KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI – GHANA

INSTITUTE OF DISTANCE LEARNING

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MATH 251: ENGINEERING MATHEMATICS IV

(CREDITS: 4)

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1. Icons
The following icons have been used to give readers a quick access to where similar information may be found in the text of this course material. Writer may use them as and when necessary in their writing. Facilitator and learners should take note of them.

Icon #1  Learning Objective	Icon #2  Activity	Icon #3  Introduction	Icon #4  Information	Icon #5 Summary
Icon #6 Time For Activity	Icon #7  Self Assessment	Icon #8 Group Discussion	Icon #9  Read	Icon #10 New Terms
Icon #11 Well Done	Icon #12  Note/Learning Tip	Icon #13  Pause	Icon #14  Question	Icon #15 Online

2. Guidelines for making use of e-learning support

This course material is also available online at the virtual classroom (v-classroom) Learning Management System. You may access it at www.kvcit.org

Course Writer

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He entered the university in 1997 for undergraduate degree and completed in 2001. After National Service in 2002, he enrolled for his MSc. Mathematics programme. He completed his MSc. Thesis in 2004 and defended in the same year.

He was a demonstrator in the university from 2002 to 2004. He was employed as a Teaching Assistant from 2004 to 2005. From 2005 to date, he has been teaching at his alma mater, Kwame Nkrumah University of Science and Technology, Kumasi at the Department of Mathematics as a Lecturer.

Course Introduction

Differential Equation is among the lynchpins of modern mathematics, which along with matrices, are essential for analyzing and solving complex problems in engineering, the natural sciences, economics, actuarial sciences, and even business.

In the first section, we begin with general differential equation and classifying them in terms of type, order, degree and linearity. We then narrowed to one type of differential equation, which is ordinary differential equation.

We move on to look at first order ordinary differential equation types, put them in classes and find an appropriate methods of solving most of them, if not all, and its applications to practical problems.

We then consider higher order linear ordinary differential equation, that is, orders greater than one. Initially, we concentrate on second – order homogeneous linear ordinary differential equation with constant coefficient because their theory is typical of that of linear differential equations of any order n (but involves much simpler formulas), so that the transition to higher order n needs only very few new ideas. The Wronskian is used to determine the independences of solutions to a particular equation. Elementary methods such as undetermined coefficient, variation of parameter and linear differential operator are used to solve second – order non – homogeneous linear ordinary differential equations with constant coefficient.

Systems of differential equations occur in various applications. Linear systems are best treated by the use of matrices and vectors, of which, however, only modest knowledge will be needed here. We will deal with first – order linear systems with constant coefficient and draw some analogies from the first order ordinary differential equation.

Course Objectives

On completion of the course, it is hoped that you would be able to:

- Classify first order ordinary differential equations in terms of type, order, degree and linearity.
- Have the technique to solve most, if not all, first order differential equations
- Be able to solve differential equation of higher order with constant coefficient using the appropriate technique for each kind of higher order equations.
- Apply the technique in solving the differential equation in solving differential equation appears in actuarial science, economics and finance.
- Reduce higher order linear differential equations to system of first order differential equations and solve.
- Transform system of equation to normal form and put it in matrix and vector to solve using eigenvalues and eigenvectors.

Course Outline

Unit 1 Basic Concepts:

- o Definition of Differential Equation
- o Classification of Differential Equations
- o Solution of a Differential Equation
- o General Solution
- o Initial – Value and Boundary – Value Problems

Unit 2 Ordinary Differential Equation of First Order:

- o General and Particular Solution of a First – Order Differential Equation.
  - Method Separation of Variables
  - Reduction to Separable Form
  - Exact Equation
  - Integrating Factors
    - Linear Equations and Those Reducible to that form
    - Linear Equations
    - Reducible to Linear Forms by change of Variable
    - Bernoulli’s Equation

            - Riccati's Equation
    - Applications of First – Order Ordinary Differential Equation

Unit 3 <u>Linear Differential Equation of Higher Order</u>:

- Homogenous Equation with constant coefficient
- Theory of solution of Linear Differential Equation
    - Independent and Dependent function
    - Wronskian
- Linear Non – Homogeneous with constant coefficients
    - Method of Undetermined Coefficient
    - Variation of Parameter
    - Linear Differential Operator

Unit 4 <u>LAPLACE TRNSFORM:</u>

- Definition of Laplace Transform
- Find Laplace Inverse
- D
    - The Differential Operator
    - Reduction of n th Order Equation to a System of First – Order Equations.
    - The Matrix Method
        - The Eigenvalue Method of Homogeneous Linear System
        - Non – Homogeneous Linear System.

Mode of Assessment

This would involve a combination of continuous assessment (30%) and end of semester examination (70%). The continuous assessment would include handed in exercises and mid-semester examination.

Test your understanding

Within each unit/session are exercises. These are meant to help you assess your understanding of the unit or course. It is vital that you take time to complete or find solutions to these exercises as they occur in the study material. Make sure you go through them. I recommend you have a notebook/exercise book specifically for this purpose and keep it with your study materials as record of your work.

References

1. Frank Ayers, JR.: *Theory and Problems of Differential Equations*, McGraw – Hill Inc.
2. Earl D. Rainville and Philip E. Bedient: *Elementary Differential Equations*, The Macmillan Company
3. Richard Bronson and Gabriel Costa (2006):*Differential Equation* (Third Edition), McGraw – Hill Inc.
4. Erwin Kreyszig(2002): *Advanced Engineering Mathematics* (Eighth Edition), John Wiley and Sons, Inc.
5. William E. Boyce and Richard C. DiPrima: *Elementary Differential Equations and Boundary Value Problems*, John Wiley and Sons, Inc.
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UNIT 1

BASIC CONCEPTS

Introduction

The laws of the universe are written largely in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting naturally phenomena involve change and are best described by equations that relate changing quantities.

Many important and significant problems in engineering, the physical sciences, and the social sciences such as economics and business when formulated in mathematical terms require the determination of a function satisfying an equation containing the derivatives of unknown function. Such equations are called *differential equation*. Perhaps the most familiar example is Newton's law:

$$m\frac{d^2x}{dt^2}=F \qquad (1)$$

for the position $x(t)$ of a particle acted on by a force F. In general F will be a function of time t, the position x, and the velocity dx/dt. To determine the motion of a particle acted on by a given force F it is necessary to find a function $x(t)$ satisfying Eq. (1). If the force is that due to gravity, then $F=-mg$ and

$$m\frac{d^2x}{dt^2}=-mg \qquad (2)$$

On integrating Eq. (2) we have:

$$\frac{dx}{dt}=-gt+c_1$$

$$x(t)=-\tfrac{1}{2}gt^2+c_1t+c_2 \qquad (3)$$

where c_1 and c_2 are constants. To determine $x(t)$ completely it is necessary to specify two additional conditions, such as the position and velocity of the particle at some instant of time. These conditions can be used to determine the constants c_1 and c_2.

This unit introduces us to the differential equation in general and helps to differentiate between various kinds and associated solutions.

Learning Objectives

After going through this section session, you would be able to:

- know what is a differential equation, and state the difference between the independent and dependent variables.
- classify differential equations in terms of types, order, degree and linearity.
- differentiate between general and particular solution.
- derive differential equation from general solutions that is; elimination of arbitrary constant.
- differentiate between boundary and initial conditions and be able to solve problems with such conditions.

Session 1-1 Differential Equations

The laws of the universe are written largely in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting naturally phenomena involve change and are best described by equations that relate changing quantities.

Many important and significant problems in engineering, the physical sciences, and the social sciences such as economics and business when formulated in mathematical terms require the determination of a function satisfying an equation containing the derivatives of unknown function. Such equations are called *differential equation*. Perhaps the most familiar example is Newton's law:

$$m\frac{d^2x}{dt^2} = F \qquad (1)$$

for the position $x(t)$ of a particle acted on by a force F. In general F will be a function of time t, the position x, and the velocity dx/dt. To determine the motion of a particle acted on by a given force F it is necessary to find a function $x(t)$ satisfying Eq. (1). If the force is that due to gravity, then $F = -mg$ and

$$m\frac{d^2x}{dt^2} = -mg \qquad (2)$$

On integrating Eq. (2) we have:

$$\frac{dx}{dt} = -gt + c_1$$

$$x(t) = -\tfrac{1}{2}gt^2 + c_1t + c_2 \qquad (3)$$

where c_1 and c_2 are constants. To determine $x(t)$ completely it is necessary to specify two additional conditions, such as the position and velocity of the particle at some instant of time. These conditions can be used to determine the constants c_1 and c_2.

Differential equations are an important part of the calculus, the fundamentals of which are presented here. In developing the theory of differential equations in a systematic manner it is helpful to classify different types of equations.

There are two types of derivatives which usually are (and always can be) interpreted as rates. For example, the *ordinary derivative* dy/dx is the rate of change of y with respect to x (independent variable), and the *partial derivative* $\partial u/\partial x$ is the rate of change of u with respect to x when all independent variables except x are given fixed values. And the study of differential equations has two principal goals:

1. To discover differential equation that describes a physical situation;
2. To find the appropriate solution of that equation.

The solution of differential equations plays an important role in the study of the motions of heavenly bodies such as planets, moons, and artificial satellites.

1-1.1 Definition of Differential Equation

A ***differential equation*** is an equation which involves one or more derivatives, or differentials of an unknown function.

Example 1.1

The following are examples of differential equations involving their respective unknown functions:

$$\frac{dy}{dx} = 3x^2 \qquad 1.1$$

$$\frac{dR}{dt} + kR = 0 \qquad 1.2$$

$$\frac{d^2y}{dx^2} + k^2 y = 0 \qquad 1.3$$

$$\left(x^2 + y^2\right)dx - 2xydy = 0 \qquad 1.4$$

$$\frac{\partial v}{\partial t} = h^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad 1.5$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\cos\omega t \qquad 1.6$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad 1.7$$

$$\left(\frac{d^2w}{dx^2}\right)^3 - xw\frac{dw}{dx} + w = 0 \tag{1.8}$$

$$\frac{d^3x}{dy^3} + x\frac{dx}{dy} - 4xy = 0 \tag{1.9}$$

$$\frac{d^2y}{dx^2} = \frac{m}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{1.10}$$

$$\frac{d^3y}{dx^3} = \left[\left(\frac{dy}{dx}\right)^3 + e^x\right]^{\frac{3}{2}} \tag{1.11}$$

$$\frac{d^2y}{dx^2} + \frac{g}{l}\sin y = 0 \tag{1.12}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x \tag{1.13}$$

When an equation involves one or more derivatives with respect to a particular variable, that variable is called an ***independent*** variable. A variable is called ***dependent*** if a derivative of that variable occurs.

In the equation:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\cos\omega t \tag{1.6}$$

i is the *dependent* variable, t the *independent* varible, and L, R, C, E, and ω are called *parameters.*

The equation :

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{1.7}$$

has one dependent variable v, and two independent variables x and y.

Since the equation

$$\left(x^2 + y^2\right)dx - 2xydy = 0$$

may be written

$$x^2 + y^2 - 2xy\frac{dy}{dx} = 0$$

or

$$\left(x^2 + y^2\right)\frac{dx}{dy} - 2xy = 0$$

we may consider either variable to be dependent, the other being the independent one

ACTIVITY **Oral Exercise 1**

Identify the independent variables, the dependent variables, and the parameters in the equations given as examples in this section.

1-1.2 Classification of Differential Equations

Recall that a differential equation is an equation (has an equal sign, but not an identity) that involves derivatives. Just as biologists have a classification system for life, mathematicians have a classification system for differential equations.

Differential equations are classified as to:

(a) type (namely, *ordinary* or *partial*)
(b) order
(c) degree, and
(d) linearity

(a) Type

We can place all differential equations into two types: *ordinary differential equation* and *partial differential equations*.

An ***Ordinary Differential Equation,*** **(ODE)** is one containing ordinary derivatives of one or more unknown functions (dependent variable) with respect to a single independent variable. Examples are equations: (1.1), (1.2), (1.3), (1.4), (1.6), (1.8), (1.9), (1.10), (1.11), (1.12) and (1.13). So also is each of the two equations with more than one dependent variable:

$$a\frac{dx}{dt}+b\frac{dy}{dt}=c \qquad (1.14)$$

$$d\frac{dx}{dt}+e\frac{dy}{dt}=f \qquad (1.15)$$

A ***Partial Differential Equation*** is one involving partial derivatives of one or more dependent variables with respect to one or more of the independent variables. Examples are equation: (1.5) and (1.7).

Note:

The systematic treatment of ***partial differential equation*** lies beyond the scope of this book. We treat only ***ordinary differential equation*** with constant coefficient in this book.

(b) Order

Definition 1.1

The ***order*** of a differential equation is the order of the highest-ordered derivative appearing in the equation. For instance,

$$\frac{d^2y}{dx^2}+2b\left(\frac{dy}{dx}\right)^3+y=0 \qquad (1.16)$$

is an equation of "order two." It is also referred at as a "second-order ordinary differential equation."

