DC TRANSIENTS IN RLC CIRCUITS

- Consider a series R-L-C circuit as shown in Fig. 3.18, excited with a DC voltage source V_S
- Applying KVL around the closed path for t>0

$$L\frac{di(t)}{dt} + Ri(t) + V_C(t) = V_S \qquad 3.39$$

- The current through the capacitor can be written as

$$i(t) = C\frac{dV_C(t)}{dt}$$

- Substituting the current ‘i(t)’ expression in equation (3.39) and rearranging the terms,

$$LC\frac{d^2V_C(t)}{dt^2} + RC\frac{dV_C(t)}{dt} + V_C(t) = V_S \qquad 3.40$$

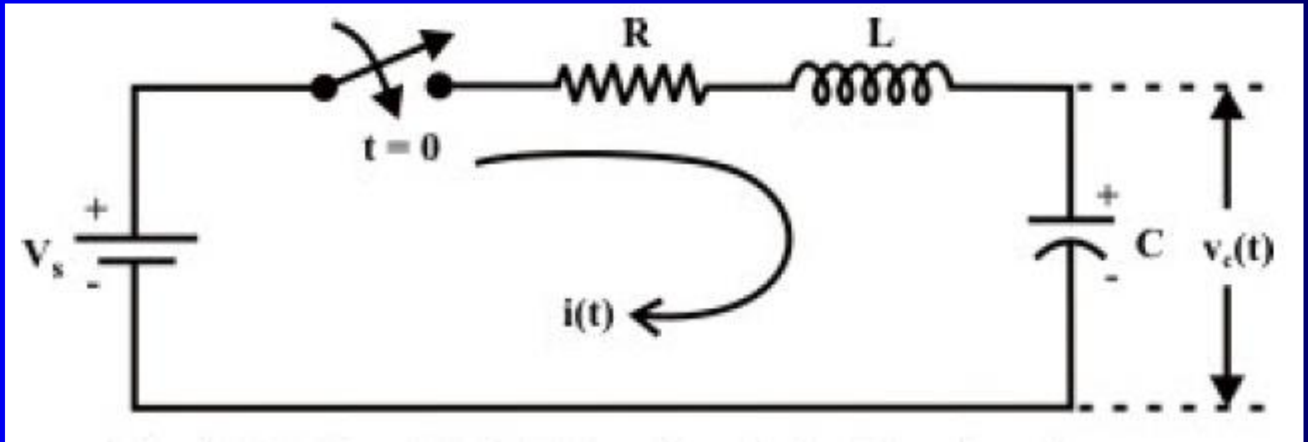


Fig. 3.18: A Simple R-L-C circuit excited with a DC voltage source

- Equation (3.40) is a 2nd-order linear differential equation
- The complete solution of the above differential equation has two components; the transient response and the steady state response.
- Mathematically, one can write the complete solution as

$$V_C(t) = V_{cn}(t) + V_{cf}(t) = (A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}) + A \qquad 3.41$$

- The nature of the steady state response is same as that of the forcing function (input voltage) and it is given by a constant value *A*.
- The natural or transient response of second order differential equation can be obtained from the homogenous equation (i.e. from force free system) that is expressed by

$$LC\frac{d^2V_C(t)}{dt^2} + RC\frac{dV_C(t)}{dt} + V_C(t) = 0$$

$$\Rightarrow \frac{d^2V_C(t)}{dt^2} + \frac{R}{L}\frac{dV_C(t)}{dt} + \frac{1}{LC}V_C(t) = 0$$

$$\Rightarrow a\frac{d^2V_C(t)}{dt^2} + b\frac{dV_C(t)}{dt} + cV_C(t) = 0 \quad \left(where\ a = 1,\ b = \frac{R}{L}\ \ and\ c = \frac{1}{LC}\right) \qquad 3.42$$

⇒ The characteristic equation of the above homogenous differential equation (using the operator $\alpha = \frac{d}{dt}$, $\alpha^2 = \frac{d^2}{dt^2}$ and $V_C(t) \neq 0$) is given by

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$$

$$\Rightarrow a\alpha^2 + b\alpha + c = 0 \qquad 3.43$$

⇒ The solution of the above equation (3.43) has the solution α_1 *and* α_2, which are associated with the exponential terms associated with the transient part of the complete solution (eqn. 3.41) and they are given below

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)$$

$$\alpha_2 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)$$

The roots of the characteristic equation (3.43) determine the kind of response depending upon the values of the parameters R,L, and C of the circuit

Case A (Overdamped Response):

- When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$, this implies that the roots are distinct with negative real parts.
- Under this situation, the natural or transient part of the complete solution is written as

$$V_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \qquad 3.44$$

and each term of the above expression decays exponentially and ultimately reduces to zero as $t \rightarrow \infty$ and it is termed as overdamped response of input free system.

