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ME 362 VIBRATIONS I

Lecture 4

TORSIONAL SYSTEMS
SINGLE- AND MULTI- ROTOR SYSTEMS

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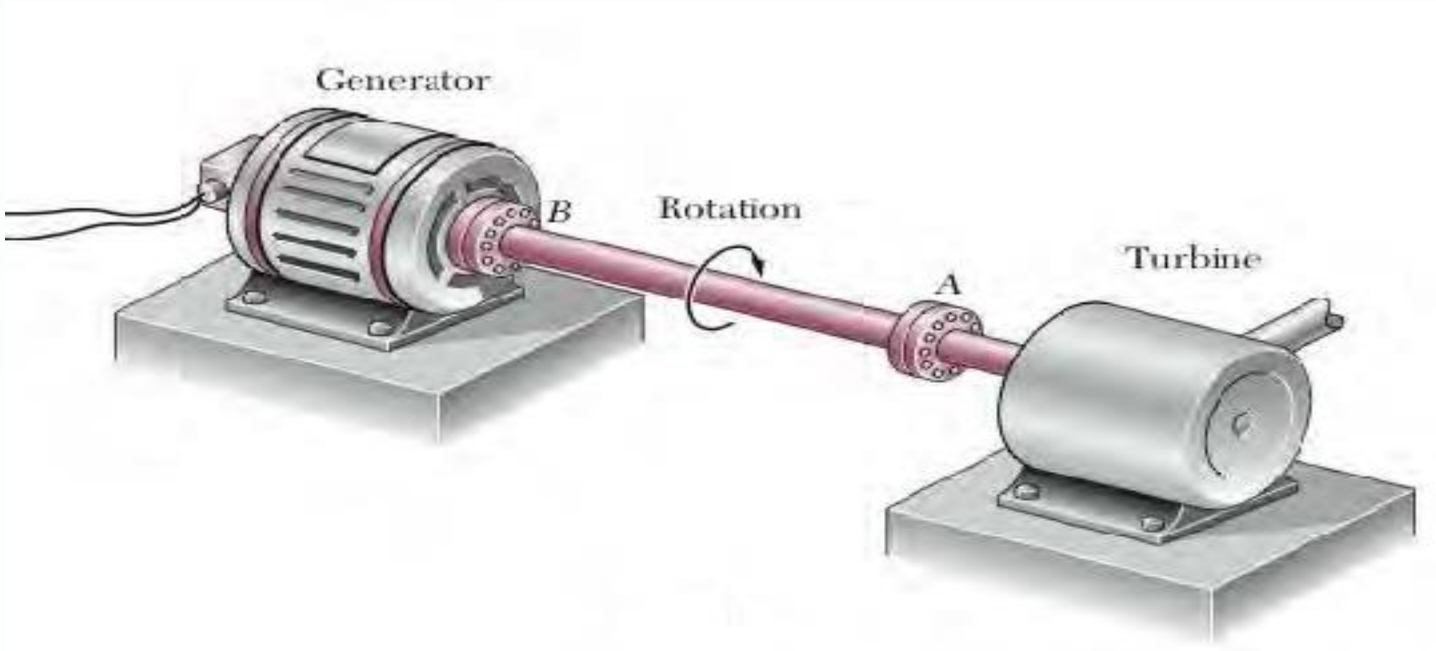
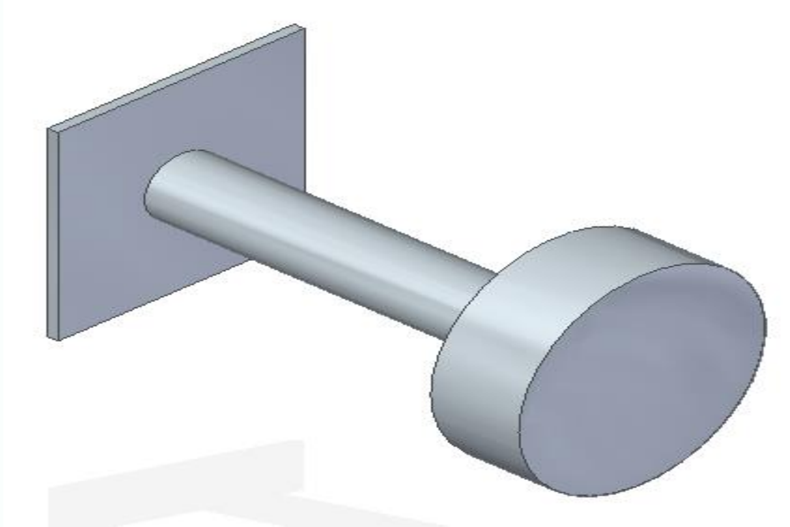
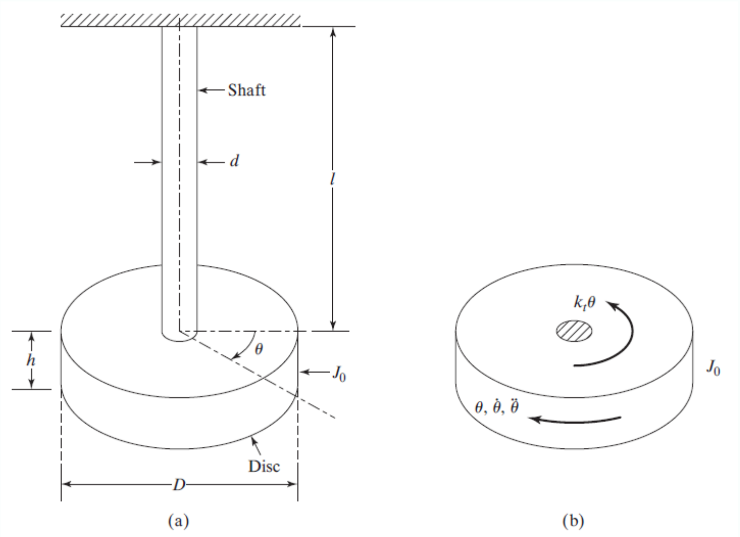
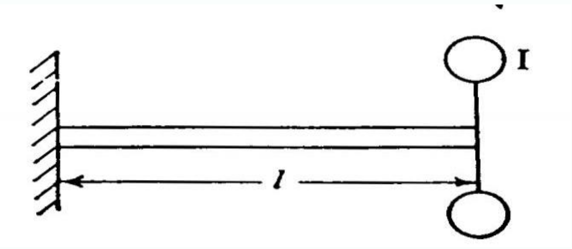
TORSIONAL SYSTEMS
SINGLE- AND MULTI- ROTOR SYSTEMS



