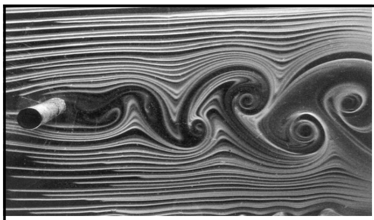
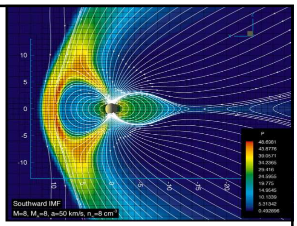
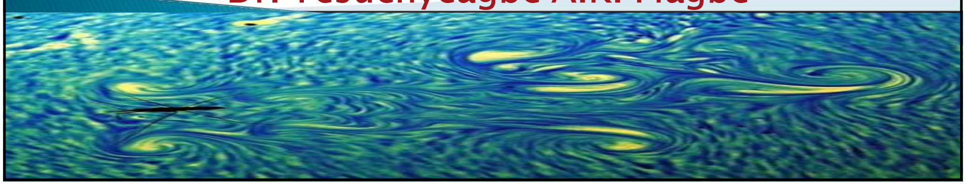




ME 252 Fluid Dynamics 1

Introduction

Dr. Yesuenyeagbe A.K. Fiagbe



Course Objectives

- Understand Fluid Kinematics
- Understand the Equation of motion
- Appreciate Bernoulli Equation and its applications.
- Understand the Equation of Energy.
- Momentum equation and its application.
- Appreciate incompressible flow in pipes and ducts.
- Appreciate issues with Open channel flow

2

Areas to Cover

- Fluid kinematics
- Mass conservation
- Equations of motion and energy.
  - Bernoulli equation
  - Energy Conservation
- Momentum equation and its applications.
- Moment of momentum equations
- Introduction to incompressible flow in pipes and ducts
- Open Channel flow

3

Instruction Format

Lecture

Mechanical – 4gsgai5	Thursday 303 5 – 7 p.m.
Automobile – 2vwgjag	Tuesday PB001 8-10 am
Marine – zp2rdmu	Tuesday PB001 8-10 am
Industrial – vjirubm	Tuesday PB001 8-10 am

Laboratory session will be taken in ME 296

4

Schedule

Week	Date	Subject Category		Remark
1		Introduction		
2		Fluid Kinematics		
3				
4		Bernoulli Equation		
5				
6		Applications of Bernoulli Equation		
7				Mid Sem Exam
8		Energy Equation and its application		
9				
10		Incompressible flow in pipes and ducts		
11				
12				
13		Review		
14		End of Semester Exam		
15				

5

Assessment

- Random and Unannounced Quizzes
- Assignments
- Attendance
- Mid Semester Examination
- End of Semester Examination

6

Class Regulations

- No lateness beyond 10 minutes: Student will be turned out.
- No mobile phone use in class: the phone will be confiscated for one week on first offence and for the semester on second offence.
- Decent conduct (both in dressing and demeanour).

7

Useful Reference Books

1. Cengel Y.A. and Cimbala J.M (2006) Fluid Mechanics: Fundamentals and Application, McGraw-Hill Companies Inc
2. Cengel, Y. A., & Turner, R. A. (2001). *Fundamentals of Thermal-Fluid Sciences.* New York: McGraw-Hill Companies Inc.
3. Crowe, C., & Elger, D. (2009). *Engineering Fluid Mechanics.* John Wiley & Sons, Inc.
4. Fox, R. W., McDonald, A. T., & Pritchard, P. J. (2004). *Introduction to Fluid Mechanics* (Sixth Edition ed.). Bogota: John Wiley & Sons, Inc.
5. NAKAYAMA, Y. (2000). *Introduction to Fluid Mechanics.* Tokyo: Yokendo Co. Ltd.

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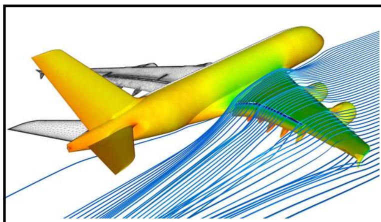
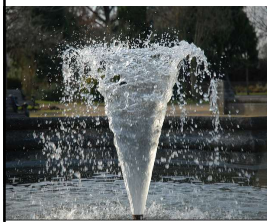


Kinematics of Fluid flow



Dr. Yesuenyeagbe A.K. Fiagbe

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Areas to Cover

- Kinematics of Fluid Flow
  - Methods of describing fluid motion
  - Types of Fluid flow
  - Rate of flow or discharge
  - Continuity Equation: Velocity and Acceleration
  - Ideal Fluid flow: Velocity potential function and stream function

10

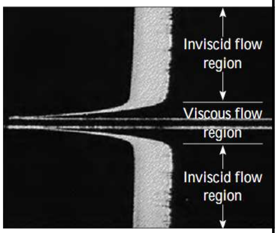
Objectives

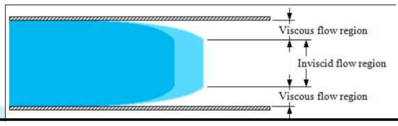
- Understand the role of the Material Derivative in transforming between Lagrangian and Eulerian descriptions
- Have an appreciation for the many ways that fluids move and deform
- Distinguish between rotational and irrotational regions of flow based on the flow property vorticity
- Understand the Conservation of Mass

CLASSIFICATION OF FLUID FLOWS

Viscous versus Inviscid Regions of Flow

- When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer.
- This internal resistance to flow is quantified by the fluid property **viscosity**, (a measure of internal stickiness of the fluid).
- Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases.
- Flows in which the frictional effects are significant are called **viscous flows.**
- In many flows of practical interest, there are regions (typically regions not close to solid surfaces) where viscous forces are negligibly small compared to inertial or pressure forces.
- Neglecting the viscous terms makes the flow an **inviscid flow** regions.





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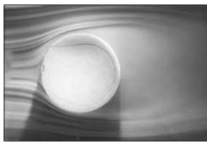
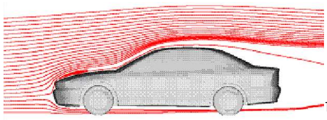
Internal versus External Flow

- The flow of an unbounded fluid over a surface such as a plate, a wire, or a pipe is **external flow.**
  - airflow over a ball or over an exposed pipe during a windy day is external flow.
- The flow in a pipe or duct is **internal flow** if the fluid is completely bounded by solid surfaces.
  - Water flow in a pipe, for example, is internal flow.
- The flow of liquids in a duct is called **open-channel flow** if the duct is only partially filled with the liquid and there is a free surface.
  - The flows of water in rivers and irrigation ditches.
- Internal flows are dominated by the influence of viscosity throughout the flow field.
- In external flows the viscous effects are limited to boundary layers near solid surfaces and to wake regions downstream of bodies.





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Compressible versus Incompressible Flow

- A flow is classified as being compressible or incompressible, depending on the level of **variation of density** during flow.
- A flow is said to be incompressible if the density remains nearly constant throughout.
- The densities of liquids are essentially constant, and thus the flow of liquids is typically incompressible.
  - liquids are usually referred to as incompressible substances.
- Gases are highly compressible.
  - A pressure change of just 0.01 atm, for example, causes a change of 1 percent in the density of atmospheric air.

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Compressible versus Incompressible Flow

- Compressibility of flow is expressed in terms of the dimensionless **Mach number** defined as

$$\mathrm{Ma} = \frac{V}{c} = \frac{\text{Speed of flow}}{\text{Speed of sound}}$$

- where c = speed of sound whose value is 346 m/s in air at room temperature at sea level.
- A flow is
  - sonic when Ma =1,
  - subsonic when Ma < 1,
  - supersonic when Ma > 1, and
  - hypersonic when Ma >> 1.

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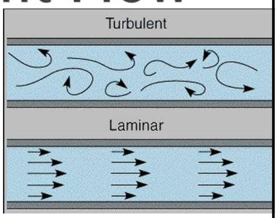
Compressible versus Incompressible Flow

- Gas flows can often be approximated as incompressible if the density changes are under about 5 percent, which is usually the case when Ma < 0.3. Therefore, the compressibility effects of air can be neglected at speeds under about 100 m/s.
- Note that the flow of a gas is not necessarily a compressible flow.

Laminar versus Turbulent Flow

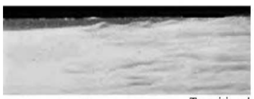


- The highly ordered fluid motion characterized by smooth layers of fluid is called **laminar**.
- The highly disordered fluid motion that typically occurs at high velocities and is characterized by velocity fluctuations is called **turbulent.**
- A flow that alternates between being laminar and turbulent is called **transitional**.
- Dimensionless **Reynolds number, Re**, is used as a key parameter in determining the flow regime in pipes.

$$Re = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{V_{avg}D}{\nu} = \frac{\rho V_{avg}D}{\mu}$$

$Re \lesssim 2300$	laminar flow
$2300 \lesssim Re \lesssim 4000$	transitional flow
$Re \gtrsim 4000$	turbulent flow

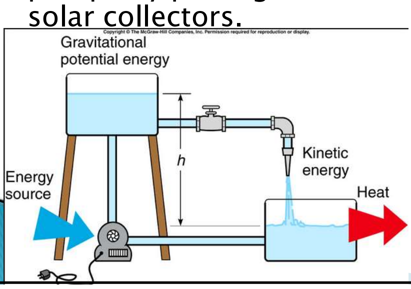

Laminar


Transitional


Turbulent

Natural (or Unforced) versus Forced Flow

- In forced flow, a fluid is forced to flow over a surface or in a pipe by **external means** such as a pump or a fan.
- In natural flows, any fluid motion is due to **natural means.**
  - In solar hot-water systems, for example, the thermosiphoning effect is commonly used to replace pumps by placing the water tank sufficiently above the solar collectors.



