FLUID TRANSPORT

Lecture notes prepared by
Zsuzsanna Momade

KWAME NKRUMAH UNIVERSITY OF
SCIENCE AND TECHNOLOGY
DEPARTMENT OF CHEMICAL ENGINEERING

TABLE OF CONTENTS

REFERENCES

C.J. Geankoplis (2003): Transport Processes and Unit Operations

C.O. Bennett, J.E. Myers (1982): Momentum, heat, and mass transfer

W.L. McCabe, J.C. Smith, P. Harriott (1985): Unit Operations of Chemical Engineering

R.B. Bird, W.E. Stewart, E.N. Lightfoot (1960): Transport Phenomena

A.S. Foust, *at al*. (1980): Principles of Unit Operations

M.R. Spiegel (1974): Theory and Problems of Vector Analysis

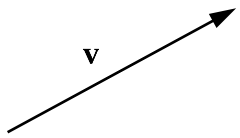
N. de Nevers (2005): Fluid Mechanics for Chemical Engineers

1 MATHEMATICAL REVIEW

1.1 Summary of vector and tensor notation

Physical quantities can be:
1) Scalars (zero-order tensors): temperature, energy, volume, time, ...
2) Vectors (first-order tensors): velocity, momentum, force, acceleration, ...
3) Tensors (second-order tensors): shear stress or momentum flux.

Vector (definition): it is a quantity of a given magnitude and direction.
In a 3-dimensional space it may be resolved into 3 components, thus 3 scalars determine any vector quantity.



$$\bar{v} = \bar{v}_x + \bar{v}_y + \bar{v}_z$$

where $\bar{v}_x, \bar{v}_y, \bar{v}_z$ are components or projections of $\bar{v}$

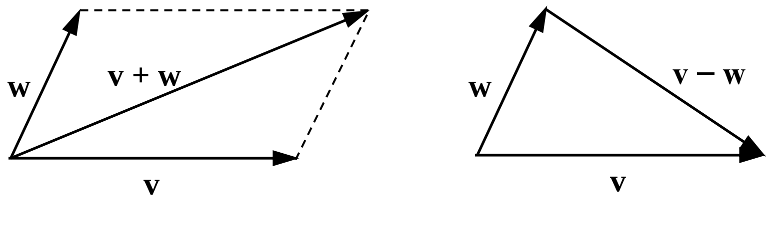
Unit vectors: **i**, **j**, **k** or **δ1, δ2, δ3**.

$\bar{v}_x = v_x i, \quad \bar{v}_y = v_y j,$ and $\bar{v}_z = v_z k$ where v_x, v_y, v_z are the magnitude of $\bar{v}$ vector's projections.

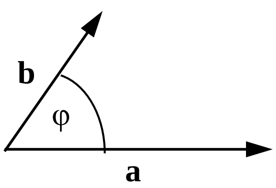
So a vector can be written as $\bar{v} = v_x \bar{i} + v_y \bar{j} + v_z \bar{k} = \sum_i \bar{\delta}_i v_i$

The **magnitude** of a vector |**v**| = v = $\sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\sum_i v_i^2}$

Addition and subtraction of 2 vectors



Scalar or dot product of 2 vectors **a ≅ b**



a ≅ b = a b cosφ $\qquad 0 \leq \varphi \leq \pi$

Example: If work (W) is done by force **F** when it acts on a particle and moves the particle through a distance **d**, then **W = F · d =** F d cosφ.

If $\quad \mathbf{a} \perp \mathbf{b} \;\Rightarrow\; \varphi = 90° \;\Rightarrow\; \cos\varphi = 0 \;\Rightarrow\; \mathbf{a}\cdot\mathbf{b} = 0$

$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a^2$

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$

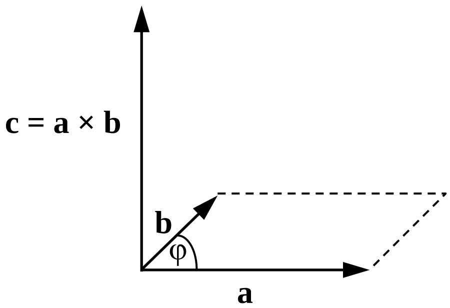
If $\left.\begin{aligned}\bar{a} &= a_x i + a_y j + a_z k\\ \bar{b} &= b_x i + b_y j + b_z k\end{aligned}\right\}\qquad \boxed{\bar{a}\cdot\bar{b} = a_x b_x + a_y b_y + a_z b_z}$

$$\bar{a}\cdot\bar{b} = \sum_i a_i b_i$$

Properties of scalar products:

commutative	$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
not associative	$(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \neq \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$
distributive	$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{c})$

Vector or cross product of 2 vectors $\mathbf{a}\times\mathbf{b}$



$\mathbf{c} = \mathbf{a}\times\mathbf{b} = (a\, b \sin\varphi)\, \mathbf{n}_{ab}$

where $\mathbf{n}_{ab}$ =the unit vector normal to the plane containing **a** and **b**.

$|\mathbf{c}| = |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\varphi$

The length of $\mathbf{a}\times\mathbf{b}$ equals the area of the parallelogram. The direction of $\mathbf{a}\times\mathbf{b}$ is normal to the plane containing **a** and **b** such that **a**, **b**, and **c** form a right-handed triad.

Example: In case of a particle in space with angular velocity $\boldsymbol{\omega}$, $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = \omega r \sin\varphi$.

If $\quad \mathbf{a} \parallel \mathbf{b} \quad \varphi = 0 \;\Rightarrow\; \mathbf{a}\times\mathbf{b} = 0$

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$

If $\quad \mathbf{a} \perp \mathbf{b} \;\Rightarrow\; \varphi = 90° \;\Rightarrow\; \sin\varphi = 1 \;\Rightarrow\; |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|$

$\mathbf{i}\times\mathbf{j} = \mathbf{k}$		$\mathbf{j}\times\mathbf{i} = -\mathbf{k}$
$\mathbf{j}\times\mathbf{k} = \mathbf{i}$	and	$\mathbf{k}\times\mathbf{j} = -\mathbf{i}$
$\mathbf{k}\times\mathbf{i} = \mathbf{j}$		$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$

Properties of vector products:

not commutative	$\mathbf{a}\times\mathbf{b} = -[\mathbf{b}\times\mathbf{a}]$
not associative	$\mathbf{a}\times[\mathbf{b}\times\mathbf{c}] \neq [\mathbf{a}\times\mathbf{b}]\times\mathbf{c}$
distributive	$[\mathbf{a}+\mathbf{b}]\times\mathbf{c} = [\mathbf{a}\times\mathbf{c}] + [\mathbf{b}\times\mathbf{c}]$

The cross product may be evaluated from the following determinant:

$$\bar{c} = \bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \left(a_y b_z - a_z b_y\right)\bar{i} - \left(a_x b_z - a_z b_x\right)\bar{j} + \left(a_x b_y - a_y b_x\right)\bar{k}$$

1.2 Ordinary derivatives of vectors

If **v**(t) is a vector depending on a single scalar variable t (time), then

$$\boxed{\frac{d\bar{v}}{dt} = \frac{dv_x}{dt}\bar{i} + \frac{dv_y}{dt}\bar{j} + \frac{dv_z}{dt}\bar{k}} \qquad \text{a vector}$$

The time derivative of the velocity vector d**v**/dt represents acceleration.

1.3 The vector differential operations

The vector differential operator is Λ called "del" or "nabla".

It is defined in rectangular coordinates as:

$$\boxed{\nabla = \frac{\partial}{\partial x}\ \bar{i} + \frac{\partial}{\partial y}\ \bar{j} + \frac{\partial}{\partial z}\ \bar{k}} \qquad \text{or} \qquad \nabla = \bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}$$

The symbol Λ is a vector-operator; it has components like a vector: $\nabla = [\ \frac{\partial}{\partial x}\ ,\ \ \frac{\partial}{\partial y}\ ,\ \ \frac{\partial}{\partial z}\ ]$

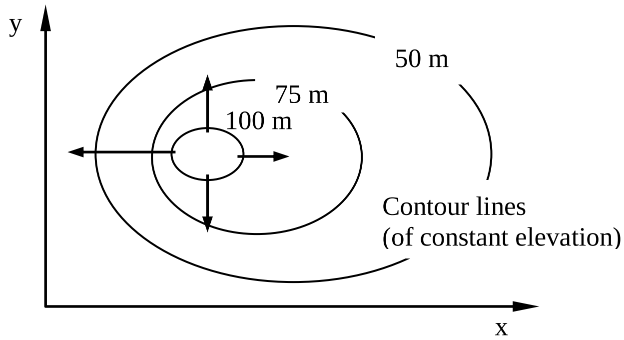
It cannot stand alone but must operate on a scalar, vector, or tensor function.
It has dimensions of 1/length and an SI unit of 1/m.

The gradient of a scalar field Λs or grad s

If s(**r**) is a scalar-valued function of the position, where **r** is the position vector, then the operation of Λ on s in rectangular coordinates is:

$$\boxed{\nabla s = \frac{\partial s}{\partial x}\ \bar{i} + \frac{\partial s}{\partial y}\ \bar{j} + \frac{\partial s}{\partial z}\ \bar{k}} \qquad \text{a vector}$$

The gradient vector has a simple geometrical interpretation. The components of Λs in the direction of **i**, **j**, and **k** give the rate of change of s in that direction. Λs is normal to the surface of constant s. This vector points "downhill" (i.e., perpendicular to contour lines of constant elevation) and has a magnitude equal to the "steepness" of the slope in the "downhill" direction.



On a 2-dimensional contour map, the contours represent lines of equal elevation (which is a scalar). There is a gradient line (a vector) through every point. Its direction is downhill, perpendicular to the contour lines, and its magnitude is proportional to the steepness of the gradient. Only 4 such vectors are shown, as representative examples.

Properties of the gradient operation:

not commutative	Λs ≠ sΛ
not associative	[Λm]s ≠ Λ(ms)
distributive	Λ (m+s) = Λm + Λs

The divergence of a vector field Λ · **v** or div **v**

The dot product of Λ with a vector is a scalar called the divergence. If **v**(**r**) is a vector-valued function of position, where **r** is the position vector, then the scalar product may be formed with the operator Λ:

$$\boxed{\nabla \cdot \bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}} \qquad \text{a scalar}$$

Properties of the divergence operation:

not commutative	Λ · **v** ≠ **v** · Λ
not associative	Λ · s**v** ≠ Λs · **v**
distributive	Λ · [**v** + **w**] = Λ · **v** + Λ · **w**

Λ·**v** is a measure of the rate of loss of fluid per unit volume. Hence, if a fluid is incompressible, then can be neither a gain nor loss: Λ·**v** = 0 . This is the statement of conservation of mass.

The divergence of the gradient of a scalar Λ · Λs = Λ^2s or div grad s

The divergence of the gradient of a scalar function s in rectangular coordinates:

$$\boxed{\nabla \cdot \nabla s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \nabla^2 s} \qquad \text{a scalar}$$

The result is defined as the Laplacian of s and is written as Λ^2s.

The Laplacian operator is the dot product of Λ with itself.:

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

The Laplacian operator has only the distributive property, just as the gradient and divergent.

The Laplacian of a vector field $\Lambda^2\mathbf{v} = \Lambda \cdot \Lambda\mathbf{v}$

In rectangular coordinates:

$$\nabla^2\bar{v} = \bar{i}\ \nabla^2\bar{v}_x + \bar{j}\ \nabla^2\bar{v}_y + \bar{k}\ \nabla^2\bar{v}_z$$

$$\nabla^2\bar{v} = \bar{i}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \bar{j}\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \bar{k}\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

The gradient of a vector field $\Lambda\mathbf{v}$ or grad $\mathbf{v}$

If $\mathbf{v}(\mathbf{r})$ is a vector-valued function of position, then $\Lambda\mathbf{v}$ produces a tensor (a nine-member array of derivatives). A tensor is a generalization of a vector.

In rectangular coordinates:

$$\nabla\bar{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

This is seldom used by itself. Most often it is seen in the following dot product in fluid mechanics:

$$\bar{v}\cdot\nabla\bar{v} = v_x\frac{\partial\bar{v}}{\partial x} + v_y\frac{\partial\bar{v}}{\partial y} + v_z\frac{\partial\bar{v}}{\partial z} =$$

$$= \bar{i}\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) + \bar{j}\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) + \bar{k}\left(v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right)$$

1.4 Vector and tensor components in curvilinear coordinates

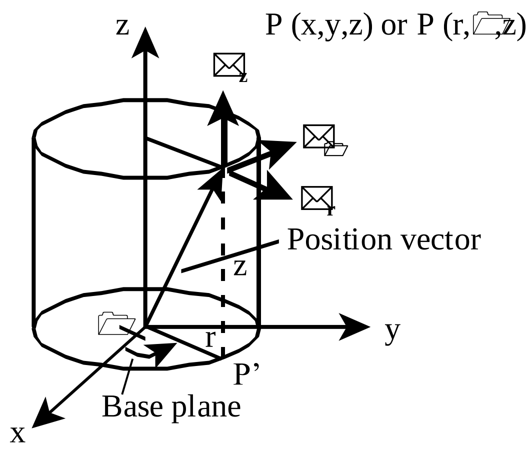
Thus far we have considered only rectangular coordinates x, y, and z. For working problems it is often better to use curvilinear coordinates:
– cylindrical coordinates, and
– spherical coordinates.

We have to know how to write various differential operations in curvilinear coordinates. We can do this simply, if we know 2 things:

(1) the expression for Λ in curvilinear coordinates,

(2) the spatial derivatives of the unit vectors in curvilinear coordinates.

Cylindrical coordinates



The position vector is from the coordinate system origin to the point P.

Θ direction is ζ to r direction.

The range of variables are:

0 # r # 4

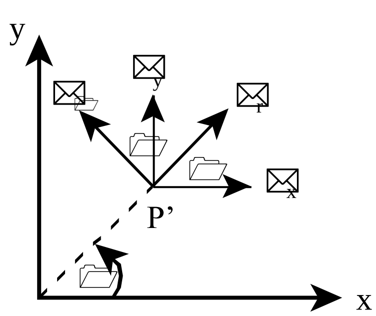
0 # Θ # 2π

!4 # z # 4

We locate the point P by giving the values of r, Θ, and z instead of x, y, and z. The cylindrical coordinates are related to the rectangular coordinates by:

$$\left.\begin{aligned} x &= r\cos\Theta \\ y &= r\sin\Theta \\ z &= z \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} r &= +\sqrt{x^2+y^2} \\ \Theta &= \arctan(y/x) \\ z &= z \end{aligned}\right.$$

Unit vectors



The unit vectors $\boldsymbol{\delta}_x$, $\boldsymbol{\delta}_y$, and $\boldsymbol{\delta}_z$ are independent of position, that is independent of x, y, and z. In cylindrical coordinates the unit vectors $\boldsymbol{\delta}_r$ and $\boldsymbol{\delta}_\Theta$ will depend on position.

$\boldsymbol{\delta}_r$ unit vector is a vector of unit length in the direction of increasing r.

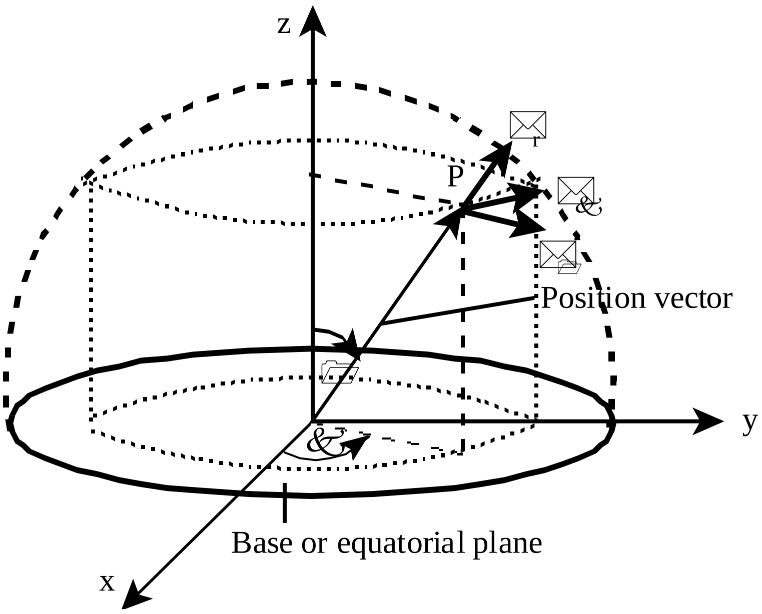
$\boldsymbol{\delta}_\Theta$ unit vector is a vector of unit length in the direction of increasing Θ.

As P’ the projection of point P moves around on the xy-plane, the direction of $\boldsymbol{\delta}_r$ and $\boldsymbol{\delta}_\Theta$ change.

Vectors and tensors can be decomposed into components with respect to cylindrical coordinates just as with respect to rectangular coordinates and the various dot and cross product operations performed as described earlier (but not the differential operations!).
e.g. $\mathbf{a} \cdot \mathbf{b} = a_r\, b_r + a_\Theta\, b_\Theta + a_z\, b_z$

To convert derivatives of scalars with respect to x, y, and z into derivatives with respect to r, Θ, and z we use the chain rule of partial differentiation.

