Introduction to Numerical Solutions of Differential Equations

Differential Equations

- There are ordinary differential equations - functions of one variable

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

- And there are partial differential equations - functions of multiple variables

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y} = 0$$

Order of Differential Equations

- 1st order (falling parachutist)

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

- 2nd order (mass-spring system with damping)

$$m\frac{d^2x}{dt^2} - c\frac{dx}{dt} + kx = 0$$

etc.

Higher Order ODE's

- Can always turn a higher order ODE into a set of 1st order ODE's
- Example: $$a\frac{d^3x}{dt^3}+b\frac{d^2x}{dt^2}+c\frac{dx}{dt}=d(t)$$
- Let $y=\frac{dx}{dt},\ z=\frac{d^2x}{dt^2}$ then $$\begin{cases} \frac{dz}{dt}=\frac{1}{a}\left[d(t)-bz-cy\right] \\ \frac{dy}{dt}=z \\ \frac{dx}{dt}=y \end{cases}$$
- **So solutions to first-order ODEs are important**

Linear and Non-Linear ODE's

- Linear: No multiplicative mixing of dependent variables, no nonlinear functions
- Nonlinear: anything else

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}$$

ODE's

ODE's show up everywhere in engineering:

- Dynamics (Newton's 2nd law) $$\frac{dv}{dt} = \frac{F}{m}$$
- Heat conduction (Fourier's law) $$q = -k\frac{dT}{dx}$$
- Diffusion (Fick's law) $$J = -D\frac{dc}{dx}$$

Ordinary Differential Equations

So…….What is an ODE?

First order ODE's relate the first derivative of a function (say, time rate of change) with the function itself. The ODE is first order if only the first derivative of the function is included.

$$\frac{dy(t)}{dt} = \alpha y(t) \qquad y(t) = y(0)e^{\alpha t}$$

Exponential growth or decay is governed by this simple ODE. Plug in *y(t)* to check.

Ordinary Differential Equations

How does an equation of form $\frac{dy}{dx} = f(x,y)$ arise?

How do we use it?

Consider the following:

How does a Differential Equation Arise?

Water flows out of the bottom of a tank of muddy water through a small tap at a rate proportional to the volume.

(The more water in the tank the faster the flow.)

However, with time, the tap slowly clogs up and so the rate of flow of water is inversely proportional to time.

$$\frac{dV}{dt} \propto -V\frac{1}{t}$$

Experiment gives $V' = -\frac{0.5V}{t}$

The tap is turned on at midnight and by 10AM the tank holds 24000 litres. How much water is left in the tank at 3pm?

Notice we need some starting information $V(10) = 24000$

We want to find $V(15)$ i.e. at 3pm

Initial Value Problem

Notice the problem can be abstractly stated as:

Given $V' = -\dfrac{0.5V}{t}$ and $V(10) = 24000$

Find $V(15)$

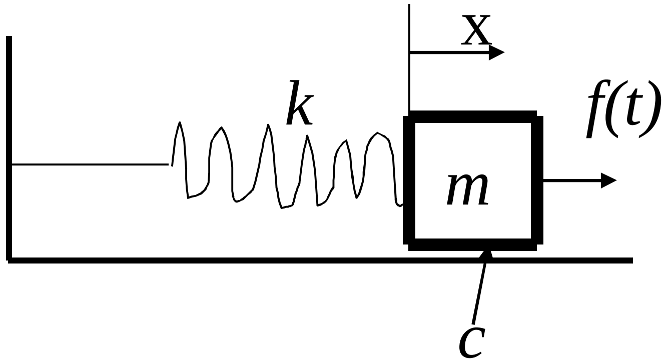
The general form of equations we will consider is

Given $\dfrac{dy}{dx} = f(x, y), \quad y(a) = b$

Find $y(c)$

Another Example

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

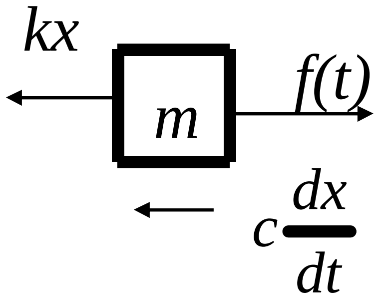


What order is this ODE?

If $f(t) = 0$, ***ODE*** is ***homogenous*****.**

If $f(t)$ is not equal to 0,
ODE is ***non-homogenous*****.**

Free-body diagram



$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx + f(t)$$

Solutions of ODE's

The solution for the homogenous ODE.

$$x(t) = C_1 S_1(t) + C_2 S_2(t)$$

C_1 & C_2 are two arbitrary constants and

$S_1(t)$ & $S_2(t)$ are two general solutions.

