Note minor changes of syllabus and notation between 2003 and 2004. These solutions use the old notation.

Linear Algebra I exam 2003 solutions

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SECTION A

A1. U is a subspace of V if $0_V \in U$ and $au_1 + bu_2 \in U$ $\forall$ $u_1, u_2 \in U$, $a, b \in \mathbb{F}$.

(a) Any plane through the origin of $\mathbb{R}^3$ is a subspace, eg the (x,y)-plane $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$.

(b) $0_V = (0,0) \in U$.

If $u_1, u_2 \in U$ then $u_1 = u_2 = (0,0)$ and $au_1 + bu_2 = a(0,0) + b(0,0) = (0,0) \in U$ $\forall$ $a, b \in \mathbb{R}$. Therefore U is a subspace of V.

A2. (a) Spanning set but not linearly independent [S clearly contains the standard basis for $\mathbb{R}^3$ but 4 vectors in $\mathbb{R}^3$ cannot be lin. ind.]

(b) Not spanning and not linearly independent. [$(2,0,4) + 4(0,1,0) = 2(1,2,2)$.]

(c) Not spanning but linearly independent. [2 vectors cannot span $\mathbb{R}^3$.]

A3. (a) Linearly independent if

$$\alpha(2+a, 1, 3)^T + \beta(b, b, -1)^T + \gamma(0, a, 0) = 0,$$

$$\alpha, \beta, \gamma \in \mathbb{R}, \Rightarrow \alpha = \beta = \gamma = 0$$

i.e.

$$\left.\begin{aligned} \alpha(2+a) + \beta b &= 0 \\ \alpha + \beta b + \gamma a &= 0 \\ 3\alpha - \beta &= 0 \end{aligned}\right\} \Rightarrow \beta = 3\alpha$$

Hence

$$\left.\begin{aligned} \alpha(2+a+3b) &= 0 \\ \alpha(1+3b) + \gamma a &= 0 \end{aligned}\right\}$$

If $2 + a + 3b \neq 0$ then $\alpha = 0 \Rightarrow \beta = 0$

and if $a \neq 0$ then $\gamma = 0$.

Hence the vectors are linearly dependent if
$2 + a + 3b = 0$ or $a = 0$.

(b) Linearly independent if
$\alpha(1, a, b, c) + \beta(0, 0, 2, d) + \gamma(0, 0, 0, 3) = 0$,
$\alpha, \beta, \gamma \in \mathbb{R}$, $\Rightarrow \alpha = \beta = \gamma = 0$

i.e.
$$\left.\begin{aligned} \alpha &= 0 \\ \alpha a &= 0 \\ \alpha b + 2\beta &= 0 \\ \alpha c + \beta d + 3\gamma &= 0 \end{aligned}\right\} \quad \begin{aligned} &\\ &\\ &\Rightarrow \beta = 0 \\ &\Rightarrow \gamma = 0. \end{aligned}$$

Hence $\alpha = \beta = \gamma = 0 \ \forall\, a, b, c, d$.

A spanning set for $\mathbb{R}^4$ must contain at least 4 vectors.

A4. $\alpha : U \to V$ is linear iff
$\alpha(au_1 + bu_2) = a\alpha(u_1) + b\alpha(u_2)$
$\forall\, u_1, u_2 \in U$ and $\forall\, a, b \in \mathbb{F}$.

(a) $\alpha\left(a(x_1, y_1, z_1) + b(x_2, y_2, z_2)\right)$
$= \alpha(ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)$
$= (az_1 + bz_2,\ ax_1 + bx_2,\ ay_1 + by_2)$
$= a(z_1, x_1, y_1) + b(z_2, x_2, y_2)$
$= a\alpha(x_1, y_1, z_1) + b\alpha(x_2, y_2, z_2)$
$\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$, $a, b \in \mathbb{R}$.
Hence α is linear.

(b) $\alpha((1,0,0) + (-1,0,0)) = \alpha(0,0,0) = 0$
$\neq \alpha(1,0,0) + \alpha(-1,0,0) = 1 + 1 = 2$.
Hence α is not linear.

A5. $\operatorname{rank}(\alpha) + \operatorname{nullity}(\alpha) = \dim U$.
$\operatorname{Ker}(\alpha)$ has $x_1 + x_2 = 0$, $x_1 = 0$, $x_3 = 0$
$\Rightarrow x_2 = 0$, x_4 unconstrained.
Thus $\operatorname{Ker}(\alpha)$ is the x_4-axis with
$x_1 = x_2 = x_3 = 0$.