More generally, the equation

$$F\left(x,y,y',y'',...,y^{(n)}\right)=0 \qquad (1.17)$$

is called an "nth - order'' ordinary differential equation. Equation (1.17) represents a relation between the $n+2$ variables $x,y,y',...,y^{(n)}$ which under suitable conditions can be solved for $y^{(n)}$ in terms of the other variables:

$$y^{(n)}=f\left(x,y,y',...,y^{(n-1)}\right) \qquad (1.18)$$

For the purpose of this book we shall assume that this is always possible. Otherwise, an equation of the form of equation (1.17) may actually represent more than one equation of the form of equation (1.18).

For example, the equation $x^2\left(y'\right)^2-3y'+2x=0$ actually represents the two different equations,

$$y'=\frac{3+\sqrt{9-8x^3}}{2x^2} \qquad \text{or} \qquad y'=\frac{3-\sqrt{9-8x^3}}{2x^2}$$

Note: Eq.(1.17) is called the ***general form***.

Note also that Eq. (1.17) is an n^{th} order differential equation because:

$$y^{(n)}=\frac{d^ny}{dx^n} \text{ ……………………} \qquad (1.19)$$

Example 1.2

1. $L\dfrac{di}{dt}+Ri=E$ (order 1)
2. $yy''=x$ (order 2)
3. $\dfrac{\partial^2 z}{\partial x^2}+3\dfrac{\partial^2 z}{\partial y^2}=0$ (order 2)

(c) Degree

Definition 1.2

The ***degree*** of the differential equation is the power (exponent) or the index that its highest-ordered derivative is raised, if the equation is rationalized or cleared of fractions with regard to the dependent variable and its derivatives involve in it.

From equation (1.11), squaring both sides results in:

$$\left(\frac{d^3y}{dx^3}\right)^2=\left[\left(\frac{dy}{dx}\right)^3+e^x\right]^3 \text{, degree } 2$$

The differential equation $\left(\dfrac{d^3y}{dx^3}\right)^2+\left(\dfrac{d^2y}{dx^2}\right)^5+\dfrac{y}{x^2+1}=e^x$ is an ordinary differential equation, of order three and degree two.

ACTIVITY Oral Exercise 2

State the **order** and **degree** of equations 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15 and 1.16

(d) Linear Differential Equation

A differential equation is ***linear*** if it can be put in the form:

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_2(x)y''+a_1(x)y'+a_0(x)y=f(x) \quad (1.20)$$

where a_n is not identically zero and also the subscripted (or indexed) a's are functions of independent variable (x) only.

The conditions for a linear differential equation are as follows:

(i) The dependent variable and all its derivatives occur only in the first degree (or to the first power).

(ii) No product of the dependent variable, say y, and/or any of its derivatives present.

(iii) No transcendental function (trigonometric, logarithmic or exponential) of the dependent variable and/or its derivatives occurs.

Example 1.3 (linear)

1. $$\frac{dy}{dx} + y = x^2$$

It is linear, it does not matter that the independent variable x is raised to the power 2, the dependent and derivative are not.

2. $$3x^2 y'' + 2\ln(x)\, y' + e^x y = 3x\cos x$$

This is a second order linear ordinary differential equation

Example 1.4 (Non - linear)

1. $$4yy''' - x^3 y' + \cos y = e^{2x}$$

This is not a linear differential equation because of the $4yy''$ and the $\cos y$ terms.

Other examples are:

(i) $$\frac{dy}{dx} + y^2 = 0$$

(ii) $$\left(\frac{dy}{dx}\right)^2 + 3y = 0$$

(iii) $$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} = e^x.$$

Remark: A **linear differential equation** is always in the **first degree** of the dependent variable (variables) and the derivatives.

Oral Exercise 3

For each of the following, state whether the equations are ordinary or partial, linear or non-linear, and give its order and degree.

1. $$\frac{d^2y}{dx^2} + k^2 x = 0$$

2. $$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$$

3. $$\left(2x^2 + y^2\right)dx + 2xydy = 0$$

4. $$y' + p(x)y = q(x)$$

5. $y''-3y'+2y=0$

6. $\frac{dy}{dx}+xy^2=0$

7. $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$

8. $\frac{d^4 y}{dx^4}=\omega(x)$

9. $\frac{d^2 y}{dx^2}+\sin(x+y)=\sin x$

10. $x^2 y''+yy'=0$

11. $(x+y)dx+(3x^2-1)dy=0$

12. $x(y'')^3+(y')^4-y=0$

13. $\left(\frac{d^3 w}{dx^3}\right)^2-2\left(\frac{dw}{dx}\right)^4+xw=0$

14. $\frac{dy}{dx}=1-xy+y^2$

15. $y''+2y'-8y=x^2+\cos x$

16. $\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0$

1-1.3 Notation

The expressions $y', y'', y''', y^{(4)}, \ldots, y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, ..., nth derivatives of y with respect to the independent variable under consideration. Thus, y'' represents $\frac{d^2y}{dx^2}$ if the independent variable is x, but represents $\frac{d^2y}{dp^2}$ if the independent variable is p. Observe that parentheses are used in $y^{(4)}$ and $y^{(n)}$ to distinguish it from the nth power, y^4 and y^n respectively. In mechanics, if the independent variable is time, usually denoted by t, primes are often replaced by dots. Thus, $\dot{y}, \ddot{y}$, and $\dddot{y}$ represent $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, and $\frac{d^3y}{dt^3}$, respectively

Session 2-1 Solution of a Differential Equation

Unlike algebra, in which we seek the unknown **numbers** that satisfy an equation such as $x^3+7x^2-11x+41=0$. In solving a differential equation we are challenged to find the unknown **functions**, say $y=f(x)$, for which an identity such as $f'(x)-2xf(x)=0$ – in Leibniz notation or $\frac{dy}{dx}-2xy=0$ holds on some interval numbers.

Ordinarily, we will want to find all solutions of the differential equation if possible. The solution of differential equations plays an important role in the study of the motions of heavenly bodies such as planets, moons and artificial satellites. Two questions that we will be asking repeatedly of a differential equation in this course are:

1. Is there a solution to the differential equation?
2. Is the solution given unique?

2-1.1 Definition 1.3

A ***solution*** of a differential equation is any function, say ϕ, satisfying the given differential equation on a specified interval, I.

A solution may be defined on the whole real line $(-\infty,\infty)$ or on only a part of the line often an interval (a,b). The n derivatives of the function must exist on the interval, say $a<x<b$ such that $\phi^{(n)}(x)=f\left(x,\phi(x),\phi'(x),\ldots,\phi^{(n-1)}(x)\right)$ for every x in $a<x<b$.

Thus if ϕ is a solution of some first order differential equation, say $F\left(x,y,\frac{dy}{dx}\right)=0$ on an interval I (real), then it implies $F\left(x,\phi,\frac{d\phi}{dx}\right)=0$.

Thus to test whether a given function solves a particular differential equation, we substitute the function ϕ and its derivatives into the differential equation. If the equation reduces to identity (0) , then the function ϕ solves the equation otherwise it does not.

It is also important to note that since solutions are often accompanied by intervals and these intervals can impart some important information about the solution.

Example 1.5

Show that for any values of the arbitrary constant c_1 and c_2 the function $\phi=c_1\cos x+c_2\sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2}+y=0$

Solution:

We will need a second derivative of the solution function, ϕ. If ϕ is a solution to $F\left(y,\frac{d^2y}{dx^2}\right)=0$, then $F\left(\phi,\frac{d^2\phi}{dx^2}\right)=0$. If ϕ proves otherwise then it is not a solution to $F\left(y,\frac{d^2y}{dx^2}\right)=0$. We differentiate ϕ twice to obtain $\frac{d^2\phi}{dx^2}$.

$$\frac{d\phi}{dx} = -c_1 \sin x + c_2 \cos x \quad \Rightarrow \frac{d^2\phi}{dx^2} = -c_1 \sin x + c_2 \cos x$$

$$\therefore \frac{d^2 y}{dx^2} + y = 0 \Rightarrow \frac{d^2\phi}{dx^2} + \phi = 0$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \phi = \left(-c_1 \cos x - c_2 \sin x\right) + \left(c_1 \cos x + c_2 \sin x\right) = 0$$

This implies we have an identity. Hence ϕ is a solution to the given ODE and that the formula: $\phi = c_1 \cos x + c_2 \sin x$ gives all possible solution of the equation $\frac{d^2 y}{dx^2} + y = 0$.

Since $\sin x$ and $\cos x$ are continuous in the entire real line, the solution is defined in the entire real line $(-\infty, \infty)$ for any arbitrary constant c_1 and c_2.

Example 1.6

1. Show that $y = \frac{1}{x^2 - 1}$ is a solution of $y' + 2xy^2 = 0$ on $\mathrm{I} = (-1,1)$ but not on any larger interval containing I.

Solution:

We will need a first derivative of the solution function.

$y = \frac{1}{x^2 - 1}$ and $y' = \frac{-2x}{\left(x^2 - 1\right)^2}$ are well-defined functions on $(-1,1)$.

Imitating the LHS of the differential equation $y' + 2xy^2 = 0$, we have:

$y' + 2xy^2 = -\frac{2x}{\left(x^2 - 1\right)^2} + 2x\left[\frac{1}{x^2 - 1}\right]^2 = 0$ Thus, $y = \frac{1}{x^2 - 1}$ is a solution of $\mathrm{I} = (-1,1)$.

Note, however, that $\frac{1}{x^2 - 1}$ is not defined at $x = \pm 1$ and therefore could not be a solution on any interval containing either of these two points.

2. Show that $y = \ln x$ is a solution of $xy'' + y' = 0$ on $I = (0, \infty)$ but is not a solution on $I = (-\infty, \infty)$.

Solution:

We will need the first and second derivative of the solution function.

$y = \ln x, y' = \frac{1}{x}$ and $y'' = -\frac{1}{x^2}$ are well-defined functions on $(0, \infty)$.

Imitating the LHS of $xy'' + y' = 0$,we have: $x\left(-\frac{1}{x^2}\right)+\frac{1}{x}=0$

Thus, $y = \ln x$ is a solution on $(0,\infty)$.

Note that $y = \ln x$ could not be a solution on $(-\infty,\infty)$, since the logarithm is undefined for negative numbers containing.

3. Prove that $y = e^{-x} + \sin x$ is a solution of $\frac{d^2y}{dx^2} + y = 2e^{-x}$

Solution:

We will need a second derivative of solution function to do this.

$y = e^{-x} + \sin x$, $y' = -e^{-x} + \cos x$ and $y'' = e^{-x} - \sin x$

Imitating the LHS of differential equation $\frac{d^2y}{dx^2} + y = 2e^{-x}$

$$e^{-x} - \sin x + e^{-x} + \sin x = 2e^{-x}$$

4. Show that the two functions $y = \left(c^2 - x^2\right)^{\frac{1}{2}}$ and $y = -\left(c^2 - x^2\right)^{\frac{1}{2}}$ are both solutions of the equation $x + y\frac{dy}{dx} = 0$, $-c < x < c$.

Solution:

We will need a first derivative to do this.

$$y = \pm\left(c^2 - x^2\right)^{\frac{1}{2}}, \quad y' = \mp x\left(c^2 - x^2\right)^{-\frac{1}{2}}$$

$$x + \left[\pm\left(c^2 - x^2\right)^{\frac{1}{2}}\right]\left[\mp x\left(c^2 - x^2\right)^{-\frac{1}{2}}\right] = 0$$

5. Show that $y = x^{-\frac{3}{2}}$ is a solution of $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution:

We will need the first and second derivative of the solution function to do this.

$y = x^{-\frac{3}{2}}, y' = -\frac{3}{2}x^{-\frac{5}{2}}$ and $y'' = \frac{15}{4}x^{-\frac{7}{2}}$ are well – defined functions on $x > 0$ or $(0,\infty)$.

Imitating the LHS of $4x^2y'' + 12xy' + 3y = 0$, we have:

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right)+12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right)+3x=0$$

Thus, $y=x^{-\frac{3}{2}}$ is a solution of $x>0$.

Note, however, that $x^{-\frac{3}{2}}=\frac{1}{\sqrt{x^3}}$ could not be a solution on $(-\infty,0]$, since zero and any negative real number plug into it would give an undefined number and complex number respectively, which is not what we are looking for.

Exercise 1.1

In each of assignment 1 – 12, verify by substitution that each given function is a solution of the given differential equation and also state the interval in which the solution exists.

	Solution(s)	**Differential Equation**
1.	$y=2e^{-x}+xe^{-x}$	$y''+2y'+y=0$
2.	$y=1$	$y''+2y'+y=x$
3.	$(y-c)^2=cx$, $c=$ constant	$4x(y')^2+2xy'-y=0$
4.	$y=e^x-e^{-x}$	$y'=y+2e^{-x}$
5.	$y_1=e^{-2x}$, $y_2=xe^{-2x}$	$y''+4y'+4y=0$
6.	$y_1=x$, $y_2=0$	$y''-xy'+y=0$
7.	$y=\left(x^8-x^4\right)^{\frac{1}{4}}$	$\frac{dy}{dx}=\frac{2y^4+x^4}{xy^3}$
8.	$y=\frac{1}{1+x^2}$	$y'+2xy^2=0$
9.	$y_1=x-\ln x$, $y_2=\frac{1}{x}-\ln x$	$x^2y''+xy'-y=\ln x$
10.	$y_1=x\cos(\ln x)$, $y_2=x\sin(\ln x)$	$x^2y''-xy'+2y=0$

Session 3-1 General Solution

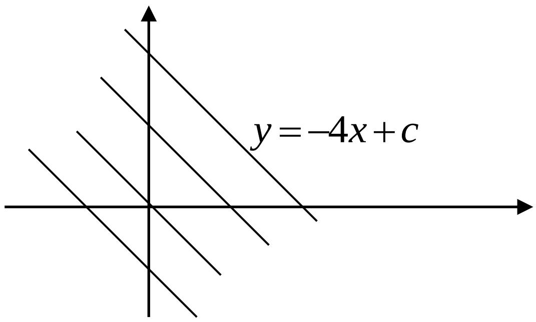
3-1.1 Definition 1.4

A formula that gives all solutions of a differential equation is called the **general solution** of the equation. A general solution to an n **th order** DE generally involves n **independent arbitrary constants**, each admitting a range of real values.