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Steady versus Unsteady Flow

- The term **steady** implies no change at a point with time.
- The opposite of steady is **unsteady.**
- The term **uniform** implies no change with location over a specified region.
- The terms **unsteady** and **transient** are often used interchangeably, but are not synonyms.
- In fluid mechanics, unsteady is the most general term that applies to any flow that is not steady, but transient is typically used for developing flows.
  - When a rocket engine is fired up, for example, there are transient effects (the pressure builds up inside the rocket engine, the flow accelerates, etc.) until the engine settles down and operates steadily.
- The term **periodic** refers to the kind of unsteady flow in which the flow oscillates about a steady mean.

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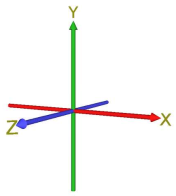
Steady versus Unsteady Flow

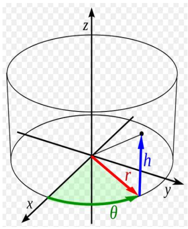
- During steady flow, the fluid properties can change from point to point within a device, but at any fixed point they remain constant.
- The volume, the mass, and the total energy content of a steady-flow device or flow section remain constant in steady operation.
- Steady-flow conditions can be closely approximated by devices that are intended for continuous operation such as turbines, pumps, boilers, condensers, and heat exchangers of power plants or refrigeration systems.
- Some cyclic devices, such as reciprocating engines or compressors, do not satisfy the steady-flow conditions since the flow at the inlets and the exits is pulsating and not steady. However, the fluid properties vary with time in a periodic manner, and the flow through these devices can still be analyzed as a steady-flow process by using time-averaged values for the properties.

21

One-, Two-, and Three-Dimensional Flows

- A flow field is best characterized by the velocity distribution, and thus a flow is said to be one-, two-, or three-dimensional if the flow velocity varies in one, two, or three primary dimensions, respectively.
- A typical fluid flow involves a three-dimensional geometry, and the velocity may vary in all three dimensions, rendering the flow three-dimensional [V(x, y, z) in rectangular or V(r, θ, z) in cylindrical coordinates].
- However, the variation of velocity in certain directions can be small relative to the variation in other directions and can be ignored with negligible error. In such cases, the flow can be modeled conveniently as being one- or two-dimensional, which is easier to analyze





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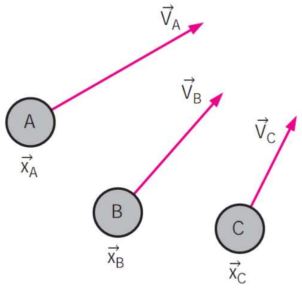
Fluid Kinematics

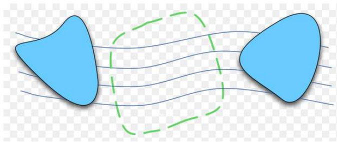
LAGRANGIAN AND EULERIAN DESCRIPTIONS

- Kinematics concerns the study of motion.
- In fluid dynamics, fluid kinematics is the study of how fluids flow and how to describe fluid motion.
- From a fundamental point of view, there are two distinct ways to describe fluid motion.
- *Lagrangian Description and Eulerian Description*

Lagrangian Description

- In the *Lagrangian Description,* individual fluid particles are tracked and followed
- The individual fluid particles are "marked," and their positions, velocities, etc. are described as a function of time
- The kinematics of such experiments involves keeping track of the position vector of each object, x_A, x_B, ... , and the velocity vector of each object, V_A, V_B, . . . , as functions of time.





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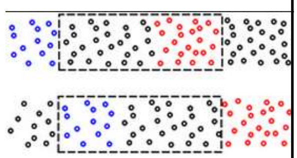
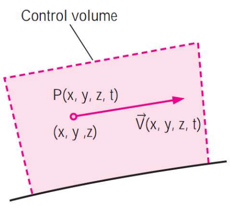
Eulerian Description

- In the *Eulerian Description,* a control volume is defined, within which fluid flow properties of interest are expressed as *fields*
- The individual fluid particles are not identified and tracked
- Pressure, velocity, acceleration, and all other flow properties are described as *fields* within the control volume.





$$P = P(x, y, z, t)$$
$$\vec{V} = \vec{V}(x, y, z, t)$$
$$\vec{a} = \vec{a}(x, y, z, t)$$

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Velocity Field

- The velocity field can be expanded in Cartesian coordinates (x, y, z), (i, j, k) as

$$\vec{V} = (u, v, w) = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

In the Eulerian description we don't really care what happens to individual fluid particles; rather we are concerned with the pressure, velocity, acceleration, etc., of whichever fluid particle happens to be at the location of interest at the time of interest.

Example 1

A steady, incompressible, two-dimensional velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$

where the *x*- and *y*-coordinates are in meters and the magnitude of velocity is in m/s. A **stagnation point** is defined as *a point in the flow field where the velocity is identically zero*.

(a) Determine if there are any stagnation points in this flow field and, if so, where?

(b) Sketch velocity vectors at several locations in the domain between $x = -2$ m to 2 m and $y = 0$ m to 5 m; qualitatively describe the flow field.

SOLUTION For the given velocity field, the location(s) of stagnation point(s) are to be determined. Several velocity vectors are to be sketched and the velocity field is to be described.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional, implying no *z*-component of velocity and no variation of u or v with *z*.

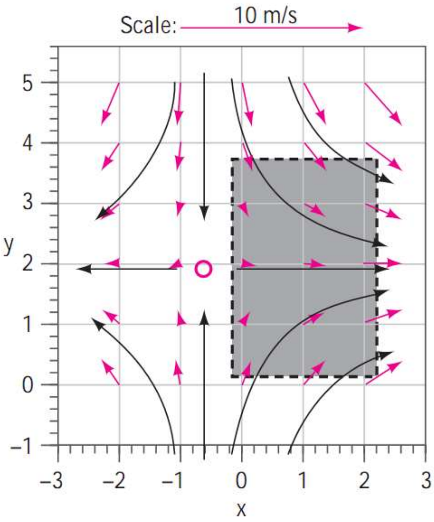
Analysis (*a*) Since $\vec{V}$ is a vector, *all* its components must equal zero in order for $\vec{V}$ itself to be zero. Using Eq. 4–4 and setting Eq. 1 equal to zero,

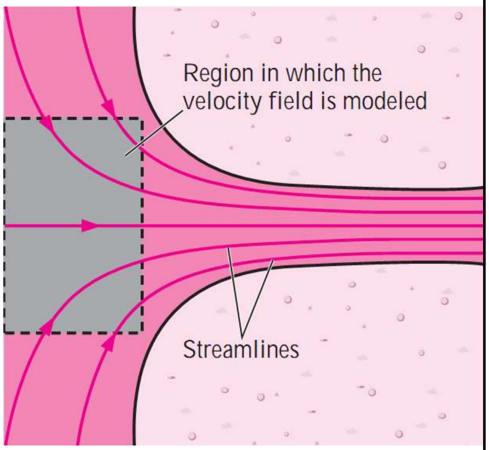
$$\text{Stagnation point:} \quad \begin{aligned} u &= 0.5 + 0.8x = 0 &\rightarrow\quad x &= -0.625\text{ m} \\ v &= 1.5 - 0.8y = 0 &\rightarrow\quad y &= 1.875\text{ m} \end{aligned}$$

Yes. There is one stagnation point located at $x = -0.625$ m, $y = 1.875$ m.

(*b*) The *x*- and *y*-components of velocity are calculated from Eq. 1 for several (*x*, *y*) locations in the specified range. For example, at the point ($x = 2$ m, $y = 3$ m), $u = 2.10$ m/s and $v = -0.900$ m/s. The magnitude of velocity (the *speed*) at that point is 2.28 m/s. At this and at an array of other locations, the velocity vector is constructed from its two components, the results of which are shown in Fig. 4–4. The flow can be described as stagnation point flow in which flow enters from the top and bottom and spreads out to the right and left about a horizontal line of symmetry at $y = 1.875$ m. The stagnation point of part (*a*) is indicated by the blue circle in Fig. 4–4.

If we look only at the shaded portion of Fig. 4–4, this flow field models a converging, accelerating flow from the left to the right. Such a flow might be encountered, for example, near the submerged bell mouth inlet of a hydroelectric dam (Fig. 4–5). The useful portion of the given velocity field may be thought of as a first-order approximation of the shaded portion of the physical flow field of Fig. 4–5.





