EE 287 CIRCUIT THEORY

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What to expect?

Networks Theorems → **Maximum Power Transfer Theorem** → **Impedance Matching**

1

NETWORK THEOREMS IN AC NETWORKS

Introduction

AC circuit

- A sinusoid is a signal that has the form of the **sine or cosine function.**
- A sinusoidal current is usually referred to as **alternating current (AC).**
- Circuits driven by sinusoidal current or voltage sources are called **AC circuits.**

Phasor

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.

Introduction

AC circuit analysis

- When we limit the input of an electric circuit to sines and cosines, we can develop AC analysis methods to figure out what happens in circuits with changing signals.

- When voltages, currents and impedances are treated as complex numbers or phasors. The solution of AC circuits becomes the same as that of **dc circuits.**

Analyzing AC Circuits

Analyzing AC circuits usually requires three steps:

- Transform the circuit to the phasor or frequency domain.
- Solve the problem using circuit techniques.
- Transform the resulting phasor to the time domain.

Note:

- Step 1 is not necessary if the problem is specified in the frequency domain.
- In step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Time Domain to Phasor Domain

TIME DOMAIN	PHASOR DOMAIN
$V_m\cos(\omega t + \Phi)$	$V_m \angle \Phi$
$V_m\sin(\omega t + \Phi)$	$V_m \angle \Phi - 90°$
$I_m\cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m\sin(\omega t + \theta)$	$I_m \angle \theta - 90°$

Impedances of Passive Elements

Resistors

Impedance: $Z = R$

Capacitors

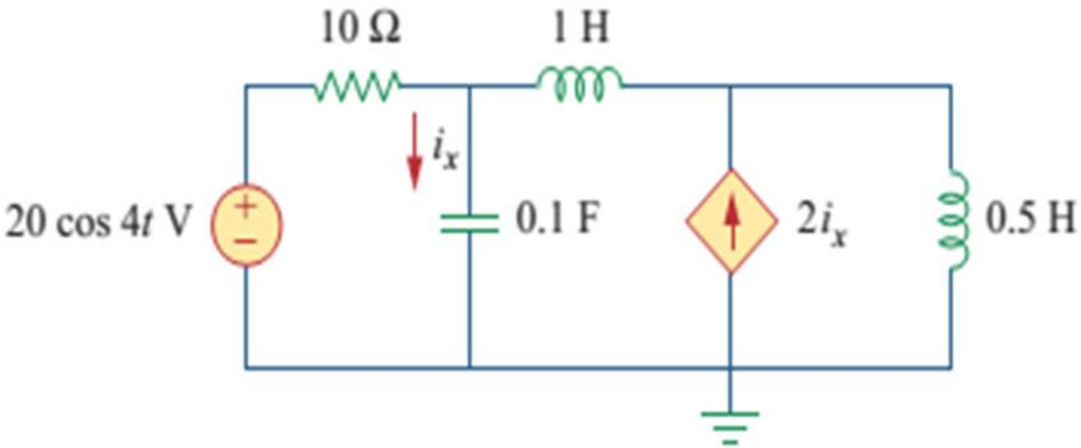
Impedance: $Z = \frac{1}{j\omega C}$

Inductors

Impedance: $Z = j\omega L$

Example 1

Transform the circuit below to the phasor/frequency domain.



Solution

Converting the elements to frequency domain

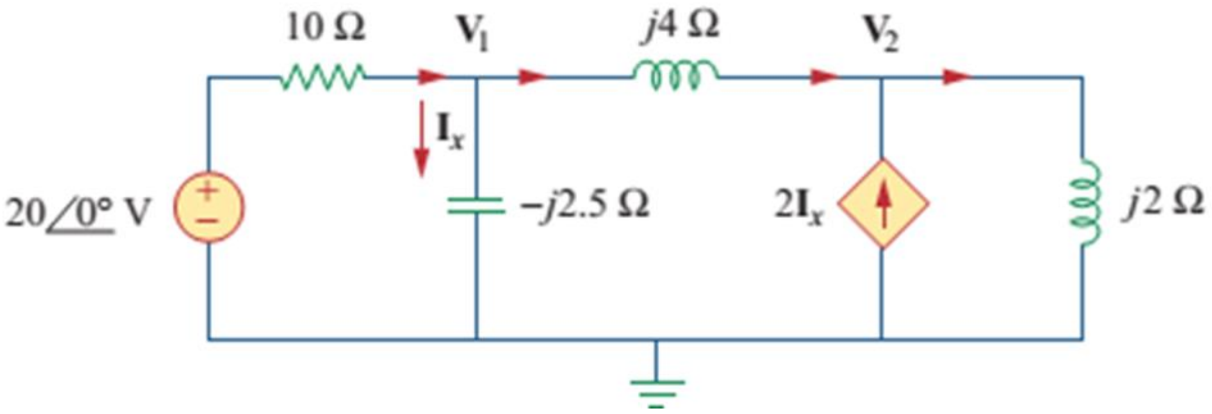
$20\cos 4t = 20\angle 0°, \;\; w = 4 \text{ rad/s}$

$1\text{H} = jwL = j4$

$0.5\text{H} = jwL = j2$

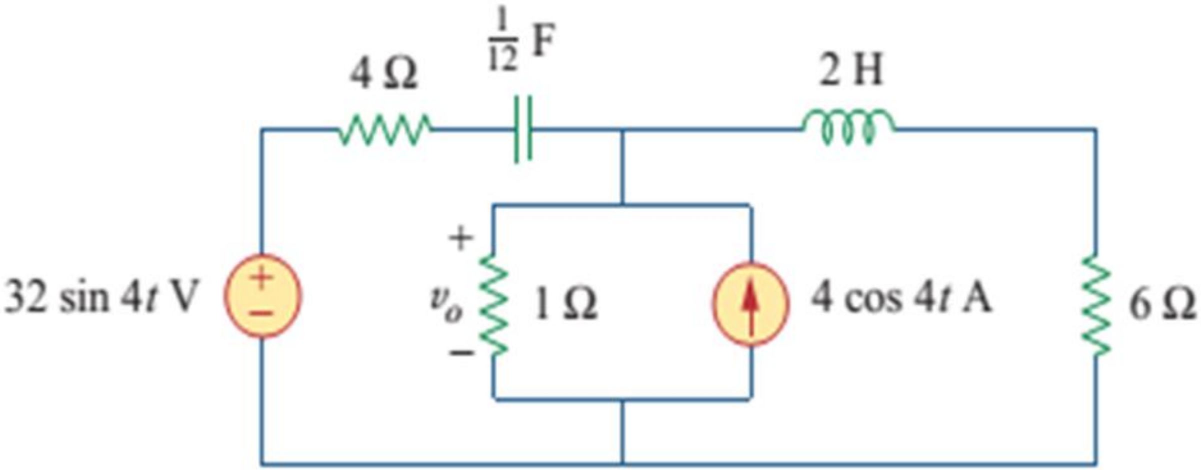
$0.1\text{F} = \frac{1}{jwC} = -j2.5$

Therefore, the frequency domain equivalent circuit is as shown below:



Practice Question

Transform the circuit below to the frequency domain.



1. SUPERPOSITION THEOREM

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Since AC circuits are linear, superposition theorem applies to AC circuits the same way it applies to dc circuits.

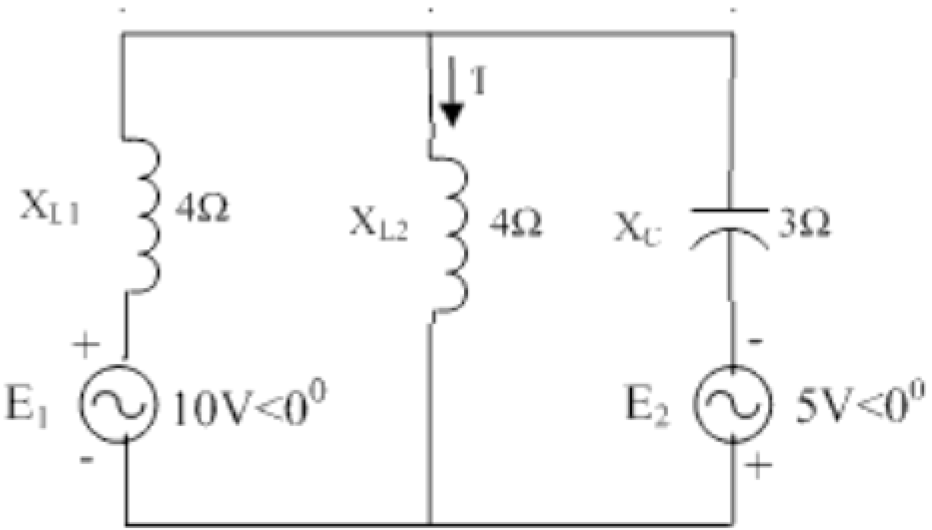
Superposition Theorem

- The theorem becomes important if the circuit has sources operating at different frequencies.
- In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each frequency.
- The total response would be obtain by adding the individual responses in the time domain.
- Superposition can be used when the circuit has **only DC sources**, **only AC sources** or **both**.

Superposition with only AC Sources

Example 1

Find the current I by the superposition theorem.



Voltage Source E_1 acting alone

Effective impedance of parallel branch:

$$X_{L2} \,||\, X_C = \frac{(j4)(-j3\,)}{j4 - j3} = -j12$$

Total impedance $= j4 - j12$

$= -j8\ \Omega$

$= 8\ \Omega\ \angle -90°$

$$\text{Total current} = \frac{10\ \angle 0°}{8\ \angle -90°}$$

$= 1.25\ \text{A}\ \angle 90°$ and

$I' =$ Voltage across parallel branch / X_{L2}

$$= \frac{12\ \angle -90° \times 1.25\ \angle 90°}{4\ \angle 90°}$$

$= 3.75\ \text{A}\ \angle -90°$

Voltage source E_2 acting alone

Impedance of parallel branch,

$X_{L1} \,||\, X_{L2} = j4 \,||\, j4 = j2$

Total Impedance $= j2 - j3 = -j1 = 1\,\angle{-90°}$

Total Current $= \dfrac{5}{1\,\angle{-90°}} = 5A\,\angle 90°$

$I'' = \dfrac{5\,\angle 90°}{2} = 2.5A\,\angle 90°$ because $X_{L1} = X_{L2}$

Actual Current, $I = I' - I''$

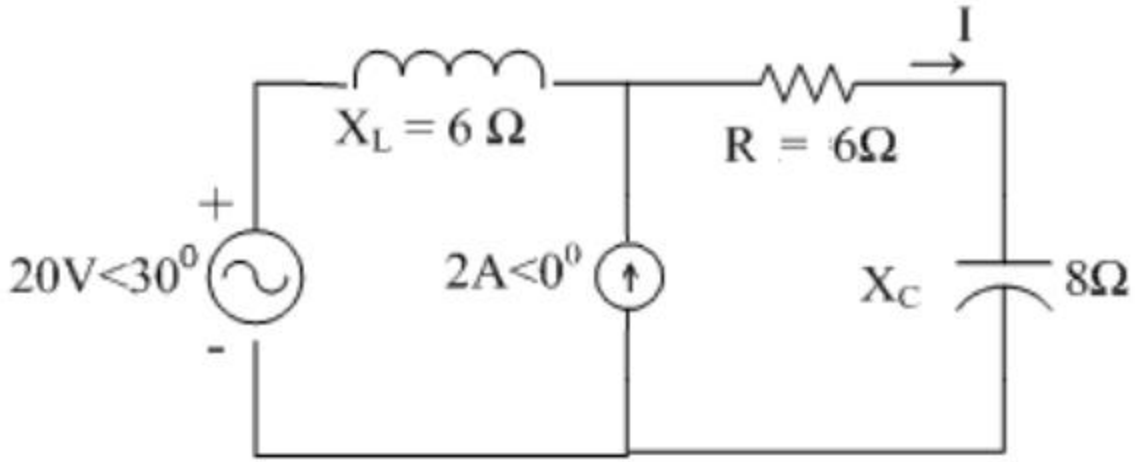
$= 3.75A\,\angle{-90°} - 2.5A\,\angle 90°$