The solution for the non-homogenous ODE

$$x(t) = C_1 S_1(t) + C_2 S_2(t) + P(t)$$

$P(t)$ is the particular solutions.

The **arbitrary constants** $\boldsymbol{C_1}$ **&** $\boldsymbol{C_2}$ are determined by the ***Initial-value or Boundary-value conditions.***

Initial and Boundary Value Conditions

The I-V Conditions à All conditions are given at the ***same value*** of the ***independent variable*.**

For example at $t = 0, x = 0$ & $\dfrac{dx}{dt} = 0.$

The two conditions are all given at $t = 0.$

The B-V Conditions à Conditions are given at the ***different values*** of the ***independent variable***.

For example at $t = 0, x = 1$ & at $t = 1, x = 0.5.$

The ***<u>numerical schemes</u>*** for solving ***Initial-value and boundary-value*** are ***<u>different</u>*.**

Numerical Solutions of ODE's

Initial Value Problems

- **Euler's and Heun's methods**
- **Runge-Kutta methods**
- **Adaptive Runge-Kutta**
- **Multistep methods**
- **Adams-Bashforth-Moulton methods**

We are going to
look at these

A Specific Example

- Consider the differential equation $\frac{dy}{dx} = 3x^2$ with $y(0) = 0$

How can we find $y(2)$?

This is a simple equation which we can integrate to get $y = x^3$

So $y(2) = 8$

- What if the expression was too complicated to integrate, but we still needed to find $y(2)$?

In this case a numerical method is needed. There are a number of such methods. We will be considering one called **Euler's method**.

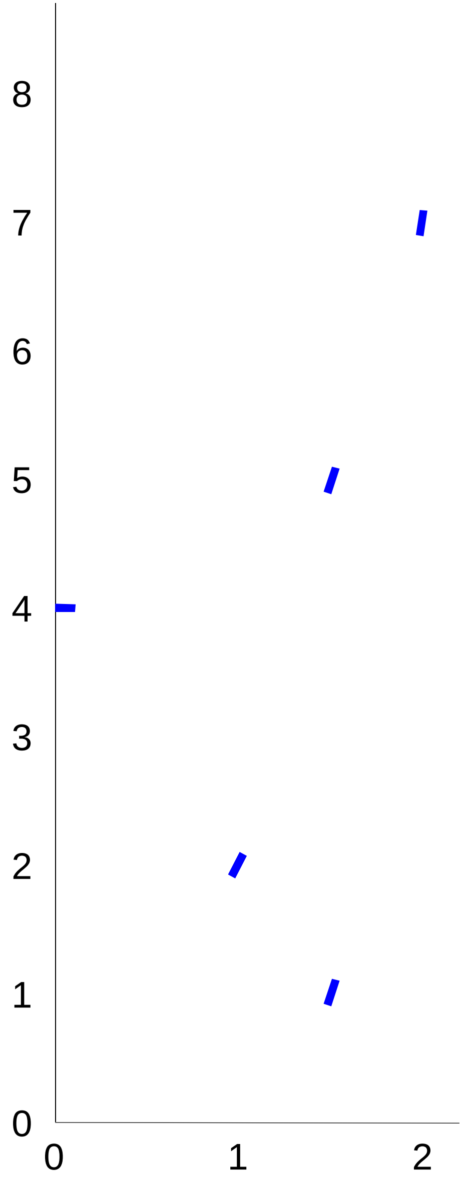
Euler's Method-1

Again consider $\dfrac{dy}{dx} = 3x^2$

This allows us to calculate a gradient at any point (x,y)

à Here the gradient depends only on x

At	(1,2)	Gradient: $3\times1^2 = 3$
	(0,4)	$3\times0^2 = 0$
	(1.5,1)	$3\times1.5^2 = 6.75$
	(1.5,5)	$3\times1.5^2 = 6.75$
	(2,7)	$3\times2^2 = 12$