Spherical coordinates



P(x,y,z) or P(r,Θ,κ) or P(**r**)

The ranges of variables are:
0 # r # 4
0 # Θ # 2π
0 # κ # 2π

The relation of spherical coordinates to rectangular coordinates:

$$\left.\begin{array}{l} x = r \sin\Theta \cos\kappa \\ y = r \sin\Theta \sin\kappa \\ z = r \cos\Theta \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l} r = +\sqrt{x^2 + y^2 + z^2} \\ \Theta = \arctan\left(\sqrt{x^2 + y^2}\,/\,z\right) \\ \kappa = \arctan\,(y/x) \end{array}\right.$$

1.5 Coordinate systems S Time derivatives

When a continuum is in motion, quantities (such as velocity, temperature, concentration) of a material particle change with time. We can describe these changes according to the observer in different kind of coordinate systems.

Fixed coordinate system — Partial time derivative $\dfrac{\partial}{\partial t}$

Suppose we stand on a bridge and note how the concentration of fish just below us changes with time. We are observing then how the concentration changes with time at a fixed position in space.

$$\left.\frac{\partial c}{\partial t}\right|_{x,y,z\ fixed}$$

Coordinate system following the motion — Substantial time derivative $\frac{D}{Dt}$

Suppose we float along in a boat counting fish. Now the velocity of the observer is the same as the velocity of the stream: **v**. When we report the change of fish concentration with respect to time, the numbers depend on the local stream velocity. The substantial time derivative is sometimes called the "derivative following the motion".

$$\boxed{\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z}}$$

This expression is valid only for rectangular coordinates.

With vector notation:

$$\boxed{\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \overline{v} \cdot \nabla c}$$

This expression is valid for all coordinates.

Free-moving coordinate system — Total time derivative $\frac{d}{dt}$

Suppose we get in a motor boat and speed around on the river (sometimes going upstream, sometimes across the current and sometimes downstream). When we report the change of fish concentration with respect to time, the numbers we report also reflect the motion of the boat.

$$\boxed{\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x}\frac{dx}{dt} + \frac{\partial c}{\partial y}\frac{dy}{dt} + \frac{\partial c}{\partial z}\frac{dz}{dt}}$$

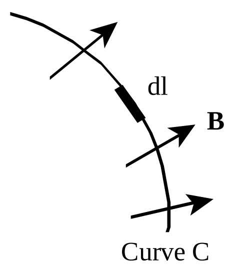
where $\frac{dx}{dt}, \frac{dy}{dt},$ and $\frac{dz}{dt}$ are components of the velocity vector of the boat (observer) $\mathbf{v}_{observer}$.

With vector notation: $\frac{dc}{dt} = \frac{\partial c}{\partial t} + \overline{v}_{observer} \cdot \nabla c$

If $\mathbf{v} = \mathbf{v}_{observer}$ then the substantial time derivative is the same as the total time derivative.

1.6 Integral theorems

Line integral



The line integral of a vector **B** over path C is defined as the integral of the scalar product of **B** and the differential distance **dl**

$$\int_C \overline{B} \cdot \overline{dl}$$

dl is a differential length along the contour C having length of dl and a + direction along the contour such that in passing around the contour the surface S is always on the left.

When $\int_C \overline{B} \cdot \overline{dl}$ is integrated over a closed loop, the line integral is denoted by $\oint_C \overline{B} \cdot \overline{dl}$ which is known as the circulation of vector **B**.

Let $\mathbf{B} = \boldsymbol{\delta}_x B_x + \boldsymbol{\delta}_y B_y + \boldsymbol{\delta}_z B_z$
and $\mathbf{dl} = \boldsymbol{\delta}_x dx + \boldsymbol{\delta}_y dy + \boldsymbol{\delta}_z dz$ then $\int_C \bar{B} \cdot \overline{dl} = \int_C (B_x dx + B_y dy + B_z dz)$

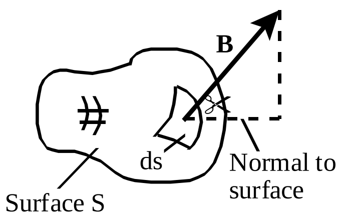
Example: If **B** vector is the force **F** on a particle moving along C, this line integral represents the work done by the force.

The value of a line integral may or may not depend on the path of integration. When a vector field can be expressed as the gradient of a scalar field, the line integral of the vector taken between two points is independent of the path followed and is equal to the difference between the values of the scalar at its ends.

$$\int_A^B \nabla s \cdot d\bar{l} = \int_A^B ds = s_B - s_A$$

If $\Lambda \times \mathbf{B} = 0$ then the line integral of **B** between 2 points is independent of the path.

Surface integral



The surface integral expresses the flow rate of a quantity through a surface and is defined as the integral of the normal component of vector **B** multiplied by the differential surface area element ds.

Example for **B** vector: ρ**v** mass flux

n is a unit vector normal to a given surface and is directed outward. To find a unit vector **n** normal to the surface S, we first write the equation of S in the form of f (x, y, z) = constant, and then use

$$\boxed{\bar{n} = \pm \frac{\text{grad} f}{|\text{grad} f|}}$$

The scalar product of **B** and **n** (**B** · **n**) is the normal component of vector **B**:

$$\boxed{\bar{B} \cdot \bar{n} = B(1) \cos \alpha_{Bn}}$$

Mathematically the surface integral is defined as:

$$\boxed{\iint_S \bar{B} \cdot \bar{n}\ ds \equiv \iint_S B \cos \alpha\ ds}$$

The surface integral is sometimes written as:

$$\iint_S \bar{B} \cdot \overline{dS}\ , \quad \text{where} \quad \overline{dS} = \bar{n}\ dS$$

The $\overline{dS}$ vector has a length of dS and its direction is perpendicular to the surface, therefore $\overline{dS} = \overline{n}\ dS$.

If the surface integral is over a closed contour, then

$$\oint_S \overline{B} \cdot \overline{n}\ ds = \oint_S \overline{B} \cdot \overline{dS}$$

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes.

1.7 Accuracy and precision

First we should make a clear distinction between accuracy and precision. **Accuracy** is a measure of how close a given value is to the “true” value, whereas **precision** is a measure of the uncertainty in the value or how “reproducible” the value is.

For example, if we were to measure the width of an A4 paper using a ruler, we might find that it is 21.5 cm, give or take 1 mm = 0.1 cm. The “give or take” (i.e., the uncertainty) value of 0.1 cm is the precision of the measurement, which is determined by how close we are able to reproduce the measurement with the ruler. However, it is possible that when the ruler is compared with a “standard” unit of measure it is found to be in error by, say, 2 mm = 0.2 cm. Thus, the “accuracy” of the ruler is limited, which contributes to the uncertainty of the measurement, although we may not know what this limitation is unless we can compare our “instrument” to one we know to be true.

Thus, the accuracy of a given value may be difficult to determine, but the precision of a measurement can be determined by the evaluation of reproducibility if multiple repetitions of the measurement are made. Unfortunately, when using values or data provided by others from handbooks, textbooks, journals, and so on, we do not usually have access to either the “true” value or information on the reproducibility of the measured values. However, we can make use of both common sense and convention to estimate the implied precision of a given value. The number of decimal places when the value is represented in scientific notation, or the number of digits, should be indicative of its precision.

For example, if the distance from Kumasi to Accra is 270 km and we drive at 65 km/h, should we say that it would take us (270/65=) 4.153846 hours for the trip? This number implies that we can determine the answer to a precision of 0.0000005 h, which is 0.0018 s, or less than 2 milliseconds! This is obviously ludicrous, because the mileage value is nowhere near that precise (is it ±1 km, ±5 km?—exactly where did we start and end?), nor can we expect to drive at a speed having this degree of precision (e.g., 65 ± 0.000005 km/h, or about ± 20 mm/s!). It is conventional to assume that **the precision of a given number is comparable to the magnitude of the last digit** to the right in that number. That is, we assume that the value of 270 km implies 270 ± 1 km (or perhaps ± 0.5 km). However, unless the numbers are always given in scientific notation, so that the least significant digit can be associated with a specific decimal place, there will be some uncertainty, in which case common sense (judgment) should prevail.

In general, the number of decimal digits that are included in reported data, or the precision to which values can be read from graphs or plots, should be consistent with the precision of the data. Therefore, **answers calculated from data with limited precision will likewise be limited in precision.**

Significant figure is a digit which denotes the amount of the quantity in the place in which it stands. 0 is used in two ways. It may be used as a significant figure or it may be used to locate the decimal point. 0 is a significant figure except when it is the first figure in a number.

Examples:

0.0025 kg	*— the zeros are not significant figures, they serve only to locate the decimal point and can be omitted by proper choice of units: 2.5 g. This number has only 2 significant figures*
356000 m	*— the zeros are not significant, this number has 3 significant figures. If the number has NO decimal point, trailing zeros are not significant.*
2.150 g	*— it means that the mass has been determined to the nearest mg, and is between 2.151 g and 2.149 g. If the number has a decimal point, trailing zeros are significant.*
$1.2x10^{-4}$	*— is 0.00012 and has 2 significant figures*

Rules for calculations to preserve the correct number of significant figures

(a) Data and results

Retain as many significant figures in data and in a result as will give only 1 uncertain figure.

(b) Addition or subtraction

The sum or difference of 2 or more quantities cannot be more precise than the quantity having the largest uncertainly, that is the quantity having the least precision. **Round the answer to the least precise place.**

Example: *394.5g + 91.47g + 1.4051g = 487.3751 ≈ 487.4 g (4 sig. fig.)*
0.0121 + 25.64 + 1.05782 = 26.70992 ≈ 26.71 (4 sig. fig.)

(c) Multiplication or division

The answer should contain only as many significant figures as the smallest number of figures in any factor, unless the factor is exactly known. **Round the answer to the least number of significant figures.**

Example:

1.26 x 1.236 x 0.6834 x 24.865 = 26.46381512376 ≈ 26.5 (3 sig. fig.)
493.15 x 32 = 15 780.8 ≈ 16 000 (2 sig. fig.)
782 x 3.778 = 2 954.396 ≈ 2 950 (3 sig. fig.)

(d) Rounding a 5

One problem which arises is whether a 5 should be rounded up or down. For example, if 9.65 is rounded to 1 decimal place, should it become 9.6 or 9.7? **The result will be biased if a 5 is always rounded up**; this bias can be avoided by **rounding the 5 to the nearest even number** giving, in this case 9.6. Analogously 4.75 is rounded to 4.8.

When several measured quantities are to be used to calculate a final result, these quantities should not be rounded-off too much or needless loss of precision will result. A good rule is to keep 1 digit beyond the last significant figure and leave further rounding until the final result is reached.

If the calculation used physical property data, which is accurate to no more than ± 5%, then it makes no sense to report the answer to more than **3 significant figures**. When the actual precision of data or other information is uncertain, **a general rule of thumb is to report numbers to no more than three significant digits**, this corresponds to an uncertainty of somewhere between 0.05 – 0.5% (which is actually much greater precision than can be justified by most engineering data). Inclusion of more than three digits in your answer implies a greater precision than this and should be justified.

Those who report values with a large number of digits that cannot be justified are usually making the implied statement "I just wrote down the numbers—I really didn't think about it." This is most unfortunate, because if these people don't think about the numbers they write down, how can we be sure that they are thinking about other critical aspects of the problem?

When combining values, each of which has a finite precision or uncertainty, it is important to be able to estimate the corresponding uncertainty of the result. A very simple method that gives good results as long as the relative uncertainty is a small fraction of the value is to use the approximation (which is really just the first term of a Taylor series expansion)

$$\mathbf{A(1 \pm a)^x \cong A(1 \pm xa + \ldots)}$$

which is valid for any value of x if a < 0.1 (about). This assumes that the relative uncertainty of each quantity is expressed as a fraction of the given value, e.g., the fractional uncertainty in the value **A** is **a** or, equivalently, the percentage error in A is 100a.

Example:
Suppose we wish to calculate the shear stress on the bob surface in a cup-and-bob viscometer from a measured value of the torque or moment on the bob. The equation for this is

$$\tau_{r\theta} = \frac{T}{2\pi R_i^2 L}$$

If the torque (T) can be measured to ± 5%, the bob radius (R_i) is known to ± 1%, and the length (L) is known to ± 3%, the corresponding uncertainty in the shear stress can be determined as follows:

$$\tau_{r\theta} = \frac{T(1 \pm 0.05)}{2\pi R_i^2 (1 \pm 0.01)^2 L(1 \pm 0.03)} =$$

$$= \frac{T}{2\pi R_i^2 L}[1 \pm (0.05) \pm (2)(0.01) \pm (0.03)] = \frac{T}{2\pi R_i^2 L}(1 \pm 0.1)$$

That is, there would be a 10% error, or uncertainty.

"The perfect is the enemy of the good!" — Voltaire

2 INTRODUCTION TO FLUID FLOW

Chemical industries operated long before the profession of chemical engineer was recognized. The technology of each industry was regarded as a special branch of knowledge, and the people who did the jobs were chemists, mechanical engineers, and technologists. Based on the knowledge of the chemical technology, an independent applied science developed called **unit operations** at the beginning of the 20^{th} century (examples: evaporation, drying, distillation, gas absorption, crushing and grinding, crystallization, filtration, mixing, ...).

As the unit operations became better understood, it was apparent that they were not distinct entities but they involved momentum, heat, and mass transfer. In the 50's from the unit operation developed another independent engineering science called **transport phenomena**. Transport phenomena are the basis of the unit operations. Transport phenomena deals with the transport or movement of a given property by molecular movement through a system. The transported property can be the mass, the momentum, and the thermal energy (heat).

The study of momentum transport, or fluid mechanics as it is often called, can be divided into 2 branches:
– fluid statics (fluids at rest), and
– fluid dynamics (fluids in motion).

Some of the subdivisions and applications of fluid mechanics are:
(1) Hydraulics: the flow of water in rivers, pipes, canals, pumps, turbines.
(2) Aerodynamics: the flow of air around airplanes, rockets, structures.
(3) Meteorology: the flow of the atmosphere.
(4) Particle dynamics: the flow of fluids around particle, the interactions of particles and fluids (i.e., dust settling, slurries, pneumatic transport, fluidized beds, air pollutant particles, corpuscles in our blood).
(5) Hydrology: the flow of water and water-born pollutants in the ground.
(6) Reservoir mechanics: the flow of oil, gas, and water in petroleum reservoirs.
(7) Multiphase flow: coffee percolators, oil wells, carburetors, fuel injectors, combustion chambers, sprays.
(8) Combinations of fluid flow: with chemical reactions in combustion, with mass transport in distillation or drying.
(9) Viscosity-dominated flows: lubrication, injection moulding, wire coating, lava, and continental drift.

In the process industries, many of the materials are in fluid form and must be stored, handled, pumped, and processed. Most of the important unit operations are concerned with the behavior of fluids in process equipment.

The mathematical analysis of transport phenomena is important in order to develop a mathematical model of the process for process automation and computer control of a plant.

We have 2 types of equations:
(1) **Balance equations** or conservation equations of mass (equation of continuity), momentum (equation of motion), energy, and chemical component. These various conservation equations are called the equation of change, because they describe the change of density (ρ), velocity (v), temperature (T) and concentration (c_i) with respect to time and position in the system.

(2) **Phenomenological equations (rate equations)**
The phenomenological method of describing a natural phenomenon (process) ignores the microscopic structure of a substance and considers it as a continuous medium (continuum). It relates the properties characterizing the phenomena. The phenomenological equations are the rate - driving force relationships necessary to write the balance equations. The coefficients are determined directly by experiments.

$$\text{rate of transpor process} = \frac{\text{driving force}}{\text{resistance}} \quad \text{or} \quad \text{rate} = \text{coefficient} \times \text{driving force}$$

Fluid moving equipment are fans, pumps, compressors, pipes, valves ...

2.1 Fluid properties

The effect of shear stress (shear force/unit area) distinguishes fluids and non-fluids. A shear stress is created whenever a tangential force acts on a substance.

Fluid (definition) is a substance that can sustain no static shear stress or, as a consequence of this definition, a substance that will deform continuously under any applied shear load. When a fluid is at rest, there can be no shear stress.

Fluids are of 2 types: liquids and gases. On the molecular level they are quite different. In liquids, the molecules are close together and are held together by significant forces of attraction; in gases, the molecules are relatively far apart and have very weak forces of attraction. As the temperature and pressure increase, these differences become less and less, until the liquid and gas become identical at the critical temperature and pressure.

The tendency for continuous deformation of a substance is termed as **fluidity** and the act of continuous deformation is termed as **flow**. Solids resist deformation by shear stress so that when it is removed, the solid returns to its original size provided the elastic limit was not exceeded.

For flow relationships and calculations, we assume that the fluid is a **continuum**, that is it can be continually subdivided without thought of a molecular structure. This approach avoids the difficulty of dealing with the complexity of molecular motion itself. A particle in the continuum is an infinitesimal volume of material. A fluid particle consists of a large number of molecules moving together.

