14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

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So, in this chapter, we:

- **OBJECTIVES**
- Turn our attention to functions of several variables.
- Extend the basic ideas of differential calculus to such functions.

14.1
Functions of Several Variables

In this section, we will learn about:

Functions of two or more variables and how to produce their graphs.

FUNCTIONS OF SEVERAL VARIABLES

In this section, we study functions of two or more variables from four points of view:

- Verbally (a description in words)
- Numerically (a table of values)
- Algebraically (an explicit formula)
- Visually (a graph or level curves)

FUNCTIONS OF TWO VARIABLES

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

- We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y).
- We indicate this functional dependence by writing:

$$T = f(x, y)$$

The volume V of a circular cylinder depends on its radius r and its height h.

- In fact, we know that $V = \pi r^2 h$.
- We say that V is a function of r and h.
- We write $V(r, h) = \pi r^2 h$.

FUNCTION OF TWO VARIABLES

A function *f* of two variables is a rule that assigns to each ordered pair of real numbers (*x*, *y*) in a set *D* a unique real number denoted by (*x*, *y*).

- The set *D* is the domain of *f.*
- Its range is the set of values that *f* takes on, that is,

{*f*(*x*, *y*) | (*x*, *y*) ™ *D*}

FUNCTIONS OF TWO VARIABLES

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y).

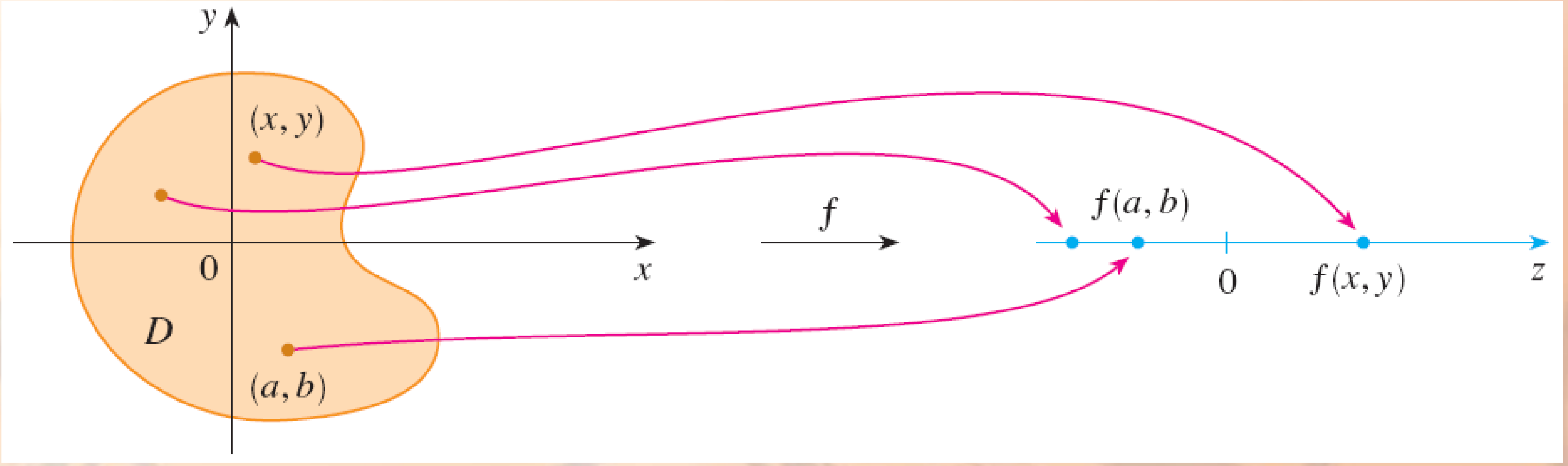
- The variables x and y are independent variables.
- z is the dependent variable.
- Compare this with the notation $y = f(x)$
  for functions of a single variable.

A function of two variables is just a function whose:

- Domain is a subset of P^2
- Range is a subset of P

FUNCTIONS OF TWO VARIABLES

One way of visualizing such a function is by means of an arrow diagram, where the domain *D* is represented as a subset of the *xy*-plane.



FUNCTIONS OF TWO VARIABLES

If a function *f* is given by a formula and no domain is specified, then the domain of *f* is understood to be:

- The set of all pairs (x, y) for which the given expression is a well-defined real number.

Example 1

For each of the following functions, evaluate $f(3, 2)$ and find the domain.

a. $$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

b. $$f(x, y) = x \ln(y^2 - x)$$

 Example 1 a

$$f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$$

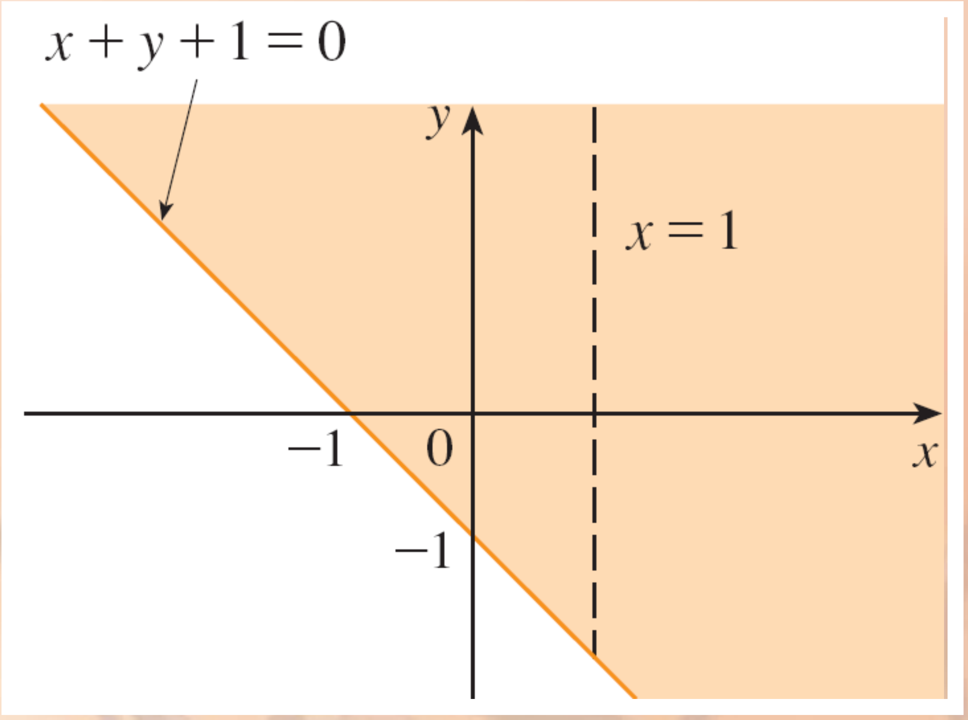
- The expression for *f* makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative.

- So, the domain of *f* is:

$$D = \{(x, y) \mid x + y + 1 \geq 0,\ x \neq 1\}$$

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$

- $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain.



FUNCTIONS OF TWO VARIABLES **Example 1 b**

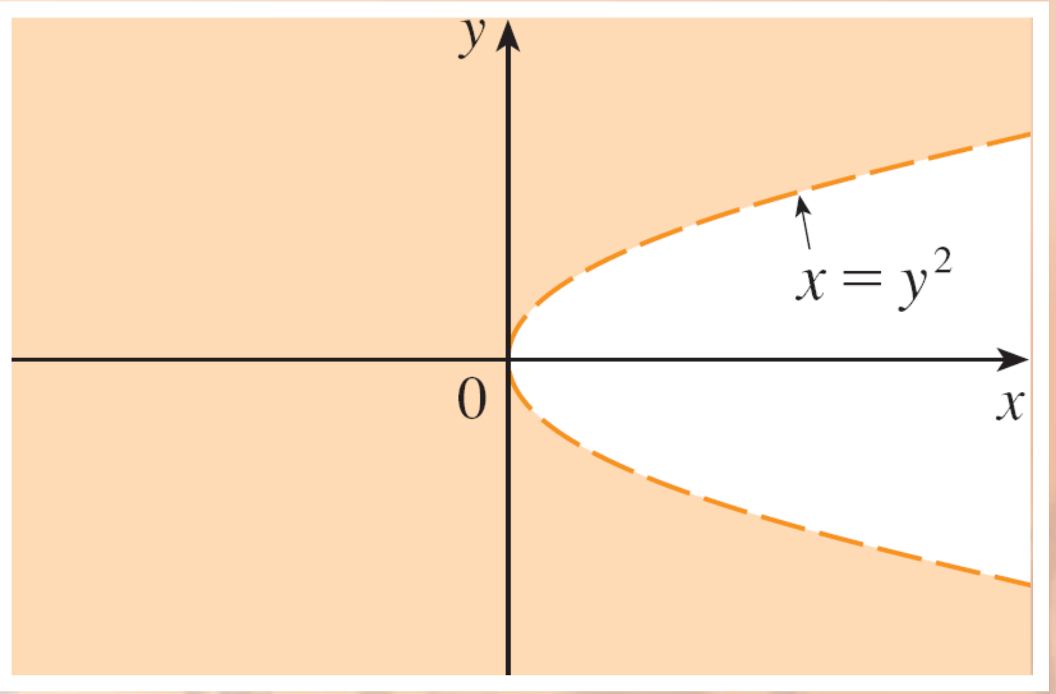
$$f(3, 2) = 3 \ln(2^2 - 3)$$
$$= 3 \ln 1$$
$$= 0$$

- Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is:

$$D = \{(x, y)|\ x < y^2$$

 Example 1 b

This is the set of points to the left of the parabola $x = y^2$.



Not all functions are given by explicit formulas.

- The function in the next example is described verbally and by numerical estimates of its values.

In regions with severe winter weather, the wind-chill index is often used to describe the apparent severity of the cold.

- This index W is a subjective temperature that depends on the actual temperature T and the wind speed v.

- So, W is a function of T and v, and we can write:

$$W = f(T, v)$$

 Example 2

The following table records values of *W* compiled by the NOAA National Weather Service of the US and the Meteorological Service of Canada.

FUNCTIONS OF TWO VARIABLES **Example 2**

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

For instance, the table shows that, if the temperature is –5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about –15°C with no wind.

- Therefore,

$$f(-5, 50) = -15$$

In 1928, Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922.

FUNCTIONS OF TWO VARIABLES **Example 3**

They considered a simplified view in which production output is determined by the amount of labor involved and the amount of capital invested.

- While there are many other factors affecting economic performance, their model proved to be remarkably accurate.

E. g. 3—Equation 1

The function they used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

E. g. 3—Equation 1

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

- P is the total production
  (monetary value of all goods produced in a year)
- L is the amount of labor
  (total number of person-hours worked in a year)
- K is the amount of capital invested
  (monetary worth of all machinery, equipment, and buildings)

 Example 3

In Section 14.3, we will show how the form of Equation 1 follows from certain economic assumptions.

FUNCTIONS OF TWO VARIABLES Example 3

Cobb and Douglas used economic data published by the government to obtain this table.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Example 3

They took the year 1899 as a baseline.

- *P*, *L*, and *K* for 1899 were each assigned the value 100.
- The values for other years were expressed as percentages of the 1899 figures.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

E. g. 3—Equation 2

Cobb and Douglas used the method of least squares to fit the data of the table to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

- See Exercise 75 for the details.

Example 3

Let's use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920.

We get:

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25}$$

$$\approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25}$$

$$\approx 235.8$$

- These are quite close to the actual values, 159 and 231.

COBB-DOUGLAS PRODN. FUNCN. **Example 3**

The production function (Equation 1) has subsequently been used in many settings, ranging from individual firms to global economic questions.

- It has become known as the Cobb-Douglas production function.

Its domain is:

$$\{(L, K) \mid L \geq 0, K \geq 0\}$$

- This is because *L* and *K* represent labor and capital and so are never negative.

Find the domain and range of:

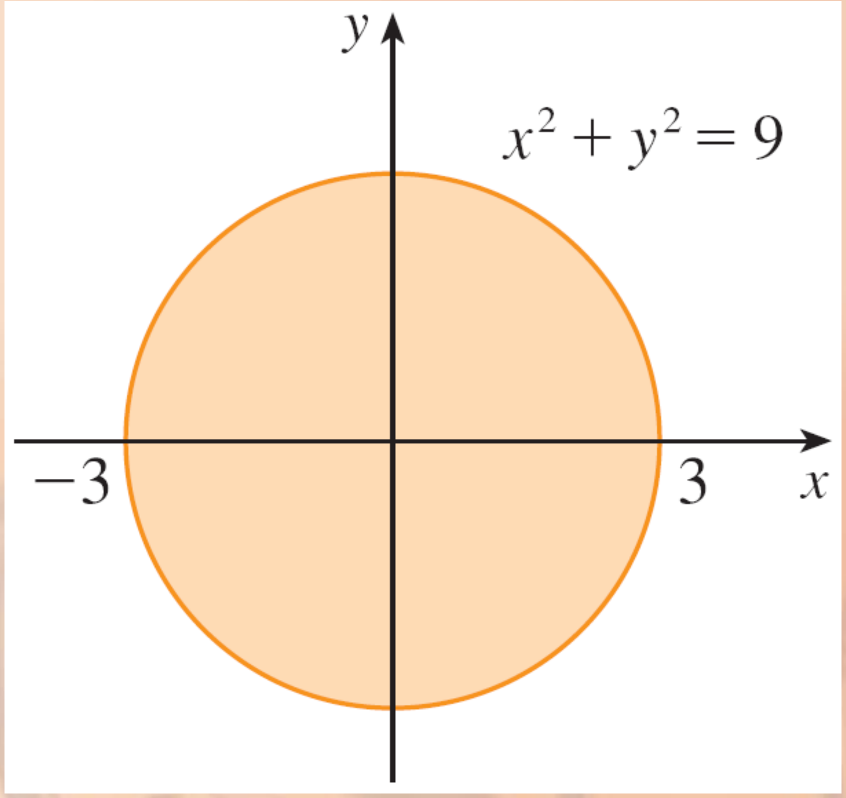
$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

- The domain of *g* is:

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\}$$
$$= \{(x, y) \mid x^2 + y^2 \leq 9\}$$

 Example 4

This is the disk with center (0, 0) and radius 3.