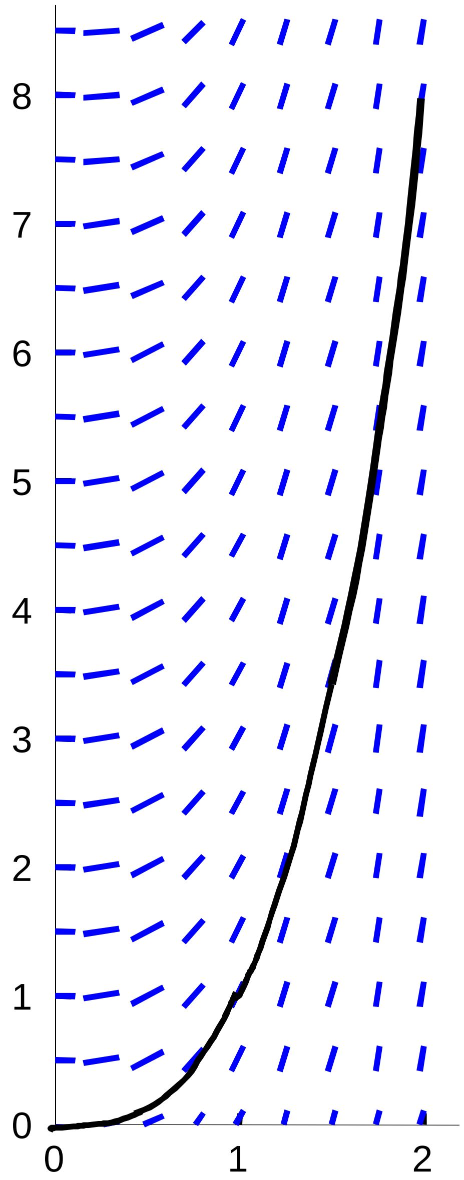
Euler's Method-2

Continuing on with $\dfrac{dy}{dx} = 3x^2$ we obtain the gradient at each point.

The graph now allows us to visualize a set of solution curves, $y = f(x)$, that start on the y axis and follow a flow of the dashes.

In fact, we know that this curve is $y = x^3 + c$ by integrating $\dfrac{dy}{dx} = 3x^2$

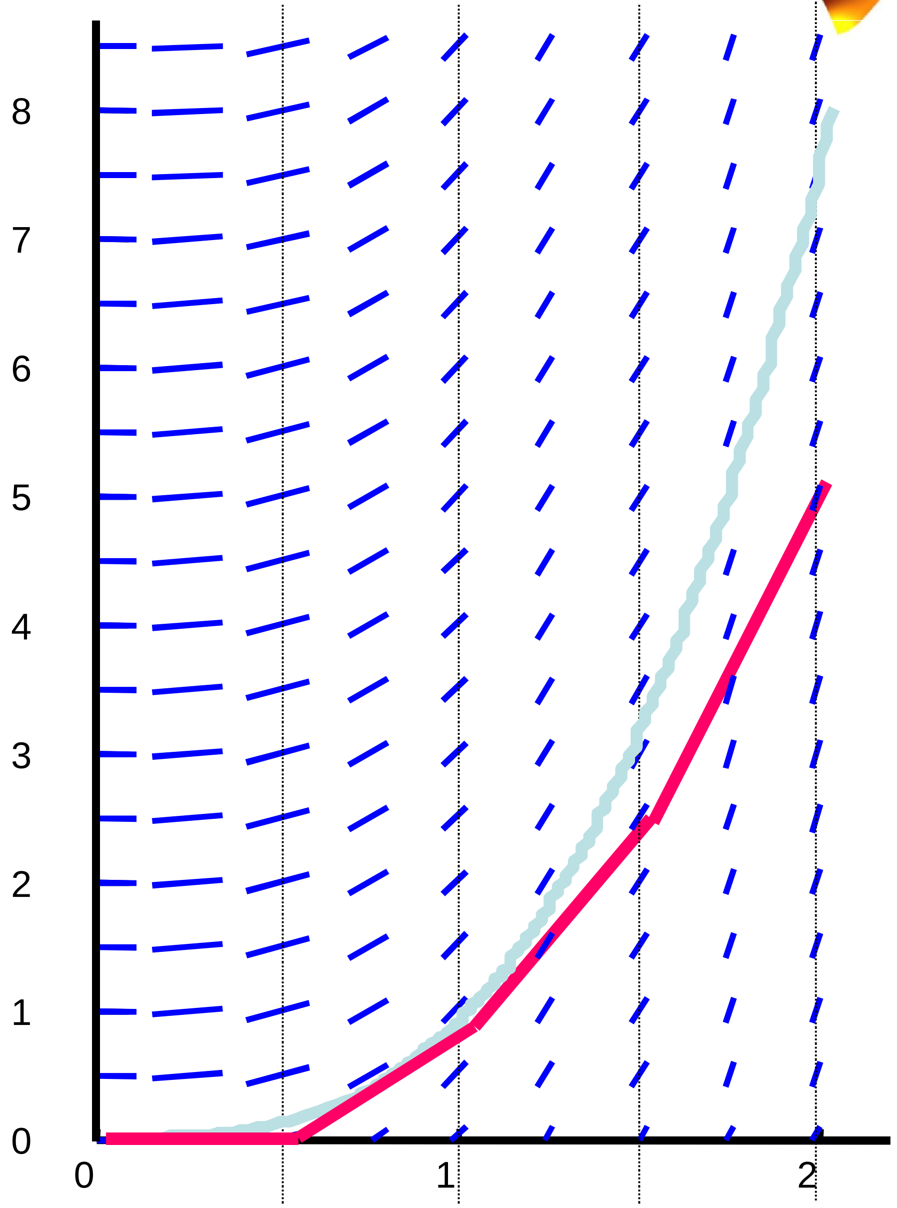
Euler's method works by approximating the curve by a series of straight line segments