$= -j3.75 - j2.5$

$= -j6.25 = 6.25\text{ A}\,\angle{-90°}$

Example 2

Using superposition, find the current I



With the current source acting alone, the current through the resistor by current divider rule is:

$$I' = \frac{j6}{6 + j(6-8)} \times 2 = \frac{j12}{6 - j2}$$

$$I' = \frac{12\angle 90°}{6.32\angle -18.43°} = 1.9A\angle 108.43°$$

With the voltage source acting alone,

$$I'' = \frac{20\angle 30°}{6 + j(6-8)} = \frac{20\angle 30°}{6 + j2}$$

$$= \frac{20\angle 30°}{6.32\angle -18.43°} = 3.16A\angle 48.43°$$

Actual Current,

$$I = I' + I'' = 1.9\angle 108.43° + 3.16\angle 48.43°$$

$$I = I' + I'' = (-0.60 + j1.80) + (2.10 + j2.36)$$

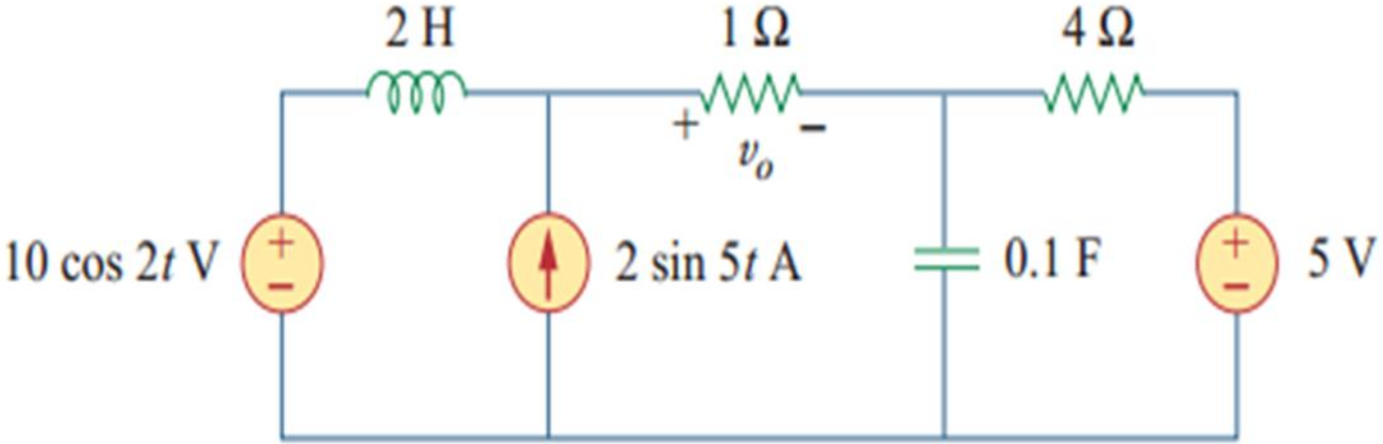
$$I = I' + I'' = 1.50 + j4.16 = 4.42A\angle 70.2°$$

Superposition with both AC and DC sources

One of the most frequent applications of the superposition theorem is to electronic systems in which the dc and ac analysis are treated separately and the total solution is the sum of the two.

Example 3

Find v_o in the circuit below using the superposition theorem.



Solution

The circuit operates at three different frequencies

Let

$v_o = v_1 + v_2 + v_3$

Where

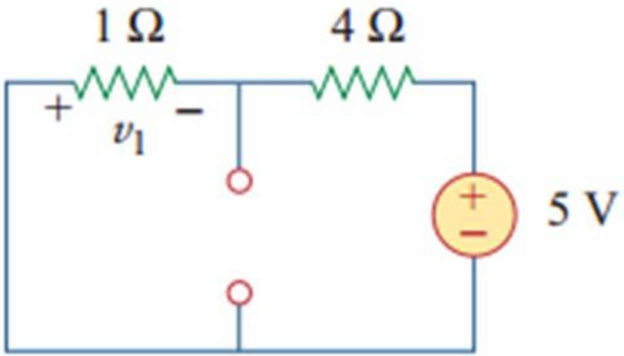
v_1 is due to the 5 V dc source

v_2 is due to the 10cos(2t) V voltage source

v_3 is due to the 2 sin(5t) A current source

- To find v_1, we set to zero all sources except the 5 V dc source.

- At steady state, a capacitor is an open circuit to dc while an inductor is a short circuit to dc.



- By voltage division:

$$-v_1 = \frac{1}{1+4} \times 5 = 1\ \text{V}$$

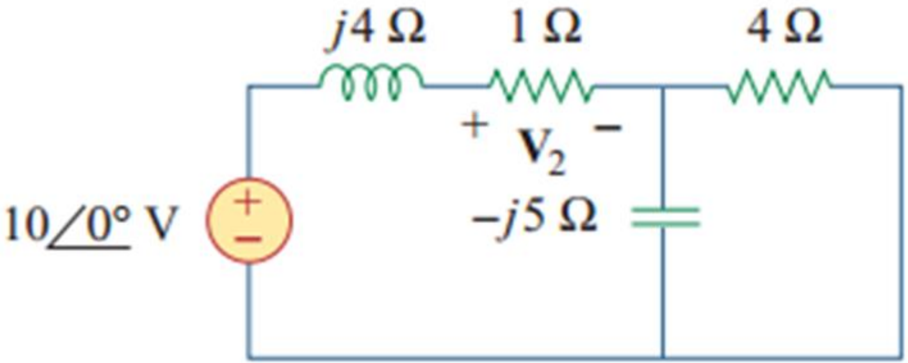
To find v_2:

We set to zero all sources except the 10 cos2t V source and transform the circuit to frequency domain.

$$10\cos 2t = 10\angle 0°, \quad w = 2 \text{ rad/s}$$

$$2\text{ H} = jwL = j4\ \Omega$$

$$0.1\text{F} = \frac{1}{jwC} = -j5\ \Omega$$



Let

$$Z = -j5 \parallel 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

By voltage division

$$V_2 = \frac{1}{1+j4+Z} \times (10 \angle 0°) = \frac{10}{3.439 + j2.049} = 2.498 \angle{-30.79°}$$

In the time domain

$$v_2 = 2.498\cos(2t - 30.79°)$$

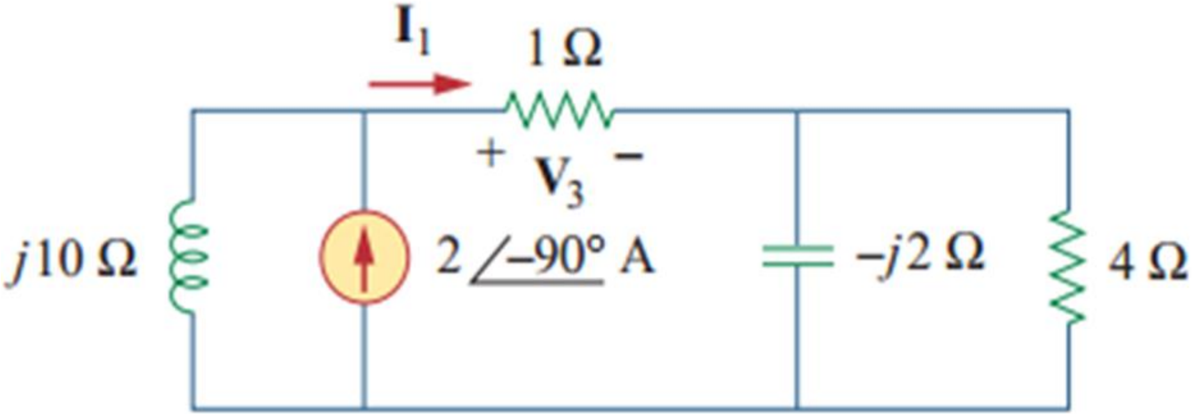
To obtain v_3:

We set the voltage sources to zero and transform what is left to the frequency domain.

$2\sin 5t = 2\angle{-90°}$, $w = 5$ rad/s

$2H = jwL = j10\ \Omega$

$$0.1F = \frac{1}{jwC} = -j2\ \Omega$$



Let

$$Z = -j2\ ||\ 4 = \frac{-j2 \times 4}{4-j2} = 0.8 - j1.6\ \Omega$$

By current division:

$$I_1 = \frac{j10}{j10+1+Z_1} \times (2\,\angle{-90°})\ \text{A}$$

$$V_3 = I_1 \times 1 = \frac{j10}{1.8 + j8.4} \times -j2 = 2.328\text{V}\,\angle{-80°}$$

In the time domain

$$v_3 = 2.33\cos(5t - 80°) = 2.33\sin(5t + 10°)\ \text{V}$$

We have:

$$v_o(t) = -1 + 2.498\cos(2t - 30.79°) + 2.33\sin(5t + 10°)\ \text{V}$$

2. THEVENIN'S THEOREM

Thevenin's theorem states that any two-terminal linear dc network can be replaced by an equivalent circuit consisting of a voltage source, V_{TH} and a series resistance R_{TH}.

Thevenin's theorem is applied to ac circuits in the same way as they are to dc circuits.

Thevenin's Theorem

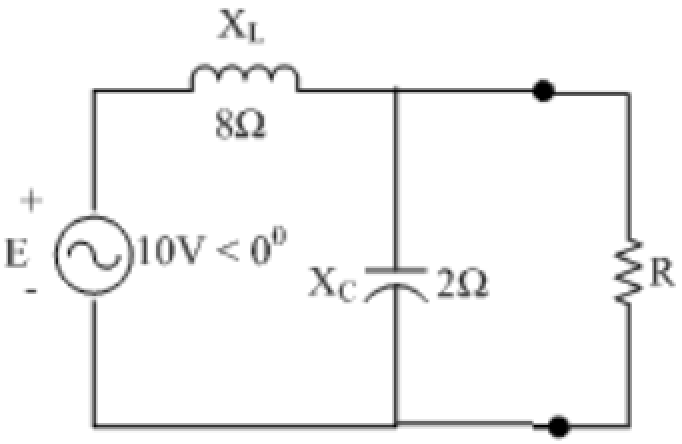
- The only change is replacement of the term resistance with impedance.

- Unlike the superposition, it is applicable to only one frequency since reactance is frequency dependent.

- Since the impedances are frequency dependent (X_L and X_C), a single Thevenin equivalent circuit cannot be created with multiple frequencies.

