# LECTURE FIVE/SIX
# System of non-linear equations

## Objectives

At the end of this section you will learn how to solve non-linear systems using

1. functional fixed-point iteration,
2. the Gauss-Seidel(Extra Reading) and
3. Newton's methods.

## Introduction

Solutions $x = x_0$ to equations of the form $f(x) = 0$ are often required where it is impossible or infeasible to find an analytical expression for the vector $\boldsymbol{x}$. If the scalar function $f$ depends on $n$ independent variables $x_1, x_2, \ldots, x_n$, then the solution $\boldsymbol{x}_0$ will describe a surface in $n - 1$ dimensional space. Alternatively we may consider the vector function $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$, the solutions of which typically collapse to particular values of $\boldsymbol{x}$. For this course we restrict our attention to $n$ independent variables $x_1, x_2, \ldots, x_n$ and seek solutions to $F(\mathrm{x}) = 0$ where $F$ is vector valued.

## *Fixed Point Method for Functions of Several Variables*

The general of a system of nonlinear equations is

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

where each function $f_i$ maps n-dimensional space, $R^n$, into the real line $R$. The above system can be defined alternatively by defining the function $F(\mathrm{x}) = 0$, where $F : R^n \to R^n$, $\mathrm{x} = (x_1, x_2, \ldots, x_n)$ and

$$F(x_1, x_2, \ldots, x_n) = \left( f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n) \right).$$

### 1.1 Functional or Fixed Point Iteration

Suppose a nonlinear system of the form $F(\mathrm{x}) = 0$ has been transformed into an equivalent fixed point problem $G(\mathrm{x}) = \mathrm{x}$. The functional or fixed point iteration process applied to $G$ is as follows:

1. Select $\mathrm{x}^{(0)} = \left( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \right)$.
2. Generate the sequence of vectors $\mathrm{x}^{(k)} = \left( x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)} \right)$ by

   $\mathrm{x}^{(k)} = \mathrm{G}\left( \mathrm{x}^{(k-1)} \right)$ for each $i = 1, 2, 3, \ldots$ or, component-wise,

$$\begin{aligned}
x_1^{(k)} &= g_1\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right)\\
x_2^{(k)} &= g_2\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right)\\
\vdots \quad & \qquad \vdots\\
x_n^{(k)} &= g_n\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right)
\end{aligned}$$

The following theorem provides conditions for the iterative process to converge.

***Theorem***

Let $D = \{(x_1, x_2, \ldots, x_n) : a_i \le x_i \le b_i, \text{ for each } i = 1, 2, \ldots, n\}$, for some collection of constants $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$. Suppose G is a continuous function with continuous first partial derivatives from $D \subset R^n$ into $R^n$ with the property that $G(\mathrm{x}) \in D$ whenever $\mathrm{x} \in D$. If a constant $K < 1$ exists with

$$\left|\frac{\partial g_i(\mathrm{x})}{\partial x_j}\right| \le \frac{K}{n} \quad \text{whenever} \quad \mathrm{x} \in D$$

for each $j = 1, 2, \ldots, n$ and each component function $g_i$, then the sequence $\left\{\mathrm{x}^{(k)}\right\}_{k=0}^{\infty}$ defined by $\mathrm{x}^{(k)} = \mathrm{G}\left(\mathrm{x}^{(k-1)}\right)$ for each $i = 1, 2, 3, \ldots$ converges to the unique fixed point $\mathrm{p} \in D$, for any $\mathrm{x}^{(0)}$ in $D$, and $\left\|\mathrm{x}^{(j)} - \mathrm{p}\right\|_\infty \le \dfrac{K^j}{1-K}\left\|\mathrm{x}^{(1)} - \mathrm{x}^{(0)}\right\|_\infty$.

