SUPPLEMENTARY MATERIAL

Gauss Quadrature Rule of Integration

After reading this lecture notes, you should be able to:

1. *derive the Gauss quadrature method for integration and be able to use it to solve problems, and*
2. *use Gauss quadrature method to solve examples of approximate integrals.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.
Here, we will discuss the Gauss quadrature rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,
$a =$ lower limit of integration
$b =$ upper limit of integration

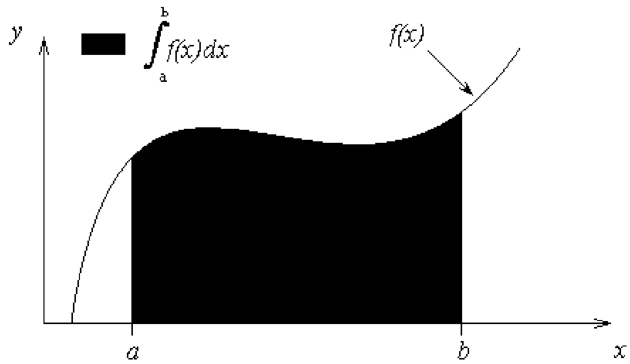

Figure 1 Integration of a function.

Gauss Quadrature Rule

Background:
To derive the trapezoidal rule from the method of undetermined coefficients, we approximated

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) \tag{1}$$

Let the right hand side be exact for integrals of a straight line, that is, for an integrated form of

$$\int_a^b (a_0 + a_1 x)dx$$

So

$$\int_a^b (a_0 + a_1 x)dx = \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \tag{2}$$

But from Equation (1), we want

$$\int_a^b (a_0 + a_1 x)dx = c_1 f(a) + c_2 f(b) \tag{3}$$

to give the same result as Equation (2) for $f(x) = a_0 + a_1 x$.

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0(c_1 + c_2) + a_1(c_1 a + c_2 b) \tag{4}$$

Hence from Equations (2) and (4),

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = a_0(c_1 + c_2) + a_1(c_1 a + c_2 b)$$

Since a_0 and a_1 are arbitrary constants for a general straight line

$$c_1 + c_2 = b - a \tag{5a}$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \tag{5b}$$

Multiplying Equation (5a) by a and subtracting from Equation (5b) gives

$$c_2 = \frac{b-a}{2} \tag{6a}$$

Substituting the above found value of c_2 in Equation (5a) gives

$$c_1 = \frac{b-a}{2} \tag{6b}$$

Therefore

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \tag{7}$$

Derivation of two-point Gauss quadrature rule

<u>Method 1:</u>

The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as a and b, but as unknowns x_1 and x_2. So in the two-point Gauss quadrature rule, the integral is approximated as

$$I = \int_a^b f(x)dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2)$$

There are four unknowns x_1, x_2, c_1 and c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right) \tag{8}$$

The formula would then give

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) =$$

$$c_1\left(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3\right) + c_2\left(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3\right) \tag{9}$$

Equating Equations (8) and (9) gives

$$
\begin{aligned}
&a_0(b-a)+a_1\left(\frac{b^2-a^2}{2}\right)+a_2\left(\frac{b^3-a^3}{3}\right)+a_3\left(\frac{b^4-a^4}{4}\right)\\
&=c_1\left(a_0+a_1x_1+a_2{x_1}^2+a_3{x_1}^3\right)+c_2\left(a_0+a_1x_2+a_2{x_2}^2+a_3{x_2}^3\right)\\
&=a_0(c_1+c_2)+a_1(c_1x_1+c_2x_2)+a_2\left(c_1{x_1}^2+c_2{x_2}^2\right)+a_3\left(c_1{x_1}^3+c_2{x_2}^3\right) \qquad (10)
\end{aligned}
$$

Since in Equation (10), the constants a_0, a_1, a_2, and a_3 are arbitrary, the coefficients of a_0, a_1, a_2, and a_3 are equal. This gives us four equations as follows.

$$
\begin{aligned}
&b-a=c_1+c_2\\
&\frac{b^2-a^2}{2}=c_1x_1+c_2x_2\\
&\frac{b^3-a^3}{3}=c_1{x_1}^2+c_2{x_2}^2\\
&\frac{b^4-a^4}{4}=c_1{x_1}^3+c_2{x_2}^3 \qquad (11)
\end{aligned}
$$

