Chapter 1

INTRODUCTION TO NUMBER THEORY

1.1 REAL NUMBERS

(a) NATURAL NUMBER

Natural numbers is 1,2,3,$\cdots$, also called positive integers, are used in counting members of a set.The symbols varied with the times, e.g., the Romans used I, II, III, IV,$\cdots$
The sum $a + b$ and product $a \cdot b$ or ab of any two natural numbers a and b is also a natural number. This is often expressed by saying that the set of natural numbers is closed under the operations of addition and multiplication, or satisfies the closure property with respect to these operations.
For any natural numbers a,b,c, we have

- $a + (b + c) = (a + b) + c$
- $a + b = b + a$
- $a(b + c) = ab + ac$
- $a(bc) = (ab)c$ Associativity
- $ab = ba$ Commutativity
- $(a + b)c = ac + bc$ Distributive

(b) INTEGERS

Negative integers and zero denoted by -1,-2,-3,$\cdots$ and 0,respectively, arose to permit solutions of equations such as $x + b = a$, where a and b are any are any natural numbers. This leads to the operation of subtraction, or inverse of addition, and we write $x = a - b$.
The set of positive and negative integers and zero is called the set of integers.

PROPERTIES OF INTEGERS

- Associativity
- Commutativity
- Distributive
- $a + 0 = a = 0 + a$
- $a \cdot 1 = a = 1 \cdot a$
- Transitivity: $a > b$ and $b > c \Rightarrow a > c$.
- Trichotomy:$a > b$,$a < b$ or $a = b$.
- Cancellation law: If $a \cdot c = b \cdot c$ and $c \neq 0$ then $a = b$.

(c) RATIONAL NUMBERS

Rational numbers or fractions such as $\frac{2}{3}$,$\frac{-5}{4}$,$\cdots$ arose to permit solutions of equations such as $bx = a$ for all integers a and b, where $b \neq 0$.This leads to the operation of division, or inverse of multiplication, and we write $x = \frac{a}{b}$ or $a \div b$ where a is the numerator and b the denominator.The set of integers is a subset of the rational numbers, since integers correspond to rational numbers where $b = 1$.

(d) IRRATIONAL NUMBERS

Irrational numbers such $\sqrt{2}$ and π are numbers which are not rational, i.e., they cannot be expressed as $\frac{a}{b}$ (called the quotient of a and b),where a and b are integers and $b \neq 0$.
The set of rational and irrational numbers is called the set of real numbers.

1.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

The principle of mathematical induction is an important property of the positive integers. It is especially useful in proving statements involving all positive integers when it is known for example that the statements are valid for n=1,2,3 but it is suspected or conjectured that they hold for all positive integers. The method of proof consists of the following steps:

1. Prove the statement for $n = 1$ (some other positive integer).

2. Assume the statement true for $n = k$; where k is any positive integer.

3. From the assumption in 2 prove that the statement must be true for $n = k + 1$. This is part of the proof establishing the induction and may be difficult or impossible.

4. Since the statement is true for $n = 1$ [from step 1] it must [from step 3] be true for $n = 1 + 1 = 2$ and from this for $n = 2 + 1 = 3$, and so on, and so must be true for all positive integers. (This assumption, which provides the link for the truth of a statement for a finite number of cases to the truth of that statement for the infinite set, is called The Axiom of Mathematical Induction).

EXAMPLES:

1. Prove that $1 + 2 + ... + n = \frac{n(n+1)}{2}$

   PROOF:

   This formula is easily verified for small numbers such as $n = 1, 2, 3,$ or 4, but it is impossible to verify for all natural numbers on a case-by-case basis.
   To prove the formula true in general, a more generic method is required.
   Suppose we have verified the equation for the first k cases.
   We will attempt to show that we can generate the formula for the $(k + 1)$th case from this knowledge.
   The formula is true for

   $$n_0 = 1, \text{ since } 1 = \tfrac{1(1+1)}{2}$$

If we have verified the first k cases, then

$$\begin{aligned} 1+2+...+k+(k+1) &= \frac{k(k+1)}{2}+k+1 \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

This is exactly the formula for the $(k+1)$th case.

2. Prove that for all integers $n \geq 3$, $2^n > n+4$.

   PROOF:

   Since $8 = 2^3 > 3+4 = 7$; the statement is true for $n_0 = 3$.

   Assume that $2^k > k+4$ for $k \geq 3$.

   then ,$2^{k+1} = 2 \cdot 2^k > 2(k+4)$

   But $2(k+4) = 2k+8 > k+5 = (k+1)+4$.

   Since k is positive. Hence, by induction, the statement holds for all integers $n \geq 3$.

1.3 DEFINITION 1:

A non-empty subset S of $\mathbb{Z}$ is well-ordered if S contains a least element.Notice that the set $\mathbb{Z}$ is not well-ordered since it does not contain a smallest element. However, the natural numbers are well-ordered.

1.4 PRINCIPLE OF WELL ORDERING:

Every non-empty subset of the natural numbers is well-ordered.
The Principle of Well-Ordering is equivalent to the Principle of Mathematical Induction.

1.5 THEOREM 1:

- If $x \in \mathbb{R}$, $y \in \mathbb{R}$,and $x > 0$,then there is a positive integer n such that
$$nx > y$$
- If $x \in \mathbb{R}$, $y \in \mathbb{R}$,and $x < y$,then there is a $p \in \mathbb{Q}$ such that $x < p < y$. The first part is usually referred to as the archimedean property of $\mathbb{R}$.Second Part may be stated by saying that $\mathbb{Q}$ is dense in $\mathbb{R}$: Between any two real numbers there is a rational one.

EXERCISE:

1. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
2. Prove that $2^n < n!$ for $n \geq 4$
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
4. Show that $n^3 + 2n$ is divisible by 3.
5. Prove Bernoullis inequality $(1 + x)^n > 1 + nx$ for $n = 2, 3, \cdots$
if $x > -1$,$x \neq 0$

1.6 ALGEBRA OF COMPLEX NUMBERS

Although complex numbers occur in many branches of mathematics, they arise most directly out of solving polynomial equations.We examine a specific quadratic equation as an example.
Consider the quadratic equation

$$z^2 - 4z + 5 \tag{1.1}$$

Equation (1.1) has two solutions, z_1 and z_2, such that

$$(z - z_1)(z - z_2) = 0 \tag{1.2}$$

Using the familiar formula for the roots of a quadratic equation,the solutions z_1 and z_2,written in brief as

$$z_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2} = 2 \pm \frac{\sqrt{-4}}{2} \tag{1.3}$$

Both solutions contain the square root of a negative number. However, it is not true to say that there are no solutions to the quadratic equation. The fundamental theorem of algebra states that a quadratic equation will always have two solutions and these are in fact given by (1.2).The second term on the RHS of (1.3) is called an imaginary term since it contains the square root of a negative number; the first term is called a real term. The full solution is the sum of a real term and an imaginary term and is called a complex number.The choice of the symbol z for the quadratic variable was not arbitrary; the conventional representation of a complex number is z, where z is the sum of a real part x and i times an imaginary part y, i.e.

$$z = x + iy, \tag{1.4}$$

where i is used to denote the square root of -1. The real part x and the imaginary part y are usually denoted by Rez and Imz respectively. We note at this point that some physical scientists, engineers in particular, use j instead of i. However,for consistency, we will use i this book. In our particular example $\sqrt{-4} = 2\sqrt{-1} = 2i$,and hence the two solutions of (1.3) are

$$z_{1,2} = 2 \pm \frac{2i}{2} = 2 \pm i \tag{1.5}$$

Thus, here $x = 2$ and $y = \pm 1$ For compactness a complex number is sometimes written in the form

$$z = (x, y),$$

where the components of z may be thought of as coordinates in an xy-plot. Such a plot is called an Argand diagram and is a common representation of complex numbers.

Our particular example of a quadratic equation may be generalised readily to polynomials whose highest power (degree) is greater than 2, e.g. cubic equations (degree 3), quartic equations (degree 4) and so on. For a general polynomial $f(z)$,of degree n, the fundamental theorem of algebra states that the equation $f(z) = 0$ will have exactly n solutions. The remainder of this chapter deals with: the algebra and manipulation of complex numbers; their polar representation, which has advantages in many circumstances; complex exponentials and logarithms; the use of complex numbers in finding the roots of polynomial equations.

(a) ADDITION AND SUBTRACTION

The addition of two complex numbers z_1 and z_2 ,in general gives another complex number. The real components and the imaginary components are added separately and in a like manner to the familiar addition of real numbers:

$$\begin{aligned} z_1 + z_2 &= (x_1 + y_1 i) + (x_2 + y_2 i) \\ &= (x_1 + x_2) + (y_1 + y_2)i, \end{aligned}$$

or in component notation

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is itself commutative and associative, i.e.

$$\begin{gathered} z_1 + z_2 = z_2 + z_1, \\ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3. \end{gathered}$$

Thus it is immaterial in what order complex numbers are added.

EXAMPLE:

(a) Sum the complex numbers $1 + 2i$, $3 - 4i$, $-2 + i$.
Summing the real terms we obtain

$$1 + 3 - 2 = 2,$$

and summing the imaginary terms we obtain

$$2i - 4i + i = -i.$$

Hence

$$(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i.$$

(b) $(2 + 3i) + (3 - 4i) = 2 + 3i + 3 - 4i = 5 - i$

(c) $(2 + 3i) - (3 - 4i) = 2 + 3i - 3 + 4i = -1 + 7i$

(d) $(-4 + 7i) + (5 - 10i) = 1 - 3i$

(e) $(4 + 12i) - (3 - 15i) = 4 + 12i - 3 + 15i = 1 + 27i$

(f) $5i - (-9 + i) = 5i + 9 - i = 9 + 4i$

The subtraction of complex numbers is very similar to their addition. As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

(b) EQUALITY

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be complex numbers, then:
$z_1 = z_2$ If and only if $x_1 = x_2$ and $y_1 = y_2$

EXERCISE:

Given $z_1 = 2 + 4i$, $z_2 = 3 - i$ and $z_3 = -3i$
Determine:

(a) $z_1 + z_2$
(b) $z_1 - z_2$
(c) $z_2 + z_3$
(d) $z_2 - z_3$
(e) $z_1 + z_3$
(f) $z_1 - z_3$

(c) MODULUS AND ARGUMENT

The modulus of the complex number z is denoted by $\mid z \mid$ and is defined as

$$\mid z \mid = \sqrt{x^2 + y^2}. \tag{1.6}$$

Hence the modulus of the complex number is the distance of the corresponding point from the origin in the Argand diagram. The argument of the complex number z is denoted by arg z and is defined as

$$argz = tan^{-1}\left(\frac{y}{x}\right) \tag{1.7}$$

It can be seen that arg z is the angle that the line joining the origin to z on the Argand diagram makes with the positive $x - axis$. The anti-clockwise direction is taken to be positive by convention.Account must be taken of the signs of x and y individually in determining in which quadrant arg z lies. Thus, for example, if x and y are both negative then arg z lies in the range $-\pi <$arg z$< \frac{-\pi}{2}$ rather than in the first quadrant

$0 <$arg z$< \frac{\pi}{2}$,though both cases give the same value for the ratio of y to x.

EXAMPLES:

(a) Find the modulus and the argument of the complex number $z = 2 - 3i$.
Using (1.6), the modulus is given by

$$| z |= \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$$

Using (1.7), the argument is given by

$$\text{arg z} = \theta = tan^{-1}\left(\frac{-3}{2}\right) = -56.31^o$$

Since $x = 2$ and $y = -3$, z clearly lies in the fourth quadrant;(i.e. the argument must be measured from the positive real axis). therefore arg z $= 56.31^o$ is the appropriate answer.

(b) Determine the modulus and argument of the complex number $z = 3 + 4i$

$$| z |= r = \sqrt{(3)^2 + (4)^2} = 5$$

$$\text{arg z} = \theta = tan^{-1}(\tfrac{4}{3}) = 53.13^o$$

Since $x = 3$ and $y = 4$, z clearly lies in the first quadrant

(c) Determine the modulus and argument of the complex number $z = -3 + 4i$

$$| z |= r = \sqrt{(-3)^2 + (4)^2} = 5$$

$$\text{arg z} = \theta = tan^{-1}\left(\frac{4}{3}\right) = 53.13^o$$

Since $x = -3$ and $y = 4$, z clearly lies in the second quadrant; Argument$= 180 - 53.13 = 126.87$ (i.e. the argument must be measured from the positive real axis).

(d) Determine the modulus and argument of the complex number $z = -3 - 4i$

$$\mid z \mid = r = \sqrt{(-3)^2 + (-4)^2} = 5$$

$$\arg \text{z} = \theta = tan^{-1}\left(\frac{4}{3}\right) = 53.13^o$$

Since $x = -3$ and $y = -4$, z clearly lies in the third quadrant; Argument$= 180 + 53.13 = 233.13$ (i.e. the argument must be measured from the positive real axis).

(d) MULTIPLICATION

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers z_1 and z_2 is found by multiplying them out in full and remembering that $i^2 = -1$,i.e.

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i)$$

$$= x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2$$

$$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

EXAMPLE:

(a) Multiply the complex numbers $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.
By direct multiplication we find

$$z_1 z_2 = (3 + 2i)(-1 - 4i) = -3 - 2i - 12i - 8i^2 = 5 - 14i$$

(b) If $z_1 = 7i$ and $z_2 = -5 + 2i$.
Determine $z_1 z_2$

$$7i(-5 + 2i) = -35i + 14(-1) = -14 - 35i$$

(c) If $z_1 = 1 - 5i$ and $z_2 = -9 + 2i$.
Determine $z_1 z_2$

$$(1-5i)(-9+2i) = -9+2i+45i-10i^2 = -9+47i-10(-1) = 1+47i$$

(d) Find $(1-8i)(1+8i)$

$$(1-8i)(1+8i) = 1+8i-8i-64i^2 = 1+64 = 65$$

The multiplication of complex numbers is both commutative and associative i.e.

$$z_1 z_2 = z_1 z_2 \tag{1.8}$$
$$(z_1 z_2)z_3 = z_1(z_2 z_3) \tag{1.9}$$

The product of two complex numbers also has the simple properties

$$\mid z_1 z_2 \mid=\mid z_1 \mid\mid z_2 \mid \tag{1.10}$$
$$arg(z_1 z_2) = arg z_1 + arg z_2. \tag{1.11}$$

EXAMPLE:

Verify that (1.10) holds for the product of $z_1 = 3+2i$ and $z_2 = -1-4i$

SOLUTION:

$$\mid z_1 z_2 \mid=\mid 5-14i \mid= \sqrt{5^2+(-14)^2} = \sqrt{221}$$

We also find

$$\mid z_1 \mid= \sqrt{3^2+2^2} = \sqrt{13}$$

$$\mid z_2 \mid= \sqrt{(-1)^2+(-4)^2} = \sqrt{17}$$

and hence

$$\mid z_1 \mid\mid z_2 \mid= \sqrt{13}\sqrt{17} = \sqrt{221} =\mid z_1 z_2 \mid$$

(e) COMPLEX CONJUGATE

If z has the convenient form $x + yi$ then the complex conjugate, denoted by $\overline{z}$ may be found simply by changing the sign of the imaginary part, i.e. if $z = x + yi$ then $\overline{z} = x - yi$. More generally, we may define the complex conjugate of z as the (complex) number having the same magnitude as z that when multiplied by z leaves a real result, i.e. there is no imaginary component in the product. In the case where z can be written in the form $x + yi$ it is easily verified, by direct multiplication of the components, that the product $z\overline{z}$ gives a real result:

$$\begin{aligned} z\overline{z} &= (x + yi)(x - yi) \\ &= x^2 - xyi + xyi - y^2 i^2 \\ &= x^2 + y^2 = \mid z \mid^2 \end{aligned}$$

EXAMPLE:

Find the complex conjugate of $z = a + 2i + 3ib$

SOLUTION:

The complex number is written in the standard form

$$z = a + i(2 + 3b);$$

Then, replacing i by $-i$, we obtain

$$\overline{z} = a - i(2 + 3b)$$

In some cases, however, it may not be simple to rearrange the expression for z into the standard form $x + yi$.Nevertheless, given two complex numbers, z_1 and z_2,it is straightforward to show that the complex conjugate of their sum (or difference) is equal to the sum (or difference) of their complex conjugates, i.e.$\overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_1}$.Similarly, it may be shown that the complex conjugate of the product (or quotient) of z_1 and z_2 is equal to the product (or quotient) of their complex conjugates, i.e.$\overline{(z_1 z_2)} = \overline{z_1}\ \overline{z_2}$ and $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_1}$. Using these results, it can be deduced that, no matter how complicated the expression, its complex conjugate may always be found by replacing every i by $-i$. To apply this rule, however, we must always ensure that all complex parts are first written out in full, so that no i's are hidden.

EXAMPLE:

Find the complex conjugate of the complex number $z = w^{(3y+2ix)}$ where $w = x + 5i$

SOLUTION:

In this case w itself contains real and imaginary components and so must be written out in full, i.e.

$$z = w^{(3y+2ix)} = (x + 5i)^{3y+2ix}.$$

Now we can replace each i by $-i$ to obtain

$$\overline{z} = (x - 5i)^{3y-2ix}.$$

It can be shown that the product $z\overline{z}$ is real, as required. The following properties of the complex conjugate are easily proved and others may be derived from them. If $z = x + yi$
then

(a) $\overline{\overline{z}} = z$,

(b) $z + \overline{z} = 2\,Re\,z = 2x$,

(c) $z - \overline{z} = 2i\,Im\,z = 2yi$,

(d) $\frac{z}{\overline{z}} = \left(\frac{x^2-y^2}{x^2+y^2}\right) + i\left(\frac{2xy}{x^2+y^2}\right)$.

