Part II Numerical Analysis (J6) Lent Term, 2000

Exercise Sheet 4 [1]

36. Let A be the $(N-1)^2 \times (N-1)^2$ matrix that is the subject of Lemma 4.11, so it is the matrix that occurs in the five point difference method for Laplace's equation on a square grid. By applying the orthogonal similarity transformation of Hockney's method, find a tridiagonal matrix, T say, that is similar to A, and derive expressions for each element of T. Hence deduce the eigenvalues of T. Verify that they agree with the eigenvalues of Lemma 4.11.

37. Let $\beta_0=2$, $\beta_1=0$, $\beta_2=6$, $\beta_3=-2$, $\beta_4=6$, $\beta_5=0$, $\beta_6=6$ and $\beta_7=2$. By applying the FFT method, calculate $\sum_{j=0}^{7} \beta_j \, e^{2\pi i jk/8}$, $k=0,2,4,6$. Check your results by direct calculation.
Hint: Because all values of k are even, you can omit some parts of the usual FFT algorithm.

38. Let $u(x,t) : \mathcal{R}^2 \rightarrow \mathcal{R}$ be an infinitely differentiable solution of the diffusion equation $u_t = u_{xx} - b u_x$, where the subscripts denote partial derivatives and where b is a positive constant, and let $u(x,0)$, $0 \le x \le 1$, $u(0,t)$, $t>0$, and $u(1,t)$, $t>0$, be given. A difference method sets $h=1/M$ and $k=T/N$, where M and N are positive integers and T is a fixed bound on t. Then it calculates the estimates $U_m^n \approx u(mh,nk)$, $1 \le m \le M-1$, $1 \le n \le N$, by applying the formula

$$U_m^{n+1} = U_m^n + (k/h^2) \left[U_{m-1}^n - 2\, U_m^n + U_{m+1}^n \right] - b\,(k/2h) \left[U_{m+1}^n - U_{m-1}^n \right],$$

the values of U_m^n being set to $u(mh,nk)$ when (mh,nk) is on the boundary. Show that the local truncation error of the formula is $\mathcal{O}(k^2 + kh^2)$.

Let $\varepsilon(h,k)$ be the greatest of the errors $|u(mh,nk) - U_m^n|$, $1 \le m \le M-1$, $1 \le n \le N$. Prove that, if h and k tend to zero in any way such that $k \le \frac{1}{2} h^2$, then $\varepsilon(h,k)$ also tends to zero.
Hint: Relate the maximum error at each time level to the maximum error at the previous time level.

39. Let $v(x,y)$ be a solution of Laplace's equation $v_{xx} + v_{yy} = 0$ on the unit square $0 \le x, y \le 1$, and let $u(x,y,t)$ solve the diffusion equation $u_t = u_{xx} + u_{yy}$, where the subscripts denote partial derivatives. Further, let u satisfy the boundary conditions $u(\xi,\eta,t) = v(\xi,\eta)$ at all points (ξ,η) on the boundary of the unit square for all $t \ge 0$. Prove that, if u and v are sufficiently differentiable, then the integral

$$\phi(t) = \int_0^1 \int_0^1 [\, u(x,y,t) - v(x,y) \,]^2 \, dx\, dy, \qquad t \ge 0,$$

has the property $\phi'(t) \le 0$. Then prove that $\phi(t)$ tends to zero as $t \rightarrow \infty$.
Hint: In the first part, try to replace u_{xx} and u_{yy} when they occur by $u_{xx} - v_{xx}$ and $u_{yy} - v_{yy}$.

40. Let $u(x,t) : \mathcal{R}^2 \rightarrow \mathcal{R}$ be a many times differentiable function that satisfies the diffusion equation $u_t = u_{xx}$, and let θ be a positive constant. Using the notation $U_m^n \approx u(mh,nk)$, where $\mu = k/h^2$ is constant, we consider the implicit difference equation

$$U_m^{n+1} - \tfrac{1}{2}(\mu - \theta) [\, U_{m-1}^{n+1} - 2\, U_m^{n+1} + U_{m+1}^{n+1} \,] = U_m^n + \tfrac{1}{2}(\mu + \theta) [\, U_{m-1}^n - 2\, U_m^n + U_{m+1}^n \,].$$

Show that its local truncation error is $\mathcal{O}(h^4)$, unless $\theta = \frac{1}{6}$ (the Crandall method), which makes the local truncation error of order h^6, or is it possible for the order to be even higher?

41. The Crank–Nicolson formula is applied to the diffusion equation $u_t = u_{xx}$ on a rectangular mesh (mh,nk), $m = 0,1,\ldots,M$, $n = 0,1,2,\ldots$, where $h = 1/M$. Let the boundary conditions include $u(0,t) = u(1,t) = 0$ for all $t \ge 0$. Prove that the estimates $U_m^n \approx u(mh,nk)$ satisfy the equation

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$$\sum_{m=1}^{M-1} \left[(U_m^{n+1})^2 - (U_m^n)^2\right] = -\tfrac{1}{2}(k/h^2) \sum_{m=1}^{M} \left[\, U_m^{n+1} + U_m^n - U_{m-1}^{n+1} - U_{m-1}^n \,\right]^2, \qquad n=0,1,2,\ldots .$$

