**Problem Set 11**
**Solutions**

1) Differentiate the two quantities with respect to time, use the chain rule and then the rigid body equations..

**17.6.18** Find a parametric representation for the surface which is the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$

The lower half of the ellipsoid is given by

$$z = -\sqrt{1 - 2x^2 - 4y^2}.$$

Let us choose $x$ and $y$ as parameters.

$$x = x, \quad y = y, \quad z = -\sqrt{1 - 2x^2 - 4y^2}.$$

Then, the vector equation is obtained as

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - \sqrt{1 - 2x^2 - 4y^2}\mathbf{k}.$$

**17.6.20** Find a parametric representation for the surface which is the part of the elliptic paraboloid $x + y^2 + 2z^2 = 4$ that lies in front of the plane $x = 0$

If you regard $y$ and $z$ as parameters, then the parametric equations are

$$x = 4 - y^2 - 2z^2, \quad y = y, \quad z = z, \quad y^2 + 2z^2 \leq 4.$$

The vector equation is obtained as

$$\mathbf{r}(y, z) = (4 - y^2 - 2z^2)\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

where $y^2 + 2z^2 \leq 4$.

**17.6.36** Find the area of the surface which is the part of the plane with vector equation $\mathbf{r}(u,v) = \langle 1+v, u-2v, 3-5u+v\rangle$ for $0 \le u \le 1$, $0 \le v \le 1$

$$\begin{aligned}
\mathbf{r}_u &= \left\langle \frac{\partial(1+v)}{\partial u}, \frac{\partial(u-2v)}{\partial u}, \frac{\partial(3-5u+v)}{\partial u} \right\rangle = \langle 0, 1, -5\rangle, \\
\mathbf{r}_v &= \left\langle \frac{\partial(1+v)}{\partial v}, \frac{\partial(u-2v)}{\partial v}, \frac{\partial(3-5u+v)}{\partial v} \right\rangle = \langle 1, -2, 1\rangle.
\end{aligned}$$

The area $A(S)$ is obtained as

$$\begin{aligned}
A(S) &= \int_0^1 \int_0^1 |\mathbf{r}_u \times \mathbf{r}_v| \, dv du \\
&= \int_0^1 \int_0^1 |\langle -9, -5, -1\rangle| \, dv du \\
&= |\langle -9, -5, -1\rangle| \\
&= \sqrt{107}.
\end{aligned}$$

**17.6.44** Find the area of the surface of the helicoid (or spiral ramp) with vector equation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$

$$\begin{aligned}
\mathbf{r}_u &= \left\langle \frac{\partial u\cos v}{\partial u}, \frac{\partial u\sin v}{\partial u}, \frac{\partial v}{\partial u} \right\rangle = \langle \cos v, \sin v, 0\rangle, \\
\mathbf{r}_v &= \left\langle \frac{\partial u\cos v}{\partial v}, \frac{\partial u\sin v}{\partial v}, \frac{\partial v}{\partial v} \right\rangle = \langle -u\sin v, u\cos v, 1\rangle.
\end{aligned}$$

The area $A(S)$ is obtained as

$$\begin{aligned}
A(S) &= \int_0^1 \int_0^1 |\mathbf{r}_u \times \mathbf{r}_v| \, dv du \\
&= \int_0^1 \int_0^1 |\langle \sin v, -\cos v, u\rangle| \, dv du \\
&= \int_0^1 \int_0^1 \sqrt{u^2+1} dv du.
\end{aligned}$$

Let us introduce $t$ ($u = \sinh t$). Note that by setting $\sinh^{-1}(1) = \ln s$, we obtain $s = 1 + \sqrt{2}$.