(f) DIVISION

The division of two complex numbers z_1 and z_2 bears some similarity to their multiplication. Writing the quotient in component form we obtain

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} \tag{1.12}$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator. By definition, this process will leave the denominator as a real quantity.

$$\frac{z_1}{z_2} = \frac{(x_1 + y_1 i)(x_2 + y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} = \frac{(x_1 x_2 + y_1 y_2) + (x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \tag{1.13}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \tag{1.14}$$

Hence we have separated the quotient into real and imaginary components, as required.

EXAMPLE 1 :

Express z in the form $x + iy$,when

$$z = \frac{3 - 2i}{-1 + 4i}$$

SOLUTION:

Multiplying numerator and denominator by the complex conjugate of the denominator we obtain

$$z = \frac{(3 - 2i)(-1 - 4i)}{(-1 + 4i)(-1 - 4i)} = \frac{-11 - 10i}{17}$$

$$= \frac{-11}{17} - \frac{10i}{17}$$

PROPERTIES

(a) $\mid \frac{z_1}{z_2} \mid = \frac{|z_1|}{|z_2|}$,

(b) $\arg(\frac{z_1}{z_2})$ =arg z_1-arg z_2

EXAMPLE 2:
Write each of the following in standard form

(a) $\frac{3-i}{2+7i}$

(b) $\frac{3}{9-i}$

(c) $\frac{8i}{1+2i}$

(d) $\frac{6-9i}{2i}$

SOLUTION:
So,in each case we are really looking at the division of two complex numbers. The main idea here is that we want to write them in standard form. standard form does not allow for any is to be in the denominator.So, we want to get the i's out of the denominator. This is actually fairly simple if we recall that a complex number times its conjugate is real number. So, if we multiply the numerator and denominator by the conjugate of the denominator we will be able to eliminate the i from the denominator. Now that we've figured out how to do these let's go ahead and work the problems.

(a) $\frac{3-i}{2+7i} = \frac{(3-i)(2-7i)}{(2+7i)(2-7i)} = \frac{6-23i+7i^2}{2^2+7^2} + = \frac{-1-23i}{53} = -\frac{1}{53} - \frac{23}{53}i$

Notice that to officially put the answer in standard form we broke up the fraction into the real and imaginary parts.

(b) $\frac{3}{9-i} = \frac{3(9+i)}{(9-i)(9+i)} = \frac{27+3i}{9^2+1^2} = \frac{27}{82} + \frac{3}{82}i$

(c) $\frac{8i}{1+2i} = \frac{8i(1-2i)}{(1+2i)(1-2i)} = \frac{8i-16i^2}{1^2+2^2} = \frac{16+8i}{5} = \frac{16}{5} + \frac{8}{5}i$

(d) This one is a little different from the previous ones since the denominator is a pure imaginary number. It can be done in the same manner as the previous ones, but there is a slightly easier way to do the problem.
First,we break up the fraction as follows.

$$\frac{6-9i}{2i} = \frac{6}{2i} - \frac{9i}{2i} = \frac{3}{i} - \frac{9}{2}$$

Now, we want the i out of the denominator and since there is only an i in the denominator of the first term we will simply multiply the numerator and denominator of the first term by an i.

$$\frac{6-9i}{2i} = \frac{3(i)}{i(i)} - \frac{9}{2} = \frac{3i}{-1} - \frac{9}{2} = \frac{-9}{2} - 3i$$

(g) POLAR REPRESENTATION OF COMPLEX NUMBERS

Although considering a complex number as the sum of a real and an imaginary part is often useful, sometimes the polar representation proves easier to manipulate.This makes use of the complex exponential function, which is defined by

$$e^z = \exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots . \tag{1.15}$$

Strictly speaking it is the function exp$z$ that is defined by (1.15).The number e is the value of exp(1), i.e. it is just a number. However, it may be shown that e^z and exp$z$ are equivalent when z is real and rational and mathematicians then define their equivalence for irrational and complex z. For the purposes of this book we will not concern ourselves further with this mathematical nicety but,rather, assume that(1.15),is valid for all z. We also note that, using(1.15),by multiplying together the appropriate series we may show that

$$e^{z_1}e^{z_2} = e^{z_1+z_2} \tag{1.16}$$

which is analogous to the familiar result for exponentials of real numbers. From (1.15),It immediately follows that for $z = i\theta$, θ real,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \tag{1.17}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots) \tag{1.18}$$

and hence that

$$e^{i\theta} = cos\theta + isin\theta \tag{1.19}$$

where the last equality follows from the series expansions of the sine and cosine functions.This last relationship is called Euler's equation. It also follows from (1.19)

$$e^{in\theta} = cosn\theta + isin\, n\theta \tag{1.20}$$

for all n. From Euler's equation (1.19) we deduce that

$$re^{i\theta} = r(cos\theta + isin\theta) \tag{1.21}$$

$$= x + yi \tag{1.22}$$

Thus a complex number may be represented in the polar form

$$z = re^{i\theta} \tag{1.23}$$

Now we can identify r with $\mid z \mid$ and θ with arg$z$. The simplicity of the representation of the modulus and argument is one of the main reasons for using the polar representation. The angle θ lies conventionally in the range $-\pi < \theta \leq \pi$,but, since rotation by θ is the same as rotation by $2n\pi + \theta$, where n is any integer,

$$re^{i\theta} \equiv re^{i(\theta+2n\pi)} \tag{1.24}$$

The algebra of the polar representation is different from that of the real and imaginary component representation, though, of course, the results are identical.Some operations prove much easier in the polar representation, others much more complicated. The best representation for a particular problem must be determined by the manipulation required.

(h) MULTIPLICATION AND DIVISION IN POLAR FORM

Multiplication and division in polar form are particularly simple. The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ is given by

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} \tag{1.25}$$

$$= r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{1.26}$$

$$= r_1 r_2 [cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)] \tag{1.27}$$

The relations $\mid z_1 z_2 \mid = \mid z_1 \mid\mid z_2 \mid$ and arg $(z_1 z_2)$ = arg z_1+argz_1 follow immediately. Division is equally simple in polar form; the quotient of z_1 and z_2 is given by

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 \theta_2)} \tag{1.28}$$

$$= \frac{r_1}{r_2} [cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)] \tag{1.29}$$

The relations $\mid z_1/z_2 \mid = \mid z_1 \mid / \mid z_2 \mid$ and arg (z_1/z_2) = arg z_1-argz_1 are again immediately apparent.

(i) De MOIVRE'S THEOREM

We now derive an extremely important theorem. Since $(e^{i\theta})^n = e^{in\theta}$, we have

$$(cos\theta + isin\theta)^n = cosn\theta + isinn\theta \tag{1.30}$$

where the identity $e^{in\theta} = cosn\theta + isinn\theta$ follows from the series definition of $e^{in\theta}$.This result is called de Moivres theorem and is often used in the manipulation of complex numbers. The theorem is valid for all n whether real,imaginary or complex. There are numerous applications of de Moivres theorem but this section examines just three: proofs of trigonometric identities; finding the nth roots of unity; and solving polynomial equations with complex roots.

(a) TRIGONOMETRIC IDENTITIES

The use of de Moivres theorem in finding trigonometric identities is best illustrated by example. We consider the expression of a multiple-angle function in terms of a polynomial in the single-angle function, and its converse.

EXAMPLES:

i. Express $sin3\theta$ and $cos3\theta$ in terms of powers of $cos\theta$ and $sin\theta$. Using de Moivres theorem,

$$cos3\theta + isin3\theta = (cos\theta + isin\theta)^3$$
$$= (cos^3\theta - 3cos\theta sin^2\theta) + i(3sin\theta cos^2\theta - sin^3\theta)$$

We can equate the real and imaginary coefficients separately,i.e.

$$\begin{aligned} cos3\theta &= cos^3\theta - 3cos\theta sin^2\theta \\ &= 4cos^3\theta - 3cos\theta \end{aligned}$$

and

$$\begin{aligned} sin3\theta &= 3sin\theta cos^2\theta - sin^3\theta \\ &= 3sin\theta - 4sin^3\theta. \end{aligned}$$

This method can clearly be applied to finding power expansions of $cosn\theta$ and $sinn\theta$ for any positive integer n. The converse process uses the following properties of $z = e^{i\theta}$,

$$z^n + \frac{1}{z^n} = 2cosn\theta, \tag{1.31}$$

$$z^n + \frac{1}{z^n} = 2isin\theta. \tag{1.32}$$

These equalities follow from simple applications of de Moivres theorem, i.e.

$$\begin{aligned}
z^n + \frac{1}{z^n} &= (cos\theta + isin\theta)^n + (cos\theta + isin\theta)^{-n} \\
&= cosn\theta + isin\ n\theta + cos(-n\theta) + isin(-n\theta) \\
&= cosn\theta + isin\ n\theta + cosn\theta - isin\ n\theta = 2cosn\theta
\end{aligned}$$

and

$$\begin{aligned}
z^n - \frac{1}{z^n} &= (cos\theta + isin\theta)^n - (cos\theta + isin\theta)^{-n} \\
&= cosn\theta + isin\ n\theta + cos(-n\theta) + isin(-n\theta) \\
&= cosn\theta + isin\ n\theta - cosn\theta + isinn\theta = 2isin\ n\theta
\end{aligned}$$

In the particular case where $n = 1$,

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2cos\theta. \tag{1.33}$$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2isin\theta. \tag{1.34}$$

ii. Find an expression for $cos^3\theta$ in terms of $cos3\theta$ and $cos\theta$.

$$\begin{aligned}
cos^3\theta &= \frac{1}{2^3}(z + \frac{1}{z})^3 \\
&= \frac{1}{8}(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}) \\
&= \frac{1}{8}(z^3 + \frac{1}{z^3}) + \frac{3}{8}(z + \frac{1}{z}).
\end{aligned}$$

we find

$$cos^3\theta = \frac{1}{4}cos3\theta + \frac{3}{4}cos\theta$$

(b) FINDING THE NTH ROOTS OF UNITY

The equation $Z^2 = 1$ has the familiar solutions $z = \pm 1$.However, now that we have introduced the concept of complex numbers we can solve the general equation $z^n = 1$.Recalling the fundamental theorem of algebra, we know that the equation has n solutions. In order to proceed we rewrite the equation as

$$z^n = e^{2ik\pi}, \tag{1.35}$$

where k is any integer. Now taking the nth root of each side of the equation we find

$$z^n = e^{\frac{2ik\pi}{n}}, \tag{1.36}$$

Hence, the solutions of $z^n = 1$ are

$$z_{1,2,\cdots,n} = 1, e^{\frac{2i\pi}{n}}, \cdots, e^{\frac{2i(n-1)\pi}{n}}, \tag{1.37}$$

corresponding to the values $0, 1, 2, \cdots, n-1$ for k. Larger integer values of k do not give new solutions, since the roots already listed are simply cyclically repeated for $k = n$, $n + 1$, $n + 2$, etc.

EXAMPLES:

i. Find the solutions to the equation $z^3 = 1$.

SOLUTION:
By applying the above method we find

$$z = e^{\frac{2ik\pi}{3}}$$

Hence the three solutions are $z_1 = e^{0i} = 1$, $z_2 = e^{\frac{2i\pi}{3}}$,$z_3 = e^{\frac{4i\pi}{3}}$. We note that, as expected,the next solution, for which $k = 3$,gives $z_4 = e^{\frac{6i\pi}{3}} = 1 = z_1$,so that there are only three separate solutions. Not surprisingly, given that $| z^3 |=| z |^3$ all the roots of unity have unit modulus, i.e. they all lie on a circle in the Argand diagram of unit radius

(c) SOLVING POLYNOMIAL EQUATIONS

A third application of de Moivres theorem is to the solution of polynomial equations. Complex equations in the form of a polynomial relationship must first be solved for z in a similar fashion to the method for finding the roots of real polynomial equations. Then the complex roots of z may be found.

i. Solve the equation $z^6 - z^5 + 4z^4 - 6z^3 + 2z^2 - 8z + 8 = 0$.

SOLUTION:
We first factorise to give

$$(z^3 - 2)(z^2 + 4)(z - 1) = 0$$

Hence $z^3 = 2$ or $z^2 = -4$ or $z = 1$.The solutions to the quadratic equation are $z = \pm 2i$;to find the complex cube roots, we first write the equation in the form

$$z^3 = 2 = 2e^{2ik\pi},$$

where k is any integer. If we now take the cube root, we get

$$z = 2^{1/3} e^{\frac{2ik\pi}{3}}.$$

To avoid the duplication of solutions, we use the fact that
$-\pi <$ arg z$\leq \pi$ and find

$$z_1 = 2^{1/3}$$

$$z_2 = 2^{1/3} e^{\frac{2\pi i}{3}} = 2^{1/3} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i \right),$$

$$z_3 = 2^{1/3} e^{\frac{-2\pi i}{3}} = 2^{1/3} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right).$$

The complex numbers z_1,z_2, and z_3 together with $z_4 = 2i$, $z_5 = -2i$ and $z_6 = 1$ are the solutions to the original polynomial equation. As expected from the fundamental theorem of algebra, we find that the total number of complex roots (six, in this case) is equal to the largest power of z in the polynomial. A useful result is that the roots of a polynomial with real coefficients occur in conjugate pairs (i.e. if z_1 is a root, then $\overline{z_1}$ unless z_1 is real). This may be proved as follows. Let the polynomial equation of which z is a root be

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Taking the complex conjugate of this equation,

$$\overline{a_n}(\overline{z})^n + \overline{a_{n-1}}(\overline{z})^{n-1} + \cdots + \overline{a_1}\ \overline{z} + \overline{a_0} = 0.$$

But the a_n are real, and so $\overline{z}$ satisfies

$$a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 = 0,$$

and is also a root of the original equation.

(j) COMPLEX LOGARITHM AND COMPLEX POWERS

The concept of a complex exponential has already been introduced, where it was assumed that the definition of an exponential as a series was valid for complex numbers as well as for real numbers. Similarly we can define the logarithm of a complex number and we can use complex numbers as exponents. Let us denote the natural logarithm of a complex number z by $w = Lnz$, where the notation Ln will be explained shortly. Thus, w must satisfy

$$z = e^w \tag{1.38}$$

we see that

$z_1 z_2 = e^{w_1} e^{w_2} = e^{w_1 + w_2}$, and taking logarithms of both sides we find

$$Ln(z_1 z_2) = w_1 + w_2 = Lnz_1 + Lnz_2, \tag{1.39}$$

which shows that the familiar rule for the logarithm of the product of two real numbers also holds for complex numbers. We may (1.42) to investigate further the properties of ln z. We have already noted that the argument of a complex number is multivalued,i.e. arg $z = \theta + 2n\pi$ where n is any integer. Thus, in polar form, the complex number z should strictly be written as

$$z = re^{i(\theta + 2n\pi)}.$$

Taking the logarithm of both sides, and using (1.42), we find

$$Lnz = lnr + i(\theta + 2n\pi) \tag{1.40}$$

where lnr is the natural logarithm of the real positive quantity r and so is written normally. Thus from (1.43) we see that lnz is itself multivalued. To avoid this multivalued behaviour it is conventional to define another function lnz, the principal value of lnz, which is obtained from lnz by restricting the argument of z to lie in the range $-\pi < \theta <\leq \pi$.

(a) Evaluate $Ln(-i)$
By rewriting i as a complex exponential, we find

$$Ln(-i) = Ln[e^{i(-\pi/2+2n\pi)}] = i(-\pi/2 + 2n\pi),$$

where n is any integer. Hence $Ln(-i) = -\frac{\pi}{2}, \frac{3i\pi}{2}, \cdots$ We note that $ln(i)$, the principal value of $Ln(i)$,is given by $ln(i) = -\frac{\pi}{2}$.

(b) Simplify the expression $z = i^{-2i}$.
Firstly we take the logarithm of both sides of the equation to give

$$Lnz = -2iLni$$

Now inverting the process we find

$$e^{Lnz} = z = e^{-2iLni}$$

We can write $i = e^{i(-\pi/2+2n\pi)}$,where n is any integer, and hence

$$\begin{aligned} Lni &= Ln[e^{i(\pi/2+2n\pi)}] \\ &= i(\pi/2 + 2n\pi) \end{aligned}$$

We can now simplify z to give

$$\begin{aligned} i^{-2i} &= e^{-2i\times i(\frac{\pi}{2}+2n\pi)} \\ &= e^{\pi+4n\pi} \end{aligned}$$

which, perhaps surprisingly, is a real quantity rather than a complex one.

