \{\sin x (-\cos x)\}_0^a + \int_0^a \cos^2 x \mathrm{d}x$$
$$= -\sin a (\cos a) + \int_0^a (1 - \sin^2 x) \mathrm{d}x = -\sin a (\cos a) + a - \int_0^a \sin^2 x \mathrm{d}x,$$

or

$$\int_0^a \sin^2 x \mathrm{d}x = \frac{a - \sin a \cos a}{2}.$$

The coordinate $y_C$ is

$$y_C = \frac{M_x}{A} = \frac{a - \sin a \cos a}{4(1 - \cos a)}.$$

The MATLAB program is given by

```
syms a x y
% dA = dx dy
% A = int dx dy ; 0<x<a 0<y<sin(x)
% Ay = int dy ; 0<y<sin(x)
Ay=int(1,y,0,sin(x))
% A = int Ay dx ; ; 0<x<a
A=int(Ay,x,0,a)

% My = int x dx dy ; 0<x<a 0<y<sin(x)
% Qyy = int dy ; 0<y<sin(x)
Qyy=int(1,y,0,sin(x))
% My = int x Qyy dx ; 0<x<a
My=int(x*Qyy,x,0,a)
```

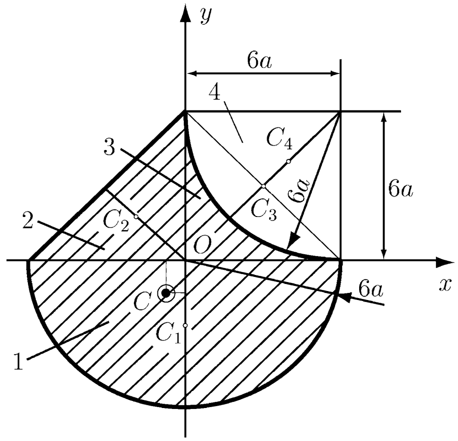



**Fig. 2.28** Example 2.5

```
% Mx = int y dx dy ; 0<x<a 0<y<sin(x)
% Qxy = int y dy ; 0<y<sin(x)
% Mx = int Qxy dx ; 0<x<a
Qxy=int(y,y,0,sin(x))
Mx=int(Qxy,x,0,a)

xC=My/A;
yC=Mx/A;
pretty(xC)
pretty(yC)
```

*Example 2.5.* Find the position of the mass center for a homogeneous planar plate ($a = 1$ m), with the shape and dimensions given in Fig. 2.28.

*Solution*
The plate is composed of four elements: the circular sector area 1, the triangle 2, the square 3, and the circular area 4 to be subtracted. Using the decomposition method, the positions of the mass center $x_i$, the areas $A_i$, and the first moments with respect to the axes of the reference frame $M_{y_i}$ and $M_{x_i}$, for all four elements are calculated. The results are given in the following table:

| $i$ | $x_i$ | $y_i$ | $A_i$ | $M_{y_i} = x_i A_i$ | $M_{x_i} = y_i A_i$ |
|---|---|---|---|---|---|
| Circular sector 1 | $0$ | $-\dfrac{8}{\pi}a$ | $18\pi a^2$ | $0$ | $-144a^3$ |
| Triangle 2 | $-2a$ | $2a$ | $18a^2$ | $-36a^3$ | $36a^3$ |
| Square 3 | $3a$ | $3a$ | $36a^2$ | $108a^3$ | $108a^3$ |
| Circular sector 4 | $6a - \dfrac{8a}{\pi}$ | $6a - \dfrac{8a}{\pi}$ | $-9\pi a^2$ | $-54\pi a^3 + 72a^3$ | $-54\pi a^3 + 72a^3$ |
| $\Sigma$ | – | – | $9(\pi + 6)a^2$ | $18\,(8 - 3\pi)\,a^3$ | $9\,(8 - 6\pi)\,a^3$ |

The $x$ and $y$ coordinates of the mass center $C$ are

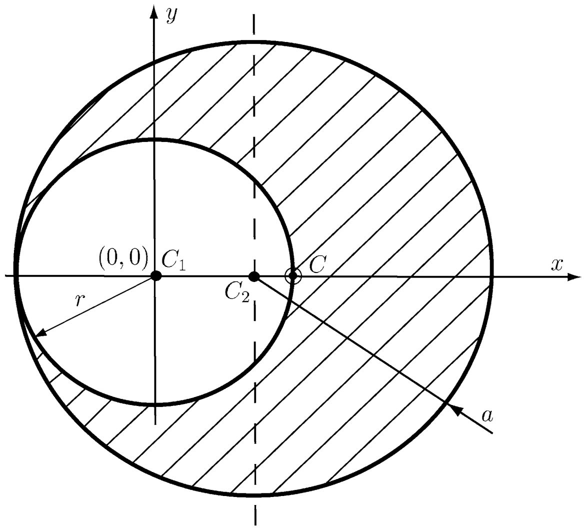
$$x_c = \frac{\sum x_i A_i}{\sum A_i} = \frac{2\,(8-3\pi)}{\pi+6} a = -0.311\,a = -0.311\text{ m},$$

$$y_c = \frac{\sum y_i A_i}{\sum A_i} = \frac{8-6\pi}{\pi+6} a = -1.186\,a = -1.186\text{ m}.$$

The MATLAB program is

```
syms a

x1=0;
y1=-8*a/pi;
A1=18*pi*a^2;
x1A1=x1*A1;
y1A1=y1*A1;
fprintf('x1 A1 = %s \n', char(x1A1))
fprintf('y1 A1 = %s \n', char(y1A1))

x2=-2*a;
y2=2*a;
A2=18*a^2;
x2A2=x2*A2;
y2A2=y2*A2;
fprintf('x2 A2 = %s \n', char(x2A2))
fprintf('y2 A2 = %s \n', char(y2A2))

x3=3*a;
y3=3*a;
A3=36*a^2;
x3A3=x3*A3;
y3A3=y3*A3;
fprintf('x3 A3 = %s \n', char(x3A3))
fprintf('y3 A3 = %s \n', char(y3A3))

x4=6*a-8*a/pi;
y4=x4;
A4=-9*pi*a^2;
x4A4=simplify(x4*A4);
y4A4=simplify(y4*A4);
fprintf('x4 A4 = %s \n', char(x4A4))
fprintf('y4 A4 = %s \n', char(y4A4))

xC=(x1*A1+x2*A2+x3*A3+x4*A4)/(A1+A2+A3+A4);
xC=simplify(xC);
```

**Fig. 2.29** Example 2.6




```
yC=(y1*A1+y2*A2+y3*A3+y4*A4)/(A1+A2+A3+A4);
yC=simplify(yC);

fprintf('xC = %s = %s \n', char(xC), char(vpa(xC,6)))
fprintf('yC = %s = %s \n', char(yC), char(vpa(yC,6)))
```

*Example 2.6.* The homogeneous plate shown in Fig. 2.29 is delimitated by the hatched area. The circle, with the center at $C_1$, has the unknown radius $r$. The circle, with the center at $C_2$, has the given radius $a = 2$ m, $(r < a)$. The position of the mass center of the hatched area, $C$, is located at the intersection of the circle, with the radius $r$ and the center at $C_1$, and the positive $x$-axis. Find the radius $r$.

*Solution*

The $x$-axis is a symmetry axis for the planar plate, and the origin of the reference frame is located at $C_1 = O(0, 0)$. The $y$-coordinate of the mass center of the hatched area is $y_C = 0$. The coordinates of the mass center of the circle with the center at $C_2$ and radius $a$ are $C_2(a - r, 0)$. The area of the of circle $C_1$ is $A_1 = -\pi r^2$, and the area of the circle $C_2$ is $A_2 = \pi a^2$. The total area is given by

$$A = A_1 + A_2 = \pi\left(a^2 - r^2\right).$$

The coordinate $x_C$ of the mass center is given by

$$x_C = \frac{x_{C_1}A_1 + x_{C_2}A_2}{A_1 + A_2} = \frac{x_{C_2}A_2}{A_1 + A_2} = \frac{x_{C_2}A_2}{A} = \frac{(a - r)\pi a^2}{\pi(a^2 - r^2)}$$

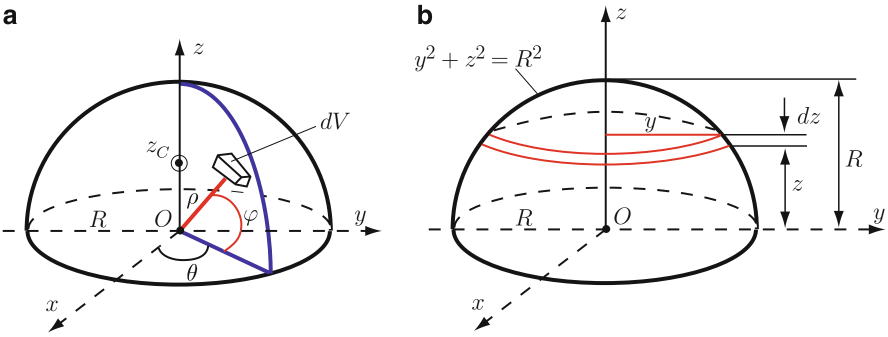



**Fig. 2.30** Example 2.7

or

$$x_C\left(a^2-r^2\right)=a^2(a-r).$$

If $x_C=r$, the previous equation gives

$$r^2+ar-a^2=0,$$

with the solutions

$$r=\frac{-a\pm a\sqrt{5}}{2}.$$

Because $r>0$, the correct solution is

$$r=\frac{a\left(\sqrt{5}-1\right)}{2}\approx 0.62\,a=0.62\,(2)=1.24\text{ m}.$$

*Example 2.7.* Locate the position of the mass center of the homogeneous volume of a hemisphere of radius $R$ with respect to its base, as shown in Fig. 2.30.