***Example 1***

$$\begin{aligned}
3x_1 - \cos(x_2x_3) \qquad\qquad & - \tfrac{1}{2} &= 0\\
x_1^2 - 81(x_2+0.1)^2 + \sin x_3 & + 1.06 &= 0\\
e^{-x_1x_2} \qquad\qquad + 20x_3 & + \tfrac{10\pi-3}{3} &= 0
\end{aligned}$$

$$\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 3x_1 - \cos(x_2x_3) - \tfrac{1}{2}\\
f_2(x_1, x_2, \ldots, x_n) &= x_1^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06\\
f_3(x_1, x_2, \ldots, x_n) &= e^{-x_1x_2} + 20x_3 + \tfrac{10\pi-3}{3}
\end{aligned}$$

$$\begin{aligned}
F(x_1, x_2, \ldots, x_n) &= \left(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)\right)\\
&= \left(3x_1 - \cos(x_2x_3) - \tfrac{1}{2}, x_1^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06, e^{-x_1x_2} + 20x_3 + \tfrac{10\pi-3}{3}\right)
\end{aligned}$$

***Example 2***

$$\begin{aligned} 3x_1 - \cos(x_2x_3) \quad\quad & -\tfrac{1}{2} = 0 \\ x_1^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06 &= 0 \\ e^{-x_1x_2} \quad\quad + 20x_3 & + \tfrac{10\pi-3}{3} = 0 \end{aligned}$$

If the *i*th equation is solved for $x_i$, the system can be changed into the fixed point problem

$$\begin{aligned} x_1 &= \tfrac{1}{3}\cos(x_2x_3) + \tfrac{1}{6}, \\ x_2 &= \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ x_3 &= -\tfrac{1}{20}e^{-x_1x_2} - \tfrac{10\pi-3}{60}. \end{aligned}$$

Let $G: R^3 \to R^3$ be defined by $G(\mathrm{x}) = \left(g_1(\mathrm{x}), g_2(\mathrm{x}), g_3(\mathrm{x})\right)$ where

$$\begin{aligned} g_1(x_1,x_2,x_3) &= \tfrac{1}{3}\cos(x_2x_3) + \tfrac{1}{6}, \\ g_2(x_1,x_2,x_3) &= \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ g_3(x_1,x_2,x_3) &= -\tfrac{1}{20}e^{-x_1x_2} - \tfrac{10\pi-3}{60}. \end{aligned}$$

$$\begin{aligned} \left|g_1(x_1,x_2,x_3)\right| &\le \tfrac{1}{3}\left|\cos(x_2x_3)\right| + \tfrac{1}{6} \le \tfrac{1}{2}, \\ \left|g_2(x_1,x_2,x_3)\right| &= \left|\tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1\right| \\ &\le \tfrac{1}{9}\sqrt{1 + \sin 1 + 1.06} - 0.1 < 0.90, \\ \left|g_3(x_1,x_2,x_3)\right| &= \tfrac{1}{20}e^{-x_1x_2} + \tfrac{10\pi-3}{60} \\ &\le \tfrac{1}{20}e + \tfrac{10\pi-3}{60} < 0.61 \end{aligned}$$

.

so $-1 \le g_i(x_1,x_2,x_3) \le 1$, for each $i = 1,2,3$. Thus, $G(\mathrm{x}) \in D$ whenever $\mathrm{x} \in D$.
Finding bounds on the partial derivatives on $D$ gives the following:

$$\left|\frac{\partial g_1}{\partial x_1}\right| = 0, \quad \left|\frac{\partial g_1}{\partial x_2}\right| \le \tfrac{1}{3}|x_3|\left|\sin(x_2x_3)\right| \le \tfrac{1}{3}\sin 1 = 0.281, \quad \left|\frac{\partial g_1}{\partial x_3}\right| \le \tfrac{1}{3}|x_2|\left|\sin(x_2x_3)\right| \le \tfrac{1}{3}\sin 1 = 0.281,$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = \frac{|x_1|}{9\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{9\sqrt{0.218}} < 0.238, \quad \left|\frac{\partial g_2}{\partial x_2}\right| \le 0,$$

$$\left|\frac{\partial g_2}{\partial x_3}\right| \le \frac{|\cos x_3|}{18\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{18\sqrt{0.218}} < 0.119,$$

$$\left|\frac{\partial g_3}{\partial x_1}\right| = \frac{|x_2|}{20} e^{-x_1 x_2} \le \tfrac{1}{20} e = 0.14, \quad \left|\frac{\partial g_3}{\partial x_2}\right| = \frac{|x_1|}{20} e^{-x_1 x_2}$$

Since the partial derivatives are bounded on *D*, the above Theorem implies that these functions are continuous on *D*. Consequently, *G* is continuous on *D*. Moreover, for every $x \in D$
$\left|\dfrac{\partial g_i(\mathrm{x})}{\partial x_j}\right| \le 0.281$ for each $i = 1, 2, 3$ and $j = 1, 2, 3,$ and the condition in the second part of Theorem 9.7 holds for $K = 0.843$. It can be shown that $\partial g_i(\mathrm{x}) / \partial x_j$ for each $i = 1, 2, 3$ and $j = 1, 2, 3$ is continuous on *D*. Consequently, *G* has a unique fixed point on *D* and the nonlinear system has a solution in *D*.