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Acceleration Field

- The equations of motion for fluid flow (such as Newton's second law) are written for an object of fixed identity, taken as a small fluid parcel.
- If we were to follow a particular fluid particle as it moves around in the flow, we would be employing the Lagrangian description, and the equations of motion would be directly applicable.
  - For example, we would define the particle's location in space in terms of a material position vector
    $(x_{particle}(t), y_{particle}(t), z_{particle}(t))$.
- Newton's second law applied to fluid particle,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

32

Acceleration Field

- By definition, the acceleration of the fluid particle is the time derivative of the particle's velocity,

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

$$\vec{V}_{particle}(t) \equiv \vec{V}(x_{particle}(t),\ y_{particle}(t),\ z_{particle}(t),\ t)$$

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{particle}, y_{particle}, z_{particle}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{particle}}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y_{particle}}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z_{particle}}\frac{dz_{particle}}{dt}$$

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- At any instant in time under consideration, the material position vector ($x_{particle}$,$y_{particle}$,$z_{particle}$) of the fluid particle in the Lagrangian frame is equal to the position vector (x, y, z) in the Eulerian frame. Then

$$\vec{a}_{particle}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

Gradient or del operator: $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

The first term on the right-hand side is called the **local acceleration** and is nonzero only for unsteady flows.
The second term is called the **advective acceleration** or the **convective acceleration**); this term can be nonzero even for steady flows.

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Acceleration Field

- In Cartesian coordinates then, the components of the acceleration vector are

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

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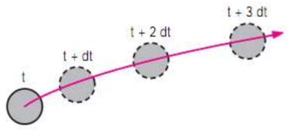
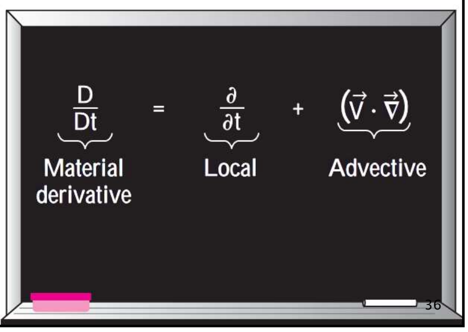
Material Derivative



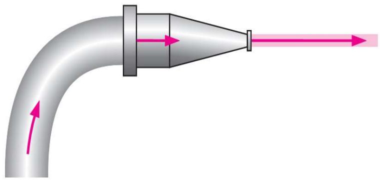
- The total derivative operator d/dt or D/Dt, is call the material derivative;
- it is formed by following a fluid particle as it moves through the flow field.

$$\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\vec{\nabla})$$



36

Nadeen is washing her car, using a nozzle. The nozzle is 3.90 in (0.325 ft) long, with an inlet diameter of 0.420 in (0.0350 ft) and an outlet diameter of 0.182 in (see Figure). The volume flow rate through the garden hose (and through the nozzle) is V = 0.841 gal/min (0.00187 ft³/s), and the flow is steady. Estimate the magnitude of the acceleration of a fluid particle moving down the centerline of the nozzle.



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SOLUTION The acceleration following a fluid particle down the center of a nozzle is to be estimated.

Assumptions **1** The flow is steady and incompressible. **2** The *x*-direction is taken along the centerline of the nozzle. **3** By symmetry, $v = w = 0$ along the centerline, but u increases through the nozzle.

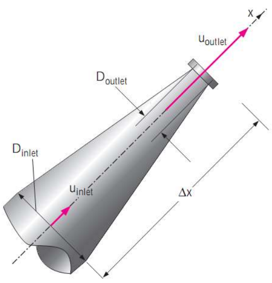
Analysis The flow is steady, so you may be tempted to say that the acceleration is zero. However, even though the local acceleration $\partial\vec{V}/\partial t$ is identically zero for this steady flow field, the advective acceleration $(\vec{V} \cdot \vec{\nabla})\vec{V}$ is *not* zero. We first calculate the average *x*-component of velocity at the inlet and outlet of the nozzle by dividing volume flow rate by cross-sectional area:

Inlet speed:

$$u_{\text{inlet}} \cong \frac{\dot{V}}{A_{\text{inlet}}} = \frac{4\dot{V}}{\pi D_{\text{inlet}}^2} = \frac{4(0.00187\ \text{ft}^3/\text{s})}{\pi(0.0350\ \text{ft})^2} = 1.95\ \text{ft/s}$$

Similarly, the average outlet speed is $u_{\text{outlet}} = 10.4$ ft/s. We now calculate the acceleration two ways, with equivalent results. First, a simple average value of acceleration in the *x*-direction is calculated based on the change in speed divided by an estimate of the **residence time** of a fluid particle in the nozzle, $\Delta t = \Delta x/u_{\text{avg}}$ (Fig. 4–10). By the fundamental definition of acceleration as the rate of change of velocity,

Method A: $$a_x \cong \frac{\Delta u}{\Delta t} = \frac{u_{\text{outlet}} - u_{\text{inlet}}}{\Delta x/u_{\text{avg}}} = \frac{u_{\text{outlet}} - u_{\text{inlet}}}{2\,\Delta x/(u_{\text{outlet}} + u_{\text{inlet}})} = \frac{u_{\text{outlet}}^2 - u_{\text{inlet}}^2}{2\,\Delta x}$$



The second method uses the equation for acceleration field components in Cartesian coordinates, Eq. 4–11,

Method B:

$$a_x = \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{Steady}}^{0} + u\frac{\partial u}{\partial x} + \underbrace{v\cancel{\frac{\partial u}{\partial y}}}_{v\,=\,0\text{ along centerline}}^{0} + \underbrace{w\cancel{\frac{\partial u}{\partial z}}}_{w\,=\,0\text{ along centerline}}^{0} \cong u_{avg}\frac{\Delta u}{\Delta x}$$

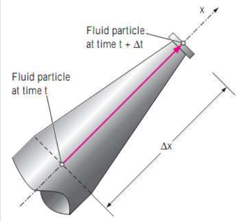
Here we see that only one advective term is nonzero. We approximate the average speed through the nozzle as the average of the inlet and outlet speeds, and we use a **first-order finite difference approximation** (Fig. 4–11) for the average value of derivative $\partial u/\partial x$ through the centerline of the nozzle:

$$a_x \cong \frac{u_{outlet} + u_{inlet}}{2}\,\frac{u_{outlet} - u_{inlet}}{\Delta x} = \frac{u_{outlet}^2 - u_{inlet}^2}{2\,\Delta x}$$

The result of method B is identical to that of method A. Substitution of the given values yields

Axial acceleration:

$$a_x \cong \frac{u_{outlet}^2 - u_{inlet}^2}{2\,\Delta x} = \frac{(10.4\text{ ft/s})^2 - (1.95\text{ ft/s})^2}{2(0.325\text{ ft})} = \mathbf{160\ ft/s^2}$$



Discussion Fluid particles are accelerated through the nozzle at nearly five times the acceleration of gravity (almost five g's)! This simple example clearly illustrates that the acceleration of a fluid particle can be nonzero, even in steady flow. Note that the acceleration is actually a **point function,** whereas we have estimated a simple average acceleration through the entire nozzle.

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Consider the steady, incompressible, two-dimensional velocity field of Example 1. (a) Calculate the material acceleration at the point (x=2 m, y =3 m). (b) Sketch the material acceleration vectors at the same array of x- and y-values as in Example 1.

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$

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SOLUTION For the given velocity field, the material acceleration vector is to be calculated at a particular point and plotted at an array of locations in the flow field.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional, implying no *z*-component of velocity and no variation of u or v with z.

Analysis (*a*) Using the velocity field of Eq. 1 of Example 4–1 and the equation for material acceleration components in Cartesian coordinates (Eq. 4–11), we write expressions for the two nonzero components of the acceleration vector:

$$a_x = \frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x}} + \underbrace{v\frac{\partial u}{\partial y}} + w\frac{\partial u}{\partial z}$$

$$= 0 + (0.5 + 0.8x)(0.8) + (15 - 0.8y)(0) + 0 = (0.4 + 0.64x)\ \text{m/s}^2$$

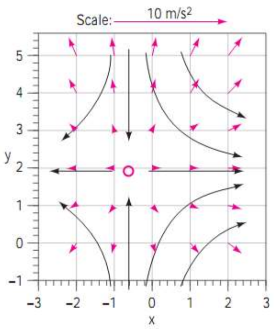
and

$$a_y = \frac{\partial v}{\partial t} + \underbrace{u\frac{\partial v}{\partial x}} + \underbrace{v\frac{\partial v}{\partial y}} + w\frac{\partial v}{\partial z}$$

$$= 0 + (0.5 + 0.8x)(0) + (1.5 - 0.8y)(-0.8) + 0 = (-1.2 + 0.64y)\ \text{m/s}^2$$

At the point (x = 2 m, y = 3 m), a_x = **1.68 m/s^2** and a_y = **0.720 m/s^2**.

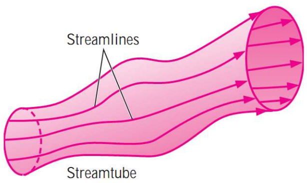
(*b*) The equations in part (*a*) are applied to an array of *x*- and *y*-values in the flow domain within the given limits, and the acceleration vectors are plotted in Fig. 4–14.



Discussion The acceleration field is nonzero, even though the flow is *steady*. Above the stagnation point (above y = 1.875 m), the acceleration vectors plotted in Fig. 4–14 point upward, increasing in magnitude away from the stagnation point. To the right of the stagnation point (to the right of x = −0.625 m), the acceleration vectors point to the right, again increasing in magnitude away from the stagnation point. This agrees qualitatively with the velocity vectors of Fig. 4–4 and the streamlines sketched in Fig. 4–14; namely, in the upper-right portion of the flow field, fluid particles are accelerated in the upper-right direction and therefore veer in the counterclockwise direction due to **centripetal acceleration** toward the upper right. The flow below y = 1.875 m is a mirror image of the flow above this symmetry line, and flow to the left of x = −0.625 m is a mirror image of the flow to the right of this symmetry line.

