SUPPLEMENTARY LECTURE NOTES

LU Decomposition

After this lecture you should be able to:

1. State when LU Decomposition is numerically more efficient than Gaussian Elimination,
2. Decompose a non-singular matrix into LU,
3. Show how LU Decomposition is used to find matrix inverse

1.0. Introduction

We already studied two numerical methods of finding the solution to simultaneous linear equations – Naive Gauss Elimination and Gaussian Elimination with Partial Pivoting. To appreciate why *LU* Decomposition could be a better choice than the Gaussian Elimination techniques in some cases, let us discuss first what *LU* Decomposition is about.

For any non-singular matrix A on which one can conduct Naive Gaussian Elimination or forward elimination steps, one can always write it as $A = LU$
where

L = Lower triangular matrix (specifically, a unit lower triangular Matrix)

U = Upper triangular matrix

Then if one is solving a set of equations $Ax = b$, it will imply that $LUx = b$ since $A = LU$.
Multiplying both side by L^{-1}, we have

$L^{-1}LUx = L^{-1}b$

$\Rightarrow IUx = L^{-1}b$ since $\left(L^{-1}L = I\right)$,

$\Rightarrow Ux = L^{-1}b$ since $\left(IU = U\right)$

Let $L^{-1}b = z$ then $Lz = b$ (1)

And $Ux = z$ (2)

So we can solve equation (1) first for z and then use equation (2) to calculate x.

The computational time required to decompose the A matrix to LU form is proportional to $\frac{n^3}{3}$, where *n* is the number of equations (size of A matrix). Then to solve the $Lz = b$, the computational time is proportional to $\frac{n^2}{2}$. Then to solve the $Ux = z$, the computational time is proportional to $\frac{n^2}{2}$. So the total computational time to solve a set of equations by *LU* decomposition is proportional to $\frac{n^3}{3}+n^2$.

In comparison, Gaussian elimination is computationally more efficient. It takes a computational time proportional to $\frac{n^3}{3}+\frac{n^2}{2}$, where the computational time for forward elimination is proportional to $\frac{n^3}{3}$ and for the back substitution the time is proportional to $\frac{n^2}{2}$.

Finding the inverse of the matrix A reduces to solving *n* sets of equations with the *n* columns of the identity matrix as the RHS vector. For calculations of each column of the inverse of the A matrix, the coefficient matrix A matrix in the set of equation $Ax=b$ does not change. So if we use *LU* Decomposition method, the $A=LU$ decomposition needs to be done only once and the use of equations (1) and (2) still needs to be done '*n*' times.

So the total computational time required to find the inverse of a matrix using *LU* decomposition is proportional to $\frac{n^3}{3}+n(n^2)=\frac{4n^3}{3}$.

In comparison, if Gaussian elimination method were applied to find the inverse of a matrix, the time would be proportional to $n\left(\frac{n^3}{3}+\frac{n^2}{2}\right)=\frac{n^4}{3}+\frac{n^3}{2}$.

For large values of *n*, $\frac{n^4}{3}+\frac{n^3}{2}\rangle\rangle\frac{4n^3}{3}$

1.1. Decomposing a non-singular matrix A into the form A=LU.

***L U* Decomposition Algorithm**:

In these methods the coefficient matrix *A* of the given system of equation $Ax=b$ is written as a product of a Lower triangular matrix *L* and an Upper triangular matrix *U*, such that $A=LU$ where the elements of $L=(l_{ij}=0;\ \text{for } i<j)$ and the elements of $U=(u_{ij}=0;\ \text{for } i>j)$ that is,

$$L=\begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \text{ and } U=\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}.$$

Now using the rules of matrix multiplication $a_{ij}=\sum_{k=1}^{\min(i,j)} l_{ik}u_{kj},\quad i,j=1,\ldots,n$

This gives a system of n^2 equations for the n^2+n unknowns (the non-zero elements in *L* and *U*). To make the number of unknowns and the number of equations equal one can fix the diagonal element either in *L* or in *U* such as '1's then solve the n^2 equations for the remaining n^2 unknowns in *L* and *U*. This leads to the following algorithm:

Algorithm

The factorization $A = LU$, where $L = \left(l_{ij}\right)_{n\times n}$ is a lower triangular and $U = \left(u_{ij}\right)_{n\times n}$ an upper triangular, can be computed directly by the following algorithm (provided zero divisions are not encountered):

Algorithm

For $k = 1 \text{ to } n$ *do specify* $\left(l_{kk} \text{ or } u_{kk}\right)$ *and compute the other such that* $l_{kk}u_{kk} = a_{kk} - \sum_{m=1}^{k-1} l_{km}u_{mk}$.