EXERCISES:

1. Two complex numbers z and w are given by $z = 3 + 4i$ and $w = 2 - i$.Evaluate the following

    (a) z+w

    (b) $w - z$

    (c) wz

    (d) z/w

    (e) lnz

    (f) $(1 + z + w)^{1/2}$

    (g) $\overline{z}w + \overline{w}z$

2. By writing $\pi/12 = (\pi/3) - (\pi/4)$ and considering $e^{i\pi/12}$,evaluate $cot(\pi/12)$ and $tan(\pi/12)$.

3. Use de Moivres theorem with n = 4 to prove that

$$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$$

4. Use de Moivres theorem to prove that

$$tan5\theta = \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$$

where $t = tan\theta$

5. Evaluate

   (a) i^i
   (b) $\mathrm{Im}2^{i+3}$
   (c) $exp(i^3)$
   (d) $(-1 + \sqrt{3i})^{1/2}$
   (e) $e^{3i\pi} + 1$
   (f) $(1 + i)^{10000}$

6. Find the cubic roots of a complex number $z = -1 + i$

7. Find the cubic roots of unity

8. Show that log z^n =log $[(re^{i\theta})^n]$

9. Show that log $(e^z) = z$

10. If $z = a - ib$, what is i^3z

11. Simplify $z = (25i)(3 + i)/(3i)$ and find the modulus and argument of the result.

12. If $w = z^2$ and $z = x + iy$ and $w = u + iv$, find u and v.

13. If $i^2 = -1$, what are i^3,i^4,i^5,i^6.

14. If $z = 3 + 4I$ find z^2 and the modulus and argument of z^2

15. Find z^n if $\theta = \pi$ and $r = 2$.

16. Write $(\sqrt{5})e^{itan^{-1}(\frac{1}{2})}$ in the form $a + bi$.

Chapter 2

VECTOR ALGEBRA

The term vector is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by the letter $\overrightarrow{v}$.
For instance, suppose a particle moves along a line segment from point A to point B.The corresponding displacement vector has initial point A (the tail) and terminal point B(the tip) and we indicate this by writing $v = \overrightarrow{AB}$. We say that u and v are equivalent (or equal) and we write $u = v$.The zero vector denoted by 0, has length 0. It is the only vector with no specific direction.

2.1 Combining Vectors

Suppose a particle moves from A to B,so its displacement vector is $\overrightarrow{AB}$.Then the particle changes direction and moves from B to C,with displacement vector $\overrightarrow{BC}$.The combined effect of these displacements is that the particle has moved from A to C.The resulting displacement vector $\overrightarrow{AC}$ is called the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

In general, if we start with vectors u and v, we first move v so that its tail coincides with the tip of u and define the sum of u and v as follows.

2.2 DEFINITION OF VECTOR ADDITION

If u and v are vectors positioned so the initial point of v is at the terminal point of u,then the sum $u + v$ is the vector from the initial point of u to the terminal point of v. If we place u and v so they start at the same point, then $u+v$ lies along the diagonal of the parallelogram with u and v as sides.(This is called the Parallelogram Law.)

2.3 DEFINITION OF SCALAR MULTIPLICATION

If is a scalar and is a vector, then the scalar multiple cv is the vector whose length is $\mid c \mid$ times the length of v and whose direction is the same as v if $c > 0$ and is opposite to v if $c < 0$.If $c = 0$ or $v = 0$,then $cv = 0$. If two non-zero vectors are parallel if they are scalar multiples of one another. In particular, the vector $-v = (-1)v$ has the same length as v but points in the opposite direction.We call it the negative of v.By the difference $u - v$ of two vectors we mean

$$u - v = u + (-v)$$

There are several nice applications of scalar multiplication that we should now take a look at.The first is parallel vectors. This is a concept that we will see quite a bit over the next couple of sections. Two vectors are parallel if they have the same direction or are in exactly opposite directions.When we performed scalar multiplication we generated new vectors that were parallel to the original vectors (and each other for that matter).
So, let's suppose that a and b are parallel vectors.If they are parallel then there must be a number c so that ,

$$a = cb$$

So, two vectors are parallel if one is a scalar multiple of the other.

2.4 COMPONENTS

For some purposes its best to introduce a coordinate system and treat vectors algebraically.If we place the initial point of a vector a at the origin of a rectangular coordinate system, then the terminal point of a has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3),depending on whether our coordinate system is two- or three-dimensional. These coordinates are called the components of and we write

$$a = \langle a_1, a_2 \rangle \text{ or } a = \langle a_1, a_2, a_3 \rangle$$

We use the notation $a = \langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

In three dimensions, the vector $a = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the position vector of the point $P(a_1, a_2, a_3)$.Let's consider any other representation $\overrightarrow{AB}$ of a,where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$.Then we must have $x_1 + a_1 = x_2$,$y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$,$a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$.Thus we have the following result. Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$,the vector with representation $\overrightarrow{AB}$ is

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

EXAMPLE 1:

Find the vector represented by the directed line segment with initial point $A\,(2, -3, 4)$ and terminal point $B\,(-2, 1, 1)$

SOLUTION:

the vector corresponding to $\overrightarrow{AB}$ is

$$a = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle$$

The magnitude or length of the vector is the length of any of its representations and is denoted by the symbol $\mid v \mid$. By using the distance formula to compute the length of a segment OP,we obtain the following formulas.

The length of the two-dimensional vector $a = \langle a_1, a_2 \rangle$ is

$$\mid a \mid = \sqrt{a_1^2 + a_2^2}$$

The length of the two-dimensional vector $\langle a_1, a_2, a_3 \rangle$ is

$$| a |= \sqrt{a_1^2 + a_2^2 + a_3^3}$$

If $a = \langle a_1, a_2 \rangle$ and $b = \langle a_1, a_2, a_3 \rangle$,then

$$a + b = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$a - b = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$ca = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle &= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle &= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle \\ c \langle a_1, a_2, a_3 \rangle &= \langle ca_1, ca_2, ca_3 \rangle \end{aligned}$$

EXAMPLE 2:

If $a = \langle 4, 0, 3 \rangle$ and $b = \langle -2, 1, 5 \rangle$,Find $| a |$ and the vectors $a+b$,$a-b$,$3b$ $2a+5b$.

SOLUTION:

1. $| a |= \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

2. $a + b = \langle 4, 0, 0 \rangle + \langle -2, 1, 5 \rangle$

   $= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle$

3. $a - b = \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle$

4. $3b = 3 \langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$

5. $2a + 5b = 2 \langle 4, 0, 3 \rangle + 5 \langle -2, 1, 5 \rangle$

   $= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle$

2.5 PROPERTIES OF VECTORS

If a,b and c are vectors in V_n and c and d are scalars,then

(a) $a + b = b + a$

(b) $a + (b + c) = (a + b) + c$

(c) $a + (-a) = 0$

(d) $c(a + b) = ca + cb$

(e) $a + 0 = a$

(f) $(c + d)a = ca + da$

(g) $(cd)a = c(da)$

(h) $1a = a$

These eight properties of vectors can be readily verified either geometrically or algebraically.(it's equivalent to the Parallelogram Law) or as follows for the case $n = 2$

$$\begin{aligned} a + b = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ = \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle = b + a \end{aligned}$$

Three vectors in V_3 play a special role.Let

$$i = \langle 1, 0, 0 \rangle \quad j = \langle 0, 1, 0 \rangle \quad k = \langle 0, 0, 1 \rangle$$

These vectors i,j and k are called the standard basis vectors. They have length 1 and point in the directions of the positive x-, y-, and z-axes.Similarly, in two dimensions we define $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$

If

$$\begin{aligned} a = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ = a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle = a_1 i + a_2 j + a_3 k \end{aligned}$$

Thus any vector in V_3 can be expressed in terms of i,j,k and . For instance,

$$\langle 1, -2, 6 \rangle = i - 2j + 6k$$

Similarly, in two dimensions, we can write

$$a = \langle a_1, a_2 \rangle = a_1 i + a_2 j$$

EXAMPLE 3:

If $a = i + 2j - 3k$ and $b = 4i + 7k$,express the vector $2a + 3b$ in terms of i,j,k.

SOLUTION:

Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$2a + 3b = 2(i + 2j - 3k) + 3(4i + 7k)$$
$$= 2i + 4j - 6k + 12i + 21k = 14i + 4j + 15k$$

2.6 UNIT VECTOR

A unit vector is a vector whose length is 1.For instance,i ,j and k and are all unit vectors In general, if $a \neq 0$,then the unit vector that has the same direction as a is

$$u = \frac{1}{|a|}a = \frac{a}{|a|}$$

In order to verify this, we let $c = 1/ \mid a \mid$.Then $u = ca$ and c is a positive scalar, so u has the same direction as a. Also

$$\mid u \mid = \mid ca \mid = \mid c \mid\mid a \mid = \frac{1}{|a|} \mid a \mid = 1$$

EXAMPLE 4:

Find the unit vector in the direction of the vector $2i - j - 2k$

SOLUTION:

The given vector has length

$$\mid 2i - j - 2k \mid = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

So,the unit vector with the same direction is

$$\frac{1}{3}(2i - j - 2k) = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k.$$

EXERCISE:

1. Find $a + b$,$2a + 3b$,$| a |$, and $| a - b |$.
   (a) $a = \langle 5, 12\rangle$, $b = \langle -3, -6\rangle$
   (b) $a = 4i + j$, $b = i - 2j$
   (c) $a = i + 2j - 3k$,$b = -2i - j + 5k$
   (d) $a = 2i - 4j + 4k$, $b = 2j - k$
2. Find a unit vector that has the same direction as the given vector.
   (a) $-3i + 7j$
   (b) $8i - j + k$
   (c) $\langle -4, 2, 4\rangle$
3. Determine the magnitude of each of the following vectors.
   (a) $a = \langle 3, -5, 10\rangle$
   (b) $b = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$
   (c) $u = \langle 1, 0, 0\rangle$
4. Determine if the sets of vectors are parallel or not.
   (a) $a = \langle 2, -4, 1\rangle$,$b = \langle -6, 12, -3\rangle$
   (b) $a = \langle 4, 10\rangle$,$b = \langle 2, -9\rangle$

2.7 DOT PRODUCT

So far we have added two vectors and multiplied a vector by a scalar. The question arises:Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows. Another is the cross product, which is discussed in the next section.

2.8 DEFINITION 1

If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$,then the dot product of a and b is the number $a \cdot b$ given by

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of a and b,we multiply corresponding components and add. The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the scalar product (or inner product). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \,.\, \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

EXAMPLE 5:

1. $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$
2. $\langle -1, 7, 4 \rangle \cdot \left\langle 6, 2, -\frac{1}{2} \right\rangle = (-1)(6) + 7(2) + 4(-\frac{1}{2}) = 6$
3. $(i + 2j - 3k) \cdot (2j - k) = 1(0) + 2(2) + (-3)(-1) = 7$

The dot product obeys many of the laws that hold for ordinary products of real numbers.These are stated in the following theorem.

2.9 PROPERTIES OF THE DOT PRODUCT

If a,b,and c are vectors in V_3 and c is a scalar,then

1. $a \cdot a =| a |^2$
2. $a \cdot b = b \cdot a$
3. $a.(b + c) = a \cdot b + a \cdot c$
4. $0 \cdot a = 0$
5. $(ca).b = c(a \cdot b) = a \cdot (cb)$
6. $a \cdot b = 0$,a and b are mutually perpendicular or $a = 0$ or $b = 0$.

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $a \cdot a = a_1^2 + a_2^2 + a_3^2 =| a |^2$
2. $a.(b + c) = \langle a_1, a_2, a_3\rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3\rangle$
   $= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$
   $= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$
   $= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$
   $= a \cdot b + a \cdot c$

The proofs of the remaining properties are left as exercises.

The dot product $a \cdot b$ can be given a geometric interpretation in terms of the angle θ between a and b,which is defined to be the angle between the representations of a and b that start at the origin, where $0 \leq \theta \leq \pi$.
In other words, θ is the angle between the line segment $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
Note that if a and b are parallel vectors, then $\theta = 0$ or $\theta = \pi$.
The formula in the following theorem is used by physicists as the definition of the dot product.

2.10 THEOREM 1

If θ is the angle between the vectors a and b ,then

$$a \cdot b =\mid a \mid\mid b \mid cos\theta$$

EXAMPLE 6:

If the vector a and b have lengths 4 and 6, and the angle between them is $\pi/3$,find $a \cdot b$

SOLUTION :

Using Theorem 1,we have

$$a \cdot b =\mid a \mid\mid b \mid cos(\pi/3) = 4 \cdot 6 \cdot \tfrac{1}{2} = 12$$

The formula in Theorem 1 also enables us to find the angle between two vectors.

2.11 COROLLARY 1

If θ is the angle between the non-zero vectors a and b,then

$$cos\theta = \frac{a \cdot b}{\mid a \mid\mid b \mid}$$

EXAMPLE 7:

Find the angle between the vectors $a = \langle 2, 2, -1 \rangle$ and $\langle 5, -3, 2 \rangle$.

SOLUTION:

Since

$$\mid a \mid = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \text{ and } \mid b \mid = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$a \cdot b = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 1

$$cos\theta = \frac{a \cdot b}{\mid a \mid\mid b \mid} = \frac{2}{3\sqrt{38}}$$

So the angle between a and b is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) = 84^o$$

Two non-zero vectors a and b are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$

$$a \cdot b = \mid a \mid\mid b \mid cos(\pi/2) = 0$$

and conversely if

$$a \cdot b = 0$$

then $cos\theta = 0$,so $\theta = \pi/2$.The zero vector is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

2.12 THEOREM 2

Two vectors a and b are orthogonal if and only if $a \cdot b = 0$

EXAMPLE 8:

Show that $2i + 2j - k$ is perpendicular to $5i - 4j + 2k$

SOLUTION:

since
$(2i + 2j - k) \cdot (5i - 4j + 2k) = 2(5) + 2(-4) + (-1)(2) = 0$ these vectors are perpendicular.

2.13 PROJECTION

The scalar projection of b onto a (also called the component of b along a) is defined to be the signed magnitude of the vector projection, which is the number $\mid b \mid cos\theta$, where θ is the angle between a and b.This is denoted by $comp_a b$.Observe that it is negative if $\pi/2 < \theta \leq \pi$.The equation

$$a \cdot b = \mid a \mid\mid b \mid cos\theta = \mid a \mid (\mid b \mid cos\theta)$$

shows that the dot product of a and b can be interpreted as the length of a times the scalar projection of b onto a.Since

$$\mid b \mid cos\theta = \frac{a \cdot b}{\mid a \mid} = \frac{a}{\mid a \mid} \cdot b$$

the component of b along a can be computed by taking the dot product of b with the unit vector in the direction of a.We summarize these ideas as follows.

1. Scalar projection of b onto a:

$$comp_a b = \frac{a \cdot b}{\mid a \mid}$$

2. Vector projection of b onto a:

$$proj_a b = \left(\frac{a \cdot b}{\mid a \mid}\right) \frac{a}{\mid a \mid} = \frac{a \cdot b}{\mid a \mid^2} a$$

Notice that the vector projection is the scalar projection times the unit vector in the direction of a.

EXAMPLE 9:

Find the scalar projection and vector projection of $b = \langle 1, 1, 2 \rangle$ onto $a = \langle -2, 3, 1 \rangle$.

SOLUTION:

Since $\mid a \mid = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of b onto a is

$$comp_a b = \frac{a \cdot b}{\mid a \mid} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of a:

$$proj_a b = \frac{3}{\sqrt{14}} \frac{a}{\mid a \mid} = \frac{3}{14} a = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

EXERCISE:

1. Find $a \cdot b$
   (a) $a = \langle -2, 3\rangle$, $b = \langle 0.7, 1.2\rangle$
   (b) $a = \langle -2, 1/3\rangle$,$b = \langle -5, 12\rangle$
   (c) $a = \langle 6, -2, 3\rangle$,$b = \langle 2, -5, -1\rangle$
   (d) $a = \langle p, -p, 2p\rangle$, $b = \langle 2q, q, -q\rangle$
   (e) $a = \langle 4, 1, 1/4\rangle$, $b = \langle 6, -3, -8\rangle$
   (f) $a = 2i + j$, $b = i - j + k$
   (g) $a = 3i + 2j - k$, $b = 4i + 5k$
   (h) $\mid a \mid= 6$, $\mid b \mid= 5$,the angle between a and b is $2\pi/3$
   (i) $\mid a \mid= 3$, $\mid b \mid= \sqrt{6}$,the angle between a and b is 45^o
2. Determine whether the given vectors are orthogonal, parallel, or neither.
   (a) $a = \langle -5, 3, 7\rangle$,$b = \langle 6, -8, 2\rangle$
   (b) $a = \langle -3, 9, 6\rangle$,$b = \langle 4, -12, -8\rangle$
3. Find the values of such that the angle between the vectors $a = \langle 2, 1, -1\rangle$ and $b = \langle 1, x, 0\rangle$ is 45^o.
4. Find the angle between the vectors.
   (a) $a = \langle 4, 3\rangle$ and $b = \langle 2, -1\rangle$
   (b) $a = \langle -2, 5\rangle$ and $b = \langle 5, 12\rangle$
   (c) $a = \langle 3, -1, 5\rangle$ and $b = \langle -2, 4, 3\rangle$
   (d) $a = \langle 4, 0, 2\rangle$ and $b = \langle 2, -1, 0\rangle$
5. Find the scalar and vector projections of b onto a.
   (a) $a = \langle -5, 12\rangle$ and $b = \langle 4, 6\rangle$
   (b) $a = \langle 1, 4\rangle$ and $b = \langle 2, 3\rangle$
   (c) $a = \langle 3, 6, -2\rangle$ and $b = \langle 1, 2, 3\rangle$
   (d) $a = \langle -2, 3, -2\rangle$ and $b = \langle 5, -1, 4\rangle$
   (e) $a = 2i - j + 4k$, $b = i - j + k$
   (f) $a = i + j + k$, $b = i - j + k$

2.14 THE CROSS PRODUCT

Given two nonzero vectors $a = \langle a_1, a_2, a_3\rangle$ and $b = \langle b_1, b_2, b_3\rangle$,it is very useful to be able to find a nonzero vector that is perpendicular to both a and b.If $c = \langle c_1, c_2, c_3\rangle$ is such a vector, then $a \cdot c = 0$ and $b \cdot c = 0$ and so

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \tag{2.1}$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \tag{2.2}$$

To eliminate c_3 we multiply (2.1) by b_3 and (2.2) a_3 and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \tag{2.3}$$

Equation (2.3) has the form $pc_1 + qc_2 = 0$,for which an obvious solution is $c_1 = q$ and $c_2 = -p$.So a solution of (2.3) is

$$c_1 = a_2b_3 - a_3b_2 \quad c_2 = a_3b_1 - a_1b_3 \tag{2.4}$$

Substituting these values into (2.1) and (2.2),we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both a and b is

$$\langle c_1, c_2, c_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

The resulting vector is called the cross product of a and b and is denoted by $a \times b$.