Example 1

Find the Thevenin equivalent circuit for the network external to R



Solution

With R disconnected, the circuit becomes:

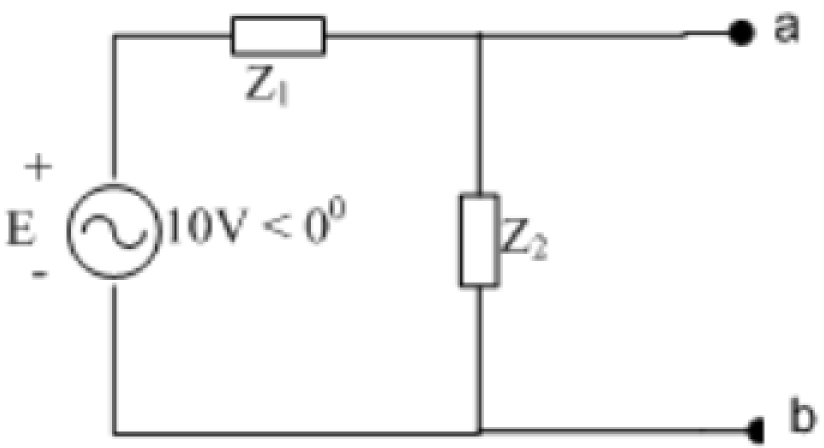
Where $Z_1 = j8$ and $Z_2 = -j2$

Using voltage divider rule:

$$E_{TH} = V_{ab} = \frac{Z_2}{Z_1 + Z_2} \times E$$

$$= -j2 \times 10\angle 0° = \frac{-j2\ 0}{j6}\ V$$

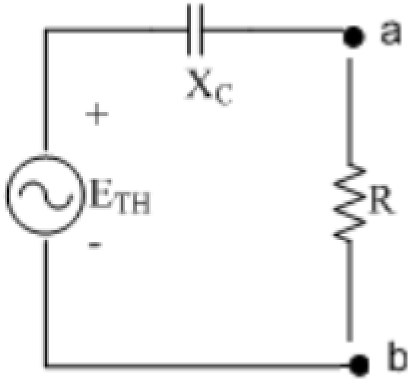
$$= 3.33\ V\angle 180°$$



$$Z_{TH} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(j8)(-j2\ )}{(j8\ -j2)} = \frac{16}{j6}$$

$$= \frac{16}{6\angle 90°} = 2.67\ \Omega\ \angle{-90°}$$

The Thevenin Equivalent Circuit is shown below:

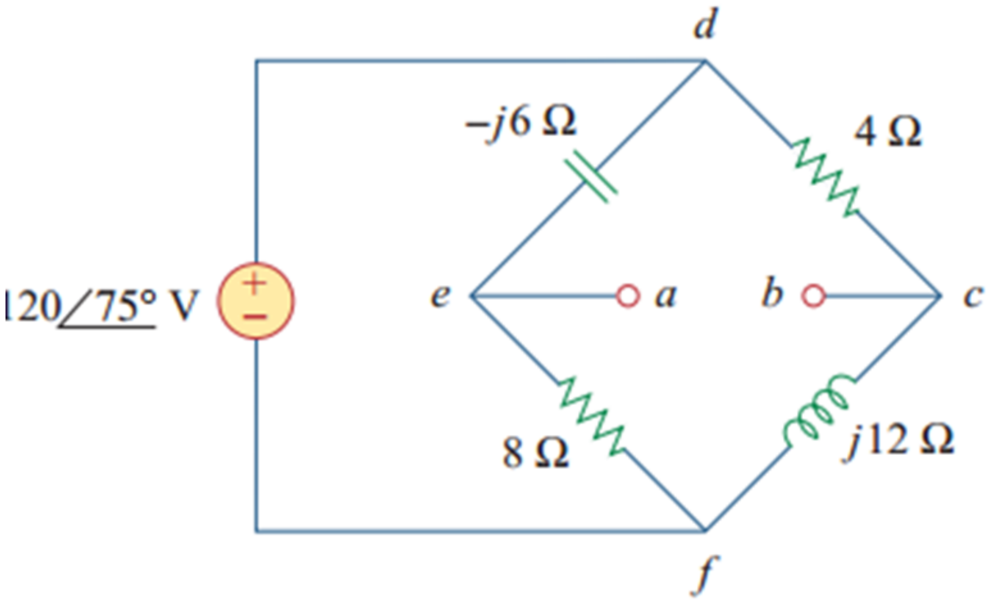


$E_{TH} = 3.33\ V\ \angle 180°$

$X_c = Z_{TH} = 2.67\ \Omega\ \angle{-90°}$

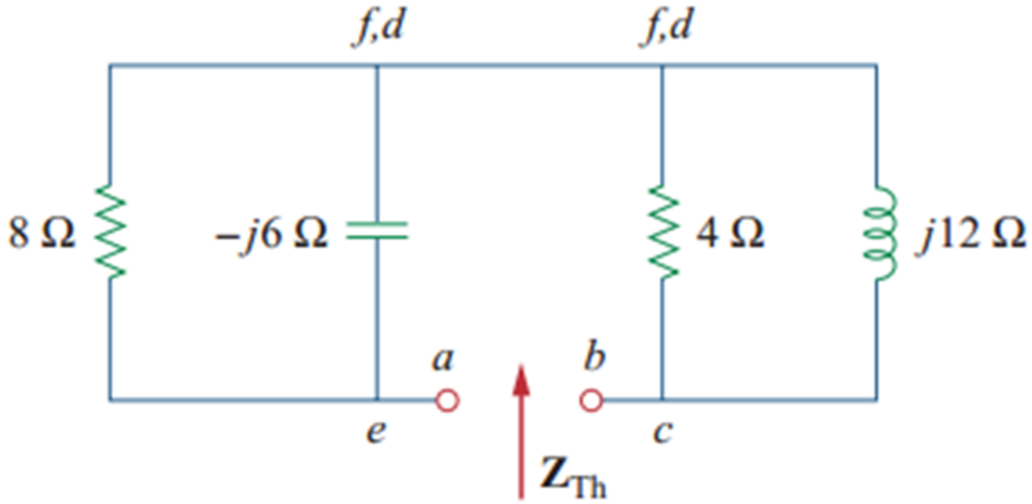
Example 2

Obtain the Thevenin equivalent at terminals a-b of the circuit below:



Solution

We find Z_{TH} by setting the voltage source to zero as shown in the circuit below.
The 8-Ω is now in parallel with the –j6 resistance.



The parallel resistors combined to:

$$Z_1 = j6||8 = \frac{-j6 \times 8}{8 - j6} = 2.88 – j3.84\ \Omega$$

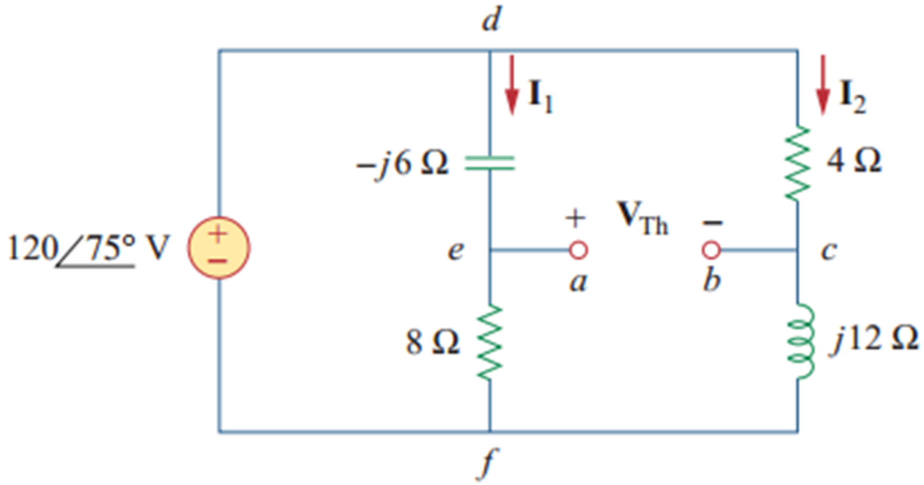
Similarly the 4-Ω resistance is in parallel with the j12 resistance, and their combination gives:

$$Z_2 = 4||12 = \frac{4 \times j12}{4 + j12} = 3.6 + j1.2\ \Omega$$

The Thevenin impedance is the series combination of Z_1 and Z_2. That is:

$$Z_{TH} = Z_1 + Z_2 = 6.48 – j2.64\ \Omega$$

To find V_{TH}, consider the circuit below:



Currents I_1 and I_2 are obtained as:

$$I_1 = \frac{120\ \angle 75°}{8 - j6}\ \text{A}$$

and

$$I_2 = \frac{120\ \angle 75°}{4 + j12}\ \text{A}$$

Applying KVL around bcdeab:

$$V_{TH} - 4I_2 + (-j6)I_1 = 0$$

or

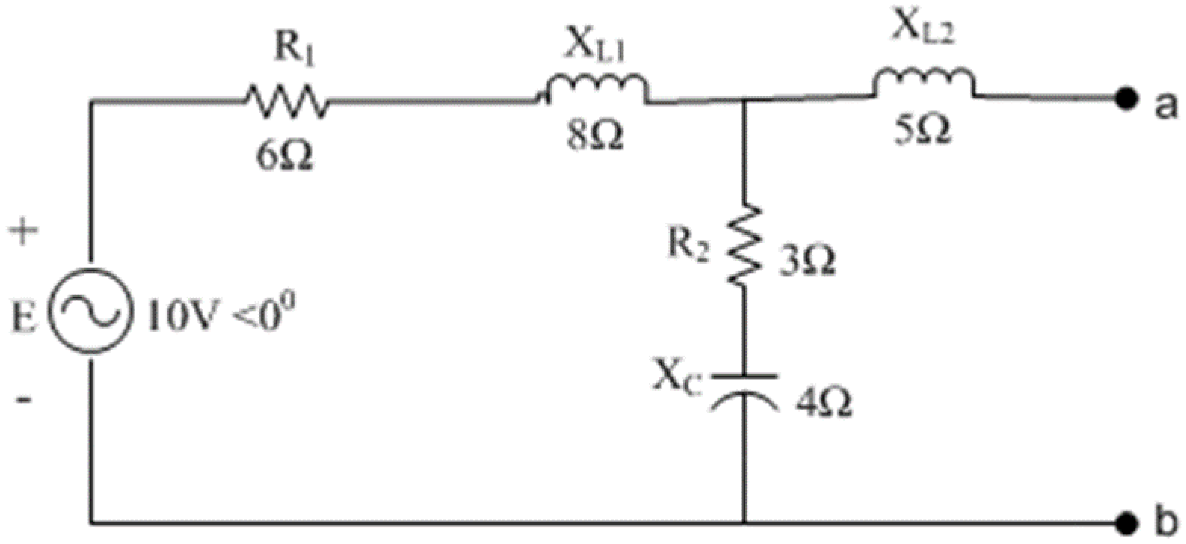
$$V_{TH} = 4I_2 + j6I_1 = \frac{480\ \angle 75°}{4 + j12} + \frac{720\ \angle 75°+90°}{8 - j6}$$

$$V_{TH} = 37.95\ \angle 3.43° + 72\ \angle 201.87° = -28.936 - j24.55$$

$$V_{TH} = 37.92V\ \angle 220.31°$$

Example 3

Find the Thevenin equivalent circuit as seen at terminals a-b of the circuit below:



Solution

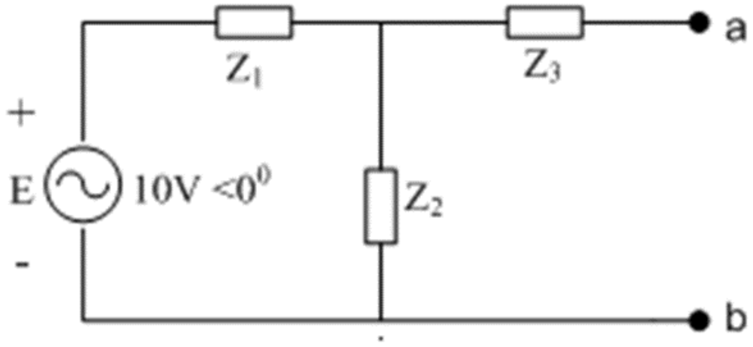
Let

$Z_1 = R_1 + jX_{L1} = 6 + j8 = 10\ \Omega\ \angle 53.13°$

$Z_2 = R_2 - jXc = 3 - j4 = 5\ \Omega\ \angle{-53.13°}$

$Z_3 = jX_{L2} = j5$

With these impedances, the circuit becomes:



Thevenin's Resistance:

$$Z_{TH} = Z_3 + \frac{Z_1 Z_2}{Z_1+Z_2}$$

$$= j5 + \frac{10\ \angle 53.13° \times 5\ \angle{-53.13°}}{(6+j8) + (3 - j4)}$$

$$= j5 + \frac{50\angle 0°}{9 + j4} = \frac{50\angle 0°}{9.85\ \angle 23.96°} + j5$$

$$= 4.64 + j2.94 = 5.49\Omega\ \angle 32.36°$$

Thevenin's Voltage:

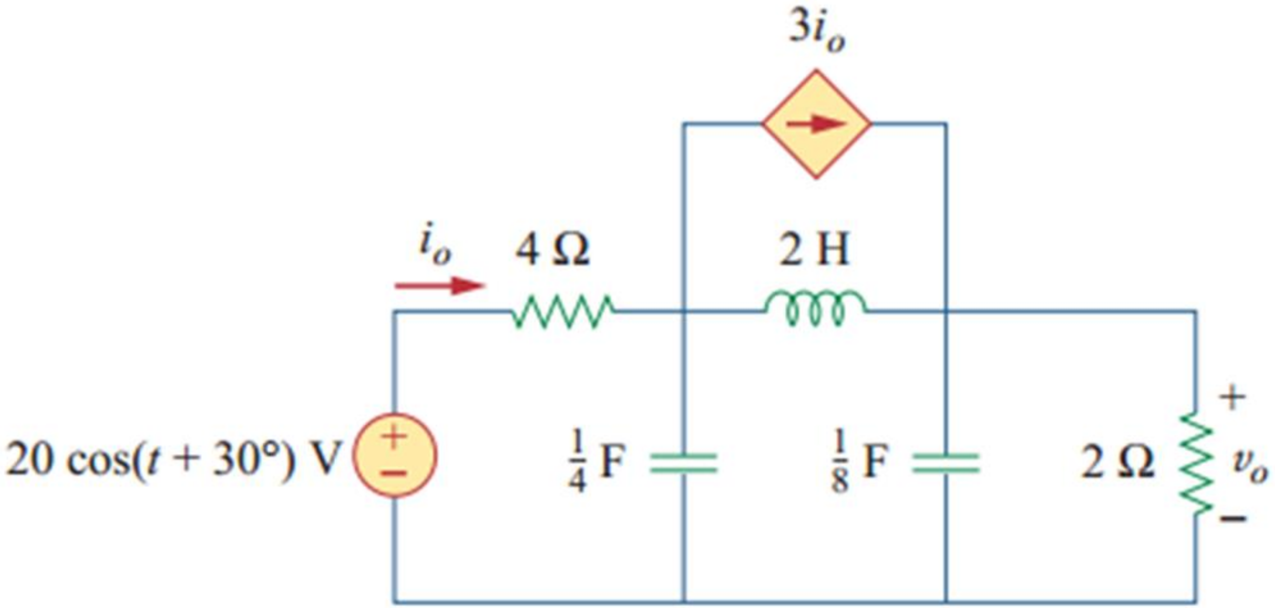
$$E_{TH} = \frac{Z_2}{Z_1+Z_2} \times E$$

$$= \frac{10\ \angle 0° \times 5\ \angle{-53.13°}}{9.85\ \angle\ 23.96°}$$

$$= 5.08\text{V}\ \angle{-77.09°}$$

Practice Question

Using Thevenin's theorem, find v_o in the circuit below



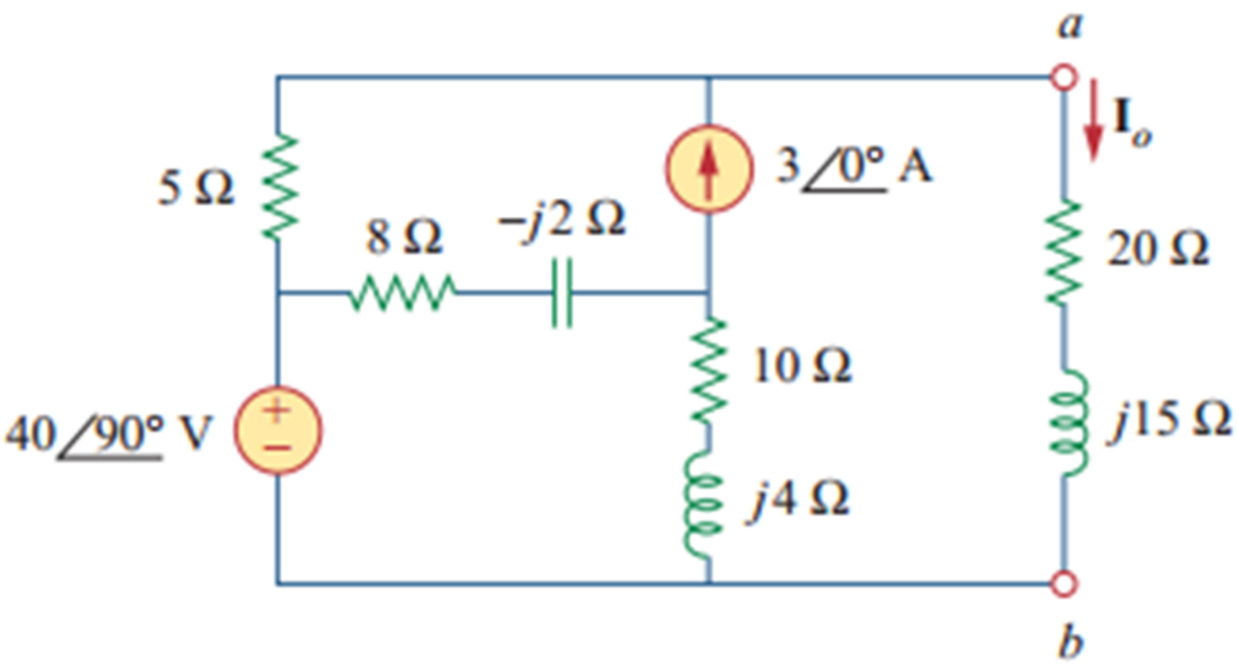
3. Norton's Theorem

Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N.

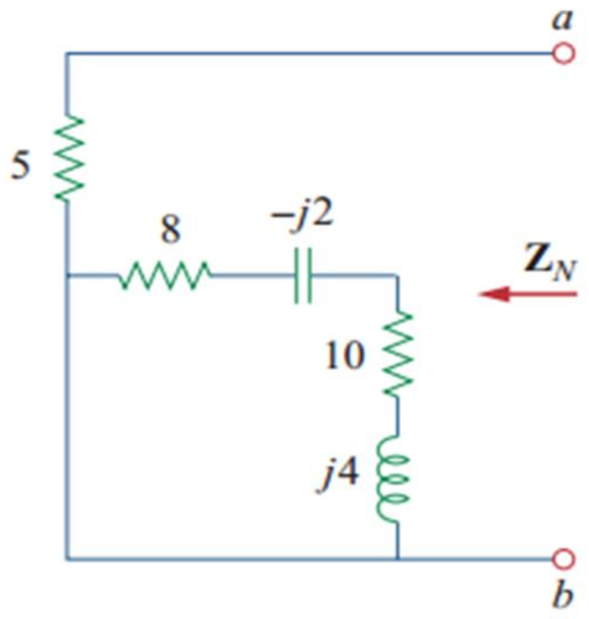
Here too, resistance is replaced by impedance.
It is applicable to only one frequency since reactance is frequency dependent.

Example 1

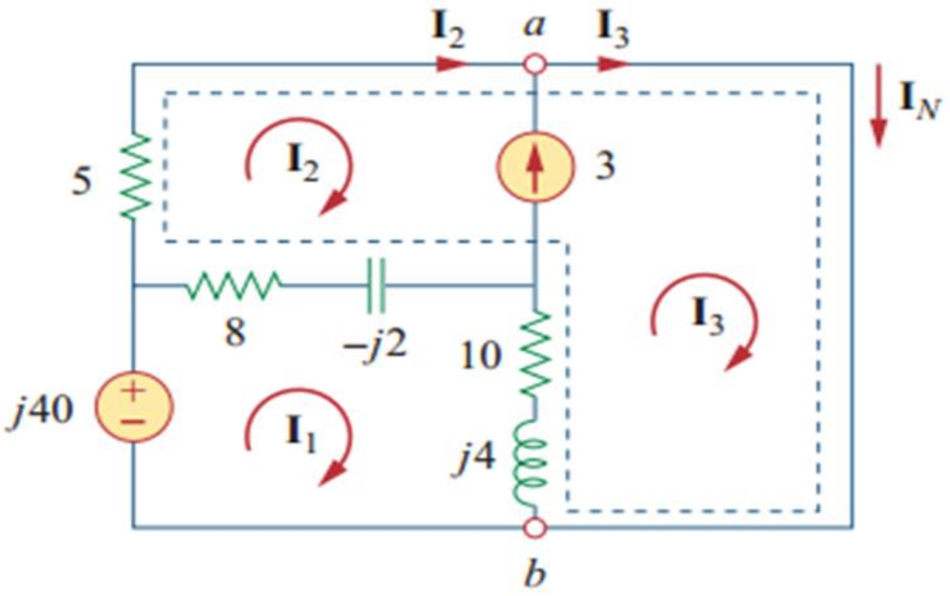
Obtain I_o in the figure below using Norton's theorem



- Z_N is found in the same way as Z_{TH}
- We set the sources to zero as shown in the circuit below



- As evident from the figure, the (8 - j2) and (10 + j4) impedances are short-circuited, so that $Z_N = 5\Omega$
- To get I_N, we short circuit terminal a-b, and apply mesh analysis to the circuit below:



Notice that meshes 2 and 3 form a supermesh because of the current source linking them

For mesh 1

$$-j40 + (18 + j2)I_1 - (8 - j2)I_2 - (10 + j4)I_3 = 0 \qquad (1)$$

For the supermesh

$$(13 - j2)I_2 + (10 + j4)I_3 - (18 + j2)I_1 = 0 \qquad (2)$$

KCL at node give

$$I_3 = I_2 + 3 \qquad (3)$$

Adding (1) and (2) gives

$-40j + 5I_2 = 0,\ I_2 = j8$

From (3)

$I_3 = I_2 + 3 = 3 + j8$

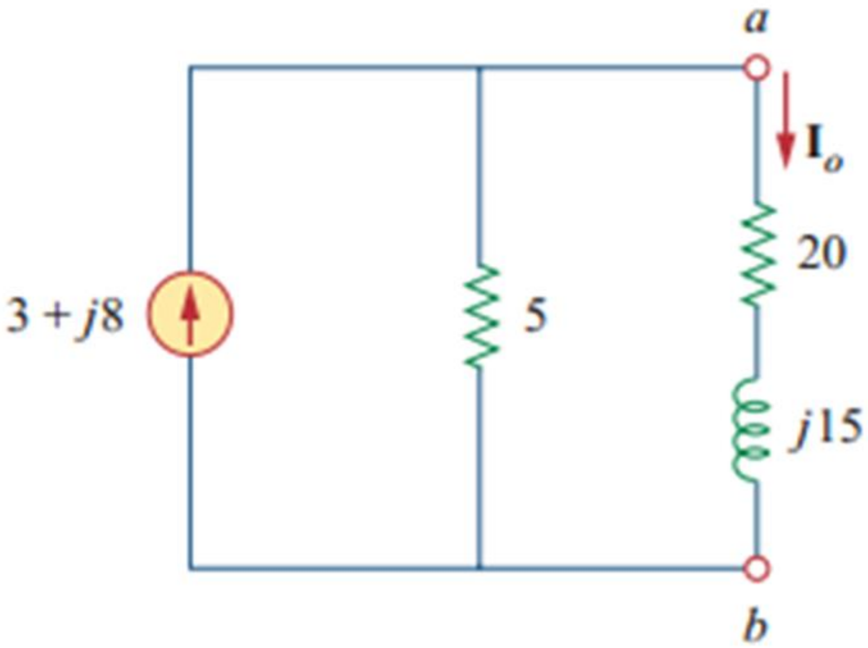
The Norton Current is

$I_N = I_3 = 3 + j8\ \text{A}$

By current division

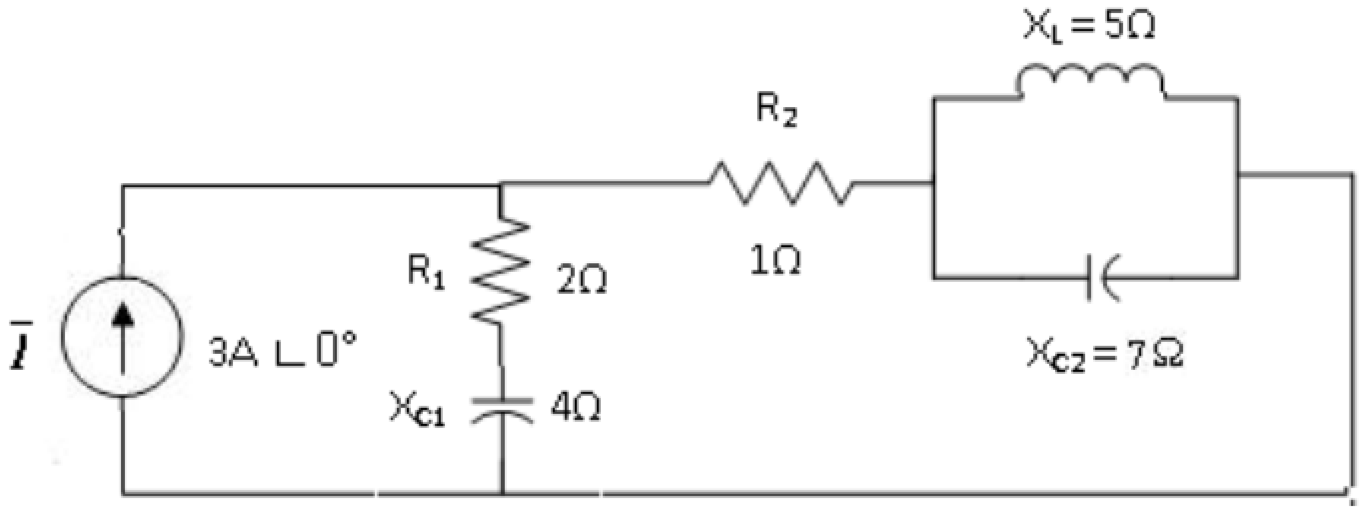
$$I_o = \frac{5}{5+20+j15} \times I_N = \frac{3+j8}{5+j3} = 1.465\ \text{A}\ \angle 38.48°$$