*Solution*
The reference frame is selected as shown in Fig. 2.30a, and the $z$-axis is the symmetry axis for the body: $x_C=0$ and $y_C=0$. Using the spherical coordinates, $z=\rho\sin\varphi$, and the differential volume element is $\mathrm{d}V=\rho^2\cos\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi$. The $z$ coordinate of the mass center is calculated from

$$z_C\iiint\rho^2\cos\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi=\iiint\rho^3\sin\varphi\cos\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi,$$

or

$$z_C\int_0^R\rho^2\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\cos\varphi\mathrm{d}\varphi=\int_0^R\rho^3\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\sin\varphi\cos\varphi\mathrm{d}\varphi,$$

or

$$z_C = \frac{\int_0^R \rho^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \sin\varphi \cos\varphi \mathrm{d}\varphi}{\int_0^R \rho^2 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \cos\varphi \mathrm{d}\varphi}. \tag{2.64}$$

From (2.64), after integration, it results

$$z_C = \frac{3R}{8}.$$

Another way of calculating the position of the mass center $z_C$ is shown in Fig. 2.30b. The differential volume element is

$$\mathrm{d}V = \pi y^2 \,\mathrm{d}z = \pi (R^2 - z^2) \,\mathrm{d}z,$$

and the volume of the hemisphere of radius $R$ is

$$V = \int_V \mathrm{d}V = \int_0^R \pi (R^2 - z^2) \,\mathrm{d}z = \pi \left( R^2 \int_0^R \mathrm{d}z - \int_0^R z^2 \,\mathrm{d}z \right) = \pi \left( R^3 - \frac{R^3}{3} \right) = \frac{2\pi R^3}{3}.$$

The coordinate $z_C$ is calculated from the relation

$$z_C = \frac{\int_V z \mathrm{d}V}{V} = \frac{\pi \left( R^2 \int_0^R z \,\mathrm{d}z - \int_0^R z^3 \,\mathrm{d}z \right)}{V} = \frac{\pi}{V} \left( R^2 \frac{R^2}{2} - \frac{R^4}{4} \right)$$
$$= \frac{\pi R^4}{4V} = \frac{\pi R^4}{4} \left( \frac{3}{2\pi R^3} \right) = \frac{3R}{8}.$$

```
syms rho theta phi R real

% dV = rho^2 cos(phi) drho dtheta dphi
% 0<rho<R 0<theta<2pi 0<phi<pi/2
Ir = int(rho^2, rho, 0, R);
It = int(1, theta, 0, 2*pi);
Ip = int(cos(phi), phi, 0, pi/2);
V = Ir*It*Ip;
fprintf('V = %s \n', char(V))

% dMz = rho rho^2 cos(phi) drho dtheta dphi
% 0<rho<R 0<theta<2pi 0<phi<pi/2
Mr = int(rho^3, rho, 0, R);
Mt = int(1, theta, 0, 2*pi);
Mp = int(sin(phi)*cos(phi), phi, 0, pi/2);
Mz = Mr*Mt*Mp;
fprintf('Mz = %s \n', char(Mz))
```

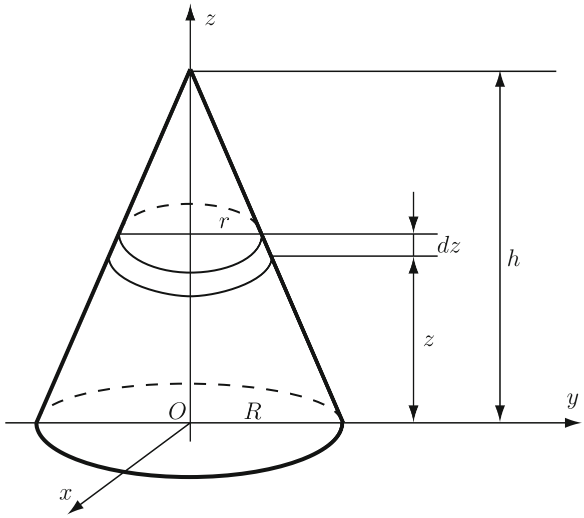



**Fig. 2.31** Example 2.8

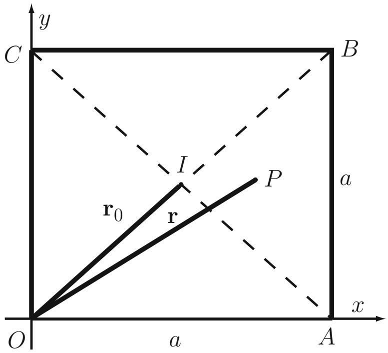
```
zC = Mz/V;
fprintf('zC = Mz/V = %s \n', char(zC))

fprintf('another method \n')
syms y z R real
% y^2 + z^2 = R^2
% dV = pi y^2 dz = pi (R^2-z^2) dz
V = int(pi*(R^2-z^2), z, 0, R);
fprintf('V = %s \n', char(V))
Mxy = int(z*pi*(R^2-z^2), z, 0, R);
fprintf('Mxy = %s \n', char(Mxy))
zC = Mxy/V;
fprintf('zC = Mxy/V = %s \n', char(zC))
```

*Example 2.8.* Find the position of the mass center for a homogeneous right circular cone, with the base radius $R$ and the height $h$, as shown in Fig. 2.31.

*Solution*
The reference frame is shown in Fig. 2.31. The $z$-axis is the symmetry axis for the right circular cone. By symmetry, $x_C = 0$ and $y_C = 0$. The volume of the thin disk differential volume element is $\mathrm{d}V = \pi r^2 \,\mathrm{d}z$. From geometry

$$\frac{r}{R} = \frac{h - z}{h},$$

**Fig. 2.32** Example 2.9




or

$$r = R\left(1 - \frac{z}{h}\right).$$

The $z$ coordinate of the centroid is calculated from

$$z_C \pi R^2 \int_0^h \left(1 - \frac{z}{h}\right)^2 \mathrm{d}z = \pi R^2 \int_0^h z\left(1 - \frac{z}{h}\right)^2 \mathrm{d}z,$$

or

$$z_C = \frac{\displaystyle\int_0^h z\left(1 - \frac{z}{h}\right)^2 \mathrm{d}z}{\displaystyle\int_0^h \left(1 - \frac{z}{h}\right)^2 \mathrm{d}z} = \frac{h}{4}.$$

The MATLAB program is

```
syms z h real
V = int((1-z/h)^2, z, 0, h);
Mxy = int(z*(1-z/h)^2, z, 0, h);
zC=Mxy/V;

fprintf('V = %s  \n',char(V))
fprintf('Mxy = %s  \n',char(Mxy))
fprintf('zC = %s  \n',char(zC))
```

*Example 2.9.* The density of a square plate with the length $a$ is given as $\rho = k\,r$, where $k =$ constant and $r$ is the distance from the origin $O$ to a current point $P(x, y)$ on the plate as shown in Fig. 2.32. Find the mass $M$ of the plate.

*Solution*
The mass of the plate is given by

$$M = \iint \rho \,\mathrm{d}x\,\mathrm{d}y.$$

The density is

$$\rho = k\sqrt{x^2+y^2}, \tag{2.65}$$

where

$$\rho_0 = k\,r_0 = k\,a\frac{\sqrt{2}}{2} \Rightarrow k = \frac{\sqrt{2}}{a}\rho_0. \tag{2.66}$$

Using (2.65) and (2.66), the density is

$$\rho = \frac{\sqrt{2}}{a}\rho_0\sqrt{x^2+y^2}. \tag{2.67}$$

Using (2.67), the mass of the plate is

$$\begin{aligned}
M &= \frac{\sqrt{2}}{a}\rho_0\int_0^a \mathrm{d}x\int_0^a \sqrt{x^2+y^2}\mathrm{d}y \\
&= \frac{\sqrt{2}}{a}\rho_0\int_0^a \left[\frac{y}{2}\sqrt{x^2+y^2}+\frac{x^2}{2}\ln\left(y+\sqrt{x^2+y^2}\right)\right]\mathrm{d}x \\
&= \frac{\sqrt{2}}{a}\rho_0\int_0^a \left[\frac{a}{2}\sqrt{x^2+a^2}+\frac{x^2}{2}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)\right]\mathrm{d}x \\
&= \frac{\sqrt{2}}{a}\rho_0\left\{\frac{a}{2}\int_0^a \sqrt{x^2+a^2}\mathrm{d}x+\frac{1}{2}\int_0^a x^2\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)\mathrm{d}x\right\} \\
&= \left\{\frac{\sqrt{2}}{a}\rho_0\frac{a}{2}\left[\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(a+\sqrt{x^2+a^2}\right)\right]\right\}_0^a \\
&\quad+\left\{\frac{1}{2}\frac{\sqrt{2}}{a}\rho_0\left[\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)+\frac{a}{3}\frac{x}{2}\sqrt{x^2+a^2}-\frac{a}{3}\frac{a^2}{2}\ln\left(a+\sqrt{x^2+a^2}\right)\right]\right\}_0^a \\
&= \frac{1}{2}\frac{\sqrt{2}}{a}\rho_0\left[\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)+\frac{a}{3}\frac{x}{2}\sqrt{x^2+a^2}-\frac{a}{3}\frac{a^2}{2}\ln\left(a+\sqrt{x^2+a^2}\right)\right] \\
&= \frac{\rho_0 a^2}{3}\left[2+\sqrt{2}\ln\left(1+\sqrt{2}\right)\right].
\end{aligned}$$

*Example 2.10.* Revolving the circular area of radius $R$ through 360° about the $x$-axis, a complete torus is generated. The distance between the center of the circle and the $x$-axis is $d$, as shown in Fig. 2.33. Find the surface area and the volume of the obtained torus.

*Solution*
Using the Guldinus–Pappus formulas

$$A = 2\pi y_C L,$$

**Fig. 2.33** Example 2.10

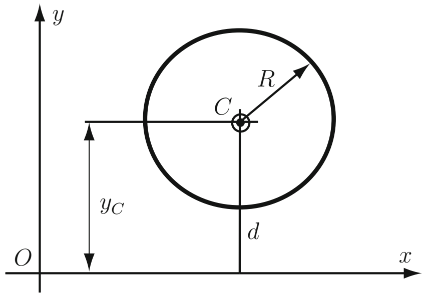
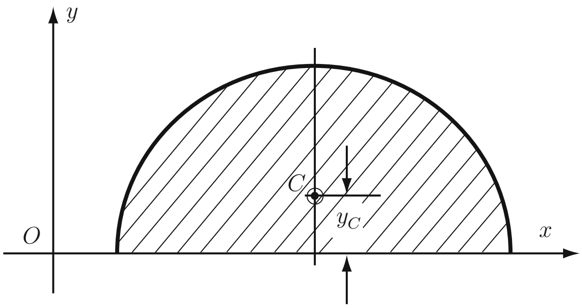



**Fig. 2.34** Example 2.11




$$V = 2\pi y_C A,$$

and with $y_C = d$ the area and the volume are

$$A = (2\pi d)(2\pi R) = 4\pi^2 R d,$$

$$V = (2\pi d)\left(\pi R^2\right) = 2\pi^2 R^2 d.$$

*Example 2.11.* Find the position of the mass center for the semicircular area shown in Fig. 2.34.