NOTE:

From the problems of Exercise 1.1, it is clear that the number of distinct solutions depend on the order of the differential equation, for which they are equal.

It is worth mentioning that, the differential equation $\frac{dy}{dx} = -4$ describes a straight line with a constant gradient of -4. There are however infinitely constant many straight lines that take the equation $y = -4x + c$, where c is an arbitrary constant. Since $c \in \mathbf{R}$, y is a constantly many hence the differential equation $\frac{dy}{dx} = -4$ has many solutions, because c can assume any real number and the differential equation would be satisfied, hence the name **arbitrary constant**. The general solution is known as **family of solutions**.



3-1.2 Determining a Differential Equation from a General Solution: The Elimination of Arbitrary Constant

To determine a differential equation from a general solution, we need to eliminate the arbitrary constant. Methods for the elimination of arbitrary constants vary with the way in which the constants enter the relation. A method which is efficient for one problem may be poor for another. One fact persists throughout. Since each differentiation yields a new relation, number of derivatives that need be used is the same as the number of arbitrary constants to be eliminated.

We shall in each case determine the differential equation that is:

(a) Of order equal to the number of arbitrary constants in the given relation
(b) Consistent with that relation;
(c) Free from arbitrary constants.

Example 1.7

Find the differential equation whose general solution is $y = c_1e^{-2x} + c_2e^{3x}$

Eliminating the arbitrary constants c_1 and c_2 from the relation:

$$y = c_1e^{-2x} + c_2e^{3x} \qquad \text{(I)}$$

Since two constants are to be eliminated, obtain the two derivatives:

$$y' = -2c_1e^{-2x} + 3c_2e^{3x} \qquad \text{(II)}$$

$$y'' = 4c_1e^{-2x} + 9c_2e^{3x} \qquad \text{(III)}$$

The elimination of c_1 from (II) and (III) yields: $y'' + 2y' = 15c_2e^{3x}$

The elimination of c_1 from equations (I) and (II) yields: $y' + 2y = 5c_2e^{3x}$

Hence $y'' + 2y' = 3(y' + 2y)$ Or $y'' - y' - 6y = 0$.

Alternatively, (another) method for obtaining the differential equation in this example proceeds as follows. We know from a theorem in algebra (MATH 152) that three equations (I),(II) and (III) considered as equations in the two unknowns c_1 and c_2 can have solutions only if:

$$\begin{vmatrix} -y & e^{-2x} & e^{3x} \\ -y' & -2e^{-2x} & 3e^{3x} \\ -y'' & 4e^{-2x} & 9e^{3x} \end{vmatrix} = 0 \qquad \text{(IV)}$$

Since e^{-2x} and e^{3x} cannot be zero for any $x \in \Re$, equation (IV) may be rewritten, with the factors e^{-2x} and e^{3x} removed, as:

$$\begin{vmatrix} y & 1 & 1 \\ y' & -2 & 3 \\ y'' & 4 & 9 \end{vmatrix} = 0$$

From which the differential equation: $y'' - y' - 6y = 0$ follows immediately.

Example 1.8

1. Find the differential equation whose general solution is $y = c\sin x$, where c is an arbitrary constant.

Solution:

$$y = c\sin x \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{(I)}$$

There is one arbitrary constant, we then find the first derivative:

$$\frac{dy}{dx} = c\cos x \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{(II)}$$

From (I) and (II): $c = \dfrac{y}{\sin x} = \dfrac{y'}{\cos x}$

$$\Rightarrow y'\sin x = y\cos x$$

$$\Rightarrow y'\sin x - y\cos x = 0$$

Alternatively, from (I) let: $c = \dfrac{y}{\sin x}$

Differentiating w.r.t. x: $0 = \dfrac{y'\sin x - y\cos x}{\sin^2 x}$

$$\Rightarrow y'\sin x - y\cos x = 0$$

2. Determine the differential equation whose general solution is $y = c_1e^x + c_2e^{-x} + 2x$.

Solution:

There are two arbitrary constants, so we get equation of order 2. Hence we differentiate twice.

Let $y = c_1e^x + c_2e^{-x} + 2x$ .... (I)

$$\Rightarrow \frac{dy}{dx} = c_1e^x - c_2e^{-x} + 2 \text{ .... (II)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = c_1e^x + c_2e^{-x} \text{ .... (III)}$$

Add (I) and (II): $\Rightarrow \dfrac{dy}{dx} + y = 2c_1e^x + 2x + 2$ .... (IV)

Add (II) and (III): $\Rightarrow \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 2c_1e^x + 2$ .... (V)

The two equations above have the same term on the RHS, hence we equate the LHS, and we get: $\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{dy}{dx}+y-2x$

This simplifies as: $\frac{d^2y}{dx^2}-y+2x=0$

3. Find the differential equation whose general solution is $y=c_0+c_1x+c_2x^2+c_3x^3$.

Solution:

The equation has four arbitrary constants (c_0,c_1,c_2,c_3); hence we need to differentiate four times to get a differential equation of the fourth order.

Let $y=c_0+c_1x+c_2x^2+c_3x^3$

$$\frac{dy}{dx}=c_1+2c_2x+3c_3x^2 \Rightarrow \frac{d^2y}{dx^2}=2c_2+6c_3x \Rightarrow \frac{d^3y}{dx^3}=6c_3$$

$\Rightarrow \frac{d^4y}{dx^4}=0$, which is the differential equation that we need.

4. Eliminate the constant c from the equation $(x-c)^2+y^2=c^2$

Solution:

Direct differentiation of the relation yields: $2(x-c)+2yy'=0$ from which $c=x+yy'$. Therefore, using the original equation, we find that:

$(yy')^2+y^2=(x+yy')^2$, or $y^2=x^2+2xyy'$, which may be written in the form:

$(x^2-y^2)dx+2xydy=0$

Alternatively:

The equation $(x-c)^2+y^2=c^2$ may be put in the form: $x^2+y^2-2cx=0$ or $\frac{x^2+y^2}{x}=2c$ then differentiation of both sides leads to:

$$\frac{x(2xdx+2ydy)-(x^2+y^2)dx}{x^2}=0 \text{ or } (x^2-y^2)dx+2xydy=0$$

5. Eliminate B and α from the relation $x = B\cos(\omega t+\alpha)$ in which ω is a parameter (not to be eliminated)

Solution:
First we obtain two derivatives of x with respect to t:

$$\frac{dx}{dt} = -\omega B\sin(\omega t+\alpha) \text{ and } \frac{d^2x}{dt^2} = -\omega^2 B\cos(\omega t+\alpha)$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x, \text{ since } x = B\cos(\omega t+\alpha)$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

6. Eliminate c from the equation $cxy + c^2x + 4 = 0$

Solution:
At once we get: $c(y+xy') + c^2 = 0$ Since $c \neq 0$, $c = -(y+xy')$ and substitution into the original equation leads us to the result: $x^3(y')^2 + x^2yy' + 4 = 0$

Remark:
The general solution of an n th order ordinary differential equation could be expected to have n arbitrary constants.

Exercise 1.2

Assuming that $a, b, c, c_1, c_2, A, \beta, B$ are arbitrary constants, show that each function or equation on the left satisfies the differential equation written opposite it. Remember that we differentiate and substitute.

	Solution	Differential Equation
1.	$x^2 + y^2 = c$	$yy' + x = 0$
2.	$y = cx + c^4$	$y = xy' + (y')^4$
3.	$y = c_1x + c_2e^x$	$(x-1)y'' - xy' + y = 0$
4.	$x^3 - 3x^2y = c$	$(x-2y)dx - xdy = 0$
5.	$y\sin x - xy^2 = c$	$y(\cos x - y)dx + (\sin x - 2xy)dy = 0$

6.	$x^2 y = 1 + cx$	$\left(x^2 y + 1\right)dx + x^3 dy = 0$
7.	$cy^2 = x^2 + y$	$2xydx - \left(y + 2x^2\right)dy = 0$
8.	$x = A\sin\left(\omega t + \beta\right)$, ω: parameter	$\dfrac{d^2x}{dt^2} + \omega^2 x = 0$
9.	$x = c_1 \cos\omega t + c_2 \sin\omega t$, ω: parameter	$\dfrac{d^2x}{dt^2} + \omega^2 x = 0$
10.	$y = cx + c^2 + 1$	$y = xy' + \left(y'\right)^2 + 1$
11.	$y = cx + \dfrac{\omega}{c}$	$y = xy' + \dfrac{\omega}{y'}$
12.	$y^2 = 4ax$	$2xdy - ydx = 0$
13.	$y = c_1 e^{px} + c_2 e^{-px}; p \in \mathbf{R}$	$y'' - p^2 y = 0$
14.	$u = \tan^{-1}\dfrac{y}{x}$	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
15.	$u = \log\left(x^2 + y^2\right)^{\frac{1}{2}}$	$u_{xx} + u_{yy} = 0$
16.	$y = x\left(c - \cos x\right)$	$xy' - y = x^2 \sin x$
17.	$y = e^x\left(1 + x\right)$	$y'' - 2y' + y = 0$
18.	$y = ax^2 + bx + c$	$y''' = 0$
19.	$y = c_1 + c_2 e^{2x}$	$y'' - 2y' = 0$
20.	$y = 4 + c_1 e^{2x}$	$y' - 2y = -8$
21.	$y = c_1 e^{-x} + c_2 e^{-3x}$	$y'' + 4y' + 3y = 0$
22.	$y = c_1 e^{-x} + c_2 x e^{-x}$	$y'' + 2y' - 2y = 0$
23.	$y = x^2 + c_1 e^{x} + c_2 e^{-2x}$	$y'' + y' - 2y = 2\left(1 + x - x^2\right)$
24.	$y = c_1 e^{2x}\cos 3x + c_2 e^{2x}\sin 3x$	$y'' - 4y' + 13y = 0$
25.	$y = c_1 x^2 + c_2 e^{2x}$	$x\left(1 - x\right)y'' + \left(2x^2 - 1\right)y' - 2\left(2x - 1\right)y = 0$

Session 4-1 Initial-value and Boundary-value Problems

4-1.1 Definition 1.5: Subsidiary Conditions

Subsidiary Condition(s) is/are condition(s), or set of conditions, on the differential equation that will allow us to determine which solution that we are after.

4-1.2 Definition 1.6: Initial–Value Problem

Initial-Value Problem is the differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable. The subsidiary conditions are *initial conditions*.

4-1.3 Definition 1.7: Boundary–Value Problem

Boundary-Value Problem is the differential equation along with subsidiary conditions on the unknown function and its derivatives, which are given at more than one value of the independent variable. The subsidiary conditions are *boundary conditions*.

Theorem 1.1

An **initial value problem** (IVP) for an nth order DE includes a specification of the solution's value and $n-1$ derivatives at some point or a set of n algebraic conditions at a common point:

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, \quad \ldots, \quad \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$$

Generally in applications, an IVP has a **unique solution** on some interval containing the initial value point.

Example 1.9

The problem $y'' + 2y' = e^x; y(\pi) = 1, y'(\pi) = 2$ is an initial – value problem, because the two subsidiary conditions are both given at $x = \pi$.

The problem $y'' + 2y' = e^x; y(0) = 1, y'(1) = 1$ is a boundary – value problem, because the two subsidiary conditions are given at the different values $x = 0$ and $x = 1$.

If an ***initial condition*** is given along with the differential equation, that is, a constraint of the form $y = y_0$ when $x = x_0$, then this information can be used to determine the particular value of c. In this way, one ***particular solution*** (**actual solution**) can be selected from the family of solutions – the one that satisfies both the differential equation and the initial conditions.

The geometric interpretation of the general solution of a first order differential equation $F\left(x, y, \frac{dy}{dx}\right) = 0$ is an infinitely many family curves; one for each value of real number. These curves are called integral curves, sometimes it is important to pick one particular member of the family of integral curves. This is done by identifying a particular point (x_0, y_0) through which the graph of the solution is required to pass. This requirement is usually written as $y(x_0) = y_0$.

This is referred to as an initial value condition and the problem of the form:

$$F\left(x, y, \frac{dy}{dx}\right) = 0; y(x_0) = y_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.21)$$

is called ***initial value problem***.

Note: If initial condition(s) is used to solve a differential equation, the solution is then called ***particular solution*** (**actual solution**).

Example 1.10

1. $y = c_1 e^{-x} + c_2 e^{3x}$ is a general solution of the differential equation:

$$y'' - 2y' - 3y = 0$$

Determine the particular solution of the initial conditions $y = 3$ when $x = 0$ and $y' = 4$ when $x = 0$.

