SUPPLEMENTARY MATERIAL

Simpson 3/8 Rule for Integration

After reading this lecture notes, you should be able to

1. *derive the formula for Simpson's 3/8 rule of integration,*
2. *use Simpson's 3/8 rule it to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 3/8 rule of integration,*
4. *use multiple-segment Simpson's 3/8 rule of integration to solve integrals,*
5. *compare true error formulas for multiple-segment Simpson's 1/3 rule and multiple-segment Simpson's 3/8 rule, and*
6. *use a combination of Simpson's 1/3 rule and Simpson's 3/8 rule to approximate integrals.*

Introduction

The main objective of this lecture is to develop appropriate formulas for approximating the integral of the form

$$I = \int_a^b f(x)dx \tag{1}$$

Most (if not all) of the developed formulas for integration are based on a simple concept of approximating a given function $f(x)$ by a simpler function (usually a polynomial function) $f_i(x)$, where i represents the order of the polynomial function. In the previous method, Simpsons 1/3 rule for integration was derived by approximating the integrand $f(x)$ with a 2nd order (quadratic) polynomial function. $f_2(x)$

$$f_2(x) = a_0 + a_1x + a_2x^2 \tag{2}$$

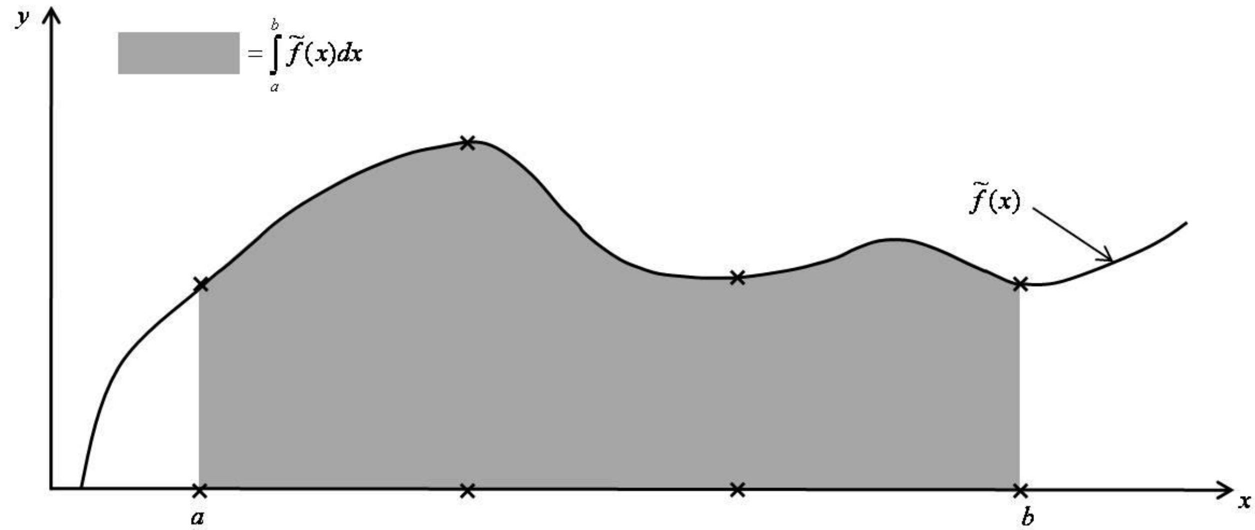

Figure 1 $\tilde{f}(x)$ Cubic function.

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial $f_3(x)$

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}\right\} \tag{3}$$

which can also be symbolically represented in Figure 1.

<u>Method 1</u>

The unknown coefficients a_0, a_1, a_2 and a_3 in Equation (3) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}$ into Equation (3) as follows.

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^2 \\ f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^2 \end{aligned}\right\} \tag{4}$$

Equation (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \tag{5}$$

The above Equation (5) can symbolically be represented as

$$[A]_{4\times 4} \vec{a}_{4\times 1} = \vec{f}_{4\times 1} \tag{6}$$

Thus,

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [A]^{-1} \times \vec{f} \tag{7}$$

Substituting Equation (7) into Equation (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \tag{8}$$

As indicated in Figure 1, one has

$$\left.\begin{aligned} x_0 &= a \\ x_1 &= a + h \\ &= a + \frac{b-a}{3} \\ &= \frac{2a+b}{3} \\ x_2 &= a + 2h \\ &= a + \frac{2b-2a}{3} \\ &= \frac{a+2b}{3} \\ x_3 &= a + 3h \\ &= a + \frac{3b-3a}{3} \\ &= b \end{aligned}\right\} \tag{9}$$

Note that you may use any known symbolic tool(eg. MATLAB, MATHEMATICA) to solve the unknown vector $\vec{a}$ (shown in Equation 7).