The range of *g* is:

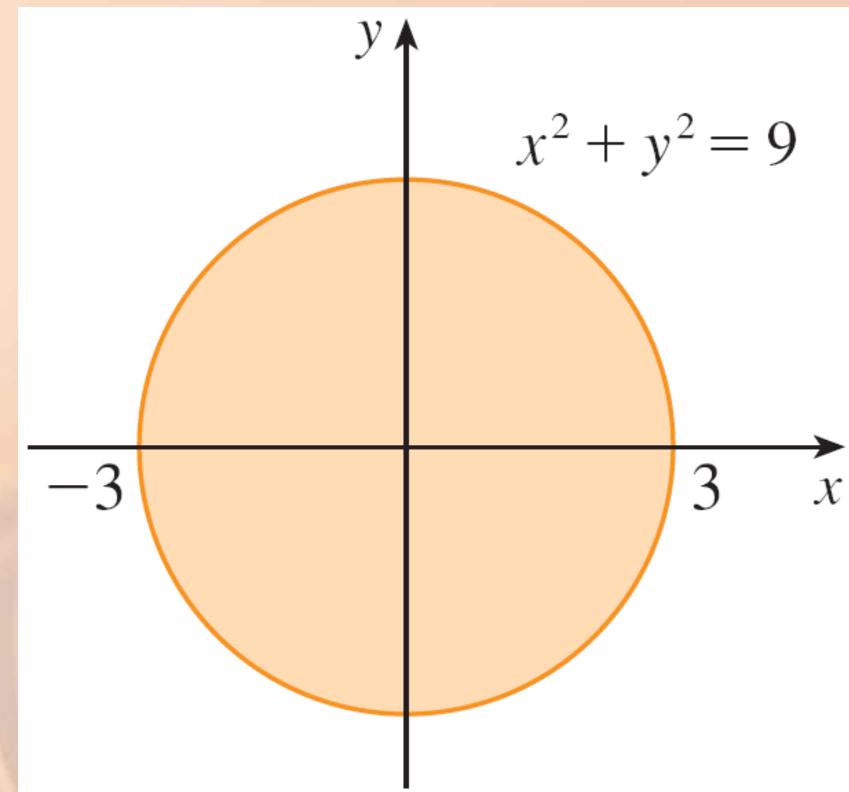
$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

- Since z is a positive square root, $z \geq 0$.
- Also,

$$9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

So, the range is:

$$\{z|\ 0 \le z \le 3\} = [0, 3]$$



GRAPHS

Another way of visualizing the behavior of a function of two variables is to consider its graph.

GRAPH

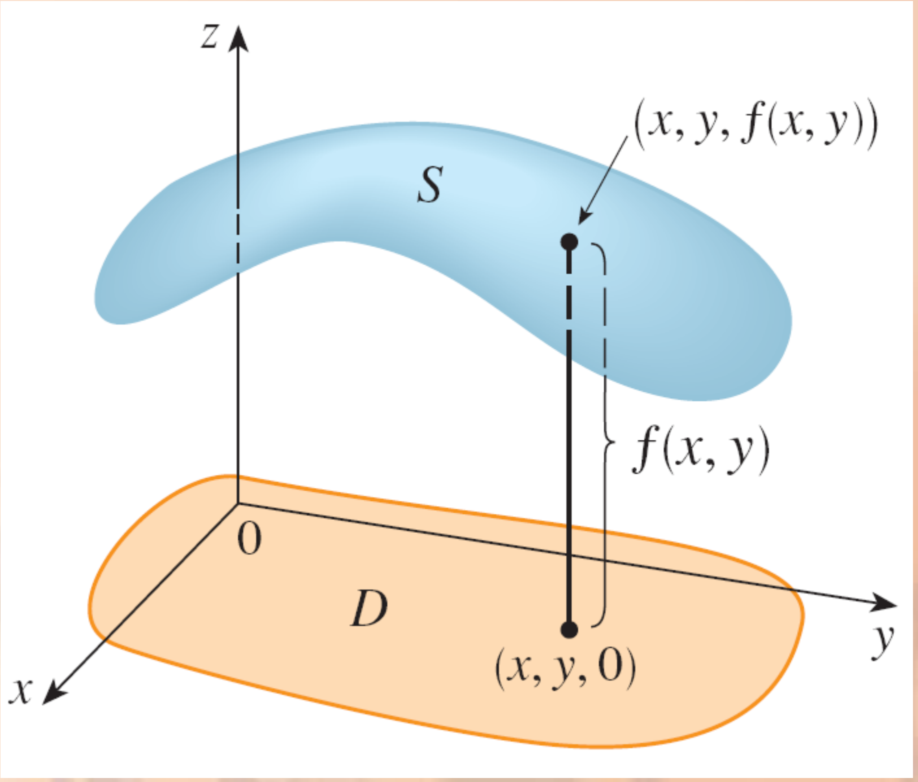
If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in P^3 such that $z = f(x, y)$ and (x,y) is in D.

GRAPHS

Just as the graph of a function *f* of one variable is a curve *C* with equation $y = f(x)$, so the graph of a function *f* of two variables is:

- A surface *S* with equation $z = f(x, y)$

We can visualize the graph *S* of *f* as lying directly above or below its domain *D* in the *xy*-plane.



Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y$$

- The graph of f has the equation

$$z = 6 - 3x - 2y$$

or

$$3x + 2y + z = 6$$

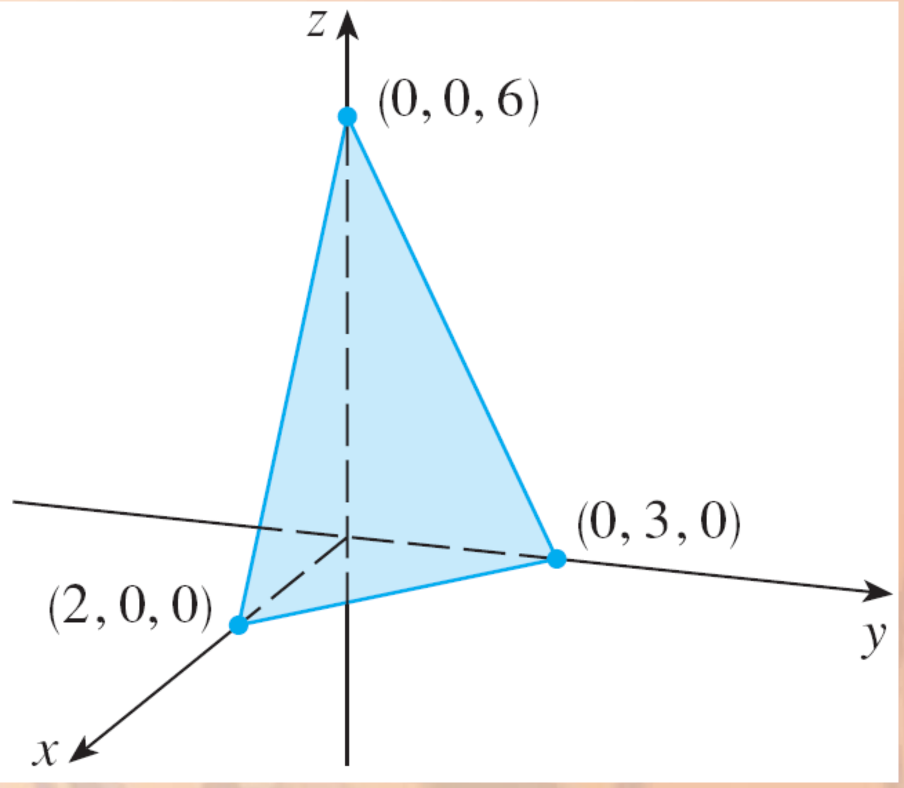
- This represents a plane.

To graph the plane, we first find the intercepts.

- Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept.

- Similarly, the y-intercept is 3 and the z-intercept is 6.

This helps us sketch the portion of the graph that lies in the first octant.



LINEAR FUNCTION

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

- It is called a linear function.

The graph of such a function has the equation

$$z = ax + by + c$$

or

$$ax + by - z + c = 0$$

- Thus, it is a plane.

LINEAR FUNCTIONS

In much the same way that linear functions of one variable are important in single-variable calculus, we will see that:

- Linear functions of two variables play a central role in multivariable calculus.

Sketch the graph of

$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

- The graph has equation

$$z = \sqrt{9 - x^2 - y^2}$$

We square both sides of the equation to obtain:

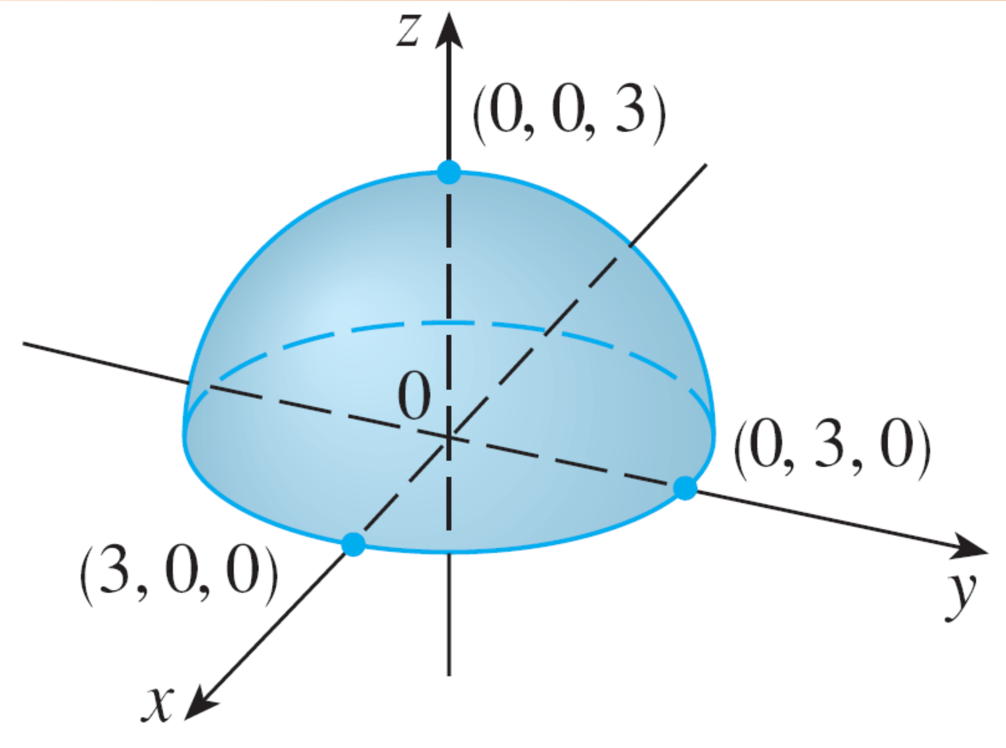
$$z^2 = 9 - x^2 - y^2$$

or

$$x^2 + y^2 + z^2 = 9$$

- We recognize this as an equation of the sphere with center the origin and radius 3.

However, since $z \geq 0$, the graph of g is just the top half of this sphere.



Note

An entire sphere can't be represented by a single function of *x* and *y*.

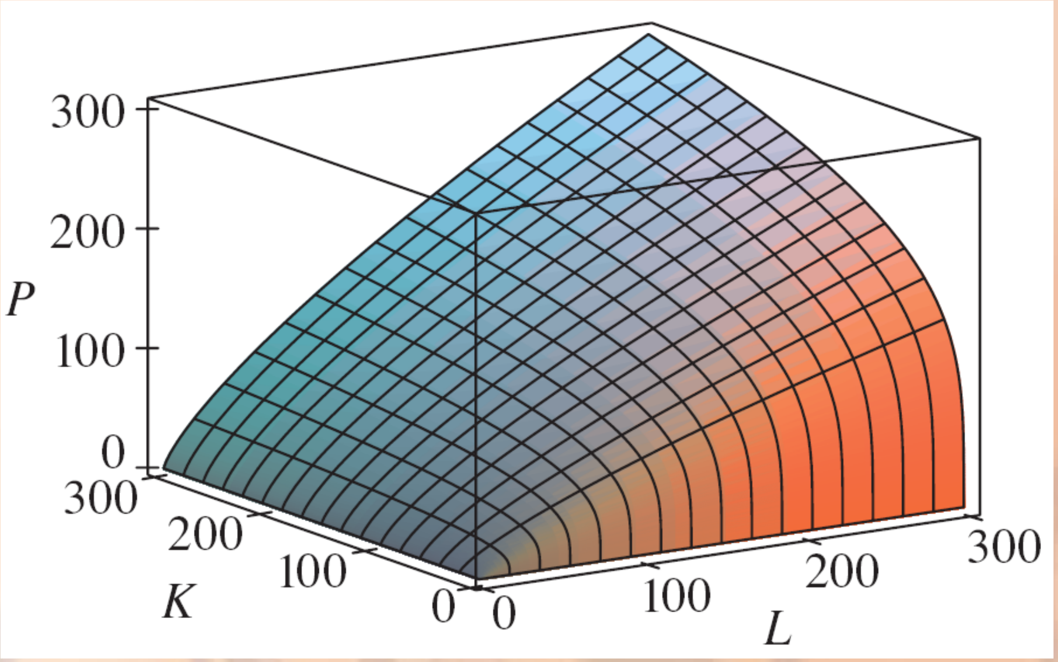
- As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function
$$g(x, y) = \sqrt{9 - x^2 - y^2}$$
- The lower hemisphere is represented by the function
$$h(x, y) = -\sqrt{9 - x^2 - y^2}$$

Use a computer to draw the graph of the Cobb-Douglas production function

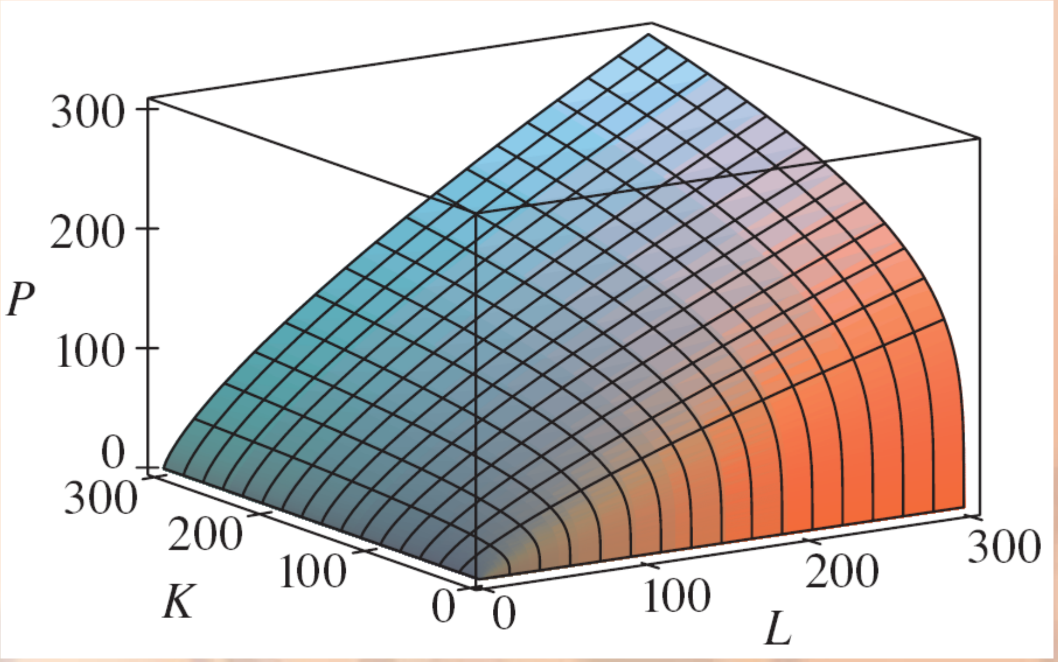
$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

The figure shows the graph of P for values of the labor L and capital K that lie between 0 and 300.

- The computer has drawn the surface by plotting vertical traces.



We see from these traces that the value of the production P increases as either L or K increases—as is to be expected.



Find the domain and range and sketch the graph of

$$h(x, y) = 4x^2 + y^2$$

Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x,y).

- So, the domain is P^2, the entire *xy*-plane.