Streamlines

- A streamline is a curve that is everywhere tangent to the instantaneous local velocity vector.
- Useful as indicators of the instantaneous direction of fluid motion throughout the flow field.
  - For example, regions of recirculating flow and separation of a fluid off of a solid wall are easily identified by the streamline pattern.
- Mathematically, however, we can write a simple expression for a streamline based on its definition.



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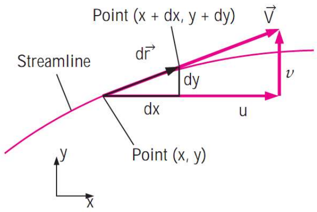
Streamlines

- Consider an infinitesimal arc length **dr=dxi+dyj+dzk** along streamline;
- dr must be parallel to the local velocity vector V =**ui+vj+wk** by definition of the streamline.
- By simple geometric arguments using similar triangles, we know that the components of dr must be proportional to those of V Hence,



Equation for a streamline: $\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

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Streamlines

- In two dimensions, (x, y), (u, v), the following differential equation is obtained:

Streamline in the xy-plane: $$\left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u}$$

For the steady, incompressible, two-dimensional velocity field of Example 1, plot several streamlines in the right half of the flow (x > 0)

SOLUTION An analytical expression for streamlines is to be generated and plotted in the upper-right quadrant.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional, implying no *z*-component of velocity and no variation of *u* or *v* with *z*.

Analysis Equation 4–16 is applicable here; thus, along a streamline,

$$\frac{dy}{dx} = \frac{1.5 - 0.8y}{0.5 + 0.8x}$$

We solve this differential equation by separation of variables:

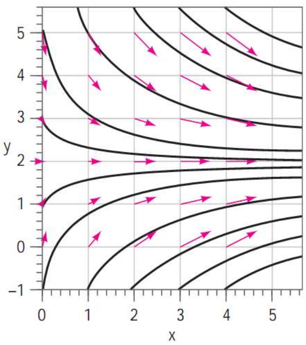
$$\frac{dy}{1.5 - 0.8y} = \frac{dx}{0.5 + 0.8x} \quad \rightarrow \quad \int \frac{dy}{1.5 - 0.8y} = \int \frac{dx}{0.5 + 0.8x}$$

After some algebra (which we leave to the reader), we solve for *y* as a function of *x* along a streamline,

$$y = \frac{C}{0.8(0.5 + 0.8x)} + 1.875$$

where *C* is a constant of integration that can be set to various values in order to plot the streamlines. Several streamlines of the given flow field are shown in Fig. 4–17.

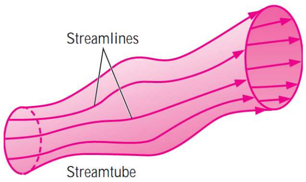
Discussion The velocity vectors of Fig. 4–4 are superimposed on the streamlines of Fig. 4–17; the agreement is excellent in the sense that the velocity vectors point everywhere tangent to the streamlines. Note that speed cannot be determined directly from the streamlines alone.



$$\int \frac{1}{a+bu} du = \frac{1}{b}\ln(a+bu) + C$$

Streamtubes

- A streamtube consists of a bundle of streamlines, much like a communications cable consists of a bundle of fiber-optic cables.
- Since streamlines are everywhere parallel to the local velocity, fluid cannot cross a streamline by definition.
- By extension, fluid within a streamtube must remain there and cannot cross the boundary of the streamtube.
- Both streamlines and streamtubes are instantaneous quantities, defined at a particular instant in time according to the velocity field at that instant.



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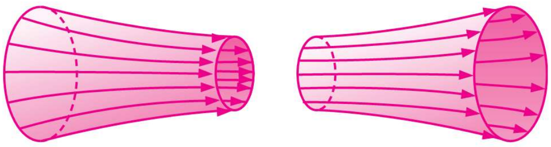
Streamtubes

- In an unsteady flow, the streamline pattern may change significantly with time.
- Nevertheless, at any instant in time, the mass flow rate passing through any cross-sectional slice of a given streamtube must remain the same.
  - For example, in a converging portion of an incompressible flow field, the diameter of the streamtube must decrease as the velocity increases so as to conserve mass (Fig. a).
  - Likewise, the streamtube diameter increases in diverging portions of the incompressible flow (Fig. b).



(a) (b)

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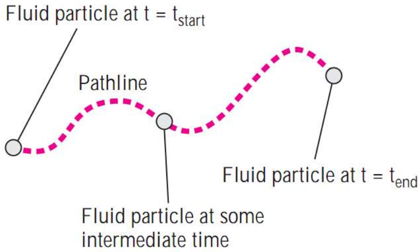
Pathlines

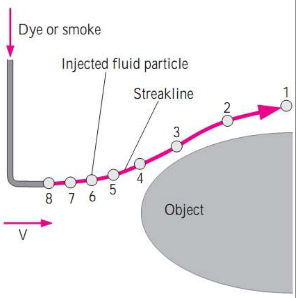


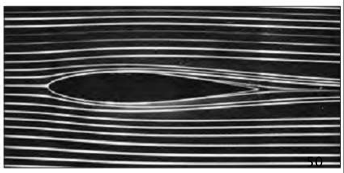
- A pathline is the actual path traveled by an individual fluid particle over some time period.
- A pathline is a Lagrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.
- Thus, a pathline is the same as the fluid particle's material position vector $(x_{particle(t)}, y_{particle(t)}, z_{particle(t)})$, traced out over some finite time interval.

Tracer particle location at time t: $\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V}\, dt$

Streaklines

- A streakline is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Streaklines are the most common flow pattern generated in a physical experiment.
- If you insert a small tube into a flow and introduce a continuous stream of tracer fluid (dye in a water flow or smoke in an airflow), the observed pattern is a streakline.





Streaklines

- For a known velocity field, a streakline can be generated numerically, although with some difficulty.
- One needs to follow the paths of a continuous stream of tracer particles from the time of their injection into the flow until the present time.
- Mathematically, the location of a tracer particle is integrated over time from the time of its injection t_{inject} to the present time $t_{present}$

$$\vec{x} = \vec{x}_{injection} + \int_{t_{inject}}^{t_{present}} \vec{V}\,dt$$

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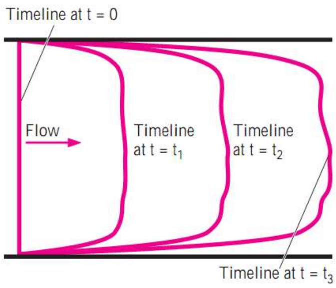
Timelines

- A timeline is a set of adjacent fluid particles that were marked at the same (earlier) instant in time.
- Timelines are particularly useful in situations where the uniformity of a flow (or lack thereof) is to be examined.



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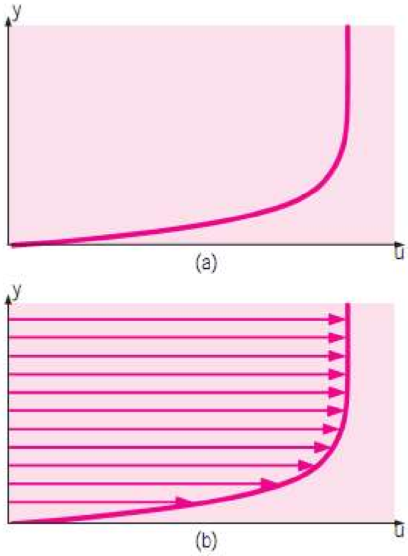
PLOTS OF FLUID FLOW DATA

- A **profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.



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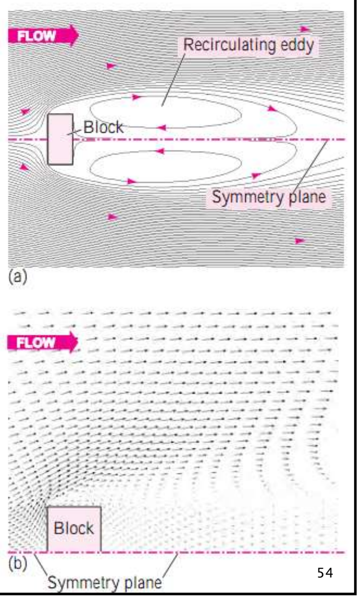
PLOTS OF FLUID FLOW DATA

- A **vector plot** is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.



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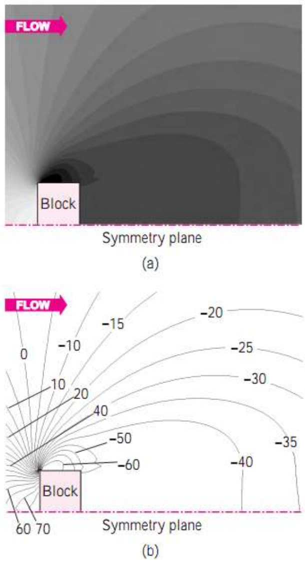
PLOTS OF FLUID FLOW DATA

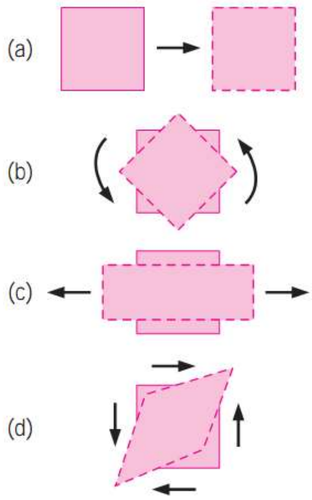
- A **contour plot** shows curves of constant values of a scalar property (or magnitude of a vector property) at an instant in time.