- A system that is overdamped responds slowly to any change in excitation.
- It may be noted that the exponential term $A_1 e^{\alpha_1 t}$ takes longer time to decay its value to zero than the term $A_2 e^{\alpha_2 t}$.
- One can introduce a factor ξ that provides an information about the speed of system response and it is defined by damping ratio

$$\xi = \frac{Actual\ damping}{critical\ damping} = \frac{R/L}{2/\sqrt{LC}} > 1 \qquad 3.45$$

Case B (Critically damped Response):

- When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$, this implies that the roots of eq.(3.43) are same with negative real parts.
- Under this situation, the form of the natural or transient part of the complete solution is written as

$$V_{cn}(t) = (A_1 t + A_2)e^{\alpha t} \qquad \left(where\ \alpha = -\frac{R}{2L}\right) \qquad 3.46$$

where the natural or transient response is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term.

- The response is the speediest response possible without any overshoot .
- The response of such a second order system is defined as a critically damped system's response.
- In this case damping ratio

$$\xi = \frac{Actual\ damping}{critical\ damping} = \frac{R/L}{2/\sqrt{LC}} = 1 \qquad 3.47$$

Case C (Underdamped Response):

- When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$, this implies that the roots of eq.(3.43) are complex conjugates and they are expressed as

$$\alpha_1 = \left(-\frac{R}{2L} + j\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \beta + j\gamma\,;$$

$$\alpha_2 = \left(-\frac{R}{2L} - j\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \beta - j\gamma$$

- Under this situation, the form of the natural or transient part of the complete solution is written as

$$V_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2 e^{(\beta - j\gamma)}$$
$$= e^{\beta t}[(A_1 + A_2)cos(\gamma t) + j(A_1 - A_2)sin(\gamma t)]$$
$$= e^{\beta t}[B_1 cos(\gamma t) + B_2 sin(\gamma t)] \qquad where\ B_1 = A_1 + A_2;\ B_2 = j(A_1 - A_2) \qquad 3.48$$

- The equation (3.48) further can be simplified as:

$$e^{\beta t}Ksin(\gamma t + \theta) \qquad 3.49$$

- Where β = real part of the root, γ = complex part of the root, $K = \sqrt{B_1^2 + B_2^2}$ and $\theta = tan^{-1}\left(\frac{B_1}{B_2}\right)$
- The values of K and **θ** can be calculated using the initial conditions of the circuit
- The system response exhibits oscillation around the steady state value when the roots of characteristic equation are complex and results in an under-damped system's response.
- This oscillation will die down with time if the roots are with negative real parts.
- In this case the damping ratio

$$\xi = \frac{Actual\ damping}{critical\ damping} = \frac{R/L}{2/\sqrt{LC}} < 1 \qquad 3.50$$

Finally, the response of a second order system when excited with a dc voltage source is presented in Fig. 3.19 for different cases, i.e., (i) under-damped (ii) over-damped (iii) critically damped system response.