The range of h is the set $[0, \infty)$ of all nonnegative real numbers.

- Notice that $x^2 \geq 0$ and $y^2 \geq 0$.
- So, $h(x, y) \geq 0$ for all x and y.

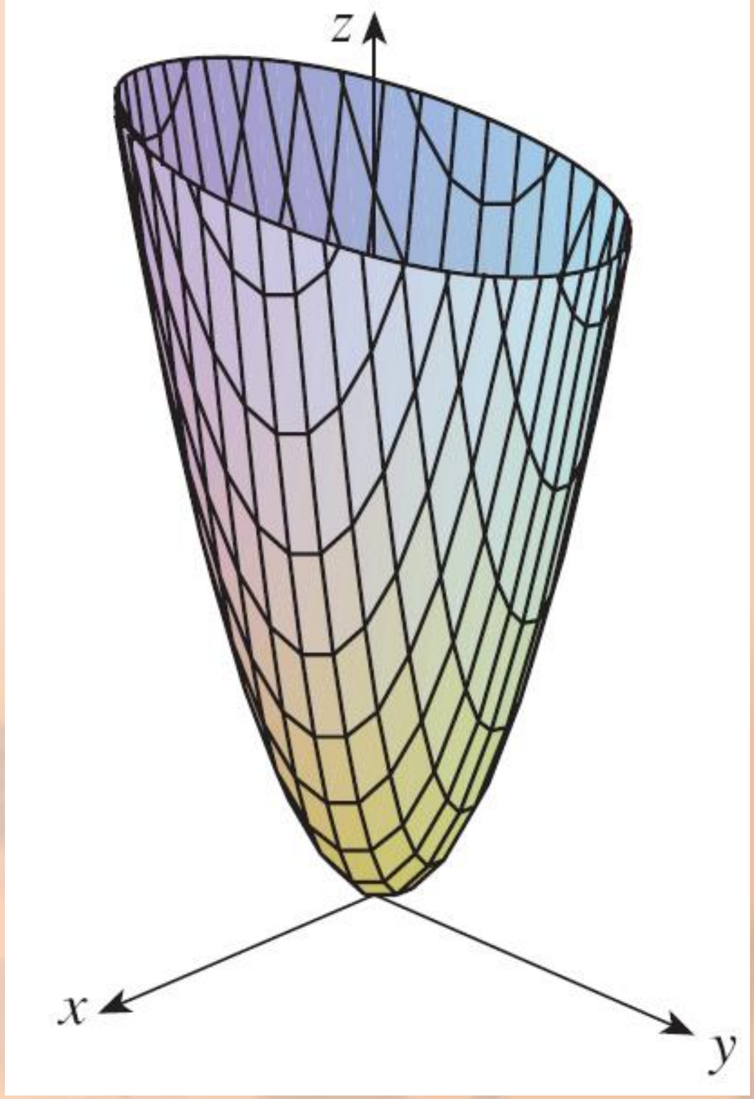
The graph of h has the equation

$$z = 4x^2 + y^2$$

- This is the elliptic paraboloid that we sketched in Example 4 in Section 12.6

Example 8

Horizontal traces are ellipses and vertical traces are parabolas.



GRAPHS BY COMPUTERS

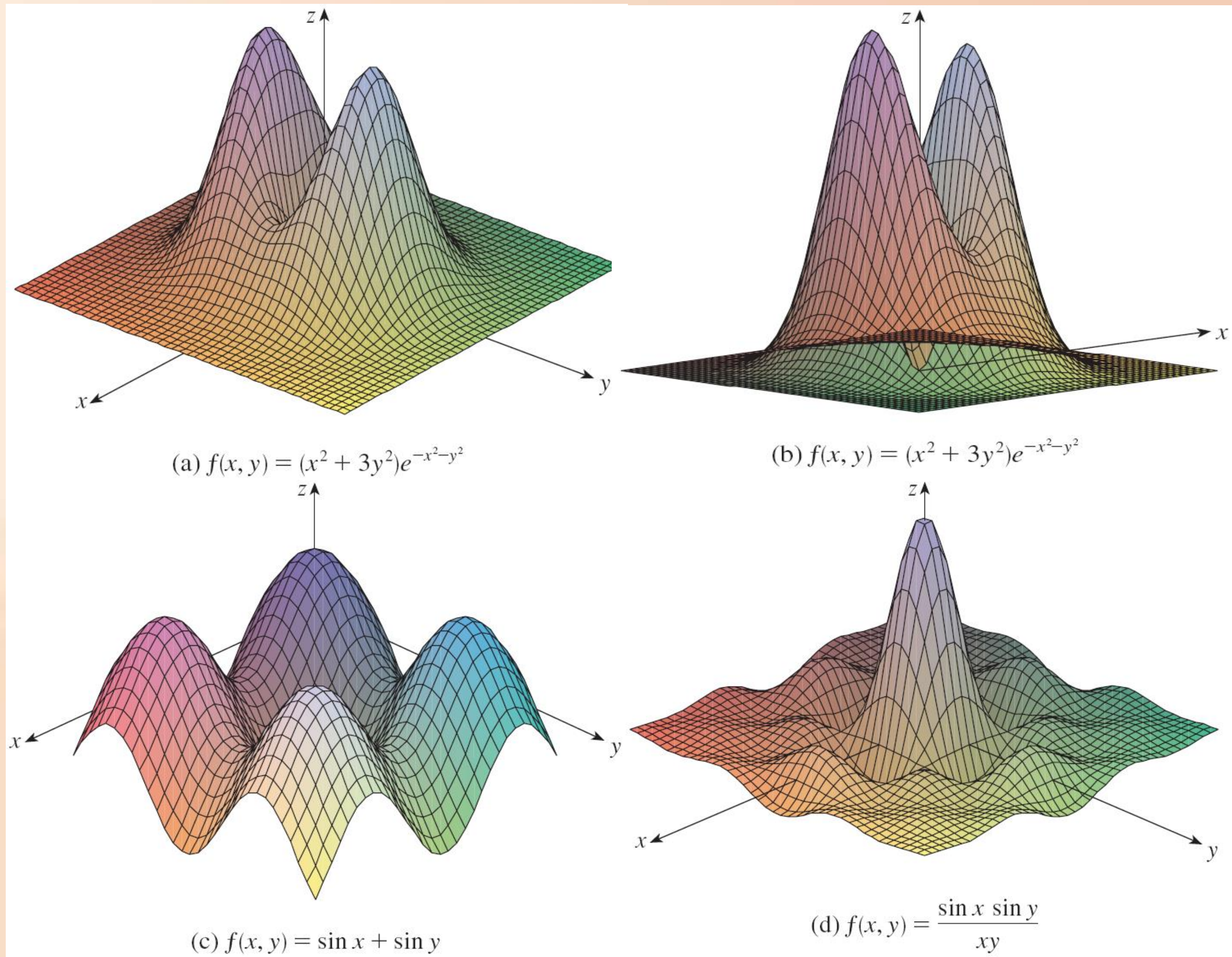
Computer programs are readily available for graphing functions of two variables.

In most such programs,

- Traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.
- Parts of the graph are eliminated using hidden line removal.

GRAPHS BY COMPUTERS

The figure shows computer-generated graphs of several functions.



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

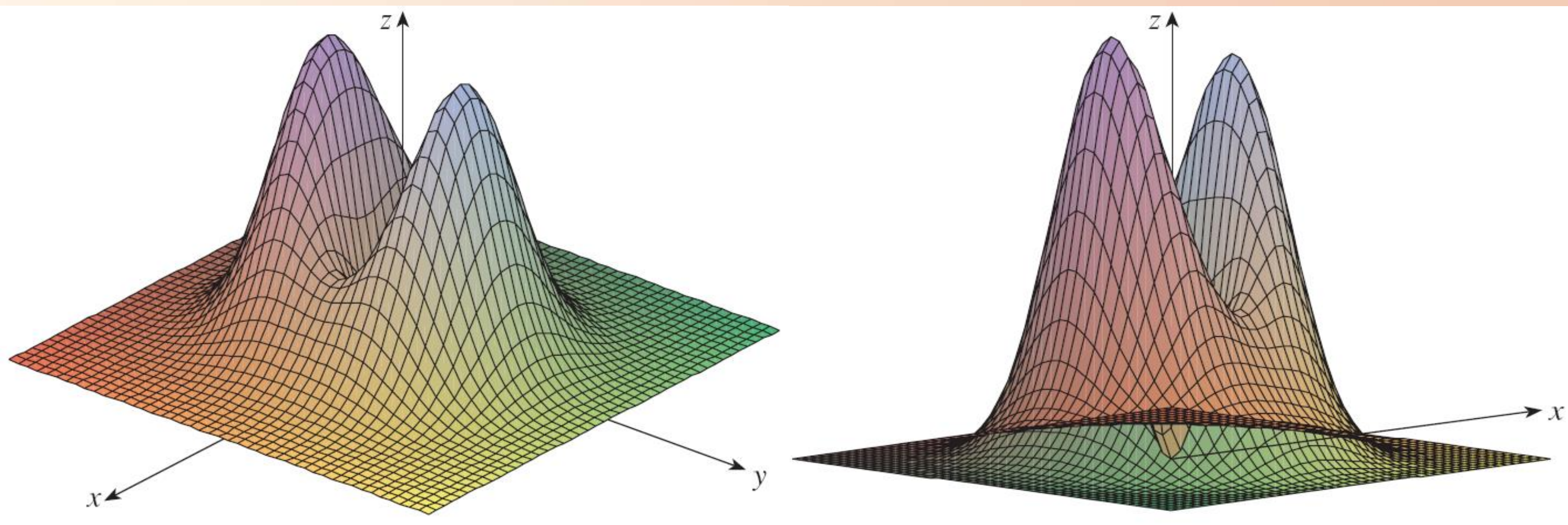
(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(c) $f(x, y) = \sin x + \sin y$

(d) $f(x, y) = \dfrac{\sin x \, \sin y}{xy}$

GRAPHS BY COMPUTERS

Notice that we get an especially good picture of a function when rotation is used to give views from different vantage points.



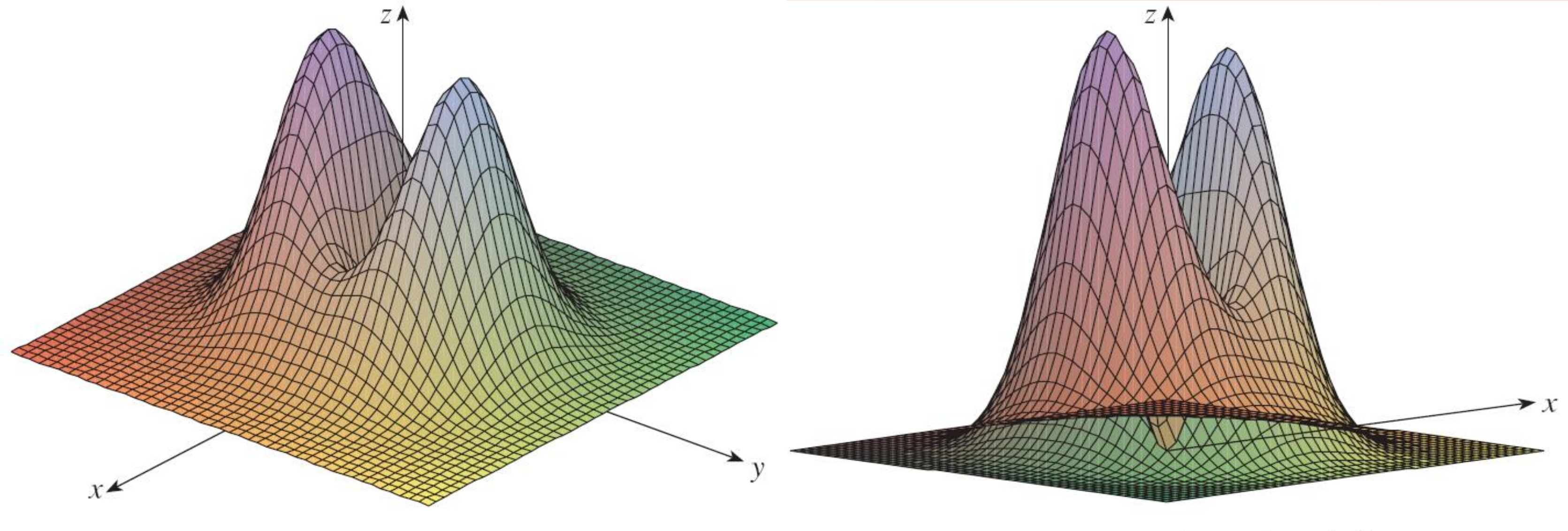
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

GRAPHS BY COMPUTERS

In (a) and (b), the graph of *f* is very flat and close to the *xy*-plane except near the origin.

- This is because $e^{-x^2-y^2}$ is small when *x* or *y* is large.



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

LEVEL CURVES

So far, we have two methods for visualizing functions, arrow diagrams and graphs.

- A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form contour curves, or level curves.

LEVEL CURVES **Definition**

The level curves of a function f of two variables are the curves with equations

$$f(x, y) = k$$

where k is a constant

(in the range of f).

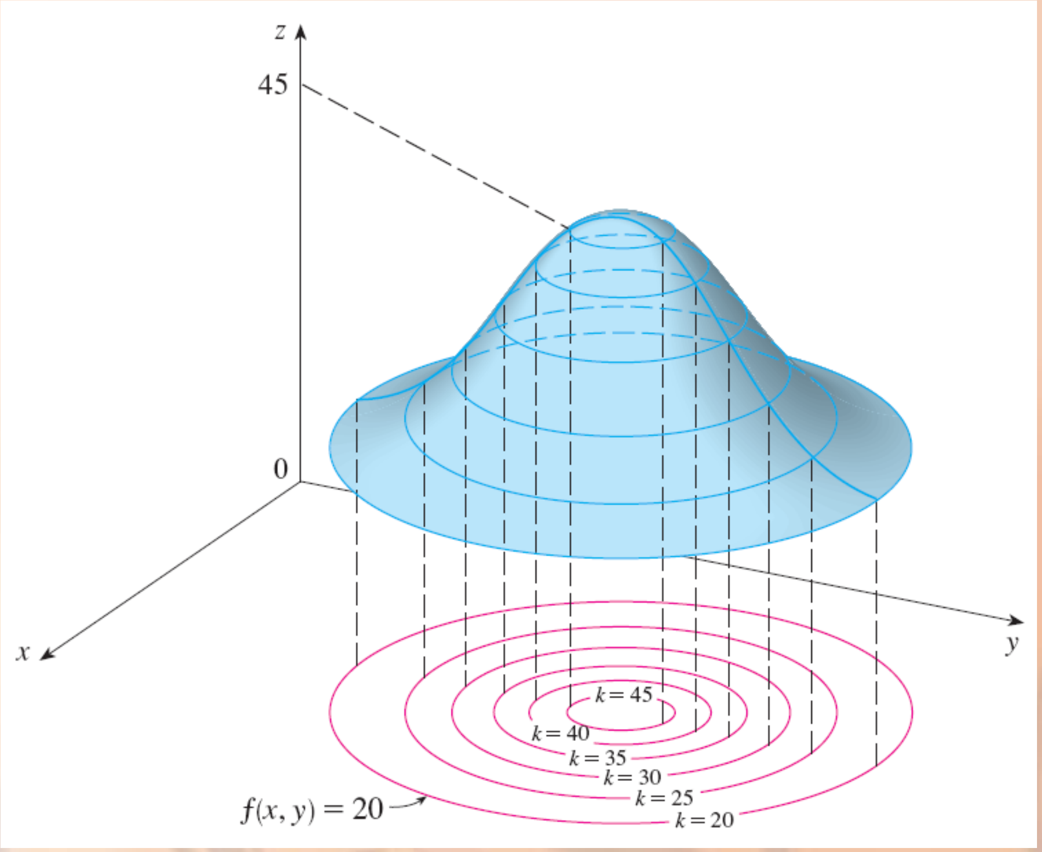
LEVEL CURVE

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k.