Types of Motion or Deformation of Fluid Elements

- In fluid mechanics, as in solid mechanics, an element may undergo four fundamental types of motion or deformation, illustrated in two dimensions as:

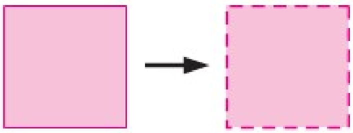
(a) Translation,
(b) Rotation,
(c) Linear strain (sometimes called extensional strain), and
(d) Shear strain.



Rate of Translation

- The rate of translation vector is described mathematically as the velocity vector.
- In Cartesian coordinates,

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$



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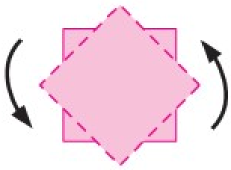
Rate of rotation

- Rate of rotation (angular velocity) at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point.
- The angle between any two initially perpendicular lines on fluid element remains at 90º since solid body rotation is illustrated in the figure.
- Therefore, both lines rotate at the same rate, and the rate of rotation in the plane is simply the component of angular velocity in that plane.



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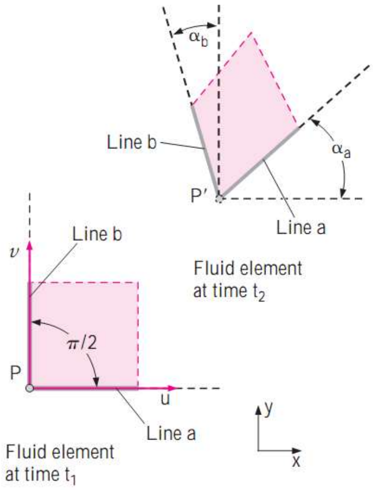
Rate of rotation

- The fluid particle translates and deforms as it rotates.
- we begin at time t_1 with two initially perpendicular lines (lines a and b) that intersect at point P in the xy-plane.
- We follow these lines as they move and rotate in an infinitesimal increment of time $dt=t_2-t_1$.
- At time t_2, line 'a' has rotated by angle α_a, and line 'b' has rotated by angle α_b, and both lines have moved with the flow as sketched.
- The average rotation angle is thus $(\alpha_a+\alpha_b)/2$, and the rate of rotation or angular velocity in the xy-plane is equal to the time derivative of this average rotation angle



$$\omega = \frac{d}{dt}\left(\frac{\alpha_a + \alpha_b}{2}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

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Rate of rotation

- The rate of rotation vector is equal to the angular velocity vector and is expressed in Cartesian coordinates as

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

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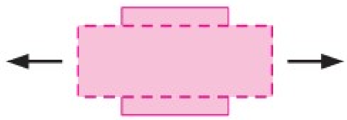
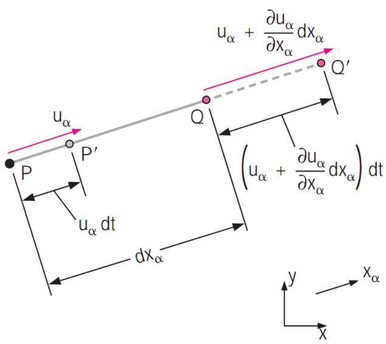
Linear strain rate

- Linear strain rate is defined as the rate of increase in length per unit length.
- Mathematically, the linear strain rate of a fluid element depends on the initial orientation or direction of the line segment upon which we measure the linear strain.
- Thus, it cannot be expressed as a scalar or vector quantity. Instead, we define linear strain rate in some arbitrary direction, which we denote as the x_α-direction.



Linear strain rate

- Line segment PQ has an initial length of dx_α, and it grows to line segment P'Q'.
- From the given definition and using the lengths marked, the linear strain rate in the x_α-direction is



$$\varepsilon_{\alpha\alpha} = \frac{d}{dt}\left(\frac{P'Q' - PQ}{PQ}\right)$$

$$\cong \frac{d}{dt}\left(\frac{\overbrace{\left(u_\alpha + \frac{\partial u_\alpha}{\partial x_\alpha} dx_\alpha\right) dt + dx_\alpha - u_\alpha\, dt}^{\text{Length of P'Q' in the } x_\alpha\text{-direction}} - \overbrace{dx_\alpha}^{\text{Length of PQ in the } x_\alpha\text{-direction}}}{\underbrace{dx_\alpha}_{\text{Length of PQ in the } x_\alpha\text{-direction}}}\right) = \frac{\partial u_\alpha}{\partial x_\alpha}$$

Linear strain rate

- Linear strain rate

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

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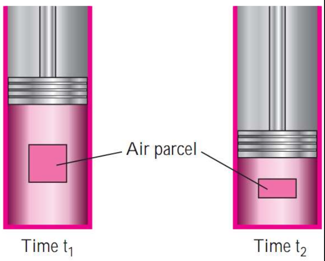
volumetric strain rate

- **Volumetric strain rate or bulk strain rate** is the rate of increase of volume of a fluid element per unit volume.
- It is defined as positive when the volume increases.
- Another synonym of volumetric strain rate is rate of volumetric dilatation,
- The volumetric strain rate is the sum of the linear strain rates in three mutually orthogonal directions.
- In Cartesian coordinates, the volumetric strain rate is thus



$$\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\frac{dV}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The volumetric strain rate is zero in an incompressible flow.

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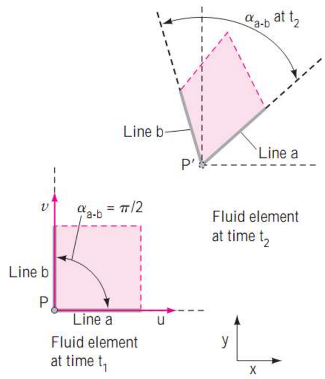
Shear strain rate

- A full mathematical description of shear strain rate requires its specification in any two mutually perpendicular directions.
- In Cartesian coordinates, the axes themselves are the most obvious choice.
- Consider a fluid element in two dimensions in the xy-plane.
- The element translates and deforms with
- Two initially mutually perpendicular lines (lines 'a' and 'b' in the x- and y-directions, respectively) are followed.
- The angle between these two lines decreases from $\pi/2$ (90°) to the angle marked α_{a-b} at t_2 in the sketch.
- The shear strain rate at point P for initially perpendicular lines in the x- and y-directions is given by



$$\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{a\text{-}b} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

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Shear strain rate

- Shear strain rate at a point is defined as half of the rate of decrease of the angle between two initially perpendicular lines that intersect at the point.

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

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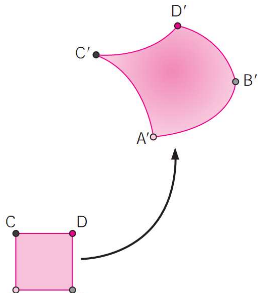
Linear strain & shear strain rate

- Finally, it turns out that we can mathematically combine linear strain rate and shear strain rate into one symmetric second-order tensor called the strain rate tensor as



$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

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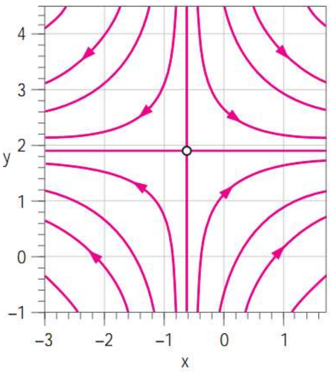
Consider the steady, two-dimensional velocity field of Example 1:

$$\vec{V} = (u, v) = (0.5 + 0.8\,x)\vec{i} + (1.5 - 0.8\,y)\vec{j}$$

where lengths are in units of m, time in s, and velocities in m/s. There is a stagnation point at (-0.625, 1.875) as shown in Figure. Streamlines of the flow are also plotted. Calculate the various kinematic properties, namely, the rate of translation, rate of rotation, linear strain rate, shear strain rate, and volumetric strain rate. Verify that this flow is incompressible.



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SOLUTION We are to calculate several kinematic properties of a given velocity field and verify that the flow is incompressible.
Assumptions **1** The flow is steady. **2** The flow is two-dimensional, implying no *z*-component of velocity and no variation of u or v with z.
Analysis By Eq. 4–19, the rate of translation is simply the velocity vector itself, given by Eq. 1. Thus,

Rate of translation: $u = 0.5 + 0.8x \qquad v = 1.5 - 0.8y \qquad w = 0$ **(2)**

The rate of rotation is found from Eq. 4–21. In this case, since $w = 0$ everywhere, and since neither u nor v vary with z, the only nonzero component of rotation rate is in the *z*-direction. Thus,

Rate of rotation: $$\vec{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \frac{1}{2}(0 - 0)\vec{k} = 0 \qquad (3)$$

In this case, we see that there is no net rotation of fluid particles as they move about. (This is a significant piece of information, to be discussed in more detail later in this chapter and also in Chap. 10.)

Linear strain rates can be calculated in any arbitrary direction using Eq. 4–22. In the *x*-, *y*-, and *z*-directions, the linear strain rates of Eq. 4–23 are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0.8\ s^{-1} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -0.8\ s^{-1} \qquad \varepsilon_{zz} = 0 \qquad (4)$$

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Thus, we predict that fluid particles *stretch* in the *x*-direction (positive linear strain rate) and *shrink* in the *y*-direction (negative linear strain rate). This is illustrated in Fig. 4–41, where we have marked an initially square parcel of fluid centered at (0.25, 4.25). By integrating Eqs. 2 with time, we calculate the location of the four corners of the marked fluid after an elapsed time of 1.5 s. Indeed this fluid parcel has stretched in the *x*-direction and has shrunk in the *y*-direction as predicted.