Solution:

Since the differential equation is second order, we have a specification of the solution value: $y(0) = 3$ and one derivative at a point: $y'(0) = 4$, that is 2 algebraic conditions at a common point: $x = 0$

let $\quad y = c_1 e^{-x} + c_2 e^{3x}$ when $x = 0$, $y = 3$

$$\Rightarrow 3 = c_1 e^{(0)} + c_2 e^{3(0)} = c_1 + c_2 \text{\_\_\_\_\_\_(I)}$$

$$y' = -c_1 e^{-x} + 3c_2 e^{3x} \text{ when } x = 0,\ y' = 4,$$

$$\Rightarrow 4 = -c_1 e^{-(0)} + 3c_2 e^{(0)}$$

$$\Rightarrow 4 = -c_1 + 3c_2 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \qquad \text{(II)}$$

Solving equation (I) and (II) we have, $c_1 = \frac{5}{4}$ and $c_2 = \frac{7}{4}$

$\therefore$ the particular solution is given by:

$$y=\frac{5}{4}e^{-x}+\frac{7}{4}e^{3x} \text{ or } y=\frac{1}{4}\left(5e^{-x}+7e^{3x}\right)$$

2. Find a solution to the boundary – value problem $y''+4y=0;$ $y(\pi/8)=0, y(\pi/6)=1$, if the general solution to the differential equation is $y=c_1\sin 2x+c_2\cos 2x$.

Solution:

For $x=\frac{\pi}{8}$, $y=c_1\sin 2\left(\frac{\pi}{8}\right)+c_2\cos 2\left(\frac{\pi}{8}\right)=c_1\sin\frac{\pi}{4}+c_2\cos\frac{\pi}{4}$

$$\Rightarrow y=c_1\frac{\sqrt{2}}{2}+c_2\frac{\sqrt{2}}{2}$$

To satisfy the condition $y(\pi/8)=0$, we have:

$$c_1\frac{\sqrt{2}}{2}+c_2\frac{\sqrt{2}}{2}=0\text{…………………(I);}$$

For $x=\frac{\pi}{6}$, $y=c_1\sin 2\left(\frac{\pi}{6}\right)+c_2\cos 2\left(\frac{\pi}{6}\right)=c_1\sin\frac{\pi}{3}+c_2\cos\frac{\pi}{3}$

$$\Rightarrow y=c_1\frac{\sqrt{3}}{2}+c_2\frac{1}{2}$$

To satisfy the condition $y(\pi/6)=1$, we have:

$$c_1\frac{\sqrt{3}}{2}+c_2\frac{1}{2}=1\text{……………………..(II)}$$

Solving (I) and (II) simultaneously, we get:

$$c_1=\frac{2}{\sqrt{3}-1} \text{ and } c_2=-\frac{2}{\sqrt{3}-1}.$$

Hence a particular solution is obtained:

$$y=\frac{2}{\sqrt{3}-1}(\sin 2x-\cos 2x)$$

as the solution of the boundary – value problem.

3. Find a solution to the boundary – value problem $y''+4y=0; y(0)=1,$ $y(\pi/2)=2$, if the general solution to the differential equation is known to be $y=c_1\sin 2x+c_2\cos 2x$.

Solution:

For $x=0$, $y=c_1\sin 0+c_2\cos 0=c_2$

To satisfy the condition $y(0)=1$, we have:

$$c_2=1 \text{……………………......................(I).}$$

For $x=\pi/2$, $y=c_1\sin\left(\frac{\pi}{2}\right)+c_2\cos\left(\frac{\pi}{2}\right)=-c_2$

To satisfy the condition $y(\pi/2)=2$, we have:

$$-c_2=2 \Rightarrow c_2=-2 \text{………………….(II).}$$

Thus, to satisfy both boundary conditions simultaneously, we must require c_2 to equal both 1 and -2, which is impossible.

Therefore, there is no solution to this problem.

Exercise 1.3

1. Verify that the following are solution of the given differential equations.

a)	$y=2\sin 2x$	for	$y''+4y=0$
b)	$y=e^{3x}$	for	$y'-3y=0$
c)	$y=\dfrac{1}{1-x}$	for	$y'-y^2=0$
d)	$y=e^x\cos x$	for	$y''-y'+2y=0$
e)	$y=2\sqrt{x}-\sqrt{x}\ln x$	for	$4x^2y''+y=0$

2. Determine c_1 and c_2 that $y=c_1\sin 2x+c_2\cos 2x+1$ will satisfy the conditions $y(\pi/8)=0$ and $y'(\pi/8)=\sqrt{2}$.

3. Determine c_1 and c_2 that $y=c_1e^{2x}+c_2e^{x}+2\sin x$ will satisfy the conditions $y(0)=0$ and $y'(0)=1$.

4. In Problems 4 a) through e), find values c_1 and c_2 so that the given functions will satisfy the given conditions. Determine whether the given conditions are initial conditions or boundary conditions:

a)	$y=c_1e^x+c_2e^{-x}+4\sin x$;	$y(0)=1, y'(0)=-1$
b)	$y=c_1x+c_2+x^2-1$;	$y(1)=1, y'(1)=2$
c)	$y=c_1\sin x+c_2\cos x$	$y(0)=1, y\left(\frac{\pi}{2}\right)=1$
d)	$y=c_1\sin x+c_2\cos x$	$y\left(\frac{\pi}{4}\right)=0, y\left(\frac{\pi}{6}\right)=1$
e)	$y=c_1e^x+c_2xe^x+x^2e^x$	$y(1)=1, y'(1)=-1$

Example 1.12

Show that the integral curves of the differential equation: $\left(y - x^3\right)dx + \left(y^3 + x\right)dy = 0$ are given by the family $y^4 + 4xy - x^4 = c$.

Solution:

Apply implicit differentiation to the proposed family of integral curves to find y ;

$$y^4 + 4xy - x^4 = c \quad \Rightarrow 4y^3y' + \left(4xy' + 4y\right) - 4x^3 = 0$$

$$\Rightarrow 4y^3y' + \left(4xy' + 4y\right) - 4x^3 = 0 \quad \Rightarrow y'\left(y^3 + x\right) + \left(y - x^3\right) = 0$$

$$\Rightarrow \frac{dy}{dx}\left(y^3 + x\right) + \left(y - x^3\right) = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{\left(y - x^3\right)}{y^3 + x}$$

This last equation can then be written in terms of the differentials dx and dy

$$\left(y^3 + x\right)dy = -\left(y - x^3\right)dx \quad \Rightarrow \left(y - x^3\right)dx + \left(y^3 + x\right)dy = 0$$

Notice that the last equation can be put in the form $M(x,y)dx + N(x,y)dy = 0$, which is simply an alternate form of the equation:

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

where $M(x,y) = y - x^3$ and $N(x,y) = y^3 + x$.

Summary

- Definition of Differential Equation: A ***differential equation*** is an equation which involves one or more derivatives, or differentials of an unknown function.
- Classification of Differential Equation:
  - Types
  - Order
  - Degree
  - Linearity

- A differential equation is ***linear*** if it can be put in the form:

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_2(x)y''+a_1(x)y'+a_0(x)y=f(x)$$

where $a_n(x)$ is not identically zero and also the subscripted (or indexed) a's are functions of independent variable (x) only.

The conditions for a linear differential equation are as follows:

- The dependent variable and all its derivatives occur only in the first degree (or to the first power).
- No product of the dependent variable, say y , and/or any of its derivatives present.
- No transcendental function (trigonometric, logarithmic or exponential) of the dependent variable and/or its derivatives occurs.

UNIT 2

ORDINARY DIFFERENTIAL EQUATION OF FIRST ORDER

Introduction

In this unit we begin our program of solving differential equations. The first – order differential equations involve only the first derivative of the unknown function. Our usual notation for the unknown function will be $y(x)$.

Learning Objectives

After going through this unit, you would be able to:

- put first order ordinary differential equation in general, standard and differential forms
- find the difference between all first all the differential equation.
- describe the methods of solving most first order differential equations.

Session 1-2 First Order Differential Equations

First order differential equations are often used to model the behaviour of engineering systems. For example, in a radioactive decay, mass is converted to energy by radiation. It has been observed that the rate of change of the mass is proportional to mass itself. That is if $N(t)$ is the mass at time t then: $\frac{dN(t)}{dt} \alpha N(t) \Rightarrow \frac{dN(t)}{dt} = kN(t)$ for some k that depends on the element. Such an equation is called a **Model** in mathematics.

1-2.1 Definition 2.1

A first order differential equation is an equation containing only the first differential. In its general form the 1st order differential equation is given by: $F\left(x, y, \frac{dy}{dx}\right) = 0$ or

$F(x, y, y') = 0$

The above equation has x called the independent variable, the unknown function y which depends on x and the first derivative of y (i.e. y') with respect to x.

Examples of first-order differential equations

1. $\frac{dy}{dx}-5x=0$
2. $\frac{dy}{dx}-4xy=x^3e^x$
3. $x\frac{dy}{dx}-y=0$
4. $3\frac{dy}{dx}=4xy$

Standard form for a first order – differential equation in the unknown function y is:

$$y'=f(x,y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad (2.1)$$

where the derivative y' appears only on the left side of (2.1). Many, but not all, first – order differential equations can be written in standard form by algebraically solving for y' and then setting $f(x,y)$ equal to the right side of the resulting equation.

Example 2.1

1. Write the differential equation $xy'-y^2=0$, which is in the general form, in standard form.

Solution:

Solving for y', we obtain: $y'=y^2/x$ as standard form with $f(x,y)=y^2/x$

2. Write the differential equation $e^xy'+e^{2x}y=\sin x$ general form and standard form.

Solution:

Shifting everything to the left of equation we have: $e^xy'+e^{2x}y-\sin x=0$ as general form and solving for y', we obtain: $y'=-e^xy+e^{-x}\sin x$ as standard form, with $f(x,y)=-e^xy+e^{-x}\sin x$.

3. Write the differential equation $(y'+y)^5=\sin(y'/x)$ in standard form.

Solution:

This equation cannot be solved algebraically for y', and cannot be written in standard form.

***Differential form*:** The right side of (2.1) can always be written as a quotient of two other functions $M(x,y)$ and $-N(x,y)$. Then (2.1) becomes: $y' = M(x,y)/-N(x,y)$, or $\dfrac{dy}{dx} = \dfrac{M(x,y)}{-N(x,y)}$ which is equivalent to the *differential form*:

$$M(x,y)dx + N(x,y)dy = 0 \text{ ……………………….. (2.2)}$$

where we call the functions $M(x,y)$ and $N(x,y)$ the coefficients of dx and dy respectively.

NOTE:

We will put most (if not all) first order ordinary differential equation in the form of equation 2.2 before solving.

Example 2.2

Write the differential equation $y(yy'-1) = x$ in the differential form.

Solution:

Solving for y', we have: $y^2y' - y = x \Rightarrow y' = \dfrac{x+y}{y^2}$, which is the standard form

with $f(x,y) = \dfrac{(x+y)}{y^2}$ ……………………………….. (I)

There are **infinitely many** different differential forms associated with (I), which are as follows:

b) Take $M(x,y) = x+y, N(x,y) = -y^2$.

Then $\dfrac{dy}{dx} = \dfrac{M(x,y)}{-N(x,y)} = \dfrac{x+y}{-(-y^2)} = \dfrac{x+y}{y^2}$ the equivalent differential form is:

$$(x+y)dx + (-y^2)dy = 0$$

c) Take $M(x,y) = -1, N(x,y) = \dfrac{y^2}{x+y}$.

Then $\dfrac{dy}{dx} = \dfrac{M(x,y)}{-N(x,y)} = \dfrac{-1}{-y^2/(x+y)} = \dfrac{x+y}{y^2}$ the equivalent differential form is:

$$(-1)dx + \left(\frac{y^2}{x+y}\right)dy = 0$$

d) Take $M(x,y)=\dfrac{x+y}{2}, N(x,y)=\dfrac{-y^2}{2}$.

Then $\dfrac{dy}{dx}=\dfrac{M(x,y)}{-N(x,y)}=\dfrac{(x+y)/2}{-(-y^2/2)}=\dfrac{x+y}{y^2}$ the equivalent differential form is:

$$\left(\frac{x+y}{2}\right)dx+\left(\frac{-y^2}{2}\right)dy=0$$

e) Take $M(x,y)=\dfrac{-x-y}{x^2}, N(x,y)=\dfrac{y^2}{x^2}$.

Then $\dfrac{dy}{dx}=\dfrac{M(x,y)}{-N(x,y)}=\dfrac{(-x-y)/x^2}{-y^2/x^2}=\dfrac{x+y}{y^2}$ the equivalent differential form is:

$$\left(\frac{-x-y}{x^2}\right)dx+\left(\frac{y^2}{x^2}\right)dy=0$$

Exercise 2.1

1. Write the differential equation $\dfrac{dy}{dx}=\dfrac{y}{x}$ in differential form.
2. Write the differential equation $(xy+3)dx+(2x-y^2+1)dy=0$ in standard form.
3. Write the differential equation $xy'+y^2=0$ in differential form.
4. Write the differential equation $(y')^2-5y'+6=(x+y)(y'-2)$ in standard form.
5. Write the differential equation $e^{(y'+y)}=x$ in differential form.
6. Write the differential equation $(y')^3+y^2+y=\sin x$ in differential form.

