LECTURE NOTES SEVEN

Direct Method of Interpolation

After reading this lecture notes, you should be able to:

1. *apply the direct method of interpolation,*
2. *solve problems using the direct method of interpolation, and*
3. *use the direct method interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead it's called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this lecture.

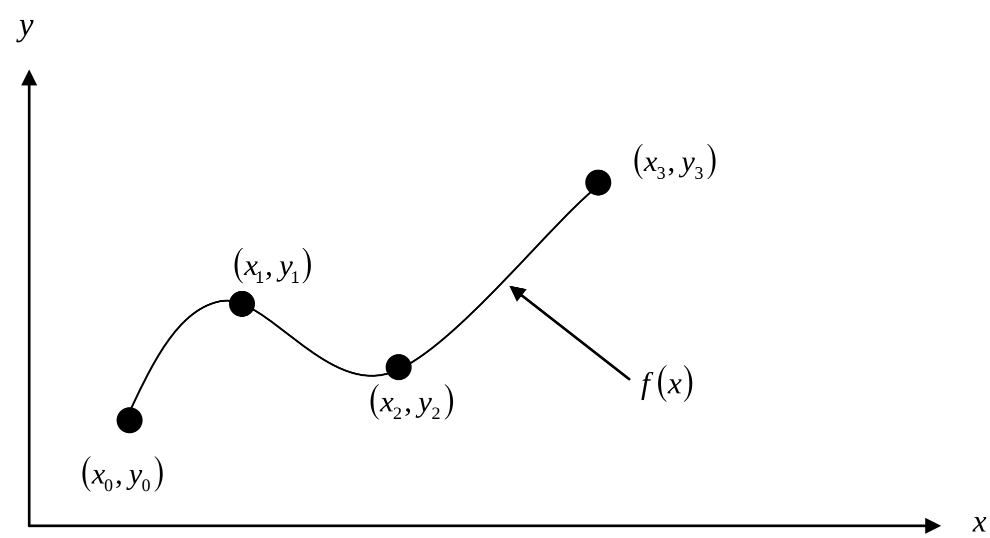

Figure 1 Interpolation of discrete data.

Direct Method

The direct method of interpolation is based on the following premise. Given $n+1$ data points, fit a polynomial of order n as given below

$$y = a_0 + a_1 x + ... + a_n x^n \quad (1)$$

through the data, where $a_0, a_1, ..., a_n$ are $n+1$ real constants. Since $n+1$ values of y are given at $n+1$ values of x, one can write $n+1$ equations. Then the $n+1$ constants, $a_0, a_1, ..., a_n$ can be found by solving the $n+1$ simultaneous linear equations. To find the value of y at a given value of x, simply substitute the value of x in Equation 1.

But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

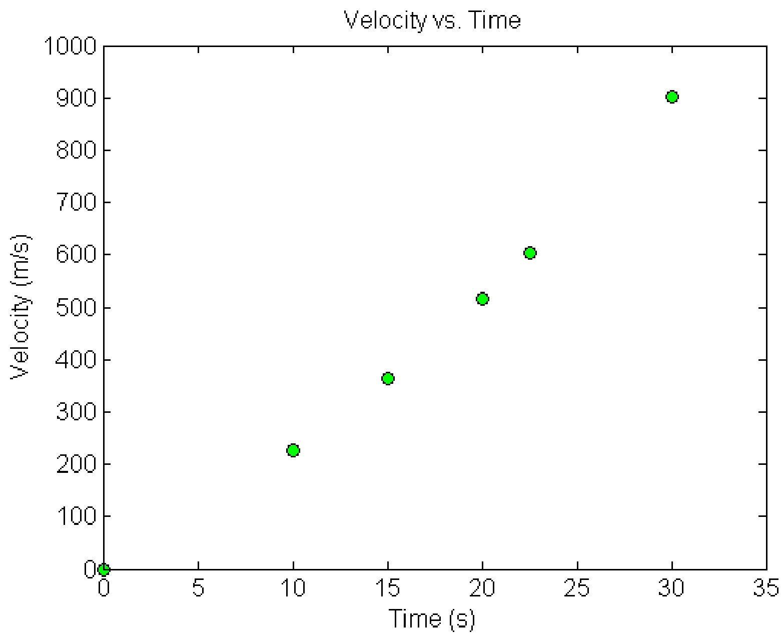

Figure 2 Graph of velocity vs. time data for the rocket example.