Shear strain rate is determined from Eq. 4–26. Because of the two-dimensionality, nonzero shear strain rates can occur only in the *xy*-plane. Using lines parallel to the *x*- and *y*-axes as our initially perpendicular lines, we calculate ε_{xy} from Eq. 4–26:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0 + 0) = 0 \qquad (5)$$

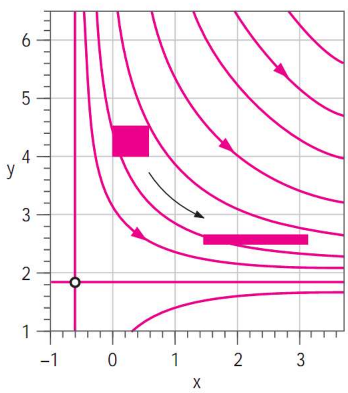
Thus, there is no shear strain in this flow, as also indicated by Fig. 4–41. Although the sample fluid particle deforms, it remains rectangular; its initially 90° corner angles remain at 90° throughout the time period of the calculation.

Finally, the volumetric strain rate is calculated from Eq. 4–24:

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = (0.8 - 0.8 + 0)\ s^{-1} = 0 \qquad (6)$$

Since the volumetric strain rate is zero everywhere, we can say definitively that fluid particles are neither dilating (expanding) nor shrinking (compressing) in volume. Thus, **we verify that this flow is indeed incompressible**. In Fig. 4–41, the area of the shaded fluid particle remains constant as it moves and deforms in the flow field.

Discussion In this example it turns out that the linear strain rates (ε_{xx} and ε_{yy}) are nonzero, while the shear strain rates (ε_{xy} and its symmetric partner ε_{yx})



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Discussion In this example it turns out that the linear strain rates (ε_{xx} and ε_{yy}) are nonzero, while the shear strain rates (ε_{xy} and its symmetric partner ε_{yx}) are zero. This means that *the x- and y-axes of this flow field are the principal axes.* The (two-dimensional) strain rate tensor in this orientation is thus

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} 0.8 & 0 \\ 0 & -0.8 \end{pmatrix} s^{-1} \quad (7)$$

If we were to rotate the axes by some arbitrary angle, the new axes would *not* be principal axes, and all four elements of the strain rate tensor would be nonzero. You may recall rotating axes in your engineering mechanics classes through use of Mohr's circles to determine principal axes, maximum shear strains, etc. Similar analyses can be performed in fluid mechanics.

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Vorticity and Rotationality

- A closely related kinematic property of great importance to the analysis of fluid flows is vorticity.
- The vorticity vector is defined mathematically as the curl of the velocity vector V.

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = \text{curl}(\vec{V})$$

Rate of rotation vector: $\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\text{curl}(\vec{V}) = \frac{\vec{\zeta}}{2}$

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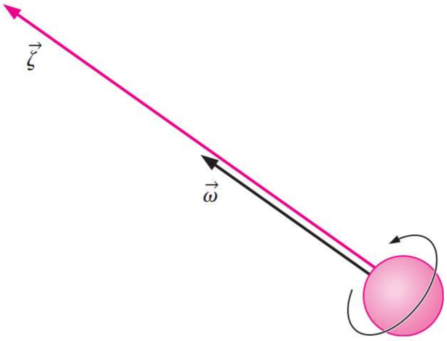
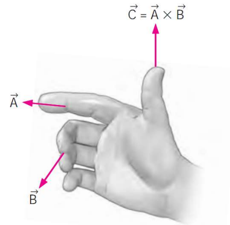
Vorticity and Rotationality

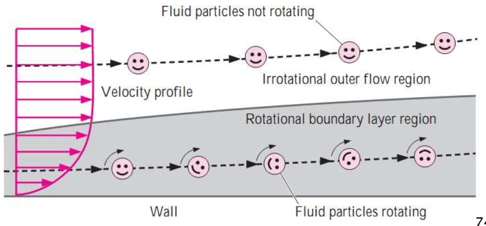
- Thus, vorticity is a measure of rotation of a fluid particle.
- The Vorticity is equal to twice the angular velocity of a fluid particle



Vorticity and Rotationality

- If the vorticity at a point in a flow field is **nonzero**, the fluid particle that happens to occupy that point in space is rotating; the flow in that region is called **rotational**.
- Likewise, if the vorticity in a region of the flow is **zero** (or negligibly small), fluid particles there are not rotating; the flow in that region is called **irrotational**.
- Physically, fluid particles in a rotational region of flow rotate end over end as they move along in the flow.
  - For example, fluid particles within the viscous boundary layer near a solid wall are rotational (and thus have nonzero vorticity), while
  - fluid particles outside the boundary layer are irrotational (and their vorticity is zero).





Vorticity and Rotationality

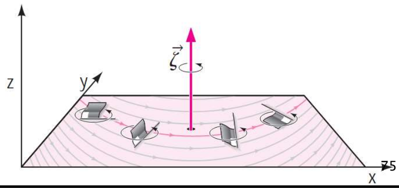
- In Cartesian coordinates, (i, j, k), (x, y, z), and (u, v, w), then: Vorticity vector is:

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Two-dimensional flow in Cartesian coordinates:

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Note that if a flow is two-dimensional in the xy-plane, the vorticity vector must point in either the z- or -z-direction



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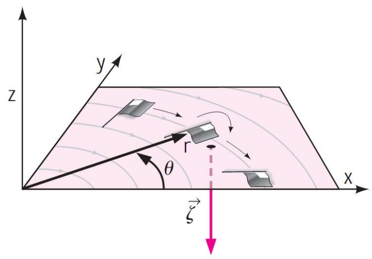
Vorticity and Rotationality

Vorticity vector in cylindrical coordinates:

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{k}$$



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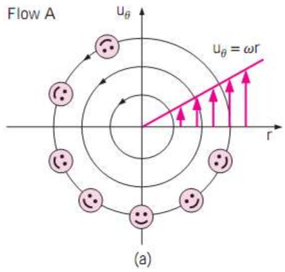
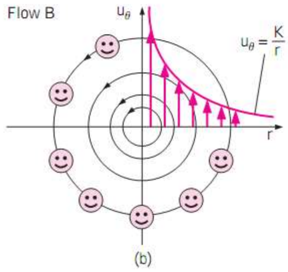
Comparison of Two Circular Flows

- Not all flows with circular streamlines are rotational.
- To illustrate this point, we consider two incompressible, steady, two-dimensional flows, both of which have circular streamlines in the rθ-plane:



Flow A—solid-body rotation: $u_r = 0$ and $u_\theta = \omega r$

Flow B—line vortex: $u_r = 0$ and $u_\theta = \dfrac{K}{r}$



where ω and K are constants.
(u_θ in flow B is infinite at r=0, which is of course physically impossible; we ignore the region close to the origin to avoid this problem.) Since the radial component of velocity is zero in both cases, the streamlines are circles about the origin.

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- We now calculate and compare the vorticity field for each of these flows

Flow A—solid-body rotation: $$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{k} = 2\omega\vec{k}$$

Flow B—line vortex: $$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{k} = 0$$

Not surprisingly, the vorticity for solid-body rotation is nonzero.
In fact, it is a constant of magnitude twice the angular velocity and pointing in the same direction. **Flow A is rotational**. Physically, this means that individual fluid particles rotate as they revolve around the origin.
By contrast, the vorticity of the line vortex is identically zero everywhere (except right at the origin). **Flow B is irrotational**. Physically, fluid particles do not rotate as they revolve in circles about the origin.

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Consider the following steady, incompressible, two-dimensional velocity field:

$$\vec{V} = (u, v) = x^2\vec{i} + (-2xy - 1)\vec{j}$$

Is this flow rotational or irrotational? Sketch some streamlines in the first quadrant and discuss.

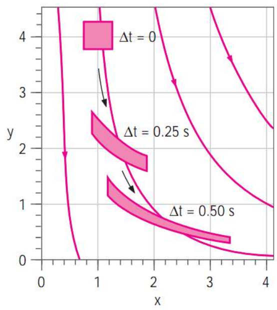
SOLUTION We are to determine whether a flow with a given velocity field is rotational or irrotational, and we are to draw some streamlines in the first quadrant.

Analysis Since the flow is two-dimensional, Eq. 4–31 is valid. Thus,

Vorticity: $$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = (-2y - 0)\vec{k} = -2y\vec{k} \qquad (2)$$

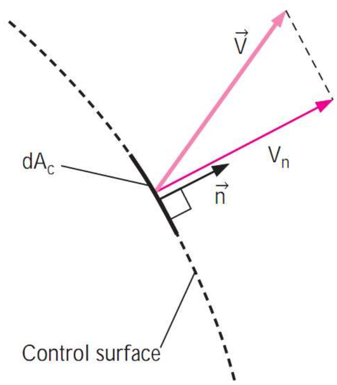
Since the vorticity is nonzero, this flow is **rotational.** In Fig. 4–47 we plot several streamlines of the flow in the first quadrant; we see that fluid moves downward and to the right. The translation and deformation of a fluid parcel is also shown: at $\Delta t = 0$, the fluid parcel is square, at $\Delta t = 0.25$ s, it has moved and deformed, and at $\Delta t = 0.50$ s, the parcel has moved farther and is further deformed. In particular, the right-most portion of the fluid parcel moves faster to the right and faster downward compared to the left-most portion, stretching the parcel in the *x*-direction and squashing it in the vertical direction. It is clear that there is also a net *clockwise* rotation of the fluid parcel, which agrees with the result of Eq. 2.

