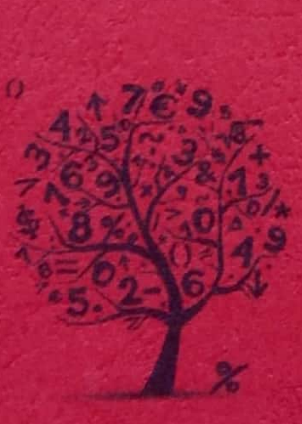
Kwame Nkrumah University of Science and Technology

MATHS 252: SOLUTION MANUAL



"Calculus is one of the most beautiful subject ever created by human mind"

~Kalmanson & Kenschaft

~M_AD

Call 0501591594 / WhatsApp 0273526472 for Copies

OR OBIRI : 0545708411

2018 MID-SEM

Dr. Barnes

1. Which of the following quadric surface represents hyperboloid of one sheet?

A. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

B) $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$ ✓

C. $z = 4x^2 + y^2$

D. $z = y^2 - x^2$

2. Which of the following quadric surface represents elliptic paraboloid?

A. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

B. $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$

C. $z = 4x^2 + y^2$ ✓

D. $z = y^2 - x^2$

3. Which of the following quadric surface represents hyperbolic paraboloid?

A. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

B. $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$

C) $z = 4x^2 + y^2$

D. $z = y^2 - x^2$ ✓

4. Find $\frac{\partial f}{\partial y}$, if $f(x, y) = xye^{-x^2-y^4}$.

A. $xe^{-x^2-y^4}\left(1 - 4x^2\right)$

B) $xe^{-x^2-y^4}\left(1 - 4y^3\right)$

C. $ye^{-x^2-y^4}\left(1 - 2x^2\right)$

D. $xe^{-x^2-y^4}\left(1 - 4y^4\right)$ ✓

5. Given that $f(x, y) = x\ln(y^2 - x)$, find the domain of $f(x, y)$

(A) $\left\{(x, y)|x < y^2\right\}$ ✓

B. $\left\{(x, y)|x > y^2\right\}$

C. $\left\{(x, y)|x = y^2\right\}$

D. $\{(x, y) | x \geq y^2\}$

6. Find the range of $g(x, y) = \sqrt{25 - x^2 - y^2}$

A. $[0, 5]$
B. $[-5, 0]$
C. $(-\infty, \infty)$
D. $[0, \infty)$

7. Evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$.

A. does not exist
B. 2
C. 1
D. 0

8. Which of the following is a critical point of $f(x, y) = \sin(\frac{x}{1+y})$?

A. $(\frac{\pi}{2}, 0)$
B. $(\frac{3\pi}{2}, 1)$
C. $(\frac{\pi}{2}, \pi)$
D. $(\frac{\pi}{4}, \frac{\pi}{4})$

9. Calculate f_{xxyz}, if $f(x, y) = \sin(3x + yz)$.

A. $-9z\cos(3x + yz) + 9yz\sin(3x + yz)$
B. $-9\cos(3x + yz) + 9yz\sin(3x + yz)$
C. $-9\cos(3x + yz) + 3y\sin(3x + yz)$
D. $9\cos(3x + yz) - 9yz\sin(3x + yz)$

10. Find the equation of tangent to $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

A. $z = 4x + 2y - 3$
B. $z = 2x - 3y + 2$
C. $z = 2x + 4y + 3$
D. $z = -3x + 2y - 4$

11. If $z = x^2 + 3xy - y^2$, find dz.

$z = x^2 + 3xy - y^2$

A. $(2x - 3y)dx + (2x + 3y)dy$

B. $(2x - 3y)dx + (2x - 3y)dy$

C. $(3x - 2y)dx + (2x + 3y)dy$

D. $(2x + 3y)dx + (3x - 2y)dy$

12. The base radius and height of a right circular cone are measured 10cm and 25cm, respectively, with a possible error in measurement of as much as 0.1cm in each. Calculate the maximum error in the calculated volume of the cone.

A. $5\pi cm^3$

B. $10\pi cm^3$

C. $20\pi cm^3$

D. $30\pi cm^3$

13. If $z = x^2y + 3xy^4$, where $x = \sin(2t)$, and $y = \cos(t)$, find $\frac{dz}{dt}$ when $t = 0$.

A. 0

B. 3

C. 6

D. 9

14. The pressure P (in kilopascals), volume V (in litres), and temperature T (in Kelvins) of a mole of an ideal gas are related by $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is $300K$ and is increasing at a rate of $0.1K/s$ and the volume is $100L$, and increasing at a rate of $0.2L/s$.

A. $0.0416kPa/s$

B. $-0.0416kPa/s$

C. $-4.1600kPa/s$

D. $4.1600kPa/s$

15. Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$

No point, No vector.

A. $13 - 2\sqrt{3}$

B. $13 - \frac{1}{2}\sqrt{3}$

C. $13 - 3\sqrt{3}$

D. $\frac{1}{2}\left(13 - 3\sqrt{3}\right)$

16. Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at a point $(2, -1)$ in the direction of the vector $v = 2i + 5j$

A. $\frac{48}{\sqrt{29}}$

B. $\frac{40}{\sqrt{29}}$

C. $\frac{32}{\sqrt{29}}$

D. $\frac{-8}{\sqrt{29}}$

17. Given that the temperature T at a point (x, y, z) is given by $T = \frac{100}{x^2+y^2+z^2}$. Find the rate of change of T with respect to the distance at the point $(1, 3-2)$ in the direction of the vector $v = i - j + k$.

A. 2.4

B. 2.7

C. 3.0

D. 3.3

18. Which of the following point is a local minimum of $f(x, y) = x^4 + y^4 - 4xy + 1$?

A. $(0, 0)$

B. $(0, 1)$

C. $(2, 2)$

D. $(-1, -1)$

19. A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.

A. $2m^3$

B. $4m^3$

C. $6m^3$

D. $8m^3$

20. Given that $f(x, y) = \sin(x + y)$, which of the following is the Taylor polynomial of degree less or equal to 2 for $f(0, 0)$?

A. $1+x^2+y^2$

B. xy

C. $x^2+xy+2y$

D. $x+y$

21. Given that $F = xi + yj + kz$, which of the following statement defines the $curl F$

A. $\nabla \cdot F$

B. $\nabla \times F$

C. ∇F

D. $\nabla + F$

22. If $f(x,y) = 4x^2+y^2+5z^2$, find the point on the plane $3x+3y+4z=12$, at which $f(x,y,z)$ has its least value?

A. $\left(\frac{4}{11}, \frac{30}{11}, \frac{9}{11}\right)$

B. $\left(\frac{4}{11}, \frac{15}{11}, \frac{6}{11}\right)$

C. $\left(\frac{5}{11}, \frac{30}{11}, \frac{8}{11}\right)$

D. $\left(\frac{5}{11}, \frac{20}{11}, \frac{7}{11}\right)$

23. Evaluate $\int_1^4 \int_{-1}^2 (2x+6x^2y)dydx$.

A. 234

B. 240

C. 48

D. 192

24. Evaluate $\int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)dxdy$.

A. $\frac{1}{4}\sin(8)$

B. $\frac{1}{2}\sin(16)$

C. $\frac{1}{10}\sin(32)$

D. $\frac{1}{5}\sin(24)$

25. Find the area A of the region in the xy−plane bounded by the graphs of $2y = 16 - x^2$, and $x + 2y = 4$.

A. 20.2 sq. units
B. 28.6 sq. units
C. 30.4 sq. units
D. 40.8 sq. units

26. Find the area of the region in the xy−plane bounded by the graphs of $x = y^3$, $x + y = 2$ and $y = 0$.

A. $\frac{5}{4}$
B. 1
C. $\frac{3}{4}$
D. $\frac{1}{2}$

27. Evaluate $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left(x^2+y^2\right)^{\frac{3}{2}}dydx$.

A. πa
B. πa^5
C. $\frac{1}{3}\pi a^5$
D. $\frac{1}{5}\pi a^5$

28. Which of the following operator is Laplacian?

A. $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

B. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

C. $\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3}$

D. $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}$

29. Which of the following operator is biharmonic?

A. $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

B. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

C. $\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3}$

D. $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}$

30. Which of the following operator is gradient?

A. $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

B. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

C. $\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3}$

D. $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}$

[0] Benedict Barnes

2018 MIDSEM

① Eqn format for hyperboloid of one sheet is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$$

ANS. B

② Eqn format for elliptic paraboloid is

$$z = Ax^2 + By^2$$

where A & B have the same sign

ANS C

③ Eqn format for hyperbolic paraboloid is

$$z = Ax^2 + By^2$$

where A & B have different signs

ANS. D

④ $x\left[-ye^{-(x^2+y^4)}\right]$

$$f_y = x\left[e^{-(x^2+y^4)} - 4y^4e^{-(x^2+y^4)}\right]$$

$$f_y = xe^{-(x^2+y^4)}\left[1 - 4y^4\right]$$

ANS D

⑤ ANS A

⑥ ANS A

⑦ fixing (0, 0) into the fxn will yield indeterminate. use squeeze theorem and you will obtain 0

ANS D

⑧ $f_x = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$

$$f_y = \frac{-x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$$

at the critical points

$f_x = 0$ $f_y = 0$

(To easy around this problem will be for you to test with the points given)

A test with $(\pi/2, 0)$ will give ~~both~~ zero in both f_x and f_y

ANS. A

⑨ $f_x = 3\cos(3x + yz)$

$f_{xx} = -9\sin(3x + yz)$

$f_{xxy} = -9z\cos(3x + yz)$

~~f_{xxz}~~ $= -9\left[z\cos(3x + yz)\right]$

$f_{xxz} = -9\left[\cos(3x + yz) - yz\sin(3x$

$f_{xxz} = -9\cos(3x + yz) + 9yz\sin(3x$

ANS. B

$f_x = 4x \quad = 4$

$f_y = 2y \quad = 2$

$f_z = -1 \quad = -1$

$(1,1,3)$

$4(x-1) + 2(y-1) - 1(z-3)$

$4x + 2y - 3 = z$

ANS. A

(11) $\frac{\delta z}{\delta x} = 2x + 3y$

$\frac{\delta z}{\delta y} = 3x - 2y$

$dz = (2x+3y)dx + (3x-2y)dy$

ANS. D

(12) Volume of a cone

$V = \pi r^2 \frac{h}{3}$

$\frac{\partial V}{\partial r} = \frac{2\pi r h}{3}$

$\frac{\partial V}{\partial h} = \frac{\pi r^2}{3}$

$dV = \frac{\partial V}{\partial h} \cdot dh + \frac{\partial V}{\partial r} \cdot dr$

$dh \simeq \Delta h = 0.1$

$dr \simeq \Delta r = 0.1$

$\Rightarrow dV = \frac{2\pi r h}{3}(0.1) + \frac{\pi r^2}{3}(0.1)$

$dV = \frac{2\pi \cdot 10 \cdot (25)}{3}(0.1) + \frac{\pi(100)}{3}(0.1)$

$= \frac{50\pi}{3} + \frac{10\pi}{3} = \frac{60\pi}{3} = 20\pi$

ANS. C

(13) $\frac{dz}{dt} = \left[\frac{\partial z}{\partial x} \cdot \frac{dx}{dt}\right] + \left[\frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\right]$

$\frac{\partial z}{\partial x} = 2xy + 3y^4$

$\frac{\partial z}{\partial y} = x^2 + 12xy^3$

$\frac{dx}{dt} = 2\cos 2t$

$\frac{dy}{dt} = \sin t$

at $t=0$; $x=0$, $y=1$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 0$

$\frac{\partial z}{\partial x} = 3 \qquad \frac{\partial z}{\partial y} = 0$

$\Rightarrow \frac{dz}{dt} = 6$

ANS. C

(14) $P = \frac{8.31T}{V}$

$\frac{\partial P}{\partial V} = -\frac{8.31T}{V^2}$

$\frac{\partial P}{\partial T} = \frac{8.31}{V}$

$$\frac{dP}{dt} = \left[\frac{\partial P}{\partial V}\cdot\frac{dV}{dt}\right] + \left[\frac{\partial P}{\partial T}\cdot\frac{dT}{dt}\right]$$

$$= \left[-0.2 \times \frac{8.31T}{V^2}\right] + \left[\frac{8.31}{V} \times 0.1\right]$$

$$= \left[\frac{-0.2 \times 8.31\,(300)}{(100)^2}\right] + \left[\frac{8.31}{100} \times 0.1\right]$$

$$= -0.0416 \text{ kPa/s}$$

ANS B

(15) Bonus (no points and vector given)

(16) $f_x = 2y^3x \quad = -4$

$f_y = 3y^2x^2 - 4 \quad = 8$

$\Big|_{(2,-1)}$

$$\hat{v} = \frac{2}{\sqrt{29}}i + \frac{5}{\sqrt{29}}j$$

$$\nabla f(2,-1)\cdot\hat{v} = \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}}$$

$$= \frac{32}{\sqrt{29}}$$

ANS C

(17) $T = 100(x^2+y^2+z^2)^{-1}$

$T_x = -200x\,(x^2+y^2+z^2)^{-2}$

$T_y = -200y\,(x^2+y^2+z^2)^{-2}$

$T_z = -200z\,(x^2+y^2+z^2)^{-2}$

evaluate at $(1,3,-2)$

$T_x = -1.02 \qquad T_z = 2.041$

$T_y = -3.06$

$$\hat{v} = \frac{i}{\sqrt{3}} - \frac{j}{\sqrt{3}} + \frac{k}{\sqrt{3}}$$

$$\nabla T(1,3,-2)\cdot\hat{v} = \frac{-1.02}{\sqrt{3}} + \frac{3.06}{\sqrt{3}} + \frac{2.041}{\sqrt{3}}$$

$$\simeq 2.4$$

ANS A

(18) $f_x = 4x^3 - 4y \qquad f_{xx} = 12x^2$

$f_y = 4y^3 - 4x \qquad f_{yy} = 12y^2$

$f_{xy} = -4 \qquad (f_{xy})^2 = 16$

$f_{xx}\cdot f_{yy} = 144x^2y^2$

at the critical points $f_x = 0$

$f_y = 0$

$4x^3 - 4y = 0 \quad$ -- (1)

$y = x^3 \quad$ -- (3)

$4x^9 - 4x = 0 \quad$ [(3) into (2)]

$x^8 = 1$

$x = \pm 1$

critical pts $\left\{ (0,0),\ (1,1),\ (-1,-1) \right.$

$D(x,y) = f_{xx}\cdot f_{yy} - (f_{xy})^2$

$= 144x^2y^2 - 16$

$D(0,0) = -16 \quad \Rightarrow$ Saddle

$D(-1,-1) = 128 \Rightarrow$ extrema
$D(1,1) = 128 \Rightarrow$ extrema

Type of extrema
$$\left.\begin{array}{l} f_{xx}(-1,-1) = 12 \\ f_{xx}(1,1) = 12 \end{array}\right\} \text{minimum}$$

41
3

ANS D

9) let x, y, z be the 3 sides of the rectangular box

$$A = 2xz + 2yz + xy = 12$$

$$V = xyz$$

where A and V are area and volume of the rectangular box with open lid.

