SOLUTIONS TO HOMEWORK ASSIGNMENT # 6

1. Find a parametric representation of the following surfaces:

   (a) that part of the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ with $y \geq 0$, where a, b, c are positive constants.

   (b) that part of the elliptical paraboloid $x + y^2 + 2z^2 = 4$ that lies in front of the plane $x = 0$.

   (c) that part of the surface $z^2 = x^2 - y^2$ that lies in the first octant.

   Solution:

   (a) A particular parametrization is

   $$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{z}{c}\right)^2}, \text{ where } \left(\frac{x}{a}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$$

   (b) We must have $0 \leq x \leq 4$. A particular parametrizatrion is

   $$y = \sqrt{4-x}\cos\theta,\ z = \frac{\sqrt{4-x}}{\sqrt{2}}\sin\theta,\ \text{where } 0 \leq x \leq 4,\ 0 \leq \theta \leq 2\pi.$$

   (c) We must have $x \geq 0,\ y \geq 0,\ z \geq 0$ and $x \geq y$. A particular parametrization is

   $$x = r\cos\theta,\ y = r\sin\theta, z = r\sqrt{\cos 2\theta},\ 0 \leq r < \infty, 0 \leq \theta \leq \pi/4.$$

2. Find an equation of the tangent plane of the given parametric surfaces at the point indicated:

   (a) $\mathbf{r} = u^2\mathbf{i} + v^2\mathbf{j} + uv\mathbf{k}, u = 1, v = 1.$

   (b) the surface that you get by rotating $z = e^{-y},\ 0 < y < \infty$, about the z-axis, at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{e}\right)$. Hint: polar coordinates.

   Solution:

   (a) $\mathbf{r}_u = 2u\mathbf{i} + v\mathbf{k} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{r}_v = 2v\mathbf{j} + u\mathbf{k} = 2\mathbf{j} + \mathbf{k}$, and so $\mathbf{r}_u \times \mathbf{r}_v = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. An equation of the tangent plane is therefore $-(x-1) - (y-1) + 2(z-1) = 0$, or $2z - x - y = 0$.

   (b) A particular parametrization is

   $$x = \rho\cos\theta,\ y = \rho\sin\theta,\ z = e^{-\rho}, 0 < \rho < \infty, 0 \leq \theta \leq 2\pi.$$

The point $(x, y, z) = (1/2, \sqrt{3}/2, e^{-1})$ corresponds to $\theta = \pi/3, \rho = 1$. Thus

$$\begin{aligned}
\mathbf{r} &= \rho\cos\theta\mathbf{i} + \rho\sin\theta\mathbf{j} + e^{-\rho}\mathbf{k} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + e^{-1}\mathbf{k} \\
\mathbf{r}_\rho &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} - e^{-\rho}\mathbf{k} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - e^{-1}\mathbf{k} \\
\mathbf{r}_\theta &= -\rho\sin\theta\mathbf{i} + \rho\cos\theta\mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \\
\mathbf{r}_\rho \times \mathbf{r}_\theta &= \frac{1}{2}e^{-1}\mathbf{i} + \frac{\sqrt{3}}{2}e^{-1}\mathbf{i} + \mathbf{k}
\end{aligned}$$

Thus the tangent plane is $x + \sqrt{3}y + 2ez = 4$, after some algebra.

3. Find the surface area of the following surfaces.

   (a) The part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0), (0, 1), (2, 1)$.

   (b) The spiral ramp $\mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}, 0 \le u \le 1, 0 \le v \le \pi$.

Solution:

(a) Let $D = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le 2y\}$. Then the surface area is

$$\begin{aligned}
S &= \int\int_D \sqrt{1 + 9 + 16y^2}dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y}\sqrt{10 + 16y^2}dxdy \\
&= \int_{y=0}^{y=1} 2y\sqrt{10 + 16y^2}dy = \frac{1}{24}(10 + 16y^2)^{3/2}\Big|_{y=0}^{y=1} \\
&= \frac{1}{24}(26^{3/2} - 10^{3/2})
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{r}_u &= \cos v\mathbf{i} + \sin v\mathbf{j} \\
\mathbf{r}_v &= -u\sin v\mathbf{i} + u\cos v\mathbf{j} + \mathbf{k} \\
\mathbf{r}_u \times \mathbf{r}_v &= \sin v\mathbf{i} - \cos v\mathbf{j} + u\mathbf{k} \\
|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{1 + u^2}
\end{aligned}$$

Therefore the surface area is

$$\begin{aligned}
S &= \int_{v=0}^{v=\pi}\int_{u=0}^{u=1}\sqrt{1 + u^2}dudv = \pi\int_{u=0}^{u=1}\sqrt{1 + u^2}du \\
&= \frac{\pi}{2}\left(u\sqrt{u^2 + 1} + \ln(u + \sqrt{u^2 + 1})\right)\Big|_{u=0}^{u=1} \\
&= \frac{\pi}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))
\end{aligned}$$

4. Evaluate the following surface integrals.

   (a) $\int\int_S yzdS$, where S is the first octant part of the plane $x+y+z=\lambda$, where λ is a positive constant.