Euler's Method-3

Start at x	Gradient
0	0
0.5	0.75
1	3
1.5	6.75

Our estimate of y(2) is 5.25 which is not too good.

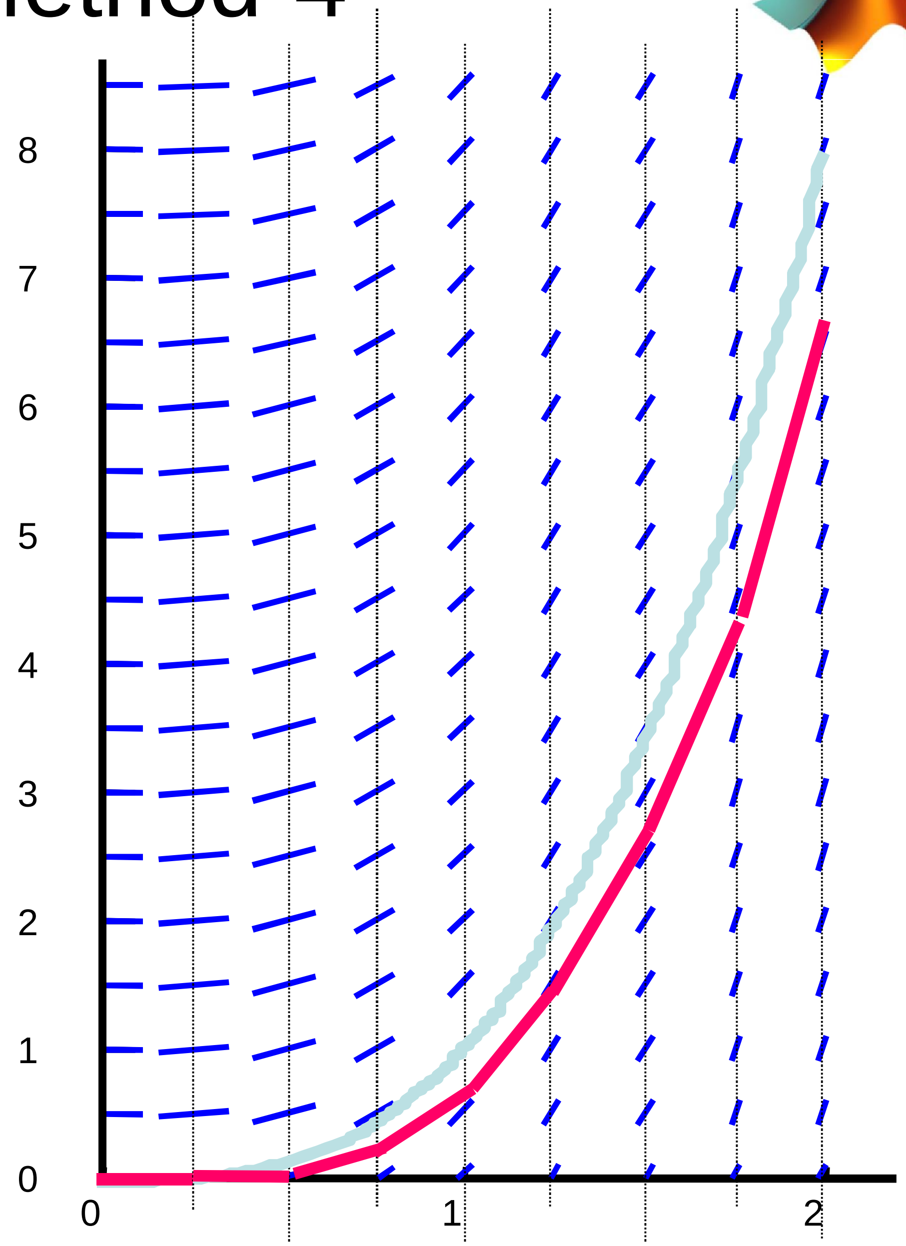


Euler's Method-4

Repeat with smaller steps

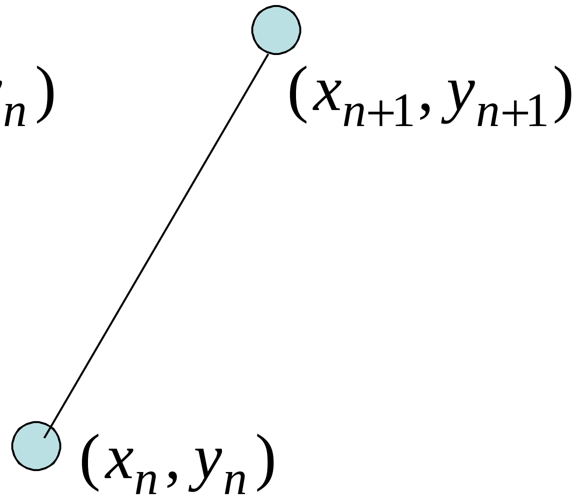
Start at *x*	Gradient
0	0
0.25	0.2
0.5	0.8
0.75	1.7
1	3
1.25	4.7
1.5	6.8
1.75	9.2

Our estimate of y(2) is 6.6 which is improving.



Euler's Method-5

Assume we have already applied the method and have joined up the points (x_n, y_n) and (x_{n+1}, y_{n+1})



Since $\dfrac{dy}{dx} = f(x, y)$, grad of line $= f(x_n, y_n)$

So $\dfrac{y_{n+1} - y_n}{x_{n+1} - x_n} = f(x_n, y_n)$

ie $y_{n+1} = (x_{n+1} - x_n) f(x_n, y_n) + y_n$

If the steps along the x-axis are kept fixed at *h*, this gives the formulae which tell how to get from one point to the next

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$

Euler’s Method-6

Procedure

- Start from the given point

- Calculate the next point using

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$

- Continue

Notice each time you continue the calculated point becomes the new starting point.

Euler's Method-7

Given $\dfrac{dy}{dx} = 3x^2, \quad y(0) = 0$