Using Lagrange's multipliers

$$V_x = yz$$
$$V_y = xz$$
$$V_z = xy$$

$$A_x = 2z + y \; ; \; \lambda A_x = \lambda(2z+y)$$
$$A_y = 2z + x \; ; \; \lambda A_y = \lambda(2z+x)$$
$$A_z = 2x + 2y \; ; \; \lambda A_z = \lambda(2x+2y)$$

$$V_x = \lambda A_x \quad -(1)$$
$$V_y = \lambda A_y \quad -(2)$$
$$V_z = \lambda A_y \quad -(3)$$
and A $\quad -(4)$

Solve (1), (2), (3), & (4) simultaneously

$$yz = \lambda(2z+y) \quad -(1)$$
$$xz = \lambda(2z+x) \quad -(2)$$
$$xy = \lambda(2x+2y) \quad -(3)$$
$$2xz + 2yz + xy = 12 \quad -(4)$$

from (1)

$$\lambda = \frac{yz}{(2z+y)} \quad -(5)$$

(5) into (2)

$$xz(2z+y) = yz(2z+x)$$
$$2xz^2 + xyz = 2yz^2 + xyz$$
$$2xz^2 = 2yz^2$$
$$x = y \quad -(6)$$

(6) and (5) into (3)

$$xy(2z+y) = yz(2x+2y)$$
$$x^2(2z+x) = xz(2x+2x)$$
$$2zx^2 + x^3 = 4x^2z$$
$$x^3 = 2x^2z$$
$$x = 2z \quad -(7)$$

(7) into (4) ~~with (6)~~

$$2z(2z) + 2(2z)z + (2z)(2z) = 12$$
$$12z^2 = 12$$
$$z = 1$$

$$\Rightarrow x = 2$$
$$y = 2 \qquad (2, 2, 1)$$

$\Rightarrow V = (2)(2)(1) = 4m^3$

ANS. B

(20) $f_x = \cos(x+y) \Big|_{(0,0)} = 1$

$f_y = \cos(x+y) \Big|_{(0,0)} = 1$

$1(x-0) + 1(y-0)$

$x + y$

ANS D

(21) ANS. B

(22) $f_x = 8x$

$f_y = 2y$

$f_z = 10z$

$g_x = 3 \quad ; \lambda g_x = 3\lambda$

$g_y = 3 \quad ; \lambda g_y = 3\lambda$

$g_z = 4 \quad ; \lambda g_z = 4\lambda$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$8x = 3\lambda$

$x = \frac{3}{8}\lambda$

$y = \frac{3}{2}\lambda$

$z = \frac{2}{5}\lambda$

$3\left(\frac{3}{8}\lambda\right) + 3\left(\frac{3}{2}\lambda\right) + 4\left(\frac{2}{5}\lambda\right) = 12$

$\frac{9\lambda}{8} + \frac{9}{2}\lambda + \frac{8\lambda}{5} = 12$

$45\lambda + 180\lambda + 64\lambda = 480$

$\lambda = \frac{480}{289}$

$\Rightarrow x = \frac{3}{8}\left(\frac{480}{289}\right) = \frac{180}{289}$

$\Rightarrow y = \frac{3}{2}\left(\frac{480}{289}\right) = \frac{720}{289}$

$\Rightarrow z = \frac{2}{5}\left(\frac{480}{289}\right) = \frac{192}{289}$

$\left(\frac{180}{289}, \frac{720}{289}, \frac{192}{289}\right)$

(since the ans. is not in the options) you may go ahead and test which of the points will give 12 or approximately 12 when you plug them into the plane,

ANS C will give 11.72

$\Rightarrow$ ANS C

(23) $\int_1^4 \int_{-1}^2 (2x + 6x^2y)\, dy\, dx$

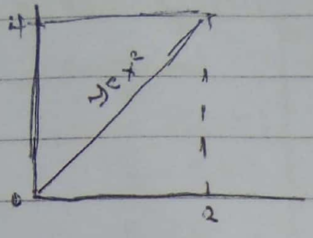
$\int_1^4 2xy + 3x^2y^2 \Big|_{-1}^{2} dx$

$\int_1^4 4x + 2x + 12x^2 - 3x^2\, dx$

$\int_1^4 6x + 9x^2\, dx = 234$

ANS A

apply reversing of order for Q.24



const. limits for x

$0 \le x \le 2$

$0 \le y \le x^2$

$$\int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)\,dx\,dy = \boxed{\int_0^2 \int_0^{x^2} y\cos(x^5)\,dy\,dx}$$

$$\int_0^2 \int_0^{x^2} y\cos(x^5)\,dy\,dx$$

$$\int_0^2 \frac{y^2}{2}\cos(x^5) \Big|_0^{x^2} dx$$

$$\int_0^2 \frac{x^4}{2}\cos(x^5)\,dx$$

let $u = x^5$

$dx = \dfrac{du}{5x^4}$

change limits

$x = 2 \Rightarrow u = 32$

$x = 0 \Rightarrow u = 0$

$$\int_0^{32} \frac{1}{10}\cos(u)\,du$$

$$\frac{1}{10}\Big[\sin u\Big]_0^{32}$$

$\frac{1}{10}\sin(32)$ ANS C

(25) $y_1 = \dfrac{16-x^2}{2}$ $y_2 = \dfrac{4-x}{2}$

at the pt. of intersection,

$y_1 = y_2$

$16 - x^2 = 4 - x$

$x^2 - x - 12 = 0$

$(x+3)(x-4) = 0$

$-3 \le x \le 4$

set;

$$\int_{-3}^{4} \int_{\frac{4-x}{2}}^{\frac{16-x^2}{2}} dy\,dx$$

$$\int_{-3}^{4} \frac{1}{2}\Big[16 - x^2 + x - 4\Big]dx$$

$$\frac{1}{2}\int_{-3}^{4} -x^2 + x + 12\,dx$$

(compute using calculator)

This will yield 28.6

ANS. B

(26) $x_1 = y^3$ $x_2 = 2-y$

$y^3 = 2 - y$

$y^3 + y - 2 = 0$

$y = 1$ (and some complx nos)

set

$$\int_0^1 \int_{y^3}^{2-y} dx\,dy$$

$$\int_0^1 2 - y - y^3\,dy = \frac{5}{4}$$

ANS A.

(27) $$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\, dy\,dx$$

The nature of this integral is to be taken in polar coordinates

$0 \leq r \leq a$

$x^2+y^2 = r^2$

$dy\,dx = r\,dr\,d\theta$

$0 \leq \theta \leq \pi$

set

$$\int_0^{\pi}\int_0^{a} (r^2)^{3/2} \cdot r\, dr\, d\theta$$

$$\int_0^{\pi}\int_0^{a} r^4\, dr\, d\theta$$

$$\int_0^{\pi} \left.\frac{r^5}{5}\right|_0^a d\theta$$

$$\int_0^{\pi} \frac{a^5}{5}\, d\theta$$

$$\frac{a^5}{5}\pi$$ ANS. D

(28) B

(29) ANS. D

(30) ANS. A

END OF SEMESTER

APRIL, 2012

(1) $\vec{r}(t) = \langle \sin t, 2t, t^2 \rangle$ when $t=0$

component	differential	$t=0$
i ($\sin t$)	$\cos t$	1
j ($2t$)	2	2
k (t^2)	$2t$	0

$T = i + 2j + 0k$

$|T| = \sqrt{5}$

$\hat{T} = \frac{T}{|T|} = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j + 0k$

$\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \rangle$

ANS. D

(2) ANS. C

(3) ANS. A

(4) NOT CLEAR (can't see the question)

(5) E

(6) A

(7) $f(x,y) = e^{2x+y^2}$

$f_x = 2e^{2x+y^2}$

$f_{xy} = 2(2y)e^{2x+y^2}$

$f_{xy} = 4ye^{2x+y^2}$

$f_{xyy} = 4[y(2y)e^{2x+y^2} + e^{2x+y^2}]$

$f_{xyy}(0,0) = 4[0+1]$

$= 4$

ANS C

(8) $f(x,y) = \tan^{-1}(x/y)$

$f_y(1,2)$

$f(x,y) = \tan^{-1}(xy^{-1})$

$f_y = \frac{-xy^{-2}}{1+(xy^{-1})^2}$

$= \frac{-x}{x^2+y^2}\Big|_{(1,2)}$

$= \frac{-1}{1+4} = -1/5$

ANS. E
(None of the above)

(9) $f(x,y,z) = e^{2x+2y} - z$

$f_x = 2e^{2x+2y} \Big| = 2$

$f_y = 2e^{2x+2y} \Big| = 2$

$f_z = -1 \Big| = -1$

$\Big|_{(0,0,1)}$

Eqn of tangent plane

$2(x-0) + 2(y-0) - 1(z-1) = 0$

$2x + 2y - z + 1 = 0$

$2x + 2y + 1 = z$

ANS. A

(10) $f(x,y,z) = x^2 + y^2 - z$

$f_x = 2x \Big| = 2$

$f_y = 2y \Big| = 2$

$f_z = -1 \Big| = -1$

$\Big|_{(1,1,2)}$

Eqn of tangent plane

$2(x-1) + 2(y-1) - 1(z-2) = 0$

$2x + 2y - 2 - 2 + 2 = z$

$2x + 2y - 2 = z$

ANS D

(12) $f(x,y,z) = \ln(2x+y) - z$

$$f_x = \frac{2}{2x+y} \quad = 2$$

$$f_y = \frac{1}{2x+y} \quad = 1$$

$$f_z = -1 \quad = -1$$

$(-1, 3, 0)$

$$2(x+1) + 1(y-3) - 1(z-0) = 0$$

$$2x + y - z = -2 + 3$$

$$2x + y - z = 1$$

ANS C.

(13) You may go ahead to use pythagoras theorem which will give you 5.10

ANS. D.

(14) $dw = \frac{\delta w}{\delta x} dx + \frac{\delta w}{\delta y} dy + \frac{\delta w}{\delta z} dz$

$$\frac{\delta w}{\delta x} = y^2 z^3 - z e^{xz} y$$

$$\frac{\delta w}{\delta y} = 2xyz^3 - e^{xz}$$

$$\frac{\delta w}{\delta z} = 3xy^2z^2 - xe^{xz}y$$

$$dw = (y^2z^3 - ze^{xz}y)dx + (2xyz^3 - e^{xz})dy + (3xy^2z^2 - xe^{xz}y)dz$$

ANS. C

(16) $f(x,y,z) = xyz - 2$

$$f_x = yz \quad = 2$$

$$f_y = xz \quad = -2$$

$$f_z = xy \quad = -1$$

$(1, -1, -2)$

Eqn of tangent plane

$$2(x-1) + (-2)(y+1) - 1(z+2) =$$

$$2x - 2y - z = 2 + 2 + 2$$

$$2x - 2y - z = 6$$

ANS A

(17) $z = e^{x^2} \sin y \quad x(1) = 0 \quad x'(1) =$

$y(1) = 0 \quad y'(1) =$

$\frac{\delta z}{\delta t}$ at $t = 1$

$$\frac{\delta z}{\delta t} = \left[\frac{\delta z}{\delta x} \cdot \frac{\delta x}{\delta t}\right] + \left[\frac{\delta z}{\delta y} \cdot \frac{\delta y}{\delta t}\right]$$

$$\frac{\delta z}{\delta x} = 2xe^{x^2} \sin y \quad = 0 \text{ at } t =$$

$$\frac{\delta z}{\delta y} = e^{x^2} \cos y = 1 \text{ at } t = 1$$

$$\frac{\delta x}{\delta t} = x'(1) = 3 \qquad \frac{\delta y}{\delta t} = y'(1) = 4$$

$$\frac{\delta z}{\delta t} = [0 \cdot 3] + [1 \cdot 4]$$

$$= 4$$

ANS. C

(18) Similar approach as in (17)

$$\frac{\delta z}{\delta s} = \left[\frac{\delta z}{\delta x} \cdot \frac{\delta x}{\delta s}\right] + \left[\frac{\delta z}{\delta y} \cdot \frac{\delta y}{\delta s}\right]$$

$$\frac{\delta z}{\delta x} = 2xe^{y^2} \sin y \quad \frac{\delta z}{\delta y} = e^{y^2} \cos y$$

$\frac{\delta z}{\delta x} = 0$ when $(s,t) = (0,0)$

$\frac{\delta z}{\delta y} = 1$ when $(s,t) = (0,0)$

$\frac{\delta x}{\delta s}$ is given as 3

$\frac{\delta xy}{\delta s}$ is given as 4

$\Rightarrow \frac{\delta z}{\delta s} = [0 \times 3] + [4 \times 1]$

$= 4$

ANS. A

(1) $z = xy + x^2y$

$\frac{\delta z}{\delta s} = \left[\frac{\delta z}{\delta x} \cdot \frac{\delta x}{\delta s}\right] + \left[\frac{\delta z}{\delta y} \cdot \frac{\delta y}{\delta s}\right]$

$\frac{\delta z}{\delta x} = y + 2xy$

which is 6 at $(s,t) = (0,0)$

$\frac{\delta z}{\delta y} = x + x^2$

which is 2 at $(s,t) = (0,0)$

$\frac{\delta x}{\delta s}$ and $\frac{\delta y}{\delta s}$ is 3 and 4 resp.

$\Rightarrow \frac{\delta z}{\delta s} = [6 \cdot 3] + [2 \cdot 4]$

$18 + 8 = 26$

ANS. B

(20) $f_x = 2x \Big|_{(1,2)} = 2$

$f_y = 2y \Big|_{(1,2)} = 4$

$2i + 4j$

~~$j = \sin\theta j$~~

given vector $\langle \cos\theta i, \sin\theta j \rangle$

$\langle \cos \pi/2\, i, \sin \pi/2\, j \rangle$

$v = \langle 0, 1 \rangle$

~~$|v| = 1$~~

$\langle f_x, f_y \rangle \cdot \langle 0, 1 \rangle$

$(2i + 4j) \cdot (0 + j)$

4

ANS. A

(21) $f_x = 2x \Big|_{(1,1)} = 2$

$f_y = 2y \Big|_{(1,1)} = 2$

$\nabla f(1,1) = 2i + 2j$

given unit vector

$\vec{v} = (\cos\theta i + \sin\theta j)$

$\vec{v} = \cos\frac{\pi}{4} i + \sin \pi/4\, j$

$\vec{v} = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j$

$\nabla f(1,1) \cdot \vec{v}$

$\sqrt{2} + \sqrt{2} = 2\sqrt{2}$

ANS. C

(22) $f_x = 2x \Big|_{(1,\sqrt{3})} = 2$

$f_y = 2y \Big|_{(1,\sqrt{3})} = 2\sqrt{3}$

$\nabla f(1, \sqrt{3}) = 2i + 2\sqrt{3} j$

given unit vector

$\hat{v} = (\cos\theta i + \sin\theta j)$

$= \cos\frac{\pi}{6} i + \sin\frac{\pi}{6} j$

$= \frac{\sqrt{3}}{2} i + \frac{1}{2} j$

$\nabla f(1,\sqrt{3}) \cdot \hat{v}$

$\sqrt{3} + \sqrt{3} = 2\sqrt{3}$

ANS. B

(23) $f_x = e^{xy/z} + \frac{xy}{z} e^{xy/z}$ $= 1$

$f_y = \frac{x^2}{z} e^{xy/z}$ $= 9$

$f_z = -\frac{x^2 y}{z^2} e^{xy/z}$ $= 0$

$(3,0,1)$

$\nabla f(3,0,1) = i + 9j$

$\vec{u} = -\frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k$

$\nabla f(3,0,1) \cdot \vec{u}$

$-\frac{1}{3} + 6 = \frac{17}{3}$

ANS. D

(24) $f_x = \frac{yz}{2}(xyz)^{-1/2}$ $= 1$

$f_y = \frac{xz}{2}(xyz)^{-1/2}$ $= \frac{1}{2}$

$f_z = \frac{xy}{2}(xyz)^{-1/2}$ $= 1$

$(2,4,2)$

$\nabla f(x,y,z) = \cancel{\frac{i}{2} + j + k}\; i + \frac{1}{2}j + k$

$|u| = \sqrt{16+4+16} = 6$

$\hat{u} = \frac{u}{|u|} = \frac{2}{3} i + \frac{1}{3} j - \frac{2}{3} k$

$\nabla f(x,y,z) \cdot \hat{u} = \cancel{\frac{1}{3} + \frac{1}{3} - \frac{2}{3}} = \frac{2}{3} + \frac{1}{6} - \frac{2}{3}$