<u>Method 2</u>

Using Lagrange interpolation, the cubic polynomial function $f_3(x)$ that passes through 4 data points (see Figure 1) can be explicitly given as

$$\begin{aligned} f_3(x) = {} & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_3) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3) \end{aligned} \tag{10}$$

Simpsons 3/8 Rule for Integration

Substituting the form of $f_3(x)$ from Method (1) or Method (2),

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &\approx \int_a^b f_3(x)dx \end{aligned}$$

$$= (b-a)\times\frac{\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\}}{8} \quad (11)$$

Since

$$h=\frac{b-a}{3}$$
$$b-a=3h$$

and Equation (11) becomes

$$I\approx\frac{3h}{8}\times\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\} \quad (12)$$

Note the 3/8 in the formula, and hence the name of method as the Simpson's 3/8 rule.
The true error in Simpson 3/8 rule can be derived as [Ref. 1]

$$E_t=-\frac{(b-a)^5}{6480}\times f''''(\zeta)\text{, where } a\le\zeta\le b \quad (13)$$

Example 1

The vertical distance covered by a rocket from $x=8$ to $x=30$ seconds is given by

$$s=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8x\right)dx$$

Use Simpson 3/8 rule to find the approximate value of the integral.

Solution

$$h=\frac{b-a}{n}$$
$$=\frac{b-a}{3}$$
$$=\frac{30-8}{3}$$
$$=7.3333$$

$$I\approx\frac{3h}{8}\times\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\}$$

$$x_0=8$$

$$f(x_0)=2000\ln\left(\frac{140000}{140000-2100\times8}\right)-9.8\times8$$
$$=177.2667$$

$$\left\{\begin{aligned} x_1&=x_0+h\\ &=8+7.3333\\ &=15.3333\\ f(x_1)&=2000\ln\left(\frac{140000}{140000-2100\times15.3333}\right)-9.8\times15.3333\\ &=372.4629\end{aligned}\right.$$

$$\begin{cases} x_2 = x_0 + 2h \\ \quad = 8 + 2(7.3333) \\ \quad = 22.6666 \\ f(x_2) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 22.6666}\right) - 9.8 \times 22.6666 \\ \qquad = 608.8976 \end{cases}$$

$$\begin{cases} x_3 = x_0 + 3h \\ \quad = 8 + 3(7.3333) \\ \quad = 30 \\ f(x_3) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 \\ \qquad = 901.6740 \end{cases}$$

Applying Equation (12), one has

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}$$
$$= 11063.3104$$

The exact answer can be computed as

$$I_{exact} = 11061.34$$

Multiple Segments for Simpson 3/8 Rule

Using n = number of equal segments, the width h can be defined as

$$h = \frac{b-a}{n} \tag{14}$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.
The integral shown in Equation (1) can be expressed as

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &\approx \int_a^b f_3(x)dx \\ &\approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + ... + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \end{aligned} \tag{15}$$

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$I = \frac{3h}{8}\left\{\begin{matrix} f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) + f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6) \\ + ... + f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n) \end{matrix}\right\} \tag{16}$$

$$= \frac{3h}{8}\left\{ f(x_0) + 3\sum_{i=1,4,7,..}^{n-2} f(x_i) + 3\sum_{i=2,5,8,..}^{n-1} f(x_i) + 2\sum_{i=3,6,9,..}^{n-3} f(x_i) + f(x_n) \right\} \quad (17)$$

Example 2

The vertical distance covered by a rocket from $x = 8$ to $x = 30$ seconds is given by

$$s = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8x \right) dx$$

Use Simpson 3/8 multiple segments rule with six segments to estimate the vertical distance.

Solution

In this example, one has (see Equation 14):

$$h = \frac{30-8}{6} = 3.6666$$

$$\{x_0, f(x_0)\} = \{8, 177.2667\}$$

$$\{x_1, f(x_1)\} = \{11.6666, 270.4104\} \text{where } x_1 = x_0 + h = 8 + 3.6666 = 11.6666$$

$$\{x_2, f(x_2)\} = \{15.3333, 372.4629\} \text{ where } x_2 = x_0 + 2h = 15.3333$$

$$\{x_3, f(x_3)\} = \{19, 484.7455\} \text{ where } x_3 = x_0 + 3h = 19$$

$$\{x_4, f(x_4)\} = \{22.6666, 608.8976\} \text{ where } x_4 = x_0 + 4h = 22.6666$$

$$\{x_5, f(x_5)\} = \{26.3333, 746.9870\} \text{ where } x_5 = x_0 + 5h = 26.3333$$

$$\{x_6, f(x_6)\} = \{30, 901.6740\} \text{ where } x_6 = x_0 + 6h = 30$$

Applying Equation (17), one obtains:

$$I = \frac{3}{8}(3.6666)\left\{ 177.2667 + 3\sum_{i=1,4,..}^{n-2=4} f(x_i) + 3\sum_{i=2,5,..}^{n-1=5} f(x_i) + 2\sum_{i=3,6,..}^{n-3=3} f(x_i) + 901.6740 \right\}$$

$$= (1.3750)\left\{ \begin{array}{l} 177.2667 + 3(270.4104 + 608.8976) + \\ 3(372.4629 + 746.9870) + 2(484.7455) + 901.6740 \end{array} \right\}$$

$$= 11{,}601.4696$$

Example 3

Compute

$$I = \int_{a=8}^{b=30} \left\{ 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100x} \right) - 9.8x \right\} dx,$$

using Simpson 1/3 rule (with $n_1 = 4$), and Simpson 3/8 rule (with $n_2 = 3$).