Compute the k^{th} *column of* L *using* $l_{ik} = \dfrac{1}{u_{kk}}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im}u_{mk}\right) \quad \left(k < i \le n\right)$, *and compute the* k^{th} *row of* U *using* $u_{kj} = \dfrac{1}{l_{kk}}\left(a_{kj} - \sum_{m=1}^{k-1} l_{km}u_{mj}\right) \quad \left(k < j \le n\right)$

End

Note: The procedure is called ***Doolittle*** or ***Crout*** Factorization when $l_{ii} = 1 \quad \left(1 \le i \le n\right)$ or $u_{jj} = 1 \quad \left(1 \le j \le n\right)$ respectively.

If forward elimination steps of Naive Gauss elimination methods can be applied on a non-singular matrix, then A can be decomposed into *LU* as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \ell_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \ell_{n1} & \cdots & \ell_{n-1,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & \mathrm{u}_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix} = LU$$

1. The elements of the U matrix are exactly the same as the coefficient matrix one obtains at the end of the forward elimination steps in Naive Gauss Elimination.
2. The lower triangular matrix L has 1 in its diagonal entries. The non-zero elements below the diagonal in L are multipliers that made the corresponding entries zero in the upper triangular matrix U during forward elimination.

1.2. Solving Ax=b in pure matrix Notations

Solving systems of linear equations (AX=b) using LU factorization can be quite cumbersome, although it seem to be one of the simplest ways of finding the solution for the system, Ax=b. In pure matrix notation, the upper triangular matrix, U, can be calculated by constructing specific permutation matrices and elementary matrices to solve the Elimination process with both partial and complete pivoting.

The elimination process is equivalent to pre multiplying A by a sequence of lower-triangular matrices M_k as follows:

$$M_{k-1}M_{k-2}\ldots M_1A = U$$

Where $M_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & 0 & 1 \end{bmatrix}$, $\mathrm{M}_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & m_{n2} & \cdots & 1 \end{bmatrix}$ with $m_{ij} = -\dfrac{a_{ij}^{(j-1)}}{a_{jj}}$ known as the multiplier

In solving Gaussian elimination without partial pivoting to triangularize *A,* the process yields the factorization, $MA = U$. In this case, the system *Ax* = *b* is equivalent to the triangular system $Ux = Mb = b'$ where $M = M_{k-1}M_{k-2}M_{k-3}\ldots M_1$

The elementary matrices M_1 and M_2 are called first and second Gaussian transformation matrix respectively with M_k being the k^{th} Gaussian transformation matrix.
Generally, to solve $Ax = b$ using Naive Gaussian elimination without partial pivoting by this approach, a permutation matrix is introduced to perform the pivoting strategies:

First We find the factorization $MA = U$ by the triangularization algorithm using partial pivoting. We then solve the triangular system by back substitution as follows $Ux = Mb = b'$.

Note that $M = M_{n-1}P_{n-1}M_{n-2}P_{n-2}\cdots M_2P_2M_1P_1$
The vector $b' = Mb = M_{n-1}P_{n-1}M_{n-2}P_{n-2}\cdots M_2P_2M_1P_1b$
Generally if we set $s_1 = b = \left(b_1, b_2, \ldots, b_n\right)^T$
Then For $k = 1, 2, \ldots, n-1$ do

$$s_{k+1} = M_kP_ks_k$$

Example

If $n = 3$, $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}$, then $s_2 = M_1 P_1 s_1 = \begin{pmatrix} s_1^{(2)} \\ s_2^{(2)} \\ s_3^{(2)} \end{pmatrix}$

If $P_1 s_1 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then the entries of $s^{(2)}$ are given by:

$$s_1^{(2)} = b_1$$
$$s_2^{(2)} = m_{21} b_1 + b_3$$
$$s_3^{(2)} = m_{31} b_1 + b_2$$

In the same way, to solve **Ax = b** Using Gaussian Elimination with Complete Pivoting, we modify the previous construction to include another permutation matrix, Q such that when post multiplied by A, we can perform column interchange. This results in $M(PAQ) = U$

Example 1

Find the *LU* decomposition of the matrix

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The U matrix is the same as found at the end of the forward elimination of Naive Gauss elimination method, that is