$= \frac{1}{6}$

ANS. C

(26) $f_x = 2x + y$ $\quad f_{xy} = 1$

$f_y = 2y + x$

$f_{xx} = 2 \quad f_{yy} = 2$

at the critical points

$\cancel{f_x = f_y}$ $\quad f_x = 0 \quad f_y = 0$

$\cancel{2x + y = 2y + x}$

$2x + y = 0$

$y = -2x$

$\cancel{f_y}$ $\quad 2(-2x) + x = 0$

$-4x + x = 0$

$x = 0$

$y = 0$

critical pts $= (0,0)$

Nature

use discriminant

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$

[every term on the R.H.S should be evaluated at determined critical point(s)]

$D(x,y) = 4 - 1 = 3$

$D > 0 \Rightarrow$ extrema

Type of extrema
$f_{xx}(x,y) = 2$
+ve means minimum

Hence ANS. B

(27) $f_x = 2x + 3y$ $f_{xy} = 3$
$f_{xx} = 2$
$f_y = 2y + 3x$
$f_{yy} = 2$
at the critical pt(s)
$f_x = 0$; $f_y = 0$
$2x + 3y = 0$
$y = \frac{-2x}{3}$
$2\left(\frac{-2x}{3}\right) + 3x = 0$
$x = 0$
$y = 0$
Critical pt. $(0,0)$

Nature
Use discriminant
$D(0,0) = 2 \cdot 2 - 9$
$= 4 - 9 = -5$
$D < 0 \Rightarrow$ saddle pt

Hence ANS B

(28) $f_x = 2x + y + 1$
$f_{xy} = 1$
$f_{xx} = 2$
$f_y = 2y + x$
$f_{yy} = 2$

at the critical pt(s)
$f_x = 0$; $f_y = 0$
$2x + y + 1 = 0$
$y = -2x - 1$
$2(-2x-1) + x = 0$
$-4x - 2 + x = 0$
$-3x = 2$
$x = -2/3$
$\Rightarrow y = -2(-2/3) - 1$
$= 4/3 - 1 = 1/3$
Critical pt(s). $(-2/3, 1/3)$

Nature
Use discriminant
$D(-2/3, 1/3) = 4 - 1$
$= 3$
$D > 0 \Rightarrow$ extrema
Type of extrema
$f_{xx}(-2/3, 1/3) = 2$
$\Rightarrow$ minimum

hence ANS. D

(29) $f_x = 2x + 4xy$
$f_y = 2y + 2x^2$

at the stationary pt(s)
$f_x = 0$ $f_y = 0$

$2x + 4xy = 0$ (1)
$4xy = -2x$
$y = -1/2$ $(0, -1/2)$

$2y + 2x^2 = 0$

$y = -x^2 \quad - ①$

$2x + 4x(-x^2) = 0$

$2x - 4x^3 = 0 \quad ②$

$2x = 4x^3 \quad (0,0)$

$2 = 4x^2$

$1 = x^2$

$x = \pm 1$

$\Rightarrow y = \pm 1 \quad (1,-1) - ③$

$(-1,1) - ④$

Hence 4 critical pt(s)

ANS. D

㉚ $f_x = 2x + 2$ | $f_{xy} = 0$

$f_y = 2y - 2$ | $f_{xx} = 2$

$f_{yy} = 2$

at the critical points

$f_x = 0; \; f_y = 0$

$2x + 2 = 0$

$x = -1 \quad (-1,0)$

$2y - 2 = 0$

$y = 1 \quad (0,1)$

$D(-1,0) = 4 - 0 = 4$

$D(-1,0) > 0$

$\Rightarrow$ etrema

$D(0,1) = 4 - 0 = 4$

$D(0,1) > 0$

$\Rightarrow$ extrema

$f_{xx}(0,1) = 2$

$f_{xx}(-1,0) = 2$

$\Rightarrow$ both are minimum pts.

sticking any of the points into the fxn itself yields -1

ANS. B

㉛ $f_x = y - 2xy - y^2$

$f_{xy} = 1 - 2x - 2y$

$f_{xx} = -2y$

$f_y = x - x^2 - 2xy$

$f_{yy} = -2x$

at the critical pt(s) $f_x = 0; \; f_y = 0$

$y - 2xy - y^2 = 0$

$x = \dfrac{y - y^2}{2y}$

$\dfrac{y - y^2}{2y} - \dfrac{(y^2 - 2y(y^3) + y^4)}{4y^2} - \dfrac{y^2 \ldots}{}$

$0 = \dfrac{y - y^2}{2y} - \dfrac{(y^2 - 2y^3 + y^4)}{4y^2} - \dfrac{(y^2 - y^3)}{y}$

$0 = \dfrac{2y(y - y^2) - y^2 + 2y^3 - y^4 - 4y(y^2 - y^3)}{4y^2}$

$0 = \dfrac{2y^2 - 2y^3 - y^2 + 2y^3 - y^4 - 4y^3 + 4y^4}{4y^2}$

$\dfrac{y^2 - 4y^3 + 3y^4}{4y^2} = 0$

$\dfrac{\cancel{y^2}(1 - 4y + 3y^2)}{4\cancel{y^2}} = 0$

$1 - 4y + 3y^2 = 0$

$y = 1 \quad y = 1/3$

$\Rightarrow x = 0 \quad x = 1/3$

$(0,1); \; (1/3, 1/3)$

$\Rightarrow$ ANS. D.

(32) $f(x,y) = xy + (x+y)(120-x-y)$

$= xy + 120x - x^2 - xy + 120y - xy - y^2$

$= 120x - x^2 - xy + 120y - y^2$

$f_x = 120 - 2x - y$

$f_y = 120 - 2y - x$

$f_x = 0$; $f_y = 0$

$y = 120 - 2x$

$120 - 2(-2x+120) - x = 0$

$120 + 4x - 240 - x = 0$

$\frac{120}{3} = x = 40$

$\Rightarrow y = 120 - 80 = 40$

Hence the max. value occur at $x = 40$, $y = 40$

ANS. A

(33) $f(x,y) = x^3 - 3xy - y^3$

$f_x = 3x^2 - 3y$

$f_y = -3y^2 - 3x$

$f_{xx} = 6x$ $\quad f_{yy} = -6y$

$f_{xy} = -3$

$f_x = 0$; $f_y = 0$ @ stat. pt(s)

$\frac{3x^2}{3} = x^2 = y \quad -(1)$

$-3x^4 - 3x = 0$

$x^4 = -x \Rightarrow x = 0$

$x^3 = -1$

when $x = 0$; $y = 0$

$x = -1$ $y = 1$ from (1)

Critical pt(s) $(0,0)$ and $(-1,1)$

Nature

$D(0,0) = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2$

$= 0 \cdot 0 - (9)$

$= -9$

$D(0,0) < 0 \Rightarrow$ saddle point

$D(-1,1) = f_{xx}(-1,1) \cdot f_{yy}(-1,1) - [f_{xy}(-1,1)]^2$

$= (-6) \cdot (-6) - 9$

$= 36 - 9 = 27$

$D(-1,1) > 0 \Rightarrow$ extrema

type of extrema

$f_{xx}(-1,1) = -6 \Rightarrow$ maximum

ANS. C

34. $f_x = 2x$ $\quad f_y = 2y$

$g_x = y$ $\quad g_y = x$

$\lambda g_x = \lambda y$ $\quad \lambda g_y = \lambda x$

where λ is Lagrange multiplier.

$f_x = \lambda g_x$ $\quad f_y = \lambda g_y$

and $g(x,y) = 0$ will be solved simultaneously

$2x = \lambda y \quad -(1)$ $\quad 2y = \lambda x \quad -(2)$

$xy - 2 = 0 \quad -(3)$

~~$y = \frac{2x}{\lambda}$~~ from (1)

$\lambda = \frac{2x}{y} \quad -(4)$

(4) into (2)

$x = y \quad -(5)$

(5) into 3

$x = y = \pm\sqrt{2}$

$(\sqrt{2}, \sqrt{2})$

$(-\sqrt{2}, -\sqrt{2})$

for the value;

pluck any of the points obtained into the main fxn.

$f(\sqrt{2}, \sqrt{2}) = 2 + 2$

$= 4$

ANS. B

(36) $f_x = y \qquad f_y = x$

$g_x = 2x \qquad g_y = 2y$

$\lambda g_x = \lambda 2x \qquad \lambda g_y = \lambda 2y$

λ is the Lagrange multiplier

$f_x = \lambda g_x \quad -(1)$

$f_y = \lambda g_y \quad -(2)$

$g(x,y) \quad -(3)$

$\Rightarrow \quad y = \lambda 2x \quad -(1)$

$x = \lambda 2y \quad -(2)$

$x^2 + y^2 = 2 \quad -(3)$

from (1)

$\frac{y}{2x} = \lambda \quad -(4)$

(4) into (2)

$x = \frac{2y^2}{2x}$

$\pm x = y \quad -(5)$

(5) into (3)

$2x^2 = 2$

$x = \pm 1$

$\Rightarrow (1,1) \quad (-1,-1)$

$(1,-1) \quad (-1,1)$

Checking for minimum and maximum values for pt(s) of f.

Stick all the pts into $f(x,y)$

$(1,1)$ yields 1 $\rightarrow$ max value

$(-1,-1)$ yields 1 $\rightarrow$ max value

$(-1,1)$ yields -1 $\rightarrow$ min value

$(1,-1)$ yields -1 $\rightarrow$ min value

ANS. C

(37) from Q.36.

max value is 1

$\Rightarrow$ ANS. E

None of the a

(38) $f_x = y^2 e^{xy^2} \Big|_{(1,1)} = e.$

$f_y = 2xy\, e^{xy^2} \Big|_{(1,1)} = 2e$

$\Rightarrow \nabla f = e i + 2e j$

ANS. C

(39) $f(x,y) = \frac{-1}{x^2+y^2}$

$f(x,y) = -(x^2+y^2)^{-1}$

$f_x = \frac{2x}{(x^2+y^2)^2} \Big|_{(1,2)} = 0.08$

$f_y = \frac{2y}{(x^2+y^2)^2} \Big|_{(1,2)} = 0.16$

$\Rightarrow \nabla f = 0.08i + 0.16j$

ANS. B

(40) $\int_c x\,ds$; $x = t \quad y = t^2$

$\frac{dx}{dt} = x' = 1 \quad \frac{dy}{dt} = y' = 2t$

$ds = \sqrt{(x')^2 + (y')^2}\,dt = \sqrt{1+4t^2}\,dt$

$\Rightarrow \int_c x\,ds = \int_0^1 t\sqrt{1+4t^2}\,dt$

let $u = 1+4t^2$

change limits

$t = 1 \Rightarrow u = 5$

$t = 0 \Rightarrow u = 1$

$\frac{du}{dt} = 8t$

$\Rightarrow \int_0^1 t\sqrt{1+4t^2} = \int_1^5 t\,u^{1/2}\,\frac{du}{8t}$

$= \frac{1}{8}\int_1^5 u^{1/2}\,du$

$= \frac{1}{8} \cdot \frac{2u^{3/2}}{3} \Big|_1^5$

$= \frac{1}{12}(\sqrt{125} - 1)$

ANS. C

(41) $\int_c x\,dx + y\,dy = \int_c x\,dx + \int_c y\,dy$

$x = t$; $y = t^3$

$x' = 1 \quad y' = 3t^2$

$dx = dt \quad dy = 3t^2\,dt$

$\int_1^2 t\,dt + \int_1^2 t^3 \cdot 3t^2\,dt$

$\frac{t^2}{2}\Big|_1^2 + \frac{3t^6}{6}\Big|_1^2$

$\frac{4-1}{2} + \frac{64-1}{2}$

$\frac{3}{2} + \frac{33}{2} = \frac{36}{2} = 18$

ANS. B

(42) $\int_c xy\,dx - x\,dy$

$y = 1 - x^2$.

$\int_c x(1-x^2)\,dx - \int_c x\,dy$

$\int_c x - x^3\,dx - \int_c x\,dy$

$(1,0)$ to $(0,1)$

parameterize pts

$\vec{r}(t) = (1-t)\langle 1,0\rangle + t\langle 0,1\rangle$

$x = 1-t \quad x' = -1$

$y = t \quad y' = 1$

$dx = -dt \quad$ for $0 \le t \le 1$

$dy = dt$

$\int_0^1 -(1-t)\,dt + \int_0^1 (1-t)^3\,dt - \int_c (1-t)\,dt$

$-\int_0^1 (1-t)\,dt + \int_0^1 (1-t)^3\,dt - \int_0^1 (1-t)\,dt$

$$-\left[t-\frac{t^2}{2}\right]_0^1 = -\left[1-\frac{1}{2}\right] = -1/2$$

$$-\left[t-\frac{t^2}{2}\right]_0^1 = -\left[1-1/2\right] = -1/2$$

$$\int_0^1 (1-t)^3\,dt = \int_0^1 (1-3t+3t^2-t^3)\,dt$$

$$= t-\frac{3t^2}{2}+t^3-\frac{t^4}{4}\Big|_0^1$$

$$= \frac{1}{4}$$

$$-1/2+\frac{1}{4}-1/2 = -3/4$$

ANS. E

None of the above.

(43) $\int_C F\,dr$; $\vec{F}(x,y,z) = yz\,i + xz\,j + xy\,\vec{k}$

$\vec{r}(t) = t^2 i + t j + t^3 K$

$\vec{F}(\vec{r}(t)) = t^4 i + t^5 j + t^3 K$

$r'(t) = 2t i + j + 3t^2 K$

$\vec{F}(\vec{r}(t))\cdot r'(t) =$

$2t^5 + t^5 + 3t^5 = 6t^5$

$$\int_C F\,dr = \int_0^1 6t^5\,dt$$

$$= t^6\Big|_0^1 = 1$$

ANS. C

(44) Integrate in polar coordinates

$dA = r\,dr\,d\theta$

$r^2 = x^2+y^2$ $0 \le r \le 1$

$r = \sqrt{x^2+y^2}$ $0 \le \theta < 2\pi$

$$\int_0^{2\pi}\int_0^1 r\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta$$

$$= 2\pi/3$$

ANS A

(46) Same as Q44

ANS. E

None of the

(47) $x^2+y^2 = r^2 = z_1$

$z_2 = 1-r^2$

$z_1 = z_2$ @ the point of in

$1-r^2 = r^2$

$r = \frac{\sqrt{2}}{2}$ $r \ge 0$

we can see that the solid which i obtained is nothing but a cyli

we set;

$dV = r\,dz\,dr\,d\theta$

$r^2 \le z \le 1-r^2$

$0 \le r \le \frac{\sqrt{2}}{2}$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\sqrt{2}/2}\int_{r^2}^{1-r^2} r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}/2} r(1-r^2-r^2)\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}/2} r-2r^3\,dr\,d\theta$$

$$\int_0^{2\pi} \frac{r^2}{2}-\frac{r^4}{2}\Big|_0^{\sqrt{2}/2}\,d\theta$$

$$\int_0^{2\pi} \frac{1}{8}\,d\theta$$

$$V = 1/4 \times \pi = \pi/4$$

ANS. B

(48) $z_1 = x^2+y^2 = r^2$

$z_2 = 4$

$z_1 = z_2$ at the pt of intersection

$r^2 = 4$

$r = 2 \quad r \geq 0$

$\therefore \quad r^2 \leq z \leq 4 \qquad 0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$

~~set~~ $dV = r\,dz\,dr\,d\theta$

set;

