COURSE STUDY GUIDE

This provides a weekly schedule of your studies

Week #	Chapters	Quiz/exams
1-2	Chapter 1	Assignments / class work
3-5	Chapter 1 & 2	Assignments/ class work
6-7	Chapter 2	Assignments/ class work
8-9	Chapter 3	Assignments/ class work
10	Midsem	Midsem
11	REST /FUN	-
12-13	Chapter 3 & 4	Assignments/ class work
14-15	Chapter 4	Assignments/ class work
15-16	Exams	Exam

GRADING

Continuous assessment: 30%
End of semester examination: 70%

References

i) Massey, B.S. *Mechanics of Fluids.* London : CHAPMAN & HALL, sixth edition. Swanson, W. M. *Fluid Mechanics.* United States of America: Holt, Rinehart and Winston, Inc., 1970.

ii) Janna, William S. *Introduction to FLUID MECHANICS.* Belmont, California: Brooks/Cole Engineering Division, A Division of Wadsworth Inc, 1983.

iii) Bansal, Dr. R. K. *A TEXTBOOK OF FLUID MECHANICS AND HYDRAULIC MACHINES(In S.I. UNITS).* NEW DELHI: LAXMI PUBLICATIONS (P) LTD, 2002.

iv) Bruce R. Munson, Donald F. Young, Theodore H. Okiishi. *Fundamentals of Fluid Mechanics.* United States: John Wiley & Sons, Inc, 1990.

Chapter **page**

Chapter 1

1 THE MOMENTUM EQUATION AND ITS APPLICATIONS

1.1 The Momentum Equation

The momentum equation is based on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in the direction of the force. The force acting on a fluid of mass 'm' is given by the Newton's second law of motion:

F = m x a,

where a is the acceleration acting in the same direction as the force F.

Now, $a = \frac{d\,v}{dt}$

$F = m\frac{d\,v}{dt} \quad = \frac{d(m\,v)}{dt}$

(where m is constant and taken inside the differential).

$$F = \frac{d(m\,v)}{dt} \qquad (1.1)$$

Equation (1.1) is called the momentum principle.

Equation (1.1) can be written as F.dt = d(mv), which is known as the *Impulse-momentum equation* and it states that the impulse of a force, F, acting on a fluid mass, m, in a short time interval dt is equal to the change of momentum d(mv) in the direction of the force. The momentum equation applies in the determination of the forces created on a solid body by fluid flowing over it or jets of fluid impinging on it, forces on pipe-bend caused by flowing fluid within it, thrust on a propeller, lift and drag forces on an aircraft wing, etc.

1.1.1 Force Exerted by a Flowing Fluid on a Pipe Bend

The impulse-momentum equation is used to determine the resultant force exerted by a flowing fluid on a pipe – bend.

Consider two sections (1) and (2) as shown in Fig.1.1.

Let V_1=velocity of flow at section (1)

P_1=pressure intensity at section (1)

A_1=area of cross section of pipe at section (1)

V_2, P_2, A_2 = corresponding values of velocity, pressure and area at section (2)

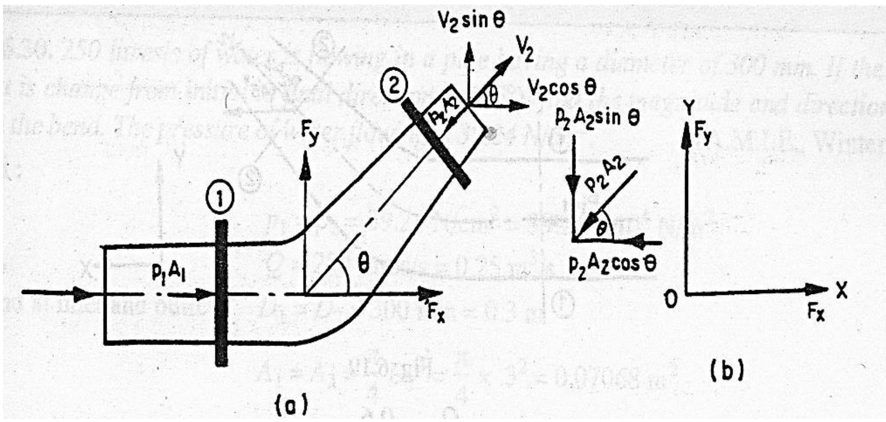


Fig. 1.1

Let F_x and F_y be the components of the forces exerted by the flowing fluid on the bend in the *x* and *y* –directions respectively. Hence the component of the force exerted by the bend on the fluid in the direction of *x* and *y* will be equal to F_x and F_y but in the opposite directions.

i.e in x – direction = - F_x

y – direction = - Fy

The other external forces acting on the fluid are p_1A_1 and p_2A_2 on the sections (1) and (2) respectively. The Momentum equation in the *x* – direction is given by

Net force acting on the fluid in the direction of x = Rate of change of momentum in the x-direction.

$$\therefore\ p_1A_1 - p_2A_2\cos\theta - F_x = (Mass\ per\ second)(change\ of\ velocity)$$

$$= \rho Q(final\ velocity\ in\ the\ direction\ of\ x - initial\ velocity\ in\ the\ direction\ of\ x)$$

$$= \rho Q(V_2\cos\theta - V_1)$$

$$\therefore F_x = \rho Q(V_1 - V_2\cos\theta) + p_1A_1 - p_2A_2\cos\theta$$

Similarly the momentum equation in *y*-direction gives

$$0 - p_2A_2\sin\theta - F_y = \rho Q(V_2\sin\theta - 0)$$

$$\therefore F_y = \rho Q(-V_2\sin\theta) - p_2A_2\sin\theta$$

Now the resultant force F_R acting on the bend is given as

$$F_R = \sqrt{{F_x}^2 + {F_y}^2}$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan\theta = \frac{F_y}{F_x}$$

Problem 1.1 A nozzle of diameter 20 mm discharging to the atmosphere, is fitted to a pipe of diameter 40 mm. Find the force exerted by the nozzle on the water which is flowing through the pipe at a rate of 1.2 m^3/minute

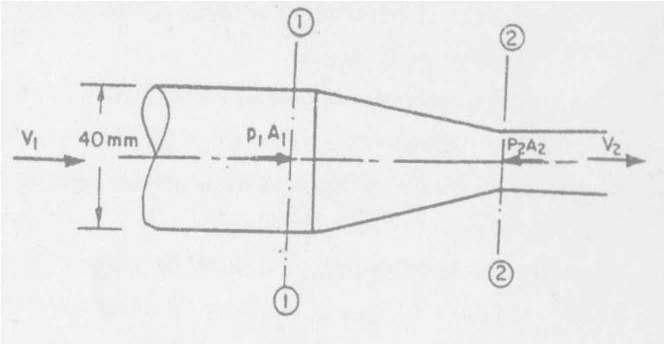


Fig. 1.2

Solution

Given D_1 = 40 mm = 0.04 m

$\therefore Area, \qquad A_1 = \frac{\pi {D_1}^2}{4} = \frac{\pi (0.04)^2}{4} = 0.001256\, m^2$

Diameter of nozzle, $D_2 = 20\ mm = 0.02\ m$

$\therefore Area, \qquad A_2 = \frac{\pi {D_2}^2}{4} = \frac{\pi (0.02)^2}{4} = 0.000314\, m^2$

Discharge, $Q = 1.2\ m^3/minute = \frac{1.2}{60} = 0.02\, m^3/s$

Applying continuity equation at (1) and (2), $A_1V_1 = A_2V_2 = Q$

$\therefore V_1 = \frac{Q}{A_1} = \frac{0.02}{0.001256} = 15.92\ m/s$

$V_2 = \frac{Q}{A_2} = \frac{0.02}{0.000314} = 63.69\ m/s$

Applying Bernoulli's equation at sections (1) and (2), we get

$\frac{p_1}{\rho g} + \frac{{V_1}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{{V_2}^2}{2g} + z_2$

Now $z_1 = z_2, \frac{p_2}{\rho g}$ = atmospheric pressure = 0

$\therefore \frac{p_1}{\rho g} + \frac{{V_1}^2}{2g} += \frac{{V_2}^2}{2g}$

$\therefore \frac{p_1}{\rho g} = \frac{{V_2}^2}{2g} - \frac{{V_1}^2}{2g} = \frac{(63.69)^2}{2\times 9..81} - \frac{(15.92)^2}{2\times 9..81} = 206.749 - 12.918 = 193.83\, m\ of\ water$

$\therefore p_1 = 193.83 \times 1000 \times 9.81 \frac{N}{m^2} = 1901472\ \frac{N}{m^2}$

Let the force exerted by the nozzle on water = F_x

Net force in the direction of *x* = Rate of change of momentum in the *x*-direction

$p_1A_1 - p_2A_2 + F_x = \rho Q(V_2 - V_1)$

Where $atmospheric\ pressure = 0\ and\ \rho = 1000$

$\therefore 1901472 \times 0.001256 - 0 + F_x = 1000 \times 0.02(63.69 - 15.92)" \Rightarrow 2388.24 + F_x = 955.4$

$\therefore F_x = -2388.24 + 916.15 = -1432.84N\ Ans.$

-ve sign indicates that the force exerted by the nozzle on the water is acting from right to left.

Problem 1.2 A pipe of 300 mm diameter conveying 0.3 m^3/s of water has a right angled bend in a horizontal plane. Find the resultant force exerted on the bend if the pressure at inlet and outlet of the bend are 24.525 N/cm^2 and 23.544 N/cm^2 respectively.

Solution

Given;

Diameter of bend, D = 300 mm = 0.3 m

Area = A_1 = A_2 = $\frac{\pi}{4}(0.3)^2 = 0.07068m^2$

Discharge, Q = 0.3 m^3/s

Velocity V=$V_1 = V_2$=$\frac{Q}{A} = \frac{0.30}{0.07068} = 4.244\ m/s$

The momentum equation gives

$\sum F_x = \rho Q \Delta V_x$

Thus

$(P_1A_1)_x - (P_2A_1)_x - F_x = \rho Q(V_{2x} - V_{1x})$

$F_x = (P_1A_1)_x + \rho Q(V_1 - V_2)_x$

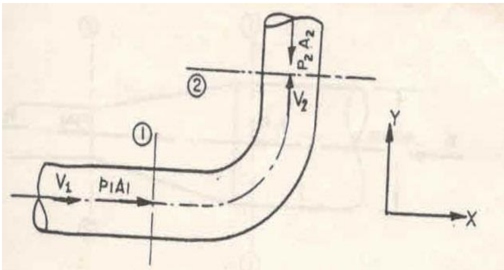


Fig. 1.3

Angle of bend, θ = 90° p_1 = 24.525 N/cm^2 = $24.524 \times 10^4\ N/m^2 = 245250\ N/m^2$

P_2 = 23.544 N/cm^2= $23.544 \times 10^4\ N/m^2 = 235440\ N/m^2$

Force on bend along x-axis $F_x = \dot{m}[V_{1x} - V_{2x}] + (p_1A_1)_x + (p_2A_2)_x$

Where ρ = 1000, V_{1x} = V_1 =4.244 m/s, V_{2x} = 0

$\dot{m}$= ρQ

$(p_1A_1)_x = p_1A_1 = 245250 \times 0.07068$

$p_2A_2 = 0$

$\therefore F_x = 1000 \times 0.30[4.244 - 0] + 245250 \times 0.07068 + 0$

$= 1273.2 + 17334.3 = 18607.5\ N$

Forces in y-direction.

$(P_1A_1)_y - (P_2A_2)_y - F_y = \rho Q(V_{2y}-V_{1y})$

$(-P_2A_2)_y - \rho QV_{2y} = F_y$

Force on bend along y-axis $F_y = \dot{m}[V_{1y} - V_{2y}] + (p_1A_1)_y + (p_2A_2)_y$

Where ρ = 1000, V_{1y} =0 =, V_{2y} = V_2 = 4.244 m/s

$(p_1A_1)_y = 0,\ (p_2A_2)_y = -p_2A_2 = -235440 \times 0.7068 = -16640.9$

$\therefore F_y = 1000 \times 0.30[0 - 4.244] + 0 - 16640.9$

$= -1273.2 - 16640.9 = -17914.1\ N$

$\therefore Resultant\ force, F_R = \sqrt{{F_x}^2 + {F_y}^2}$

$\sqrt{(18607.5)^2 + (17914.1)^2} = 25829.3\ N$

$tan\theta = \frac{17914.1}{18607.5} = 0.9627$

$\theta = 43.9°. \quad Ans$

Problem 1.3 The diameter of a pipe gradually reduces from 1 m to 0.7 m as shown. The pressure intensity at the centre-line of the 1 m section is 7.848 kN/m^2 and the rate of flow of the water through the pipe is 600 litres/s. Find the intensity of pressure at the centre-line of 0.7 m section. Also determine the force exerted by the flowing water on transition of the pipe.

Solution

Given

$area = A_1 \frac{\pi}{4}(1)^2 = 0.7854m^2$

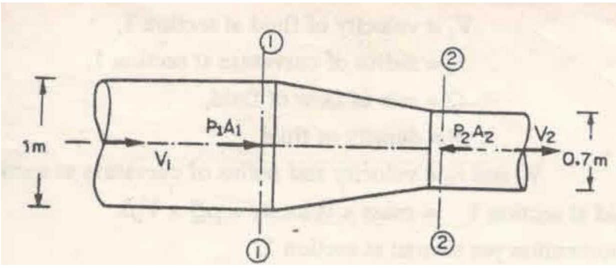


Fig. 1.4

Diameter of the pipe at section 2, D_2 = 0.7 m

Area = A_1 =A_2 = $\frac{\pi}{4}(0.7)^2 = 0.3848m^2$

Pressure at section 1, p_1 = 7.848 kN/m^2 = 7848 N/m^2

Discharge, $Q = 600\ litres/s = \frac{600}{1000} = 0.6\ m^3/s$

Applying continuity equation $A_1V_1 = A_2V_2 = Q$

$V_1 = \frac{Q}{A_1} = \frac{0.6}{0.7848} = 0.764\ m/s$

$V_2 = \frac{Q}{A_2} = \frac{0.6}{00.3854} = 1.55\ m/s$

Applying Bernoulli's equation at sections (1) and (2),

$\frac{p_1}{\rho g} + \frac{{V_1}^2}{2g} = \frac{p_2}{\rho g} + \frac{{V_2}^2}{2g}\ \{\therefore pipe\ is\ horizontal\ \therefore\ z_1 = z_2\}$

$\frac{7848}{1000\times 9.81} + \frac{(0.764)^2}{2\times 9.81} = \frac{p_2}{\rho g} + \frac{(1.55)^2}{2\times 9.81}$

$\frac{p_2}{\rho g} = 0.8 + \frac{(0.764)^2}{2\times 9.81} - \frac{(1.55)^2}{2\times 9.81}$

$0.8 + 0.0297 - 0122 = 0.7077\ m\ of\ water$

$p_2 = 0.7077\ \times 9.81 \times 1000 = \frac{6.942kN}{m^2}\ Ans$

Let F_x = the force exerted by pipe transition on the flowing water in the direction of flow.

Then net force in the direction of flow = rate of change of momentum in the direction of flow

$p_1A_1 - p_2A_2 + F_x = Q\rho(V_2 - V_1)$

$\therefore 7848 \times 0.7854 - 694254 \times 0.3848 + F_x = 1000 \times 0.6[1.55 - 0.764]$

$6163.8 - 2671.5 + F_x = 471.56$

$F_x = 471.56 - 6163.8 + 2671.5 = -3020.74\ N$

$\therefore The\ force\ exerted\ by\ the\ water\ on\ pipe\ transition$

$= -F_x = -(-3020.74) = 3020.74\ N\ Ans.$

Problem 1.4 The horizontal nozzle in Fig. 1.5 has D_1 = 0.254 m and D_2 = 0.1524 m. The inlet pressure p_1 = 413.6854 kPa and the exit velocity V_2 = 25.908 m/s. Compute the tensile force in the flange bolts. Assume incompressible steady flow.

Solution:

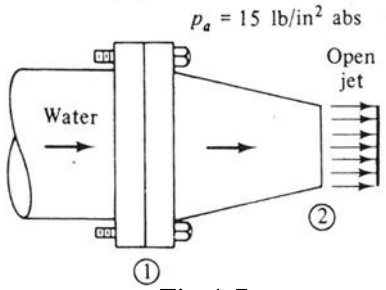


Fig 1.5

$(P_1A_1)_x - (P_2A_2)_x - F_x = \dot{m}\,(V_{2x}-V_{1x})$

Force of water on nozzle

$$\sum F = \rho A v_1(v_2 - v_1)$$

$$A_1 = \frac{\pi}{4} \times (0.254)^2 = 0.05067\ m^2$$

$$V_1 = \left(\frac{\frac{\pi}{4}D_2^2}{\frac{\pi}{4}D_1^2}\right) \times V_2 = \left(\frac{D_2}{D_1}\right)^2 \times V_2 = \left(\frac{0.1524}{0.254}\right)^2 \times 25.908 = 9.327\ m/s$$

$$(p_1 - p_2)A_1 - F_{bolt} = \rho A_1 v_1(v_2 - v_1)$$

$\Rightarrow$

$$0.05067(413.6854 - 103.4214) - F_{bolt} = 1000 \times 0.05067 \times 9.327(25.908 - 9.327)$$

$$15.7211 \times 10^3 - F_{bolt} = 7836.1655$$

$F_{bolt} = 7884.9345\ N = 7.885\ kN$

Problem 1.5 A 45 ° reducing bend, with 0.6096 m diameter upstream and 0.3048 m diameter downstream, has water flowing through it at the rate of 0.4445 m^3/s under a pressure of 144.7899 kPa. Neglecting any loss in the bend, calculate the force exerted by the water on the reducing bend

Solution

$$A_1 = \frac{\pi}{4} \times d_1^2 = \frac{\pi}{4} \times (0.6096)^2 = 0.2919\ m^2$$

$$A_2 = \frac{\pi}{4} \times d_2^2 = \frac{\pi}{4} \times (0.3048)^2 = 0.07297\ m^2$$

$$V_1 = \frac{Q}{A_1} = \frac{0.4445}{0.2919} = 1.524\ m/s\ and\ V_2 = \frac{0.4445}{0.07297} = 6.096\ m/s$$

Applying Bernoulli's equation to section 1 and section 2, we have

$$\left(\frac{p_1}{\rho g}\right) + \frac{V_1^2}{2g} + z_1 - neglegible\ lost\ head = \left(\frac{p_2}{\rho g}\right) + \frac{V_2^2}{2g} + z_2, but\ z_1 = z_2 = 0$$

$$\left(\frac{144.7899}{9810}\right) + \frac{1.524^2}{2 \times 9.81} + 0 - neglegible\ lost\ head = \left(\frac{p_2}{\rho g}\right) + \frac{6.096^2}{2 \times 9.81} + 0$$

$$\frac{p_2}{\rho g} = 12.9837\ m$$

$$p_2' = 127.37058\ kPa$$

Let the force $F_1 = p_1 A_1 = 144.7899 \times 0.2919 = 42.264\ kN$

And $F_2 = p_2 A_2 = 127.3706 \times 0.07297 = 9.294\ kN$

Also $F_{2_x} = F_{2_y} = 9.294 \times 0.707 = 6.57\ kN$

Because $F_x = F_2 \cos 45$ and $F_y = F_2 \sin 45$

In the X direction, $MV_{x_1} + \sum(forces\ in\ X\ direction\ )(1) = MV_{x_2}$

$$F_{1x} - F_{2_x} - F_x = \rho A_1 V_1 [V_2 \cos 45 - V_1]$$

Thus $= 10^3(42.264 - 6.57) - F_x = 1000 \times 0.2919 \times 1.524[(6.096 \cos 45) - 1.524]\ and$

$$F_x = 34.45\ kN\ to\ the\ left$$

In the Y direction,

$$F_y - F_{2y} = \rho Q(V_{2y} - V_{1y})$$

$+F_y - 9.294 sin 45 = \rho A_2 V_2 [V_2 \sin 45 - V_1]$ *[As $F_{2y} = F_2 sin45$]*

$$+F_y - (6.57 \times 10^3) = 1000 \times 6.096 \times 0.07297[(6.096 \sin 45)] - 0$$

$$F_y = (6.57 \times 10^3) + 1000 \times 6.096 \times 0.07297[(6.096 \sin 45)]$$

$$= 8.487\ kN$$

The force exerted by the water on the reducing bend is

$$F = \sqrt{(34.45)^2 + (8.487)^2} = 35.48\ kN\ to\ the\ right\ downward\ at\ an\ angle$$

$$\theta_x = \tan^{-1}\frac{8.487}{34.45} = 13.84°$$

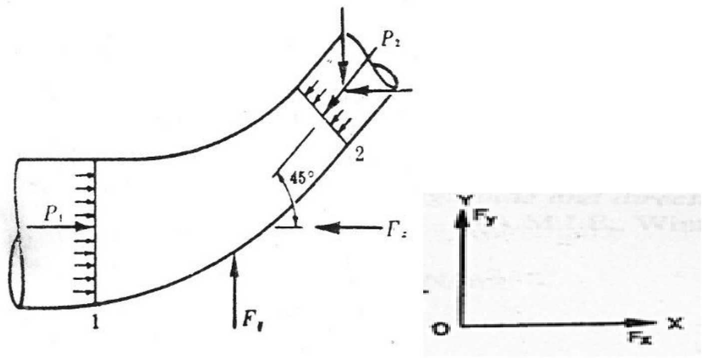


Fig. 1.6

Problem 1.6 A jet craft travelling at a velocity of 179.832 m/s, takes in air at a rate of 22.6205 kg/s. The air: fuel ratio is 25:1, and the exhaust velocity relative to the jet is 594.36 m/s. Find the thrust produced by the jet engine.

Solution

$$F = \rho_a A v_1 (v_2 - v_1) + \rho_f Q_f v_2$$

$$where\ \dot{m} = \dot{\rho}_a A v_1$$

$$\rho_a Q_a = \dot{m}_a = 22.6205\ \text{kg/s}$$

Since the air: fuel ratio is 25:1,

$$m_f = \rho_f Q_f = \frac{22.6205}{25} = 0.90482\ kg/s$$

$$22.6205(594.36 - 179.832) + (0.90482 \times 594.36) = 9.915\ kN$$

Problem 1.7 A rocket engine burns 8.5 kg of fuel and oxidizer per second, and combustion gases exit the rocket at a velocity of 565 m/s, relative to the rocket, at a pressure approximately equal to the surrounding atmospheric pressure. Find the thrust produced by the rocket engine.

Solution:

Thrust, $F = \rho A V_2 = m \times V_2 = 8.5 \times 565 = 4802\ N$

Problem 1.8 A jet craft travelling at a velocity of 213.36 m/s, takes in air at a rate of 24.5178 kg/s. The air: fuel ratio is 20:1, and the exhaust velocity relative to the jet is 641.604 m/s. Find the thrust produced by the jet engine.

Solution

Thrust, $F = \rho_a A v_1 (v_2 - v_1) + \rho_f Q_f v_2$

$$where\ \dot{m} = \rho_a A v_1$$

$$\rho_a Q_a = \dot{m}_a = 22.6205\ \text{kg/s}$$

Since the air: fuel ratio is 20:1,

$$\rho_f Q_f = \frac{24.5178}{20} = 1.2259 kg/s$$

$$24.5178(641.604 - 213.36) + (1.2259 \times 641.604) = 11.29\ kN$$

Problem 1.9 A jet is being tested in the laboratory. The engine consumes 22.6796 kg/s of air and 0.226796 kg/s of fuel. If the exit velocity of the gases is 457.2 m/s, what is the thrust? (NB. Thrust = Force)

Solution

Thrust, $F = \rho_a A v_1 (v_2 - v_1)$ where $\dot{m} = \rho_a A v_1$

$\Longrightarrow$ $F = (22.6796 + 0.226796)(457.2 - 0) = 10.47\ kN$

1.2 Moment of momentum Equation

The moment of momentum equation is derived from moment of momentum principle which states that the resulting torque acting on a rotating fluid is equal to the rate of change of moment of momentum.

Let V_1 = velocity of the fluid at section (1)

r_1 =radius of curvature at section (1)

Q = rate of flow of the fluid

ρ = density of the fluid

V_2 and r_2 = velocity and radius of curvature respectively at section (2).

Momentum of fluid at section (1) = mass x velocity per second $= \rho Q \times V_1\ per\ second$

Moment of momentum per second at section (1) $= \rho Q \times V_1 \times r_1$

Similarly, moment of momentum per second of fluid at section (2) $= \rho Q \times V_2 \times r_2$

Rate of change of moment of momentum $= \rho Q V_2 r_2 - \rho Q V_1 r_1 = \rho Q [V_2 r_2 - V_1 r_1]$

According to the moment of momentum principle,

Resultant torque = rate of change of moment of momentum

$T = \rho Q[V_2 r_2 - V_1 r_1]$ (1.)

This equation is known as moment of momentum equation. It is applied

(1) For analysis of flow problems in turbines and pumps

(2) For finding torque exerted by water on sprinkler.

Problem 1.10 A lawn sprinkler with two nozzles of diameter 4 mm each is connected across a tap of water as shown below. The nozzles are at a distance of 30 cm and 20 cm from the center of the tap. The rate of flow of water through the tap is 120 cm^3/s. The nozzles discharge water in the downward direction. Determine the angular velocity at which the sprinkler rotates freely.

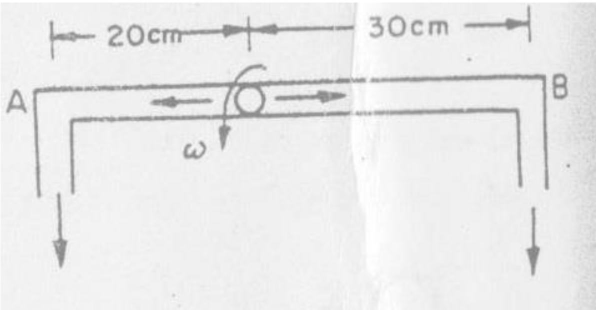


Fig. 1.7

Solution

Diameter of each nozzle D = D_A = D_B = 4 mm = 0.004 m

Area of nozzle $A = \frac{\pi(0.004)^2}{4} = 0.000001256\ m^2$

Discharge $Q = 120\ cm^3/s$

Assuming the discharge to be equally divided between the two nozzles, we have

$$Q_A = Q_B = \frac{Q}{2} = \frac{120}{2} = 60\ cm^3/s = 60 \times 10^{-6}\ m^3/s$$

∴ Velocity of water at the outlet of each nozzle, ∴

$$V_A = V_B = \frac{Q_A}{A} = \frac{60 \times 10^{-6}}{0.00001256} = 4.777\ m/s$$

The jet of water coming out from nozzles A and B is having velocity 4.777 m/s. These jets of water will exert a reaction force in the opposite direction, that is, force exerted by the jets will be in the upward direction. The torque exerted will also be in the opposite direction. Hence torque at B will be in the anticlockwise direction while torque at A will be in the clockwise direction. But torque at B is more than torque at A and hence the sprinkler, if free, will rotate in the anti-clockwise direction as shown.

Let ω = angular velocity of sprinkler

Then absolute velocity of water at A

$V_1 = V_A + \omega \times r_A$

Where r_A= distance of nozzle A from the centre of tap = 0.2 m

ωr_A= tangential velocity due to rotation

$\Rightarrow V_1 = (4.777 + \omega \times 0.2)\, m/s$

Similarly absolute velocity of water at B

$V_2 = V_B - tangential\ velocity\ due\ to\ rotation$

$V_2 = 4.777 - \omega r_B = 4.777 - \omega \times 0.3\ \{where\ r_B = 30\ cm = 0.3\ m\}$

Applying the moment of momentum equation, we get

$$\sum T = \rho Q(V_2 r_2 - V_1 r_1)$$

$$=\rho Q_A(V_2 r_B - V_1 r_A) \qquad |Here\ r_2 = r_B, r_1 = r_A|$$

$=1000 \times 60 \times 10^{-6}[(4.777 - \omega \times 0.3) \times 0.3 - (4.777 + \omega \times 0.2) \times 0.2]$

The moment of momentum of the fluid entering the sprinkler is given zero and also there is no external torque applied on the sprinkler. Hence resultant torque is zero, i.e., T = 0

$\therefore 1000 \times 60 \times 10^{-6}[(4.777 - \omega \times 0.3)0.3 - (4.777 + \omega \times 0.2) \times 0.2] = 0$

$[(4.777 - \omega \times 0.3) \times 0.3 - (4.777 + \omega \times 0.2) \times 0.2] = 0$

$$4.777 \times 0.3 - 0.9\omega - 4.777 \times 0.2 - 0.04\omega = 0$$

$$0.1 \times 4.777 = (0.09 + 0.04)\omega = 0.13\omega$$

$$\omega = \frac{0.477}{0.13} = 3.6746\, rad/s\ \ Ans.$$

1.3 Applications of linear momentum equation

1.3.1 Force Exerted by a Jet on a Stationary Vertical Plate

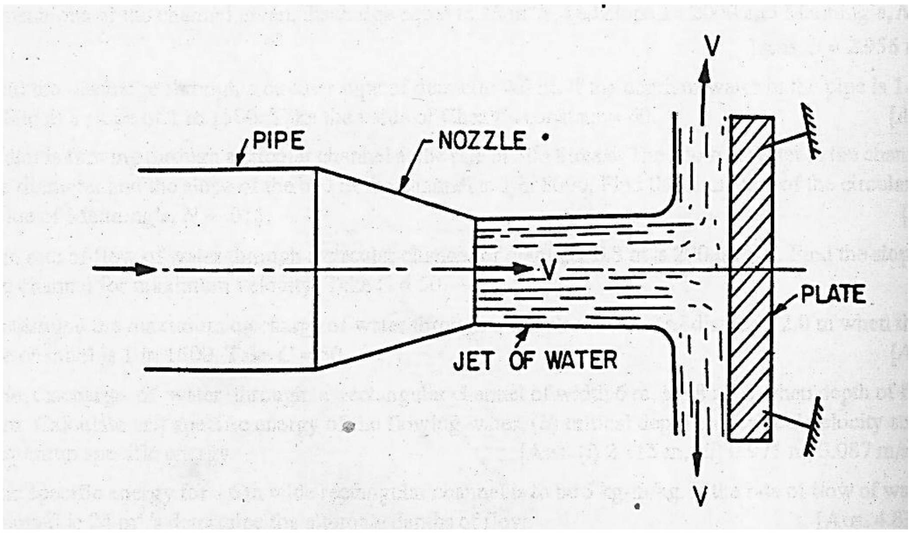


Fig.1.8 Force exerted by a jet on vertical plate

Consider a jet of water coming out from the nozzle, strikes a flat vertical plate as shown in Fig1.8

Let V=velocity of jet, *d* =diameter of the jet, *a* =area of jet cross-section = $\frac{\pi d^2}{4}$

The jet after striking the plate will move along the plate. The plate is at right angles to the jet. Therefore the jet after striking the plate will be deflected through 90^o.

Hence the component of velocity, in the direction of the jet, after striking will be zero.

The force exerted by the jet on the plate in the direction of jet

(i) Let F_x = Rate of change of momentum in the direction of the force

$$= \dot{m}(v_{1x} - v_{2x})$$

Where, V_{1x}=*initial velocity in the direction of jet*

V_{2x}=*final velocity*

$$= \frac{Mass}{Time}[Initial\ velocity - Final\ velocity]$$

$$(Mass/sec) \times (velocity\ of\ jet\ before\ striking - final\ velocity\ of\ jet\ after)$$

$$= \rho a V[V - 0]$$

$$= \rho a V^2$$

In deriving the above equation we have taken initial velocity minus final velocity and not final velocity minus initial velocity. If the force exerted on the jet is to be calculated, then final velocity minus initial velocity is taken. But if the force exerted by the jet on the plate is to be calculated, then initial velocity minus final velocity is taken.

1.3.2 **Force Exerted by Jet on a Stationary Inclined Plate**

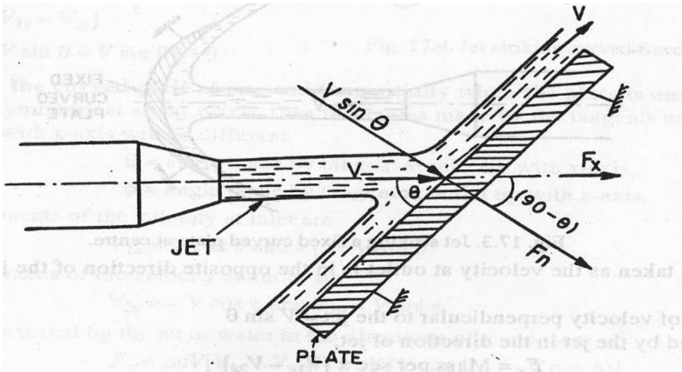


Fig. 1.9. Jet striking stationary inclined plate

The jet of water coming out from the nozzle strikes an inclined flat plate as shown in Fig. 1.9

Let V= Velocity of jet in the direction of *x*,

θ = Angle between the jet and plate,

a=Area of cross-section of the jet

Then mass of water per second striking the plate= ρaV

If the plate is smooth and if it is assumed that there is no loss of energy due to impact of the jet, then the jet will move over the plate after striking with a velocity equal to initial velocity *i.e.*, with a velocity V. Let us find the force exerted by the jet on the plate in the direction normal to the plate. Let this force be presented by F_n

Then F_n = mass of jet striking per second × $(v_{1x} - v_{2x})$

$$= \rho aV[V\sin\theta - 0] = \rho aV^2 \sin\theta$$

This force can be resolved into two components, one in the direction of the jet and the other perpendicular to the direction of flow. Then we have

F_x = component of F_n in the direction of flow

$= F_n \cos(90^\circ - \theta) = F_n sin\theta = \rho aV^2 sin\theta \times sin\theta$

$F_x = \rho aV^2 sin^2\theta$ And

F_y= component of F_n, perpendicular to flow

$F_y = F_n \sin(90^\circ - \theta) = F_n cos\theta = F_y = \rho aV^2 sin\theta cos\theta$

1.3.3 **Jet Striking a Curved Plate at the Centre**

Let a jet of water strike a fixed curved plate in the centre as shown. Fig. 1.10. The jet after striking the plate, comes out with the same velocity if the plate is smooth and there is no loss of energy due to impact of the jet, in the tangential direction of the curved plate. The velocity at outlet of the plate can be resolved into two components, one in the direction of the jet and the other perpendicular to the direction of the jet.

Component of velocity in the direction of jet $= -V\cos\theta$

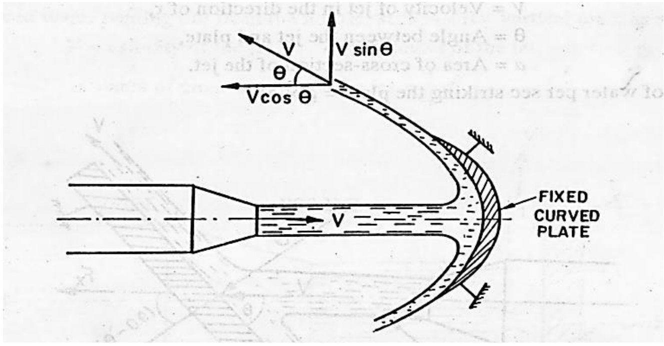


Fig. 1.10 Jet striking a fixed curved plate at centre

(-ve sign is taken as the velocity at the outlet and is in the opposite direction of the jet of water coming out from nozzle).

Component of velocity perpendicular to jet $= V\sin\theta$

Force exerted by the jet in the direction of jet,

$F_x = Mass\ per\ second \times [V_{1x} - V_{2x}]$

Where V_{1x} = Initial velocity in the direction of jet = V

V_{2x} = Final velocity in the direction of jet $= -V\cos\theta$

$$F_x = \rho a V[V - (-V cos\theta)] = \rho a V[V + V cos\theta]$$
$$= \rho a V^2[1 + \cos\theta]$$

Similarly, $F_y = Mass\ per\ second \times [V_{1y} - V_{2y}]$

Where V_{1y} = Initial velocity in the direction of y = 0

V_{2y} = Final velocity in the direction of $y = -V\sin\theta$

$F_y = \rho a V[0 - V sin\theta] = -\rho a V^2 sin\theta$

-ve sign means the force is acting in the downward direction. In this case the angle of the jet

$$= 180 - \theta$$

Problem 1.11 A hose and nozzle discharge a horizontal water jet against a nearby vertical plate as shown in Fig. 1.11. The flow rate of water is 0.025 m^3/s, and the diameter of the nozzle tip is 30 mm. Find the horizontal force necessary to hold the plate in place.

Solution

$F = \rho A v_1(v_1 - v_2) = \dot{m}(v_1 - v_2)$

The net external force acting on the fluid (F in the equation) is the horizontal force necessary to hold the plate in place (R). Assume it acts toward the left, as shown in the figure, and this direction is taken to be positive.

$$v_1 = \frac{Q}{A_1}$$
$$A_1 = \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.030)^2 = 7.0685 \times 10^{-4}$$
$$v_1 = \frac{0.025}{7.0685\times10^{-4}} = 35.37\ m/s,\ v_2 = 0$$
$$R = 1000 \times 7.0685 \times 10^{-4} \times 35.37[0 - (-35.37)] = 884\ N\ left$$

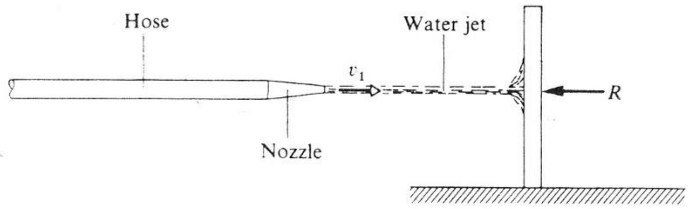


Fig. 1.11

Problem 1.12 Water flows from a large tank through an orifice of 0.0762 m diameter against a block, as shown in Fig. 1.12. The water jet strikes the block at the vena contracta. The block weighs 222.411 N, and the coefficient of friction between the block and floor is 0.57. The orifice's coefficient of discharge, C, is 0.60 and its coefficient of contraction, C_c, is 0.62. What is the minimum height to which water must rise in the tank (*y* in Fig. 1.12) in order to start the block moving to the right.

Solution

$$F = \rho A v_1 (v_1 - v_2) = \dot{m}(v_1 - v_2)$$

The force caused by the water striking the block must equal (or slightly exceed) the friction force between the block and the floor (F_f). In other words the net external force acting on the fluid, (F) in the above equals (F_f) when the block begins to move. Hence

$$F = F_f = 0.57 \times 222.411 = 126.7743\ N$$

$$a = \frac{\pi(0.0762)^2}{4} = 4.5604 \times 10^{-3}\ m^2$$

$$F = \rho A v_1 (v_1 - 0) = \rho A v_1^2$$

$$126.7743 = 1000 \times 4.5604 \times 10^{-3} v_1^2$$

$$v_1 = 5.2725\ m/s$$

But actual velocity, $v = C\sqrt{2gh},$ $\qquad where\ h = y - 0.3048$

$$5.2725 = 0.60\sqrt{2 \times 9.81 \times (y - 0.3048)}$$

$$y = 4.2357\ m$$

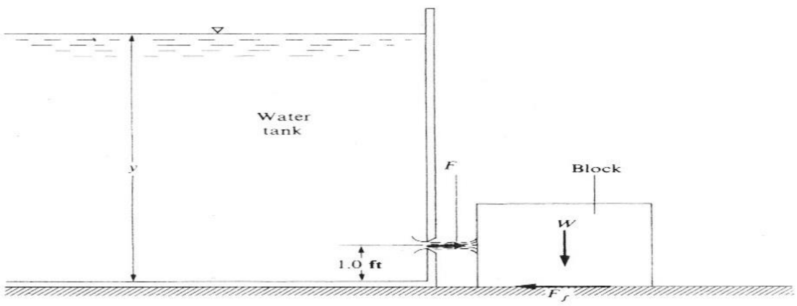


Fig. 1.12

Problem 1.13 A hose and nozzle discharge a horizontal water jet against a nearby vertical plate as shown in Fig. 1.13. The flow rate of water is 0.043 m^3/s, and the diameter of the nozzle tip is 50 mm. Find the horizontal force necessary to hold the plate in place.

Solution

$$F = \rho A_1 v_1 (v_1 - v_2)$$

$$v_1 = \frac{Q}{A_1}$$

$$A_1 = \frac{\pi}{4} d^2 = \frac{\pi}{4}(0.030)^2 = 1.9635 \times 10^{-3}\ m^2$$

$$v_1 = \frac{0.043}{7.0685\times10^{-4}} = 21.899\ m/s\ but\ v_2 = 0$$

$\Longrightarrow$

$$F = \rho A v_1(v_1 - 0) = \rho A v_1^2$$

$$R = 1000 \times 1.9635 \times 10^{-3} \times (21.899)^2 = 941.628\ N\ left$$

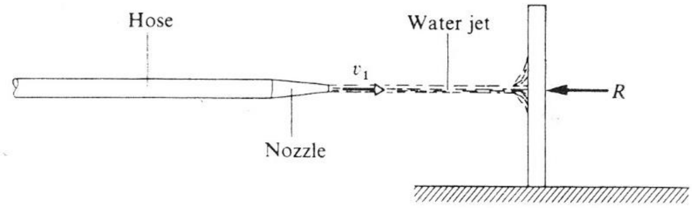


Fig. 1.13

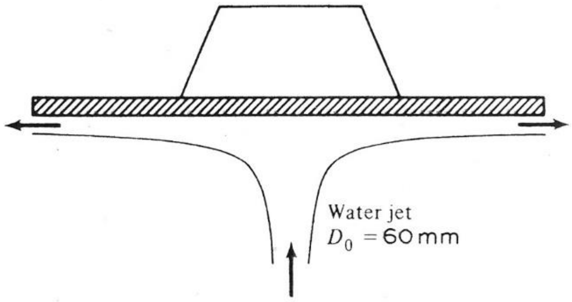


Fig 1.14.

Problem 1.14 In Fig. 1.14 a small ingot and platform rest on a steady water jet. If the total weight supported is 825 N, what is the jet velocity?

Solution

$$F = \rho A v_1(v_1 - v_2)$$

$$A_1 = \frac{\pi(0.060)^2}{4} = 2.827 \times 10^{-3}\ m^2$$

$$v_2 = 0$$

$$825 = \rho A v_1(v_1)$$

$$825 = 1000 \times (0.002827 v_1)(v_1),$$

$$v_1 = 17.08\ m/s$$

Problem 1.15 The water jet in Fig. 1.15, moving at 13.716 m/s, is split so that one-third of the water moves toward A. Calculate the magnitude and direction of the force on the stationary splitter. Assume ideal flow in a horizontal plane.

Solution

$$F = \rho A v_1(v_2 - v_1)$$

$$-F_x = \rho Q_A[(v_x)_2 - (v_x)_1] + \rho Q_B[(v_x)_2 - (v_x)_1]$$

$$F_y = \rho Q_A\left[(v_y)_2 - (v_y)_1\right] + \rho Q_B\left[(v_y)_2 - (v_y)_1\right]$$

$$Q = AV = 0.01395 \times 13.719 = 0.19138\ m^3/s$$

$$Q_A = 0.19138/3 = 0.06379\ m^3/s$$

$$Q_B = 0.19138 - 0.06379 = 0.12759\ m^3/s$$

$$-F_x = 1000 \times 0.06379(-13.716\cos 60 - 13.716)$$
$$+ 1000 \times 0.12759(13.716\cos 60 - 13.716)$$
$$F_x = 2187.42768\ N$$
$$F_y = 1000 \times 0.06379(-13.716\cos 30^0 - 0)$$
$$+ 1000 \times 0.12759(13.716\cos 30^0 - 0) = 757.8422\ N$$
$$F_r = \sqrt{(2187.42768)^2 + (757.8422)^2} = 2314.99\ N$$
$$\tan\alpha = \frac{757.8422}{2187.42768} = 0.3465 \qquad \alpha = 19.11^0$$

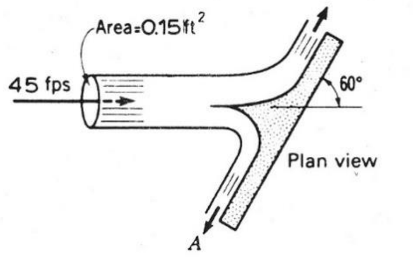


Fig. 1.15

Problem 1.15b Find the force exerted by a jet of water of diameter 100mm on a stationary flat plate when the jet strikes the plate normally with a velocity of 30m/s.

$$Fx = \rho AV[V - 0]$$
$$A = \frac{\pi d^2}{4} = \frac{\pi \times 0.1^2}{4}$$
$$V = \frac{30m}{s}$$
$$F_x = 1000 \times \frac{\pi \times 0.1^2}{4} \times 30^2 = 7068.6N$$

Problem 1.16 The force exerted by a 0.0254 m.diameter stream of water against a flat plate held normal to the stream's axis is 644.9921 N. What is the flow rate of the water?

Solution

$$F = \rho A v_1(v_1 - v_2)$$
$$A = \frac{\pi(0.0254)^2}{4} = 5.067 \times 10^{-4}\ m^2$$
$$v_2 = 0$$

$\Longrightarrow$

$$F = \rho A v_1(v_1 - 0) = \rho A v_1^2$$
$$644.9921 = 1000 \times 5.067 \times 10^{-4} \times v_1^2$$

$\Longrightarrow$

$$644.9921 = 0.5067 \times v_1^2$$

$\therefore$

$$v_1 = 35.678\ m/s$$
$$Q = Av_1 = 5.067 \times 10^{-4} \times 35.678$$
$$Q = 0.01808\ m^3/s \quad or\ 18.08\ l/s$$

Problem 1.17 A jet of water flowing freely in the atmosphere is deflected by a curved vane as shown in Fig. 1.17. If the water jet has a diameter of 0.0381 m and a velocity of 7.7724 m/s, what is the force required to hold the vane in place?

Solution

$$F_x = \rho A v_1[(v_x)_2 - (v_x)_1]$$
$$A = \frac{\pi}{4} \times (0.0381)^2 = 1.14 \times 10^{-3}\ m^2$$
$$(v_x)_2 = 0$$
$$R_x = 1000 \times 1.14 \times 10^{-3} \times 7.7724[0 - (-7.7724)] = 68.8676\ N\ (leftward)$$
$$F_y = \rho A v_1\left[\left(v_y\right)_2 - \left(v_y\right)_1\right], \left(v_y\right)_1 = 0$$
$$R_y = 1000 \times 1.14 \times 10^{-3} \times 7.7724[(7.7724) - 0] = 68.8676\ N\ (upward)$$
$$R = \sqrt{68.8676^2 + 68.8676^2} = 97.3935\ N$$
With equal x and y components, the direction of R is at a 45°

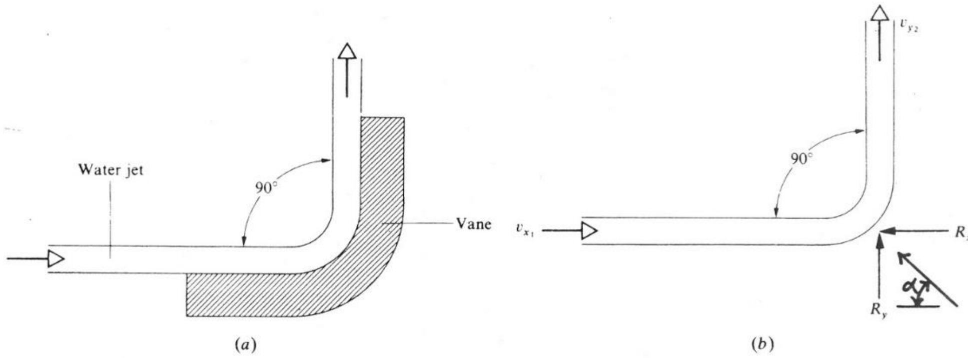


Fig. 1.17

Problem 1.18 A gardener is squirting the side of a house with a hose. The nozzle produces a 0.0127 m diameter jet having a velocity of 1.524 m/s. Determine the force exerted by the jet on the wall when the angle between the jet and the house is 90°.

Solution

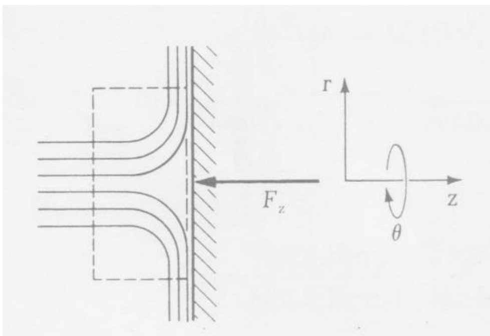


Fig. 1.18

$$\sum F_z = \rho_1 A_1 V_1 (0 - V_1) = -\rho_1 A_1 V_1^2$$
$$A_1 = \frac{\pi}{4} \times 0.0127^2 = 1.2668 \times 10^{-4}\ m^2$$
$$-F_z = -1000 \times 1.2668 \times 10^{-4} \times (1.524)^2$$

Solving, we get

$$F_z = 0.2936\ N$$

Problem 1.19 A water jet of velocity 3.6576 m/s and a cross-sectional area of 0.4645 m^2 strikes a curved vane as shown in Fig. 1.19. The vane is moving at a velocity of 0.9144 m/s in the positive *x* direction, and it deflects the jet through an angle of 60°. Assuming no frictional losses between the jet and the surface, determine the reaction forces F_x and F_y.

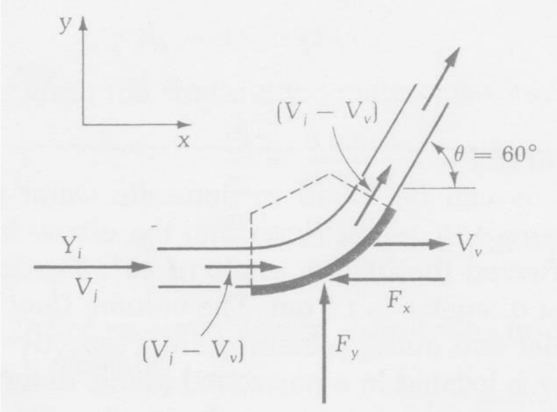


Fig. 1.19

Solution

$$-F_x = \dot{m}(V_{x\,out} - V_{\,xin})$$
$$= \rho A(V_j - V_v)[(V_j - V_v)\cos\theta - (V_j - V_v)]$$
$$= \rho A(V_j - V_v)^2(\cos\theta - 1)$$

By substitution,

$$-F_x = 1000 \times 0.4645(3.6576 - 0.9144)^2(\cos 60^0 - 1)$$

Solving, we obtain

$$F_x = 1747.715\ N$$

Applying the momentum equation in the *y* direction, we get

$$F_y = \dot{m}(V_{y\,out} - V_{\,yin})$$
$$= \rho A(V_j - V_v)[(V_j - V_v)\sin\theta - 0]$$
$$= \rho A(V_j - V_v)^2\sin\theta$$

By substitution,

$$F_y = 1000 \times 0.4645(3.6576 - 0.9144)^2 \times \sin 60^0$$

Solving, we obtain

$$F_y = 3027.132\ N$$

1.3.4 Force exerted by a jet on a moving plate in direction of jet

Fig. 1.20 shows a jet of water striking a flat vertical plate moving with a uniform velocity away from the jet.

Let V = Velocity of the jet (absolute)

a= Area of cross-section of the jet,

u = Velocity of the flat plate.

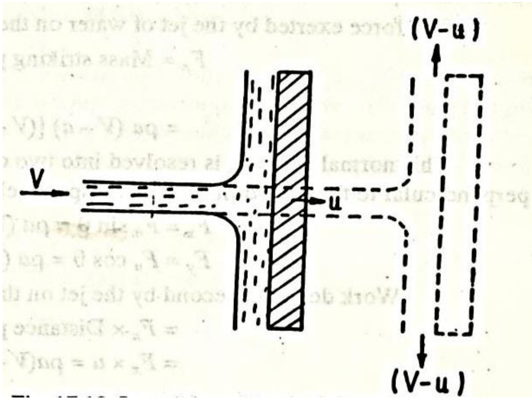


Fig. 1.20 jet striking a flat vertical moving plate.

In this case, the jet does not strike the plate with a velocity V, but it strikes it with a relative velocity which is equal to the absolute velocity of jet of water minus the velocity of the plate.

Hence relative velocity of the jet with respect to plate = (*V-u)*

Mass of water striking the plate per second

= ρ × Area of jet × velocity with which jet strikes the plate

$= \rho a \times [V - u]$

∴ Force exerted by the jet on the moving plate in the direction of the jet

F_x = Mass of water striking per second × [Initial velocity with which water strikes-Final velocity]

$F_x = \rho a(V - u)[(V - u) - 0]$ {∴ Final velocity in the direction of jet is zero}

$= \rho a(V - u)^2$

In this case, the work will be done by the jet on the plate, as the plate is moving. For the stationary plates the work down is zero

∴ Work done per second by the jet on the plate (W).

$= Force \times \frac{Distance\ in\ the\ direction\ of\ force}{Time}$

$= F_x \times u = \rho a(V - u)^2 \times u$

1.3.5 **Force Exerted by a Jet on Moving Inclined Plate in the Direction of the Jet**

Let a jet of water strike an inclined plate, which is moving with uniform velocity in the direction of the jet as shown in Fig. 1.21

Let V = Velocity of the jet (absolute)

a= Area of cross-section of the jet,

u = Velocity of the flat plate.

θ = Angle between jet and plate

Relative velocity of jet of water = (V-u)

∴ The velocity with which jet strikes = (V-u)

Mass of water striking per second

$= \rho \times a \times (V - u)$

If the plate is smooth and loss of energy due to impact of the jet is assumed zero, the jet of water will leave the inclined plate with a velocity equal to (V-u)

The force exerted by the jet of water on the plate in the direction normal to the plate is given as

F_n = Mass striking per second × [Initial velocity in the normal direction with which jet strikes-Final velocity]

$$= \rho a(V - u)[(V - u)\sin\theta - 0] = \rho a(V - u)^2 \, sin\,\theta$$

This normal force F_n is resolved into two components namely F_x and F_y in the direction of the jet and perpendicular to the direction of the jet respectively.

$\therefore F_x = F_n \sin\theta = \rho a(V - u)^2 sin^2\theta$

$F_y = F_n \cos\theta = \rho a(V - u)^2 \sin\theta\cos\theta$

In this case work will be done by the jet on the plate, as the plate is moving.

NB: For stationary plates the work done is zero.

∴ Work done per second by jet on the plate

$= F_x \times distance\ per\ second\ in\ the\ direction\ of\ x$

$= F_x \times u = \rho a(V - u)^2 \times sin^2\theta \times u = \rho a(V - u)^2 u \times sin^2\theta \ \ Nm/s$

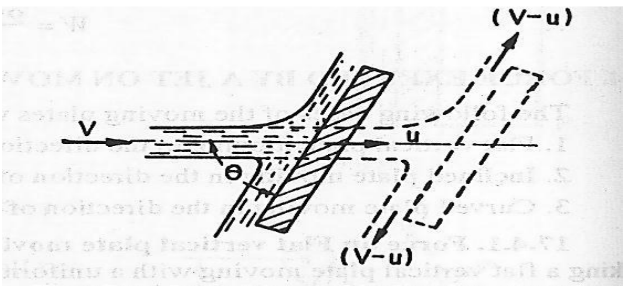


Fig. 1.21 Jet striking and inclined moving plate

1.3.6 **Force Exerted by Jet of Water on an Un–symmetrical Moving Curved Plate When Jet Strikes Tangentially at One of the Tips.**

Fig. 1.22 shows a jet of water striking a moving curved plate (also called vane) tangentially, at one of its tips. As the jet strikes tangentially, the loss of energy due to impact of the jet will be zero. In this case as the plate is moving, the velocity with which the jet of water strikes is equal to the relative velocity of the jet with respect to the plate. Also as the plate is moving in different direction of the jet, the relative velocity at inlet will be equal to the vector difference of the velocity of jet and velocity of the plate at inlet.

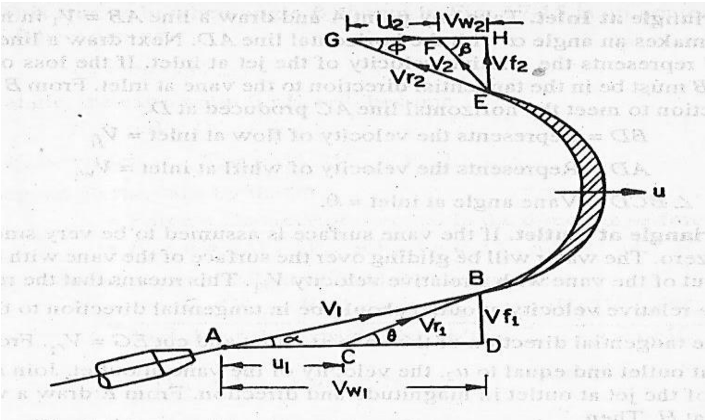


Fig. 1.22 Jet striking a moving curved vane at one of the tips.

Let V_1= Velocity of the jet at inlet

$V_{r_1} = Relative\ velocity\ of\ jet\ to\ plate\ at\ inlet$

u_1= Velocity of the plate (vane) at inlet

α = Angle between the direction of the jet and direction of the plate, also called guide blade angle

θ = Angle made by the relative velocity (V_{r1}) with the direction of motion at inlet also called vane angle at inlet

V_{w1}/ Vu_1 = the components of the velocity of the jet V_2, in the direction of motion

V_{f1}= the perpendicular velocity to the direct of motion of the vane.

V_2 = Velocity of the jet, leaving the vane or velocity of jet at outlet of the vane

u_2 = velocity of the vane at outlet.

V_{r2} = relative velocity of the jet with respect to the vane at outlet

β= Angle made by the velocity V_2 with the direction of motion of the vane at outlet.