Disc of moment of inertia, I, attached to shaft







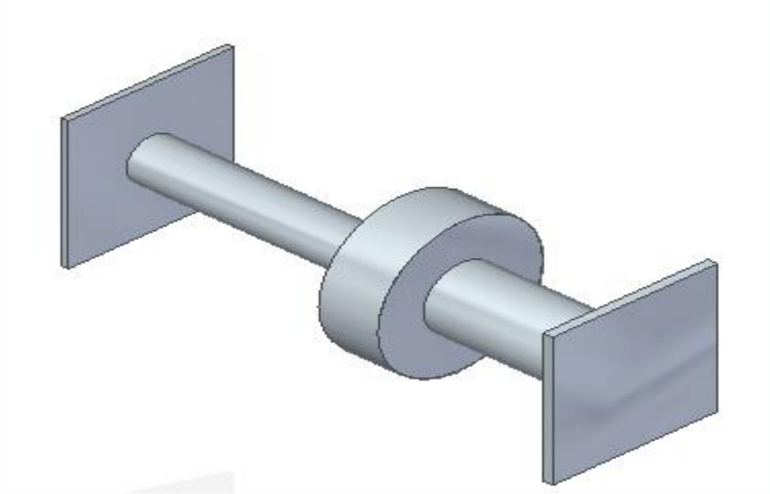
$$I\ddot{\theta} + k\theta = 0$$

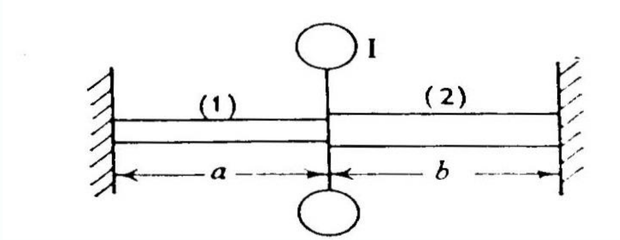
$$\ddot{\theta} + \omega^2\theta = 0$$

$$\omega^2 = \frac{k}{I}; \quad k = \frac{GJ}{l}; J = \frac{\pi d^4}{32}$$

$$\omega = \sqrt{\frac{GJ}{Il}} \text{rad/s}$$

Rotor attached to shaft between two fixed ends





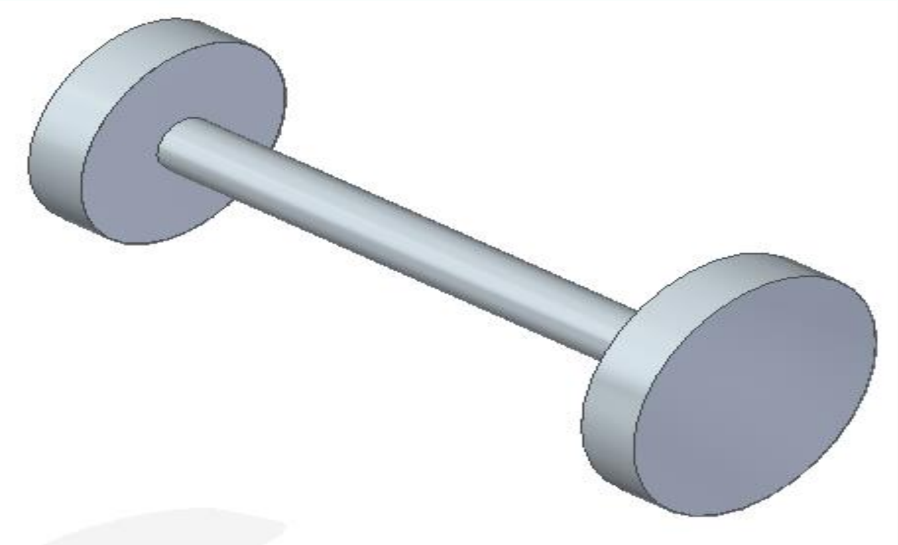
$$I\ddot{\theta} + (k_1 + k_2)\theta = 0$$

