Part II Numerical Analysis (J6) Lent Term, 2000

Exercise Sheet 2 [1]

13. Let A be an $n \times n$ symmetric tridiagonal matrix that is not deflatable (i.e. all the elements of A that are adjacent to the diagonal are nonzero). Prove that A has n different eigenvalues. Prove also that, if A has an eigenvalue of zero and if one iteration of the QR algorithm is applied to A, then the resultant tridiagonal matrix is deflatable.
Hint: In the second part deduce that a diagonal element of R is zero.

14. Let A be a 2×2 symmetric matrix whose trace does not vanish, let $A_1 = A$, and let the sequence of matrices $\{A_k : k = 2, 3, \ldots\}$ be calculated by applying the QR algorithm to A_1 (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of A_k. Show that, except in the special case when A is already diagonal, the sequence $\{(A_k)_{1,1} : k = 1, 2, \ldots\}$ converges monotonically to the eigenvalue of A of larger modulus.
Hint: The sign of this eigenvalue is the sign of the trace of A.

15. Apply a single step of the QR method to the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & \varepsilon \\ 0 & \varepsilon & 0 \end{pmatrix}.$$

You should find that the $(2,3)$ element of the new matrix is $\mathcal{O}(\varepsilon^3)$ and that the new matrix has exactly the same trace as A.

16. (For those who like analysis). Let A be a real 4×4 upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices A_k, $k = 1, 2, 3, \ldots$, are calculated from A by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{3,2}$ converges to zero as $k \to \infty$.

17. Apply a single iteration of the QR algorithm with double shifts to the matrix

$$A = \begin{pmatrix} 0 & 2 & -1 & -1 \\ -1 & 1 & 0 & 2 \\ 0 & \varepsilon & -1 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

assuming $|\varepsilon|$ is so small that $\mathcal{O}(\varepsilon^2)$ terms are negligible. You should find that the first column of $(A - s_1 I)(A - s_2 I)$, where s_1 and s_2 are the shifts, has the elements 1, 0, $-\varepsilon$ and 0. Further, you should find that the iteration provides a matrix that is deflatable, because its $(3,2)$ element is of magnitude ε^2.

18. Let $h = 1/N$, where N is an integer, and let Euler's method be applied to calculate the estimates $\{y_n : n = 1, 2, \ldots, N\}$ of $y(nh)$ for each of the differential equations

$$y'(t) = f(t, y) = -y \,/\, (1+t), \qquad 0 \le t \le 1,$$
$$y'(t) = f(t, y) = 2y \,/\, (1+t), \qquad 0 \le t \le 1,$$

starting with $y_0 = y(0) = 1$ in both cases. By using induction analytically, and by cancelling as many terms as possible in the resultant products, deduce simple expressions for y_n, $n = 1, 2, \ldots, N$, which should be free from summations and products of n terms. Hence deduce the exact solutions of the equations from the limit $h \to 0$. Verify that the magnitudes of the errors $y(nh) - y_n$, $n = 1, 2, \ldots, N$, are at most $\mathcal{O}(h)$.

[1] Please send any corrections and comments by e-mail to mjdp@cam.ac.uk

19. The "trapezoidal rule" for solving $y'(t) = f(t, y)$, $a \le t \le b$, generates the estimates $y_{n+1} \approx y(t_{n+1})$, $n=0, 1, \ldots, N-1$, by satisfying the equation

$$y_{n+1} = y_n + \tfrac{1}{2}h\,\{f(t_n, y_n) + f(t_{n+1}, y_{n+1})\},$$

which defines y_{n+1} for sufficiently small h. Here $h=(b-a)/N$, $t_n=a+nh$ and $y_0=y(a)$. Apply the method of analysis of Euler's method, given in the lectures, to this calculation, assuming the usual Lipschitz condition on f and that the true solution $y(t)$, $a \le t \le b$, has a bounded third derivative. You should find an upper bound on $\max\{\,|y(t_n)-y_n| : n=0, 1, \ldots, N\}$ that is of magnitude h^2.

20. The k-step Adams–Bashforth method is of order k and has the form

$$y_{n+k} - y_{n+k-1} = h \,\textstyle\sum_{j=0}^{k-1} \beta_j\, f_{n+j}.$$

Calculate the actual values of the coefficients of the two-step and three-step formulae.

Prove that there is at most one k-step Adams–Bashforth method of order k, where k is any positive integer. Thus deduce the existence of the method for every positive integer k, using the remark that the coefficients β_j, $j = 0, 1, \ldots, k-1$, provide the required order if and only if they satisfy a linear system of equations.

21. By solving a three term recurrence relation, calculate analytically the sequence of values $\{y_k : k=2, 3, 4, \ldots\}$ that is generated by the mid-point rule

$$y_{n+2} = y_n + 2hf_{n+1}$$

when it is applied to the differential equation $dy/dt = -y$, $t \ge 0$. Starting from the values $y_0 = 1$ and $y_1 = 1-h$, show that the sequence diverges as $k \to \infty$. There is a theorem, however, that consistency, zero stability and suitable starting conditions provide convergence to the true solution on a *finite* interval as $h \to 0$. Prove that your analytic implementation of the mid-point rule is consistent with this theorem.
Hint: In the last part, relate the roots of the recurrence relation to $\pm e^{\mp h} + \mathcal{O}(h^3)$.

22. Show that the multistep method

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + y_{n+3} = h\,[\,\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}\,]$$

is fourth order only if the conditions $\alpha_0+\alpha_2 = 8$ and $\alpha_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and zero stable.

23. Let an s-stage explicit Runge–Kutta method of order s with constant stepsize $h>0$ be applied to the differential equation $y'(t)=\lambda y$, $t \ge 0$. Beginning with $n=1$, prove the identity

$$y_n = \left(\textstyle\sum_{\ell=0}^{s} (\ell!)^{-1} (\lambda h)^{\ell}\right)^n y_0, \qquad n=1, 2, 3, \ldots .$$

24. The following four-stage Runge–Kutta method has order four:

$$\begin{aligned}
k_1 &= f(t_n,\, y_n)\\
k_2 &= f(t_n + \tfrac{1}{3}h,\, y_n + \tfrac{1}{3}hk_1)\\
k_3 &= f(t_n + \tfrac{2}{3}h,\, y_n - \tfrac{1}{3}hk_1 + hk_2)\\
k_4 &= f(t_n + h,\, y_n + hk_1 - hk_2 + hk_3)\\
y_{n+1} &= y_n + h\,(\tfrac{1}{8}k_1 + \tfrac{3}{8}k_2 + \tfrac{3}{8}k_3 + \tfrac{1}{8}k_4).
\end{aligned}$$

By considering the equation $dy/dt = f(t, y) = y$, show that the order is at most four. Then prove that the order is at least four in the easy case when $f(t, y)$ is independent of y, and that the order is at least three in the relatively easy case when $f(t, y)$ is independent of t. Therefore you are not expected to derive all of the details that occur when $f(t, y)$ depends on both t and y.