ϕ = Angle made by the relative velocity V_{r2}, with the direction of motion of the vane at outlet and also called vane angle at outlet

V_{a2} and V_{f2} = components of the velocity V_2, in the direction of motion of vane and perpendicular to the direction of motion of vane at outlet

V_{w2}/V_{u2} = component of V_2 in the direction of motion of the vane

V_{f2} = component of V_2 perpendicular to the direction of motion of the vane at outlet

The triangles ABD and EGH are called velocity triangles at inlet and outlet

$V_{r1} = V_1 - u$

F_x = Mass of water striking per sec x (initial velocity with which jet strikes in the direction of motion -final velocity of jet in the direction of motion).

$$F_x = \rho a V_{r1}[(V_{w1} - u_1)] - \{-(u_2 + V_{u2})\},$$

$$= \rho a V_{r1}(V_{w1} - u_1 + u_2 + V_{w2}), \quad u_1 = u_2$$

$$F_x = \rho a V_{r1}(V_{w1} + V_{w2})$$

Note if $\beta = 90^{\circ} = \frac{\pi}{2}; V_{w2} = 0$

$F_x = \rho a V_{r1}[V_{w1}]$

If β is an obtuse angle, the expression for F_x will become

$F_x = \rho a V_{r1}(V_{w1} - V_{w2})$

Thus in general, F_x is written as $F_x = \rho a V_{r1}(V_{w1} \pm V_{w2})$

Work done per second on the vane by the jet

= Force × Distance per second in the direction of force

$= F_x \times u = \rho a V_{r1}(V_{w1} \pm V_{w2}) \times u$

Workdone per second per unit weight of fluid striking per second

$$\frac{\rho a V_{r1}[(V_{w1} \pm V_{w2})] \times u}{Weight\ of\ fluid\ striking\ per\ second} = \frac{\rho a V_{r1}[(V_{w1} \pm V_{w2})] \times u}{\rho a V_{r1} \times g} \qquad \text{in metres(m)}$$

$$F_x = \frac{[(V_{w1} \pm V_{w2})] \times u}{g}$$

Workdone per second per unit mass of fluid striking per second

$$\frac{\rho a V_{r1}[(V_{u1} \pm V_{u2})] \times u}{Mass\ of\ fluid\ striking\ per\ second} = \frac{\rho a V_{r1}[(V_{u1} \pm V_{u2})] \times u}{\rho a V_{r1}}, \text{ in } \frac{m^2}{s^2}.$$

$$Workdone\ per\ unit\ mass = \frac{\rho a V_{r_1}\left[V_{u_1} \pm V_{u_2}\right] \times u}{\rho a V_{r_1}} = \left[V_{u_1} \pm V_{u_2}\right] \times u\ (\frac{Nm}{kg}) or (\frac{kJ}{kg})$$

Problem 1.20 A jet of water having a velocity of 20 m/s strikes a curved vane which is moving with a velocity of 10 m/s. The jet makes an angle of 20 ^{o} with the direction of motion of the vane at inlet and leaves at an angle of 130 ^{o} to the direction of motion of vane at outlet. Calculate

i) vane angle, so that the water enters and leaves the vane without shock

ii) Work done per second per unit weight of water striking (or work done / unit weight of water striking) the vane per second.

Solution

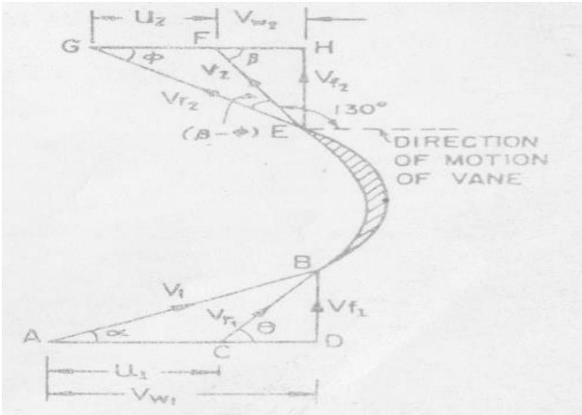


Fig. 1.23

NB: V_{u2} in the inlet diagram here should be V_{u1}

Given:

$V_1 = 20$ m/s, $u_1 = 10$ m/s, $\alpha_1 = 20^0$,

Angle made by the leaving jet, with direction of motion $=130^0$

$\beta = 180° - 130° = 50°$

$u_{1=}u_2 = 10\, m/s$

$V_{r1} = V_{r2}$

(i) Vane angle means angle made by the relative velocities at inlet and outlet, i.e. θ and ∅

From Fig. 1.10, in Δ ABD, $tan\ \theta = \frac{BD}{CD}$

$= \frac{V_{f1}}{AD-AC} = \frac{V_{f1}}{V_{w1}-u_1}$

$where\ V_{f1} = V_1 sin\alpha = 20 \times sin20 = 6.84\, m/s$

$V_{w1} = V_1 cos\alpha = 20 \times \cos 20 = 18.794 m/s$

$\Longrightarrow tan\ \theta = \frac{6.84}{18.794-10} = 0.7778$

$\theta = 37.875°$

From ΔABC, $sin\theta = \frac{V_{f1}}{V_{r1}} = V_{r1} = \frac{V_{f1}}{sin\theta} = \frac{6.84}{\sin 37.875} = 11.14$

$V_{r1} = V_{r2} = 11.14\, m/s$

From ΔEFG, applying sine rule, we have

$\frac{V_{r2}}{\sin(180°-\beta)} = \frac{u_2}{\sin(\beta-\emptyset)}$

$\frac{11.14}{\sin\beta} = \frac{10}{\sin(\beta-\emptyset)} = or\ \frac{11.14}{\sin\beta} = \frac{10}{\sin(50-\emptyset)}$

$\sin(50° - \emptyset) = \frac{10\times\sin 50}{11.14} = 0.6876$

$(50° - \emptyset) = sin^{-1}\,(0.6876)$

$(50° - \emptyset) = 43.44°$

$\Longrightarrow \emptyset = 50° - 43.44° = 6.56°$

(ii) Work done per second per unit weight of the water striking the vane per second is given by

$= \frac{1}{g}[V_{u1} + V_{u2}] \times u$ (+ sign is taken as β is an acute angle)

$V_{u2} = GH - GF = V_{r2}cos\emptyset - u_2 = 11.14 \times \cos(6.56) - 10 = 1.067\ m/s$

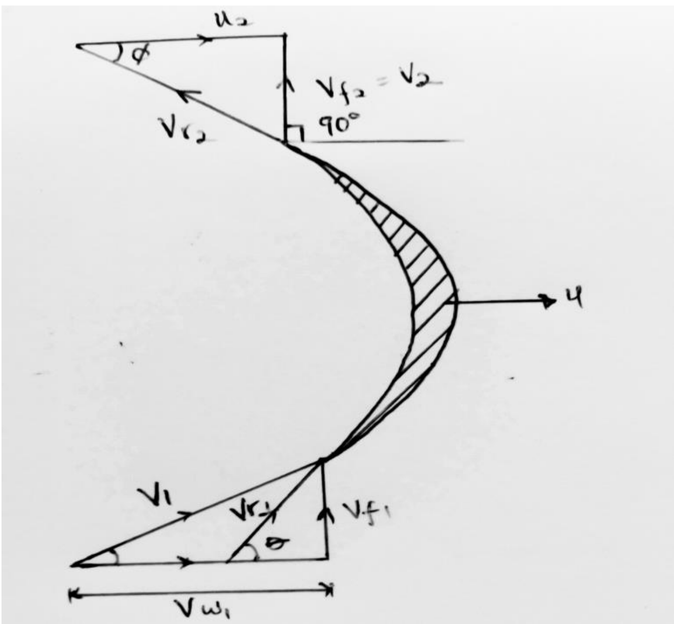
∴ Work done per unit weight of water

$= \frac{1}{9.81}[18.794 + 1.067] \times 10 = 20.24\ Nm/N$ or 20.24 m

Problem 1.21

1) A jet of water having a velocity of 40m/s strikes a curved vane which is moving with a velocity of 20m/s. the jet makes an angle of 30^0 with the direction of motion of the vane at the inlet and leaves at an angle of 90^0 to the direction of motion of vane at outlet.

Draw the velocity diagrams at the inlet and outlet and determine the vane angles, θ at inlet and Ø at outlet so that the water enters and leaves without shock.



v_1 =40m/s, u_1 = 20m/s

$\alpha = 30^0, \beta = 90^0$

$$u_1 = u_2 = u$$

$$\sin\alpha = \frac{v_{f1}}{v_1}$$

$$v_{f1} = v_1 \sin\alpha = 40\sin 30 = 20 \text{m/s}$$

$$\theta = \tan^{-1}\frac{v_{f1}}{v_{w1} - u_1} = \tan^{-1}\frac{20}{v_{w1} - 20}$$

$$v_{w1} = v_1 \cos 30 = 40\cos 30 = 34.64°$$

$$\theta = \tan^{-1}\frac{20}{34.64 - 20} = \tan^{-1}1.366 = 53.79°$$

$$\text{Also}, \sin\theta = \frac{v_{f1}}{v_{r1}}$$

$$v_{r1} = \frac{v_{f1}}{\sin\theta} = \frac{20}{\sin 53.79} = 24.78 \text{m/s}$$

$$v_{r1} = v_{r2}$$

$$\Rightarrow \cos\varphi = \frac{u_2}{v_{r2}} = \frac{20}{24.78} = 0.8071$$

$$\varphi = \cos^{-1}0.8071 = 36.18°$$

1.3.6.1 Force Exerted by a jet of water on a Series of Vanes

The force exerted by a jet of water on a single moving plate (which may be flat or curved) is not practically feasible. This case is only a theoretical one. In actual practice, a large number of plates are mounted on the circumference of a wheel at a fixed distance apart as shown in the figure below. The jet strikes a plate and due to the force exerted by the jet on the plate, the wheel starts moving and the second plate mounted on the wheel appears before the jet, which again exerts the force on the second plate. Thus each plate appears successively before the jet and the jet exerts force on each plate. The wheel start moving at a constant speed

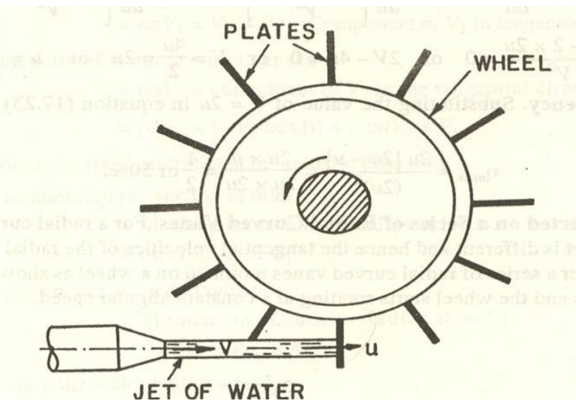


Fig. 1.24 Jet striking a series of vanes.

Let V = Velocity of jet,

d = Diameter of jet,

a = Cross sectional area of jet = $\frac{\pi}{4}d^2$

u = Velocity of vane

In this case the mass of water coming out from the nozzles is always in contact with the plates, when all the plates are considered. Hence mass of water/s striking the series of plates = ρaV.

Also the jet strikes the plate with a velocity $= (V - u)$

After striking, the jet moves tangentially to the plate and hence the velocity component in the direction of motion of plate is equal to zero.

Therefore the force exerted by the jet in the direction of motion of the plate is given by

F_x = Mass per second [initial velocity – final velocity]

$= \rho aV[(V-u) - 0] = \rho aV(V-u)$

Work done by the jet on the series of plates per second

=Force x distance per second in direction of force

$F_x \times u = \rho aV[(V-u)] \times u$

Kinetic energy of the jet per second $= \frac{1}{2}mV^2 = \frac{1}{2}(\rho aV) \times V^2 = \frac{1}{2}\rho aV^3$

$\therefore$ Efficiency $\eta = \frac{Workdone}{kinetic\ energy} = \frac{\rho aV(V-u)\times u}{\frac{1}{2}\rho aV^3} = \frac{2u(V-u)}{V^2}$

1.3.7 **Force Exerted On A Series Of Radial Curved Vanes**

For a radial curved vane the radius of the vane at inlet and outlet is different and hence the tangential velocities of the radial vane at inlet and outlet will not be equal. Consider a series of radial curved vanes mounted on a wheel as shown in Fig. 1.25. The jet of water strikes the vanes and the wheel starts rotating at a constant angular speed.

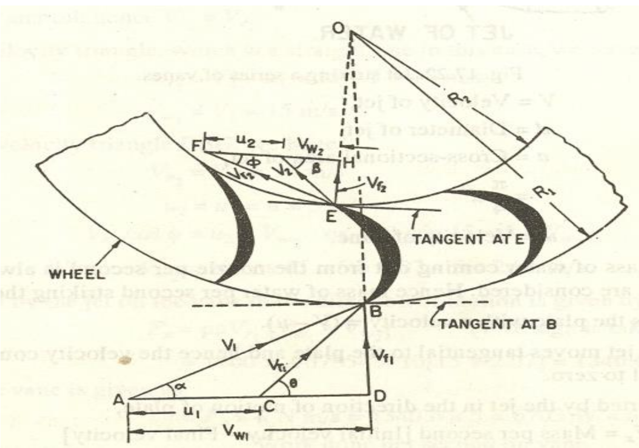


Fig. 1.25 Series of Radial Curved Vanes mounted on a wheel

R_1 =Radius of wheel at inlet of the vane

R_2 =Radius of the wheel at the outlet of the vane

$\omega = angular\ speed\ of\ the\ wheel$

u_1=$u_1 = \omega R_1$, to $u_2 = \omega R_2$

The mass of water striking per second for a series of vanes is =mass of water coming out from nozzle per second $= \rho a V_1$

Momentum of water striking the vanes in the tangential direction per second = Mass of water per second × component of V_1 in tangential direction

$= \rho a V_1 \times V_{u1}$

Similarly momentum at outlet

$= \rho a V_1$ × Component of V_2 in the tangential direction

=$\rho a V_1 \times (-V_2 \cos \beta) = -\rho a V_1 V_{u2}$

-ve sign is taken as the velocity V_2 at outlet is in the opposite direction.

Now angular momentum per second at inlet = momentum at inlet × Radius at inlet

$= \rho a V_1 \times V_{u1} \times R_1$

Angular momentum per second at outlet

= Momentum at inlet × Radius at outlet

$= -\rho a V_1 \times V_{u2} \times R_2$

Torque exerted by water on the wheel

T = Rate of change of angular momentum

= [Initial angular momentum per second – Final angular Momentum per second]

$= \rho a V_1 \times V_{u1} \times R_1 - (-\rho a V_1 \times V_{u2} \times R_2) = \rho a V_1 [V_{u1} R_1 + V_{u2} R_2]$

Work done per second on the wheel

= Torque × angular velocity $T \times \omega$

$= \rho a V_1 [V_{u1} R_1 + V_{u2} R_2] \times \omega = \rho a V_1 [V_{u1} R_1 \times \omega + V_{u2} R_2 \times \omega]$

$= \rho a V_1 [V_{u1} u_1 + V_{u2} u_2$

If β is an obtuse angle, the work done is $= \rho a V_1 [V_{u1} u_1 - V_{u2} u_2]$

∴ The general expression for the work done per second on the wheel

$= \rho a V_1 [V_{u1} u_1 \pm V_{u2} u_2]$

If the discharge is radial at outlet, then $\beta = 90^{\circ}$and the workdone becomes

$= \rho a V_1 [V_{u1} \times u_1]$

Problem 1.22 A jet of water having a velocity of 30m/s strikes a series of radial curved vanes mounted on a wheel which is rotating at 200 rpm. The jet makes an angle of 20° with the tangent to the wheel at inlet and leaves the wheel with a velocity of (V) = 5 m/s, at an angle of 130° to the tangent to the wheel at outlet. Water is flowing from outwards in a radial direction. The outer and inner radii of the wheel are 0.5 m and 0.25 m respectively.

Determine:

i. The vane angle at inlet and outlet.

ii. Work done per unit weight of the water.

iii. Efficiency of the wheel.

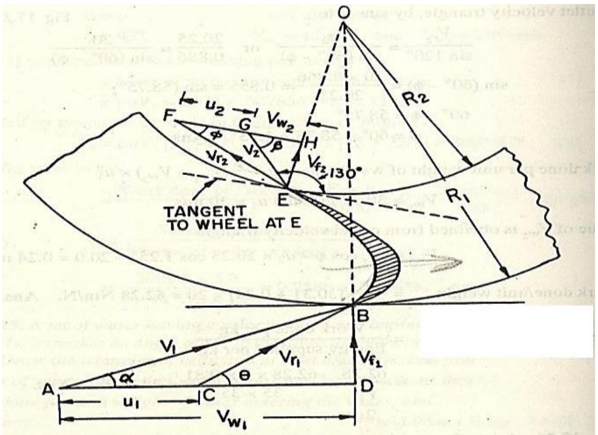


Fig. 1.26

Solution

Velocity of jet, V_1 = 30 m/s

Speed of wheel, N = 200 rpm

$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.94 \frac{rad}{s}$

$u_{1=}\omega R_{1=} 20.94 \times 0.5 = 10.47 \frac{m}{s}$

$u_{2=}\omega R_{2=} 20.94 \times 0.25 = 5.235 \frac{m}{s}$

i) Vane angles at inlet and outlet mean the angle made by the relative velocities V_{r1} and V_{r2}, that is θ and $\emptyset$.

In Δ CBD $tan\theta = \frac{BD}{CD} = \frac{V_{f1}}{AD-AC} = \frac{10.26}{V_{w1-u_1}} = \frac{10.26}{28.19-10.47} = 0.579$

$\theta = tan^{-1} 0.579 = 30.07°$

From outlet velocity Δ, $V_{w2} = V_2 cos\beta = 5 \times cos 50 = 3.214\ m/s$

$V_{f2} = V_2 sin\beta = 5 \times sin 50 = 3.83\ m/s$

In Δ EFH, $tan\emptyset = \frac{V_{f2}}{u_{2+V_{w2}}} = \frac{3.83}{5.235-3.214} = 0.453$

$\emptyset = tan^{-1}0.453 = 24.385°$

ii) Worked done per second by the water is given by

$W = \rho a V_1[V_{w1}u_1 + V_{w2}u_2]$

$(+ve\ sign\ is\ taken\ as\ \beta\ is\ acte\ angle\ in\ the\ figure)$

$\leftarrow Work\ done\ per\ unit\ weight\ of\ water = \frac{\rho a V_1[V_{w1}u_1+V_{w2}u_2]}{\rho a V_1 \times g}$

$\frac{1}{g}[V_{w1}u_1 + V_{w2}u_2] = \frac{1}{9.81}[28.19 \times 10.47 + 3.214 \times 5.235]$

$= 31.8\frac{Nm}{N}\ or\ (m)$

$iii)\ \eta = \frac{[V_{w1}u_1+V_{w2}u_2]}{\frac{V^2{}_1}{2}} = \frac{(28.19\times10.47+3.214\times5.235)}{30^2/_2}$

$\eta = \frac{295.15+16.82}{450} = 69.32\%$

Problem 1.23 A jet of water having a velocity of 35 m/s impinges on a series of vanes moving with a velocity of 20 m/s. The jet makes an angle of 30° to the direction of motion of vanes when entering and leaves at an angle of 120°. Draw the triangles of velocities at inlet and outlet and find:

a) The angles of vane tips so that water enters and leaves without shock,
b) The work done per unit weight of water entering the vanes, and
c) The efficiency.

Solution

Given:

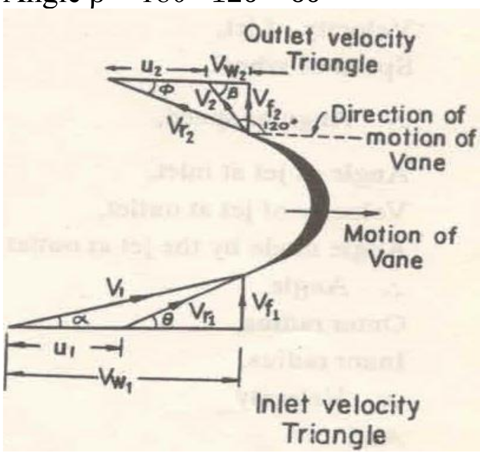
Velocity of jet V_1 = 35 m/s

Velocity off vane, = $u_1 = u_2$ = 20 m/s

Angle of jet at inlet α = 30°

Angle made jet the at outlet with the direction of motion of vanes = 120°

Angle β = 180°-120°=60°



a) Angles of vanes tips.

From inlet velocity triangle

$V_{w1} = V_1 cos\alpha = 35\ cos30 = 30.31\ m/s$

$V_{f1} = V_1 sin\alpha = 35 sin30° = 17.50\ m/s$

$tan\theta = \frac{V_{f1}}{V_{w1} - u_1} = \frac{17.50}{30.31 - 20} = 1.697$

$\theta = tan^{-1} 1.697 = 60°\ Ans.$

$by\ sine\ rule, \frac{V_{r1}}{\sin 90°} = \frac{V_{f1}}{sin\theta}\ or\ \frac{V_{r1}}{1} = \frac{17.50}{sin60°}$

$V_{r1} = \frac{17.50}{0.866} = 20.25 \frac{m}{s}$

Now $V_{r2} = V_{r1} = 20.25\ m/s$

From outlet velocity triangle, by sine rule $\frac{V_{r2}}{\sin 120°} = \frac{u_2}{\sin (60° - \varphi)}\ or\ \frac{20.25}{0.886} = \frac{20}{\sin (60° - \varphi)}$

$\sin(60° - \varphi) = \frac{20 \times 0.866}{20.25} = 0.855 = \sin (58.75°)$

$60° - \varphi = 58.75°$

$\varphi = 1.25°\ Ans.$

b) Work done per unit weight of water entering = $\frac{1}{g}(V_{w1} + V_{w2}) \times u_1$

$V_{w1} = 30.31 \frac{m}{s}\ and\ u_1 = 30\ m/s$

The value of V_{w2} is obtained from outlet velocity triangle

$V_{w2} = V_{r2} cos\varphi - u_1 = 20.25 cos 1.25° - 20.0 = 0.24\ m/s$

$\therefore workdone/unit\ weight = \frac{1}{9.81}[30.31 + 0.24] \times 20 = 62.28\ \ Nm/N\ Ans.$

c) Efficiency $\frac{Workdone\ per\ kg}{Energy\ supplied\ per\ kg}$

$\frac{62.28}{\frac{V_1^2}{2g}} = \frac{62.28 \times 2 \times 9.81}{35 \times 35} = 99.74\%\ Ans.$

1.4 JET PROPULSION

Jet propulsion means the propulsion or movement of the bodies such as ships aircrafts, rocket etc. with the help of jet. The reaction of the jet coming out from the orifice provided in the bodies is used to move the bodies. A jet of fluid coming out from an orifice or nozzle when it strikes a plate, exerts a force on the plate. The magnitude of the force exerted on the plate can be determined depending upon whether the plate is flat, inclined, curved, stationary or moving. This force exerted by the jet on the plate is known as "ACTION OF THE JET. From Newton's third law of motion, every action is accompanied by an equal and opposite reaction, hence the jet while coming out of the orifice or nozzle, exerts a force on the orifice or nozzle in the opposite direction in which the jet is coming out. The magnitude of the force exerted is equal to the action of the jet. This force which is acting on the orifice or nozzle in the opposite direction is called "REACTION OF THE JET. This principle is used in the following cases.

1) Jet propulsion of a tank to which orifice is fitted
2) Jet propulsion of ships

1.4.1 JET PROPULSION OF A TANK WITH AN ORIFICE

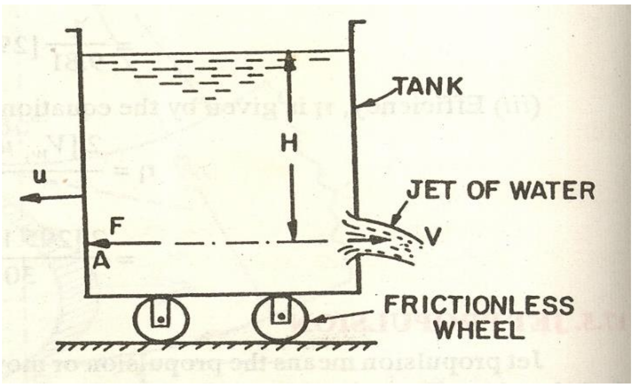


Fig. 1.28 jet Propulsion of a tank with an orifice

H=constant head of water in tank from the centre of orifice

a= area of orifice

v = velocity of jet of water

Cv = co-efficient of the velocity of orifice

Then, $V = C_v\sqrt{2gH}$

Mass of water coming out from the orifice per second=ρaV

=ρ × Volume per second = ρ × (Area × Velocity)

$= \rho \times a \times V$

Force acting on the water is equal to the rate of change of momentum

F = Mass per second × [Change of velocity]

= Mass per second × [Final velocity – Initial velocity]

Initial velocity of water in the tank is zero and final velocity of water when it comes out of the jet is equal to V

$\therefore F = \rho aV[V - 0] = \rho AV^2$

$F = mass/s\,x\ [change of\ velocity]$

$F = \rho aV[V - o] = \rho aV^2$

If u = velocity of tank, then, $V - \{-u\} = V_r$

V = absolute velocity of jet

$V_r = V + u$

Mass of water coming out from the orifice per second

=ρa x velocity with which water comes out

$\rho aV_r = \rho a(V + u)$

Force exerted on the tank is given as

F_x = Mass of water coming out from orifice per second× [Change of velocity]

$= \rho a(V+u) \times [(V+u) - u] = \rho a(V+u)V$

$= \rho a(V+u) \times V$

Thus the force given by the equation below is used for propelling the tank

∴ Work done on the moving tank by jet per second

$F_x \times u = \rho a(V+u)V \times u$

∴ Efficiency of propulsion is given as,

$$\eta = \frac{Workdone\ per\ second}{Kinetic\ energy\ of\ the\ issuing\ jet\ per\ second}$$

$$\eta = \frac{\rho a(V+u) \times V \times u}{\frac{1}{2}(mass\ of\ water\ per\ second) \times Velocity\ of\ issuing\ jet}$$

$$= \frac{\rho a(V+u) \times V \times u}{\frac{1}{2}[\rho a(V+u)] \times (V+u)^2} = \frac{2Vu}{(V+u)^2}$$

Condition for Maximum Efficiency and Expression for Maximum Efficiency

For a given value of V, the efficiency will be maximum when $\frac{d\eta}{du} = 0$

$\frac{d\eta}{du} = \frac{d}{du}\left[\frac{2Vu}{(V+u)^2}\right] = \frac{d}{du}[2Vu \times (V+u)^{-2}] = 0$

$= 2Vu \times (-2)(V+u)^{-3} + (V+u)^{-2} \times 2V = 0$

$= -\frac{4Vu}{(V+u)^3} + \frac{2V}{(V+u)^2} = 0$ 0r $= -4Vu + 2V(V+u) = 0$

$Dividing\ by\ 2V$

$= -2u + (V+u) = 0$

$\Rightarrow -u + V = 0\ or\ V = u$

$$\eta_{max} = \frac{2u \times u}{(u+u)^2} = \frac{2u^2}{4u^2} = \frac{1}{2} = \eta_{max} = 50\%$$

1.4.2 JET PROPULSON OF SHIPS

By the application of jet propulsion principle, a ship is driven through water. A jet of water which is discharge at the back (also called the stern) of the ship, exerts a propulsive force on the ship. The ship carries centrifugal pumps which draws water from the surrounding sea. This water is discharged through the orifice provided at the back of the ship in the form of a jet. The reaction of the jet coming out at the back of the ship propels the ship in the opposite direction of the jet. The water from the surrounding sea by the centrifugal pump is taken by the following two ways:

1. Through inlet orifices which are at right angles to the direction of the motion of the ship, and
2. Through inlet orifices which are facing the direction of motion of the ship.

1.4.2.1 Case 1. Through inlet orifices which are at right angles to the direction of the motion of the ship

Fig. 1.29 shows which is having the inlet orifices at right angles to its direction.

V=Absolute velocity of jet of water coming at the back of the ship.

u= Velocity of the ship $V_r = Relative\ velocity\ of\ jet\ with\ respect\ to\ ship = V + u$

As the velocity V and u are in the opposite direction and hence the relative velocity will be equal to sum of these two velocities.

Mass of water issuing from the orifice at the back of the ship=$\rho a(V + u)$

Propulsion force exerted on the ship

$F = mass\ of\ water\ issuing\ per\ per\ second \times change\ of\ velocity$

$= \rho a(V + u)(V_r - u) = \rho a(V + u)[(V + u) - u] = \rho a(V + u) \times V$

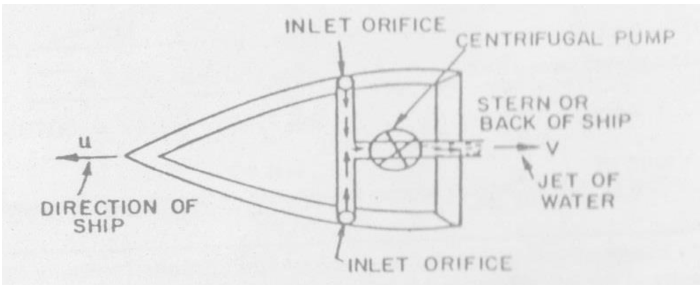


Fig. 1.29 Inlet orifices are at right angles

$Workdone\ per\ second = F \times u = \rho a(V + u) \times V \times u$

ɳ of propulsion, condition of maximum ɳ of expression for maximum ɳ are given by the same equation for the jet propulsion.

1.4.2.2 Case 2. Jet propulsion of ship when the inlet orifices face the direction of motion of the ship

Fig. 1.16 shows a ship which is having the inlet orifices facing the direction of motion of the ship. In this case the expression for propelling force and workdone per second will be same as the 1st case in which the orifices are at right angles to the ship. But the energy supplied by the jet will be different as in this case the water enters with a velocity equal to the velocity of the ship, i.e., with velocity *u.*

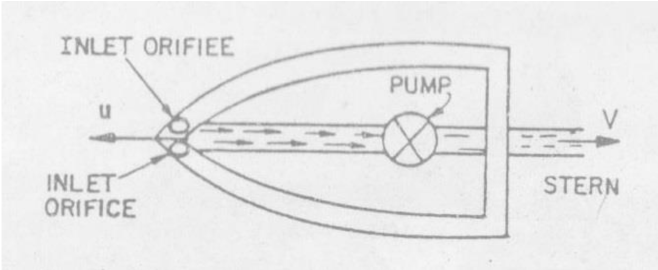


Fig. 30 Inlet orifices facing the direction of ship

Hence the expression for the energy supplied by the jet

$= \frac{1}{2}(mass\ of\ water\ supplied\ per\ second) \times ({V_r}^2 - u^2)$

$= \frac{1}{2}\rho a V_r \times (V_r - u^2)$

$where\ V_r = V + u$

$\Longrightarrow K.E\ supplied\ by\ jet = \frac{1}{2}\rho a(V+u)[(V+u)^2 - u^2]$

$\Longrightarrow$ Efficiency of propulsion, $\eta = \frac{workdone\ per\ sec}{Energy\ supplied\ by\ jet}$

$= \frac{\rho a(V+u)\times V\times u}{\frac{1}{2}\rho a(V+u)[(V+u)^2-u^2]} = \frac{2u}{V+2u}$

Problem 1.24 The head of water from the centre of the orifice which is fitted to one side of the tank is maintained at 2 m of water. The tank is not allowed to move and the diameter of orifice is 100 mm. find the force exerted by the jet of water on the tank. Take $C_v = 0.97$

Solution

Given:

Head of water, H = 2 m

Diameter of orifice, d= 100 mm = 0.1 m

Area $= \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.1)^2 = 0.007854\ m^2$

Value of $C_v = 0.97$

$\therefore Velocity\ of\ jet, V = C_v \times \sqrt{2gH} = 0.97 \times \sqrt{2 \times 9.81 \times 2.0} = 6.07\ m/s$

Force exerted on the tank is given by $F = \rho a V(V - 0)$

$\Longrightarrow \quad F = \rho a V^2 = 1000 \times 0.007854 \times 6.07^2 = 289.3\ N$

1.5 UNIT QUANTITITES

In order to predict the behavior of fluid machines as those discussed above including the hydraulic turbines working under varying conditions of head, speed, output and gate opening, the results are expressed in terms of quantities which may be obtained when the head on the turbine is reduced to unity. The conditions of the turbine under unit head are such that the efficiency of the turbine remains unaffected. The followings are the three important unit quantities which must be studied under unit head.

1. Unit speed 2. Unit power 3. Unit discharge

Unit speed is defined as the speed of a turbine working uner a unit head (i.e. under a head of 1 m). It is denoted by N_u and it is obtained as follows:

Let,

N = speed of a turbine under a head H,

H =head under which a turbine is working

u= tangential velocity

The tangential velocity, absolute velocity of water and head on the turbine are related as

$$u \propto V \quad where\ V \propto \sqrt{H}$$

Also tangential velocity (u) is given by

$$u = \frac{\pi DN}{60} \text{ Where, D = diameter of turbine}$$

For a given turbine, the diameter (D) is constant.

$$\therefore u \propto N \ or\ N \propto u \ or\ N \propto \sqrt{H}, N = k_1\sqrt{H}$$

Where k is a constant of proportionality.

If head on the turbine becomes unity, the speed becomes unit speed or when H = 1, N= N_u:

Substituting these values in equation (ii),

$$N = N_u\sqrt{H} \ or\ N_u = \frac{N}{\sqrt{H}} \ \ldots\ldots \text{(a)}$$

Unit discharge: It is defined as the discharge passing through a turbine, which is working under a unit head (i.e. 1 m). it is denoted by symbol $'Q_u'$. The expression for unit discharge is given as follow:

Let H = head water on the turbine

Q= discharge passing through turbine when head is H on the turbine,

a = Area of flow of water.

The discharge passing through a given turbine under a head 'H' is given by,

Q = Area of flow X *Velocity*

But for a turbine, area of flow is constant and velocity is proportional to $\sqrt{H}$.

$$Q \propto Velocity \propto \sqrt{H}$$

$$Q = k_2\sqrt{H} \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

Where k_2 is a constant of proportionality

$$if\ H = 1, Q = Q_w$$

Substituting these values in equation (iii), we get

$$Q_u = k_2\sqrt{1} = k_2$$

Substituting the value of k_2 in equation (iii), we get

$$Q = Q_u\sqrt{H}, Q_u = \frac{Q}{\sqrt{H}} \ldots\ldots \text{(b)}$$

Unit Power: It is defined as the power developed by a turbine, working under a unit head (i.e., under a unit head of 1 m). It is denoted by the symbol, P_u. The expression for unit power is obtained as follows:

Let H = head of water on the turbine

P = power developed by the turbine under a head of H, and is given as P=ρgQH.

Q = discharge through turbine under a head H.

Therefore, $P \, \alpha \, QH$, $\Longrightarrow$ $\quad P \, \alpha \, \sqrt{H}\, H$

$P \, \alpha \, H^{3/2}$

The overall efficiency η_o is given as

$$\propto Q \times H$$

$$\propto \sqrt{H} \times H$$

$$\propto H^{3/2}$$

$$P = k_3 H^{3/2} \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Where k_3 is a constant of proportionality.

When H = 1m , P = P_u

$\Longrightarrow$ $$P_u = k_3(1)^{3/2} = k_3$$

Substituting the value of k_3 in equation (iv), we get

$$P = P_u H^{3/2}$$

$$\therefore P_u = \frac{P}{H^{3/2}} \text{……………….. (c)}$$

Use of Unit Quantities($\boldsymbol{N_u}, \boldsymbol{Q_u}, \boldsymbol{P_u}$). If a turbine is working under different heads, the behavior of the turbine can be easily known from the values of the unit quantities, i.e., from the values of unit speed, unit discharge and unit power.

$$Let\ H_1, H_2 \ldots\ldots be\ the\ heads\ under\ which\ a\ turbine\ works,$$
$$N_1, N_2 \ldots\ldots are\ the\ corresponding\ speeds$$
$$Q_1, Q_2 \ldots\ldots are\ the\ discharge$$
$$P_1, P_2 \ldots\ldots are\ the\ power\ developed\ by\ the\ turbine$$

Using equations (a, b, c) respectively,

$$N_u = \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$$
$$Q_u = \frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$$
$$P_u = \frac{P}{H_1^{3/2}} = \frac{P}{H_2^{3/2}}$$

1.6 QUESTIONS

1. A 30cm diameter pipe carries water under a head of 15 metres with a velocity of 4m/s. if the axis of the pipe turns through 45°, find the magnitude and direction of the resultant force at the bend. **[8717.5 N, $\boldsymbol{\theta = 67°30'}$]**
2. A pipe of 20cm diameter conveying $0.20\, m^3/s$ of water has a right angled bend in the horizontal plane. Find the resultant force exerted on the bend if the pressure at inlet and outlet of the bend are $22.563 \frac{N}{cm^2}$ and $21.582 \frac{N}{cm^2}$ respectively. **[11604.7 N, $\boldsymbol{\theta = 43°54.2'}$]**
3. A 45° reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 40cm and 20cm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet of the bend is $21.58\, N/cm^2$. the rate of flow of water is 500litres/s [$\mathbf{22696.5N, 20°35'}$]
4. A nozzle of diameter 30mm is fitted to a pipe of 60mm diameter. Find the force exerted by the nozzle on the water which is flowing through the pipe at the rate of $\frac{40m^3}{min}$ $\boldsymbol{7057.7N}$]
5. A lawn sprinkler with two nozzles of diameters 3mm is connected across a tap of water. The nozzles are at a distance of 40cm and 30cm from the centre of the tap. The rate of water through the tap is $100\, cm^3/s$. The nozzle discharges water in the downward directions. Determine the angular speed at which the sprinkler will rotate free **[2.83rad/s]**
6. A lawn sprinkler has two nozzles of diameters 8mm each at the end of a rotating arm and the velocity of flow of water from each nozzle is 12m/s. One nozzle discharges water in the downward direction, while the other nozzle discharges water vertically up. The nozzles are at a distance of 40cm from the centre of the rotating arm. Determine the torque required to hold the rotating arm stationary. Also determine the constant speed of the rotation of arm if it is free to move. **[5.78Nm, 30rad/s]**
7. Find the force exerted by a jet of water of diameter 100mm on a stationary flat plate, when the jet strikes the plate normally with a velocity of 30m/s **[7068.6N]**
8. A jet of water of diameter 50mm moving with a velocity of 20m/s strikes a fixed plate in such a way that the angle between the jet and the plate is 60°. Find the force exerted by the jet on the plate (i) in the direction of the jet (ii) in the direction normal to the plate. **[589N, 680.13N]**
9. A jet of water of diameter 100mm moving with a velocity of 30m/s strikes a curved fixed symmetrical plate at the centre. Find the force exerted by the jet of water in the direction of the jet, if the jet is deflected through an angle of 120° at the outlet of the curved plate. **[10602.7N]**.
10. A jet of water of diameter of 100mm moving with a velocity of 20m/s strikes a curved fixed plate tangentially a one end at an angle of 30° to the horizontal. The jet leaves the plate at an angle of 20° to the horizontal. Find the force exerted by the plate in horizontal and vertical directions. **[5672.34N, 496.3N].**

11. A jet of water 30mm diameter, moving with a velocity of 15m/s, strikes a hinged square plate of weight 245.25N at the centre of the plate. The plate is of uniform thickness. Find the angle through which the plate will swing **[θ = 40^o 25.6']**
12. A jet of water of diameter 100mm strikes a curved plate at its centre with a velocity of 15m/s. The curved plate is moving with a velocity of 7m/s in the direction of the jet. The jet is deflected through an angle of 150^o. Assuming the plate is smooth, find: (i) force exerted on the plate in the direction of the jet. (ii) power of the jet (iii) efficiency **[938 N, 6.56 kW, 49.53%]**
13. A jet of water having a velocity of 30m/s strikes a curved vane, which is moving with a velocity of 15m/s. The jet makes an angle of 30^0 with the direction of motion of vane at inlet and leaves at angle of 120^0 to the direction of motion of vane at outlet. Calculate: (i) vane angles, if the water enters and leaves the vane without shock (ii) work done per second per unit weight of water striking the vanes per second. **[(i) 53^0 47.7', 15^0 41', (ii) 44.15 Nm/N]**

CHAPTER 2

2 DIMENSIONAL AND MODEL ANALYSIS

2.1 INTRODUCTION

Dimensional analysis is a mathematical technique used for studying the relationships between the physical parameters governing a fluid phenomenon. All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length, (L) Mass, (M) and Time, (T) are three fixed dimensions which are of importance in Fluid Mechanics. If in any problem of fluid mechanics, heat is involved, then temperature is also taken as fixed dimension. These fixed dimensions are called Fundamental dimensions or Fundamental quantities.

2.2 SECONDARY OR DERIVED QUANTITIES

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example velocity is denoted by distance per unit time (L/T), density by mass per unit volume $\left(\frac{M}{L^3}\right)$and acceleration by distance per second squared$\left(\frac{L}{T^2}\right)$. Then velocity, density and acceleration become secondary or derived quantities. The expressions (L/T), $\frac{M}{L^3}$ and $\frac{L}{T^2}$ are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanic are given in Table 12.1

S. No.	*Physical quantity*	*Symbol*	*Dimensions*	*SI units*
	(a) Fundamental			
	Length	L	L	m
2	Mass	M	M	kg
3	Time	T	T	s
	b) Geometric			
	4. Area	A	L^2	m^2
	5. Volume	∀	L^3	m^3
	c) Kinematic			
	6. Velocity	v	LT^{-1}	ms^{-1}
	7. Angular Velocity	ω	T^{-1}	rad/s
	8. Acceleration	a	LT^{-2}	m/s^2

	9. Angular acceleration	$\propto$	T^{-2}	rad/s^2
	10. Discharge	Q	L^3T^{-1}	m^3/s
	11. Acceleration due to gravity	g	LT^{-2}	m/s^2
	12. Kinematic viscosity	ν	L^2T^{-1}	m^2/s
	d) Dynamic quantities			
	13. Force	F	MLT^{-2}	N
	14. Weight	W	MLT^{-2}	N
	15. Density	ρ	ML^{-3}	kg/m^3
	16. Dynamic viscosity	μ	$ML^{-1}T^{-1}$	Pa.s
	17. Pressure	p	$ML^{-1}T^{-2}$	Pa
	18. Modules of elasticity	E, K	$ML^{-1}T^{-2}$	Pa
	19. Surface tension	σ	MT^{-2}	N/m
	20. Specific weight	w	$ML^{-2}T^{-2}$	N/m^3
	e) Physical quantity			
	21. Shear stress	τ	$ML^{-1}T^{-2}$	Pa
	22. Work, Energy	W or E	ML^2T^{-2}	Joule (J)
	23. Power	P	MLT^{-3}	Watt (W)
	24. Torque	T	ML^2T^{-2}	Nm
	25. Momentum	M	MLT^{-1}	kgm/s

Problem 2.1. Determine the dimensions of the quantities given below:

(i) Angular velocity (ii) angular acceleration, (iii) Discharge (iv) kinematic viscosity (v) Force (vi) Specific weight (vii) dynamic viscosity

Solution

(i) Angular velocity $= \frac{Angle\ covered\ in\ radians}{Times} = \frac{1}{T} = T^{-1}$

(ii) Angular acceleration $= rad/s^2 = \frac{rad}{T^2} = T^{-2}$

(iii) Discharge = Area × velocity $= L^2 \times \frac{L}{T} = \frac{L^3}{T} = L^3T^{-1}$

(iv) Kinematic viscosity (v) $=\frac{\mu}{\rho}$ where μ is given by $\tau = \mu\frac{\partial u}{\partial y}$

$\therefore\ \mu = \frac{\tau}{\frac{\partial u}{\partial y}} = \frac{Shear\ stress}{\frac{L}{T}\times\frac{1}{T}} = \frac{\frac{Force}{Area}}{\frac{1}{T}}$

$$= \frac{Mass \times Acceleration}{Area \times Time} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}} = \frac{ML}{L^2T^2 \times \frac{1}{T}} = \frac{M}{LT} = ML^{-1}T^{-1}$$

And $\rho = \frac{Mass}{Volume} = \frac{M}{L^3} = ML^{-3}$

$\therefore Kinematic\ viscosity(\nu) = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1}$

(v) Force = Mass × Acceleration $= M \times \frac{Length}{(Time)^2} = \frac{ML}{T^2} = MLT^{-2}$

(vi) Specific weight = $\frac{Weight}{Volume} = \frac{Force}{Volume} = \frac{MLT^{-2}}{L^3} = ML^{-2}T^{-2}$

(vii) Dynamic viscosity,μ is derived in (iv) as $\mu = ML^{-1}T^{-1}$

2.3 DIMENSIONAL HOMOGENEITY:

Dimensional homogeneity means the dimensions of each term in an equation on both sides are equal. Thus if the dimensions of each term on both sides of the equation are the same, the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions (i.e., L, M, T) on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S = Dimension of R.H.S = LT^{-1}

∴ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

2.4 METHODS OF DIMENSIONAL ANALYSIS

If the number of variables involved in a physical phenomenon is known, then the relation among the variables can be determined by the following two methods:

i) Rayleigh’s method

ii) Buckingham’s π-theorem

2.4.1 Rayleigh’s Method.

This method is used for determining the expression for a variable which depends on maximum three or four variables only. If the number of independent variables becomes more than four then it is very difficult to find the expression for the dependent variable.

Let X be a variable, which depends on X_1, X_2, and X_3 variables. Then according to Rayleigh's method X is a function of X_1, X_2, and X_3 and mathematically, it is written as

$$X = f(X_1,X_2,X_3).$$

This can also be written as $X= KX_1^a.\ X_2^b.\ X_3^c$.

where K is a constant and a, b and c are arbitrary powers.

The values of a, b and c are obtained by comparing the powers of the fundamental dimensions on both sides. Thus the expression is obtained for the dependent variable.

Problem 2.2. The time period (t) of a pendulum depends on the length (L) of the pendulum and acceleration due to gravity. Derive an expression for the time period.

Solution

Time period T is a function of (i) L and (ii) g

∴ $t = KL^a.g^b$ where K is a constant. …(i)

Substituting the dimensions on both sides $T^1= KL^a.\ (LT^{-2})^b$

Equating the powers of M,L, and T on both sides we have

Powers of T 1= -2b, ∴ $b = -\frac{1}{2}$

Powers of L 0= a+b ∴ $a= -b = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

Substitute the values of a and b in equation (i),

$$t = KL^{1/2}.\ g^{-1/2}= K\sqrt{\frac{L}{g}}$$

The value of K is determined from experiment from which is given as

$$K= 2\pi$$

∴ $$t = \mathbf{2\pi}\sqrt{\frac{L}{g}}$$ **Ans.**

Problem 2.3. Find an expression for the drag force on a smooth sphere of diameter D, moving with uniform velocity V in a fluid of density ρ and dynamic viscosity, μ.

Solution Drag force is a function of

(i) Diameter D, (ii) Velocity V (iii) Density ρ (iv) dynamic viscosity μ

∴ $$F = K\, D^a V^b \rho^c \mu^d$$

Where K is a non-dimensional factor.

Substituting the dimensions on both sides we have

$$MLT^{-2} = KL^a(LT^{-1})^b.(ML^{-3})^c.(ML^{-1}T^{-1})^d$$

Power of M, $1 = c + d$

Power of L, $1 = a + b - 3c - d$

Power of T, $-2 = -b - d$

There are four unknowns (a, b, c, d) but the equations are three. Hence it is not possible to find the values of a, b, c, and d. But three of them can be expressed in terms of the fourth variable which is most important. Here viscosity is having a vital role and hence a, b, c are expressed in terms of d which is the power to viscosity.

$$c = 1 - d$$

$$b = 2 - d$$

$$a = 1 - b + 3c + d = 1 - (2 - d) + 3(1 - d) + d$$

$$= 1 - 2 + d + 3 - 3d + d = 2 - d$$

Substituting the values of a,b and c in (i) we get

$$F = KD^{2-d}\, V^{2-d}\, \rho^{1-d} \mu^{d}$$

$$F = KD^2\, V^2\, \rho\, (D^{-d}\, V^{-d}\, \rho^{-d} \mu^{d}) = K\rho D^2\, V^2 \left(\frac{\mu}{\rho V D}\right)^{d}$$

$$= \mathbf{K\rho D^2\, V^2}\, \boldsymbol{\phi}\, \boldsymbol{\left(\frac{\mu}{\rho V D}\right)} \quad \textbf{Ans.}$$

Problem 2.4. Find the expression for the power P, developed by a pump when P depends on the head H, discharge Q and specific weight w of the fluid.

Solution

Power P is a function of

(i) Head, H (ii) Discharge, Q

(iii) Specific weight, w

$\therefore$ $$P = KH^{a}.Q^{b}.\omega^{c} \quad \text{(i)}$$

Where K = Non-dimensional constant

Substituting the dimensions on both sides of equation (i)

$$ML^2T^{-3} = K\, L^a.\, (L^3T^{-1})^b.\, (ML^{-2}T^{-2})^c$$

Equating the powers of M, L, and T on both sides,

Power of M, $1 = c$, $\therefore \quad c = 1$

Power of L, $2 = a + 3b - 2c,\;\; a = 2 - 3b - 2c = 2 - 3 + 2 = 1$

Power of T, $-3 = -b - 2c,\quad \therefore\; b = 3 - 2c = 3 - 2 = 1$

Substituting the values of a, b, and c in (i)

$$P = KH^1.Q^1.w^1 = \boldsymbol{KHQw} \quad \textbf{Ans.}$$

Problem 2.5 The efficiency Ƞ of a fan depends on the densityρ, the dynamic viscosity μ of the fluid, the angular velocity ω, diameter D of the rotor and the discharge Q. Express Ƞin terms of dimensionless parameters.

Solution

The efficiency Ƞ depends on

i. Density, ρ ii. Viscosity, μ

iii. Angular velocity, ω iv. Diameter, D

v. discharge, Q

$\therefore$ $$Ƞ = \mathrm{K}\rho^{a}.\mu^{b}.\omega^{c}.D^{d}.Q^{e} \quad \text{.... (i)}$$

Where K= non dimensional constant.

Substituting the dimensions on both sides of the equation (i)

$$M^0L^0T^0 = (ML^{-3}).(ML^{-1}T^{-1})^b.(T^{-1})^c.(L)^d.(L^3T^{-1})^e$$

Equating powers of M, L,T on both sides,

Power of M, $\quad 0 = a + b$

Power of L, $\quad 0 = -3a - b + d + 3e$

Power of T, $\quad 0 = -b - c - e.$

There are five unknowns but equations are three. Express the three unknowns in terms of the other two unknowns which are more important. Viscosity and discharge are more important in this problem. Hence expressing a, c and d in terms of b and e, we get

$$a = -b$$

$$c = (b + e)$$

$$d = 3a + b - 3e = -3b + b - 3e = -2b - 3e$$

Substituting these values in equation (i), we get

$$Ƞ = \mathrm{K}\rho^{-b}.\mu^{b}.\omega^{-(b+e)}.D^{-2b-3e}.Q^{e}$$

$$= \mathrm{K}\rho^{-b}.\mu^{b}.\omega^{-e}.\omega^{-e}.D^{-2b}.D^{-3e}.Q^{e}$$

$$= K\left(\frac{\mu}{\rho\omega D^2}\right)^b.\left(\frac{Q}{\omega D^3}\right)^e = \phi\left[\left(\frac{\mu}{\rho\omega D^2}\right).\left(\frac{Q}{\omega D^3}\right)\right]$$ ..**Ans.**

Problem 2.6. The resisting force Rof a supersonic plane during flight can be considered as dependent upon the length of the aircraft, l, velocity, V, air viscosity, μ, air density, ρ, and bulk modulus of air, K. express the functional *relationship between these variables and the resisting force.*

Solution

The resisting force R depends upon

(i) Length, l   (ii) velocity, V

(ii) Viscosity, μ (iv) Density, ρ

(v) Bulk modulus, K.

$\therefore$ $$R = Al^a.V^b.\mu^c.\rho^d.K^e \quad \ldots.(i)$$

Where A is the non- dimensional constant.

Substituting the dimensions on both sides of the equation (i),

$$MLT^{-2} = AL^a.(LT^{-1})^b.(ML^{-1}T^{-1})^c.(T^{-1})^c.(ML^{-3})^d.(ML^{-1}T^{-2})^e$$

Equating the powers of M, L,T on both sides,

Powers of M, $1 = c + d + e$

Powers of L, $1 = q + b - c - 3d - e$

Powers of T, $-2 = -b - c - 2e.$

There are five unknowns but the equations are only three. Expressing the three unknowns in terms of two unknowns (μ and K).

$\therefore$ Express the values of a, b and d in terms of c and e.

Solving, $d = 1 - c - e$

$b = 2 - c - 2e$

$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e$

$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.$

Substituting these values in (i), we get

$$R = Al^{2-c}.V^{2-c-2e}.\mu^c.\rho^{1-c-e}.K^e$$

$$= Al^2.V^2.\rho(l^{1-c}V^c\mu^c\rho^{-c}).(V^{-2e}.\rho^{-e}.K^e)$$

$$= Al^2V^2\rho\left(\frac{\mu}{\rho VL}\right)^e.\left(\frac{K}{\rho V^2}\right)^e$$

$$= A\rho l^2V^2\phi\left[\left(\frac{\mu}{\rho VL}\right).\left(\frac{K}{\rho V^2}\right)\right].$$ **Ans**

Problem 2.7. *A partially submerged body is towed in water. The resistance R to its motion depends on the densityρ, the viscosity μ of water, length of the body, velocity v of the body and the acceleration due to gravity g. show that the resistance to the motion can be expressed in the form*

$$R = \rho l^2V^2\phi\left[\left(\frac{\mu}{\rho VL}\right),\left(\frac{lg}{V^2}\right)\right]$$

Solution

The resistance R depends on

(i) Density ρ, (ii) viscosity μ,

(iii) Length l, (iv) velocity v,

(v) Acceleration g.

$\therefore \qquad R = K\rho^a.\mu^b.l^c.V^d.g^e \qquad$ …(i)

Where K =non-dimensional constant.

Substituting the dimensions on both sides of the equation (i),

$MLT^{-2} = K(ML^{-3})^a.(ML^{-1}T^{-1})^b.L^c.(LT^{-1})^d.(LT^{-2})^e$

Equating the powers of M, L, T on both sides

Power of M, $\qquad 1 = a + b$

Power of L, $\qquad 1 = -3a - b + c + d + e$

Power of T, $\qquad -2 = -b - d - 2e.$

There are five unknowns and equations are only three. Expressing the three unknowns in terms of two unknowns (μ and g). Hence express a, c and b in terms of b and e. Solving, we get

$$a = 1 - b$$

$$d = 2 - b - 2e$$

$$c = 1 + 3a + b - d - e = 1 + 3(1 - b) + b - (2 - b - 2e) - e$$

$$= 1 + 3 - 3b + b - 2 + b + 2e - e = 2 - b + e.$$

Substituting the values in equation (i), we get

$$R = K\rho^{1-b}.\mu^b.l^{2-b+c}.V^{2-b-2e}.g^e$$

$$= K\rho l^2.V^2.(\rho^{-b}\mu^b l^{-}V^{-b}).(l^e.V^{-2e}.g^e). \quad \text{b=?}$$

$$= K\rho l^2V^2.\left(\frac{\mu}{\rho Vl}\right)^b.\left(\frac{lg}{V^2}\right)^e = \boldsymbol{\rho l^2V^2\phi\left[\left(\frac{\mu}{\rho VL}\right),\left(\frac{lg}{V^2}\right)\right]} \qquad \textbf{Ans}$$

2.4.2 BUCKINGHAM'S π-THEOREM

The Rayleigh's method becomes difficult if the variables are more than the fundamental dimensions (M, L, T). This difficulty is overcome by Buckingham's π-Theorem, which states "if there are *n* variables (independent plus dependent variables) in a physical phenomenon and if these variables contain m fundamental dimensions (M, L, T) then the variables are arranged into *(n-m)* dimensionless terms. Each term is called π-term. Each π-term contains m+1 variable.

Let X_1, X_2, X_3….. Xn be the variables involved in a physical problem. Let X_1 be the dependent variable and X_2, X_3…..Xn be the independent variables on which X_1 depends. Then X_1 is a function of X_2,X_3……Xn and mathematically this is expressed as

$$X_1=f\,(X_2,X_3\ldots\ldots Xn) \qquad \text{….(2.1)}$$

Equation (2.1) can be also be written as

$$f(X_1,X_2,X_3\ldots\ldots Xn) =0 \qquad \text{…(2.2)}$$

Equation (2.2) is dimensionally homogeneous. It contains ***n*** variables. If there are m fundamental dimensions then according to Buckingham's π theorem, equation (2.2) can be written in terms of the number of dimensionless groups or π terms in which the number of π-terms is equal to *(n-m)*. Hence, equation (2.2) can become

$$f(\pi_1, \pi_2, \ldots \ldots \pi_{n-m}) = 0$$

Each term is dimensionless and is independent of the system. Division or multiplication by a constant does not change the character of the term π-term. Each π-term contains m+1 variables. Where m is the number of fundamental dimensions and is also called the **repeating variables.** Let in the above case X_2, X_3, and X_4 be the repeating variables if the fundamental dimension m (M,L,T) =3. Then each π-term is

$$\pi_1 = {X_2}^{a_1}.{X_3}^{b_1}.{X_4}^{c_1}.X_1$$
$$\pi_2 = {X_2}^{a_2}.{X_3}^{b_2}.{X_4}^{c_2}.X_5 \qquad \ldots..(2.4)$$
$$\pi_{n-m} = {X_2}^{a_{n-m}}.{X_3}^{b_{n-m}}.{X_4}^{c_{n-m}}.X_n$$

Each equation is solved by the principle of dimensional homogeneity and the values of a_1, b_1, c_1, etc. are obtained. These values are substituted in equation (2.4) and values of $\pi_1, \pi_2, \ldots., \pi_{n-m}$ are obtained. These values of πs are substituted in equation (2.3). The final equation for the phenomenon is obtained by expressing any one of the π-terms as a function of the others as

$$\pi_1 = \phi\,(\pi_2, \pi_3, \ldots \ldots \pi_{n-m})$$

$$\ldots..(2.5)$$

$$\pi_2 = \phi_1(\pi_1, \pi_3, \ldots \ldots \pi_{n-m})$$

2.4.3 Method of Selecting Repeating Variables.

The number of repeating variables is equal to the number of fundamental dimensions of the problem. The choice of repeating variables is governed by the following consideration:

1. As far as possible, the dependent variable should not be selected as a repeating variable.
2. The repeating variable should be chosen in such a way that one variable contains geometric property, another variable contains flow property and the third variable contains fluid property.