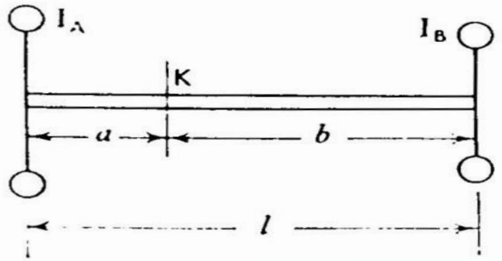
$$\ddot{\theta} + \omega^2\theta = 0$$

$$\omega^2 = \frac{k_1 + k_2}{I}; k_1 = \frac{GJ_1}{a}; k_2 = \frac{GJ_2}{b}; J_1 = \frac{\pi {d_1}^4}{32}; J_2 = \frac{\pi {d_2}^4}{32}$$

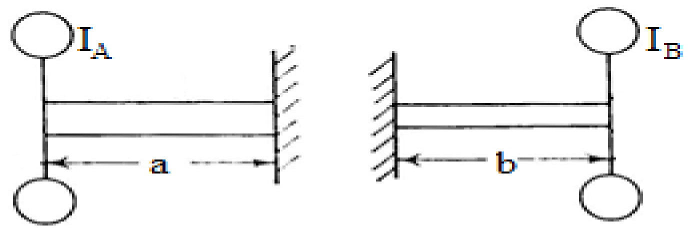
$$\omega = \sqrt{\frac{G}{I}\left(\frac{J_1}{a} + \frac{J_2}{b}\right)}\ \text{rad/s}$$

Two-rotor system





K, is the position of the node.
A ***node*** is the point where amplitude is minimal or tends to zero