Without proof (see Example 1 for proof of a related problem), we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$
\begin{aligned}
&c_1=\frac{b-a}{2}\\
&c_2=\frac{b-a}{2}\\
&x_1=\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\\
&x_2=\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2} \qquad (12)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_a^b f(x)dx &\approx c_1 f(x_1)+c_2 f(x_2)\\
&=\frac{b-a}{2}\, f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)+\frac{b-a}{2}\, f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right) \qquad (13)
\end{aligned}
$$

<u>Method 2:</u>

We can derive the same formula by assuming that the expression gives exact values for the individual integrals of $\int_a^b 1dx$, $\int_a^b xdx$, $\int_a^b x^2dx$, and $\int_a^b x^3dx$. The reason the formula can also be derived using this method is that the linear combination of the above integrands is a general third order polynomial given by $f(x)=a_0+a_1x+a_2x^2+a_3x^3$.

These will give four equations as follows

$$\int_a^b 1dx = b - a = c_1 + c_2$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1x_1 + c_2x_2$$

$$\int_a^b x^2dx = \frac{b^3 - a^3}{3} = c_1{x_1}^2 + c_2{x_2}^2$$

$$\int_a^b x^3dx = \frac{b^4 - a^4}{4} = c_1{x_1}^3 + c_2{x_2}^3 \quad (14)$$

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \quad (15)$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) \quad (16)$$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

Higher point Gauss quadrature formulas

For example

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) \quad (17)$$

is called the three-point Gauss quadrature rule. The coefficients c_1, c_2 and c_3, and the function arguments x_1, x_2 and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5\right)dx .$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n) \quad (18)$$

Arguments and weighing factors for *n*-point Gauss quadrature rules

In handbooks (see Table 1), coefficients and arguments given for n-point Gauss quadrature rule are given for integrals of the form

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} c_i g(x_i) \tag{19}$$

Table 1 Weighting factors c and function arguments x used in Gauss quadrature formulas

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$
5	$c_1 = 0.236926885$	$x_1 = -0.906179846$
	$c_2 = 0.478628670$	$x_2 = -0.538469310$
	$c_3 = 0.568888889$	$x_3 = 0.000000000$
	$c_4 = 0.478628670$	$x_4 = 0.538469310$
	$c_5 = 0.236926885$	$x_5 = 0.906179846$
6	$c_1 = 0.171324492$	$x_1 = -0.932469514$
	$c_2 = 0.360761573$	$x_2 = -0.661209386$
	$c_3 = 0.467913935$	$x_3 = -0.238619186$
	$c_4 = 0.467913935$	$x_4 = 0.238619186$
	$c_5 = 0.360761573$	$x_5 = 0.661209386$
	$c_6 = 0.171324492$	$x_6 = 0.932469514$

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$?

The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$. Let

$$x = mt + c \tag{20}$$

If $x = a$, then $t = -1$

If $x = b$, then $t = +1$

such that

$$a = m(-1) + c$$

$$b = m(1) + c \tag{21}$$

Solving the two Equations (21) simultaneously gives

$$m = \frac{b-a}{2}$$

$$c = \frac{b+a}{2} \tag{22}$$

Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2}dt$$

Substituting our values of x and dx into the integral gives us

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx \tag{23}$$

Example 1

For an integral $\int_{-1}^{1} f(x)dx,$ show that the two-point Gauss quadrature rule approximates to

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

where

$$c_1 = 1$$

$$c_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Solution

Assuming the formula

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) \quad \text{(E1.1)}$$

gives exact values for integrals $\int_{-1}^{1} 1dx,\ \int_{-1}^{1} xdx,\ \int_{-1}^{1} x^2dx,$ and $\int_{-1}^{1} x^3dx$. Then

$$\int_{-1}^{1} 1dx = 2 = c_1 + c_2 \quad \text{(E1.2)}$$

$$\int_{-1}^{1} xdx = 0 = c_1x_1 + c_2x_2 \quad \text{(E1.3)}$$

$$\int_{-1}^{1} x^2dx = \frac{2}{3} = c_1x_1^{\ 2} + c_2x_2^{\ 2} \quad \text{(E1.4)}$$

$$\int_{-1}^{1} x^3dx = 0 = c_1x_1^{\ 3} + c_2x_2^{\ 3} \quad \text{(E1.5)}$$

Multiplying Equation (E1.3) by $x_1^{\ 2}$ and subtracting from Equation (E1.5) gives

$$c_2x_2\left(x_1^{\ 2} - x_2^{\ 2}\right) = 0 \quad \text{(E1.6)}$$

The solution to the above equation is

$c_2 = 0,$ or/and

$x_2 = 0,$ or/and

$x_1 = x_2,$ or/and

$x_1 = -x_2.$

I. $c_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 = 2,\ \ c_1x_1 = 0,$ $c_1x_1^2 = \frac{2}{3},$ and $c_1x_1^3 = 0$. But since $c_1 = 2$, then $x_1 = 0$ from $c_1x_1 = 0$, but $x_1 = 0$ conflicts with $c_1x_1^2 = \frac{2}{3}$.