The Norton equivalent circuit along with the impedance at terminal a-b is shown below:

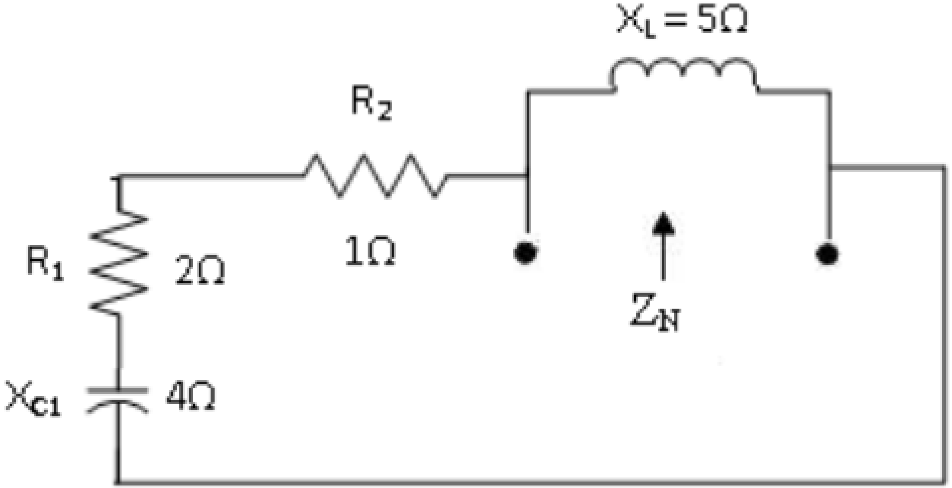


Example 2

Find the Norton equivalent circuit for the network external to the 7-Ω capacitive reactance.



Norton's Resistance



$$Z_N = jX_L \mid\mid (R_1 + R_2 - jX_c)$$

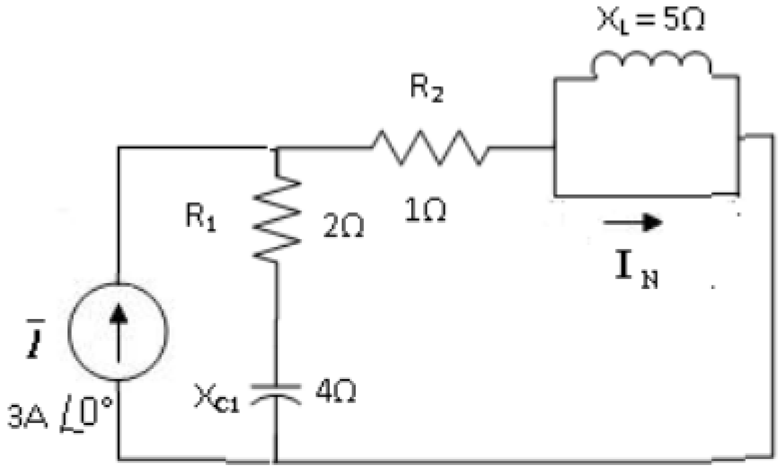
$$= \frac{(j5)(3 - j4)}{3+j}$$

$$= 7.91\ \Omega\ \angle 18.44°$$

$$= 7.50 + j2.50\ \Omega$$

Short-Circuit Current

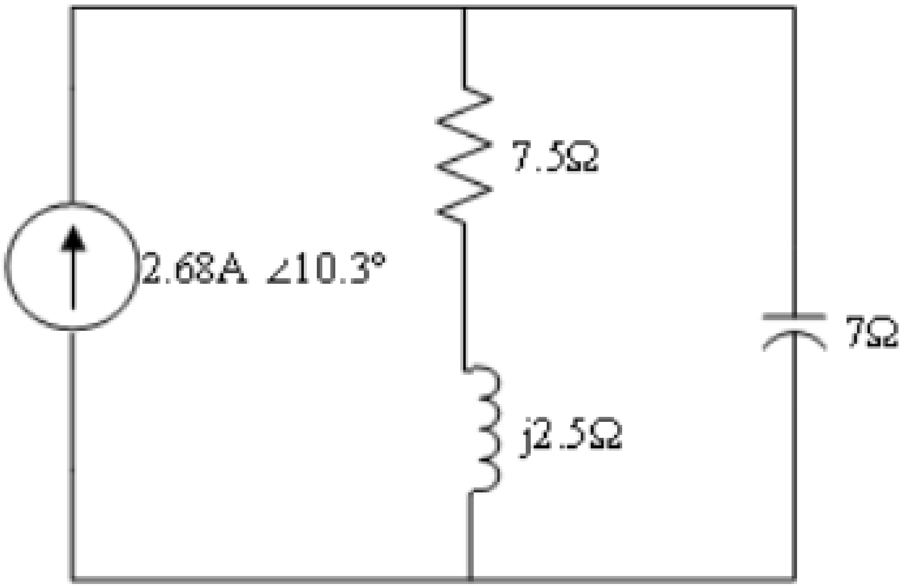
X_L is short–circuited and the parallel branch consists of $(R_1 + jX_{C1})$ // R_2.



Using the current divider rule, the Norton's current = Current through R_2 is given by:

$$I_N = \left[\frac{R_1 - jXC_1}{R_1 + R_2 - jXC_1}\right] \times I = 2.68A \angle 10.3°$$

Equivalent Circuit



2

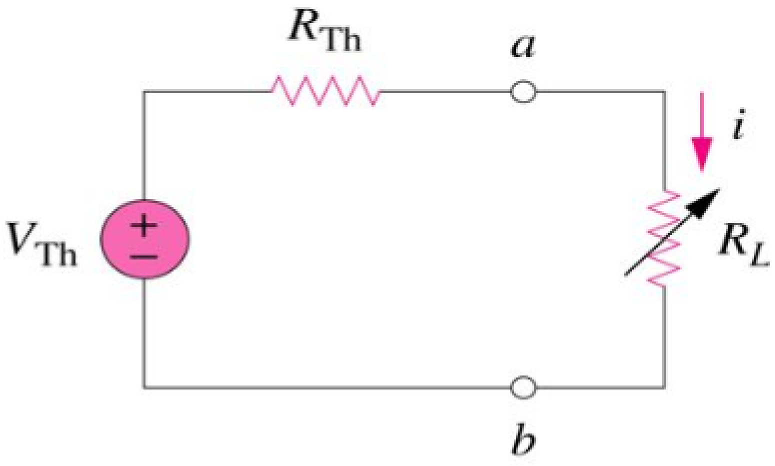
MAXIMUM POWER TRANSFER THEOREM

Maximum Power Transfer Theorem

Maximum Power Transfer Theorem

Maximum Power is transferred from the source to the load when the load impedance is equal to the Thevenin's equivalent impedance.

i.e. $\mathbf{Z_{TH} = Z_L}$



Introduction

- This theorem describes the condition for maximum power transfer from **an active network** to an external load.

- Very often we come across various real time circuits that works based on maximum power transfer theorem.

- To effectively connect a source to a load, an impedance matching transformer is used.

- In case of transmission lines, the distortion and reflections are avoided by making source and load impedances match the characteristic impedance of the line.

Introduction - Cont

- In the case of solar photovoltaic (PV) systems, Maximum Power Point Tracking (MPPT) is achieved with incremental conductance method (ICM) in which **the load resistance** must be equal to the **output resistance of the PV panel and solar cell**.

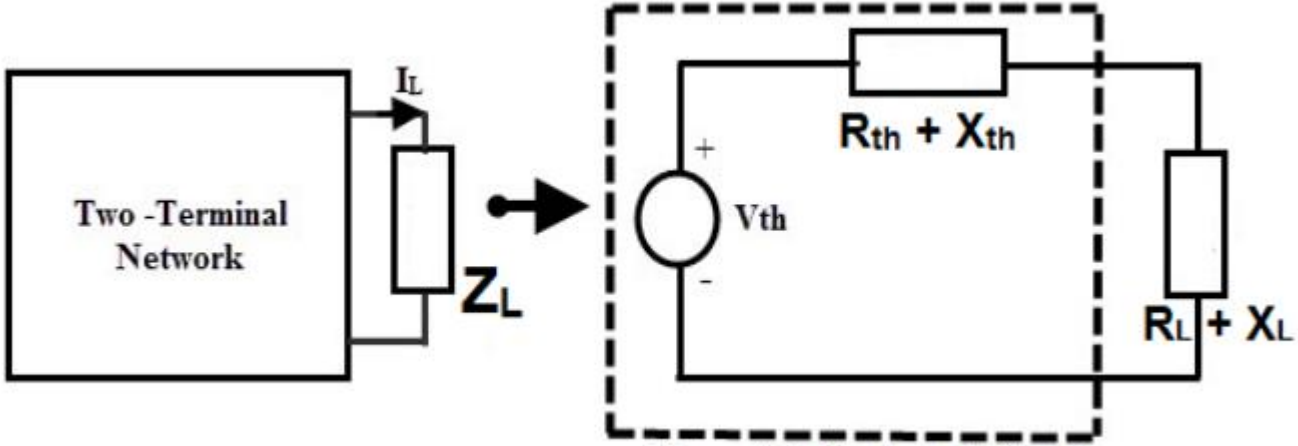
- So there are several cases or applications that use maximum power transfer theorem for effectively connecting the source to a load.

- This theorem can be applied for both DC and AC circuits.

- A network that contains linear impedances and one or more voltage or current sources can be reduced to a Thevenin equivalent circuit as shown before.

- When a load is connected to the terminals of this equivalent circuit, power is transferred from the source to the load.
- A Thevenin equivalent circuit is shown below with source internal impedance,

$$Z_{th} = (R_{th} + jX_{th})\ \Omega \text{ and complex load } Z_L = (R_L + jX_L)\ \Omega$$



Thevenin's Equivalent Circuit

Condition 1

“

The load consists of pure variable resistance, R_L.
Hence, $X_L = 0$

Let the load consist of a pure variable resistance R_L (i.e. let $X_L = 0$). Then current I_L in the load is given by:

$$I_L = \frac{V_{th}}{(R_{th}+RL) + jXth}$$

And the magnitude of current,

$$|I_L| = \frac{V_{th}}{\sqrt{(Rth+RL)^2 + {X_{th}}^2}}$$

The active power, P delivered to load R_L is given by:

$$P = |I_L|^2 R_L = \frac{{V_{th}}^2 R_L}{(R_{th}+RL)^2 + {X_{th}}^2}$$

To determine the value of R_L for maximum power transferred to the load, P is differentiated with respect to R_L and then equated to zero.