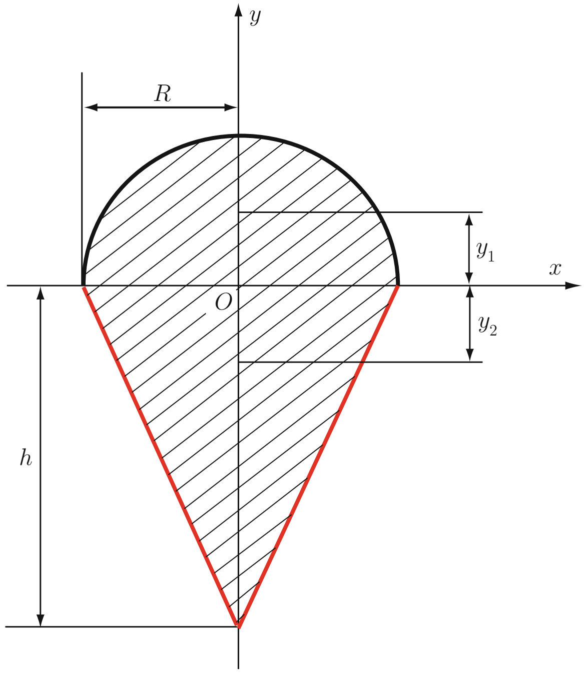
*Solution*
Rotating the semicircular area with respect to $x$-axis, a sphere is obtained. The volume of the sphere is given by

$$V = \frac{4\pi R^3}{3}.$$

The area of the semicircular area is

$$A = \frac{\pi R^2}{2}.$$

**Fig. 2.35** Example 2.12




Using the second Guldinus–Pappus theorem, the position of the mass center is

$$y_C = \frac{V}{2\pi A} = \frac{\dfrac{4\pi R^3}{3}}{2\pi\dfrac{\pi R^2}{2}} = \frac{\dfrac{4\pi R^3}{3}}{\pi^2 R^2} = \frac{4R}{3\pi}.$$

*Example 2.12.* Find the first moment of the area with respect to the $x$-axis for the surface given in Fig. 2.35.

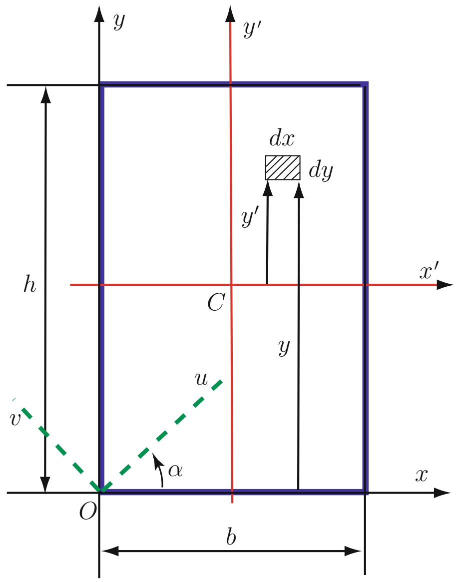
*Solution*
The composite area is considered to be composed of the semicircular area and the triangle. The mass center of the semicircular surface is given by

$$y_1 = \frac{2}{3}\frac{R\sin\dfrac{\pi}{2}}{\dfrac{\pi}{2}} = \frac{4R}{3\pi}.$$

The mass center of the triangle is given by

$$y_2 = -\frac{b}{3}.$$

**Fig. 2.36** Example 2.13




The first moment of area with respect to $x$-axis for the total surface is given by

$$\mathrm{M}_x = \sum_{i=1}^{2} y_i A_i = \frac{4R}{3\pi}\frac{\pi R^2}{2} - \frac{h}{3}\frac{2Rh}{2},$$

or

$$M_x = \frac{R}{3}\left(2R^2 - h^2\right).$$

*Example 2.13.* A rectangular planar plate with the sides $b = 1$ $m$ and $h = 2$ $m$ is shown in Fig. 2.36.

(a) Find the product of inertia and the moments of inertia with respect to the axes of the reference frame $xy$ with the origin at $O$.
(b) Determine the product of inertia and the moments of inertia with respect to the centroidal axes that are located at the mass center $C$ of the rectangle and are parallel to its sides.
(c) Another reference $uv$ with the same origin $O$ is rotated with an angle $\alpha = 45°$ from $xy$ (counterclockwise as positive). Find the inertia matrix of the plate with respect to $uv$ axes.
(d) Find the principal moments and the principal directions with the reference frame $xy$ with the origin at $O$.

*Solution*

(a) The differential element of area is $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$. The product of inertia of the rectangle about the $xy$ axes is

$$I_{xy} = \int_A xy\,\mathrm{d}A = \int_0^h \int_0^b xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^b x\,\mathrm{d}x \int_0^h y\,\mathrm{d}y = \frac{b^2}{2}\frac{h^2}{2} = \frac{b^2h^2}{4} = \frac{1^2\,(2^2)}{4} = 1\ \mathrm{m}^4.$$

The moment of inertia of the rectangle about $x$-axis is

$$I_{xx} = \int_A y^2\,\mathrm{d}A = \int_0^h \int_0^b y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^b \mathrm{d}x \int_0^h y^2\,\mathrm{d}y = b\frac{h^3}{3} = \frac{bh^3}{3} = \frac{(1)\,2^3}{3} = 2.666\ \mathrm{m}^4.$$

The moment of inertia of the rectangle about $y$-axis is

$$I_{yy} = \int_A x^2\,\mathrm{d}A = \int_0^h \int_0^b x^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^b x^2\,\mathrm{d}x \int_0^h \mathrm{d}y = \frac{b^3}{3}h = \frac{hb^3}{3} = \frac{(2)\,1^3}{3} = 0.666\ \mathrm{m}^4.$$

The moment of inertia of the rectangle about $z$-axis (the polar moment about $O$) is

$$I_O = I_{zz} = I_{xx} + I_{yy} = \frac{A}{3}\left(b^2 + h^2\right) = \frac{1\,(2)}{3}\left(1^2 + 2^2\right) = 3.33\ \mathrm{m}^4.$$

The inertia matrix of the plane figure with respect to $xy$ axes is represented by

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \dfrac{bh^3}{3} & \dfrac{b^2h^2}{4} & 0 \\ \dfrac{b^2h^2}{4} & \dfrac{hb^3}{3} & 0 \\ 0 & 0 & \dfrac{A}{3}(b^2+h^2) \end{bmatrix}$$

$$= \begin{bmatrix} 2.666 & 1 & 0 \\ 1 & 0.666 & 0 \\ 0 & 0 & 3.33 \end{bmatrix}.$$

(b) The product of inertia of the rectangle about the $x'y'$ axes is

$$I_{x'y'} = \int_A xy\,\mathrm{d}A = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} xy\,\mathrm{d}x\,\mathrm{d}y = \int_{-h/2}^{h/2} x\,\mathrm{d}x \int_{-b/2}^{b/2} y\,\mathrm{d}y = 0.$$

The same results is obtained using the parallel-axis theorem

$$I_{xy} = I_{x'y'} + \frac{b}{2}\frac{h}{2}A,$$

or

$$I_{x'y'} = I_{xy} - \frac{bh}{4}(bh) = \frac{b^2h^2}{4} - \frac{b^2h^2}{4} = 0.$$

The moment of inertia of the rectangle about $x'$ axis is

$$I_{x'x'} = \int_A y^2\,\mathrm{d}A = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y^2\,\mathrm{d}x\,\mathrm{d}y = \int_{-h/2}^{h/2}\mathrm{d}x\int_{-b/2}^{b/2} y^2\,\mathrm{d}y$$
$$= \{x\}_{-h/2}^{h/2}\left\{\frac{y^3}{3}\right\}_{-b/2}^{b/2} = \frac{b\,h^3}{12} = \frac{(1)\,2^3}{12} = 0.666\ \mathrm{m}^4.$$

Using the parallel-axis theorem, the moment of inertia of the rectangle about $y'$ axis is

$$I_{y'y'} = I_{yy} - \left(\frac{b}{2}\right)^2 A = \frac{h\,b^3}{3} - \frac{h\,b^3}{4} = \frac{h\,b^3}{12} = \frac{(2)\,1^3}{12} = 0.166\ \mathrm{m}^4.$$

The moment of inertia of the rectangle about $z'$ axis (the centroid polar moment) is

$$I_C = I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{A}{12}(b^2+h^2) = \frac{1(2)}{12}(1^2+2^2) = 0.833\ \mathrm{m}^4.$$

The inertia matrix of the plane figure with respect to centroidal axes $x'y'$ is represented by

$$[I_C] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} \frac{b\,h^3}{12} & 0 & 0 \\ 0 & \frac{h\,b^3}{12} & 0 \\ 0 & 0 & \frac{A}{12}(b^2+h^2) \end{bmatrix}$$
$$= \begin{bmatrix} 0.666 & 0 & 0 \\ 0 & 1.666 & 0 \\ 0 & 0 & 0.833 \end{bmatrix}.$$

(c) The moment of inertia of the rectangle about $u$-axis is

$$I_{uu} = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha - I_{xy}\sin 2\alpha$$
$$= \frac{2.666+0.666}{2} + \frac{2.666-0.666}{2}\cos 2(45^\circ) - (1)\sin 2(45^\circ) = 0.666\ \mathrm{m}^4.$$

The moment of inertia of the rectangle about $v$-axis is

$$I_{vv} = \frac{I_{xx}+I_{yy}}{2} - \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha + I_{xy}\sin 2\alpha$$
$$= \frac{2.666+0.666}{2} - \frac{2.666-0.666}{2}\cos 2(45^\circ) + (1)\sin 2(45^\circ) = 2.666\ \mathrm{m}^4.$$

The product of inertia of the rectangle about *uv* axes is

$$I_{uv} = \frac{I_{xx} - I_{yy}}{2}\sin 2\alpha + I_{xy}\cos 2\alpha$$
$$= \frac{2.666 - 0.666}{2}\sin 2(45^\circ) + (1)\cos 2(45^\circ) = 1 \text{ m}^4.$$

The polar moment of inertia of the rectangle about $O$ is

$$I_O = I_{zz} = I_{uu} + I_{vv} = I_{xx} + I_{yy} = 0.666 + 2.666 = 3.33 \text{ m}^4.$$