***Example 3***

$$\begin{aligned} 3x_1 - \cos(x_2 x_3) \quad\quad\quad - \tfrac{1}{2} &= 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0 \\ e^{-x_1 x_2} \quad\quad + 20x_3 + \tfrac{10\pi - 3}{3} &= 0 \end{aligned}$$

***Solution***

If the i<sup>th</sup> equation is solved for $x_i$, the system can be changed into the fixed point problem as

$$\begin{aligned} x_1 &= \tfrac{1}{3}\cos(x_2 x_3) + \tfrac{1}{6}, \\ x_2 &= \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ x_3 &= -\tfrac{1}{20} e^{-x_1 x_2} - \tfrac{10\pi - 3}{60}. \end{aligned}$$

and write the iterative process as

$$\begin{aligned} x_1^{(k)} &= \tfrac{1}{3}\cos\left(x_2^{(k-1)} x_3^{(k-1)}\right) + \tfrac{1}{6}, \\ x_2^{(k)} &= \tfrac{1}{9}\sqrt{\left(x_1^{(k-1)}\right)^2 + \sin x_3^{(k-1)} + 1.06} - 0.1, \\ x_3^{(k)} &= -\tfrac{1}{20} e^{-x_1^{(k-1)} x_2^{(k-1)}} - \tfrac{10\pi - 3}{60}. \end{aligned}$$

$$\left\|\mathrm{x}^{(k)} - \mathrm{x}^{(k-1)}\right\|_\infty < 10^{-5}$$

| $k$ | $x_1^{(k)}$ | $x_1^{(k)}$ | $x_1^{(k)}$ | $\left\|\mathrm{x}^{(k)} - \mathrm{x}^{(k-1)}\right\|_\infty$ |
|---|---|---|---|---|
| 0 | 0.10000000 | 0.10000000 | –0.10000000 | -- |
| 1 | 0.49998333 | 0.00944115 | –0.52310127 | 0.423 |
| 2 | 0.49999593 | 0.00002557 | –0.52336331 | $9.4 \times 10^{-3}$ |
| 3 | 0.50000000 | 0.00001234 | –0.52359814 | $2.3 \times 10^{-4}$ |
| 4 | 0.50000000 | 0.00000003 | –0.52359847 | $1.2 \times 10^{-5}$ |
| 5 | 0.50000000 | 0.00000002 | –0.52359877 | $3.1 \times 10^{-7}$ |

$$\left\| \mathrm{x}^{(5)} - \mathrm{p} \right\|_\infty \le \frac{(0.843)^5}{1-0.843} 0.423 < 1.15$$

$$\mathrm{x}^{(3)} = \left(0.50000000, 1.234\times10^{-5}, -0.52359814\right)$$

$$\left\| \mathrm{x}^{(5)} - \mathrm{p} \right\|_\infty \le \frac{(0.843)^5}{1-0.843} (1.20\times10^{-5}) < 5.5\times10^{-5}$$

$$\mathrm{p} = \left(0.5, 0, -\tfrac{\pi}{6}\right) \approx \left(0.5, 0, -0.5235987757\right)$$

$$\left\| \mathrm{x}^{(5)} - \mathrm{p} \right\|_\infty \le 2\times10^{-8}$$

$$\begin{aligned}
x_1^{(k)} &= \tfrac{1}{3}\cos\left(x_2^{(k-1)} x_3^{(k-1)}\right) + \tfrac{1}{6}, \\
x_2^{(k)} &= \tfrac{1}{9}\sqrt{\left(x_1^{(k)}\right)^2 + \sin x_3^{(k-1)} + 1.06} - 0.1, \\
x_3^{(k)} &= -\tfrac{1}{20} e^{-x_1^{(k)} x_2^{(k)}} - \tfrac{10\pi - 3}{60}.
\end{aligned}$$