Using the quotient rule of differentiation,

$$\frac{\mathrm{dP}}{\mathrm{dR_L}} = \mathrm{V_{th}}^2 \left\{ \frac{\left[(\mathrm{R_{th}}+\mathrm{RL})^2 + \mathrm{X_{th}}^2\right](1) - (\mathrm{RL})(2)(\mathrm{Rth}+\mathrm{RL})}{[(\mathrm{R_{th}}+\mathrm{RL})^2 + \mathrm{X_{th}}^2]^2} \right\} = 0$$

Hence

$$(\mathrm{R_{th}} + \mathrm{RL})^2 \ + \ \mathrm{X_{th}}^2 - 2\ \mathrm{R_L}(\mathrm{Rth} + \mathrm{RL}) = 0$$

$$\mathrm{R_{th}}^2 + 2\mathrm{R_{th}}\mathrm{R_L} + \mathrm{R_L}^2 + \mathrm{X_{th}}^2 - 2\mathrm{R_L}\mathrm{R_{th}} - 2\mathrm{R_L}^2 = 0$$

From which,

$$\mathrm{R_{th}}^2 + \mathrm{X_{th}}^2 = \mathrm{R_L}^2$$

Finally

$$\mathbf{R_L = \sqrt{R_{th}{}^2 \ + \ X_{th}{}^2} = |Z_L|}$$

Thus, with a variable purely resistive load, the maximum power is delivered to the load if the **load resistance R_L** is made equal to the **magnitude of the source impedance**.

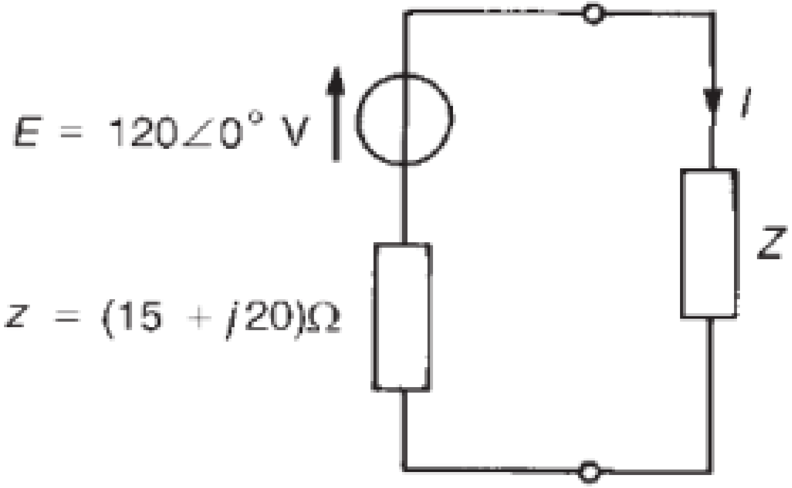
Example 1

For the circuit shown below, the load impedance Z is a pure resistance, R.

Determine

(a) the value of R for maximum power to be transferred from the source to the load.

(b) the value of the maximum power delivered to R.



Solution

(a) From Condition 1

Maximum power transfer occurs when $R = |Z|$

$R = 15 + j20 = \sqrt{15^2 + 20^2} = 25\ \Omega$

(b) Current I flowing in the load is given by $I = \frac{E}{Z_T}$

Where the total circuit impedance $Z_T = Z + R = 15 + j20 + 25 = (40 + j20)$

$Z_T = 44.72\ \angle 26.57°\ \Omega$

Hence $I = \frac{120\ \angle 0°}{44.72\ \angle 26.57°} = 2.683A\ \angle{-26.57°}$

Thus Maximum Power Delivered, $P = I^2R = (2.683)^2(25) = 180W$

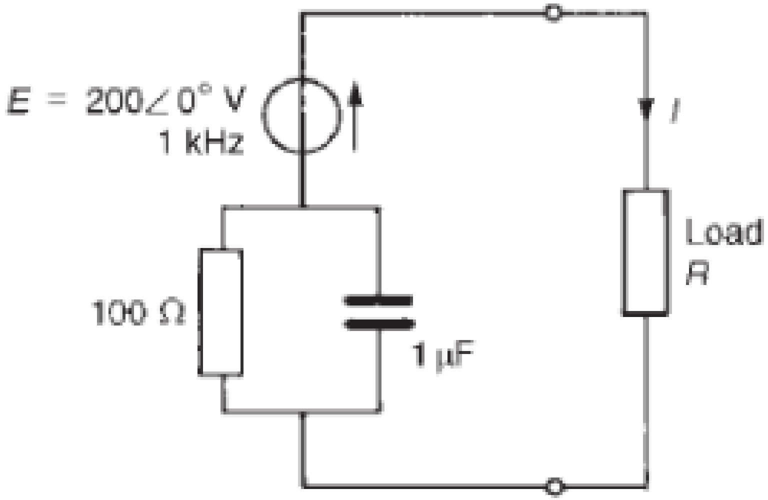
Example 2

For the network shown below,

Determine

(a) the value of the load resistance R required for maximum power transfer.

(b) the value of the maximum power transferred.



Solution

(a) This problem is an example of condition 1, where maximum power transfer is achieved when $R_L = Z_{th}$.

Source impedance Z_{th} is composed of a 100 Ω resistance in parallel with a 1 uF capacitor.

Capacitive reactance, $X_C = \dfrac{1}{2\pi fC} = \dfrac{1}{2\pi(1000)(1 \times 10^{-6})} = 159.15\ \Omega$

Hence source impedance,

$$Z_{th} = \frac{(100)(-j159.15)}{(100 - j159.15)} = \frac{159.15 \angle -90^\circ}{187.96 \angle -57.86^\circ} = 84.67\ \Omega \angle -32.14^\circ$$

Thus the value of load resistance for maximum power transfer is 84.67 Ω

(b) With $Z_{th} = 84.67\ \Omega\ \angle{-32.14°}$ and $R = 84.67\ \Omega$ for maximum power transfer, the total circuit impedance,

$$Z_T = 71.69 + 84.67 - j45.04 = (156.36 - j45.04)\ \Omega \text{ or } 162.72\ \Omega\ \angle{-16.07°}$$

Current flowing in the load

$$I = \frac{E}{Z_T} = \frac{200\ \angle 0°}{162.72\ \angle{-16.07°}} = 1.23\text{A}\ \angle 16.07°$$

Thus the maximum power transferred

$$P = I^2R = (1.23)^2(84.67) = 128\text{W}$$

Condition 2

"

The load and source impedance are purely resistive

Hence, $X_L = X_{th} = 0$

Let both the load and the source impedance be purely resistive (i.e. $X_{th} = X_L = 0$)

From ${R_{th}}^2 + {X_{th}}^2 = {R_L}^2$

Maximum Power Transfer Equation becomes, $\mathbf{R_{th} = R_L}$

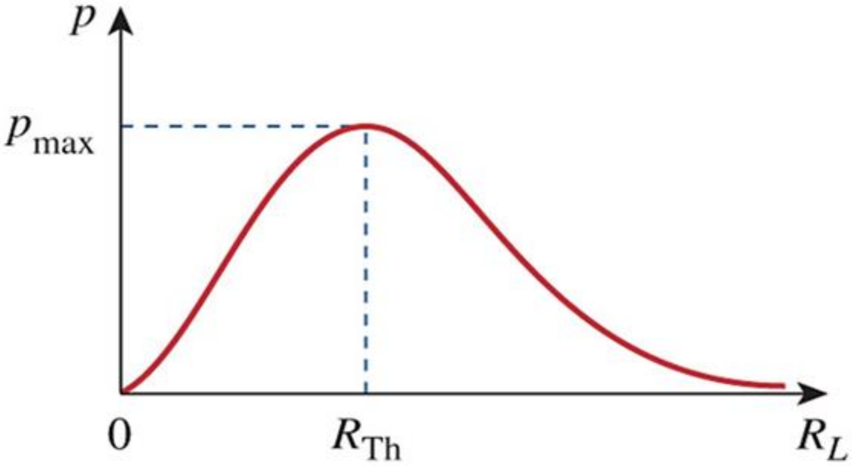
Under this condition, the maximum power delivered to the load is

$$P_{max} = \frac{{V_{th}}^2}{4RL}$$

The figure on the next slide shows a curve of power delivered to the load with respect to the load resistance.

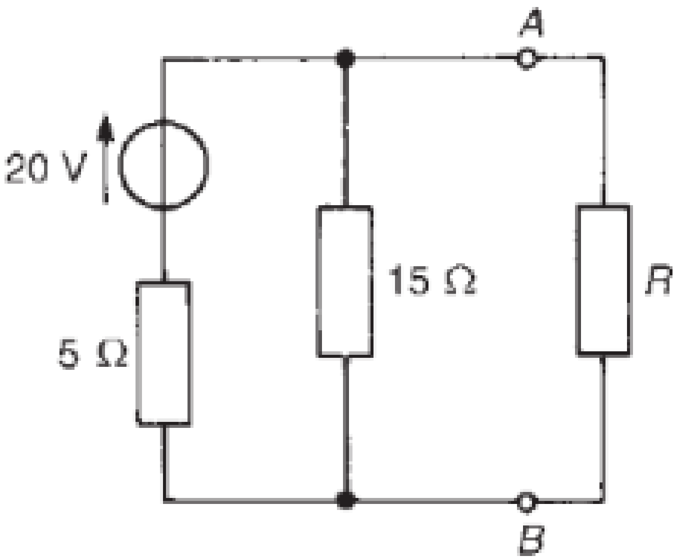
Note

- The power delivered is zero when the load resistance is zero as there is no voltage drop across the load during this condition.
- Also, the power will be maximum, when the load resistance is equal to the internal resistance of the circuit (or Thevenin's equivalent resistance).
- Again, the power is zero as the load resistance reaches to infinity as there is no current flow through the load.



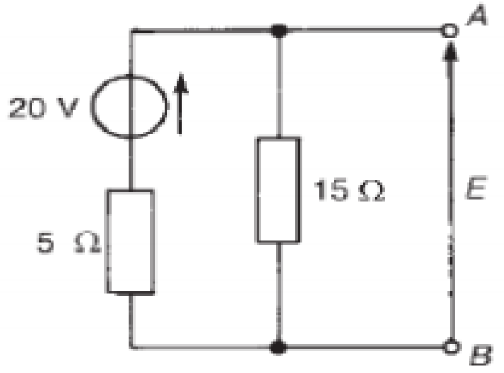
Example 1

Determine the value of the load resistance R that gives maximum power dissipation and calculate the value of this power.