The inertia matrix of the plane figure with respect to *uv* axes is

$$[I_\alpha] = \begin{bmatrix} I_{uu} & -I_{uv} & 0 \\ -I_{vu} & I_{vv} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.666 & 1 & 0 \\ 1 & 2.666 & 0 \\ 0 & 0 & 3.33 \end{bmatrix}.$$

(d) The maximum or minimum moment of inertia for the area is

$$I_{1,2} = I_{\max,\min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2},$$

$$I_1 = I_{\max} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$
$$= \frac{2.666 + 0.666}{2} + \sqrt{\left(\frac{2.666 - 0.666}{2}\right)^2 + 1^2} = 3.080 \text{ m}^4,$$

$$I_2 = I_{\min} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$
$$= \frac{2.666 + 0.666}{2} - \sqrt{\left(\frac{2.666 - 0.666}{2}\right)^2 + 1^2} = 0.252 \text{ m}^4.$$

The polar moment of inertia of the rectangle about $O$ is

$$I_O = I_{zz} = I_1 + I_2 = I_{uu} + I_{vv} = I_{xx} + I_{yy} = 3.0806 + 0.252 = 3.33 \text{ m}^4.$$

The principal directions are obtained from

$$\tan 2\alpha_0 = \frac{2I_{xy}}{I_{yy} - I_{xx}},$$

or

$$\alpha_0 = \frac{1}{2}\tan^{-1}\frac{2I_{xy}}{I_{yy}-I_{xx}} = \frac{1}{2}\tan^{-1}\frac{2(1)}{0.666-2.666} = -22.5^\circ.$$

The principal directions are

$$\alpha_1 = -22.5^\circ \text{ and } \alpha_2 = \alpha_1 + \pi/2 = 67.5^\circ.$$

The MATLAB program for this example is

```
syms b h x y real

list={b, h};
listn={1, 2}; % m

% dA = dx dy
% A = int dx dy ; 0<x<b 0<y<h
% Ax = int dx ; 0<x<b
% Ay = int dy ; 0<y<h

Ax=int(1,x,0,b);
Ay=int(1,y,0,h);
A=Ax*Ay;

% Ixy = int x y dx dy ; 0<x<b 0<y<h
% Ixy1 = int x dx ; 0<x<b
% Ixy2 = int y dy ; 0<y<h
Ixy1=int(x, x, 0, b);
Ixy2=int(y, y, 0, h);
Ixy=Ixy1*Ixy2;
Ixyn=subs(Ixy, list, listn);
fprintf('Ixy = %s = %g (m^4) \n',char(Ixy), Ixyn);

% Ixx = int y^2 dx dy ; 0<x<b 0<y<h
% Ixx1 = int dx ; 0<x<b
% Ixx2 = int y^2 dy ; 0<y<h
Ixx1=int(1, x, 0, b);
Ixx2=int(y^2, y, 0, h);
Ixx=Ixx1*Ixx2;
Ixxn=subs(Ixx, list, listn);
fprintf('Ixx = %s = %g (m^4) \n',char(Ixx), Ixxn);

% Iyy = int x^2 dx dy ; 0<x<b 0<y<h
% Iyy1 = int dx ; 0<x<b
% Ixy2 = int y^2 dy ; 0<y<h
```

```
Iyy1=int(x^2, x, 0, b);
Iyy2=int(1, y, 0, h);
Iyy=Iyy1*Iyy2;
Iyyn=subs(Iyy, list, listn);
fprintf('Iyy = %s = %g (m^4) \n',char(Iyy), Iyyn);

IO=Ixx+Iyy;
IOn=Ixxn+Iyyn;
fprintf('IO = %s = %g (m^4) \n',char(IO), IOn);

% Ixxp = int y^2 dx  dy ; -b/2<x<b/2 ; -h/2<y<h/2
% Ixx1p = int dx ; -b/2<x<b/2
% Ixx2p = int y^2 dy ; -h/2<y<h/2
Ixx1p=int(1, x, -b/2, b/2);
Ixx2p=int(y^2, y, -h/2, h/2);
Ixxp=Ixx1p*Ixx2p;
Ixxpn=subs(Ixxp, list, listn);
fprintf('Ixxp = %s = %g (m^4) \n',char(Ixxp),
  Ixxpn);

Iyyp=Iyy-(b*h)*(b/2)^2;
Iyypn=subs(Iyyp, list, listn);
fprintf('Iyyp = %s = %g (m^4) \n',char(Iyyp),
  Iyypn);

Ixyp=Ixy-(b*h)*(-b/2)*(-h/2);
Ixypn=subs(Ixyp, list, listn);
fprintf('Ixyp = %s = %g (m^4) \n',char(Ixyp),
  Ixypn);

IC=Ixxp+Iyyp;
ICn=Ixxpn+Iyypn;
fprintf('IC = %s = %g (m^4) \n',char(IC), ICn);

fprintf('\n');
alpha=pi/4;
fprintf('alpha = %g (degree) \n', alpha*180/pi);
fprintf('\n');

Ixa=...
(Ixx+Iyy)/2+(Ixx-Iyy)/2*cos(2*alpha)
  -Ixy*sin(2*alpha);
Ixan=subs(Ixa, list, listn);
fprintf('Iuu45 = %g (m^4) \n', Ixan);
```

```
Iya=...
(Ixx+Iyy)/2-(Ixx-Iyy)/2*cos(2*alpha)
  +Ixy*sin(2*alpha);
Iyan=subs(Iya, list, listn);
fprintf('Ivv45 = %g (m^4) \n', Iyan);

Ixya=(Ixx-Iyy)/2*sin(2*alpha)+Ixy*cos(2*alpha);
Ixyan=subs(Ixya, list, listn);
fprintf('Iuv45 = %g (m^4) \n', Ixyan);

IOa=Ixan+Iyan;
fprintf('IO45 = Iuu45+Ivv45 = %g (m^4) \n', IOa);

fprintf('\n');
I1=(Ixx+Iyy)/2+sqrt((Ixx-Iyy)^2/4+Ixy^2);
I1n=subs(I1, list, listn);
I2=(Ixx+Iyy)/2-sqrt((Ixx-Iyy)^2/4+Ixy^2);
I1n=subs(I1, list, listn);
I2n=subs(I2, list, listn);
fprintf('I1 = Imax = %g (m^4) \n', I1n);
fprintf('I2 = Imin = %g (m^4) \n', I2n);
fprintf('I1+I2 = %g (m^4) \n', I1n+I2n);

fprintf('\n');
tanalpha0=simplify(2*Ixy/(Iyy-Ixx));
fprintf('tan (2 alpha0) = %s \n',char(tanalpha0));
alpha0n=atan(2*Ixyn/(Iyyn-Ixxn));
fprintf('alpha0 = %g (degrees) \n',
  alpha0n/2*180/pi);
```

*Example 2.14.* Determine the moment of inertia for the slender rod, shown in Fig. 2.37, with respect to axes of reference with the origin at the end $O$ and with respect to centroidal axes. The length of the rod is $l$, the density is $\rho$, and the cross-sectional area ia $A$. Express the results in terms of the total mass, $m$, of the rod.

*Solution*
The mass of the rod is $m = \rho\, l A$, and the density will be $\rho = m/(lA)$. The differential element of mass is $\mathrm{d}m = \rho A\,\mathrm{d}x$. The moment of inertia of the slender rod about the $y$- or $z$-axes is

$$I_{yy} = I_{zz} = \int_0^l x^2 \rho A\,\mathrm{d}x = \int_0^l \frac{m}{lA} A x^2\,\mathrm{d}x = \frac{m}{l}\int_0^l x^2\,\mathrm{d}x = \frac{ml^2}{3}.$$

The $x$-axis is a symmetry axis and that is why $I_{xx} = 0$. The moment of inertia of the slender rod about the centroidal axes $y'$ or $z'$ is calculated with the parallel-axis theorem

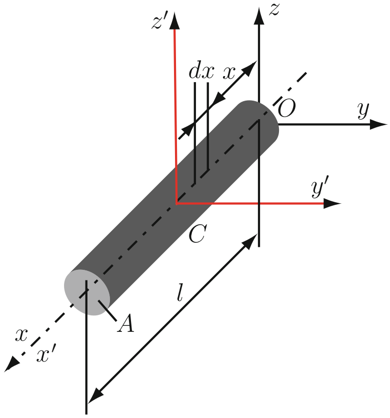



**Fig. 2.37** Example 2.14

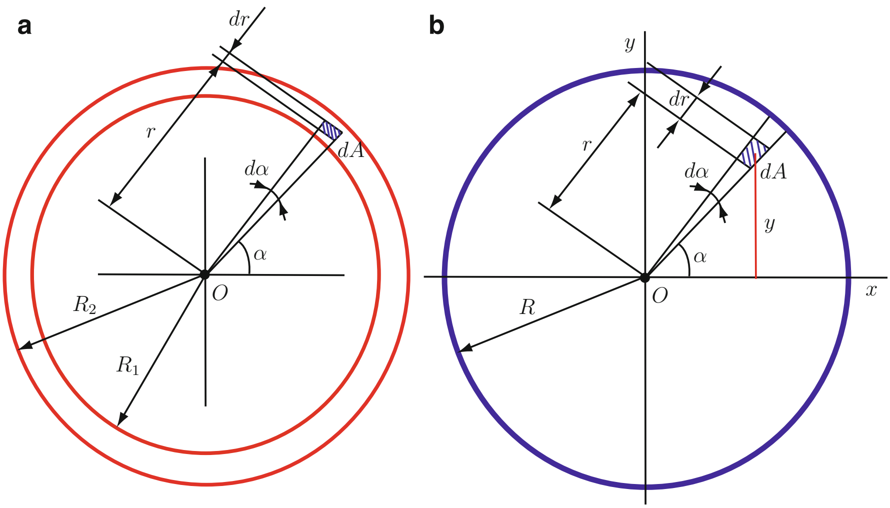



**Fig. 2.38** Example 2.15

$$I_{y'y'} = I_{z'z'} = I_{yy} - \left(\frac{l}{2}\right)^2 m = \frac{ml^2}{3} - \frac{ml^2}{4} = \frac{ml^2}{12}.$$

*Example 2.15.* Find the polar moment of inertia of the planar flywheel shown in Fig. 2.38a. The radii of the wheel are $R_1$ and $R_2$ ($R_1 < R_2$). Calculate the moments of inertia of the area of a circle with radius $R$ about a diametral axis and about the polar axis through the center as shown in Fig. 2.38b.