| $k$ | $x_1^{(k)}$ | $x_1^{(k)}$ | $x_1^{(k)}$ | $\left\| \mathrm{x}^{(k)} - \mathrm{x}^{(k-1)} \right\|_\infty$ |
|---|---|---|---|---|
| 0 | 0.10000000 | 0.10000000 | –0.10000000 | -- |
| 1 | 0.49998333 | 0.02222979 | –0.52304613 | 0.423 |
| 2 | 0.49997747 | 0.00002815 | –0.52359807 | $2.2\times10^{-2}$ |
| 3 | 0.50000000 | 0.00000004 | –0.52359877 | $2.8\times10^{-5}$ |
| 4 | 0.50000000 | 0.00000000 | –0.52359877 | $3.8\times10^{-8}$ |

## ***Newton's Method***

In order to construct an algorithm that led to an appropriate fixed-point method

$$A(\mathrm{x}) = \begin{pmatrix} a_{11}(\mathrm{x}) & a_{12}(\mathrm{x}) & \cdots & a_{1n}(\mathrm{x}) \\ a_{21}(\mathrm{x}) & a_{22}(\mathrm{x}) & \cdots & a_{2n}(\mathrm{x}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\mathrm{x}) & a_{n2}(\mathrm{x}) & \cdots & a_{nn}(\mathrm{x}) \end{pmatrix}$$

where each of the entries $a_{ij}(\mathrm{x})$ is a function from $R^n \to R$. The procedure requires that $A(\mathrm{x})$ be found so that $G(\mathrm{x}) = \mathrm{x} - A(\mathrm{x})^{-1} F(\mathrm{x})$ gives quadratic convergence to the solution $F(\mathrm{x}) = 0$, provided that $A(\mathrm{x})$ is nonsingular at the fixed point.

**Theorem**

Suppose $\mathrm{p}$ is a solution of $G(\mathrm{x}) = \mathrm{x}$ for some function $G = (g_1, g_2, \ldots, g_n)$, mapping $R^n$ into $R^n$. If a number $\delta > 0$ exists with the property that

i) $\partial g_i / \partial x_j$ is continuous on $N_\delta = \{\mathrm{x} : \|\mathrm{x} - \mathrm{p}\| < \delta\}$ for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$,

ii) $\partial^2 g_i(\mathrm{x}) / \partial x_j\, \partial x_k$ is continuous, and $\left\|\partial^2 g_i(\mathrm{x}) / \partial x_j\, \partial x_k\right\| \le M$ for some constant $M$ whenever $\mathrm{x} \in N_\delta$ for each $i = 1, 2, \ldots, n,$ $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$,

iii) $\partial g_i(\mathrm{p}) / \partial x_j = 0$ for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$,

then the sequence generated by $\mathrm{x}^{(k)} = G\left(\mathrm{x}^{(k-1)}\right)$ converges quadratically to $\mathrm{p}$ for any choice of $\mathrm{x}^{(0)} \in N_\delta$ and $\left\|\mathrm{x}^{(k)} - \mathrm{p}\right\|_\infty \le \dfrac{n^2 M}{2} \left\|\mathrm{x}^{(k-1)} - \mathrm{p}\right\|_\infty^2$ for each $k \ge 1$.

Since $G(\mathrm{x}) = \mathrm{x} - A(\mathrm{x})^{-1} F(\mathrm{x})$, $g_i(\mathrm{x}) = x_i - \sum_{j=1}^{n} b_{ij}(\mathrm{x}) f_j(\mathrm{x})$;

so
$$\frac{\partial g_i(\mathrm{x})}{\partial x_k} = \begin{cases} 1 - \sum_{j=1}^{n} \left( b_{ij}(\mathrm{x}) \dfrac{\partial f_j}{\partial x_k}(\mathrm{x}) + \dfrac{\partial b_{ij}}{\partial x_k}(\mathrm{x}) f_j(\mathrm{x}) \right), & \text{if } i = k, \\ -\sum_{j=1}^{n} \left( b_{ij}(\mathrm{x}) \dfrac{\partial f_j}{\partial x_k}(\mathrm{x}) + \dfrac{\partial b_{ij}}{\partial x_k}(\mathrm{x}) f_j(\mathrm{x}) \right), & \text{if } i \ne k. \end{cases}$$