2.15 DEFINITION 4

If $a = \langle a_1, a_2, a_3\rangle$ and $b = \langle b_1, b_2, b_3\rangle$,then the cross product of a and b is the vector

$$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

Notice that the cross product $a \times b$ of two vectors a and b,unlike the dot product, is a vector. For this reason it is also called the vector product. Note that $a \times b$ is defined only when and are three-dimensional vectors. In order to make Definition 4 easier to remember, we use the notation of determinants. A determinant of order 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 8 - (-6) = 14$$

A determinant of order 3 can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{2.5}$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors i,j and k, we see that the cross product of the vectors $a = a_1 i + a_2 j + a_3 k$ and $a = a_1 i + a_2 j + a_3 k$ is

$$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k \tag{2.6}$$

In view of the similarity between Equations (2.5) and (2.6), we often write

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{2.7}$$

Although the first row of the symbolic determinant in Equation (2.7) consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation (2.5), we obtain Equation (2.6). The symbolic formula in Equation (2.7) is probably the easiest way of remembering and computing cross products.

1. EXAMPLE 1:

   If $a = \langle 1, 3, 4 \rangle$ and $b = \langle 2, -7, -5 \rangle$,then

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} k$$
$$= (-15 - 28)i - (-5 - 8)j + (7 - 6)k = -43i + 13j + k$$

2. EXAMPLE 2:

   Show that $a \times a = 0$ for any vector in v_3.

   SOLUTION:

   If $a = \langle a_1, a_2, a_3 \rangle$,then

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (a_2a_3 - a_3a_2)i - (a_1a_3 - a_3a_1)j + (a_1a_2 - a_2a_1)k$$
$$= 0i - 0j + 0k = 0$$

We constructed the cross product $a \times b$ so that it would be perpendicular to both a and b. This is one of the most important properties of a cross product, so let's emphasize and verify it in the following theorem and give a formal proof.

2.16 THEOREM 3

The vector $a \times b$ is orthogonal to a and b.

PROOF:

In order to show that $a \times b$ is orthogonal to a, we compute their dot product as follows:

$$(a \times b) \cdot a = a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$
$$= a_1 (a_2b_3 - a_3b_2) - a_2 (a_1b_3 - a_3b_1) + a_3 (a_1b_2 - a_2b_1)$$
$$= a_1a_2b_3 - a_1b_1a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 = 0$$

A similar computation shows that $(a \times b) \cdot b = 0$.Therefore $a \times b$ is orthogonal to both a and b.

2.17 THEOREM 4

If θ is the angle between a and b (so $0 \leq \theta \leq \pi$),then

$$| a \times b |=| a || b | sin\theta$$

2.18 COROLLARY

Two nonzero vectors a and b are parallel if and only if

$$a \times b = 0$$

PROOF:

Two nonzero vectors a and b are parallel if and only if $\theta = 0$ or π.In either case $sin\theta = 0$,so $| a \times b |= 0$ and therefore $a \times b = 0$.

2.19 LEMMA 1(GEOMETRICAL APPLICATION OF VECTOR PRODUCT)

1. The area of a parallelogram with adjacent sides a and b is given by $| a \times b |$.
2. The area of a triangle with adjacent sides a and b is given by $\frac{1}{2} | a \times b |$
3. Two vectors a and b are collinear (linear dependent) if $a \times b = 0$

EXAMPLE 3:

(a) Find a vector perpendicular to the plane that passes through the points $P\,(1, 4, 6)$,$Q\,(-2, 5, -1)$,$R\,(1, -1, 1)$.

SOLUTION:

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P,Q,and R.

$$\overrightarrow{PQ} = (-2-1)i + (5-4)j + (-1-6)k = -3i + j - 7k$$
$$\overrightarrow{PR} = (1-1)i + (-1-4)j + (1-6)k = -5j - 5k$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = (-5-35)i - (15-0)j + (15-0)k = -40i - 15j + 15k.$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$,is also perpendicular to the plane.

(b) Find the area of the triangle with vertices $P\,(1,4,6)$,$Q\,(-2,5,-1)$,$R\,(1,-1,1)$.

SOLUTION:

In the previous example we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$.The area of the parallelogram with adjacent sides PQ and PQ is the length of this cross product:

$$\mid \overrightarrow{PQ} \times \overrightarrow{PR} \mid = \sqrt{(-40)^2 + (-15)^2 + (15)^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is,$\frac{5}{2}\sqrt{82}$.

If we apply Theorems 3 and 4 to the standard basis vectors i,j,and k using $\theta = \pi/2$,we obtain

$$\begin{array}{rrr} i \times j = k & j \times k = i & k \times i = j \\ j \times i = -k & k \times j = i & i \times k = -j \end{array}$$

Observe that

$$i \times j \neq j \times i$$

1. Thus the cross product is not commutative. Also

$$i \times (i \times j) = i \times k = -j$$

whereas

$$(i \times i) \times j = 0 \times k = 0$$

2. So the associative law for multiplication does not usually hold; that is, in general,

$$(a \times b) \times c \neq a \times (b \times c)$$

However, some of the usual laws of algebra do hold for cross products. The following theorem summarizes the properties of vector products.

2.20 THEOREM 5

If a,b,and c are vectors and k is a scalar,then

1. $a \times b = -b \times a$
2. $(ka) \times b = k(a \times b) = a \times (kb)$
3. $a \times (b + c) = a \times b + a \times c$
4. $(a + b) \times c = a \times c + b \times c$
5. $a \cdot (b \times c) = (a \times b) \cdot c$
6. $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

2.21 TRIPLE PRODUCTS

The product $a \cdot (b \times c)$ is called the scalar triple product of the vectors a,b,and c.We can write the scalar triple product as a determinant:

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{2.8}$$

2.22 DEFINITION 5

1. Three vectors a,b,and c (in the given order)are said to constitute a right-handed system if the scalar triple product $a \cdot (b \times c) > 0$
2. Three vectors a,b,and c (in the given order)are said to constitute a left-handed system if the scalar triple product $a \cdot (b \times c) < 0$

2.23 CRITERION FOR LINEAR INDEPENDENCE/DEPENDENCE

1. The vectors a,b,and c are linearly independent or non-coplanar if and only if $a \cdot (b \times c) \neq 0$.
2. The vectors a,b,and c are linearly dependent or coplanar or even collinear if and only if $a \cdot (b \times c) = 0$.

2.24 GEOMETRICAL APPLICATION OF SCALAR TRIPLE APPLICATION

1. The volume of a parallelepiped with adjacent edges a,b,and c is given by $\mid a \cdot (b \times c) \mid$.

2. The volume of a tetrahedron with adjacent edges a,b,and c is given by $\frac{1}{6} \mid a \cdot (b \times c) \mid$.

EXAMPLE 4:

Use the scalar triple product to show that the vectors $a = \langle 1, 4, -7 \rangle$,$b = \langle 2, -1, 4 \rangle$,and $c = \langle 0, -9, 18 \rangle$ are coplanar.

SOLUTION:

compute their scalar triple product:

$$a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 1(18) - 4(36) - 7(-18) = 0$$

Therefore,the volume of the parallelepiped determined by a,b,and c is 0.This means that a,b,and c are coplanar.

EXERCISES:

1. Find the cross product $a \times b$ and verify that it is orthogonal to both a and b.
   (a) $a = \langle 6, 0, -2\rangle$, $b = \langle 0, 8, 0\rangle$
   (b) $a = \langle 1, 1, -1\rangle$, $b = \langle 2, 4, 6\rangle$
   (c) $a = \langle t, 1, 1/t\rangle$, $b = \langle t^2, t^2, 1\rangle$
   (d) $a = i + 3j - 3k$,$b = -i + 5k$
   (e) $j + 7k$, $b = 2i - j + 4k$
   (f) $a = ti + costj + sintk$, $b = i - sintj + costk$
2. If $a = \langle 2, -1, 3\rangle$ and $b = \langle 4, 2, 1\rangle$,find $a \times b$ and $b \times a$.
3. If $a = \langle 1, 0, 1\rangle$,$b = \langle 2, 1, -1\rangle$,and $c = \langle 0, 1, 3\rangle$,show that $(a \times b) \times c \neq a \times (b \times c)$
4. Find the volume of the parallelepiped determined by the vectors a,b,and c.
   (a) $a = \langle 1, 2, 3\rangle$ and $b = \langle -1, 1, 2\rangle$, $c = \langle 2, 1, 4\rangle$.
   (b) $a = i + j$, $b = j + k$,$c = i + j + k$
5. Find a nonzero vector orthogonal to the plane through the points P,Q,and R and the area of the triangle PQR
   (a) $P(1, 2, 3)$,$Q = (-2, 1, 3)$,$R = (4, 2, 5)$
   (b) $P(0, 0, -3)$,$Q = (4, 2, 0)$,$R = (3, 3, 1)$
   (c) $P(0, -2, 0)$,$Q = (4, 1, -2)$,$R = (5, 3, 1)$
   (d) $P(-1, 3, 1)$,$Q = (0, 5, 2)$,$R = (4, 3, -1)$

2.25 EQUATION OF LINES AND PLANES

A line in the $xy-$plane is determined when a point on the line and the direction of the line(its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.
Likewise, a line L in three-dimensional space is determined when we know a point $P_o(x_0, y_o, z_o)$ on L and the direction of L.In three dimensions the direction of a line is conveniently described by a vector, so we let v be a vector parallel to L.Let $P(x, y, x)$ be an arbitrary point on L and let r_0 and r be the position vectors of P_o and P (that is, they have representations $\overrightarrow{OP_o}$ and $\overrightarrow{OP}$).If a is the vector with representation $\overrightarrow{P_oP}$,then the Triangle Law for vector addition gives $r = r_o + tv$.But, since a and v are are parallel vectors,there is a scalar t such that $a = tv$.Thus

$$r = r_o + tv \tag{2.9}$$

which is a vector equation of L.Each value of the parameter t gives the position vector r of a point on L.In other words, as t varies, the line is traced out by the tip of the vector r.If the vector v that gives the direction of the line L is written in component form as $v = \langle a, b, c\rangle$,then we have $tv = \langle ta, tb, tc\rangle$.We can also write $r = \langle x, y, z\rangle$ and $r_o = \langle x_o, y_o, z_o\rangle$, so the vector equation (2.9) becomes

$$\langle x, y, z\rangle = \langle x_o + ta, y_o + tb, z_o + ct\rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x_o + ta \quad y_o + tb \quad z_o + ct \tag{2.10}$$

where $t \in \mathbb{R}$.These equations are called parametric equations of the line L through the point $P_o(x_0, y_o, z_o)$ and parallel to the vector $v = \langle a, b, c\rangle$.Each value of the parameter t gives a point (x, y, x) on L.

EXAMPLE 1:

(a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $i + 4j - 2k$.

(b) Find two other points on the line.

SOLUTION:

(a) Here $r_o = \langle 5, 1, 3\rangle = 5i + j + 3k$ and $v = i + 4j - 2k$,so the vector equation becomes

$$r = (5i + j + 3k) + t(i + 4j - 2k)$$
$$r = (5 + t)i + (1 + 4t)j + (3 - 2t)k$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$,$y = 5$,and $z = 1$,so (6,5,1)is a point on the line. Similarly, $t = -1$ gives the point (4,-3,5).

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of (5,1,3),we choose the point (6,5,1),hen the parametric equations of the line become

$$x = 6 + t \quad y = 5 + 4t \quad z = 1 - 2t$$

Or, if we stay with the point (5,1,3) but choose the parallel vector $2i + 8j - 4k$,we arrive at the equations

$$x = 5 + 2t \quad y = 1 + 8t \quad z = 3 - 4t$$

In general, if a vector $v = \langle a, b, c\rangle$ is used to describe the direction of a line L,then the numbers a,b,and c are called direction numbers of L.Since any vector parallel to v could also be used, we see that any three numbers proportional to a,b,and c could also be used as a set of direction numbers for L. Another way of describing a line L is to eliminate the parameter t

Equation(2.10).If none of a,b,or c is 0, we can solve each of these equations for t,equate the results, and obtain

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c} \tag{2.11}$$

These equations are called symmetric equations of L.Notice that the numbers a,b,and c that appear in the denominators of Equations(2.11)are direction numbers of L,that is, components of a vector parallel to L.If one of a,b,or c is 0,we can still eliminate t.For instance, if $a=0$,we could write the equations of L as

$$x-x_0=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

This means that L lies in the vertical plane $x-x_0$.

EXAMPLE 2:

(a) Find parametric equations and symmetric equations of the line that passes through the points $A\,(2,4,-3)$ and $B\,(3,-1,1)$.

(b) At what point does this line intersect the $xy-$plane?

SOLUTION:

(a) We are not explicitly given a vector parallel to the line, but observe that the vector v with representation $\overrightarrow{AB}$ is parallel to the line and

$$v=\langle 3-2,-1-4,1-(-3)\rangle=\langle 1,-5,4\rangle$$

Thus direction numbers are $a=1$,$b=-5$,and $c=4$.Taking the point (2,4,-3) as P_o we see that parametric equations (2.10)are

$$x=2+t \qquad y=4-5t \qquad z=-3+4t$$

and symmetric equations (2.11) are

$$\frac{x-2}{1}=\frac{y-4}{-5}=\frac{z+3}{4}$$

(b) The line intersects the $xy-$plane when $z = 0$ in the symmetric equations and obtain

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$,so the line intersects the $xy-$plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_o(x_0, y_o, z_o)$ and $P_1(x_1, y_1, z_1)$ are $x - x_0 = y - y_0 = z - z_0$ nd so symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Often, we need a description, not of an entire line, but of just a line segment. How, for instance, could we describe the line segment AB in Example 2? If we put $t = 0$ in the parametric equations in Example 2(a),we get the point (2,4,-3)and if we put $t = 1$ we get (3,-1,1).So the line segment AB is described by the parametric equations

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t \qquad 0 \leq t \leq 1.$$

or by the corresponding vector equation

$$r(t) = \langle 2 + t, 4 - 5t, -3 + 4t\rangle \qquad 0 \leq t \leq 1.$$

In general, we know from Equation (2.9) that the vector equation of a line through the (tip of the) vector r_o in the direction of a vector v is $r = r_o + tv$.If the line also passes through(the tip of) r_1,then we can take $r_1 - r_o$ and so its vector equation is

$$r = r_o + t(r_1 - r_o) + tr_1$$

The line segment from r_o to r_1 is given by the parameter interval $0 \leq t \leq 1$. The line segment from r_o to r_1 is given by the parameter interval

$$r = r_o + t(r_1 - r_o) + tr_1 \qquad 0 \leq t \leq 1 \tag{2.12}$$

EXAMPLE 3:

Show that the lines L_1 and L_2 with parametric equations

$$x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$
$$x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are skew lines; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION:

The lines are not parallel because the corresponding vectors $\langle 1, 3, -1\rangle$ and $\langle 2, 1, 4\rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$,and these values don't satisfy the third equation. Therefore there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect.Thus L_1 and L_2 are skew lines.

2.26 PLANES

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_o(x_0, y_o, z_o)$ in the plane and a vector n that is orthogonal to the plane. This orthogonal vector n is called a normal vector. Let $P(x, y, z)$ be an arbitrary point in the plane, and let r_o and r be the position vectors of P_o and P. Then the vector $r - r_o$ is represented by $\overrightarrow{P_0P}$.The normal vector n is orthogonal to every vector in the given plane. In particular, n is orthogonal to $r - r_o$ and so we have

$$n \cdot (r - r_o) = 0 \tag{2.13}$$

which can be rewritten as

$$n \cdot r = n \cdot r_o \tag{2.14}$$

Either Equation (2.13) or Equation (2.14) is called a vector equation of the plane. To obtain a scalar equation for the plane, we write $n = \langle a, b, c\rangle$,$r = \langle x, y, z\rangle$ and $r_0 = \langle x_o, y_o, z_o\rangle$ Then the vector equation (2.13) becomes

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{2.15}$$

Equation (2.15) is the scalar equation of the plane through $P_o(x_0, y_o, z_o)$ with normal vector $n = \langle a, b, c\rangle$

EXAMPLE 4:

Find an equation of the plane through the point (2,4,-1) with normal vector $n = \langle 2, 3, 4\rangle$.Find the intercepts.

SOLUTION:

Putting $a = 2$,$b = 3$,$c = 4$,$x_o = 2$,$y_o = 4$,and $z_o = -1$ in Equation (2.15),we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the $x-$ intercept we set $y = z = 0$ in this equation and obtain $x = 6$.Similarly, the $y-$ intercept is 4 and $z-$ intercept is 3. By collecting terms in Equation (2.15) as we did in Example 4, we can rewrite the equation of a plane as

$$ax + by + cz + d = 0 \tag{2.16}$$

where $d = -(ax_o + by_o + cz_o)$.Equation (2.16) is called a linear equation in x,y,and z.Conversely,it can be shown that if a,b,and c are not all 0, then the linear equation (2.16) represents a plane with normal vector $\langle a, b, c\rangle$.