*Solution*
The polar moment of inertia is given by the equation

$$I_O = \int_A r^2 \, \mathrm{d}A,$$

where $r$ is the distance from the pole $O$ to an arbitrary point on the wheel, and the differential element of area is

$$\mathrm{d}A = r \, \mathrm{d}\alpha \, \mathrm{d}r.$$

The polar moment of inertia is

$$I_O = \int_{R_1}^{R_2} \int_0^{2\pi} r^3 \, \mathrm{d}r \, \mathrm{d}\alpha = \int_{R_1}^{R_2} r^3 \, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\alpha = \frac{R_2^4 - R_1^4}{4}(2\pi) = \frac{R_2^4 - R_1^4}{2}\pi.$$

The area of the wheel is $A = \pi(R_2^2 - R_1^2)$ and

$$I_O = A\frac{R_2^2 + R_1^2}{2}. \tag{2.68}$$

If $R_1 = 0$ and $R_2 = R$, the polar moment of inertia of the circular area of radius $R$ is, Fig. 2.38b,

$$I_O = \frac{A\,R^2}{2} = \frac{\pi\,R^4}{2}.$$

By symmetry for the circular area, shown in Fig. 2.38b, the moment of inertia about a diametral axis is $I_{xx} = I_{yy}$ and $I_O = I_{xx} + I_{yy} \Longrightarrow I_{xx} = I_{yy} = I_O/2 = \pi\,R^4/4$. The results can be obtained using the integration

$$I_{xx} = \int_A y^2 \, \mathrm{d}A = \int_0^R \int_0^{2\pi} (r \sin\alpha)^2 \, r \, \mathrm{d}r \, \mathrm{d}\alpha = \int_0^{2\pi} \frac{R^4}{4} (\sin\alpha)^2 \, \mathrm{d}\alpha$$
$$= \frac{R^4}{4}\frac{1}{2}\left\{\alpha - \frac{\sin 2\alpha}{2}\right\}_0^{2\pi} = \frac{\pi R^4}{4}.$$

*Example 2.16.* Find the moments of inertia and products of inertia for the area shown in Fig. 2.39a, with respect to the $xy$ axes and with respect to the centroidal $x'y'$ axes that pass through the mass center $C$. Find the principal moments of inertia for the area and the principal directions.

*Solution*
The plate is composed of two element area: the rectangular area 1 and the rectangular area 2, Fig. 2.39b. The $x$ and $y$ coordinates of the mass center $C$ are

$$x_C = \frac{x_{C_1} A_1 + x_{C_1} A_1}{A_1 + A_2} = \frac{(a/2)(2\,a^2) + (2\,a)(2\,a^2)}{2\,a^2 + 2\,a^2} = \frac{5\,a}{4}$$

$$y_C = \frac{y_{C_1} A_1 + y_{C_1} A_1}{A_1 + A_2} = \frac{(a)(2\,a^2) + (a/2)(2\,a^2)}{2\,a^2 + 2\,a^2} = \frac{3\,a}{4}.$$

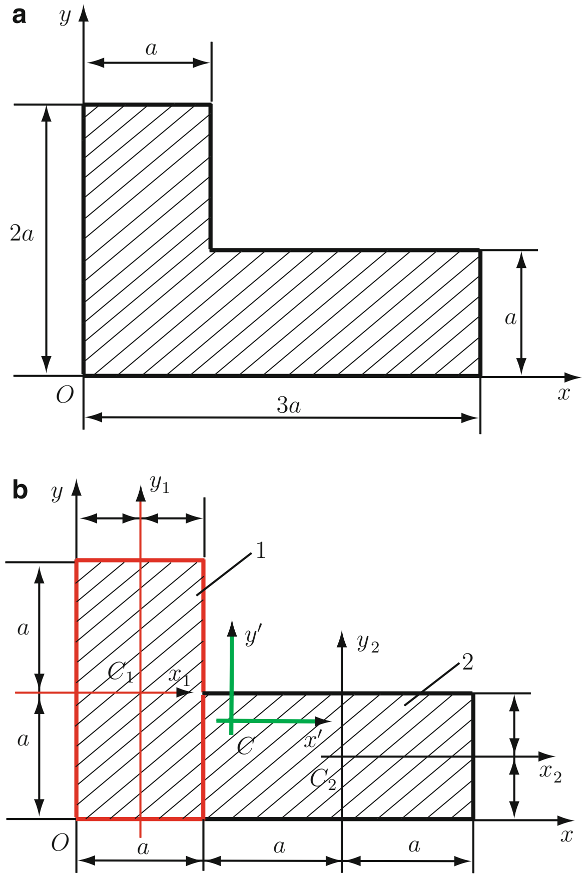



**Fig. 2.39** Example 2.16

The product of inertia for the area shown in Fig. 2.39a about $xy$ axes is given by

$$
\begin{aligned}
I_{xy} &= \int_0^a \int_0^{2a} xy\,\mathrm{d}x\mathrm{d}y + \int_a^{3a} \int_0^a xy\,\mathrm{d}x\mathrm{d}y = \int_0^a x\,\mathrm{d}x \int_0^{2a} y\,\mathrm{d}y + \int_a^{3a} x\,\mathrm{d}x \int_0^a y\,\mathrm{d}y \\
&= \frac{a^2\left(4a^2\right)}{4} + \frac{\left(9a^2 - a^2\right)\left(a^2\right)}{4} = 3a^4.
\end{aligned}
$$

The same result is obtained if parallel-axis theorem is used:

$$
\begin{aligned}
I_{xy} &= I_{C_1x_1y_1} + x_{C_1}y_{C_1}A_1 + I_{C_2x_2y_2} + x_{C_2}y_{C_2}A_2 \\
&= 0 + \left(\frac{a}{2}\right)(a)(2a^2) + 0 + (2a)\left(\frac{a}{2}\right)(2a^2) = 3a^4.
\end{aligned}
$$

The product of inertia for the area about $xz$ and $yz$ axes are $I_{xz} = I_{yz} = 0$.

The moment of inertia of the figure with respect to $x$-axis is

$$I_{xx} = I_{C_1x_1x_1} + (y_{C_1})^2 A_1 + I_{C_2x_2x_2} + (y_{C_2})^2 A_2$$
$$= \frac{a(2a)^3}{12} + a^2(2a^2) + \frac{2a(a)^3}{12} + \left(\frac{a}{2}\right)^2(2a^2) = \frac{10a^4}{3}.$$

The moment of inertia of the figure with respect to $y$-axis is

$$I_{yy} = I_{C_1y_1y_1} + (x_{C_1})^2 A_1 + I_{C_2y_2y_2} + (x_{C_2})^2 A_2$$
$$= \frac{(2a)a^3}{12} + \left(\frac{a}{2}\right)^2(2a^2) + \frac{a(2a)^3}{12} + (2a)^2(2a^2) = \frac{28a^4}{3}.$$

The moment of inertia of the area with respect to $z$-axis is

$$I_{zz} = I_{xx} + I_{yy} = \frac{10a^4}{3} + \frac{28a^4}{3} = \frac{38a^4}{3}.$$

The inertia matrix of the plane figure is represented by the matrix

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{10a^4}{3} & -3a^4 & 0 \\ -3a^4 & \frac{28a^4}{3} & 0 \\ 0 & 0 & \frac{38a^4}{3} \end{bmatrix}.$$

Using the parallel-axis theorem, the moments of inertia of the area with respect to the centroidal axes $x'y'z'$ are

$$I_{x'x'} = I_{xx} - (x_C)^2 A = \frac{10a^4}{3} - \left(\frac{3a}{4}\right)^2(4a^2) = \frac{13a^4}{12},$$
$$I_{y'y'} = I_{yy} - (y_C)^2 A = \frac{28a^4}{3} - \left(\frac{5a}{4}\right)^2(4a^2) = \frac{37a^4}{12},$$
$$I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{13a^4}{12} + \frac{37a^4}{12} = \frac{25a^4}{6},$$
$$I_{x'y'} = I_{xy} - (x_C)(y_C)A = 3a^4 - \left(\frac{3a}{4}\right)\left(\frac{5a}{4}\right)(4a^2) = -\frac{3a^4}{4},$$
$$I_{x'z'} = I_{y'z'} = 0.$$

The centroidal inertia matrix of the plane figure is

$$[I'] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} \dfrac{13a^4}{12} & \dfrac{3a^4}{4} & 0 \\ \dfrac{3a^4}{4} & \dfrac{37a^4}{12} & 0 \\ 0 & 0 & \dfrac{25a^4}{6} \end{bmatrix}.$$

The principal moments of inertia for the area are

$$I_{1,2} = I_{\max,\min} = \frac{I_{x'x'} + I_{y'y'}}{2} \pm \sqrt{\left(\frac{I_{x'x'} - I_{y'y'}}{2}\right)^2 + I_{x'y'}^2}$$
$$= \frac{25a^4}{12} \pm \frac{a^4}{2}\sqrt{4+9/4} = \frac{25a^4}{12} \pm \frac{5a^4}{4}.$$
$$I_1 = \frac{10a^4}{3} \quad \text{and} \quad I_2 = \frac{5a^4}{6}.$$

The invariant of the system is $I_{x'x'} + I_{y'y'} = I_1 + I_2$. The principal directions are obtained from

$$\tan 2\alpha = \frac{2I_{x'y'}}{I_{y'y'} - I_{x'x'}} = \frac{-2\dfrac{3a^4}{4}}{\dfrac{24a^4}{12}} = -\frac{3}{4}$$
$$\Longrightarrow \alpha_1 = \frac{1}{2}\tan^{-1}(-3/4) = -36.869^\circ \text{ and } \alpha_2 = \frac{\pi}{2} + \frac{1}{2}\tan^{-1}(-3/4) = 53.131^\circ.$$

*Example 2.17.* Find the inertia matrix of the area delimited by the curve $y^2 = 2\,px$, from $x = 0$ to $x = a$ as shown in Fig 2.40a, about the axes of the Cartesian frame with the origin at $O$. Calculate the centroidal inertia matrix.