Im (α): Map the standard basis to give as a spanning set $\{(0,1,1,0), (0,1,0,0), (0,0,0,1), (0,0,0,0)\}$.
Hence a basis for Ker(α) is $\{(0,0,0,1)\}$ and a basis for Im (α) is $\{(0,1,1,0), (0,1,0,0), (0,0,0,1)\}$, so nullity $(\alpha) = 1$ and rank $(\alpha) = 3$.

A6. $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

$U = \{(x, y, -x-y) \mid x, y \in \mathbb{R}\}$
$= \{x(1,0,-1) + y(0,1,-1) \mid x, y \in \mathbb{R}\}$

Hence a basis for U is $\{(1,0,-1), (0,1,-1)\}$

$W = \{(x,y,x) \mid x, y \in \mathbb{R}\}$
$= \{x(1,0,1) + y(0,1,0) \mid x, y \in \mathbb{R}\}$

Hence a basis for W is $\{(1,0,1), (0,1,0)\}$

$U + W = \langle (1,0,-1), (0,1,-1), (1,0,1), (0,1,0) \rangle$
$= \langle (1,0,0), (0,1,0), (0,0,1) \rangle$

i.e. a basis for $U+W$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$

$\dim U = 2$, $\dim W = 2$, $\dim(U+W) = 3$

Hence $\dim(U \cap W) = 2 + 2 - 3 = 1$.

[In fact, $U \cap W = \{(x, -2x, x) \mid x \in \mathbb{R}\}$.]

A7.
$$\begin{aligned}
\alpha(1,0,0) &= (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1) \\
\alpha(0,1,0) &= (-1,1,0) = -1(1,0,0) + 1(0,1,0) + 0(0,0,1) \\
\alpha(0,0,1) &= (0,-1,1) = 0(1,0,0) - 1(0,1,0) + 1(0,0,1)
\end{aligned}$$

Hence
$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\alpha(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3) = (y_1, y_2, y_3)$

Solve for x_i as functions of y_j.

$$\left.\begin{array}{l} x_1 - x_2 = y_1 \\ x_2 - x_3 = y_2 \\ x_3 = y_3 \end{array}\right\} \begin{array}{l} \Rightarrow x_1 = y_1 + y_2 + y_3 \\ \Rightarrow x_2 = y_2 + y_3 \\ \end{array}$$

Thus $\beta(y_1, y_2, y_3) = (x_1, x_2, x_3) = (y_1 + y_2 + y_3, y_2 + y_3, y_3)$

$$\beta(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$
$$\beta(0,1,0) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$$
$$\beta(0,0,1) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

Hence $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

AB must be the identity matrix.

Check: $AB = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

A8. The identity map Id_V maps every element of V to itself, i.e. $Id_V(v) = v \ \forall v \in V$.

P is the matrix of Id_V wrt. ordered basis $\mathcal{B}$ in its domain and ordered basis $\bar{\mathcal{B}}$ in its codomain. i.e. $P = (Id_V, \mathcal{B}, \bar{\mathcal{B}})$.

$$Id_V(1,3) = (1,3) = 1(1,0) + 3(0,1)$$
$$Id_V(2,-1) = (2,-1) = 2(1,0) - 1(0,1)$$

Hence $P = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$. $P^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$.

B1. (a) Solve for α, β, γ:

(i) $\alpha(1,2,3) + \beta(4,5,6) + \gamma(7,8,9) = (x,y,z)$

$$\begin{aligned} \alpha + 4\beta + 7\gamma &= x \quad \Rightarrow \alpha + \beta + \gamma = y - x \\ 2\alpha + 5\beta + 8\gamma &= y \quad \Rightarrow \alpha + \beta + \gamma = z - y \\ 3\alpha + 6\beta + 9\gamma &= z \end{aligned}$$

Therefore a solution for α, β, γ exists only if $y - x = z - y$ and not $\forall x, y, z$. Hence the vectors do not span $\mathbb{R}^3$.

(ii) $\alpha(1,2,3) + \beta(4,5,6) + \gamma(7,8,10) = (x,y,z)$

$$\begin{aligned} \alpha + 4\beta + 7\gamma &= x \quad \Rightarrow \alpha + \beta + \gamma = y - x \\ 2\alpha + 5\beta + 8\gamma &= y \quad \Rightarrow \alpha + \beta + 2\gamma = z - y \\ 3\alpha + 6\beta + 10\gamma &= z \end{aligned}$$

So $\gamma = (z-y) - (y-x) = z - 2y + x$

$\alpha + \beta = 2(y-x) - (z-y) = 3y - 2x - z$

$\alpha + 4\beta = x - 7(z - 2y + x) = -7z + 14y - 6x$

$3\beta = (-7z + 14y - 6x) - (3y - 2x - z)$

$= -6z + 11y - 4x$

This leads to a solution for α, β, γ $\forall x, y, z$. Hence the vectors span $\mathbb{R}^3$.