A **field** is a region within which physical or mathematical quantities can be either specified or measured at every point. A field is specified by giving the value of the physical quantity as a function of the 3 independent space coordinates.

The quantities of pressure p, density ρ, and temperature T are scalar quantities; their distribution in space is termed a **scalar field**. Velocity **v** is a vector quantity; its distribution in space is termed a **vector field**. The properties of a given point in the field may also be functions of time. A **flow field** is any region in space where a fluid flow exists.

The flow is **uniform**, if the velocity at a given instant is the same in magnitude and direction at every points in the fluid.

If, at a given instant, the velocity changes from point to point, the flow is **non-uniform**.

Steady-state flow: when the velocity at each location is constant, the flow pattern does not vary with time, so the flow is invariant with respect to time or the flow field is termed stationary. Or with other words, a steady flow is one in which the velocity, pressure and cross-section of the stream may vary from point to point but do not change with time.

If at a given point conditions do change with time, the flow is described as **unsteady**.

Physical properties may be
- **extensive properties**: that depend on the amount (mass) of the substance e.g. volume, energy, momentum, ...
- **intensive properties**: that are independent of the amount of the substance e.g. temperature, pressure, viscosity, ...
- **specific properties**: extensive properties may be made intensive by dividing the property by the mass of the substance e.g. specific volume, specific energy, ...

Density (ρ)

$$\rho = \frac{m}{V} \quad , \quad \text{kg/m}^3$$

Density may vary from point to point in a not homogeneously mixed fluid and may also vary with respect to time

$$\rho = \lim_{\Delta V \to \delta V} \frac{\Delta m}{\Delta V}$$

where Δm is the mass contained in a volume ΔV, and δV is the smallest volume surrounding the point for which statistical averages are meaningful.

Fluids whose densities remain almost constant over wide ranges of pressure and temperature are usually termed **incompressible**. The density of liquids is only slightly dependent on either temperature or pressure and the variation can generally be ignored (incompressible fluids) e.g. density of water □ 1000 kg/m ^{3}. (We can treat liquids and gases as incompressible when the velocities are less than 30% of the speed of sound and the pressure does not vary greatly.)

The density of a gas varies significantly with both temperature and pressure. At low pressure the densities of most gases are well approximated by the universal gas law:

$$pV = nRT \quad \Rightarrow \quad \rho = \frac{pM}{RT}$$

where T = absolute temperature, K
M = the average molecular mass of the gas
p = pressure, Pa
R = 8.314 J/mol K = 8.314 m^3 Pa/mol K

Specific weight (γ)
The specific weight is the weight per unit volume: $\gamma = \rho g$

Specific volume (v_s)
It is the reciprocal of the density: $v_s = 1/\rho$

Specific gravity (sp. gr. or SG)
It is the ratio of the density of a substance to the density of water at some specific temperature, usually at 4°C — the standard specific gravity is measured at 1 atm and 4°C.
E.g. if the specific gravity of an oil is 0.98, then the density of the oil is 980 kg/m^3.

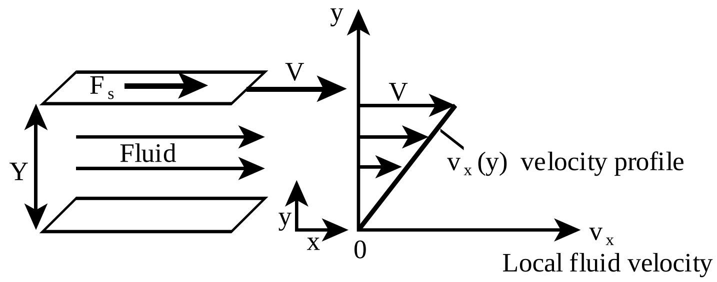
Viscosity (μ — read "mu")

Real fluids resist any force tending to cause one layer to move over another, the resistance is offered only while the movement is taking place. This resistance to the movement of one layer of fluid over an adjacent one is the viscosity of the fluid.

Viscosity is a measure of internal, frictional resistance to flow; or it is essentially a measure of the fluid's ability to transmit shear stresses.

Consider a thin film of fluid confined between 2 large (infinite) parallel plates, the lower one being at rest and the upper one is moving parallel to the bottom plate at a constant velocity V. The plates are Y apart. (This apparatus is easy to grasp conceptually and mathematically but difficult to use, because the fluid leaks out at the edges and gravity pulls the two plates together. Other devices are actually used to measure viscosities).

Imagine the fluid as being subdivided into infinitesimally thin layers parallel to the plates (like a deck of playing cards). Each layer of liquid moves in the x direction. The layer adjacent to the moving plate will have velocity V. The layer below is at a slightly slower velocity, each layer moving at a slower velocity as we go down in the !y direction. The layer adjacent to the stationary plate will be at rest. This velocity profile is linear with y direction.



A force $\mathbf{F_s}$ must be exerted on the upper plate to maintain the motion, and an equal and opposite force must be exerted on the stationary plate. It is convenient to use not the total shear force (F_s), but the shear force per unit area of the plate (A_s) called shear stress (τ— read "tau"):

$$\boxed{\tau_{yx} = \frac{F_s}{A_s}}$$

τ_{yx} shear stress has 2 indexes. The first index y shows, that the shear stress varies with distance y. The second index x shows the direction in which the shear force and shear stress acts.

Experiments indicate the direct proportionality between the shear force (F_s) and the velocity (V) and the area of the plane (A_s):

$$\frac{F_s}{A_s} = -\mu \frac{\Delta V}{\Delta y}$$

If we let Y approach zero, then $\boxed{\tau_{yx} = -\mu \frac{dv_x}{dy}}$ **Newton's law of viscosity**

where τ_{yx} = shear stress exerted in the x-direction on a fluid surface of constant y, N/m^2
v_x = the x-component of the fluid velocity vector, m/s

μ = dynamic viscosity, Pa·s = kg/m·s
1 P (Poise) = 1 g/cm·s = 0.1 Pa·s

1 cP (centi Poise) **= 10^{-3} Pa·s**

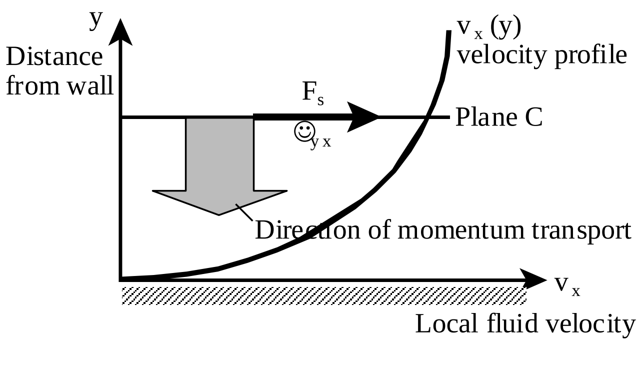
Above empirical relation is known as Newton's law of viscosity. It states that the shear force per unit area (F_s/A_s) is proportional to the negative of the local velocity gradient (dv_x/dy), and the proportionality constant is called the dynamic viscosity (μ— read "mu"). The velocity gradient is often called shear rate, so the shear stress (τ) is proportional to the shear rate (dv_x/dy).

Fluids that obey this equation are called **Newtonian fluids**. The viscous shear stress in a fluid is primarily a result of the intermolecular forces.

Examples of Newtonian fluids:
– all gases,
– all liquids for which simple chemical formulae can be written (such as water, ethyl alcohol, hexane, carbon tetrachloride, …) and their molecular mass is < 5000,
– most dilute solutions of simple molecules in water or organic solvents (such as solutions of inorganic salts or sugar in water or benzene).

The variation of fluid viscosity with temperature and pressure is quite complicated. It must be determined experimentally. In general:
– the viscosity of a gas increases with increasing temperature,
– the viscosity of a liquid decreases with increasing temperature,
– the viscosity of an ideal gas is independent of pressure,
– the viscosities of real gases and liquids usually increase with pressure.

Another interpretation of τ_{yx}
The shear stress τ_{yx} can also be interpreted as a flux of x-momentum in the y direction, which is the rate of flow of momentum per unit area



F_s – is required to maintain the motion,
– acts parallel to the plane of the shear,
– is exerted by the fluid outside of plane C on the fluid between plane C and the wall.

The moving fluid just above plane C has slightly more momentum in the x-direction than the fluid just below this plane. The random motions of the molecules in the faster-moving layer send some of the molecules into the slower-moving layer, where they collide with the slower-moving molecules and tend to speed them up or increase their momentum in the x-direction.

Similarly, molecules in the slower layer tend to retard those in the faster layer. This exchange of molecules between layers produces a transport or flux of x-direction momentum across plane C from the high-velocity to the slow-velocity layers.

Therefore we can say that the x-direction momentum is transported in the ! y direction, or the momentum tends to go in the direction of decreasing velocity. A velocity gradient can thus be thought of as a "driving force" for momentum transport.

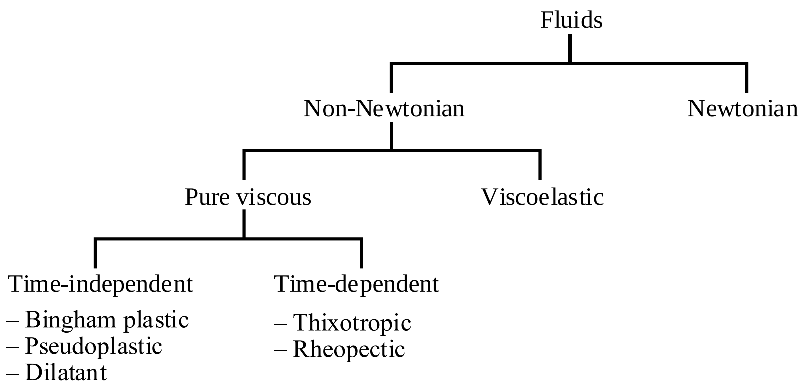
Kinematic viscosity (ν — read "nu")
In the formulation of the equations of fluid dynamics, the ratio of dynamic viscosity to density frequently occurs. To save writing kinematic viscosity is defined:

$$\boxed{\nu = \frac{\mu}{\rho}} \quad , \ \mathrm{m^2/s} \qquad\qquad 1\mathrm{St\ (Stokes)} = 10^{-4}\ \mathrm{m^2/s}$$

2.2 Rheology of non-Newtonian fluids

Rheology is the science of deformation and flow of materials. The rate and nature of deformation, which occurs when a shear stress is applied on a material, can be expressed by its rheological properties. An important branch of rheology concerns the behavior of non-Newtonian fluids. Most non-Newtonian fluids are mixtures with constituents of different sizes.

Whenever there is a fluid flow, there must exist a velocity gradient. So we can use that to distinguish between various types of fluids.



Newtonian fluids
The shear stress (τ_{yx}) is directly proportional to the shear rate (dv_x/dy):

$$\tau_{yx} = -\mu_a \frac{dv_x}{dy}$$

where μ_a = apparent viscosity in general
$\mu_a = \mu$ for Newtonian fluids

Non-Newtonian fluids
There is no such linear relationship between shear stress and shear rate. The apparent viscosity is not a property of the fluid in the same way as Newtonian viscosity; it depends on the shear force exerted on the fluid. Most non-Newtonian fluids have apparent viscosities, which are relatively high compared with the viscosity of water. In a plot of shear stress and shear rate, we obtain a curve known as a **flow curve**, **shear diagram**, or **rheogram**.

Time-independent fluids are those whose properties are independent of time or duration of shear. **Time-dependent fluids** are those whose properties are dependent upon duration of shear. Their apparent viscosity either decreases (thixotropic fluids) or increases (rheopectic fluids) with time. Type of this behavior is complicated, and little progress has been made toward its analytical representation.

(1) Bingham plastic fluids

Bingham plastic fluids do not produce motion until some finite **yield stress** (τ_o) has been applied. At stresses below τ_o they behave like solids and at stresses above τ_o a plot of τ_{yx} vs. dv_x/dy is linear:

$$\tau = \tau_o + k_p\left(-\frac{dv_x}{dy}\right) \qquad \tau \geq \tau_o$$

where τ = shear stress

τ_o = yield stress (initial stress needed for flow to start)

k_p = constant ratio that exists between change in shear stress and change in shear rate (It is analogous to the viscosity of a Newtonian fluid)

Examples of Bingham plastic fluids are sewage sludge; drilling muds; suspensions of regular, granular solids; some plastic melts; cooking fats; tomato ketchup; toothpaste; some paper pulps, mayonnaise, butter, peanut butter ...

2 parameter models

Pseudoplastic and dilatant fluids obey the Power law:

$$\tau = k\left(-\frac{dv_x}{dy}\right)^n$$

where τ = shear stress, $kg/ms^2 = N/m^2$

k = consistency index, kg/ms^{n-2}

n = flow behaviour index, dimensionless

$-dv_x/dy$ = shear rate, 1/s

when $n = 1 \Rightarrow$ the fluid is Newtonian,

$n < 1 \Rightarrow$ the fluid is pseudoplastic,

$n > 1 \Rightarrow$ the fluid is dilatant.

(2) Pseudoplastic fluids

This is the most common type of non-Newtonian behaviour.

Flow behaviour index: $n < 1$

Apparent viscosity: $\mu_a = k\,|-dv_x/dy|^{n-1}$

The apparent viscosity decreases with increasing shear rate; these fluids are said to exhibit **shear thinning**. The slope of the flow curve often approaches unity at very high values of the rate of shear; that is, the fluids become more Newtonian.

Examples of pseudoplastic fluids are rubber latex, adhesives, polymer solutions or melts and other large elongated molecules, colloidal or ordinary suspensions of asymmetric particles, cellulose acetate, suspensions of paper pulps or pigments, some greases, biological fluids like milk or blood, gelatine, fruit juice concentrates, shampoo, liquid cement ...

(3) Dilatant fluids

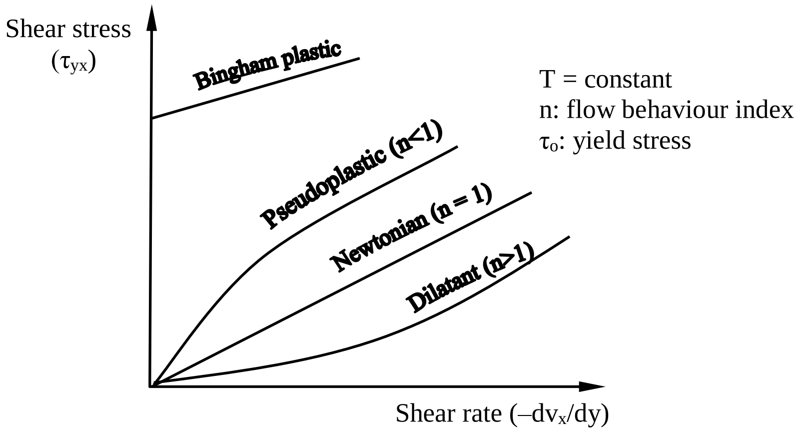
They exhibit rheological behaviour opposite to that of pseudoplastics. They are uncommon.

Flow behaviour index: $n > 1$

Apparent viscosity: $\mu_a = k\,|-dv_x/dy|^{n-1}$

The apparent viscosity increases with increasing shear rate; these fluids are **shear thickening**. The flow curve is concave upward at low shears and become linear at high shears.

Examples of dilatant fluids are some cornflour and sugar solutions, concentrated starch suspensions, wet cement aggregate, wet beach sand, quick sand, mica suspension in water...



Flow diagram for purely viscous time-independent fluids

(4) Thixotropic fluids

They possess a structure, the break down of which is a function of time as well as shear rate. As the structure breaks down with constant shear rate, shear stress decreases. Their apparent viscosity decreases with time. This structure can rebuild itself if not prevented from doing so by externally applied forces.

Many thixotropic fluids are known, almost all of which are slurries or solutions of polymers.

Examples of thixotropic fluids are mayonnaise, many paints, inks, drilling muds, cultures containing fungal mycelia or extracellular microbial polysaccharides, many gels and colloids...

(5) Rheopectic fluids

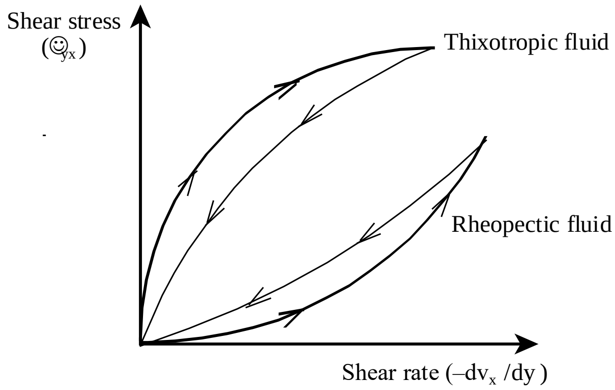
They are very rare in occurrence. Their apparent viscosity increases very rapidly upon being rhythmically shaken or tapped.

Examples of rheopectic fluids are gypsum pastes, slurries.