*Solution*
From Fig. 2.40a, when $x = a$, the value of $y$-coordinate is $y = b$ and $b^2 = 2\,pa \Longrightarrow 2\,p = b^2/a$. The expression of the function is

$$y^2 = 2\,px = \frac{b^2}{a}x.$$

The differential element of area is $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, and the area of the figure is

$$A = \int_A \mathrm{d}x\,\mathrm{d}y = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} \mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} \mathrm{d}y$$
$$= \int_0^a \mathrm{d}x\,\{y\}_{-\sqrt{2px}}^{\sqrt{2px}} = \int_0^a 2\sqrt{2px}\,\mathrm{d}x = 2\sqrt{2p}\int_0^a x^{1/2}\,\mathrm{d}x = \frac{2b}{\sqrt{a}}\left\{\frac{x^{3/2}}{3/2}\right\}_0^a = \frac{4\,ab}{3}.$$

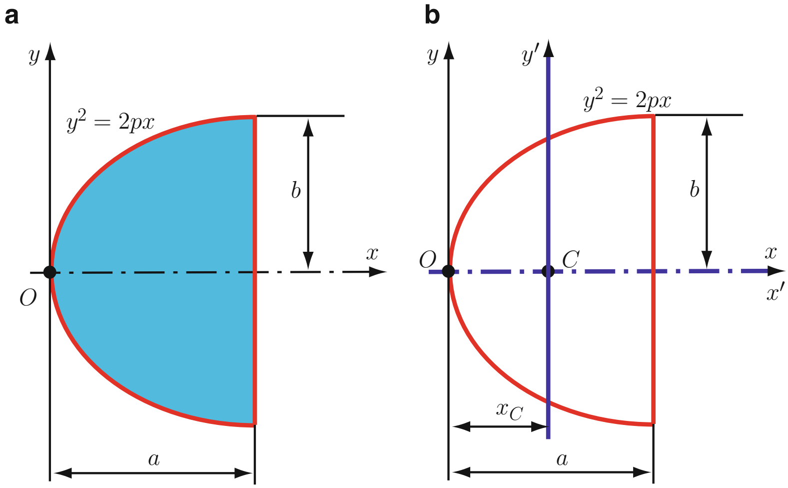



**Fig. 2.40** Example 2.17

The moment of inertia of the area with respect to *x*-axis is

$$I_{xx} = \int_A y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} y^2\mathrm{d}y$$

$$= \int_0^a \mathrm{d}x \left\{\frac{y^3}{3}\right\}_{-\sqrt{2px}}^{\sqrt{2px}} = \frac{2}{3}\int_0^a (2px)^{3/2}\,\mathrm{d}x$$

$$= \frac{2}{3}(2p)^{3/2}\int_0^a x^{3/2}\,\mathrm{d}x = \frac{2}{3}(2p)^{3/2}\left\{\frac{x^{5/2}}{5/2}\right\}_0^a = \frac{4\,a\,b^3}{15} = \frac{4\,a\,b}{3}\frac{b^2}{5},$$

or

$$I_{xx} = \frac{b^2 A}{5}.$$

The moment of inertia of the area with respect to *y*-axis is

$$I_{yy} = \int_A x^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} x^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^a x^2\,\mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} \mathrm{d}y$$

$$= \int_0^a x^2\,\mathrm{d}x\,\{y\}_{-\sqrt{2px}}^{\sqrt{2px}} = 2\int_0^a x^2\,(2px)^{1/2}\,\mathrm{d}x$$

$$= \frac{2b}{\sqrt{a}}\int_0^a x^{5/2}\,\mathrm{d}x = \frac{2b}{\sqrt{a}}\left\{\frac{x^{7/2}}{7/2}\right\}_0^a = \frac{4\,a^3\,b}{7} = \frac{4\,a\,b}{3}\frac{3a^2}{7},$$

or

$$I_{yy} = \frac{3a^2 A}{7}.$$

The moment of inertia of the area with respect to $z$-axis is

$$I_{zz} = I_{xx} + I_{yy} = \frac{b^2 A}{5} + \frac{3a^2 A}{7} = A\left(\frac{b^2}{5} + \frac{3a^2}{7}\right).$$

The product of inertia of the area with respect to $xy$ axes is

$$\begin{aligned} I_{xy} &= \int_A x y \,\mathrm{d}x \,\mathrm{d}y = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} x y \,\mathrm{d}x \,\mathrm{d}y = \int_0^a x \,\mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} y \,\mathrm{d}y \\ &= \int_0^a x \,\mathrm{d}x \left\{\frac{y^2}{2}\right\}_{-\sqrt{2px}}^{\sqrt{2px}} = 0. \end{aligned}$$

The products of inertia of the area with respect to $xz$ and $yz$ axes are $I_{xz} = I_{yz} = 0$. The inertia matrix of the plane figure is

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \dfrac{b^2 A}{5} & 0 & 0 \\ 0 & \dfrac{3a^2 A}{7} & 0 \\ 0 & 0 & A\left(\dfrac{b^2}{5} + \dfrac{3a^2}{7}\right) \end{bmatrix}.$$

The first moment of the area $A$ with respect to $y$-axis is

$$\begin{aligned} M_y &= \int_A x \,\mathrm{d}x \,\mathrm{d}y = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} x \,\mathrm{d}x \,\mathrm{d}y = \int_0^a x \,\mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} \mathrm{d}y \\ &= \int_0^a x \,\mathrm{d}x \,\{y\}_{-\sqrt{2px}}^{\sqrt{2px}} = 2\int_0^a x\sqrt{2px}\,\mathrm{d}x \\ &= \frac{2b}{\sqrt{a}} \int_0^a x^{3/2} \,\mathrm{d}x = \frac{2b}{\sqrt{a}} \left\{\frac{x^{5/2}}{5/2}\right\}_0^a = \frac{2b}{\sqrt{a}} \frac{a^{5/2}}{5/2} = \frac{4ba^2}{5}. \end{aligned}$$

The $x$-coordinate of the mass center, Fig 2.40b, is

$$x_C = \frac{M_y}{A} = \frac{4ba^2}{5} \frac{3}{4ab} = \frac{3a}{5}.$$

The first moment of the area $A$ with respect to $x$-axis is $M_x = 0$ and $y_C = M_x/A = 0$.

Using the parallel-axis theorem, the moments of inertia of the area with respect to the centroidal axes $x'y'z'$, Fig 2.40b, are

$$
\begin{aligned}
I_{x'x'} &= I_{xx} - d^2 A = I_{xx} = \frac{b^2 A}{5},\\
I_{y'y'} &= I_{yy} - (x_C)^2 A = I_{yy} - \left(\frac{3a}{5}\right)^2 A = \frac{3a^2 A}{7} - \frac{9a^2 A}{25} = \frac{12a^2 A}{175}\\
I_{z'z'} &= I_{x'x'} + I_{y'y'} = \frac{b^2 A}{5} + \frac{12a^2 A}{175} = A\left(\frac{b^2}{5} + \frac{a^2}{175}\right),\\
I_{x'y'} &= I_{xy} - (x_C)(0)A = 0,\\
I_{x'z'} &= I_{y'z'} = 0.
\end{aligned}
$$

The centroidal inertia matrix of the plane figure is

$$
[I'] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} \frac{b^2 A}{5} & 0 & 0 \\ 0 & \frac{12a^2 A}{175} & 0 \\ 0 & 0 & A\left(\frac{b^2}{5} + \frac{a^2}{175}\right) \end{bmatrix}.
$$

## 2.4 Problems

2.1 Find the $x$-coordinate of the centroid of the indicated region where $A = 2$ m and $k = \pi/8$ m$^{-1}$ (Fig. 2.41).

2.2 Find the $x$-coordinate of the centroid of the shaded region shown in the figure. The region is bounded by the curves $y = x^2$ and $y = \sqrt{x}$. All coordinates may be treated as dimensionless (Fig. 2.42).

2.3 The region shown is bounded by the curves $y = b$ and $y = k\,|x|^3$, where $b = k\,a^3$. Find the coordinates of the centroid (Fig. 2.43).

2.4 Determine the centroid of the area where $A(a, a)$. Use integration (Fig. 2.44).

2.5 Locate the center of gravity of the volume where $x^2 = a^2 y/b$, $a = b = 1$ m. The material is homogeneous (Fig. 2.45).

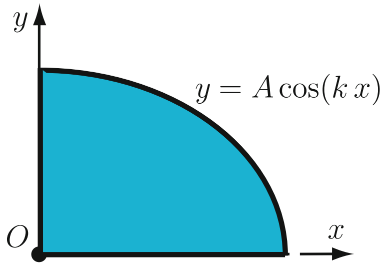



**Fig. 2.41** Problem 2.1

**Fig. 2.42** Problem 2.2

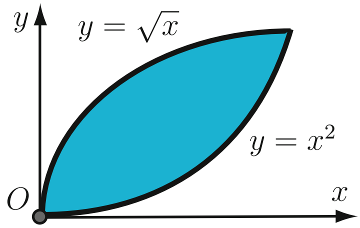



**Fig. 2.43** Problem 2.3

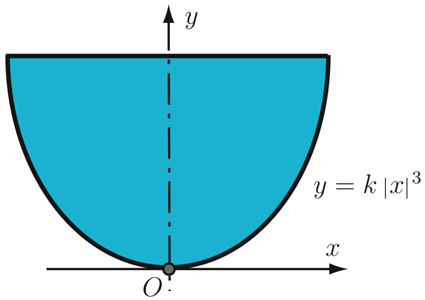



**Fig. 2.44** Problem 2.4

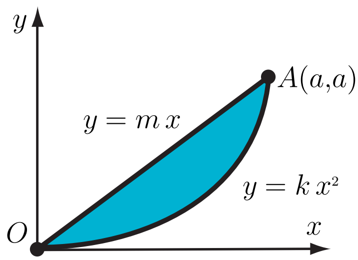



**Fig. 2.45** Problem 2.5

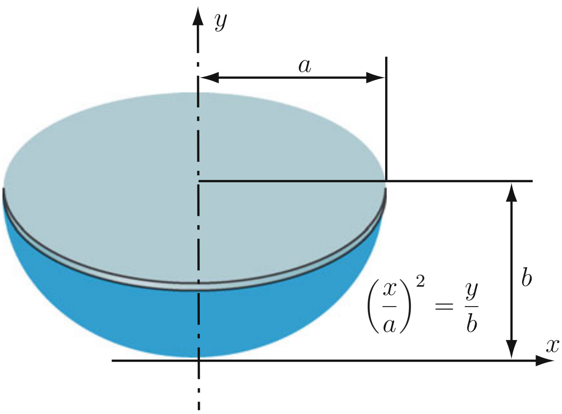


**Fig. 2.46** Problem 2.6

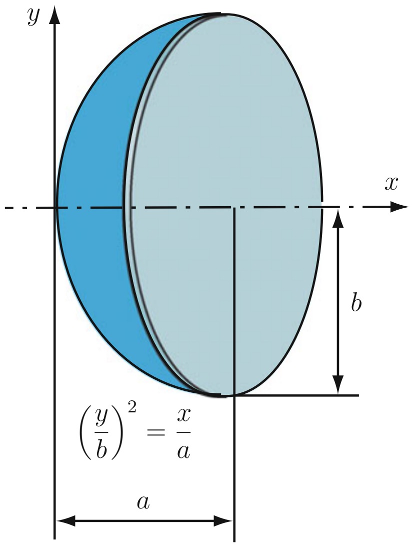
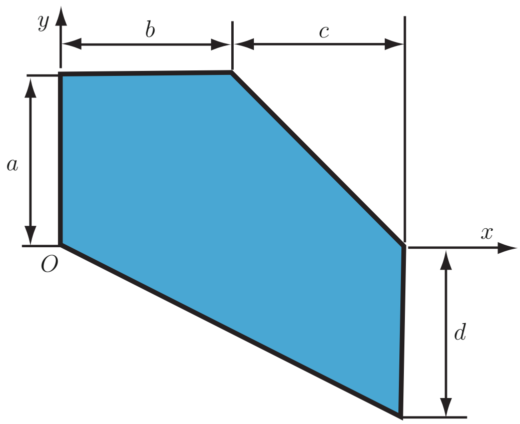



**Fig. 2.47** Problem 2.7




2.6 Locate the centroid of the paraboloid, shown in Fig. 2.46 defined by the equation $x/a = (y/b)^2$ where $a = b = 5$ m. The material is homogeneous.