(iii) Reduce the set to a linearly independent set by elementary row operations. The result must be a basis for the span of the vectors.

$$\begin{matrix} 1 & 1 & 4 \\ 2 & 1 & 5 \\ 0 & 1 & 3 \\ 3 & 2 & 9 \\ 1 & 1 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 4 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -3 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 4 \\ 0 & -1 & -3 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & -3 \end{matrix}$$

These three vectors span $\mathbb{R}^3$, therefore so did the original set.

For (i), the set span the plane with equation
$y - x = z - y$ or $x - 2y + z = 0$

(b) Reduce to echelon form

$$\begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{matrix}$$

$$\rightarrow \quad \begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Hence only 3 linearly independent vectors, so the set does not span $\mathbb{R}^4$.

Alternative solution to part (a) (i).
Reduce to row echelon form:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{matrix}$$

Hence the set does not span $\mathbb{R}^3$.
But a basis for the space spanned is
$\{(1,2,3), (0,1,2)\}$. The general vector in this subspace of $\mathbb{R}^3$ has the form
$(x, y, z) = \alpha(1,2,3) + \beta(0,1,2), \quad \alpha, \beta \in \mathbb{R}.$
Thus $x = \alpha$, $y = 2\alpha + \beta$, $z = 3\alpha + 2\beta$.
Eliminate α & β: $y = 2x + \beta$, $z = 3x + 2\beta$,
i.e. $2y - z = 4x - 3x = x$
or $x - 2y + z = 0$.
This is the equation of the plane spanned.

B2. A vector space has dimension n if any basis for the vector space contains n vectors.

Let $v = \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n$

Then $(\alpha_1 - \beta_1) v_1 + \dots + (\alpha_n - \beta_n) v_n = 0$

But $\{v_1, \dots, v_n\}$ is a basis and therefore linearly independent.

Therefore $\alpha_1 - \beta_1 = \dots = \alpha_n - \beta_n = 0$

so the expansion of v is unique.

The standard basis for $\mathbb{R}^n$ is the ordered list of n-tuples $e_1, e_2, \dots, e_n$ where $e_i \in \mathbb{R}^n$ and every element of e_i is zero except for the ith element, which is one.

A basis is a linearly independent spanning set. Let $(x, y, z) = \alpha(2,1,0) + \beta(3,0,1) + \gamma(0,1,1)$.

$$\Rightarrow \left.\begin{aligned} x &= 2\alpha + 3\beta \\ y &= \alpha + \gamma \\ z &= \beta + \gamma \end{aligned}\right\} \text{ Solve for } \alpha, \beta, \gamma.$$

$y - z = \alpha - \beta$

$x - 2(y - z) = 3\beta + 2\beta = 5\beta$

i.e. $5\beta = x - 2y + 2z$

$5\gamma = 5z - 5\beta = 5z - (x - 2y + 2z)$

i.e. $5\gamma = -x + 2y + 3z$

$5\alpha = 5y - 5\gamma = 5y - (-x + 2y + 3z)$

i.e. $5\alpha = x + 3y - 3z$

Thus, for any x, y, z, the coeffs α, β, γ can be found, so the set $\mathcal{B}$ is a spanning set.

Setting $x = y = z = 0 \Rightarrow \alpha = \beta = \gamma = 0$ uniquely

Thus $\mathcal{B}$ is also a linearly independent set, and so $\mathcal{B}$ is a basis.

Let $v = (2,-1,-1) = \alpha(2,1,0) + \beta(3,0,1) + \gamma(0,1,1)$.
Then setting $x = 2$, $y = -1$, $z = -1$ in the previous solution gives

$$5\alpha = 2 - 3 + 3 = 2$$
$$5\beta = 2 + 2 - 2 = 2$$
$$5\gamma = -2 - 2 - 3 = -7$$

Thus the coordinates of v wrt $\mathcal{B}$ are $(2/5, 2/5, -7/5)$.

mod 5 the equations above reduce to the single equation $x + 3y + 2z = 0$, so the vectors span the subspace satisfying this equation and not $\mathbb{F}_5^3$.