- That is, it shows where the graph of f has height k.

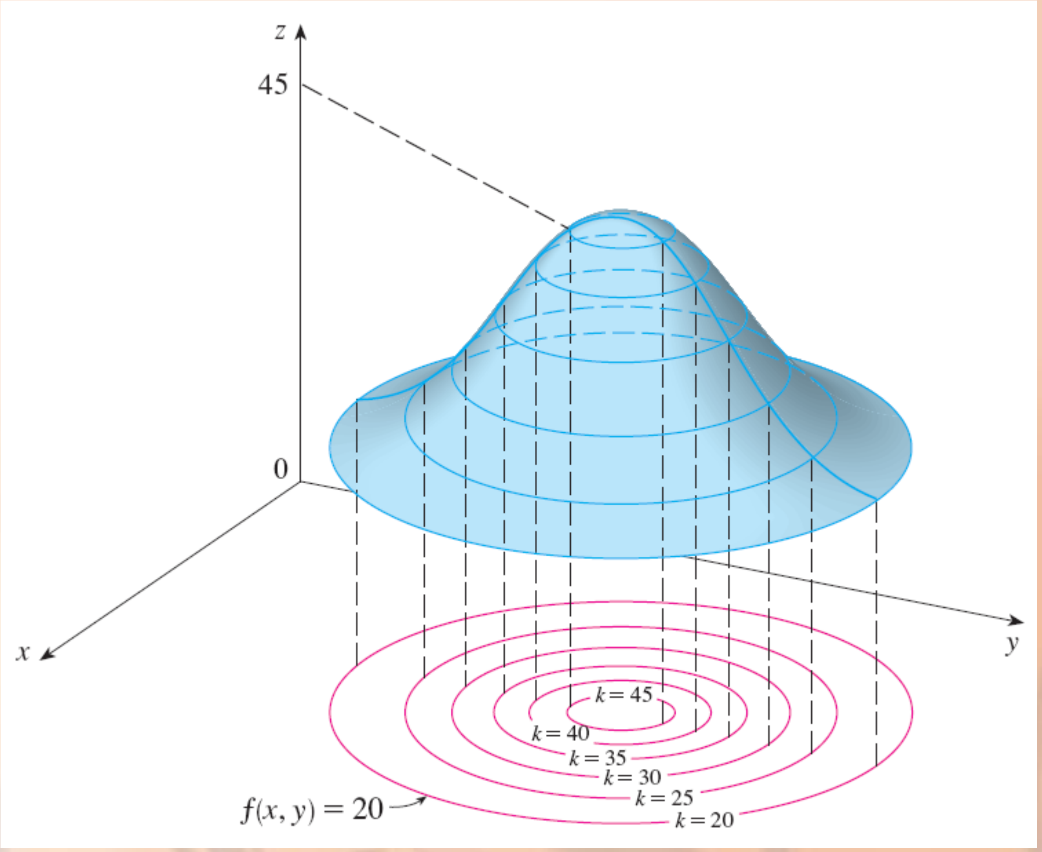
LEVEL CURVES

You can see from the figure the relation between level curves and horizontal traces.



LEVEL CURVES

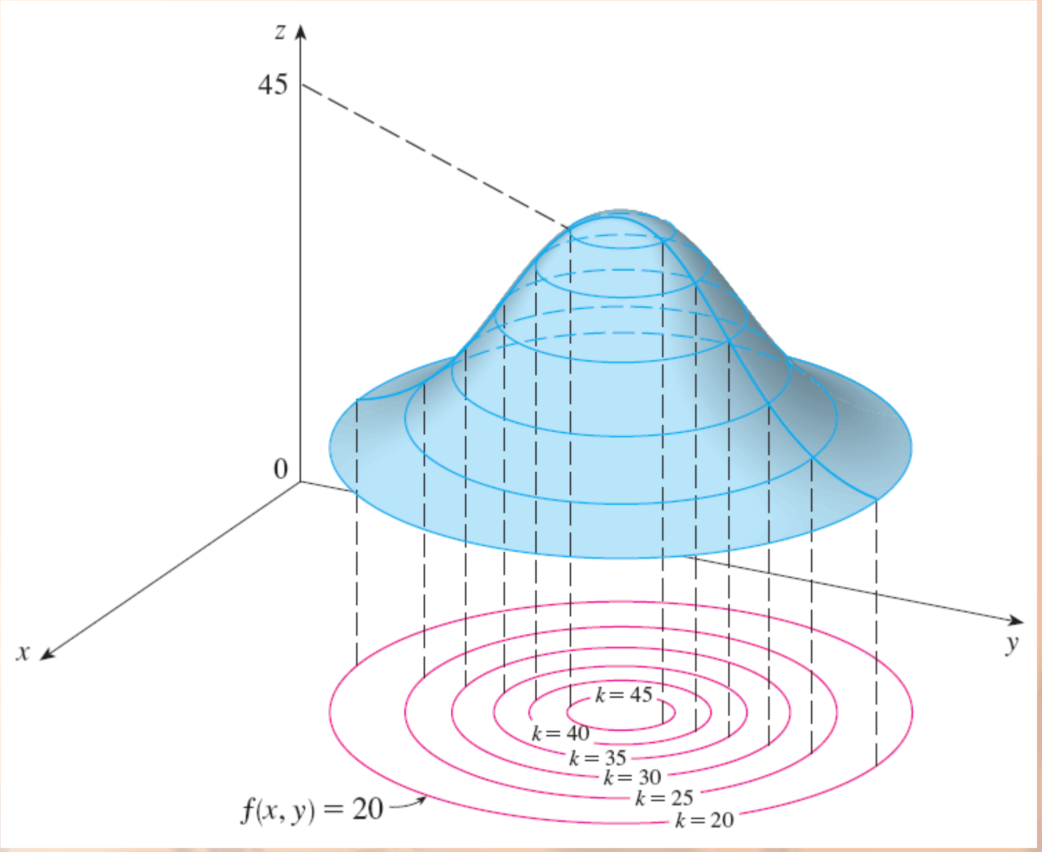
The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.



LEVEL CURVES

So, suppose you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height.

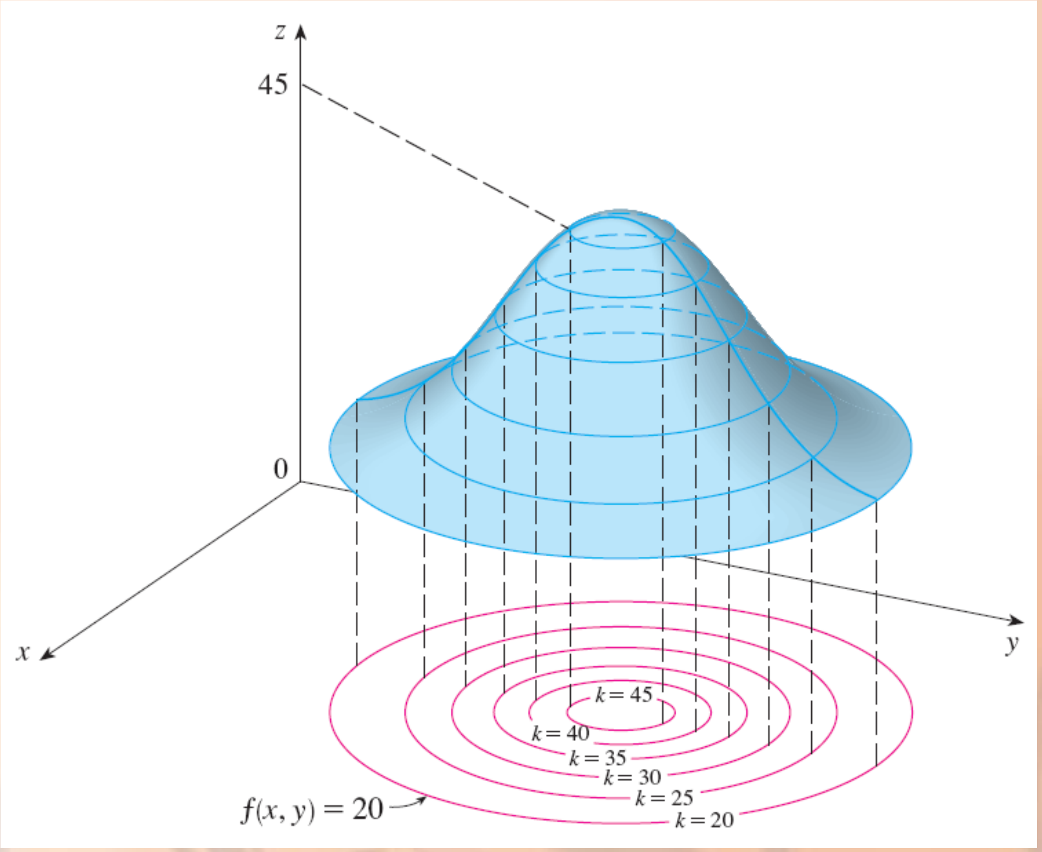
- Then, you can mentally piece together a picture of the graph.



LEVEL CURVES

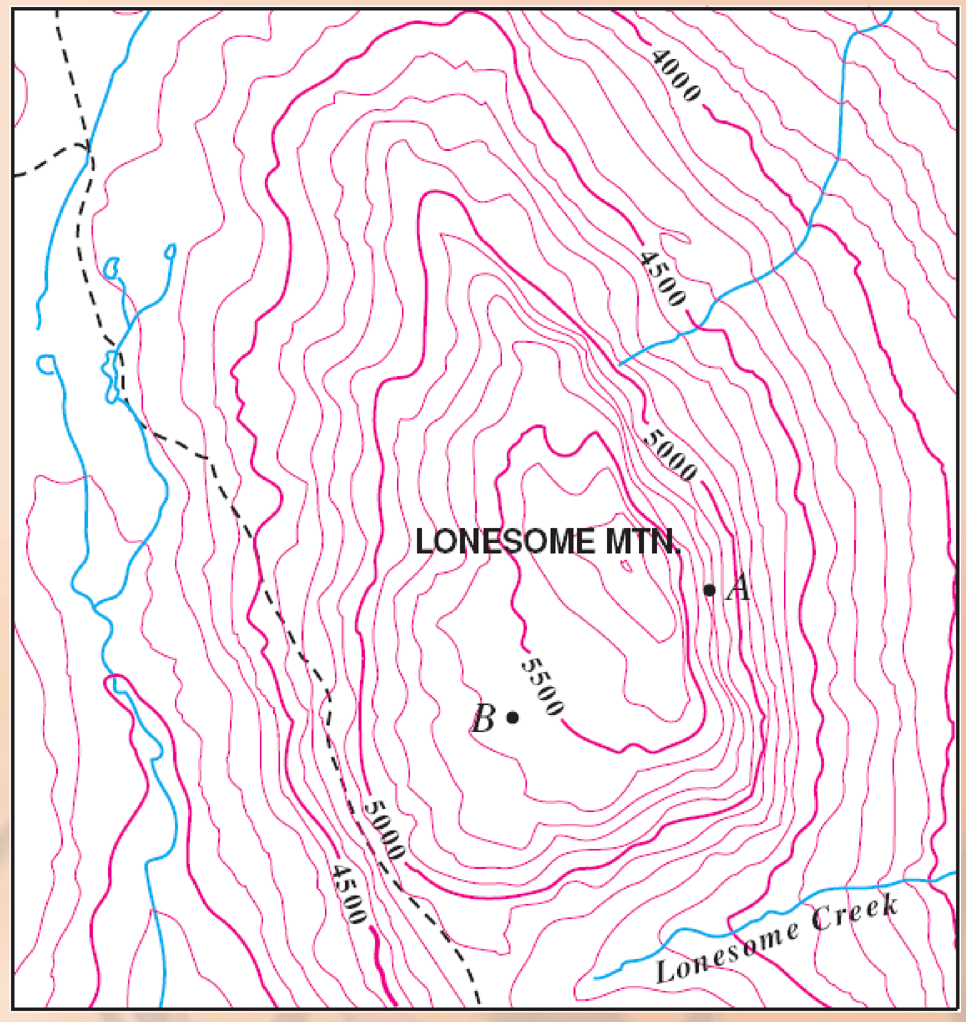
The surface is:

- Steep where the level curves are close together.
- Somewhat flatter where the level curves are farther apart.



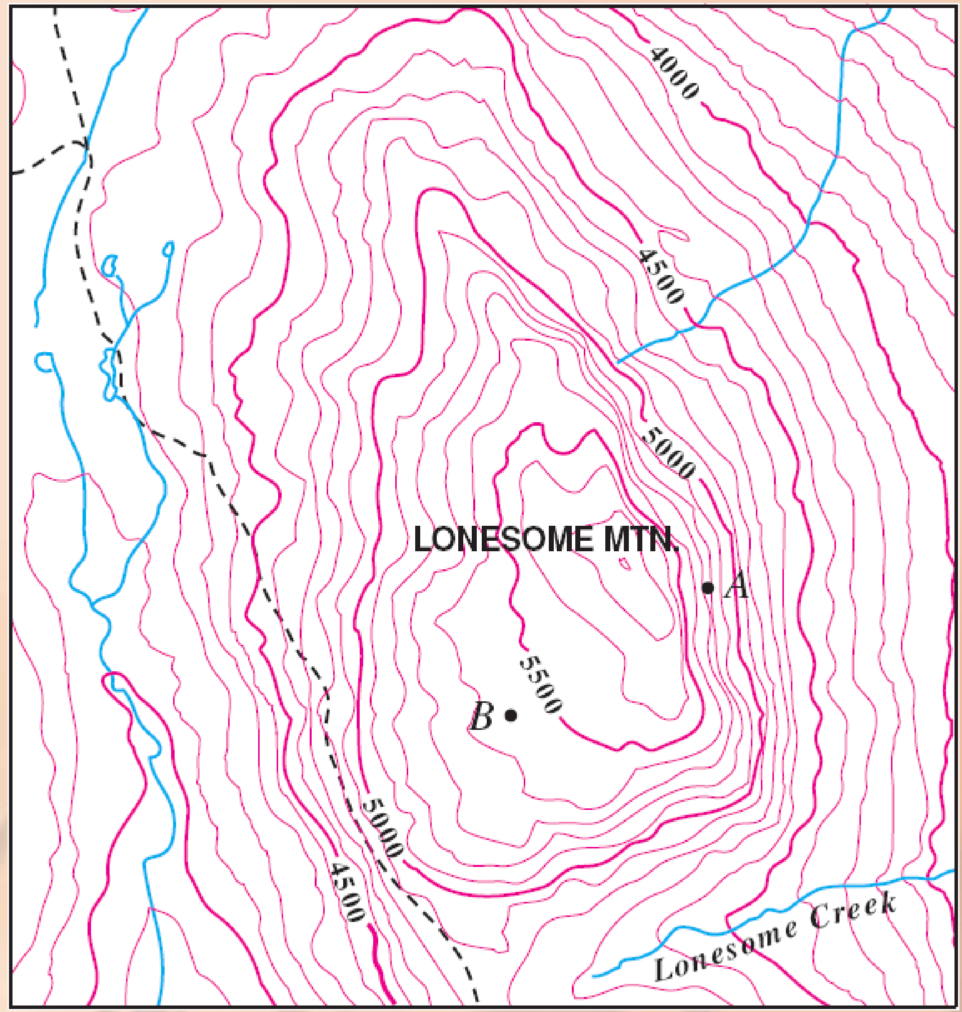
LEVEL CURVES

One common example of level curves occurs in topographic maps of mountainous regions, such as shown.