Solution

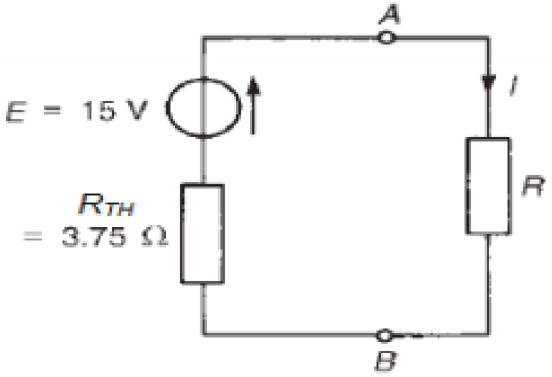
Using Thevenin's Theorem, R is removed from the network as shown:



- Voltage across AB, $E = \frac{15}{15+5} \times 20 = 15V$

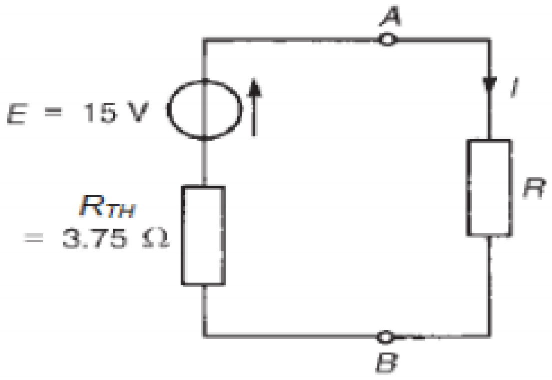
- Impedance looking in at terminals AB with the 20V source removed is given by:

$$R_{th} = \frac{15}{15+5} \times 15 = 3.75\ \Omega$$



- The equivalent Thevenin circuit supplying terminals AB is shown.
- From condition 2, for maximum power transfer,

$$R = R_{th} = 3.75\ \Omega$$



Current

$$I = \frac{E}{R + Rth} = \frac{15}{3.75 + 3.75}$$

$$I = 2A$$

Maximum Power dissipated in the load

$$P = I^2 R = (2)^2 (3.75)$$

$$P = 15W$$

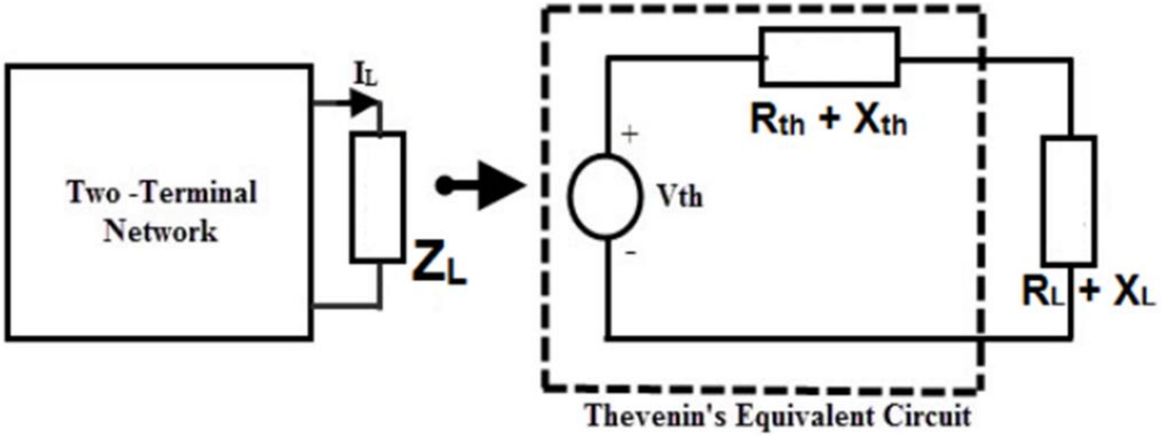
Condition 3

"

The load impedance Z_L has both variable resistance R_L and variable reactance X.

Let the load Z_L have both variable resistance R_L and variable reactance X.

From the diagram,



$$\text{Current, } I_L = \frac{V_{th}}{(R_{th}+RL)+j(X_{th}+XL)}$$

$$|I_L| = \frac{V_{th}}{\sqrt{(Rth+RL)^2+(Xth+XL)^2}}$$

The active power P delivered to the load is given by:

$$P = |I_L|^2 R_L = \frac{{V_{th}}^2 R_L}{(R_{th}+RL)^2 + (X_{th}+XL)^2}$$

If X_L is adjust such that $X_L = -X_{th}$, then the value of power is a maximum.

If $X_L = -X_{th}$, then

$$P = \frac{{V_{th}}^2 R_L}{(R_{th}+RL)^2}$$

For maximum value,

$$\frac{dP}{dR_L} = {V_{th}}^2 \left\{ \frac{\left[(R_{th}+RL)^2\right](1) - (RL)(2)(Rth+RL)}{(R_{th}+RL)^4} \right\} = 0$$

Hence

$$(R_{th} + RL)^2 - 2\, R_L(Rth + RL) = 0$$

i.e.

$$R_{th}^{2} + 2RLRth + R_{th}^{2} - 2RthRL - 2RL^{2} = 0$$

From which,

$$R_{th}^{2} - R_{L}^{2} = 0 \qquad \text{and} \qquad R_L = R_{th}$$

Thus with the load impedance Z consisting of variable resistance R and variable reactance X, maximum power is delivered to the load when:

$$X_L = -X_{th} \text{ and } R_L = R_{th}$$

i.e. when

$$R_L + jX_L = R_{th} - jX_{th}$$

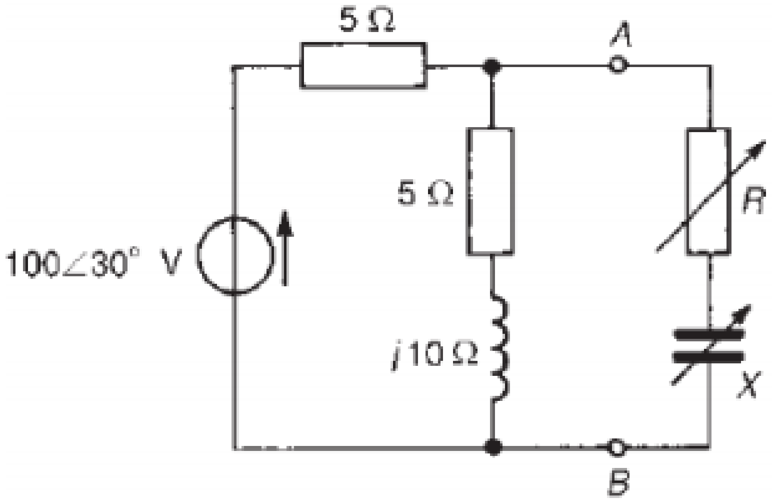
Hence maximum power is delivered to the load when the **load impedance** is the **complex conjugate** of the **source impedance**.

Example 1

Determine, for the network shown below,

(a) The values of R and X that will result in maximum power being transferred across terminals AB

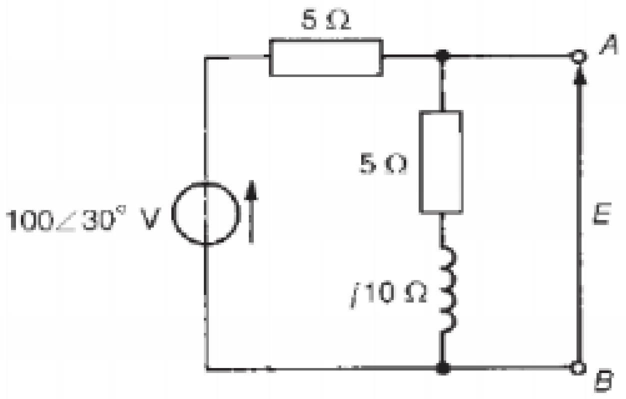
(b) The value of the maximum power.



Solution

(a) Using Thevenin's Theorem:

Resistance R and reactance X are removed from the network as shown



Voltage across AB

$$E = \left(\frac{5+j10}{5+j10+5}\right) \times (100\ \angle 30°)$$

$$E = \frac{(11.18\ \angle 63.43°)}{14.14\ \angle 45°} \times (100\ \angle 30°)$$

$$E = 79.07V\ \angle 48.43°$$

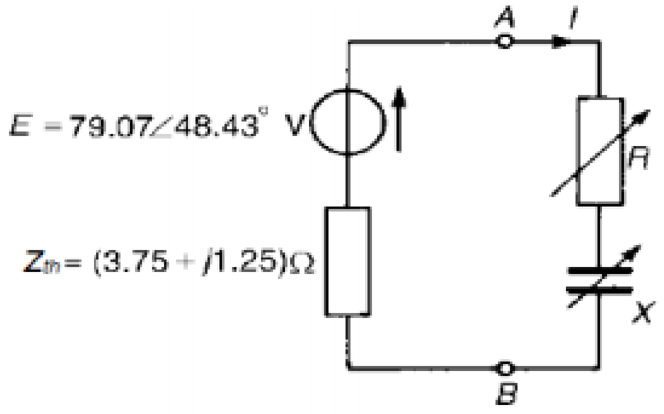
With the 100 ∠30° V source removed, the impedance $Z_{th,}$ looking in at terminals AB is given by:

$$Z_{th} = \frac{(5+j10)(5)}{(5+j10+5)} = \frac{(11.18\ \angle 63.43°)(5)}{(14.14\ \angle 45°)} = 3.953\ \Omega\ \angle 18.43° \text{ or } (3.75 + j1.25)\ \Omega$$

The equivalent Thevenin circuit is shown below.

From condition 3, maximum power transfer is achieved when $X = X_{th}$ and $R = R_{th}$

i.e. in the case when $X = -1.25\ \Omega$ and $R = 3.75\ \Omega$



Current

$$I = \frac{E}{Z_L + ZTH} = \left(\frac{79.07\angle 48.43°}{(3.75 + j1.25) + (3.75 - j1.25)}\right)$$

$$I = \frac{79.07\angle 48.43°}{7.5} = 10.54\ \angle 48.43°$$

Thus the maximum power transferred

$P = I^2R = (10.543)^2(3.75) = 417\text{W}$

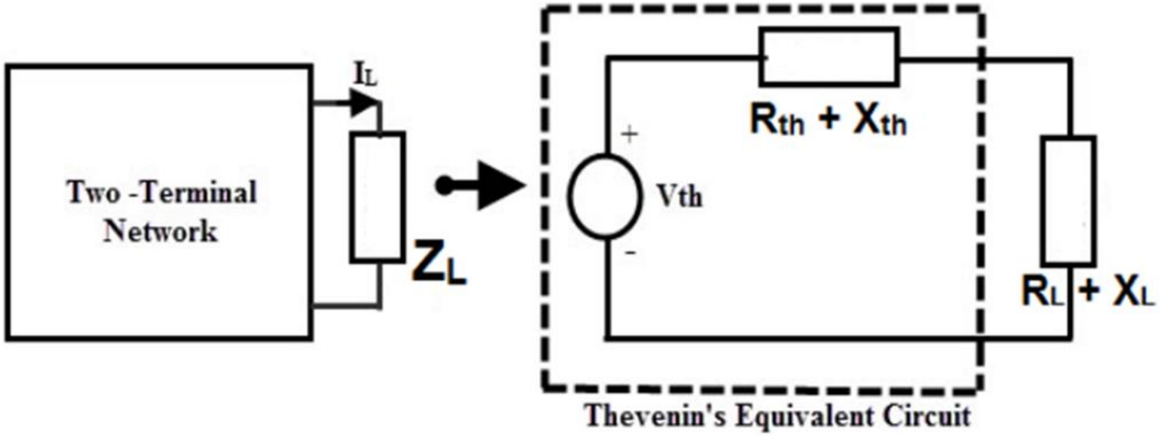
Condition 4

“

The load impedance Z has variable resistance R and fixed reactance X

Let the load Z have both variable resistance R and fixed reactance X.