1-2.2 General and Particular Solutions of a First Order Ordinary Differential Equation

In this section, we examine methods of solving a first order differential equation. The methods or differential equation kinds to consider are as follows;

(I) Method of separation of variables

(II) Reduction to Separable Form

- Homogeneous equation
- Transformations (coefficients linear in the two variables)

(III) Exact Differential Equations

(IV) Method of Integrating – factor used for the following:

a) Non – exact differential equation

b) Linear ordinary differential equation of first order and those reducible to that form:

(i) Bernoulli's Equation

(ii) Riccati's Equation

Theorem 2.1: An Existence Theorem for Equations of Order One

Consider the equation of order one

$$\frac{dy}{dx} = f(x,y) \qquad \text{(2.1)}$$

Let T denote the rectangular region defined by: $|x-x_0| \le a$ and $|y-y_0| \le b$, where $a,b \in \mathbf{R}$, a region with the point (x_0, y_0) at its centre. Suppose that f and $\partial f/\partial y$ are continuous functions of x and y in T.

Under the conditions imposed on $f(x,y)$ above, there exists an interval about x_0, $|x-x_0| \le h$, and a function $y(x)$ which has the properties:

(a) $y = y(x)$ is a solution of equation (2.1) on the interval $|x-x_0| \le h$;

(b) On the interval $|x-x_0| \le h$, $y(x)$ satisfies the inequality $|y(x)-y_0| \le b$;

(c) At $x = x_0$, $y = y(x_0) = y_0$;

(d) $y(x)$ is unique on the interval $|x-x_0| \le h$ in the sense that it is the only function that has all of the properties (a) , (b), and (c).

The interval $|x-x_0| \le h$ may or may not need to be smaller than the interval $|x-x_0| \le a$ over which conditions were imposed upon $f(x,y)$.

In other sense, the theorem states that $f(x,y)$ is sufficiently well behaved near the point (x_0, y_0) then the differential equation

$$\frac{dy}{dx} = f(x,y) \text{.......................................}(2.1)$$

has a solution that passes through the point (x_0, y_0) and that the solution is unique near (x_0, y_0).

Session 2-2 Method of Separation of Variables

Some differential equations have a special form that allows us to separate them. Such equations can be solved by using the method in this section.

A **separable differential equation** is a first-order ordinary equation that is algebraically reducible to a standard differential form in which each of the non-zero terms contains exactly one variable. Solutions to this kind of equation are usually quite straightforward.

2-2.1 Definition 2.2 Separable Differential Equation

A first order differential equation is said to be separable if it can be written in the form:

$\frac{dy}{dx} = f(x,y) = G(x)H(y)$ or equivalent form:

$$M(x,y)dx + N(x,y)dy = 0 \text{------------- } (2.2)$$

If (2.2) is separable, then:

$$\left.\begin{array}{l} M(x,y) = m(x)p(y) \\ \text{and} \\ N(x,y) = n(x)q(y) \end{array}\right\} \text{---------} \qquad (2.3)$$

Then (2.2) is called separable differential equation;

$$m(x)p(y) + n(x)q(y)\frac{dy}{dx} = 0 \text{...................... } (2.4)$$

We then divide throughout by $p(y)n(x)$, we get:

$$\Rightarrow \frac{m(x)}{n(x)} + \frac{p(y)}{q(y)}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{m(x)}{n(x)}dx + \frac{p(y)}{q(y)}dy = 0 \text{.................................. } (2.5),$$

now it is separated in each variable x and y (hence the method's name)

Integrating both sides:

$$\Rightarrow \int \frac{m(x)}{n(x)}dx + \int \frac{p(y)}{q(y)}dy = c \text{ ....} \qquad (2.6)$$

where c is arbitrary constant. (2.6) is then the general solution of the differential equation. The evaluated term on the left-hand side of (2.6) is a function F whose total differential is the left-hand side of (2.5)

ALGORITHM 1

***Separation of Variables*:** To solve the initial-value problem

$$y' = f(x,y); y(x_0) = y_0 \text{ ....} \qquad (2.7)$$

(a) Separate the variables, so that all of the x-dependence is on one side of the equation, and all of the y-dependence is on the other side of the equation.

(b) Integrate both sides of the equation.

(c) Substitute the initial condition to solve for the constant of integration.

Example 2.3

1. Solve the following differential equations by separation of variables:

(a) $(x+1)\dfrac{dy}{dx} = x\left(y^2+1\right)$

(b) $\dfrac{dy}{dx} = \dfrac{x^2}{y\left(1+x^3\right)}$

Solution:

(a) We change to differential form, separate the variables, and integrate:

$$(x+1)dy = x\left(y^2+1\right)dx \Rightarrow \frac{x}{x+1}dx - \frac{1}{y^2+1}dy = 0$$

Assuming $x \neq -1$, integrating both sides: $\int \frac{x}{x+1}dx - \int \frac{1}{y^2+1}dy = c$ we get:

$x - \log|x+1| - \tan^{-1} y = c$, as our general solution.

Note:

$\ln$ is the same as $\log$

(b) The ODE $\frac{dy}{dx}=\frac{x^2}{y\left(1+x^3\right)}$ becomes: $ydy=\frac{x^2}{1+x^3}dx$ and assuming that $x\neq -1$, then:

$$\int ydy=\int\frac{x^2}{1+x^3}dx \Rightarrow \frac{y^2}{2}=\frac{1}{3}\ln\left(1+x^3\right)+c$$

$$y=\pm\sqrt{\frac{2}{3}\ln\left(1+x^3\right)+2c}=\pm\sqrt{\frac{2}{3}\ln\left(1+x^3\right)+c_1}$$

Since c is arbitrary constant, any number multiplying or dividing it will not have any effect, it will still be arbitrary constant. We replace $2c$ with c_1 to bring a distinction but no effect in general, so one can write the general solution as: $y=\pm\sqrt{\frac{2}{3}\ln\left(1+x^3\right)+c}$

2. Solve the equation (IVP) $\left(1+x^2\right)-x(1+2y)y'=0 \quad y(1)=0$

Solution:

$$\left(1+x^2\right)-x(1+2y)y'=0$$

Divide throughout by $x \Rightarrow \frac{\left(1+x^2\right)}{x}-(1+2y)\frac{dy}{dx}=0$

Assuming that $x\neq 0$, the general solution is:

$$\Rightarrow \int\frac{\left(1+x^2\right)}{x}dx-\int(1+2y)dy=c$$

$$\Rightarrow \ln x+\frac{x^2}{2}-y+y^2=c$$

To solve for the arbitrary c, we substitute $x=1, y=0$;

$$\Rightarrow \ln 1+\frac{1^2}{2}-0+0^2=c \Rightarrow c=\frac{1}{2}$$

$$\Rightarrow \ln x+\frac{x^2}{2}-y+y^2=\frac{1}{2}$$: as particular(actual)solution.

3. Find a particular solution of $\frac{dy}{dx}=\frac{x^2}{1+y^2}$, $y(2)=1$

Solution:

The variables are separable, since the equation can be written as: $x^2dx-\left(1+y^2\right)dy=0$

Integrating both sides: $\int x^2dx-\int\left(1+y^2\right)dy=0$, we get:

$$\frac{x^3}{3}-y-\frac{y^3}{3}=c\Rightarrow x^3-3y-y^3=3c=c$$

Note the replacement of $3c$ with c

Solving for arbitrary constant, c: $2^3-3-1=c\Rightarrow c=4$

Then particular solution is given by: $x^3-3y-y^3=4$.

Exercise 2.2

	Solve the differential equations	Ans.
1.	$9yy'+4x=0$	$\frac{x^2}{9}+\frac{y^2}{4}=c \ \left(c=\frac{c*}{18}\right)$
2.	$\ln x\frac{dx}{dy}=\frac{x}{y}$	$\frac{1}{2}\left(\ln\lvert x\rvert\right)^2=\ln\lvert y\rvert+c$
3.	$x\cos x dx+\left(1-6y^2\right)dy=0; y(\pi)=0$	$x\sin x+\cos x+1=y^6-y$
4.	$mydx=nxdy$	$x^m=cy^n$
5.	$xy'+y=0; y(1)=1$	$y=1/x$
6.	$\sqrt{1+x^2}dy+\sqrt{y^2-1}dx=0$	$\cosh^{-1}y+\sinh^{-1}x=c$
7.	$\frac{dy}{dx}=e^{x-y}$	$e^y=e^x+c$
8.	$xe^ydy+\frac{x^2+1}{y}dx=0$	$(y-1)e^y+\frac{x^2}{2}+\ln\lvert x\rvert=c$
9.	$\frac{dV}{dP}=-\frac{V}{P}$	$PV=C$
10.	$\frac{dr}{dt}=-4rt$; when $t=0, r=r_0$	$r=r_0\exp\left(-2t^2\right)$
11.	$xy^2dx+e^xdy=0$; when $x\to\infty, y\to\frac{1}{2}$	$y=e^x/\left(2e^x-x-1\right)$
12.	$\upsilon\frac{d\upsilon}{dx}=g$; when $x=x_0, \upsilon=\upsilon_0$	$\upsilon^2-\upsilon_0{}^2=2g(x-x_0)$

2-2.2 Reduction to Separable Form

Certain differential equations are not separable but can be made separable by the introduction of a new unknown function. We illustrate the idea of this method by some typical cases, i.e. **Homogeneous forms** and **Non-Homogeneous forms**, where the **coefficients are linear in the two variables**, transform to Homogeneous form.

a) Homogeneous Functions

Polynomials in which all terms are of the same degree, such as:

$$x^2 - 3xy + 4y^2,$$
$$x^3 + y^3,$$
$$x^4 y + 7y^5.$$

are called ***homogeneous*** polynomials. We wish now to extend that concept of homogeneity so it will apply to functions other than polynomials.

A formal definition of homogeneity is: *A function of two variables, $f(x, y)$, is said to be* ***homogeneous of degree*** n *if there is constant* n *such that*:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.8)$$

for all λ, x and y for which both sides are defined.

Example 2.4

1. $f(x, y) = x^4 - x^3 y$

$$f(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3 (\lambda y) = \lambda^4 x^4 - \lambda^4 x^3 y$$
$$= \lambda^4 \left(x^4 - x^3 y\right)$$
$$f(\lambda x, \lambda y) = \lambda^4 f(x, y)$$

Thus the equation of is homogeneous with degree 4.

2. $f(x, y) = e^{y/x} + \tan\frac{y}{x}$

$$\Rightarrow f(\lambda x, \lambda y) = e^{\lambda y/\lambda x} + \tan\frac{\lambda y}{\lambda x}$$
$$= e^{y/x} + \tan\frac{y}{x}$$
$$\Rightarrow f(\lambda x, \lambda y) = \lambda^0 f(x, y)$$

This function is homogeneous of degree zero.

3. $f(x,y)=x^2+\sin x\cos y$

$$\Rightarrow f(\lambda x,\lambda y)=(\lambda x)^2+\sin\lambda x\cos\lambda y$$

$$=\lambda^2x^2+\sin\lambda x\cos\lambda y$$

$$\Rightarrow f(\lambda x,\lambda y)=\lambda^2\left(x^2+\frac{1}{\lambda^2}\sin\lambda x\cos\lambda y\right)$$

$$\Rightarrow f(\lambda x,\lambda y)\neq\lambda^2 f(x,y)$$

This function is **not** homogenous.

4. $f(x,y)=\dfrac{2x+y}{x^2y^2}$

$$f(\lambda x,\lambda y)=\frac{2(\lambda x)+(\lambda y)}{(\lambda x)^2(\lambda y)^2}=\frac{2\lambda x+\lambda y}{\lambda^2x^2\lambda^2y^2}=\frac{\lambda(2x+y)}{\lambda^4x^2y^2}$$

$$\Rightarrow f(\lambda x,\lambda y)=\lambda^{-3}\left(\frac{2x+y}{x^2y^2}\right)$$

$$\Rightarrow f(\lambda x,\lambda y)=\lambda^{-3}f(x,y)$$

This function is homogeneous of degree -3.

Equations with homogeneous coefficients

A differential equation of the form (differential form):

$$M(x,y)dx+N(x,y)dy=0 \text{ .... (2.9)}$$

is **homogeneous** if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the **same degree**. Or if the equation in standard form:

$$y'=f(x,y) \text{ .... (2.10)}$$

depends only on the ratio $\dfrac{y}{x}$ or $\dfrac{x}{y}$, then it is said to be **homogenous**.

To solve homogeneous equations, turn them into separable ones using the substitution:

$$y=x\upsilon \text{ .... (2.11),}$$

where υ is a function of x; also:

$$\frac{dy}{dx}=\upsilon+x\frac{d\upsilon}{dx} \text{ .... (2.12)}$$

or $dy=xd\upsilon+\upsilon dx$

Example 2.5

1. Solve the differential equation $(x^2+y^2)dx-2xydy=0$

Solution:

Notice that the function $M(x,y)=x^2+y^2$ and $N(x,y)=-2xy$ are both homogeneous of degree 2; therefore, the differential equation is homogeneous.