Find $y(2)$ using a step size of $h = 0.5$

x_n	y_n	$f(x_n, y_n) = 3x_n^{\ 2}$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	0	0	0+0.5×0 = 0
0.5	0	0.75	0+0.5×0.75 = 0.375
1	0.375	3	0.375+0.5×3 = 1.875
1.5	1.875	6.75	1.875+0.5×6.75 = 5.25
2	5.25		

Euler's Method-8

Another Example:

Given $\dfrac{dy}{dx} = x + y, \quad y(1) = 3$

Find $y(3)$

Use a step size of $h = 1$

x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_{n+1} = y_n + hf(x_n, y_n)$
1	3	4	$3+1\times4 = 7$
2	7	9	$7+1\times9 = 16$
3	16		

Euler's Method-9

Back to the original problem:

Given $V' = -\dfrac{0.5V}{t}$ and $V(10) = 24000$

Find $V(15)$ using a step size of $h = 1$

t_n	V_n	$f(t_n,V_n) = -\dfrac{0.5V_n}{t_n}$	$V_{n+1} = V_n + hf(t_n,V_n)$
10	24000	-1200	22800
11	22800	-1036	21764
12	21764	-907	20857
13	20857	-802	20055
14	20055	-716	19399
15	19399		

Euler’s Method-10

A non clogging tap is fitted to the tank. Solve:

Given $V' = -0.5V$ and $V(10) = 24000$

Find $V(15)$ using a step size of $h = 1$

t_n	V_n	$f(t_n, V_n) = 0.5V_n$	$V_{n+1} = V_n + hf(t_n, V_n)$
10	24000	-12000	12000
11	12000	-6000	6000
12	6000	-3000	3000
13	3000	-1500	1500
14	1500	-750	750
15	750		