LEVEL CURVES

The level curves are curves of constant elevation above sea level.



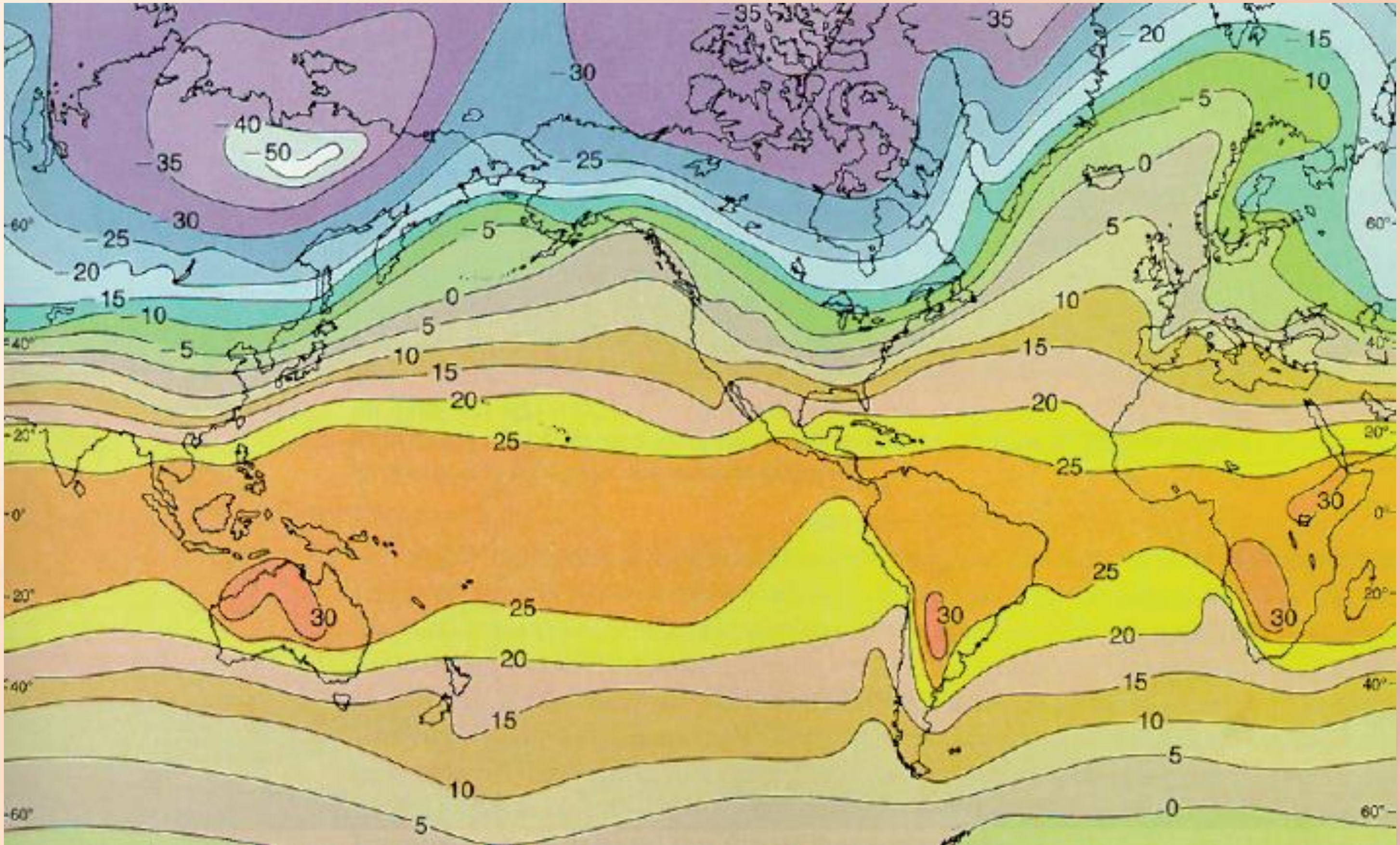
- If you walk along one of these contour lines, you neither ascend nor descend.

Another common example is
the temperature function introduced
in the opening paragraph of the section.

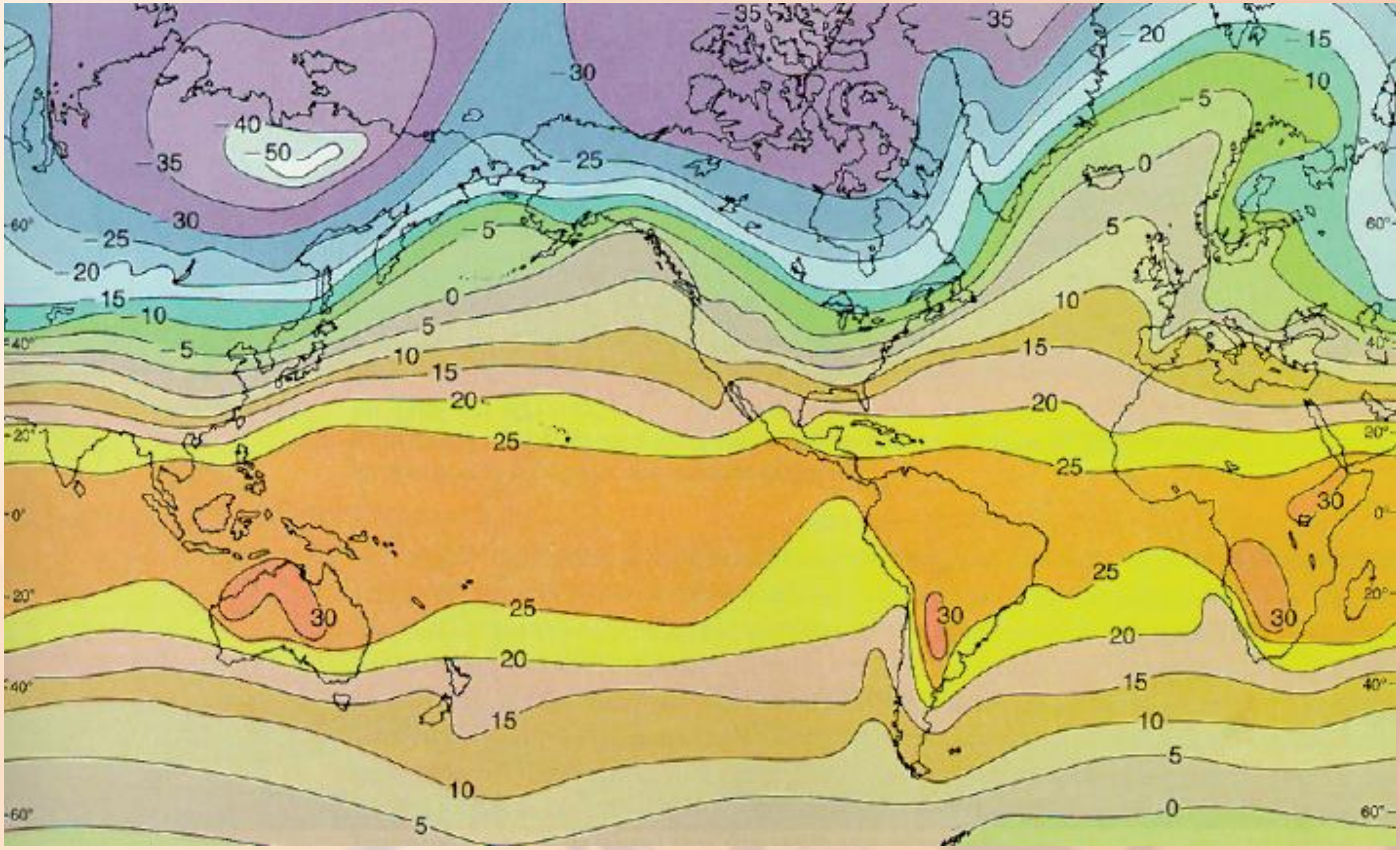
- Here, the level curves are called isothermals.
- They join locations with the same temperature.

LEVEL CURVES

The figure shows a weather map of the world indicating the average January temperatures.

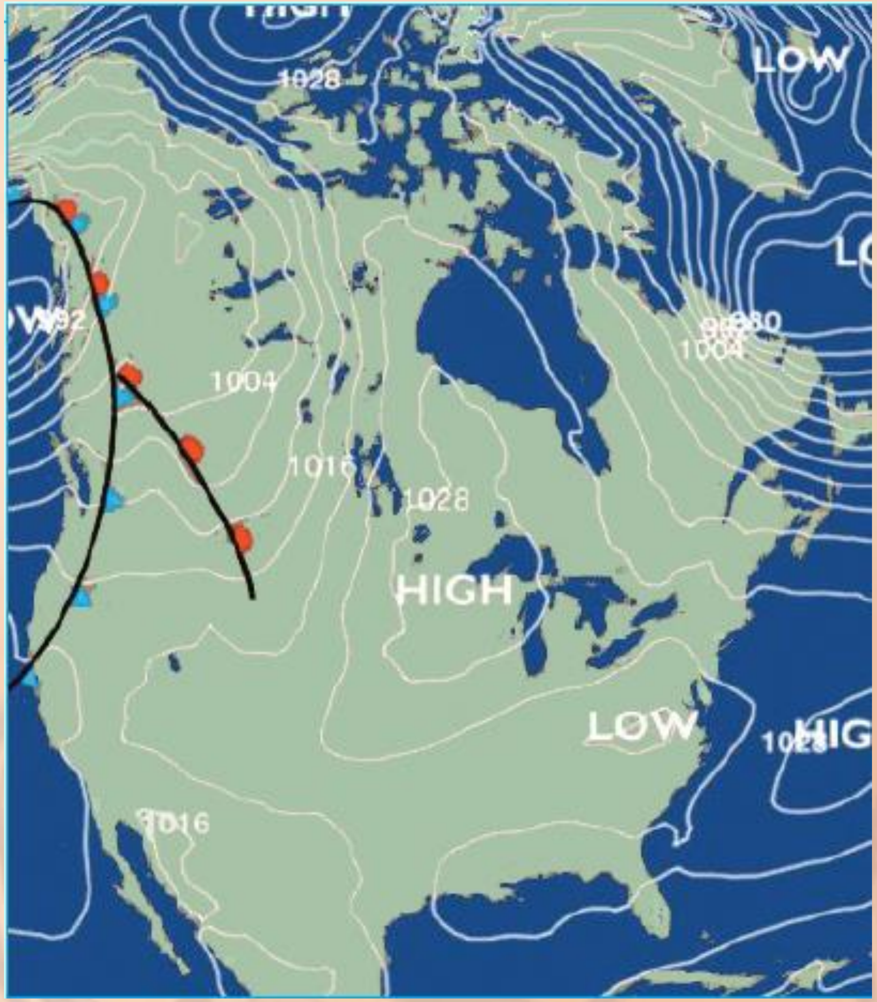


The isothermals are the curves that separate the colored bands.



LEVEL CURVES

The isobars in this atmospheric pressure map provide another example of level curves.

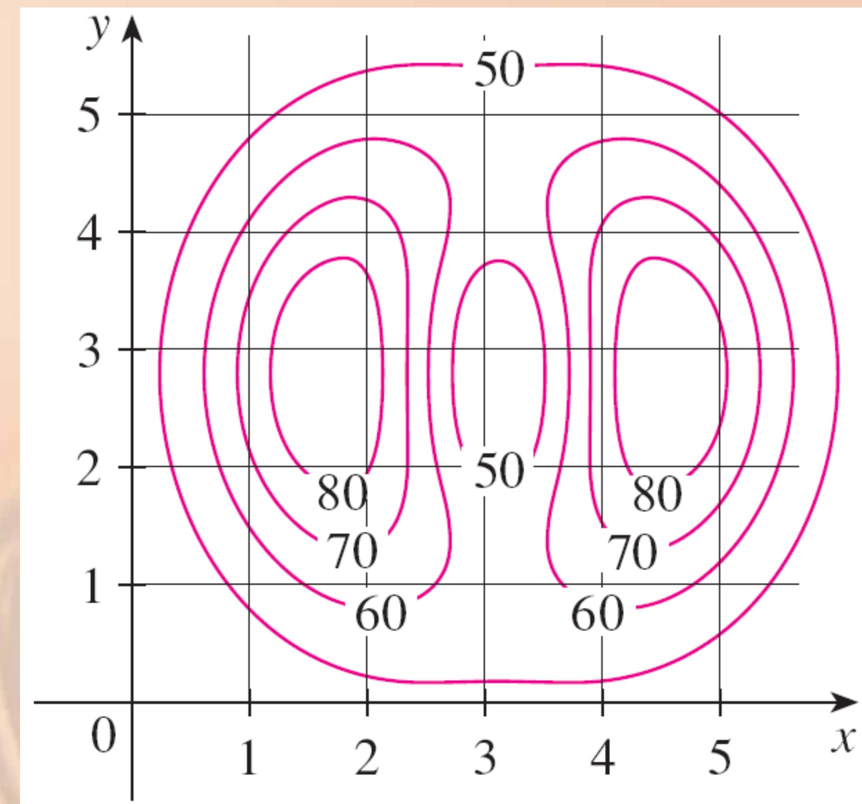


LEVEL CURVES

Example 9

A contour map for a function *f* is shown.

- Use it to estimate the values of *f*(1, 3) and *f*(4, 5).