Discussion From Eq. 4–29, individual fluid particles rotate at an angular velocity equal to $\vec{\omega} = -y\vec{k}$, half of the vorticity vector. Since $\vec{\omega}$ is not constant, this flow is *not* solid-body rotation. Rather, $\vec{\omega}$ is a linear function of *y*. Further analysis reveals that this flow field is incompressible; the shaded



Mass and Volume Flow Rates

- The amount of mass flowing through a cross section per unit time is called the mass flow rate.
- A fluid flows into or out of a control volume, usually through pipes or ducts.
- The differential mass flow rate of fluid flowing across a small area element dA_c in a cross section of the pipe is proportional to dA_c itself,
- The fluid density ρ, and the component of the flow velocity normal to dA_c, which we denote as V_n, and is expressed as



$$\delta\dot{m} = \rho V_n \, dA_c$$

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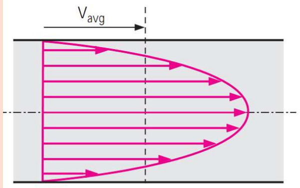
Mass and Volume Flow Rates

- The mass flow rate through the entire cross-sectional area of a pipe or duct is obtained by integration:

$$\dot{m} = \int_{A_c} \delta \dot{m} = \int_{A_c} \rho V_n \, dA_c \qquad \text{(kg/s)}$$

In a general compressible flow, both ρ and V_n vary across the pipe.
In many practical applications, however, the density is essentially uniform over the pipe cross section, and ρ can be taken outside the integral.
Velocity, however, is never uniform over a cross section of a pipe because of the no-slip condition at the walls.



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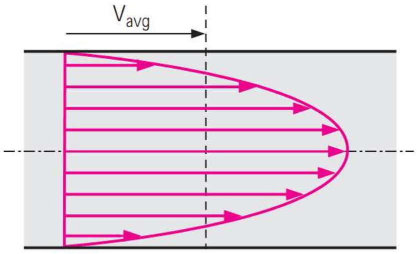
Mass and Volume Flow Rates

- The velocity varies from zero at the walls to some maximum value at or near the centerline of the pipe.
- Average velocity V_{avg} is the average value of V_n across the entire cross section of the pipe.

$$V_{avg} = \frac{1}{A_c} \int_{A_c} V_n \, dA_c$$



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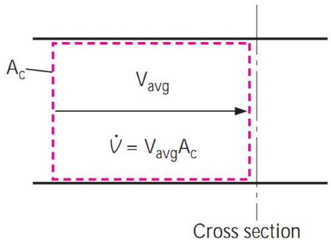
Mass and Volume Flow Rates

- The volume of the fluid flowing through a cross section per unit time is called the volume flow rate $\dot{V}$ given by



$$\dot{V} = \int_{A_c} V_n \, dA_c = V_{avg} A_c = VA_c \qquad (m^3/s)$$

Note that many fluid mechanics textbooks use Q instead of $\dot{V}$. for volume flow rate.

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Mass and Volume Flow Rates

- The mass and volume flow rates are related

$$\dot{m} = \rho\dot{V} = \frac{\dot{V}}{v}$$

Where v is the specific volume.
This is the relation between the mass and the volume of a fluid in a container.

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Conservation of Mass Principle

- The conservation of mass principle for a control volume can be expressed as:
  - The net mass transfer to or from a control volume during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the control volume during Δt.
  - That is,

$$\begin{pmatrix}\text{Total mass entering} \\ \text{the CV during } \Delta t\end{pmatrix} - \begin{pmatrix}\text{Total mass leaving} \\ \text{the CV during } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Net change in mass} \\ \text{within the CV during } \Delta t\end{pmatrix}$$

$$m_{in} - m_{out} = \Delta m_{CV} \qquad \text{(kg)}$$

$$\dot{m}_{in} - \dot{m}_{out} = dm_{CV}/dt \qquad \text{(kg/s)}$$

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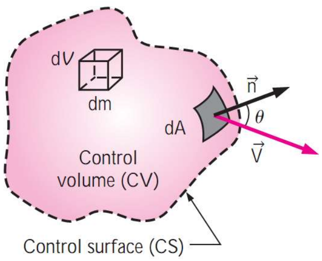
Conservation of Mass Principle

- Consider a control volume of arbitrary shape.
- The mass of a differential volume dV within the control volume is **dm** = ρdV.
- The total mass within the control volume at any instant in time, t, is determined by integration to be



$$m_{CV} = \int_{CV} \rho \, dV$$

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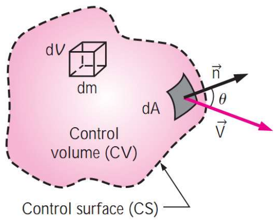
Conservation of Mass Principle

- Rate of change of mass within the CV:

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho \, dV$$

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$



For mass flow into or out of the control volume through a differential area dA on the control surface of a fixed control volume.
Let n be the outward unit vector of dA normal to dA and V be the flow velocity at dA relative to a fixed coordinate system.
In general, the velocity may cross dA at an angle θ off the normal of dA, and the mass flow rate is proportional to the normal component of velocity V !n ! Vcos θ ranging from a maximum outflow of V for u = 0 (flow is normal to dA) to a minimum of zero for u = 90° (flow is tangent to dA) to a maximum inflow of V for u =180° (flow is normal to dA but in the opposite direction).

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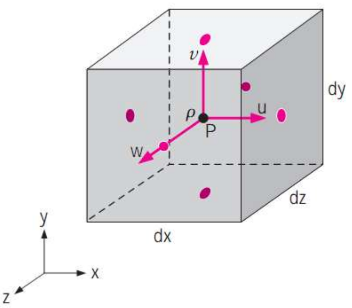
Conservation of Mass: Continuity Equation

- Consider a box of CV with dimensions dx, dy, and dz, and the center of the box at some arbitrary point P from the origin.
- At the center of the box we define the density as ρ and the velocity components as u, v, and w.



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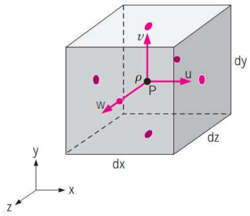
Conservation of Mass: Continuity Equation

- At locations away from the center of the box, we use a Taylor series expansion about the center of the box (point P).
  - For example, the center of the right-most face of the box is located a distance dx/2 from the middle of the box in the x-direction; the value of ρu at that point is



$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots$$

As the control volume shrinks to a point, however, second-order and higher terms become negligible.

For example, suppose $dx/L = 10^{-3}$, then $(dx/L)^2 = 10^{-6}$, a factor of a thousand less than dx/L.

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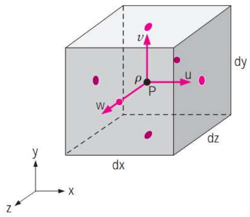
- Applying this truncated Taylor series expansion to the density times the normal velocity component at the center point of each of the six faces of the box, we have



Center of right face: $(\rho u)_{\text{center of right face}} \cong \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$

Center of left face: $(\rho u)_{\text{center of left face}} \cong \rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$

Center of front face: $(\rho w)_{\text{center of front face}} \cong \rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}$

Center of rear face: $(\rho w)_{\text{center of rear face}} \cong \rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}$

Center of top face: $(\rho v)_{\text{center of top face}} \cong \rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}$

Center of bottom face: $(\rho v)_{\text{center of bottom face}} \cong \rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}$

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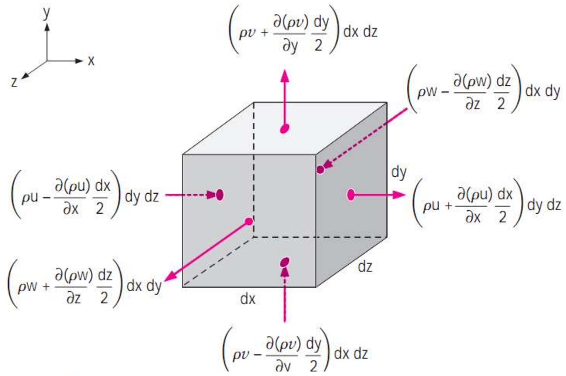

Net mass flow rate into CV:

$$\sum_{in} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{rear face}}$$

Net mass flow rate out of CV:

$$\sum_{out} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{front face}}$$

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Conservation of Mass: Continuity Equation

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

Rate of change of mass within CV:

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV \cong \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

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Conservation of Mass: Continuity Equation

$$\frac{\partial \rho}{\partial t} dx\,dy\,dz = -\frac{\partial(\rho u)}{\partial x} dx\,dy\,dz - \frac{\partial(\rho v)}{\partial y} dx\,dy\,dz - \frac{\partial(\rho w)}{\partial z} dx\,dy\,dz$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

For Steady State
$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Incompressible continuity equation in Cartesian coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

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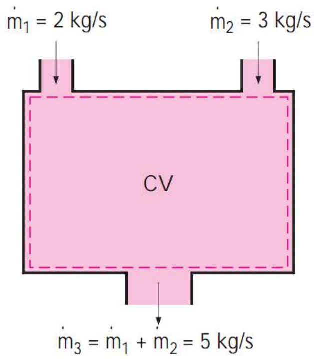
Mass Balance for Steady-Flow Processes