$$
\begin{aligned}
A(S) &= \int_0^{\sinh^{-1}(1)} \sqrt{\sinh^2 t + 1}\cosh t dt = \int_0^{\ln(1+\sqrt{2})} \cosh^2 t dt \\
&= \left[\frac{t}{2} + \frac{\sinh(2t)}{4}\right]_0^{\ln(1+\sqrt{2})} \\
&= \frac{\ln(1+\sqrt{2})}{2} + \frac{1}{2}\sinh\left(\sinh^{-1}(1)\right)\sqrt{1+\sinh^2\left(\sinh^{-1}(1)\right)} \\
&= \frac{1}{2}\left[\ln(1+\sqrt{2}) + \sqrt{2}\right].
\end{aligned}
$$

**17.7.6** Evaluate the surface integral $\iint_S xy\, dS$, where $S$ is the triangular region with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$

Let $P$, $Q$, and $R$ be vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$. Points in the triangle are expressed as

$$
\mathbf{r}(u,v) = \overrightarrow{OP} + u\overrightarrow{PQ} + v\left(\overrightarrow{PR} - \overrightarrow{PQ}\right) = (1-u)\mathbf{i} + (2u-2v)\mathbf{j} + 2v\mathbf{k},
$$

where $0 \le u \le 1$ and $0 \le v \le u$. We obtain

$$
\begin{aligned}
\mathbf{r}_u &= \left\langle \frac{\partial(1-u)}{\partial u}, \frac{\partial(2u-2v)}{\partial u}, \frac{\partial 2v}{\partial u} \right\rangle = \langle -1, 2, 0\rangle, \\
\mathbf{r}_v &= \left\langle \frac{\partial(1-u)}{\partial v}, \frac{\partial(2u-2v)}{\partial v}, \frac{\partial 2v}{\partial v} \right\rangle = \langle 0, -2, 2\rangle,
\end{aligned}
$$

and $\mathbf{r}_u \times \mathbf{r}_v = \langle 4, 2, 2\rangle$. The surface integral is calculated as

$$
\begin{aligned}
\iint_S xy\, dS &= \int_0^1 \int_0^u (1-u)(2u-2v)2\sqrt{6} dv du \\
&= 2\sqrt{6}\int_0^1 \left[(2u-2u^2)v + (u-1)v^2\right]_{v=0}^{v=u} du \\
&= 2\sqrt{6}\int_0^1 (u^2-u^3) du \\
&= \frac{1}{\sqrt{6}}.
\end{aligned}
$$

**17.7.14** Evaluate the surface integral $\iint_S xyz\, dS$, where $S$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Using the spherical coordinates, let us write the parametric equation as

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$

where $0 \le \phi \le \pi/4$ and $0 \le \theta \le 2\pi$ (we used $\rho = 1$). That is

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}.$$

We obtain

$$\begin{aligned}
\mathbf{r}_\phi &= \left\langle \frac{\partial \sin\phi\cos\theta}{\partial\phi}, \frac{\partial \sin\phi\sin\theta}{\partial\phi}, \frac{\partial\cos\phi}{\partial\phi} \right\rangle = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle, \\
\mathbf{r}_\theta &= \left\langle \frac{\partial \sin\phi\cos\theta}{\partial\theta}, \frac{\partial \sin\phi\sin\theta}{\partial\theta}, \frac{\partial\cos\phi}{\partial\theta} \right\rangle = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle.
\end{aligned}$$

Thus,

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi.$$

The surface integral is calculated as follows.

$$\begin{aligned}
\iint_S xyz\, dS &= \int_0^{2\pi}\int_0^{\pi/4} (\sin\phi\cos\theta)(\sin\phi\sin\theta)(\cos\phi)\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4} \cos\theta\sin\theta\sin^3\phi\cos\phi d\phi d\theta \\
&= \int_0^{2\pi} \frac{1}{2}\sin(2\theta)d\theta \left[\frac{\sin^4\phi}{4}\right]_0^{\pi/4} \\
&= 0.
\end{aligned}$$

**17.7.20** Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for vector field $\mathbf{F} = xy\mathbf{i} + 4x^2\mathbf{j} + yz\mathbf{k}$ and the oriented surface $S$ that is the surface $z = xe^y$, $0 \le x \le 1$, $0 \le y \le 1$, with upward orientation. In other words, find the flux of $\mathbf{F}$ across $S$.