Variables with geometric property are:

i) Length, *l* (ii) diameter, *d* (iii) Height, *H* etc

Variables with flow property are:

i) Velocity, v (ii) acceleration etc.

Variables with fluid property are

(i) μ (ii) ρ(iii) w, etc.

3. The repeating variables selected should not form a dimensionless group.
4. The repeating variables together must have the same number of fundamental dimensions.
5. No two repeating variables should have the same dimensions.

   In most fluid mechanics problems, the choice of repeating variables may be

   i) d,v,ρ or (ii)l, v, ρ or (iii)l, v, μ or (iv) d, v ,μ

2.4.4 Procedure for Solving Problem by Buckingham's π-theorem

The procedure for solving Buckingham's π-theorem is explained by considering the problem 2.6 which is also solved by the Rayleigh's method.

Problem 2.8. The resisting force R of a supersonic plane during flight can be considered as dependent on the length of the aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force.

Solution

Step 1.The resisting force R depends on (i) , (ii) V, (iii) μ, (iv) ρ and (v) K. Hence R is a function for L, V, μ, ρ and K. Mathematically,

$$R = f(L, V, \mu, \rho, K) \qquad \ldots(i)$$

Or it can be written as $f_1(R, L, V, \mu, \rho, K) = 0$ …(ii)

∴Total number of variables, n = 6

Number of fundamental dimensions, m = 3.

m is obtained by writing out the dimensions of each variable as $R=MLT^{-2}$, $V=LT^{-1}$, $M=ML^{-1}T^{-1}$, $\rho = ML^{-3}$, $K=ML^{-1}T^{-2}$. Thus fundamental dimensions are M, L, T and hence m = 3.

Number of dimensionless π - terms =n-m = 6-3 =3.

Thus three π –terms can be formed as π_1, π_2, π_3. Hence equation (ii) is written as

$$f_1(\pi_1,\pi_2, \pi_3)=0 \qquad \ldots..(iii)$$

Step 2. Each π-term =m +1 variables where m is equal to 3 and also called repeating variables.

Out of the six variables R, l, μ, V, ρ and K, three variables are to be selected as repeating variables. The dependent variable, R should not be selected as a repeating variable. Out of the five remaining variables, one variable should have geometric property, the second variable should have flow property and the third fluid property. These requirements are fulfilled by selecting l, V and ρ as repeating variables. The repeating variables themselves should not form a dimensionless term and should have themselves fundamental dimensions equal to m, i.e., 3 here. Dimensions of l, V and ρ are L, LT^{-1}, ML^{-3} and hence the three fundamental dimensions exist in l, V and ρ and they themselves do not form dimensionless group.

Step 3. Each π-term is written as according to equation (2.4)

$$\pi_1 = l^{a1}.V^{b1}.\rho^{c1}.R$$
$$\pi_2 = l^{a2}.V^{b2}.\rho^{c2}.\mu \qquad \ldots(iv)$$
$$\pi_3 = l^{a3}.V^{b3}.\rho^{c3}.K$$

Step 4. Each π term is solved by using the principle of dimensional homogeneity. For the first term, we have

$$\pi_1 = M^oL^oT^o = L^{a_1}.(LT^{-1})^{b_1}.(ML^{-3})^{c_1}.MLT^{-2}$$

Equating the powers of M,L,T on both sides, we get

Powers of M, $\quad 0 = c_1 + 1 \quad \therefore \; c_1 = -1$

Powers of L, $\quad 0 = a_1+b_1-3c_1+1,$

$\therefore \quad a_1=-b_1+3c_1-1=2-3-1=-2$

Powers of T, $\quad 0 = -b_1-2 \quad \therefore b_1=-2$

Substituting the values for a_1, b_1 and $c_1$in equation (iv),

$$\pi_1 = L^{-2}\,V^{-2}.\,\rho^{-1}.R$$

Or $\quad \pi_1=\frac{R}{l^2V^2\rho} = \frac{R}{\rho l^2 V^2}$

Similarly for 2nd π- term, we get $\pi_2 = M^0L^0T^0 = L^{a_2}.(LT^{-1})^{b_2}.\,(ML^{-3})^{b_2}.ML^{-1}T^{-1}$

Equating the powers of M, L, T on both sides

Powers of M, $\quad 0=c_2+1 \quad \therefore c_2=-1$

Powers of L, $\quad 0=a_2+b_2-3c_2-1 \quad a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Powers of T, $\quad 0=-b_2-1 \quad \therefore b_2=-1$

Substituting the values of a_2, b_2, c_2 in π_2 of (iv)

$$\pi_2 = l^{-1}.V^{-1}.\rho^{-1}.\mu = \frac{\mu}{lV\rho}$$

3rd π- term,

$$\pi_3=l^{a_3}.\,V^{b_3}.\,\rho^{c_3}.\,K$$

Or $\quad M^0L^0T^0=L^{a_3}.\,(LT^{-1})^{b_3}.\,(ML^{-3})^{c_3}.\,ML^{-1}T^{-2}$

Equating the powers of M, L, T on both sides, we have

Power of M, $0=c_3$ $\therefore c_3 = -1$

Powers of L, $0=a_3+b_3-3c_3-1$ $\therefore a_3 = -b_3 + 3c_3 + 1 = 2 - 3 + 1 = 0$

Powers of T, $0=-b_3-2$ $b_3=-2$

Substituting the values of a_3, b_3, c_3 in π_3- terms

$$\pi_3 = l^0.V^{-2}.\rho^{-1}.K = \frac{k}{v^2\rho}$$

Step 5 Substituting values of π_1, π_2, π_3 in equation (ii), we get

$$f_1\left(\frac{R}{\rho l^2 V^2}, \frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right) = 0 \quad \text{or} \quad \left(\frac{R}{\rho l^2 V^2}\right) = \phi\left[\frac{\mu}{lV\rho}, \frac{k}{V^2\rho}\right]$$

$$R = \rho l^2 V^2 \ \phi\left(\frac{\mu}{\ell V\rho}, \frac{k}{\rho V^2}\right)$$ ….. **Ans**

Problem 2.9. (a) State Buckingham's π-theorem

(b) The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Solution

(a) Statement of Buckingham's π-theorem is given in section 2.3.2.

(b) Given: η is a function of ρ, μ, ω, D *and* Q

$\therefore \quad \eta = f(\rho, \mu, \omega, D, Q) \quad or \quad f_1 = (\eta, \rho, \mu, \omega, D, Q) = 0$ …(i)

Hence the total number of variables, η=6.

The value of m, i.e., number of fundamental dimensions for the problem is obtained by writing dimensions of each variable. Dimensions of each variable are

η= Dimensionless, $\rho = ML^{-3}$, $\mu = ML^{-1}T^{-1}$, $\omega = T^{-1}$, $D=L$ and $Q= L^{-3}T^{-1}$

$\therefore$ m = 3

Number of π- terms = n-m = 6-3 = 3

Equation (i) is written as $f_1(\pi_1, \pi_{2,}\pi_3) = 0$ ….(ii)

Each π-term contains $m + 1$ variables, where m is equal to three and is also a repeating variable. Choosing D, ω *and* ρ as repeating variables, we have

$$\pi_1 = D^{a_1}.\omega^{b_1}.\rho^{c_1}.\eta$$

$$\pi_2 = D^{a_2}.\omega^{b_2}.\rho^{c_2}.\mu$$

$$\pi_3 = D^{a_3}.\omega^{b_3}.\rho^{c_3}.Q$$

First $\boldsymbol{\pi}$-term (π_1). $\pi_1 = D^{a_1}.\omega^{b_1}.\rho^{c_1}.\eta$

Substituting dimensions on both sides of π_1

$$M^0L^0T^0 = L^{a_1}.(T^{-1})^{b_1}.(ML^{-3})^{c_1}.M^0L^0T^0$$

Equating power of M, L, T on both sides

Power of M, $0 = c_1 + 0 \quad \therefore c_1 = 0$

Power of L, $0 = a_1 + 0 \quad \therefore a_1 = 0$

Power of T, $0 = -b_1 + 0 \quad \therefore b_1 = 0$

Substituting the values of $a_1, b_1 and\ c_1\ in\ \pi_1$, we get

$$\pi_1 = D^0\omega^0\rho^0.\eta = \eta$$

[If a variable is dimensionless, it itself is a π-term. Here, the variable η is dimensionless and hence η is a π- term. As it exists in first π-term and hence $\pi_1 = \eta$. Then there is no need of equating the powers. Directly the value can be obtained.]

2nd π-term. $\pi_2 = D^{a_2}.\omega^{b_2}.\rho^{c_2}.\mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = (L)^{a_2}.(T^{-1})^{b_2}.(ML^{-3})^{c_2}.ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_2 + 1 \quad \therefore c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1 \quad \therefore a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of T, $0 = -b_2 - 1 \quad \therefore b_2 = -1$

Substituting the values of $a_2, b_2\ and\ c_2\ in\ \pi_2$,

$$\pi_2 = D^{-2}.\omega^{-1}.\rho^{-1}.\mu = \frac{\mu}{D^2\omega\rho}$$

3rd π- term. $\pi_3 = D^{a_3}.\omega^{b_3}.\rho^{c_3}.Q$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_3}.(T^{-1})^{b_3}.(ML^{-3})^{c_3}.L^3T^{-1}$$

Equating the powers of M, L and T on both sides

Power of M, $0 = c_3, \quad \therefore\ c_3 = 0$

Power of L, $0 = a_3 - 3c_3 + 3 \quad \therefore\ a_3 = 3c_3 - 3 = -3$

Power of T, $0 = -b_3 - 1 \quad \therefore b_3 = -1$

Substituting values of $a_3, b_3\ and\ c_3\ in\ \pi_3$

$$\pi_3 = D^{-3}.\omega^{-1}.\rho^0.Q = \frac{Q}{D^3\omega}$$

Substituting the values of $\pi_1, \pi_2 and\ \pi_3$ in equation (ii)

$$f_1\left(\eta, \frac{\mu}{D^2\omega\rho}, \frac{Q}{D^3\omega}\right) = 0 \quad \text{0r} \quad \eta = \phi\left[\frac{\mu}{D^2\omega\rho}, \frac{Q}{D^3\omega}\right] \text{.........Ans.}$$

Problem 2.10. Using Buckingham's π-theorem, show that the velocity through a circular rifice given by $V = \sqrt{2gH}\phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right]$, where H is the head causing flow, D is the diameter of the orifice, μ is the coefficient of viscosity,ρ is the mass density and g is the acceleration due to gravity.

Solution

Given:

V is a function of H, D, μ, ρ and g

$\therefore$ $$V = f(H, D, \mu, \rho, g) \quad \text{or} \quad f_1(V, H, D, \mu, \rho, g) = 0$$

$\therefore$ Total number of variable, n =6

Writing dimension of each variable, we have

$$V = LT^{-1}, H = L, D = L, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, g = LT^{-2}$$

Thus number of fundamental dimensions, m=3

$\therefore$ Number of π-terms = n-m = 6-3 = 3

Equation (i) can be written as $f_1(\pi_1, \pi_2, \pi_3) = 0$

…(ii)

Each π- term contains m+1 variables, where m = 3 and is equal to repeating variables. Hence V is a dependent variable, we get three π- terms as

$$\pi_1 = H^{a_1}.g^{b_1}.\rho^{c_1}.V$$

$$\pi_2 = H^{a_2}.g^{b_2}.\rho^{c_2}.D$$

$$\pi_3 = H^{a_3}.g^{b_3}.\rho^{c_3}.\mu$$

First π- term $\quad \pi_1 = H^{a_1}.g^{b_1}.\rho^{c_1}.V$

Substituting dimensions on both sides

$$M^0L^0T^0 = L^{a_1}.(LT^{-2})^{b_1}.(MT^{-3})^{c_1}.(LT^{-1})$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1 \quad \therefore\ c_1 = 0$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1 \quad \therefore\ a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Power of T, $\quad 0 = -2b_1 - 1 \quad \therefore b_1 = -\frac{1}{2}$

Substituting the values ofa_1, b_1 and $c_1 in\ \pi_1$,

$$\pi_1 = H^{-\frac{1}{2}}.g^{-\frac{1}{2}}.\rho^0.V = \frac{V}{\sqrt{gH}}$$

Second π- term $\quad \pi_2 = H^{a_2}.g^{b_2}.\rho^{c_2}.D$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2}.(LT^{-2})^{b_2}.(ML^{-3})^{c_2}.L$$

Equating the powers of M, L, T,

Power of M, $\quad 0 = c_2 \quad \therefore\ c_2 = 0$

Power of L, $0 = a_2 + b_2 - 3c_2 + 1 \quad a_2 = -b_2 - 3c_2 - 1 = -1$

Power of T, $0 = -2b_2 \quad \therefore\ b_2 = 0$

Substituting the values of a_2, b_2 and $c_2 in\, \pi_2$,

$$\pi_2 = H^{-1}.g^6.\rho^0.D = \frac{D}{H}$$

Third π- term $\pi_3 = H^{a_3}.g^{b_3}.\rho^{c_3}.\mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_3}.(LT^{-2})^{b_3}.(ML^{-3})^{c_3}.(ML^{-1}T^{-1})$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_3 + 1 \quad \therefore c_3 = -1$

Power of L, $0 = a_3 + b_3 - 3c_3 - 1 \quad \therefore\ a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{1}{2}$

Power of T, $0 = -2b_3 - 1 \quad \therefore b_3 = -\frac{1}{2}$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = H^{-3/2}.g^{-1/2}.\rho^{-1}.\mu = \frac{\mu}{H^{3/2}\sqrt{g\rho}}$$

$$= \frac{\mu}{H\rho\sqrt{gH}} = \frac{\mu V}{H\rho V\sqrt{gH}} \qquad \text{[Multiply and Divide by V]}$$

$$= \frac{\mu}{H\rho V}\ .\pi_1 \qquad \left\{\because \frac{V}{\sqrt{gH}}\right\} = \pi_1$$

Substituting the values of π_1, π_2, π_3 in equation (ii),

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{\mu}{H\rho V}, \frac{D}{H}\right) = 0 \quad \text{or} \quad \frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \frac{\mu}{H\rho V}\right] \qquad \textbf{Ans}$$

Multiplying by a constant does not change the character of π-terms.

Problem 2.11. The pressure difference ΔP in a pipe of diameter D and length l due to turbulent flow depends on the velocity V, viscosity μ, density, ρ and roughness K. Using Buckingham's π-theorem, obtain an expression for ΔP.

Solution

Given:

ΔP is a function of D, l, V, μ, ρ, k

$\therefore$ $\Delta\rho = f(D, l, V, \mu, \rho, k)$ or $f_1(\Delta\rho, D, l, V, \mu, \rho, k) = 0$ …(i)

$\therefore$ Total number of variables, n= 7.

Writing dimensions of each variable,

Dimension of $\Delta\rho$ = dimension of pressure = $ML^{-1}T^{-2}$

$$D = L, l = L, V = LT^{-1}, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, k = L$$

$\therefore$ Number of fundamental dimensions, m=3

Number of π- terms = n-m =7 – 3 = 4.

Now equation can be grouped in 4π-terms as

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \qquad \text{....(ii)}$$

Each π-term contains m + 1 or 3 +1 = 4 variables. Out of four variables, three are repeating variables. Choosing D, V, ρ as the repeating variables, we have the four π-terms as

$$\pi_1 = D^{a_1}.V^{b_1}.\rho^{c_1}.\Delta\rho$$

$$\pi_2 = D^{a_2}.V^{b_2}.\rho^{c_2}.\mathrm{l}$$

$$\pi_3 = D^{a_3}.V^{b_3}.\rho^{c_3}.\mu$$

$$\pi_4 = D^{a_4}.V^{b_4}.\rho^{c_4}.\mathrm{k}$$

First π-term $\quad \pi_1 = D^{a_1}.V^{b_1}.\rho^{c_1}.\Delta\rho$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1}.(LT^{-1})^{b_1}.(ML^{-3})^{c_1}.ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1 + 1 \quad \therefore c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 - 1 \quad \therefore a_1 = -b_1 + 3c_1 + 1 = 2 - 3 + 1 = 0$

Power of T, $\quad 0 = -b_1 - 2 \quad \therefore b_1 = -2$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^0.V^{-2}.\rho^{-1}.\Delta\rho = \frac{\Delta\rho}{\rho V^2}$$

Second π-term $\quad \pi_2 = D^{a_2}.V^{b_2}.\rho^{c_2}.\mathrm{l}$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_2}.(LT^{-1})^{b_2}.(ML^{-3})^{c_2}.L$$

Equating powers of M, L, T on both sides,

Power of M, $\quad 0 = c_2 \quad \therefore c_2 = 0$

Power of L, $\quad 0 = a_2 - b_2 - 3c_2 + 1, \quad \therefore a_2 = b_2 + 3c_2 - 1 = -1$

Power of T, $\quad 0 = -b_2, \quad \therefore b_2 = 0$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = D^{-1}.V^0.\rho^0.\mathrm{l} = \frac{\mathrm{l}}{\mathrm{D}}$$

Third π-term $\quad \pi_3 = D^{a_3}.V^{b_3}.\rho^{c_3}.\mu$

Substituting dimensions on both sides

$$M^0L^0T^0 = L^{a_3}.(LT^{-1})^{b_3}.(ML^{-3})^{c_3}.ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_3 + 1 \quad \therefore c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1 \quad \therefore a_3 = -b_3 + 3c_3 + 1 = 1 - 3 + 1 = -1$

Power of T, $\quad 0 = -b_3 - 1 \quad \therefore b_3 = -1$

Substituting the values ofa_3, b_3 *and* c_3 in π_3,

$$\pi_3 = D^{-1}.V^{-1}.\rho^{-1}.\pi = \frac{\mu}{DV\rho}$$

Fourth π-term $\quad \pi_4 = D^{a_4}.V^{b_4}.\rho^{c_4}.\mathrm{k}$

$M^0L^0T^0 = L^{a_4}.(LT^{-1})^{b_4}.(ML^{-3})^{c_4}.L$ {Dimension of k = L}

Equating the power of M, L, T on both sides,

Power of M, $\quad 0 = c_4 \quad \therefore c_4 = 0$

Power of L, $\quad 0 = a_4 - b_4 - 3c_4 + 1 \quad \therefore a_4 = b_4 + 3c_4 - 1 = -1$

Power of T, $\quad 0 = -b_4 \quad \therefore b_4 = 0$

Substituting the values of a_4, b_4 *and* c_4 in π_4,

$$\pi_4 = D^{-1}.V^0.\rho^0.\mathrm{k} = \frac{\mathrm{k}}{\mathrm{D}}$$

Substituting the values ofπ_1, π_2, π_3 *and* $\pi_4$in (ii), we get

$f_1\left(\frac{\Delta\rho}{\rho V^2}, \frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right) = 0$ or $\frac{\Delta\rho}{\rho V^2} = \phi\left[\frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right]$. **Ans**

Expression for h_f(Difference of pressure head). From experiments, it was observed that pressure difference, $\Delta\rho$ is a linear function of $\frac{l}{D}$ and hence it is taken out of the function

$$\frac{\Delta\rho}{\rho V^2} = \frac{l}{D}\phi\left[\frac{\mu}{DV\rho}, \frac{k}{D}\right]$$

$$\frac{\Delta\rho}{\rho} = V^2.\frac{l}{D}\phi\left[\frac{\mu}{DV\rho}, \frac{k}{D}\right]$$

Dividing by g to both sides, we have $\frac{\Delta\rho}{\rho g} = \frac{V^2.l}{g.D}\phi\left[\frac{\mu}{DV\rho}, \frac{k}{D}\right]$.

Now $\phi\left[\frac{\mu}{DV\rho}, \frac{k}{D}\right]$ contains two terms. Find one is $\frac{\mu}{DV\rho}$ which is $\frac{1}{Reynold\ number}$ or $\frac{1}{Re}$ and second one is $\frac{k}{D}$ which is called roughness factor. Now $\phi\left[\frac{1}{Re}, \frac{k}{D}\right]$is put equal to f, where f is the co efficient of friction which is a function of Reynold number and roughness factor.

$$\therefore \qquad \frac{\Delta\rho}{\rho g} = \frac{\Delta f}{2}.\frac{V^2 l}{gD}$$

Multiplying or dividing by any constant does not change the character of π-terms

$$\frac{\Delta\rho}{\rho g} = hf = \frac{\boldsymbol{4f.LV^2}}{\boldsymbol{Dx2g}}.\qquad \textbf{Ans.}$$

Problem 2.12. The pressure difference Δp in apipe of diameter D and length l due to viscous flow depends on the velocity V, viscous μ and density ρ. Using Buckingham'sπ-theorem, obtain an expression for Δp.

Solution

This problem is similar to problem 2.10. The only difference is that Δp is to be calculated for viscous flow. Then in the repeating variable instead of ρ, the fluid property μ is to be chosen.

Now Δp is a function of D, l, V, μ, ρ or Δp = f(D, l, V, μ, ρ)

Or $\quad f_1$ =(Δp, D, l, V, μ, ρ) …(i)

Total number of variables, n = 6

Number of fundamental dimensions, m = 3

Number of π-terms = n – 3 = 6 – 3 = 3

Hence equation (i) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ …(ii)

Each π-term contains m +1 variables, i.e., 3+1 = 4 variables. Out of four variables, three are repeating variables.

Choosing D, V, μ as repeating variables, we have π-terms as

$$\pi_1 = D^{a_1}.V^{b_1}.\mu^{c_1}.\Delta\rho$$

$$\pi_2 = D^{a_2}.V^{b_2}.\mu^{c_2}.\mathrm{l}$$

$$\pi_3 = D^{a_3}.V^{b_3}.\mu^{c_3}.\rho$$

First $\boldsymbol{\pi}$-term $\quad \pi_1 = D^{a_1}.V^{b_1}.\mu^{c_1}.\Delta\rho$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_1}.(LT^{-1})^{b_1}.(ML^{-1}T^{-1})^{c_1}.ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1 + 1 \quad \therefore c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - c_1 - 1 \quad \therefore a_1 = -b_1 + c_1 + 1 = 1 - 1 + 1 = 0$

Power of T, $\quad 0 = -b_1 - c_1 - 2 \quad \therefore b_1 = -c_1 - 2 = 1 - 2 = -1$

Substituting the values ofa_1, b_1 and c_1 in π_1,

$$\pi_1 = D^1.V^{-1}.\mu^{-1}.\Delta\rho = \frac{D\Delta\rho}{\mu V^2}$$

Second $\boldsymbol{\pi}$-term $\quad \pi_2 = D^{a_2}.V^{b_2}.\mu^{c_2}.\mathrm{l}$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_2}.(LT^{-1})^{b_2}.(ML^{-1}T^{-1})^{c_2}.L$$

Equating powers of M, L, T on both sides, Power of M, $0 = c_2$ $\therefore c_2 = 0$

Power of L, $0 = a_2 + b_2 - c_2 + 1,$ $\therefore a_2 = -b_2 + c_2 - 1 = -1$

Power of T, $0 = -b_2 - c_2,$ $\therefore b_2 = -c_2 = 0$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = D^{-1}.V^0.\mu^0.\mathrm{l} = \frac{\mathrm{l}}{\mathrm{D}}$$

Third π-term $\pi_3 = D^{a_3}.V^{b_3}.\mu^{c_3}.\rho$

Substituting dimensions on both sides

$$M^0L^0T^0 = L^{a_3}.(LT^{-1})^{b_3}.(ML^{-1}T^{-1})^{c_3}.ML^{-3}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_3 + 1$ $\therefore c_3 = -1$

Power of L, $0 = a_3 + b_3 - c_3 - 3$ $\therefore a_3 = -b_3 + c_3 + 3 = 1 - 1 + 3 = 1$

Power of T, $0 = -b_3 - c_3$ $\therefore b_3 = c_3 = -(-1) = 1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^{-1}.V^{-1}.\mu^{-1}.\rho = \frac{\rho DV}{\mu}.$$

Substituting the values of π_1, π_2 and π_3 in (ii),

$f_1\left(\frac{D\Delta\rho}{\mu V^2}, \frac{l}{D}, \frac{\rho DV}{\mu}\right) = 0$ or $\frac{\Delta\rho}{\mu V^2} = \phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right]$. or $\Delta p = \frac{\mu V}{D}\phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right]$

Experiments show that the pressure Δp is a linear function $\frac{l}{D}$. Hence $\frac{l}{D}$ can be taken out of the functional as

$$\Delta p = \boldsymbol{\frac{\mu V}{D}} \times \boldsymbol{\frac{L}{D}\phi\left[\frac{\rho DV}{\mu}\right]}. \qquad \textbf{Ans}$$

Expression for difference of pressure head for viscous flow

$$h_f = \frac{\Delta p}{\rho g} = \frac{\mu V}{D} \times \frac{L}{D} \times \frac{1}{\rho g}\phi[R_e] \qquad \left\{\because \frac{\rho DV}{\mu} = R_e\right\}$$

$$h_f = \boldsymbol{\frac{\mu VL}{wD^2}\phi[R_e]} \qquad \textbf{Ans.}$$

Problem 2.13. Derive on the basis of dimensional analysis suitable parameters to present the thrust develop by a propeller. Assume that thrust P depends upon the angular velocity ω, speed of advance (V), diameter D, dynamic viscosity μ, mass density ρ, elasticity of the fluid medium which can be noted by the speed of sound in the medium C.

Solution

Thrust P is function of ω, V, D, μ, ρ, C

Or $P = f(\omega, V, D, \mu, \rho, C)$

Or $f_1 = (P, \omega, V, D, \mu, \rho, C) = 0$ …(i)

$\therefore$ Total number of variables, n = 7

Writing dimensions of each variable, we have

$P = MLT^{-2}, \omega = T^{-1}, V = LT^{-1}, D = L, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, C = LT^{-1}$

$\therefore$ Number of fundamental dimensions, m = 3

$\therefore$Number of π terms =n-m = 7-3 = 4

Hence equation (i) can be written as $f_1(\pi_{1,}\pi_2, \pi_{3,}\pi_4) = 0$ …(ii)

Each π term contains m+1, i.e., 3+1 = 4 variables. Out of four variables, three are repeating variables.

Choosing D, V, ρ as repeating variables, we get π-terms as

$$\pi_1 = D^{a_1}.V^{b_1}.\rho^{c_1}.P$$
$$\pi_2 = D^{a_2}.V^{b_2}.\rho^{c_2}.\omega$$
$$\pi_3 = D^{a_3}.V^{b_3}.\rho^{c_3}.\mu$$
$$\pi_4 = D^{a_4}.V^{b_4}.\rho^{c_4}.C$$

First π-term $\pi_1 = D^{a_1}.V^{b_1}.\rho^{c_1}.P$

Writing dimensions on both sides,

$$M^0L^0T^0 = L^{a_1}.(LT^{-1})^{b_1}.(ML^{-3})^{c_1}.MLT^{-2}.$$

Equating powers of M, L T on both sides

Powers of M, $0 = c_1 + 1,$ $\therefore c_1 = -1$

Power of L, $0 = a_1 + b_1 - 3c_1 + 1$ $a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Power of T, $0 = -b_1 - 2,$ $\therefore b_1 = -2$

Substituting the values of $a_1, b_1, and\ c_1\ in\ \pi_1$

$$\pi_1 = D^{-2}.V^{-2}.\rho^{-1}.P = \frac{P}{D^2.V^2.\rho}$$

Second π-term $\pi_2 = D^{a_2}.V^{b_2}.\rho^{c_2}.\omega$

Writing dimensions on both sides

$$M^0L^0T^0 = L^{a_2}.(LT^{-1})^{b_2}.(ML^{-3})^{c_2}.T^{-1}$$

Equating powers of M, L T on both sides

Powers of M, $0 = c_2,$ $\therefore$ $c_2 = 0$

Power of L, $0 = a_2 + b_2 - 3c_2$ $\therefore a_2 = -b_2 + 3c_2 = 1 + 0 = 1$

Power of T, $0 = -b_2 - 1,$ $\therefore b_2 = -1$

Substituting the values of $a_2, b_2, and\ c_2\ in\ \pi_2$

$$\pi_2 = D^1.V^{-1}.\rho^0.\omega = \frac{D\omega}{V}$$

Third π-term $\pi_3 = D^{a_3}.V^{b_3}.\rho^{c_3}.\mu$

Writing dimensions on both sides,

$$M^0L^0T^0 = L^{a_3}.(LT^{-1})^{b_3}.(ML^{-3})^{c_3}.ML^{-1}T^{-1}.$$

Equating powers of M, L T on both sides

Powers of M, $0 = c_3 + 1, \quad \therefore c_3 = -1$

Power of L, $0 = a_3 + b_3 - 3c_3 - 1$

$a_1 = -b_3 + 3c_3 + 1 = 1 - 3 + 1 = -1$

Power of T, $0 = -b_3 - 1, \quad \therefore b_3 = -1$

Substituting the values of $a_3, b_3, and\ c_3\ in\ \pi_3$

$$\pi_3 = D^{-1}.V^{-1}.\rho^{-1}.\mu = \frac{\mu}{DV\rho}$$

Fourth π-term $\pi_4 = D^{a_4}.V^4.\rho^{c_4}.C$

Writing dimensions on both sides

$$M^0L^0T^0 = L^{a_4}.(LT^{-1})^{b_4}.(ML^{-3})^{c_4}.LT^{-1}$$

Equating powers of M, L T on both sides

Powers of M, $0 = c_4, \therefore c_4 = 0$

Power of L, $0 = a_4 + b_4 - 3c_4 + 1$

$$\therefore a_4 = -b_4 + 3c_4 - 1 = 1 + 0 - 1 = 0$$

Power of T, $0 = -b_4 - 1, \quad \therefore b_1 = -1$

Substituting the values of $a_4, b_4, and\ c_4\ in\ \pi_4$

$$\pi_4 = D^0.V^{-1}.\rho^0.C = \frac{C}{V}$$

Substituting the values of $\pi_1, \pi_2,\ \pi_3\ and\ \pi_4$ in equation (ii),

$$f_1\left(\frac{P}{D^2.V^2.\rho}, \frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right) = 0 \quad \text{or} \quad \frac{P}{D^2.V^2.\rho} = f_2\left(\frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right)$$

$$P = \boldsymbol{D^2V^2\rho f_2\left(\frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right)} \qquad \textbf{...Ans}$$

Problem 2.14. The frictional torque T of a disc of diameter D rotating at a speed N in a fluid of viscosity μ and density ρ in a turbulent flow is given by $T = D^5N^2\rho\phi\left[\frac{\mu}{D^2N\rho}\right]$.

Prove this by the method of dimensions.

Solution

Given: $T = f(D, N, \mu, \rho)$ or $f_1(T, D, N, \mu, \rho) = 0$

$\therefore$ Total number of variable, n=5

Dimensions of each variable are expressed as

$$T = ML^2T^{-2}, D = L, N = T^{-1}, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}$$

$\therefore$ Number of fundamental dimensions, m = 3

Number of π-term = n – m = 5 – 3 = 2

Hence equation (i) can be written as $f_1(\pi_1, \pi_2) = 0$

Each π-term contains m + 1 variable, i.e. 3 +1 = 4 variables. Three variables are repeating variables. Choosing D,N, ρ as repeating, the π-terms are

$$\pi_1 = D^{a_1}.N^{b_1}.\rho^{c_1}.T$$

$$\pi_2 = D^{a_2}.N^{b_2}.\rho^{c_2}.\mu$$

Dimensional Analysis of π_1

$$\pi_1 = D^{a_1}.N^{b_1}.\rho^{c_1}.T$$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1}.(T^{-1})^{b_1}.(ML^{-3})^{c_1}.ML^2T^{-2}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1 + 1 \quad \therefore\ c_1 = -1$

Power of L, $\quad 0 = a_1 - 3c_1 + 2, \quad \therefore a_1 = 3c_1 - 2 = -3 - 2 = -5$

Power of T, $\quad 0 = -b_1 - 2 \quad \therefore b_1 = -2$

Substituting the values of a_1, b_1 *and* c_1 *in* π_1,

$$\pi_1 = D^{-5}.N^{-2}.\rho^{-1}.T = \frac{T}{D^5N^2\rho}$$

Dimensional analysis of π_2

$$\pi_2 = D^{a_2}.N^{b_2}.\rho^{c_2}.\mu$$

Substituting dimension on both sides,

$$M^0L^0T^0 = L^{a_2}.(T^{-1})^{b_2}.(ML^{-3})^{c_2}.ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_2 + 1 \quad \therefore\ c_2 = -1$

Power of L, $\quad 0 = a_2 - 3c_2 - 1 \quad \therefore a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of T, $\quad 0 = -b_2 - 1 \quad \therefore b_2 = -1$

Substituting the values of a_2, b_2 *and* c_2 *in* π_2,

$$\pi_2 = D^{-2}.N^{-1}.\rho^{-1}.\mu = \frac{\mu}{D^2N\rho}$$

Substituting the values of π_1, π_2 in equation (ii),

$f_1\left(\frac{T}{D^2N^2\rho}, \frac{\mu}{D^2N\rho}\right) = 0 \quad$ or $\quad \frac{T}{D^2N^2\rho} = \phi\left(\frac{\mu}{D^2N\rho}\right)$

Or $\quad T = D^5N^2\rho\ \phi\left[\frac{\mu}{D^2N\rho}\right]. \quad$ **Ans**

Problem 2.15. Using Buckingham's π-theorem, shown that the discharge Qconsumed by an oil ring is given by

$$Q = Nd^3\phi\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{\rho N^2 d^2}, \frac{w}{\rho N^2 d}\right]$$

Where d is the internal diameter of the ring, N is rotational speed, ρ is density, μ is viscosity, σ is surface tension and w is the specific weight of oil.

Sol. Given: $Q = f(d, N, \rho, \mu, \sigma, w)$ or $f_1(Q, d, N, \rho, \mu, \sigma, w) = 0$ …(i)

∴ Total number of variables, n=7

Dimensions of each variables are

$Q = L^3T^{-1}$, d = L, $N = T^{-1}$, $\rho = ML^{-3}$, $\mu = ML^{-1}T^{-1}$, $\sigma = MT^{-2}$, w= $ML^{-2}T^{-2}$

∴ Total number of fundamental dimensions, m = 3

∴ Total number of π- terms = n – m = 7 – 3 = 4

∴ Equation (i) becomes as $f_1(\pi_1,\ \pi_2,\ \pi_3,\ \pi_4) = 0$

Choosing d, N, ρ as repeating variables, the π-term are

$$\pi_1 = d^{a_1}.N^{b_1}.\rho^{c_1}.Q$$

$$\pi_2 = d^{a_2}.N^{b_2}.\rho^{c_2}.\mu$$

$$\pi_3 = d^{a_3}.N^{b_3}.\rho^{c_3}.\sigma$$

$$\pi_4 = d^{a_4}.N^{b_4}.\rho^{c_2}.w$$

First π-term $\pi_1 = d^{a_1}.N^{b_1}.\rho^{c_1}.Q$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1}.(T^{-1})^{b_1}.(ML^{-3})^{c_1}.L^3T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_1$ $\therefore\ c_1 = 0$

Power of L, $0 = a_1 - 3c_1 + 3$ $\therefore a_1 = 3c_1 - 3 = -3$

Power of T, $0 = -b_1 - 1$ $\therefore b_1 = -1$

Substituting $a_1, b_1, c_1 in\, \pi_1$, $\pi_1 = d^{-3}.N^{-1}.\rho^0.Q = \frac{Q}{d^3N}$

Second π-term $\pi_2 = d^{a_2}.N^{b_2}.\rho^{c_2}.\mu$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2}.(T^{-1})^{b_2}.(ML^{-3})^{c_2}.ML^{-1}T^{-1}$$

Equating powers of M, L, T on both sides,

Power of M, $0 = c_2 + 1$ $\therefore\ c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1$ $\therefore a_2 = -3 + 1 = -2$

Power of T, $0 = -b_2 - 1$ $\therefore b_2 = -1$

Substituting the values of $a_2, b_2, c_2 in\ \pi_2$

$$\pi_2 = d^{-2}.N^{-1}.\rho^{-1}.\mu = \frac{\mu}{d^2N\rho} or \frac{\mu}{\rho N d^2}$$

Third π-term $\pi_3 = d^{a_3}.N^{b_3}.\rho^{c_3}.\sigma$

Substituting dimensions on both sides,

Power of M $0 = c_3 + 1$ $\therefore\ c_3 = -1$

Power of L, $0 = a_3 - 3c_3,$ $\therefore a_2 = 3c_3 = -3$

Power of T, $0 = -b_3 - 1$ $\therefore b_3 = -1$

Substituting the values of $a_3, b_3, c_3 in\ \pi_3,$

$$\pi_3 = d^{-3}.N^{-2}.\rho^{-1}.\sigma = \frac{\sigma}{d^3N^2\rho}$$

Fourth π-term

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_4}.(T^{-1})^{b_4}.(ML^{-3})^{c_4}.ML^{-2}T^{-2}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_4 + 1$ $\therefore\ c_4 = -1$

Power of L, $0 = a_4 - 3c_4 - 2,$ $\therefore a_4 = 3c_4 + 2 = -3 + 2 = -1$

Power of T, $0 = -b_4 - 2$ $\therefore b_4 = -2$

Substituting the values of $a_4, b_4, c_4 in\ \pi_4,$

$$\pi_4 = d^{-1}.N^{-2}.\rho^{-1}.w = \frac{w}{dN^2\rho}$$

Now substituting the values of $\pi_1, \pi_2, \pi_3, \pi_4$ in (ii),

$f\left(\frac{Q}{d^3N}, \frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3N^2}, \frac{w}{dN^2\rho}\right) = 0$ or $\frac{Q}{d^3N} = f_1\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3N^2}, \frac{w}{dN^2\rho}\right]$

Or $\frac{Q}{d^3N} = d^3N\phi\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3N^2}, \frac{w}{dN^2\rho}\right].$ Ans.

Problem 2.16*:* A thin rectangular plate having a width of w and height h is located so that it is normal to a moving stream of fluid. Assume the drag force D exerted by the fluid on the plate is a function of w and h, the fluid viscosity and density, μ and ρ respectively and the velocity V of the fluid approaching the plate. Determine a suitable set of π – terms to study this problem experimentally.

D = $f(w, h, \mu, \rho, V)$

This equation expresses the general functional relationship between the drag and the several variables that will affect it. The dimensions of the variables (using the MLT system)

$D = MLT^{-2}$
$w = L$
$h = L$
$\mu = ML^{-1}T^{-1}$
$\rho = ML^{-3}$
$V = LT^{-1}$

We see that all three basic dimensions are required to define the six variables so that the Buckingham pi theorem tells us that three pi terms will be needed (six variables minus three reference dimensions, k - r = 6 - 3)

We will next select three repeating variables such as *w, V, ρ.* A quick inspection of these three reveals that they are dimensionally independent, since each one contains a basic dimension not included in the others. Note that it would be incorrect to use both *w* and *h* as repeating variables since they have the same dimensions.

Starting with the dependent variable *D*, the first pi term can be formed by combining *D* with the repeating variables such that

$\Pi_1 = Dw^a V^b \rho^c$

And in terms of dimensions

$(MLT^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c = M^0 L^0 T^0$

Thus, for Π_1 to be dimensionless it follows that

1 + c = 10 (for M)

1 + a + b – 3c = 0 (for L)

-2 – b = 0 (for T)

And therefore, a= -2, b= -2 and c= -1. The pi term then becomes

$$\Pi_1 = \frac{D}{w^2 V^2 \rho}$$

Next the procedure is repeated with the second nonrepeating variable, h, so that

$\Pi_2 = hw^a V^b \rho^c$

It follows that

$(L)(L)^a (LT^{-1})^b (ML^{-3})^c = M^0 L^0 T^0$

c = 0 (for M)

1 + a + b – 3c = 0 (for L)

b=0 (for T)

so that a = -1, b = 0, c=0, and therefore

$$\Pi_2 = \frac{h}{w}$$

The remaining nonrepeating variable is μ so that

$\Pi_3 = \mu w^a V^b \rho^c$ with

$(ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c = M^0 L^0 T^0$ and therefore

1+c=0 (for M)

-1 + a + b - 3c = 0 (for L)

-1 – b = 0 (for T)

Solving for the exponents we obtain a = -1, b = -1, c = -1 so that

$$\Pi_3 = \frac{\mu}{wV\rho}$$

Now that we have the three required pi terms we should check to make sure they are dimensionless. To make this check we use *F,L, and T*, which will also verify the correctness of the original dimensions used for the variables. Thus,

$$\Pi_1 = \frac{D}{w^2V^2\rho} = \frac{(F)}{(L^2)(LT^{-1})(FL^{-4}T^2)} = F^0L^0T^0$$

$$\Pi_2 = \frac{h}{w} = \frac{L}{L} = F^0L^0T^0$$

$$\Pi_3 = \frac{\mu}{wV\rho} = \frac{FL^{-2}T}{(L)(LT^{-1})(FL^{-4}T^2)} = F^0L^0T^0$$

If these do not check, go back to the original list of variables and make sure you have thecorrect dimensions for each of the variables, and then check the algebra you used to obtain the exponents a, b, and c.

Finally, we can express the results for the dimensional analysis in the form

$$\frac{D}{w^2V^2\rho} = \phi\left(\frac{h}{w}, \frac{\mu}{wV\rho}\right)$$

2.5 MODEL ANALYSIS

For predicting the performance of the hydraulic structures (such as dams, spill ways etc.), before actually constructing or manufacturing, models of the structures or machines are, made and tests are performed on them to obtain the desired information.

The **model** is a small scale replica of the actual structure or system or machine. The actual system or machine is called **Prototype.** It is not necessary that the models should be smaller than the prototypes (In most cases it is), they may be larger than the prototype. The study of models of actual machines is called model analysis. **Model analysis** is an experimental method of finding solution of complex flow problems. Exact analytical solutions are possible only for a limited number of flow problems. The following are the advantages of the dimensional and model analysis:

1. The performance of the actual fluid system can be easily predicted in advance from its model.
2. With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests of the model.
3. The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be finally adopted.
4. The tests performed on the models can be utilized for obtaining, in advance, useful information about the performance of the prototypes only if a complete similarity exists between the model and the prototype.

2.6 SIMILITUDES

Similitude is defined as the similarity between the model and its prototype in every respect, which means that the model and prototype have similar properties or the model and prototype are completely similar. Three types of similarities must exist between the model and prototype. They are:

1. Geometric similarity 2. Kinematic similarity, and 3.Dynamic similarity

1 Geometric Similarity:

The geometric similarity is said to exist between the model and the prototype, if the ratio of all corresponding linear dimensions in the model and prototype are equal.

Let, L_m =length of model b_m=Breadth of model

D_m=Diameter of model A_m = area of model

$\forall_m$= Volume of model

And L_p,D_p, b_p, A_p, $\forall_p$ = corresponding values for the prototype.

For geometric similarity between model and prototype, we must have the relation,

$$\frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{D_p}{D_m} = L_r \qquad (2.6)$$

Where L_r is called the length ratio.

For area's ratio and volume's ratio the relation should be as given below:

$$\frac{A_p}{A_m} = \frac{L_p \times b_p}{L_m \times b_m} = L_r \times L_r = {L_r}^2 \quad (2.7)$$

And $$\frac{\forall_p}{\forall_m} = \left(\frac{L_p}{L_m}\right)^3 = \left(\frac{b_p}{b_m}\right)^3 = \left(\frac{D_p}{D_m}\right)^3 = {L_r}^3 \quad (2.8)$$

2. **Kinematic Similarity**

Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios of the velocity and acceleration at the corresponding points in the model and at the corresponding points in the prototype are the same. Since velocity and acceleration are vector quantities, hence not only the ratio of magnitude of velocity and acceleration at the corresponding points in the model and prototype should be the same; but the directions of velocity and acceleration at the corresponding points in the model and prototype also should be parallel.

Let V_{p1}= Velocity of fluid at point 1 in prototype,

V_{p2}= Velocity of fluid at point 2 in prototype,

a_{p1} = Acceleration of fluid at point 1 in prototype,

a_{p2}= Acceleration of fluid at point 2 in prototype, and

$V_{m_1}, V_{m_2}, a_{m_1}, a_{m_2}$ = Corresponding values at the corresponding points of fluid velocity and acceleration in the model

For kinematic similarity, we must have

$$\frac{V_{p_1}}{V_{m_1}} = \frac{V_{p_2}}{V_{m_2}} = V_r, \quad (2.9)$$

Where V_r is the Velocity ratio

For acceleration, we must have $$\frac{a_{p_1}}{a_{m_1}} = \frac{a_{p_2}}{a_{m_2}} = a_r \ , \quad (2.10)$$

Where $a_r$is the acceleration ratio.

Also, the directions of the velocities in the model and prototype should be the same.

3. **Dynamic Similarity.** Dynamic similarity means the similarity of forces between model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratio of the corresponding forces acting at the corresponding points are equal. Also the directions of the corresponding forces at the corresponding points should be the same.

   Let $(F_i)_p$ = Inertia force at a point in prototype

$(F_v)_p$ = Viscous force at a point in the prototype

$(F_g)_p$ = Gravity force at a point in the prototype

And $(F_i)_m$, $(F_v)_m (F_g)_m$ = Corresponding values of forces at the corresponding points in the model

Then for dynamic similarity, we have

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_v)_p}{(F_v)_m} = \cdots = F_r \,. \qquad \text{Where } F_r \text{ is the force ratio.}$$

Also the directions of the corresponding forces at the corresponding points in the model and prototype should be the same.

2.7 TYPES OF FORCES ACTING IN A MOVING FLUID

For the fluid flow problems, the forces acting on a fluid mass may be any one, or a combination of the several of the following forces:

1. Inertia forces, Fi
2. Viscous force, Fv
3. Gravity force, Fg
4. Pressure force, Fp
5. Surface tension forces, Fs
6. Elastic force, Fe

1. **Inertia force** (Fi). It is equal to the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration. It always existing in the fluid flow problems.
2. **Viscous force**, (Fv). It is equal to the product of shear stress (τ) due to viscosity and surface area of the flow. It is present in flow where viscosity is having an important role to play.
3. **Gravity force** (Fg); It is equal to the product of mass and acceleration due to gravity of the flowing fluid. It is present in case of open surface flow.
4. **Pressure force** (Fp). It is equal to the product of pressure intensity and cross-sectional area of the flowing fluid. It is present in case of pipe-flow.
5. **Surface tension force (Fs).** It is equal to the product of surface tension and length of surface of the flowing fluid.
6. **Elastic force (Fe).** It is equal to the product of elastic stress and area of the flowing fluid.

For flowing fluid, the above mentioned may not all always be present. And so the forces which are present in a fluid flow problem, are not of equal magnitude. There are always one or two forces which dominate the other forces. These dominating forces govern the flow of fluid.

2.8 DIMENSIONLESS NUMBERS

Dimensionless numbers are those numbers which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force. As this is a ratio of one force to the other force, it will be a dimensionless number. These dimensionless numbers are also called non-dimensional parameters. The following are the important dimensionless numbers:

1. Reynold's number, Re
2. Froude's number, Fe
3. Euler's number, Eu
4. Weber's number, We
5. Mach's number, M

2.6.1 **Reynold's Number(Re).** It is defined as the ratio of inertia force of a flowing fluid to the viscous force of the fluid. The expression for Reynold's number is obtained as

Inertia force (F_i) = Mass × Acceleration of flowing fluid.

$$= \rho \times Volume \times \frac{Velocity}{Time} = \rho \times \frac{Volume}{Time} \times Velocity$$

$$= \rho \times AV \times V \qquad \{\therefore Volume\ per\ sec. = Area \times Velocity = A \times V\}$$

$$= \rho AV^2 \qquad (2.11)$$

Viscous force (Fv) = shear stress × Area $\left\{\therefore \tau = \mu \frac{du}{dy} \ \therefore Force = \tau \times Area\right\}$

$$= \tau \times A$$

$$= \left(\mu \frac{du}{dy}\right) \times A = \mu . \frac{V}{L} \times A \qquad \left\{\therefore \frac{du}{dy} = \frac{V}{L}\right\}$$

By definition, Reynold's number

$$Re = \frac{F_i}{F_v} = \frac{\rho AV^2}{\mu \frac{V}{L} \times A} = \frac{\rho VL}{\mu}$$

$$= \frac{V \times L}{(\mu/\rho)} = \frac{V \times L}{v} \qquad \left\{\because \frac{\mu}{\rho} = v = \right.$$

$$\left. Kinematic\ viscosity \right\}$$

In case of pipe flow, the linear dimension L is taken as diameter d. Hence Reynold's number for pipe flow,

$$\text{Re} = \frac{V \times d}{v} \qquad \text{or} \qquad \frac{\rho V d}{\mu}$$

…(2.12)

2.8.1 Froude's Number (Fe).

The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as

$$Fe = \sqrt{\frac{F_i}{F_g}}$$

Where F_i from equation (2.11) $= \rho A V^2$

And $F_g = Force\ due\ to\ gravity$

$$= \text{Mass} \times \text{Acceleration due to gravity}$$

$$= \rho \times Volume \times g = \rho \times L^3 \times g \qquad \{\because Volume = L^3\}$$

$$= \rho \times L^2 \times L \times g = \rho \times A \times L \times g \qquad \{\because L^2 = A = Area\}$$

$$\therefore \qquad Fe = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}} \qquad (2.13)$$

2.8.2 Euler's Number (E_u).

It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as

$$E_u = \sqrt{\frac{F_i}{F_p}}$$

Where F_p = intensity of pressure ×Area = p × A

And $F_i = \rho A V^2$

$$\therefore \qquad E_u = \sqrt{\frac{\rho A V^2}{p \times A}} = \sqrt{\frac{V^2}{p/\rho}} \frac{V}{\sqrt{p/\rho}} \qquad (2.14)$$

2.8.3 $\textbf{Weber}'s$ **number** (**We**).

It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

Weber's Number, $$We = \sqrt{\frac{F_i}{F_s}} = \sqrt{\frac{inertia\ force}{surface\ tension}}$$

Where F_i = Inertia force = $\rho A v^2$

And F_s= Surface tension force

= Surface tension per unit length × Length = $\sigma \times L$

$$\therefore \quad We = \sqrt{\frac{\rho AV^2}{\sigma L}} = \sqrt{\frac{\rho L^2 \times V^2}{\sigma \times L}} \qquad \{\therefore A = L^2\}$$

$$= \sqrt{\frac{\rho V^2 L}{\sigma}} = \sqrt{\frac{V^2}{\sigma/\rho L}} = \frac{V}{\sqrt{\sigma/\rho L}} \qquad (2.15)$$

2.8.4 **Mach's Number (M).**

Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. Mathematically, it is defined as

$$M = \sqrt{\frac{inertia\ force}{Elastic\ force}} = \sqrt{\frac{F_i}{F_e}}$$

Where $F_i = \rho AV^2$

And F_e = Elastic force = Elastic stress × Area

$= K \times A = = K \times L^2$ $\qquad \{\because K = Elastic\ stress\}$

$$\therefore M = \sqrt{\frac{\rho AV^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 \times V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}$$

But $$\sqrt{\frac{K}{\rho}} = C = velocity\ of\ sound\ in\ the\ fluid$$

$$\therefore M = \frac{V}{C}$$

2.9 **MODEL LAWS**

For dynamic similarity between the model and the prototype, the ratio of the corresponding forces acting at the corresponding points in the model and the prototype should be equal. The ratios of the forces are dimensionless numbers. It means for dynamic similarity between the model and prototype, the dimensionless numbers should be the same for the model and the prototype. But it is quite difficult to satisfy the condition that all the dimensionless numbers (i.e., Re, Fe, We, Eu and M) are the same for the model to the prototype. Hence models are designed on the basis of ratio of the force, which is dominating in the phenomenon. The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The following are the model laws:

1. Reynold's model law
2. Froude model law
3. Euler model law
4. Weber model law
5. Mach model law

2.9.1 **Reynold's Model Law.**

Reynold's model law is the law in which models are based on Reynold's number. Models base on Reynold's number include:

i) Pipe flow, boundary layers, jets, wakes, etc…

ii) Resistance experienced by sub-marines, airplanes or fully immersed bodies etc.

As defined earlier that Reynold's number is the ratio of inertia force to viscous force, and hence fluid flow problems where viscous alone are predominant, the models are designed for dynamic similarity on Reynold's law, which states that, the Reynolds number for the model must be equal to the Reynold number for the prototype

Let V_m= Velocity of fluid in model

ρ_m= Density of fluid in model

L_m= Length or linear dimension of the model

μ_m= Viscosity of fluid in model

and V_p, ρ_p, L_p, μ_p are the corresponding values of velocity, density, linear dimension and viscosity of fluid in the prototype. Then according to Reynold's model laws,

$$(Re)_m = (Re)_p \qquad \text{or} \qquad \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \qquad ...(2.16)$$

$$\frac{\rho_p.V_p.L_p}{\rho_m.V_m.L_m} \times \frac{1}{\frac{\mu_P}{\mu_m}} = 1 \qquad \text{or} \qquad \frac{\rho_r.V_r.L_r}{\mu_r} = 1$$

$\left\{where\ \rho_r = \frac{\rho_p}{\rho_m}, = V_r = \frac{V_p}{V_m} and = L_r = \frac{L_p}{L_m}, \frac{\mu_P}{\mu_m} = \mu_r\right\}$ And also ρ_r, V_r, L_r, μ_r are called the scale ratios for density, velocity, linear dimension and viscosity. The scale ratio for time, acceleration, force and discharge for Reynold's model law are obtain as

$$t_r = Time\ scale\ ratio = \frac{L_r}{V_r} \qquad \left\{\therefore V = \frac{L}{t} \therefore t = \frac{L}{V}\right\}$$

$$a_r = accleration\ scale\ ratio = \frac{V_r}{t_r}$$

$$F_r = Force\ scale\ ratio = (Mass \times Acceleration)_r$$
$$= m_r \times a_r = \rho_r A_r V_r \times a_r$$
$$= \rho_r {L_r}^2 V_r \times a_r \text{and}$$
$$Q_r = \text{Discharge scale ratio} = (\rho AV)_r$$
$$= \rho_r A_r V_r = \rho_r . {L_r}^2 . V_r$$

Problem 2.17. A pipe of diameter 1.5m is required to transport an oil of specific gravity 0.9 and viscosity 3×10^{-2} poise at the rate of 3000 L/s. Tests were conducted on a 15cm diameter pipe using water at 20^0C. Find the velocity and rate of flow in the model. Take viscosity of water at 20^0C =0.01 poise.

Solution

Given

Dia. of prototype $D_p = 1.5$ m

Viscosity of fluid, $\mu_p = 3 \times 10^{-2}$poise

Q for prototype, $Q_p = 3000$ litres/s $= 3.0\ m^3/s$

Sp. gr. of oil, $S_p = 0.9$

∴ Density of oil, $\rho_p = S_p \times 1000 = 0.9 \times 1000 = 900\ kg/m^3$

Diameter of model, $D_m = 15$ cm $= 0.15$ m

Viscosity of water at 20 ^{0}C = 0.01 poise = 1×10^{-2}poise or $\mu_m = 1 \times 10^{-2}$poise

Density of water or $\rho_m = 1000\ kg/m^3$

For pipe flow, the dynamic similarity will be obtained if the Reynold's number in the model and prototype are equal

Hence using equation (2.17), $\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p}$ {For pipe linear dimension is D}

$$\therefore \quad \frac{V_m}{V_p} = \frac{\rho_p}{\rho_m} . \frac{D_p}{D_m} . \frac{\mu_m}{\mu_p}$$

$$= \frac{900}{1000} \times \frac{1.5}{0.15} \times \frac{1\times10^{-2}}{3\times10^{-2}} = \frac{900}{1000} \times 10 \times \frac{1}{3} = 3.0$$

But $V_p = \frac{Rate\ of\ flow\ in\ prototype}{area\ of\ prototype} = \frac{3}{\frac{\pi}{4}{D_p}^2} = \frac{3}{\frac{\pi}{4}(1.5)^2} = \frac{3\times4}{\pi\times2.25} = 1.698\ m/s$

∴ $V_m = 3 \times V_p = 3 \times 1.698 = \mathbf{5.091\ m/s.}$ **Ans.**

Rate of flow through model, $Q_m = A_m \times V_m = \frac{\pi}{4}(D_m)^2 \times V_m = \frac{\pi}{4}(0,15)^2 \times 5.091$

$$= 0.0899\ m^3/s = 0.0899 \times 1000\ lit/s = \mathbf{89.9\ lit/s}$$

Problem 2.18. Water is flowing through a pipe of diameter 30 cm at a velocity of 4 m/s. Find the velocity of oil flowing in another pipe of diameter 10 cm, if the condition of dynamic similarity is satisfied between the two pipes. The viscosity of water and oil is given as 0.01 poise and 0.025 poise. Take the specific gravity of oil = 0.8

Solution

Given

Two pipes have different liquids

Let for pipe 1, Liquid = Water

Diameter of pipe, d_1 = 30 cm = 0.3 m

Velocity of flow, V_1 = 4 m/s

Viscosity, μ_1 = 0.01 poise = $\frac{0.01}{10}$

Density, ρ_1 = 1000 kg/m^3

For pipe 2, Liquid = Oil

Diameter of pipe, d_2 = 10 cm = 0.1 m

Velocity of flow, V_2= ?