$$\omega_1 = \sqrt{\frac{GJ}{I_A a}} rad/s$$

$$\omega_2 = \sqrt{\frac{GJ}{I_B b}}\ \text{rad/s}$$

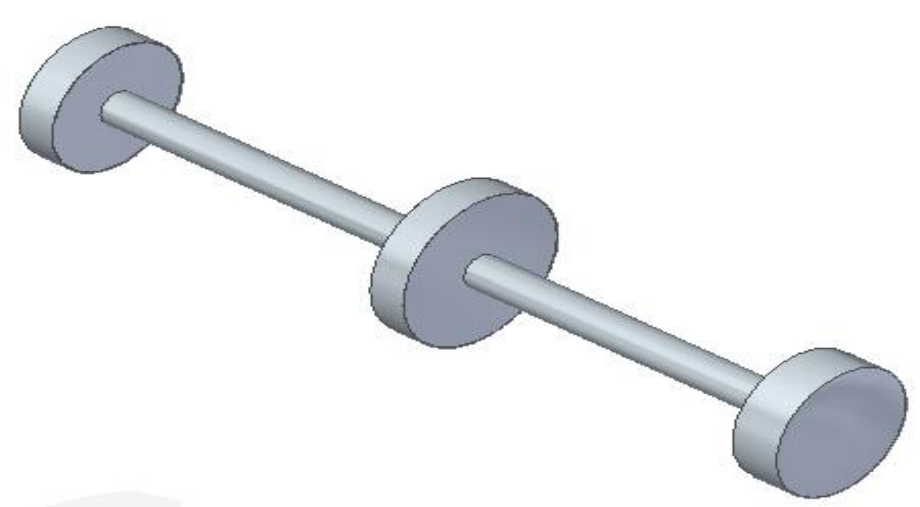
At point K, i.e., node

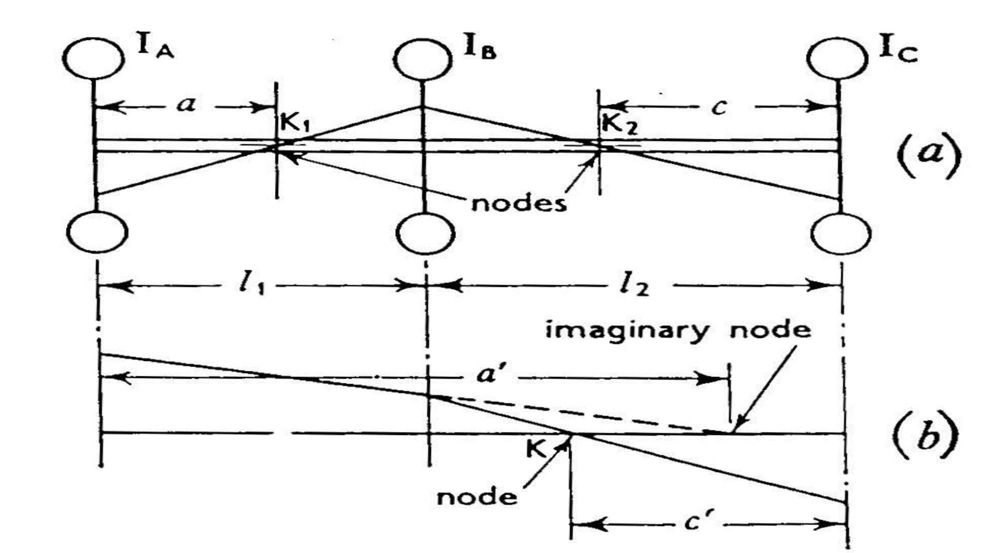
$$\omega_1 = \omega_2$$
$$aI_A = bI_B \ldots . i$$
$$a + b = l \ldots . ii$$

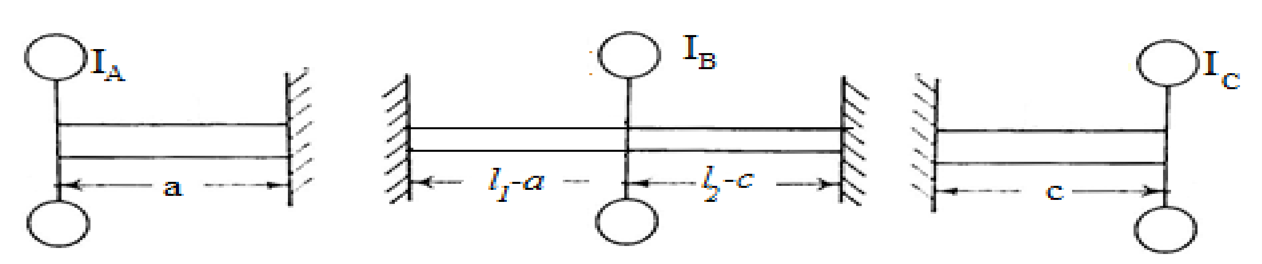
Solving (i) and (ii) simultaneously gives a and b, the location of the node, from which the natural frequency y of the system can be found.

Three-rotor system

Here there are two possible modes of vibration, having tw nodes as shown





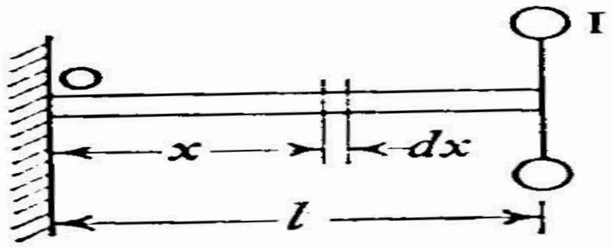


$$\omega_1 = \sqrt{\frac{GJ}{I_A a}} rad/s; \quad \omega_3 = \sqrt{\frac{GJ}{I_c c}} \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{G}{I_B}\left(\frac{1}{l_1 - a} + \frac{1}{l_2 - b}\right)}$$

The frequency of vibration of A, B and C must be the same so that *a* and *c* may be determined by equating ω_1, ω_2 and ω_3. The resulting quadratic will give two values for both *a* and *c*; one pair will give the positions of the nodes for the two-node vibration, but of the other pair only one value will give a node lying within the limits of the corresponding part of the shaft. This will be the node for the single-node vibration.

Effect of Inertia of Shaft; Torsionally Equivalent Shaft



Let the instantaneous angular velocity of the rotor in the above Figure be Ω.

Then angular velocity of element at distance *x* from O = $\frac{x}{l}\Omega$

If I_s is the polar moment of inertia of the shaft, then (kinetic energy) K.E. of element of length dx

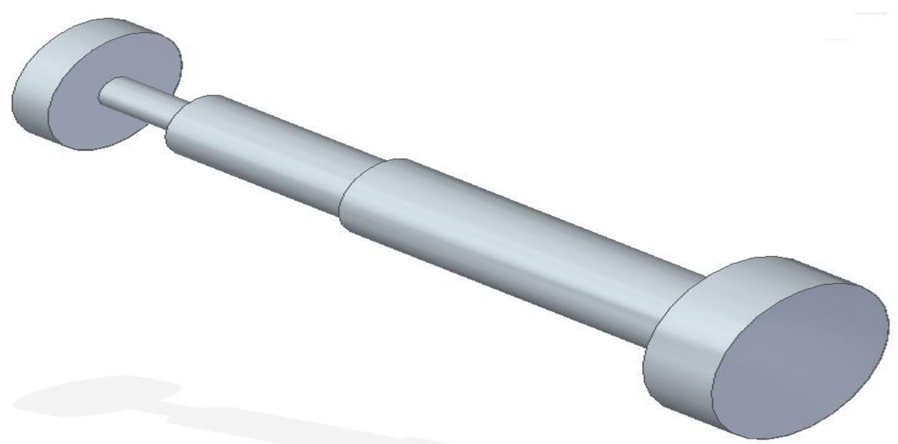
$$= \frac{1}{2} I_s \frac{dx}{l} \times \left(\frac{x}{l}\Omega\right)^2$$

$$\therefore \text{total K. E. of shaft} = \frac{I_s}{2l^3}\Omega^2 \int_0^1 x^2 dx = \frac{I_s}{2}\frac{\Omega^2}{3}$$

$$\therefore \text{total K. E. of system} = \frac{1}{2}\left\{I + \frac{I_s}{3}\right\}\Omega^2$$