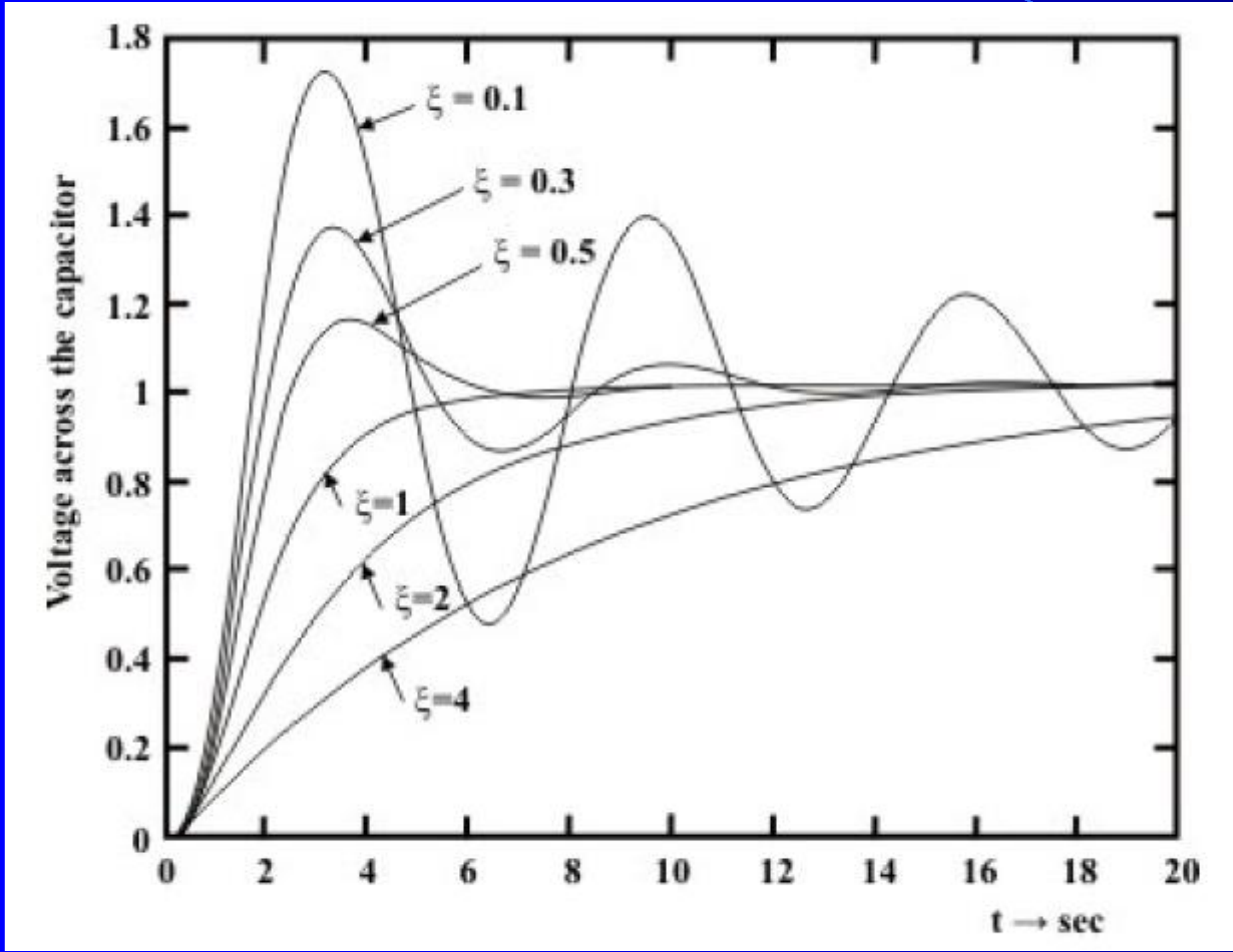
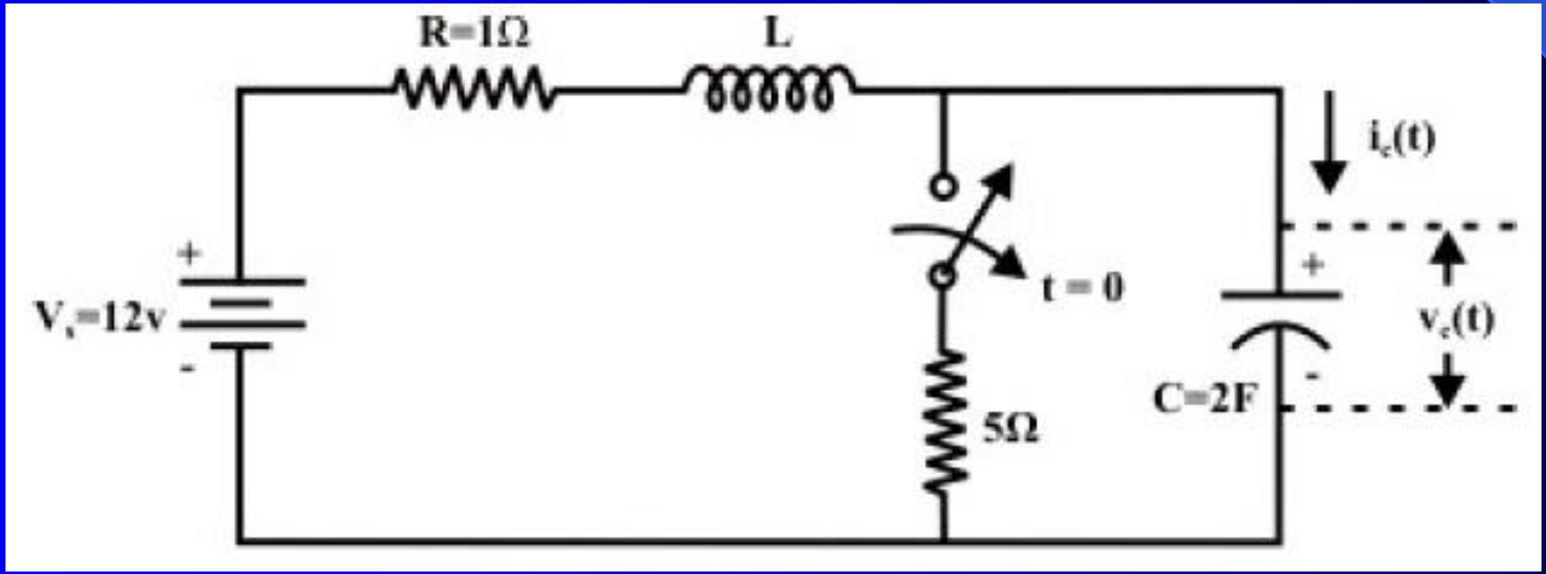