Graphical Interpretation of the Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

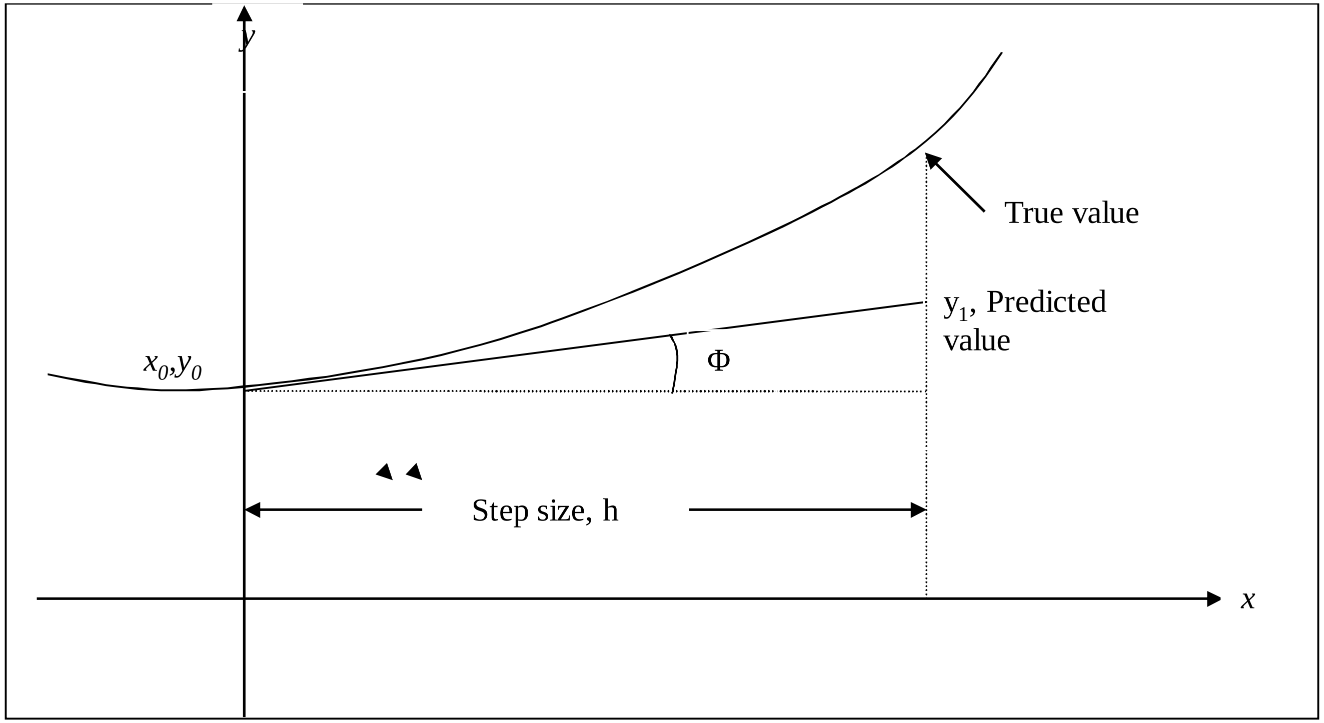
$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