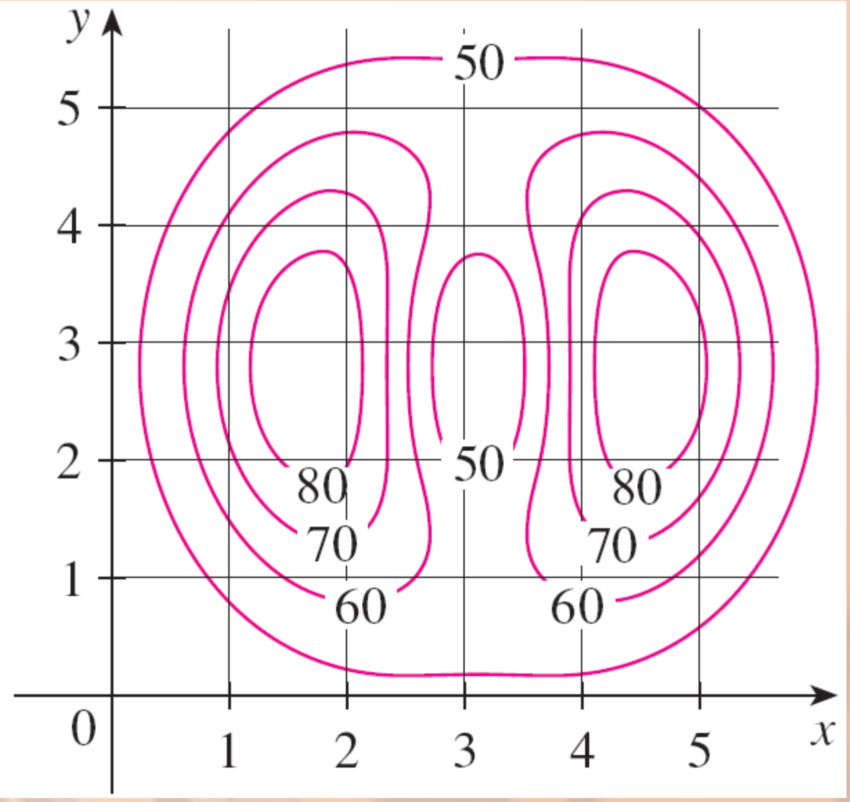
The point (1, 3) lies partway between the level curves with *z*-values 70 and 80.

- We estimate that:

  $f(1, 3) \approx 73$

- Similarly, we estimate that:

  $f(4, 5) \approx 56$



Sketch the level curves of the function

$$f(x, y) = 6 - 3x - 2y$$

for the values

$$k = -6, 0, 6, 12$$

The level curves are:

$$6 - 3x - 2y = k$$

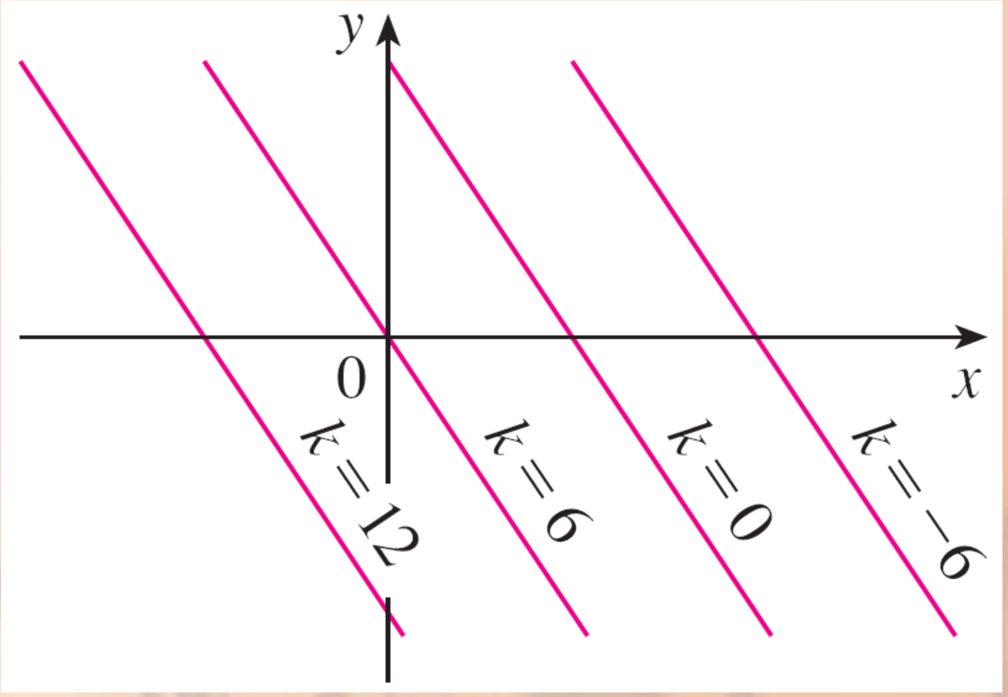
or

$$3x + 2y + (k - 6) = 0$$

This is a family of lines with slope –3/2.

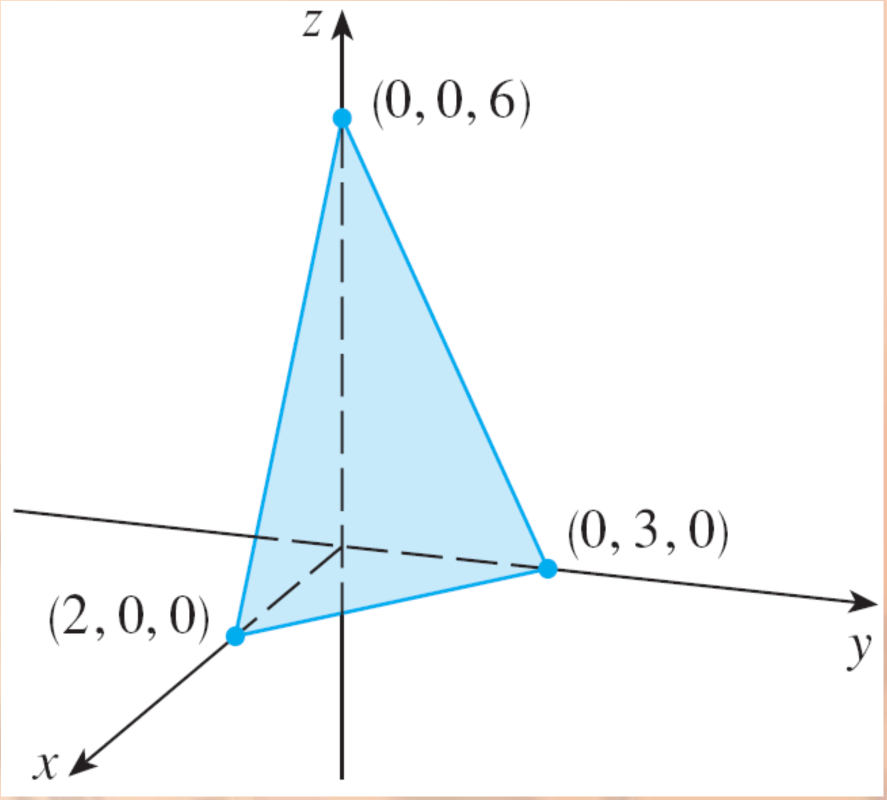
The four particular level curves with k = –6, 0, 6, 12 are:

- $3x + 2y - 12 = 0$
- $3x + 2y - 6 = 0$
- $3x + 2y = 0$
- $3x + 2y + 6 = 0$



Example 10

The level curves are equally spaced parallel lines because the graph of *f* is a plane.



Sketch the level curves of the function

$$g(x,y)=\sqrt{9-x^2-y^2}$$

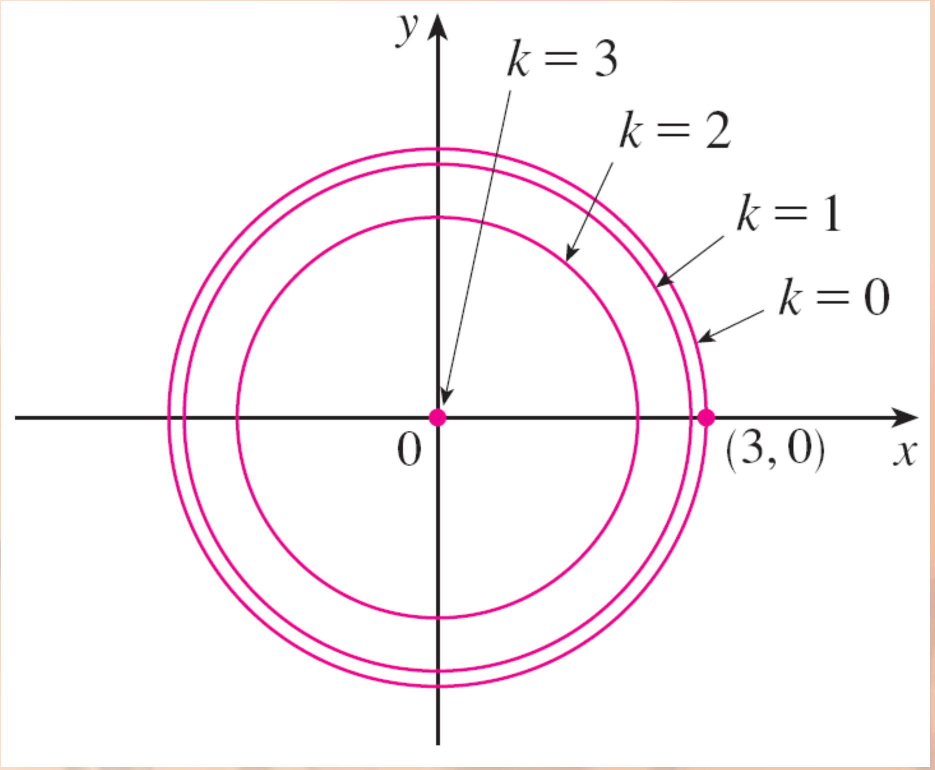
for k = 0, 1, 2, 3

- The level curves are:

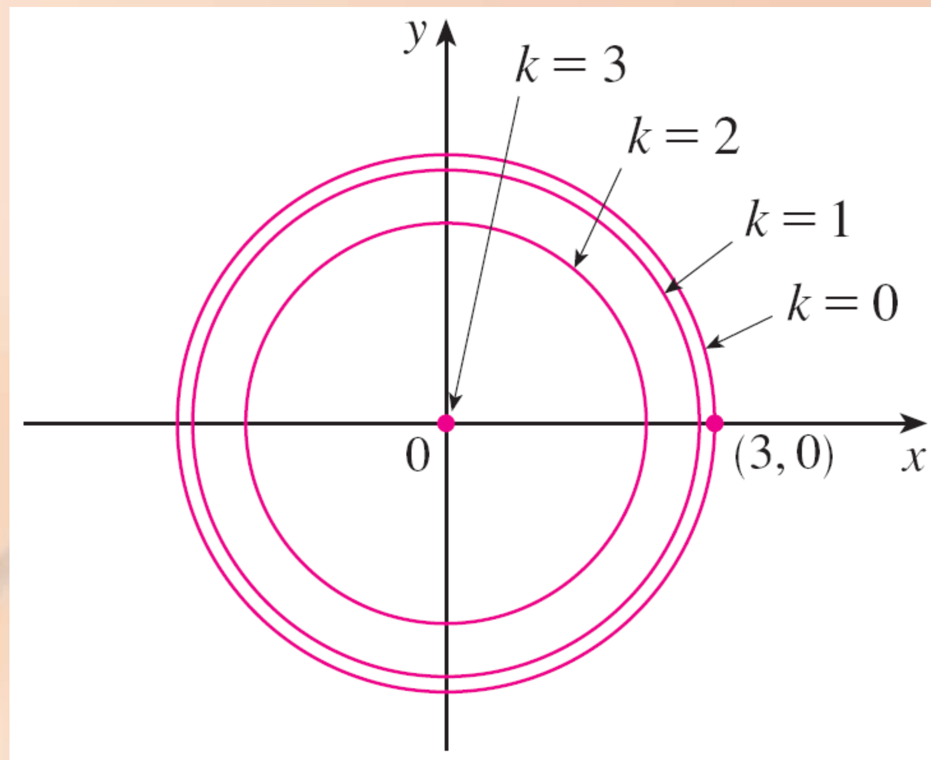
$$\sqrt{9-x^2-y^2}=k \quad \text{or} \quad x^2+y^2=9-k^2$$

This is a family of concentric circles with center (0, 0) and radius $\sqrt{9-k^2}$

- The cases k = 0, 1, 2, 3 are shown.



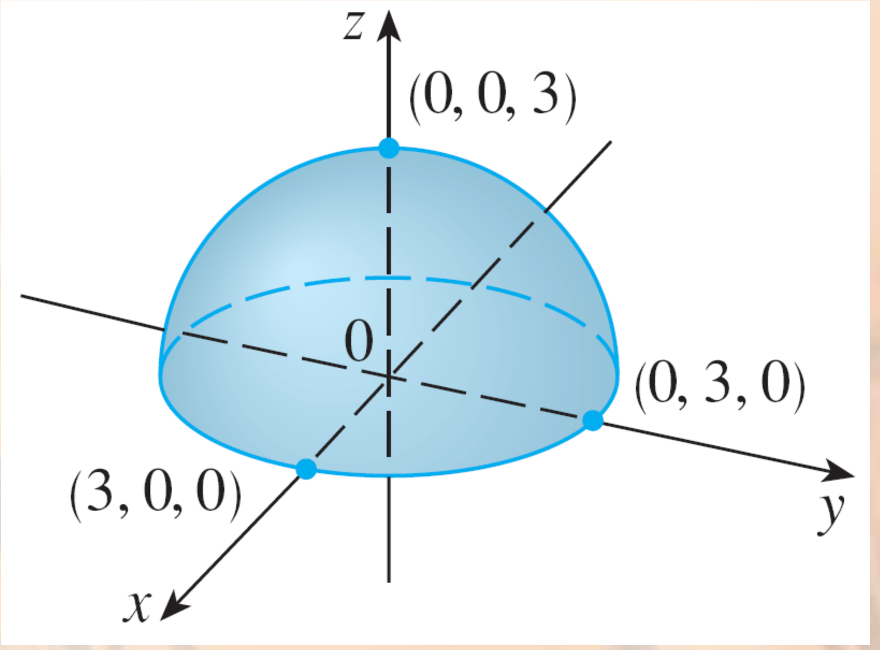
Try to visualize these level curves lifted up to form a surface.

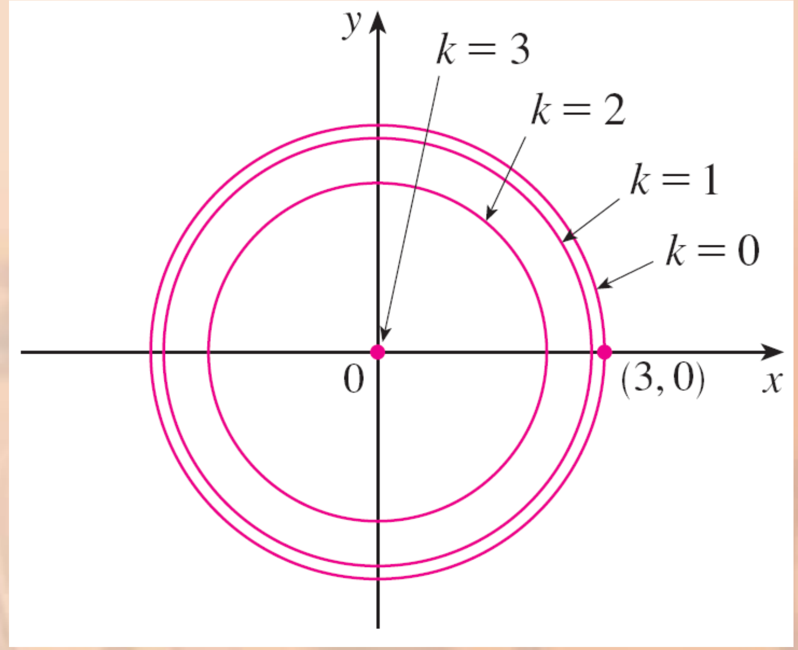


LEVEL CURVES

Example 11

Then, compare the formed surface with the graph of *g* (a hemisphere), as in the other figure.