Shaft inertia may therefore be allowed for by adding a third of its inertia to that of the rotor.

Torsionally equivalent shaft
Step Shafts



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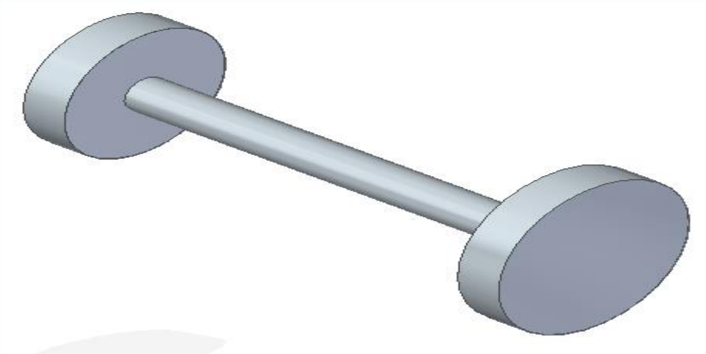
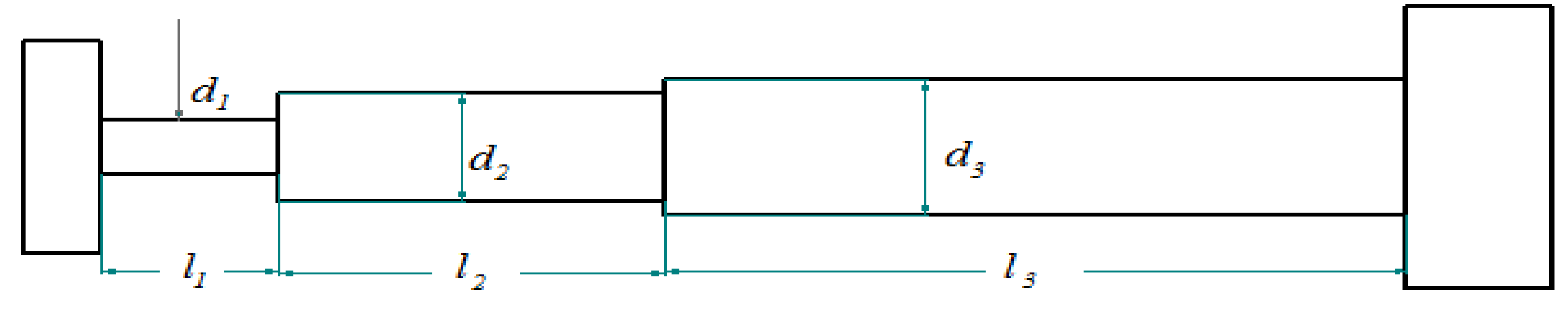




$$\theta = \theta_A + \theta_B + \theta_C$$

$$\Rightarrow \frac{Tl}{GJ} = \frac{Tl_1}{GJ_1} + \frac{Tl_2}{GJ_2} + \frac{Tl_3}{GJ_3}$$

$$l = l_1\left(\frac{d}{d_1}\right)^4 + l_2\left(\frac{d}{d_2}\right)^4 + l_3\left(\frac{d}{d_3}\right)^4 = \sum_{i=1}^{n} l_i\left(\frac{d}{d_i}\right)^4$$

MODELLING OF GEARED SYSTEMS

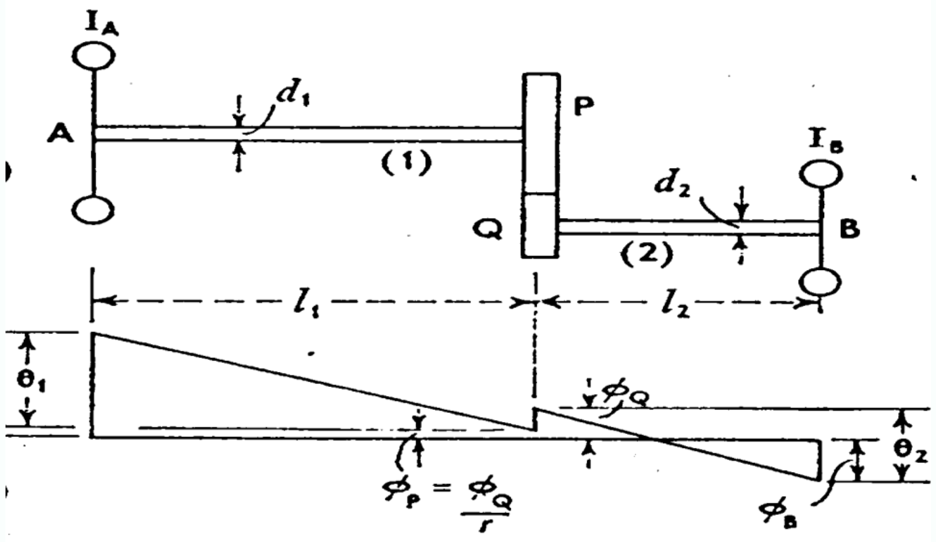
- Gears are used in order to increase or decrease the rotational speed and torque of a motor.
- Two systems are said to be equivalent if their kinetic and strain energies are the same.
- The kinetic energy id given by