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^{4} r\,dz\,dr\,d\theta$$

$$I = \int_0^{2\pi}\int_0^2 4r-r^3\,dr\,d\theta$$

$$\int_0^{2\pi} 2r^2-\frac{r^4}{4}\Big|_0^2\,d\theta$$

$$\int_0^{2\pi} 8-4\,d\theta = \int_0^{2\pi} 4\,d\theta$$

$$= 4\theta\Big|_0^{2\pi}$$

$$= 8\pi$$

ANS. D

(49) $x = r\cos\theta$

$\sqrt{x^2+y^2} = r$

$dA = r\,dr\,d\theta = r\,d\theta\,dr$

$x^2+y^2 = r^2 = 4$

$r = 2 \quad r \geq 0$

$\Rightarrow$ limits of r

$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

$$\int_0^2\int_0^{2\pi} r \cdot r\cos\theta \; r\,d\theta\,dr$$

$$\int_0^2\int_0^{2\pi} r^3\cos\theta\,d\theta\,dr$$

ANS. A

(51) $\frac{\delta x}{\delta u} = \sin v \qquad \frac{\delta x}{\delta v} = u\cos v$

$\frac{\delta y}{\delta u} = \cos v \qquad \frac{\delta y}{\delta v} = -u\sin v$

set

$$J = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix}$$

$J = -u\sin^2 v - u\cos^2 v$

$= -u(\sin^2 v + \cos^2 v)$

$\sin^2 v + \cos^2 v = 1$

$\Rightarrow J = -u$

ANS A

(52) change variable and find jacobians

$u = xy$ — (1) $\quad v = xy^2$ — (2)

$x = \frac{u}{y}$ — (3) $\quad$ (3) into (2)

$v = uy$

$y = \frac{v}{u}$ — (4)

(4) into (3)

$x = \frac{u^2}{v}$

$\Rightarrow x = \frac{u^2}{v} \quad y = \frac{v}{u}$

$\frac{\delta x}{\delta u} = \frac{2u}{v} \quad \frac{\delta y}{\delta u} = \frac{-v}{u^2}$

$\frac{\delta x}{\delta v} = \frac{-u^2}{v^2} \quad \frac{\delta y}{\delta v} = \frac{1}{u}$

$$J = dx\,dy = \begin{vmatrix} \frac{2u}{v} & \frac{-v}{u^2} \\ \frac{-u^2}{v^2} & \frac{-1}{u} \end{vmatrix}^T du\,dv$$

$= -\frac{2}{v} - \frac{1}{v} = -\frac{3}{v}$

$dx\,dy = \frac{3}{v}\,du\,dv$

(due to the transpose nature of Jacobian)

$y^2 = \frac{v^2}{u^2}$

$\Rightarrow$ we set

$$\int_1^2 \int_1^2 \frac{v^2}{u^2} \frac{3}{v}\, du\, dv$$

$$\int_1^2 \int_1^2 \frac{3v}{u^2}\, du\, dv$$

$$\int_1^2 -\frac{3v}{u}\Big|_1^2 dv$$

$$\int_1^2 -\frac{3}{2}v + 3v \; dv$$

direct computation gives

(If you had use $\frac{-3}{v}du\,dv$, your results would have been $-9/4$. which is also right)

ANS. E

(None of the

(53) $$\int_0^1 \int_0^1 \int_0^y x^2\, dz\, dy\, dx$$

$$\int_0^1 \int_0^1 x^2 y \; dy\, dx$$

$$\int_0^1 \frac{x^2 y^2}{2}\Big|_0^1 =$$

$$\int_0^1 \frac{x^2}{2}\; dx$$

$$V = \int \frac{x^3}{6}\Big|_0^1 = 1/6$$

ANS. D

55. $x+2y+3z=6$

$z = \frac{6-x-2y}{3}$

set $z=0$

$x = 6-2y$

set $x=0$

$y=3$

limits

$0 \leq z \leq \frac{6-x-2y}{3}$

$0 \leq x \leq 6-2y$

$0 \leq y \leq 3$

$$\int_0^3 \int_0^{6-2y} \int_0^{\frac{6-x-2y}{3}} dz\,dx\,dy$$

$$= \int_0^3 \int_0^{6-2y} \frac{6-x-2y}{3}\,dx\,dy$$

$$= \frac{1}{3}\int_0^3 6x - \frac{x^2}{2} - 2xy \Big|_0^{6-2y}$$

$$V = \frac{1}{3}\int_0^3 6(6-2y) - \frac{(36-24y+4y^2)}{2} - 2y(6-2y)\,\boxed{dy}$$

$$= \frac{1}{3}\int_0^3 \frac{4y^2-24y+36}{2}\,dy$$

(direct computation yields 6)

ANS. C

Q56 $x+y+z=3$

$z = 3-x-y$

set $z=0$

$y = 3-x$

set $y=0$

$x=3$

limits

$0 \leq z \leq 3-x-y$

$0 \leq y \leq 3-x$

$0 \leq x \leq 3$

$$\int_0^3 \int_0^{3-x} \int_0^{3-x-y} x\,dz\,dy\,dx$$

$$\int_0^3 \int_0^{3-x} 3x - x^2 - xy\,dy\,dx$$

$$\int_0^3 3xy - x^2y - \frac{xy^2}{2} \Big|_0^{3-x} dx$$

$$\int_0^3 3(3-x)x - x^2(3-x) - \frac{x(9-6x-x^2)}{2}\,dx$$

$$\int_0^3 9x - 3x^2 - 3x^2 + x^3 - \frac{9x+6x^2+x^3}{2}\,dx$$

$$V = \frac{1}{2}\int_0^3 18x - 6x^2 - 6x^2 + 2x^3 - 9x + 6x^2 + x$$

$$\frac{1}{2}\int_0^3 9x - 12x^2 + 6x^2 + 3x^3\,dx$$

$$\frac{1}{2}\int_0^3 3x^3 - 6x^2 + 9x\,dx$$

(direct computation yields $\frac{189}{8}$)

ANS. E (None of the above)

(57) $\int_0^1 \int_{x^2}^1 \int_0^{3y} (y + 2x^2 z)\, dz\, dy\, dx$

$\int_0^1 \int_{x^2}^1 yz + x^2 z^2 \Big|_0^{3y} dy\, dx$

$\int_0^1 \int_{x^2}^1 3y^2 + x^2 9y^2 \; dy\, dx$

$\int_0^1 y^3 + 3x^2 y^3 \Big|_{x^2}^{1} dx$

$\int_0^1 [1 + 3x^2] - [x^6 + 3x^8]\, dx$

$\int_0^1 -3x^8 - x^6 + 3x^2 + 1 \; dx$

(direct computation on your calculator will yield $32/21$

ANS B

(58) $z = x^2 + y^2$

$y = 1 - x$

set $y = 0$

$x = 1$

limits

$0 \leq x \leq 1$

$0 \leq y \leq 1 - x$

$\int_0^1 \int_0^{1-x} x^2 + y^2 \; dy\, dx$

$\int_0^1 x^2 y + \frac{y^3}{3} \Big|_0^{1-x}$

$\int_0^1 x^2 - x^3 + \frac{1 - 3x + 3x^2 - x^3}{3}$

$\frac{1}{3} \int_0^1 3x^2 - 3x^3 + 1 - 3x + 3x^2 - x^3$

$\frac{1}{3} \int_0^1 6x^2 - 4x^3 - 3x + 1 \; dx$

direct computation on your calculator is $\frac{1}{6}$

ANS. A

(59) Hint: Over here, you wi[ll] have to change the whole integral to cylindrical co-ordin[ates]

$\Rightarrow 0 \leq z \leq r$

$0 \leq r \leq 5$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

set:

$\int_{-\pi/2}^{\pi/2} \int_0^5 \int_0^r \frac{1}{r}\, r\, dz\, dr\, d\theta$

$\int_{-\pi/2}^{\pi/2} \int_0^5 r\, dr\, d\theta$

$\int_{-\pi/2}^{\pi/2} \frac{r^2}{2} \Big|_0^5 d\theta$

$\int_{-\pi/2}^{\pi/2} \frac{25}{2}\, d\theta = \frac{25\pi}{2}$

ANS B

[into polar co-ordinates]

$$\int_{-\pi/2}^{\pi/2}\int_0^2 \sqrt{9-r^2}\, r\,dr\,d\theta$$

integrate w.r.t r by using substitution

let $u = 9 - r^2$

$\frac{du}{dr} = -2r$

change limits

$r = 2 \Rightarrow u = 5$

$r = 0 \Rightarrow u = 9$

$$\int_{-\pi/2}^{\pi/2} d\theta \left[-\int_9^5 \frac{\sqrt{u}}{2}\, du\right]$$

(swap limits and negate integral)

$$\int_{-\pi/2}^{\pi/2} d\theta \int_5^9 \frac{\sqrt{u}}{2}\, du$$

$$\int_{-\pi/2}^{\pi/2} \frac{u^{3/2}}{3}\Big|_5^9 \, d\theta$$

$$\int_{-\pi/2}^{\pi/2} \frac{9-5^{3/2}}{3}\, d\theta$$

$$\frac{\pi}{2}\left[\frac{9-5^{3/2}}{3}\right] + \frac{\pi}{2}\left[\frac{9-5^{3/2}}{3}\right]$$

$$\pi\left[\frac{9-5^{3/2}}{3}\right]$$

ANS A

(60) $\iiint r^2\, dV$

dV in sphereical coordinates is $r^2 \sin\phi\, dr\, d\theta\, d\phi$

$0 \leq \phi \leq \frac{\pi}{2}$ $\quad 0 \leq \theta \leq 2\pi$

$0 \leq r \leq 1$

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r^2 \cdot r^2 \sin\phi\, dr\, d\theta\, d\phi$$

$$\int_0^{\pi/2}\int_0^{2\pi} \frac{r^5}{5}\Big|_0^1 \sin\phi\, d\theta\, d\phi$$

$$\int_0^{\pi/2}\int_0^{2\pi} \frac{\sin\phi}{5}\, d\theta\, d\phi$$

$$\int_0^{\pi/2} \frac{2\pi}{5}\sin\phi\, d\phi$$

$$\frac{2\pi}{5}\left[-\cos\phi\right]_0^{\pi/2}$$

$$= \frac{2\pi}{5}$$

ANS A

(61.) ANS A

NOTE

1. The curl of a grad is $\vec{0}$
2. The divergence of a curl is 0

These are the two null identities.

(62) $r(t) = xi + yj$

$x = t \qquad y = 2t^2$

$r(t) = ti + 2t^2 j$

$r'(t) = i + 4t j$

$F(r(t)) = 3(t)(2t^2) - (2t^2)^2$

$= 6t^3 i - 4t^4 j$

$F(r(t)) \cdot r'(t) = 6t^3 - 16t^5$

$\int_0^1 6t^3 - 16t^5 \, dt$

$= \frac{3t^4}{2} - \frac{8}{3}t^6 \Big|_0^1$

$\frac{3}{2} - \frac{8}{3} = -7/6$ ANS A

11th MARCH, 2015
MID-SEM EXAMS

(1.) Surface is $z+1 = xe^y\cos z$

$f(x,y,z) = xe^y\cos z - z - 1$

$f_x = e^y\cos z \quad = 1$

$f_y = xe^y\cos z \quad = 1$

$f_z = -xe^y\sin z - 1 \quad = -1 \quad |_{(1,0,0)}$

Eqn of tangent

$f_x(x-1) + f_y(y-0) + f_z(z-0) = 0$

$(x-1) + (y-0) + (-1)(z-0) = 0$

$x+y-z = 1$

ANS. C

(2.) $f(x,y,z) = x\sin yz$

$\text{grad} f = \langle f_x, f_y, f_z \rangle$

$f_x = \sin yz$

$f_y = xz\cos yz$

$f_z = xy\cos yz$

$\Rightarrow \text{grad} f(x,y,z) = \langle \sin yz, xz\cos yz, xy\cos yz \rangle$

ANS. A

(3) Directional derivative

$\text{grad} f(x_0, y_0, z_0) \cdot \hat{y}$

$(x_0, y_0, z_0) = (1, 3, 0)$

$y = i + 2j - k$

$|y| = \sqrt{1^2 + 2^2 + (-1)^2}$

$= \sqrt{6}$

$\hat{y} = \frac{i}{\sqrt{6}} + \frac{2j}{\sqrt{6}} - \frac{k}{\sqrt{6}}$

from Q2

$\text{grad} f \quad f_x = \sin yz \quad = 0i$

$f_y = xz\cos yz \quad = 0j$

$f_z = xy\cos yz \quad = 3k \quad |_{(1,3,0)}$

directional derivative

$\text{grad} f(1,3,0) \cdot \hat{y}$

$= \frac{-3}{\sqrt{6}} = \frac{-\sqrt{3}}{\sqrt{2}}$

ANS. D

(4) $\frac{\delta^3 u}{\partial x \partial y \partial z}$ if $u = \ln(x+2y^2+3z^3)$

$\frac{\partial u}{\partial x} = \frac{1}{x+2y^2+3z^3} = (x+2y^2+3z^3)^{-1}$

$\frac{\partial u}{\partial x \partial y} = 4y(-1)(x+2y^2+3z^3)^{-2}$

$= -4y(x+2y^2+3z^3)^{-2}$

$\frac{\partial u}{\partial x \partial y \partial z} = -4y(9z^2)(-2)(x+2y^2+3z^3)^{-3}$

$= \frac{72yz^2}{(x+2y^2+3z^3)^3}$

ANS. A

(5) $\frac{\partial z}{\partial s}$ if $z = e^r\cos\theta$

$r = st$

$\theta = \sqrt{s^2+t^2}$

$\frac{\partial z}{\partial s} = \left[\frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial s}\right] + \left[\frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial s}\right]$

$\frac{\partial z}{\partial r} = e^r\cos\theta \qquad \frac{\partial z}{\partial \theta} = -e^r\sin\theta$

$\frac{\partial r}{\partial s} = t$ $\qquad \frac{\partial \theta}{\partial s} = \frac{1}{2} 2s(s^2+t^2)^{-1/2}$

$= s(s^2+t^2)^{-1/2}$

$\Rightarrow \frac{\partial z}{\partial s} = te^r\cos\theta + \left(-e^r\sin\theta \cdot \frac{s}{\sqrt{s^2+t^2}}\right)$

$= \frac{\partial z}{\partial s} = e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$

ANS. A

⑥ $z = 6 - xy$ plane $x = 2$, $x = -2$

$0 \le y \le 3$

$\int_{-2}^{2}\int_{0}^{3} 6 - xy \, dy\, dx$

$\int_{-2}^{2} 6y - \frac{xy^2}{2}\Big|_0^3 dx$

$\int_{-2}^{2} 18 - \frac{9x}{2} dx$

$18x - \frac{9x^2}{4}\Big|_{-2}^{2}$

$(36 - 18) - (-36 - 18)$

$= 72$

ANS A

⑦ $\frac{\delta^3 f}{\delta y \delta x^2}$ if $f(x,y) = e^{xy}$

$\frac{\delta f}{\delta y} = xe^{xy}$

$\frac{\delta f}{\partial y \partial x} = xye^{xy} + e^{xy}$

$\frac{\delta f}{\delta y\delta x\delta x} = \frac{\delta f}{\delta y\delta x^2} = y[xye^{xy} + e^{xy}] + ye^{xy}$

$= \cancel{xy^2e^{xy} + ye^{xy} + y^2e^{xy}}$

$= y^2xe^{xy} + ye^{xy} + ye^{xy}$

$ye^{xy}[2 + xy]$

ANS B

⑧ ANS $x^2 + y^2 + z^2 < 4$

ANS. D

⑨ $x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$ but $r \ge 0$

$\Rightarrow r = \sqrt{x^2+y^2}$

$r \to 0$ IF $(x,y) \to (0,0)$

meaning

$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$ can

be written as

$\lim_{r\to 0^+} r^2 \ln r^2$

Apply L'hospital's rule

$\lim_{r\to 0} r^2\ln r^2 = \frac{2\ln r}{1/r^2} = \lim_{r\to 0} \frac{2/r}{-2/r^3}$

$= \lim_{r\to 0}(-r^2) = 0$

ANS. A

(11) $\iint_D (x+2y)\,dA$ where D is parabolas $y=2x^2$ and $y=1+x^2$

$2x^2 = 1+x^2$

$x^2 = 1 \qquad x = \pm 1$

$-1 \leq x \leq 1$

we any no b/n the limits of x to determine for the limits of y (taking $1/2$ in this case)

$y_1 = 1/2 \qquad y_2 = 5/4$

$\Rightarrow \quad 2x^2 \leq y \leq 1+x^2$

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y \; dy\,dx$$

$$\int_{-1}^{1} xy + y^2 \Big|_{2x^2}^{1+x^2} dx$$

$$\int_{-1}^{1} (x+x^3+1+2x^2+x^4) - (2x^3+4x^4)\,dx$$

$$\int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \; dx$$

(Direct computation on your calculator yields $\frac{32}{15}$)

ANS. B

(11) This question may seem tedious at first sight.