Let $g(x,y) = xe^y$ and $f(x,y,z) = z - g(x,y)$. We have $f(x,y,z) = 0$ on the surface $S$. We obtain an upward unit normal vector as

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{(1+x^2)e^{2y}+1}} \langle -e^y, -xe^y, 1 \rangle .$$

Therefore,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} dS \\
&= \iint_D \langle xy, 4x^2, yz \rangle \cdot \frac{\langle -e^y, -xe^y, 1 \rangle}{\sqrt{(1+x^2)e^{2y}+1}} \sqrt{(1+x^2)e^{2y}+1} dA \\
&= \iint_D \left(-4x^3e^y\right) dA = -\int_0^1 \int_0^1 4x^3 e^y \, dydx \\
&= -\left[x^4\right]_0^1 \left[e^y\right]_0^1 \\
&= 1 - e.
\end{aligned}$$

**17.7.22** Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$ and the oriented surface $S$ that is the part of the cone $z = \sqrt{x^2+y^2}$ beneath the plane $z = 1$ with downward orientation. In other words, find the flux of $\mathbf{F}$ across $S$.

Using spherical coordinates, the cone is expressed as

$$x = \frac{\rho}{\sqrt{2}} \cos\theta, \quad y = \frac{\rho}{\sqrt{2}} \sin\theta, \quad z = \frac{\rho}{\sqrt{2}}.$$

Note that $\phi = \pi/4$. We obtain a vector equation

$$\mathbf{r}(\theta, \rho) = \frac{\rho}{\sqrt{2}} \cos\theta \mathbf{i} + \frac{\rho}{\sqrt{2}} \sin\theta \mathbf{j} + \frac{\rho}{\sqrt{2}} \mathbf{k},$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \rho \leq \sqrt{2}$. We obtain

$$\mathbf{r}_\theta = \left\langle -\frac{\rho}{\sqrt{2}} \sin\theta, \frac{\rho}{\sqrt{2}} \cos\theta, 0 \right\rangle, \quad \mathbf{r}_\rho = \left\langle \frac{1}{\sqrt{2}} \cos\theta, \frac{1}{\sqrt{2}} \sin\theta, \frac{1}{\sqrt{2}} \right\rangle,$$

Thus,

$$\mathbf{n} = \frac{\mathbf{r}_\theta \times \mathbf{r}_\rho}{|\mathbf{r}_\theta \times \mathbf{r}_\rho|} = \frac{\frac{1}{2}\langle \rho\cos\theta, \rho\sin\theta, -\rho \rangle}{|\frac{1}{2}\langle \rho\cos\theta, \rho\sin\theta, -\rho \rangle|}.$$

Finally,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_S \mathbf{F}\cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_\rho}{|\mathbf{r}_\theta \times \mathbf{r}_\rho|} dS \\
&= \iint_D \mathbf{F}\cdot (\mathbf{r}_\theta \times \mathbf{r}_\rho) dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} \langle x, y, z^4\rangle \cdot (\mathbf{r}_\theta \times \mathbf{r}_\rho) d\rho d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(\frac{\rho^2}{2\sqrt{2}} - \frac{\rho^5}{8}\right) d\rho d\theta = 2\pi \int_0^{\sqrt{2}} \left(\frac{\rho^2}{2\sqrt{2}} - \frac{\rho^5}{8}\right) d\rho \\
&= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}.
\end{aligned}$$

**17.8.4** Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = x^2y^3z\mathbf{i} + \sin(xyz)\mathbf{j} + xyz\mathbf{k}$, and $S$ is the part of the cone $y^2 = x^2 + z^2$ that lies between the planes $y = 0$ and $y = 3$, oriented in the direction of the positive $y$-axis.