$$T = \frac{1}{2} I\omega^2$$

- And the strain energy is given by

$$U = \frac{1}{2}\frac{T^2 l}{GJ}$$

MODELLING OF GEARED SYSTEMS



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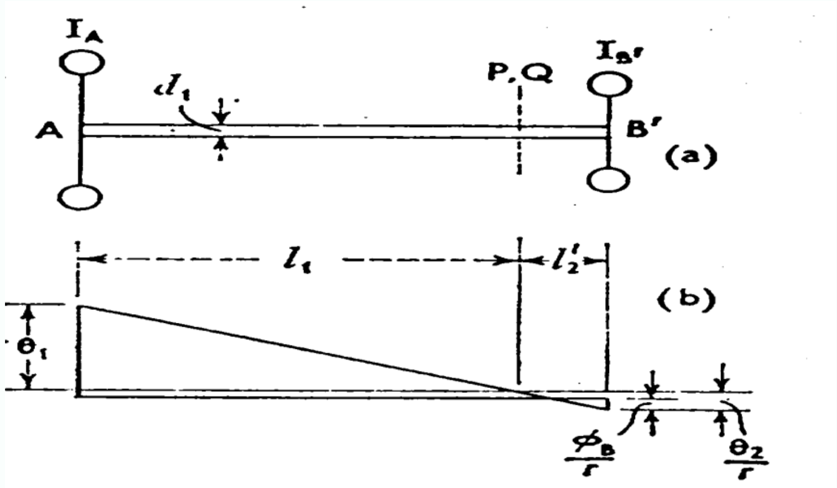


$$K.E_1 + K.E_2 = K.E$$

Neglecting the effect of the Inertias of the gears

$$\frac{1}{2}I_A{\omega_A}^2 + \frac{1}{2}I_B{\omega_B}^2 = \frac{1}{2}I\omega^2$$

$$If\ \omega = \omega_A$$

$$\frac{1}{2}\left(I_A + I_B\left(\frac{\omega_B}{\omega_A}\right)^2\right){\omega_A}^2 = \frac{1}{2}I\omega^2$$

$$I = I_A + I_B(r)^2; \quad r = \frac{\omega_B}{\omega_A}$$

$$U_1 + U_2 = U$$

$$\frac{1}{2}\frac{{T_A}^2 l_A}{GJ_A} + \frac{1}{2}\frac{{T_B}^2 l_B}{GJ_B} = \frac{1}{2}\frac{{T_A}^2}{GJ_A}\left(l_A + l_B\left(\frac{T_B}{T_A}\right)^2\left(\frac{J_A}{J_A}\right)\right)$$

$$U = \frac{1}{2}\frac{{T_A}^2}{GJ_A}(l_A + l_{B'})$$

$$\therefore I_{B'} = I_B(r)^2; l_{B'} = \frac{l_2}{r^2}\left(\frac{d}{d_2}\right)^4$$

Example 1

A steel disc 300 mm diameter, of mass 29 kg, is suspended from the end of a wire 2·5 mm diameter and 1·5 m long which is clamped into a central hole in the disc, the upper end of the wire being rigidly supported. When the disc is set in torsional vibration it is found to make 10 complete oscillations in 78·2 s. Find the modulus of rigidity of the wire, and calculate the amplitude of the oscillation which may be allowed if the maximum permissible intensity of shearing stress in the wire is 140 MN/m^2.

Solution 1

Solution. For the wire,

$$J = \frac{\pi d^4}{32} = \frac{\pi \times 2.5^4}{32 \times 10^{12}} = 3.84 \times 10^{-12}\mathrm{m}^4$$

For the disc,

$$I = \frac{mR^2}{2} = \frac{29 \times 0.15^2}{2} = 0.326\ \mathrm{kgm}^2$$

$$n = \frac{1}{2\pi}\sqrt{\frac{GJ}{Il}}$$

i.e.

$$\frac{1 \times 10}{78.2} = \frac{1}{2\pi}\sqrt{\left[\frac{G \times 3.84 \times 10^{-12}}{0.326 \times 1.5}\right]}$$