Because the right hand side is nonpositive, it follows that $\sum_{m=1}^{M-1}(U_m^n)^2$ is a monotonically decreasing function of n. We see that this property is analogous to part of Exercise 39 if $v\equiv 0$ there.
Hint: Substitute the value of $U_m^{n+1}-U_m^n$ *that is given by the Crank–Nicolson formula into the elementary equation* $\sum_{m=1}^{M-1}\left[(U_m^{n+1})^2 - (U_m^n)^2\right] = \sum_{m=1}^{M-1}(U_m^{n+1}-U_m^n)\,(U_m^{n+1}+U_m^n)$. *It is also helpful occasionally to change the index* m *of the summation by one.*

42. Apply the von Neumann stability test to the difference equation

$$U_m^{n+1} = \tfrac{1}{2}\,(2-5\mu+6\mu^2)\,U_m^n + \tfrac{2}{3}\mu\,(2-3\mu)\,(U_{m-1}^n + U_{m+1}^n) - \tfrac{1}{12}\mu\,(1-6\mu)\,(U_{m-2}^n + U_{m+2}^n).$$

Deduce that the test is satisfied if and only if $0\le\mu\le\frac{2}{3}$.

43. A square grid is drawn on the region $\{(x,t) : 0\le x\le 1,\ t\ge 0\}$ in $\mathcal{R}^2$, the grid points being (mh, nh), $0\le m\le M$, $n=0,1,2,\ldots$, where $h=1/M$ and M is even. Let $u(x,t)$ be an exact solution of the wave equation $u_{tt}=u_{xx}$ and let the boundary values $u(x,0)$, $0\le x\le 1$, $u(0,t)$, $t>0$, and $u(1,t)$, $t>0$, be given. Further, an approximation to $\partial u/\partial t$ at $x=0$ allows each of the function values $u(mh,h)$, $m=1,2,\ldots,M-1$, to be estimated to accuracy ε. Then the difference equation

$$U_m^{n+1} = U_{m+1}^n + U_{m-1}^n - U_m^{n-1}$$

is applied to estimate u at the remaining grid points. Prove that all of the moduli of the errors $|U_m^n - u(mh,nh)|$ are bounded above by $\frac{1}{2}\varepsilon M$, even when n is very large.
Hint: Let the error in $u(mh,h)$ *be* $\delta_{mj}\,\varepsilon$, $m=1,2,\ldots,M-1$, *where* δ_{mj} *is the Kronecker delta and where* j *is any integer that you choose from* $[1,M-1]$. *Draw a diagram that shows the contribution from this error to* U_m^n *for every* m *and* n.

44. A rectangular grid is drawn on $\mathcal{R}^2$, with grid spacing h in the x-direction and k in the t-direction. Let the difference equation

$$\begin{aligned} U_m^{n+1} - 2\,U_m^n + U_m^{n-1} \;&=\; (k/h)^2\Big(a\,(U_{m-1}^{n+1} - 2\,U_m^{n+1} + U_{m+1}^{n+1}) \\ &\qquad + b\,(U_{m-1}^n - 2\,U_m^n + U_{m+1}^n) + c\,(U_{m-1}^{n-1} - 2\,U_m^{n-1} + U_{m+1}^{n-1})\,\Big) \end{aligned}$$

be used to approximate solutions of the wave equation $u_{tt}=u_{xx}$. Deduce that the local truncation error is $\mathcal{O}(h^4+k^4)$ if and only if the parameters a, b and c satisfy $a=c$ and $a+b+c=1$. Show also that, if these conditions hold, then the von Neumann stability condition is achieved for all values of k/h if and only if the parameters also satisfy $|b|\le 2a$.

45. Let the split form of the Crank–Nicolson formula, namely

$$(1-\tfrac{1}{2}\mu\Delta_x^2)\,(1-\tfrac{1}{2}\mu\Delta_y^2)\,U_{\ell m}^{n+1} = (1+\tfrac{1}{2}\mu\Delta_x^2)\,(1+\tfrac{1}{2}\mu\Delta_y^2)\,U_{\ell m}^n,$$

be applied to the diffusion equation $u_t=u_{xx}+u_{yy}$, $(x,y)\in\mathcal{S}$, $t\ge 0$, in order to generate estimates $U_{\ell m}^n \approx u(\ell h, mh, nk)$ of the function $u(x,y,t)$, $(x,y)\in\mathcal{S}$, $t\ge 0$. Further, let $\mathcal{S}$ be the L-shaped region $\{(x,y) : -1\le x\le 1,\ -1\le y\le 1,\ \max[x,y]\ge 0\}$, and let $h=1/M$ for some positive integer M. Therefore the range of ℓ and m in the formula that is displayed above is all pairs of integers that satisfy the strict inequalities $-M<\ell<M$, $-M<m<M$ and $\max[\ell,m]>0$. We assume as usual that, if $(\ell h, mh)$ is on the boundary of $\mathcal{S}$, then $U_{\ell m}^n$ is set to $u(\ell h, mh, nk)$ for all $n\ge 0$. Find a splitting method that calculates all the unknowns $U_{\ell m}^{n+1}$ at time $t=nk+k$ from the estimates $U_{\ell m}^n$ at time $t=nk$. The main task of your method should be the solution of tridiagonal systems of linear equations, the dimensions of each system being either $(M-1)\times(M-1)$ or $(2M-1)\times(2M-1)$.