II. $x_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1x_1 = 0,$ $c_1x_1^2 = \frac{2}{3},$ and $c_1x_1^3 = 0$. Since $c_1x_1 = 0$, then c_1 or x_1 has to be zero but this violates $c_1x_1^2 = \frac{2}{3} \neq 0$.

III. $x_1 = x_2$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1x_1 + c_2x_1 = 0,\ c_1x_1^2 + c_2x_1^{\ 2} = \frac{2}{3},$ and $c_1x_1^3 + c_2x_1^3 = 0$. If $x_1 \neq 0$, then $c_1x_1 + c_2x_1 = 0$ gives $c_1 + c_2 = 0$ and that violates $c_1 + c_2 = 2$. If $x_1 = 0$, then that violates $c_1x_1^2 + c_2x_1^{\ 2} = \frac{2}{3} \neq 0$.

That leaves the solution of $x_1 = -x_2$ as the only possible acceptable solution and in fact, it does not have violations (see it for yourself)

$$x_1 = -x_2 \quad \text{(E1.7)}$$

Substituting (E1.7) in Equation (E1.3) gives

$$c_1 = c_2 \tag{E1.8}$$

From Equations (E1.2) and (E1.8),

$$c_1 = c_2 = 1 \tag{E1.9}$$

Equations (E1.4) and (E1.9) gives

$$x_1^{\ 2} + x_2^{\ 2} = \frac{2}{3} \tag{E1.10}$$

Since Equation (E1.7) requires that the two results be of opposite sign, we get

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Hence

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) \tag{E1.11}$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Example 2

For an integral $\int_a^b f(x)dx,$ derive the one-point Gauss quadrature rule.

Solution

The one-point Gauss quadrature rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1) \tag{E2.1}$$

Assuming the formula gives exact values for integrals $\int_{-1}^{1} 1dx,$ and $\int_{-1}^{1} xdx$

$$\int_a^b 1dx = b - a = c_1$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1 \tag{E2.2}$$

Since $c_1 = b - a,$ the other equation becomes

$$(b-a)x_1 = \frac{b^2 - a^2}{2}$$

$$x_1 = \frac{b+a}{2} \tag{E2.3}$$

Therefore, one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{b+a}{2}\right) \qquad \text{(E2.4)}$$

Example 3

What would be the formula for

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b)$$

if you want the above formula to give you exact values of $\int_a^b \left(a_0 x + b_0 x^2\right)dx,$ that is, a linear combination of x and x^2.

Solution

If the formula is exact for a linear combination of x and x^2, then

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 a + c_2 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 b^2 \qquad \text{(E3.1)}$$

Solving the two Equations (E3.1) simultaneously gives

$$\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{b^2 - a^2}{2} \\ \dfrac{b^3 - a^3}{3} \end{bmatrix}$$

$$c_1 = -\frac{1}{6} \frac{-ab - b^2 + 2a^2}{a}$$

$$c_2 = -\frac{1}{6} \frac{a^2 + ab - 2b^2}{b} \qquad \text{(E3.2)}$$

So

$$\int_a^b f(x)dx = -\frac{1}{6} \frac{-ab - b^2 + 2a^2}{a} f(a) - \frac{1}{6} \frac{a^2 + ab - 2b^2}{b} f(b) \qquad \text{(E3.3)}$$

Let us see if the formula works.

Evaluate $\int_2^5 \left(2x^2 - 3x\right)dx$ using Equation(E3.3)

$$\int_2^5 \left(2x^2 - 3x\right)dx \approx c_1 f(a) + c_2 f(b)$$

$$= -\frac{1}{6} \frac{-(2)(5) - 5^2 + 2(2)^2}{2} \left[2(2)^2 - 3(2)\right] - \frac{1}{6} \frac{2^2 + 2(5) - 2(5)^2}{5} [2(5)^2 - 3(5)]$$

$$= 46.5$$

The exact value of $\int_2^5 (2x^2 - 3x)dx$ is given by

$$\int_2^5 (2x^2 - 3x)dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2}\right]_2^5$$
$$= 46.5$$

Any surprises?