Hence $G = 82.4\ \mathrm{GN/m^2}$

If τ is the shear stress in the wire at a radius *r*, then $\frac{\tau}{r} = \frac{G\theta}{l}$, i.e.,

$$\frac{140 \times 10^6}{0.00125} = 82.4 \times 10^9 \times \theta$$

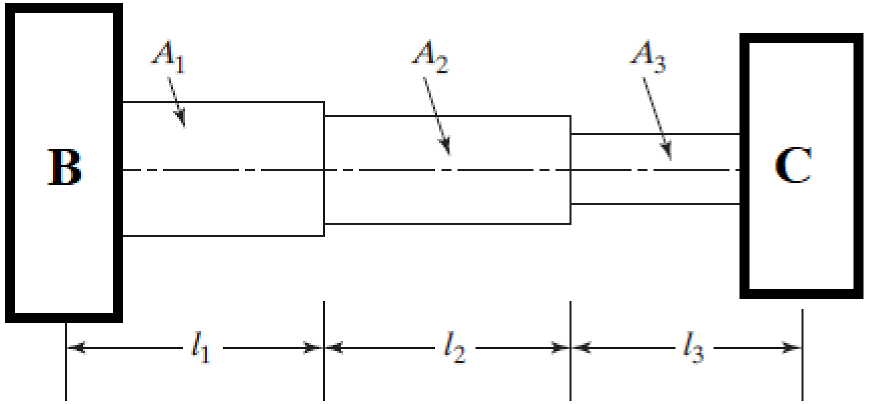
$$\therefore \theta = 2.04\ \mathrm{rad}\ \text{ or }\ 117^0$$

Example 2

A stepped shaft shown, has diameters and length as shown in the table below, it is fitted with two flywheels. The flywheel *B* has a mass of 22 kg and a radius of gyration of 450 mm and flywheel C has a mass of 27 kg and radius of gyration 0.6 m. The modulus of rigidity of the material of the shafts is 80 GN /m^2.

a. If the flywheel *B* is rotated relatively to the flywheel C and released, find the frequency of the resulting vibrations and the position of the node.

b. If the maximum angle of rotation of C relative to B is 1.2°, find the angular displacement of B.



	All dimensions are in mm		
D	$D_1 = 30$	$D_2 = 25$	$D_3 = 20$
l	$l_1 = 250$	$l_2 = 200$	$l_3 = 100$

Solution 2

The stepped shaft is equivalent to a solid shaft of diameter 30 mm and length

$$l = l_1\left(\frac{d}{d_1}\right)^4 + l_2\left(\frac{d}{d_2}\right)^4 + l_3\left(\frac{d}{d_3}\right)^4$$

$$l = 250 + 200\left(\frac{30}{25}\right)^4 + 100\left(\frac{30}{20}\right)^4 = 1.17\ m$$

The given system may therefore be replaced by the two-rotor system shown above, the shaft being 30 mm diameter throughout and 1.17 m long. If *K* is the node,

then $$I_B a = I_C b$$

i.e. $4.455\ a = 9.72b$...... 1

Also $a + b = 1.17$ 2

$\therefore a = 0.80$ m and $b = 0.37$ m

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{GJ}{I_B a}} = \frac{1}{2\pi}\sqrt{\left[\frac{80\times 10^9\times\pi\times 0.03^4}{4.455\times 32\times 0.37}\right]} = 62.12\ \text{Hz}$$

Solution 2

$$\frac{\theta_B}{\theta_C} = \frac{a}{b} = 2.16 \quad \ldots\ldots 3$$

$$\theta_B + \theta_C = 1.2° \ldots\ldots 4$$

$$\theta_C = 0.38° \quad and \quad \theta_C = 0.82°$$

Example 3

A uniform shaft 85 mm diameter carries three rotors *A*, *B* and *C* having moments of inertia of 17, 40 and 24 kg m^2 respectively. The distance between *A* and *B* is 0·75 m and between *B* and *C* is 1.35 m. Find the frequencies of the free torsional vibration. If the rotor *A* has an amplitude of 1^0 in each case, find the amplitudes of *B* and *C*. The modulus of rigidity of the shaft is 80 GN/m^2.

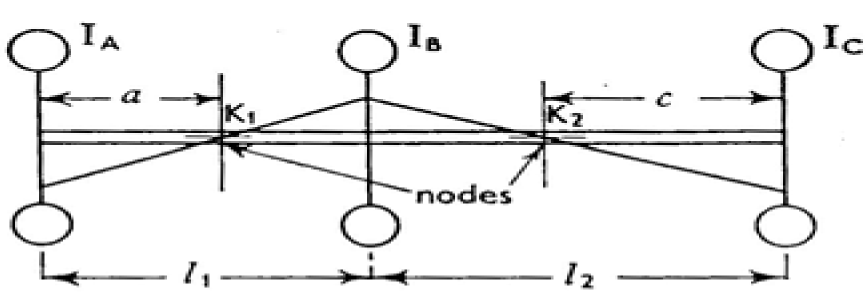
Solution 3

Referring to Figure, I_A = 17 kg m^2, I_B = 40 kg m^2 and I_C = 24 kg m^2 and, by $I_A a = I_C c = \frac{I_B}{\frac{1}{l_1 - a} + \frac{1}{l_2 - c}}$