Determine the value of the velocity at $t = 16$ seconds using the direct method of interpolation and a first order polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), the velocity given by

$$v(t) = a_0 + a_1 t$$

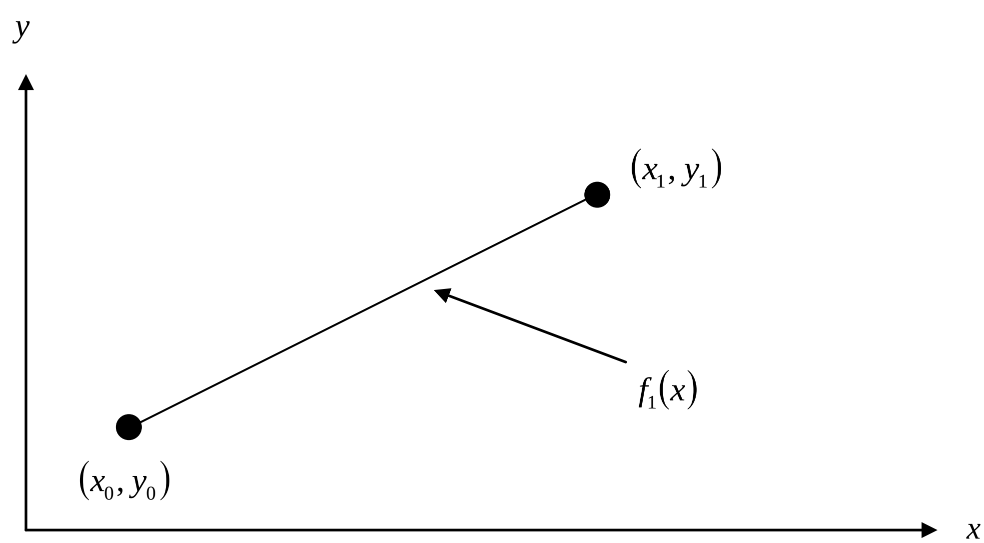

Figure 3 Linear interpolation.

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15, \ v(t_0) = 362.78$$
$$t_1 = 20, \ v(t_1) = 517.35$$

gives

$$v(15) = a_0 + a_1(15) = 362.78$$
$$v(20) = a_0 + a_1(20) = 517.35$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.78 \\ 517.35 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = -100.93$$
$$a_1 = 30.914$$

Hence

$$v(t) = a_0 + a_1 t$$
$$= -100.93 + 30.914t, \ 15 \leq t \leq 20$$

At $t = 16$,

$$v(16) = -100.92 + 30.914 \times 16$$
$$= 393.7 \text{ m/s}$$

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using the direct method of interpolation and a second order polynomial.

Solution

For second order polynomial interpolation (also called quadratic interpolation), the velocity is given by

$$v(t) = a_0 + a_1 t + a_2 t^2$$

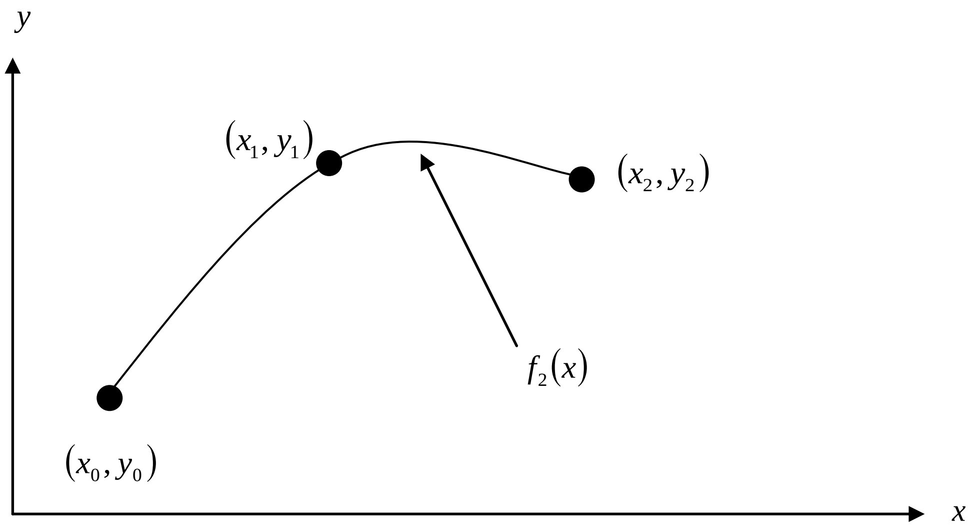

Figure 4 Quadratic interpolation.