EXAMPLE 5:

1. Find an equation of the plane that passes through the points $P\,(1,3,2)$,$Q\,(3,-1,6)$,and $R\,(5,2,0)$.

   SOLUTION:
   The vectors a and b corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

   $$a = \langle 2,-4,4\rangle \qquad b = \langle 4,-1,-2\rangle$$

   Since both a and b lie in the plane, their cross product $a\times b$ is orthogonal to the plane and can be taken as the normal vector. Thus

   $$n = a\times b = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12i + 20j + 14k$$

   With the point $P\,(1,3,2)$ and the normal vector n, an equation of the plane is

   $$12(x-1) + 20(y-3) + 14(z-2) = 0$$

   or

   $$6x + 10y + 7z = 50$$

2. Find the point at which the line with parametric equations $x = 2 + 3t$,$y = -4t$,$z = 5 + t$ intersect the plane $4x + 5y - 2z = 18$.

   SOLUTION:

   We substitute the expressions for x,y,and z from the parametric equations into the equation of the plane:

   $$4(2+3t) + 5(-4t) - 2(5+t) = 18$$

   This simplifies to $-10t = 20$,so $t = -2$.Therefore the point of intersection occurs when the parameter value is $t = -2$.Then $x = 2 + 3(-2) = -4$,$y = -4(-2) = 8$,$z = 5 - 2 = 3$ and so the point of intersection is $(-4,8,3)$. Two planes are parallel if their normal vectors are parallel. For instance, the planes $x+2y-3z = 4$ and $2x+4y-6z = 3$ are parallel because their normal vectors are $n_1 = \langle 1,2,-3\rangle$ and $n_2 = \langle 2,4,-6\rangle$ and $n_2 = 2n_1$.If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

3. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$ and Find symmetric equations for the line of intersection L of these two planes.

   SOLUTION:

   (a) The normal vectors of these planes are

   $$n_1 = \langle 1, 1, 1 \rangle \qquad n_2 = \langle 1, -2, 3 \rangle$$

   and so, if θ is the angle between the planes,gives

   $$cos\theta = \frac{n_1 \cdot n_2}{\mid n_1 \mid\mid n_2 \mid}$$
   $$= \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$
   $$\theta = cos^{-1}\left(\frac{2}{\sqrt{42}}\right) = 72^o$$

   (b) We first need to find a point on L.For instance, we can find the point where the line intersects the $xy-$plane by setting $z = 0$ in the equations of both planes. This gives the equations $x + y = 1$ and $x - 2y = 1$,whose solution is $x = 1$,$y = 0$.So the point (1,0,0) lies on L. Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector v parallel to L is given by the cross product

   $$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5i - 2j - 3k$$

   and so the symmetric equations of L can be written as

   $$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

4. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$

   SOLUTION:

   Let $P_o(x_o, y_o, z_o)$ be any point in the given plane and let b be the vector corresponding to $\overrightarrow{P_oP_1}$.Then

   $$b = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

We can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of b onto the normal vector $n = \langle a, b, c \rangle$

$$\begin{aligned} D = \mid comp_n b \mid &= \frac{\mid n \cdot b \mid}{\mid n \mid} \\ &= \frac{\mid a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) \mid}{\sqrt{a^2 + b^2 + C^2}} \\ &= \frac{\mid (ax_1 + by_1 + cz_1) - (ax_o + by_0 + cz_0) \mid}{\sqrt{a^2 + b^2 + C^2}} \end{aligned}$$

Since P_o lies in the plane, its coordinates satisfy the equation of the plane and so we have $ax_o + by_0 + cz_0 + d = 0$. Thus the formula for D can be written as

$$D = \frac{\mid (ax_1 + by_1 + cz_1) \mid}{\sqrt{a^2 + b^2 + C^2}} \tag{2.17}$$

5. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION:

First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $(1/2, 0, 0)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{\mid 5(1/2) + 1(0) - 1(0) - 1}{\sqrt{5^2 + 1^2 + (-1)^1}} = \frac{3/2}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$.

EXERCISE:

1. Determine whether each statement is true or false.
   (a) Two lines parallel to a third line are parallel.
   (b) Two lines perpendicular to a third line are parallel
   (c) Two planes parallel to a third plane are parallel.
   (d) Two planes perpendicular to a third plane are parallel.
   (e) Two lines parallel to a plane are parallel
   (f) Two lines perpendicular to a plane are parallel.
   (g) Two planes parallel to a line are parallel.
   (h) Two planes perpendicular to a line are parallel.
   (i) Two planes either intersect or are parallel.
   (j) Two lines either intersect or are parallel.
   (k) A plane and a line either intersect or are parallel.
2. Find a vector equation and parametric equations for the line.
   (a) The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -2/3 \rangle$
   (b) The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3i + 3j - k$.
   (c) The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$
3. Find parametric equations and symmetric equations for the line.
   (a) The line through the origin and the point $(4, 3, -1)$
   (b) The line through the points $(0, 1/2, 1)$ and $(2, 1, -3)$
   (c) The line through the points $(1.0, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$
   (d) The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$
   (e) The line of intersection of the planes $x+2y+3z = 1$ and $x-y+z = 1$.

4. Find an equation of the plane.

   (a) The plane through the origin and perpendicular to the vector $\langle 1, -2, 5 \rangle$.

   (b) The plane through the point $(5, 3, 5)$ and with normal vector $2i + j - k$

   (c) The plane through the point $(-1, 1/2, 3)$ and with normal vector $i + 4j + k$

5. Find the point at which the line with parametric equations $x = 2 + 3t$,$y = -4t$,$z = 5 + t$ intersect the plane $2x + y - z = 18$.

6. Find the angle between the planes $x + y - z = 1$ and $x - 2y - 3z = 1$

2.27 VECTOR SPACES

Let V be a set on which addition and scalar multiplication are defined (this means that if u and v are objects in V and c is a scalar then we've defined $u + v$ and cu in some way). If the following axioms are true for all objects u, v, and w in V and all scalars c and k then V is called a vector space and the objects in V are called vectors.

(a) $u + v$ is in $V-$ This is called closed under addition

(b) cu is in $V-$ This is called closed under scalar multiplication.

(c) $u + v = v + u$

(d) $u + (v + w) = (u + v) + w$

(e) There is a special object in V, denoted 0 and called the zero vector, such that for all u in V we have $u + 0 = 0 + u = u$.

(f) For every u in V there is another object in V, denoted $-u$ and called the negative of u, such that $u - u = u + (-u) = 0$.

(g) $c(u + v) = cu + cv$

(h) $(c + k)u = cu + ku$

(i) $c(ku) = (ck)u$

(j) $1u = u$

2.28 LINEAR COMBINATION

We say the vector w from the vector space V is a linear combination of the vectors $v_1, v_2, \cdots, v_n$,all from V if there are scalars $c_1, c_2, \cdots, c_n$ so that w can be written

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

EXAMPLE 1:
Determine if the vector is a linear combination of the two given vectors.

1. Is $w = (-12, 20)$ a linear combination of $v_1 = (-1, 2)$ and $v_2 = (4, -6)$?
2. Is $w = (1, -4)$ a linear combination of $v_1 = (2, 10)$ and $v_2 = (-3, -15)$?

SOLUTION:

1. In each of these cases we'll need to set up and solve the following equation,

$$\begin{aligned} w &= c_1 v_1 + c_2 v_2 \\ (-12, 20) &= c_1 (-1, 2) + c_2 (4, -6) \end{aligned}$$

   Then set coefficients equal to arrive at the following system of equations

$$\begin{aligned} -c_1 + 4c_2 &= -12 \\ 2c_1 - 6c_2 &= 20 \end{aligned}$$

   If the system is consistent (i.e. has at least one solution then w is a linear combination of the two vectors. If there is no solution then w is not a linear combination of the two vectors. We'll leave it to you to verify that the solution to this system is $c_1 = 4$ and $c_2 = -2$ Therefore,w is a linear combination of v_1 and v_2 and we can write $w = 4v_1 - 2v_2$.

2. Here is the system we'll need to solve for this part.

$$\begin{aligned} 2c_1 - 3c_2 &= 1 \\ 10c_1 - 15c_2 &= -4 \end{aligned}$$

   This system does not have a solution and so w is not a linear combination of v_1 and v_2.

2.29 DEFINITION 6

Suppose $S = \{v_1, v_2, \cdots, v_n\}$ is a non-empty set of vectors and form the vector equation,

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

This equation has at least one solution, namely $c_1 = 0, c_2 = 0, \cdots,$
$c_n = 0$.This solution is called the trivial solution.If the trivial solution is the only solution to this equation then the vectors in the set S are called linearly independent and the set is called a linearly independent set.
If there is another solution then the vectors in the set S are called linearly dependent and the set is called a linearly dependent set.

EXERCISE:
Determine if each of the following sets of vectors are linearly independent or linearly dependent

1. $v_1 = (3, -1)$ and $v_2 = (-2, 2)$
2. $v_1 = (12, -8)$ and $v_2 = (-9, 6)$
3. $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$,and $v_3 = (0, 0, 1)$.
4. $v_1 = (2, -2, 4)$, $v_2 = (3, -5, 4)$,and $v_3 = (0, 1, 1)$.

EXAMPLES:

1. (Simplest linear independent set)
   A single nonzero vector v is linearly independent.

   PROOF:
   indeed, suppose $kv = 0$,then $k = 0$. since $v \neq 0$. Thus according to the above definition, v must be linearly independent.

2. (Simplest linear dependent set)
   The zero vector 0 is linear dependent.

   PROOF:
   $c0 = 0$,does not necessarily imply that $k = 0$. Thus according to the above definition, 0 must be linearly dependent.

2.30 DEFINITION 8

Vectors $v_1, v_2, \cdots, v_n$,all from V are said to constitute a basis for w if

1. They are linearly independent

2. For every vector $w \in V$,there exist scalars $c_1, c_2, \cdots, c_n$ so that w can be written

$$w = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

$c_1, c_2, \cdots, c_n$ are called coordinates of w relative to the given basis.

Chapter 3

MATRIX ALGEBRA

A matrix is rectangular array of numbers arranged in m horizontal rows and n vertical columns,For example,

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

We say A is an $m \times n$ matrix.a_{ij} in the ith row and jth column of A is called the (i, j)th element of A and we write the matrix $A = (a_{ij})_{m\times n}$. Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ are equal if $a_{ij} = b_{ij}$ for all i, j. If $m = n$,then A is called a square matrix of order n.a_{ii},$i = 1, 2, \cdots, n$ are called the diagonal elements of the square matrix.A square matrix $A = [a_{ij}]_{n\times m}$ such that $a_{ij} = 0$ for $i \neq j$ is called a diagonal matrix. A square matrix $A = (a_{ij})_{m\times n}$ such that $a_{ij} = 0$ for $i \neq j$ is called a diagonal matrix.

3.1 ADDITION OF MATRICES

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices.The sum is defined as

$$A + B = C$$

where $C = [c_{ij}]_{m\times n}$ and $c_{ij} = a_{ij} + b_{ij}$.That is C is obtained by adding corresponding elements of A and B.For example,

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 3 \end{bmatrix} \, and \;\; B = \begin{bmatrix} 3 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

Then

$$A+B=\begin{bmatrix}5 & 7 & 1\\5 & -1 & 3\end{bmatrix}$$

observe that the matrix sum or difference is defined only when the two matrices have the same number of rows and same number of columns.
A matrix such that all its entries are zero is called a zero matrix and is denoted by 0.That is in a zero matrix $a_{ij}=0$ for all i and j.

3.2 PROPERTIES OF MATRIX ADDITION

Let A,B,C are three matrix of the order $m\times n$ is a null matrix of the same order.Then

1. $A+B=B+A$
2. $(A+B)+C=A+(B+C)$
3. $A+0=0+A=A$

3.3 MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A=[a_{ij}]_{m\times n}$ and α is a scalar(a real number).Then $\alpha A=[\alpha a_{ij}]_{m\times n}$ is called the scalar multiplication of A by α.

3.4 MULTIPLICATION OF MATRICES

Let $A=[a_{ij}]$ be an $m\times p$ matrix and $B=[b_{ij}]$ be a $p\times m$ matrix.Then the product of A and B denoted by AB is the $m\times n$ matrix $C=[c_{ij}]$,where

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ip}b_{pj}$$

for $1\leq i\leq m$,$1\leq j\leq n$.c_{ij} can be computed in the following way

1. Select ith row in A and jth column in B and place them side by side.
2. Multiply the corresponding entries and add all the products

B is defined only when the number of columns in A is equal to the number of rows in B.We write $A \cdot B$ only when it is defined .If $A \cdot B$ is defined then $B \cdot A$ may not be defined.

EXAMPLE 1:

If A and B be two matrices of order 2×3 and 3×2 respectively, such as

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix} \; and \;\; B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 7 & 13 \\ -3 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 4 & 1 & -2 \\ 10 & 3 & -1 \\ 16 & 5 & 0 \end{bmatrix}$$

3.5 PROPERTIES OF MATRIX MULTIPLICATION

Let A ,B,C be three matrices of order $m \times n$.Then

1. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
2. $A \cdot (B + C) = A \cdot B + A \cdot C$
3. $(A + B) \cdot C = A \cdot C + B \cdot C$ Then $n \times n$ diagonal matrix ,where $a_{ij} = 1$ for all i is called the identity matrix of order n and is denoted by I_n.For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\; and \;\; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. If A is $n \times n$ matrix then $A \cdot I_n = I_n \cdot = A$.

5. If A is a square matrix of order n then we define $A^p = A \cdot A, \cdots, A$ (p factor) and $A^0 = I$.Then we have

$$A^p A^q = A^{p+q}$$

for non-negative integers P, q.

3.6 TRANSPOSE OF A MATRIX

If $A = [a_{ij}]$ is an $m \times n$ matrix then the $n \times m$ matrix $A^T = [a_{ij}^T]$,where $a_{ij}^T = a_{ji}$,$1 \leq i \leq m$,$1 \leq j \leq n$ is called the transpose of A.For example,if

$$\begin{bmatrix} 1 & -3 & 4 \\ 5 & 6 & -2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 5 \\ -3 & 6 \\ 4 & -2 \end{bmatrix}$$

That is A^T is obtained by interchanging rows and column of A.

3.7 PROPERTIES OF TRANSPOSE

If A and B are two matrices then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$

3.8 TRACE

If A is a square matrix,then the trace of A,denoted by $tr(A)$ is defined by the entries of the main diagonal of A.The trace of A is undefined if A is not a square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad tr(A) = a_{11} + a_{22} + a_{33}$$

EXAMPLE 2:
If

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 10 & 3 & -1 \\ 16 & 5 & 0 \end{bmatrix} \quad , \quad then \;\; tr(A) = 4 + 3 + 0 = 7$$

EXERCISE 1: If

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -4 & 2 & 5 \\ 2 & 2 & 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

,the evaluate the following where possible

1. $A + B$
2. $B - A$
3. $A + C$
4. $A - B$
5. $B - A$
6. $A - C$
7. $B - C$
8. $tr(C)$

3.9 SYMMETRIC MATRIX

A matrix $A = [a_{ij}]$ is called symmetric if $A = A^T$.That is A is symmetric if and only if $a_{ij} = a_{ji}$ for $1 \leq i \leq m$,$1 \leq j \leq n$.If $A = -A^T$ then A is called skew symmetric.

EXAMPLE 3:

1. $A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$
2. $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

are symmetric.Note if A is symmetric,then for all positive integers n ,A^n is symmetric.

1. $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

2. $B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

are skew symmetric.

3.10 THEOREM 2

Let A and B be symmetric matrices of the same size and C is any scalar then,

1. $A + B$ is symmetric.
2. $A - B$ is symmetric.
3. AB is symmetric if and only if $AB = BA$.
4. cA is symmetric.
5. A^T is symmetric

3.11 THEOREM 3

Let A and B be skew symmetric matrices and scalar k,then

1. $A + B$ is skew symmetric.
2. $A - B$ is skew symmetric.
3. kA is skew symmetric.
4. If A is $n \times n$ matrix,then $A - A^T$ is skew symmetric.
5. $A^T = -A$, then BAB^T is skew symmetric matrix for all B.

3.12 UNITARY MATRIX

A square matrix A with the property that $A^{-1} = A^T$ is said to be unitary.

EXAMPLE 4:

$$A = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$$

3.13 SKEW HERMITIAN MATRIX

A matrix is said to be skew hermitian if $\overline{A}^T = -A$

EXAMPLE 5:

$$A = \begin{bmatrix} i & 2+i & 3+2i \\ -2+i & 3i & -3i \\ -3+2i & -3i & 0 \end{bmatrix}$$

3.14 ORTHOGONAL MATRIX

A square matrix with the property that $A^{-1} = A^T$ is said to be orthogonal.It follows from this definition that a square matrix A is orthogonal if and only if $AA^T = A^TA = I$.
EXAMPLE 3:

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$$

3.15 HERMITIAN MATRICES

A matrix is said to be hermitian if $A = \overline{A}^T$

$$A = \begin{bmatrix} 2 & 1-i & 0 \\ 1+i & -1 & i \\ 0 & -i & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 3 & 2+i \\ 2-i & -12 \end{bmatrix}$$

3.16 INVERSE MATRICES

DEFINITION:If A is a square matrix and we can find another matrix of the same size, say B, such that

$$AB = BA = I$$

then we call A invertible and we say that B is an inverse of the matrix A.If we can't find such a matrix B we call A a singular matrix ($det(A) = 0$).

3.17 THEOREM 1

Suppose that A is invertible and that both B and C are inverses of A. Then $B = C$ and we will denote the inverse as A^{-1}.

EXAMPLE 1:Given the matrix A verify that the indicated matrix is in fact the inverse.