Let $(x,y,z) = \alpha(2,1,0) + \beta(3,0,1) + \gamma(1,0,1)$ mod 5

$$\Rightarrow \left.\begin{aligned} x &= 2\alpha + 3\beta + \gamma \\ y &= \alpha \\ z &= \beta + \gamma \end{aligned}\right\} \Rightarrow x + 3y = 3\beta + \gamma$$

$$x + 3y - z = 2\beta \Rightarrow 3x + 9y - 3z = 6\beta$$
$$\Rightarrow \beta = 3x + 4y + 2z \mod 5$$
$$\gamma = z - \beta = z - (3x + 4y + 2z)$$
$$= -3x - 4y - z = 2x + y + 4z \mod 5$$

i.e.
$$\left.\begin{aligned} \alpha &= y \\ \beta &= 3x + 4y + 2z \\ \gamma &= 2x + y + 4z \end{aligned}\right\} \mod 5.$$

$x = y = z = 0 \Rightarrow \alpha = \beta = \gamma = 0$ uniquely
therefore the set is a basis for $\mathbb{F}_5^3$.

<u>NB</u>: The only finite field considered from 2003–4 onwards is $\mathbb{F}_2 = \{0,1\}$, in which the arithmetic is slightly simpler: in particular 1 is its own inverse under both addition and multiplication.

B3. $\mathrm{Ker}(\alpha) = \{v \in V \mid \alpha(v) = 0\}$
$\mathrm{Im}(\alpha) = \{w \in W \mid \exists\, w' \in V \text{ st } \alpha(w') = w\}$

$0_V \in \mathrm{Ker}(\alpha)$ since $\alpha(0_V) = 0_W$.
Let $u, v \in \mathrm{Ker}(\alpha)$. Then $\alpha(u) = \alpha(v) = 0$.
$\alpha(au + bv) = a\alpha(u) + b\alpha(v) = 0 \ \forall a, b \in \mathbb{K}$
where $\mathbb{K}$ is the field of scalars.
Hence $au + bv \in \mathrm{Ker}(\alpha)$ $\forall u, v \in \mathrm{Ker}(\alpha)$
and $\forall a, b \in \mathbb{K}$.
Therefore $\mathrm{Ker}(\alpha)$ is a vector subspace of V.

$0_W \in \mathrm{Im}(\alpha)$ since $\alpha(0_V) = 0_W$.
Let $u, v \in \mathrm{Im}(\alpha)$. Then $\exists\, u', v' \in U$ st.
$\alpha(u') = u$, $\alpha(v') = v$.
$\alpha(au' + bv') = a\alpha(u') + b\alpha(v') = au + bv \in \mathrm{Im}(\alpha)$
$\forall u, v \in \mathrm{Im}(\alpha)$ and $\forall a, b \in \mathbb{K}$.
Therefore $\mathrm{Im}(\alpha)$ is a vector subspace of W.

Formula: $\dim V = \dim \mathrm{Ker}(\alpha) + \dim \mathrm{Im}(\alpha)$.
Proof: Let $\{v_1, \dots v_m\}$ be a basis for $\mathrm{Ker}(\alpha)$
and extend it to a basis $\{v_1, \dots, v_m, v_{m+1}, \dots v_{m+n}\}$
for V. Then $\{\alpha(v_{m+1}), \dots, \alpha(v_{m+n})\}$ is a basis
for $\mathrm{Im}(\alpha)$
Span: For any $w \in \mathrm{Im}(\alpha)$ $\exists\, w' \in V$ st $\alpha(w') = w$.
Let $w' = a_1 v_1 + \dots + a_{m+n} v_{m+n}$
Then $w = \alpha(w') = a_1 \alpha(v_1) + \dots + a_{m+n}\alpha(v_{m+n})$
$= a_{m+1}\alpha(v_{m+1}) + \dots + a_{m+n}\alpha(v_{m+n})$
since $\alpha(v_1) = \dots = \alpha(v_m) = 0$ since $v_1, \dots v_m \in \mathrm{Ker}(\alpha)$.
Lin Ind: Suppose $a_{m+1}\alpha(v_{m+1}) + \dots + a_{m+n}\alpha(v_{m+n}) = 0$
Then $\alpha(a_{m+1} v_{m+1} + \dots + a_{m+n} v_{m+n}) = 0$
$\Rightarrow a_{m+1} v_{m+1} + \dots + a_{m+n} v_{m+n} \in \mathrm{Ker}(\alpha)$
So $a_{m+1} v_{m+1} + \dots + a_{m+n} v_{m+n} = b_1 v_1 + \dots + b_m v_m$
for some $b_i \in \mathbb{K}$

i.e. $a_{m+1} v_{m+1} + \dots + a_{m+n} v_{m+n} - b_1 v_1 - \dots - b_m v_m = 0$

But $\{v_1, \dots v_m, v_{m+1}, \dots v_{m+n}\}$ are a basis (for V) and so are linearly independent.
Therefore $a_{m+1} = \dots = a_{m+n} = 0$ (and $b_1 = \dots = b_m = 0$).