The curve $C$ is given by

$$\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\mathbf{j} + 3\cos t\mathbf{k},$$

where $0 \le t \le 2\pi$. Thus,

$$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 3\sin t\mathbf{k}.$$

We obtain

$$
\begin{aligned}
\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_0^{2\pi} \left\langle 3^6\sin^2 t\cos t, \sin(3^3\sin t\cos t), 3^3\sin t\cos t\right\rangle \cdot \langle 3\cos t, 0, -3\sin t\rangle\, dt \\
&= \int_0^{2\pi} \left(3^7\sin^2 t\cos^2 t - 3^4\sin^2 t\cos t\right) dt \\
&= \int_0^{2\pi} \left(3^7\left(\frac{\sin(2t)}{2}\right)^2 - 3^4\sin t\frac{\sin(2t)}{2}\right) dt \\
&= \int_0^{2\pi} \left[\frac{3^7}{8}\left(1-\cos(4t)\right) + \frac{3^4}{4}\left(\cos(3t)-\cos t\right)\right] dt \\
&= \frac{3^7}{8}\left[t - \frac{1}{4}\sin(4t)\right]_0^{2\pi} + \frac{3^4}{4}\left[\frac{1}{3}\sin(3t) - \sin t\right]_0^{2\pi} \\
&= \frac{3^7}{8}(2\pi) = \frac{2187}{4}\pi.
\end{aligned}
$$

**17.8.8** Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}(x,y,z) = e^{-x}\mathbf{i} + e^x\mathbf{j} + e^z\mathbf{k}$. Here, $C$ is the boundary of the part of the plane $2x+y+2z=2$ in the first octant, and is oriented counterclockwise as viewed from above.

Let $S$ denote the surface bounded by $C$. We call the intercepts $P(1,0,0)$, $Q(0,2,0)$, and $R(0,0,1)$. The surface $S$ is given by

$$\mathbf{r}(u,v) = \overrightarrow{OP} + u\overrightarrow{PQ} + u\left(\overrightarrow{PR} - \overrightarrow{PQ}\right) = \langle 1-u, 2u-2v, v\rangle,$$

where $0 \le u \le 1$ and $0 \le v \le u$. Thus,

$$\mathbf{r}_u = \langle -1, 2, 0\rangle, \quad \mathbf{r}_v = \langle 0, -2, 1\rangle, \quad \mathbf{r}_u \times \mathbf{r}_v = \langle 2, 1, 2\rangle.$$

Therefore,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} \\
&= \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}dS \\
&= \iint_D \nabla\times\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)dA \\
&= \iint_D \langle 0,0,e^x\rangle\cdot\langle 2,1,2\rangle\, dA \\
&= \int_0^1\int_0^u 2e^{1-u}dvdu \\
&= \int_0^1 \left[2e^{1-u}v\right]_{v=0}^{v=u} du \\
&= \left[-2(1+u)e^{1-u}\right]_0^1 \\
&= 2e-4.
\end{aligned}$$

(Alternative Solution)

We have

$$\nabla\times\mathbf{F} = \langle 0,0,e^x\rangle, \quad \mathbf{n} = \frac{\langle 2,1,2\rangle}{|\langle 2,1,2\rangle|} = \left\langle \frac{2}{3},\frac{1}{3},\frac{2}{3}\right\rangle.$$

Since $z = 1-x-y/2$, we have

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1} = \frac{3}{2}.$$

Using these things, we obtain

$$\begin{aligned}
\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} &= \iint_D e^x\, dA \\
&= \int_0^1\int_0^{2-2x} e^x\, dydx \\
&= 2e-4.
\end{aligned}$$

**17.8.15** Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and surface $S$ which is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive $y$-axis.