Now evaluate $\int_2^5 3dx$ using Equation (E3.3)

$$\int_2^5 3dx \approx c_1 f(a) + c_2 f(b)$$
$$= -\frac{1}{6}\frac{-2(5) - 5^2 + 2(2)^2}{2}(3) - \frac{1}{6}\frac{2^2 + 2(5) - 2(5)^2}{5}(3)$$
$$= 10.35$$

The exact value of $\int_2^5 3dx$ is given by

$$\int_2^5 3dx = [3x]_2^5$$
$$= 9$$

Because the formula will only give exact values for linear combinations of x and x^2, it does not work exactly even for a simple integral of $\int_2^5 3dx$.

Do you see now why we choose $a_0 + a_1 x$ as the integrand for which the formula

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

gives us exact values?

Example 4

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8,\ 30]$ to $[-1,\ 1]$ using Equation(23) gives

$$\int_8^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f\left(11x + 19\right)dx$$

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$$c_1 = \ \ 1.000000000.$$
$$x_1 = -0.577350269$$
$$c_2 = \ \ 1.000000000$$
$$x_2 = \ \ 0.577350269$$

Now we can use the Gauss quadrature formula

$$11\int_{-1}^{1} f\left(11x + 19\right)dx \approx 11\left[c_1 f\left(11x_1 + 19\right) + c_2 f\left(11x_2 + 19\right)\right]$$
$$= 11\left[f\left(11(-0.5773503) + 19\right) + f\left(11(0.5773503) + 19\right)\right]$$
$$= 11\left[f(12.64915) + f(25.35085)\right]$$
$$= 11\left[(296.8317) + (708.4811)\right]$$
$$= 11058.44 \text{ m}$$

since

$$f(12.64915) = 2000\ln\left[\frac{140000}{140000 - 2100(12.64915)}\right] - 9.8(12.64915)$$
$$= 296.8317$$
$$f(25.35085) = 2000\ln\left[\frac{140000}{140000 - 2100(25.35085)}\right] - 9.8(25.35085)$$
$$= 708.4811$$

The absolute relative true error, $\left|\in_t\right|$, is (True value = 11061.34 m)

$$\left|\in_t\right| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100$$
$$= 0.0262\%$$

Example 5

Use three-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation (23) gives

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f\left(11x + 19\right)dx$$

The weighting factors and function argument values are

$$c_1 = \quad 0.555555556$$
$$x_1 = -0.774596669$$
$$c_2 = \quad 0.888888889$$
$$x_2 = \quad 0.000000000$$
$$c_3 = \quad 0.555555556$$
$$x_3 = \quad 0.774596669$$

and the formula is

$$11\int_{-1}^{1} f\left(11x + 19\right)dx \approx 11\left[c_1 f\left(11x_1 + 19\right) + c_2 f\left(11x_2 + 19\right) + c_3 f\left(11x_3 + 19\right)\right]$$

$$= 11\left[\begin{array}{l} 0.5555556 f\left(11(-.7745967) + 19\right) + 0.8888889 f\left(11(0.0000000) + 19\right) \\ + 0.5555556 f\left(11(0.7745967) + 19\right) \end{array}\right]$$

$$= 11\left[0.55556 f(10.47944) + 0.88889 f(19.00000) + 0.55556 f(27.52056)\right]$$

$$= 11\left[0.55556 \times 239.3327 + 0.88889 \times 484.7455 + 0.55556 \times 795.1069\right]$$

$$= 11061.31 \text{ m}$$

since

$$f(10.47944) = 2000\ln\left[\frac{140000}{140000 - 2100(10.47944)}\right] - 9.8(10.47944)$$
$$= 239.3327$$

$$f(19.00000) = 2000\ln\left[\frac{140000}{140000 - 2100(19.00000)}\right] - 9.8(19.00000)$$
$$= 484.7455$$

$$f(27.52056) = 2000\ln\left[\frac{140000}{140000 - 2100(27.52056)}\right] - 9.8(27.52056)$$
$$= 795.1069$$

The absolute relative true error, $\left|\in_t\right|$, is (True value = 11061.34 m)

$$\left|\in_t\right| = \left|\frac{11061.34 - 11061.31}{11061.34}\right| \times 100$$
$$= 0.0003\%$$