Viscosity, μ_2 = 0.025 poise = $\frac{0.025}{10}$

Density, ρ_1 = 1000 kg/m^3

Specific gravity of oil = 0.8

∴ Density, ρ_p = 0.8 × 1000 = 800 kg/m^3

If the pipes are dynamically similar, the Reynold's number for both pipes should be the same.

∴ $\frac{\rho_1 V_1 d_1}{\mu_1} = \frac{\rho_2 V_2 d_2}{\mu_2}$ or $V_2 = \frac{\rho_1}{\rho_2} . \frac{d_1}{d_2} . \frac{\mu_2}{\mu_2} V_1$

$$= \frac{1000}{800} \times \frac{0.3}{0.1} \times \frac{\frac{0.025}{10}}{\frac{0.01}{10}} \times 4.0 = \frac{1000}{800} \times 3 \times \frac{0.025}{0.01} \times 4$$

= $\mathbf{37.5}\ \boldsymbol{m/s}$ **Ans.**

Problem 2.19. The ratio of length of a sub-marine and its model 30:1. The speed of sub-marine (prototype) is 10 m/s. The model is to be tested in a wind tunnel. Find the speed of air in the wind tunnel. Also determine the ratio of the drag (resistance) between the model and its prototype. Take the value of kinematic viscosity of sea water and air as 0.12 stokes and 0.016 stokes respectively. The densities for sea-water and air are given as 1030 kg/m^3 and 1.24 kg/m^3 respectively.

Solution

Given

Prototype (sub-marine) and its model.

For prototype, Speed $V_p = 10$ m/s

Fluid = Sea-water

Kinematic viscosity $\nu_p = 0.012$ stokes = 0.012 cm^2/s

$= 0.012 \times 10^{-4}$ m^2/s $\{\therefore 1 \text{ stoke} = 1\ cm^2/s\}$

Density, $\rho_p = 1030$ kg/m^3

For model Fluid = Air

Kinematic viscosity, $\nu_m = 0.016$ stokes = 0.012 cm^2/s = 0.016×10^{-4} m^2/s

Density, $\rho_m = 1.24$ kg/m^3

Also $\frac{Length\ of\ prototype}{Length\ of\ model} = \frac{L_p}{L_m} = 30.0$

Let the velocity of air in the model = V_m

For dynamic similarity between model and sub-marine, the viscous resistance is to be overcome and hence for fully submerged sub-marine, the Reynold's number for model and prototype should be the same.

$$\therefore \quad \frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p} \text{ or } \frac{V_p D_p}{(\mu/\rho)_p} = \frac{V_m D_m}{(\mu/\rho)_m}; \frac{V_p D_p}{v_p} = \frac{V_m D_m}{v_m}$$

$$\therefore \quad V_m = \frac{v_m}{v_p} \times \frac{D_p}{D_m} \times V_p$$

$$= \frac{0.016 \times 10^{-4}}{0.012 \times 10^{-4}} \times 30 \times 10 \qquad \left\{\therefore \frac{D_p}{D_m} = \frac{L_p}{L_m} = 30\right\}$$

$$= \frac{0.016}{0.012} \times 30 \times 10 = 400\ m/s$$

Ratio of drag force (resistance):

Drag force = Mass × Acceleration

$$= \rho L^3 \times \frac{V}{t} = \rho . L^2 . \frac{L}{t} \times V = \rho L^2 V^2 \qquad \left\{\therefore \frac{L}{t} = V\right\}$$

Let F_p and F_m denote the drag force for the prototype and for the model respectively then,

$$\frac{F_p}{F_m} = \frac{\rho_p {L_p}^2 {V_p}^2}{\rho_m {L_m}^2 {V_m}^2} = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2$$

$$= \frac{1030}{1.24} \times 30^2 \times \left(\frac{10}{400}\right)^2 = \mathbf{467.22} \qquad \textbf{Ans}$$

Problem 2.20. A ship 300 m long moves in sea-water, whose density 1030 kg/m^3. A 1: 100 model of this ship is to be tested in a wind tunnel. The velocity of air in the wind tunnel around the model is 30m/s and the resistance of the model is 60 N. Determine the velocity of the ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as 1.24 kg/m^3. Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively.

Solution

Given:

For prototype

Length, L_p= 300m

Fluid = Sea-water

Density of water = 1030 kg/m^3

kinematic viscosity, v_p= 0.012 stokes= $0.012\times 10^{-4}\ m^2/s$

Let velocity of ship =V_p

Resistance =F_p

For model: Length, $L_m = \frac{1}{100}\times 300 = 3\ m$

Velocity, $V_m = 30\ m/s$

Resistance, $F_m = 60N$

Density of air, ρ_m= 1.24 kg/m^3

Kinematic viscosity of air, $v_m = 0.018\ stokes = 0.018\times 10^{-4}\ m^2/s.$

For dynamic similarity between the prototype and its model, Reynolds number for both of them should be equal.

$$\therefore \quad \frac{V_p L_p}{v_p} = \frac{V_m L_m}{v_m} \quad \text{Or} \quad V_p = \frac{v_p}{v_m}\times\frac{L_m}{L_p}\times V_m$$

$$= \frac{0.012\times10^{-4}}{0.018\times10^{-4}}\times\frac{3}{300}\times 30 = \frac{1}{1.5}\times\frac{1}{100}\times 30 = \mathbf{0.2\ m/s} \quad \textbf{Ans.}$$

Resistance = Mass$\times$ Acceleration

$$= \rho L^3\times\frac{V}{t} = \rho L^2\times\frac{V}{1}\times\frac{L}{t} = \rho L^2 V^2$$

Then $\frac{Fp}{Fm} = \frac{(\rho L^2V^2)_p}{(\rho L^2V^2)_m} = \frac{\rho_p}{\rho_m}\left(\frac{L_p}{L_m}\right)^2\left(\frac{V_p}{V_m}\right)^2$

But $\frac{\rho_p}{\rho_m} = \frac{1030}{1.24}$

$$\therefore \quad \frac{F_p}{F_m} = \frac{1030}{1.24}\times\left(\frac{300}{3}\right)^2\times\left(\frac{0.2}{30}\right)^2 = 369.17$$

$$\therefore \quad F_p = 369.17\times F_m = 369.17\times 60 = \mathbf{22150.2\ N} \quad \textbf{Ans}$$

2.9.1.1 Froude Model Law.

Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal. Froude model law is applicable when the gravity force is the only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems:

1. Free surface flows such as flow over spill ways , weirs, sluices, channels etc
2. Flow of jet from an orifice or nozzle
3. Where waves are likely to be formed on surface,
4. Where fluids of different densities flow over one another.

Let V_m = Velocity of fluid in model,

$L_m =$ Linear dimension of length of model,

$g_m =$ Acc. due to gravity at a place where model is tested,

And V_p, L_p and g_p are the corresponding values of the velocity, length and acceleration due to gravity for prototype. Then according to Froude model law,

$$(Fe)_{model} = (Fe)_{prototype} \quad \text{or} \quad \frac{Vm}{\sqrt{g_m L_m}} = \frac{Vp}{\sqrt{g_p L_p}} \qquad \ldots(2.18)$$

If tests on the model are performed at the same place where prototype is to operate, then $g_m = g_p$ and equation (2.18) becomes

$$\frac{Vm}{\sqrt{L_m}} = \frac{Vp}{\sqrt{L_p}} \qquad \ldots(2.19)$$

Or

$$\frac{V_p}{V_m} \, x \, \frac{1}{\sqrt{\frac{L_m}{L_p}}} = 1$$

$$\frac{V_p}{V_m} = \sqrt{\frac{L_p}{L_m}} = \sqrt{L_r} \qquad \ldots(2.20)$$

Where L_r = scale ratio for length

$\frac{V_p}{V_m}$ = V_r = scale ratio for velocity.

$$\therefore \quad \frac{V_p}{V_m} = V_r = \sqrt{L_r} \qquad \ldots(2.21)$$

Scale ratios for various physical quantities based on Froude model law are:

(a) **Scale ratio time,**

As time = $\frac{Length}{Velocity}$

Then the ratio of time for prototype and model is

$$T_r = \frac{T_p}{T_m} = \frac{\left(\frac{L}{V}\right)_P}{\left(\frac{L}{V}\right)_m} = \frac{\frac{L_p}{V_p}}{\frac{L_m}{V_m}} = \frac{L_p}{L_m}\ x\ \frac{V_m}{V_p} = L_p\ x\ \frac{1}{\sqrt{L_r}}$$

$$\because \left\{\frac{V_p}{V_m} = \sqrt{L_r}\right\}$$

(b) **Scale ratio for acceleration**

Acceleration $= \frac{V}{T}$

$\therefore$ $a_r = \frac{\left(\frac{V}{T}\right)_p}{\left(\frac{V}{T}\right)_m} = \frac{V_p}{T_p}.\frac{T_m}{V_m} = \frac{V_p}{V_m}.\frac{T_m}{T_p}$

$= \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} \qquad = 1$ …….(2.22)

(c) **Scale ratio for discharge**

$$Q = AV = L^2 \times \frac{L}{T} = \frac{L^3}{T}$$

$$Q_r = \frac{Q_p}{Q_m} = \frac{\left(\frac{L^3}{T}\right)_p}{\left(\frac{L^3}{T}\right)_m} = \left(\frac{L_p}{L_m}\right)^3.\left(\frac{T_m}{T_p}\right) = {L_r}^3.\frac{1}{\sqrt{L_r}} = {L_r}^{2.5} \qquad (2.23)$$

(d) **Scale ratio for force**

As F$= mass \times acceleration = PL^3\ x\ \frac{V}{T} = PL^2.\frac{L}{T}.V = \rho L^2V^2$

$\therefore$ Ratio for force, $F_r = \frac{F_p}{F_m} = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2.$

If fluid used in model of prototype is the same, then

$\frac{\rho p}{\rho m} = 1$ or $\rho_p = \rho_m$

And hence $F_r = \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2 = {L_r}^2.\left(\sqrt{L_r}\right) = {L_r}^3.$ (2.24)

(e) **Scale ratio for pressure intensity**

As P$= \frac{force}{area} = \frac{\rho L^2V^2}{L^2} = \rho V^2$

$\therefore$ Pressure ratio, $P_r = \frac{P_p}{P_m} = \frac{\rho_P {V_P}^2}{\rho_m {V_m}^2}$

If the fluid is same, $\rho_m = \rho_p$

$$P_r = \frac{{V_P}^2}{{V_m}^2} = \left(\frac{V_P}{V_m}\right)^2 = L_r \qquad (2.25)$$

(f) **Scale ratio for work, energy, torque, moment, etc.**

Torque = Force x Distance = F x L

$\therefore$ Torque ratio, $T_r *= \frac{T_p*}{T_m*} = \frac{(F\times L)_p}{(F\times L)_m} = F_r \times L_r = {L_r}^3 \times L_r = {L_r}^4$ (2.26)

(g) Scale ratio for power

As Power = Work done / time

$$= \frac{F\times L}{T}$$

$\therefore$ Power ratio, $P_r = \frac{P_p}{P_m} = \frac{\frac{F_p \times L_p}{T_p}}{\frac{F_m \times L_m}{T_m}} = \frac{F_p}{F_m} \times \frac{L_p}{L_m} \times \frac{1}{\frac{T_p}{T_m}}$

$$= F_r . L_r . \frac{1}{T_r} = {L_r}^3 . L_r . \frac{1}{\sqrt{L_r}} = {L_r}^{3.5}. \qquad \ldots(2.27)$$

Problem 2.21. In 1 in 40 model of spillway, the velocity and discharge are 2m/s and 2.5m/s. Find the corresponding velocity and discharge in the prototype.

Solution

Given:

Scale ratio of length, $L_r = 40$

Velocity in model, $V_m = 2\, m/s$

Discharge in model, $Q_m = 2.5\, m^3/s$

Let V_p and Q_p are the velocity and discharge in prototype.

Using equation (2.20) for velocity ratio, $\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{40}$

$\therefore$ $V_p = V_m \times \sqrt{40} = 2\ x\ \sqrt{40} = \mathbf{12.65 m/s}$ **Ans**.

Using equation (2.23) for discharge ratio,

$$\frac{Q_p}{Q_m} = {L_r}^{2.5} = (40)^{2.5}$$

$\therefore$ $Q_p = Q_m\ x\ (40)^{2.5} = 2.5\ x\ (40)^{2.5} = \mathbf{25298.2\, m^3/s}$ **Ans**

Problem 2.22. A ship model of scale 1/50 is towed through sea water at a speed of 1 m/s. A force of 2 N is required to tow the model. Find the speed of ship and the propulsive force on the ship if the prototype is subjected to wave resistance only.

Solution

Given:

Scale ratio of length, $L_r = 50$

Speed of model, $V_m = 1\ m/s$

Force required for model, $F_m = 2\ N$

Speed of ship $= V_p$

And the propulsive force for ship $= F_p$

As prototype is subjected to wave resistance only for dynamic similarity, the Froude number should be same for model and prototype. Hence for velocity ratio, for Froude model law using equation (2.20), we have

$\therefore \quad \frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{50}$

$\therefore \quad V_p = \sqrt{50} \times V_m = \sqrt{50} \times 1 = \mathbf{7.071\ m/s\ Ans.}$

Force scale ratio is given by equation (2.24),

$\therefore \quad F_r = \frac{F_p}{F_m} = {L_r}^3$

$\therefore \quad F_p = F_m \times {L_r}^3 = 2 \times (50)^3 = \mathbf{250000N.}$ **Ans**

Problem 2.23. In the model test of a spillway the discharge and velocity of flow over the model were $2\ \mathrm{m^3/s}$ and $1.5\ \mathrm{m/s}$ respectively. Calculate the velocity and discharge over the prototype which is 36 times the model size.

Solution

Given:

Discharge over model, $Q_m = 2\, m^3/s$

Velocity over model, $V_m = 1.5\, m/s$

Linear scale ratio, $L_r = 36$

For dynamic similarity, Froude model is used. Using equation (2.20), we have

$$\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{36} = 6.0$$

$\therefore \quad V_p$ = velocity over prototype = $V_m\ x\ 6.0 = 1.5\ x\ 6.0 = 9\, m/s$

For discharge, using equation (12.23), we get

$$\frac{Q_p}{Q_m} = {L_r}^{2.5} = (36)^{2.5}$$

$\therefore \quad Q_p = Q_m\ x\ (36)^{2.5} = 2\ x\ 36^{2.5} = 15552\, m^3/s$

Problem 2.24. In a geometrically similar model of spillway the discharge per metre length is $\frac{1}{6}\mathrm{m^3/s}$. If the scale of the model is $\frac{1}{36}$, find the discharge per metre run of the prototype.

Solution

Given:

Discharge per metre length model, $q_m = \frac{1}{6} m^3/s$

Linear scale ratio, $L_r = 36$

Discharge per metre length for prototype, $q_p = ?$

The discharge ratio for spillway is given by the equation (12.23), $\frac{Q_p}{Q_m} = {L_r}^{2.5}$

But discharge ratio per metre length is given as

$$\frac{q_p}{q_m} = \frac{Q_p/L_p}{Q_m/L_m} = \frac{Q_p}{Q_m}\ x\ \frac{L_m}{L_p} = {L_r}^{2.5}\ x\ \frac{1}{L_r} = {L_r}^{1.5}$$

$\therefore$ $$q_p = q_m\ x\ {L_r}^{1.5} = \frac{1}{6}\ x\ (36)^{1.5} = \frac{1}{2}\ x\ 6^{2\ x\ 1.5}$$

$$= 6^{3-1} = 6^2 = 36\, m^3/s$$ Ans.

Problem 2.25. A spillway model is to be built to a geometrically similar scale of $\frac{1}{50}$ across a flame of 600 mm width. The prototype is 15 m high and maximum head on it is expected to be 1.5m. (i) What height of model and what head on the model should be used? (ii) if the flow over model at a particular head is 12 litres per second, what flow per metre length of the prototype is expected? (iii) if the negative pressure in the model is 200mm, what is the negative pressure in prototype? Is it possible?

Solution

Given:

Scale ratio for length, $L_r = 50$

Width of model, $B_m = 600mm = 0.6m$

Flow over model, $Q_m = 12\ litres/s$

Pressure in model, $h_m = -200mm\ of\ water = -.2m$

Height of prototype, $H_p = 15m$

Head of prototype, ${H_p}^* = 1.5m$

(i) Let the height of model $= H_m$

And head on model $= H_m *$

Linear scale ratio, $L_r = \frac{H_p}{H_m} = \frac{H_p*}{H_m*} = 50$

$\therefore$ Height of model, $H_m = \frac{H_p}{50} = \frac{15}{50} = \mathbf{0.3\ m.}$ **Ans.**

And height on model, ${H_m}^* = \frac{H_p*}{50} = \frac{1.50}{50} = \mathbf{0.03\ m}$ **Ans.**

Width of prototype, $B_p = L_r \times B_m = 50 \times 0.6 = 30\ m.$

(ii) Discharge ratio is given by equation (2.23) as

$$\frac{Q_p}{Q_m} = L_r{}^{2.5} = (50)^{2.5} = 17677.67$$

$\therefore$ $$Q_p = Q_m \times 17677.67 = 12 \times 17677.67$$

$$= 212132.04\ lit/s$$

Discharge per metre length of prototype = $\frac{Q_p}{length\ of\ prototype} = \frac{212132.04}{width\ of\ prototype}$

$$= \frac{212132.04}{30} = \mathbf{7071.078\ litres\ /s}$$ **. .Ans**

(iii) Negative pressure head in prototype,

$$h_p = L_r x\ h_m = 50(-0.2) = \mathbf{-10.0m}.$$ **Ans.**

Negative pressure is not practicable. Maximum practicable negative pressure head is -7.50 m of water.

Problem 2.26 In a 1 in 20 model of stilling basin, the height of the hydraulic jump in the model is observed to be 0.20 metre. What is the height of the hydraulic jump in the prototype? If the energy dissipated in the model is $\frac{1}{10}$kW, what is the corresponding value in prototype?

Solution

Given:

Linear scale ratio, $L_r = 20$

Height of hydraulic jump in model, $h_m = 0.20\ m$

Energy dissipated in model, $P_m = \frac{1}{10} kW$

(i) Let the height of hydraulic jump in the prototype = h_p

Then $$\frac{h_p}{h_m} = L_r = 20$$

$\therefore$ $$h_p = h_p\ x\ 20 = 0.20\ x\ 20 = 4\ m$$

(ii) Let the energy dissipated in prototype = F_p

Using equation (12.27) for power ratio, $\frac{P_p}{P_m} = L_r{}^{3.5} = 20^{3.5} = 35777.088$

$\therefore$ $$P_p = P_m\ x\ 35777.088 = \frac{1}{10}\ x\ 35777.088 = \mathbf{3577.708kW.}$$ **Ans**

Problem 2.27. The characteristics of the spillway are studied by means of a geometrically similar model constructed to the scale ratio of 1:10.

(i) If the maximum rate of flow of the prototype is 28.3 m^3/s, what will be the corresponding flow in model?

(ii) If the measured velocity in the model at appoint on the spillways is 2.4 m/s, what will be the corresponding velocity in prototype?

(iii) If the hydraulic jump at the foot of the model is 50 mm high, what will be the height of jump in prototype?

(iv) If the energy dissipated per second in the model is 3.5 Nm, what energy is dissipated per second in the prototype?

Solution

Given:

$$\frac{Linear\ dimension\ of\ model}{linear\ dimension\ of\ prototype} = \frac{1}{10}$$

Scale ratio, $L_r = 10.$

(i) Discharge in prototype, $Q_P = 28.3\, m^3/s$

Let Q_m = Discharge in model

For discharge using equation (2.23), we get

$$\frac{Q_p}{Q_m} = L_r^{\ 2.5}$$

∴ $Q_m = \frac{Q_p}{L_r^{\ 2.5}} = \frac{28.3}{10^{2.5}} = \mathbf{0.0895\ m^3/s.}$ **Ans**

(ii) Velocity in the model, $V_m = 2.4\, m/s$

Let V_p =Velocity in the prototype

For velocity using equation (12.20), we get

$$\frac{V_p}{V_m} = \sqrt{L_r}$$

∴ $V_p = V_m\ x\ \sqrt{L_r} = 2.4\ x\ \sqrt{10} = \mathbf{7.589\ m/s.}$

Ans

(iii) Hydraulic jump in model, $H_m = 50mm$

Let H_p = Hydraulic jump in prototype

New scale ratio $= \frac{H_p}{H_m}$

∴ $H_p = H_m x$ Scale ratio $= 50\ x\ 10 = \mathbf{500\ mm}$. **Ans**.

(iv) Energy dissipated/s in model, $E_m = 3.5 Nm/s$

Let E_p = Energy dissipated/s in prototype

Now using equation (2.27), we get $\frac{E_p}{E_m} = L_r^{\ 3.5}$

$\therefore$ $$E_p = E_m \, x \, {L_r}^{3.5} = 3.5 \, x \, 10^{3.5} = \mathbf{11067.9}\boldsymbol{Nm/s}. \quad \textbf{Ans}$$

Problem 2.28. A 1:64 model is constructed of an open channel in concrete which has Manning's N= 0.014. Find the value of N for the model.

Solution

Given:

Linear scale ratio, $L_r = 64$

Value of N for prototype, $N_P = 0.014$

Let N_m = value of N for model.

The Manning's formula * is given by, $V = \frac{1}{N} m^{3/2} . i^{1/2}$

In which m = Hydraulic mean depth in m

i = Slope of the bed of the channel

Now for the model, the Manning's formula is written as

$$V_m = \frac{1}{N_m} . (m_m)^{2/3} . (i_m)^{1/2}$$

…(i)

And for the prototype, the Manning's formula is written as

$$V_p = \frac{1}{N_p} . \left(m_p\right)^{2/3} . \left(i_p\right)^{1/2} \quad \text{…(ii)}$$

Dividing equation(ii) by (i), we get

$$\frac{V_p}{V_m} = \frac{\frac{1}{N_p} . (m_p)^{2/3} . (i_p)^{1/2}}{\frac{1}{N_m} . (m_m)^{2/3} . (i_m)^{1/2}} = \frac{N_m}{N_p} . \left(\frac{m_p}{m_p}\right)^{2/3} . \left(\frac{i_p}{i_m}\right)^{1/2} \quad \text{…(iii)}$$

For dynamic similarity, Froude model law is used. Using equation (12.20), we have

$$\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{64} = 8$$

But $\frac{m_p}{m_p} = L_r$ and $\frac{i_p}{i_m} = 1$ as i_p and i_m are dimensionless.

Substituting these values in equation (iii), we get

$$8 = \frac{N_m}{N_p} \, x \, (L_r)^{2/3} \, x \, 1 = \frac{N_m}{0.014} \, x \, (64)^{2/3}$$

$$N_m = \frac{8 \, x \, 0.014}{64^{2/3}} = \frac{8 \, x \, 0.014}{16} = \mathbf{0.007}. \quad \textbf{Ans}$$

Problem 2.29. A 7.2 m high and 15m long spillway discharges 94 m^3/s discharge under a head of 2.0 m. if a 1:9 scale model of this spillway is to be constructed, determine model dimensions, head over spillway model and the model discharge. If model experiences a force of 7500N (764.53kgf), determine force on the prototype.

Solution

Given:

For prototype: height $h_p = 7.2\ m$

Length, $L_p = 15m$

Discharge, $Q_p = 94\ m^3/s$

Head, H_p=2.0m

Size of model = $\frac{1}{9}$ of the size of prototype.

∴ Linear scale ratio, $L_r = 9$

Force experienced by model, $F_p = 7500N$

Find: (i) Model dimensions i.e. height and length of model (h_m *and* L_m)

(ii) Head over model i.e. H_m

(iii)Discharge through model i.e. Q_m

(iv)Force on prototype (i.e. F_p)

(i) Model dimensions (h_m *and* L_m)

$$\frac{h_p}{h_m} = \frac{L_p}{L_m} = L_r = 9$$

∴ $$h_m = \frac{h_p}{9} = \frac{7.2}{9} = 0.8\ m.$$

And $$L_m = \frac{L_p}{9} = \frac{15}{9} = 1.67\ m.$$ Ans.

(ii) Head over model (H_m)

$$\frac{h_p}{H_m} = L_r = 9$$

∴ $$H_m = \frac{H_p}{9} = \frac{2}{9} = 0.222\ m.$$ Ans.

(iii) Discharge through model (Q_m)

Using equation (2.23), we get $\frac{Q_p}{Q_m} = {L_r}^{2.5}$

∴ $$Q_m = \frac{Q_p}{{L_r}^{2.5}} = \frac{94}{9^{2.5}} = \frac{94}{243} = 0.387\ m^3/s.$$ Ans

(iv) Force on the prototype (F_p)

Using equation (2.24), we get $F_r = \frac{F_p}{F_m} = {L_r}^3$

∴ $$F_p = F_m \times {L_r}^3 = 7500 \times 9^3 = 5467500N.$$ Ans

2.9.3. Euler's Model law. Euler's model law is the law in which the models are designed on Euler's number which means for dynamic similarity between the model and prototype, the Euler number for model and prototype should be equal. Euler's model law is applicable when the pressure force is the only predominant force in addition to the inertia force. According to this law:

$$(E_u)_{model} = (E_u)_{prototype} \tag{2.28}$$

If V_m = Velocity of fluid in model,

P_m = Pressure of fluid in model,

ρ_m =Density of fluid in model,

And V_p, P_p, ρ_p = Corresponding values in prototype, then

Substituting these values in equation (2.28), we get

$$\frac{V_m}{\sqrt{P_m/\rho_m}} = \frac{V_p}{\sqrt{P_p/\rho_p}} \tag{2.29}$$

If the fluid is the same in model and prototype, then equation (2.29) becomes

$$\frac{V_m}{\sqrt{P_m}} = \frac{V_p}{\sqrt{P_p}} \tag{2.30}$$

Euler's model law is applied for fluid flow problems where flow is taking place in a closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension force is absent. This law is also used where the phenomenon of cavitation takes place.

2.9.4. Weber Model Law. Weber model law is the law in which models are based on Weber's number, which is the ratio of the square root of inertia force to surface tension force. Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between the model and prototype is obtained by equating the Weber number of the model and its prototype. Hence according to this law:

$(W_e)_{model} = (W_e)_{prototype}$ where W_e is Weber's number and $= \frac{V}{\sqrt{\sigma/\rho L}}$

If V_m = Velocity of fluid in model,

σ_m = Surface tensile force in model,

ρ_m =Density of fluid in model,

L_m = Length of surface in model

And $V_p, \sigma_p, \rho_p, L_p$ = Corresponding values of fluid in prototype,

Then according to Weber's law, we have $\frac{V_m}{\sqrt{\sigma_m/\rho_m L_m}} = \frac{V}{\sqrt{\sigma_p/\rho_p L_p}}$ (2.31)

Weber model law is applied in following cases:

1. Capillary rise in narrow passages
2. Capillary movement of water in soil,
3. Capillary waves in channels;
4. Flow over weirs for small heads.

2.9.5. Mach Model law. Mach model law is the law in which models are designed on Mach number, which is the ratio of square root of inertia force to elastic force of a fluid. Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the Mach number of the model and its prototype. Hence according to this law:

$$(M_e)_{model} = (M_e)_{prototype}$$

Where M = Mach number = $\frac{V}{\sqrt{K/\rho}}$

If V_m = Velocity of fluid in model,

K_m = Elastic stress for model,

ρ_m =Density of fluid in model,

And V_p, K_p, ρ_p = Corresponding values of prototype. Then according to Mach law,

$$= \frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V}{\sqrt{K_p/\rho_p}} \qquad (2.32)$$

Mach model law is applied in the following cases:

1. Flow of aero plane and projectile through air at supersonic speed, i.e., at a velocity more than the velocity of sound,
2. Aerodynamic testing,
3. Under water testing of torpedoes,
4. Water-hammer problems.

Problem 2.30. The pressure drop in an airplane model of size $\frac{1}{10}$ of its prototype is 80 N/cm^2. The model is tested in water. Find the corresponding pressure drop in the prototype. Take density of air = 1.24kg/m^3. The viscosity of water is 0.01 poise while the viscosity of air is 0.00018 poise.

Solution

Given:

Pressure drop in model, $P_m = 80\, N/cm^2 = 80\, x\, 10^4\, N/m^2$

Linear scale ratio, $L_r = 40$

Fluid in model, $= Water, while\ in\ prototype = Air$

Viscosity of water, $\mu_m = 0.01$ poise

Density of water, $\rho_m = 1000 kg/m^3$

Viscosity of air, $\mu_p = 0.00018$ poise

Density of air, $\rho_p = 1.24 kg/m^3$

Let the corresponding pressure drop in prototype = P_p.

As the problem involves pressure force and viscous force and hence for dynamic similarity between the model and prototype, Euler's number and Reynold's number should be considered. Making first of all Reynold's number equal, we get from equation (2.17)

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \quad \text{or} \quad \frac{V_m}{V_p} = \frac{\rho_p}{\rho_m} \times \frac{L_p}{L_m} \times \frac{\mu_m}{\mu_p}$$

But $\quad \frac{\rho_p}{\rho_m} = \frac{1.24}{1000}$

$$\frac{L_p}{L_m} = L_r = 40,\ \frac{\mu_m}{\mu_p} = \frac{0.01}{0.00018}$$

$$\therefore \quad \frac{V_m}{V_p} = \frac{1.24}{1000} \times 40 \times \frac{0.01}{0.00018} = 2.755.$$

Now making Euler's number equal, we get from equation (12.29) as

$$\frac{V_m}{\sqrt{\frac{P_m}{\rho_m}}} = \frac{V_p}{\sqrt{\frac{P_p}{\rho_p}}} \quad \text{or} \quad \frac{V_m}{V_p} = \frac{\sqrt{P_m \rho_m}}{\sqrt{P_p/\rho_p}} = \sqrt{\frac{P_m}{P_p}}\ x\ \sqrt{\frac{\rho_p}{\rho_m}}$$

But $\quad \frac{V_m}{V_p} = 2.755$ and $\frac{\rho_p}{\rho_m} = \frac{1.24}{1000}$

$$\therefore \quad \sqrt{\frac{P_m}{P_p}} = \frac{2.755}{0.0352} = 78.267$$

$$\therefore \quad \frac{P_m}{P_p} = (78.267)^2\ or\ P_p = \frac{P_m}{(78.267)^2} = \frac{80}{(78.267)^2}$$

$$= 0.01306\, N/cm^2. \qquad \text{Ans.}$$

2.10. MODEL TESTING OF PARTIALLY SUB-MERGED BODIES

Let us consider the testing of a ship model (ship is a partially sub-merged body) in a water-tunnel in order to find the drag force F or resistance experienced by a ship. The drag experienced by a ship consists of:

1. The wave resistance, which is the resistance offered by the waves on the free sea-surface, and

2. The frictional or viscous resistance, which is offered by the water on the surface in contact of the ship with water.

Thus in this case, three forces namely inertia, gravity and viscous forces are present. Then for dynamic similarity between the model and its prototype, the Reynold's number (which is ratio of inertia to viscous force) and the Froude number (which is the ratio of inertia force to gravity force) should be taken into account. This means that in this case, the Reynold model law and Froude model law should be applied.

But for Reynold model law, the condition is

Reynold's number of model = Reynold's number of prototype

Or $$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

If the fluid is the same for model and prototype, then $\rho_m = \rho_p$ and $\mu_m = \mu_p$

$$\therefore \qquad V_m L_m = V_p L_p$$

$$V_m = \frac{V_p L_p}{L_m} = L_r V_p \qquad \left\{\because \frac{L_p}{L_m} = L_r\right\} \qquad \ldots(2.31)$$

For Froude model law, we have from equation (2.18) as $\frac{V_m}{\sqrt{g_m/L_m}} = \frac{V_p}{\sqrt{g_p/L_p}}$

If the fluid is the same for model and prototype and test is conducted at the same place where prototype is to operate, then $g_m = g_p$

$$\therefore \qquad \frac{V_m}{\sqrt{L_m}} = \frac{V}{\sqrt{L_p}}$$

$$\therefore \qquad V_m = \sqrt{\frac{L_m}{L_p}} \times V_p = V_p \times \frac{1}{\sqrt{\frac{L_p}{L_m}}} = V_p \times \sqrt{L_r} \qquad \left\{\because \frac{L_p}{L_m} = L_r\right\} \qquad \ldots(2.32)$$

From equation (2.31) and (2.32), we observe that the velocity of fluid in model for Reynold model law and Froude model law is different. Thus it is quite impossible to satisfy both the laws together, which means the dynamic similarity between the model and its prototype will not exist. To overcome this difficulty, the method suggested by William Froude is adopted for testing the ship model (or partially submerged bodies) as:

Step 1. The total resistance experienced by a ship is equal to the wave resistance plus frictional or viscous resistance.

Let (R_p) = total resistance experienced by prototype,

$(R_w)_p$ = Wave resistance experienced by prototype

$(R_f)_p$ = Frictional resistance experienced by prototype, and

$(R)_m, (R_w)_m, (R_f)_m$ = Corresponding values for model.

Then, we have for the prototype, $(R)_p = (R_w)_p + (R_f)_p$ …(2.33)

And for the model, $(R)_m = (R_w)_m + (R_f)_m$ …(2.34)

Step 2. The frictional resistances for the model and the ship $\left[i.e., (R_f)_m and\ (R_f)_p\right]$ are calculated from the expressions given below:

$$(R_f)_p = f_p A_p {V_p}^n \quad \text{…(2.35)}$$

And

$$(R_f)_m = f_m A_m {V_m}^n \quad \text{…(2.36)}$$

Where f_p = Frictional resistances per unit area per unit velocity of prototype,

A_p= Wetted surface area of the prototype,

$V_p =$ Velocity of prototype,

n = Constant, and

$f_m, A_m, V_m =$ Corresponding values of frictional resistance, wetted area and velocity of model.

The values of f_p and f_m are determined from experiments.

Step 3. The model is tested by towing it in water contained in a towing tank such that the dynamic similarity for Froude number is satisfied i.e., $(F_e)_m = (F_e)_p$. The total resistance of the model (R_m) is measured for this condition.

Step 4. The total resistance (R_m) for the model is known from step 3 and frictional resistance of the model $(R_f)_m$ is calculated from equation (12.67) . then the wave resistance for the model is known from equation (12.34) as

$$(R_w)_m = R_m - (R_f)_m \quad \text{…(2.37)}$$

Step 5. The resistance experienced by a ship of length L, flowing with velocity V in fluid of viscosity μ, density ρ depends upon g, the gravity. By dimensional analysis, the expression for resistance is given by

$$\frac{R}{\rho L^2 V^2} = \phi\left[\frac{\rho VL}{\mu}, \frac{V^2}{gL}\right] = \phi\left[R_e, {F_e}^2\right]$$

Thus resistance is a function of Reynold's number (R_e) and Froude number (F_e).

For dynamic similarity for model and prototype for wave resistance only, we have

$$\frac{(R_w)_p}{\rho_p {L_p}^2 {V_p}^2} = \frac{(R_w)_m}{\rho_m {L_m}^2 {V_m}^2}$$

Or wave resistance for prototype is given as

$$(R_w)_p = \frac{\rho_p}{\rho_m} \times \frac{{L_p}^2}{{L_m}^2} \times \frac{{V_p}^2}{{V_m}^2} \times (R_w)_m \quad \text{…(2.38)}$$

But from step 3, $(F_e)_m = (F_e)_p$ or $\frac{V_m}{\sqrt{L_m g_m}} = \frac{V_p}{\sqrt{L_p g_p}}$

If the model and ship are at the same place, $g_m = g_p$

$$\therefore \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}} \quad \text{or} \quad V_m = \sqrt{\frac{L_m}{L_p}}.V_p$$

Substituting the value of V_m in equation (12.38), we have

$$(R_w)_P = \frac{\rho_p}{\rho_m} \times \frac{{L_p}^2}{{L_m}^2} \times \frac{{V_p}^2}{V_p \times \frac{L_m}{L_p}} \times (R_w)_m$$

$$= \frac{\rho_p}{\rho_m} \times \frac{{L_p}^3}{{L_m}^3} \times (R_w)_m \qquad \text{...(2.39)}$$

Step 6. The total resistance of the ship is given by adding $(R_w)_P$ from equation (2.39) to $\left(R_f\right)_P$ given by equation (2.35) as

$$R_p = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^3 \times (R_w)_m + f_p A_p {V_p}^2 \qquad \text{...(2.40)}$$

Problem 2.31. A I in 20 of a naval ship having a sub-merged surface area of $5m^2$ and length 8 m has a total drag of 20N when towed through water at a velocity of 1.5m/s. calculate the total drag on the prototype when moving at the corresponding speed. Use the relation $\mathrm{F_f} = \frac{1}{2}\mathrm{C_f}\rho \mathrm{A}\mathrm{V}^2$ for calculating the skin (frictional) resistance. The value of $\mathrm{C_f} = \frac{0.0735}{(\mathrm{R_e})^{1.5}}$

Take kinematic viscosity of water (or sea-water) as 0.01 stokes and density of water (sea-water) as $1000 kg/m^3$.

Solution

Given:

Linear scale ratio, $L_r = 20$

Sub-merged area of model, $A_m = 5.0 m^2$

Length of model, $L_m = 8.0\ m$

Total drag of model, $R_m = 20N$

Velocity of model, $V_m = 1.5 m/s$

Let A_p, L_p, R_p, V_p = corresponding values for prototype.

Fluid in model is the same as in prototype and is sea-water.

Kinematic viscosity of sea water, $V_m = V_p = 0.01\ stokes = 0.01\ cm^2/s = 0.01\ x\ 10^{-4}\ m^2/s$

Density of water, $\rho_m = 1000 kg/m^3$

The skin (frictional) resistance of model is given by

$$\left(F_f\right)_m = \frac{1}{2} C_{f_m} \rho_m A_m {V_m}^2 \qquad \text{(i)}$$

Where $C_{f_m} = \frac{0.0735}{[(R_e)_m]^{1/5}}$

Where $(R_e)_m$ = Reynolds number for model

$$= \frac{\rho_m V_m L_m}{\mu_m} \quad \text{or} \quad \frac{V_m L_m}{V_m} \qquad \left\{\because v = \frac{\mu}{\rho}\right\}$$

$$= \frac{1.5\ x\ 8.0}{0.01\ x\ 10^{-4}} = 1.2\ x\ 10^7.$$

Substituting this value in equation (ii), we get

$$C_{f_m} = \frac{0.0735}{(1.2\ x\ 10^7)^{1/5}} = \frac{0.0735}{26.0517} = 2.82\ x\ 10^{-3}$$

Substituting the value of C_{f_m} in equation (i), we get

$$\left(F_f\right)_m = \frac{1}{2} \times 2.82\ x\ 10^{-3} \times 1000\ \times 5.0 \times (1.5)^2 = 15.8617 = 15.862N$$

Using equation (12.34), we get $R_m = (R_w)_m + \left(R_f\right)_m$

Where $\left(R_f\right)_m = \left(F_f\right)_m = 15.862$ or $20 = (R_w)_m + 15.862$

∴ Wave resistance for model, $(R_w)_m = 20 - 15.862 = 4.138N$ ..(iii)

The wave resistance experienced by ship is given by equation (12.39) as

$$(R_w)_p = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^3 \times (R_w)_m$$

$$= 1 \times {L_r}^3 \times 4.138N \qquad \left\{\because \frac{\rho_p}{\rho_m} = 1\ same\ fluid\right\}$$

$$= 1 \times 20^3 \times 4.138 = 33104N$$

And skin (frictional) resistance of prototype is given by

$$\left(R_f\right)_p = \left(F_f\right)_p = \frac{1}{2} C_{f_p}\ x\ \rho_p\ x\ A_p\ x\ {V_p}^2 \qquad \text{…(iv)}$$

Where V_p is the velocity of prototype and is given by Froude model law,

i.e., $(F_e)_m = (F_e)_p$ or $\frac{V_m}{\sqrt{L_m g}} = \frac{V}{\sqrt{L_p g}}\ or\ \frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}}$

∴ $V_p = \sqrt{\frac{L_p}{L_m}} \times V_m = \sqrt{L_r} \times V_m \qquad \left\{\because \frac{L_p}{L_m} = L_r\right\}$

$$= \sqrt{20} \times 1.5 = 6.708 m/s$$

Now $\frac{A_p}{A_m} = {L_r}^2 = 20^2$

∴ $A_p = A_m \times 20^2 = 5 \times 400 = 2000m^2$

and $L_p = L_m \times L_r = 8 \times 20 = 160m$

in equation (iv), the value of C_{f_p} is given by $C_{f_p} = \frac{0.0735}{[(R_e)_p]^{1/5}}$

where $(R_e)_p$ = Reynolds number for prototype

$$= \frac{V_p \times L_p}{V_p} = \frac{6.708 \ x \ 160}{0.01 \ x \ 10^{-4}} = 1.073 \ x \ 10^{9}$$

$\therefore$ $$C_{f_p} = \frac{0.0735}{(1.073 \ x \ 10)^{91/5}} = \frac{0.0735}{63.99} = 1.1486 \ x \ 10^{-3}$$

Substituting this value of C_{f_p} in equation (iv), we get

$$(R_f)_p = C_{f_p} = \frac{1}{2} \times 1.1486 \times 10^{-3} \times 1000 \times 2000 \times (6.708)^2 = 51683.8N$$

$\therefore$ Total drag on prototype is obtained by using equation (12.33).

$$R_p = (R_w)_p + (R_f)_p = 33104 + 51683.8 = 84787.8N. \quad \text{Ans.}$$

Problem 2.32 A 1:15 model of a flying boat is towed through water. The prototype is moving in sea-density 1024kg/m^3 at a velocity of 20m/s. find the corresponding speed of the model. Also determine the resistance due to waves on model if the resistance due to waves of prototype is 600N.

Solution

Given:

Linear scale ratio, $L_r = 15$

Velocity of prototype, $V_p = 20m/s$

Fluid in prototype is sea-water while in model it is water

Density of sea-water, $\rho_p = 1024kg/m^3$

Density of water, $\rho_m = 1000kg/m^3$

Resistance due to waves for prototype is, $(R_w)_p = 600N.$

Find V_m and $(R_w)_m$.

(i) The velocity, V_m from model is given by Froude model law,

$\therefore$ $$\frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p \ x \ g}}$$

$$V_m = \sqrt{\frac{L_m}{L_p}} \times V_p = \frac{V_p}{\sqrt{L_p/L_m}} = \frac{20}{\sqrt{15}} \qquad \left\{\because \frac{L_p}{L_m} = L_r = 15\right\}$$

$$= \frac{20}{3.872} = \mathbf{5.165m/s.} \qquad \textbf{Ans}$$

(ii) For dynamic similarity between model and its prototype for wave resistance only, we have equation (2.39) as $(R_w)_p = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^3 \times (R_w)_m$

Substituting the known values, $600 = \frac{1024}{1000} \times {L_r}^3 \times (R_w)_m = \frac{1024}{1000} \times 15^3 \times (R_w)_m$

$\therefore$ $$(R_w)_m = \frac{600 \times 1000}{1024 \times 15^3} = \mathbf{0.1736N}.\quad \textbf{Ans}.$$

Problem 2.33. A 1:40 model of an ocean tanker is dragged through fresh water at 2m/s with a total measured drag of 12N. The skin (frictional) drag co-efficient 'f' for model and prototype are 0.03 and 0.002 respectively in the equation $\mathrm{R_f = f.AV^2}$. The wetted surface area of the model is 25m^2. Determine the total drag on the prototype and the power required to drive the prototype.

Take $\rho_p = 1030kg/m^3$ *and* $\rho_m = 1000kg/m^3$.

Solution

Given:

Linear scale ratio, $L_r = 40$

Velocity of model, $V_m = 2m/s$

Total drag of model, $R_m = 12N$

Wetted area of model, $A_m = 25m^2$

Co-efficient of friction for model, $f_m = 0.03$

for prototype, $f_p = 0.002$.

Let the total drag on prototype = R_p

And power required to drive the prototype = P

Frictional drag on model, $\left(R_f\right)_m = f_m A_m {V_m}^2 = 0.03 \times 25 \times 2^2 = 3N$

$\therefore$ Wave drag on model, $(R_w)_m = R_m - \left(R_f\right)_m = 12 - 3 = 9N.$

The wave drag on prototype is obtained from equation (12.39) as

$$(R_w)_p = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^3 \times (R_w)_m = \frac{1030}{1000} \times {L_r}^3 \times 9 \qquad \left\{\because \frac{L_p}{L_m} = L_r = 40\right\}$$

$$= \frac{1030}{1000} \times 40^3 \times 9 = 593291.8N \qquad ..(i)$$

The frictional drag on prototype is given by

$$\left(R_f\right)_p = f_p A_p {V_p}^2 \qquad \ldots(ii)$$

Where the velocity of prototype V_p is obtained from Froude model law as

$$\frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p \; x \; g}} \quad \text{or} \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}}$$

$\therefore$ $$V_p = \sqrt{\frac{L_p}{L_m}} \times V_m = \sqrt{L_r} \times V_m = \sqrt{40} \times 2 = 12.65m/s$$

And $\frac{A_p}{A_m} = {L_r}^2 = 40\ x\ 40\ or\ A_p = 40\ x\ 40\ x\ A_m$

$= 40\ x\ 40\ x\ 25 = 40000 m^2.$

Substituting these values in (ii), we get

$\left(R_f\right)_p = 0.002 \times 40000 \times (12.65)^2 = 12801.8N$ …(iii)

Total drag on the prototype is obtained by adding equations (i) and (ii) as

$$R_p = (R_w)_p + \left(R_f\right)_p$$

$$= 593291.8 + 12801.8 = \mathbf{606093.6N}$$ **Ans.**

Power required to drive the prototype,

$$P = \frac{(Total\ drag\ on\ prototype) \times Velocity\ of\ prototype}{1000}$$

$$= \frac{606093.6 \times 12.65}{1000} = 7667kW.$$ Ans.

Problem 2.34. Resistance R, to the motion of a completely sub-merged body is given by

$$\mathrm{R} = \rho \mathrm{V}^2 \mathrm{l}^2 \phi \left(\frac{\mathrm{Vl}}{\mathrm{v}}\right),$$

Where ρ and v are density and kinematic viscosity of the fluid while l is the length of the body and V is the velocity of flow. If the resistance of a one-eight scale air-ship model when tested in water of 12m/s is 22N, what will be the resistance in air of the air-ship at the corresponding speed? Kinematic viscosity of air is 13 times that of water and density of water is 810 times of air.

Solution

Given:

Linear scale ratio, $L_r = 8$

Velocity of model, $V_m = 12 m/s$

Resistance to model, $R_m = 22N$

The fluid for model is water and for prototype the fluid is air.

Kinematic viscosity of air $= 13 \times \text{Kinematic viscosity of water}$

∴ $v_p = 13 \times v_m$

Density of water $= 810 \times \text{density of air}$

∴ $\rho_m = 810 \times \rho_p$

Let $V_p = \text{Velocity of air} - \text{ship(prototype)}$

$R_p =$ Resistance of the air-ship

The resistance, R, is given by $R = \rho V^2 l^2 \phi\left(\frac{Vl}{v}\right)$

$\therefore$ The non-dimensional terms $\frac{R}{\rho V^2 l^2}$ and $\frac{Vl}{v}$ should be same for the prototype and its model.

$\therefore \left(\frac{Vl}{v}\right)_{prototype} = \left(\frac{Vl}{v}\right)_{model}$ or $\frac{V_p l_p}{v_p} = \frac{V_m l_m}{v_m}$

$\therefore \quad V_p = V_m \frac{l_m}{l_p} \times \frac{V_p}{V_m} = 12 \times \frac{1}{L_r} \times 13 \qquad \left\{\because \frac{l_p}{l_m} = L_r\right\}$

$$= 12 \times \frac{1}{8} \times 13 = 19.5 m/s$$

Also $\left(\frac{R}{\rho V^2 l^2}\right)_{prototype} = \left(\frac{R}{\rho V^2 l^2}\right)_{model}$ or $\frac{R_p}{\rho_p {V_p}^2 {l_p}^2} = \frac{R_m}{\rho_m {V_m}^2 c}$

$\therefore \quad R_p = R_m \times \frac{\rho_p}{\rho_m} \times \frac{{V_p}^2}{{V_m}^2} \times \frac{{l_p}^2}{{l_p}^2}$

$$= 22 \times \frac{1}{810} \times \frac{(19.5)^2}{12^2} \times 8^2$$

$$= \mathbf{4.59N.}$$ **Ans**

Problem 2.35: A ship 250 m long moves in sea-water, whose density is 1030kg/m^3. A 1:125 model of this ship is to be tested in wind tunnel. The velocity of air in the wind tunnel around the model is 20m/s and the resistance of the model is 50N. Determine the velocity of ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as 1.24kg/m^3. Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively.

Solution

Given:

$L_p = 250m \quad \rho_p = 1030\, kg/m^3 \quad V_p = ? \quad R_p = ? \quad$ v $= 0.012 stokes \quad {L_m}/{L_p} = 1:125$

$R_m = 50N \quad V_m = 20ms^{-1} \quad \rho_m = 1.241 kgm^{-3}$ v $= 0.018 stokes$

$\frac{L_p V_p}{\vartheta_p} = \frac{L_m V_m}{\vartheta_m} \qquad V_p = \frac{L_m V_m}{\mathrm{v}_m} \times \frac{\mathrm{v}_p}{L_p} = \frac{20 \times 1 \times 0.012}{0.018 \times 125} = 0.1066 ms^{-1}$

$$\frac{R_p}{\rho_p L_p^2 V_p^2} = \frac{R_m}{\rho_m L_m^2 V_m^2}$$

$$R_p = \frac{R_m}{\rho_m L_m^2 V_m^2} \times \rho_p L_p^2 V_p^2 = \frac{50 \times 0.1066^2 \times 125^2 \times 1030}{1.24 \times 20^2} = 18228.767N$$

2.11 CLASSIFICATION OF MODELS

The hydraulic models are classified as:

1. Undistorted models, and
2. Distorted models.

2.11.1 Undistorted models. Undistorted models are those models which are geometrically similar to their prototype or in other words if the scale ratio for the linear dimensions of model and its prototype is the same, the model is called undistorted model. The behavior of the prototype can be easily predicted from the results of undistorted model.

2.11.2 Distorted models. A model is said to be distorted if it is not geometrically similar to its prototype. For a distorted model, different scale ratios for the linear dimensions are adopted. For example, in case of rivers, harbors, reservoirs, etc. two different scale ratios, one for horizontal dimensions and other for vertical dimensions are taken. Thus the model of rivers, harbors, reservoirs will become as distorted models. If for the river, the horizontal and vertical scale ratios are taken to be the same so that the model is undistorted, then the depth of water in the model of river will be very small which may not be measured accurately. The following are the advantages of distorted models:

1. The vertical dimensions of the model can be measured accurately.
2. The cost of the model can be reduced.
3. Turbulent flow in the model can be maintained.

Though there are some advantages of the distorted model, yet the results of the distorted model cannot be directly transferred to its prototype. But sometimes from the distorted models, very useful information can be obtained.

2.11.3. Scale ratios For Distorted Models. As mentioned above, two different scale ratios, one for horizontal dimensions and other for vertical dimensions, are taken for distorted models.

Let $(L_r)_R$ = Scale ratio for horizontal dimension

$$= \frac{L_p}{L_m} = \frac{B_p}{B_m} = \frac{Linear\ horizontal\ dimension\ of\ prototype}{Linear\ horizontal\ dimension\ of\ model}$$

$(L_r)_p$ = Scale ratio for vertical dimension

$$= \frac{Linear\ vertical\ dimension\ of\ prototype}{Linear\ vertical\ dimension\ of\ model} = \frac{h_p}{h_m}$$

Then the scale ratios of velocity, area of low, discharge etc. in terms of $(L_r)_H$ and $(L_r)_P$ can be obtained for distorted models as given as:

1. **Scale ratio for velocity**

Let $V_P =$ velocity in prototype

$V_m =$ velocity in model

Then $\frac{V_P}{V_m} = \frac{\sqrt{2gh_p}}{\sqrt{2g}h_m} = \sqrt{\frac{h_p}{h_m}} = \sqrt{(L_r)_V}$ $\left\{\because \frac{h_p}{h_m} = (L_r)_V\right\}$

2. **Scale ratio for area of flow**

Let $A_P =$ Area of flow in prototype = $B_P \; x \; h_p$

$A_m =$ Area of flow in model = $B_m \; x \; h_m$

$\therefore$ $\frac{A_P}{A_m} = \frac{B_P \; x \; h_p}{B_m \; x \; h_m} = \frac{B_P}{B_m} \times \frac{h_p}{h_m} = (L_r)_H \times (L_r)_V$

3. **Scale ratio for discharge**

Let $Q_p =$ Discharge through prototype = $A_P \times V_P$

$Q_m =$ Discharge through model = $A_m \times V_m$

$\therefore$ $\frac{Q_p}{Q_m} = \frac{A_P \times V_P}{A_m \times V_m} = (L_r)_H \times (L_r)_V \times \sqrt{(L_r)_V} = (L_r)_H \times [(L_r)_P]^{3/2}$. (2.41)

Problem 2.36. The discharge through a weir is 1.5 m^3/s. find the discharge through the model of the weir if the horizontal dimension of the model = $\frac{1}{50}$ the horizontal dimension of the prototype and vertical dimension of the model = $\frac{1}{10}$ the vertical dimension of the prototype.

Solution

Given:

Discharge through the weir (prototype), $Q_p = 1.5 \, m^3/s$

Horizontal dimension of model $= \frac{1}{50}$ x horizontal dimension of prototype

$\therefore$ $\frac{\text{Horizontal dimension of prototype}}{\text{Horizontal dimension of model}} = 50$ or $(L_r)_H = 50$

Vertical dimension of model = $\frac{1}{10}$ x Vertical dimension of prototype

$\therefore$ $\frac{\text{Vertical dimension of prototype}}{\text{Vertical dimension of model}} = 10$

$\therefore (L_r)_V = 10$

Using equation (12.41), we get $\frac{Q_p}{Q_m} = (L_r)_H \times [(L_r)_P]^{3/2} = 50 \times 10^{3/2} = 1581.14$

$Q_m = \frac{Q_p}{1581.14} = \frac{1.50}{1581.14} = 0.000948 \, m^3/s = \mathbf{0.948 litres/s.}$ **Ans**

2.14 HIGHLIGHTS

1. Dimensional analysis is a method of dimensions, in which the fundamental dimensions are M, L and T.
2. Dimensional analysis is performed by two methods namely Rayleigh's method and Buckingham's π- theorem.
3. Rayleigh's method is used for finding an expression for a variable which depends on maximum three or four variables while there is no restriction on the number of variables for Buckingham's π- theorem.
4. Model analysis is an experiment method of finding solutions to complex flow problems. A model is a small scale replica of the actual machine or structure. The actual machine or structure is called prototype.
5. Three types of similarities must exist between the model and the prototype. They are; (i) Geometric Similarity, (ii) Kinematic Similarity, and (iii) Dynamic Similarity.
6. For geometric similarity, the ratio of all linear dimensions of model and prototype should be equal.
7. Kinematic similarity means the similarity of motion between model and prototype.
8. Dynamic similarity means the similarity of forces between the model and prototype.
9. Reynolds number is defined as the ratio of inertia force to viscous force of fluid. It is given by: $R_e = \frac{\rho VL}{\mu} = \frac{VL}{v} = \frac{V \times d}{v} for\ pipe\ flow$

   $w$here V = Velocity of flow, d = Diameter of pipe

   v = Kinematic viscosity of fluid
10. Froude's number is the ratio of the square root of inertia force to gravity force and is given by

$$Fe = \sqrt{\frac{F_i}{F_g}} = \frac{V}{\sqrt{Lg}}.$$

11. Euler's number is the ratio of the square root of inertia force to pressure force and is given by, $E_u = \sqrt{\frac{F_i}{F_p}} = \frac{V}{\sqrt{P/\rho}}$
12. Mach number is the ratio of the square root of inertia force to elastic force is given by

$$M = \sqrt{\frac{F_i}{F_e}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}.$$

13. The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The model laws are (i) Reynolds Model law, (ii) Froude's Model law, (iii) Euler Model law, (iv) Weber Model law, (v) Mach Model law.
14. The drag experienced by a ship model (or partially sub-merged body) is obtained by Froude's method.
15. Hydraulic models are classified as (i) undistorted models and distorted models.