Example 12

Sketch some level curves of the function

$$h(x, y) = 4x^2 + y^2$$

- The level curves are:

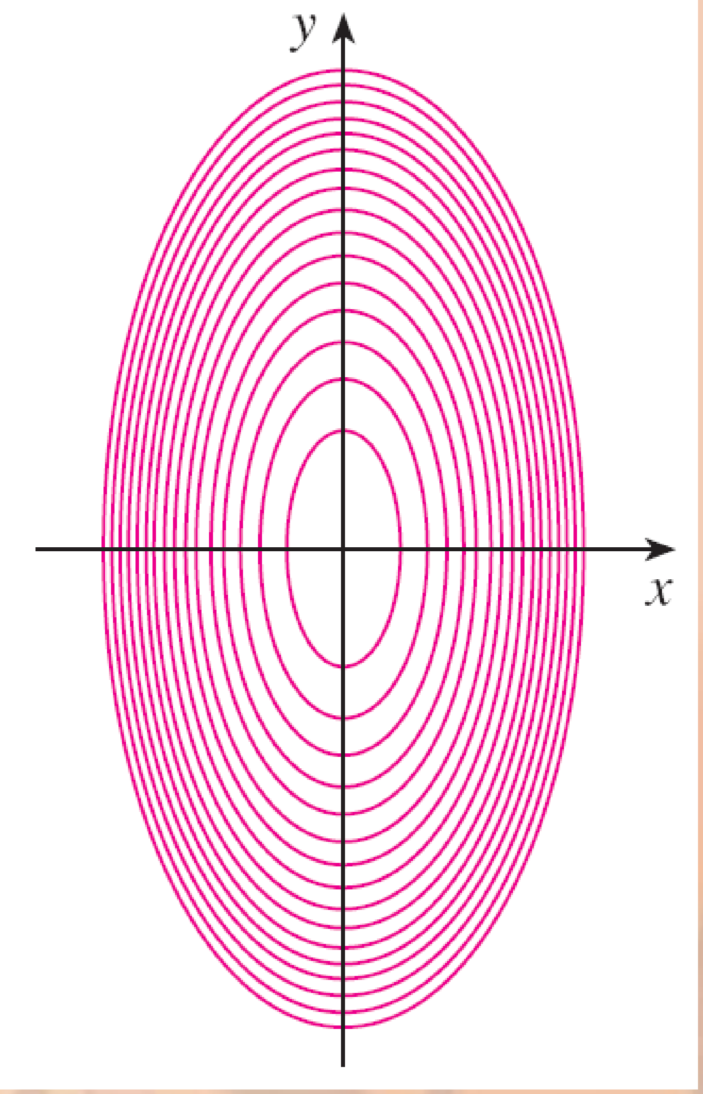
$$4x^2 + y^2 = k \quad \text{or} \quad \frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

- For $k > 0$, this describes a family of ellipses with semiaxes $\sqrt{k}/2$ and $\sqrt{k}$.

Example 12

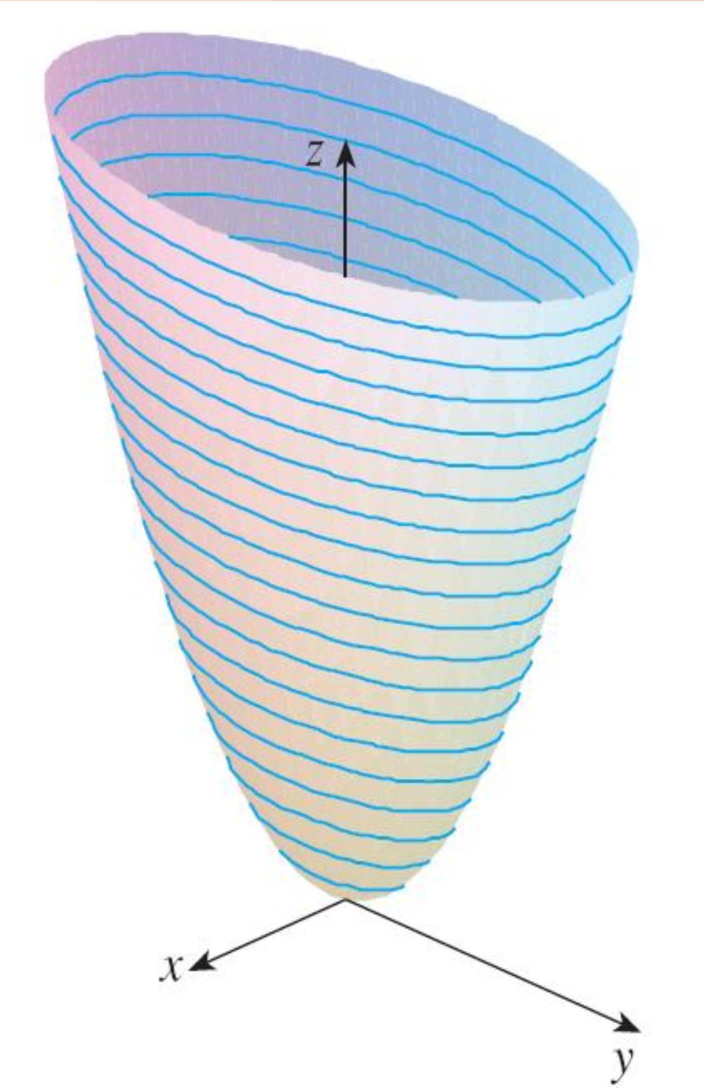
The figure shows a contour map of *h* drawn by a computer with level curves corresponding to:

k = 0.25, 0.5, 0.75, . . . , 4



Example 12

This figure shows those level curves lifted up to the graph of *h* (an elliptic paraboloid) where they become horizontal traces.

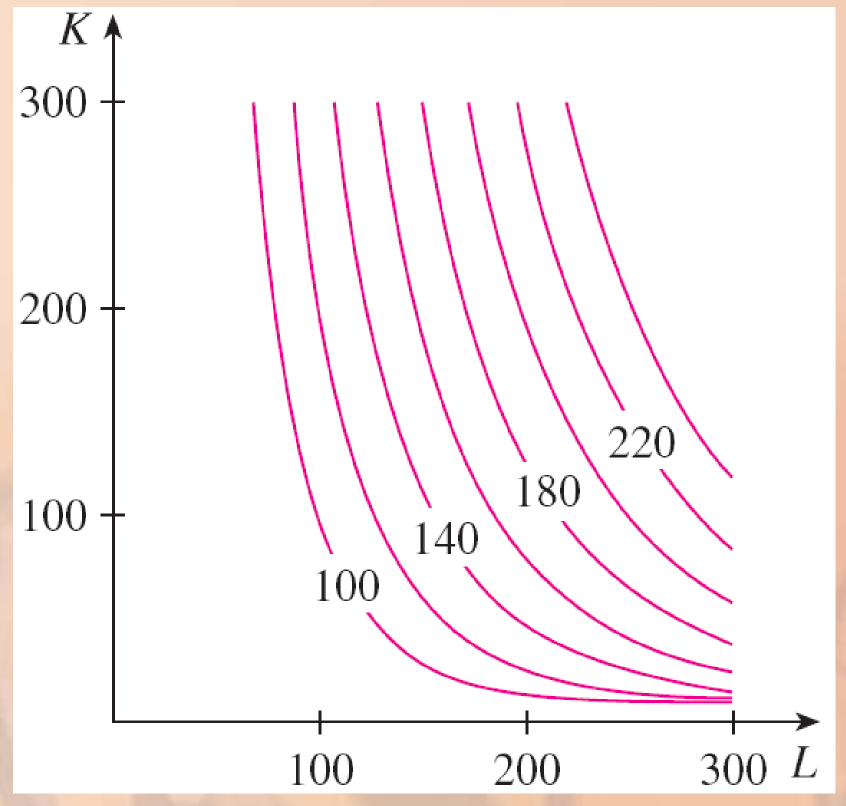


Example 13

Plot level curves for the Cobb-Douglas production function of Example 3.

Example 13

Here, we use a computer to draw a contour plot for the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$

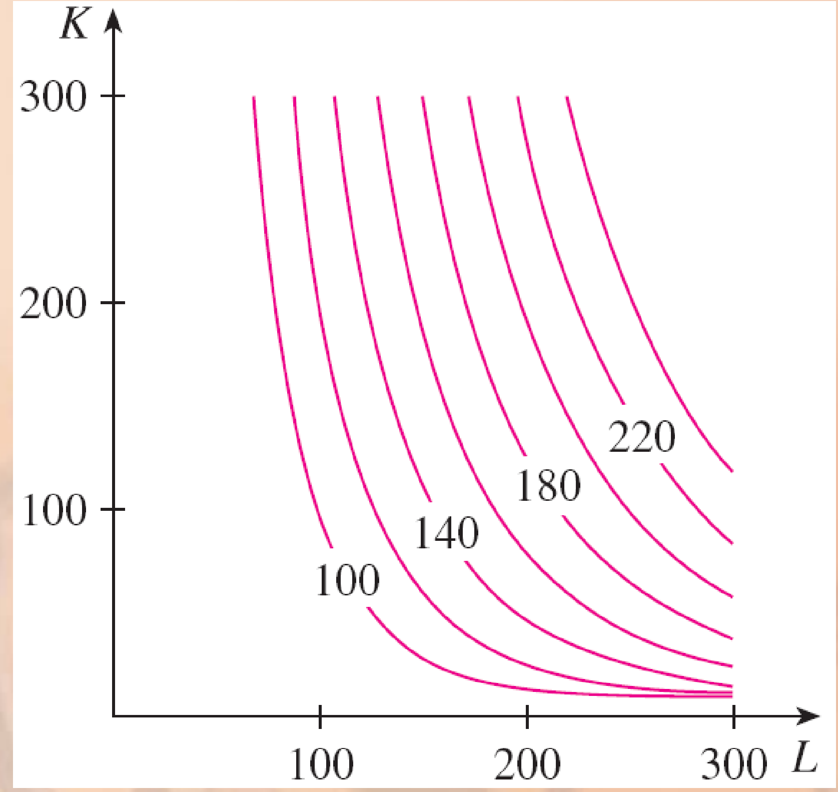


LEVEL CURVES

Example 13

Level curves are labeled with the value of the production *P*.

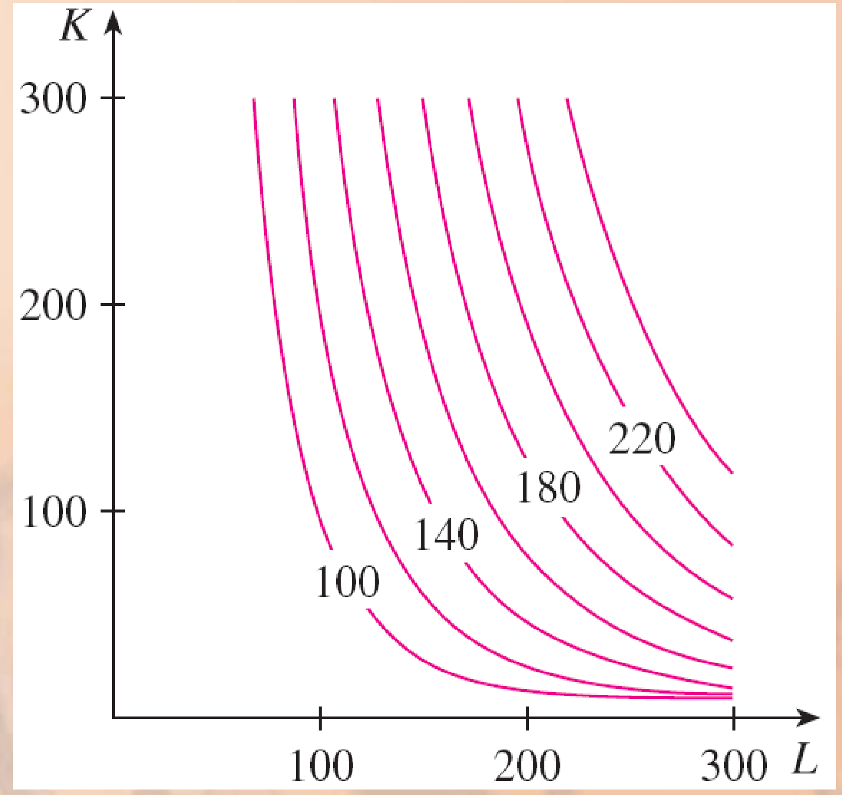
- For instance, the level curve labeled 140 shows all values of the labor *L* and capital investment *K* that result in a production of *P* = 140.



Example 13

We see that, for a fixed value of *P*,

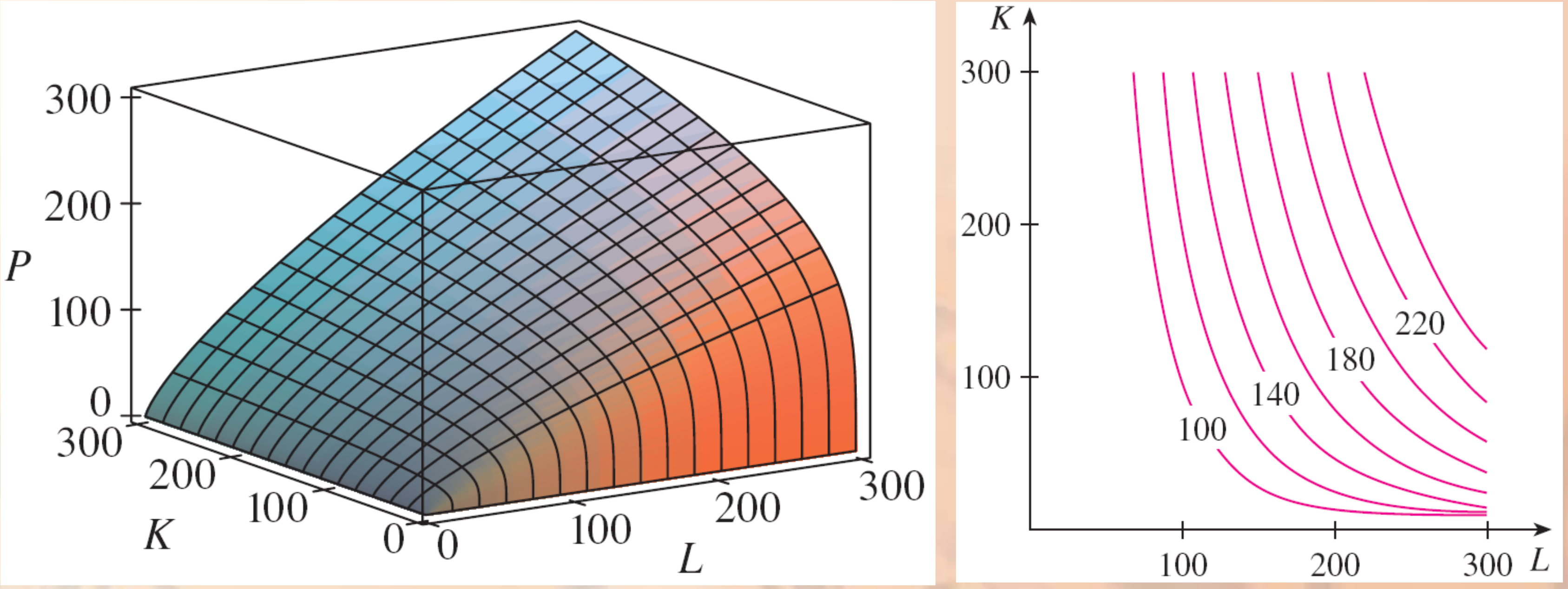
as *L* increases *K* decreases, and vice versa.