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{bmatrix}$$

SOLUTION:

To verify that we do in fact have the inverse we'll need to check that

$$AA^{-1} = A^{-1}A = I$$

This is easy enough to do and so we'll leave it to you to verify the multiplication.

$$AA^{-1} = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3.18 DEFINITION 2

If A is a square matrix and $n > 0$ then,

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1}$$

3.19 THEOREM 1

Suppose that A and B are invertible matrices of the same size. Then,

(a) AB is invertible and $(AB)^{-1} = B^{-1}A_1$

(b) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(c) For $n = 0, 1, 2, \cdots$ A^n is invertible and $(A^n) = A^{-n} = (A^{-1})^n$.

(d) If c is any non-zero scalar then cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.

(e) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

3.20 FINDING INVERSE MATRICES

If A is an $n \times n$ matrix then the following statements are equivalent.

1. A is invertible.
2. The only solution to the system $Ax = 0$ is the trivial solution.
3. A is row equivalent to I_n.
4. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.
5.
6. $Ax = b$ is consistent for every $n \times 1$ matrix b.

3.21 THEOREM 2

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $ad - bc \neq 0$ and singular if $ad - bc = 0$.If the matrix is invertible its inverse will be,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EXAMPLE 4:Use the fact to show that

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$$

is an invertible matrix and find its inverse.

SOLUTION:

We've already looked at this one above, but let's do it here so we can contrast the work between the two methods. First, we need,

$$ad - bc = (-4)(5) - (5)(-2) = -10 \neq 0$$

So, the matrix is in fact invertible by the fact and here is the inverse,

$$A^{-1} = \frac{1}{-10}\begin{bmatrix} 5 & 2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{bmatrix}$$

. EXAMPLE 5:Determine if the following matrix is singular.

$$A = \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix}$$

SOLUTION:

Not much to do with this one.

$$(-4)(3) - (-2)(6) = 0$$

So, by the fact the matrix is singular.

3.22 DETERMINANT

DEFINITION:If A is square matrix then the determinant function is denoted by det and $det(A)$ is defined to be the sum of all the signed elementary products of A.
Note that often we will call the number $det(A)$ the determinant of A. Also, there is some alternate notation that is sometimes used for determinants. We will sometimes denote determinants as $det(A) = \mid A \mid$ and this is most often done with the actual matrix instead of the letter representing the matrix.

1. Determinant function for a 2×2 matrix

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

2. Determinant function for a 3×3 matrix

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

EXAMPLE 7:Compute the determinant of each of the following matrices.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & 5 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{bmatrix}$

SOLUTION:

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & 5 \end{bmatrix}$, $det(A) = (3)(5) - (2)(-9) = 33$

(b) $B = \begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}, det(B) = 3 \begin{bmatrix} -1 & 8 \\ 1 & 7 \end{bmatrix} - 5 \begin{bmatrix} -2 & 8 \\ -11 & 7 \end{bmatrix} + 4 \begin{bmatrix} -2 & -1 \\ -11 & 1 \end{bmatrix}$
$= 3(-7-8) - 5(-14+88) + 4(-2-11) = -467$

(c) $C = \begin{bmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{bmatrix}, det(C) = 2 \begin{bmatrix} -8 & 3 \\ 1 & 1 \end{bmatrix} + 6 \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2 & -8 \\ -3 & 1 \end{bmatrix}$
$= 2(-8-3) + 6(2+9) + 2(2-24) = 0$

3.23 PROPERTIES OF DETERMINANT

In this section we'll be taking a look at some of the basic properties of determinants and towards the end of this section we'll have a nice test for the invertibility of a matrix. In this section we'll give a fair number of theorems (and prove a few of them) as well as examples illustrating the theorems.
Most of the theorems in this section will not help us to actually compute determinants in general.Most of these theorems are really more about how the determinants of different matrices will relate to each other. We will take a look at a couple of theorems that will help show us how to find determinants for some special kinds of matrices.

3.24 THEOREM 1

let A be an $n \times n$ matrix and c be a scalar then,

$$det(cA) = c^n det(A)$$

EXAMPLE 1:For the given matrix below compute both $det(A)$ and $det(2A)$.

$$A = \begin{bmatrix} 4 & -2 & 5 \\ -1 & -7 & 10 \\ 0 & 1 & -3 \end{bmatrix}$$

SOLUTION:

We'll leave it to you to verify all the details of this problem. First the scalar multiple.

$$2A = \begin{bmatrix} 8 & -4 & 10 \\ -2 & -14 & 20 \\ 0 & 2 & -6 \end{bmatrix}$$

The determinants.

$$det(A)45 \qquad\qquad det(2A) = 2^3 det(A)$$

Now, let's investigate the relationship between $det(A)$, $det(B)$ and $det(A+B)$.

3.25 THEOREM 2

Suppose that A, B, and C are all $n \times n$ matrices and that they differ by only a row,say the k^{th} row. Let's further suppose that the k^{th} row of C can be found by adding the corresponding entries from the k^{th} rows of A and B. Then in this case we will have that

$$det(C) = det(A) + det(B)$$

EXAMPLE 3:Consider the following three matrices.

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 6 & 1 & 7 \\ -1 & -3 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -5 & 3 \\ -1 & -3 & 9 \end{bmatrix}; \qquad C = \begin{bmatrix} 4 & 2 & -1 \\ 4 & -4 & 10 \\ -1 & -3 & 9 \end{bmatrix}$$

First, notice that we can write C as,

$$C = \begin{bmatrix} 4 & 2 & -1 \\ 4 & -4 & 10 \\ -1 & -3 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -1 \\ 6+(-2) & 1+(-5) & 7+3 \\ -1 & -3 & 9 \end{bmatrix}$$

All three matrices differ only in the second row and the second row of C can be found by adding the corresponding entries from the second row of A and B. The determinants of these matrices are,

$$det(A) = 15,\ det(B) = -115 \text{ and } det(C) = -100 = 15 + (-115)$$

3.26 THEOREM 3

If A and B are matrices of the same size then

$$det(AB) = det(A)det(B)$$

EXAMPLE 4:For the given matrices compute $det(A)$, $det(B)$, and $det(AB)$.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 7 & 4 \\ 3 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 8 \\ 4 & -1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

SOLUTION:

Here's the product of the two matrices.

$$AB = \begin{bmatrix} -8 & 12 & 15 \\ 28 & 7 & 35 \\ 4 & 14 & 37 \end{bmatrix}$$

Here are the determinants.

$$det(A) = -41,\ det(B) = 84 \text{ and}$$
$$det(AB) = -3444 = (-41)(84) = det(A)det(B)$$

3.27 THEOREM 4

A square matrix A is invertible if and only if $det(A) \neq 0$. A matrix that is invertible is often called non-singular and a matrix that is not invertible is often called singular.

3.28 THEOREM 5

If A is a square matrix then,

$$det(A) = det(A^T)$$

EXAMPLE 5:Compute $det(A)$ and $det(A^T)$ for the following matrix.

$$A = \begin{bmatrix} 5 & 3 & 2 \\ -1 & -8 & -6 \\ 0 & 1 & 1 \end{bmatrix}$$

SOLUTION:

We'll leave it to you to verify that

$$det(A) = det(A^T) = -9$$

3.29 THEOREM 6

If A is a square matrix with a row or column of all zeroes then

$$det(A) = 0$$

and so A will be singular.
It is actually very easy to compute the determinant of any triangular (and hence any diagonal) matrix. Here is the theorem that tells us how to do that.

3.30 THEOREM 7

Suppose that A is an $n \times n$ triangular matrix then,

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

EXAMPLE 6:Compute the determinant of each of the following matrices

(a) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, det(A) = -60$

(b) $B = \begin{bmatrix} 6 & 0 \\ 2 & -1 \end{bmatrix}, det(B) = -6$

(c) $C = \begin{bmatrix} 10 & 5 & 1 & 3 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}, det(C) = 0$

3.31 THE METHOD OF COFACTORS

In this section we're going to examine one of the two methods that we're going to be looking at for computing the determinant of a general matrix. We'll also see how some of the ideas we're going to look at in this section can be used to determine the inverse of an invertible matrix.
So, before we actually give the method of cofactors we need to get a couple of definitions taken care of.

3.32 DEFINITION 1

If A is a square matrix then the minor of a_{ij},denoted by M_{ij},is the determinant of the submatrix that results from removing the i^{th} row and j^{th} column of A.

3.33 DEFINITION 2

If A is a square matrix then the cofactor of a_{ij},denoted by C_{ij},is the number $(-1)^{i+j}M_{ij}$.
It turns out that we can also use cofactors to determine the inverse of an invertible matrix. To see how this is done we'll first need a quick definition.

3.34 DEFINITION 3

lET A be an $n \times n$ matrix and C_{ij} be the cofactors of a_{ij}.The matrix of cofactors from A is,

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{23} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The adjoint of A is the transpose of the matrix of cofactors and is denoted by $adj(A)$.

EXAMPLE 1:Compute the adjoint of the following matrix.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

SOLUTION:

We need the cofactors for each of the entries from this matrix.

$$A_{11} = (-1)^2 \begin{bmatrix} 6 & 3 \\ -4 & 0 \end{bmatrix} = 12, A_{12} = (-1)^3 \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = 6, A_{13} = (-1)^4 \begin{bmatrix} 1 & 6 \\ 2 & -4 \end{bmatrix} = -16,$$
$$A_{21} = (-1)^3 \begin{bmatrix} 2 & -1 \\ -4 & 0 \end{bmatrix} = 4, A_{22} = (-1)^4 \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = 2, A_{23} = (-1)^5 \begin{bmatrix} 3 & 2 \\ 2 & -4 \end{bmatrix} = 16,$$
$$A_{31} = (-1)^4 \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix} = 12, A_{32} = (-1)^5 \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = -10, A_{33} = (-1)^6 \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} = 16$$

The matrix of the cofactors is

$$A = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & -10 \\ 12 & -10 & 16 \end{bmatrix}$$

The Adjoint of A is the transpose of the cofactors

$$adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

3.35 THEOREM 8

If A is an invertible matrix then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

EXAMPLE 2:Use the adjoint matrix to compute the inverse of the following matrix.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

SOLUTION:

$$det(A) = 64$$

$$adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Therefore,the inverse of the matrix is,

$$A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

3.36 THEOREM 9

Let A be a square matrix.

(a) If B is the matrix that results from multiplying a row or column of A by a scalar, c, then $det(B) = cdet(A)$.

(b) If B is the matrix that results from interchanging two rows or two columns of A then $det(B) = -det(A)$.

(c) If B is the matrix that results from adding a multiple of one row of A onto another row of A or adding a multiple of one column of A onto another column of A then $det(B) = det(A)$.

(d) If A is a square matrix and that two of its rows are proportional or two of its columns are proportional. Then $det(A) = 0$.

3.37 SYSTEM OF EQUATIONS

Let's start off this section with the definition of a linear equation. Here are a couple of examples of linear equations.

$$6x_8y + 10z = 3 \qquad 7x_1 - \frac{5}{9}x_2 = -1$$

In the second equation note the use of the subscripts on the variables. This is a common notational device that will be used fairly extensively here. It is especially useful when we get into the general case(s) and we won't know how many variables (often called unknowns) there are in the equation.So, just what makes these two equations linear? There are several main points to notice. First,the unknowns only appear to the first power and there aren't any unknowns in the denominator of a fraction. Also notice that there are no products and/or quotients of unknowns. All of these ideas are required in order for an equation to be a linear equation. Unknowns only occur in numerators, they are only to the first power and there are no products or quotients of unknowns.The most general linear equation is,

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b \tag{3.1}$$

where there are n unknowns, $x_1 + x_2 + \cdots, x_n$ and $a_1 + a_2 + \cdots, a_n$,b are all known numbers.Next we need to take a look at the solution set of a single linear equation. A solution set (or often just solution) for (3.1) is a set of numbers $t_1 + t_2 + \cdots, t_n$ so that if we set $x_1 = t_1 + x_2 = t_2 + \cdots, x_n = t_n$ then (3.1) will be satisfied. By satisfied we mean that if we plug these numbers into the left side of (3.1) and do the arithmetic we will get b as an answer.The first thing to notice about the solution set to a single linear equation that contains at least two variables with non-zero coefficients is that we will have an infinite number of solutions. We will also see that while there are infinitely many possible solutions they are all related to each other in some way.Note that if there is one or less variables with non-zero coefficients then there will be a single solution or no solutions depending upon the value of b. Let's find the solution sets for the two linear equations given at the start of this section.

EXAMPLE 1:

Find the solution set for each of the following linear equations.

(a) $7x_1 - \frac{5}{9}x_2 = -1$

(b) $6x_8y + 10z = 3$

SOLUTION:

(a) $7x_1 - \frac{5}{9}x_2 = -1$
The first thing that we'll do here is solve the equation for one of the two unknowns. It doesn't matter which one we solve for, but we'll usually try to pick the one that will mean the least amount (or at least simpler) work. In this case it will probably be slightly easier to solve for x_1 so let's do that.

$$\begin{aligned} 7x_1 - \frac{5}{9}x_2 &= -1 \\ 7x_1 &= \frac{5}{9}x_2 - 1 \\ x_1 &= \frac{5}{63}x_2 - \frac{1}{7} \end{aligned}$$

Now, what this tells us is that if we have a value for x_2 then we can determine a corresponding value for x_1.Since we have a single linear equation there is nothing to restrict our choice of x_2 and so we we'll let x_2 be any number. We will usually write this as $x_2 = t$,where t is any number. Note that there is nothing special about the t, this is just the letter that I usually use in these cases. Others often use s for this letter and, of course, you could choose it to be just about anything as long as it's not a letter representing one of the unknowns in the equation (x in this case). Once we've "chosen" x_2 we'll write the general solution set as follows,

$$x_1 = \frac{5}{63}x_2 - \frac{1}{7} \qquad x_2 = t$$

So, just what does this tell us as far as actual number solutions go? We'll choose any value of t and plug in to get a pair of numbers x_1 and x_2 that will satisfy the equation. For instance picking a couple of values of t completely at random gives,

$$t = 0: \qquad x_1 = -\frac{1}{7} \qquad x_2 = 0$$
$$t = 27: \qquad x_1 = \frac{5}{63}(27) \qquad x_2 = 27$$

We can easily check that these are in fact solutions to the equation by plugging them back into the equation.

$$t = 0: \qquad 7(-\frac{1}{7}) - \frac{5}{9}(0) = -1$$
$$t = 27: \qquad 7(2) - \frac{5}{9}(27) = -1$$

So, for each case when we plugged in the values we got for x_1 and x_2 we got -1 out of the equation as we were supposed to. Note that since there an infinite number of choices for t there are in fact an infinite number of possible solutions to this linear equation.

(b) $6x_8y + 10z = 3$ We'll do this one with a little less detail since it works in essentially the same manner. The fact that we now have three unknowns will change things slightly but not overly much. We will first solve the equation for one of the variables and again it won't matter which one we chose to solve for.

$$10z = 3 - 6x_8y$$
$$z = \frac{3}{10} - \frac{3}{5}x + \frac{4}{5}y$$

In this case we will need to know values for both x and y in order to get a value for z. As with the first case, there is nothing in this problem to restrict out choices of x and y. We can therefore let them be any number(s). In this case we'll choose $x = t$ and $y = 5$. Note that we chose different letters here since there is no reason to think that both x and y will have exactly the same value(although it is possible for them to have the same value).
The solution set to this linear equation is then,

$$x = t \qquad y = s \qquad z = \frac{3}{10} - \frac{3}{5}x + \frac{4}{5}y$$

So, if we choose any values for t and s we can get a set of number solutions as follows.

$$\begin{aligned} x &= 0 & y &= -2 & z &= \frac{3}{10} - \frac{3}{5}(0) + \frac{4}{5}(-2) = -\frac{13}{10} \\ x &= -\frac{3}{2} & y &= 5 & z &= \frac{3}{10} - \frac{3}{5}(-\frac{3}{2}) + \frac{4}{5}(-\frac{3}{2}) = \frac{26}{5} \end{aligned}$$

As with the first part if we take either set of three numbers we can plug them into the equation to verify that the equation will be satisfied. We'll do one of them and leave the other to you to check.

$$6\left(\frac{-3}{2}\right) - 8\left(5\right) + 10\left(\frac{26}{5}\right) = -9 - 40 + 52 = 3$$

The variables that we got to choose values for (x_2 in the first example and x and y in the second)are sometimes called free variables.We now need to start talking about the actual topic of this section, systems of linear equations. A system of linear equations is nothing more than a collection of two or more linear equations.Here are some examples of systems of linear equations.

$$\begin{aligned} 2x + 3y &= 9 \\ x - 2y &= -13, \end{aligned}$$

$$\begin{aligned} 2x_1 + 3x_2 + 3x_3 &= 9 \\ x_1 + 9x_2 + x_3 &= 26 \\ 7x_1 - 5x_2 + 2x_3 &= 29 \end{aligned}$$

3.38 THEOREM 1

Given a system of n equations and m unknowns there will be one of three possibilities for solutions to the system.

1. There will be no solution.
2. There will be exactly one solution.
3. There will be infinitely many solutions

If there is no solution to the system we call the system inconsistent and if there is at least one solution to the system we call it consistent.
Now that we've got some of the basic ideas about systems taken care of we need to start thinking about how to use linear algebra to solve them. Actually that's not quite true. We're not going to do any solving until the next section. In this section we just want to get some of the basic notation and ideas involved in the solving process out of the way before we actually start trying to solve them.
We're going to start off with a simplified way of writing the system of equations. For this we will need the following general system of n equations and m unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

In this system the unknowns are $x_1 + x_2 + \cdots, x_n$ and a_{ij} and b_i are known numbers. Note as well how we've subscripted the coefficients of the unknowns (the a_{ij}).The first subscript, i,denotes the equation that the subscript is in and the second subscript, j, denotes the unknown that it multiplies. For instance,a_{36} would be in the coefficient of x_6 in the third equation.
Any system of equations can be written as an augmented matrix. A matrix is just a rectangular array of numbers and we'll be looking at these in great detail in this course so don't worry too much at this point about what a matrix is. Here is the augmented matrix for the general system in

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{bmatrix}$$

Each row of the augmented matrix consists of the coefficients and constant on the right of the equal sign from a given equation in the system. The first row is for the first equation, the second row is for the second equation etc. Likewise each of the first m columns of the matrix consists of the coefficients from the unknowns.The first column contains the coefficients of x_1,the second column contains the coefficients of x_2,etc. The final column contains all the constants on the right of the equal sign. Note that the augmented part of the name arises because we tack the b_i's onto the matrix. If we don't tack those on and we just have

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

and we call this the coefficient matrix for the system.