(6) Viscoelastic fluids

Viscoelastic fluids exhibit an elastic response to changes in shear stress. In the flow of these fluids, normal stresses in addition to the usual tangential stresses are build up. These normal stresses cause for example, the fluid to climb up a shaft rotating in the fluid. When shear forces are removed from a moving viscoelastic fluid, the direction of flow may be reversed. Most viscoelastic fluids are pseudoplastic and may exhibit other rheological characteristics such as yield stress. For the steady-state flow of viscoelastic fluids, the equations developed for pseudoplastic fluids apply.

Examples of viscoelastic fluids are flour dough, egg white, and some polymer solutions like nylon, bitumen.



Flow diagram for purely viscous time-dependent fluids

These strange types of fluid behaviour are of considerable practical use. A good toothpaste should be a Bingham fluid, so it can easily be squeezed out of the tube but will not drip off the toothbrush. A good paint should be a thixotropic Bingham fluid, so that in the can it will be very viscous and the pigment will not settle to the bottom, but when it is stirred, it will become less viscous and can be easily brushed onto the surface. In addition, the brushing should temporarily reduce the viscosity so that the paint will flow sideways and fill in the brush marks; then as it stands, its viscosity should increase, so that it will not form drops and run down the wall.

Viscosities of some non-Newtonian fluids

Material	**Viscosity, cP**
Blood, kerosene	10
Ethylene glycol (anti-freeze)	15
Motor oil SAE 10, Mazola corn oil	50 – 100
Motor oil SAE 30, Maple syrup	150 – 200
Motor oil SAE 40, Castor oil	250 – 500
Motor oil SAE 60, Glycerin	1000 – 2000
Honey	2000 – 3000
Heinz Ketchup, French Mustard	50,000 – 70,000
Tomato paste, Peanut butter	150,000 – 250,000
Window putty	100,000,000

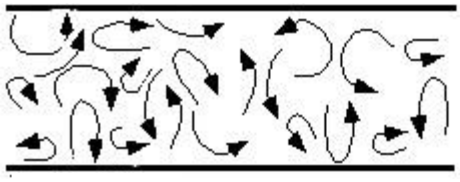
2.3 Types of fluid flow

When a fluid is flowing through a tube or over a surface, the pattern of the flow varies with the velocity, the physical properties of the fluid, and the geometry of the surface. This problem was first examined by Reynolds (1883).

A fluid can flow through a pipe in 2 different ways: in laminar or turbulent fashion. There is a region of transition from the one type of flow to the other, called transition region.

Laminar flow
The layers of fluid (laminae) slide over each other smoothly with no macroscopic mixing perpendicular to the axis of the pipe and Newton's law of viscosity holds. It is also called **viscous flow** or streamlined flow.

Turbulent flow
At higher velocities, the flow becomes turbulent characterized by rapid, chaotic motion in all directions in the pipe superimposed on the overall axial motion. There is mixing by eddy motion between the layers, and even in overall steady flow the velocity at a point fluctuates about some mean value.



Eddies are small pockets of fluid particles moving in all directions and at all angles to the normal line of flow. Turbulent flow consists of a mass of eddies of various sizes coexisting in the flowing stream. Large eddies are continually formed. They break down into smaller eddies, which in turn evolve still smaller ones. Finally the smallest eddies disappear. The size of the largest eddy is comparable with the smallest dimension of the turbulent stream; the diameter of the smallest eddies is about 1 mm and contain 10^{16} molecules, so all eddies are of macroscopic size, and the turbulent flow is not a molecular phenomena! The flow within an eddy is laminar.

Any given eddy possesses a definite amount of mechanical energy. The energy of the largest eddies are supplied by the kinetic energy of the bulk flow of the fluid. The energy is transferred as energy of rotation along a continuous series of smaller eddies. Mechanical energy is not appreciably dissipated into heat during the break up of large eddies into smaller ones, but such energy is not available for maintaining pressure or overcoming resistance to flow and is worthless for practical purposes. This mechanical energy is finally converted into heat when the smallest eddies are destroyed (stopped) by viscous action.

Turbulence may be generated from
- contact of the flow stream with solid boundaries (**wall turbulence**),
- contact between 2 layers of fluid moving at different velocities (**free turbulence**).

Wall turbulence appears when the fluid flows through closed or open channels or past solid shapes immersed in the stream. In turbulent flow in pipes, the largest eddies have a length of 25% of the pipe diameter. In the wakes of ships and airplanes the largest eddies will be a similar fraction of the size of the ship or airplane.

Free turbulence appears in the flow of a jet into a mass of stagnant fluid or when boundary layer separates from a solid wall and flows through the bulk of the fluid. It is especially important in mixing.

Transition region
Laminar flow can exist in conditions in which it is not the stable form, but it fails to switch to turbulent flow unless some outside disturbance such as microscopic roughness on the pipe wall or very small vibrations in the equipment triggers the transition. Thus, in the transition region the flow can be laminar or turbulent, and the pressure drop or flow rate can suddenly

change by a factor of 2. Under some circumstances the flow can alternate back and forth being laminar and turbulent, causing the pressure drop to oscillate between a higher and lower value; or for a constant pressure drop, the velocity can oscillate between a higher and a lower value.

Almost all flows of gases and liquids in ordinary-sized pipes are turbulent. The only exceptions are flows of fluids much more viscous than water, such as asphalt or polymer solutions. However, in very small tubes or other flow passages the flow is normally laminar. The flow in the heart and the major arteries near it in our body is turbulent. The rest of the blood flow in our body is laminar, as is the flow in filters, in groundwater, and in oil fields. River flows are mostly turbulent, and the main flows of the atmosphere are turbulent.

Ideal flow (potential flow)
An ideal fluid is an incompressible fluid, which has zero viscosity. There are 2 important characteristics of potential flow:

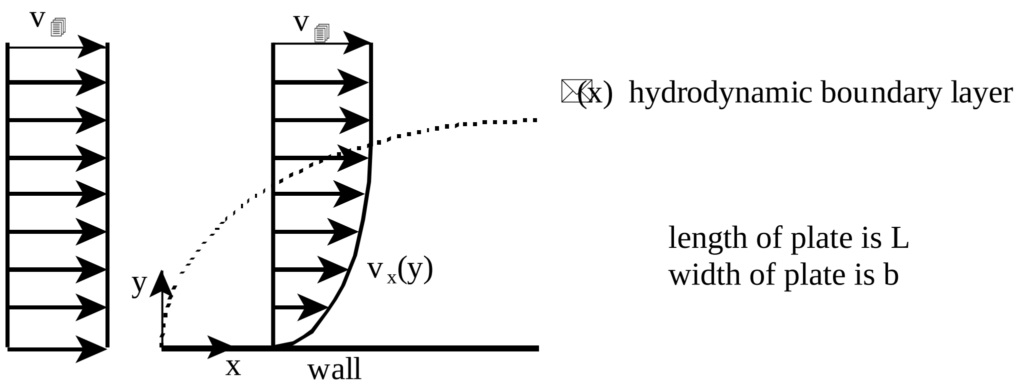
(1) Neither circulations nor eddies can form within the stream, so potential flow is also called **irrotational flow**,
(2) Friction cannot develop, so that there is no dissipation of mechanical energy into heat.

Potential flow can exist at distances not far from a solid boundary. Prandtl (1904) stated that except for fluids moving at low velocities or possessing high viscosities, the effect of the solid boundary on the flow is confined to a layer of fluid immediately adjacent to the solid wall. This layer is called **boundary layer**. Outside the boundary layer, potential flow survives.

Boundary layer flow
Boundary layer is the region near a solid where the fluid motion is affected by the solid boundary. When a fluid flows over a surface the elements in contact with the surface will be brought to rest and the adjacent layers retarded by the viscous drag of the fluid. Thus, the velocity in the neighborhood of the surface will change with distance perpendicular to the flow.

Suppose a fluid with uniform velocity (v_4) approaches a plane surface. When the fluid reaches the surface, a velocity gradient is set up because of the viscous forces acting within the fluid. The fluid in contact with the surface will be brought to rest and will gradually approach the free stream velocity (v_4) at some distance from the surface.

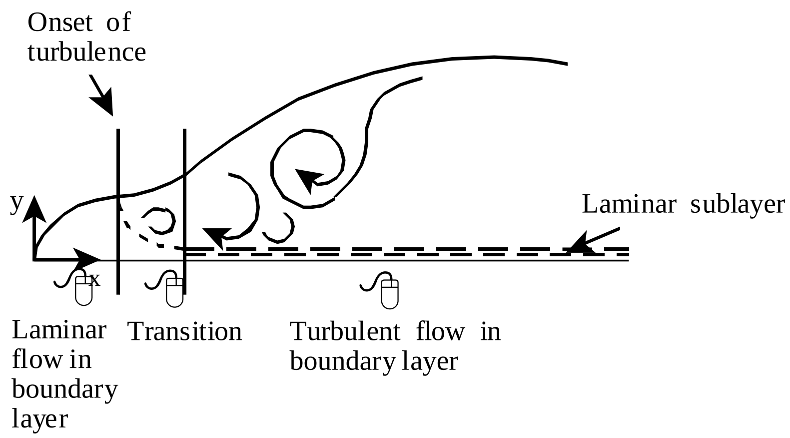
We divide the flow region into 2 parts:
(1) A non-viscous region away from any solid surfaces (ideal fluid: $\mu = 0$, ρ = constant)
(2) A boundary layer (the fluid adheres to the surface due to viscous effects)



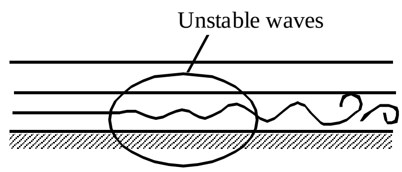
The thickness of the layer in which the fluid is retarded becomes greater with distance in the direction of flow. This layer was termed boundary layer by Prandtl.

Anywhere along the plate, the velocity profile changes from the uniform free-stream profile. The velocity increases from zero at the surface to nearly the free-stream value at some distance away. The free stream velocity is approached asymptotically, and therefore the boundary layer strictly has no precise outer limit. However, it is convenient to define the boundary layer thickness such that the velocity at its outer edge equals 99% of the free stream velocity. That is the **hydrodynamic boundary layer** thickness δ is where $v_x \cong 0.99v_4$ Within the boundary layer the velocity in x direction varies only with y: $v_x = v_x(y)$. Typically, δ is a few μm thick. Despite its size, it is important in determining drag forces.

As the boundary layer thickens, at distances farther from the leading edge, a point is reached where turbulence appears. The onset of turbulence is characterized by a sudden rapid increase in the thickness of the boundary layer. In the turbulent region, however, there is still a **laminar sublayer** near the surface. The change from laminar to turbulent flow in the boundary layer will occur at different distances downstream depending on the roughness of the surface and the physical properties of the fluid.



How do turbulent boundary layers form?

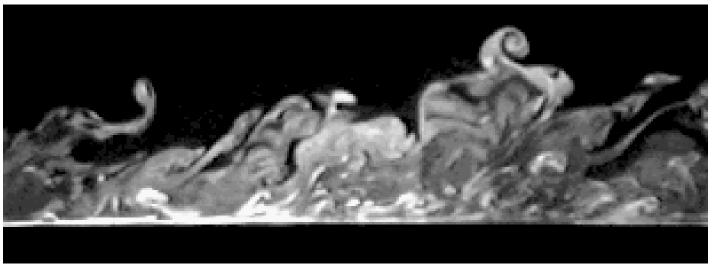


Laminar boundary layers become unstable due to some disturbance. The disturbance forms unstable waves in the laminar boundary layer, which grows into eddies and a turbulent boundary layer. Some common disturbances include pressure gradients, surface roughness, heat transfer, body forces, and free stream disturbances. The flow disturbances may grow to form a turbulent boundary layer.

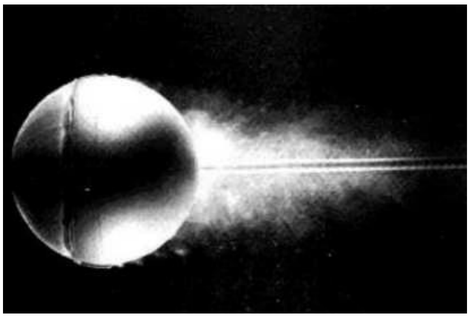
The turbulent boundary layer is divided into 3 regions; they are the viscous (or laminar) sublayer, the buffer layer and the turbulent layer. The viscous sublayer represents a very small portion in the turbulent boundary layer. The buffer layer represents –15% and the turbulent layer represents –85% of the turbulent boundary layer. Each region has its own velocity profile.

Most technical flow processes are best studied by considering the fluid stream as 2 parts, the boundary layer and the remaining fluid. (In flow through pipes, the boundary layer fills the entire channel; in flow in a converging nozzle, the boundary layer may be neglected.)

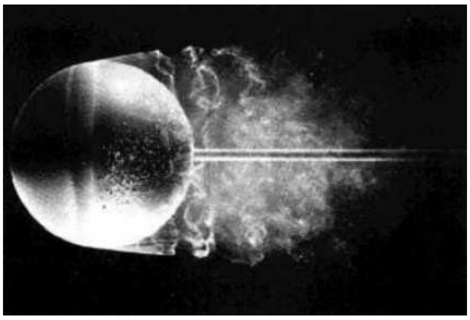
Boundary layer separation occurs when the boundary layer travels far enough against an adverse pressure gradient that the speed of the boundary layer relative to the object falls almost to zero. The fluid flow becomes detached from the surface of the object, and instead takes the forms of eddies and vortices. Flow separation can often result in increased drag.



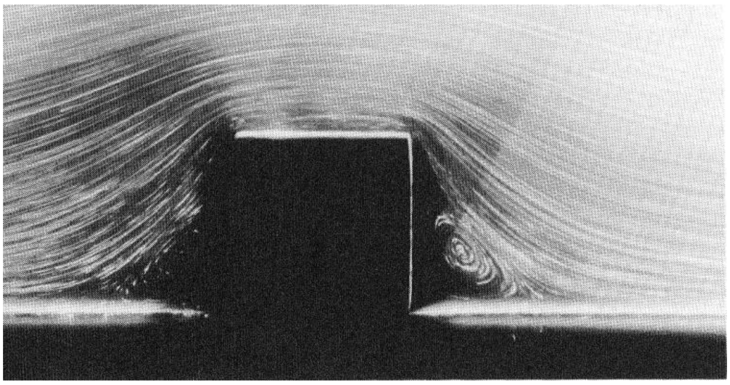
Laser-induced fluorescence image of an incompressible turbulent boundary layer



Laminar flow



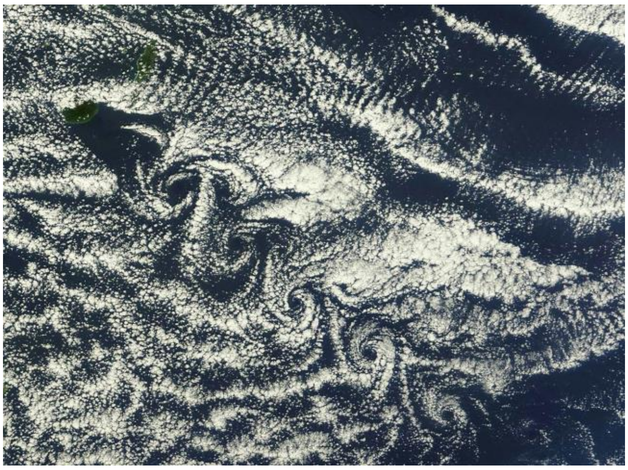
Turbulent flow



Boundary layer separation

Kármán vortex street behind a cylinder placed in uniform flow (Re ~ 300)

Cars in wind-tunnel



Yakushima Island von Kármán vortex street

Vapour trail behind an airplane

2.4 Reynolds number (Re)

Reynolds studied the conditions under which one type of flow changes into another, and found that the **critical velocity** at which laminar flow changes into turbulent flow, depends on 4 quantities:

- the diameter of the tube (D) — characteristic length,
- the viscosity (μ),
- the density (ρ), and
- the average linear velocity (v_{avg} or v) of the fluid.

He found that these factors can be combined into a dimensionless group and the change in kind of flow occurs at a definite value of the group

$$\boxed{Re = \frac{\rho v D}{\mu}}$$

The transition from laminar flow to turbulent flow may occur over a wide range of Reynolds numbers depending on the conditions.

Under ordinary conditions of **flow in pipes**:

laminar flow	$Re < 2100$
transition flow	$2100 < Re < 4000$
turbulent flow	$Re > 4000$

For **flow over a flat plate**, transition usually occurs over a range between $2x10^5 \le Re_x \le 3x10^6$ and not at a single point. For purposes of calculation it is customarily assumed that transition occurs at $Re_x = 5x10^5$.

For **flow around a spherical particle** the flow is laminar if Re < 1.

2.5 The physical meaning of Reynolds number

The transition to turbulent flow and the degree of turbulence intensity depend on:

(1) The kinetic energy of the mean flow, which is an indication of the ability of the flow to supply the energy that makes up the random turbulent motion.

(2) The dissipative effects of the viscous forces (the viscous work done on the mass). The viscous forces damp or dissipate the turbulent eddy motion.

(3) The physical confinement of the fluid: the boundaries within which the flow occurs.