$$0 = 1 - \sum_{j=1}^{n} b_{ij}(\mathrm{p}) \frac{\partial f_j}{\partial x_k}(\mathrm{p})$$

so $\displaystyle\sum_{j=1}^{n} b_{ij}(\mathrm{p}) \frac{\partial f_j}{\partial x_k}(\mathrm{p}) = 1$

$$0 = -\sum_{j=1}^{n} b_{ij}(\mathrm{p}) \frac{\partial f_j}{\partial x_k}(\mathrm{p})$$

So $\displaystyle\sum_{j=1}^{n} b_{ij}(\mathrm{p}) \frac{\partial f_j}{\partial x_k}(\mathrm{p}) = 0$

$$J(\mathrm{x}) = \begin{pmatrix} \frac{\partial f_1(\mathrm{x})}{\partial x_1} & \frac{\partial f_1(\mathrm{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathrm{x})}{\partial x_n} \\ \frac{\partial f_2(\mathrm{x})}{\partial x_1} & \frac{\partial f_2(\mathrm{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathrm{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathrm{x})}{\partial x_1} & \frac{\partial f_n(\mathrm{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathrm{x})}{\partial x_n} \end{pmatrix}$$

$$A(\mathrm{p})^{-1} J(\mathrm{p}) = I$$

so $J(\mathrm{p}) = A(\mathrm{p})$

$$G(\mathrm{x}) = \mathrm{x} - J(\mathrm{x})^{-1} F(\mathrm{x})$$

$$\mathrm{x}^{(k)} = G\left(\mathrm{x}^{(k\text{-}1)}\right) = \mathrm{x}^{(k\text{-}1)} - J\left(\mathrm{x}^{(k\text{-}1)}\right)^{-1} F\left(\mathrm{x}^{(k\text{-}1)}\right)$$

***Example 1***

Solve the nonlinear system

$$\begin{aligned} 3x_1 - \cos(x_2 x_3) \qquad\qquad - \tfrac{1}{2} &= 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0 \\ e^{-x_1 x_2} \qquad\qquad + 20x_3 + \tfrac{10\pi - 3}{3} &= 0 \end{aligned}$$

***Solution***

The Jacobian matrix for the system is given by

$$J(x_1, x_2, x_3) = \begin{pmatrix} 3 & x_3 \sin x_2 x_3 & x_2 \sin x_2 x_3 \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix} = \begin{pmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{pmatrix} + \begin{pmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{pmatrix}$$

where

$$\begin{pmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{pmatrix} = -\left[ J\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)\right]^{-1} F\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right).$$

Thus, at the $k^{\text{th}}$ step, the linear system

$$\begin{pmatrix} 3 & x_3^{(k-1)}\sin x_2^{(k-1)}x_3^{(k-1)} & x_2\sin x_2^{(k-1)}x_3^{(k-1)} \\ 2x_1^{(k-1)} & -162(x_2^{(k-1)}+0.1) & \cos x_3^{(k-1)} \\ -x_2^{(k-1)}e^{-x_1^{(k-1)}x_2^{(k-1)}} & -x_1^{(k-1)}e^{-x_1^{(k-1)}x_2^{(k-1)}} & 20 \end{pmatrix}\begin{pmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{pmatrix}$$
$$= \begin{pmatrix} 3x_1^{(k-1)} - \cos(x_2^{(k-1)}x_3^{(k-1)}) - \frac{1}{2} \\ \left(x_1^{(k-1)}\right)^2 - 81(x_2^{(k-1)}+0.1)^2 + \sin x_3^{(k-1)} + 1.06 \\ e^{-x_1^{(k-1)}x_2^{(k-1)}} + 20x_3^{(k-1)} + \frac{10\pi-3}{3} \end{pmatrix}$$

must be solved. The results obtained using the above iterative procedure is as shown below