But when you check through the answers, you will recognize that the denominator changes.

· Sticking the limit (i.e 0,0) into f_2 (i.e $e^{\sqrt{x^2+y^2}}$) yields 1 (...)

implies ANS B

(you may refer to Handout (pg. 11) for detailed solution).

(12) ANS. D

(13) $z = \tan^{-1}(2x+y)$

$x = s^2 t$

$y = s \ln t$

$$\frac{\delta z}{\delta t} = \left[\frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta t}\right] + \left[\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta t}\right]$$

$$\frac{\delta z}{\delta x} = \frac{2}{1+(2x+y)^2} \qquad \frac{\delta z}{\delta y} = \frac{1}{1+(2x+y)^2}$$

$$\frac{\delta x}{\delta t} = s^2 \qquad \frac{\delta y}{\delta t} = \frac{s}{t}$$

$$\frac{\delta z}{\delta t} = \frac{2s^2 + (s/t)}{1+(2x+y)^2}$$

ANS. B.

(14) $\frac{\delta z}{\delta y}$ if $xe^y + yz = -ze^x + 100$

(using implicit partial differentiation)

$$xe^y + y\frac{dz}{\delta y} + z = -e^x\frac{\delta z}{\delta y}$$

$$\frac{\delta z}{\delta y} = \frac{-(xe^y+z)}{y+e^x}$$

ANS E (None of the above)

(15) ANS. B

(16) $z = x^2 \tan^{-1}(y/x)$ $\quad \frac{\delta^2 z}{\delta x \delta y}$

$$\frac{\delta z}{\delta x} = 2x \tan^{-1}(y/x) + x^2\left(\frac{-y}{x^2+y^2}\right)$$

$$\frac{\delta z}{\delta x \delta y} = 2x\left[\frac{x^2}{x^2+y^2}\right] + \frac{2y^2x^2}{(x^2+y^2)^2} \frac{-x^2}{(x^2+y^2)}$$

$$\frac{2x^3(x^2+y^2) + 2y^2x^2 - x^2(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{2x^3(x^2+y^2) + 2y^2x^2 - x^4 - x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^3(x^2+y^2) + y^2x^2 - x^4}{(x^2+y^2)^2}$$

ANS. E (None)

(17) $u = x^2 e^{y/x}$

$$du = \frac{\delta u}{\delta x}dx + \frac{\delta u}{\delta y}dy$$

$$\frac{\delta u}{\delta x} = -x^2 y x^{-2} e^{y/x} + 2x e^{y/x}$$

$$= -ye^{y/x} + 2xe^{y/x}$$

$$\frac{\delta u}{\delta y} = xe^{y/x}$$

$$\Rightarrow du = \left[2xe^{y/x} - ye^{y/x}\right]dx + xe^{y/x}dy$$

ANS. C

(18) $\frac{\delta h}{\delta s}$, $h(s,t) = \log_t s^2$

Using change of base

$$\log_t s^2 = \frac{\ln s^2}{\ln t}$$

$$\frac{\delta h}{\delta s} = \frac{2s\left(\frac{1}{s^2}\right)}{\ln t} = \frac{2}{s \ln t}$$

ANS. D

(20) $h(s,t) = t^\alpha \ln(s^2) + e^{t^2} - \frac{t}{\sqrt[5]{s^4}}$

$$\frac{\delta h}{\delta t} = \alpha t^{\alpha-1}\ln(s^2) + 2te^{t^2} - \frac{1}{\sqrt[5]{s^4}}$$

ANS. D

(21) ANS C

(22) ANS. C

(23) ANS A

(24) at the point of intersection

$y_1 = y_2$

$2x = x^2$

$x = 0, \quad x = 2$

$0 \leq x \leq 2$

(use any n° within the limits to check for the limits of y) using 1 in this case

$y_1 = 2 \quad y_2 = 1 \quad \Rightarrow \cancel{x^2} \le y \le 2x$

$\Rightarrow \int_2^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$

$\int_0^2 x^2y + \frac{y^3}{3} \Big|_{x^2}^{2x} dy\,dx$

$\int_0^2 \left[2x^3 + \frac{8x^3}{3}\right] - \left[x^4 + \frac{x^6}{3}\right] dx$

$\int_0^2 \frac{14x^3}{3} - x^4 - \frac{x^6}{3}\,dx$

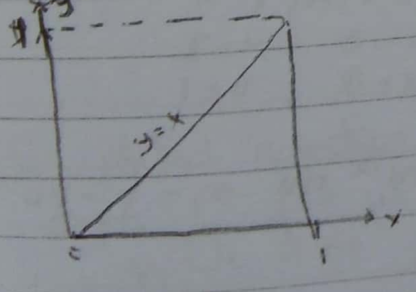
(direct computation on your calculator yields $\frac{216}{35}$)

ANS. D

(25) $\int_0^1 \int_x^1 \sin y^2\,dy\,dx$

(doing 'direct' integration will not be fruitful)

we employ 'change of order' in solving it



cont. limits for y. $0 \le y \le 1$

'variable' limits for x $0 \le x \le y$

$\Rightarrow \int_0^1 \int_0^y \sin y^2\,dx\,dy$

$\int_0^1 x\sin y^2 \Big|_0^y dy$

$\int_0^1 y\sin y^2\,dy$

(using integration by substitution)

$y^2 = u$

$\frac{du}{dy} = 2y \Rightarrow dy = \frac{du}{2y}$

limits

$1^2 = u = 1$

$0^2 = u = 0$

$\int_0^1 \frac{y}{2y} \sin u\,du$

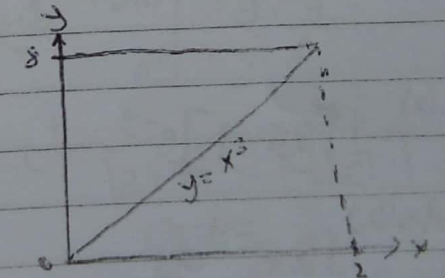
$\frac{1}{2}\int_0^1 \sin u\,du$

$\frac{1}{2}\Big[-\cos u\Big]_0^1$

$\frac{1}{2}\Big[(-\cos 1 + 1)\Big]$

$\frac{1}{2}\Big[1 - \cos 1\Big]$ ANS. C

(26) we employ 'change of order'



const limits for x, $0 \le x \le 2$

variable limits for y, $0 \le y \le x^3$

$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$

$\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 ye^{x^4} \Big|_0^{x^3} dx$

$= \int_0^2 x^3 e^{x^4}\,dx$

using integration by substitution,

$x^4 = u$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

<u>limits</u>

in u x^4.

upper limit $(2)^4 = 16$

lower limits $(0)^4 = 0$

$\Rightarrow \frac{1}{4}\int_0^{16} e^u \, du$

$\frac{1}{4} e^u \Big|_0^{16}$

$\frac{1}{4}(e^{16} - 1)$ ANS A

(27) $\int_0^{\ln(2)} \int_0^{\ln(5)} e^{2x-y} \, dx \, dy$

$\int_0^{\ln(2)} \frac{e^{2x-y}}{2} \Big|_0^{\ln(5)} dy$

$\int_0^{\ln(2)} \frac{e^{2x} \cdot e^{-y}}{2} \Big|_0^{\ln(5)} dy$

$\int_0^{\ln(2)} \frac{1}{2}[25 - 1] e^{-y} \, dy$

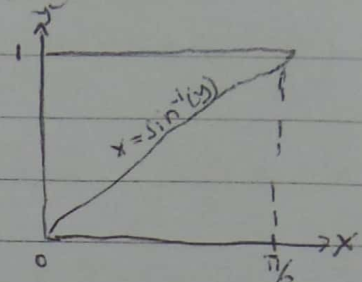
$\int_0^{\ln(2)} 12 e^{-y} \, dy$

(direct computation on your calculator will yield 6)

$\Rightarrow$ ANS C

(28) $\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \cos x (1 + \cos^2 x)^{1/2} \, dx \, dy$

we employ reverse order



Const limits for x $0 < x < \pi/2$

variable limits for y $0 < y < \sin x$

$\Rightarrow \int_0^{\pi/2} \int_0^{\sin x} \cos x (1 + \cos^2 x)^{1/2} \, dy \, dx$

$\int_0^{\pi/2} y \cos x (1 + \cos^2 x)^{1/2} \Big|_0^{\sin x} dx$

$\int_0^{\pi/2} \sin x \cos x (1 + \cos^2 x)^{1/2} \, dx$

let $u = 1 + \cos^2 x$

$\frac{du}{dx} = -2\cos x \sin x$

changing limits

$u = 1 + \cos^2 x$

for $x = 0$, $u = 2$

$x = \pi/2$, $u = 1$

$\Rightarrow \int_2^1 -\frac{1}{2} u^{1/2} \, du = -\int_1^2 -\frac{1}{2} u^{1/2} \, du$

$= \frac{1}{2} \int_1^2 u^{1/2} \, du$

$= \frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_1^2$

$\frac{1}{3}(\sqrt{8} - 1)$

$\frac{2\sqrt{2} - 1}{3}$

ANS E (None of the above)

(29) $r^2 = x^2+y^2$

integrating in polar co-ordinates

$dxdy = r\,dr\,d\theta$

$0 \le r \le 1/2 \quad 0 \le \theta \le 2\pi$

$\iint_D e^{(x^2+y^2)} = \int_0^{2\pi}\int_0^{1/2} e^{r^2} r\,dr\,d\theta$

$\int_0^{2\pi}\int_0^{1/2} e^{r^2} r\,dr\,d\theta$

let $u = r^2$

changing limits

$r = 1/2 \quad u = \frac{1}{4}$

$r = 0 \quad u = 0$

$\frac{du}{dr} = 2r \; ; \; dr = \frac{du}{2r}$

$\int_0^{2\pi}\int_0^{1/4} \frac{e^u}{2}\,du\,d\theta$

$\frac{1}{2}\int_0^{2\pi} e^{1/4} - 1 \, d\theta$

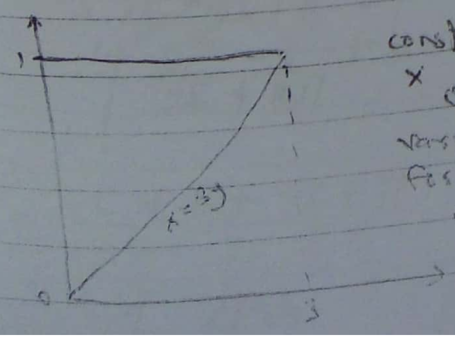
$\frac{1}{2}\, 2\pi \left[e^{1/4} - 1\right]$

$\pi\left[e^{1/4} - 1\right]$

ANS C

(30) Actual question should have been $\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy$

using reversing order as requested



const limit for x $0 \le x \le 3$

variable limit for y $0 \le y \le \frac{x}{3}$

$\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx$

R.H.S

$\int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 y e^{x^2}\Big|_0^{x/3} dx$

$= \int_0^3 \frac{x}{3} e^{x^2}\,dx$

(use the integration by substitution as described in several questions)

$\frac{1}{6}\int_0^9 e^u\,du$

$\frac{1}{6}\left(e^9 - 1\right)$

ANS A

(31) $\int_1^3\int_0^1 (1+4xy)\,dx\,dy$

$\int_1^3 x + 2x^2 y \Big|_0^1 dy$

$\int_1^3 (1+2y)\,dy$

$y + y^2 \Big|_1^3$

$[3+9] - [1+1]$

$12 - 2 = 10$

ANS B

(33) ANS A

(34) Integrating in polar co-ordinates

$z = r^2 = x^2+y^2$

$dxdy = rdrd\theta$

$0 \leq r \leq 3 \quad 0 < \theta < 2\pi$

$$\int_0^{2\pi}\int_0^3 r^2 \, r dr d\theta = \int_0^{2\pi}\int_0^3 r^3 dr d\theta$$

$$\int_0^{2\pi} \frac{r^4}{4}\Big|_0^3 d\theta = \int_0^{2\pi} \frac{81}{4} d\theta$$

$$= \frac{81(2\pi)}{4} = \frac{81\pi}{2}$$

ANS B

(35) $\iint_R \frac{xy^2}{x^2+1} dA$

[it will be wise to integrate wrt x first]

$$\int_{-3}^{3}\int_0^1 \frac{xy^2}{x^2+1} dx\, dy$$

$$\int_{-3}^{3} y^2 \int_0^1 \frac{x}{x^2+1} dx\, dy$$

let $u = x^2+1$

$\frac{du}{dx} = 2x \; ; \; dx = \frac{du}{2x}$

(change limits)

$x = 1 \Rightarrow u = 2$

$x = 0 \Rightarrow u = 1$

$$\int_{-3}^{3} y^2 \frac{1}{2}\int_1^2 u^{-1} du\, dy$$

$$\frac{1}{2}\int_{-3}^{3} y^2 \ln u \Big|_1^2 dy$$

$$\frac{1}{2}\ln 2 \int_{-3}^{3} y^2 dy$$

$$\frac{\ln 2}{2} \cdot \frac{y^3}{3}\Big|_{-3}^{3}$$

$$\frac{\ln 2}{6}\left[27+27\right] = 9\ln 2$$

ANS D

(36) $\int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y \, dx\, dy$

$$\int_0^{\pi/2} \cos y \cdot -\cos x \Big|_0^{\pi/2} dy$$

$$\int_0^{\pi/2} \cos y \left[-0 - -1\right] dy$$

$$\int_0^{\pi/2} \cos y \, dy$$

$$\sin y \Big|_0^{\pi/2}$$

$1 - 0 = 1$

ANS A

(38) $\int_{-1}^{1}\int_0^{\pi} 1 + e^x \sin y \, dy\, dx$

$$\int_{-1}^{1} y + (-e^x \cos y)\Big|_0^{\pi} dx$$

$$\int_{-1}^{1} (\pi + e^x + e^x) dx$$

$$\int_{-1}^{1} \pi + 2e^x dx$$

$$\pi x + 2e^x \Big|_{-1}^{1}$$

$\pi + 2e + \pi + \frac{2}{e}$

ANS B

$$\int_0^1\int_0^1 x(x^2+y)^{1/2}\,dy\,dx$$

(it will be very wise to integrate wrt y first)

$$\int_0^1 \left.\frac{2x(x^2+y)^{3/2}}{3}\right|_0^1 dx$$

$$\frac{2}{3}\int_0^1 x\left[(x^2+1)^{3/2}-x^3\right]dx$$

$$\frac{2}{3}\left[\int_0^1 x(x^2+1)^{3/2}dx-\int_0^1 x^4dx\right] - (1)$$

for $\int_0^1 x^4dx = \frac{1}{5}$

for $\int_0^1 x(x^2+1)^{3/2}dx$

let $x^2+1 = u$ $\quad dx = \frac{du}{2x}$

change limit

$x=1 \quad u=2$

$x=0 \quad u=1$

$$\frac{1}{2}\int_1^2 u^{3/2}\,du$$

$$\frac{1}{2}\left[\frac{2u^{5/2}}{5}\right]_1^2$$

$$\frac{1}{2}\left[\frac{2\cdot2^{5/2}}{5}-\frac{2}{5}\right]$$

$$\frac{2^{5/2}}{5}-\frac{1}{5}$$

Into (1)

$$\frac{2}{3}\left[\frac{2^{5/2}}{5}-\frac{1}{5}-\frac{1}{5}\right]$$

$$\frac{2}{3}\left[\frac{2^{5/2}}{5}-\frac{2}{5}\right] = 0.4875$$

NO ANSWER

(19) $f(x,y) = 6\sqrt{x^4+y-10^{xy}}$

$= 6(x^4+y-10^{xy})^{1/2}$

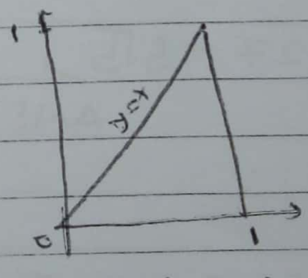
$$\frac{\delta f}{\delta y} = \frac{1}{2}6(1-x10^{xy}\ln 10)(x^4+y-10^{xy})^{-1/2}$$

$$= \frac{3-3x10^{xy}\ln 10}{\sqrt{x^4+y-10^{xy}}}$$

ANS. E (None of the above)