2.7 Determine the location of the centroid $C$ of the area where $a = 6$ m, $b = 6$ m, $c = 7$ m, and $d = 7$ m, as shown in Fig. 2.47.

2.8 The solid object, shown in Fig. 2.48 it consists of a solid half-circular base with an extruded ring. The bottom of the object is half-circular in shape with a thickness, $t$, equal to 10 mm. The half-circle has a radius, $R$, equal to 30 mm. The coordinate axes are aligned so that the origin is at the bottom of the object at the center point of the half-circle. This half-ring has an inner radius, $r$, of 20 mm, and an outer radius, $R$, of 30 mm. The extrusion height for the

**Fig. 2.48** Problem 2.8

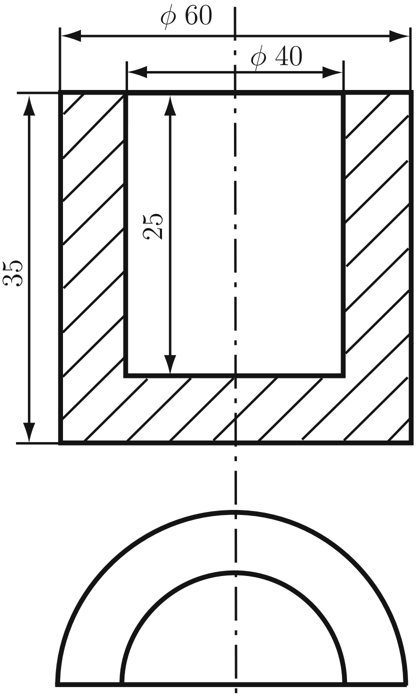
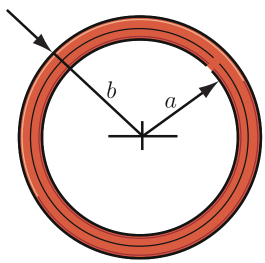



**Fig. 2.49** Problem 2.9




half-ring, $h$, is equal to 25 mm. The density of the object is uniform and will be denoted $\rho$. Find the coordinates of the mass center.

2.9 The ring with the circular cross section has the dimensions $a = 3$ m and $b = 5$ m. Determine the surface area of the ring (Fig. 2.49).

2.10 The belt shown in Fig. 2.50 has the dimensions of the cross area: $a = 27$ mm, $b = 66$ mm, and $c = 58$ mm. The radius of the belt is $r = 575$ mm. Find the volume of the belt (Fig. 2.50).

2.11 Determine the moment of inertia about the $x$-axis of the shaded area shown in Fig. 2.51 where $m = h/b$ and $b = h = 2$ m. Use integration.

2.12 Determine the moment of inertia about the $y$-axis of the area shown in Fig. 2.52 where $a = 2$ m and $b = 6$ m. Use integration.

**Fig. 2.50** Problem 2.10

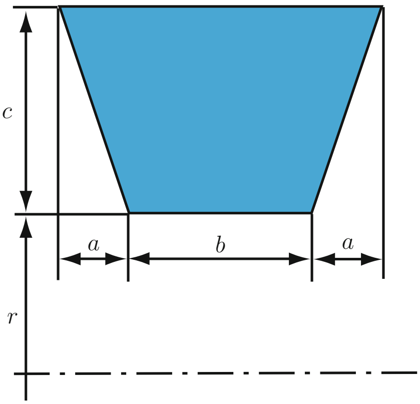



**Fig. 2.51** Problem 2.11

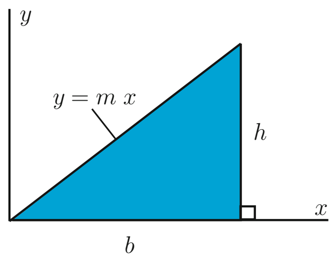
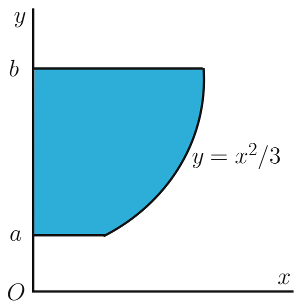



**Fig. 2.52** Problem 2.12




2.13 Determine the moment of inertia about the $x$-axis of the area shown in Fig. 2.53 where $a = 5$ m and $b = 3$ m. Use integration.
2.14 Determine the moment of inertia about the centroidal axes of the area shown in Fig. 2.54 where $r = 1$ m.
2.15 Determine the moments of inertia and the products of inertia about the centroidal axes of the shaded area shown in Fig. 2.55, where $a = 1$ in. Find the centroid polar moment of inertia. The mass center of the shaded area is at $C$.

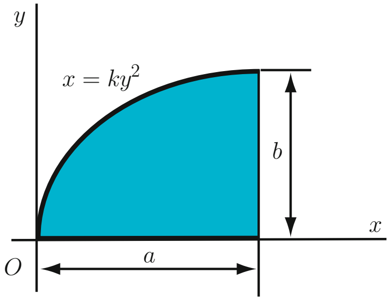



**Fig. 2.53** Problem 2.13

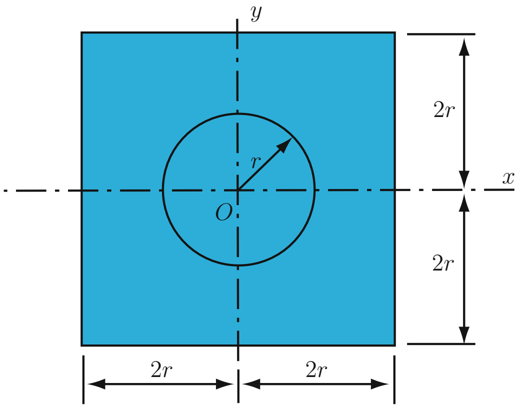



**Fig. 2.54** Problem 2.14

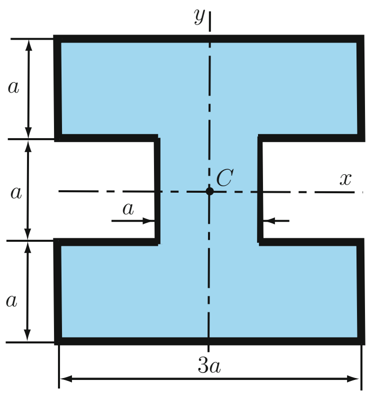



**Fig. 2.55** Problem 2.15

2.16 Determine the moments of inertia about the centroidal axes of the area shown in Fig. 2.56, where $a = 140$ mm, $b = 90$ mm, and the uniform thickness is $t = 18$ mm.

2.17 The region shown in Fig. 2.57 bounded by the curves $y = x$ and $y = x^2/a$, where $a = 10$ cm. Find the area moments of inertia about the $x$ and $y$ axes.

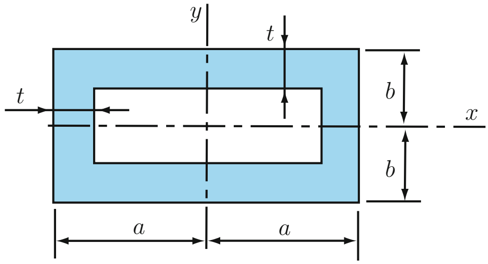



**Fig. 2.56** Problem 2.16

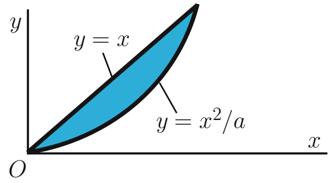



**Fig. 2.57** Problem 2.17

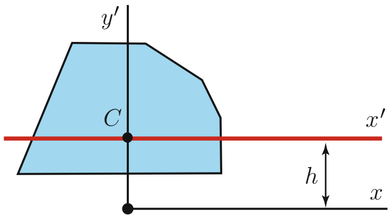



**Fig. 2.58** Problem 2.18

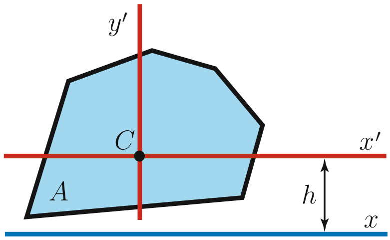



**Fig. 2.59** Problem 2.19

2.18 The polar moment of inertia of the area shown in Fig. 2.58 is $I_{Czz}$ about the $z$-axis passing through the centroid $C$. The moment of inertia about the $y'$ axis is $I_{y'y'}$ and the moment of inertia about the $x$-axis is $I_{xx}$, determine the area $A$. Numerical application: $I_{Czz} = 500 \times 10^6$ mm$^4$, $I_{y'y'} = 300 \times 10^6$ mm$^4$, $I_{xx} = 800 \times 10^6$ mm$^4$, and $h = 200$ mm.

2.19 The polar moment of inertia of the area, $A$, about the $z$-axis passing through the centroid $C$ is $I_{Czz}$, see Fig. 2.59. The moment of inertia of the area about the $x$-axis is $I_{xx}$. Find the moment of inertia about the $y'$ axis, $I_{y'y'}$. Numerical application: $I_{Czz} = 20$ in$^4$, $I_{xx} = 30$ in$^4$, $A = 5$ in$^2$, and $h = 2.5$ in.