The substitution $y=x\upsilon$ implies that $dy=xd\upsilon+\upsilon dx$, and given equation is transformed into:

$$\left[x^2+(xv)^2\right]dx-2x(x\upsilon)(xd\upsilon+vdx)=0$$

After little algebra, this becomes the separable equation:

$$\frac{1}{x}dx=\frac{2\upsilon}{1-\upsilon^2}d\upsilon$$

Integrating both sides gives: $\log|x|=-\log\left|1-\upsilon^2\right|+\log|c|$, in which we wrote the arbitrary constant of integration as $\log|c|$ because it makes the next step neater:

$$\log|x|=\log\left|\frac{c}{1-\upsilon^2}\right|$$

Finally, to write the solution in terms of the original variables, x and y, we replace υ by y/x:

$$x=\frac{c}{1-(y/x)^2}\Rightarrow x^2-y^2=cx$$

2. Solve the differential equation $xdy-ydx-\sqrt{x^2-y^2}dx=0$

Solution:

The differential equation is clearly homogeneous degree 1. Using the transformation $y=x\upsilon$,$\Rightarrow dy=\upsilon dx+xd\upsilon$ and dividing by x, we have:

$$\upsilon dx+xd\upsilon-\upsilon dx-\sqrt{1-\upsilon^2}dx=0$$

$$\Rightarrow xd\upsilon-\sqrt{1-\upsilon^2}dx=0\Rightarrow\frac{d\upsilon}{\sqrt{1-\upsilon^2}}-\frac{dx}{x}=0$$

Integrating both sides: $\Rightarrow\int\frac{d\upsilon}{\sqrt{1-\upsilon^2}}=\int\frac{dx}{x}$

We get: $\sin^{-1}\upsilon=\ln x+\ln c$

And returning to the original variables, using $\upsilon=y/x$

We get: $\sin^{-1}\frac{y}{x} = \ln xc \Rightarrow xc = \exp\sin^{-1}(y/x)$

3. Solve the differential equation $xy' = (y-x)^3 + y$.

Solution:

Express the differential equation in form of $\frac{y}{x}$: $y' = x^2\left(\frac{y}{x} - 1\right)^3 + \frac{y}{x}$

Substitute $\upsilon = \frac{y}{x}$ and $y' = \upsilon + x\upsilon'$ by equation (2.11) and (2.12) on page 34, we get:

$$\upsilon + x\upsilon' = x^2(\upsilon - 1) + \upsilon \Rightarrow x\frac{dv}{dx} = x^2(\upsilon - 1)$$

$$\Rightarrow \int \frac{1}{\upsilon - 1}dv = \int x dx \Rightarrow \log|\upsilon - 1| = \frac{x^2}{2} + c$$

Returning back to the original variables: $\upsilon = \frac{y}{x}$, we get:

$$\Rightarrow \frac{y}{x} - 1 = \exp\left(\frac{x^2}{2} + c\right)$$

$$y = x\left[1 + \exp\left(\frac{x^2}{2} + c\right)\right]$$

Exercise 2.3

Solve the following differential equations (IVP) Ans.

1. $xy\frac{dy}{dx} = x^2 + y^2; y(1) = -2$; $y^2 = x^2\ln x^2 + 4x^2$ or $-\sqrt{x^2\ln x^2 + 4x^2}$

Note the negative square root is taken to be consistent with the initial condition

2. $xy' = y + x$ $y = x\ln xc$
3. $xy' = y + x^2\sec\left(\frac{y}{x}\right)$; $y(1) = \pi$
4. $xy' = y + 3x^4\cos^2(y/x)$; $y(1) = 0$
5. $xyy' = 2y^2 + 4x^2$; $y(2) = 4$

A Transformation (Equations in which $M(x, y)$ and $N(x, y)$ are linear but not homogeneous)

Another type of equation that can be reduced to a more basic type by means of a suitable transformation is an equation of the form:

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0 \text{ .... (2.13)}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants where

$$M(x, y) = a_1x + b_1y + c_1 \text{ and } N(x, y) = a_2x + b_2y + c_2$$

written as $M(x, y) + N(x, y)y' = 0$, which are not homogeneous.

However the differential equation can be made homogeneous and solved accordingly provided;

Case I

If $\dfrac{a_2}{a_1} \neq \dfrac{b_2}{b_1}$, then the transformation: $x = \mathbf{X} + h$ and $y = \mathbf{Y} + k$, where (h, k) is the solution of the system $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, that is $x = h$ and $y = k$, which reduces (2.13) to a homogeneous equation:

$$(a_1\mathbf{X} + b_1\mathbf{Y})d\mathbf{X} + (a_2\mathbf{X} + b_2\mathbf{Y})d\mathbf{Y} = 0 \text{ .....(2.14)}$$

in the variables $\mathbf{X}$ and $\mathbf{Y}$.

Case II

If $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = k$, then the transformation $z = a_1x + b_1y$ reduces the equation (2.13) to a separable equation in the variables x and z:

$$\frac{dz}{dx} = a_1 + b_1\frac{dy}{dx} \text{ .... (2.15)}$$

Example 2.6

1. Solve the differential equation:

$$(x - 2y + 1)dx + (4x - 3y - 5)dy = 0 \text{ .... (I)}$$

Solution:

Here $a_1 = 1, b_1 = -2, a_2 = 4, b_2 = -3$ and so $\dfrac{a_2}{a_1} = 4$ and $\dfrac{b_2}{b_1} = \dfrac{3}{2} \Rightarrow \dfrac{a_2}{a_1} \neq \dfrac{b_2}{b_1}$, case I transformation is applied.

Hence, we then solve: $x-2y+1=0,\ 4x-3y-5=0$ simultaneously to obtain: $x=3, y=2$.

This implies the solution of the system is $h=3, k=2$, and so the transformation is: $x=\mathbf{X}+3$ and $y=\mathbf{Y}+2$

This reduces the (I) to the homogeneous equation:

$$(\mathbf{X}-\mathbf{2Y})d\mathbf{X}+(4\mathbf{X}-\mathbf{3Y})d\mathbf{Y}=0 \text{ .... (I)}$$

$$\Rightarrow \frac{d\mathbf{Y}}{d\mathbf{X}}=\frac{1-2\left(\frac{\mathbf{Y}}{\mathbf{X}}\right)}{3\left(\frac{\mathbf{Y}}{\mathbf{X}}\right)-4}$$

Let $\mathbf{Y}=\upsilon\mathbf{X}$ to obtain:

$$\upsilon+\mathbf{X}\frac{d\upsilon}{d\mathbf{X}}=\frac{1-2\upsilon}{3\upsilon-4} \Rightarrow \frac{(3\upsilon-4)d\upsilon}{1+2\upsilon-3\upsilon^2}=\frac{d\mathbf{X}}{\mathbf{X}} \text{ ..... (II)}$$

Integrating both side: $\Rightarrow \int\frac{(3\upsilon-4)d\upsilon}{1+2\upsilon-3\upsilon^2}=\int\frac{d\mathbf{X}}{\mathbf{X}}$

$$\Rightarrow \int\frac{(3\upsilon-4)d\upsilon}{(1+3\upsilon)(1-\upsilon)}=\int\frac{-15}{4(1+3\upsilon)}d\upsilon+\int\frac{-1}{4(1-\upsilon)}d\upsilon=\int\frac{d\mathbf{X}}{\mathbf{X}}$$

$$\Rightarrow -\frac{5}{4}\ln|1+3\upsilon|+\frac{1}{4}\ln|1-\upsilon|=\ln\mathbf{X}+\ln c_1$$

$$\Rightarrow -5\ln|1+3\upsilon|+\ln|1-\upsilon|=4\ln\mathbf{X}+4\ln c_1$$

$$\Rightarrow \frac{1-\upsilon}{(1+3\upsilon)^5}=\mathbf{X}^4c_1^4$$

$$\Rightarrow 1-\upsilon=c(1+3\upsilon)^5\mathbf{X}^4, \text{ where } c=c_1^4$$

These are the solutions of the separable equation (II).

Now, substituting $\upsilon=\frac{\mathbf{Y}}{\mathbf{X}}$, we obtain the solutions of the homogeneous equation (I) in the form: $\mathbf{X}-\mathbf{Y}=c(\mathbf{X}+3\mathbf{Y})^5$

Finally, replacing $\mathbf{X}$ by $x-3$ and $\mathbf{Y}$ by $y-2$, from the original transformation, we obtain the solution of the differential equation in the form:

$$x-3-(y-2)=c[(x-3)+3(y-2)]^5 \Rightarrow x-y-1=c(x+3y-9)^5$$

2. If $(x+2y+3)dx+(2x+4y-1)dy=0$, determine the general solution.

Solution:

Here $a_1 = 1$, $b_1 = 2$, $a_2 = 2$, and $b_2 = 4 \Rightarrow \dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = 2$,

therefore it follows case (II)

So, let $z = x + 2y$, $\Rightarrow z' = 1 + 2y'$

The transformations give us:

$$(z+3)dx + (2z-1)\frac{(dz-dx)}{2} = 0$$

$$\Rightarrow 7dx + (2z-1)dz = 0$$

This is separable.

$$\int 7dx + \int (2z-1)dz = c \Rightarrow 7x + z^2 - z = c$$

Now, replacing z by $x + 2y$, we obtain the solutions as: $7x + (x+2y)^2 - (x-2y) = c$

Exercise 2.4

Solve each differential equation by making a suitable transformation.

1. $(5x+2y+1)dx + (2x+y+1)dy = 0$
2. $(3x-y+1)dx + -(6x-2y-3)dy = 0$
3. $(x-2y-3)dx + (2x+y-1)dy = 0$
4. $(10x-4y+12)dx - (x-5y+3)dy = 0$

Solve the following initial-value problems

5. $(6x+4y+1)dx + (4x+2y+2)dy = 0;\ y\left(\frac{1}{2}\right) = 3$
6. $(3x-y-5)dx + (x+y+2)dy = 0;\ y(2) = -2$
7. $(2x+3y+1)dx + (4x+6y+1)dy = 0;\ y(-2) = 2$
8. $(4x+3y+1)dx + (x+y+1)dy = 0;\ y(3) = -4$

2-2.3 Exact Equation

Given a function $f(x,y)$, its **total differentials**, df, is defined as:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \text{ ---------------------}(\cdot)$$

This shows that the family of curves (or general solution) $f(x,y)=c$ satisfies the differential equation $df=0$.

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \text{ ----------------------}(\cdot\,\cdot)$$

So if there exists a function $f(x,y)$ such that:

$$M(x,y) = \frac{\partial f}{\partial x} \text{ and } N(x,y) = \frac{\partial f}{\partial y}$$

then $M(x,y)dx + N(x,y)dy$ is called an **exact differential**, and the equation:

$$M(x,y)dx + N(x,y)dy = 0 \text{ ....}(2.2)$$

is said to be an **exact** equation, whose solution is the family $f(x,y)=c$.

How can you tell when a differential equation is exact? Recall that in partial derivatives if $f(x,y)$ continuous second partial derivatives, then: $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$ (that is, the order of differentiation does not matter).

So if $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$, then: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (which is $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$)

Not only is this condition necessary for exactness, it also determines exactness.

Example 2.6

1. Solve the differential equation $(1-2xy)dx + (4y^3 - x^2)dy = 0$.

Solution:

With $M(x,y) = 1-2xy$ and $N(x,y) = 4y^3 - x^2$, we notice that: $\frac{\partial M}{\partial y} = -2x = \frac{\partial N}{\partial x}$

So the given functions M and N pass the test for exactness. All that is left to do is to find the function $f(x,y)$ such that: $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

The way to accomplish this is to integrate M with respect to x, integrate N with respect to y, and then merge the results:

$$\int M(x,y)dx = \int (1-2xy)dx = x - x^2y + \text{(a function of } y \text{ alone)}$$

$$\int N(x,y)dy = \int \left(4y^3 - x^2\right)dy = y^4 - x^2y + \text{(a function of } x \text{ alone)}$$

These calculations imply that a function $f(x,y)$ which satisfies both:

$$\frac{\partial f}{\partial x} = M = 1-2xy \text{ and } \frac{\partial f}{\partial y} = N = 4y^3 - x^2 \text{ is: } f(x,y) = x - x^2y + y^4$$

Therefore, the exact equation we were given is satisfied by the family of curves:

$$x - x^2y + y^4 = c$$

2. Solve the differentiation equation $\left(y\cos x + 2xe^y\right) + \left(\sin x + x^2e^y + 2\right)\frac{dy}{dx} = 0$

Solution:

$$M(x,y) = y\cos x + 2xe^y,\ N(x,y) = \sin x + x^2e^y + 2$$

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y,\ \frac{\partial N}{\partial x} = \cos x + 2xe^y$$

Therefore, the differentiation equation is exact.

We know that $\frac{\partial f}{\partial x} = M(x,y)$; $\frac{\partial f}{\partial y} = N(x,y)$. Thus there is an $f(x,y)$, such that:

$$\frac{\partial f}{\partial x} = y\cos x + 2xe^y, \text{ and } \frac{\partial f}{\partial y} = \sin x + x^2e^y + 2 \quad \ldots\text{(I)}$$

Integrating the first of these equations: $\int df = \int \left(y\cos x + 2xe^y\right)dx$, keeping y constant gives: $f(x,y) = y\sin x + x^2e^y + \psi(y)$ …………….(II)

and hence differentiating the new function w.r.t y keeping x constant:

$$\frac{\partial f}{\partial y} = \sin x + x^2e^y \frac{d\psi}{dy} = \sin x + x^2e^y + 2,$$

i.e. the differential is equal to the second equation in (I).