Solution

The segment width is

$$h = \frac{b-a}{n}$$

$$= \frac{b-a}{n_1 + n_2}$$

$$= \frac{30-8}{(4+3)}$$

$$= 3.1429$$

$$\left.\begin{aligned} x_0 &= a = 8 \\ x_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\ x_2 &= x_0 + 2h = 8 + 2(3.1429) = 14.2857 \\ x_3 &= x_0 + 3h = 8 + 3(3.1429) = 17.4286 \\ x_4 &= x_0 + 4h = 8 + 4(3.1429) = 20.5714 \end{aligned}\right\} \text{Simpson's 1/3 rule}$$

$$x_5 = x_0 + 5h = 8 + 5(3.1429) = 23.7143$$

$$x_6 = x_0 + 6h = 8 + 6(3.1429) = 26.8571$$

$$x_7 = x_0 + 7h = 8 + 7(3.1429) = 30$$

$$f(x_0 = 8) = 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100 \times 8}\right) - 9.8 \times 8 = 177.2667$$

Similarly:

$$f(x_1 = 11.1429) = 256.5863$$

$$f(x_2) = 342.3241$$

$$f(x_3) = 435.2749$$

$$f(x_4) = 536.3909$$

$$f(x_5) = 646.8260$$

$$f(x_6) = 767.9978$$

$$f(x_7) = 901.6740$$

For multiple segments $(n_1 = \text{first 4 segments})$, using Simpson 1/3 rule, one obtains (See Equation 19):

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\sum_{i=1,3,\ldots}^{n_1-1=3} f(x_i) + 2\sum_{i=2,\ldots}^{n_1-2=2} f(x_i) + f(x_{n_1}) \right\}$$

$$= \left(\frac{3.1429}{3}\right)\{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$= 4364.1197$$

For multiple segments $(n_2 = \text{last 3 segments})$, using Simpson 3/8 rule, one obtains (See Equation 17):

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f(x_0) + 3\sum_{i=1,3,\ldots}^{n_2-2=1} f(x_i) + 3\sum_{i=2,\ldots}^{n_2-1=2} f(x_i) + 2\sum_{i=3,6,\ldots}^{n_2-3=0} f(x_i) + f(x_{n_1}) \right\}$$

$$= \left(\frac{3}{8} \times 3.1429\right)\{177.2667 + 3(256.5863) + 3(342.3241) + (\text{no contribution}) + 435.2749\}$$

$$= 6697.2748$$

The mixed (combined) Simpson 1/3 and 3/8 rules give

$$I = I_1 + I_2$$
$$= 4364.1197 + 6697.2748$$
$$= 11{,}061.3946$$

Comparing the truncated error of Simpson 1/3 rule

$$E_t = -\frac{(b-a)^5}{2880} \times f''''(\zeta) \tag{18}$$

With Simpson 3/8 rule (See Equation 12), it seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significantly higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).

The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2, 4, 6, 8, … (any even numbers).

$$I_1 = \left(\frac{h}{3}\right)\{f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \ldots.. + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\}$$

$$= \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,\ldots}^{n-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n-2} f(x_i) + f(x_n)\right\} \tag{19}$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be certain odd or even numbers that are multiples of 3).

If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments) would be appropriate.

Computer Algorithm for Mixed Simpson 1/3 and 3/8 Rule for Integration

Based on the earlier discussion on (single and multiple segments) Simpson 1/3 and 3/8 rules, the following "pseudo" step-by-step mixed Simpson rules can be given as

<u>Step 1</u>

User inputs information, such as

$f(x)$ = integrand

n_1 = number of segments in conjunction with Simpson 1/3 rule (a multiple of 2 (any even numbers)

n_2 = number of segments in conjunction with Simpson 3/8 rule (a multiple of 3)

<u>Step 2</u>

Compute

$$n = n_1 + n_2$$
$$h = \frac{b-a}{n}$$

$x_0 = a$

$x_1 = a + 1h$

$x_2 = a + 2h$

.

.

$x_i = a + ih$

.

.

$x_n = a + nh = b$

Step 3

Compute result from multiple-segment Simpson 1/3 rule (See Equation 19)

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\sum_{i=1,3,\ldots}^{n_1-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n_1-2} f(x_i) + f(x_{n_1}) \right\} \qquad \text{(19, repeated)}$$

Step 4

Compute result from multiple segment Simpson 3/8 rule (See Equation 17)

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f(x_0) + 3\sum_{i=1,4,7\ldots}^{n_2-2} f(x_i) + 3\sum_{i=2,5,8\ldots}^{n_2-1} f(x_i) + 2\sum_{i=3,6,9,\ldots}^{n_2-3} f(x_i) + f(x_{n_2}) \right\} \qquad \text{(17, repeated)}$$

Step 5

$$I \approx I_1 + I_2 \qquad (20)$$

and print out the final approximated answer for I .