From the diagram,



$$|I_L| = \frac{V_{th}}{\sqrt{(Rth+RL)^2 + (Xth+XL)^2}}$$

The power dissipated by the load is,

$$P = \frac{{V_{th}}^2 R_L}{(R_{th}+RL)^2 + (Xth+XL)^2}$$

For maximum value

$$\frac{dP}{dR_L} = V_{th}{}^2 \left\{ \frac{\left[(R_{th}+RL)^2+(Xth+XL)^2\right] - (RL)(2)(Rth+RL)}{[(R_{th}+RL)^2+(Xth+XL)^2]^2} \right\} = 0$$

Hence

$$(R_{th} + RL)^2 + (Xth + XL)^2 - 2R_L(Rth + RL) = 0$$

$$R_{th}{}^2 + 2RLRth + R_L{}^2 + (Xth + XL)^2 - 2RthRL - 2RL^2 = 0$$

From which

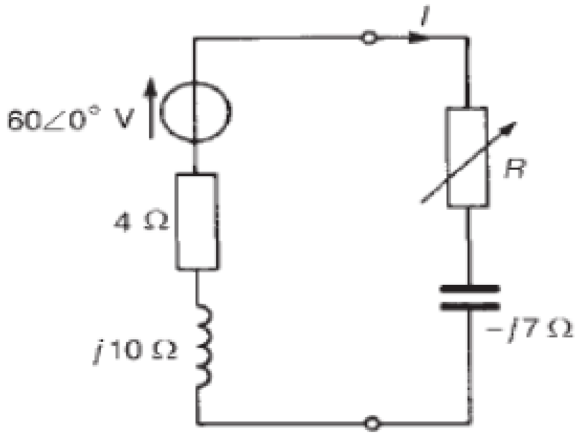
$$R_L{}^2 = R_{th} + (Xth + XL)^2$$ and $$R_L = \sqrt{R_{th}{}^2 + (Xth + XL)^2}$$

Example 1

In the network shown below, the load consists of a fixed capacitive reactance of 7 Ω and a variable resistance R. Determine

(a) The value of R for which the power transferred to the load is a maximum

(b) The value of the maximum power



Solution

From condition 4

Maximum power transfer is achieved when

$$R_L = \sqrt{{R_{th}}^2 \ + (Xth + XL)^2} = \sqrt{4^2 + (10 \ - 7)^2} = 5\Omega$$

Current

$$I = \frac{60 \angle 0°}{(4+j10) + (5 - j7)} = \frac{60 \angle 0°}{(9 + j3)} = \frac{60 \angle 0°}{9.487\angle 18.43°} = 6.324A \angle -18.43°$$

Thus the maximum power transferred,

$$P = I^2 R = 6.324^2(5) = 200W$$

Summary

- When the load is purely resistive (i.e., $X_L = 0$) and adjustable, maximum power transfer is achieved when $\mathbf{R_L = Z_{TH} = \sqrt{R_{th}{}^2 \ + \ X_{\mathit{th}}{}^2}}$

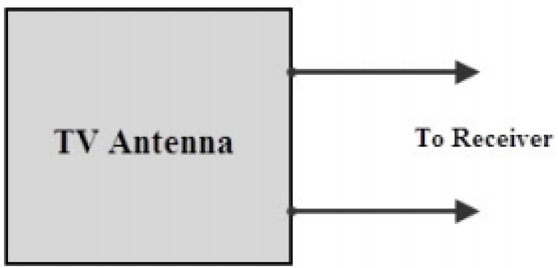
- When both the load and the source impedance are purely resistive (i.e., $X_L = X_{th} = 0$), maximum power transfer is achieved when $\mathbf{R_L = R_{TH}}$

- When the load resistance R_L and reactance X_L are both independently adjustable, maximum power transfer is achieved when the load impedance is the complex conjugate of the source impedance.

- When the load resistance R_L is adjustable with reactance X_L fixed, maximum power transfer is achieved when $\mathbf{R_L = \sqrt{R_{th}{}^2 \ + (X_{th} + X_L)^2}}$

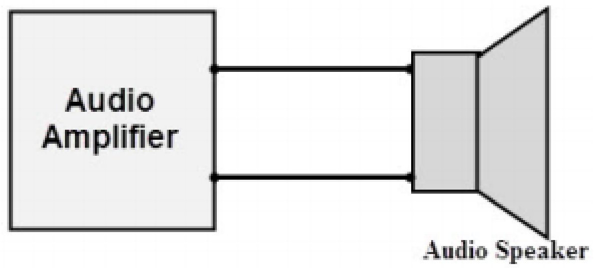
Applications of Maximum Power Transfer Theorem

- In electronic circuits, especially in communication system the signal present at the receiving antenna is of low strength. In order to receive the maximum signal from the antenna, impedance of (TV) receiver and (TV) antenna should be matched.



- In an audio amplifier with audio speaker arrangement in public addressing systems, speaker resistance must be equal to the amplifier resistance in order to transfer maximum power from amplifier to the speaker.

Applications of Maximum Power Transfer Theorem - Cont



- In case of a car engine starting system, starter motor resistance must be matched with internal resistance of the battery. If the battery if full and these resistances are matched, maximum power will be transferred to the motor to turn ON the engine.

3
IMPEDANCE MATCHING

Introduction

- It is seen from the previous section that when it is necessary to obtain the maximum possible amount of power from a source, it is advantageous if the circuit components can be adjusted to give equality of impedances.

- This adjustment is called **impedance matching** and is an important consideration in electronic and communications devices which normally involve small amounts of power.

- Examples where matching is important include coupling an aerial to a transmitter or receiver, or coupling a loudspeaker to an amplifier.

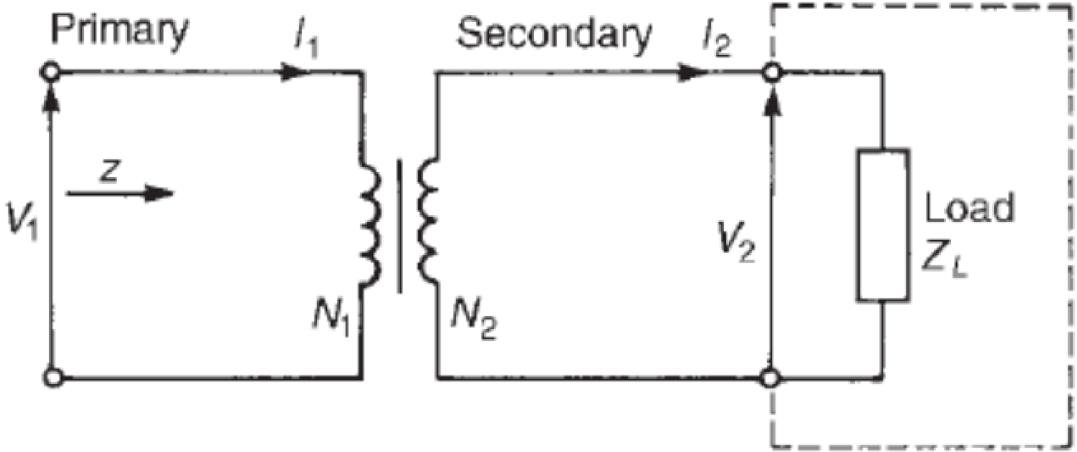
Introduction - Cont

- The mains power supply is considered as infinitely large compared with the demand upon it, and under such conditions it is unnecessary to consider the conditions for maximum power transfer.

- With d.c. generators, motors or secondary cells, the internal impedance is usually very small and in such cases, if an attempt is made to make the load impedance as small as the source internal impedance, overloading of the source results.

- A method of achieving maximum power transfer between a source and a load is to adjust the value of the load impedance to match the source impedance, which can be done using a '**matching transformer**'.

- A transformer is represented in the next slide supplying a load impedance $Z_{L.}$



Small transformers used in low power networks are usually regarded as ideal (i.e., losses are negligible), such that

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = \frac{I_2}{I_1}$$

From the figure, the primary input impedance Z is given by:

$$|Z| = \frac{V_1}{I_1} = \frac{(N_1/N_2)V_2}{(N_2/N_1)I_2} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_2}$$

Since the load impedance $|Z_L| = \frac{V_2}{I_2}$

$$= \left(\frac{N_1}{N_2}\right)^2 Z_L$$

If the input impedance of the figure above is purely resistive (say, r) and the load impedance is purely resistive (say, R_L), then the above equation becomes

$$r = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Thus by varying the value of the transformer turns ratio, the equivalent input impedance of the transformer can be 'matched' to the impedance of a source to achieve maximum power transfer.

Example 1

Determine the optimum value of load resistance for maximum power transfer if the load is connected to an amplifier of output resistance 448 Ω through a transformer with a turns ratio of 8:1.

Solution

The equivalent input resistance r of the transformer must be 448Ω for maximum power transfer.

From the equation, $$r = \left(\frac{N_1}{N_2}\right)^2 R_L$$

The load resistance is determined as:

$$R_L = r\left(\frac{N_2}{N_1}\right)^2 = 448\left(\frac{1}{8}\right)^2 = 7\ \Omega$$

Example 2

A generator has an output impedance of $450 + j60\ \Omega$. Determine the turns ratio of an ideal transformer necessary to match the generator to a load of $40 + j19\ \Omega$ for maximum transfer of power.

Solution

Let the output impedance of the generator be z, where

$z = (450 + j60)\Omega$ or $453.98\ \angle 7.59°\ \Omega$

and the load impedance be Z_L, where $Z_L = (40 + j19)\Omega$ or $44.28\ \angle 25.41°\ \Omega$.

$|Z| = \left(\frac{N_1}{N_2}\right)^2 Z_L$, it implies transformer turns ratio

$$\left(\frac{N1}{N2}\right) = \sqrt{\frac{z}{Z_L}} = \sqrt{\frac{453.98}{44.28}} = \sqrt{(10.25)} = 3.20$$

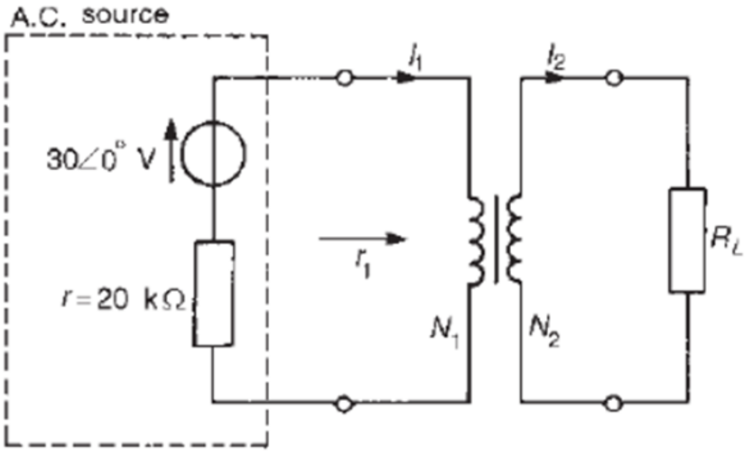
Example 3

An ac. source of 30 $\angle 0°$ V and internal resistance 20 kΩ is matched to a load by a 20:1 ideal transformer. Determine for maximum power transfer

(a) the value of the load resistance, and

(b) the power dissipated in the load.

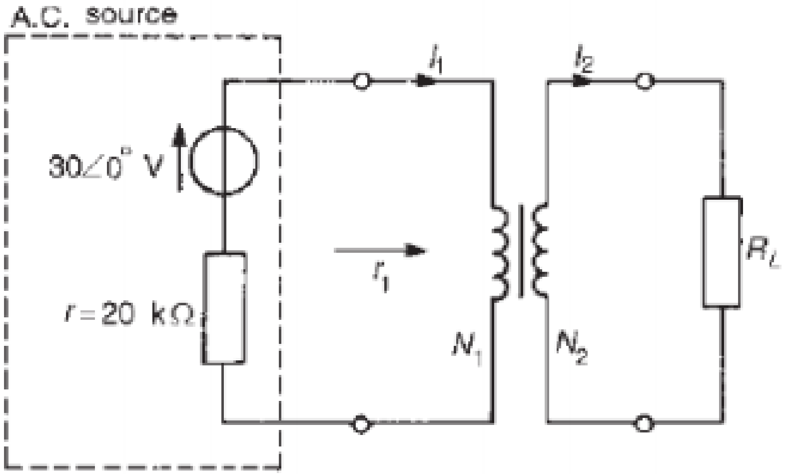
Solution



(a) For maximum power transfer, r_1 must be equal to 20 kΩ. From impedance matching equation, $r_1 = \left(\frac{N_1}{N_2}\right)^2 R_L$ from which, load resistance

$$R_L = r_1\left(\frac{N_1}{N_2}\right)^2 = (20000)\left(\frac{1}{20}\right)^2 = 50\Omega$$

(b) The total input resistance when the source is connected to the matching transformer is $(r + r_1)$, i.e., $20\ k\Omega + 20\ k\Omega = 40\ k\Omega$



Primary Current

$$I_1 = \frac{V}{4000} = \frac{30}{40000} = 0.75\ mA$$

$\frac{N_1}{N_2} = \frac{I_2}{I_1}$ from which $I_2 = I_1\left(\frac{N_1}{N_2}\right) = (0.75 \times 10^{-3})\left(\frac{20}{1}\right) = 15\ mA$

Power dissipated in load resistance R_L is given by:

$$P = I_2{}^2 R_L = (15 \times 10^{-3})^2 (50) = 0.75W$$

Thanks!

Any questions?