Consider the rate of kinetic-energy propagation through an area normal to the pipe axis:

$$\frac{1}{2}\dot{m}\,v^2 \qquad \text{where} \qquad \dot{m} = \rho A v = \text{mass flow rate}$$

The work rate of viscous forces (the shear stress work rate) is:

$$F_s\,v = \tau A\,v \qquad \text{and}\ \tau = \mu v/L$$

where all quantities are not specifically defined flow field quantities but are quantities characteristic of the entire flow.

$$\frac{\text{kinetic}-\text{energy rate}}{\text{shear stress work rate}} = \frac{(1/2)\,\dot{m}v^2}{\tau A v} = \frac{\rho A v^3}{(\mu\,v/L)Av} = \frac{\rho v L}{\mu} = Re$$

In case of fluid flow through a tube

the only possible characteristic dimension L is the diameter (D)

the characteristic velocity is the average velocity (v_{avg}).

Another interpretation for the physical meaning:

$$Re = \frac{\text{inertia force}}{\text{viscous force}}$$

3 THE OVERALL MASS BALANCE

Much of engineering is simply careful accounting of mass, momentum, energy, chemical components, etc. The accountings are called balances. Chemical engineers use some form of the balance equation in almost every problem they encounter.

The general balance equation:

Accumulation = Production – Destruction + Flow in – Flow out

A balance is made over an identifiable set of boundaries. The set of boundaries need not be fixed, but they must be identifiable. Whatever is inside a set of boundaries is called a **system**. Everything that is outside the boundaries is called **surrounding**.

A **closed system** is one, which has an identified amount or quantity of fluid. The system therefore contains a prescribed mass of fluid. It may be either infinitesimal or large. A system may change shape, position, and thermal conditions but must always entail the same amount of matter. There is no flow in or out of a closed system and the balance equation reduces to

Accumulation = Production – Destruction

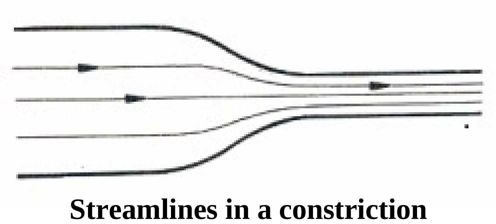
An **open system** is usually a kind of container or vessel that has flow in and out across its boundaries at some number of places.

If we choose as a system some arbitrary region of space that can have flow in or out over the entire boundary, then this system is called a **control volume** (C.V.). Therefore, a control volume is a region in space through which the fluid flows. The mathematical equations in fluid dynamics are written for the control volume. The control volume may be large for which the overall relationships are to be determined, or it may be a differential volume element in which case differential equations are determined. The shape of a control volume is chosen arbitrarily; it should depend on the geometry of the problem. The C.V. does not need to be fixed in space; it can also be moving. The **control surface** (C.S.) is the boundary of the control volume.

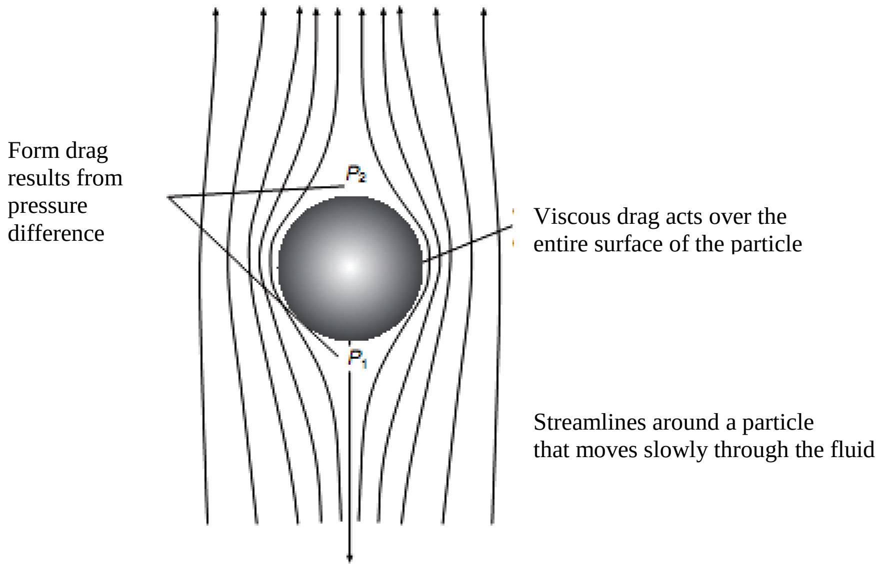
The balance equation deals with changes in the quantity being accounted for, not with the total amount present. Balance equations can be applied to any extensive property.

The balance equations may be written in differential form, showing conditions at a point within a volume element of a fluid, or in integrated form applicable to a finite volume or mass of fluid.

Streamline (definition) is an imaginary curve in a mass of flowing fluid so drawn that at every point on the curve the net velocity vector is **tangent** to the streamline. Since the velocity vector is tangent to it, no matter can cross it. In steady flow, the path of a fluid particle follows a streamline, hence a streamline is the trajectory of an element of fluid in such a situation. A **streamtube** is a tube having the surface made up of streamlines.



Streamlines in a constriction



Preliminary steps when setting up equations that mathematically describe the physical behavior of a flow field:

(1) Locate and specify a coordinate system origin.

(2) Choose a coordinate system that best matches the characteristics of the physical boundaries of the problem.

(3) Specify the flow region (that is the flow field) to be considered — control volume

(4) Make a sketch of the problem that includes the above quantities.

The most important chemical engineering balance is the mass balance.

3.1 Reynolds' transport theorem

We need to formulate and use equations appropriate to a convenient control volume (C.V.). However, the basic principles of physics are stated for a system (closed system!). (e.g. Newton's 2 nd law permits calculation of the force which must be applied to a system or fixed mass to give it a specified acceleration.) The Reynolds' transport theorem (or equation) provides the necessary relationship between the system and the control volume approach.

Using a general property, which will be taken as mass, momentum or energy:

N = total amount of a quantity (mass, momentum, energy)

n = amount per unit mass (n = N/m)

It is assumed that n varies through the system both in position and time.

dV = differential volume element

The total amount of n within the system (which is N) may be determined from

$$N = \iiint_{System} n\rho \, dV$$

We need to express the rate at which the quantity of N within the system changes with time.

$\dfrac{DN}{Dt}$ = rate of change (increase) of N of a moving, constant mass of fluid within the system with respect to an observer following the fluid (substantial time derivative)

We want this expression to be in terms of a control volume.

$\dfrac{\partial N}{\partial t} = \dfrac{\partial}{\partial t}\iiint_{C.V.} n\rho \, dV$ it is the increase with time of the amount of N within the C.V.

$\iint_{C.S.} n\rho \bar{v} \cdot \overline{dA}$ it is the net flow rate of N through the control surface

$$\left[\begin{array}{l} n\,\rho\, v\, dA = \dfrac{N}{m}\ \dfrac{m}{V}\ \dfrac{\dot{V}}{A}\ dA = \dfrac{N}{m}\ \dfrac{m}{V}\ \dfrac{V}{tA}\ dA = \dfrac{N}{tA} dA \\ \Downarrow \\ \iint_{C.S.} \dfrac{N}{tA} dA = \dfrac{N}{t} \qquad \textit{flow rate of } N \end{array}\right]$$

Therefore

$$\boxed{\frac{DN}{Dt} = \frac{\partial}{\partial t}\iiint_{C.V.} n\rho \, dV + \iint_{C.S.} n\rho \, \bar{v} \cdot \overline{dA}}$$

This is Reynolds' transport theorem

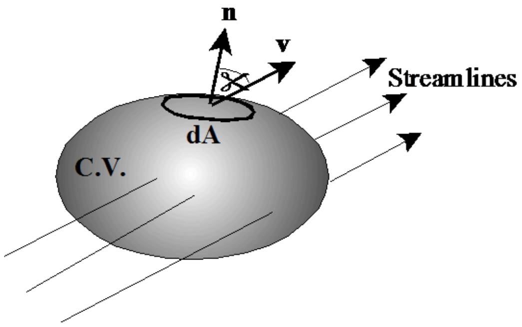
Reynolds' transport theorem says that the rate at which the property N increases with time within the system is equal to the time rate of increase of N in the control volume plus the net rate of flow of N through the control surface.

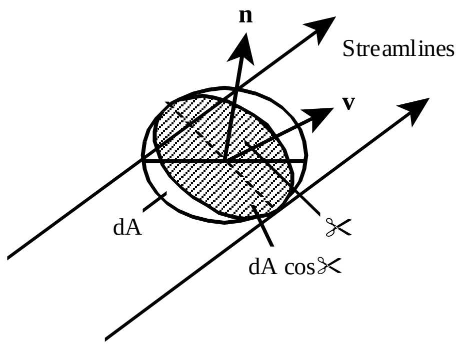
3.2 Overall mass balance equation

The overall mass balance for the control volume is:

$$\underbrace{\left\{\begin{array}{c}\text{Rate of mass}\\ \textbf{out}\end{array}\right\} - \left\{\begin{array}{c}\text{Rate of mass}\\ \textbf{in}\end{array}\right\}}_{\text{net rate of mass } \textbf{efflux}} + \left\{\begin{array}{c}\text{Rate of mass}\\ \textbf{accumulation}\end{array}\right\} = 0$$

Consider a general control volume located in a fluid flow field. dA is a small element of area on the control surface. The orientation of this surface is defined by an outward unit vector, **n**, in the normal direction. dA cosα is the projection of the area dA in a direction (or plane) normal to the velocity vector **v**.





α is the angle between a line normal to the surface dA and directed outward and a line representing the direction of velocity. dA has been chosen as a sufficiently small portion of the cross sectional area for flow, so that the fluid passing through dA cosα may be treated as having a constant velocity.

The rate of flow passing a given cross section (dA cosα) may be determined from the velocity distribution across the section.

In a time interval dt the fluid will travel a distance	ds = v dt.
The volume passing through dAcosα during time dt will be	dV = v dt dA cosα
The corresponding mass must be	dm = ρ dV = ρ vdt dAcosα
The rate of flow of mass in the control volume	dm/dt = ρv dA cosα

Therefore the **net outward mass flow rate** across the surface dA is given by integrating this mass flow over the cross sectional area (surface integral!)

$$\iint_{C.S.} \frac{dm}{dt} = \iint_{C.S.} \rho v \cos\alpha \; dA$$

$$\left[\text{Other representations of the same surface integral are: } \iint_{C.S.} \rho\,(\bar{v}\cdot\bar{n})\,dA = \iint_{C.S.} \rho\,(\bar{v}\cdot\overline{dA}) \right]$$

The total mass accumulation rate in the control volume is:

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\iiint_{C.V.} \rho \; dV$$

Substituting above terms into the verbal mass balance equation, we get the general form of the overall mass balance for a control volume as:

$$\boxed{\iint_{C.S.} \rho v \cos\alpha \; dA + \frac{\partial}{\partial t}\iiint_{C.V.} \rho \; dV = 0} \qquad \text{kg/s}$$

In vector notation:

$$\iint_{C.S.} \rho(\bar{v}\cdot\bar{n})\; dA + \frac{\partial}{\partial t}\iiint_{C.V.} \rho \; dV = 0$$

3.3 Application of Reynolds' transport theorem

Applying Reynolds' transport equation for the overall mass balance: N = m, and n = N/m = 1

The amount of mass within the system:

$$m_{system} = \iiint_{System} (1)\,\rho\, dV \quad , \quad \text{kg}$$

The mass accumulation rate in the C.V.:

$$\dot{m}_{control\ volume} = \frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \iiint_{C.V.} \rho\, dV \quad , \quad \text{kg/s}$$

The mass flow rate through the control volume:

$$\iint_{C.S.} \rho\, \bar{v} \cdot \bar{n}\, dA$$

The change (increase) in the amount of mass in the system:

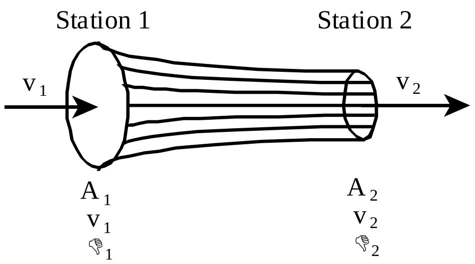
$$\left(\frac{Dm}{Dt}\right)_{\text{system}} = \frac{\partial}{\partial t} \iiint_{C.V.} \rho\, dV + \iint_{C.S.} \rho\, \bar{v} \cdot \overline{dA}$$

The law of conservation of mass says that the mass within a system cannot change with time, therefore Dm/Dt = 0 and the conservation of mass, as applied to a control volume becomes:

$$\iint_{C.S.} \rho\, v \cos\alpha\, dA + \frac{\partial}{\partial t} \iiint_{C.V.} \rho\, dV = 0$$

3.4 Flow through a conduit

Let's apply the overall mass balance equation for flow through a conduit.



Control volume (streamtube)

v_1 and v_2 are the average fluid velocities

The **average fluid velocity** (v_{avg}) or **bulk velocity** is defined by

$$\boxed{v_{avg} = \frac{1}{A} \iint_{A} v\, dA}$$

Assumptions:

- steady-state (dm/dt = 0, that is there is no mass accumulation in the control volume)
- one-dimensional flow
- all the flow inward is normal to A_1 and outward normal to A_2
- ρ = constant over the area of integration

Steady state does not mean nothing is changing; it means nothing is changing with respect to time at a particular point.

Mass crosses the control surface at stations (1) and (2) only.
When the velocity v_2 is normal to A_2 then $\alpha_2 = 0$ and $\cos\alpha_2 = 1$.
Where v_1 is directed inward, then $\alpha_1 > 90°$, and for this case $\alpha_1 = 180°$ and $\cos\alpha_1 = !\ 1$

$$\Rightarrow \quad \iint_{C.S.} \rho v \cos\alpha \; dA = \iint_{A_2} \rho v \cos\alpha_2 \; dA + \iint_{A_1} \rho v \cos\alpha_1 \; dA = \rho_2 v_2 A_2 - \rho_1 v_1 A_1 = 0$$

$\boxed{\dot{m} = \rho v A = \text{constant}}$ This is the **overall mass balance equation**

The **mass velocity** or **mass flux** is defined as: $\mathbf{G} = \rho\mathbf{v}$, kg/m^{2}s

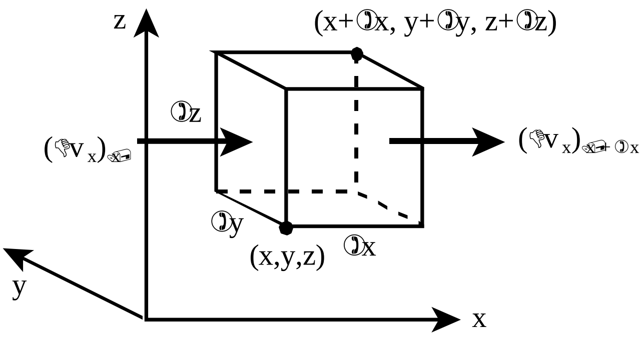
Physically $\rho\mathbf{v}$ represents the amount of mass flowing through a unit cross-sectional area per unit time.

The overall balances are powerful because they do not require knowledge of details inside the control volume (the flow situation inside the control volume need not to be specified).

4 THE DIFFERENTIAL MASS BALANCE; EQUATION OF CONTINUITY

The equation of continuity is a mass balance over a stationary differential volume element (control volume) $\Delta x \Delta y \Delta z$ through which the fluid is flowing.

$$\left\{ \begin{array}{c} \text{Rate of mass} \\ \textbf{accumulation} \end{array} \right\} = \left\{ \begin{array}{c} \text{Rate of mass} \\ \textbf{in} \end{array} \right\} - \left\{ \begin{array}{c} \text{Rate of mass} \\ \textbf{out} \end{array} \right\}$$



Rate of mass in
through the face at x $\qquad (\rho v_x)|_x \, \Delta y \Delta z$

Rate of mass out
through the face at x + Δx $\qquad (\rho v_x)|_{x+\Delta x} \, \Delta y \Delta z$

Similar expressions may be written for the other 2 pairs of faces.

Rate of mass accumulation
within the volume element $\qquad$ (Mρ/Mt) $\Delta x \Delta y \Delta z$

So the mass balance:

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \, [(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}] + \Delta x \Delta z \, [(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}] + \Delta x \Delta y \, [(\rho v_z)|_z - (\rho v_z)|_{z+\Delta z}]$$

Divide by the volume of the element ($\Delta x \Delta y \Delta z$) and take the limit as $\Delta x, \Delta y, \Delta z \rightarrow 0$

$$\frac{\partial \rho}{\partial t} = -\lim_{\Delta x \to 0} \frac{(\rho v_x)|_{x+\Delta x} - (\rho v_x)|_x}{\Delta x} - \lim_{\Delta y \to 0} \frac{(\rho v_y)|_{y+\Delta y} - (\rho v_y)|_y}{\Delta y} - \lim_{\Delta z \to 0} \frac{(\rho v_z)|_{z+\Delta z} - (\rho v_z)|_z}{\Delta z}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x} \rho v_x + \frac{\partial}{\partial y} \rho v_y + \frac{\partial}{\partial z} \rho v_z \right)}$$

This is the **equation of continuity**

It describes the rate of change of density at a fixed point resulting from the changes in the mass velocity vector ρ**v**.