Fig. 3.19: System response for series R-L-C circuit:
(a) underdamped
(b) critically damped
(c) overdamped system

Example 3.5:

The switch S1 has been closed for a sufficiently long time and then it is opened at t=0. Find the expression for (a) $v_c(t)$, (b) $i_c(t), t > 0$ for inductor values of (i) L=0.5 H (ii) L=0.2 H (iii) L=0.1 H and plot $v_c(t) - vs - t$ and $i(t) - vs - t$ for each case.



Solution:

- At $t = 0^-$(before the switch is opened) the capacitor acts as an open circuit or block the current through it but the inductor acts as short circuit.
- Using the properties of inductor and capacitor, one can find the current in inductor at time $t = 0^+$ as $i_L(0^+) = i_L(0^-) = \frac{12}{1+5} = 2\ A$ (note inductor acts as a short circuit) and voltage across the 5 Ω resistor = 2 x 5= 10 volts.
- The capacitor is fully charged with the voltage across the 5 Ω resistor and the capacitor voltage at $t = 0^+$ is given by $v_c(0^+) = v_c(0^-) = 10\ volts$
- The circuit is opened at time t=0 and the corresponding circuit diagram is shown in Fig. 3.21

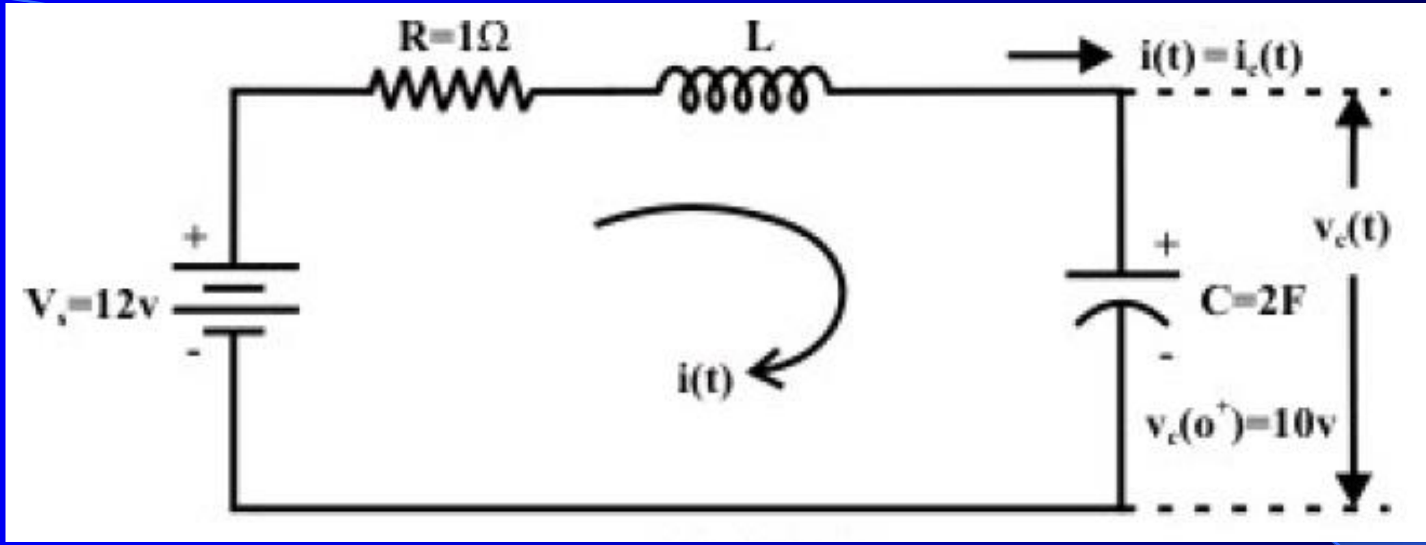


Fig. 3.21

Case 1: $L = 0.5\ H, R = 1\Omega\ and\ C = 2F$

Let us assume the current flowing through the circuit is i(t) and apply KVL equation around the closed path