EXAMPLE 2:

Write down the augmented matrix for the following system.

$$\begin{aligned} 3x_1 - 10x_2 + 6x_3 - x_4 &= 3 \\ x_1 + 9x_3 - 5x_4 &= -12 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= 7 \end{aligned}$$

SOLUTION:

There really isn't too much to do here other than write down the system.

$$\begin{bmatrix} 3 & -10 & 6 & -1 & 3 \\ 1 & 0 & 9 & -5 & -12 \\ -4 & 1 & -9 & 2 & 7 \end{bmatrix}$$

Notice that the second equation did not contain an $x_2$and so we consider its coefficient to be zero.
Note as well that given an augmented matrix we can always go back to a system of equations.

EXAMPLE 3:

For the given augmented matrix write down the corresponding system of equations.

$$\begin{bmatrix} 4 & -1 & 1 \\ -5 & -8 & 4 \\ 9 & 2 & -2 \end{bmatrix}$$

SOLUTION:

So since we know each row corresponds to an equation we have three equations in the system.Also, the first two columns represent coefficients of unknowns and so we'll have two unknowns while the third column consists of the constants to the right of the equal sign. Here's the system that corresponds to this augmented matrix.

$$\begin{aligned} 4x_1 - x_2 &= 1 \\ -5x_1 - 8x_2 &= 4 \\ 9x_1 + 2x_2 &= -2 \end{aligned}$$

There is one final topic that we need to discuss in this section before we move onto actually solving systems of equation with linear algebra techniques. In the next section where we will actually be solving systems our main tools will be the three elementary row operations. Each of these operations will operate on a row (which shouldn't be too surprising given the name) in the augmented matrix and since each row in the augmented matrix corresponds to an equation these operations have equivalent operations on equations. Here are the three row operations, their equivalent equation operations as well as the notation that we'll be using to denote each of them.

Row Operation	Equation Operation	Notation
Multiply row i by the constant c	Multiply equation i by the constant c	cR_i
Interchange rows i and j	Interchange equations i and j	$R_i \leftrightarrow R_j$
Add c times row i to row j	Add c times equations i to equation j	$R_j + cR_i$

3.39 SOLVING SYSTEMS OF EQUATIONS

In this section we are going to take a look at using linear algebra techniques to solve a system of linear equations. Once we have a couple of definitions out of the way we'll see that the process is a fairly simple one. Well, it's fairly simple to write down the process anyway. Applying the process is fairly simple as well but for large systems it can take quite a few steps. So, let's get the definitions out of the way. A matrix (any matrix, not just an augmented matrix) is said to be in reduced row-echelon form if it satisfies all four of the following conditions.

1. If there are any rows of all zeros then they are at the bottom of the matrix.

2. If a row does not consist of all zeros then its first non-zero entry (i.e. the left most non-zero entry) is a 1. This 1 is called a leading 1.

3. In any two successive rows, neither of which consists of all zeroes, the leading 1 of the lower row is to the right of the leading 1 of the higher row.

4. If a column contains a leading 1 then all the other entries of that column are zero.

A matrix (again any matrix) is said to be in row-echelon form if it satisfies items 1-3 of the reduced row-echelon form definition.
Notice from these definitions that a matrix that is in reduced row-echelon form is also in row echelon form while a matrix in row-echelon form may or may not be in reduced row-echelon form

EXAMPLE 1:

The following matrices are all in row-echelon form.

1. $\begin{bmatrix} 1 & -6 & \underline{9} & \underline{1} & 0 \\ 0 & 0 & 1 & \underline{-4} & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & \underline{5} \\ 0 & 1 & \underline{3} \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & \underline{-8} & 10 & \underline{5} & -3 \\ 0 & 1 & 13 & \underline{9} & 12 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

None of the matrices in the previous example are in reduced row-echelon form. The entries that are preventing these matrices from being in reduced row-echelon form are underlined In order for these matrices to be in reduced row-echelon form all of these underlined entries would need to be zeroes. Notice that we didn't underline the entries above the 1 in the fifth column of the third matrix.Since this 1 is not a leading 1 (i.e. the leftmost non-zero entry) we don't need the numbers above it to be zero in order for the matrix to be in reduced row-echelon form.

EXAMPLE 2:

The following matrices are all in row-echelon form.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -7 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 9 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In the second matrix on the first row we have all zeroes in the entries. This is perfectly acceptable and so don't worry about it. This matrix is in reduced row-echelon form, the fact that it doesn't have any non-zero entries does not change that fact since it satisfies the conditions.Also, in the second matrix of the second row notice that the last column does not have zeroes above the 1 in that column. That is perfectly acceptable since the 1 in that column is not a leading 1 for the fourth row.

Notice from Examples 1 and 2 that the only real difference between row-echelon form and reduced row-echelon form is that a matrix in row-echelon form is only required to have zeroes below a leading 1 while a matrix in reduced row-echelon from must have zeroes both below and above a leading 1.

Okay, let's now start thinking about how to use linear algebra techniques to solve systems of linear equations. The process is actually quite simple. To solve a system of equations we will first write down the augmented matrix for the system. We will then use elementary row operations to reduce the augmented matrix to either row-echelon form or to reduced row-echelon form. Any further work that we'll need to do will depend upon where we stop.

If we go all the way to reduced row-echelon form then in many cases we will not need to do any further work to get the solution and in those times where we do need to do more work we will generally not need to do much more work. Reducing the augmented matrix to reduced row-echelon form is called Gauss-Jordan Elimination.

If we stop at row-echelon form we will have a little more work to do in order to get the solution,but it is generally fairly simple arithmetic. Reducing the augmented matrix to row-echelon form and then stopping is called Gaussian Elimination.

At this point we should work a couple of examples.

EXAMPLE 3:

Use Gaussian Elimination and Gauss-Jordan Elimination to solve the following system of linear equations.

$$\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 + x_3 &= -1 \end{aligned}$$

SOLUTION:

Since we're asked to use both solution methods on this system and in order for a matrix to be in reduced row-echelon form the matrix must also be in row-echelon form. Therefore, we'll start off by putting the augmented matrix in row-echelon form, then stop to find the solution. This will be Gaussian Elimination. After doing that we'll go back and pick up from row-echelon form and further reduce the matrix to reduced row echelon form and at this point we'll have performed Gauss-Jordan Elimination.
So, let's start off by getting the augmented matrix for this system.

$$\begin{bmatrix} \underline{-2} & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

As we go through the steps in this first example we'll mark the entry(s) that we're going to be looking at in each step underlined so that we don't lose track of what we're doing. We should also point out that there are many different paths that we can take to get this matrix into row-echelon form and each path may well produce a different row-echelon form of the matrix. Keep this in mind as you work these problems. The path that you take to get this matrix into row-echelon form should be the one that you find the easiest and that may not be the one that the person next to you finds the easiest. Regardless of which path you take you are only allowed to use the three elementary row operations that we looked in the previous section.
So, with that out of the way we need to make the leftmost non-zero entry in the top row a one. In this case we could use any three of the possible row operations. We could divide the top row by -2 and this would certainly change the red -2 into a one. However, this will also introduce fractions into the matrix and while we often can't avoid them let's not put them in before we need to.

Next, we could take row three and add it to row one, or we could take three times row 2 and add it to row one. Either of these would also change the red -2 into a one. However, this row operation is the one that is most prone to arithmetic errors so while it would work let's not use it unless we need to.
This leaves interchanging any two rows. This is an operation that won't always work here to get a 1 into the spot we want, but when it does it will usually be the easiest operation to use. In this case we've already got a one in the leftmost entry of the second row so let's just interchange the first and second rows and we'll get a one in the leftmost spot of the first row pretty much for free.Here is this operation.

$$\begin{bmatrix} \underline{-2} & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & 3 & 13 \\ \underline{-2} & 1 & -1 & 4 \\ \underline{-3} & 0 & 1 & -1 \end{bmatrix}$$

Now, the next step we'll need to take is changing the two numbers in the first column under the leading 1 into zeroes. Recall that as we move down the rows the leading 1 MUST move off to the right. This means that the two numbers under the leading 1 in the first column will need to become zeroes. Again, there are often several row operations that can be done to do this. However, in most cases adding multiples of the row containing the leading 1 (the first row in this case) onto the rows we need to have zeroes is often the easiest. Here are the two row operations that we'll do in this step.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ \underline{-2} & 1 & -1 & 4 \\ \underline{-3} & 0 & 1 & -1 \end{bmatrix} R_2 + 2R_1 \ , \ R_3 - 3R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & \underline{5} & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix}$$

Notice that since each operation changed a different row we went ahead and performed both of them at the same time. We will often do this when multiple operations will all change different rows.
We now need to change the red "5" into a one. In this case we'll go ahead and divide the second row by 5 since this won't introduce any fractions into the matrix and it will give us the number we're looking for.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & \underline{5} & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix} \frac{1}{5}R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & \underline{-6} & -8 & -40 \end{bmatrix}$$

Next, we'll use the third row operation to change the red "-6" into a zero so the leading 1 of the third row will move to the right of the leading 1 in the second row. This time we'll be using a multiple of the second row to do this.

Here is the work in this step.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & \underline{-6} & -8 & -40 \end{bmatrix} R_3 + 6R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & \underline{-2} & -4 \end{bmatrix}$$

Notice that in both steps were we needed to get zeroes below a leading 1 we added multiples of the row containing the leading 1 to the rows in which we wanted zeroes. This will always work in this case. It may be possible to use other row operations, but the third can always be used in these cases.

The final step we need to get the matrix into row-echelon form is to change the red "-2" into a one. To do this we don't really have a choice here. Since we need the leading one in the third row to be in the third or fourth column (i.e. to the right of the leading one in the second column) we MUST retain the zeroes in the first and second column of the third row.

Interchanging the second and third row would definitely put a one in the third column of the third row, however, it would also change the zero in the second column which we can't allow.Likewise we could add the first row to the third row and again this would put a one in the third column of the third row, but this operation would also change both of the zeroes in front of it which can't be allowed.

Therefore, our only real choice in this case is to divide the third row by -2. This will retain the zeroes in the first and second column and change the entry in the third column into a one. Note that this step will often introduce fractions into the matrix, but at this point that can't be avoided.Here is the work for this step.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & \underline{-2} & -4 \end{bmatrix} -\frac{1}{2}R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

At this point the augmented matrix is in row-echelon form. So if we're going to perform Gaussian Elimination on this matrix we'll stop and go back to equations. Doing this gives,

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

At this point solving is quite simple. In fact we can see from this that $x_3 = 2$.Plugging this into the second equation gives $x_2 = 4$.Finally, plugging both of these into the first equation gives $x_1 = -1$.Summarizing up the solution to the system is,

$$x_1 = -1, x_2 = 4, x_3 = 2$$

This substitution process is called back substitution. Now, let's pick back up at the row-echelon form of the matrix and further reduce the matrix into reduced row-echelon form. The first step in doing this will be to change the numbers above the leading 1 in the third row into zeroes. Here are the operations that will do that for us.

$$\begin{bmatrix} 1 & 2 & \underline{3} & 13 \\ 0 & 1 & \underline{1} & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_1 - 3R_3 \ , \quad R_2 - R_3 \rightarrow \begin{bmatrix} 1 & \underline{2} & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The final step is then to change the red "2" above the leading one in the second row into a zero.Here is this operation

$$\begin{bmatrix} 1 & \underline{2} & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_1 - 3R_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We are now in reduced row-echelon form so all we need to do to perform Gauss-Jordan Elimination is to go back to equations.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\Rightarrow$

$$x_1 = -1, x_2 = 4, x_3 = 2$$

We can see from this that one of the nice consequences to Gauss-Jordan Elimination is that when there is a single solution to the system there is no work to be done to find the solution. It is generally given to us for free. Note as well that it is the same solution as the one that we got by using Gaussian Elimination as we should expect.

EXAMPLE 4:

Solve the following system of linear equations.

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= 1 \end{aligned}$$

SOLUTION:

First, the instructions to this problem did not specify which method to use so we'll need to make a decision. No matter which method we chose we will need to get the augmented matrix down to row-echelon form so let's get to that point and then see what we've got. If we've got something easy to work with we'll stop and do Gaussian Elimination and if not we'll proceed to reduced row-echelon form and do Gauss-Jordan Elimination.
So, let's start with the augmented matrix and then proceed to put it into row-echelon form and again we're not going to put in quite the detail in this example as we did with the first one. So,here is the augmented matrix for this system.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{bmatrix}$$

and here is the work to put it into row-echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{bmatrix} R_2 + R_1 \ , \ R_3 - 2R_1 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{bmatrix} - R_2 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{bmatrix}$$

$$R_3 - 3R_2 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{bmatrix} \frac{1}{8} R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Okay, we're now in row-echelon form. Let's go back to equation and see what we've got.

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 - x_3 &= -1 \\ 0 &= 1 \end{aligned}$$

That last equation doesn't look correct. We've got a couple of possibilities here.We've either just managed to prove that 0=1 (and we know that's not true), we've made a mistake (always possible, but we haven't in this case) or there's another possibility we haven't thought of yet.
Recall from Theorem 1 in the previous section that a system has one of three possibilities for a solution. Either there is no solution, one solution or infinitely many solutions. In this case we've no solution. When we go back to equations and we get an equation that just clearly can't be true such as the third equation above then we know that we've no solution.
Note as well that we didn't really need to do the last step above. We could have just as easily arrived at this conclusion by looking at the second to last matrix since 0=8 is just as incorrect as 0=1.
So, to close out this problem, the official answer is that there is no solution to this system.

EXERCISES: Solve the following system of linear equations.

1. $x_1 - 2x_2 + 3x_3 = -2$
   $-x_1 + x_2 - 2x_3 = 3$
   $2x_1 - x_2 + 3x_3 = -7$

2. $3x_1 - 4x_2 = 10$
   $-5x_1 + 8x_2 = -17$
   $-3x_1 + 12x_2 = -12$

3. $7x_1 + 2x_2 - 2x_3 - 4x_4 + 3x_5 = 8$
   $-3x_1 - 3x_2 + 2x_4 + x_5 = -1$
   $4x_1 - x_2 - 8x_3 + 20x_5 = 1$

3.40 HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

We've got one more topic that we need to discuss briefly in this section. A system of n linear equations in m unknowns in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0 \end{aligned}$$

is called a homogeneous system. The one characteristic that defines a homogeneous system is the fact that all the equations are set equal to zero unlike a general system in which each equation can be equal to a different (probably non-zero) number. Hopefully, it is clear that if we take

$$x_1 = 0, x_2 = 0, \cdots, x_m = 0$$

we will have a solution to the homogeneous system of equations. In other words, with a homogeneous system we are guaranteed to have at least one solution. This means that Theorem 1 from the previous section can then be reduced to the following for homogeneous systems.

3.41 THEOREM 2

Given a homogeneous system of n equations and m unknowns there will be one of two possibilities for solutions to the system.

1. There will be exactly one solution,$x_1 = 0, x_2 = 0, \cdots, x_m = 0$.This solution is called the trivial solution.

2. There will be infinitely many non-zero solutions in addition to the trivial solution.

Note that when we say non-zero solution in the above fact we mean that at least one of the x_i s in the solution will not be zero. It is completely possible that some of them will still be zero, but at least one will not be zero in a non-zero solution.

We can make a further reduction to Theorem 1 from the previous section if we assume that there are more unknowns than equations in a homogeneous system as the following theorem shows.

3.42 THEOREM 3

Given a homogeneous system of n linear equations in m unknowns if $m > n$ (i.e.there are more unknowns than equations) there will be infinitely many solutions to the system.

3.43 THE RANK OF MATRIX

DEFINITION:The rank of a matrix A,denoted $rank(A)$ is the maximum number of linearly independent rows vectors or columns in the matrix. If a vector is linearly independent of a set of other vectors that means it cannot be written as a linear combination of them.

3.44 THEOREM 4

If A is any matrix,then $rank(A) = rank(A^T)$.

3.45 THEOREM 5

If A is an $m \times n$ matrix,then:

(a) $rank(A)$=the number of leading variable in the solution of $Ax = 0$.

(b) $nullity(A)$=the number of parameters in the solution $Ax = 0$.

3.46 THEOREM 6

If A is a matrix with n columns then,$n = rank(A) + nullity(A)$

EXAMPLE 1:
Find the rank and nullity of

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -1 \\ 3 & 10 & -6 & -5 \end{bmatrix}$$

SOLUTION:

By using the elementary row operation we have

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -1 \\ 3 & 10 & -6 & -5 \end{bmatrix}$$

to

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix}$$

to

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Rank(A) = 2, nullity(A) = 4 - 2 = 2.$$

EXERCISE:
Find the rank and nullity of

1. $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

2. $B = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

3. $C = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$

4. $D = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix}$

5. $E = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$

3.47 THE CONSISTENT THEOREM

If $Ax = b$ is a linear system of m equations in n unknowns,then the following are equivalent

(a) $Ax = b$ is consistent

(b) The coefficient matrix A and the augmented matrix $[A \mid b]$ have the same rank.In other words,A linear system is said to be consistent if and only if the rank is equal of the augmented matrix $[A \mid b]$.