<u>Forward Elimination of Unknowns:</u> Since there are three equations, there will be two steps of forward elimination of unknowns.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

First step: Divide Row 1 by 25 and then multiply it by 64 and subtract the results from Row 2

$$Row\ 2-\left[\frac{64}{25}\right]\times(Row\ 1)=\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 144 & 12 & 1\end{bmatrix}$$

Here the multiplier , $m_{21}=-\frac{64}{25}$

Divide Row 1 by 25 and then multiply it by 144 and subtract the results from Row 3

$$Row\ 3-\left[\frac{144}{25}\right]\times(Row\ 1)=\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & -16.8 & -4.76\end{bmatrix}$$

Here the multiplier , $m_{31}=-\frac{144}{25}$, hence the first Gaussian transformation matrix is given by:

$$M_1=\begin{bmatrix}1 & 0 & 0\\ m_{21} & 1 & 0\\ m_{31} & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2.56 & 1 & 0\\ -5.76 & 0 & 1\end{bmatrix}$$ And the corresponding product is given by:

$$A^{(1)}=M_1A=\begin{bmatrix}1 & 0 & 0\\ -2.60 & 1 & 0\\ -5.76 & 0 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}=\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & -16.8 & -4.76\end{bmatrix}$$ (by a single multiplication)

Second step: We now divide Row 2 by -4.8 and then multiply by -16.8 and subtract the results from Row 3

$$Row\ 3-\left[\frac{-16.8}{-4.8}\right]\times(Row\ 2)=\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$ which produces $U=\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$

Here the multiplier, $m_{32}=-\frac{-16.8}{-4.8}$, hence the 2nd Gaussian transformation matrix is given by:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3.5 & 1 \end{bmatrix}$$ And the corresponding product is given by:

$$A^{(2)} = M_2 A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = U \text{ (by a single multiplication)}$$

To find ℓ_{21} and ℓ_{31}, what multiplier was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination of Naive Gauss Elimination Method It was

$$\ell_{21} = -m_{21} = \frac{64}{25} = 2.56$$

$$\ell_{31} = -m_{31} = \frac{144}{25} = 5.76$$

To find ℓ_{32}, what multiplier was used to make a_{32} element zero. Remember a_{32} element was made zero in the second step of forward elimination. The A matrix at the beginning of the second step of forward elimination was

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

So

$$\ell_{32} = -m_{32} = \frac{-16.8}{-4.8} = 3.5$$

Hence

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Confirm $LU = A$.

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Example 2

Use *LU* decomposition method to solve the following linear system of equations.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution

Recall that $Ax = b$ and if $A = LU$ then first solving $Lz = b$ and then $Ux = z$ gives the solution vector x.

Now in the previous example, we showed

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

First solve $Lz = b$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

to give

$$\begin{aligned} z_1 \qquad\qquad &= 106.8 \\ 2.56z_1 + \quad z_2 \qquad &= 177.2 \\ 5.76z_1 + 3.5z_2 + z_3 &= 279.2 \end{aligned}$$

Forward substitution starting from the first equation gives

$$\begin{aligned} z_1 &= 106.8 \\ z_2 &= 177.2 - 2.56z_1 = 177.2 - 2.56(106.8) = -96.2 \\ z_3 &= 279.2 - 5.76z_1 - 3.5z_2 = 279.2 - 5.76(106.8) - 3.5(-96.21) = 0.735 \end{aligned}$$

Hence

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

This matrix is same as the right hand side obtained at the end of the forward elimination steps of Naive Gauss elimination method. Is this a coincidence?

Now solve $Ux = z$, i.e.,

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

$$\begin{aligned} 25x_1 + \ \ 5x_2 + \ \ \ \ x_3 &= 106.8 \\ -4.8x_2 - 1.56x_3 &= -96.21 \\ 0.7x_3 &= 0.735 \end{aligned}$$

From the third equation $0.7x_3 = 0.735 \ \Rightarrow \ x_3 = \dfrac{0.735}{0.7} = 1.050$

Substituting the value of a_3 in the second equation,

$$-4.8x_2 - 1.56x_3 = -96.21 \ \Rightarrow \ x_2 = \frac{-96.21 + 1.56x_3}{-4.8} = \frac{-96.21 + 1.56(1.050)}{-4.8} = 19.70$$

Substituting the value of x_2 and x_3 in the first equation,

$$\begin{aligned} 25x_1 + 5x_2 + x_3 = 106.8 \ \Rightarrow \ x_1 &= \frac{106.8 - 5x_2 - x_3}{25} \\ &= \frac{106.8 - 5(19.70) - 1.050}{25} = 0.2900 \end{aligned}$$

The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Example 3

Solve

$$Ax = b \text{ with } A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

(a) using partial pivoting and (b) using complete pivoting.

