Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNITS

MAS212 Linear Algebra I

Tuesday 8 May 2007, 2:30 pm – 4:30 pm

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Show your working in full; marks will be awarded for method.

If not specified then assume that the field of scalars is the field of rational numbers $\mathbb{Q}$.

You must not remove this question paper from the examination room.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A

This section carries 56 marks and each question carries 7 marks. You should attempt ALL 8 questions. ***Do* not *begin each answer in this section on a fresh page. Write the number of the question in the* left *margin.***

A1. Let $A = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. For each of the following matrix products, either evaluate it or state that is does not exist: AB, BA, A^TC, B^TC, CB^T.

A2. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. Compute the determinant of A and, if it exists, compute *and check* the inverse of A.

A3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Use only rank computations to determine (a) the dimension of the solution space of the matrix equation $Ax = 0$ and (b) whether the matrix equation $Ax = b$ has a solution. [You may solve the equations explicitly to check your answer, but no marks will be awarded for explicit solutions.]

A4. (a) For the three-dimensional Euclidean vector space $\mathbb{R}^3$, give precise definitions of the set of vectors and the operations of vector addition and scalar multiplication.

(b) Let U be a subset of a vector space V over a field $\mathbb{K}$ that inherits the operations defined on V. Give a complete but minimal set of explicitly testable conditions for U to be a vector subspace of V.

(c) Give an example of a two-dimensional vector subspace of $\mathbb{R}^3$ that does not contain any of the standard basis vectors.

A5. Let V be a vector space over a field $\mathbb{K}$.

(a) Define a *linear combination* of vectors in V.

(b) Define a *spanning set* for V.

(c) Prove that $\{(1,2,3),(2,3,1),(3,1,2),(1,1,1)\}$ is a spanning set for the vector space $\mathbb{R}^3$.

A6. Let V be a vector space over a field $\mathbb{K}$ and let S be a set of vectors in V.

(a) Define *linear dependence* of S.

(b) Prove that S is linearly dependent if and only if one vector in S is a linear combination of the others.

(c) Prove that $\{(1,2,3),(2,3,1),(3,1,2),(1,1,1)\}$ is a linearly dependent set of vectors in the vector space $\mathbb{R}^3$.

[Next question overleaf]

A7. (a) Let $\alpha : U \to V$ be a map between two vector spaces U, V over a field $\mathbb{K}$. Give a complete but minimal set of conditions for α to be a *linear map*.

(b) Let $\beta : S \to T$ be a linear map between two vector spaces S, T over the field $\mathbb{K}$. Define the composed map $\beta\alpha$ (or $\beta \circ \alpha$) and state the condition for it to exist.

(c) Prove that $\beta\alpha$ is a linear map, assuming it exists.

A8. (a) Define the standard inner product and the Euclidean norm on the three-dimensional Euclidean vector space $\mathbb{R}^3$, i.e. $\langle x, y\rangle$ and $||x||$ for $x, y \in \mathbb{R}^3$.

(b) Define an *orthonormal set* of vectors.

(c) In $\mathbb{R}^3$, let $u = (1, 1, 1)$. By finding the condition for the general vector (x, y, z) to be orthogonal to u, find a vector v that is orthogonal to u. Then, by finding the condition for the general vector (x, y, z) to be orthogonal to v, find a vector w that is orthogonal to *both* u and v. Hence construct an orthonormal set of vectors that contains a scalar multiple of $(1, 1, 1)$.

[Next section overleaf]

SECTION B

This section carries 44 marks and each question carries 22 marks. You may attempt all 4 questions but, except for the award of a bare pass, only marks for the best 2 questions will be counted. ***Begin each answer in this section on a fresh page. Write the number of the question at the* top *of* each *page.***

B1. (a) [8 marks] Let $\alpha : U \to V$ be a linear map between two vector spaces U, V over a field $\mathbb{K}$. Define the *kernel* $\ker(\alpha)$ and *image* $\operatorname{im}(\alpha)$ of α. Prove that $\ker(\alpha)$ and $\operatorname{im}(\alpha)$ are both vector subspaces. You may assume that $\alpha(0) = 0$.

(b) [2 marks] Let S, T be vector subspaces of a vector space U. Define the sum $S + T$ and the direct sum $S \oplus T$.