(c) The system has a solution if and only if $rank(A) = rank[A \mid b]$. If the rank is equal to n, then the system has a unique solution.If the rank is equal to n, then the system has a unique solution. If $rank(A) = rank[A \mid b]$ but the $rank < n$, there are an infinite number of solutions. If we denote the rank by r, then r of the unknown variables can be expressed as linear combinations of nr of the other variables.

EXAMPLE 2:
Find k such that the following system is consistent

$$\begin{aligned} x - 2y - z &= 1 \\ -x - 3y + z &= -1 \\ -5x - 8y + 5z &= k \end{aligned}$$

SOLUTION:

We first put the above system in the augmented form.Thus

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ -5 & -5 & 5 & k \end{bmatrix}$$

We can then apply row operations and reduced the augmented matrix to echelon form.This

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & k+3 \end{bmatrix}$$

From the above results, the rank of A is 2 and for consistency,the rank 0f $[A \mid B]$.This is possible $k + 3 = 0 \Rightarrow k = -3$.

$$\begin{aligned} x - 2y - z &= 1 \\ -x - 3y + z &= -1 \\ -5x - 8y + 5z &= k \end{aligned}$$

is consistent if and only if $k = -3$.

EXAMPLE 3:
Show that

$$\begin{aligned} x_1 - 2x_2 - 3x_3 + x_4 &= -4 \\ -3x_1 + 7x_2 - x_3 + x_4 &= -3 \\ 2x_1 - 5x_2 + 4x_3 - 3x_4 &= 7 \\ -3x_1 + 6x_2 + 9x_3 - 6x_4 &= -1 \end{aligned}$$

is inconsistent.

SOLUTION:

The augment form is

$$\begin{bmatrix} 1 & -2 & -3 & 2 & | & -4 \\ -3 & 7 & -1 & 1 & | & -3 \\ 2 & -5 & 4 & -3 & | & 7 \\ -3 & 6 & 9 & -6 & | & -1 \end{bmatrix}$$

which has a reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -23 & 16 & | & 0 \\ 0 & 1 & -10 & 7 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

the system is inconsistent because $rank(A) \neq rank[A \mid b]$

EXAMPLE 3:
Find the general solution

$$\begin{aligned} 4x + 3y + 2z &= 0 \\ -x + 2y + 2z + w &= 0 \\ 3x + 5y + 4z + w &= 0 \end{aligned}$$

The coefficient matrix is

$$\begin{bmatrix} 4 & 3 & 2 & 0 \\ -1 & 2 & 2 & 1 \\ 3 & 5 & 4 & 1 \end{bmatrix}$$

$R_2 = R_2 - 4R_1$,$R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 11 & 10 & 4 \\ 0 & 11 & 10 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 11 & 10 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & 10/11 & 4/11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is equivalent to

$$\begin{aligned} x - 2y - 2z - w &= 0 \\ y + 10/11z + 4/11w &= 0 \end{aligned}$$

Let $z = 11s$ and $w = 11t$ where s and t arbitrary real numbers then we have

$$\begin{aligned} x - 2y &= 22s + 11t = 0 \\ y &= -10s - 4t = 0 \\ x &= -20s - 8t + 22s + 11t = 2s + 3t \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} s \begin{pmatrix} 2 \\ -10 \\ 11 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -4 \\ 0 \\ 11 \end{pmatrix}$$

EXERCISE:
Solve the systems

1. $x + 3y - 3z = 6$
   $2x - y + 4z = 2$
   $4x + 3y - 3z = 14$

2. $x + y - z = 1$
   $2x + 3y + 9z = -3$
   $x + 3y + 3z = 2$

3. $x - y + 2z = 4$
   $2x - 3y - 11z = -6$
   $x - 2y + 7z = 7$

Find all values of a for which the resulting linear system has

(a) no solution

(b) a unique solution

(c) infinitely many solutions

of the following

1. $x + y - z = 2$
   $x + 2y + z = 3$
   $x + y + 86z = a$

2. $x + y = 3$
   $x + (9^2 - 8)y = a$

3. $x + y - z = 2$
   $x + 2y + z = 3$
   $x + y + (9^2 - 5)z = a$

4. $x + y + z = 2$
   $2x + 3y + 2z = 5$
   $2x + 3y + (9^2 - 1)z = a + 1$

Chapter 4

EIGENVALUES AND EIGENVECTORS

4.1 DEFINITION 1

Suppose that A is an $n \times n$ matrix.Also suppose that x is a non-zero vector from $\mathbb{R}^n$ and that λ is any scalar (this can be zero) so that,

$$Ax = \lambda x$$

We then call x an eigenvector of A and λ an eigenvalue of A.
We will often call x the eigenvector corresponding to or associated with λ and we will often call λ the eigenvalue corresponding to or associated with x.
Note that eigenvalues and eigenvectors will always occur in pairs. You can't have an eigenvalue without an eigenvector and you can't have an eigenvector without an eigenvalue.

4.2 FINDING EIGENVALUES

EXAMPLE 1:
Suppose $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ then $x = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$ is an eigenvector with corresponding eigenvalue 4 because

$$Ax = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} -8 \\ 1 \end{bmatrix} = \begin{bmatrix} -32 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} -8 \\ 1 \end{bmatrix} = \lambda x$$

Okay, what we need to do is figure out just how we can determine the eigenvalues and eigenvectors for a given matrix. This is actually easier to do than it might at first appear to be. We'll start with finding the eigenvalues for a matrix and once we have those we'll be able to find the eigenvectors corresponding to each eigenvalue.
Let's start with $Ax = \lambda x$ and rewrite it as follows,

$$Ax = \lambda I x$$

Note that all we did was insert the identity matrix into the right side. Doing this will allow us to further rewrite this equation as follows,

$$\begin{aligned} Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

Now, if λ is going to be an eigenvalue of A this system must have a non-zero solution, x, since we know that eigenvectors associated with λ cannot be the zero vector.
The Fundamental Subspaces section tells us that this system will have a non-zero solution if and only if

$$det(A - \lambda I)x = 0$$

So, eigenvalues will be scalars,λ for which the matrix $A - \lambda I$ will be singular, i.e. $det(A - \lambda I)x = 0$.

4.3 DEFINITION 2

Suppose A is an $n \times n$ matrix then, $det(A - \lambda I)x = 0$ is called the characteristic equation of A.When computed it will be an n^{th} degree polynomial in λ of the form,

$$p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

called the characteristic polynomial of A.

4.4 DEFINITION 3

Suppose A is an $n \times n$ matrix and that $\lambda_1, \lambda_2, \cdots, \lambda_n$ is the complete list of all the eigenvalues of A including repeats. If λ occurs exactly once in this list then we call λ a simple eigenvalue.If λ occurs $k \geq 2$ times in the list we say that λ has multiplicity of k.

EXAMPLE 2:
Find all the eigenvalues for the given matrices.

1. $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}$

3. $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

SOLUTION:

1. $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$
   We'll do this one with a little more detail than we'll do the other two. First we'll need the matrix $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 6 - \lambda & 16 \\ -1 & -4 - \lambda \end{bmatrix}$$

   Next we need the determinant of this matrix, which gives us the characteristic polynomial.

$$det(A - \lambda I) = (6 - \lambda)(-4 - \lambda) + 16 = \lambda^2 - 2\lambda - 8$$

   Now, set this equal to zero and solve for the eigenvalues

$$\lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) \Rightarrow \lambda_1 = -2, \lambda_2 = 4$$

   So, we have two eigenvalues and since they occur only once in the list they are both simple eigenvalues.

2. $A = \begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}$

Here is the matrix $A - \lambda I$ and its characteristic polynomial.

$$A - \lambda I = \begin{bmatrix} -4-\lambda & 2 \\ 3 & -5-\lambda \end{bmatrix} \qquad det(A - \lambda I) = \lambda^2 + 9\lambda + 14$$

We'll leave it to you to verify both of these. Now, set the characteristic polynomial equal to zero and solve for the eigenvalues.

$$\lambda^2 + 9\lambda + 14 = (\lambda + 7)(\lambda + 2) \;\Rightarrow\; \lambda_1 = -7, \lambda_2 = -2$$

3. $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

   Here is the matrix $A - \lambda I$ and its characteristic polynomial.

$$A - \lambda I = \begin{bmatrix} 7-\lambda & -1 \\ 4 & 3-\lambda \end{bmatrix} \qquad det(A - \lambda I) = \lambda^2 - 10\lambda + 25$$

   We'll leave it to you to verify both of these. Now, set the characteristic polynomial equal to zero and solve for the eigenvalues

$$= \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \;\Rightarrow\; \lambda_{1,2} = 5$$

   In this case we have an eigenvalue of multiplicity two. Sometimes we call this kind of eigenvalue a double eigenvalue. Notice as well that we used the notation $\lambda_{1,2}$ to denote the fact that this was a double eigenvalue.

Now, let's take a look at some 3×3 matrices.
If $A = [a_{ij}]$ is a 3×3 ,then the calculation of $det(A - \lambda I)$ can be done directly,or the following formulas for the coefficients can be used.By collecting terms we can put the determinant of $(A - \lambda I)$ into form

$$det(A - \lambda I) = -\lambda^3 + tr(A)\lambda^2 - pm(A)\lambda + det(A)$$

Where the coefficients of λ^2 and λ are as follows:

$$tr(A) = a_{11} + a_{22} + a33$$

is the sum of the diagonal entries,called the *trace* of A, and

$$pm(A) = M_{11} + M_{22} + M_{33} = det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If the sum of the minors of the main diagonal entries, called the principal minors of A.

EXAMPLES:

1. Find the Eigenvalues of the matrix A.
$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$
SOLUTION:
The characteristic equation of A is

$$det(A - \lambda I) = det \begin{bmatrix} 3-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix} = -\lambda^3 + 5\lambda^2 - 2\lambda - 2 = 0$$

multiplying by -1, we get the cubic equation $\lambda^3 - 5\lambda^2 + 2\lambda + 2 = 0$
Testing the possible integer roots $\pm 1, \pm 2$,we find that $\lambda = 1$ is a root. Dividing by $\lambda - 1$, we obtain the factored form

$$(\lambda - 1)(\lambda^2 - 4\lambda - 2 = 0)$$

Using the quadratic formula on the second factor, we get $\lambda = 2 \pm \sqrt{6}$.Thus the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2 + \sqrt{6}, \lambda_3 = 2 - \sqrt{6}$.

2. $A = \begin{bmatrix} 6 & 3 & -8 \\ -0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$
Here is $(A - \lambda I)$ and the characteristic polynomial for this matrix.

$$det(A - \lambda I) = \begin{bmatrix} 6-\lambda & 3 & -8 \\ -0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{bmatrix} = \lambda^3 - \lambda^2 - 16\lambda - 20$$

Now, in this case the list of possible integer solutions to the characteristic polynomial are,

$$\pm 1, \pm 2, \pm 4, \pm 10,\ \pm 20$$

Again, if we start with the smallest integers in the list we'll find that -2 is the first integer solution. Therefore,$\lambda - (-2) = \lambda + 2$ must be a factor of the characteristic polynomial. Factoring this out of the characteristic polynomial gives,

$$\lambda^3 - \lambda^2 - 16\lambda - 20 = (\lambda + 2)(\lambda^2 - 3\lambda - 10)$$

Finally, factoring the quadratic and setting equal to zero gives us,

$$(\lambda + 2)^2(\lambda - 5) \Rightarrow \lambda_{1,2} = -2, \lambda_3 = 5$$

So, we have one double eigenvalue $\lambda_{1,2} = -2$ and one simple eigenvalue $\lambda_3 = 5$.

3. $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$
Here is $(A - \lambda I)$ and the characteristic polynomial for this matrix.

$$det(A - \lambda I) = \begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = \lambda^3 - 9\lambda^2 - 27\lambda - 27$$

Okay, in this case the list of possible integer solutions is,

$$\pm 1, \pm 3, \pm 9, \pm 27$$

The smallest integer that will work in this case is 3. We'll leave it to you to verify that the factored form of the characteristic polynomial is,

$$\lambda^3 - 9\lambda^2 - 27\lambda - 27 = (\lambda - 3)^3$$

and so we can see that if we set this equal to zero and solve we will have one eigenvalue of multiplicity 3 (sometimes called a triple eigenvalue),

$$\lambda_{1,2,3} = 3$$

4.5 FINDING EIGENVECTORS

We now turn to the problem of finding the eigenvectors associated with the eigenvalues of a matrix.If λ is an eigenvalue of a matrix A, then its associated eigenvectors are the non zero solutions of the homogeneous system of equations.

$$(A - \lambda I)x = 0$$

The solution space of the system, which is the same as the null space of the matrix $A - \lambda I$, is a subspace consisting of the eigenvectors associated with λ together with the zero vector. This subspace is called the Eigen space for the Eigenvalue λ .
To find the Eigenvector for a given eigenvalue,we simply apply the method of solving a homogeneous system and represent the solution vectors as linear combinations of the basis vectors.
The technique is illustrated in the example that follow.Since a matrix will usually have more than one eigenvalue, the procedure must be applied to each of the eigenvalues in turn.
EXAMPLE 5:

1. The Eigenvalues of the matrix

   $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

   are $\lambda_1 = 5$ and $\lambda_2 = -2$.Find the associated eigenvectors for each eigenvalue.

   SOLUTION:

   First we take $\lambda_1 = 5$, which gives us the homogeneous system,

   $$(A - \lambda I)x = \begin{bmatrix} 1-5 & 4 \\ 3 & 2-5 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   The solution space consists of all vectors $x = (x_1, x_2)$ such that

   $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   When $\lambda_1 = -2$ the solution space consists of all vectors $x = (x_1, x_2)$ such that

   $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

2. The Eigenvalues of the matrix

   $A = \begin{bmatrix} 7 & -1 \\ 3 & 2 \end{bmatrix}$
   is $\lambda_{1,2} = 5$ .Find the associated eigenvector.

$$\begin{bmatrix} 2 & -1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   The solution to this system is,

$$x_1 = \tfrac{1}{2}t \text{ and } x_2 = t$$

   The general eigenvector and a basis for the eigenspace corresponding $\lambda_{1,2} = 5$ is then,

$$x = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \text{ and } v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

   In this case we get only a single eigenvector and so a good eigenvalue/eigenvector pair is,

$$\lambda_{1,2} = 5 \; v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

EXAMPLE 6:

1. Find the Eigenvalues and the corresponding Eigenvectors of the matrix.

$$A = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & 2 \\ 3 & -3 & -2 \end{bmatrix}$$

   SOLUTION:

   We calculate the determinant

$$det(A - \lambda I) = det \begin{bmatrix} 3-\lambda & -2 & -2 \\ -2 & 2-\lambda & 2 \\ 3 & -3 & -2-\lambda \end{bmatrix} = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

Then multiplying by -1, we get cubic equation

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

Testing the numbers $\pm 1, \pm 2$ for roots, we find that $\lambda_1 = 1$ and $\lambda_2 = 2$ are roots, and we obtain the factored form

$$(\lambda - 1)^2(\lambda - 2) = 0$$

Thus we have the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$,where $\lambda_1 = 1$ has multiplicity two.

(a) For $\lambda_1 = 1$
We get the system
$(A - 1I)x = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ 3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ The coefficient matrix has the echelon form

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore we may set $x_3 = t$,$x_2 = s$ and get $x_1 = s + t$.
Thus the eigenvector for For $\lambda_1 = 1$ consists of all vectors of the form

$$x = s \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvector is therefore 2-dimensional.

(b) For $\lambda_2 = 2$
We get the system
$(A - 2I)x = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & 2 \\ 3 & -3 & -4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We leave it to the reader to verify that the solutions are all vectors of the form.

$$x = t \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

2. $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

$$det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 3 & 5 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

Thus the characteristic equation, which can be written as

$$(\lambda - 2)^2(\lambda - 3) = 0$$

The eigenvalues for this matrix are $\lambda_{1,2} = 2$ and $\lambda_3 = 3$ so well have three eigenvectors to find.

(a) Starting with $\lambda_1 = 2$ we'll need to find the solution to the following system,

$$(A - 2I)x = \begin{bmatrix} 0 & 3 & 5 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced echelon form of the coefficient matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $x_3 = 0$,x_2, and $x_1 = t$ is arbitrary.Thus the eigenspace for $\lambda_1 = 2$ consist of all vectors

$$x = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

consequently,the eigenspace has dimension 1 even though $\lambda_1 = 2$ has multiplicity two as a root of the characteristic equation.

(b) For $\lambda_3 = 3$ will necessarily have a 1-dimensional eigenspace. The homogeneous system

$$(A - 3I)x = \begin{bmatrix} -1 & 3 & 5 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Has a coefficient matrix whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus if we let $x_3 = t$,we get $x_2 = -t$, and $x_1 = 2t$,so that the solutions are given by

$$x = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus the eigenspce for $\lambda_3 = 3$ is 1-dimensional.

EXERCISES:

Determine the characteristic equation, the eigenvalues, and the eigenvector for each eigenvalue of the given matrix.

1. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 1 & -1 \\ 4 & 4 & -4 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 5 \\ -10 & 12 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 4 & -1 \\ 6 & -2 & 3 \\ 10 & -8 & - \end{bmatrix}$

4. $\begin{bmatrix} -2 & 5 & 1 \\ -2 & 4 & 1 \\ 6 & -9 & 2 \end{bmatrix}$