If the models are geometrically similar to its prototype, the models are known as undistorted model. And if the models are having different scale ratio for horizontal and vertical dimensions, the models are known as distorted model

2.12 QUESTIONS

1. a. Define the terms dimensional analysis and model analysis.
2. Give the dimensions of: (i) Force (ii) Viscosity (iii) Power and (iv) Kinematic viscosity.
   $[\boldsymbol{Ans}. (i) MLT^{-2}, ML^{-1}T^{-1}, ML^{2}T^{-3}, L^{-2}T^{-1}]$
3. The variables controlling the motion of a floating vessel through water are drag force *F*, the speed V, the length L, the density ρ and dynamic viscosity μ of water and acceleration due to gravity g. Derive an expression for *F* by dimensional analysis.
   $\left[Ans. F = \rho L^2 V^2 \phi \left[\frac{\mu}{\rho V L}, \frac{Lg}{V^2}\right]\right]$
4. The resistance R, to the motion of s completely sub-merged body depends upon the length of the body L, velocity of flow V, mass density of fluid v and kinematic viscosity of fluid v. By dimensional analysis prove that

$$R = \rho L^2 V^2 \phi \left(\frac{VL}{v}\right).$$

5. What do you mean by fundamental units and derived units? Give examples.
6. Explain the term, 'dimensionally homogenous equation'
7. a. What are the methods of dimensional analysis? Describe the Rayleigh's method for dimensional analysis.
   b. State Buckingham's π- theorem. Why this theorem is considered superior over Rayleigh's method for dimensional analysis?
   c. What do you mean by repeating variables? How are the repeating variables selected for dimensional analysis?
8. In 1:30 model of spillway, the velocity and discharge are 1.5m/s and 2.0m^3/s. Find the corresponding velocity and the discharge in the prototype.
   $[\text{Ans.}\, 8.216\, \text{m/s}, 9859\, \text{m}^3/\text{s}]$
9. A ship model of scale $\frac{1}{60}$ is towed through sea-water at a speed of 0.5m/s. A force of 1.5N is required to tow the model. Determine the speed of the ship and propulsive force on the ship, if prototype is subjected to wave resistance only.
   $[\boldsymbol{Ans}. 3.873\, \text{m/s}, 324000\text{N}]$
10. a. Explain the different types of hydraulic similarities that must exist between a prototype and its model
    b. What do you mean by dimensionless number? Name any four dimensionless numbers.

c. Define and explain Reynolds number, Froude's number and Mach's number, what are their significances for fluid flow problems?

d. What is meant by geometric, kinematic and dynamic similarities? Are these similarities truly attainable? If not, why?

11. A model of a sub-marine of scale $\frac{1}{40}$ is tested in a wind tunnel. Find the seed of air in wind tunnel if the speed of submarine in sea- water is 15 m/s. Also find the ratio of the resistance between the model and its prototype. Take the values of kinematic viscosities for sea-water and air as 0.012 stokes and 0.016 stokes respectively. The densities of sea-water and of air are given as 1030 kg/m^3 and 1.24 kg/m^3 respectively.
$\left[\text{Ans.}\,800\,\text{m/s}\,,\frac{\text{F}_\text{m}}{\text{F}_\text{p}}=0.00214\right]$
12. A ship 250 m long moves in sea-water, whose density is 1030kg/m^3. A 1:125model of the ship is to be tested in wind tunnel. The velocity of air in the wind tunnel around the model is 20m/s and the resistance of the model is 50N. Determine the velocity of ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as 1.24kg/m^3. Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively. $[\text{Ans.}\,0.106\,\text{m/s}\,,18228.7\text{N}]$
13. A pipe of diameter 1.8m is required to transport an oil of sp. gr. 0.8 and viscosity 0.04 poise at the rate of 4m^3/s. tests were conducted on a 20cm diameter pipe using water at $20°\text{C}$. Find the velocity and rate of flow in the model. Viscosity of water at $20°\text{C} =$ 0.01poise. $[\textbf{Ans.}\,2.829\,\text{m/s}\,,88.8\,\text{liters/s}]$
14. What are the different laws on which models are designed for dynamic similarity? Where are they used?
15. The drag force exerted by a flowing fluid on solid body depends upon the length of the body, L, velocity of flow V, density of fluid ρ, and viscosity μ. Find an expression for drag force using Buckingham's theorem.
16. Explain the terms: distorted models and undistorted models. What is the use of distorted models?
17. Prove that the scale ratio for discharge for a distorted model is given as

$$\frac{Q_p}{Q_m} = (L_r)_H \times [(L_r)_v]^{3/2}$$

Where Q_p = Discharge through prototype, Q_m = Discharge through model $(L_r)_H$ = Horizontal scale ratio and $(L_r)_v$ = Vertical scale ratio.

18. State Buckingham's π- theorem. What do you mean by repeating variables? How are the repeating variables selected in dimensional analysis?
19. A fluid of density ρ and viscosity μ, flows at an average velocity V through a circular pipe of diameter D. show by dimensional analysis that the shear stress at the pipe wall is given as $\tau_0 = \rho V^2 \phi \left[\frac{\rho V}{\mu}\right]$.
20. The efficiency of η of geometrically similar fans depends upon the mass density of air ρ, its viscosity μ, speed of fan N(revolutions per sec), diameter of blades D and discharge Q. Perform dimensional analysis.

$$\begin{bmatrix} \textit{Hint}.\ \text{Take D, N, }\rho\text{ as repeating variable. Then three }\pi\text{ terms will be} \\ \pi_1 = \mathrm{D}^{a_1}.\mathrm{N}^{b_1}.\rho^{c_1}.\eta = \eta; \\ \pi_2 = \mathrm{D}^{a_2}.\mathrm{N}^{b_2}.\rho^{c_2}.\mu = \dfrac{\mu}{\mathrm{D}^2\mathrm{NP}} \text{ and } \pi_3 = \mathrm{D}^{a_3}.\mathrm{N}^{b_3}.\rho^{c_3}.\mathrm{Q} = \dfrac{\mathrm{Q}}{\mathrm{D}^3\mathrm{N}} \\ \therefore \eta = \phi(\pi_2, \pi_3) = \phi\left(\dfrac{\mu}{\mathrm{D}^2\mathrm{NP}}, \dfrac{\mathrm{Q}}{\mathrm{D}^3\mathrm{N}}\right) \quad \textbf{Ans.} \end{bmatrix}$$

21. The discharge through an orifice depends on the dia. D of the orifice, head H over the orifice, density ρ of liquid and acceleration g due to gravity. Using dimensional analysis, find an expression for the discharge. Hence find the dimensionless parameters on which the discharge co-efficient of an orifice meter depend.

Hint. Q = f(D,H,ρ, μ,g). Hence N =6, m = 3 and no. of π-terms = 3. Take ρ,D,g as repeating variables. The n

$\pi_1 = \rho^{a_1}.D^{b_1}g^{c_1}Q$, $\pi_2 = \rho^{a_2}.D^{b_2}.g^{c_2}.\mu$ and $\pi_3 = \rho^{a_1}.D^{b_3}.g^{c_3}.H$.

Find π_1, π_2, π_3. They will be $\pi_1 = \frac{Q}{D^{2.5}\ x\ g^{1/2}}$, $\pi_2 = \frac{\mu}{\rho.D^{3/2}.g^{1/2}}$ and $\pi_3 = \frac{H}{D}$.

Hence $\qquad Q = D^{2.5}.g^{1/2}.\left(\frac{H}{D}, \frac{\mu}{\rho.D^{3/2}.g^{1/2}}\right) =$

$D^2.g^{1/2}.H^{1/2}\left(\frac{H}{D}, \frac{\mu}{\rho.D^{3/2}.g^{1/2}}\right)$.

Also $\qquad Q = C_d\ x\ A\ x\ \sqrt{2gh}$

Comparing the two values of Q.

We get $C_d = \phi\left(\frac{H}{D}, \frac{\mu}{\rho.D^{3/2}.g^{1/2}}\right)$. Hence C_d depends upon $\frac{H}{D}$ and $\frac{\mu}{\rho.D^{3/2}.g^{1/2}}$

22. The force exerted by a flowing fluid on a stationary body depends upon the length (L) of the body, velocity (V) of the fluid, density (ρ) of fluid, viscosity (μ) of the fluid and acceleration (g) due to gravity. Find an expression for the force using dimensional analysis. $\left[\textbf{Ans.}\ F = \rho L^2 V^2 \phi\left[\frac{\mu}{\rho V L}, \frac{L\ x\ g}{V^2}\right]\right]$

23. A spillway model is to be built to a geometrically similar scale of $\frac{1}{40}$ across a flume of 50cm width. The prototype is 20m high and maximum head on it is expected to be 2m. (i) What height of model and what head on the model should be used? (ii) if the flow over model at a particular head is 10 litres/s, what flow per meter length of the prototype is expected? (iii) If the negative pressure in the model is 150mm, what is the negative pressure in the prototype? Is it practicable?

    [**Ans.** (i)$0.5\text{m}, 0.05\text{m}$, (ii)$5059.64\ \text{litres/s}$ (iii)$6\text{m}.\text{Yes}$]

24. The pressure drop in an aero plane model of size $\frac{1}{50}$ of its prototype is 4N/cm^2. The model is tested in water. Find the corresponding pressure drop in prototype. Take density of air =1.24kg/m^3. The viscosity of water is 0.01poise while the viscosity of air is 0.00018 poise. (Ans. $0.00042\ \text{N/cm}^2$)

25. A 1:20 model of a flying boat is towed through water. The prototype is moving in sea-water of density 1024 kg/m^3 at a velocity of 1.5m/s. Find the corresponding speed of the model. Also determine the resistance due to wave on model, if the resistance due to waves on prototype is 500 N. [**Ans.** $3.354\ \text{m/s}, 0.061\text{N}$]

26. A 1:50 model of an ocean tanker is dragged through fresh water at 1.5m/s with total measured drag of 10 N. the frictional drag co-efficient 'f' for model and prototype are 0.03 and 0.02 respectively in the equation, $R_f = f.A.V^2$. The wetted surface area of the model is 20m^2. Determine the total drag on the prototype and the power required to derive the prototype. Take density of sea-water and of fresh water as 1024kg/m^3 and 1000kg/m^3 respectively. [**Ans.** $1118436\text{N}, 158072\text{h.p.}$]

27. If model prototype ratio is 1:75, show that the ratio of discharges per unit width of spillway is given by $\left(\frac{1}{75}\right)^{3/2}$

3. VISCOUS FLOW

3.1 INTRODUCTION

This chapter deals with flow of fluids which are viscous and flowing at very low velocity. At low velocity the fluid moves in layers. Each layer of fluid slides over the adjacent layer. Due to relative velocity between two layers the velocity gradient $\frac{du}{dy}$ exists and hence a shear stress $\tau = \mu \frac{du}{dy}$ acts on the layers.

The following cases will be considered in this chapter:

1. Flow of viscous fluid through circular pipe,
2. Flow of viscous fluid between two parallel plates.
3. Kinetic Energy correction and momentum correction factors.
4. Power absorbed in viscous flow through
   (a) Journal Bearings, (b) Foot-step Bearings and (c) Collar Bearings.

3.2 FLOW OF VISCOUS FLUID THROUGH CIRCULAR PIPE

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum velocity to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through the circular pipe will be viscous or laminar, if the Reynolds number (R_e) is less than 2000. The expression for Reynolds number is given by

$$R_e = \frac{\rho V D}{\mu}$$

Where $\rho =$ Density of fluid flowing through pipe

V = Average velocity of fluid

D = Diameter of pipe and

$\mu =$ Viscosity of fluid.

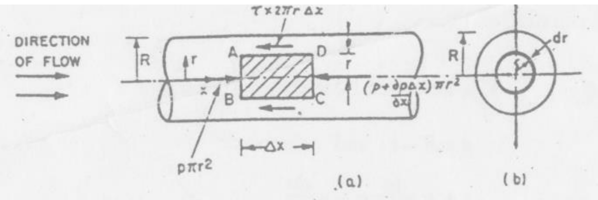


Figure 3.1. Viscous flow through a pipe

Consider a horizontal pipe of radius R. The viscous fluid is flowing from left to right in the pipe as shown in Fig 3.1(a). Consider a fluid element of radius r, sliding in a cylindrical fluid element of radius (r + dr). Let the length of fluid element be Δx. If 'p' is the pressure on the face AB, then the pressure on face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$. Then the forces acting on the fluid element are:

1. The pressure force, $p \times \pi r^2$ on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2$on face CD.
3. The shear force, $\tau \times 2\pi r\Delta x$ on the surface of fluid element. Because flow is steady, there is no acceleration hence the summation of all forces in the direction of flow must be zero i.e.,

$$p\pi r^2 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

Or $$-\frac{\partial p}{\partial x}\Delta x \pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

Or $$-\frac{\partial p}{\partial x}.r - 2\tau = 0$$

∴ $$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \qquad \ldots(3.1)$$

The shear stress τ across a section varies with 'r' as $\frac{\partial p}{\partial x}$across a section is constant. $\frac{2\tau}{r}$ is independent of r_1. At centre of the pipe, r = 0, $\tau = 0$ and at the wall r = r_o, the wall shear stress is $\tau_o$which is maximum. Hence shear stress distribution across a section is a linear function of r as shown in Fig 3.2(a)

τ can be written as:

$$\tau = \frac{2\tau_o}{r_o}r$$

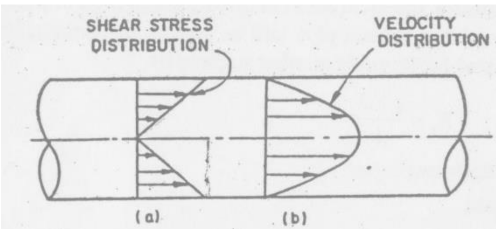


Fig. 3.2. Shear stress and velocity distribution across a section

Equation 3.1 shows a fully developed laminar flow in a horizontal pipe is a balance between pressure and viscous forces; the pressure difference acting on the ends of the pipe area πr^2 and the shear stress acting on the lateral surfaces of the pipe of area $2\pi rL$. This basic balance of forces can be written as: $\frac{\tau}{r/2} = \frac{\Delta p}{L}$

(i) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \frac{du}{dy}$ is substituted in equation (3.1).

But in the relation, $\tau = \mu \frac{du}{dy}$, is measured from pipe wall. Hence

$$y = R - r \text{ and } dy = -dr$$

$$\therefore \qquad \tau = \mu \frac{du}{-dr} = -\mu \frac{du}{dr}$$

Substituting this value in (3.1), we get

$$-\mu \frac{du}{dr} = -\frac{\partial p}{\partial x}\frac{r}{2} \qquad \text{or} \qquad \frac{du}{dr} = \frac{1}{2\mu}\frac{\partial p}{\partial x} r \qquad (3.1a)$$

Integrating this equation w.r.t., ‘r’, we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 + C \qquad (3.2)$$

Where C is the constant of integration and its value is obtained from a boundary condition that at r = R, u = 0 at the wall.

$$\therefore \qquad 0 = \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 + C$$

$$\therefore \qquad C = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

Substituting this value of C in equation (3.2), we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \qquad (3.3)$$

In equation (3.3), values of $\mu, \frac{\partial p}{\partial x}$ and R are constant, which means the velocity, u varies with the square of r. Thus equation (3.3) is an equation of parabola. This is shows that the velocity distribution across the section of a pipe is parabolic. This velocity distribution is shown in Fig. 3.2(b).

(ii) **Ratio of maximum Velocity to Average Velocity**

The velocity is maximum, when $r = 0$ in equation (3.3). Thus maximum velocity, u_{max} is obtained as

$$u_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 \qquad \ldots(3.4)$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge of the fluid across the section by the area of the pipe (πR^2). The discharge (Q) across the section is obtained by considering the flow through a circular ring element of radius r and thickness dr as shown in Fig 3.1(b). the fluid flowing per second through this elementary ring dQ = velocity at a radius r x area of ring element

$$= u \times 2\pi r\, dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2]x\ 2\pi r\, dr$$

$$Q = \int_0^R dQ = \int_0^R -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \times\ 2\pi r\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times\ 2\pi \int_0^R (R^2 - r^2) r dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times\ 2\pi \int_0^R (R^2 r - r^3) dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times\ 2\pi \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^2}{2} - \frac{R^4}{4}\right]$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times\ 2\pi\ \times\ \frac{R^4}{4} = \frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^4$$

∴ Average velocity, $\bar{u} = \frac{Q}{Area} = \frac{\frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^4}{\pi R^2}$

Or $\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^2$ …(3.5)

Dividing equation (3.4) by equation (3.5)

$$\frac{u_{max}}{\bar{u}} = \frac{-\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2}{\frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^2} = 2.0$$

∴ Ratio of maximum velocity to average velocity = 2.0

(iii) **Pressure Drop for a given length (L) of a pipe due to friction**

From equation (3.5), we have

$\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^2$ or $\left(\frac{-\partial p}{\partial x}\right) = \frac{8\mu\bar{u}}{R^2}$

Integrating the above equation w.r.t. x, we get

$$-\int_2^1 dp = \int_2^1 \frac{8\mu\bar{u}}{R^2} dx$$

∴ $-[p_1 - p_2] = \frac{8\mu\bar{u}}{R^2}[x_1 - x_2]$ or $(p_1 - p_2) = \frac{8\mu\bar{u}}{R^2}[x_2 - x_1]$

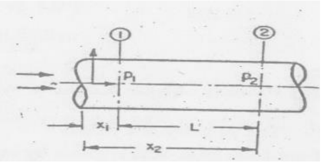


Fig. 3.3

$$= \frac{8\mu\bar{u}}{R^2} L\{\because x_2 - x_1 = L\ from\ Fig. 3.3\}$$

$$= \frac{8\mu\bar{u}L}{(D/2)^2} \qquad \left\{\because R = \frac{D}{2}\right\}$$

Or $(p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2}$, where $p_1 - p_2$ is the drop of pressure.

$\therefore$ Loss of pressure head $= \frac{p_1 - p_2}{\rho g}$

$$\therefore \quad \frac{p_1 - p_2}{\rho g} = h_f = \frac{32\mu\bar{u}L}{\rho g D^2} \qquad (3.6)$$

Equation (3.6) is called **Hagen Poiseuille Formula.**

Problem 3.1. A crude oil of viscosity 0.97 poise and relative density 0.9 is flowing through a horizontal circular pipe of diameter 100 mm and of length 10 m. Calculate the difference of pressure at the two ends of the pipe, if 100 kg of the oil is collected in a tank in 30 seconds.

Solution

Given: $\mu = 0.97\ poise = \frac{0.97}{10} = 0.097\ Ns/m^2$

Relative Density $= 0.9$

$\therefore \rho_0, or\ density$ $= 0.9\ x\ 1000 = 900\ kg/m^3$

Dia. Of pipe, $D = 100\ mm = 0.1\ m$

Length $L = 10\ m$

Mass of oil collected, $M = 100kg$

Time $t = 30\ seconds$

The difference of pressure $(p_1 - p_2)$ for viscous or laminar flow is given by

$(p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2}$ where $\bar{u}$ = average velocity = $\frac{Q}{Area}$

Now, mass of oil/sec $= \frac{100}{30} kg/s$

$$= \rho_0 \times Q = 900 \times Q \qquad (\because \rho_0 = 900)$$

$$\therefore \quad \frac{100}{30} = 900 \times Q$$

$$\therefore \quad Q = \frac{100}{30} \text{ x } \frac{1}{900} = 0.0037\ m^3/s$$

$$\therefore \quad \bar{u} = \frac{Q}{Area} = \frac{0.0037}{\frac{\pi}{4}D^2} = \frac{0.0037}{\frac{\pi}{4}0.1^2} = 0.471 m/s$$

For laminar or viscous flow, the Reynolds number (R_e) is less than 2000. Let us calculate the Reynolds number for this problem.

Reynolds number, $R_e = \frac{\rho V D}{\mu}$

Where $\rho = \rho_0 = 900, V = \bar{u} = 0.471, D = 0.1\ m, \mu = 0.097$

$\therefore \qquad R_e = 900 \times \frac{.471 \times 0.1}{0.097} = 436.91$

As Reynolds number is less than 2000, the flow is laminar.

$\therefore \qquad (p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2} = \frac{32 \times 0.097 \times .471 \times 10}{(.1)^2} N/m^2$

$$= 1462.28\, N/m^2 = 1462.28 \times 10^{-4}\, N/cm^2 = \mathbf{0.1462\, N/cm^2.}$$

Problem 3.2. An oil of viscosity 0.1Ns/m^2 and relative density 0.9 is flowing through a circular pipe of diameter 50 mm and of length 300 m. The rate of flow of fluid through the pipe is 3.5 litre/s. find the pressure drop in a given length of 300 and also the shear stress at the pipe wall.

Solution

Given: viscosity = $\mu = 0.1\, Ns/m^2$

Relative Density $= 0.9$

$\therefore \rho_0, or\ density \quad = 0.9 \times 1000 = 900\, kg/m^3 \qquad (\because Density\ of\ water = 1000\, kg/m^3)$

$$D = 50\, mm = 0.05\, m$$

$$L = 300\, m$$

$$Q = 3.5\, litres/s = \frac{3.5}{1000} = 0.0035\, m^3/s$$

Pressure drop ($(p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2}$ where $\bar{u} = \frac{Q}{Area} = \frac{.0035}{\frac{\pi}{4}D^2} = \frac{.0035}{\frac{\pi}{4}(0.05)^2} = 1.782 m/s$

The Reynolds number (R_e) is given by, $R_e = \frac{\rho V D}{\mu}$

Where $\rho = 900\, kg/m^3, V = average\ velocity = \bar{u} = 1.782\, m/s,$

$\therefore \qquad R_e = 900 \times \frac{1.782 \times 0.05}{0.1} = 801.9$

As Reynolds number is less than 2000, the flow is viscous and laminar

$$p_1 - p_2 = \frac{32\mu\bar{u}L}{D^2}$$

$\therefore \qquad p_1 - p_2 = \frac{32 \times 0.1 \times 1.782 \times 3000}{(0.05)^2}$

= 684288 N/m^2 = 682428 x 10^{-4} N/cm^2 = **68.43 N/cm^2. Ans.**

Shear stress at pipe wall (τ_0)

The shear stress at any radius r is given by the equation (3.1)

i.e., $\qquad \tau = -\frac{\partial p}{\partial x}\frac{r}{2}$

$\therefore$ Shear stress at pipe wall, where r =R is given by

$$\tau_0 = \frac{-\partial p}{\partial x}\frac{R}{2}$$

Now $\frac{-\partial p}{\partial x} = \frac{-(p_2-p_1)}{x_2-x_1} = \frac{p_1-p_2}{x_2-x_1} = \frac{p_1-p_2}{L}$

$$= \frac{684288}{300}\frac{\mathrm{N}/m^2}{m} = 2280.96\ \mathrm{N}/m^3$$

And $R = \frac{D}{2} = \frac{.05}{2} = .025\ m$

$$\tau_0 = 2280.96 \text{ x } \frac{.025}{2}\frac{N}{m^2} = \mathbf{28.512\ N/m^2}.$$ **Ans.**

Problem 3.3. A laminar flow is taking place in a pipe of diameter of 200 mm. the maximum velocity is 1.5 m/s. Find the mean velocity and the radius at which this occurs. Also calculate the velocity at 4 cm from the wall of the pipe

Solution

Given: $D = 200\ mm = 0.2\ m$

$$u_{max} = 1.5\ m/s$$

(i) **Mean Velocity,** $\bar{u}$

Ratio of $\frac{u_{max}}{\bar{u}} = 2.0$ or $\frac{1.5}{\bar{u}} = 2.0$

$$\bar{u} = \frac{1.5}{2.0} = \mathbf{0.75\ m/s.}$$ **Ans**

(ii) **Radius at which** $\bar{u}$ occurs

The velocity, u, at any radius 'r' is given by (3.3)

$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2\left[1 - \frac{r^2}{R^2}\right]$$

But from equation (3.4) u_{max} is given by

$$u_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2 \qquad \therefore u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Now the radius r at which $u = \bar{u} = 0.75 m/s$

$\therefore \quad 0.75 = 1.5\left[1 - \left(\frac{r}{D/2}\right)^2\right]$

$\therefore \quad = 1.5\left[1 - \left(\frac{r}{0.2/2}\right)^2\right] = 1.5\left[1 - \left(\frac{r}{0.1}\right)^2\right]$

$\therefore \quad \frac{0.75}{1.50} = 1 - \left(\frac{r}{0.1}\right)^2$

$\therefore \quad \left(\frac{r}{0.1}\right)^2 = 1 - \frac{0.75}{1.50} = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \quad \frac{r}{0.1} = \sqrt{\frac{1}{2}} = \sqrt{0.5}$

$\therefore \quad r = 0.1\sqrt{0.5} = 0.1\ x\ .707 = .0707\ m = \mathbf{70.7\ mm}.$ **Ans**

(iii) **Velocity at 4 cm from the wall**

$$r = R - 4.0 = 10 - 4.0 = 6.0\ cm = 0.06\ m$$

∴ The velocity at a radius = 0.06 m

Or 4cm from pipe wall is given by equation (1)

$$u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] = 1.5\left[1-\left(\frac{.06}{0.1}\right)^2\right]$$

$$= 1.5[1.0 - .36] = 1.5\ x\ .64 = \mathbf{0.96\ m/s.} \quad \textbf{Ans}$$

Problem 3.4. Crude oil of μ = 1.5 poise and relative density 0.9 flows through a 20 mm diameter vertical pipe. The pressure gauges fixed 20 m apart read 58.86 N/cm^2 and 19.62 N/cm^2 as shown in Fig. 3.5. Find the direction and rate of flow through the pipe.

Solution

Given: $\mu = 1.5\ poise = \frac{1.5}{10} = 0.15\ Ns/m^2$

Relative Density $= 0.9$

∴ Density of oil $= 0.9\ x\ 1000 = 900\ kg/m^3$

Dia. of pipe, $D = 20\ mm = 0.02m$

$$L = 20\ m$$

$$P_A = 58.86N/cm^3 = 58.86\ \times\ 10^4 N/m^2$$

$$P_B = 19.62N/cm^3 = 19.62 \times\ 10^4 N/m^2$$

i. Direction of flow.

To find the direction of flow, the total energy $\left(\frac{P}{\rho g} + \frac{v^2}{2g} + Z\right)$ at the lower end A and at the upper end B is to be calculated. The direction of flow will be given from the higher energy to the lower energy. As here the diameter of the pipe is same and hence kinetic energy at A and B will be same. Hence to find the direction of flow, calculate $\left(\frac{P}{\rho g} + Z\right)$ at A and B.

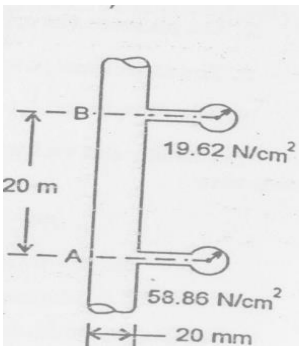


Fig. 3.5

Taking the level at A as datum. The value of $\left(\frac{P}{\rho g} + Z\right)$ at

$$A = \frac{P_A}{\rho g} + Z_A = \frac{58.86 \times 10^4}{900 \times 9.81} = 66.67m$$

$B = \frac{P_B}{\rho g} + Z_B = \frac{19.62 \times 10^4}{900 \times 9.81} + 20 = 42.22m \quad \{\because \rho = 900 kg/cm^3\}$

The value of $\left(\frac{P}{\rho g} + Z\right)$is higher at A and hence flow takes place from A to B. **Ans**

ii. **Rate of flow.**

The loss of pressure head for viscous flow through circular pipe is given by $h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$

For vertical pipe, h_f = loss of peizometric head

$$= \left(\frac{P_A}{\rho g} + Z_A\right) - \left(\frac{P_B}{\rho g} + Z_B\right) = 66.67 - 42.22 = 24.45\, m$$

$\therefore \qquad 24.45 = \frac{32 \times 0.15 \times \bar{u} \times 20.0}{900 \times 9.81 \times (.02)^2}$

$$\bar{u} = \frac{900 \times 9.81 \times 0.0004 \times 24.45}{32 \times 0.15 \times 20.0} = 0.889 = 0.9\, m/s$$

The Reynolds number should be calculated. If Reynolds number is less than 2000, the flow will be laminar and the above expression for loss of pressure head for laminar flow can be used.

Now Reynolds number $= \frac{\rho V D}{\mu}$

Where $\rho = 900\, kg/m^3$ and $V = \bar{u}$

$\therefore$ Reynolds number $= 900 \frac{0.9 \,x\, 0.02}{.15} = 108$

As the Reynolds number is less than 2000, the flow is laminar.

$\therefore$ rate of flow = average velocity x area

$$= \bar{u} \times \frac{\pi}{4} D^2 = 0.9 \times \frac{\pi}{4} .02^2\, m^3/s = 2.827\, x\, 10^{-4}\, m^3/s$$

$$= \mathbf{0.2827 litres/s}$$ **Ans.**

Problem 3.5. A fluid of viscosity 0.7 $Ns/m^2$and specific gravity 1.3 is flowing through a circular pipe of diameter 100 mm. The maximum shear stress at the pipe wall is given as 196.2 N/m^2, find (a) the pressure gradient (b) the average velocity and (c) Reynolds number of the flow.

Solution

Given: $\mu = 0.7\, Ns/m^2$

Specific Gravity $= 1.3$

$\therefore$ Density $= 1.3 \times 1000 = 1300\, kg/m^3$

Dia. of pipe, $D = 100mm = 0.1\, m$

Shear stress $\tau_0 = 196.2\, N/m^2$

(i) Pressure gradient $\boldsymbol{\frac{dp}{dx}}$

The maximum shear stress is given by

$$\tau_0 = \frac{-\partial p}{\partial x}\frac{R}{2} \quad \text{or} \quad 196.2 = \frac{-\partial p}{\partial x}\ x\ \frac{D}{4} = \frac{-\partial p}{\partial x}\ x\ \frac{0.1}{4}$$

$$\therefore \quad \frac{\partial p}{\partial x} = -\frac{196.2 \times 4}{0.1} = -7848\ N/m^2\ per\ m$$

$\therefore$ Pressure Gradient $= \boldsymbol{-7848\ N/m^2\ per\ m}$ **Ans**

(ii) Average velocity $\bar{u}$

$$\bar{u} = \frac{1}{2}u_{max}\frac{1}{2}\left[-\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2\right]$$

$$\left\{\because u_{max} = -\frac{1}{8\mu}\left(\frac{\partial p}{\partial x}\right)R^2\right\}; \qquad \left\{\because R = \frac{D}{2} = \frac{0.1}{2} = 0.05m\right\}$$

$$= \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2 = \frac{1}{8\ x\ 0.7} \times (7848) \times (0.05)^2$$

$$= 3.50\, m/s$$

(iii) Reynolds number, R_e

$$R_e = \frac{\bar{u} \times D}{v} = \frac{\bar{u} \times D}{\mu/\rho} = \frac{\rho \times \bar{u} \times D}{\mu}$$

$$= 1300 \times \frac{3.50\ x\ 0.1}{0.7} = \mathbf{650.00}$$ **Ans.**

Problem 3.6. What power is required per kilometer of a line to overcome the viscous resistance to the flow of glycerine through a horizontal pipe of diameter 100 mm at the rate of 10 litres/s? Take μ = 8 poise and kinematic viscosity (v) = 6.0 stokes.

Solution

Given:

Length of pipe, $L = 1km = 1000\, m$

Dia. of pipe, D = 100mm = 0.1 m

Discharge, $Q = 10\, litres/s = \frac{10}{1000} m^3/s = .01\, m^3/s$

Viscosity, $\mu = 8\, poise = \frac{8}{10} Ns/m^2$ $\left(\therefore 1poise = \frac{1}{10} Ns/m^2\right)$

Kinematic viscosity, $v = 6.0\, stokes$

$$= 6.0\, cm^2/s = 6.0 \times 10^{-4}\, m^2/s$$

Loss of pressure head is given by equation (9.6) as $h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$

Power required $= W \times h_f\ watts$ …(i)

Where W = weight of oil flowing per sec = $\rho g \times Q$

Substituting the values of W and h_f in equation (1),

Power required $= (\rho g \times Q)\ \frac{32\mu\bar{u}L}{\rho g D^2} watts = \frac{Q\ x\ 32\mu\bar{u}L}{D^2}$ cancelling ρg)

But $\bar{u} = \frac{Q}{Area} = \frac{.01}{\frac{\pi}{4}D^2} = \frac{.01\ x\ 4}{\pi\ x\ (.1)^2} = 1.273\ m/s$

∴ Power required $= \frac{.01 \times 32 \times 0.8 \times 1.273 \times 1000}{(0.1)^2}$ = 32588.8 W = **32.588 kW.** **Ans**

3.3 FLOW OF VISCOUS FLUID BETWEEN TWO PARALLEL PLATES

In this case also, the shear stress distribution, the velocity distribution across a section; the ratio of maximum velocity to average velocity and difference of pressure head for a given length of parallel plates, are to be calculated.

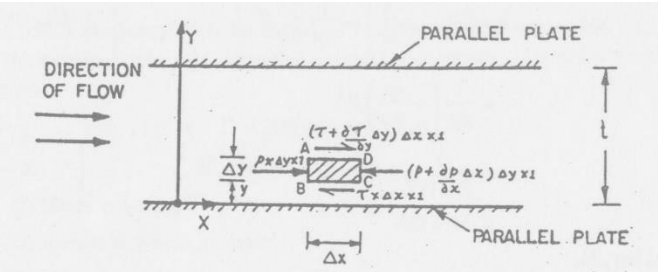


Fig.3.6 Viscous flow between two parallel plates

Consider two parallel fixed plates kept at a distance ‘t’ apart as shown in Fig. 3.6. A viscous fluid is flowing between these two plates from left to right. Consider a fluid element of length Δx and thickness Δy at a distance y from the lower fixed plate. If p is the intensity of pressure on the face AB of the fluid element then intensity of pressure on the face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$. Let τ be the shear stress acting on the face BC then the shear stress on the face AD will be $\left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)$. If the width of the element in the direction perpendicular to the paper is unity then the forces acting on the fluid element are:

1. The pressure force, p x Δy x 1 on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y \times 1$ on face CD.
3. The shear force, $\tau \times \Delta x \times 1$on face BC

4. The shear force, $\left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)\Delta x \times 1$ on face AD.

For steady and uniform flow, there is no acceleration and hence the resultant force in the direction of flow is zero.

$\therefore$ $$p\Delta y \times 1 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y \times 1 - \tau\Delta x \times 1 + \left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)\Delta x \times 1 = 0$$

Or $$-\frac{\partial p}{\partial x}\Delta x\Delta y + \frac{\partial \tau}{\partial x}\Delta y\Delta x = 0$$

Dividing by $\Delta x\Delta y$, we get $-\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = 0$ or $\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y}$ (3.7)

(i) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu\frac{du}{dy}$ from Newton's law of viscosity for laminar flow is substituted in equation (3.7).

$\therefore$ $$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu\frac{du}{dy}\right) = \mu\frac{\partial^2 u}{\partial y^2}$$

$\therefore$ $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating the equation w.r.t.y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1 \qquad \left\{\because \frac{\partial p}{\partial x} is\ constant\right\}$$

Integrating again $u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + C_1 y + C_2$ ..(3.8)

Where C_1 and C_2 are constants of integrations. Their values are obtained from the two boundary conditions that is (i) at y = 0, u = 0 (ii) at y = t, u = 0.

The substitution of at y = 0, u = 0 in equation (3.8) gives

$$0 = 0 + C_1\ x\ 0 + C_2\ or\ C_2 = 0$$

The substitution of at y = t, u = 0 in equation (3.8) gives

$$0 = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{t^2}{2} + C_1 \times\ t + 0$$

$\therefore$ $$C_1 = -\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{t^2}{2t} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}t$$

Substituting the values of C_1 and C_2 in equation (3.8)

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + y\left(-\frac{1}{2\mu}\frac{\partial p}{\partial x}t\right)$$

Or $u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2]$ …..(3.9)

In the above equation, μ, $\frac{\partial p}{\partial x}$ and t are constant. It means u varies with the square of y. hence equation (3.9) is an equation of a parabola. Hence velocity distribution across a section of the parallel plate is parabolic. This velocity distribution is shown in Fig. 3.7(a).

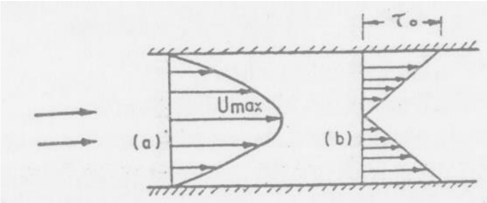


Fig. 3.7 Velocity distribution and shear stress distribution across a section of parallel plates

(ii) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum when y = t/2.

Substituting this value in equation (3.9), we get

$$u_{max} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[t \times \frac{t}{2} - \left(\frac{t}{2}\right)^2\right]$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{t^2}{2} - \frac{t^2}{4}\right] = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{t^2}{4} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2 \qquad ..(3.10)$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge (Q) across the section by the area of the section (t x 1). And the discharge Q is obtained by considering the rate of flow of fluid through the strip of thickness dy and integrating it. The ration of flow through strip is

$dQ =$ velocity at a distance y x area of strip

$$dQ = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2] \times dy \times 1$$

$$Q = \int_0^t dQ = \int_0^t -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2]\, dy$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{ty^2}{2} - \frac{y^3}{3}\right]_0^t = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{t^3}{2} - \frac{t^3}{3}\right]$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{t^3}{6} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^3$$

$$\therefore \qquad \bar{u} = \frac{Q}{Area} = \frac{-\frac{1}{12\mu}\frac{\partial p}{\partial x}t^3}{t\ x\ 1} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2 \qquad ..... (3.11)$$

Dividing equation (3.10) by equation (3.11), we get

$$\frac{u_{max}}{\bar{u}} = \frac{-\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2}{-\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2} = \frac{12}{8} = \frac{3}{2} \qquad \text{... (3.12)}$$

(iii) **Drop of Pressure head for a given length.** From equation (3.11) we have

$$\bar{u} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2 \qquad \text{or} \qquad \frac{\partial p}{\partial x} = -\frac{12\mu\bar{u}}{t^2}$$

Integrating this equation w.r.t.x. we get

$$\int_2^1 dp = \int_2^1 -\frac{12\mu\bar{u}}{t^2}dx$$

Or $$p_1 - p_2 = -\frac{12\mu\bar{u}}{t^2}[x_1 - x_2] = \frac{12\mu\bar{u}}{t^2}[x_2 - x_1]$$

Or $$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2} \qquad \{\because x_2 - x_1 = L\}$$

If h_f is the drop of pressure head, then

$$h_f = \frac{p_1 - p_2}{\rho g} = \frac{12\mu\bar{u}L}{\rho g t^2} \qquad \text{...(3.13)}$$

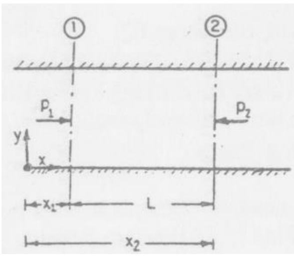


Fig. 3.8

(iv) **Shear stress Distribution.** It is obtained by substituting the value of u from equation (3.9) into

$$\tau = \mu\frac{\partial u}{\partial y}$$

$$\therefore \qquad \tau = \mu\frac{\partial u}{\partial y} = u\frac{\partial}{\partial y}\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2]\right] = \mu\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}[t - 2y]\right]$$

$$\tau = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[t - 2y] \qquad \text{...(3.14)}$$

In equation (3.14), $\frac{\partial p}{\partial x}$ and t are constant. Hence τ varies linearly with y. The shear stress distribution is shown in Fig. 3.7 (b). Shear stress is maximum when y = 0 or t that at the walls of the plates. Shear stress is zero when y = t/2 that is at centre –line between two plates. Max. shear stress (τ_0) is given by

$$\tau_0 = -\frac{1}{2}\frac{\partial p}{\partial x}t. \qquad \text{...(3.15)}$$

Problem 3.7. Calculate: (a) the pressure gradient along flow, (b) the average velocity, and (c) the discharge for an oil of viscosity 0.02 $\mathrm{Ns/m^2}$ flowing between two stationary parallel plates 1 m wide maintained 10 mm apart. The velocity midway between the plates is 2 m/s.

Solution

Given:

Viscosity, $\mu = .02\, Ns/m^2$

Width, $b = 1\, m$

Distance between plates, $t = 10\, mm = .01\, m$

Velocity midway between the plates, $U_{max} = 2\, m/s$

(i) Pressure Gradient $(\frac{dp}{dx})$

Using equation (3.10), $U_{max} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2$ or $2.0 = -\frac{1}{8 \times .02}\left(\frac{dp}{dx}\right)(.01)^2$

$$\frac{dp}{dx} = -\frac{2.0 \times 8 \times .02}{.01 \times .01} = \mathbf{-3200\, N/m^3} \textbf{ per m.} \qquad \textbf{Ans}$$

(ii) Average velocity($\bar{u}$)

Using equation (3.12), $\frac{U_{max}}{\bar{u}} = \frac{3}{2}$

$$\bar{u} = \frac{2U_{max}}{3} = \frac{2 \times 2}{3} = \mathbf{1.33\, m/s.} \qquad \textbf{Ans}$$

(iii) Discharge (Q) $= Area\ of\ flow \times \bar{u} = b \times t \times \bar{u} = 1 \times .01 \times 1.33 = \mathbf{0.0133\, m^3/s.}$ **Ans**

Problem 3.8 Determine (a) the pressure gradient, (b) the shear stress at the two horizontal parallel plates and (c) the discharge per metre width for laminar flow of oil with a maximum velocity of 2 m/s between two horizontal parallel fixed plates which are 100 mm apart. Given $\mu = 2.4525\, Ns/m^2$.

Solution

Given:

$$U_{max} = 2\, m/s\, , t = 100\, mm = 0.1m, \mu = 2.4525\, Ns/m^2$$

(i) **Pressure gradient**

Maximum Velocity, U_{max} is given by equation (3.10)

$$U_{max} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2$$

Substituting the values

Or $$2.0 = -\frac{1}{8 \times 2.4525}\frac{\partial p}{\partial x}(.1)^2$$

$\therefore$ $$\frac{\partial p}{\partial x} = -\frac{2.0 \times 8 \times 2.4525}{.1 \times .1} = \boldsymbol{-3924\, N/m^2\, per\, m}.$$ **Ans.**

(ii) Shear stress at the wall τ_0

τ_0 is given equation (3.15) as $\tau_0 = -\frac{1}{2}\frac{\partial p}{\partial x} \times\ t = -\frac{1}{2}(-3924) \times\ 0.1 = \mathbf{196.2\ N/m^2}$

(iii) Discharge per meter width, Q

$$= Mean\, Velocity\ x\ Area$$

$$= \frac{2}{3}U_{max}\ \times (t\ \times 1) = \frac{2}{3}\ \times 2.0\ \times .1\ \times\ 1 = \mathbf{0.133\, m^3/s.}$$ **Ans.**

Problem 3.9. An oil of viscosity 10 poise flows between two parallel fixed plates which are kept at a distance of 50 mm apart. Find the rate of flow between the plates if the drop of pressure in a length of 1.2 m be 0.3 N/cm^2. The width of the plates is 200 mm.

Solution

Given:

μ= 10 pois $= \frac{10}{10}Ns/m^2 = 1\, Ns/m^2$ $\left\{\because 1\, poise = \frac{1}{10}Ns/m^2\right\}$

$t = 50\ mm = 0.05\ m p_1 - p_2 = 0.3\, N/cm^2 = 0.3\ x\ 10^4\ N/m^2$

$L = 120\ m$, Width, $B = 200mm = 0.20\ m$

The difference of pressure is given by equation (3.13)

$$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2}$$

Substituting the values, we get

$$0.3\ \times 10^4 = 12\ \times\ 1.0 \times \frac{\bar{u}\ \times 1.2}{.05\ \times .05}$$

$\therefore$ $$\bar{u} = \frac{0.3\ x\ 10^4\ x\ 1.0\ x\ .05\ x\ .05}{12\ x\ 1.20} = 0.52 m/s$$

$\therefore$ Rate of flow $= \bar{u}\ \times\ Area = 0.52 \times (B \times t)$

$$= 0.52\ \times\ 0.20\ \times\ .05\ \ m^3/s = .0052\, m^3/s$$

$$= 0.0052 \times 10^3\ liter/s = \mathbf{5.2 litre/s.}$$ **Ans**

Problem 3.10. Water at 15℃ flows between two large parallel plates at a distance of 1.6 mm apart. Determine (a) the maximum velocity (b) the pressure drop per unit length and (c) the shear stress at the walls of plates if the average velocity is 0.2m/s. the viscosity of water at 15℃ is given by 0.01 poise.

Solution

Given:

$t = 1.6mm = 1.6\ x 10^{-3}\ m$

$= 0.0016\ m$

$\bar{u} = 0.2\ m/sec,\ \mu = .01 poise = \frac{.01}{10} Ns/m^2 = 0.001\ Ns/m^2$

(i) **Maximum Velocity,** U_{max} is given by equation(3.12)

i.e., $U_{max} = \frac{3}{2}\bar{u} = 1.5\ x\ 0.2 = \mathbf{0.3\ m/s}$ **Ans.**

(ii) **The pressure drop, (**$p_1 - p_2$**)** is given by equation (3.13)

$$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2}$$

Or pressure drop per unit length $= \frac{12\mu\bar{u}}{t^2}$

Or $\frac{\partial p}{\partial x} = 12\ \times\ \frac{.01}{10} \times\ \frac{.2}{(0.0016)^2} = 937.44\ N/m^2$ per m.

(iii) Shear stress at the walls is given by equation (3.15)

$$\tau_0 = -\frac{1}{2}\frac{\partial p}{\partial x}\ \times\ t = -\frac{1}{2}\ \times 937.44\ \times .0016 = \mathbf{0.749\ N/m^2}$$ **Ans.**

Problem 3.11. There is a horizontal crack 40 mm wide and 2.5 mm deep in a wall of thickness 100 mm. water leaks through the crack. Find the rate of leakage of water through the crack if the difference of pressure between the two ends of the crack is 0.02943 N/cm^2. Take the viscosity of water equal to 0.01 poise.

Solution

Given:

Width of crack, $b = 40mm = 0.04\ m$

Depth of crack, $t = 2.5mm = .0025\ m$

Length of crack, $L = 100m = 0.1m$

$$p_1 - p_2 = 0.02943\ N/cm^2 = 0.02943\ x\ 10^4\ N/m^2 = 294.3\ N/m^2$$

$$\mu = .01\ poise = \frac{.01}{10}\frac{Ns}{m^2}$$

Find the rate of leakage (Q)

$p_1 - p_2$ is given by equation (9.13)

$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2}$ or $294.3 = 12\ x \frac{.01}{10}\ x\ \frac{\bar{u}\ x\ 0.1}{(.0025\ x\ .0025)}$

$$\bar{u} = \frac{294.3\ \times\ 10\ \times (.0025\ \times\ .0025)}{12\ \times .01\ \times\ .1} = 1.5328 m/s$$

∴ Rate of leakage $\bar{u}\ x\ area\ of\ cross\ section\ of\ crack$

$$= 1.538 \times (b \times t)$$

$$= 1.538 \times .04 \times .0025\, m^3/s = 1.538 \times 10^{-4}\, m^3/s$$

$$= 1.538 \times 10^{-4} \times 10^3\, liter/s = \mathbf{0.1538}\, \boldsymbol{liter/s} \qquad \textbf{Ans.}$$

Problem 3.12. The radial clearance between a hydraulic plunger and the cylinder walls is 0.1 mm, the length of the plunger is 300 mm and diameter 100 mm. find the velocity of leakage and rate of leakage past the plunger at an instant when the difference of the pressure between the two ends of the plunger is 9 m of water. Take μ = 0.0127 poise

Solution

Given:

The flow through the clearance area will be the same as the flow between two parallel surfaces,

$$t = 0.1mm = 0.0001\, m$$

$$L = 300\, mm = 0.3\, m$$

Diameter $D = 100\, mm = 0.1m$

Difference of pressure $= \frac{p_1 - p_2}{\rho g} = 9\, m\ of\ water$

$$p_1 - p_2 = 9 \times 1000 \times 9.81\ N/m^2 = 88290\, N/m^2$$

Viscosity, $\mu = ,0127\, poise\ = \frac{.0127}{10}\frac{Ns}{m^2}$

(i). **Velocity of leakage** ($\bar{u}$). The average velocity ($\bar{u}$)is given by equation (3.11)

$$\bar{u} = \frac{1}{12\mu}\frac{\partial p}{\partial x}t^2$$

$$= \frac{1}{12 \times \frac{.0127}{10}} \times \frac{p_1 - p_2}{L} \times (.0001) \times (.0001)$$

$$= \frac{1}{12 \times \frac{.0127}{10}} \times \frac{88290}{0.3} \times (.0001) \times (.0001)$$

$$= .193\ m/s = \mathbf{19.3}\boldsymbol{cm/s} \qquad \textbf{Ans}$$

(ii) Rate of leakage, Q

$$Q = \bar{u}\ x\ Area\ of\ flow$$

$$= 0.193 \times \pi D \times t\ m^3/s = 0.193 \times \pi \times .1 \times .0001\, m^3/s$$

$$= 6.06 \times 10^{-6}\, m^3/s = 6.06 \times 10^{-6} \times 10^3 litre/s$$

$$= \mathbf{6.06} \times \mathbf{10^{-3}}\ \boldsymbol{litre/s} \quad \textbf{Ans.}$$

3.4 KINETIC ENERGY CORRECTION AND MOMENTUM CORRECTION FACTORS

Kinetic Energy correction factor is defined as the ratio of the kinetic energy of the flow per second based on actual velocity across a section to the kinetic energy of the flow per second based on average velocity across the same section. It is denoted by

$$\alpha = \frac{K.E/\sec \textit{ based on actual velocity}}{K.E/\sec \textit{ based on average velocity}} \qquad \ldots(3.16)$$

Momentum Correction Factor. It is defined as the ratio of momentum of flow per second based on actual velocity to the momentum of the flow per second based on average velocity across a section. It is denoted by β. Hence mathematically

$$\beta = \frac{\textit{momentum per second based on actual velocity}}{\textit{momentum}/\sec \textit{ based on average velocity}} \qquad \ldots(3.17)$$

Problem 3.13. Shows that the momentum correction factor and energy correction factor for laminar flow through a circular pipe are 4/3 and 2.0 respectively.

Solution.

(i) **Momentum Correction Factor or $\boldsymbol{\beta}$**

The velocity distribution through a circular pipe for laminar flow at any radius r is given by equation (3.3)

$$u = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)(R^2 - r^2) \qquad \ldots(1)$$

Consider an elementary area dA in the form of a ring at a radius r and width dr, then

$$dA = 2\pi r dr$$

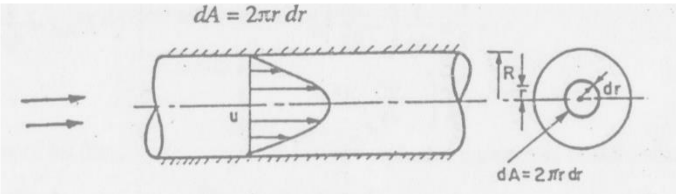


Fig. 3.9

Rate of fluid flowing through the ring

$$= dQ = velocity \times area\ of\ ring\ element$$

$$= u \times 2\pi r\ dr$$

Momentum of the fluid through ring per second

$$= mass \times velocity$$

$$= \rho \times dQ \times u = \rho \times 2\pi r\ dr \times u \times u = 2\pi\rho u^2 r\ dr$$

$\therefore$ Total actual momentum of the fluid per second across the section

$$= \int_0^R 2\pi\rho\, u^2 r\, dr$$

Substituting the value of u from (1),

$$= 2\pi\rho \int_0^R \left[\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)(R^2 - r^2)\right]^2 rdr$$

$$= 2\pi\rho \left[\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\right]^2 \int_0^R (R^2 - r^2)^2\, rdr$$

$$= 2\pi\rho \frac{1}{(16\mu^2)}\left(\frac{\partial p}{\partial x}\right)^2 \int_0^R (R^4 r + r^5 - 2R^2 r^3)\ dr$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \left[\frac{R^4 r^2}{2} + \frac{r^6}{6} - \frac{2R^2 r^4}{4}\right]_0^R = \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \left[\frac{R^6}{2} + \frac{R^6}{6} - \frac{2R^6}{4}\right]$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \frac{6R^6 + 2R^6 - 2R^6}{12}$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \times \frac{R^6}{6} = \frac{\pi\rho}{48\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6 \qquad \ldots(2)$$

Momentum of the fluid per second based on average velocity

$$= \frac{mass\ of\ fluid}{sec} \times\ average\ velocity$$

$$= \rho A\bar{u}\ \times \bar{u} = \rho A\bar{u}^2$$

Where A=area of cross-section = πR^2, $\bar{u} = average\ velocity = \frac{U_{max}}{2}$

$$= \frac{1}{2} \times \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) R^2 \qquad \left\{\because U_{max} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) R^2\right\}$$

$$= \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2$$

Momentum/ sec based on average velocity

$$= \rho\ \times \pi R^2\ \times \left[\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2\right]^2 = \rho\ \times\ \pi R^2 \frac{1}{64\mu^2}\left(-\frac{\partial p}{\partial x}\right)^2 R^4$$

$$= \frac{\rho\pi\left(-\frac{\partial p}{\partial x}\right)^2 R^6}{64\mu^2} \qquad \ldots(3)$$

$$\beta = \frac{momentum/\ sec\ based\ on\ actual\ velocity}{momentum/\ \mathrm{sec}\ based\ on\ average\ velocity}$$

$$= \frac{\frac{\pi\rho}{48\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6}{\frac{\pi\rho}{64\mu^2}\left(-\frac{\partial p}{\partial x}\right)^2 R^6} = \frac{64}{48} = \frac{4}{3}. \qquad \textbf{Ans}$$

(ii). **Energy Correction Factor,** α. Kinetic energy of the fluid flowing through the elementary ring of radius ‘r’ and width ‘dr’ per sec

$$= \frac{1}{2} \times\ mass \times\ u^2 = \frac{1}{2} \times\ \rho dQ\ \times\ u^2$$

$$= \frac{1}{2}\ \times \rho\ \times (u\ \times 2\pi r\ dr)\ \times u^2 =\ \frac{1}{2} \rho \times\ 2\pi r u^3\ dr = \pi \rho r u^3\ dr$$

∴ Total actual kinetic energy of flow per second

$$\int_0^R \pi \rho r u^3\ dr = \int_0^R \pi \rho r \left[\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)(R^2 - r^2)\right]^3 dr$$

$$= \pi \rho\ x\ \left[\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\right]^3 \int_0^R [R^2 - r^2]^3\ rdr$$

$$= \pi \rho\ x\ \frac{1}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \int_0^R (R^6 - r^6 - 3R^4 r^2 + 3R^2 r^4) r\ dr$$

$$= \frac{\pi \rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \int_0^R (R^6 r - r^7 - 3R^4 r^3 + 3R^2 r^5)\, dr$$

$$= \frac{\pi \rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \left[\frac{R^6 r^2}{2} - \frac{r^8}{8} - \frac{3R^4 r^4}{4} + \frac{3R^2 r^6}{6}\right]_0^R$$

$$= \frac{\pi \rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \left[\frac{R^8}{2} - \frac{R^8}{8} - \frac{3R^8}{4} + \frac{3R^8}{6}\right]$$

$$= \frac{\pi \rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 R^8 \left[\frac{12 - 3 - 18 + 12}{24}\right]$$

$$= \frac{\pi \rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \frac{R^8}{8} \qquad \ldots(4)$$

Kinetic energy of the flow based on average velocity

$$\frac{1}{2}\ \times\ mass\ \times\ \bar{u}^2 = \frac{1}{2}\rho A \bar{u}\ \times\ \bar{u}^2 = \frac{1}{2} \times\ \rho A \bar{u}^3$$

Substituting the value of A = (πR^2)

And $\qquad \bar{u} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2$

∴ Kinetic energy of the flows/sec

$$= \frac{1}{2}\ \times \rho\ \times\ \pi R^2\ \times\ \left[\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2\right]^3$$

$$= \frac{1}{2}\ \times \rho\ \times\ \pi R^2\ \times \frac{1}{64\ \times\ 8\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \times\ R^6$$

$$= \frac{\rho \pi}{128 \times 8\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3\ \times R^8 \qquad \ldots(5)$$

∴ $$\alpha = \frac{K.E/\sec based\ on\ actual\ velocity}{K.E/\sec based\ on0\ average\ velocity} = \frac{equation\ (4)}{equation\ (5)}$$

$$= \frac{\frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \frac{R^8}{8}}{\frac{\rho\pi}{128\ x 8\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3\ x\ R^8} = \frac{128\ x\ 8}{64\ x\ 8} = \mathbf{2.0} \qquad \textbf{Ans}.$$

3.5 POWER ABSORBED IN VISCOUS FLOW

For the lubrication of machine parts, an oil is used. Flow of oil in bearings is an example of viscous flow. If a highly viscous oil is used for lubrication of bearings, it will offer great resistance and thus a greater power loss will take place. But if a light oil is used, a required film between the rotating part and stationary metal surface will take place. Hence oil of correct viscosity should be used for lubrication. The power required to overcome the viscous resistance in the following cases will be determined:

1. Viscous resistance of journal Bearings,
2. Viscous resistance of Foot-step Bearings,
3. Viscous resistance of collar bearings.

3.5.1 Viscous Resistance of Journal Bearings. Consider a shaft of diameter D rotating in a journal bearing, the clearance between the shaft and journal bearing is filled with a viscous oil. The oil film in contact with journal bearing is stationary. Thus the viscous resistance will be offered by the oil to the rotating shaft.

Let N = speed of shaft in r.p.m.

t = thickness of oil film

L = length of oil film

∴ Angular speed of the shaft, $\omega = \frac{2\pi N}{60}$

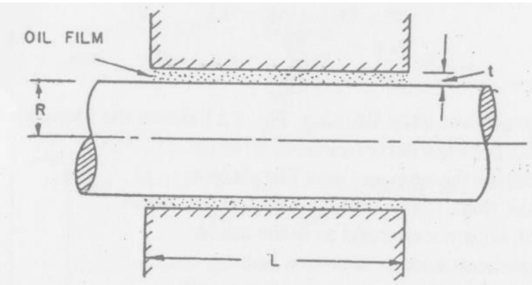


Fig. 3.10 Journal bearing

∴ Tangential speed of the shaft = $\omega \times R$ or $V = \frac{2\pi N}{60} \times \frac{D}{2} = \frac{\pi N D}{60}$

The shear stress in the oil is given by, $\tau = \mu \frac{du}{dy}$

As the thickness of oil film is very small, the velocity distribution in the oil film can be assumed as linear.