Solution:

(a) Partial pivoting:

We compute *U* as follows:

$Step 1:$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_1A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix};$$

$$A^{(1)} = M_1P_1A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix};$$

$Step 2:$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_2A^{(1)} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}; \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix};$$

$$U = A^{(2)} = M_2P_2A^{(1)} = M_2P_2M_1P_1A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$Note$: Defining $P = P_2P_1$ and $L = P(M_2P_2M_1P_1)^{-1}$, we have $PA = LU$.

We compute b' as follows:

$Step 1:$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_1b = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}; \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \quad M_1P_1b = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix};$$

$Step 2:$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_2M_1P_1b = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}; \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \quad b' = M_2P_2M_1P_1b = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix};$$

The solution of the system

$$Ux = b' \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \text{ and } x_1 = x_2 = x_3 = 1$$

(b) Complete pivoting: We compute *U* as follows:

$Step 1$:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad P_1AQ_1 = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}; \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix};$$

$$A^{(1)} = M_1P_1AQ_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$Step 2$:

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad P_2A^{(1)}Q_2 = \begin{pmatrix} 3 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix};$$

$$U = A^{(2)} = M_2P_2A^{(1)}Q_2 = M_2P_2M_1P_1AQ_1Q_2 = \begin{pmatrix} 3 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$Note$: Defining $P = P_2P_1$, $Q = Q_1Q_2$ and $L = P(M_2P_2M_1P_1)^{-1}$, we have $PAQ = LU$.

We compute b' as follows:

$Step 1$:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad P_1b = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}; \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}; \quad M_1P_1b = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

$Step 2$:

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad P_2M_1P_1b = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}; \quad M_1P_1b = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix};$$

$$b' = M_2P_2M_1P_1b = \begin{pmatrix} 6 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

The solution of the system

$$Uy = b' \quad \Rightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

is $y_1 = y_2 = y_3 = 1$. Because $\{x_k\},\ k = 1,2,3$ is simply the rearrangement of $\{y_k\}$, we have $x_1 = x_2 = x_3 = 1$.

1.4 Finding the inverse of a square matrix using LU Decomposition

A matrix B is the inverse of A if $AB = I = BA$. First assume that the first column of B (the inverse of A is $[b_{11}\ b_{21}\ \cdots\ b_{n1}]^T$ then from the above definition of inverse and definition of matrix multiplication.

$$A\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly the second column of B is given by

$$A\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly, all columns of B can be found by solving *n* different sets of equations with the column of the right hand sides being the *n* columns of the identity matrix.

Example 3

Use *LU* decomposition to find the inverse of

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Knowing that

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

We can solve for the first column of $B = A^{-1}$ by solving for

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First solve $Lz = c$, that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

to give

$$\begin{aligned} z_1 \qquad\qquad &= 1 \\ 2.56 z_1 + \quad z_2 \qquad &= 0 \\ 5.76 z_1 + 3.5 z_2 + z_3 &= 0 \end{aligned}$$

Forward substitution starting from the first equation gives

$$\begin{aligned} &\mathrm{z}_1 = 1 \\ &\mathrm{z}_2 = 0\text{-}2.56\mathrm{z}_1 = 0 - 2.56(1) = -256 \\ &z_3 = 0 - 5.76 z_1 - 3.5 z_2 = 0 - 5.76(1) - 3.5(-2.56) = 3.2 \end{aligned}$$

Hence

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Now solve $Ux = z$, that is

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix} \Rightarrow \begin{aligned} 25b_{11} + 5b_{21} + b_{31} &= 1 \\ -4.8b_{21} - 1.56b_{31} &= -2.56 \\ 0.7b_{31} &= 3.2 \end{aligned}$$

Backward substitution starting from the third equation gives

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560b_{31}}{-4.8} = \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25} = \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

Hence the first column of the inverse of A is

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Similarly by solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

and solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} \mathrm{b}_{13} \\ \mathrm{b}_{23} \\ \mathrm{b}_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} 0.4762 & 0.08333 & 0.0357 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.050 & 1.429 \end{bmatrix}$$

Exercise: Show that $AA^{-1} = I = A^{-1}A$ for the above example.