We will show

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

The curve $C$ is given by

$$\mathbf{r}(t) = \langle \sin t, 0, \cos t \rangle,$$

where $0 \leq t \leq 2\pi$. Thus,

$$\begin{aligned}
\text{LHS} &= \int_0^{2\pi} \langle y, z, x \rangle \cdot \langle \cos t, 0, -\sin t \rangle \, dt \\
&= \int_0^{2\pi} (-\sin^2 t) dt = \int_0^{2\pi} \frac{\cos(2t) - 1}{2} dt \\
&= \left[ \frac{\sin(2t)}{4} - \frac{t}{2} \right]_0^{2\pi} = -\pi.
\end{aligned}$$

Using spherical coordinates, the hemisphere is expressed as

$$\mathbf{r}(\phi, \theta) = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq \pi$ $(\rho = 1)$. Thus,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \sin^2\phi \cos\theta, \sin^2\phi \sin\theta, \sin\phi \cos\phi \rangle, \quad |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi.$$

Therefore,

$$\mathbf{n} = \begin{cases} \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle & 0 \leq \phi \leq \pi/2, \\ \langle -\sin\phi \cos\theta, -\sin\phi \sin\theta, -\cos\phi \rangle & \pi/2 < \phi \leq \pi. \end{cases}$$

We obtain

$$\begin{aligned}
\text{RHS} &= \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_S \langle -1, -1, -1 \rangle \cdot \mathbf{n} dS \\
&= \iint_D \langle -1, -1, -1 \rangle \cdot \mathbf{n} \sin\phi \, dA \\
&= \int_0^\pi \int_0^{\pi/2} (-\sin^2\phi\cos\theta - \sin^2\phi\sin\theta - \cos\phi\sin\phi) d\phi d\theta \\
&+ \int_0^\pi \int_{\pi/2}^{\pi} (\sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \cos\phi\sin\phi) d\phi d\theta \\
&= [\sin\theta]_0^\pi \left[\frac{\sin(2\phi)}{4} - \frac{\phi}{2}\right]_0^{\pi/2} + [\cos\theta]_0^\pi \left[\frac{\sin(2\phi)}{4} - \frac{\phi}{2}\right]_0^{\pi/2} + [\theta]_0^\pi \left[\frac{\cos(2\phi)}{4}\right]_0^{\pi/2} \\
&+ [\sin\theta]_0^\pi \left[\frac{\sin(2\phi)}{4} - \frac{\phi}{2}\right]_{\pi/2}^{\pi} - [\cos\theta]_0^\pi \left[\frac{\sin(2\phi)}{4} - \frac{\phi}{2}\right]_{\pi/2}^{\pi} - [\theta]_0^\pi \left[\frac{\cos(2\phi)}{4}\right]_{\pi/2}^{\pi} \\
&= -\pi.
\end{aligned}$$

Therefore,

$$\text{LHS} = \text{RHS}.$$

The Stokes' Theorem is verified.

**17.8.18** Evaluate $\int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz$, where $C$ is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leq t \leq 2\pi$. [Hint: Observe that $C$ lies on the surface $z = 2xy$.]

Let us define

$$\mathbf{F} = \langle y + \sin x, z^2 + \cos y, x^3 \rangle.$$

We also define the surface $S$ bounded by $C$, which is given by

$$\mathbf{r}(x, y) = \langle x, y, 2xy \rangle.$$

Here, $x, y \in D$, where

$$D = \left\{ (x, y) | 0 \leq x^2 + y^2 \leq 1 \right\}.$$

Note that

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 0, 2y\rangle \times \langle 0, 1, 2x\rangle = \langle -2y, -2x, 1\rangle.$$

We obtain

$$\begin{aligned}
\int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz &= \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS \\
&= \iint_D \nabla \times \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) dA \\
&= \iint_D \langle -2z, -3x^2, -1\rangle \cdot \langle -2y, -2x, 1\rangle \, dA \\
&= \iint_D \left(6x(x^2 + y^2) - 1\right) \, dA \\
&= \int_0^{2\pi} \int_0^1 \left(6r^3 \cos\theta - 1\right) r dr d\theta \\
&= \int_0^{2\pi} \left[\frac{6}{5} r^5 \cos\theta - \frac{r^2}{2}\right]_{r=0}^{r=1} d\theta \\
&= \int_0^{2\pi} \left(\frac{6}{5}\cos\theta - \frac{1}{2}\right) d\theta \\
&= \left[\frac{6}{5}\sin\theta - \frac{\theta}{2}\right]_0^{2\pi} \\
&= \pi.
\end{aligned}$$

**17.9.4** Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ on the region E which is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane.