Hence $\frac{du}{dy} = \frac{V-0}{t} = \frac{V}{t} = \frac{\pi ND}{60\,x\,t}$

∴ $\tau = \mu \frac{\pi ND}{60 \times t}$

∴ Shear force or viscous resistance = τ × Area of surface of shaft

$$= \frac{\mu\pi ND}{60\,t} \times \pi DL = \frac{\mu\pi^2 D^2 NL}{60\,t}$$

∴ Torque required to overcome the viscous resistance,

T = Viscous Resistance x $\frac{D}{2}$

$$= \frac{\mu\pi^2 D^2 NL}{60\,t} \times \frac{D}{2} = \frac{\mu\pi^2 D^3 NL}{120\,t}$$

∴ Power absorbed in overcoming the viscous resistance

$$* P = \frac{2\pi NT}{60} = \frac{2\pi N}{60} \times \frac{\mu\pi^2 D^3 NL}{120\,t}$$

$$= \frac{\mu\pi^3 D^3 N^2 L}{60 \times 60\,t} Watts. \qquad \ldots(3.18)$$

Problem 3.14. A shaft having a diameter of 50 mm rotates centrally in a journal bearing having a diameter of 50.15 mm and length 100 mm. The angular space between the shaft and the bearing is filled with oil having viscosity of 0.9 poise. Determine the power absorbed in the bearing when the speed of rotation is 600 r.p.m.

Solution

Given:

Dia. of shaft, $D = 50\,mm\ or\ .05\,m$

Dia. of bearing, $D_1 = 50.15\,mm\ or\ .05015\,m$

Length, $L = 100\,mm\ or\ 0.1m$

$\mu\ of\ oil = 0.9\,poise = \frac{0.9}{10}\frac{Ns}{m^2}$

$N = 600 r.p.m$

∴ Thickness of oil film, $t = \frac{D_1 - D}{2} = \frac{50.15-50}{2}$

$$= \frac{0.15}{2} = 0.075\,mm = 0.075 \times 10^{-3} m$$

Tangential speed of shaft, $V = \frac{\pi ND}{60} = \frac{\pi\ x\ 0.05\ x\ 600}{60} = 0.5 \times \pi\ m/s$

Shear stress $\tau = \mu\frac{du}{dy} = \mu\frac{V}{t} = \frac{0.9}{10} \times \frac{0.5 \times \pi}{0{,}075 \times 10^{-3}} = 1883.52\,N/m^2$

∴ Shear force (F) $= \tau \times Area = 1883.52 \times \pi D \times L$

$$= 1883.52 \times \pi \times .05 \times 0.1 = 29.586\,N$$

Resistance Torque $T = F \times \frac{D}{2} = 29.586 \times \frac{.05}{2} = 0.7387\ Nm$

Power $= \frac{2\pi NT}{60} = \frac{2\pi\ x\ 600\ x\ 0.7387}{60} = \mathbf{46.41\ W.}$ **Ans.**

Problem 3.15 A shaft of 100 mm, diameter rotates at 60 r.p.m in a 200 mm long bearing. Taking that the two surfaces are uniformly separated by a distance of 0.5 mm and taking linear velocity distribution in the lubricating oil having dynamic viscosity of 4 centipoises, find the power absorbed in the bearing.

Solution

Given:

Dia. of shaft, $D = 100\ mm = 0.1\ m$

Length of bearing, $L = 200\ mm = 0.2\ m$

$t = 0.5mm = .5 \times 10^{-3}m$

$$\mu = 4\ centpoise = 0.04\ poise = \frac{0.04}{10}\frac{Ns}{m^2}$$

$$Nn = 60\ r.p.m$$

Using the equation (3.18), $P = \frac{\mu\pi^3 D^3 N^2 L}{60\times 60\ t}$

$= \frac{0.04}{10} \times \frac{\pi^3(.1)^3 \times (60)^2 \times 0.2}{60\times 60\times 0.5\times 10^{-3}} = \mathbf{4.961 \times 10^{-2}\ W.}$ **Ans**

Problem 3.16. A sleeve, in which a shaft of diameter 75 mm, is running at 1200 r.p.m., is having a radial clearance of 0.1 mm. calculate the torque resistance if length of sleeve is 100 mm and the space is filled with oil of dynamic viscosity 0.96 poise.

Solution

Given:

Dia. of shaft, $D = 75mm = 0.075m$

$N = 1200\ r.p.m.$

$t = 0.1\ mm = 0.1 \times 10^{-3}\ m$

Length of sleeve, $L = 100mm = 0.1\ m$

$$\mu = 0.96\ poise = \frac{0.96}{10}\frac{Ns}{m^2}$$

Tangential velocity of shaft, $V = \frac{\pi ND}{60} = \frac{\pi\ x.075\ x\ 1200}{60} = 4.712m/s$

Shear stress, $\tau = \mu\frac{V}{t} = \frac{.96}{10} \times \frac{4.712}{.1\times 10^{-3}} = 4523.5\ N/m^2$

Shear force, $F = \tau \times \pi DL$

$= 4523.5\pi \times .075 \times .1 = 106.575\ N$

$\therefore$ Torque resistance $= F \times \frac{D}{2}$

$$= 106.575 \times \frac{.075}{2} = \mathbf{3.996\ Nm.} \qquad \textbf{Ans.}$$

Problem 3.17. A shaft 100 mm diameter runs in a bearing of length 200 mm with a radial clearance of 0.025 mm at 30 r.p.m. Find the viscosity of the oil, if the power required to overcome the viscous resistance is 183.94 Watts.

Solution

Given:

$$D = 100\ mm = 0.1m$$

$$L = 200mm = 0.2\ m$$

$$t = .025\ mm = 0.025\ x\ 10^{-3}m$$

$$N = 30\ r.p.m.; H.P = 0.25$$

The h.p. is given by equation (3.18) as

$$P = \frac{\mu\pi^3 D^3 N^2 L}{60\ x\ 60\ t} \qquad \text{or} \qquad 183.94 = \frac{\mu\pi^3 \times (.1)^3 \times (30)^2 \times 0.2}{60 \times 60 \times 0.025 \times 10^{-3}}$$

$$\mu = \frac{183.94 \times 60 \times 60 \times 0.025 \times 10^{-3}}{\pi^3 \times .001 \times 900 \times 0.2} \frac{Ns}{m^2}$$

$$= 2.96 \frac{Ns}{m^2} = 2.96 \times 10 = \mathbf{29.6\ poise.} \qquad \textbf{Ans.}$$

3.5.2. Viscous Resistance of Foot-step Bearing. Fig. 3.11 shows the foot-step bearing, in which a vertical shaft is rotating. An oil film between the bottom surface of the shaft and bearing is provided, to reduce the wear and tear. The viscous resistance is offered by the oil to the shaft. In this case the radius of the surface of the shaft in contact with oil is not constant as in the case of the journal bearing. Hence, viscous resistance in foot- step bearing is calculated by considering an elementary circular ring of radius r and thickness dr as shown in Fig. 3.11.

Let N = speed of the shaft

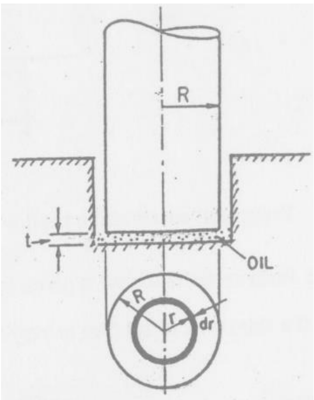


Fig. 3.11 Foot-step bearing

t = thickness of oil film

R= radius of the shaft

Area of elementary ring = $2\pi r dr$

Now shear stress is given by

$$\tau = \mu \frac{du}{dy} = \mu \frac{V}{t}$$

Where V is tangential velocity of shaft at radius r and is equal to

$$\omega . r = \frac{2\pi N}{60} r$$

∴ Shear force on the ring $= dF = \tau \times area\ of\ elementary\ ring$

$$= \mu \times \frac{2\pi N}{60} \times \frac{r}{t} \times 2\pi r dr = \frac{\mu}{15} \frac{\pi^2 N r^2}{t} dr$$

∴ Torque required to overcome the viscous resistance,

$$dT = dF\ x\ r$$

$$= \frac{\mu}{15t} \pi^2 N r^2 dr \times r = \frac{\mu}{15t} \pi^2 N r^3 dr \qquad \text{…(3.19 A)}$$

∴ Total torque required to overcome the viscous resistance,

$$T = \int_0^R dT = \int_0^R \frac{\mu}{15t} \pi^2 N r^3 dr$$

$$= \frac{\mu}{15t} \pi^2 N \int_0^R r^3 dr = \frac{\mu}{15t} \pi^2 N \left[\frac{r^4}{4}\right]_0^R = \frac{\mu}{15t} \pi^2 N \frac{R^4}{4}$$

$$= \frac{\mu}{60t} \pi^2 N R^4 \qquad \text{…(3.19 B)}$$

∴ Power absorbed, $P = \frac{2\pi N}{60} \times \frac{\mu}{60t} \pi^2 N R^4 = \frac{\mu \pi^2 N^2 R^4}{60\ x\ 30t}$. …(3.20)

Problem 3.18. Find the torque required to rotate a vertical shaft of diameter 100 mm at 750 r.p.m. The lower end of the shaft rests in a foot-step bearing. The end of the shaft and surface of the hearing are both flat and are separated by an oil film of thickness 0.5 mm. The viscosity of the oil is given 1.5 poise.

Solution

Given:

Dia. of shaft, $D = 100\ mm = 0.1\ m$

∴ $R = \frac{D}{2} = \frac{0.1}{2} = 0.05\ m$

$N = 750\ r.p.m$

Thickness of oil film, $t = 0.5\ mm = 0.0005\ m$

$$\mu = 1.5 poise = \frac{1.5}{10}\frac{Ns}{m^2}$$

The torque required is given by equation (3.19) or

$$T = \frac{\mu}{60t}\pi^2 N R^4 \; Nm$$

$$= \frac{1.5}{10} \times \frac{\pi^2 \, x \, 750 \, x \, (.05)^4}{60 \, x \, .0005} = \mathbf{0.2305 Nm}. \qquad \textbf{Ans.}$$

Problem 3.19. Find the power required to rotate a circular disc of diameter 200 mm at 1000 r.p.m. The circular disc has a clearance of 0.4 mm from the bottom flat plate and the clearance contains oil of viscosity 1.05 poise.

Solution

Given:

Dia. of disc, $D = 200 \, mm = 0.2 \, m$

∴ $R = \frac{D}{2} = \frac{0.2}{2} = 0.1 \, m$

$N = 1000 \, r.p.m$

Thickness of oil film, $t = 0.4 \, mm = 0.0004 \, m$

$$\mu = 1.05 poise = \frac{1.05}{10}\frac{Ns}{m^2}$$

The power required to rotate the disc is given by equation (3.20) or

$$P = \frac{\mu \pi^3 N^2 R^4}{60 \times 30 \times t} \; watts$$

$$= \frac{1.05}{10} \times \frac{\pi^3 \, x \, 1000^2 \, x \, (0.1)^4}{60 \, x \, 30 \, x \, .0004} = \mathbf{452.1 \, W}.$$

3.5.3 Viscous Resistance of Collar bearing. Fig. 3.12 shows the collar bearing, where the face of the collar is separated from the bearing surface by an oil film of uniform thickness.

Let N = speed of the shaft in r.p.m.

R_1 = Internal radius of the collar

R_2 = External radius of the collar

t = Thickness of oil film.

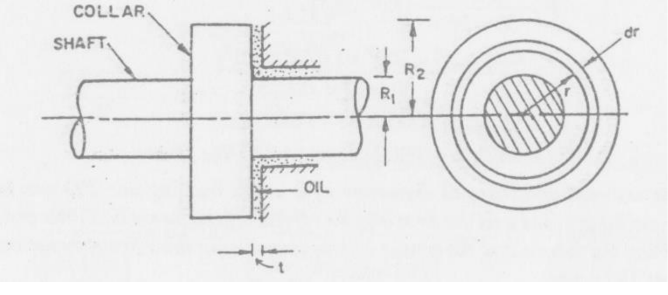


Fig. 3.12 Collar bearing.

Consider an elementary circular ring of radius ‘r’ and width dr of the bearing surface. Then the torque (dT) required to overcome the viscous resistance on the elementary circular ring is the same as given by equation (3.19A) or

$$dT = \frac{\mu}{15t}\pi^2 N r^3 dr$$

∴ Total torque, required to overcome the viscous resistance, on the whole collar is

$$T = \int_{R_1}^{R_2} dT = \int_{R_1}^{R_2} \frac{\mu}{15t}\pi^2 N r^3 dr = \frac{\mu}{15t}\pi^2 N \left[\frac{r^4}{4}\right]_{R_1}^{R_2}$$

$$= \frac{\mu}{15t\times 4}\pi^2 N\left[{R_2}^4 - {R_1}^4\right] = \frac{\mu}{60t}\pi^2 N\left[{R_2}^4 - {R_1}^4\right] \qquad \ldots(3.21)$$

∴ Power absorbed in overcoming viscous resistance

$$P = \frac{2\pi NT}{60} = \frac{2\pi N}{60} \times \frac{\mu}{60t}\pi^2 N\left[{R_2}^4 - {R_1}^4\right]$$

$$= \frac{\mu\pi^3 N^2}{60\times 30t}\left[{R_2}^4 - {R_1}^4\right] Watts. \qquad (3.22)$$

Problem 3.20. A collar bearing having external and internal diameters 150 mm and 100 mm respectively is used to take the thrust of a shift. An oil film of thickness 0.25 mm is maintained between the collar surface and the bearing. Find the power lost in overcoming the viscous resistance when the shaft rotates at 300 r.p.m. Take $\mu = 0.91$poise.

Solution

Given:

External Dia. Of collar, $D_2 = 150\, mm = 0.15\, m$

∴ $R_2 = \frac{D}{2} = \frac{0.15}{2} = 0.075\, m$

Internal Dia. Of collar, $D_1 = 100\, mm = 0.10\, m$

∴ $R_1 = \frac{D}{2} = \frac{0.1}{2} = 0.05\, m$

Thickness of oil film, $t = 0.25\, mm = 0.00025\, m$

$$N = 300\, r.p.m$$

$$\mu = 0.91\, poise = \frac{0.91}{10}\frac{Ns}{m^2}$$

The power required is given by equation (9.22) or

$$P = \frac{\mu\pi^3 N^2}{60\times 30t}\left[{R_2}^4 - {R_1}^4\right]$$

$$= \frac{0.91}{10}\, x\, \frac{\pi^3\, x\, 300^2\, x\, [.075^4 - .05^4]}{60\, x\, 30\, x\, .00025}$$

$$= 564314[.00003164 - .00000625]$$

$$= 564314 \times .00002539 = \mathbf{14.327W}. \qquad \textbf{Ans}$$

Problem 3.21. The external and internal diameter of a collar bearing are 200 mm and 150 mm respectively. Between the collar surface and the bearing, an oil film of thickness 0.25 mm and viscosity 0.9 poise, is maintained. Find the torque and the power lost in overcoming the viscous resistance of the oil when the shaft is running at 250 r.p.m.

Solution

Given:

$$D_2 = 200\, mm = 0.2\, m$$

$\therefore$ $$R_2 = \frac{D}{2} = \frac{0.2}{2} = 0.1\, m$$

$$D_1 = 150\, mm = 0.15\, m$$

$\therefore$ $$R_1 = \frac{D}{2} = \frac{0.15}{2} = 0.075\, m$$

$$t = 0.25\, mm = 0.00025\, m$$

$$\mu = 0.91\, poise = \frac{0.91}{10}\frac{Ns}{m^2}$$

Torque required is given by equation (9.21)

$\therefore$ $$T = \frac{\mu}{60t}\pi^2 N\left[{R_2}^4 - {R_1}^4\right] = \frac{0.9}{10} \times \frac{\pi^2\, x\, 250\left[0.1^4 - .075^4\right]}{60\, x\, 0.00025} Nm$$

$$= 14804.4[.0001 - .00003164] = \mathbf{1.0114 Nm}. \qquad \textbf{Ans.}$$

$\therefore$ Power lost in viscous resistance

$$= \frac{2\pi NT}{60} = \frac{2\pi\, x\, 250\, x\, 1.0114}{60} = \mathbf{26.48\, W}. \qquad \textbf{Ans}$$

3.6 LOSS OF HEAD DUE TO FRICTION IN VISCOUS FLOW

The loss of pressure head, h_f in a pipe of diameter D, in which a viscous fluid of viscosity μ is flowing with a velocity $\bar{u}$ is given by Hagen Poiseuille formula i.e., by equation (3.6) as

$$h_f = \frac{32\mu\bar{u}L}{\rho g D^2} \qquad \ldots\text{(i)}$$

Where $L = length\ of\ pipe$

The loss of head due to friction is given by

$$h_f = \frac{4.f.L.V^2}{D\, x 2g} = \frac{4.f.L.\bar{u}^2}{D\, x 2g} \qquad \ldots\text{(ii)}$$

$$\{\because velocity\ in\ pipe\ is\ always\ average\ velocity \because V = \bar{u}\}$$

Where f = co-efficient of friction between the pipe and fluid.

Equating (i) and (ii), we get $\frac{32\mu\bar{u}L}{\rho g D^2} = \frac{4.f.L.\bar{u}^2}{D \times 2g}$

$$f = \frac{32\mu\bar{u}L \times D \times 2g}{4.f.L.\bar{u}^2 .\rho g.D^2} = \frac{16\mu}{\bar{u}.\rho.D} \qquad \{\therefore \bar{u} = V\}$$

$$= 16 \times \frac{\mu}{\rho VD} = 16 \times \frac{1}{R_e}$$

Where $\frac{\mu}{\rho VD} = \frac{1}{R_e}$ and R_e = Reynolds number = $\frac{\rho VD}{\mu}$

$\therefore$ $$f = \frac{16}{R_e} \qquad \ldots(3.23)$$

The friction coefficient f is also often defined as λ=4f where λ is known as the friction factor.

The head lost due to the friction for a pipe length , L is given as $h_f = \frac{\lambda L\bar{u}^2}{D\times 2g}$; $\lambda = \frac{64}{Re}$

For a pipe, given $h_f = \frac{32\mu uL}{\rho g D^2}$, $\lambda = \frac{\mu}{\rho D\bar{u}}$ $or\ f = \frac{16}{Re}$. The function factor can be expressed as 3.1 by substituting pressure drop for shear stress:

$$\frac{dp}{dx} = \frac{2\tau}{r}$$

Problem 3.22. Water is flowing through a 200 mm diameter pipe with co-efficient of friction f = 0.04. The shear stress at a point 40mm from the pipe axis is 0.0981 N/cm^2. Calculate the shear stress at the pipe wall.

Solution

Given:

Diameter. of pipe, D = 200 mm =0.2m

Coeffiecient of friction, $f = 0.04$

Shear stress at r = 40 mm, $\tau = 0.000981\, N/cm^2$

Let the shear stress at pipe wall = τ_0.

First find whether the flow is viscous. The shear stress in case of viscous flow through a pipe is given by the equation (3.1) as

$$\tau_0 = -\frac{\partial p}{\partial x}\frac{r}{2}$$

But $\frac{\partial p}{\partial x}$ is constant across a section. Across a section, there is no variation of x and hence there is no variation of p.

$\therefore$ $\tau \propto r$

At the pipe wall, radius = 100 mm and shear stress is τ_0

$$\frac{\tau}{r} = \frac{\tau_0}{100}\, or\ \frac{0.00981}{40} = \frac{\tau_0}{100}$$

$\therefore$ $$\tau_0 = \frac{100\times 0.00981}{40} = 0.0245\, N/cm^2$$

Problem 3.24. A pipe of diameter 20 cm and length 10^4 m is laid at a slope of 1 in 200. An oil of sp.gr. 0.9 and viscosity 1.5 poise is pumped at a rate of 20 litres per second. Find the head lost due to friction. Also calculate the power required to pump the oil.

Solution

Given:

Diameter of pipe, $D = 20\ cm = 0.02m$

Length of pipe, $L = 10000m$

Slope of pipe, $i = 1\ in\ 200 = \frac{1}{200}$

Sp.gr. of oil, $s = 0.9$

∴ Density of oil $\rho = 0.9\ x\ 1000 = 900\ kg/m^3$

Viscosity of oil,

Discharge $Q = 20\ liter/s = .02\ m^3/s$ $\{\because 1000\ liters = 1m^3\}$

∴ Velocity of flow, $\bar{u} = \frac{Q}{Area} = \frac{0.02}{\frac{\pi}{4}D^2} = \frac{0.02}{\frac{\pi}{4}(0.20)^2} = 0.6366\ m/s$

∴ $R_e = Reynolds\ number$

$$= \frac{\rho VD}{\mu} = \frac{900\ x\ 0.6366\ x\ .2}{\frac{1.5}{10}}$$

$$= \frac{900\ x\ 0.6366\ x\ .2\ x\ 10}{1.5} \qquad \{\because V = \bar{u} = 0.6366\}$$

$$= 763.89$$

As the Reynold number is less than 2000, the flow is viscous. The co-efficient of friction for viscous flow is given by equation (3.23) as

$$f = \frac{16}{R_e} = \frac{16}{763.89} = 0.02094$$

∴ Head lost due to friction, $h_f = \frac{4.f.L.\bar{u}^2}{D\ x 2g}$

$$= \frac{4\times.02094\times10000\times(.6366)^2}{.2\times2\times9.81} m = \mathbf{86.50\ m.} \qquad \textbf{Ans.}$$

Due to slope of pipe 1 in 200, the height through which oil is to be raised by pump

$$= slope\ x\ length\ of\ pipe$$

$$= i \times L = \frac{1}{200} \times 10000 = 50\ m$$

∴ Total head against which pump is to work,

$$H = h_f + i \times L = 86.50 + 50 = 136.50\ m$$

∴ Power required to pump the oil

$$= \frac{\rho g.Q.H}{1000} = \frac{900\ x\ 9.81\ x\ 0.20\ x\ 136.50}{1000} = \mathbf{24.1 kW.} \quad \textbf{Ans.}$$

3.7 MOVEMENT OF PISTON IN DASH-POT

Consider a piston moving in a vertical dash-pot containing oil as shown in Fig. 3.13.

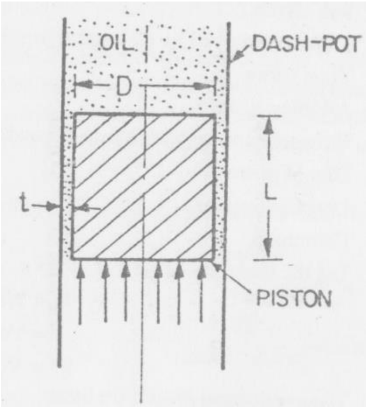


Fig. 3.13

Let $D =$ diameter of piston,

$L = length\ of\ piston,$

W=Weight of piston,

$\mu =$ Viscosity of oil,

V = Velocity of piston,

$\bar{u} = Average\ velocity\ of\ oil\ in\ the\ clearance,$

$l =$ Clearance between the dash-pot and piston,

Δp=Difference of pressure intensities between the two ends of the piston.

The flow of oil through clearance is similar to the viscous flow between two parallel plates.

The difference of pressure for parallel plates for length 'L' is given by

$$\Delta p = \frac{12\mu\bar{u}L}{t^2} \qquad \text{...(i)}$$

Also the difference of pressure at the two ends of piston is given by,

$$\Delta p = \frac{Weight\ of\ piston}{Area\ of\ piston} = \frac{W}{\frac{\pi}{4}D^2} = \frac{4W}{\pi D^2} \qquad \text{...(ii)}$$

Equating (i) and (ii), we get $\frac{12\mu\bar{u}L}{t^2} = \frac{4W}{\pi D^2}$

$$\therefore \qquad \bar{u} = \frac{4W}{\pi D^2}\ x\ \frac{t^2}{12\mu L} = \frac{Wt^2}{3\pi\mu L D^2} \qquad \text{...(iii)}$$

V is the velocity of piston or the velocity of oil in dash-pot in contact with piston. The rate of flow of oil in dash-pot

= velocity x area of dash-pot =V x $\frac{\pi}{4}D^2$

Rate of flow through clearance = velocity through clearance x area of clearance

$\therefore \qquad = \bar{u}\ x\ \pi D\ x\ t = \text{V x}\ \frac{\pi}{4} D^2$

$\therefore \qquad \bar{u} = \text{V x}\ \frac{\pi}{4} D^2 \times \frac{1}{\pi D\ x\ t} = \frac{VD}{4t}$ …(iv)

Equating the value of $\bar{u}$ from (iii) and (iv), we get

$$\frac{Wt^2}{3\pi\mu LD^2} = \frac{VD}{4t}$$

$$\mu = \frac{4t^3W}{3\pi LD^3V} = \frac{4Wt^3}{3\pi LD^3V}. \qquad ..(3.24)$$

Problem 3.25. An oil dash-pot consists of a piston moving in a cylinder having oil. This arrangement is used to damp out the vibrations. The piston falls with uniform speed and covers 5 cm in 100 seconds. If an additional weight of 1.36 N is placed on the top of the piston, it falls through 5 cm in 86 seconds with uniform speed. The diameter of the piston is 7.5cm and its length is 10cm. the clearance between the piston and the cylinder is 0.12 cm which is uniform throughout. Find the viscosity of oil.

Solution

Given:

Distance covered by piston due to self weight, = 5 cm

Time taken, = 100 seconds

Addition weight, = 1.36 N

Time taken to cover 5 cm due to addition weight, = 86 seconds

Dia. Of piston, D = 7.5 cm = 0.075 m

Length of piston, L = 10 cm = 0.1 m

Clearance, t = 0.12 cm = 0.0012 m

Let viscosity of oil $= \mu$

$W = weight\ of\ piston,$

$V = Velocity\ of\ piston\ without\ additional\ weight,$

$V^* = Velocity\ of\ piston\ with\ addition\ weight.$

Using equation (3.24), we have

$$\mu = \frac{4Wt^3}{3\pi LD^3V} = \frac{4[W + 1.36]t^3}{3\pi LD^3V^*}$$

Or $\qquad \frac{W}{V} = \frac{W+1.36}{V^*} \qquad \left(cancelling\ \frac{4Wt^3}{3\pi LD^3}\right)$

Or $\qquad \frac{V}{V^*} = \frac{W}{W+1.36}$ …(i)

But $\qquad V = Velocity\ of\ piston\ due\ to\ self\ weight\ of\ piston$

$$= \frac{distance\ covered}{time\ taken} = \frac{5}{100}\ cm/s$$

Similarly, $$V^* = \frac{Distance\ covereddue\ to\ self\ weight + additonal\ weight}{time\ taken}$$

$$= \frac{5}{86} cm/s$$

∴ $$\frac{V}{V^*} = \frac{5}{100}\ x\ \frac{86}{5} = 0.86$$

Equating (i) and (ii), we get $\frac{W}{W+1.36} = 0.86$

Or $$W = 0.86W + .86\ x\ 1.36$$

Or $$W - 0.86W = .86\ x\ 1.36$$

∴ $$W = \frac{0.86 \times 1.36}{0.14} = 8.354N$$

Using equation (3.24), we get $\mu = \frac{4Wt^3}{3\pi LD^3V}$

$$= \frac{4 \times 8.354 \times (.0012)^3}{3\pi \times (0.075)^2 \times .10 \times \left(\frac{5}{100} \times \frac{1}{100}\right)} \qquad \left\{\because V = \frac{5}{100} cm = \frac{5}{100}\ x\ \frac{1}{100} m\right\}$$

$$= 0.29\ Ns/m^2 = 0.29 \times 10\ poise = \mathbf{2.9}\ \boldsymbol{poise}.$$ **Ans**

3.8 METHODS OF DETERMINATION OF COEFFICIENT OF VISCOSITY

The following are the experimental methods of determining the co-efficient of viscosity of a liquid.

1. Capillary tube method
2. Falling sphere resistance method
3. By rotating cylinder method, and
4. Orifice type viscometer.

The apparatus used for determining the viscosity of a liquid is called viscometer.

Capillary tube method. In capillary tube method, the viscosity of a liquid is calculated by measuring the pressure difference for a given length of the capillary tube. The Hagen Poiseuille law is used for calculating viscosity.

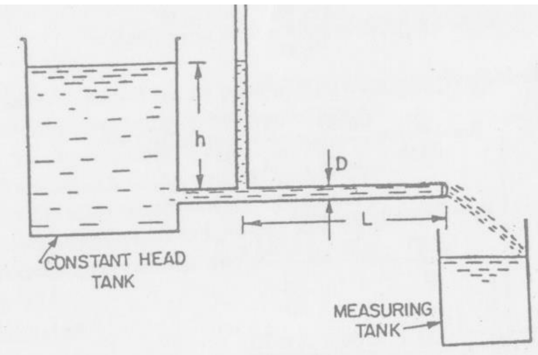


Fig. 3.14 Capillary tube viscometer

Fig. 3.14 shows the capillary tube viscometer. The liquid whose viscosity is to be determined is filled in a constant head tank. The liquid is maintained at constant temperature and is allowed to pass through the capillary tube from the constant head tank. Then, the liquid is collected in a measuring tank for a given time. Then the rate of liquid collected in the tank per second is determined. The pressure head 'h' is measured at a point far away from the tank as shown in Fig.3.14.

Then $h =$Difference of pressure head for length L.

The pressure outlet is atmospheric,

Let D= diameter of capillary tube

L = Length of tube for which difference of pressure head is known,

$\rho =$ Density of fluid,

μ = coefficient of viscosity,

Using Hagen Poiseuille's formula, $h = \frac{32\mu\bar{u}L}{\rho g\, D^2}$

But $$\bar{u} = \frac{Q}{Area} = \frac{Q}{\frac{\pi}{4}D^2}$$

Where Q is the rate of liquid flowing through the tube.

$$h = \frac{32\mu\, x \frac{D}{\frac{\pi}{4}D^2}\, x\, L}{\rho g\, D^2} = \frac{128\mu Q.L}{\pi\rho g\, D^4}$$

or $$\mu = \frac{\pi\rho g D^4}{128\mu QL} \quad \text{or} \quad \mu = \frac{\pi\rho g h\bar{D}}{128QL} \qquad \ldots(3.25)$$

Measurement of D should be done accurately.

3.8.1 Falling Sphere Resistance Method

Theory. This method is based on Stoke's law, according to which the drag force, F on a small sphere moving with a constant velocity, U through a viscous fluid of viscosity, μ for viscous condition is given by

$F = 3\pi\mu Ud$ …(i)

Where d = diameter of sphere

U = velocity of sphere

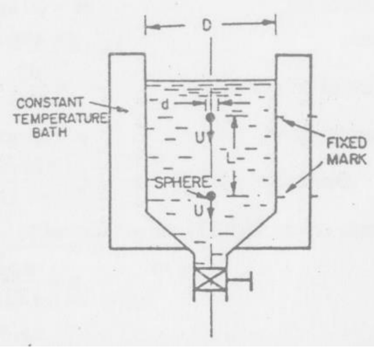


Fig. 3.15 Falling sphere resistance method.

When the sphere attains constant velocity U, the drag force is the difference between the weight of sphere and buoyant force acting on it.

Let L = distance travelled by sphere in viscous fluid,

t = time taken by sphere to cover distance L,

ρ_s = density of sphere,

ρ_f = density of fluid,

W = weight of sphere,

And F_B = buoyant force acting on sphere.

Then constant velocity of sphere, $U = \frac{L}{t}$

Weight of sphere, $W = volume \times density\ of\ sphere \times g$

$= \frac{\pi}{6} d^3\ x\ \rho_s\ x\ g$ $\left\{\because volume\ of sphere = \frac{\pi}{6} d^3\right\}$

And buoyant force, $F_B = weight\ of\ fluid\ displaced$

$= volume\ of\ liquid\ displaced \times density\ of\ fluid \times g$

$= \frac{\pi}{6} d^3 x\ \rho_f\ x\ g$ $\{volume\ of\ liquid\ displaced = volume\ of\ sphere\}$

For equilibrium,

Or Drag force = Weight of sphere – buoyant force

$$F = W - F_B$$

Substituting the values of F, W and F_B, we get

$$3\pi\mu Ud = \frac{\pi}{6} d^3 \times \rho_s \times g - \frac{\pi}{6} d^3 \times \rho_f \times g = \frac{\pi}{6} d^3\ \times\ g[\rho_s - \rho_f]$$

Or $\mu = \frac{\pi}{6} \frac{d^3 \times g[\rho_s - \rho_f]}{3\pi Ud} = \frac{gd^2}{18U}[\rho_s - \rho_f]$ …(3.26)

Where ρ_f = Density of liquid

Thus in equation (3.26), the values of d, U, ρ_s and ρ_f are known and hence the viscosity of liquid can be determined.

Method. This method consist of a tall vertical transparent cylindrical tank, which is filled with the liquid whose viscosity is to be determined. This tank is surrounded by another transparent tank to keep the temperature of liquid in the cylindrical tank constant.
A spherical ball of small diameter 'd' is placed on the surface of the liquid. Provision is made to release the drop of this ball. After a short distance of travel, the ball attains a constant velocity. The time to travel a known vertical distance between two fixed marks on the cylindrical tank is noted to calculate the constant velocity U of the ball. Then with the known values of d, ρ_s, ρ_f the viscosity μ of the fluid is calculated by using equation (3.26).

3.8.3. Rotating Cylinder Method. This method consists of two concentric cylinders of radii $R_1$and R_2 as shown in Fig.3.16. the narrow space between the two cylinders is filled with the liquid whose viscosity is to be determined. The inner cylinder is held stationary by means of a torsional spring while outer cylinder is rotated at constant angular speed ω.
The torque T acting on the inner cylinder is measured by the torsional spring. The torque on the inner cylinder is measured by the torsional spring. The torque on the inner cylinder must be equal and opposite to the torque applied on the outer cylinder.
The torque applied on the outer cylinder is due to viscous resistance provided by liquid in the annular space and at the bottom of the inner cylinder.

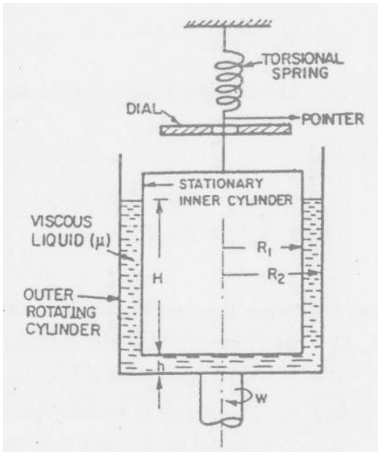


Fig. 3.16 Rotating cylinder viscometer

Let ω = angular speed of outer cylinder.

Tangential (peripheral) speed of outer cylinder

$$= \omega \; x \; R_2$$

Tangential velocity of liquid layer in contact with outer cylinder will be equal to the tangential velocity of outer cylinder.

∴ Velocity of liquid layer with outer cylinder = $\omega \; x \; R_2$

Velocity of liquid layer with inner cylinder = 0

$\{inner\ cylinder\ is\ stationary\}$

∴ Velocity gradient over the radial distance $(R_2 - R_1)$

$= \frac{du}{dy} = \frac{\omega R_2 - 0}{R_2 - R_1} = \frac{\omega R_2}{R_2 - R_1}$

∴ Shear stress (τ) $\quad = \mu \frac{du}{dy} = \mu \frac{\omega R_2}{(R_2 - R_1)}$

∴ Shear stress (F) $\quad$ = shear stress x area of cylinder

$= \tau \times 2\pi R_1 H \quad \{\because shear\ stress\ is\ acting\ on\ surface\ area = 2\pi R_1\ x\ H\}$

$$= \mu \frac{\omega R_2}{(R_2 - R_1)} \times 2\pi R_1 H$$

The torque T_1 on the inner cylinder due to shearing action of the liquid in the annular space is

T_1 = shear force x radius

$$= \mu \frac{\omega R_2}{(R_2 - R_1)} \times 2\pi R_1 H \times R_1$$

$= \frac{2\pi {R_1}^2 \mu H \omega R_2}{(R_2 - R_1)}$ …(i)

If the gap between the bottom of the two cylinders is 'h', then the torque applied on inner cylinder (T_2) is given by equation (3.21) as

$$T_2 = \frac{\mu}{60t} \pi^2 N R^4$$

But here $\quad$ R = R_1, t = h = $\frac{\mu}{60t} \pi^2 N {R_1}^4$

$$\omega = \frac{2\pi N}{60}\ or\ N = \frac{60}{2\pi}$$

∴ $\quad T_2 = \frac{\mu}{60h} \times \pi^2 \times \frac{60\omega}{2\pi} \times\ {R_1}^4 = \frac{\pi\mu\omega}{2h} {R_1}^4$ …(ii)

∴ Total torque acting on the inner cylinder is

$$T = T_1 + T_2$$

$$= \frac{2\pi {R_1}^2 \mu H \omega R_2}{(R_2 - R_1)} + \frac{\pi\mu\omega}{2h} {R_1}^4 = 2\pi\mu {R_1}^2 \left[\frac{R_2 H}{R_2 - R_1} + \frac{{R_1}^2}{2h}\right] x\ \omega$$

$\mu = \frac{2(R_2 - R_1)hT}{\pi {R_1}^2 \omega [4HhR_2 + {R_1}^2 (R_2 - R_1)]}$ …(3.27)

Where T = torque measured by the strain of torsional spring,

R_1,R_2 = radii of inner and outer cylinder,

h = clearance at the bottom of cylinders

H = height of liquid in annular space,

μ = co-efficient of viscosity to be determined.

Hence the value of μ can be calculated from equation (3.27).

3.8.4. Orifice Type Viscometer. In this method, the time taken by a certain quantity of the liquid, whose viscosity is to be determined, to flow through a short capillary tube is noted down. The coefficient of viscosity is then obtained by comparing with the co-efficient of viscosity of a liquid whose viscosity is known or by the use of conversion factors. Viscometers such as Saybolt, Red wood or Engler are usually used the principle for all the three viscometer is same. In The United Kingdom Redwood viscometer is used while in USA, Saybolt viscometer is commonly used.

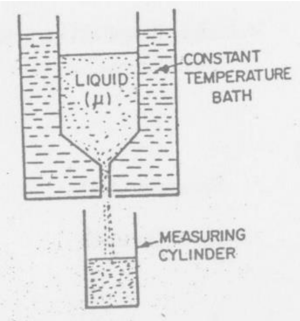


Fig. 3.17 Saybolt viscometer

Fig. 3.17 shows that Saybolt's viscometer, which consists of a tank at the bottom of which a short capillary tube is fitted. In this tank, the liquid whose viscosity is to be determined is filled. This tank is surrounded by another tank, called constant temperature bath. The liquid is allowed to flow through the capillary tube at standard temperature. The time taken by 60c.c of liquid to flow through is noted down. The initial height of liquid in the tank is previously adjusted to a standard height. From the time measurement, the kinematic viscosity of liquid is known from the relation,

$$v = At - \frac{B}{t}$$

Where A = 0.24, B = 190, t = time noted in seconds, v = kinematic viscosity in stokes.

Problem 3.26. The viscosity of an oil of sp. Gr. 0.9 is measured by a capillary tube of diameter 50 mm. the difference of pressure head between two points 2 m apart is 0.5 m of water. The mass of oil collected in a measuring tank is 0 kg in 100 seconds. Find the viscosity of oil.

Solution

Given:

Sp.gr. of oil = 0.9

Diameter. Of capillary tube, D = 50 mm = 5 cm = 0.05 m

Length of tube, $L = 2m$

Difference of pressure head, h = 0.5 m

Mass of oil, M = 60 kg

Time, t = 100 s

Mass of oil per second $= \frac{60}{100} = 0.6kg/s$

Density of oil, $\rho = sp.gr. of\ oil \times 1000 = 0.9 \times 1000 = 900\ kg/m^3$

∴ Discharge, Q = $\frac{Mass\ of\ oil/s}{density} = \frac{0.6}{900} m^3/s$

Using equation (3.25), we get viscosity,

$$\mu = \frac{\pi\rho g h D^4}{128Q.L} \qquad [here\ h = h_f = 0.5]$$

$$= \frac{\pi 900 \times 9.81 \times 0.5 \times (.05)^4}{128 \times 0.000667 \times 2.0} = 0.5075\ (SI\ Units)\ Ns/m^2$$

$$= 0.5075 \times 10\ poise = \mathbf{5.705\ poise}.$$ **Ans.**

Problem 3.27. A capillary tube of diameter 2 mm and length 100 mm is used for measuring viscosity of a liquid. The difference of pressure between the two ends of the tube is 0.6867 N/cm^2 and the viscosity of liquid is 0.25 poise. Find the rate of flow of liquid through the tube.

Solution

Given:

Diameter of capillary tube, D = 5 mm = 2 x 10^{-3} m

Length of tube, $L = 100m = 10\ cm = 0.1m$

Difference of pressure, $\Delta p = 0.6867\ N/cm^2 = 0.6867\ x\ 10^4\ N/m^2$

∴ Difference of pressure head, $h = \frac{\Delta p}{\rho g} = \frac{0.07 \times 10^4}{\rho g}$

Viscosity, $\mu = 0.25\ poise$

$$= \frac{0.25}{10}\ Ns/m^2$$

Let the rate of flow of liquid = Q

Using equation (3.25), we get $\mu = \frac{\pi\rho g h D^4}{128Q.L} = \pi\rho g \times \frac{\frac{0.6867 \times 10^4}{\rho g} \times (2\ x\ 10-3)^4}{128 \times Q \times 0.1}$

Or $\frac{0.25}{10} = \frac{\pi\ x 0.6867\ x\ 10^4\ x(2\ x\ 10-3)^4}{128\ x\ Q\ x\ 0.1}$

Or $$Q = \frac{\pi\, x 0.6867\; x\; 10^4\; x\; 2^4\; x\; 10^{-12}\;\; x\; 10}{128\; x .1\; x\; 0.25} m^3/s$$

$$= 107.86\; x\; 10^{-8}\; m^3/s = 107.86 x\; 10^{-8}\; x\; 10^6\; cm^3/s$$

$$= 107.86\; x\; 10^{-2}\; cm^3/s = \mathbf{1.078\, cm^3/s}. \qquad \textbf{ans.}$$

Problem 3.28. A sphere of diameter 2 mm falls 150 mm in 20 seconds in a viscous liquid. The density of the sphere is 7500 kg/m^3 and liquid is 900 kg/m^3. Find the coefficient of viscosity of the liquid.

Solution

Given:

Diameter of sphere, d = 2 mm = 2 x 10^{-3} m

Distance travelled by sphere = 150 mm = 0.15 m

Time taken, t = 20 seconds

Velocity sphere, $U = \frac{Distance}{time} = \frac{0.15}{20} = .0075\; m/s$

Density of sphere, $\rho_s = 7500 kg/m^3$

Density of liquid, $\rho_f = 900 kg/m^3$

Using relation (9.26), we get $\mu = \frac{gd^2}{18U}\left[\rho_s - \rho_f\right] = \frac{9.81\; x\; \left[2\; x\; 10^{-3}\right]^2}{18\; x\; 0.0075}[7500 - 900]$

$$= \frac{9.81 \times 4 \times 10^{-6} \times\; 6600}{18 \times .0075} = 1.917 \frac{Ns}{m^2}$$

$$= 1.917 \times 10 = \mathbf{19.17\, poise.} \qquad \textbf{Ans}$$

Problem 3.29. Find the viscosity of a liquid of sp.gr. 0.8, when a gas bubble of diameter 10 mm rises steadily through the liquid at a velocity of 1.2 cm/s, neglect the weight of the bubble.

Solution

Given:

Sp.gr. of liquid = 0.8

∴ Density of liquid, $\rho_f = 0.8\; 1000 = 800 kg/m^3$

Dia. Of gas bubble, D = 10 mm = 1 cm = 0.01 m

Velocity of bubble, $U = 1.2\, cm/s = .012\; m/s$

As weight of bubble is neglected and density of bubble

$$\rho_s = 0$$

Now using the relation, $\mu = \frac{gd^2}{18U}\left[\rho_s - \rho_f\right]$ which is for a falling sphere

For a rising bubble, the relation will become

$$\mu = \frac{gd^2}{18U}\left[\rho_f - \rho_s\right]$$

Substituting the values, we get $\mu = \frac{9.81\ x\ .01\ x\ .01}{18\ x\ .012}[800 - 0]\frac{Ns}{m^2} = 3.63\frac{Ns}{m^2}$

$$= 3.63 \times 10 = \mathbf{36.3\ poise}.$$

Problem 3.30. The viscosity of a liquid is determined by rotating cylinder method, in which case the inner cylinder of diameter 20 cm is stationary. The outer cylinder of diameter 20.5 cm, contains the liquid up to height of 30 cm. the clearance at the bottom of the two cylinders is 0.5 cm. the outer cylinder is rotated at 400 r.p.m. the torque registered at the torsion meter attached to the inner cylinder is 5.886 Nm. Find the viscosity of fluid.

Solution

Given:

Diameter of inner cylinder, $D_1 = 20\ cm$

∴ Radius of inner cylinder, $R_1 = 10\ cm = 0.1m$

Dia. Of outer cylinder, $D_2 = 20.5\ cm$

∴ Radius of inner cylinder, $R_2 = \frac{20.5}{2}\ cm = 10.25\ cm = 0.1025\ m$

Height of liquid from bottom of outer cylinder = 30 cm

Clearance at the bottom of two cylinders, h = 0.5 cm = 0.005 m

∴ Height of inner cylinder immersed in liquid

$$= 30 - h = 30 - 0.5 = 29.5\ cm$$

Or H = 29.5 cm = .295 m

Speed of outer cylinder, N = 400 r.p.m

∴ $\omega = \frac{2\pi N}{60} = \frac{2\ x\pi\ x\ 400}{60} = 41.88$

Torque measured T = 5.886 Nm

Using equation (3.27), we get $\mu = \frac{2(R_2 - R_1)\ x\ h\ x\ T}{\pi {R_1}^2 \omega\left[4HhR_2 + {R_1}^2(R_2 - R_1)\right]}$

$$= \frac{2(.1025 - 0.1) \times .005 \times 5.886}{\pi \times (.1)^2 \times 41.88[.0006047 - .000025]}$$

$$= 0.19286\ Ns/m^2 = 0.19286 \times 10 = \mathbf{1.9286\ poise.}$$

Problem 3.31. A sphere of diameter 1 mm falls through 335 m in 10 seconds in a viscous fluid. If the relative densities of the sphere and the liquid are 7.0 and 0.96 respectively, determine the dynamic viscosity of the liquid.

Solution

Given:

Diameter of sphere, d = 1 mm = 0.001 m

Distance travelled by sphere = 335 mm = 0.335 m

Time taken, t = 100 seconds

∴ Velocity of sphere, $U = \frac{Distance}{time} = \frac{0.335}{100} = 0.00335 m/sec$

Relative density of sphere = 7

∴ Density of sphere, $\rho_s = 7 \ x \ 1000 = 7000 \ kg/m^3$

Relative density of liquid = 0.96

∴ Density of liquid, $\rho_f = 0.96 \ x \ 1000 = 960 \ kg/m^3$

Using the relation (3.26), we get $\mu = \frac{gd^2}{18U}[\rho_s - \rho_f]$

$$= \frac{9.81 \ x \ 0.001^2}{18 \ x \ 0.00335}[7000 - 960]$$

$$= \frac{0.00000981 \ x \ 6040}{18 \ x \ 0.00335} = 0.981 \ Ns/m^2$$

$$= 0.981 \ x \ 10 = \mathbf{9.81 \mathit{poise}}.$$ **Ans.**

Problem 3.32. Determine the fall velocity of 0.06 mm sand particle (specific gravity =2.65) in water at 20℃, Take $\mu = 10^{-3}$kg/ms.

Solution

Given:

Diameter of sand particle, d = 0.06 mm = 0.06 x 10^{-3}m

Specific Gravity of sand = 2.65

∴ Density of sand, $\rho_s = 2.6 \ x \ 1000 \ kg/m^3$ $(\because \rho for \ water \ in \ S.I. unit = 1000 \ kg/m^3)$

$$= 2650 \ kg/m^3$$

Sand particle is just like a sphere.

For equilibrium of sand particle,

Drag force = Weight of sand particle – buoyant force

Or $F_D = W - F_B$ …(i)

But $F_D = 3\pi\mu \times U \times d,$ where U = Velocity of particle

$$= 3\pi \ x \ 10^{-3} \ x \ U \ x \ 0.06 \ x \ 10^{-3} N$$

W = Weight of sand particle

$$= \frac{\pi}{6} \times d^3 \times \rho_f \times g = \frac{\pi}{6} \times (0.06 \ x \ 10^{-3})^3 \times 2650 \times 9.81 N$$

F_B= Buoyant force

= Weight of sand particles

$$= \frac{\pi}{6} \times d^3 \times \rho_f \times g = \frac{\pi}{6}(0.06 \times 10^{-3})^3 \times 1000 \times 9.81$$

Substituting the above values in equation (i), we get

$3\pi \times 10^{-3} \times U \times 0.06 \times 10^{-3} = \frac{\pi}{6}(0.06 \times 10^{-3})^3 \times 2650 \times 9.81 - \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times$

2650×9.81

Cancelling $(\pi \times 0.06\ x\ 10^{-3})$ throughout, we get

$$3 \times U = \frac{1}{6}(0.06)^2 \times 10^{-3} \times 2650 \times 9.81 - \frac{1}{6} \times 0.006^2 \times 10^{-3} \times 1000 \times 9.81$$

$$= \frac{1}{6}\ x\ 0.006^2\ x 10^{-3}\ x\ 9.81(2650 - 1000)$$

$$= \frac{1}{6}\ x\ 0.0036\ x\ 10^{-3}\ x\ 9.81\ x\ 1650 = 0.009712$$

∴ $U = 0.009712/3 = \mathbf{0.00323\ m/s.}$ **Ans**

3.8.6 HIGHLIGHTS

1. A flow is said to be viscous if Reynolds number is less than 2000, or the fluid flows in layers.
2. For the viscous flow through circular pipes,

(i).Shear stress, $\tau = -\frac{\partial p}{\partial x}\frac{r}{2}$ (ii) Velocity, $u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r]$

(iii) Ratio of velocities, $\frac{U_{max}}{u} = 2.0$ (iv) Loss of pressure head, $h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$

Where $\frac{\partial p}{\partial x}$ = pressure gradient, r =radius at any point,

R = radius of the pipe, U_{max} = maximum velocity or velocity at r = 0,

$\bar{u}$ = average velocity = $\frac{Q}{\pi R^2}$, μ = co-efficient of viscosity

D = Diameter of the pipe.

3. For the viscous flow between two parallel plates,

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2]$$ ….Velocity distribution

$$\frac{U_{max}}{u} = 1.5$$ …. Ratio of maximum and average velocity

$$h_f = \frac{12\mu\bar{u}L}{\rho g t^2}$$ ….Loss of pressure head

$$\tau = -\frac{1}{2}\frac{\partial p}{\partial x}[t - 2y]$$ … Shear stress distribution

4. The kinetic energy correction factor, α for a circular pipe is given as

$$\alpha = \frac{K.E. per\ second\ based\ on\ actual\ velocity}{K.E. per\ second\ based\ on\ average\ velocity}$$

$$= 2.0 \ldots\ldots\ldots for\ a\ circular\ pipe.$$

5. Momentum correction factor β is given by

$$\beta = \frac{Momentum\ per\ second\ based\ on\ actual\ velocity}{Momentum\ per\ second\ based\ on\ average\ velocity}$$

$$= \frac{4}{3} \ldots\ldots. for\ a\ circular\ pipe.$$

6. For the viscous resistance of Journal Bearing.

$$V = \frac{\pi DN}{60}, \frac{du}{dy} = \frac{V}{t} = \frac{\pi DN}{60t}$$

$\tau = \frac{\mu\pi^2 dN}{60t}$, Shear force $= \frac{\mu\pi^2 D^2 NL}{60t}$

Torque, T = $\frac{\mu\pi^2 D^3 NL}{120t}$ and power = $\frac{\mu\pi^3 D^3 N^2 L}{60\ x\ 60\ x\ t}$

Where L = length of bearing, N = speed of shaft in rpm, t = clearance between the shaft and bearing.

7. For the Foot-Step Bearing, the shear force, torque and power absorbed are given as:

Shear force, $F = \frac{\mu}{15}\frac{\pi^2 N}{t}\frac{R^3}{3}$

Torque, $T = \frac{\mu}{60t}\pi^2 N R^4$

Power, $P = \frac{\mu\pi^3 N^2 R^4}{60\ x\ 30\ x\ t}$

Where R = radius of the shaft, N = speed of the shaft,

8. For the collar bearing on the torque and power absorbed are given as

$T = \frac{\mu}{60\ t}\pi^2 N\left({R_2}^4 - {R_1}^4\right)$, $P = \frac{\mu\pi^3 N^2}{60\ x\ 30\ x\ t}\left[{R_2}^4 - {R_1}^4\right]$

Where R_1 = Internal radius of the collar, R_2= External radius of the collar,

T = Thickness of oil film, P = Power in Watts.

9. For the viscous flow the co-efficient of friction is given by, $f = \frac{16}{R_e}$

Where R_e = The Reynolds number =$\frac{\rho VD}{\mu} = \frac{VD}{v}$.

10. The co-efficient of viscosity is determined by dash pot arrangement as $\mu = \frac{4Wt^3}{3\pi LD^3 V}$

Where W = Weight of the piston, t = Clearance between dash-pot and piston,

L = Length of the piston, D = diameter of the piston,

V = velocity of the piston.

11. The co-efficient of viscosity of a liquid is also determined experimentally by the following methods:

(i) Capillary tube method, $\mu = \frac{\pi whD^4}{128QL}$

(ii) Falling sphere method, $\mu = \frac{gd^2(\rho_s - \rho_f)}{18U}$

(iii) Rotating cylinder method, $\mu = \frac{2(R_2 - R_1)hT}{\pi {R_1}^2 \omega\left[4HhR_2 + {R_1}^2(R_2 - R_1)\right]}$

Where w = Specific weight of fluid, L = Length of the tube,

D = Diameter of the capillary tube, Q = Rate of flow of fluid through capillary tube,

d = Diameter of the sphere, ρ_S = Density of sphere,

ρ_f = Density of fluid, U = Velocity of sphere,

R_2 = Radius of outer rotating cylinder, R_1= Radius of inner stationary cylinder,

T = Torque.

3.9 EXERCISE 3

3.9.1 THEORETICAL PROBLEMS

1. Define the terms: Velocity, kinematic viscosity, velocity gradient and pressure gradient.
2. What do you mean by 'Viscous Flow'?
3. Derive an expression for the velocity distribution for viscous flow through a circular pipe. Also sketch the velocity distribution and shear stress distribution across a section of the pipe.
4. Prove that the maximum velocity in a circular pipe for viscous flow is equal to two times the average velocity of the flow.
5. Find an expression for the loss of head of a viscous fluid flowing through a circular pipe.
6. What is Hagen Poiseuille's Formula? Derive an expression for Hagen Poiseuille' formula.
7. Prove that the velocity distribution for viscous flow between two plates when both plates are fixed across a section is parabolic in nature. Also prove that maximum velocity is equal to one and a half times the average velocity.
8. Show that the difference of pressure head for a given length of the two parallel plates which are fixed and through which viscous fluid is flowing is given by

$$h_f = \frac{12\mu\bar{u}L}{\rho g t^2}$$

Where μ =Viscosity of fluid, $\bar{u} = Average\ velocity$

t = Distance between the two parallel plates $L =$ Length of the plates.

9. Define the terms: kinetic energy correction factor and momentum correction factor.
10. Prove that for viscous flow through a circular pipe the kinetic energy correction factor is equal to 2 while momentum correction factor $= \frac{4}{5}$. (Delhi University, 1992)
11. A shaft is rotating in a journal bearing. The clearance between the shaft and the bearing is filled with a viscous oil. Find an expression for the power absorbed in overcoming viscous resistance.
12. Prove that power absorbed in overcoming viscousresistance in foot-step bearing is given by

$H.P. = \frac{\mu\pi^3 N^2 R^4}{4500\ x\ 30\ \ t}$

Where R = Radius of shaft, N = Speed of the shaft

t = Clearance between shaft and foot-step bearing, $\mu =$ Viscosity of fluid.

13. Show that the value of the co-efficient of friction for viscous flow through a circular pipe given by

$f = \frac{16}{R_e}$, where R_e =Reynolds number.

14. Prove that the co-efficient of viscosity by dash-pot arrangement is given by

$$\mu = \frac{4Wt^3}{3\pi L D^3 V}$$

Where W = weight of piston, t = Clearance between dash-pot and piston

L = Length of piston, D = Diameter of piston

V =Velocity of piston

15. What are the different methods of determining the co-efficient of viscosity of a liquid. Describe any two method in details.
16. Prove that the loss of pressure head for the viscous flow through a circular pipe is given by

$$h_f = \frac{32\mu\bar{u}L}{\rho g d^2}$$

Where $\bar{u} = Average\ velocity$, w = Specific weight.

17. For a laminar steady flow, prove that the pressure in a direction of motion is equal to the shear gradient normal to the direction of motion.
18. Describe Reynolds experiments to demonstrate the two types of flow.
19. For the laminar flow through a circular pipe, prove that
    (i) The shear stress variation across the section of the pipe is linear and
    (ii) The velocity variation is parabolic.