| $k$ | $x_1^{(k)}$ | $x_1^{(k)}$ | $x_1^{(k)}$ | $\left\Vert \mathrm{x}^{(k)} - \mathrm{x}^{(k-1)} \right\Vert_\infty$ |
|---|---|---|---|---|
| 0 | 0.10000000 | 0.10000000 | –0.10000000 | -- |
| 1 | 0.50003702 | 0.01946686 | –0.52152047 | 0.422 |
| 2 | 0.50004593 | 0.00158859 | –0.52355711 | $1.79\times10^{-2}$ |
| 3 | 0.50000034 | 0.00001244 | –0.52359845 | $1.58\times10^{-3}$ |
| 4 | 0.50000000 | 0.00000000 | –0.52359877 | $1.24\times10^{-5}$ |
| 5 | 0.50000000 | 0.00000000 | –0.52359877 | 0 |

$$\mathrm{p} = \left(0.5, 0, -\tfrac{\pi}{6}\right) \approx \left(0.5, 0, -0.5235987757\right)$$

***Examples***

1. The nonlinear system

$$\underline{\mathrm{F}}(x,y) = \begin{bmatrix} x^2 - 10x + y^2 + 8 \\ xy^2 + x - 10y + 8 \end{bmatrix} = 0$$

can be transformed into the fixed point problem

$$\mathrm{G}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x^2+y^2+8}{10} \\ \frac{xy^2+x+8}{10} \end{bmatrix}$$

(a) Starting with the initial estimates $x_0 = y_0 = 0$, apply functional iteration to G to approximate the solution to an accuracy of $10^{-5}$.

(b) Does Gauss-Seidel Method accelerate convergence?

***Solution***

(a)

| i | x | y | g1(x,y) | g2(x,y) | Tol |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.8 | 0.8 | |
| 1 | 0.8 | 0.8 | 0.928 | 0.9312 | 0.8 |
| 2 | 0.928 | 0.9312 | 0.972832 | 0.97327 | 0.1312 |
| 3 | 0.972832 | 0.97327 | 0.989366 | 0.989435 | 0.044832 |
| 4 | 0.989366 | 0.989435 | 0.995783 | 0.995794 | 0.016534 |
| 5 | 0.995783 | 0.995794 | 0.998319 | 0.998321 | 0.006417 |
| 6 | 0.998319 | 0.998321 | 0.999328 | 0.999329 | 0.002536 |
| 7 | 0.999328 | 0.999329 | 0.999732 | 0.999732 | 0.00101 |
| 8 | 0.999732 | 0.999732 | 0.999893 | 0.999893 | 0.000403 |
| 9 | 0.999893 | 0.999893 | 0.999957 | 0.999957 | 0.000161 |
| 10 | 0.999957 | 0.999957 | 0.999983 | 0.999983 | 6.44E-05 |
| 11 | 0.999983 | 0.999983 | 0.999993 | 0.999993 | 2.58E-05 |
| 12 | 0.999993 | 0.999993 | 0.999997 | 0.999997 | 1.03E-05 |
| 13 | 0.999997 | 0.999997 | 0.999999 | 0.999999 | 4.12E-06 |
| 14 | 0.999999 | 0.999999 | 1 | 1 | 1.65E-06 |

(b)

| i | x | y | g1(x,y) | g2(x,y) | Tol |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.8 | 0.88 | |
| 1 | 0.8 | 0.88 | 0.94144 | 0.967049 | 0.88 |
| 2 | 0.94144 | 0.967049 | 0.982149 | 0.990064 | 0.14144 |
| 3 | 0.982149 | 0.990064 | 0.994484 | 0.99693 | 0.040709 |
| 4 | 0.994484 | 0.99693 | 0.998287 | 0.999045 | 0.012335 |
| 5 | 0.998287 | 0.999045 | 0.999467 | 0.999703 | 0.003803 |
| 6 | 0.999467 | 0.999703 | 0.999834 | 0.999907 | 0.00118 |
| 7 | 0.999834 | 0.999907 | 0.999948 | 0.999971 | 0.000367 |
| 8 | 0.999948 | 0.999971 | 0.999984 | 0.999991 | 0.000114 |
| 9 | 0.999984 | 0.999991 | 0.999995 | 0.999997 | 3.56E-05 |
| 10 | 0.999995 | 0.999997 | 0.999998 | 0.999999 | 1.11E-05 |
| 11 | 0.999998 | 0.999999 | 1 | 1 | 3.46E-06 |

(c)

From (a) and (b) it is seen that Gauss-Seidel Method accelerate convergence.