NB: $\frac{\delta(10^{xy})}{\delta y} = x10^{xy}\cdot\ln 10$

(39)



$0 \le y \le 1$ OR $0 \le x \le 1$

$0 \le x \le y \quad\quad 0 \le y \le x$

$f_x = 2x \quad\quad f_y = 2$

$$A = \iint_T \sqrt{|f_x|^2+|f_y|^2+1}\,dA$$

$$\int_0^1\int_0^x \sqrt{(2x)^2+(2)^2+1}\,dy\,dx$$

$$\int_0^1\int_0^x \sqrt{4x^2+5}\,dy\,dx$$

$$\int_0^1\int_0^x \sqrt{4x^2+5}\; dy\,dx$$

$$\int_0^1 x\sqrt{4x^2+5}\; dx$$

let $u = 4x^2+5$

$dx = \frac{du}{8x}$

change limits

$x = 1 \;,\; u = 9$

$x = 0 \;;\; u = 5$

$$\int_5^9 \frac{\sqrt{u}}{8}\, du$$

$$\frac{2\cdot u^{3/2}}{8\cdot 3}\Bigg|_5^9$$

$$\frac{1}{12}\left[9^{3/2} - 5^{3/2}\right]$$

$$\frac{1}{12}\left[27 - 5\sqrt{5}\right]$$

ANS. A

(40) $z = x^2+y^2 \quad z = 9$

$$\iint_R \sqrt{1+(f_x)^2+(f_y)^2}\; dA$$

$f_x = 2x \rightarrow (f_x)^2 = 4x^2$

$f_y = 2y \rightarrow (f_y)^2 = 4y^2$

$$\iint_R \sqrt{1+4x^2+4y^2}\; dx\,dy$$

Integrating in polar

$z = r^2 = x^2+y^2 \Rightarrow z = r^2 = 9$

$r = 3, r \geq 0$

$0 \leq r \leq 3$

$0 < \theta \leq 2\pi$

$4x^2+4y^2 = 4r^2$

$dx\,dy = r\,dr\,d\theta$

set:

$$\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

let $u = 1+4r^2$

$dx = \frac{du}{8r}$

change limits

$r = 3 \;;\; u = 37$

$r = 0 \;;\; u = 1$

$\rightarrow$ integration by subst.

reset:

$$\int_0^{2\pi}\int_1^{37} \frac{u^{1/2}}{8}\, du\,d\theta$$

$$\int_0^{2\pi} \frac{1}{12}\left[37\sqrt{37} - 1\right] d\theta$$

$$= \frac{\pi}{6}\left[37\sqrt{37} - 1\right]$$

ANS. D

2) $f_x = 2x \qquad f_x^2 = 4x^2$

$f_y = 2y \qquad f_y^2 = 4y^2$

$r = x^2+y^2 = 2\cos\theta$

$0 \leq r \leq 2\cos\theta$

set

$$\iint_R \sqrt{1+4x^2+4y^2}\,dA = \iint_R \sqrt{1+4r^2}\,dA$$

$dA = r\,dr\,d\theta$ (polar coord.)

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \sqrt{1+4r^2}\, r\,dr\,d\theta$$

$$\int_{-\pi/2}^{\pi/2} \frac{\sqrt{(1+4r^2)^3}}{12}\Bigg|_0^{2\cos\theta} d\theta$$

$$\frac{1}{12}\int_{-\pi/2}^{\pi/2} \sqrt{1+16\cos^2\theta} - 1 \; d\theta$$

direct computation yields $\simeq 7.904$

ANS. D $\simeq 7.85$ (ignore neg for absolute value)

ANS. D

(compute in radians)

END OF SEMESTER EXAMS
2010 | 2011

① $z = 10 - x^2 - y^2$

at $(1,2)$

$z = 10 - 1 - 4 = 5$

$(1,2,5)$

let $f(x,y,z) = z + x^2 + y^2 - 10$

$f_x = 2x$	$= 2$	
$f_y = 2y$	$= 4$	
$f_z = 1$	$= 1$	
		$(1,2,5)$

$\langle 2, 4, 1 \rangle$

but since y was fixed

we get $\langle 2, 2, 1 \rangle$

parameterize $\langle 1,2,5 \rangle$ to $\langle 2,2,1 \rangle$

$\vec{r}(t) = (1-t)\langle 1,2,5 \rangle + t\langle 2,2,1 \rangle$

$x = 1 - t + 2t$

$x = 1 + t$

$y = 2 - 2t + 2t$

$= 2$

$z = 5 - 5t + t$

$z = 5 - 4t$

$= (1+t, 2, 5-4t)$

ANS. E

None of the above

② $\frac{\delta u}{\delta t} = \frac{3t^2 r^6}{s^2}$

$\frac{\delta u}{\delta s} = -\frac{2t^3 r^6}{s^3}$

$\frac{\delta u}{\delta r} = \frac{6t^3 r^5}{s^2}$

$du = \frac{\delta u}{\delta t}dt + \frac{\delta u}{\delta r}dr + \frac{\delta u}{\delta s}ds$

$= \frac{3t^2 r^6}{s^2}dt - \frac{2t^3 r^6}{s^3}ds + \frac{6t^3 r^5}{s^2}dr$

ANS. A

③

$\frac{dx}{du} = -e^{-u}$	$= -1$
$\frac{dy}{du} = 2\cos u$	$= 2$
$\frac{dz}{du} = 1 + \sin u$	$= 1$
	$u = 0$

$\langle -1, 2, 1 \rangle$

It's from this vector that we are going to get our unit vector $\hat{v}$

$|v| = \sqrt{6}$

$\hat{v} = \frac{-1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j + \frac{k}{\sqrt{6}}$

also, our point P,

$x = e^{-u}$	$= 1$
$y = 2\sin u + 1$	$= 1$
$z = u - \cos u$	$= -1$
	$u = 0$

$(1, 1, -1)$

$$F = x^2yz^3$$

$$\left.\begin{aligned} F_x &= 2xyz^3 \\ F_y &= x^2z^3 \\ F_z &= 3x^2yz^2 \end{aligned}\right| \begin{aligned} &= -2 \\ &= -1 \\ &= 3 \end{aligned}$$

$(1,1,-1)$

$$\nabla F = -2i - j + 3k$$

$$\nabla f \cdot \hat{v}$$

$$= \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} + \frac{3}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

ANS. A

(5) Let $f(x,y,z) = 2x^2 + 4yz - 5z^2 + 10$

$$\left.\begin{aligned} f_x &= 4x \\ f_y &= 4z \\ f_z &= 4y - 10z \end{aligned}\right| \begin{aligned} &= 12 \\ &= 8 \\ &= -24 \end{aligned}$$

$(3,-1,2)$

$$\hat{n} = \frac{\nabla F(x_0, y_0, z_0)}{\|\nabla F(x_0, y_0, z_0)\|}$$

$$\nabla F(x_0, y_0, z_0) = 12i + 8j - 24k$$

$$= 3i + 2j - 6k$$

$$\|\nabla F(x_0, y_0, z_0)\| = \sqrt{9+4+36}$$

$$= 7$$

$$\hat{n} = \frac{1}{7}(3i + 2j - 6k)$$

ANS. C

(7) $U = (x+y)(1-xy)^{-1}$

$$U_x = (x+y)(-1)(-y)(1-xy)^{-2} + (1-xy)^{-1}$$

$$= \frac{y(x+y)}{(1-xy)^2} + \frac{1}{(1-xy)^2} =$$

$$U_y = \frac{x(x+y)}{(1-xy)^2} + \frac{1}{(1-xy)^2}$$

$$V_x = \frac{1}{1+x^2}$$

$$V_y = \frac{1}{1+y^2}$$

$$\begin{vmatrix} U_x & V_x \\ U_y & V_y \end{vmatrix}$$

$$\begin{vmatrix} \frac{y^2+1}{(1-xy)^2} & \frac{1}{1+x^2} \\ \frac{x^2+1}{(1-xy)^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$\frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

ANS A

(11)
$$\left.\begin{aligned} g_x &= e^{yz} + ye^z \\ g_y &= xze^{yz} + xe^z \\ g_z &= xye^{yz} + xye^z \end{aligned}\right| \begin{aligned} &2e \\ &-4e \\ &-4e \end{aligned}$$

$(-2,1,1)$

$\hat{v} = \frac{i}{\sqrt{14}} - \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k$

$\nabla f(-2,1,1) \cdot \hat{v}$

$\frac{2e}{\sqrt{14}} + \frac{8e}{\sqrt{14}} - \frac{12e}{\sqrt{14}}$

$\frac{-2e}{\sqrt{14}} = \frac{-e\sqrt{14}}{7}$

ANS. D

(12) $f_x = 6xy - 6x$

$f_{xx} = 6y - 6$

$f_{xy} = 6x$

$f_y = 3x^2 + 3y^2 - 6y$

$f_{yy} = 6y - 6$

at the stationary pts, $f_x = 0$
$f_y = 0$

$6xy - 6x = 0$

$y = 1 \quad (0,1)$

$3x^2 + 3 - 6 = 0$

$3x^2 = 3$

$x = \pm 1$

$(1,1)$

$(-1,1)$

$3y^2 - 6y = 0$

$y = 2$

$(0,2)$

therefore the pt(s) are
$(0,1)$; $(1,1)$; $(-1,1)$, $(0,2)$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$D(0,1) = 0 \Rightarrow$ can not conclude.

$D(1,1) = -36 \Rightarrow$ saddle

$D(-1,1) = -36 \Rightarrow$ saddle

$D(0,2) = 6 \Rightarrow$ extrema

type of extrema

$f_{xx}(0,2) = 12 \Rightarrow$ minimum

ANS. B

(13) Plug $(0,2)$ into $f(x,y)$
This yields -2

ANS C

(15) let $z = e^{kr}$

$\frac{dz}{dt} = \left[\frac{dz}{dr} \cdot \frac{dr}{dx} \cdot \frac{dx}{dt}\right] + \left[\frac{dz}{dr} \cdot \frac{dr}{dy} \cdot \frac{dy}{dt}\right]$

$\frac{dz}{dr} = ke^{kr}$

$\frac{dr}{dx} = x(x^2+y^2)^{-1/2} = \frac{x}{r}$

$\frac{dr}{dy} = \frac{y}{r}$

$\frac{dx}{dt} = Aw\cos wt \quad \frac{dy}{dt} = -Bw\sin wt$

$\frac{dz}{dt} = \frac{wke^{kr}}{r}(xA\cos wt - yB\sin wt)$

$= \frac{xywke^{kr}}{r^2}(A-B)$

ANS. C

(16) $\int_0^{\pi} \cos^4\theta \, d\theta$

This is beta function applied to trig.

It's have the format

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$$2\int_0^{\pi/2} \cos^4\theta \, d\theta$$

$\int_0^{\pi/2} \cos^4\theta \, d\theta$ being compared to the format'

$$2n-1 = 4$$
$$n = \frac{5}{2}$$

$$2m-1 = 0 \quad \text{(no sin fxn.)}$$
$$m = \frac{1}{2}$$

$$2\left[\frac{1}{2}\frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{5}{2}}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)}\right] = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma 3}$$

$\Gamma\frac{1}{2} = \sqrt{\pi}$ $\qquad \Gamma 3 = 2! = 2$

$$\Gamma\frac{5}{2} = \frac{3}{2}\Gamma\frac{3}{2}$$

$$\Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2}$$

$$\Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \Gamma\frac{5}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{2} = \frac{3\pi}{4\cdot 2} = \frac{3}{8}\pi$$

ANS: A

[NB" Γ is called gamma]
It is different from the sq. rt, $\sqrt{\ }$)

(17) $\int_0^{a} y^4 \sqrt{a^2-y^2} \, dy$

This is a beta function

$$B(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1} dy = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$$

So we'll model the given integral to suit the definition of Beta fxn.

let $u = \frac{y^2}{a^2}$ (substitution)

(change limits)

$y = a \ ; \ u = 1$

$y = 0 \ ; \ u = 0$

$y = a\sqrt{u}$

$$\frac{dy}{du} = \frac{a}{2}u^{-1/2}$$

$$dy = \frac{a}{2}u^{-1/2}\,du$$

Set;

$$\int_0^1 (a\sqrt{u})^4 \sqrt{a^2-a^2u}\;\frac{a\,u^{-1/2}}{2}\,du$$

$$\int_0^1 a^4u^2 \cdot a\sqrt{1-u}\;\frac{a}{2}u^{-1/2}\,du$$

$$\frac{a^6}{2}\int_0^1 u^{3/2}\sqrt{1-u}\,du$$

(we can call the above a def. of Beta)

by comparing;

$m-1 = \frac{3}{2}$; $m = 5/2$

$n-1 = 1/2$; $m = \frac{3}{2}$

$\Gamma m = \Gamma 5/2 = \frac{3}{4}\sqrt{\pi}$ $\quad \Gamma m+n = \Gamma \frac{5}{2}+\frac{3}{2}$
$\Gamma n = \Gamma 3/2 = \frac{\sqrt{\pi}}{2}$ $\quad = \Gamma 4$
$= 3! = 6$

(we had the above' ~~because~~ because we did similar stuff in the last question).