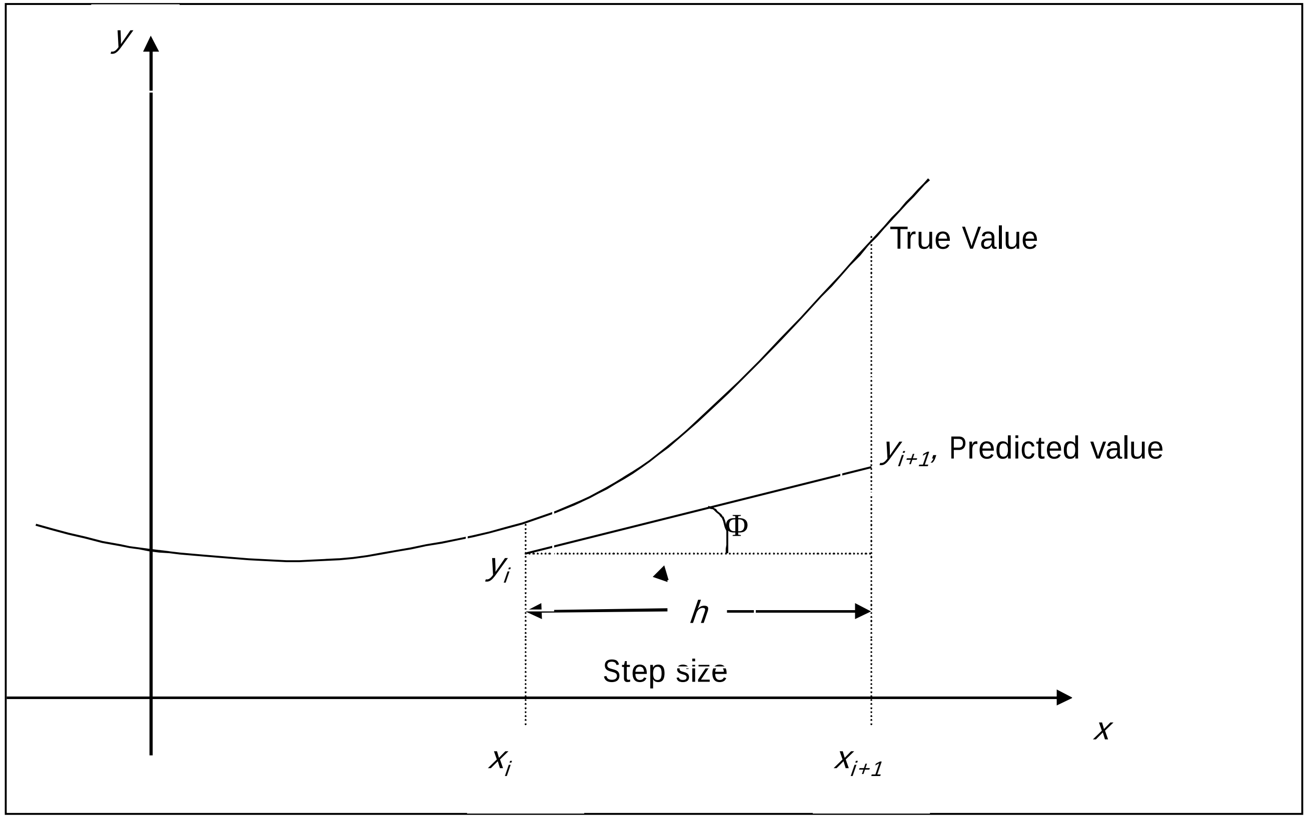
$$= y_0 + f(x_0, y_0)h$$



Graphical Interpretation of the Euler's Method-cont.

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$



Matlab Function for Euler's Method

```
Function [x, y] = Euler( f, tspan, y0, n )
% solve y' = f(x,y) with initial condition y(a) = y0
% using n steps of Euler's method; step size h = (b-a)/n
a = tspan(1); b = tspan(2); h = (b-a)/n;
x = (a+h : h : b);
y(1) = y0 + h*feval( f, a, y0);
for i = 2 : n
        y(i) = y(i-1) + h* feval(f, x(i-1), y(i-1));
end
x = [ a  x ];
y = [ y0 y ];
```

Euler's Method

```matlab
function [t, y] = Eulode(dydt, tspan, y0, h)
% [t, y] = Eulode(dydt, tspan, y0, h)
%   uses Euler's method to integrate an ODE
% input
%   dydt = name of the M-file that evaluates the ODE
%   tspan = [ti, tf] where ti and tf = initial and final
%           values of independent variable
% output
%   t = vector of independent variable
%   y = vector of solution of dependent variable

ti = tspan(1);
tf = tspan(2);
t = (ti:h:tf)';
n = length(t);
% if necessary, add an additional value of t
% so that range goes from t = ti to tf
if t(n) < tf
    t(n+1) = tf;
    n = n + 1;
end
y = y0*ones(n,1); % preallocate y to improve efficiency
for i = 1: n-1    % implement Euler's method
    y(i+1) = y(i) + feval(dydt,t(i),y(i))*(t(i+1)-t(i));
a = tspan(1); b = tspan(2); h = (b - a) / n;
end
disp('    step          t                  y')
k = 1:length(t); out = [k; t'; y'];
fprintf('%5d  %15.10f %15.10f\n',out)
```