3.9.2 NUMERICAL PROBLEMS

1. A crude oil of viscosity 0.9 poise and sp. Gr. 0.8 is flowing through a horizontal circular pipe of diameter 80 mm and of length 15m. Calculate the difference of pressure at the two ends of the pipe, if 50kg of the oil is collected in a tank in 15 seconds. [Ans. 0.559 N/cm^2]

2. A Viscous flow is taking place in a pipe of diameter 100mm. the maximum velocity is 2 m/s. find the mean velocity and the radius at which this occurs. Also calculate the velocity at 30 mm from the wall of the pipe. [Ans. 1m/s, r = 35.35mm, $\mu = 1.68m/s$]

3. A fluid of viscosity 0.5 poise and specific gravity 1.20 is flowing through a circular pipe of diameter 100mm. The maximum shear stress at the pipe wall is given as 147.15N/m^2, find : (a) the pressure gradient, (b) the average velocity, and (c) the Reynolds number of the flow.

[Ans. (a) -64746N/m^2 per m, (b) 3.678 m/s, (c) 882.72]

4. Determine (a) the pressure gradient (b) the shear stress at the two horizontal parallel plates and (c) the discharge per meter width for laminar flow of oil with a maximum velocity of 1.5m/s between two horizontal parallel fixed plates which are 80 mm apart. Take viscosity of oil as $\frac{1.962Ns}{m^2}$. [Ans. (a) -3678.7 N/m^2 per m, (b)147.15 N/m^2, (c).08m^3/s]
5. Water is flowing between two large parallel plates which are 2.0mm apart. Determine (a) maximum velocity, (b) the pressure drop per unit length and (c) the shear stress at walls of the plate if the average is 0.4m/s. Take viscosity of water as 0.01poise.

   [**Ans**. (a) 0.6 m/s, (b) 1199.7 N/m^2 per m, (c)1.199 N/m^2]
6. There is a horizontal crack 550 mm wide and 3 mm deep in a wall of thickness 150mm. water leaks through the crack. Find the rate of leakage of water through the crack if the difference of pressure between the two ends of the crack is 245.25N/m^2. Take viscosity of water as 0.01 poise. [**Ans.** 183.9cm^3/s]
7. A shaft having a diameter of 10 cm rotates centrally in a journal bearing having a diameter of 10.02 cm and length 20cm. the annular space between the shaft and the bearing is filled with oil having viscosity of 0.8 poise. Determine the power absorbed in the bearing when the speed of rotation is 500 r.p.m. [**Ans.** 343.6W]
8. A shaft 150mm diameter runs in a bearing of length 300mm, with a radial clearance of 0.04 mm at 40 r.p.m. Find the viscosity of the oil, if the power required to overcome the viscous resistance is 220.725W.

   [**Ans.** 6.32]
9. Find the torque required to rotate a vertical shaft of diameter 8 cm at 800 r.p.m. The lower end of the shaft rests in a foot-step bearing. The end of the shaft and surface of the bearing are both flat and are separated by an oil film of thickness 0.075 cm. the viscosity of the oil is given as 1.2 poise.

   [**Ans.** 0.0538Nm]
10. A collar bearing having external and internal diameters 20 cm and 10cm respectively is used to take the thrust of a shaft. An oil film of thickness 0.03 cm is maintained between the collar surface and the bearing. Find the power lost in overcoming the viscous

resistance when the shaft rotates at 250 r.p.m. take $\mu = 0.9$poise. [**Ans**. 30.165W]

11. Water is flowing through a 150 mm diameter pipe with a coefficient of friction f = 0.05. The shear stress at point 40 mm from the pipe wall is 0.01962N/cm^2. Calculate the shear stress at the pipe wall.
[**Ans.** 0.04198N/cm^2]
12. An oil dash-pot consists of a piston moving in a cylinder having oil. The piston falls with uinform speed and covers 4.5 cm in 80 seconds. If an additional weight 1.5N is placed on the top of the piston, it falls through 4.5 cm in 70 seconds with uniform speed. The diameter of the piston is 10cm and its length is 15 cm. The clearance between the piston and the cylinder is 0.15 cm, which is uniform throughout. Find the viscosity of oil.
[**Ans.** 0.177 poise]
13. The viscosity of oil of sp.gr. 0.8 is measured by a capillary tube of diameter 40 mm. The difference of pressure head between two points 1.5 m apart is 0.3 m of water. The mass of oil collected in a measuring tank is 40 kg in 120 seconds. Find the viscosity of the oil.
[**Ans.** 2.36 poise]
14. A capillary tube of diameter 4 mm and length 150 mm is used for measuring viscosity of a liquid. The difference of pressure between the two ends of the tube is 0.7848N/cm^2 and the viscosity of the liquid is 0.2 poise. Find the rate of flow of liquid through the tube.
[**Ans.** 16.43 cm^3/s]
15. A sphere of diameter 3 mm falls through 100 mm in 1.5 seconds in a viscous liquid. The density of the sphere is 7000kg/m^3 and of liquid is 800kg/m^3. Find the co-efficient of viscosity of the liquid. [**Ans.** 45.61 poise]
16. The viscosity of a liquid is determined by rotating cylinder method, in which case the inner cylinder of diameter 25 cm is stationary. The outer cylinder of diameter 25.5 cm contains the liquid up to a height of 40 cm. the clearance at the bottom of the two cylinders is 0.6 cm. the outer cylinder is rotated at 300 r.p.m. The torque registered on the torsion meter attached to the inner cylinder is 4.905 Nm. Find the viscosity of liquid.
[**Ans.** .77poise]
17. Calculate : (a) the pressure gradient along the flow, (b) the average velocity, and (c) the discharge for an oil of viscosity 0.02Ns/m^2 flowing between two stationary parallel plates 1m wide maintained 10 mm apart. The velocity midway between the plates is 2.5 m/s
[**Ans.** -4000N/m^2 per m, 1.667 m/s, .01667 m^3/s]

18. Calculate :
    (i) The pressure gradient along the flow,
    (ii) The average velocity, and
    (iii) The discharge for an oil of viscosity 0.03Ns/m^2 flowing between two stationary plates which are parallel and are at 10 mm apart. Width of plates is 2m. The velocity midway between the plates is 2.0 m/s. (Delhi University, 1988)

19. A cylinder of 100 mm diameter, 0.15 m length and weighing 10N slides axially in a vertical pipe of 104 mm dia. If the space between cylinder surface and pipe wall is filled with liquid of viscosity μ and the cylinder slides downwards at a velocity of 0.45m/s, determine μ.

    (AMIE, W 1991)

    [Hint. D = 100 mm = 0.1, L = 0.15 m, W = 10N, $D_p = 1.4\, mm = 0.104\, m,$

    V = 0.45 m/s. Hence t = (0.104-0.1)/2 = 0.002 m.

    $\mu = \frac{4Wt^3}{3\pi LD^3V} = \frac{4\,x\,10\,x\,0.002^3}{3\pi\,x\,.1^3\,x\,0.15\,x\,.45} = 503\,x\,10^{-6} Ns/m^2$]

20. A liquid is pumped through a 15 cm diameter and 300 m long pipe at the rate of 20 tonnes per hour. The density of liquid is 910kg/m^3 and kinematic viscosity = 0.002 m^2/s. Determine the power required and show that the flow is viscous.

    [Hint. D = 15 cm = 0.15 m, L = 300 m, W = 20 tonnes/hr

    $$= 20\,x\,1000\ kgf/60\,x\,60\,sec = 5.555\ kgf/sec = 5.555\,x\,9.81 N/s.$$

    $$Q = \frac{W}{\rho g} = \frac{5.555\,x\,9.81}{910\,x\,9.81} = 0.0061\, m^3/s.\ \mathrm{V} = \frac{Q}{A} = \frac{0.0061}{\frac{\pi}{4}(.15)^2}$$

    $$= 0.345 m/s, v = 0.002 m^2/s.$$

    Now R$_e$ = $\frac{\rho VD}{\mu} = \frac{V\,x\,D}{v} = \frac{0.345\,x\,0.15}{0.002} = 25.87$

    Which is less than 2000, Hence the flow is viscous.

    $h_f = 32\mu L\,V/\rho g D^2$, where $v = \frac{\mu}{\rho} \therefore\ \mu = v\,x\rho = 0.002\,x\,910 = 1.82$

    Hence, $h_f = \frac{32\,x\,1.82\,x\,300\,x\,0.345}{(910\,x\,9.81\,x\,0.15^2)} = 30$

    $\therefore$ $P = \rho g\,.Q.h_f/1000 = 910\,x\,9.81\,x\,0.0061\,x\ 30/1000 = 1.633\, kW$. **Ans.**]

21. An oil of specific gravity 0.9 and viscosity 10 poise is flowing through a pipe of diameter 110 mm. the velocity at the center is 2m/s, find: (i) pressure gradient in direction of flow, (ii) shear stress at the pipe wall; (iii) Reynolds numbers, and (iv) Velocity at a distance of 30 mm from the wall.

[Hint. $\rho = \frac{900 kg}{m^3}$; $\mu = 10\ poise = 1\ Ns/m^2$; $D = 110\ mm = 0.11 m, U_{max} = 2\ m/s$; $\bar{u} = 1\ m/s$; $U_{max} = \frac{1}{4\mu}\left(\frac{-dp}{dx}\right) R^2$

(i) $\left(\frac{-dp}{dx}\right) = \frac{4\mu\ x\ U_{max}}{R^2} = \frac{4\ x\ 1\ x\ 2}{0.055^2} = 2644.6 N/m^3$;

(ii) $\tau_0 = \left(\frac{-dp}{dx}\right)\ x\ \frac{R}{2} = 2644.6\ x\ \frac{0.055}{2} = 72.72 N/m^2$;

(iii) $R_e = \frac{\rho\ x\ \bar{u}\ x\ D}{\mu} = \frac{900\ x\ 1\ x\ 0.11}{1} = 99$; and

(iv) $u = \frac{1}{4\mu}\left(\frac{-dp}{dx}\right)(R^2 - r^2) = \frac{1}{4\ x\ 1}(2644.6)(0.055^2 - 0.25^2) = 1.586 m/s$]

4. TURBULENT FLOW

4.1 INTRODUCTION

The laminar flow has been discussed in chapter 3. In laminar flow, the fluid particles move along straight parallel paths in layers or laminae, such that the paths of individual fluid particles do not cross those of neighboring particles. Laminar flow is possible only at low velocities and when the fluid is highly viscous. But when the velocity is increased, the fluid is less viscous, the fluid particles do not move in straight paths. The fluid particles move in random manner resulting in general mixing of the particles. This type of flow is called turbulent flow.

A laminar flow changes to turbulent flow when (i) velocity is increased or (ii) diameter of a pipe is increased or (iii) the viscosity of fluid is decreased. Osborne Reynolds was first to demonstrate that the transition from laminar to turbulent depends not only on the mean velocity but on the quantity $\frac{\rho V D}{\mu}$. This quantity $\frac{\rho V D}{\mu}$ is a dimensionless quantity and is called Reynolds number (R_e). In case of a circular pipe if $R_e > 4000$, the flow is said to be turbulent. If R_e lies between 2000 to 4000, the flow changes from laminar to turbulent.

4.2 REYNOLDS EXPERIMENT

The type of flow is determined from Reynolds number i.e., $\frac{\rho V d}{\mu}$. This was demonstrated by O. Reynolds in 1883. His apparatus is shown in Fig. 4.1.

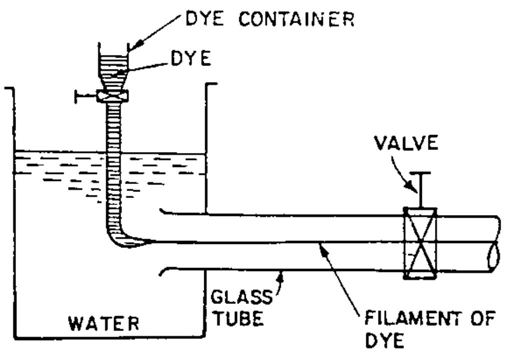


Fig. 4.1 Reynold apparatus

The apparatus consists of:

(i) A tank containing water at constant head,

(ii) A small tank containing some dye,

(iii) A glass tube having a bell- mouthed entrance at one end and a regulating value at the other ends.

The water from the tank was allowed to flow through the glass tube. The velocity of flow was varied by the regulating valve. A liquid dye having same specific weight as water was introduced into a glass tube as shown in Fig. 4.1.

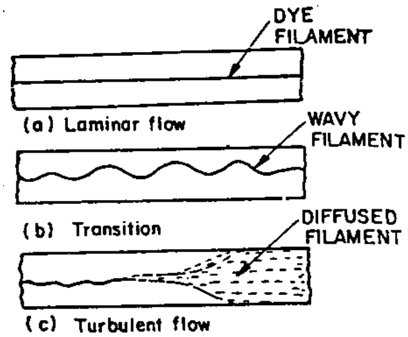


Fig. 4.2 Different stages of filament.

The following observations were made by Reynolds:

(i) When the velocity was low, the dye filament in the glass tube was in the form of a straight line. This straight line of dye filament was parallel to the glass tube, which was the case of laminar flow as shown in Fig. 4.2(a).

(ii) With the increase of velocity of flow, the dye filament was no longer a straight line but it became a wavy one as shown in Fig. 4.2(b). This shows that flow is no longer laminar.

(iii) With further increase of velocity of flow, the wavy dye filament broke up and finally diffused in water as shown in fig. 4.2(c). This means that the fluid particles of dye at this higher velocity are moving in random fashion, which shows the case of turbulent flow. Thus in case of turbulent flow, the mixing of dye filament and water is intense and the flow is irregular, random and disorderly.

In case of laminar flow, the loss of pressure head was found to be proportional to the velocity but in case of turbulent flow, Reynolds observed that the loss of the head is approximately proportional to the square of velocity. More exactly the loss of head, $h_f \propto V^n$, where n varies from 1.75 to 2.0.

4.3 FRICTIONAL LOSS IN PIPE FLOW

When a liquid is flowing through a pipe wall, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity. This viscous action causes loss of energy which is usually known as frictional loss.

On the basis of his experiments, William Froude gave the following laws of fluid friction for turbulent flow.

The friction resistance for turbulent flow is:

(i) Proportional to V^n, where n varies from 1.5 to 2.0,

(ii) Proportional to the density of fluid,

(iii) Proportional to the area of surface in contact,

(iv) Independent of pressure,

(v) Dependent on the nature of the surface in contact.

4.3.1 Expression for loss of head due to Friction in pipes. Consider a uniform horizontal pipe, having steady flow as shown in fig. 4.3. Let 1-1 and 2-2 are two sections of pipe.

Let $p_1 =$ pressure intensity at section 1-1,

$V_1 =$ velocity of flow at section 1-1,

$L =$ length of the pipe between sections 1-1 and 2-2,

$d =$ diameter in pipe,

$f^{\iota} =$ frictional resistance per unit wetted area per unit velocity squared,

$h_f =$ loss of head due to friction,

And $p_2, V_2 =$ are values of pressure intensity and velocity at section 2-2.

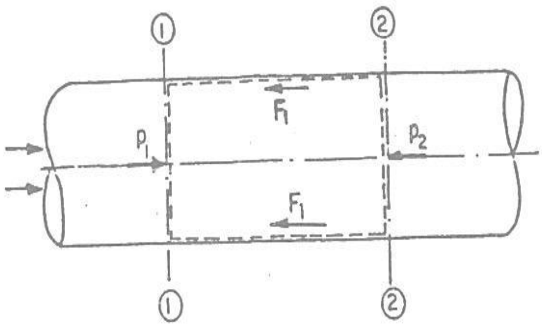


Fig. 4.3 Uniform horizontal pipe

Applying Bernoulli's equations between section 1-1 and 2-2,

Total head at 1-1 = Total head at 2-2 + loss head due to friction between 1-1 and 2-2

Or $$\frac{P_1}{\rho g} + \frac{{V_1}^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{{V_2}^2}{2g} + z_2 + h_f$$

But $z_1 = z_2$ as pipe is horizontal

$V_1 = V_2$ as dia. Of pipe is same at 1-1 and 2-2

$\therefore \quad \frac{P_1}{\rho g} = \frac{P_2}{\rho g} + h_f \quad$ or $\quad h_f = \frac{P_1}{\rho g} - \frac{P_2}{\rho g} = \frac{P_1 - P_2}{\rho g} \quad$ …(i)

But h_f is the head lost due to friction and hence intensity of pressure will be reduced in the direction of flow by frictional resistance.

Now frictional resistance = frictional resistance per unit wetted area per unit velocity squared = wetted area x velocity^2

Or $\quad F_1 = f' x\, \pi dL\, x\, V^2 \quad [\because wetted\ area = \pi d\ x\ L\ velocity = V = V_1 = V_2\,]$

$= f'\ x\ P\ x\ L\ x\ V^2 \quad [\because \pi d = Perimeter = P\,] \quad$ …(ii)

The forces acting on the fluid between sections 1-1 and 2-2 are:

1. Pressure force at section 1-1 = p_1 x A

Where A = Area of pipe

2. Pressure force at section 2-2 = p_2 x A
3. Frictional force F_1 as shown in Fig. 4.3.

Resolving all forces in the horizontal direction, we have

$$p_1 \mathrm{A} - p_2 A - F_1 = 0$$

Or $\quad (p_1 - p_2)A = F_1 = f'\ x\ P\ x\ L\ x\ V^2 \quad [\because from\ (ii), F_1 = f'PLV^2]$

Or $\quad p_1 - p_2 = \frac{f'\ x\ P\ x\ L\ x\ V^2}{A}$

But from equation (i), $\quad p_1 - p_2 = \rho g h_f$

Equating the value of $(p_1 - p_2)$, we get

$$\rho g h_f = \frac{f'\ x\ P\ x\ L\ x\ V^2}{A}$$

Or $\quad h_f = \frac{f'}{\rho g}\ x\ \frac{P}{A}\ x\ L\ x\ V^2 \quad$ …(iii)

In equation (iii), $\quad \frac{P}{A} = \frac{wetted\ perimeter}{Area} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$

$\therefore \quad h_f = \frac{f'}{\rho g}\ x\ \frac{A}{D}\ x\ L\ x\ V^2 = \frac{f'}{\rho g}\ x\ \frac{4L\ V^2}{d} \quad$ …(iv)

Putting $\quad \frac{f'}{\rho g} = \frac{f}{2}, \quad$ where f is known as co-efficient of friction.

Equation (iv), becomes $\quad h_f = \frac{4.f}{2g} \cdot \frac{LV^2}{d} = \frac{4f.L.V^2}{d\ x\ 2g} \quad$ …(4.2)

Equation (4.2) is known as Darcy-Weisbach equation. This equation is commonly used for finding loss of head due to friction in pipes.

Sometimes equation (4.2) is written as

$$h_f = \frac{f.L.V^2}{d \ x \ 2g} \qquad \text{…(4.2A)}$$

Then f is known as friction factor

4.3.2 Expression for co-efficient of friction in terms of shear stress.

The equation (4.1) gives the forces acting on a fluid between sections 1-1 and 2-2 of Fig. 4.3 in horizontal direction as

$$p_1 A - p_2 A - F_1 = 0$$

Or $$(p_1 - p_2)A = F_1 = \text{force due to shear stress } \tau_0$$

$$= \text{shear stress x surface area}$$

$$= \tau_0 \times \pi d \times L$$

Or $$(p_1 - p_2)\frac{\pi}{4}d^2 = \tau_0 \times \pi d \times L \qquad \left\{\because A = \frac{\pi}{4}d^2\right\}$$

Cancelling πd on both sides, we have

$$(p_1 - p_2)\frac{d}{4} = \tau_0 \ x \ L$$

Or $$(p_1 - p_2) = \frac{4\tau_0 \ L}{d} \qquad \text{…(4.3)}$$

Equation (4.2) can be written as $h_f = \frac{p_1 - p_2}{\rho g} = \frac{4.fL.V^2}{d \ x 2g}$

Or $$(p_1 - p_2) = \frac{4fL.V^2}{d \ x 2g} \ x \ \rho g \qquad \text{…(4.4)}$$

Equating the value of $(p_1 - p_2)$ in equations (4.3) and (4.4),

$$\frac{4\tau_0 \ L}{d} = \frac{4fL.V^2}{d \times 2g} \times \rho g$$

Or $$\tau_0 = \frac{fV^2 \times \rho g}{2g} = \frac{fV^2}{2g}\rho g$$

Or $$\tau_0 = f\frac{\rho V^2}{2} \qquad \text{…(4.5)}$$

$\therefore$ friction coefficient $f = \frac{2 \ \tau_0}{\rho V^2}$ …(4.6 A)

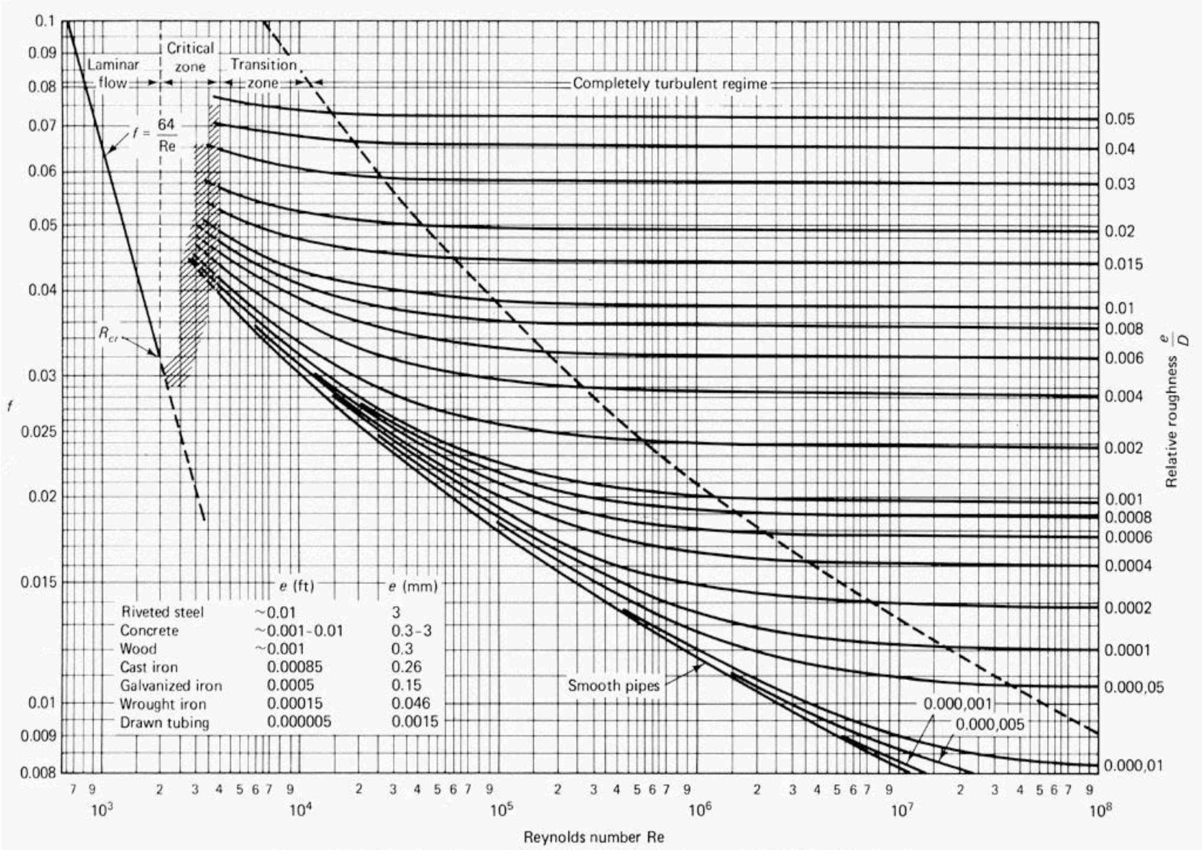
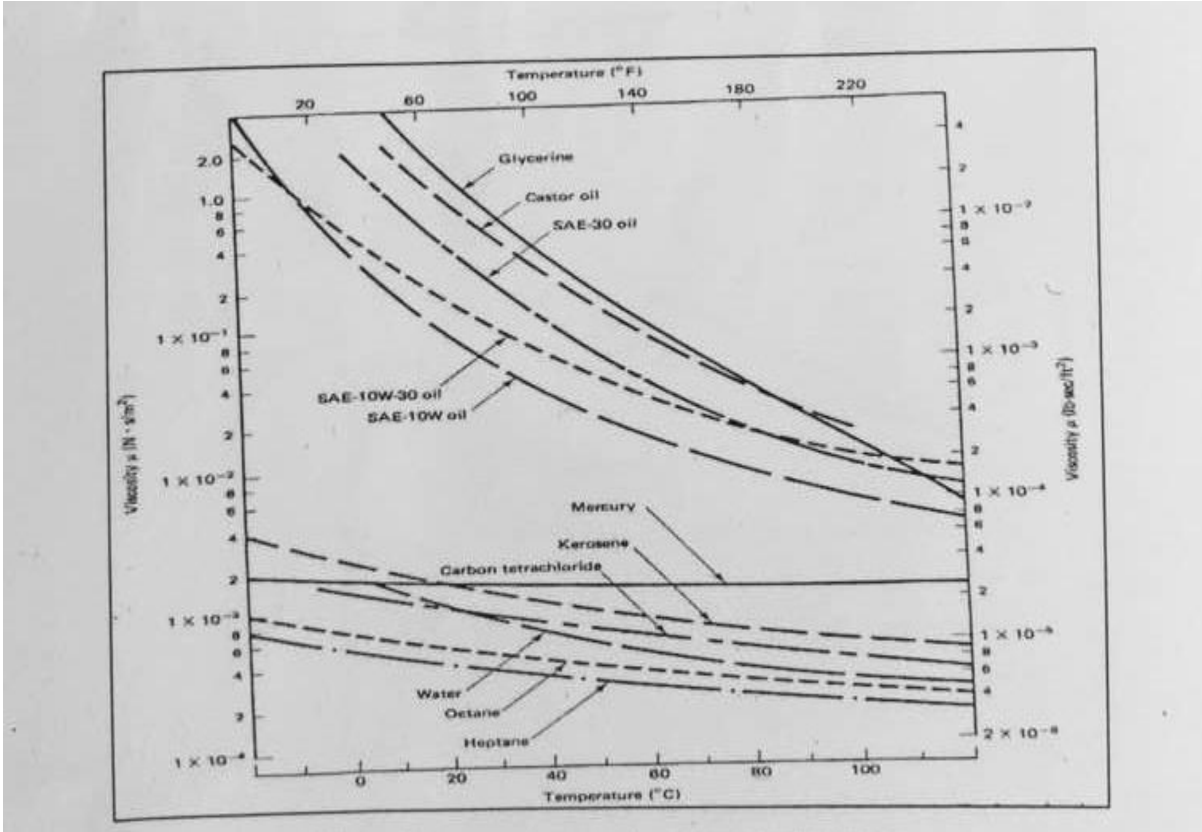


Figure 7.13 Moody diagram. (From L. F. Moody, *Trans. ASME*, Vol. 66, 1944.)

Table for Properties of water

Temperature, °C	Density ρ, kg/m³	Viscosity μ, (N · s/m²)	Kinematic viscosity ν, m²/s	Surface tension σ, N/m	Vapor pressure, kPa	Bulk modulus B, Pa
0	999.9	1.792×10^{-3}	1.792×10^{-6}	0.0762	0.588	204×10^{7}
5	1000.0	1.519	1.519	0.0754	0.882	206
10	999.7	1.308	1.308	0.0748	1.176	211
15	999.1	1.140	1.141	0.0741	1.666	214
20	998.2	1.005	1.007	0.0736	2.45	220
30	995.7	0.801	0.804	0.0718	4.30	223
40	992.2	0.656	0.661	0.0701	7.40	227
50	988.1	0.549	0.556	0.0682	12.22	230
60	983.2	0.469	0.477	0.0668	19.60	228
70	977.8	0.406	0.415	0.0650	30.70	225
80	971.8	0.357	0.367	0.0630	46.40	221
90	965.3	0.317	0.328	0.0612	68.20	216
100	958.4	0.284×10^{-3}	0.296×10^{-6}	0.0594	97.50	207×10^{7}



Viscosity versus temperature for several fluids

4.4 SHEAR STRESS IN TURBULENT FLOW

The shear stress in viscous flow is given by Newton's law of viscosity as

$$\tau_v = \mu \frac{du}{dy} \quad \text{where} \quad \tau_v = \text{shear stress due to viscosity.} \qquad (4.6\text{ B})$$

Similar to the expression for viscous shear,J. Boussinesq expressed the turbulent shear in mathematical form as

$$\tau_t = \eta \frac{du}{dy} \qquad \ldots(4.7)$$

Where $\tau_t =$ shear stress due to turbulence

η = eddy viscosity

$\bar{u}$ = average velocity at a distance y from boundary.

The ratio of η(eddy viscosity) and ρ(mass density) is known as kinematic eddy viscosity and is denoted by ε(epsilon). Mathematically it is written as

$$\varepsilon = \frac{\eta}{\rho} \qquad \ldots(4.8)$$

If the shear stress due to viscous flow is also considered, then the total shear stress becomes as

$$\tau = \tau_v + \tau_t = \mu \frac{du}{dy} + \eta \frac{d\,\bar{u}}{dy} \qquad \ldots(4.9)$$

The value of $\eta = 0$ is for laminar flow. For other cases, the value of η may be several thousand times the value of μ. To find shear stress in turbulent flow, equation (4.7) given by Boussinesq is used. But as the value of η(eddy viscosity) cannot be predicted, this equation is having limited use.

4.4.1 Reynolds Expression For Turbulent Shear Stress. Reynolds in 1886 developed an expression for turbulent shear stress between two layers of a fluid at a small distance apart, which is given as

$$\tau = \rho u' v' \qquad \ldots(4.10)$$

Where $u', v' =$ fluctuating component of velocity in the direction of x and y due to turbulence.

As u' and v' are varying and hence τ will also vary. Hence to find the shear stress, the time average on both the sides of the equation (4.10) is taken. Then equation (4.10) becomes

$$\bar{\tau} = \overline{\rho u' v'} = \overline{\rho u' v'} \qquad \ldots(4.11)$$

The turbulent shear stress given by equation (4.11) is known as Reynolds shear stress.

4.4.2 Prandtl Mixing Length theory For Turbulent Shear Stress. In equation (4.11), the turbulent shear stress can only be calculated if the value of $u'v'$ is known. But it is very difficult to measure $\overline{u'v'}$. To overcome this difficulty, L. Prandtl in 1925, presented a mixing length hypothesis which can be used to express turbulent shear stress in terms of measurable quantities.

According to Prandtl, the mixing length, *l*, is that distance between two layers in transverse direction such that the lumps of fluid particles from one layer could reach the other layer and the particles are mixed in the other layer in such a way that the momentum of particles in the direction of x is same. He also assumed that the velocity fluctuation in the x- direction u' is related to mixing length l as

$$u' = l \frac{du}{dy}$$

And v', the fluctuation component of velocity in y-direction is of the same order of magnitude as u' and hence

$$v' = l \frac{du}{dy}$$

Now, the Reynold's shear stress becomes $\overline{u'v'} = \left(l \frac{du}{dy}\right) x \left(l \frac{du}{dy}\right) = l^2 \left(\frac{du}{dy}\right)^2$

Substituting the value of $\overline{u'v'}$ in equation (4.11), we get the expression for shear stressin turbulent flow due to Prandtl as

$$\bar{\tau} = \rho l^2 \left(\frac{du}{dy}\right)^2 \qquad \ldots(4.12)$$

The total shear stress at any point in turbulent flow is the sum of shear stress due to viscous shear and turbulent shear and can be written as

$$\bar{\tau} = \mu \frac{du}{dy} + \rho l^2 \left(\frac{du}{dy}\right)^2 \qquad \ldots(4.13)$$

But the viscous shear stress is negligible except near a boundary. Equation (4.13) is used for most of turbulent flow problems for determining shear stress in turbulent flow.

4.5 VELOCITY DISTRIBUTION IN TURBULENT FLOW IN PIPES

In case of turbulent flow, the total shear stress at any point is the sum of viscous shear stress and turbulent shear stress. Also the viscous shear stress is negligible except near the boundary. Hence it can be assumed that the shear stress in turbulent flow is given by equation (4.12). From this equation, the velocity distribution can be obtained if the relation between l, the mixing length and y is known. Prandtl assumed that the mixing length, l is a linear function of the distance y from the pipe wall i.e., $l = ky$, where k is a constant, known as Karman constant and = 0.41

Substituting the value of l in equation (4.12), we get

$$\bar{\tau} \; or \; \tau = \rho \; x \; (ky)^2 \left(\frac{du}{dy}\right)^2$$

Or $$\tau = \rho \, k^2 y^2 \left(\frac{du}{dy}\right)^2 \qquad \text{or} \qquad \left(\frac{du}{dy}\right)^2 = \tau/\rho \, k^2 y^2$$

Or $$\frac{du}{dy} = \sqrt{\frac{\tau}{\rho \, k^2 y^2}} = \frac{1}{ky}\sqrt{\frac{\tau}{\rho}} \qquad \ldots(4.14)$$

For small values of y that is very close to the boundary of the pipe, Prandtl assumed shear stress τ to be constant and approximately equal to τ_0 which presents the turbulent shear stress at the pipe boundary. Substituting $\tau = \tau_0$ in equation (4.14), we get

$$\frac{du}{dy} = \frac{1}{ky}\sqrt{\frac{\tau_0}{\rho}} \qquad \ldots(4.15)$$

In equation 4.15, $\sqrt{\frac{\tau_0}{\rho}}$ has dimensions $\sqrt{\frac{ML^{-1}T^{-2}}{ML^{-3}}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$. But $\frac{L}{T}$ is velocity and hence $\sqrt{\frac{\tau_0}{\rho}}$ has dimension of velocity, which is known as shear velocity and is denoted by u_*.

Thus

$\sqrt{\frac{\tau_0}{\rho}} = u_*$, then equation (4.15) becomes $\frac{du}{dy} = \frac{1}{ky} u_*$.

For a given case of turbulent flow, u_* is constant. Hence integrating above equation, we get

$$u = \frac{u_*}{k} \log_e y + C \qquad \dots(4.16)$$

Where C = constant of integration.

Equation (4.16) shows that in turbulent flow, the velocity varies directly with the logarithm of the distance from the boundary or in other words the velocity distribution in turbulent flow is logarithmic in nature. To determine the constant of integration, C the boundary condition that at y = R (radius of pipe), $\bar{u} = u_{max}$ is substituted in equation (4.16).

Hence $\qquad u_{max} = \frac{u_*}{k} \log_e R + C \qquad \therefore C = u_{max} - \frac{u_*}{k} \log_e R$

Substituting the value of C in equation (4.16), we get

$$u = \frac{u_*}{k} \log_e y + u_{max} - \frac{u_*}{k} \log_e R = u_{max} + \frac{u_*}{k} (\log_e y - \log_e R)$$

$$= u_{max} + \frac{u_*}{0.4} \log_e(y/R) \qquad [\because k = 0.4 = karman\ constant]$$

$$= u_{max} + 2.5 u_* \log_e(y/R) \qquad \dots(4.17)$$

Equation (4.17) is called Prandtl's universal velocity distribution equation for turbulent flow in pipes. This equation is applicable to smooth as well as rough pipe boundaries. Equation (4.17) is also written as

$$u_{max} - u = -2.5 u_* \log_e(y/R) = 2.5 u_* \log_e(R/y)$$

Dividing by u*, we get

$$\frac{u_{max} - u}{u_*} = 2.5 \log_e(R/y) = 2.5\ x\ 2.3 \log_{10}(R/y)$$

$[\because \log_e(R/y) = 2.3 \log_{10}(R/y)]$

Or $$\frac{u_{max} - u}{u_*} = 5.75 \log_{10}(R/y) \qquad \dots(4.18)$$

In equation (4.18), the difference between the maximum velocity u_{max} , and local velocity u at any point i.e., $(u_{max} - u)$ is known as '**velocity defect'.**

4.5.1 Hydrodynamically Smooth and Rough Boundaries. Let k be the average height of the irregularities projecting form the surface of a boundary as shown in Fig. 4.4. If the value k is large for a boundary then the boundary is called rough boundary and if the value of k is less, then the boundary is known as smooth boundary, in general. This is the classification of rough and smooth boundaries based on boundary characteristics. But for proper classification, the flow and fluid characteristics are also to be considered.

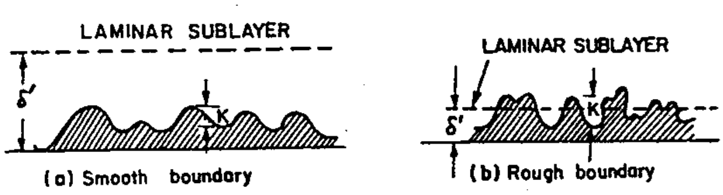


Fig. 4.4 Smooth and rough boundaries

For turbulent flow analysis along a boundary, the flow is divided in two portions. The first portion consists of a thin layer of fluid in the immediate neighborhood of the boundary where viscous shear stress predominates while the shear stress due to turbulence is negligible. This portion is known as laminar sub-layer. The height up to which the effect of viscosity predominates in this zone is denoted by δ'. The 2nd portion of flow, where shear stress due to turbulence is large as compared to viscous stress is known as turbulent zone.

If the average height k, of the irregularities, projecting from the surface of a boundary is much less than δ', the thickness of laminar sub-layer as shown in Fig. 4.4(a), the boundary is called smooth boundary. This is because, outside the laminar sub-layer the flow is turbulent and eddies of various size present in turbulent flow try to penetrate the laminar sub-layer and reach the surface of the boundary. But due to great thickness of laminar sub-layer the eddies are unable to reach the surface irregularities and hence the boundary behaves as a smooth boundary. This type of boundary is called hydrodynamically smooth boundary.

Now, if the Reynolds number of the flow is increased then the thickness of laminar sub-layer will decrease. If the thickness of laminar sub-layer becomes much smaller than the average height K of irregularities of the surface as shown in Fig. 4.4(b), the boundary will act as rough boundary. This is because the irregularities of the surface are above the laminar sub-layer and the eddies present in turbulent zone will come in contact with the irregularities of the surface and lot of energy will be lost. Such a boundary is called hydrodynamically rough boundary.

From Nikuradse's experiment:

1. If $\frac{k}{\delta'}$ is less than 0.25 or $\frac{k}{\delta'} < 0.25$, the boundary is called smooth boundary.
2. If $\frac{k}{\delta'}$ is greater than 6.0, the boundary is rough,
3. If $0.25<\left(\frac{k}{\delta'}\right)<6.0$, the boundary is in transition.

In terms of roughness, Reynolds number $\frac{u_* k}{v}$:

1. If $\frac{u_* k}{v} < 4$, boundary is considered smooth,
2. If $\frac{u_* k}{v}$ lies between 4 and 100, boundary is in transition stage, and
3. If $\frac{u_* k}{v} > 100$, the boundary is rough.

4.5.2 Velocity Distribution For turbulent Flow in Smooth Pipes.

The velocity distribution for turbulent flow in smooth or rough pipe is given by equation (4.16) as

$$u = \frac{u_*}{k}\log_e y + c$$

It may be seen that at y = 0, the velocity u at wall is $-\infty$. This means that velocity u is +ve at some distance far away from wall and $-\infty$ (minus infinity) at wall. Hence at some finite distance wall from the velocity will be equal to zero. Let this distance from pipe wall is y'. Now the constant C is determined from the boundary condition i.e., at y = y', $u = 0$. Hence above equation becomes

$$0 = \frac{u_*}{k}\log_e\ y' + c \qquad \text{or} \qquad C = -\frac{u_*}{k}\log_e\ y'$$

Substituting the values of C in the above equation, we get

$$u = \frac{u_*}{k}\log_e y - \frac{u_*}{k}\log_e\ y' = \frac{u_*}{k}\log_e(y/\ y')$$

Substituting the value of k = 0.4, we get

$$u = \frac{u_*}{0.4}\log_e(y/y') = 2.5u_* \log_e(y/y')$$

$$\frac{u}{u_*} = 2.5\ x\ 2.3\log_{10}(y/y') \qquad [\because \log_e(y/y') = 2.3\log_{10}(y/y')]$$

Or $$\frac{u}{u_*} = 5.75\log_{10}(y/y') \qquad \text{...(4.19)}$$

For the smooth boundary, there exists a laminar sub-layer as shown in Fig 4.4(a). The velocity distribution in the laminar sub-layer is parabolic in nature. Thus in the laminar sub-layer, logarithmic velocity distribution does not hold good. Thus it can be assumed that y' proportional to δ', where δ' is the thickness of laminar sub-layer. From Nikuradse's experiment the value of y' is given as

$$y' = \frac{\delta'}{107}$$

Where $\delta' = \frac{11.6v}{u_*}$, where v = kinematic viscosity of fluid.

$$\therefore \qquad y' = \frac{11.6v}{u_*}\ x\ \frac{1}{107} = \frac{0.108v}{u_*}$$

Substituting the value of y' in equation (4.19), we obtain

$$\frac{u}{u_*} = 5.75 \log_{10}\left(\frac{y}{\frac{0.108v}{u_*}}\right)$$

$$= 5.75 \log_{10}\left(\frac{yu_*}{o.108v}\right) = 5.75 \log_{10}\left(\frac{u_* y}{v}\ x\ 9.256\right)$$

$$= 5.75\log_{10}\frac{u_* y}{v} + 5.75\log_{10}9.256 \qquad \left[\because \frac{1}{0.108} = 9.256\right]$$

$$\frac{u}{u_*} = 5.75\log_{10}\frac{u_* y}{v} + 5.55 \qquad \ldots(4.20)$$

4.5.3. Velocity Distribution for Turbulent Flow in Rough Pipes.

In case of rough boundaries, the thickness of laminar sub-layer is very small as shown in Fig.4.4 (b). The surface irregularities are above the laminar sub-layer and hence the laminar sub-layer is completely destroyed. Thus y' can be considered proportional to the height of protrusions k. Nikuradse's experiment shows the value of y' for pipes coated with uniform sand (rough pipes) as $y' = \frac{k}{30}$.

$$\frac{u}{u_*} = 5.75 \log_{10}\left(\frac{y}{k/30}\right) = 5.75[\log_{10}(y/k)\ x\ 30]$$

$$\frac{u}{u_*} = 5.75\log_{10}(y/k) + 5.75\log_{10}(30.0)$$

$$\frac{u}{u_*} = 5.75\log_{10}(y/k) + 8.5 \qquad (4.21)$$

Problem 4.1. A pipe-line carrying water has average height of irregularities projecting from the surface boundary of the pipe as 0.15 mm. What type of boundary is it? The shear stress developed is 4.9 N/m^2. The kinematic viscosity of water is 0.01 stokes.

Solution

Given:

Average height of irregularities, $k = 0.15\ mm = 0.15\ x\ 10^{-3} m$

Shear stress developed, $\tau_0 = 4.9\ \ N/m^2$

Kinematic viscosity, $v = 0.01\ stokes = 0.01\ cm^2/s = 0.01\ x\ 10^{-4}\ m^2/s$

Density of water, $\rho = 1000\ kg/m^3$

Shear velocity, $u_* = \sqrt{\tau_0/\rho} = \sqrt{\frac{4.9}{1000}} = \sqrt{0.0049} = 0.07\ m/s$

Roughness Reynolds number $= \frac{u_* k}{v} = \frac{o.o7\ x\ 0.15\ x\ 10^{-3}}{0.01\ x\ 10^{-4}} = 10.5$

Since $\frac{u_* k}{v}$ lies between 4 and 100 and hence pipe surface behaves as in transition.

Problem 4.2. A rough pipe is of diameter 8.0 cm. The velocity at a point 3.0 cm from wall is 30% more than the velocity at point 1cm from pipe wall. Determine the average height of the roughness.

Solution

Given:

Diameter of rough pipe, D =8cm = 0.08m

Let velocity of flow at 1cm from pipe wall $= u$

Then velocity at 3cm from pipe wall $= 3u$

The velocity distribution for rough pipe is given by equation (4.12) as

$$\frac{u}{u_*} = 5.75 \log_{10}(y/k) + 8.5 \qquad \text{where k = height of roughness.}$$

For a point, 1cm from pipe wall, we have

$$\frac{u}{u_*} = 5.75 \log_{10}(1.0/k) + 8.5 \qquad \ldots\text{(i)}$$

For a point, 3cm from pipe wall, velocity is $1.3u$ and hence

$$\frac{1.3u}{u_*} = 5.75 \log_{10}(3.0/k) + 8.5 \qquad \ldots\text{(ii)}$$

Dividing (ii) by (i), we get $1.3 = \frac{5.75 \log_{10}(3.0/k)+8.5}{5.75 \log_{10}(1.0/k)+8.5}$

Or $\quad 1.3[5.75 \log_{10}(1.0/k) + 8.5] = 5.75 \log_{10}(3.0/k) + 8.5$

Or $\quad 7.475 \log_{10}(3.0/k) + 11.5 = 5.75 \log_{10}(3.0/k) + 8.5$

Or $\quad 7.475 \log_{10}(1/k) - 5.75 \log_{10}(3.0/k) = 8.5 - 11.5 = -2.55$

Or $\quad 7.475[\log_{10} 1.0 - \log_{10} k] - 5.75[\log_{10} 3.0 - \log_{10} k] = -2.55$

Or $\quad 7.475[0 - \log_{10} k] - 5.75[.4771 - \log_{10} k] = -2.55$

Or $\quad -7.475 \log_{10} k - 2.7433 + 5.75 \log_{10} k = -2.55$

Or $\quad$ -1.725$\log_{10} k = 2.7433 - 2.55 = 0.1933$

Or $\quad \log_{10} k = \frac{0.1933}{-1.725} = -0.1120 = \bar{1}.88$

$$\boldsymbol{k = .7726cm.} \quad \textbf{Ans.}$$

Problem 4.3. A smooth pipe of diameter 80 mm and 800 m long carries water at a rate of 0.480 m^3/minute. Calculate the loss of head, wall shearing stress, center line velocity, velocity and shear stress at 30 mm from pipe wall. Also calculate the thickness of laminar sub-layer. Take kinematic viscosity of water as 0.015 stokes. Take the value of coefficient of friction ‘f’ from the relation given as

$$f = \frac{0.0971}{(R_e)^{1/4}}, \qquad \textit{where } R_e = \textit{Reynolds number.}$$

Solution

Given:

Diameter. Of smooth pipe, $d = 80mm = .08m$

Length of pipe, $L = 800m$

Discharge, $Q = 0.048\, m^3/minute = \frac{0.48}{60} = 0.008\, m^3/s$

Kinematic Viscosity, $v = 0.015\, stokes = 0.015\, x\, 10^{-4}\, m^2/s$ [stokes = cm^2/s]

Density of water, $\rho = 1000\, kg/m^3$

Mean velocity, $V = \frac{Q}{Area} = \frac{0.008}{\frac{\pi}{4}(.08)^2} = 1.591\, m/s$

∴ Reynolds number, $R_e = \frac{V\, x\, d}{v} = \frac{1.591\, x\, 0.08}{.015\, x\, 10^{-4}} = 8.485\, x\, 10^4$

As the Reynolds number is more than 4000, the flow is turbulent.

Now the value of 'f' is given by $f = \frac{0.0791}{(R_e)^{1/4}} = \frac{0.0791}{(8.495\, x\, 10^4)^{1/4}} = 0.004636$

(i) Head lost is given by equation (4.2) as

$$h_f = \frac{4.f.L.V^2}{d\, x\, 2g} = \frac{4\, x\, 0.004636\, x\, 800\, x\, 1.591^2}{0.08\, x\, 2\, x\, 9.81} = \mathbf{23.42m} \qquad \textbf{Ans.}$$

(ii) Wall shearing stress, τ_0 is given by equation (4.5) as

$$\tau_0 = \frac{f\rho V^2}{2} = 0.004636 \times \frac{1000}{2} \times 1.591^2 = \mathbf{5.866\, N/m^2} \quad \textbf{Ans.}$$

(iii) Centre-line velocity, u_{max} for smooth pipe is given by equation (4.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{v} + 5.55 \qquad \ldots(i)$$

Where u_* is shear velocity and $\sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{5.866}{1000}} = 0.0765\, m/s$

The velocity will be maximum when $y = \frac{d}{2} = \frac{.08}{2} = 0.04m$.

Hence at $y = 0.04m, u = u_{max}$, substituting these values in (i), we get

$$\frac{u_{max}}{0.0765} = 5.75 \log_{10} \frac{0.0765 \times 0.04}{0.015 \times 10^{-4}} + 5.55$$

$$= 5.75\log_{10}(2040) + 5.55$$

$$= 5.75 * 3.309 + 5.55 = 19.03 + 5.55 = 24.58$$

$$u_{max} = 0.0765 \times 24.58 = \mathbf{1.88\, m/s}. \qquad \textbf{Ans.}$$

(iv) The shear stress, τ at any point given by

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \qquad \ldots(A)$$

Where r = distance from center pipe

And hence shear stress at pipe wall where r = R is

$$\tau_0 = -\frac{\partial p}{\partial x}\frac{R}{2} \qquad \ldots(B)$$

Dividing equation (A) by equation (B), we get

$$\frac{\tau}{\tau_0} = \frac{r}{R}$$

∴ Shear stress $\tau = \frac{\tau_0 r}{R}$

A point 30 mm from pipe wall is having r = 4-3 = 1cm =.01m

∴ τ at (r=.01 m) = $\frac{\tau_0 \times 0.01}{0.04} = \frac{5.866}{4} = \mathbf{1.4665\ N/m^2}$. **Ans.**

Velocity at a point 3 cm from pipe wall means y =3 cm = .03 m

And is given by equation (4.20) as $\frac{u}{u_*} = 5.75\log_{10}\frac{u_* y}{v} + 5.55$, where $u_* = 0.0765, y = .03$

∴ $\frac{u}{0.765} = 5.75\log_{10}\frac{.076 \times .03}{0.015 \times 10^{-4}} + 5.55$

$= 5.75\log_{10} 1530 + 5.55 = 23.86$

∴ $u = 0.0765 \times 23.86 = \mathbf{1.825\, m/s}$. **Ans.**

(v) Thickness of laminar sub-layer is given by

$$\delta' = \frac{11.6\ x\ v}{u_*} = \frac{11.6\ x\ 0.15\ x\ 10^{-4}}{.0765} = 2.274\ x\ 10^{-4} m$$

$$= 2.274\ x\ 10^{-2} cm = \mathbf{.02274\ cm}.$$ **Ans.**

Problem 4.4. Determine the wall shearing stress in a pipe of diameter 100 mm which carries water. The velocities at the pipe center and 30 mm, from the pipe center are 2 m/s and 1.5 m/s respectively. The flow in pipe is given turbulent.

Solution

Given:

Diameter of pipe, D = 100mm = 0.10m

∴ Radius, R = $\frac{0.10}{2} = 0.05m$

Velocity at center, $u_{max} = 2\, m/s$

Velocity at 30mm or 0.03m from center = 1.5m/s

∴ Velocity (at r = 0.03 m), $u = 1.5\, m/s$

Let the wall shearing stress = τ_0

For turbulent flow, the velocity distribution in terms of centerline velocity (u_{max}) is given by equation (4.18) as

$$\frac{u_{max} - u}{u_*} = 5.75\log_{10}\left(\frac{R}{y}\right)$$

Where u = 1.5 m/s at y = (R – r) = 0.05 – 0.03 =0.02m

∴ $\frac{2.0-1.5}{u_*} = 5.75\log_{10}\frac{.05}{.02} = 2.288$ or $\frac{0.5}{u_*}$ = 2.288

∴ $u_* = \frac{0.5}{2.288} = 0.2185 m/s$

Using the relation $u_* = \sqrt{\tau_0/\rho}$ where ρ for water = 1000kg/m

$\therefore$ $0.2185 = \sqrt{\frac{\tau_0}{1000}}$ or $\frac{\tau_0}{1000} = 0.2185^2 = 0.0477$

Or $\tau_0 = 0.0477\, x\, 1000 = \mathbf{47.676\ N/m^2}$. **Ans.**

4.5.4. Velocity Distribution for Turbulent Flow in Terms of Average Velocity.

The average velocity $\bar{U}$, through the pipe is obtained by first finding the total discharge Q and then dividing the total discharge by area of the pipe.

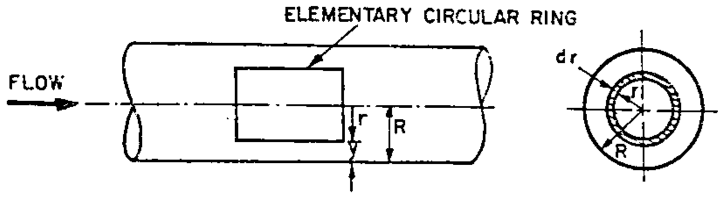


Fig. 4.5 Average velocity for turbulent flow.

Consider an elementary circular ring of radius 'r' and thickness dr as shown in Fig 4.5. The distance of the ring from pipe wall is $y = (R - r)$, where R = radius of pipe.

Then the discharge, dQ, through the ring is given by

dQ = area of ring x velocity

$$= 2\pi r dr \times u = u 2\pi r dr$$

Total discharge, $Q \int dQ = \int_0^R u \times 2\pi r dr$ ...(4.22)

(a) **For smooth Pipes,** For smooth pipes, the velocity distribution is given by equation (4.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{v} + 5.55$$

Or $u = \left[5.75 \log_{10} \frac{u_* y}{v} + 5.55\right] x\, u_*$

But $y = (R - r)$

$\therefore$ $u = \left[5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.55\right] x\, u_*$

Substituting the value of u in equation(4.22), we get

$$Q = \int_0^R \left[5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.55\right] u_* x\ 2\pi r dr$$

$\therefore$ Average velocity, $\bar{U} = \frac{Q}{Area} = \frac{Q}{\pi R^2}$

$$= \frac{1}{\pi R^2} \int_0^R \left[5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.55\right] u_* \times 2\pi r dr$$

Integration of the above equation and subsequent simplification gives the average velocity for turbulent flow in smooth pipes as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{v} + 1.75 \qquad \ldots(4.23)$$

(b) **For Rough Pipes**. For rough pipes, the velocity at any point in turbulent flow is given by equation (4.21) as

$$\frac{u}{u_*} = 5.75 \log_{10}(y/k) + 8.5$$

But $$y = (R - r)$$

$$\frac{y}{u_*} = 5.75 \log_{10}\left(\frac{R-r}{k}\right) + 8.5$$

Or $$u = u_* \left[5.75 \log_{10}\left(\frac{R-r}{k}\right) + 8.5\right]$$

Substituting the value of u in equation (4.22), we get

$$Q = \int_0^R u_* \left[5.75 \log_{10}\left(\frac{R-r}{k}\right) + 8.5\right] 2\pi r dr$$

∴ Average velocity, $$\bar{U} = \frac{Q}{\pi R^2} = \frac{\int_0^R u_* \left[5.75 \log_{10}\left(\frac{R-r}{k}\right) + 8.5\right] 2\pi r dr}{\pi R^2}$$

Integration of the above equation and subsequent simplification will give the following relation for average velocity, $\bar{U}$ for turbulent flow in pipe as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{R}{k} + 4.75 \qquad \ldots(4.24)$$

(c) **Difference of the velocity at any point and average velocity for smooth and rough pipes.** The velocity at any point for turbulent flow for smooth pipes is given by equation (4.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.5 \qquad [\because y = R - r]$$

And the average velocity is given b equation (4.23) as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{v} + 1.75$$

∴ Difference of velocity u and $\bar{U}$ for smooth pipe is obtained as

$$\frac{u}{u_*} - \frac{\bar{U}}{u_*} = \left[5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.5\right] - \left[5.75 \log_{10} \frac{u_* R}{v} + 1.75\right]$$

Or $$\frac{u-U}{u_*} = 5.75 \left[\log_{10} \frac{u_*(R-r)}{v} - \log_{10} \frac{u_* R}{v}\right] + 5.5 - 1.75$$

$$= 5.75 \log_{10} \left[\frac{u_*(R-r)}{v} \div \frac{u_* R}{v}\right] + 3.75$$

$$= 5.75 \log_{10} \left(\frac{(R-r)}{R}\right) + 3.75$$

$$= 5.75 \log_{10}(y/R) + 3.75 \qquad \ldots(4.25) \; [\because R - r = y]$$

Similarly the velocity, u at a point for rough pipe is given by equation (4.21) as

$$\frac{u}{u_*} = 5.75 \log_{10}(y/k) + 8.5$$

And average velocity is given by equation (4.24) as $\frac{\bar{U}}{u_*} = 5.75 \log_{10}(R/k) + 4.75$

∴ Difference of velocity u and $\bar{U}$ for rough pipe is given by

$$\frac{u}{u_*} - \frac{\bar{U}}{u_*} = [5.75 \log_{10}(y/k) + 8.5] - [5.75 \log_{10}(R/k) + 4.75]$$

$$= 5.75 \log_{10}[(y/k) \div (R/k)] + 8.5 - 4.75$$

Or $\qquad \frac{u-\bar{U}}{u_*} = 5.75 \log_{10}(y/R) + 3.75 \qquad \ldots(4.26)$

Equation (4.25) and (4.26) are the same. This shows that the difference of velocity at any point and the average velocity will be the same in case of smooth as well as rough pipes.

Problem 4.5. Determine the distance from the pipe wall at which the local velocity is equal to the average velocity for turbulent flow in pipes.