The equation of continuity in vector symbolism:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \bar{v}$$

ρ**v** vector is the **mass velocity** or **mass flux** vector, $kg/m^2 s$
Λ · ρ**v** scalar is the net rate of mass **efflux** per unit volume (rate of mass loss), $kg/m^3 s$

The equation of continuity simply states that the rate of increase of density within a small volume element fixed in space is equal to the net rate of mass **influx** to the element divided by its volume.

We can modify the equation of continuity by performing the indicated differentiation and collecting all derivatives of ρ on the left hand side.

$$\frac{\partial \rho}{\partial t} + \underbrace{\bar{v}_x \frac{\partial \rho}{\partial x} + \bar{v}_y \frac{\partial \rho}{\partial y} + \bar{v}_z \frac{\partial \rho}{\partial z}}_{\bar{v} \cdot \nabla \rho} = -\rho \underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)}_{\nabla \cdot \bar{v}}$$

$$\Rightarrow \qquad \boxed{\frac{D\rho}{Dt} = -\rho \, \nabla \cdot \bar{v}}$$

The equation of continuity in this form describes the rate of changes of density as seen by an observer "floating along" with the fluid.

For incompressible fluid
ρ = constant | Dρ /Dt = 0 (at steady or unsteady state):

$$\boxed{\nabla \cdot \bar{v} = 0}$$

$$\boxed{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0} \qquad \text{in rectangular coordinates}$$

The main use of the equation of continuity is to simplify the differential momentum (equation of motion) and energy equations, with which it must hold simultaneously in flow problems.

THE EQUATION OF CONTINUITY

Rectangular coordinates (x, y, z):

5 THE OVERALL ENERGY BALANCE

The energy balance equation is also called the **first law of thermodynamics** or the law of conservation of energy. This law seems intuitively obvious today. But it was far from obvious to scientist before about 1800 that the various forms of energy were all manifestations of the same quantity. Furthermore there is no satisfactory, simple definition of energy. The definition can be simple or accurate, but not both. The technically accurate definition is that energy is an abstract quantity, which appears in various forms, which can be converted from one form to another subject to some restrictions, and which appears to be conserved in all energy transitions.

All energy quantities are relative to some arbitrary datum. All energy calculations are based on changes in energy or on energies relative to some arbitrary datum.

The general balance says that

Accumulation = Creation – Destruction + Flow in – Flow out

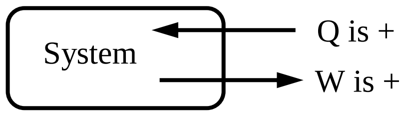
However, since energy can be neither created nor destroyed, the energy balance equation is:

Accumulation = Flow in – Flow out

The 1^{st} law of thermodynamics may be written as:

$\Delta e = Q - W$ where
- e = the total energy per unit mass, J/kg
- Q = the **heat added** to the system per unit mass, J/kg
- W = the **work done** by the system per unit mass, J/kg

Sign convention:



The energy per unit mass, e, possessed by the fluid includes:

(1) Potential energy: gz , J/kg – changes with height in a gravitational field
(2) Kinetic energy: $v^2/2$, J/kg – changes with velocity
(3) Internal energy: u , J/kg – changes with input of heat or work

The work done by the system can be:

(1) Shaft work (W_S): it is purely mechanical and is identified with a rotating shaft crossing the control surface.
A pump adds energy to a volume of fluid passing through it or a turbine removes energy from the flow.
The rate at which shaft work is done **by the system** is
+ when the system does work on a turbine, and
− when a pump does work on a system.

(**– W_S = shaft work done on the fluid by the pump**)

$\boxed{W_S \text{ is in J/kg} \qquad \dot{W}_S \text{ is in J/s}}$

(2) Flow work (or injection work): it is the net work done by the fluid as it flows into and out of the control volume.

This pressure — volume work is pV or $\int_V p\, dV$, J

or in work per unit mass pV/m = p/ρ, J/kg

(3) The work done by the viscous forces over the part of the control surface through which the fluid flows — neglected (that is assume frictionless fluid)

The 1st law of thermodynamics can be written for a **system** (having arbitrary, constant mass) with respect to an observer following the fluid, as

$$\frac{DE}{Dt} = q - \dot{W}$$

where E = the total energy, J
q = the rate of heat added, J/s
$\dot{W}$ = the work rate by the system, J/s

We need to rewrite this energy balance equation for a **control volume**:

$$\left\{\begin{matrix}\text{Rate of energy}\\ \textbf{out}\\ \text{of C.V.}\end{matrix}\right\} - \left\{\begin{matrix}\text{Rate of energy}\\ \textbf{in}\\ \text{into C.V.}\end{matrix}\right\} + \left\{\begin{matrix}\text{Rate of energy}\\ \textbf{accumulation}\\ \text{in C.V.}\end{matrix}\right\} = 0$$

By Reynolds' transport theorem:

$$\frac{DE}{Dt} = \iint_A e\rho\,\bar{v}\cdot\overline{dA} + \frac{\partial E}{\partial t}$$

The rate of energy accumulation within the control volume V:

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\iiint_V e\rho\, dV = \frac{\partial}{\partial t}\iiint_V (u + \frac{v^2}{2} + gz)\rho\, dV \quad , \quad \text{J/s}$$

The net rate of energy efflux from the control volume:

$$\iint_A e\rho\,\bar{v}\cdot\overline{dA} = \iint_A (u + \frac{v^2}{2} + gz)\rho\,\bar{v}\cdot\overline{dA}$$

The work rate by the control volume:

$$\dot{W} = \dot{W}_S + \iint_A \rho\,\frac{p}{\rho}\,\bar{v}\cdot\overline{dA}$$

Therefore, the energy balance equation for a control volume:

$$\frac{DE}{Dt} = q - \dot{W}$$

$$\iint_A (u + \frac{v^2}{2} + gz)\rho\,\bar{v}\cdot dA + \frac{\partial E}{\partial t} = q - \dot{W}_S - \iint_A \rho\,\frac{p}{\rho}\,\bar{v}\cdot\overline{dA}$$

Rearranging: $$\iint_A (u + \frac{v^2}{2} + gz)\rho\, \bar{v} \cdot dA + \iint_A \rho\ \frac{p}{\rho}\ \bar{v} \cdot \overline{dA} + \frac{\partial E}{\partial t} = q - \dot{W}_S$$

$$\iint_A (u + \frac{v^2}{2} + gz + \frac{p}{\rho})\ \rho\, \bar{v} \cdot \overline{dA} + \frac{\partial E}{\partial t} = q - \dot{W}_S$$

recall the definition of enthalpy:
H = U + pV $\Rightarrow$ h = u + (pV/m) = u + p/ρ

$$\iint_A (h + \frac{v^2}{2} + gz)\ \rho\, \bar{v} \cdot \overline{dA} + \frac{\partial E}{\partial t} = q - \dot{W}_S \quad , \quad \text{J/s}$$

Certain simplified forms of this equation are very important.

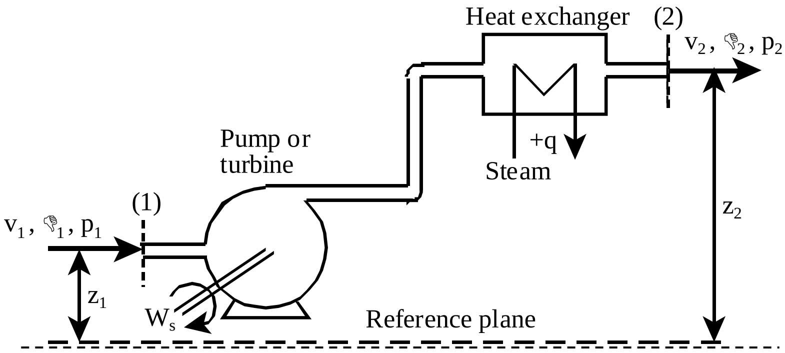
The limitation of the first law of thermodynamics is that it says nothing about the direction in which changes of energy occur. That prediction is made by the second law of thermodynamics.

5.1 Steady-state one-dimensional flow

Define a fixed C.V. encompassing the flow network between points (1) and (2).

Assumptions:

- steady-state (ME /Mt = 0)
- negligible variation of height z, density ρ, and enthalpy h across inlet or outlet area (ρ, z, h = constant through inlet or outlet area)



Note: **v · dA** = v cosα dA
at inlet α = 180°, cosα = !1
at outlet α = 0°, cosα = +1
$\dot{m} = \rho v_{avg} A$

Applying the overall energy balance equation (steady-state):

$$\iint_A (h + \frac{v^2}{2} + gz)\ \rho\, \bar{v} \cdot \overline{dA} = q - \dot{W}_S \quad , \quad \text{J/s}$$

L.H.S.:

$$\iint_A (h + \frac{v^2}{2} + gz)\, \rho\, \bar{v} \cdot dA =$$

$$= \iint_{A_2} (h_2 + \frac{v_2^2}{2} + g z_2)\, \rho_2\, v_2 \cos\alpha_2\, dA + \iint_{A_1} (h_1 + \frac{v_1^2}{2} + g z_1)\, \rho_1\, v_1 \cos\alpha_1\, dA =$$

$$= h_2\, \rho_2 v_2 A_2 - h_1\, \rho_1 v_1 A_1 + \frac{\rho_2}{2} \iint_{A_2} v_2^3\, dA - \frac{\rho_1}{2} \iint_{A_1} v_1^3\, dA + g\, z_2\, \rho_2 v_2 A_2 - g\, z_1\, \rho_1 v_1 A_1 =$$

$$= \dot{m}_2 h_2 - \dot{m}_1 h_1 + \frac{\rho_2}{2} \frac{v_{2,avg} A_2}{v_{2,avg} A_2} \iint_{A_2} v_2^3\, dA - \frac{\rho_1}{2} \frac{v_{1,avg} A_1}{v_{1,avg} A_1} \iint_{A_1} v_1^3\, dA + g\, z_2\, \dot{m}_2 - g\, z_1\, \dot{m}_1 =$$

$$= \dot{m}_2 h_2 - \dot{m}_1 h_1 + \frac{\dot{m}_2}{2} \frac{(v_2^3)_{avg}}{v_{2,avg}} - \frac{\dot{m}_1}{2} \frac{(v_1^3)_{avg}}{v_{1,avg}} + g z_2\, \dot{m}_2 - g z_1\, \dot{m}_1$$

where $\iint_A \frac{v^2}{2} \rho\, v \cos\alpha\, dA = \frac{\rho}{2} \iint_A v^3 dA = \frac{\rho}{2} \frac{v_{avg} A}{v_{avg} A} \iint_A v^3 dA = \frac{\dot{m}}{2} \frac{(v^3)_{\text{avg}}}{v_{avg}}$

Defining $(v^3)_{\text{avg}} \equiv \frac{1}{A} \iint_A v^3 dA$

Replacing $\frac{(v^3)_{\text{avg}}}{v_{avg}}$ by $\frac{v_{avg}^2}{\alpha}$ $\Rightarrow$ $\boxed{\alpha \equiv \frac{v_{avg}^3}{(v^3)_{\text{avg}}}}$

α = kinetic-energy correction factor
α = 0.9 – 0.99 for turbulent flow (precise values of α are seldom known.)
α = 0.50 for laminar flow

For steady state: $\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \Rightarrow \dot{m}_1 = \dot{m}_2 = \dot{m}$

So dividing through the whole equation by $\dot{m}$

$$\boxed{h_2 - h_1 + \frac{(v_{2,avg}^2 - v_{1,avg}^2)}{2\alpha} + g(z_2 - z_1) = Q - W_S \quad , \quad \text{J/kg}}$$

Note! $\dot{W}_S = \dot{m} W_S$, where $\dot{W}_S$ = work rate, J/s
W_S = work per unit mass, J/kg

$$\boxed{\Delta h = h_2 - h_1 = \int_{T_1}^{T_2} c_p\, dT = c_p \Delta T}$$

6 THE OVERALL MECHANICAL ENERGY BALANCE; BERNOULLI EQUATION

6.1 The overall mechanical energy balance equation

A more useful type of energy balance for flowing fluids, especially liquids, is a modification of the total energy balance to deal with mechanical energy.

It includes:
- the work term,
- the kinetic energy,
- the potential energy, and
- the flow work part of the enthalpy term (the work done by the fluid as it flows into and out of the control volume).

Mechanical energy is either work or can be directly converted into work almost completely. Energy converted to heat or internal energy is lost work or a loss in mechanical energy, which is caused by frictional resistance to flow.

The overall energy balance equation for steady-state is:

$$\iint_A \left(u + \frac{v^2}{2} + gz + \frac{p}{\rho}\right) \rho\, \bar{v} \cdot dA = q - \dot{W}_S \quad , \quad \text{J/s}$$

This equation applies to the changes from one point to the next along the direction of flow in any steady flow of a homogeneous fluid.

The first law of thermodynamics for this case is:

$$\Delta U = \hat{Q} - \hat{W}_{flow\,work} = \hat{Q} - \int_{V_1}^{V_2} p\, dV \quad , \quad \text{J}$$

Recall the definition of enthalpy: H ≡ U + pV , J

In differential form:

$$\Delta H = \Delta U + \Delta(pV) = \Delta \mathrm{U} + \int_{V_1}^{V_2} p\, dV + \int_{p_1}^{p_2} V\, dp = \hat{Q} - \cancel{\int_{V_1}^{V_2} p\, dV} + \cancel{\int_{V_1}^{V_2} p\, dV} + \int_{p_1}^{p_2} V\, dp$$

$$\Rightarrow \quad \Delta H = \hat{Q} + \int_{p_1}^{p_2} V\, dp \quad , \quad \text{J}$$

$$\Rightarrow \quad \Delta h = Q + \int_{p_1}^{p_2} \frac{1}{\rho}\, dp \quad , \quad \text{J/kg}$$ for a unit mass

substituting Δh to the overall energy balance equation:

$$\Delta h + \frac{\Delta v^2}{2\alpha} + g\Delta z = Q - W_S \quad , \quad \text{J/kg}$$

$$\cancel{Q} + \int_{p_1}^{p_2} \frac{dp}{\rho} + g\Delta z + \frac{\Delta v^2}{2\alpha} = \cancel{Q} - W_S$$

$$\Rightarrow \quad \int_{p_1}^{p_2} \frac{dp}{\rho} + g\Delta z + \frac{\Delta v^2}{2\alpha} = -W_S \quad , \quad \text{J/kg}$$

6.2 Incompressible fluid (ρ = const)

The overall mechanical energy balance equation for incompressible fluids is:

$$\frac{\Delta p}{\rho} + g\,\Delta z + \frac{\Delta v^2}{2\alpha} = -W_S \quad , \quad \text{J/kg}$$

or

$$\frac{p_2 - p_1}{\rho} + (z_2 - z_1)\text{g} + \frac{v_2^2 - v_1^2}{2\alpha} + W_S = 0 \quad , \quad \text{J/kg}$$

since $\int_{p_1}^{p_2} \frac{dp}{\rho} = \frac{\Delta p}{\rho}$ *where* $\Delta p = p_2 - p_1$

6.3 Bernoulli equation (W$_s$ = 0) **(1738)**

$$\boxed{\frac{p_1}{\rho} + z_1 g + \frac{v_1^2}{2} = \frac{p_2}{\rho} + z_2 g + \frac{v_2^2}{2} = \text{constant} \quad , \quad \text{J/kg} = \text{m}^2/\text{s}^2}$$

Assumptions:

- steady-state
- incompressible fluid (ρ = constant)
- ideal fluid (μ = 0 — no friction, therefore no momentum transport to the wall)
- isothermal fluid (Q = 0)
- no shaft work ($W_S = 0$)
- α = 1 (turbulent flow)
- It applies to a streamline, but it may also be used for streamtube. By the use of correction factors (α , h_f) it can be used where velocity variations within a cross section occur and there are friction effects.

The Bernoulli equation shows that when the velocity increases, it does so only at the expense of height or pressure. If the height is changed, compensation must be found in a change of either pressure or velocity. The Bernoulli equation is often used in conjunction with the overall mass balance equation for steady state ($\dot{m} = \rho_1 v_1 A_1 = \rho_2 v_2 A_2$).

The head form of Bernoulli equation:

$$\boxed{\frac{p_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{v_2^2}{2g} = \text{constant} \quad , \quad \text{m}}$$

All the quantities appearing in this form have the dimension of length and are known as **head**s. These heads can be regarded as quantities of energy per unit weight of the fluid.

$p/\rho g$	**pressure head**, m
z	**static head** or **elevation head**, m
$v^2/2g$	**velocity head** or **dynamic head**, m
$p/\rho g + z + v^2/2g$	**total head**, m (it is constant along any streamlines)
$p/\rho g + z$	**piezometric head**, m

Even though Bernoulli equation is valid for incompressible fluids, it can be applied for gases at velocities below 60 m/s with very small errors. Assuming incompressible flow of a gas at velocities as high as 200 m/s cause only about 5% error in calculating gas velocity from Bernoulli equation.