Set $\frac{\Gamma m \cdot \Gamma n}{\Gamma m+n} = \frac{\frac{3\pi}{8}}{6} = \frac{3}{48}\pi$

$\frac{9^6}{2}\left[\frac{3}{48}\pi\right] = \frac{9^6 \pi}{32}$

ANS. E

(18) $\int_0^2 x^2(2-x)^{-1/2}\,dx$

Beta fxn

let $n = \frac{x}{2}$; $2n = x$

Change limits
$x=2$; $n=1$
$x=0$; $n=0$
$dx = 2\,dn$

Set

$\int_0^1 4n^2(2-2n)^{-1/2}\,2dn$

$\int_0^1 \frac{8}{\sqrt{2}} n^2(1-n)^{-1/2}\,dn$

$\frac{8}{\sqrt{2}}\int_0^1 n^2(1-n)^{-1/2}\,dn$

by comparing;

$m-1 = 2$; $n-1 = -1/2$
$m = 3$ $\quad n = 1/2$

$B(m,n) = \frac{\Gamma m \; \Gamma n}{\Gamma m+n} = \frac{\Gamma 3 \cdot \Gamma 1/2}{\Gamma 7/2}$

$\Gamma 7/2 = 5/2\,\Gamma 5/2$
but $\Gamma 5/2 = \frac{3}{4}\sqrt{\pi}$ $\Rightarrow \Gamma 7/2 = \frac{15\sqrt{\pi}}{8}$

$\Gamma 1/2 = \sqrt{\pi}$
$\Gamma 3 = 2! = 2$

$B(m,n) = \frac{2\cdot\sqrt{\pi}}{\frac{15}{8}\sqrt{\pi}} = \frac{16}{15}$

~~but~~

$\Rightarrow \frac{8}{\sqrt{2}} \times \frac{16}{15} = \frac{64\sqrt{2}}{15}$ ANS. B

(19) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$

This is a gamma fxn

Def
$\Gamma m = \int_0^\infty e^{-n} n^{m-1}\,dn$

(so we'll model the given integral to form gamma) let $u = -\ln x$

$x = e^{-u}$

change limits

$x = 1 \quad ; \quad u = 0$

$x = 0 \quad ; \quad u = \infty$

$dx = -e^{-u} du$

set

$$\int_{\infty}^{0} \frac{-e^{-u}}{\sqrt{u}} du = -\int_{0}^{\infty} \frac{-e^{-u}}{\sqrt{u}} du$$

$$= \int_{0}^{\infty} e^{-u} \cdot u^{-1/2} du$$

This fit our definition for gamma

where $m-1 = -1/2$

$m = 1/2$

$\Gamma m = \Gamma 1/2 = \sqrt{\pi}$

ANS. B

(20) $\int_0^{\infty} 3^{-4x^2} dx$

This is a gamma fxn.

$e^{\ln 3} = 3$

$\Rightarrow e^{-4x^2 \ln 3} = 3^{-4x^2}$

let $4x^2 \ln 3 = u$

(L.H.S)

change limits

$x \Rightarrow \infty \quad ; \quad u = \infty$

$x = 0 \quad ; \quad u = 0$

$x = \frac{\sqrt{u}}{2\sqrt{\ln 3}}$

$dx = \frac{u^{-1/2}}{4\sqrt{\ln 3}} du$

set

$$\int_0^{\infty} e^{-u} \cdot \frac{u^{-1/2}}{4\sqrt{\ln 3}} du$$

$$\frac{1}{4\sqrt{\ln 3}} \int_0^{\infty} e^{-u} \cdot u^{-1/2} du$$

This fit our definition for gamma

where $m-1 = -1/2$

$m = 1/2$

$\Gamma m = \Gamma 1/2 = \sqrt{\pi}$

$\Rightarrow \frac{1}{4\sqrt{\ln 3}} \cdot \sqrt{\pi} = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}$

ANS. D

(21.) $f_x = y\,i$

$f_y = x\,j$

$\nabla f = y i + x j$

ANS A

(22) $f_x = -(4y - y^2)\sin x$

$f_y = (4-2y)\cos x$

$\nabla f(x,y) = \langle -(4y-y^2)\sin x, (4-2y)\cos x \rangle$

ANS. B

(23) ANS. C

refer to pg. 113 of Handout

(24) ANS. D

(25) ANS. C

(26) Parametric eqn of C

$x = 4\cos t$

$y = 4\sin t$

$\frac{dx}{dt} = -4\sin t$

$\frac{dy}{dt} = 4\cos t$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \sqrt{16\sin^2 t + 16\cos^2 t}\, dt$

$= 4dt$

set

$\int_{-\pi/2}^{\pi/2} 4\cos t\, (4\sin t)^4\, 4dt$

$4096 \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\, dt$

set $u = \sin t$

$\frac{du}{dt} = \cos t$

$\frac{du}{\cos t} = dt$

Change limits

$t = \pi/2 \quad u = 1$

$t = -\pi/2 \quad u = -1$

reset

$4096 \int_{-1}^{1} u^4\, du$

$\frac{4096}{5} u^5 \Big|_{-1}^{1}$

$\frac{4096}{5} + \frac{4096}{5} = \frac{8192}{5}$

ANS A

(27) ANS B.

refer to pg. 104 of handout.

(28) $\frac{\partial g}{\partial x_1} = \frac{x_3 e^{x_1} \sin(e^{x_1})}{x_2 + x_4}$

ANS. A

(29) $r = g \circ f = \left(2(t^2), -(1+t^3)\right)^3$

$r = (2t^2 - 1 - t^3)^3$

$\frac{dr}{dt} = 3(4t - 3t^2)(2t^2 - 1 - t^3)^2$

at $t = 2$

$= 3(8-12)(8-1-8)^2$

$= 3(-4)(\;)$

$= -12$ ANS. B

(30) $f_x = \cos y - z$ | -2

$f_y = -x\sin y$ | 0

$f_z = 9z - x$ | 8

$(1,\pi,1)$

$\nabla f(1,\pi,1) = -2i + 8K$

$\hat{v} = \frac{3}{\sqrt{17}} - \frac{2}{\sqrt{17}} + \frac{2}{\sqrt{17}}$

$\nabla f(1,\pi,1) \cdot \hat{v}$

$-\frac{6}{\sqrt{17}} + \frac{16}{\sqrt{17}} = \frac{10}{\sqrt{17}}$ A.

31. ANS D

refer to pg. 24/23 of handout.

32. $x = t$ $\quad \frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2t$

$y = t^2$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{1 + 4t^2}\, dt$$

$-1 \leq t \leq 1$

set

$$\int_{-1}^{1} t\sqrt{1+4t^2}\, dt$$

let $u = 1 + 4t^2$

$$dt = \frac{du}{8t}$$

$t = 1 \Rightarrow u = 5$

$t = -1 : u = 5$

$$\frac{\sqrt{u}\, du}{8}$$

$$\frac{1}{8}\left[\frac{2}{3 \cdot 8} u^{3/2}\right]_5^5 = 0$$

ANS. B

33. $f_x = 2x + 2y \quad f_{xx} = 2$

$f_y = 2x \quad f_{yy} = 0$

$f_x = 0$, $f_y = 0$ at crit points

$f_{xy} = 2$

$x = 0, \; y = 0 \quad (0,0)$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$D(0,0) = 0 - 4$

$-4 \Rightarrow$ saddle pt

ANS. C

34. B

35. ANS. A

refer to pg. 9 of handout.

36. $\vec{r}(t) = (1-t)\langle -1, 2, 0\rangle + t\langle 3, 0, 1\rangle$

$\vec{r}(t)$: $x(t) = 4t - 1$, $y(t) = 2 - 2t$, $z(t) = t$

$r'(t) = [x'(t) = 4\, i \quad z'(t) = 1\, k$, $y'(t) = -2\, j]$

$F(\vec{r}(t)) = [(4t-1)t]\, i - [t(2-2t)]\, k$

$= (4t^2 - t)\, i - (2t - 2t^2)\, k$

$F(\vec{r}(t)) \cdot r'(t) = 16t^2 - 4t - 2t + 2t^2$

$= 18t^2 - 6t$

set

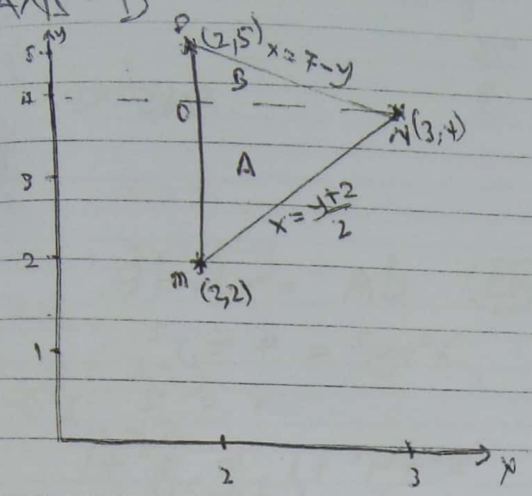
$$\int_0^1 18t^2 - 6t\, dt$$

(compute on your calculator to get 3)

ANS. B

(37) ANS. D

(38)



for MNO (aka triangle A)
we can set

$$\int_2^4 \int_2^{\frac{y+2}{2}} f(x,y)\,dx\,dy$$

for NOP (aka triangle B)
we can set

$$\int_4^5 \int_2^{7-y} f(x,y)\,dx\,dy$$

(addition of the two yields the the total region for MNP.

ANS. B.

(39) for polar cordinates

$x = r\cos\theta$

$x^2 = r^2\cos^2\theta$

$dA = r\,dr\,d\theta$

$0 \le \theta \le \frac{\pi}{2} = \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

Set

$$\int_0^9 \int_{\pi/4}^{3\pi/4} r^3\cos^2\theta\,dr\,d\theta$$

ANS. C

(40) $dA = dx\,dy \,/\, dy\,dx$

given

$x = -y$ $\quad \cancel{x} = y$

(I did because, we need const. limits for $\cancel{xy}$)

$\Rightarrow y^2 + y^2 = a^2$

$y^2 = \frac{a^2}{2}$

$y = \pm\frac{a}{\sqrt{2}}$

set

$$\int_{-a/\sqrt{2}}^{a/\sqrt{2}} \int_{-y}^{y} f(x,y)\,dx\,dy$$

where $f(x,y) = x^2$

ANS. D

(41) $y_1 = 9 - x^2$ $\quad y_2 = 1 + x^2$

$y_1 = y_2$ at the pts of intersection

$9 - x^2 = 1 + x^2$

$4 = x^2$

$\pm 2 = x$

$-2 \le x \le 2$

set

$$\int_{-2}^{2} \int_{1+x^2}^{9-x^2} y\,dy\,dx$$

$$\frac{1}{2}\int_{-2}^{2} y^2 \Big|_{1+x^2}^{9-x^2} dx$$

$$\frac{1}{2}\int_{-2}^{2} (81 - 18x^2 + x^4) - (1 + 2x^2 + x^4)\,dx$$

$$\frac{1}{2}\int_{-2}^{2} 80-20x^2 \, dx$$

Computation with your calculator yield $\frac{320}{3}$

ANS A

(42) $y_1 = 4[1-(x-2)^2]$

$y_2 = 9[1-(x-2)^2]$

at the pt(s) of intersection,

$y_1 = y_2$

$$4-4(x-2)^2 = 9-9(x-2)^2$$

$$5(x-2)^2 = 5$$

$$(x-2)^2 = 1$$

$$x^2-4x+3 = 0$$

$$x=3;\ x=1$$

$$1 \le x \le 3$$

$$4[1-(x-2)^2] \le y \le 9[1-(x-2)^2]$$

Set

$$\int_1^3 \int_{4[1-(x-2)^2]}^{9[1-(x-2)^2]} dy\,dx$$

$$\int_1^3 9-9(x-2)^2-4+4(x-2)^2 \, dx$$

$$\int_1^3 5-5(x^2-4x+4)\, dx$$

$$\int_1^3 5-5x^2+20x-20 \, dx$$

$$\int_1^3 (-5x^2+20x-15)\, dx$$

Computation with calculator yields $\frac{20}{3}$

ANS C

(43) $dA = r\,dr\,d\theta$

$x^2+y^2 = 4 = r^2$

$r = 2 \quad r \ge 0$

$x^2+y^2+1 = r^2+1$

$-2\pi \le \theta \le 2\pi$

Set

$$\int_{-2\pi}^{2\pi}\int_0^2 (r^2+1)r\,dr\,d\theta$$

let $u = r^2+1$

$dr = \frac{du}{2r}$

(Change limits)

$r=2,\quad u=5$

$r=0,\quad u=1$

$$\int_{-2\pi}^{2\pi}\int_1^5 \frac{u}{2}\,du\,d\theta$$

(you may have go ahead to multiply by the r outside the bracket, but I will like you to understand the integration by substitution, since it has been the core of this topic)

$$\int_{-2\pi}^{2\pi} \frac{u^2}{4}\Big|_1^5 \, d\theta$$

$$\int_{-2\pi}^{2\pi} \frac{25-1}{4}\, d\theta = 24\pi$$

ANS. A

(47) $\iint_S x\,dx\,dy$

$x_1 = y$

$x_2 = 2-y$

$x_1 = x_2$

$y = 2-y$

$2y = 2$;

$0 \le y \le 1$

$\int_0^1 \int_y^{2-y} x\,dx\,dy$

$\int_0^1 \frac{x^2}{2}\Big|_y^{2-y} dy$

$\int_0^1 \frac{4-4y+y^2-y^2}{2}\,dy$

$\int_0^1 2-2y\,dy$

$2y-y^2\Big|_0^1$

1

ANS. B

(48) $x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

$x = 8\cos\frac{\pi}{4}\sin\frac{3}{4}\pi = 4$

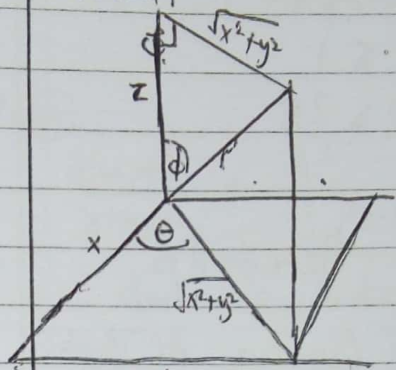
$y = 8\sin\frac{\pi}{4}\sin\frac{3}{4}\pi = 4$

$z = 8\cos\frac{3}{4}\pi = -4\sqrt{2}$

ANS. B

(49) $2 = \frac{x^2+y^2}{z^2}$

recall



$\tan\phi = \frac{\sqrt{x^2+y^2}}{z} =$

but $\frac{x^2+y^2}{z^2} = 2$

$\Rightarrow \sqrt{2} = \frac{\sqrt{x^2+y^2}}{z}$

$\Rightarrow \tan\phi = \frac{\sqrt{x^2+y^2}}{z} = \sqrt{2}$

ANS. A

(50) $x^2+y^2+(z-1)^2 = 1$

$z = x^2+y^2$

$r^2 = x^2+y^2$

$r^2 + (z-1)^2 < 1$

$(z-1)^2 = 1-r^2$

$z = \sqrt{1-r^2}+1$

$z = r^2$

$r^2 \le z \le \sqrt{1-r^2}+1$

$r^2 + (r^2-1)^2 = 1$

$(r^2-1)^2 = (1-r^2)$

$r^4 - 2r^2 + 1 = 1 - r^2$

$r^4 = r^2$

$r = 1$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

Set

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{1+\sqrt{1-r^2}} dz\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r(1+\sqrt{1-r^2}-r^2)\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r + r\sqrt{1-r^2} - r^3\, dr\, d\theta$$

$$\int_0^{2\pi} d\theta \left[\int_0^1 r\,dr + \int_0^1 r\sqrt{1-r^2}\,dr - \int_0^1 r^3\,dr\right]$$

$$\int_0^{2\pi} d\theta \left[\frac{r^2}{2}\Big|_0^1 + \frac{u^{3/2}}{3}\Big|_0^1 - \frac{r^4}{4}\Big|_0^1\right]$$

$$\int_0^{2\pi} d\theta \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right]$$

$$\int_0^{2\pi} \left[\frac{6+4-3}{12}\right] d\theta$$

$$\frac{7}{12}\cdot 2\pi$$

$$\frac{7}{6}\pi$$

ANS C

$z = x^2 + y^2$

at the intersection

$r^2 + (z-2)^2 = 4$

$(z-2)^2 = 4 - r^2$

$z = \sqrt{4-r^2} + 2$

$z = r^2$

$r^2 \leq z \leq \sqrt{4-r^2} + 2$

find for the limits of r

$r^2 + (r^2-2)^2 = 4$

$r^2 + r^4 - 2r^2 + 4 = 4$

$r^4 = r^2$

$r = 1 \;;\; r \geq 0$

$\Rightarrow 0 \leq r \leq 1$

$0 \leq \theta \leq \pi$ (above the first octant and second octant

$$\int_0^{\pi}\int_0^1\int_{r^2}^{\sqrt{4-r^2}+2} r\, dz\, dr\, d\theta$$

$$\int_0^{\pi}\int_0^1 r(\sqrt{4-r^2}+2-r^2)\, dr\, d\theta$$

$$\int_0^{\pi}\int_0^1 2r + r\sqrt{4-r^2} - r^3\, dr\, d\theta$$

$$\int_0^{\pi} r^2\Big|_0^1 + \frac{u^{3/2}}{3}\Big|_3^4 - \frac{r^4}{4}\Big|_0^1 d\theta$$

$$\int_0^{\pi} 1 + \frac{8}{3} - \frac{3^{3/2}}{3} - \frac{1}{4}\, d\theta$$

16 17 //

ANS C gives 16 16

ANS. C

(51) $x^2+y^2+z^2 = 4z$

$= x^2+y^2+(z-2)^2 = 4$

(52) $P(1,1,1)$

$Q(0,0,2)$

$R(0,1,0)$

$\overrightarrow{PQ} = (0-1, 0-1, 2-1)$
$= (-1, -1, 1)$

$\overrightarrow{PR} = (0-1, 1-1, 0-1)$
$= (-1, 0, -1)$

$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & -1 & 1 \\ -1 & 0 & -1 \end{vmatrix}$$