- During a steady-flow process, the total amount of mass contained within a control volume does not change with time (m_{CV} = constant).
- Then the conservation of mass principle requires that the total amount of mass entering a control volume equal the total amount of mass leaving it.
- For a garden hose nozzle in steady operation, for example, the amount of water entering the nozzle per unit time is equal to the amount of water leaving it per unit time.
- When dealing with steady-flow processes, we are not interested in the amount of mass that flows in or out of a device over time; instead, we are interested in the amount of mass flowing per unit time, that is, the mass flow rate m .
- The conservation of mass principle for a general steady-flow system with multiple inlets and outlets can be expressed in rate form as



Steady flow: $$\sum_{in} \dot{m} = \sum_{out} \dot{m} \qquad \text{(kg/s)}$$

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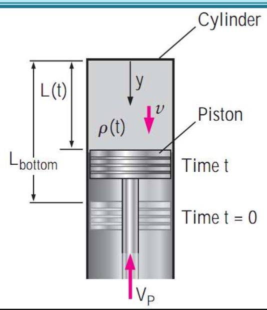
An air-fuel mixture is compressed by a piston in a cylinder of an internal combustion engine. The origin of coordinate y is at the top of the cylinder, and y points straight down as shown. The piston is assumed to move up at constant speed V_P. The distance L between the top of the cylinder and the piston decreases with time according to the linear approximation $L = L_{bottom} - V_P t$, where L_{bottom} is the location of the piston when it is at the bottom of its cycle at time $t = 0$. At $t = 0$, the density of the air-fuel mixture in the cylinder is everywhere equal to $\rho(0)$.

Estimate the density of the air-fuel mixture as a function of time and the given parameters during the piston's up stroke.



SOLUTION The density of the air–fuel mixture is to be estimated as a function of time and the given parameters in the problem statement.

Assumptions **1** Density varies with time, but not space; in other words, the density is uniform throughout the cylinder at any given time, but changes with time: $\rho = \rho(t)$. **2** Velocity component v varies with y and t, but not with x or z; in other words $v = v(y, t)$ only. **3** $u = w = 0$. **4** No mass escapes from the cylinder during the compression.

Analysis First we need to establish an expression for velocity component v as a function of y and t. Clearly $v = 0$ at $y = 0$ (the top of the cylinder), and $v = -V_P$ at $y = L$. For simplicity, we assume that v varies linearly between these two boundary conditions,

Vertical velocity component: $$v = -V_P \frac{y}{L} \qquad \textbf{(1)}$$

where L is a function of time, as given. The compressible continuity equation in Cartesian coordinates (Eq. 9–8) is appropriate for solution of this problem.

$$\frac{\partial \rho}{\partial t} + \underbrace{\cancel{\frac{\partial(\rho u)}{\partial x}}}_{0 \text{ since } u = 0} + \frac{\partial(\rho v)}{\partial y} + \underbrace{\cancel{\frac{\partial(\rho w)}{\partial z}}}_{0 \text{ since } w = 0} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial y} = 0$$

By assumption 1, however, density is not a function of y and can therefore come out of the y-derivative. Substituting Eq. 1 for v and the given expression for L, differentiating, and simplifying, we obtain

$$\frac{\partial \rho}{\partial t} = -\rho\frac{\partial v}{\partial y} = -\rho\frac{\partial}{\partial y}\left(-V_P\frac{y}{L}\right) = \rho\frac{V_P}{L} = \rho\frac{V_P}{L_{bottom} - V_P t} \qquad (2)$$

By assumption 1 again, we replace $\partial\rho/\partial t$ by $d\rho/dt$ in Eq. 2. After separating variables we obtain an expression that can be integrated analytically,

$$\int_{\rho=\rho(0)}^{\rho}\frac{d\rho}{\rho} = \int_{t=0}^{t}\frac{V_P}{L_{bottom} - V_P t}dt \quad \rightarrow \quad \ln\frac{\rho}{\rho(0)} = \ln\frac{L_{bottom}}{L_{bottom} - V_P t} \qquad (3)$$

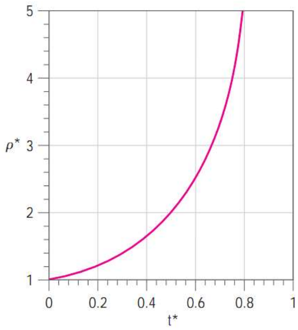
Finally then, we have the desired expression for ρ as a function of time,

$$\rho = \rho(0)\frac{L_{bottom}}{L_{bottom} - V_P t} \qquad (4)$$

In keeping with the convention of nondimensionalizing results, Eq. 4 can be rewritten as

$$\frac{\rho}{\rho(0)} = \frac{1}{1 - V_P t/L_{bottom}} \quad \rightarrow \quad \rho^* = \frac{1}{1 - t^*} \qquad (5)$$

where $\rho^* = \rho/\rho(0)$ and $t^* = V_P t/L_{bottom}$. Equation 5 is plotted in Fig. 9–8.



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THE STREAM FUNCTION

- In Cartesian Coordinates Consider the simple case of incompressible, two-dimensional flow in the xy-plane. The continuity equation in Cartesian coordinates reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

To reduce the two dependent variables (u and v) to one a variable ψ is introduced.
The ψ is define the stream function

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

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Velocity potential function

- The stream function and the corresponding velocity potential function were first introduced by the Italian mathematician Joseph Louis Lagrange (1736–1813).
- By Substitution

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \quad \text{into} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0$$

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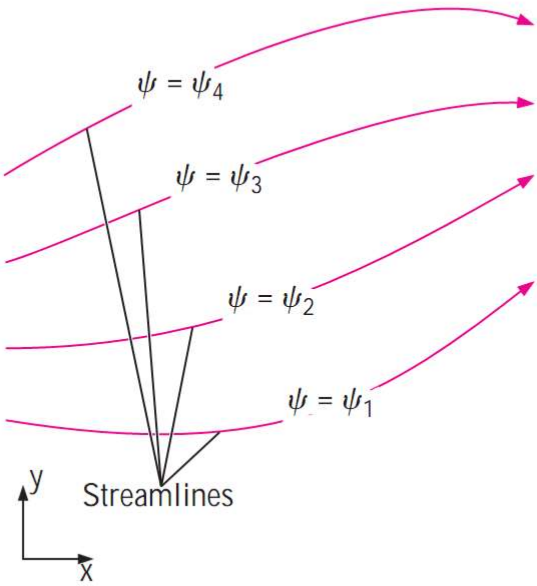

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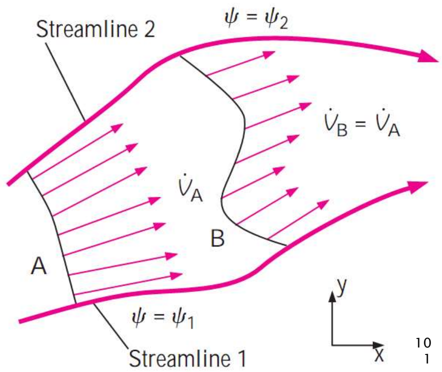
- The difference in the value of ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines.



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A steady, two-dimensional, incompressible flow field in the xy-plane has a stream function given by $\psi = ax^3 + by + cx$, where a, b, and c are constants: $a = 0.50\ (m\cdot s)^{-1}$, $b = -2.0$ m/s, and $c = -1.5$ m/s.
(a) Obtain expressions for velocity components u and v.
(b) Verify that the flow field satisfies the incompressible continuity equation.
(c) Plot several streamlines of the flow in the upper-right quadrant

SOLUTION For a given stream function, we are to calculate the velocity components, verify incompressibility, and plot flow streamlines.
Assumptions **1** The flow is steady. **2** The flow is incompressible (this assumption is to be verified). **3** The flow is two-dimensional in the *xy*-plane, implying that $w = 0$ and neither u nor v depend on z.
Analysis (*a*) We use Eq. 9–20 to obtain expressions for u and v by differentiating the stream function,

$$u = \frac{\partial \psi}{\partial y} = b \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} = -3ax^2 - c$$

(*b*) Since u is not a function of x, and v is not a function of y, we see immediately that the two-dimensional, incompressible continuity equation (Eq. 9–19) is satisfied. In fact, since ψ is smooth in x and y, the two-dimensional,

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incompressible continuity equation in the xy-plane is automatically satisfied by the very definition of ψ. We conclude that **the flow is indeed incompressible.**
(c) To plot streamlines, we solve the given equation for either y as a function of x and ψ, or x as a function of y and ψ. In this case, the former is easier, and we have

Equation for a streamline: $$y = \frac{\psi - ax^3 - cx}{b}$$

This equation is plotted in Fig. 9–19 for several values of ψ, and for the provided values of a, b, and c. The flow is nearly straight down at large values of x, but veers upward for $x < 1$ m.

Discussion You can verify that $v = 0$ at $x = 1$ m. In fact, v is negative for $x > 1$ m and positive for $x < 1$ m. The direction of the flow can also be determined by picking an arbitrary point in the flow, say ($x = 3$ m, $y = 4$ m), and calculating the velocity there. We get $u = -2.0$ m/s and $v = -12.0$ m/s at this point, either of which shows that fluid flows to the lower left in this region of the flow field. For clarity, the velocity vector at this point is also plotted in Fig. 9–19; it is clearly parallel to the streamline near that point. Velocity vectors at three other points are also plotted.