(c) [4 marks] Suppose that $\alpha : U \to U$ is a linear map from a vector space U to the *same* vector space U. Prove that $U = \ker(\alpha) \oplus \operatorname{im}(\alpha)$ provided $\ker(\alpha) + \operatorname{im}(\alpha)$ is a direct sum. State carefully any theorems you quote.

(d) [8 marks] Find basis sets for the kernel and image of the linear map

$$\alpha : \mathbb{R}^3 \to \mathbb{R}^3, \quad \alpha(x, y, z) = (x + y, y - z, z + x).$$

[You need not prove that they are basis sets.] Hence verify explicitly that $\ker(\alpha) + \operatorname{im}(\alpha)$ is a direct sum and that $\mathbb{R}^3 = \ker(\alpha) \oplus \operatorname{im}(\alpha)$.

B2. (a) [2 marks] Let U, V be vector spaces over a field $\mathbb{K}$ with ordered bases $\mathcal{B} = u_1, u_2, \ldots, u_m$ and $C = v_1, v_2, \ldots, v_n$ and let $\alpha : U \to V$ be a linear map. Define the matrix representation A of α with respect to the bases $\mathcal{B}, C$, i.e. $A = (\alpha; \mathcal{B}, C)$.

(b) [3 marks] Let U, V, W be vector spaces over a field $\mathbb{K}$ with ordered bases $\mathcal{B} = u_1, u_2, u_3$, $C = v_1, v_2, v_3$ and $\mathcal{D} = w_1, w_2$ respectively. Let $\alpha : U \to V$ and $\beta : V \to W$ be linear maps defined by

$$\alpha(u_1) = v_1 + 2v_2 + v_3, \quad \alpha(u_2) = v_1 - v_3, \quad \alpha(u_3) = -v_1 + v_2 - v_3$$

and

$$\beta(v_1) = w_1 + 2w_2, \quad \beta(v_2) = w_1 - w_2, \quad \beta(v_3) = -w_1 + w_2.$$

Construct the map $\gamma = \beta\alpha$ (in the same form that α and β are defined above).

(c) [3 marks] Use your previous definition to write down the matrix representations A, B, C of α, β, γ.

(d) [3 marks] State, and illustrate, the relationship among the matrices A, B, C that corresponds to the relationship $\gamma = \beta\alpha$ among the maps.

[This question continues overleaf . . .]

(e) [5 marks] Suppose $(x, y, z) \in \mathbb{K}^3$ is the coordinate representation of a vector $u \in U$. Use your definition of γ in part (b) above to deduce the value of $\gamma(x, y, z)$ as the coordinate representation of a vector $w \in W$ and show how the same value can be deduced using the matrix C. The bases for U, V are $\mathcal{B} = u_1, u_2, u_3$ and $C = v_1, v_2, v_3$ as defined in part (b) above.

(f) [6 marks] Suppose new bases $\mathcal{B}', C', \mathcal{D}'$ are defined in the vector spaces U, V, W. Use mapping diagrams to explain briefly how the matrix representations A, B, C of α, β, γ change to matrix representations A', B', C' and *prove* that the relationship among A', B', C' is the same as that among A, B, C.

B3. (a) [4 marks] Define the terms *basis set* and *dimension* for a vector space.

(b) [2 marks] Define the *row rank* of a matrix.

(c) [3 marks] Define the set of *elementary row operations* on a matrix.

(d) [7 marks] Prove that each elementary row operation preserves the row rank of the matrix.

(e) [6 marks] Find a basis set for the vector subspace of $\mathbb{R}^5$ spanned by the following vectors:

$$(1, 2, 4, 1, 3),$$
$$(2, -1, 3, 5, 0),$$
$$(2, 5, -5, 0, -4),$$
$$(-1, 3, 1, -4, 3),$$
$$(3, 1, 7, 6, 3).$$

B4. (a) [5 marks] Define an *orthogonal matrix*. Prove that if the same orthogonal matrix is applied to two vectors in $\mathbb{R}^n$ then their standard inner product is preserved.

(b) [5 marks] Prove that the eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues are orthogonal.

(c) [4 marks] Prove that a real symmetric matrix with distinct eigenvalues can be diagonalised by a change of basis that corresponds to an orthogonal transformation.

(d) [3 marks] Let $A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find the eigenvalues of this matrix.

(e) [5 marks] Find an orthogonal matrix R and a diagonal matrix A' such that when R is applied to the basis used to define A the transformed matrix is A'.

[End of examination paper]