Given the bases defined above,
$\dim V = m+n$ where $m = \dim \operatorname{Ker}(\alpha)$
and $n = \dim \operatorname{Im}(\alpha)$.

$$\operatorname{Ker}(\alpha): \quad \left.\begin{aligned} 2x + y &= 0 \\ 3x + y - 2z &= 0 \end{aligned}\right\}$$

$\Rightarrow y = -2x$, so $x = 2z$, $y = -4z$.
Thus $\operatorname{Ker}(\alpha) = \{(2z, -4z, z) \mid z \in \mathbb{R}\}$
and a basis for $\operatorname{Ker}(\alpha)$ is $\{(2, -4, 1)\}$

$\operatorname{Im}(\alpha)$: map the standard basis for $\mathbb{R}^3$ to give $\{(2,3), (1,1), (0,-2)\}$ as a spanning set.
This clearly contains $\{(1,0), (0,1)\}$, which is therefore a basis for $\operatorname{Im}(\alpha)$.

[NB: dim domain = 3, dim Ker = 1, dim Im = 2 and dimension Theorem is satisfied.]

B4. (a) First find the eigenvalues of A.

$$\begin{vmatrix} 1-\lambda & 0 & 4 \\ 3 & 2-\lambda & -6 \\ -2 & 0 & 7-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)\left\{(1-\lambda)(7-\lambda) - 4(-2)\right\} = 0$$

$$\Rightarrow \lambda = 2 \quad \text{or} \quad 7 - 8\lambda + \lambda^2 + 8 = 0$$

$$\text{i.e. } \lambda^2 - 8\lambda + 15 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 5) = 0$$

Thus $\lambda = 2, 3$ or 5.

Now find the eigenvectors of A for each λ.

$\lambda = 2$: $$\begin{pmatrix} -1 & 0 & 4 \\ 3 & 0 & -6 \\ -2 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{l} -x + 4z = 0 \\ 3x - 6z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = z = 0 \\ y = \text{anything} \end{array}$$

An eigenvector is $(0, 1, 0)$

$\lambda = 3$: $$\begin{pmatrix} -2 & 0 & 4 \\ 3 & -1 & -6 \\ -2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{l} -2x + 4z = 0 \\ 3x - y - 6z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = 2z \\ y = 0. \end{array}$$

An eigenvector is $(2, 0, 1)$

$\lambda = 5$: $$\begin{pmatrix} -4 & 0 & 4 \\ 3 & -3 & -6 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{l} x - y - 2z = 0 \\ -x + z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = z \\ y = -z \end{array}$$

An eigenvector is $(1, -1, 1)$.

Hence $$S^{-1} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Now invert S^{-1}. $\det(S^{-1}) = 1-2 = -1$.
Matrix of minors of S^{-1} is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & -1 & -2 \end{pmatrix}$

Hence $S = (S^{-1})^{-1} = -1 \begin{pmatrix} 1 & -1 & -2 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & 0 & 2 \end{pmatrix}$

[Can check that $SAS^{-1} = \text{diag}(2,3,5)$.]

(b) $\begin{vmatrix} 1-\lambda & 2\sqrt{2} \\ 2\sqrt{2} & -1-\lambda \end{vmatrix} = 0 \Rightarrow -(1-\lambda)(1+\lambda) - 8 = 0$
$\Rightarrow 1-\lambda^2+8 = 0$
$\Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$.

$\lambda = 3$: $\begin{pmatrix} -2 & 2\sqrt{2} \\ 2\sqrt{2} & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$
$\Rightarrow -2x + 2\sqrt{2}\,y = 0 \Rightarrow x = \sqrt{2}\,y$
An eigenvector is $(\sqrt{2}, 1)$.
Normalized this is $(\sqrt{2}, 1)/\sqrt{3}$.

$\lambda = -3$: $\begin{pmatrix} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$
$\Rightarrow 2\sqrt{2}\,x + 2y = 0 \Rightarrow y = -\sqrt{2}\,x$
An eigenvector is $(1, -\sqrt{2})$.
Normalized this is $(1, -\sqrt{2})/\sqrt{3}$.

Hence $P^T = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}$ & $P = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}$

[Check: $PAP^T = \frac{1}{3}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 3\sqrt{2} & -3 \\ 3 & 3\sqrt{2} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 & 0 \\ 0 & -9 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$]