Another possible function

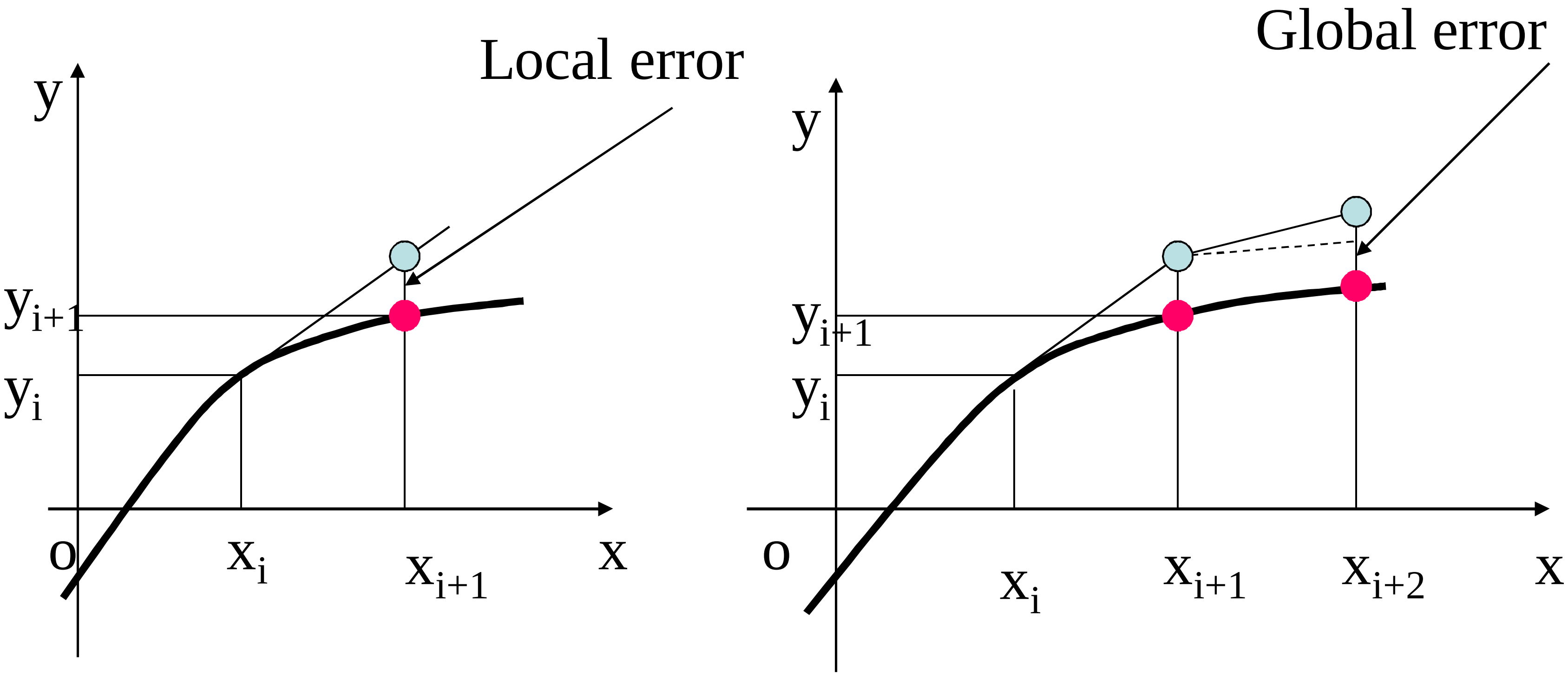
Truncation Errors

There are

- Local truncation errors - error from application at a single step
- Propagated truncation errors - previous errors carried forward

The sum is “global truncation error”

Local and Global Errors



Implicit Euler's Method

Procedure

- Start from the given point

- Calculate the next point using

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

Solve using nonlinear techniques.

- Continue

Notice each time you continue the calculated point becomes the new starting point.

Implicit Euler's Method

- The above method is difficult to implement hence the need for modification.

- Calculate the next point using

$$x_{n+1} = x_n + h$$

$$y^*_{n+1} = y_n + hf(x_n, y_n) \longleftarrow \text{Predictor}$$

$$y_{n+1} = y_n + hf(x_{n+1}, y^*_{n+1}) \longleftarrow \text{Corrector}$$

- Continue

Notice each time you continue the calculated point becomes the new starting point.

Taylor Series

- Euler's method uses Taylor series with only first order terms: Higher order methods are possible if we include more terms

- The true local truncation error is

$$E_t = \frac{f'(t_i, y_i)}{2!} h^2 + \ldots + O(h^{n+1})$$

- Approximate local truncation error - neglect higher order terms (for sufficiently small *h*)

$$\boxed{E_a = \frac{f'(t_i, y_i)}{2!} h^2 = O(h^2)}$$