Thus $\frac{d\psi}{dy} = 2 \Rightarrow \int d\psi = \int 2dy$ and $\psi = 2y$ the constant of integration can be omitted; we do not require the most general one.

Substituting for $\psi(y)$ in (II) gives: $f(x,y) = y\sin x + x^2e^y + 2y$

Before $f(x,y)=c$,constant. Hence $c = y\sin x + x^2 e^y + 2y$

3. Solve the differential equation $\left(3x^2 + 2xy\right) + \left(x + y^2\right)\dfrac{dy}{dx} = 0$

Solution:

Here $M_x = 2x,\ N_x = 1$; since $M_y \neq N_x$, the given equation is not exact. To see that it cannot be solved by the procedure above, let us suppose that there is a function $f(x, y)$ such that:

$$f_x = 3x^2 + 2xy,\ f_y = x + y^2 \text{............(I)}$$

Integrating the first of (I) gives:

$$f(x, y) = x^3 + x^2 y + \psi(y) \text{..................(II)}$$

where $\psi(y)$ is an arbitrary function of y only. To try to satisfy the second of (I) we compute f_y from (II), obtaining:

$$f_y = x^2 + \frac{d\psi}{dy} = x + y^2$$

or
$$\frac{d\psi}{dy} = x + y^2 - x^2 \text{...............(III)}$$

Since the right – hand side of (III) depends on x as well as y, it is impossible to solve (III) for $\psi(y)$. Thus there is no $f(x, y)$ satisfying both of (I)

So in the next sub – topic we try to find a factor to make the equation exact and have a function satisfying both (I) – the factor is called integrating factor.

Session 3-2 Integrating factors

3-2.1 Non- Exact Equations

If $M(x,y)dx + N(x,y)dy = 0$................. (2.2) is not exact, sometimes we can turn it into an exact differential equation by multiplying the whole equation by an appropriate factor, called an **integrating factor**, let's say $\mu = \mu(x,y)$, which is a function of x, y. For which we obtain:

$$\mu M(x,y)dx + \mu N(x,y)dy = 0 \text{---------------- (2.13)}$$

and expect it to be exact. Then for (2.13) to be exact then:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Applying product rule we get:

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

$$\Rightarrow M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \text{------------- (2.14)}$$

Simply put: $M\mu_y - N\mu_x = \mu(N_x - M_y)$

$$\Rightarrow N\mu_x - M\mu_y = \mu(M_y - N_x)$$

Remark: Finding the integrating factor for a given equation can be very difficult.

Some of the important rules and procedures follow.

Theorem 2.1

a) Assume $\mu = \mu(x) \Rightarrow \frac{\partial \mu}{\partial y} = \mu_y = 0$,then: $\mu_x = \frac{d\mu}{dx} = \frac{M_y - N_x}{N}$, which is a function of x alone, call this function $\xi(x)$.

Then $\mu(x) = e^{\int \xi(x)dx}$ is an integrating factor of (2.2).

b) Assume $\mu = \mu(y) \Rightarrow \frac{\partial \mu}{\partial x} = 0$, then: $\mu_y = \frac{d\mu}{dy} = \frac{M_y - N_x}{(-M)}$, which is a function of y alone, call this function $\psi(y)$.

Then $\psi(y) = e^{\int \psi(y)dy}$ is an integrating factor of (2.2).

c) If $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \neq 0$, then: $\mu(x,y) = \frac{1}{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}$, is the integrating factor of (2.2).

d) If $M(x,y)dx + N(x,y)dy = 0$ can be written in the form $yg(x,y)dx + xh(x,y)dy = 0$, where $f(x,y) \neq g(x,y)$, then

$$\mu(x,y) = \frac{1}{xM - yN}$$

The following observations are often helpful in finding integrating factors:

(1) If a first-order differential equation contains the combination

$xdx + ydy = \frac{1}{2}d\left(x^2 + y^2\right)$ try some function $x^2 + y^2$ as a multiplier.

(2) If a first-order differential equation contains the combination $xdy + ydx = d\left(xy\right)$, try some function xy as a multiplier.

(3) If a first-order differential equation contains the combination $xdy - ydx$, try $\frac{1}{x^2}$ or $\frac{1}{y^2}$ as a multiplier. If neither of these works, try $\frac{1}{xy}$ or $\frac{1}{x^2 + y^2}$ or some function of these expressions, as an integrating factor, remember that $d\tan^{-1}\left(\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$ and $\tan^{-1}\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$.

Sometimes an integrating factor may be found by inspection, after grouping the terms in the equation, and recognizing a certain group as being a part of an exact differential. The table below has some of the forms already listed above:

Table 2.1

Group of Terms	**Integrating factor**	**Exact differential** $df(x,y)$
$ydx - xdy$	$-\dfrac{1}{x^2}$	$\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
$ydx - xdy$	$\dfrac{1}{y^2}$	$\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$
$ydx - xdy$	$-\dfrac{1}{xy}$	$\dfrac{xdy - ydx}{xy} = d\left(\ln\dfrac{y}{x}\right)$
$ydx - xdy$	$-\dfrac{1}{x^2 + y^2}$	$\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\arctan\dfrac{y}{x}\right)$
$ydx + xdy$	$\dfrac{1}{xy}$	$\dfrac{ydx + xdy}{xy} = d(\ln xy)$
$ydx + xdy$	$\dfrac{1}{(xy)^n}, n > 1$	$\dfrac{ydx + xdy}{(xy)^n} = d\left[\dfrac{-1}{(n-1)(xy)^{n-1}}\right]$
$ydy + xdx$	$\dfrac{1}{x^2 + y^2}$	$\dfrac{ydy + xdx}{x^2 + y^2} = d\left[\dfrac{1}{2}\ln\left(x^2 + y^2\right)\right]$
$ydy + xdx$	$\dfrac{1}{\left(x^2 + y^2\right)^n}, n > 1$	$\dfrac{ydy + xdx}{\left(x^2 + y^2\right)^n} = d\left[\dfrac{-1}{2(n-1)\left(x^2 + y^2\right)^{n-1}}\right]$
$aydx + bxdy$ $(a, b \text{ constant})$	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(aydx + bxdy) = d\left(x^a y^b\right)$

NOTE: If a non – exact equation has a solution, then an integrating factor is guaranteed to exist, but that does not mean it is easy to find.

Example 2.7

1. Solve the differential equation $\left(3xy + y^2\right) + \left(x^2 + xy\right)\dfrac{dy}{dx} = 0$.

Solution:

$$M(x, y) = 3xy + y^2, N(x, y) = x^2 + xy$$

$$M_y = 3x + 2y \text{ and } N_x = 2x + y.$$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore the differential equation is not exact.

Involve (2.14): $M\mu_y - N\mu_x = \mu(N_x - M_y)$

Let μ be a function of x alone, that is, $\mu = \mu(x)$,

Applying theorem 2.1a:

$$\xi(x) = \frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}$$

$$\mu(x) = e^{\int \xi(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Multiplying $\mu(x) = x$ through the differential equation we obtain:

$$\left(3x^2 y + xy^2\right) + \left(x^3 + x^2 y\right)\frac{dy}{dx} = 0$$

which is a new equation in same form i.e.: $M(x,y) + N(x,y)y' = 0$

$$M(x,y) = \left(3x^2 y + xy^2\right), N(x,y) = x^3 + x^2 y$$

$$M_y = 3x^2 + 2xy,\ N_x = 3x^2 + 2xy \text{ ,which is exact.}$$

Let $\quad f_x = 3x^2 y + xy^2$

Integrating:

$$f(x,y) = \int \left(3x^2 y + xy^2\right)dx = x^3 y + \tfrac{1}{2}x^2 y^2 + \psi(y)$$

Find the derivative w.r.t y and equating with $N(x,y)$ we obtain:

$$\Rightarrow f_y = x^3 + x^2 y + \psi'(y) = x^3 + x^2 y$$

$$\Rightarrow \psi'(y) = 0, \qquad \Rightarrow \psi(y) = c$$

Finally $\qquad x^3 y + \tfrac{1}{2}x^2 y^2 = c.$

2. Solve $\left(y^2 - y\right)dx + xdy = 0$.

Solution:

$$M(x,y) = y^2 - y,\ N(x,y) = x$$

$$M_y = 2y - 1 \qquad N_x = 1$$

Hence the differential equation is not exact, and no integrating factor is immediately apparent. Note, however, that if terms are strategically regrouped, the differential equation can be rewritten as:

$$-(ydx - xdy) + ydx = 0 \qquad \text{(I)}$$

The group of terms in parenthesis has many integrating factors in table (2.1). Trying each integrating factor separately, we find that the only one that makes the entire equation exact is:

$$\mu(x,y) = \frac{1}{y^2}.$$

Using this integrating factor we can rewrite (I) as:

$$-\frac{ydx - xdy}{y^2} + 1dx = 0 \qquad \text{(II)}$$

Since (II) is exact, it can now be solved. Alternatively, we note from table 2.1 that (II) can be rewritten as $-d(x/y) + 1dx = 0$, or as $d\left(\frac{x}{y}\right) = 1dx$.

Integrating, we obtain the solution:

$$\frac{x}{y} = x + c \text{ or } y = \frac{x}{x+c}$$

Exercise 2.5

Solve the following differential equation	Ans.

	Equation	Ans.
1.	$2xy + (1+x^2)y' = 0$	$x^2y + y = c$
2.	$y - xy' = 0$	$y = cx$
3.	$(x + \sin y) + (x\cos y - 2y)y' = 0$	$\frac{1}{2}x^2 + x\sin y - y^2 = c$
4.	$y + xy' = 0$	$xy = c$
5.	$3x^2y^2 + (2x^3y + 4y^3)y' = 0$	$x^3y^2 + y^4 = c$
6.	$(x^2 + y + y^2)dx - xdy = 0$	$y = x\tan(x+c)$
7.	$ydx + (1-x)dy = 0$	$cy = x - 1$
8.	$2xy + y^2dy = 0$	$y^2 = 2(c - x^2)$
9.	$1 - 2xyy' = 0$	$y^2 = \ln\lvert cx\rvert$
10.	$(x^2 - y)dx - xdy = 0; y(1) = 5$	$y = \frac{1}{3}x^2 + \frac{14}{3}\left(\frac{1}{x}\right)$

3-2.2 Linear Equations and those Reducible to that form

A **first – order linear equation** is defined as a differential of the form:

$$A(x)y' + B(x)y = C(x), \text{..........................} \quad (2.15)$$

where $A(x) \neq 0$ such that the derivative of the dependent variable exist otherwise it is not a differential equation.

Then (2.14) in the standard form is written as

$$y' + p(x)y = q(x) \text{..................................} \quad (2.16)$$

$$p(x) = \frac{B(x)}{A(x)} \text{ and } q(x) = \frac{C(x)}{A(x)}$$

To divide by $A(x)$ would violate the *fundamental commandment of mathematics*, if $A(x) = 0$, which prohibits division by zero.

We recognise (2.16) as an inexact differential equation which requires an integrating factor $\mu(x)$, that is: $\frac{d\mu(x)}{dy} = 0$

Multiplying through (2.16) by $\mu(x)$ to obtain the following:

$$\mu(x)y'(x) + \mu[p(x)y(x) - q(x)] = 0 \text{.................} \quad (2.17)$$

$$M(x, y) = \mu(x)[p(x)y - q(x)] \text{ and } N(x, y) = \mu(x)$$

If (2.17) is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Thus, $$\frac{\partial}{\partial y}[\mu py - \mu q] = \frac{\partial}{\partial x}\mu(x) = \frac{d\mu(x)}{dx}$$

$$\Rightarrow \mu p = \frac{d\mu}{dx} \text{............} \quad (2.18)$$

Separating variables:

$$\int \frac{d\mu}{\mu} = \int p dx \Rightarrow \ln \mu = \int p dx$$

$$\mu = e^{\int p dx} \text{........................} \quad (2.19)$$

Without loss of operating the constant of integration of (2.19) is unity.

Consider (2.16) and multiply through by: $\mu(x) = e^{\int p(x)dx}$, then we have the exact equation:

$$\mu y' + \mu py = \mu q$$

$$\mu y' + \mu' y = \mu q \text{ by (2.18) that is } \mu p = \mu'$$

$$\frac{d}{dx}(\mu p) = \mu p$$

Integrating both sides:

$$\mu y = \int \mu q dx + c$$

$$\Rightarrow y = \mu^{-1}\int \mu q dx + \mu^{-1}c$$

$$\Rightarrow y = e^{-\int p(x)dx}\int e^{\int p(x)dx} q(x)dx + ce^{-\int p(x)dx} \quad \text{...... (2.20)}$$

The solution is in two parts in the R.H.S. The term $ce^{-\int p(x)dx} = \mu^{-1}c$, called the **complementary function** is the solution of the reduced equation: $y' + p(x)y = 0$ obtained by setting $q(x) = 0$ in (2.16).

The differential equation which has $q(x) = 0$ is called **Homogeneous** differential equation. The other term on the R.H.S of the solution is dependent on $q(x)$, that $\mu^{-1}\int \mu q dx$ is called the particular integral and it satisfies (with the complementary function) the differential equation: $y' + p(x)y = q(x)$, called the **Non – Homogeneous** differential equation. Notice that the particular integral contains no constant.