Since we want to find the velocity at $t = 16$, and we are using a second order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10,\ t_1 = 15,$ and $t_2 = 20$.

Then

$$t_0 = 10,\ v(t_0) = 227.04$$
$$t_1 = 15,\ v(t_1) = 362.78$$
$$t_2 = 20,\ v(t_2) = 517.35$$

gives

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$
$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$
$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 12.05$$
$$a_1 = 17.733$$
$$a_2 = 0.3766$$

Hence

$$v(t) = 12.05 + 17.733t + 0.3766t^2,\ 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2$$
$$= 392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$
$$= 0.38410\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds using the direct method of interpolation and a third order polynomial.
b) Find the absolute relative approximate error for the third order polynomial approximation.
c) Using the third order polynomial interpolant for velocity from part (a), find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.
d) Using the third order polynomial interpolant for velocity from part (a), find the acceleration of the rocket at $t = 16\text{s}$.

Solution

a) For third order polynomial interpolation (also called cubic interpolation), we choose the velocity given by

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

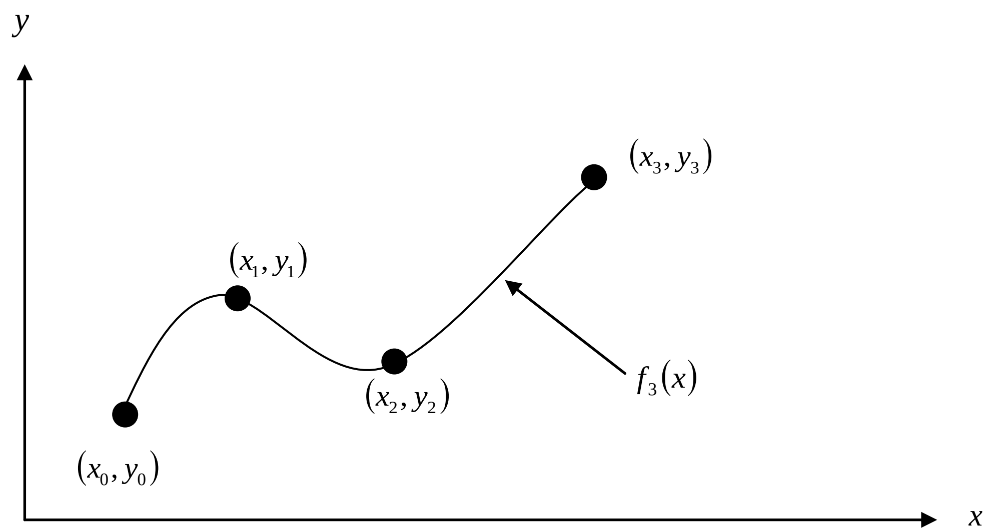

Figure 5 Cubic interpolation.

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The four points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20$ and $t_3 = 22.5$.

Then

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

gives

$$v(10) = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3 = 227.04$$
$$v(15) = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3 = 362.78$$
$$v(20) = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3 = 517.35$$
$$v(22.5) = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3 = 602.97$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 10 & 100 & 1000 \\ 1 & 15 & 225 & 3375 \\ 1 & 20 & 400 & 8000 \\ 1 & 22.5 & 506.25 & 11391 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = -4.2540$$
$$a_1 = 21.266$$
$$a_2 = 0.13204$$

$$a_3 = 0.0054347$$

Hence

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$= -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \le t \le 22.5$$
$$v(16) = -4.2540 + 21.266(16) + 0.13204(16)^2 + 0.0054347(16)^3$$
$$= 392.06 \text{ m/s}$$

b) The absolute percentage relative approximate error $|\in_a|$ for the value obtained for $v(16)$ between second and third order polynomial is

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$
$$= 0.033269\%$$

c) The distance covered by the rocket between $t = 11\text{s}$ and $t = 16\text{s}$ can be calculated from the interpolating polynomial

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of integration of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$= \int_{11}^{16} \left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right)dt$$
$$= \left[-4.2540t + 21.266\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$
$$= 1605 \text{ m}$$

d) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$

Given that

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \le t \le 22.5$$
$$a(t) = \frac{d}{dt}v(t)$$
$$= \frac{d}{dt}\left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right)$$
$$= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \le t \le 22.5$$
$$a(16) = 21.266 + 0.26408(16) + 0.016304(16)^2$$
$$= 29.665 \text{ m/s}^2$$