2. Convert the nonlinear system

$$3x - \cos(yz) - \tfrac{1}{2} = 0$$
$$x^2 - 81\left(y + \tfrac{1}{10}\right)^2 + \sin z + 1.06 = 0$$
$$e^{-xy} + 20z + \tfrac{10\pi - 3}{3} = 0$$

to a fixed point problem and use both functional iteration and the Gauss-Seidel variant of functional iteration to approximate the root to within $10^{-5}$ in the $l_\infty$ norm, starting the initial estimate $x_0 = y_0 = 0.1$ and $z_0 = -0.1$. [Note the exact root is $\left(\tfrac{1}{2}, 0, -\tfrac{\pi}{6}\right)^T$ ]

***Solution***

$$x = \frac{\cos(yz) + \frac{1}{2}}{3},\ \ y = \sqrt{\frac{x^2 + \sin z + 1.06}{81}} - \frac{1}{10},\ \ z = \frac{-e^{-xy} - \frac{10\pi - 3}{3}}{20}$$

For $k = 1, 2, \ldots$, the functional iteration and the Gauss-Seidel variant of functional iteration are given as

$$x^{(k)} = \frac{\cos(y^{(k-1)} z^{(k-1)}) + \frac{1}{2}}{3}$$

$$y^{(k)} = \sqrt{\frac{(x^{(k-1)})^2 + \sin z^{(k-1)} + 1.06}{81}} - \frac{1}{10} \quad \text{and}$$

$$z^{(k)} = \frac{-e^{-x^{(k-1)} y^{(k-1)}} - \frac{10\pi - 3}{3}}{20}$$

$$x^{(k)} = \frac{\cos(y^{(k-1)} z^{(k-1)}) + \frac{1}{2}}{3}$$

$$y^{(k)} = \sqrt{\frac{(x^{(k)})^2 + \sin z^{(k-1)} + 1.06}{81}} - \frac{1}{10} \quad \text{respectively for the above fixed point}$$

$$z^{(k)} = \frac{-e^{-x^{(k)} y^{(k)}} - \frac{10\pi - 3}{3}}{20}$$

problem.
Using the functional iteration we have the following table:

| i | x | y | z | g1(x,y,z) | g2(x,y,z) | g3(x,y,z) | Tol |
|---|---|---|---|---|---|---|---|
| 0 | 0.1 | 0.1 | -0.1 | 0.499983 | 0.009441 | -0.5231013 | |
| 1 | 0.499983 | 0.009441 | -0.523101 | 0.499996 | 2.56E-05 | -0.5233633 | 0.399983 |
| 2 | 0.499996 | 2.56E-05 | -0.523363 | 0.5 | 1.23E-05 | -0.5235981 | 0.009154 |
| 3 | 0.5 | 1.23E-05 | -0.523598 | 0.5 | 3.42E-08 | -0.5235985 | 4.07E-06 |
| 4 | 0.5 | 3.42E-08 | -0.523598 | 0.5 | 1.65E-08 | -0.5235988 | 1.2E-05 |
| 5 | 0.5 | 1.65E-08 | -0.523599 | 0.5 | 4.57E-11 | -0.5235988 | 6.95E-12 |
| 6 | 0.5 | 4.57E-11 | -0.523599 | 0.5 | 2.2E-11 | -0.5235988 | 1.6E-08 |
| 7 | 0.5 | 2.2E-11 | -0.523599 | 0.5 | 6.1E-14 | -0.5235988 | 0 |
| 8 | 0.5 | 6.1E-14 | -0.523599 | 0.5 | 2.94E-14 | -0.5235988 | 2.14E-11 |
| 9 | 0.5 | 2.94E-14 | -0.523599 | 0.5 | 0 | -0.5235988 | 0 |
| 10 | 0.5 | 0 | -0.523599 | 0.5 | 0 | -0.5235988 | 2.87E-14 |