$\vec{n} = i - 2j - k$

Eqn. of a plane

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

where $\langle a, b, c \rangle$ normal vector components

(x_0, y_0, z_0) points in the plane

taking $P(1,1,1)$

$1(x-1) - 2(y-1) - 1(z-1) = 0$

$x - 2y - z + 2 = 0$

ANS. C

(53) Recall

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \cos\phi$

$z = \rho \cos\phi$

set

$\boxed{\rho\cos\phi = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$

for $z = x^2 + y^2$

$\rho\cos\phi = \rho^2 \sin^2\phi (\cos^2\theta + \sin^2\theta)$

$\rho\cos\phi = \rho^2 \sin^2\phi$

NB $\cos^2\theta + \sin^2\theta = 1$

$1 = \rho \dfrac{\sin\phi}{\cos\phi} \cdot \sin\phi$

$1 = \rho \tan\phi \sin\phi$

ANS A

(54)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$du\,dv = 2\,dx\,dy$

but the natural form of dA in this case is $dx\,dy$

$\Rightarrow dx\,dy = dA = \dfrac{du\,dv}{2}$

Set

$$\int_{-1}^{1}\int_{-1}^{1} \frac{e^{-v} u^2\,du\,dv}{2}$$

$$\int_{-1}^{1} \frac{u^3}{2}\Big[-e^{-v}\Big]_{-1}^{1} du$$

$$\int_{-1}^{1} \frac{e - e^{-1}}{2} u^2\,du$$

$$\frac{e-e^{-1}}{2}\left[\frac{u^3}{3}\right]_{-1}^{1}$$

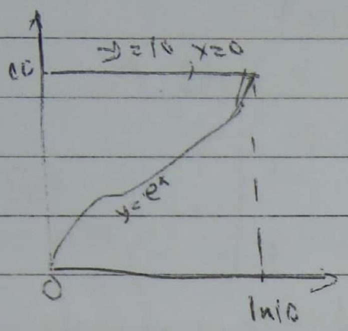
$$\frac{e-e^{-1}}{2}\left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right]$$

$$\frac{e-e^{-1}}{2}\left[\frac{2}{3}\right] = \frac{e-e^{-1}}{3}$$

ANS B

(55) $\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx$

(Change order)



constant limit for y, now and variable limit for x.

y varies from 0 to 10

$$0 \le y \le 10$$

x varies from line $x=0$ to $x=\ln y$

$$0 \le x \le \ln y$$

$$\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx = \int_0^{10}\int_0^{\ln y} \frac{1}{\ln y}\, dx\, dy$$

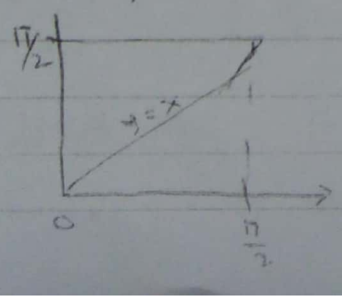
$$= \int_0^{10} dy$$

$$= 10$$

ANS. B

(56) $\int_0^{\pi/2}\int_x^{\pi/2} \frac{\cos y}{y}\, dy\, dx$

(change order)



const. limits for y, now and variable limit for x

$$0 \le y \le \pi/2$$

$$0 \le x \le y$$

set

$$\int_0^{\pi/2}\int_0^{y} \frac{\cos y}{y}\, dx\, dy$$

$$\int_0^{\pi/2} \cos y\, dy$$

$$\sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

ANS. C

(58) $\frac{\partial x}{\partial u} = 2u \qquad \frac{\partial x}{\partial v} = -2v$

$\frac{\partial y}{\partial u} = 2v \qquad \frac{\partial y}{\partial v} = 2u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$dA = dx\,dy = (4u^2 + 4v^2)\,du\,dv$$

set $\int_1^2\int_1^2 (4u^2+4v^2)\,du\,dv$

$$\int_1^2 \frac{4u^3}{3} + 4v^2u \Big|_1^2 \, dv$$

$$\int_1^2 \frac{28}{3} + 4v^2\, dv$$

$$\left[\frac{28v}{3} + \frac{4v^3}{3}\right]_1^2 = \frac{56}{3}$$

ANS A

(59) $f(x,y,z) = x^2 + 2y^2 + 3z^2 - 15$

$f_x = 2x \quad = 2$

$f_y = 4y \quad = 4$

$f_z = 6z \quad = 12$

$(1,1,2)$

Eqn

$2(x-1) + 4(y-1) + 12(z-2) = 0$

$2x + 4y + 12z - 30 = 0$

$x + 2y + 6z - 15 = 0$

ANS C

(60) Recall:

Spherical
$$\begin{cases} \rho^2 = x^2+y^2+z^2 \\ z = \rho \sin\phi \\ dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \end{cases}$$

given $a^2 = x^2+y^2+z^2$

$\Rightarrow \rho = a \qquad \rho \geq 0$

$0 \leq \rho \leq a$

Set

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} (\rho \sin\phi)^2 \rho^2 \sin\phi \, \boxed{d\rho \, d\phi \, d\theta}$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} (\rho^4 \sin^3\phi) \, d\rho \, d\phi \, d\theta$$

ANS C

(61) Recall:

Cylindrical
$$\begin{cases} x^2+y^2 = r^2 \\ dV = r \, dz \, dr \, d\theta \end{cases}$$

given,

$z^2 = x^2+y^2 = r^2$

and $z^2 = a^2 - (x^2+y^2) = a^2 - r^2$

$r \leq z \leq \sqrt{a^2-r^2}$

for r, limits

$r^2 = a^2 - r^2$

$2r^2 = a^2$

$r = \frac{a}{\sqrt{2}} \qquad r \geq 0$

$\Rightarrow 0 \leq r \leq \frac{a}{\sqrt{2}} \qquad 0 \leq \theta \leq 2\pi$

Set

$$\int_0^{2\pi}\int_0^{\frac{a}{\sqrt{2}}}\int_r^{\sqrt{a^2-r^2}} r^2 \cdot r \, dz \, dr \, d\theta$$

$$\int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}} r^3 \, dz \, dr \, d\theta$$

(62) $z_1 = \sqrt{x^2+y^2}$

$z_2 = \sqrt{a^2-x^2-y^2}$

$\sqrt{x^2+y^2} \leq z \leq \sqrt{a^2-x^2-y^2}$

~~set z into both eqns~~

$z_1 = z_2$

$x^2+y^2 = a^2-x^2-y^2$

$2y^2 = a^2 - 2x^2$ — Eqn

$y = \pm\sqrt{\frac{a^2}{2} - x^2}$

$-\sqrt{\frac{a^2}{2} - x^2} \leq y \leq \sqrt{\frac{a^2}{2} - x^2}$

set $y = 0$ in Eqn

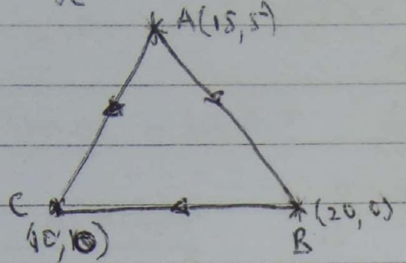
$2x^2 = a^2$

$x = \pm \frac{a}{\sqrt{2}}$

$-\frac{a}{\sqrt{2}} \leq x \leq \frac{a}{\sqrt{2}}$

ANS. B

(63) $\oint_C 3y\,dx - x^2 dy$



$\overrightarrow{AC}$

* gradient $m = -1$
  * $y = 20 - x$

$\frac{dy}{dx} = -1 \Rightarrow dy = -dx$

x moves from 15 to 20

$\Rightarrow 15 \leq x \leq 20$

integral

$$\oint_C 3y\,dx - x^2 dy = \boxed{\int_{15}^{20} 3(20-x)dx - x^2(-dx)}$$

$$= \int_{15}^{20} 60 - 3x\,dx + \int_{15}^{20} x^2 dx$$

HINT: USE YOUR CALCULATOR!! $= \frac{75}{2} + \frac{4625}{3}$

$\overrightarrow{BC}$

* zero gradient

$y = 0 \Rightarrow dy = 0$

x moves from 20 to 10

$20 \leq x \leq 10$

Integral

$$\int_{20}^{10} 3(0)dx - x^2(0) = 0$$

$\overrightarrow{CA}$

* gradient $m = 1$
  * $y = \cancel{x} - 10$

$\frac{dy}{dx} = 1 \Rightarrow dy = dx$

x moves from 10 to 15

$10 \leq x \leq 15$

integral

$$\int_{10}^{15} 3(x-10)\,dx - \int_{10}^{15} x^2 dx$$

$$\int_{10}^{15} 3x - 30\,dx - \int_{10}^{15} x^2 dx$$

$$\frac{75}{2} - \frac{2375}{3}$$

$$\oint_C 3y\,dx - x^2 dy = \frac{75}{2} + \frac{4625}{3} + 0 + \frac{75}{2} - \frac{2375}{3}$$

$= 825$

(but negate it because it was in a counterclockwise direction)

$= -825$

ANS. A

(64) $\vec{F} \cdot r'(t)$

$(3x + 3y)\,dx + (y - 3x)\,dy$

$P = 3x + 3y$ $\quad Q = y - 3x$

$\frac{\partial P}{\partial y} = 3$ $\quad \frac{\partial Q}{\partial x} = -3$

Using Green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In this case our region, R is circle centred at $\Rightarrow dA = r\,dr\,d\theta$

$$\int_{-2\pi}^{2\pi}\int_0^3 (-3-3)\,r\,dr\,d\theta = \int_{-2\pi}^{2\pi}\int_0^3 -6r\,dr\,d\theta$$

$$= \int_{-2\pi}^{2\pi} -18\,d\theta = -72\pi$$

ANS. B

(65) $u = x - 2y$ — (1) $\quad v = 2x + y$ — (2)

$u_x = 1 \qquad v_x = 2$

$u_y = -2 \qquad v_y = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} dx\,dy$$

$du\,dv = 1 + 4 = 5\,dx\,dy$

$dx\,dy = \frac{du\,dv}{5}$

$$\Rightarrow \int_{2}^{7}\int_{-3}^{5} \left[e^{u} + \left(\frac{u+2v}{5}\right)\left(\frac{v-2u}{5}\right)\right] \frac{1}{5}\, du\,dv$$

from (1) and (2)

$x = u + 2y$ — (3)

(3) into (2)

$v = 2u + 4y + y$

$\frac{v - 2u}{5} = y$ — (4)

(4) into (3)

$x = \frac{u + 2v}{5}$

ANS C

(67) $x^2 + y^2 = 2$

$g(x,y) = x^2 + y^2 - 2$

$g_x = 2x$

$g_y = 2y$

$\lambda(\nabla g(x,y))$ where λ is the Langrange's multiplier

$\langle \lambda 2x ; \lambda 2y \rangle$

$f_x = y \qquad f_y = x$

at the stationary pt(s)

$\lambda(\nabla g(x,y)) = \nabla f(x,y)$

$y = \lambda 2x$ — (1)

$x = \lambda 2y$ — (2)

$x^2 + y^2 = 2$ — (3)

Simultaneous

$\lambda = \frac{y}{2x}$ — (4)

(4) into (2)

$2x^2 = 2y^2$

$x = \pm\sqrt{y^2}$

$x = \pm y$ — (5)

$y^2 + y^2 = 2$

$2y^2 = 2$

$y = \pm 1$ — (6)

(6) into

These are the candidates (pts) for max and min values: $(1,1)\ (1,-1)\ (-1,1)\ (-1,-1)$

$(1,1)$ and $(-1,-1)$ will yield 1 when we plug it into $f(x,y)$ (max values)

$(-1,1)$ and $(1,-1)$ will yield -1, a minimum value

NONE OF THE ANS.

(68) at the pt. of intersection, let take $t = 0$

both lines gives $(-2,-3,1)$

ANS B

(69) for ~~line~~ both lines 1, let obtain pts using the direction (constants with t)

line 1 gives $(-3, 2, 1)$
line 2 gives $(-2, -1, 5)$

$$\hat{n} = \begin{vmatrix} i & j & k \\ -3 & 2 & 1 \\ -2 & -1 & 5 \end{vmatrix}$$

$$\hat{n} = 11i + 13j + 7k$$

Eqn of plane with Point P is given by

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

where x, y, z are arbitrary pts on the plane and x_0, y_0 and z_0 are the points of P

A, B and C are the i, j and k components of the normal vector respectively. Let $P = (2, 3, -1)$ ie when $t = 0$ on both lines

$$11(x-2) + 13(y-3) + 7(z+1) = 0$$
$$11x + 13y + 7z - 54 = 0$$

ANS B

(70) $f_x = 3x^2 + x$ $\quad f_{xx} = 6x$
$f_y = 3y^2 - 2y - 1$ $\quad f_{xy} = 0$
$f_{yy} = 6y - 2$

at the stationary pt(s);
$f_x = 0 \quad f_y = 0$
$3x^2 + x = 0$
$x = -1/3 \quad (-1/3, 0)$

$3y^2 - 2y - 1 = 0$ fa
$(3y+1)(y-1)$
$y = -1/3, 1$

Crit. pts $(-1/3, -1/3)$
(you do pairing up) $(-1/3, 1)$
$(-1/3, 0)$
$(0, 1)$
$(0, -1/3)$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$
$D(-1/3, -1/3) = 8 \Rightarrow$ extrema
$D(-1/3, 1) = -8 \Rightarrow$ Saddle
$D(-1/3, 0) = 4 \Rightarrow$ extrema
$D(0, 1) = 0 \Rightarrow$ no conclusion
$D(0, -1/3) = 0 \Rightarrow$ no conclusion

Checking for our extrema with
$f_{xx} = 6x$
$f_{xx}(-1/3, -1/3) = -2 \Rightarrow$ maximum
$f_{xx}(-1/3, 0) = -2 \Rightarrow$ maximum

Max pts $= (-1/3, -1/3), (-1/3, 0)$

Confirmed saddle $= (-1/3, 1)$

From this information, we can pick ANS. C though we can't conclude for $(0, 1)$ or $(0, -1/3)$ but maybe, they are minimum and saddle resp.