For some purposes, a contour map is more useful than a graph.

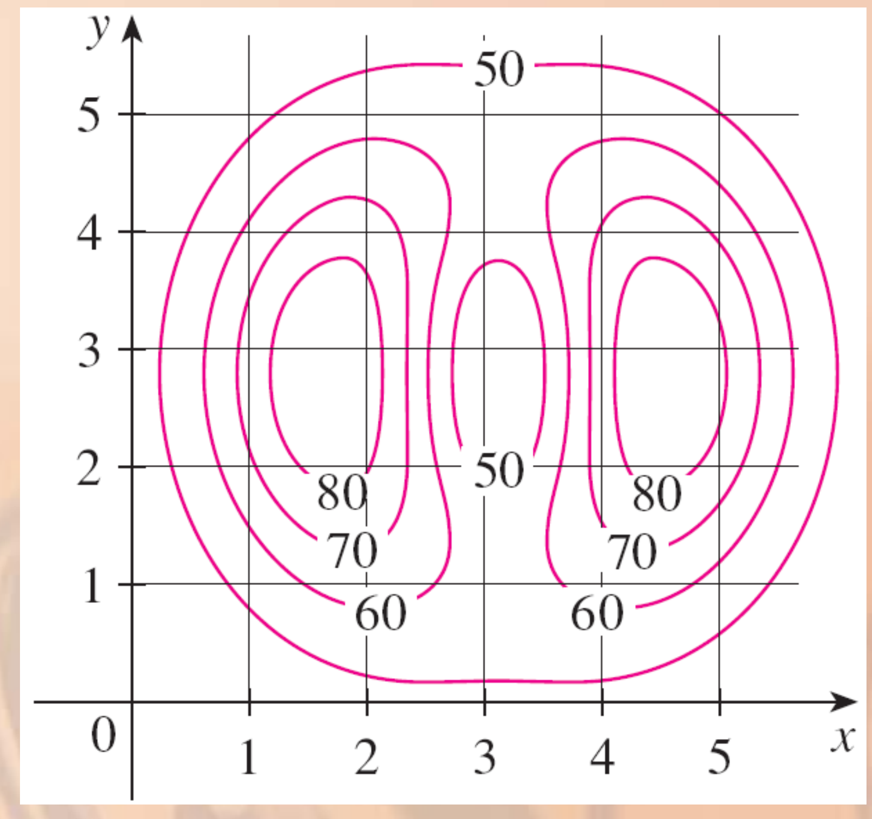
That is certainly true in Example 13.

- Compare the two figures.



LEVEL CURVES

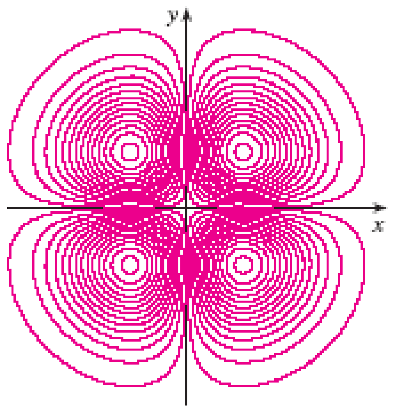
It is also true in estimating function values, as in Example 9.



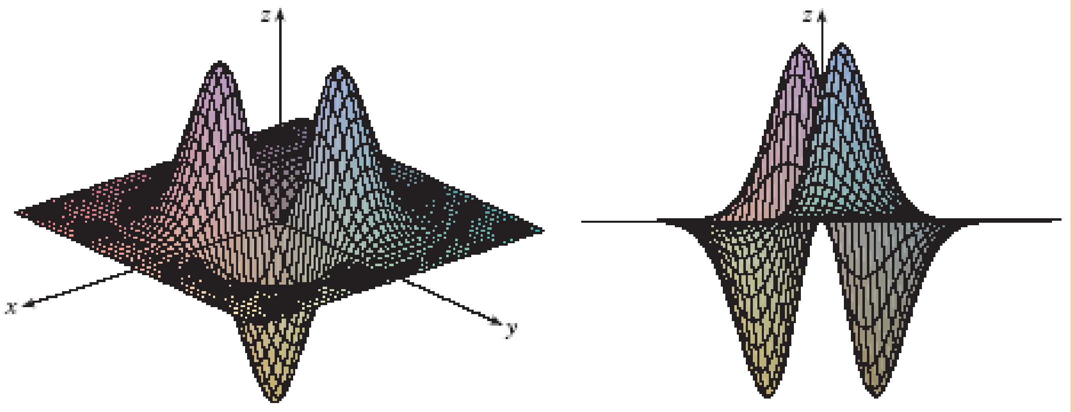
LEVEL CURVES

The following figure shows some computer-generated level curves together with the corresponding computer-generated graphs.

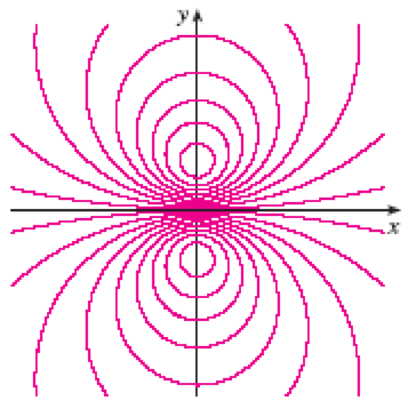
LEVEL CURVES



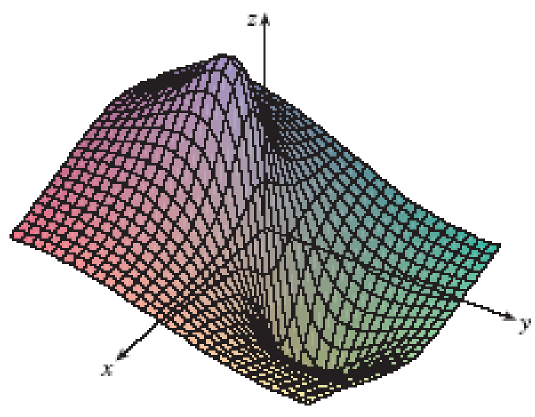
(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$



(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

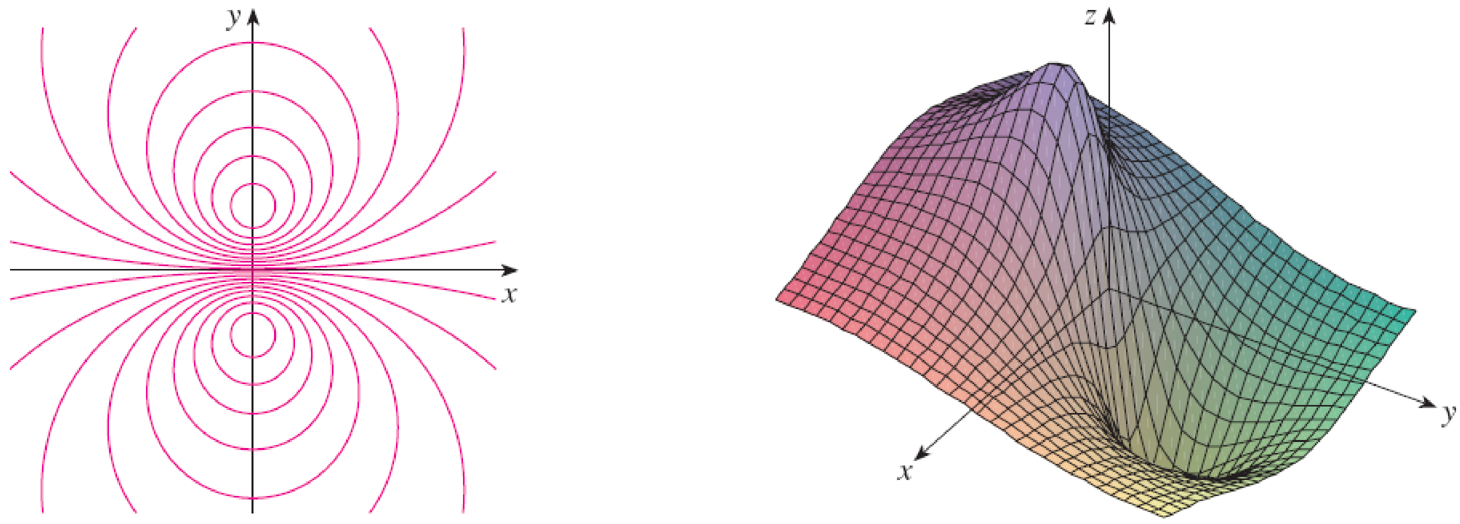


(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

LEVEL CURVES

Notice that the level curves in (c) crowd together near the origin.

- That corresponds to the fact that the graph in (d) is very steep near the origin.



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

FUNCTION OF THREE VARIABLES

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathrm{P}^3$ a unique real number denoted by $f(x, y, z)$.

FUNCTION OF THREE VARIABLES

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t.

- So, we could write:

$$T = f(x, y, t)$$

Example 14

Find the domain of *f* if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

- The expression for $f(x, y, z)$ is defined as long as $z - y > 0$.
- So, the domain of *f* is: $D = \{(x, y, z)$ ™ $\mathrm{P}^3 \mid z > y\}$

This is a half-space consisting of all points that lie above the plane $z = y$.

MULTIPLE VARIABLE FUNCTIONS

It's very difficult to visualize a function *f* of three variables by its graph.

- That would lie in a four-dimensional space.

MULTIPLE VARIABLE FUNCTIONS

However, we do gain some insight into *f* by examining its level surfaces—the surfaces with equations $f(x, y, z) = k$, where k is a constant.

- If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

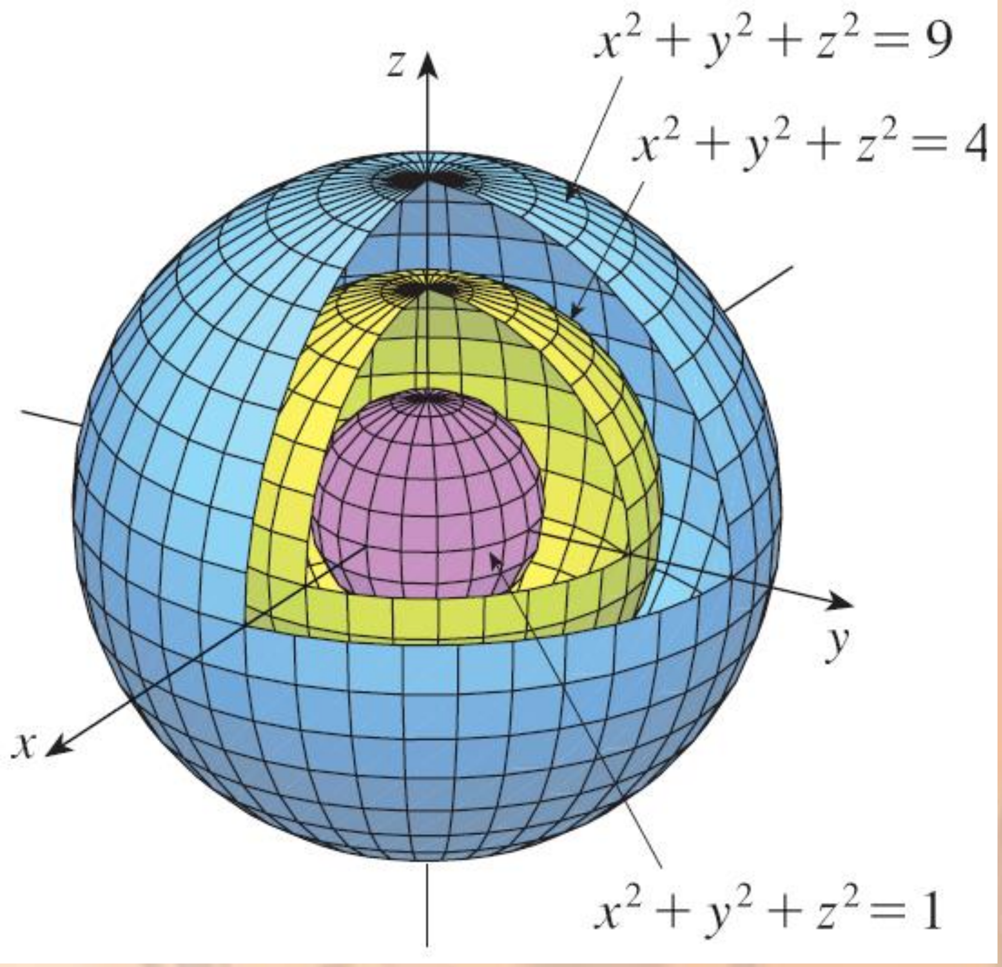
- The level surfaces are:

$$x^2 + y^2 + z^2 = k$$

where $k \geq 0$.

These form a family of concentric spheres with radius $\sqrt{k}$.

- So, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed.



MULTIPLE VARIABLE FUNCTIONS

Functions of any number of variables can be considered.

- A function of n variables is a rule that assigns a number $z = f(x_1, x_2, . . . , x_n)$ to an n-tuple $(x_1, x_2, . . . , x_n)$ of real numbers.

- We denote P^n by the set of all such n-tuples.

MULTIPLE VARIABLE FUNCTIONS

For example, suppose, for making a food product in a company,

- n different ingredients are used.
- c_i is the cost per unit of the ingredient.
- x_i units of the i th ingredient are used.

MULTIPLE VARIABLE FUNCTIONS **Equation 3**

Then, the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$C = f(x_1, x_2, \ldots, x_n)\ c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

MULTIPLE VARIABLE FUNCTIONS

The function f is a real-valued function whose domain is a subset of P^n.

Sometimes, we will use vector notation to write such functions more compactly:

- If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

MULTIPLE VARIABLE FUNCTIONS

With this notation, we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where:

- $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$
- $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors **c** and **x** in V_n

MULTIPLE VARIABLE FUNCTIONS

There is a one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in P^n and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n.

So, we have the following three ways of looking at a function f defined on a subset of P^n.

MULTIPLE VARIABLE FUNCTIONS

1. Function of real variables $x_1, x_2, \ldots, x_n$

2. Function of a single point variable

   $(x_1, x_2, \ldots, x_n)$

3. Function of a single vector variable

   $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.