Using the Gauss-Seidel variant of functional iteration we have the following table:

| i | x | y | z | g1(x,y,z) | g2(x,y,z) | g3(x,y,z) | Tol |
|---|---|---|---|---|---|---|---|
| 0 | 0.1 | 0.1 | -0.1 | 0.499983 | 0.02223 | -0.5230461 | |
| 1 | 0.499983 | 0.02223 | -0.523046 | 0.499977 | 2.82E-05 | -0.5235981 | 0.399983 |
| 2 | 0.499977 | 2.82E-05 | -0.523598 | 0.5 | 3.76E-08 | -0.5235988 | 0.02165 |
| 3 | 0.5 | 3.76E-08 | -0.523599 | 0.5 | 5.03E-11 | -0.5235988 | 2.74E-05 |
| 4 | 0.5 | 5.03E-11 | -0.523599 | 0.5 | 6.72E-14 | -0.5235988 | 3.66E-08 |
| 5 | 0.5 | 6.72E-14 | -0.523599 | 0.5 | 0 | -0.5235988 | 4.9E-11 |
| 6 | 0.5 | 0 | -0.523599 | 0.5 | 0 | -0.5235988 | 6.55E-14 |

3. Starting with the initial guess $x_0 = y_0 = 1.0$, use fixed point (functional iteration to approximate the solution to the system

$$2x^2 + y^2 = 4.32, \quad x^2 - y^2 = 0$$

by performing 5 iterations.

4. Consider the nonlinear system
$$2x + xy - 1 = 0$$
$$2y - xy + 1 = 0$$
which has a unique root $x = (1, -1)^T$. Starting with the initial estimate $x_0 = y_0 = 0$, compare the methods of functional iteration, Gauss-Seidel Newton when approximating the root of this system (perform 5 iterations in each case).
5. Use Newton's Method to approximate the solution of the nonlinear system
$$x^2 - 2x - y + \tfrac{1}{2} = 0$$
$$x^2 + 4y^2 - 4 = 0$$
starting with the initial estimate $(x_0, y_0) = (2, \tfrac{1}{4})$ and computing 3 iterations.

Solution

Using the Newton's iterative method
$x^{(k)} = x^{(k-1)} - J^{-1}(x^{(k-1)})F(x^{(k-1)}), \quad \text{for } k = 1, \ldots$ with $x^{(0)} = (x_0, y_0) = (2, \tfrac{1}{4})$, we have
$$x^{(1)} = x^{(0)} - J^{-1}(x^{(0)})F(x^{(0)}) = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8} \begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8} \begin{pmatrix} 0.75 \\ -0.50 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix}$$
$$x^{(2)} = x^{(1)} - J^{-1}(x^{(1)})F(x^{(1)}) = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \begin{pmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{pmatrix}^{-1} \begin{pmatrix} 0.00879 \\ -1.70312 \end{pmatrix}$$
$$= \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \frac{1}{8.34375} \begin{pmatrix} 2.5 & 1 \\ -3.8125 & 1.8125 \end{pmatrix} \begin{pmatrix} 0.00879 \\ -1.70312 \end{pmatrix} = \begin{pmatrix} 2.10773 \\ 0.68232 \end{pmatrix}$$
6. Use Newton's Method to approximate the two solutions of the nonlinear system
$$ye^x = 2, \quad x^2 + y^2 = 4$$
by computing 2 iterations for each of the given initial estimates
(a) $(x_0, y_0) = (-0.6, +3.7)$
(b) $(x_0, y_0) = (+1.9, +0.4)$
7. Use Newton's Method to approximate the solution of the nonlinear system
$$x^2 + y^2 + 0.6y - 0.16 = 0$$
$$x^2 - y^2 + x - 1.6y = 0$$
by computing 3 iterations with the initial estimate of $(x_0, y_0) = (0.6, 0.25)$.
Using the more accurate initial estimate of $(x_0, y_0) = (0.3, 0.1)$, repeat the process using the modified Newton's method whereby the Jacobian is evaluated and held constant for subsequent iterations. Compare the two results.
8. Use the modified Newton's method (i.e., by evaluating the Jacobian and keeping at a constant value throughout) to find the root of the system
$e^x + y = 0, \ \cosh(y) - x = \tfrac{7}{2}$ starting with the initial estimate $x = -2.4, \ y = -0.1$ computing two iterations.