Solutions to Homework 8

Section 12.4 # 6: Convert the integral to an integral in polar coordinates and then compute the integral:

$$\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$$

Solution: Begin by sketching the region. Since $0 \le x \le \sqrt{4 - y^2}$ and $0 \le y \le 2$, one determines that the region is the quarter of the disc of radius 2 centered at $(0, 0)$ which lies in the first quadrant. Choose a fixed θ value; this corresponds to a ray. If the ray intersects the region, one sees that as r varies, r ranges from 0 to 2. Then, allowing θ to vary over all rays which intersect the disc, one sees that θ ranges from 0 to $\pi/2$. Therefore, the integral in polar coordinates is

$$\int_0^{\pi/2} \int_0^2 r^3\, dr\, d\theta,$$

since $x^2 + y^2 = r^2$ and since $dx\, dy = r\, dr\, d\theta$. The computation is straightforward. Answer: 2π.

Section 12.4 # 8: Convert the integral to an integral in polar coordinates and then compute the integral:

$$\int_0^2 \int_0^x y\, dy\, dx.$$

Solution: Skecth the region. The region in the cartesian plane is the filled triangle having vertices $(0, 0)$, $(2, 0)$, $(2, 2)$. Choosing a fixed θ and letting r vary, one see that r ranges from 0 to the r-coordinate of a point on the line $x = 2$. Since this equation, in polar coordinates is $r \cos\theta = 2$, r ranges from 0 to $2 \sec\theta$. Then, allowing θ to vary, θ ranges from 0 to $\pi/4$ since the line $y = x$ makes an angle of $\pi/4$ with the positive x axis. Therefore, the integral in polar coordinates is

$$\int_0^{\pi/4} \int_0^{2\sec\theta} r^2 \sin\theta\, dr\, d\theta.$$

The computation of the innermost integral is as follows:

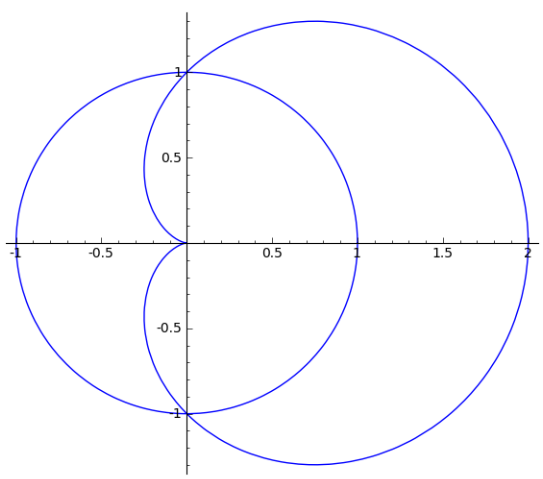
$$\frac{1}{3}r^3 \sin\theta \Big|_0^{2\sec\theta} = \frac{8}{3}\sec^3\theta \sin\theta = \frac{8}{3}\sec^2\theta \tan\theta.$$

The computation of the next integral (using the substitution $u = \tan\theta$ so that $du = \sec^2\theta$) is

$$\frac{8}{3}\frac{1}{2}\tan^2\theta \Big|_0^{\pi/4} = \frac{4}{3}$$

It is worth noting two aspects of this problem. First, you can check your work since the original integral is easy to compute. And, second, in practice you would not choose to convert this integral to polar coordinates since the original integral is already simple to compute.

Section 12.4 # 18: Find the area of the region that lies inside the cardiod $r = 1 + \cos\theta$ and outside the circle $r = 1$.



Solution: Using symmetry, calculate the area of the portion of the region which lies above the x-axis and double its value:

$$2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta.$$

The limits of integration for the integral above are clear from the sketch, but you can verify these values algebraically. The circle $r = 1$ intersects the cardiod $r = 1 + \cos\theta$ when $1 = 1 + \cos\theta$, i.e. when $\cos\theta = 0$. One such solution is when $\theta = \pi/2$.

Now for the computation. The innermost integral is equal to

$$\frac{1}{2}r^2\bigg|_1^{1+\cos\theta} = \frac{1}{2}\left((1+\cos\theta)^2 - 1)\right) = \frac{1}{2}(2\cos\theta + \cos^2\theta).$$

To compute the outer integral, use the following (or use the half-angle formula for $\cos\theta$):

$$\int \cos^2 x\, dx = \frac{1}{2}(x + \cos x \sin x) + C.$$

Therefore, the outer integral (together with the doubling factor) is equal to

$$\left(2\sin\theta + \frac{1}{2}(x + \cos x \sin x)\right)\bigg|_0^{\pi/2} = 2 + \frac{\pi}{4}$$

Section 12.5 # 22: The region of integration of the following integral is shown on p. 685 of the textbook:

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx.$$

Rewrite the integral as an equivalent interated integral for each of the other five permutations of the symbols $dz\, dy\, dx$.

Solution: Please refer to the figure in the book.

(a) $dy\, dz\, dx$: As y varies and x and z are held constant, the region is bounded below (meaning the lowest y-value, which in the figure in the book appears to be on the left) by the plane $y = -1$ and bounded above (right) by the parabolic cyllinder $z = y^2$. Thus, $-1 \le y \le -\sqrt{z}$ since $y \le 0$ in the figure. Next, consider the shadow, i.e. the projection, of the solid onto the xz-plane. This shadow is a square where $0 \le x \le 1$ and $0 \le z \le 1$. Therefore, the integral above is equivalent to

$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx.$$

(b) $dy\,dx\,dz$: As in the previous case, the shadow is a square. Therefore, the answer is

$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\,dx\,dz.$$

(c) $dx\,dy\,dz$: As x varies and y and z are held constant, the region is bounded below by the plane $x = 0$ and above by the plane $x = 1$. The shadow of the solid on the yz-plane is the region bounded by the y-axis, the parabola $z = y^2$ and the line $y = -1$ (all in the yz-plane). Therefore, the answer is

$$\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\,dy\,dz.$$

(d) $dx\,dz\,dy$: From the previous case it is easy to see that the answer is

$$\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\,dz\,dy.$$

(e) $dz\,dx\,dy$: As z varies and x and y are held contant, the region is bounded below by the plane $z = 0$ and above by the parabolic cylinder $z = y^2$. The shadow on the xy-plane is the square bounded by the axes and the lines $x = 1$ and $y = -1$ (in the xy-plane). Therefore, the answer is

$$\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\,dx\,dy.$$

Section 12.5 # 24: Compute the volume of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1$ and $y + 2z = 2$.

Solution: This problem was solved in class on 03/19. The key observation was that the solid is simple in both the x and y directions, but not in the z-direction: so do not choose the innermost integral to be with respect to z! Answer: 2/3.