Solution

Given:

Local velocity at a point = average velocity

Or $\qquad u = \bar{U}$

For a smooth or rough pipe, the difference of velocity at any point and average velocity is given by equation (4.25) or (4.26) as

$$\frac{u-\bar{U}}{u_*} = 5.75 \log_{10}(y/R) + 3.75$$

Substituting the given condition i.e., $u = \bar{U}$, we get

$$\frac{u-\bar{U}}{u_*} = 0 = 5.75 \log_{10}(y/R) + 3.75 \text{ or } 5.75 \log_{10}(y/R) = 3.75$$

Or $\qquad \log_{10}(y/R) = -\frac{3.75}{5.75} = -0.6521 = -\bar{1}.3479$

∴ $\qquad (y/R) = 0.22279 \cong 0.2228 \quad$ or $\quad y = \mathbf{.2228\, R}. \qquad$ **Ans.**

Problem 4.6. For turbulent flow in a pipe of diameter 300 mm, find the discharge when the center-line velocity is 2.0 m/s and the velocity at appoint 100 mm from the center as measured by pitot-tube is 1.6 m/s.

Solution

Given:

Diameter of pipe, D = 300mm = 0.3m

∴ Radius, $R = \frac{0.3}{2} = 0.15m$

Velocity at center, $U_{max} = 2.0\ m/s$

Velocity (at r = 100mm = 0.1m), *u*= 1.6 m/s

Now $y = R - r = 0.15 - 0.10 = 0.05m$

∴ Velocity (at r = 0.1 m or at y = 0.05m), u = 1.6 m/s

The velocity in terms of center-line velocity is given by equation (4.18) as

$$\frac{U_{max} - u}{u_*} = 5.75 \log_{10}(R/y)$$

Substituting the values, we get $\frac{2.0-1.6}{u_*} = 5.75 \log_{10} \frac{.15}{.05}$ $[\because y = .05m\ R = 0.15m]$

$$= 5.75 \log_{10} 3.0 = 2.7434$$

Or $\frac{0.4}{u_*} = 2.7434$

∴ $u_* = \frac{0.4}{2.7434} 0.1458 m/s$ …(i)

Using equation (4.26) which gives relation between velocity at any point and average velocity, we have

$$\frac{u - \overline{U}}{u_*} = 5.75 \log_{10}(R/y) + 3.75$$

At y = R, velocity *u* becomes = u_{max}

∴ $\frac{U_{max} - \overline{U}}{u_*} = 5.75 \log_{10}(R/R) + 3.75 = 5.75\ x\ 0 + 3.75 = 3.75$

But $U_{max} = 2.0$ and u_* from (i) = 0.01458

∴ $\frac{2.0 - \overline{U}}{0.1458} = 3.75$

Or $\overline{U} = 2.0 - .1458 \times 3.75 = 2.0 - .5647 = 1.4533 m/s$

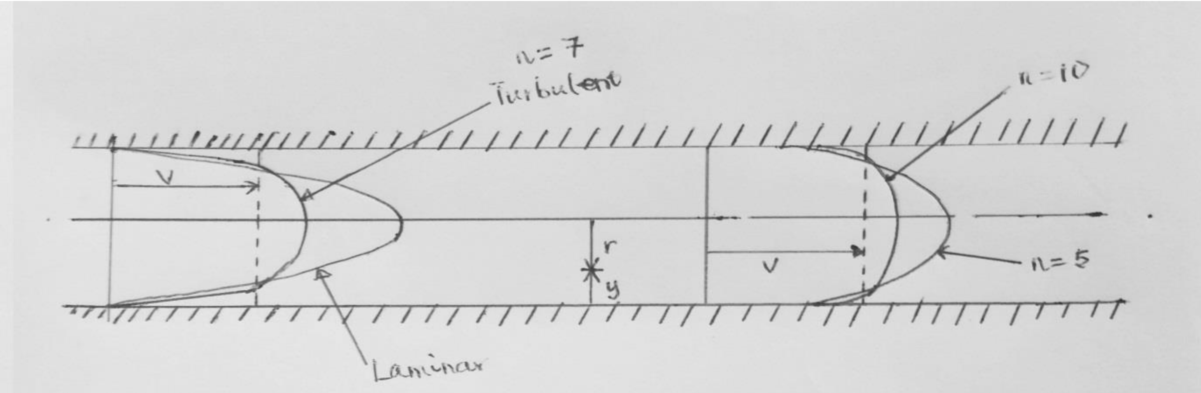
∴ Discharge, $Q = Area \times average\ velocity$

$= \frac{\pi}{4} D^2 \times \overline{U} = \frac{\pi}{4} (0.3)^2 \times 1.4533 = \boldsymbol{0.1027\ m^3/s}$ **Ans.**

4.6 POWER LAW

An alternative and simpler form of the turbulent flow velocity distribution in pipes, that adequately describes the turbulent flow in a pipe is the ***power law profile***, namely $\frac{\bar{u}}{u_o} = \left(\frac{y}{r_o}\right)^{\frac{1}{n}}$, where y is measured from the pipe wall and n is an integer between 5 and 10. Using this distribution, the average velocity is found to be:

$$V = \frac{\int \bar{u}(r) 2\pi r dr \pi}{\pi r_o^{\,2}} = \frac{2n^2}{(n+1)(2n+1)} u_{\max}$$



This distribution is compared with a laminar profile

The value of n in the exponent is related to the friction factor *f* by the empirical expression

$$n = \frac{1}{\sqrt{f}} \; or \; f = \frac{1}{n^2}$$

The constant n varies from 5 to 10 depending on the Reynolds number and the pipe wall roughness ***e/D***. For smooth pipes the exponent *n* is related to the Reynolds number as shown in Table 4.7.

Table 4.7, Exponent n for smooth pipes.

Re=VD/v	4×10^3	10^5	10^6	$> 2 \times 10^6$
n	6	7	9	10

The power-law profile cannot be used to obtain the slope at the wall since it always yield $((\frac{\partial u}{\partial y})_{wall} = \infty \; for \; all \; n.$ Thus it cannot be used to predict the wall shear stress. The wall shear stress is found by combining equations 4.6 B and 4.12.

It should be noted that the kinetic-energy-correction factor α in pipes is 1.11, 1.06 and 1.03 for n= 5, 7, and 10 respectively. Because it is close to unity, it is often set equal to unity in the energy equation when working problems involving turbulent flow.

Problem 4.7 A 4 cm diameter smooth, horizontal pipe transports $0.004\,m^3/s$ of water at 20^0C. Calculate (a) the friction factor, (b) the maximum velocity, (c) the radial position where $\bar{u} = V$, (d) the wall shear, € the pressure drop over 10m length and (f) the maximum velocity using equation 4.3.5

Solution

(a) The average velocity is calculated to be

$$V = \frac{Q}{A} = \frac{0.004}{\pi(0.04)\text{^}2} = 3.18 m/s$$

The Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{3.18 \times 0.04}{10^{-6}} = 1.27 \times 10^5$$

From Table 4.7, we see that n ≅ 7 and from equation 4.3.10,

$$f = \frac{1}{n^2} = \frac{1}{7^2} = 0.02$$

(b) The maximum velocity is found using equation(6.3.9) to be

$$u_{\max} = \frac{(n+1)(2n+1)}{2n^2} V = \frac{8 \times 15 \times 3.18}{2 \times 49} = 3.89 m/s$$

(c) The distance from the wall where $\bar{u} = V = 3.18 m/s$ is found using equation (6.3.8) as follows

$$\frac{\bar{u}}{u_{\max}} = \left(\frac{y}{r_o}\right)^{1/n}$$

$$\therefore y = \left(\frac{\bar{u}}{u_{\max}}\right)^7 = 2\left(\frac{3.18}{3.89}\right)^7 = 0.488 cm$$

The radial position is thus

$$r = r_o - y = 2 - 0.488 = 1.51 cm$$

(d) The wall shear is found using the definition of the friction factor = 4f

$$coefficient,\ f = \frac{\tau_o}{\frac{1}{8}\rho V^2} \therefore \tau_o = \frac{1}{8}\rho V^2 = \frac{1}{8} \times 1000 \times 3.18^2 \times 0.02 = 25.3 Pa$$

(e) The pressure drop is calculated using equation 6.3.7 with

$$\Delta p / L = -dp/dx$$

$$\Delta p = \frac{2\tau_o L}{r_o} = \frac{2 \times 253 \times 10}{0.02} = 25300 Pa$$

(f) To use equation (6.3.5) we must know the shear velocity. It is

$$u_T = \sqrt{\tau_o/\rho} = \sqrt{\frac{25.3}{1000}} = 0.159 m/s$$

$$u_{\max} = 0.159\left(2.4 \ln \frac{0.159 \times 0.02}{10^{-6}} + 5.7\right) = 4.04 m/s$$

Very close to that given by the power-law formula in part (b). This answer is considered to be more accurate.

4.7 RESISTANCE IN SMOOTH AND ROUGH PIPES.

The loss of head, due to friction in pipes is given by equation (4.2) as

$$h_f = \frac{4.f.L.V^2}{d \ x \ 2g}$$

In this equation, the value of co-efficient of friction, *f* should be known accurately or predicting the loss of head due to friction in pipes. On the basis of dimensional analysis, it can be shown that the pressure loss in a straight pipe of diameter D, length L, roughness k, average velocity of flow $\bar{U}$, viscosity and density of fluid μ and ρ is

$$\Delta p = \frac{\rho \bar{U}^2}{2} \phi \left[R_e, \frac{k}{D}, \frac{L}{D}\right] \ or \ \frac{\Delta p}{\frac{\rho \bar{U}^2}{2}} = \phi \left[R_e, \frac{k}{D}, \frac{L}{D}\right]$$

Experimentally it was found that pressure drop is a function of $\frac{L}{D}$ to the first power and hence

$$\frac{\Delta p}{\frac{\rho \bar{U}^2}{2}} = \frac{L}{D} \phi \left[R_e, \frac{k}{D}\right] \text{ or } \frac{\Delta p \ x \ D}{L \frac{\rho \bar{U}^2}{2}} = \phi \left[R_e, \frac{k}{D}\right]$$

The term of the right hand side is called co-efficient of friction *f*. thus $f = \phi \left[R_e, \frac{k}{D}\right]$

This equation shows that friction co-efficient is a function of Reynolds number and k/D ratio, where k is the average height of pipe wall roughness protrusions.

(a) **Variation of 'f' Laminar flow.** In viscous flow chapter, it is shown that co-efficient of friction 'f' for laminar flow in pipes is given by (3.23) as,

$$f = \frac{16}{R_e} \qquad \ldots(4.29)$$

Thus friction co-efficient is only a function of Reynolds number in case of laminar flow.it is independent of (k/D) ratio.

(b) **Variation of 'f' for turbulent flow.** For turbulent flow, the co-efficient of friction is a function of R_e and k/D ratio. For relative roughness (k/D), in the turbulent flow the boundary may be smooth or rough and hence the value of 'f' will be different for these boundaries.

(i) 'f' for smooth pipes. For turbulent flow in smooth pipes, co-efficient of friction is a function of Reynold number only. The value of laminar sub-layer in case of smooth pipe for Reynolds number varying from 4000 to 100000 is given by the relation

$$f = \frac{.0791}{(R_e)^{1/4}} \qquad \ldots(4.30)$$

The equation (4.30) is given by Blasius.

The value of 'f' for $R_e > 10^5$ is obtained from equation (4.23) which gives the velocity distribution for smooth pipe in terms of average velocity ($\bar{U}$) as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{(u_* R)}{v} + 1.75 \qquad \ldots(4.31)$$

From equation (4.6), we have $f = \frac{2\tau_0}{\rho V^2}$, where V = average velocity

$$\therefore \qquad f = \frac{2\tau_0}{\rho \bar{U}^2} = \frac{2}{\bar{U}}\left(\sqrt{\frac{\tau_0}{\rho}}\right)^2 = \frac{2}{\bar{U}}\ x\ {u_*}^2 \qquad \left[\because \sqrt{\frac{\tau_0}{\rho}} = u_*\right]$$

$$\therefore \qquad {u_*}^2 = \frac{f\bar{U}^2}{2}$$

Or

$${u_*}^2 = \bar{U}\sqrt{\frac{f}{2}} \qquad \ldots(4.31A)$$

Substituting the value of u_* in equation(4.31), we get

$$\frac{\bar{U}}{\bar{U}\sqrt{f/2}} = 5.75 \log_{10}\left(\frac{\sqrt{f/2}}{v}\right) R + 1.75 \qquad \text{or} \qquad \frac{1}{\sqrt{f/2}} = 5.75 \log_{10}\left(\frac{\bar{U}R}{v}\sqrt{f/2}\right) + 1.75$$

Taking R = D/2 and simplifying the above equation is written as

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}\left(\frac{\bar{U}D}{v}\sqrt{4f}\right) - 0.91$$

But $\frac{\bar{U}D}{v} = R_e$ and hence above equation is written as

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}\left(R_e\sqrt{4f}\right) - 0.91 \qquad \ldots(4.32)$$

Equation (4.32) is valid up to $R_e = 4\ x\ 10^6$

Nikuradse's experimental result for turbulent flow in smooth pipe for 'f' is

$$\frac{1}{\sqrt{4f}} = 0.2 \log_{10}\left(R_e\sqrt{4f}\right) - 0.8 \qquad \ldots(4.33)$$

This applicable up to $R_e = 4\ x\ 10^7$. But the equation (4.33) is solved by hit and trial method.

The value of 'f'(i.e., co-efficient of friction) can alternatively be obtained as

$$f = .0008 + \frac{.05525}{(R_e)^{0.237}} \qquad \ldots(4.34)$$

The value of 'f' (i.e, friction factor which is used in equation 4.2 A) is given by

$$f = .00032 + \frac{.221}{(R_e)^{0.237}} \qquad \ldots(4.34A)$$

(ii)**Value of 'f' for rough pipes.** For turbulent flow in rough pipes, the co-efficient of friction is a function of relative roughness (k/D) and it is independent of Reynolds number. This is because the value of laminar sub-layer for rough pipes is very small as compared to the height of surface roughness. The average velocity for rough pipes is given by (4.24) as

$$\frac{\bar{U}}{u_*} = 5.75\log_{10}(R/k) + 4.75$$

$$u_* = \bar{U}\sqrt{f/2}$$

Substituting the value of u_* in the above equation, we get

$$\frac{\bar{U}}{\bar{U}\sqrt{f/2}} = 5.75\log_{10}(R/k) + 4.75$$

Which is simplified to form as

$$\frac{1}{\sqrt{4f}} = 2.03\log_{10}(R/k) + 1.68 \qquad \ldots(4.35)$$

But Nikuradse's experimental result gave rough pipe the following relation for 'f' as

$$\frac{1}{\sqrt{4f}} = 2\log_{10}(R/k) + 1.74 \qquad \ldots(4.36)$$

(c) **Value of 'f' for commercial pipes.** The value of 'f' for commercial pipes such as pipes made of metal, concrete and wood is obtained from Nikuradse's experimental data for smooth and rough pipes. According to Colebrook, by subtracting$2\log_{10}(R/k)$ from both sides of equations (4.33) and (4.36), the value of 'f' is obtained for commercial smooth and rough pipes as:

1. **Smooth pipes**

$$\frac{1}{\sqrt{4f}} - 2\log_{10}(R/k) = 2\log_{10}\left(R_e\sqrt{4f}\right) - 0.8 - 2\log_{10}(R/k)$$

$$= 2\log_{10}\left(\frac{R_e\sqrt{4f}}{R/k}\right) - 0.8 \qquad \ldots(4.37)$$

2. **Rough pipes**

$$\frac{1}{\sqrt{4f}} - 2\log_{10}(R/k) = 2\log_{10}(R/k) + 1.74 - 2\log_{10}(R/k) = 1.74 \qquad (4.38)$$

Problem 4.8. For the problem 4.6, find the co-efficient of friction and the average height of roughness projections.

Solution.

Given:

From the solution of problem 4.6, we have

$$R = 0.15m$$

$$u_* = 0.1458\ m/s$$

$$\bar{U} = 1.4533m/s$$

For co-efficient of friction, we know that

$$u_* = \bar{U}\sqrt{f/2}$$

Or $\quad 0.1458 = 1.4533\sqrt{f/2}$

Or $\quad \sqrt{f/2} = \frac{0.1458}{1.4533} = 0.1$

$$f = 2.0\ x\ (0.1)^2 = \mathbf{.02} \qquad \textbf{Ans}.$$

Height of roughness projection is obtained from equation (4.36) as

$$\frac{1}{\sqrt{4f}} = 2\log_{10}(R/k) + 1.74$$

Substituting the values of R and *f* we get

$$\frac{1}{\sqrt{4\ x\ 0.2}} = 2\log_{10}\left(\frac{0.15}{k}\right) + 1.74 \qquad \text{or} \qquad 3.5355 = 2\log_{10}\left(\frac{.15}{k}\right) + 1.74$$

Or $\quad \log_{10}\left(\frac{.15}{k}\right) = \frac{3.5355-1.74}{2} = 0.8977 = \log_{10} 7.90$

∴ $\quad \frac{.15}{k} = 7.90$

∴ $\quad k = \frac{0.15}{7.90} = 0.01898\ m = \mathbf{18.98mm}.$ **Ans**

Problem 4.9 Water is flowing through a rough pipe of diameter 500 m and length 4000 m at the rate of 0.5 m^3/s. find the power required to maintain this flow. Take the average height of roughness as k = 0.40 mm.

Solution

Given:

Diameter of rough pipe, $\quad D = 500\ mm = 0.50m$

∴ Radius, $\quad R = \frac{D}{2} = 0.25m$

Length of pipe, $\quad L = 4000\ m$

Discharge, $\quad Q = 0.5\ m^3/s$

Average height of roughness, $k = 0.40\ mm = 0.4\ x\ 10^{-3}m$

First find the value of the co-efficient of friction. Then calculate the head lost due to friction and then power required.

For a rough pipe, the value of 'f' is given by the equation (4.36) as

$$\frac{1}{\sqrt{4f}} = 2\log_{10}(R/k) + 1.74 = 2\log_{10}\left(\frac{0.25}{0.4\ x\ 10^{-3}}\right) + 1.74$$

$$= 2\log_{10}(625.0) + 1.74 = 5.591 + 1{,}74 = 7.331$$

$$\sqrt{4f} = \frac{1}{7.331} = 0.1364 \quad \text{or } f = (0.1364)^2/4 = 0.00465$$

Also the average velocity, $\overline{U} = \frac{Discharge}{Area} = \frac{0.5}{\frac{\pi}{4}D^2} = \frac{0.5}{\frac{\pi}{4}(.5)^2} = 2.546$

∴ Head lost due to friction, $h_f = \frac{4.f.L.V^2}{d\,x\,2g} = \frac{4\,x\,.00465\,x\,4000\,x\,2.546^2}{0.5\,x\,2\,x\,9.81}$

$$= 49.16\,m \qquad [\because V = \overline{U} = 2.546, d = D = 0.5]$$

∴ Power required, $P = \frac{W\times h_f}{1000} = \frac{\rho\times g\times Q\times h_f}{1000}kW = \frac{1000\times9.81\times0.5\times49.16}{1000} = \mathbf{241.13kW}.$

Problem 4.10 A smooth pipe of diameter 400mm and length 800m caries water at the rate of 0.04 m^3/s. Determine the head lost due to friction, wall shear stress, center-line velocity ad thickness of laminar sub-layer. Take kinematic viscosity of water as 0.018 stokes.

Solution

Given:

Diameter of rough pipe, $D = 400\,mm = 0.40m$

∴ Radius, $R = \frac{D}{2} = 0.20m$

Length of pipe, $L = 800\,m$

Discharge, $Q = 0.04\,m^3/s$

Kinematic viscosity, $v = 0.018 stokes = 0.018\ cm^2/s = 0.018\,x\,10^{-4}\,m^2/s$

Average velocity, $\overline{U} = \frac{Q}{A} = \frac{0.04}{\frac{\pi}{4}(.4)^2} = 0.3183\,m/s$

∴ Reynolds number, $R_e = \frac{V\,x\,D}{v} = \frac{\overline{U}\,x\,D}{v} = \frac{0.3813\,x\,0.4}{.018\,x\,10^{-4}} = 7.073\,x\,10^4$

The flow is turbulent.

The co-efficient of friction 'f' is obtained from equation (4.30) as

$$f = \frac{.0791}{(R_e)^{1/4}} = \frac{0.0791}{(7.073\times10^4)^{1/4}} = .00485$$

(i) Head lost due friction, $h_f = \frac{4.f.L.V^2}{D\times2g} = \frac{4.f.L.\overline{U}^2}{D2g}$

$$= \frac{4\times0.00485\times800\times(.3183)^2}{.40\times2\times9.81} = \mathbf{0.20\,m}. \qquad \textbf{Ans.}$$

(ii) Wall shear stress (τ_0) is given by equation (4.5) as

$$\tau_0 = \frac{f.p.V^2}{2} = \frac{f.p.\overline{U}^2}{2}$$

$$= 0.00485\,x\,1000\,x\,\frac{(.3184)^2}{2.0}N/m^2 = \mathbf{0.245\,N/m^2} \qquad \textbf{Ans.}$$

(iii) The center-line velocity (U_{max}) for smooth pipe is given by equation (4.20) as in which u = U_{max} at y = R

$$\therefore \quad \frac{U_{max}}{u_*} = 5.75\log_{10}\frac{(u_*R)}{v} + 5.55 \qquad [Put\ in\ equation\ (4.20), u = u_{max}\ at\ y = R]$$

Where the shear velocity $u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.245}{1000}} = \sqrt{0.000245} = 0.0156\, m/s$

Substituting the values u_*, R and v in the above equation, we get

$$\frac{U_{max}}{0.0156} = 5.75 \log_{10} \frac{0.0156\, x\, 0.20}{.018\, x\, 10^{-4}} + 5.55 = 24.173$$

Or $U_{max} = 24.173 \times .0156 = \mathbf{0.377 m/s}$ **Ans.**

(iv) The thickness of laminar sub-layer (δ') is given by

$$\delta' = \frac{11.6\, x\, v}{u_*} = \frac{11.6\, x\, .081\, x\, 10^{-4}}{.0156} = .001338\, m = \mathbf{1.338\, mm}. \quad \textbf{Ans.}$$

Problem 4.11 A rough pipe of diameter 400mm and length 1000m carries water at the rate of 0.4m^3/s. The wall roughness is 0.012mm. Determine the co-efficient of friction, wall shear stress, center-line velocity and velocity at a distance of 150mm from the pipe wall.

Solution

Given:

Diameter of rough pipe, $D = 400\, mm = 0.40m$

∴ Radius, $R = \frac{D}{2} = \frac{.4}{2} 0.20m$

Length of pipe, $L = 1000\, m$

Discharge, $Q = 0.4\, m^3/s$

Wall roughness, $k = 0.012\, mm = 0.012\, x\, 10^{-3}m$

(i) The value of co-efficient of friction '*f*' for rough pipe is given by the equation (4.36) as

$$\frac{1}{\sqrt{4f}} = 2 \log_{10}(R/k) + 1.74$$

Or $\frac{1}{\sqrt{4f}} = 2 \log_{10} \left(\frac{.20}{.012 \times 10^{-3}}\right) + 1.74 = 2 \log_{10}(16666.67) + 1.74 = 10.183$

∴ $4f = \left(\frac{1}{10.183}\right)^2 = .00964$

∴ $f = \frac{.00964}{4} = \mathbf{.00241}$ **Ans.**

(ii) Center-line velocity (U_{max}) for rough pipe is given by equation (4.21) in which u is made = U_{max} at y = R and hence

$$\frac{U_{max}}{u_*} = 5.75 \log_{10}(R/k) + 8.5 \qquad \text{…(i)}$$

Where shear velocity, $u_* = \sqrt{\frac{\tau_0}{\rho}}$

And τ_0 = wall shear stress = $\frac{f.\rho.V^2}{2}$

Where V = $\frac{discharge}{area} = \frac{Q}{\frac{\pi}{4}D^2} = \frac{0.4}{\frac{\pi}{4}(.4)^2} = 3.183m/s$

(iii) ∴ $\tau_0 = \frac{f.\rho.V^2}{2} = .00241\ x\ 1000\ x\ \frac{3.183^2}{2.0} = \mathbf{12.2\ N/m^3}$ **Ans**

(iv) Velocity (u) at a distance $y = 150mm = 0.15m$

The velocity (u) at any point for rough pipe is given by equation (4.21) as

$$\frac{u}{u_*} = 5.75\log_{10}(R/k) + 8.5$$

Where $u_* = 0.11m/s$ and $y = 0.15m,\ k = 0.012\ x\ 10^{-3}m$

∵ $\frac{u}{0.11} = 5.75\log_{10}\left(\frac{0.15}{.012\ x10^{-3}}\right) + 8.5 = 32.05$

∴ $u = 32.05\ x\ 0.11 = \mathbf{3.52m/s.}$ **Ans**

Problem 4.12 A smooth pipe line of 100mm diameter carries 2.27 m^3 per minute o water at 20℃ with kinematic viscosity of 0.0098 stokes. Calculate the friction factor, maximum velocity as well as shear stress at the boundary

Solution

Given:

Diameter of pipe, $D = 100\ mm = 0.10m$

∴ Radius, $R = 0.05m$

Discharge, $Q = 2.27\ m^3/min = \frac{2.27}{60}m^3/s$

Kinematic viscosity, $v = 0.0098\ stokes = 0.0098\ cm^3/s = 0.0098\ x\ 10^{-4}\ m^3/s$

Now average velocity is given by $\bar{U} = \frac{Q}{Area} = \frac{0.0378}{\frac{\pi}{4}(.1)^2} = \frac{0.0378\ x\ 4}{\pi\ x\ 0.01} = 4.817m/s$

∴ Reynolds number is given by, $R_e = \frac{\bar{U}\ x\ D}{v} = \frac{4.817\ x\ 0.1}{0.0098\ x\ 10^{-4}} = 4.9154\ x\ 10^5.$

The flow is turbulent and R_e is more than 10^5. Hence for smooth pipe, the co-efficient of friction 'f' is obtained from equation (4.23) as

$$\frac{1}{\sqrt{4f}} = 2\log_{10}\left(R_e\sqrt{4f}\right) - 0.8$$

Or $\frac{1}{\sqrt{4f}} = 2.0\log_{10}\left(4.9154\ x\ 10^5\sqrt{4f}\right) - 0.8$

$$= 2.0\left[\log_{10} 4.9154\ x\ 10^5 - \log_{10}\sqrt{4f}\right] - 0.8$$

$$= 2.0\left[5.9615 + \log_{10}\ \sqrt{4f}\right] - 0.8 = 2\ x\ 5.6915 + 2\log_{10}\sqrt{4f} - 0.8$$

$$= 11.3830 + \log_{10}\left(\sqrt{4f}\right)^2 - 0.8 = 11.383 + \log_{10}(4f) - 0.8$$

Or $\frac{1}{\sqrt{4f}} - \log_{10} 4f = 11.383 - 0.8 = 10.583$ …(i)

(i) Friction factor

Now, friction factor (f*) $= 4\ x\ co - efficient\ of\ friction = 4f$

Substituting the value of '$4f$' in equation (i), we get

$$\frac{1}{\sqrt{f*}} - \log_{10} f* = 10.583 \qquad \ldots(ii)$$

The equation above is solved by hit and trial method.

Let f* = 0.1, then L.H.S of equation (ii), becomes

L.H.S = $\frac{1}{\sqrt{0.1}} - \log_{10} 0.1 = 3.16 - (-1.0) = 4.16$

Let f* = 0.01, then L.H.S. of equation (ii), becomes

L.H.S = $\frac{1}{\sqrt{0.01}} - \log_{10} 0.01 = 10 - (-2) = 12$

But for exact solution, L.H.S. should be 10.583, hence the value of f* lies between 0.1 and 0.001.

Let f* = 0.013 then L.H.S. of equation (ii), becomes

L.H.S = $\frac{1}{\sqrt{0.013}} - \log_{10} 0.013 = 8.77 - (-1.886) = 8.77 + 1.886 = 10.656$

Which is approximately equal to 10.583.

Hence the value of f* is equal to 0.013.

∴ Friction factor, f* = **0.013.** **Ans**

(ii) Maximum velocity (U_{max})

Now we know that f* = 4f

∴ Co-efficient of friction, $f = \frac{f*}{4} = \frac{0.013}{4} = 0.00325$

Now shear velocity (u*) in terms of co-efficient of friction and average velocity is given by equation (4.31A) as

$$u* = \bar{U}\sqrt{\frac{f}{2}} = 4.817\ x\sqrt{\frac{0.00325}{2}} = 4.817\ x\ 0.0403 = 0.194$$

For smooth pipe, the velocity at any point is given by equation (4.20)

$$u = u*\left[5.75\log_{10}\frac{(u*\ x\ y)}{v} + 5.55\right]$$

The velocity will be maximum at the centre of the pipe,

Where y = R = 0.05

i.e., radius of pipe. Hence the above equation becomes

$$u_{max} = u*\left[5.75\log_{10}\frac{(u*\times R)}{v} + 5.55\right]$$

$$= 0.194 \left[5.75 \log_{10} \frac{0.194 \ x \ 0.05}{0.0098 \ x \ 10^{-4}} + 5.55\right]$$

$$= 0.194[22.974 + 5.55] = \mathbf{5.528 \ m/s.} \qquad \textbf{Ans}$$

(iii) Shear stress at the boundary (τ_0)

We know that $u^* = \sqrt{\frac{\tau_0}{\rho}}$ or $u^{2*} = \frac{\tau_0}{\rho}$

∴ $\tau_0 = \rho u^{2*} = 1000 \ x \ 0.194^2 = \mathbf{37.63 \ N/m^2}$ **Ans**

Problem 4.13. Hydrodynamically smooth pipes carries water at the rate of 300l/s at 20℃ ($\rho = 1000 \ kg/m^3$) with a head loss of 3m in 100 m length of pipe. Determine the pipe diameter. Use $f = 0.0032 + \frac{.221}{(R_e)^{0.237}}$ equation for f, where $h_f = \frac{f.L.V^2}{D \ x \ 2g}$ and $R_e = \frac{\rho VD}{\mu}$.

Solution

Given:

Discharge, $Q = 300\frac{l}{s} = 0.3 m/s$

Density, $\rho = 1000 \ kg/m^3$

Kinematic viscosity, $v = 10^{-6} \ m^2/s$

Head loss, $h_f = 3 \ m$

Length of pipe, $L = 100m$

Value of friction factor, $f = 0.0032 + \frac{.221}{(R_e)^{0.237}}$

Reynolds number, $R_e = \frac{\rho VD}{\mu} = \frac{V \ x \ D}{v}$ $\left(\because \frac{\mu}{\rho} = v\right)$

$$= \frac{V \ x \ D}{10^{-6}} = V \ x \ D \ x \ 10^6$$

Find: diameter of pipe,

Let D = diameter of pipe

Head loss in terms of friction factor is given as

$$h_f = \frac{f \ x \ L \ x \ V^2}{D \ x \ 2g}$$

Or $3 = \frac{f \times 100 \times V^2}{D \times 2 \times 9.81}$ $\{\because h_f = 3, L = 100m\}$

Or $f = \frac{3 \ x \ D \ x \ 2 * 9.81}{100 \ x \ V^2}$ or $f = \frac{0.5886D}{V^2}$ …(i)

Now $Q = A \ x \ V$

Or $0.3 = \frac{\pi}{4} D^2 \ x \ V$ or $D^2 \ x \ V = \frac{4 \ x \ 0.3}{\pi} = 0.382$

∴ $V = \frac{0.382}{D^2}$ …(ii)

Also $$f = 0.0032 + \frac{.221}{(R_e)^{0.237}}$$

Or $$\frac{0.5886D}{V^2} = 0.0032 + \frac{.221}{(V\ X\ D\ X\ 10^6)^{0.237}}$$

$$\left(\because From\ equation(i), f = \frac{0.5886D}{V^2}\ and R_e = V\ x\ D\ x\ 10^6\ \right)$$

Or $$\frac{0.5886D}{\left(\frac{0.382}{D^2}\right)^2} = 0.0032 + \frac{.221}{\left(\frac{0.382}{D^2}\ X\ D\ X\ 10^6\right)^{0.237}}$$

$$\left(\because From\ equation\ (ii), V = \frac{0.382}{D^2}\right)$$

Or $$\frac{0.5886D^5}{0.382^2} = 0.0032 + \frac{.221}{\frac{(0.382\ \ X\ 10^6)^{0.237}}{D^{0.237}}}$$

Or $$4.033D^5 = 0.0032 + 0.0105 *\ D^{0.237}$$

Or $$4.033D^5 - +0.0105\ x\ D^{0.237} - 0.0032 = 0$$

The above equation (iii) will be solved by hit and trial method.

(i) Assume D =1 m, then L.H.S of equation (iii), becomes

L.H.S. = $4.033 \times 1^5 - +0.0105 \times 1^{0.237} - 0.0032$

$$= 4.033 - 0{,}0105 - 0.0032 = 4.0193$$

By increasing the value of D more than 1m, the L.H.S. will go on increasing. Hence decreases the value of D.

(ii) Assume D =0.3m, then L.H.S of equation (iii), becomes

L.H.S. = $4.033\ x\ 0.3^5 - +0.0105\ x\ 0.3^{0.237} - 0.0032$

$$= 0.0098 - 0.00789 - 0.0032 = -0.00129$$

As the value is negative, the value of D will be slightly more than 0.3.

(iii) Assume D =0.306 m, then L.H.S of equation (iii), becomes

L.H.S. = $4.033 \times 0.306^5 - +0.0105 \times 0.306^{0.237} - 0.0032$

$$= 0.0108 - 0.00793 - 0.0032 = -0.00033$$

This value of L.H.S. is approximately equal to zero. Actually the value of D will be slightly more than 0.306 m, say **0.308m. Ans.**

Problem 4.14. Water is flowing through a rough pipe of diameter 600 mm at the rate of 600 litres/second. The wall roughness is 3mm. find the power lost for 1 km length of pipe.

Solution

Given:

Diameter. of pipe, D = 600mm = 0.6m

∴ Radius, $R = \frac{0.06}{2} = 0.3\ m$

Discharge, $Q = 600\ litres/s = 0.6\, m^3/s$

Wall roughness, $k = 3\ mm = 3\ x\ 10^{-3} m = 0.003m$

Length of pipe, $L = 1km = 1000m$

For rough pipes, the co-efficient of friction in terms of wall roughness, k is given equation (4.36)

As

$$\frac{1}{\sqrt{4f}} = 2\log_{10}(R/k) + 1.74 = 2\log_{10}\left(\frac{0.3}{0.003}\right) + 1.74 = 5.74$$

Or $\sqrt{4f} = \frac{1}{5.74} = 0.1742$ or $4f = (0.1742)^2 = 0.03035$

The head loss due to friction is given by $h_f = \frac{4f\ x\ L\ x\ V^2}{D\ x\ 2g}$

Where $V = \frac{Q}{A} = \frac{0.6}{\frac{\pi}{4}(0.6)^2} = 2.122m/s$

$$= \frac{0.03035\ x\ 1000\ x\ 2.122^2}{0.6\ x\ 2\ x\ 9.81} = 11.6\ m$$

The power lost is given by, $P = \frac{\rho g\ x\ Q\ x\ h_f}{1000} = \frac{1000\ x\ 9{,}81\ x\ 0.6\ x\ 11.6}{1000} kW = \mathbf{68.27kW}.$ **Ans.**

4.8 DESIGN OF PIPES

Three categories of problems can be identified for turbulent flow in a pipe of length L.

Category	Known	Unknown
1	Q, D, e, v	h_L
2	D, e, v,h_L	Q
3	Q, e, v, h_L	D

A category 1 problem is straightforward and requires no iteration procedure when using the Moody diagram. Category 2 and 3 problems are more like problems encountered in engineering design situations and require an iterative trial-and-error process when using the Moody diagram. Each of these types will be illustrated with an example.

An alternate to using the Moody diagram is made possible by empirically derived formulas. The best of such formulas were presented by Swamee and Jain; an explicit expression that provides an approximate value for the unknown in each category above is as follows:

$$h_L = 1.07\frac{Q^2 L}{gD^5}\{\ln[\frac{e}{3.7D} + 4.62\left(\frac{vD}{Q}\right)^{0.9}]\}^{-2} \qquad 10^{-6} < e/D < 10^{-2},\quad 3000 < \mathrm{Re} < 3\times10^8 \qquad \text{(a)}$$

$$Q = -0.965\left(\frac{gD^5 h_L}{L}\right)^{0.5} \ln\left[\frac{e}{3.7D} + \left(\frac{3.17v^2 L}{gD^3 h_L}\right)^{0.5}\right] \qquad \mathrm{Re} > 2000 \qquad \text{(b)}$$

$$D = 0.6\left[e^{1.25}\left(\frac{LQ^2}{gh_L}\right)^{4.75} + vQ^{9.4}\left(\frac{L}{gh_L}\right)^{5.2}\right]^{0.04} \qquad \begin{matrix} 10^{-6} < e/D < 10^{-2} \\ 5000 < \text{Re} < 3\times10^8 \end{matrix} \qquad (c)$$

Equation (b) is as accurate as the Moody diagram, and equations a and c accurate to within approximately 2% of the Moody diagram. These tolerances are acceptable for engineering calculations. It is important to realize that the Moody diagram is based on experimental data that likely is accurate to within no more than 5%. Hence the foregoing three formulae of Swamee and Jain, which can easily be input on programmable hand-held calculator, are often used by design engineers.

Problem 4.15 Water at 20^0C is transported for 500m in a 4cm diameter wrought iron horizontal pipe with a flow rate of $0.003\frac{m^3}{s}$. Calculate the pressure drop over the 500m length of pipe.

Solution.

The average velocity is V = Q/A=$\frac{0.03}{\pi\times0.02^2} = 2.39m/s$

Reynolds number is Re =VD/v = $\frac{2.39\times0.04}{10^{-6}} = 9.6\times10^4$

Obtaining e, using D=40mm,

$$\frac{e}{D} = \frac{0.046}{40} = 0.00115$$

The friction factor is read from the Moody diagram to be ***f = 0.023***

The head loss is calculated as $h_L = \frac{fLV^2}{2gD} = \frac{0.023\times500\times2.39^2}{0.04\times2\times9.81} = 84m$

This answer is given to two significant numbers since the friction factor is known to at most two significant numbers. The pressure drop as found to be

$$\Delta p = \gamma h_L = 9800\times84 = 820000Pa$$

Using eqn(a) we find:

$$h_L = 1.07\frac{0.003^2\times500}{9.81\times0.04^5}\left\{\ln\left[\frac{0.00115}{3.7} + 4.62\left(\frac{10^{-6}\times0.04}{0.003}\right)^{0.9}\right]\right\}^{-2} = 1.07\times4480\times0.01731 = 83m$$

This value is within 1.2% of the value using the Moody diagram.

Problem 4.16 A pressure drop of 700kPa is measured over a 300m length of horizontal, 10cm diameter wrought iron pipe that transports oil (S= 0.9, $v = 10^{-5}\,m^2/s$). Calculate the flow rate.

Solution

The relative roughness is e/D = 0.046/100 = 0.00046

Assuming the flow rate is completely turbulent (Re is not needed), the Moody diagram gives *f=0.0165*

The head loss is found to be $h_L = \frac{\Delta p}{\gamma} = \frac{700000}{9800\times0.9} = 79.4m$

Velocity:

$$h_L = \frac{fLV^2}{2gD}, V = \sqrt{\frac{2gDh_L}{fL}} = \sqrt{\frac{2\times9.8\times0.1\times79.4}{0.0165\times300}} = 5.61m/s$$

$$\text{Re} = \frac{VD}{v} = \frac{5.61\times0.1}{10^{-5}} = 5.61\times10^4$$

Using this Reynolds number and e/D = 0.00046, the Moody diagram gives the friction factor as *f=0.023*

This corrects the original value is *f*. The velocity is recalculated to be

$$V = (\frac{2 \times 9.8 \times 0.1 \times 79.4}{0.023 \times 300})^{0.5} = 4.75m/s$$

The Reynolds number is then, Re = $\frac{4.75 \times 0.1}{10^{-5}} = 4.75 \times 10^4$

From the Moody diagram *f = 0.023* appears to be satisfactory. Thus the flow rate is Q=VA=$4.75 \times \pi \times 0.05 \times 0.05 = 0.037m^3/s$

Only two significant number are given since *f* is known to at most two significant numbers.

Using the explicit relationship (b), we can directly calculate Q to be:

$$Q = -0.965\left(\frac{9.8 \times 0.1^2 \times 79.4}{300}\right)^{0.5} ln\left[\frac{0.00046}{3.7} + \left(\frac{300 \times 3.17 \times 10^{-10}}{9.81 \times 0.1^3 \times 79.4}\right)^{0.5}\right]$$

$$= 0.038m^3/s$$

This value is within 2.5% of the value using the Moody diagram

Problem 4.17 Drawn tubing of what diameter should be selected to transport $0.002m^3/s$ of 20^0C water over a 400m length so that the head loss does not exceed 30m?

Solution

In this problem we do not know D, thus a trial-and- error solution is anticipated. The average velocity is related to D by

V=Q/A, = $\frac{0.002}{\pi D^2/4} = \frac{0.00255}{D^2}$

The friction factor and D are related as follows:

$$h_L = f\frac{LV^2}{2gD}$$

$$30 = f\frac{400\left(\frac{0.00255}{D^2}\right)^2}{D \times 2 \times 9.8}$$

$$D^5 = 4.42 \times 10^{-6} f$$

The Reynolds number is $\mathrm{Re} = \frac{VD}{\nu} = \frac{0.00255D}{D^2 \times 10^{-6}} = 2550/D$

Now, let us simply guess a value for *f* and check with the relations above the Moody diagram. The first guess of *f =0.03* and the correction is listed in Table 5.

Table 5 guessed values of *f*.

F	D(m)	Re	e/D	*F*
0.03	0.0421	6.06×10^4	0.000036	0.02
0.02	0.0388	6.57×10^4	0.000039	0.02

The value of *f=0.02* is acceptable, yielding a diameter of 3.88cm. Since this diameter would undoubtedly not be standard, a diameter of D = 4cm would be the tube size selected. This tube would have a head loss less than the limit $h_L = 30m$ imposed in the problem statement. Any larger-diameter tube would also satisfy this criterion but would be more costly, so should not be selected.

Using the explicit relationship (c), we can directly calculate D to be

$$D = 0.66\left[\left(1.5\times10^{-6}\right)^{1.25}\left(\frac{400\times0.002^2}{9.8\times30}\right)^{4.75} + 10^{-6}\times0.002^{9.4}\left(\frac{400}{9.8\times30}\right)^{5.2}\right]^{0.04}$$

$$= 0.66\left[5.163\times10^{-33} + 2.102\times10^{-31}\right]^{0.04} = 0.039m$$

Hence D =cm would be the tube size selected. This is the same tube size as that selected using the Moody diagram

Problem 4.18 Air at standard conditions is to be transported through m of a smooth horizontal 30cm x 20cm rectangular duct at a flow rate of $0.24m^3/s$. Calculate the pressure drop.

Solution*:*

$$R = \frac{A}{P} = \frac{0.3\times0.2}{2(0.3+0.2)} = 0.06m$$

The average velocity is $V = \dfrac{Q}{A} = \dfrac{0.24}{0.3\times0.2} = 4m/s$

This gives a Reynolds number of

$$\mathrm{Re} = \frac{4VR}{v} = \frac{4\times4\times0.06}{1.6\times10^{-5}} = 6\times10^4$$

Using the smooth pipe curve of the Moody diagram, we have *f*=0.0198. Hence,

$$h_L = f\frac{LV^2}{2gR} = 0.0198\frac{500}{4\times0.06}\frac{4^2}{2\times9.8} = 33.7m$$

The pressure drop is $\Delta p = \rho g\, h_L = 1.23\times9.8\times33.7 = 406$Pa

Problem 4.19 In a rotary viscometer, the radii of the cylinders are respectively 50mm and 50.5mm, and the outer cylinder is rotated steadily at 30rad/s. for a certain liquid, the torque is 0.45Nm when the depth of the liquid is 50mm and 0.81Nm when the depth is 100mm. Find the viscosity of the liquid.

Solution

Given:

a=0.0505m, b=0.05m , ω =30rad/s

a) 50mm, T=0.45Nm
b) 100mm, T=0.81Nm

dT=0.81 - 0.45=0.36Nm

$$\mu = \frac{T(a^2-b^2)}{4\pi h a^2 b^2\omega} = \frac{0.36(0.0505^2-0.05^2)}{4\pi\times0.05\times0.0505^2\times0.05^2\times30}$$

$$= 0.1505Ns/m^2$$

Problem 4.20 A steel sphere 15mm diameter and of mass 13.7mg falls steadily in oil through a vertical distance of 500mm in56s. The oil has a density of $950kg/m^3$ and is contained in a drum so large that any wall effects are negligible. Find the viscosity of the oil.

Solution

Given:

$D = 1.5mm = 0.0015m, m = 13.7mg = 1.37\times10^{-6}kg, L = 500mm = 0.5m, T = 56s, \rho = 950kg/m^3$

V=0.5/56m/s

$$volume = \frac{4\pi r^3}{3} = \frac{4\pi \times (0.0015*0.5)^3}{3} = 1.767146\times 10^{-9} m^3$$

$$\rho_s = \frac{1.37\times 10^{-6}}{1.767146\times 10^{-9}} = 7752.6 kg/m^3$$

$$\mu = \frac{d^2 g(\rho_s - \rho)}{18u} = \frac{0.0015^2 \times 9.81\times(7752.6 - 950)}{18\times 0.5/56} = 0.934 Ns/m^2$$

Problem 4.21 A straight smooth pipe 100mm diameter and 60m long is inclined at 10^0 to the horizontal. A liquid of relative density 0.9 and kinematic viscosity $120mm^2/s$ is to be pumped through it into a reservoir at the upper with a gauge pressure of 120kPa. The pipe friction coefficient is given by 16/Re for laminar flow and by $\frac{0.0791}{Re^{1/4}}$ for turbulent

a) Find the maximum pressure at the lower inlet end of the pipe if the mean shear stress at the pipe is not to exceed 200GPa.
b) The corresponding flow rate

Solution

Given:

$$d = 0.1m, P_2 = 120\times 10^3 Pa, \rho = 900 kgm^{-3}, \upsilon = 120 mm^2/s = 120\times 10^{-6} m^2/s, P_1 = ???, L = 60m, \tau = 200 N/m^2$$

$$\tau = \frac{dp}{dx} * \frac{r}{2}$$

$$dp = \frac{200\times 60\times 2}{0.05} = 480\times 10^3 Pa$$

$$h = 60\sin 10 = 10.42m$$

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f, V_1 = V_2$$

$$P_1 = P_2 + (z_2 - z_1)\rho g + dp_f$$

$$P_1 = 120\times 10^3 + 10.42\times 10^3 \times 0.9\times 9.81 + 480\times 10^3 = 692 kPa$$

$$\Delta P = 480 kPa = \rho g h_f$$

$$h_f = \frac{\Delta P}{\rho g} = \frac{480\times 10^3}{900\times 9.81} = 54.366 m = \frac{32\bar{u}\mu L}{\rho g D^2}$$

$$\bar{u} = \frac{\Delta P * D^2}{32\mu L} = \frac{480\times 10^3 \times 0.1^2}{32\times 900\times 120\times 10^{-6}\times 60} = 23.148 ms^{-1}$$

$$h_f = \frac{4fLV^2}{2gD}$$

$$\mathrm{Re} = \frac{VD}{\upsilon} = \frac{23.15\times 0.1}{120\times 10^{-6}} = 19291.7$$

The flow is turbulent

$$h_f = \frac{4LV^2}{2gD} * \frac{0.0791}{\mathrm{Re}^{1/4}} = \frac{4LV^2}{2gD} * 0.0791 * \left(\frac{\upsilon}{VD}\right)^{1/4}$$

$$V = \left(\frac{h_f * 2gD^{5/4}}{4 \times 0.0791 \times L \times \upsilon^{1/4}}\right)^{4/7} = \left(\frac{54.366 \times 2 \times 9.81 \times 0.1^{5/4}}{4 \times 0.0791 \times (120 \times 10^{-6})^{1/4} \times 60}\right)^{4/7} = 6.9636 ms^{-1}$$

$$Q = AV = \pi \times 0.25 \times 0.1^2 \times 6.9636 = 0.05469 m^3 s^{-1}$$

4.9 HIGHLIGHTS

1. If the Reynolds number is less than 2000 in a pipe, the flow is laminar while if the Reynolds number is more than 4000, the flow is turbulent in pipes.
2. Loss of pressure head in a laminar flow is proportional to the mean velocity of flow, while in case of turbulent flow, it is approximately proportional to the square of velocity.
3. Expression for head loss due to friction in pipes is given by Darcy-Weisbach equation,

$$h_f = \frac{4\ x\ f\ x\ L\ x\ V^2}{d\ x\ 2g} \qquad \text{Where } f = \text{co-efficient of friction}$$

$$= \frac{f\ x\ L\ x\ V^2}{D\ x\ 2g} \qquad \text{where } f = \text{friction factor}$$

4. Co-efficient of friction is expressed in terms of shear stress as $= \frac{2\tau_0}{\rho V^2}$

   Where V= mean velocity of flow, ρ = mass density of fluid.
5. Shear stress in turbulent flow is sum of shear stress due to viscosity and shear due to turbulence, i.e.,

   $\tau = \tau_v + \tau_t$, where τ_v = shear stress due to viscosity, τ_t= shear stress due to turbulence
6. Turbulent shear stress by Reynolds is given as $\tau = \rho u'v'$

   Where $u'\ and\ v'$ = fluctuating component of velocity.
7. The expression for shear stress in turbulent flow due to Prandtl is $\bar{\tau} = \rho l^2 \left(\frac{du}{dy}\right)^2$, where l = mixing length.
8. The velocity distribution in the turbulent flow for pipes is given by expression

$$u = u_{max} + 2.5U_* \log_e(y/R)$$

   Where u_{max} = is the center-line velocity,

   Y = distance from pipe wall

   R =radius of the pipe,

   And U_*= shear velocity which is equal to $\sqrt{\frac{\tau_0}{\rho V^2}}$.
9. Velocity defect is the difference between the maximum velocity (u_{max}) and local velocity (u) at any point and is given by $(u_{max} - u) = 5.75\ x\ u_* \log_{10}(y/R)$
10. The boundary is known as hydrodynamically smooth if k, the average height of the irregularities projecting from the surface of the boundary is small compared to the thickness of the laminar sub-layer(δ') and boundary is rough if k is large in comparison with the thickness of the sub-layer.

or if $\frac{k}{\delta'} < 0.25$, the boundary is smooth; if $\frac{k}{\delta'} > 6.0$, the boundary is rough and if $\frac{k}{\delta'}$ lies between 0.25 to 6.0, the boundary is in transition.

11. Velocity distribution for turbulent flow is

$\frac{u}{u_*} = 5.75 \log_{10} \frac{(u_* R)}{v} + 5.55$ for smooth pipes

$= 5.75 \log_{10}(y/k) + 8.5$ for rough pipes

Where u = velocity at any point in the turbulent flow,

u_* = shear velocity and = $\sqrt{\frac{\tau_0}{\rho V^2}}$, $v =$ kinematic viscosity of fluid,

$y =$ distance from pipe wall, and $k =$ roughness factor.

12. Velocity distribution in terms of average velocity is

$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{(u_* R)}{v} + 1.75$ for smooth pipes,

$= 5.75 \log_{10} R/k + 4.75$ for rough pipes.

13. Difference of local velocity and average velocity for smooth and rough pipes is

$\frac{\bar{U}}{u_*} = 5.75 \log_{10}(y/R) + 3.75.$

$*$ Power = $\rho g \ x \ Q \ x \ h_f$ Watt =$\frac{\rho g \ x \ Q \ x \ h_f}{1000}$KW

4.9.2 SOLVED EXAMPLES

4.10 QUESTIONS

4.10.2 THEORETICAL PROBLEMS

1. What do you understand by turbulent flow? What factor decides the type of flow in pipes?
2. Derive an expression for the loss of head due to friction in pipes.
3. Explain the term co-efficient of friction. On what factors does this co-efficient depend?
4. Obtain an expression for the co-efficient of friction in the terms of shear stress.
5. What do you mean by Prandtl mixing Length Theory? Find an expression for shear stress due to Prandtl
6. Derive an expression for Prandtl's universal velocity distribution for turbulent flow in pipes. Why this velocity distribution is called universal?
7. What is a velocity defect? Derive an expression for velocity defect in pipes.
8. How would you distinguish between hydrodynamically smooth and rough boundaries?
9. Obtain expression for the velocity distribution for turbulent flow in smooth pipes.
10. Show that velocity distribution for turbulent flow through rough pipe is given by $\frac{u}{u_*} = 5.75log_{10}(y/k) + 8.5$

Where u_* = shear velocity, y - distance form pipe wall, k = roughness factor.

11. Obtain an expression for velocity distribution in terms of average velocity for *(a)* smooth pipes and *(b)* rough pipes.
12. Prove that the difference of local velocity and average velocity for turbulent flow through rough or smooth pipes

is given by

$$\frac{u - \overline{U}}{u_*} = 5.75log_{10}(y/R) + 3.75$$

(B) NUMERICAL PROBLEMS

1. A pipe-line carrying water has average height of irregularities projecting from the surface of the boundary of the pipe as 0.20 mm. What type of the boundary is it? The shear stress development is 7.848 N/m. Take value of kinematic viscosity for water as 0.01 stokes. [Ans. Boundary is in transition]
2. Determine the average height of the roughness for a rough pipe of diameter 10.0 cm when the velocity at a point 4 cm from wall is 40% more than the velocity at a point 1 cm from pip wall. [Ans. 0.94 cm]
3. A smooth pipe of diameter 10 cm and 1000 m long carries water at the rate of 0.70 m^2/minute. Calculate the loss of head, wall shearing stress, centre line velocity, velocity and shear stress at 3 cm from pipe wall. Also calculate the thickness of the laminar sub-layer. Take kinematic viscosity of water as 0.015 stokes and value of co-efficient of friction f as

$$f = \frac{0.0791}{(R_e)^{1/4}}, where\ R_e = Reynold\ number$$

4. The velocities of water through a pipe of diameter 10 cm, are 4 m/s and 3.5 m/s at the centre of the pipe and 2 cm from the pipe centre respectively. Determine the wall shearing stress in the pipe for turbulent flow.

[Ans. 15.66 kgf/m2]

5. For turbulent flow in a pipe of diameter 200 mm, find the discharge when the centre-line velocity is 30 m/s and velocity at a point 80 mm from the centre as measured by pilot-tube is 2.0 m/s. [Ans. 64.9 litres/s]
6. For problem 5 find the co-efficient of friction and the average height of roughness projections.

[Ans. 0.029,25.2 mm]

7. Water is flowing through a rough pipe of diameter 40 cm and length 3000 m at the rate of 0.4 m /s. Find the power required to maintain this flow. Take the average height of roughness as K = 0.3 mm. [Ans. 278.5 kN]
8. A smooth pipe of diameter 300 mm and length 600 m carries water at rate of 0.04 m^2 /s. Determine the head lost due to friction, wall shear stress, centre-line velocity and thickness of laminar sub-layer. Take the kinematic viscosity of water as 0.018 stokes. [Ans. 0.588 m, 0.72 N/cm, 0.665 m/s, 0.779 mm]

f\

9. A rough pipe of diameter 300 mm and length 800 m carries water at the rate of 0.4 m /s. The wall roughness is 0.015 mm. Determine the co-efficient of friction, wall shear stress, centre line velocity and velocity at a distance of 100 mm from the pipe wall.
   [Ans. f = .00263, τ_0 = 42.08 N/cm^2, u_{max} = 6.457 m/s, u = 6.249 m/s]
10. Determine the distance from the centre of the pipe, at which the local velocity is equal to the average velocity for turbulent flow in pipes. [Ans. 0.7772 R]

5. APPENDIX

Terminology	Dimensions	Imperial Units (USCS)	SI-units
Acceleration due to gravity	L / T^2	ft/s^2	m/s^2
Area	L^2	ft^2	m^2
Chezy roughness coefficient	$L^{1/2} / T$	$ft^{1/2}/s$	$m^{1/2}/s$
Critical Depth	L	Ft	M
Density	$F\ T^2 / L^4$	$lb\ s^2/ ft^4$	$N\ s^2/m^4$
Depth	L	Ft	M
Depth in open channel	L	Ft	M
Diameter	L	Ft	M
Distance from solid boundary	L	Ft	M
Flow rate	L^3 / T	ft^3/s	m^3/s
Force	F	Lb	N
Force due to pressure	F	Lb	N
Hazen Williams roughness coefficient	$L^{0.37} / T$	$ft^{0.37}/s$	$m^{0.37}/s$
Head loss due to friction	L	Ft	m
Head of height	L	Ft	m
Head of weir	L	Ft	m
Height above datum	L	Ft	m
Hydraulic radius	L	Ft	m

Terminology	**Dimensions**	**Imperial Units (USCS)**	**SI-units**
Kinematic viscosity	L^2 / T	ft^2/s	m^2/s
Length	L	Ft	m
Manning's roughness coefficient	$T / L^{1/3}$	$s/ft^{1/3}$	$s/m^{1/3}$
Modulus of elasticity	F / L^2	lb/in^2 (psi)	Pa
Perimeter, Weir Height	L	Ft	m
Pressure	F / L^2	lb/ft^2	Pa
Radius	L	Ft	m
Shear stress	F / L^2	lb/ft^2	Pa
Size of roughness	L	Ft	m
Specific weight	F / L^3	lb/ft^3	kg/m^3
Surface tension	F / L	lb/ft	kg/m
Time	T	S	s
Thickness	L	Ft	m
Time	T	S	s
Total head	L	Ft	m
Unit flow rate	$L^3 / T\ L$	$ft^3/(s\ ft)$	$m^3/(s\ ft)$
Velocity	L / T	ft/s	m/s
Viscosity	$F\ T / L^2$	$lb\ s/ft$	$Pa\ s$
Weight	F	Lbf	N

6. INDEX