6.4 Fluid friction

In any real flow of the fluid viscosity tends to resist flow, thereby resulting in a transfer of momentum to the wall of the channel. The work done by shear forces in maintaining the velocity gradient is eventually converted into heat by viscous action. This heat is absorbed by the fluid itself.

Fluid friction can be defined as any conversion of mechanical energy into heat in a flowing stream. Because of this energy conversion, friction loss is also called **friction heating**. Energy converted to heat is lost work. Friction manifests itself by the disappearance (**dissipation**) of mechanical energy.

Therefore, the R.H.S. of the Bernoulli equation will be less than the L.H.S. This is corrected by adding a term h_f to the R.H.S. This term represents all the friction generated per unit mass of fluid, and therefore all the conversion of mechanical energy into heat, that occurs in the fluid between station 1 and 2.

$$\frac{p_1}{\rho} + z_1 g + \frac{v_1^2}{2} = \frac{p_2}{\rho} + z_2 g + \frac{v_2^2}{2} + h_f \quad , \quad \text{J/kg} = \text{m}^2/\text{s}^2$$

where **h_f = friction loss, J/kg**

The mechanical energy balance equation becomes:

$$\frac{p_2 - p_1}{\rho} + (z_2 - z_1)\text{g} + \frac{v_2^2 - v_1^2}{2\alpha} + W_S + h_f = 0 \quad , \quad \text{J/kg}$$

6.5 Pump work

A pump is used in a flow system to increase the mechanical energy of the flowing fluid, the increase being used to maintain flow. The value of h_f friction loss depends on the detailed nature of the flow. It can be determined from theory for flow through conduits, but not for flow through a pump.

Friction occurring within the pump is due to fluid friction and mechanical friction as well. The mechanical energy supplied to the pump as **negative shaft work** must be discounted by these frictional losses to give the net mechanical energy actually available to the flowing fluid.

It is customary to define pump efficiency η_p instead of using $h_{f\ pump}$.

$$\boxed{\eta_P \equiv \frac{\text{useful energy (given to fluid)}}{\text{total energy (supplied to pump)}} = \frac{W_P - h_{f\ pump}}{W_P}}$$

where $h_{f\ pump}$ = total friction loss in pump per unit mass of fluid, J/kg — cannot be determined

η_p = efficiency of pump, $\eta < 1$

W_p = energy supplied to pump, J/kg

W_p ! $h_{f\ pump}$ = energy delivered to fluid by pump = !W_s shaft work (the shaft work is negative for pumps), J/kg

From the definition of efficiency: $\eta_P W_P = W_P - h_{f\ pump} = -\dot{W}_S$

$$\Rightarrow \quad \boxed{\dot{W}_S = -\eta_P W_P}$$

Therefore, the Bernoulli equation corrected for fluid friction and pump work is:

$$\boxed{\frac{p_1}{\rho} + z_1 g + \frac{v_1^2}{2\alpha} + \eta_P W_P = \frac{p_2}{\rho} + z_2 g + \frac{v_2^2}{2\alpha} + h_f \quad , \quad \text{J/kg}}$$

7 FLOW MEASUREMENTS

In industrial processing plants it is often important to measure and control the pressure or the liquid level in a vessel. Also since many fluids are flowing in a pipe, it is necessary to measure the rate at which the fluid is flowing. Many of these flow meters depend on devices to measure a pressure or pressure difference.

Several important types of fluid-flow measuring devices are based on the Bernoulli equation. When the friction effects in these devices become significant, they are normally accounted for by introducing empirical coefficients and retaining the frictionless form of the Bernoulli equation, rather than by introducing the friction loss term into Bernoulli equation.

7.1 Pressure

Pressure along with velocity is one of the main parameters in fluid dynamics. It is defined as compressive stress (that is compressive force per unit area) at any point in a fluid at rest. In a stationary fluid, the pressure is the same in all direction and is referred to as **static pressure**. In a moving fluid, the static pressure is exerted on any plane parallel to the direction of motion. The pressure exerted on a plane perpendicular to the direction of flow is greater than the static pressure because the surface has, in addition, to exert sufficient force to bring the fluid to rest. This additional pressure is proportional to the kinetic energy of the fluid; it cannot be measured independently of the static pressure.

We differentiate between absolute pressure and gauge pressure. **Absolute pressure** is a pressure relative to zero pressure. **Gauge pressure** is a pressure relative to the local atmospheric pressure. Both systems of measurements are in common use. Gauge pressures are not used in the SI system.

Pressure scales

Absolute pressure		Relative or gage pressure
135.8 kPa = 19.7 psia		34.5 kPa = 5 psig ($p_{gage} = p_{abs} - p_{atm}$)
101.3 kPa = 14.7 psia	Atmospheric pressure (not a fixed reference datum)	0 kPa = 0 psig
60 kPa = 8.7 psia	Vacuum	−41.4 kPa = −6 psig
0 kPa = 0 psia	Absolute zero pressure	−101.3 kPa = −14.7 psig

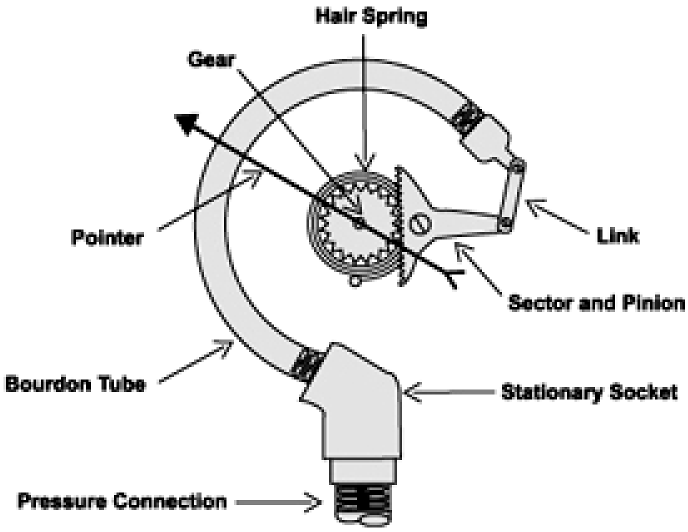
10^5 Pa = 1 bar (for pressures of gases)
1 atm = 101.325 kPa
1 atm = 760 mmHg

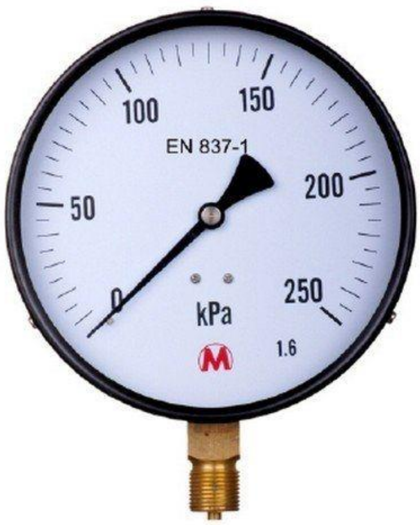
1 atm = 14.7 psia
1 atm ≅ 1bar
1 bar = 100 kPa
1 psi = 6894.7 Pa

Pressures are measured by letting them act across some area and opposing them with either a gravity force or the force of a compressed spring. The gravity-force method uses a device called a manometer.

Bourdon pressure gauge

The most common pressure-measuring device is the mechanical Bourdon-tube pressure gauge. A coiled hollow tube in the gauge tends to straighten out when subjected to internal pressure, and the degree of straightening depends on the pressure difference between the inside and outside pressure. The tube is connected to a pointer on a calibrated dial. The pressure indicated is the difference between that communicated by the system to the tube and the ambient pressure, and this is usually referred to as gauge pressure.





7.2 Manometers

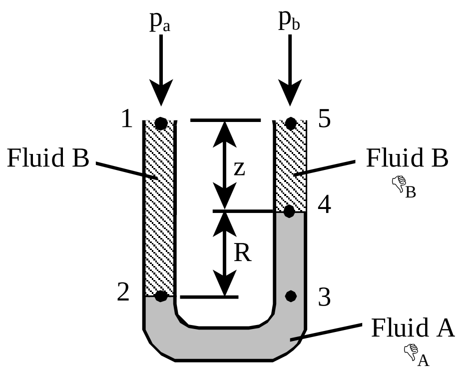
Manometers consist of 1 or more vertical or inclined tubes, usually of glass which are connected to a pipe or tank in which the pressure is to be measured. By measuring the liquid level or levels in these columns and applying the hydrostatic equations, the desired static pressures may be obtained.

The total mass of fluid is $m = \rho hA$, kg
The total force F of fluid on area A due to fluid only is $F = \rho hAg$, N
The pressure is $p = F / A = \rho hg$, Pa

The total pressure on A is the sum of the pressures on A due to the mass of fluid above and the pressure on top of the fluid (p_o) $p_{total} = \rho hg + p_o$, Pa
The pressure difference between 2 points is $\Delta p = p_2 ! p_1 = (h_2 ! h_1)\rho g$

Since is the vertical height of a fluid that determines the pressure in a fluid, the shape of the vessel does not affect the pressure.

Simple U-tube manometer



Pressure p_a is exerted on one arm (or limb) of the U-tube and p_b on the other arm ($p_a > p_b$). Both pressures p_a and p_b could be pressure taps from a fluid meter, or p_a could be a pressure tap and p_b the atmospheric pressure.

The top of the manometer is filled with liquid B, and the bottom with a more dense fluid A. Liquid A is immiscible with B. (Remember that the pressure is the same at all points at a given elevation in a single fluid.)

The pressures:

at point 1 $\quad p_1 = p_a$

at point 2 $\quad p_2 = p_a + \rho_B\, g\,(z + R)$ $\qquad$ where $\qquad$ R = reading of the manometer, m

at point 3 $\quad p_2 = p_3$

$$p_3 = p_b + z\,\rho_B\, g + R\,\rho_A\, g$$

$$\Rightarrow \quad p_a + \rho_B\, g\,(z + R) = p_b + z\,\rho_B\, g + R\,\rho_A\, g$$

$$p_a \,!\, p_b = R\,(\rho_A \,!\, \rho_B)\, g, \quad \text{Pa}$$

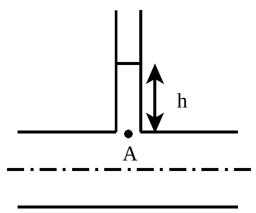
The simple way to work all manometer problems is to start with some pressure we know and work step by step to the pressure we want to find.

Very often pressures are reported in terms of manometer readings as heights (mm Hg, mm H_2O, inches of Hg and inches of H_2O).

At no place in the calculation does the cross-sectional area of the manometer tube enter. Therefore, this tube can be of any convenient size and need not be made of constant-diameter tubing. The only measurements necessary are the fluid densities, which can be looked up in handbooks, and the difference in elevation, which can be read directly with rulers. Thus, manometers require neither calibration nor testing with standards. **There is no requirement that the tubes be vertical, only that we can read the vertical distance between the horizontal liquid surfaces**.

Neither the Bourdon-tube gauge nor the manometer is suited to measuring rapidly changing pressures. For rapidly changing pressures diaphragm gauge or quartz-crystal piezometer gauge can be used.

7.3 Piezometer



A piezometer is a simple pressure-measuring device. The pressure at point A can be found simply by measuring the height h of the fluid and using the hydrostatic pressure equation:

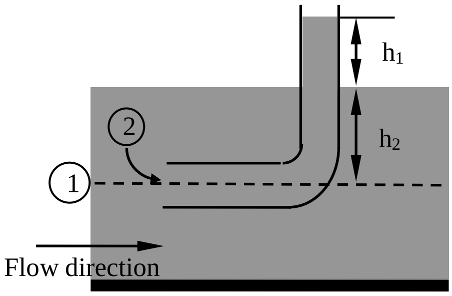
$$p_A = \rho g h + p_{atm}$$

The velocity at A is zero.

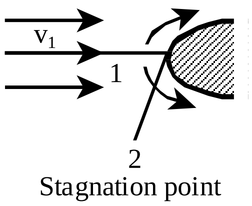
The open piezometer tube is cumbersome for use with liquids under high pressure and cannot be used for gases. Instead, U-tube manometer is used.

7.4 Pitot tube

The Pitot tube is a device to measure the local velocity at a given point in the flow stream. Pitot developed it in 1732. The simplest Pitot tube is also called an **impact tube**. It consists of a bent, transparent tube with one vertical leg projecting out of the flow and another leg pointing directly upstream in the flow.



The fluid flows in an open channel. At location 1 the flow is undisturbed by the presence of the tube and hence has the velocity that would exist at location 2 if the tube were not present. At location 2 the flow has been stopped by the tube that has been inserted. This point is called stagnation point.



Stagnation point is any point in the flow field where the velocity is brought to zero.

The Bernoulli equation between point 1 and 2:

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} = \frac{p_2}{\rho} + h_f \qquad \Rightarrow \qquad p_2 = p_1 + \frac{\rho v_1^2}{2} \quad (\text{static pr.} + \text{dynamic pr.})$$

where
p_1 = static pressure
p_2 = stagnation pressure or impact pressure (pressure at the stagnation point)
$\rho v_1^2/2$ = dynamic pressure
h_f = friction loss, J/kg — neglected

$$v_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho}}$$

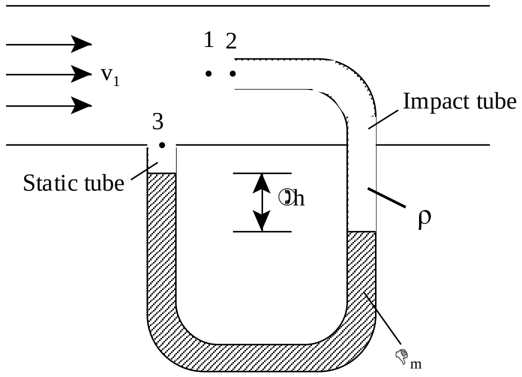
but $p_2 = p_{atm} + \rho g(h_1 + h_2)$
$p_1 = p_{atm} + \rho g\, h_2$

$$v_1 = \sqrt{2gh_1}$$

It has been found experimentally, that the friction loss is less than 1% of the total so it may be ignored. The device, exactly as shown above is used for finding velocities at various points in open-channel flow and for determining velocities for boats. It is not suitable for flow of the atmosphere or flow in pipes.

Pitot-static tube

For pipe flow measurements the impact tube is combined with a second tube, called a **static tube**. This combination is often simply called a Pitot tube. The 2 tubes are connected to the legs of a manometer or equivalent device for measuring small pressure differences. The velocity of flow is calculated from the difference in levels of the fluid in the manometer (Δh).



Simple Pitot tube

The fluid flows into the opening on the Pitot tube at point 2, pressure builds up, and then remains constant. The static pressure is measured by the static tube ($p_1 = p_3$ = static pressure). The difference between p_1 and p_2 represents the pressure rise associated with the deceleration of the fluid. The manometer measures this small pressure rise.

For incompressible fluid we apply the Bernoulli equation:

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} = \frac{p_2}{\rho} \quad \Rightarrow \quad v_1 = C_P \sqrt{\frac{2\,(p_2 - p_1)}{\rho}}$$

where C_p = a dimensionless coefficient to take into account deviations. It should be determined by calibration of the Pitot tube. It is between 0.98 ! 1.

The pressure drop p_2 ! p_1 is related to Δh manometer reading:

$$\boxed{\Delta p = \Delta h\,(\rho_m - \rho)\,g}$$

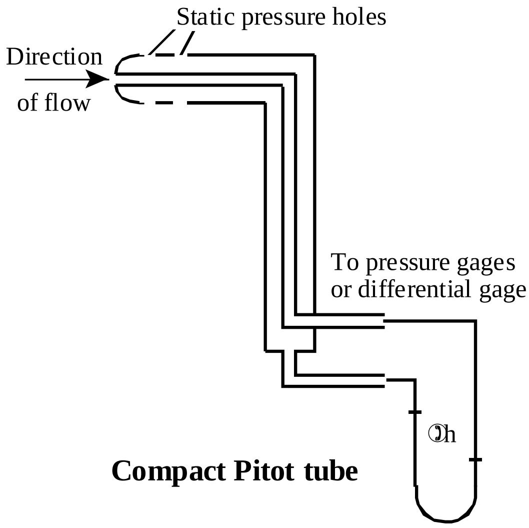
$$\boxed{v = C_P \sqrt{\frac{2g\,(\rho_{\mathrm{m}} - \rho)\Delta \mathrm{h}}{\rho}}}$$

Since the Pitot tube measures velocity at one point only in the flow, several methods can be used to obtain the average velocity in the pipe:

- measure velocity at the exact center of the tube to obtain v_{max} and calculate v_{avg},
- measure velocity at several known positions in the pipe cross section and use

$$v_{avg} = \frac{1}{A} \iint_A v\, dA \qquad \text{graphical integration}$$

In addition to the simple Pitot tube, there exist many modifications.