$$L\frac{di(t)}{dt} + Ri(t) + V_C(t) = V_S$$

$$\Rightarrow LC\frac{d^2V_C(t)}{dt^2} + RC\frac{dV_C(t)}{dt} + V_C(t) = V_S$$

The solution of the above is given by

$$V_C(t) = V_{cn}(t) + V_{cf}(t)$$

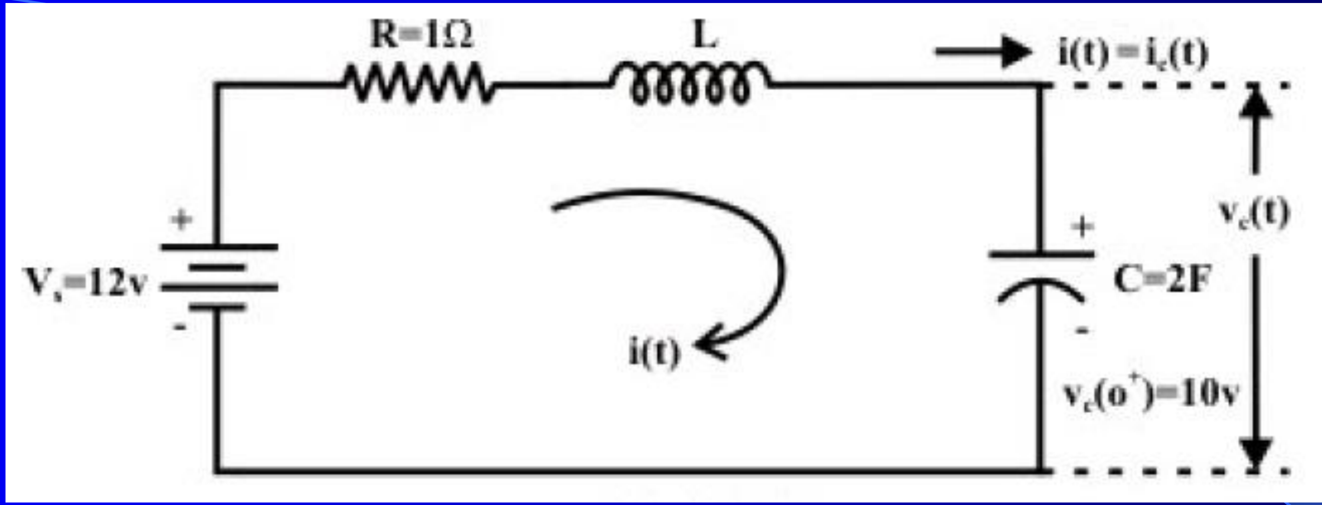


Fig. 3.21

The solution of natural or transient response $v_{cn}(t)$ is obtained from the force free equation or homogenous equation which is

$$\frac{d^2 v_c(t)}{dt^2}+\frac{R}{L}\frac{dv_c(t)}{dt}+\frac{1}{LC}v_c(t)=0$$

The characteristic equation of the above homogenous equation is written as

$$\mathbf{a}\boldsymbol{\alpha}^{\mathbf{2}}+\boldsymbol{b\alpha}+\boldsymbol{c}=\mathbf{0}$$

The roots of the characteristic equation are given as:

$$\boldsymbol{\alpha_1}=\left(-\frac{\boldsymbol{R}}{\mathbf{2}\boldsymbol{L}}+\sqrt{\left(\frac{\boldsymbol{R}}{\mathbf{2}\boldsymbol{L}}\right)^{\mathbf{2}}-\frac{\mathbf{1}}{\boldsymbol{LC}}}\right)=-\mathbf{1};\quad \boldsymbol{\alpha_2}=\left(-\frac{\boldsymbol{R}}{\mathbf{2}\boldsymbol{L}}+\sqrt{\left(\frac{\boldsymbol{R}}{\mathbf{2}\boldsymbol{L}}\right)^{\mathbf{2}}-\frac{\mathbf{1}}{\boldsymbol{LC}}}\right)=-\mathbf{1}$$