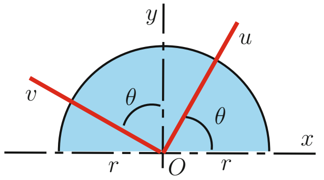



**Fig. 2.60** Problem 2.20

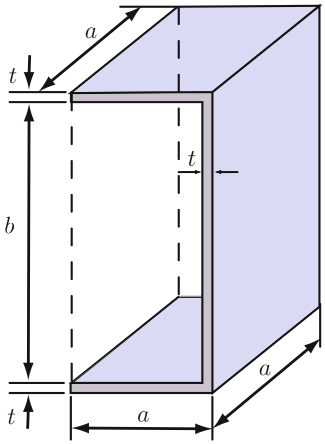



**Fig. 2.61** Problem 2.21

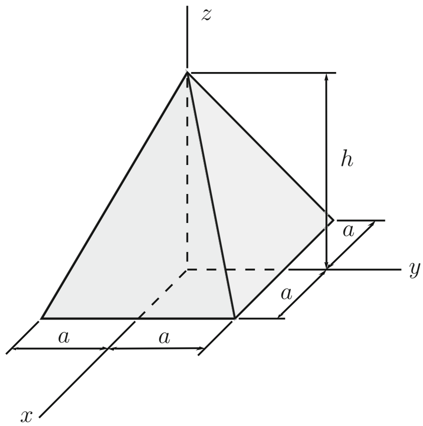



**Fig. 2.62** Problem 2.22

2.20 Determine the area moments of inertia $I_{uu}$ and $I_{vv}$ and the product of inertia $I_{uv}$ for the semicircular area with the radius $r = 70$ mm and $\theta = 35°$, as shown in Fig. 2.60.

2.21 Determine the moments of inertia and the products of inertia about the centroidal axes of the shaded figure shown in Fig. 2.61. where $a = 1$ m, $b = 3$ m, and $t = 0.3$ m. Find the centroid polar moment of inertia.

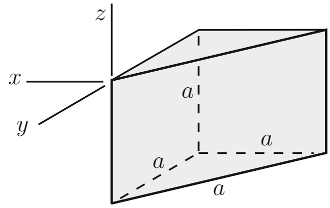



**Fig. 2.63** Problem 2.23

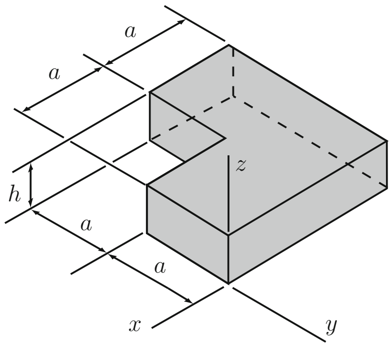



**Fig. 2.64** Problem 2.24

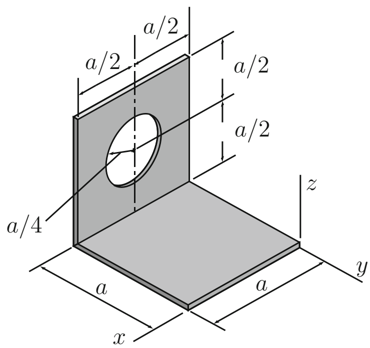



**Fig. 2.65** Problem 2.25

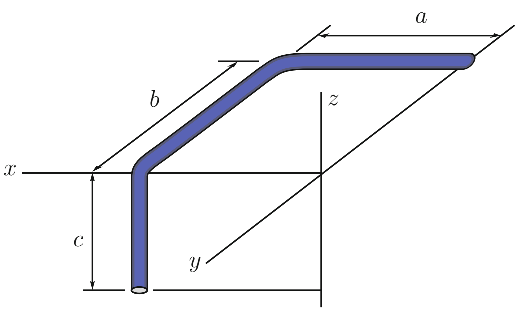
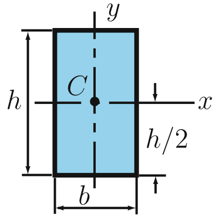



**Fig. 2.66** Problem 2.26

**Table 2.1** Area inertia properties for some common cross sections

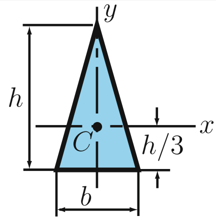



$$A = bh$$

$$I_{xx} = \frac{bh^3}{12} \qquad I_C = \frac{bh}{12}(b^2 + h^2)$$

$$I_{yy} = \frac{b^3h}{12}$$

---

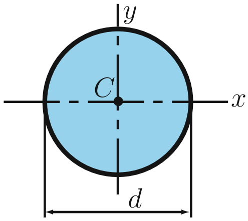



$$A = \frac{bh}{2}$$

$$I_{xx} = \frac{bh^3}{36} \qquad I_C = \frac{bh}{36}(b^2 + h^2)$$

$$I_{yy} = \frac{b^3h}{36}$$

---

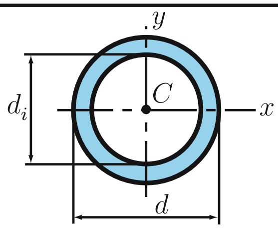



$$A = \frac{\pi d^2}{4}$$

$$I_{xx} = I_{yy} = \frac{\pi d^4}{64}$$

$$I_C = \frac{\pi d^4}{32}$$

---

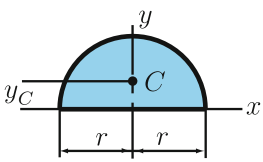



$$A = \frac{\pi}{4}(d^2 - d_i^2)$$

$$I_{xx} = I_{yy} = \frac{\pi}{64}(d^4 - d_i^4)$$

$$I_C = \frac{\pi}{32}(d^4 - d_i^4)$$

---

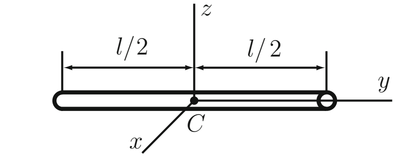



$$A = \frac{\pi r^2}{2}$$

$$I_{xx} = I_{yy} = \frac{\pi r^4}{8}$$

$$y_C = \frac{4r}{3\pi}$$

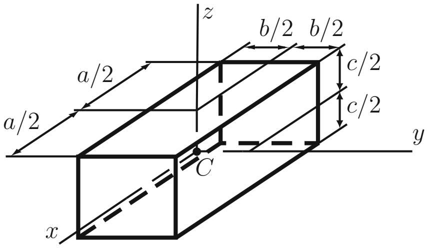
**Table 2.2** Volume inertia properties for some homogenous bodies




$$m = \rho l A$$

$$I_{xx} = I_{zz} = \frac{m}{12} l^2$$

$$I_{yy} = 0$$

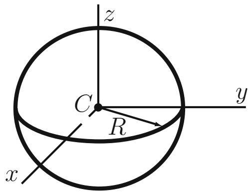



$$m = \rho abc$$

$$I_{xx} = \frac{1}{12} m(b^2 + c^2)$$

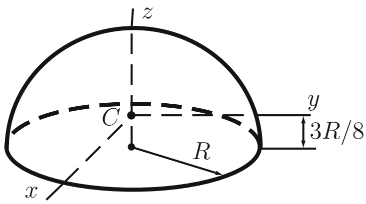
$$I_{yy} = \frac{1}{12} m(a^2 + c^2)$$

$$I_{zz} = \frac{1}{12} m(a^2 + b^2)$$




$$m = \frac{4}{3} \pi \rho R^2$$

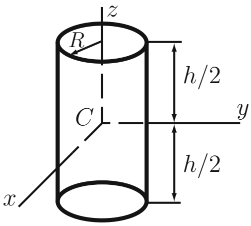
$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m R^2$$




$$m = \frac{2}{3} \pi \rho R^3$$

$$I_{xx} = I_{yy} = \frac{83}{320} m R^2$$

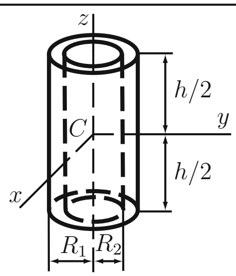
$$I_{zz} = \frac{2}{5} m R^2$$




$$m = \pi \rho R^2 h$$

$$I_{xx} = I_{yy} = \frac{1}{12} m(3R^2 + h^2)$$

$$I_{zz} = \frac{1}{2} m R^2$$




$$m = \pi \rho h(R_1^2 - R_2^2)$$

$$I_{xx} = I_{yy} = \frac{1}{12} m(3R_1^2 + 3R_2^2 + h^2)$$

$$I_{zz} = \frac{1}{2} m(R_1^2 + R_2^2)$$

2.22 Find the moment of inertia of the homogeneous pyramid with the mass $m$ and density $\rho$ about the $x$-axis and about the $z$-axis, as shown in Fig. 2.62.

2.23 Determine the products of inertia $I_{xy}$, $I_{yz}$, and $I_{xz}$ for the homogeneous prism with the density $\rho$, as shown in Fig. 2.63.

2.24 Determine the products of inertia $I_{xy}$, $I_{yz}$, and $I_{xz}$ for the homogeneous element with the material density $\rho$, as shown in Fig. 2.64.

2.25 Determine the products of inertia $I_{xy}$, $I_{yz}$, and $I_{xz}$ for the homogeneous thin body with the material mass per unit area $\rho = 30$ kg/m$^2$ and the dimension $a = 0.8$ m, as shown in Fig. 2.65.

2.26 Determine the products of inertia $I_{xy}$, $I_{yz}$, and $I_{xz}$ for the thin tube with the material mass per unit length $\rho = 5$ kg/m and the dimensions $a = 0.5$ m, $b = 0.8$ m, and $c = 0.4$ m, as shown in Fig. 2.66.

Table 2.1 gives area inertia properties for some homogeneous areas where $A$ is the area $C$ is the location of the centroid $I_{xx}$ and $I_{xx}$ are the moments of area about $x$ and $y$ axes, respectively; and $I_C$ is the polar moment of area about $C$.

Table 2.2 gives volume inertia properties for some homogeneous bodies where $A$ is the cross-sectional area, $C$ is the location of the centroid, $m$ is the mass, $\rho$ is the density, $I_{xx}$, $I_{yy}$, and $I_{zz}$ are the moments of inertia about $x$, $y$, and $z$ axes, respectively.

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