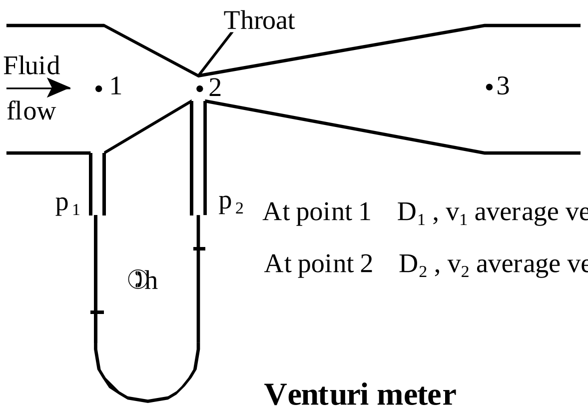


Compact Pitot tube

The compact Pitot tube is the standard device for measuring the air speed of airplanes and is often used for measuring the local velocity in pipes and ducts, particularly in air pollution sampling procedures. [One can easily identify the Pitot probes on airplanes. Multiengine planes have them near the nose, at the side below the pilot's window. Single-engine propeller planes place the probe below the wing, far enough out from the center not to be influenced by the propeller.]

7.5 Venturi meter

A Venturi meter is constructed from a converging section (of angle 15-20°), a short cylindrical portion (throat), and a long truncated cone (of angle 5-7°). The shape of the converging and diverging sections of the Venturi meter minimizes losses by eddy formation. It is usually inserted directly into a pipeline. A manometer or other device is connected to the two pressure taps, one upstream and one at the throat, and measures the pressure difference.



At point 1 D_1 , v_1 average velocity

At point 2 D_2 , v_2 average velocity

Venturi meter

The pressure drop between point 1 and 2 is $\Delta p = p_1 - p_2$.

In a Venturi meter, the velocity is increased, and the pressure decreased, in the upstream cone ($p_1 > p_2$). The pressure drop is utilized to measure the rate of flow through the instrument. The basic equation for the Venturi meter is obtained by using the Bernoulli equation for incompressible fluids between point 1 and 2. Friction is neglected and the meter is assumed to be horizontal.

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} = \frac{p_2}{\rho} + \frac{v_2^2}{2}$$

From the overall mass balance: $\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \quad \Rightarrow \quad v_1 = v_2 \frac{A_2}{A_1} = v_2 \left(\frac{D_2}{D_1}\right)^2$

Combining these 2 equations and eliminating v_1:

$$v_2^2 - v_1^2 = \frac{2\,(p_1 - p_2)}{\rho}$$

$$v_2^2 \left[1 - \left(\frac{D_2}{D_1}\right)^4\right] = \frac{2\,(p_1 - p_2)}{\rho}$$

$$v_2 = \frac{1}{\sqrt{1-(D_2/D_1)^4}} \sqrt{\frac{2\,(p_1 - p_2)}{\rho}}$$

To account for the small friction loss an experimental coefficient C_v is introduced:

$$\boxed{v_2 = \frac{C_V}{\sqrt{1-(D_2/D_1)^4}} \sqrt{\frac{2\,(p_1 - p_2)}{\rho}}}$$

The Venturi coefficient depends only on the Reynolds number.

For Re > 10^4 at point 1 $\quad C_v \approx 0.98$ for $D < 0.2$ m pipes

$C_v \approx 0.99$ for $D > 0.2$ m pipes

However, these coefficients can vary and individual calibration is recommended if the manufacturer's calibration is not available.

The friction loss from point 1 to 3 is about 10 % of $p_1 - p_2$.

A Venturi meter is often used to measure flows in large pipelines. Normally these meters are designed to operate at high velocities. Their typical accuracy is 1% of full scale.

Disadvantages of Venturi meters:

- it is expensive,
- it occupies considerable space,
- its ratio of throat diameter to pipe diameter (D_2 / D_1) cannot be changed so if the flow rate is changed considerably, the throat diameter may be too large to give an accurate reading or too small to accommodate the next maximum flow rate.

7.6 Orifice meter

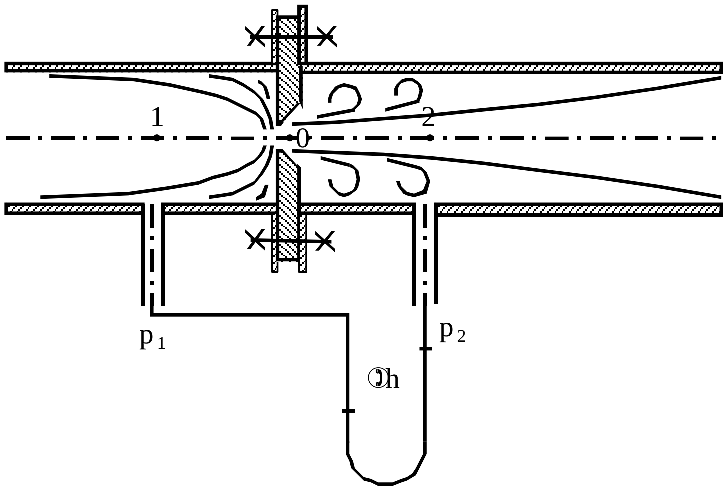
The orifice meter overcomes the disadvantages of Venturi meter at a price of a much larger head or power cost. It operates on the same principle as the Venturi meter but with some important differences:

- the orifice plate can easily be changed to accommodate widely different flow rates,
- it has a large permanent pressure loss because of the presence of eddies on the downstream side of the plate.

It consists of an accurately machined and drilled plate mounted between 2 flanges with the hole concentric with the pipe in which it is mounted. The positions of the 2 pressure taps are arbitrary, and the coefficient of the meter will depend on the position of the taps.
Because of the sharpness of the orifice, the fluid stream separates from the downstream side of the orifice plate and forms a free-flowing jet in the downstream fluid. A **vena contracta** forms. Vena contracta is the point where the flow lines reach a minimum cross sectional area. The jet is not under the control of solid walls there.

There are 3 customary positions for the pressure taps:

- Flange taps: the holes are located 1-in (2.54 cm) upstream and 1-in downstream from the faces of the orifice plate.
- Vena contracta taps: the upstream tap is 1 pipe diameter from the plate, and the downstream tap at the point of minimum pressure (0.3-0.8 pipe diameter depending on D_o / D_1).
- Pipe taps: the upstream tap is 2½ pipe diameter from the orifice plate, and the downstream tap 8 pipe diameters from the plate. The downstream tap is at or beyond the point of maximum downstream pressure.



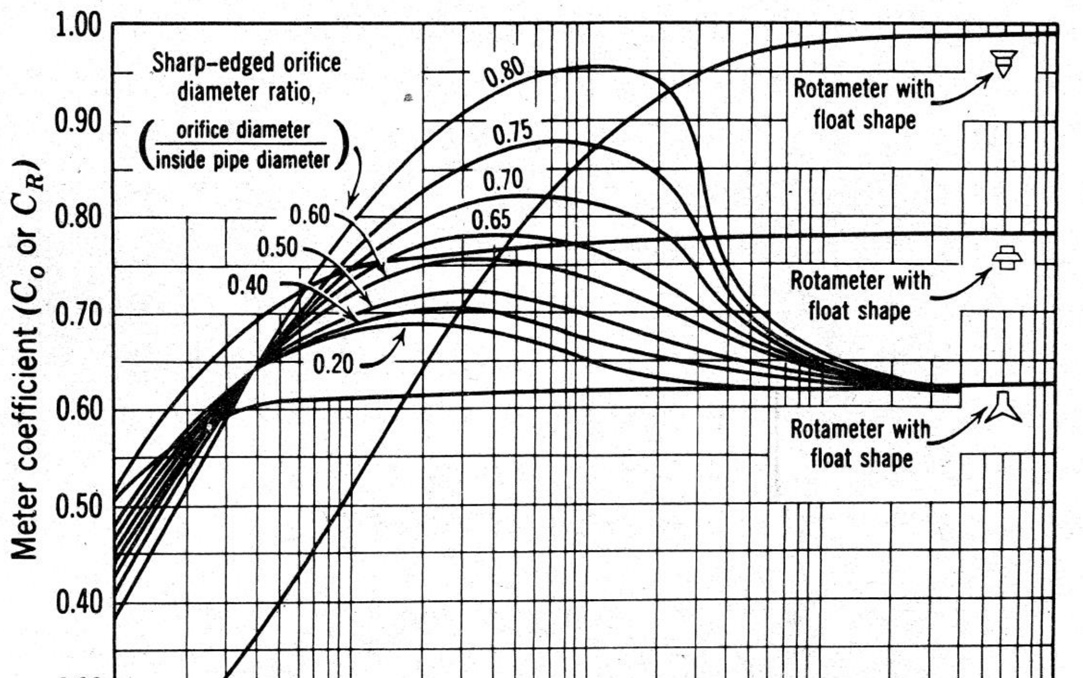
Sharp-edged orifice meter

The velocity of the jet at the downstream tap is not easily determinable, and the velocity of the jet at the downstream tap is not easily related to the diameter of the orifice. Therefore, the orifice coefficients, C_o, are more empirical than those for the Venturi. It is always determined experimentally. It varies considerably with changes in D_o/D_1 and with Reynolds number especially below 10 000.

If Re > 20 000 and $D_o/D_1 < 0.5$, then $C_o = 0.61$ can be used.

$$\boxed{v_o = \frac{C_o}{\sqrt{1-(D_o/D_1)^4}} \sqrt{\frac{2\,(p_1 - p_2)}{\rho}}}$$

The orifice meter coefficient can be read from the following chart:

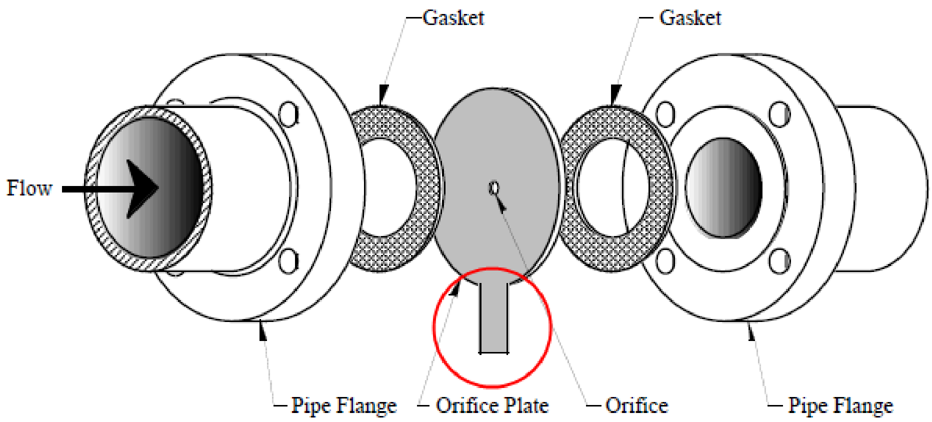


The permanent pressure loss caused by an orifice is high because of the large friction losses from the eddies generated by the reexpanding jet below the vena contracta. This loss depends on D_o / D_1.

If $D_o / D_1 = 0.5$, the loss is 73 % of $p_1 - p_2$
$D_o / D_1 = 0.65$, the loss is 56 % of $p_1 - p_2$
$D_o / D_1 = 0.8$, the loss is 38 % of $p_1 - p_2$.

Typical accuracy is 2 - 4% of full scale.

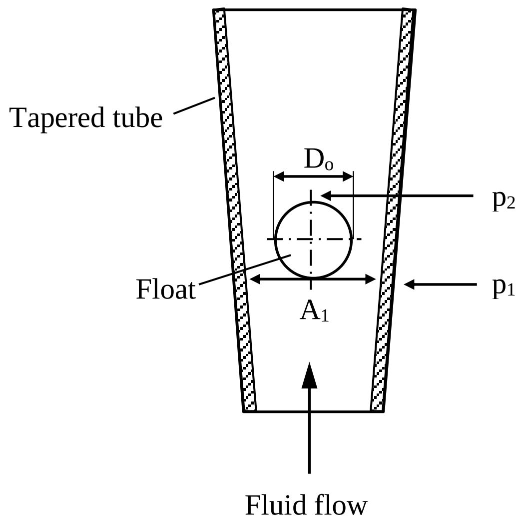
The accuracy of the flow measuring devices is very much affected by uniformity of the approaching fluid flow. Therefore, ideally there should be a straight length of piping before the flow-measuring device. It is generally accepted that for accurate flow readings there should be 50 pipe diameters of straight piping before the metering device following any pipe bend, valve, tee, reducer etc. The relevant standard provides a range of recommended minimum straight lengths depending on the nearest upstream fittings varying from 5 - 44 lengths.



Orifice plate and flange assembly

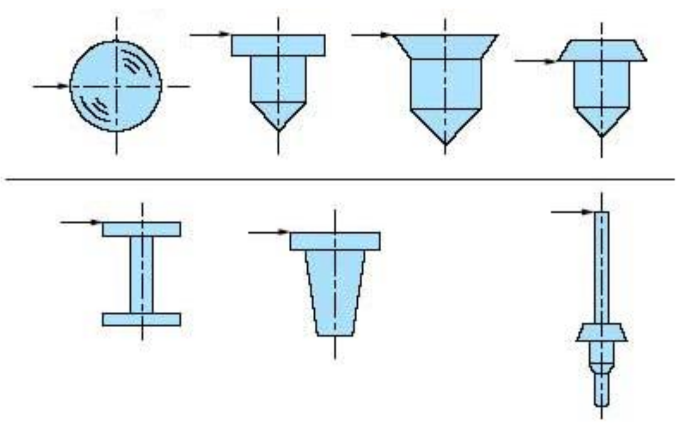
7.7 Rotameter

The previous devices discussed use a fix geometry and read a pressure difference that is proportional to the square of the volumetric flow rate. A rotameter uses a fix pressure difference, and a variable geometry, which is a simple function of the volumetric flow rate. It belongs to a class of meters called **variable area meters**.



A rotameter consists of a **tapered tube**, typically made of glass, with a **float** inside that is pushed up by flow and pulled down by gravity. The float is heavier than the fluid it replaces. It can have grooves that make it rotate in the fluid stream. The elevation of the float depends on the velocity and on the density and viscosity of the fluid. At a higher flow rate more annular area (between the float and the tube) is needed to accommodate the flow, so the float rises. When the flow rate is constant, the float stays in one position. That position is indicated on a graduated scale.

Floats are made in many different shapes, with spheres and spherical ellipses being the most common for the smallest flows and more complex float designs are used for larger flow rates. The float is shaped so that it rotates axially as the fluid passes. This allows you to tell if the float is stuck since it will only rotate if it is free. Readings are usually taken from the top of the float.



Different floats and their reading positions

Standard rotameters are produced for connection to pipes with diameters between 3-150 mm. The corresponding capacities for water are from 0.0003 m^3/h (= 0.3 P/h) to 200 m^3/h. The corresponding values for the flow of air at atmospheric pressure and room temperature are 0.005 m^3/h and 17 000 m^3/h respectively.

For a steady upward flow, when the ball is not moving we can make a force balance around the ball:

$$F_{gr} + F_{p2} = F_{p1} + F_b$$

$F_{gr} = \rho_o V_o g$ = gravity force
$F_{p2} = p_2 A_o$ = pressure force directly above the ball
$F_{p1} = p_1 A_o$ = pressure force directly under the ball
$F_b = \rho V_o g$ = buoyant force
D_o = diameter of ball
$V_o = \frac{\pi D_o^3}{6}$ = volume of ball

$$A_o = \frac{\pi D_o^2}{4} \quad = \quad \text{maximum cross section of ball}$$

ρ_o = density of ball
ρ = density of fluid

$$A_o (p_1 - p_2) = V_o (\rho_o - \rho) g$$

$$p_1 - p_2 = \frac{V_o (\rho_o - \rho) g}{A_o}$$

From Bernoulli equation:

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} = \frac{p_2}{\rho} + \frac{v_2^2}{2}$$

$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2)$$

from overall mass balance

$Q = v_1 A_1 = v_2 A_2$

where Q = volumetric flow rate
A_1 = cross section of rotameter directly under float
A_2 = cross section of float clearance (annular space between the wall and the ball)

$$\Rightarrow \quad v_1 = \frac{Q}{A_1} \qquad v_2 = \frac{Q}{A_2}$$

$$p_1 - p_2 = \frac{\rho}{2}\left[\left(\frac{Q}{A_2}\right)^2 - \left(\frac{Q}{A_1}\right)^2\right]$$

$$Q = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{\frac{2(p_1 - p_2)}{\rho}}$$

substituting $p_1 - p_2$ from force balance

$$Q = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{\frac{2g(\rho_o - \rho)}{\rho} \frac{V_o}{A_o}}$$

Accounting for frictional losses:

$$Q = C_R \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{\frac{2g(\rho_o - \rho)}{\rho} \frac{V_o}{A_o}}$$

A_1 and A_2 is determined by the position of the ball. C_R is a function of Re. So the above can be summarized as:

$$Q = C' \sqrt{\frac{2g\,(\rho_o - \rho)}{\rho} V_o} \quad or \quad Q = k\,A_2 \sqrt{\frac{2g\,(\rho_o - \rho)}{\rho}}$$

This is the basic equation for all cases. It shows that **the float response to flow rate changes is linear**.