- The roots are equal with negative real sign. The expression for the natural response is given by

$$v_{cn}(t) = (A_1 t + A_2)e^{\alpha t} \quad (\text{where } \alpha = \alpha_1 = \alpha_2 = -1)$$

- The forced or the steady state response $v_{cf}(t)$ is the form of the applied input voltage and it is constant ‘A’.
- Now the final expression for $v_c(t)$ is

$$v_c(t) = (A_1 t + A_2)e^{\alpha t} + A = (A_1 t + A_2)e^{-t} + A$$

- The initial and final conditions needed to evaluate the constants are based on

$$v_c(0^+) = v_c(0^-) = 10\ volt; \quad i_L(0^+) = i_L(0^-) = 2\ A$$

- **At $t = 0^+$;**

$$v_c(t)\Big|_{t=0^+} = A_2 e^{-1x0} + A = A_2 + A \quad \Rightarrow \quad A_2 + A = 10 \qquad (1)$$

- Also,

$$\frac{dv_c(t)}{dt} = -(A_1 t + A_2)e^{-t} + A_1 e^{-t}$$

$$=> \frac{dv_c(t)}{dt}\Big|_{t=0^+} = A_1 - A_2 \qquad (2)$$

- $V_C(\infty) = A \quad \Rightarrow \quad A = 12$

- Using the value of A in equation (1) and then solving (1) and (2) we get, $A_1 = -1$; $A_2 = -2$
- The total solution is

$$\boldsymbol{V_c(t) = -(t+2)e^{-t} + 12 = 12 - (t+2)e^{-t}}$$

$$\boldsymbol{i(t) = C\frac{dV_c(t)}{dt} = 2 \text{ x } [(t+2)e^{-t} - e^{-t}] = 2 \text{ x } (t+1)e^{-t}}$$

- **Case-2: L=0.2 H, R= 1Ω and C=2F**

- It can be noted that the initial and final conditions of the circuit are all same as in case-1 but the transient or natural response will differ.
- In this case the roots of characteristic equation are computed and the values of roots are

$$\alpha_1 = -0.563;\ \ \alpha_2 = -4.436$$

$$\boldsymbol{V_c(t) = A_1e^{-\alpha_1 t} + A_2e^{-\alpha_2 t} + A = A_1e^{-4.436t} + A_2e^{-0.563t} + A \qquad (1)}$$

$$\frac{dv_c(t)}{dt} = \alpha_1 A_1 e^{-\alpha_1 t} + \alpha_2 A_2 e^{-\alpha_2 t} = -4.435 A_1 e^{-4.436t} - 0.563 A_2 e^{-0.536t} \qquad (2)$$

- **Using the initial conditions ($v_c(0^+) = 10, \frac{dv_c(0^+)}{dt} = 1 \frac{volt}{sec}.$) and A=12 obtained in case-1 in the equations (1) and (2) above, we have**

$$A_1 = 0.032;\ A_2 = -2.032$$

- **The total response is**

$$V_c(t) = 0.032e^{-4.436t} - 2.032e^{-0.563t} + 12$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2[1.14e^{-0.563t} - 0.14e^{-4.436t}]$$

- **Case 3:**
- Again the initial and final conditions will remain same and the natural response of the circuit will be decided by the roots of the characteristic equation and they are obtained as

$$\alpha_1 = \beta + j\gamma = -0.063 + j0.243; \qquad \alpha_2 = \beta - j\gamma = -0.063 - j0.242$$

- **The expression for the total response is**

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = e^{\beta t} K sin(\gamma t + \theta) + A \qquad \textbf{(1)}$$

$$\frac{dv_c(t)}{dt} = Ke^{\beta t}[\beta \sin(\gamma t + \theta) + \gamma \cos(\gamma t + \theta)] \qquad \textbf{(2)}$$

- Again the initial conditions that were obtained in case-1 are used in above equations with A=12 (final steady state condition) and simultaneous solution gives

$$K = 4.13; \;\; \theta = -28.98^o$$

- The total response is

$$v_c(t) = e^{\beta t} K sin(\gamma t + \theta) + 12 = e^{-0.063t} 4.13 sin(0.242t - 28.99^o) + 12$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2e^{-0.063t}[0.999 * cos(0.242t - 28.99^o) - 0.26 sin(0.242t - 28.99^o)]$$