   (b) $\int\int_S (x^2z+y^2z)dS$, where S is the hemisphere $x^2+y^2+z^2=a^2, z\geq 0$.

   Solution:

   (a)

$$\begin{aligned}\int\int_S yzdS &= \int_{y=0}^{y=\lambda}\int_{x=0}^{x=\lambda-y} y(\lambda-x-y)\sqrt{3}dxdy \\ &= \sqrt{3}\int_{y=0}^{y=\lambda}\left(y(\lambda-y)^2-y\frac{(\lambda-y)^2}{2}\right)dy \\ &= \frac{\sqrt{3}}{2}\int_{y=0}^{y=\lambda} y(\lambda-y)^2dy = \frac{\sqrt{3}}{2}\int_{y=0}^{y=\lambda}\left(y\lambda^2-2\lambda y^2+y^3\right)dy \\ &= \frac{\sqrt{3}}{2}\left(\frac{\lambda^4}{2}-\frac{2\lambda^4}{3}+\frac{\lambda^4}{4}\right) = \frac{\sqrt{3}\lambda^4}{24}\end{aligned}$$

   (b) We use spherical coordinates to evaluate this integral.

$$\begin{aligned}\int\int_S (x^2z+y^2z)dS &= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}(a^2\sin^2\phi)(a\cos\phi)(a^2\sin\phi)d\phi d\theta \\ &= 2\pi a^5\int_{\phi=0}^{\phi=\pi/2}\sin^3\phi\cos\phi d\phi = 2\pi a^5\frac{\sin^4\phi}{4}\Big|_{\phi=0}^{\phi=\pi/2} = \frac{\pi a^5}{2}\end{aligned}$$

5. Let S be the surface you get by rotating the circle $(x-a)^2+z^2=b^2$, where $0<b<a$, about the z-axis.

   (a) Sketch the surface S.

   (b) Find a parametric representation of S.

   (c) Find the unit outward normal at every point of S.

   Solution:

   (a) This surface is a torus. For a sketch see question # 56 on page 1145.

   (b) Using the angles θ, α from page 1145 we see that a parametrization is

$$x=(a+b\cos\alpha)\cos\theta,\ \ y=(a+b\cos\alpha)\sin\theta,\ \ z=b\sin\alpha$$

   (c) The position vector for a point on the torus is

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(a+b\cos\alpha)\cos\theta\mathbf{i}+(a+b\cos\alpha)\sin\theta\mathbf{j}+b\sin\alpha\mathbf{k}$$

Therefore

$$\begin{aligned}
\mathbf{r}_\theta &= -(a+b\cos\alpha)\sin\theta\mathbf{i} + (a+b\cos\alpha)\cos\theta\mathbf{j} \\
\mathbf{r}_\alpha &= -b\sin\alpha\cos\theta\mathbf{i} - b\sin\alpha\sin\theta\mathbf{j} + b\cos\alpha\mathbf{k} \\
\mathbf{r}_\theta \times \mathbf{r}_\alpha &= (a+b\cos\alpha)\left(b\cos\theta\cos\alpha\mathbf{i} + b\sin\theta\cos\alpha\mathbf{j} + b\sin\alpha\mathbf{k}\right) \\
\frac{\mathbf{r}_\theta \times \mathbf{r}_\alpha}{|\mathbf{r}_\theta \times \mathbf{r}_\alpha|} &= \cos\theta\cos\alpha\mathbf{i} + \sin\theta\cos\alpha\mathbf{j} + \sin\alpha\mathbf{k}
\end{aligned}$$

Putting $\theta = 0, \alpha = 0$ gives $\dfrac{\mathbf{r}_\theta \times \mathbf{r}_\alpha}{|\mathbf{r}_\theta \times \mathbf{r}_\alpha|} = \mathbf{i}$, the unit outward normal at this point. Therefore the outward unit normal is $\mathbf{n} = \cos\theta\cos\alpha\mathbf{i} + \sin\theta\cos\alpha\mathbf{j} + \sin\alpha\mathbf{k}$.