We will show

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} dV.$$

The paraboloid is expressed as

$$\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 4 - r^2\rangle,$$

where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Note that

$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, -2r\rangle, \quad \mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle, \quad \mathbf{r}_r \times \mathbf{r}_\theta = \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle.$$

The $xy$-plane is given by

$$z = 0, \qquad \text{or} \qquad \mathbf{r}(\theta, r) = \langle r\cos\theta, r\sin\theta, 0\rangle,$$

where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Note that

$$\mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle, \quad \mathbf{r}_r = \langle \cos\theta, \sin\theta, 0\rangle, \quad \mathbf{r}_\theta \times \mathbf{r}_r = \langle 0, 0, -r\rangle.$$

The left-hand side is calculated as follows.

$$\begin{aligned}
\text{LHS} &= \iint_S \mathbf{F}\cdot\mathbf{n}dS \\
&= \iint_{\text{paraboloid}} \mathbf{F}\cdot\mathbf{n}dS + \iint_{xy\text{-plane}} \mathbf{F}\cdot\mathbf{n}dS \\
&= \int_0^{2\pi}\int_0^2 \langle x^2, xy, z\rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle\, dr d\theta \\
&+ \int_0^{2\pi}\int_0^2 \langle x^2, xy, z\rangle \cdot \langle 0, 0, -r\rangle\, dr d\theta \\
&= \int_0^{2\pi}\int_0^2 \left(2r^4\cos^3\theta + 2r^4\cos\theta\sin^2\theta + (4-r^2)r\right)\, dr d\theta \\
&+ \int_0^{2\pi}\int_0^2 0\, dr d\theta \\
&= \int_0^{2\pi} \left(\frac{2^6}{5}\cos\theta + 4\right)\, d\theta \\
&= 8\pi.
\end{aligned}$$

We have

$$\nabla\cdot\mathbf{F} = 2x + x + 1 = 3x + 1.$$

The right-hand side is calculated as follows.

$$\begin{aligned}
\text{RHS} &= \iiint_E (3x+1)dV \\
&= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\, r dz dr d\theta \\
&= \int_0^{2\pi}\int_0^2 (-3r^4\cos\theta - r^3 + 12r^2\cos\theta + 4r)\, dr d\theta \\
&= \int_0^{2\pi} \left(\frac{2^6}{5}\cos\theta + 4\right)\, d\theta \\
&= 8\pi.
\end{aligned}$$

Therefore, LHS=RHS. The Divergence Theorem is verified.

**17.9.8** Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x,y,z) = x^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + xz^4\mathbf{k}$, and $S$ is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$. That is, calculate the flux of $\mathbf{F}$ across $S$.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \nabla \cdot \mathbf{F} dV \\
&= \int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} 8xz^3 dz dy dx \\
&= \int_{-1}^{1} 2x dx \int_{-2}^{2} dy \int_{-3}^{3} 4z^3 dz \\
&= 0.
\end{aligned}$$

**17.9.20** Let $\mathbf{F}(x,y,z) = z\tan^{-1}(y^2)\mathbf{i} + z^3 \ln(x^2+1)\mathbf{j} + z\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \nabla \cdot \mathbf{F} dV \\
&= \int_0^{2\pi} \int_0^1 \int_1^{2-r^2} 1\, r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (-r^3 + r)\, dr d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\, d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

**17.9.25** Prove the identity

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

assuming that $S$ satisfies the conditions of the Divergence Theorem and the components of the vector fields have continuous second-order partial derivatives.

Since $\operatorname{div} \operatorname{curl} \mathbf{F} = \mathbf{0}$, we have

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot (\nabla \times \mathbf{F}) dV = 0.$$