Queen Mary
University of London

BSc Degree by Course Units

MAS 212 LINEAR ALGEBRA I

30th April 2003 2.30pm - 4.30pm

Duration 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

CALCULATORS MAY NOT BE USED IN THIS EXAMINATION

SECTION A *Each question carries 7 marks. You should attempt ALL questions.*

A1. Let V be a vector space over a field $\mathbb{F}$, and U a subset of V. Define what it means for U to be a *subspace* of V.
(a) Give an example of a 2-dimensional subspace U of the 3-dimensional space $V = \mathbb{R}^3$.

(b) Either show that $U = \{(0,0)\}$ is a subspace of $V = \mathbb{R}^2$, or give a reason why it is not.

A2. Consider the vector space $V = \mathbb{R}^3$. **State** for each set of vectors $S \subseteq V$ whether, or not, they form (i), a spanning set for V, (ii), a linearly independent set.

(a) $S = \{(1,0,1), (0,1,0), (0,0,1), (1,1,1)\}$;
(b) $S = \{(1,2,2), (0,1,0), (2,0,4)\}$;
(c) $S = \{(1,0,1), (0,1,1)\}$.

A3. (a) For what conditions on the real numbers a, b are the vectors $v_1 = (2+a, 1, 3), v_2 = (b, b, -1), v_3 = (0, a, 0)$ in $\mathbb{R}^3$ linearly **dependent**.

(b) Let $v_1 = (1, a, b, c), v_2 = (0, 0, 2, d), v_3 = (0, 0, 0, 3)$, where a, b, c, d are *arbitrary* real numbers, be vectors in $\mathbb{R}^4$.
Prove that $\{v_1, v_2, v_3\}$ is a linearly **independent** set.
Explain why $\{v_1, v_2, v_3\}$ is not a spanning set for $\mathbb{R}^4$.

A4. Let U and V be vector spaces over the field $\mathbb{F}$. Define what it means for the map $\alpha : U \to V$ to be *linear*.

For each of the following maps, either prove that the map is linear or show why it fails to be linear:

(a) $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(x, y, z) = (z, x, y)$;

(b) $\alpha : \mathbb{R}^3 \to \mathbb{R}$ defined by $\alpha(x, y, z) = x^2 + y^2$.

A5. Let U and V be vector spaces and $\alpha : U \to V$ a linear map.

State the relation between rank(α), nullity(α) and dim U when U is finite dimensional.

Consider the linear map $\alpha : \mathbb{R}^4 \to \mathbb{R}^4$ where $\alpha(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_1, x_3)$.

Find Ker(α) and a spanning set for Im(α), write down a basis for each of these subspaces and hence state both the nullity and rank of α.

A6. Let V be a finite-dimensional vector space, and U, W be subspaces of V. Give a formula relating the dimensions of the subspaces $U + W$ and $U \cap W$ to those of U and W.

Let the vector space V and the subspaces U and W be given as follows:
$V = \mathbb{R}^3$, $U = \{(x, y, z) : x, y, z \in \mathbb{R}, x + y + z = 0\}$, $W = \{(x, y, z) : x, y, z \in \mathbb{R}, x = z\}$.

Find a basis for each of U, W and $U+W$, and state the dimension of each of these subspaces.
Hence, or otherwise, write down the dimension of $U \cap W$.

A7. Let U and V be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$, respectively, and let $\alpha : U \to V$ be a linear map.
Write down the matrix A, with respect to the standard bases of the domain and range spaces, of the linear map $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3)$. Find the inverse $\beta : \mathbb{R}^3 \to \mathbb{R}^3$ of the map α. Write down the matrix B, with respect to the standard bases of the domain and range spaces, of the linear map β. What is the matrix AB?

A8. Define the *identity map* $\mathcal{I}d_V$ of a vector space V.

Define in terms of $\mathcal{I}d_V$ the *change of basis* matrix P from an ordered basis $\mathcal{B}$ of V to an ordered basis $\bar{\mathcal{B}}$ of V.

Find the change of basis matrix P and its inverse P^{-1} when $V = \mathbb{R}^2$, $\mathcal{B} = (1, 3), (2, -1)$ and $\bar{\mathcal{B}} = \mathcal{E}_2$, the standard basis.

Section B is overleaf

SECTION B *Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.*

B1. (a)Which of the sets of vectors (i), (ii), (iii) below span $\mathbb{R}^3 = \{(x, y, z)|x, y, z \in \mathbb{R}\}$. For those sets which do not span $\mathbb{R}^3$, find the equation of the plane or the line they span in $\mathbb{R}^3$?

(i) $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$,

(ii) $\{(1, 2, 3), (4, 5, 6), (7, 8, 10)\}$,

(iii) $\{1, 1, 4), (2, 1, 5), (0, 1, 3), (3, 2, 9), (1, 1, 1)\}$.

(b) Does the set of vectors $\{(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, -1), (-1, 0, 0, 1)\}$ span $\mathbb{R}^4$? Give reasons for your answer.

B2. Let V be a vector space over the field $\mathbb{F}$. State what is meant by the statement that V has dimension n.

Prove that if V is finite-dimensional with ordered basis $\mathcal{B} = v_1, \ldots, v_n$, then every vector v of V is a unique linear combination of $v_1, \ldots, v_n$.

Define the standard basis of $\mathbb{R}^n$.

Let $\mathcal{B} = v_1, v_2, v_3$, where $v_1 = (2, 1, 0), v_2 = (3, 0, 1)$ and $v_3 = (0, 1, 1)$. Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

Find the coordinates of $v = (2, -1, -1)$ with respect to $\mathcal{B}$.

Now let v_1, v_2, v_3 be as above, but regarded as vectors of $\mathbb{F}_5^3$, where $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, the field of integers added and multiplied mod 5. Show that $\{v_1, v_2, v_3\}$ is not a basis for $\mathbb{F}_5^3$, but that $\{v_1, v_2, (1, 0, 1)\}$ is a basis for $\mathbb{F}_5^3$.

B3. Let $\alpha : V \to W$ be a linear map between finite dimensional vector spaces V and W. Define $\text{Ker}(\alpha)$ and $\text{Im}(\alpha)$. Show that $\text{Ker}(\alpha)$ is a subspace of V and that $\text{Im}(\alpha)$ is a subspace of W. **State** and **prove** a formula relating the dimensions of V, $\text{Ker}(\alpha)$ and $\text{Im}(\alpha)$.
Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by $\alpha(x, y, z) = (2x + y, 3x + y - 2z)$, for real numbers x, y, z. Find bases for $\text{Ker}(\alpha)$ and $\text{Im}(\alpha)$.

B4. (a) Find an invertible matrix S such that SAS^{-1} is a diagonal matrix, where

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 2 & -6 \\ -2 & 0 & 7 \end{pmatrix}.$$

(b) Find a real orthogonal matrix P such that PAP^T is a diagonal matrix, where

$$A = \begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & -1 \end{pmatrix}.$$

END OF EXAM