:. 17a = 24c from which a = 1·413c

and

$$24c = \frac{40}{\frac{1}{0.75 - 1.413c} + \frac{1}{1.35 - c}}$$



$\therefore$ c = 0·342 m or 1·028 m

and a = 0·483 m or 1·454 m

Solution 3 cont'd…

The first pair of values locate the nodes for the two-node vibration and,

$$n_A = \frac{1}{2\pi}\sqrt{\left[\frac{80 \times 10^9 \times \frac{\pi}{32} \times 0.085^4}{17 \times 0.483}\right]} = 35.6 \text{ Hz}$$

The value c = 1·028m gives the position of the node for the single-node vibration, the value for *a* being greater than 0·75 m.

$$n_C = \frac{1}{2\pi}\sqrt{\left[\frac{80 \times 10^9 \times \frac{\pi}{32} \times 0.085^4}{24 \times 1.028}\right]} = 20.5 \text{ Hz}$$

If the amplitude of *A* is 1°, the other amplitudes may be found from the elastic lines shown. For-the-two-node vibration,

$$\theta_B = \frac{l_1 - a}{a} \times 1^0 = \frac{0.267}{0.483} \times 1^0 = 0.553^0$$

and $$\theta_C = \frac{c}{l_2 - c} \times 0.553^0 = \frac{0.342}{1.008} \times 0.553^0 = 0.188^0$$

For the single-node vibration,

$$\theta_B = \frac{a' - l_1}{a'} \times 1^0 = \frac{0.705}{0.454} \times 1^0 = 0.484^0$$

and $$\theta_C = \frac{c\prime}{l_2 - c\prime} \times 0.484^0 = \frac{1.028}{0.322} \times 0.484^0 = 0.545^0$$

Example 4

Two parallel shafts *A* and *B*, of diameters 50 and 75 mm respectively, are connected by a pair of gear wheels, the speed of *A* being five times that of *B*. A flywheel of mass 55 kg and radius of gyration 240 mm is mounted on shaft *A* at a distance of 0·9 m from the gears. Shaft *B* also carries a flywheel, of mass 90 kg and radius of gyration 430 mm, at a distance of 0·6 m from the gears. Neglecting the effect of the shaft and gear masses, find the fundamental frequency of free torsional oscillations, and the position of the node. Modulus of rigidity = 80 GN/m^2.

Solution 4

$$I_A = 55 \times 0.24^2 = 3{\cdot}17 \text{ kg m}^2, \quad I_B = 90 \times 0.43^2 = 16.65 \text{ kg m}^2$$

$$r = \frac{\omega_B}{\omega_A} = \frac{1}{5}$$

$I'_B = \frac{1}{5^2}$ x 16.65 = 0.666 kg m^2 and $l'_2 = 5^2 \times 0{\cdot}6$

The given system therefore reduces to a simple two-rotor system, the shaft connecting I_A and I'_B being 50 mm diameter and 3·86 m long.

If the node divides the shaft into two parts of lengths *a* and *b*, then

$$I_A a = I'_B b$$

i.e.

$$b = \frac{3.17}{0.666} a = 4.76a$$

also $a + b = 3{\cdot}86$ m

$\therefore a = 0{\cdot}67$ m

$$\therefore \quad n = \frac{1}{2\pi}\sqrt{\frac{GJ}{I_A a}} = \frac{1}{2\pi}\sqrt{\left[\frac{80 \times 10^9 \times \frac{\pi}{32} \times 0.05^4}{3.17 \times 0.67}\right]} = 24.2 \text{ Hz}$$

The node occurs at a distance of 0·67 m from rotor A.

ASSIGNMENT 1

A rotor has a mass of 225 kg and has a radius of gyration of 400 mm. It is bolted between the ends of two shafts one of which is 75 mm diameter, 0.9 m long and the other is 65 mm diameter, 0·45 m long. The other ends of the shafts are rigidly fixed in position. Find the frequency of the natural torsional vibrations of the system.

ASSIGNMENT 2

The upper end of a vertical steel wire 2 mm diameter and 2 m long is held securely. The other end is fixed central to a steel cylinder, 75 mm diameter and of density $7.8 Mg/m^3$, arranged with its axis horizontally. Find the length of the cylinder to give 0·6 torsional vibration per second. Calculate the amplitude of the vibrations when the maximum shearing stress is $120 MN/m^2$.

ASSIGNMENT 3

A solid shaft *AB*, of 100 mm diameter, is rigidly connected at its ends to two hollow shafts *AC* and *ED* (both of external diameter 138 mm and internal diameter 106 mm, At C and *D* two masses of 130 kg and 250 kg, and of radius of gyration 175 mm and 375 mm respectively, are attached to the ends of the hollow shafts. Determine from first principles the frequency of free torsional vibrations and the position of the node.