ALGORITHM

The solution process for a first order linear differential equation is as follows:

1. Put the differential equation in the correct initial form (2.16).
2. Find the integrating factor, $\mu(x)$, using (2.19).
3. Multiply everything in the differential equation by $\mu(x)$ and verify that the left side becomes the product rule $(\mu y)'$ and write it as such.
4. Integrate both sides; make sure you properly deal with the constant of integration.
5. Solve for the solution y.

Let's work a couple of examples.

Example 2.8

1. Find the solution of $\dfrac{dy}{dx} = 5x - \dfrac{3y}{x}$ such that $y(1) = 2$

Solution:

First, we rearrange the equation to put tin the standard form (2.16) of a linear equation:

$$\frac{dy}{dx} + \frac{3}{x}y = 5x$$

Multiply both sides of this by the integrating factor:

$$\mu(x) = e^{\int p(x)dx} = e^{\int \left(\frac{3}{x}\right)dx} = e^{3\log x} = x^3$$

To give the equivalent equation: $x^3 \dfrac{dy}{dx} + 3x^2 y = 5x^4$

Notice that the left – hand side is $\frac{d}{dx}\left(x^3y\right)$, so by integrating both sides, we get:

$$x^3y = \int\left(5x^4\right)dx = x^5 + c \Rightarrow y = x^5 + cx^{-3}$$

Now, to satisfy the initial condition, $y(1) = 2$, the value of c must be 1. Therefore, the solution to the IVP is: $y = x^5 + x^{-3}$.

2. Solve the differential equation $xy' + 2y = 4x^2$

Solution:

$$y' + \tfrac{2}{x}y = 4x \text{ ....}(*)$$

Integrating factor, $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.

Multiply (*) through by x^2:

$$x^2y' + 2xy = 4x^3 \Rightarrow \left(x^2y\right)' = 4x^3 \Rightarrow x^2y = \int 4x^3dx = x^4 + c$$

3-2.3 Equations Reducible to Linear Form

Other first order equations in first degree which are not linear may be reduced to the linear form by means of appropriate transformations. No general rule can be stated; in each instance, the proper transformation is suggested by the form of the equation. Examples of such equations are Bernoulli's equation, Ricatti's equation and the one considered below.

Consider

$$\{xs(y) + t(y)\}y' = \mu(y) \text{ ....} \quad (2.21)$$

in which the coefficient of y' is a linear function of x and **No** other x appears anywhere again.

Thus (2.21) can be expressed as a linear differential equation with x as the dependent variable and y the independent variable. To be able to do this, multiply (30) through by $\frac{dx}{dy}$ to obtain: $\mu(y)\frac{dx}{dy} = xs(y) + t(y)$ or $\mu(y)\frac{dx}{dy} - s(y)x = t(y)$

Thus this is a linear differential equation of x as a function of the independent variable y.

Example 2.9

Solve the differential equation $\left(3x-4y^3\right)y'+y=0$.

Solution:

Clearly the coefficient of y' is linear in x and further more x appears nowhere again.

Therefore we multiply through by $\dfrac{dx}{dy}$ to obtain:

$$\left(3x-4y^3\right)+y\frac{dx}{dy}=0 \Rightarrow y\frac{dx}{dy}+3x=4y^3$$

Whose standard form is : $\dfrac{dx}{dy}+\dfrac{3}{y}x=4y^2$

Integrating factor: $\mu(y)=e^{3\int\frac{dy}{y}}=e^{\ln y^3}=y^3 \Rightarrow y^3\dfrac{dx}{dy}+3y^2x=4y^5$

$$\int d\left(xy^3\right)=\int 4y^5 \Rightarrow xy^3=(4/6)y^6+c$$

$$\Rightarrow 3xy^3=2y^6+c$$

BERNOULLI'S EQUATION

A Bernoulli equation is a first order differential of the form:

$$y'+p(x)y=q(x)y^n \quad \ldots\ldots\ldots\ldots\ldots (2.22)$$

in which n is any real number. You would notice that the equation differs from a standardized first order linear differential equation in first degree at the right side of the equation. Here there is a multiplication of y^n, $n\neq 0, n\neq 1$, at the right hand side(RHS)

Then divide (2.22) through by y^n and substitute $z=y^{1-n}$.

$$y'y^{-n}+p(x)y^{1-n}=q(x)$$

Let $\quad z=y^{1-n} \Rightarrow z'=(1-n)y^{-n}y' \Rightarrow \dfrac{z'}{1-n}=y^n y'$

The equation becomes: $z'+(1-n)p(x)z=(1-n)q(x)$

The solution is obtained using first order linear equation in first degree solution techniques.

Example 2.10

Solve the differential equation $x^2y-x^3\dfrac{dy}{dx}=y^4\cos x$.

Solution:

Write it in the form $y' + p(x)y = q(x)y^n$

i.e. $$y' - \frac{y}{x} = -\frac{\cos x}{x^3} y^4 \qquad \ldots. \qquad \text{(I)}$$

Divide through by y^3: $y^{-4}y' - \dfrac{y^{-3}}{x} = -\dfrac{\cos x}{x^3}$

Let $z = y^{-3}$ and $z' = -3y^{-4}y'$, and substitute into (I)

We get: $z' + \dfrac{3}{x} z = \dfrac{3\cos x}{x^3}$, which is linear first order in first degree.

$$\Rightarrow \mu(x) = e^{\int p(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

Therefore, $z \cdot x^3 = \displaystyle\int \frac{3\cos x}{x^3} x^3 dx = \int 3\cos x dx \Rightarrow zx^3 = \sin x + c$

But $z = y^{-3} \Rightarrow \dfrac{x^3}{y^3} = 3\sin x + c$

Hence, $y^3 = \dfrac{x^3}{3\sin x + c}$

Exercise 2.6

Solve the following differential equation. Ans

1.	$xy' + 3y = 6x^3$	$y = x^3 + x^{-3}c$
2.	$y' + y = \sin x$	$y = ce^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$
3.	$\frac{dQ}{dt} + \frac{2}{10+2t}Q = 4; Q(2) = 100$	$Q(t) = \frac{4t^2 + 40t + 1304}{2t+10}(t > -2)$
4.	$y' + y = y^2$	$y = \frac{1}{ce^x + 1}$
5.	$y' + \frac{2}{x}y = -x^9y^5; y(-1) = 2$	$\frac{1}{y^4} = -\frac{31}{16}x^8 + 2x^{10}$

RICCATI'S EQUATION

The equation $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$ .... (2.23) is non – linear, and needs a particular solution, before one can have the chance to solve it.

Consider the new function z defined by: $z = \frac{1}{y - y_1}$, where y_1 is the particular solution.

Then easy calculations give: $\frac{dz}{dx} = -[Q(x) + 2y_1 R(x)]z - R(x)$ ...... (2.24)

Which is a linear equation satisfied by new function z. Once it is solved, we go back to via the relation: $y = y_1 + \frac{1}{z}$.

Example 2.11

Solve the equation $\frac{dy}{dx} = -2 - y + y^2$ where $y_1 = 2$ is a particular solution.

Solution:

First we verify that $y_1 = 2$ is a solution, otherwise our calculation is fruitless.

Let $\qquad z = \frac{1}{y-2} \Rightarrow y = 2 + \frac{1}{z}$

$$y' = -\frac{z'}{z^2}$$

Hence, from the equation satisfied by y, we get:

$$-\frac{z'}{z^2} = -2 - \left(2 + \frac{1}{z}\right) + \left(2 + \frac{1}{z}\right)^2 \Rightarrow -\frac{z'}{z^2} = \frac{3}{z} + \frac{1}{z^2}$$

Hence $z' = -3z - 1$.

This is a linear equation. The general solution is given by:

$$z = e^{-3x}\left(-\frac{1}{3}e^{3x} + c\right) = -\frac{1}{3} + ce^{-3x}$$

Therefore, we have: $y = 2 + \frac{1}{-\frac{1}{3} + ce^{-3x}}$

Note:

If one remembers the equation satisfied by z, then the linear equation satisfied by the functions z is: $\frac{dz}{dz} = -[Q(x) + 2y_1 R(x)]z - R(x) = -(-1+4)z - 1 = -3z - 1$, since $P(x) = -2, Q(x) = -1,$ and $R(x) = 1.$

Exercise 2.7

1. Verify that $y_1 = \sin x$ is a solution to $\frac{dy}{dx} = \frac{2\cos^2 x - \sin^2 x + y^2}{2\cos x}$ and solve the differential equation using the initial condition $y(0) = -1$

2. Solve the following;

   i) $y' = (1-x)y^2 + (2x-1)y - x$; given solution $y = 1$

   ii) $y' = -y^2 + xy + 1$; given solution $y = x$

   iii) $y' = -8xy^2 + 4x(4x+1)y - (8x^3 + 4x^2 - 1)$; given soln. $y = x$

Session 4-2 Applications of First-Order Differential Equation (Modeling)

We now move into one of the main applications of differential equations. **Modeling** is the process of writing a differential equation to describe a physical situation. Almost all of the differential equations that you will use in your job (as an engineer) are there because somebody, at some time, modeled a situation to come up with the differential equation that you are using.

This session is not intended to teach you how to go about modeling physical situations. A whole course could be devoted to the subject of modeling and still not cover everything! This session is designed to introduce you to the process of modeling and show you what is involved in modeling.

In all of these situations we will be forced to make assumptions that do not accurately depict reality in most cases, but without them the problems would be very difficult and beyond the scope of this discussion (and the course in most cases to be honest).

We will look at different situations in this session: Law of exponential change – that deals with growth and decay problems – **Temperature problems, Falling Bodies, Mixing Problems and Electrical Circuits. But we will part particular attention on electrical problems.**

4-2.7 Electrical Circuits

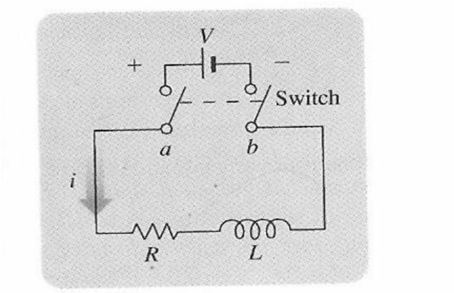


Fig. 2.2

The diagram in Figure 2.2 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, show as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source (may be emf) of E (or V) volts.

Ohm's law, $E = IR$ or $(V = IR)$, has to be modified for such a circuit. The modified form is:

$$L\frac{dI}{dt} + RI = E \text{ ……………………..(2.38)}$$

where I is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.

For an RC circuit consisting of a resistance, a capacitance C (in farads), an emf and no inductance (Fig 2.2) the equation governing the amount of electrical charge q (in coulombs) on the capacitor is:

$$\frac{dq}{dt}+\frac{1}{RC}q=\frac{E}{R} \qquad \text{..................(2.39)}$$

The relationship between q and I is:

$$I=\frac{dq}{dt} \qquad \text{.............................(2.40)}$$

Example

1. The switch in the RL circuit in (Fig 2.2) is closed at time $t=0$. How will the current flow as a function of time?

Solution:

Equation (2.38) is a liner first-order differential equation for I as a function of t. Its standard form is:

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L} \qquad \text{.......................(2.41)}$$

and the corresponding solution, given that $I=0$ when $t=0$, is:

$$I=\frac{E}{R}-\frac{E}{R}e^{-(R/L)t} \qquad \text{.......................(2.42)}$$

Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t}\to 0$ as $t\to\infty$.

Thus, $\lim\limits_{t\to\infty} I=\lim\limits_{t\to\infty}\left(\frac{E}{R}-\frac{E}{R}e^{-(R/L)t}\right)=\frac{E}{R}-\frac{E}{R}\cdot 0=\frac{E}{R}$

At any given time, the current is theoretically less than E/R, but as time passes, the current approaches the **steady-state value** E/R. According to the equation:

$$L\frac{dI}{dt}+RI=E,$$

$I=E/R$ is the current that will flow in the circuit if either $L=0$ (no inductance) or $\frac{dI}{dt}=0$ (steady current, $I=$constant).

Equation (2.42) expresses the solution of equation (2.41) as the sum of two terms: a **steady-state solution** V/R and a **transient solution** $-(V/R)e^{-(R/L)t}$ that tends to zero as $t\to\infty$.

2. An RL circuit has emf of 5 volts, a resistance of 50 ohms, an inductance of 1 henry, and no initial current. Find the current in the circuit at any time t

Solution:

Here $E=5, R=50$, and $L=1$, hence (2.41) becomes: $\frac{dI}{dt}+50I=5$

The corresponding general solution is: $I=ce^{-50t}+\frac{1}{10}$

Using the initial condition $I=0$ when $t=0$, we get

$$I=-\frac{1}{10}e^{-50t}+\frac{1}{10} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

as a particular solution.

The quantity $-\frac{1}{10}e^{-50t}$ in (I) is called the transient current, since this quantity goes to zero ("dies out") as $t\to\infty$. The quantity $\frac{1}{10}$ in (I) is called the steady – state current. As $t\to\infty$, the current I approaches the value of the steady – state current.

3. The RL circuit has an emf given (in volts) by $3\sin t$, a resistance of 10 ohms, an inductance of 0.5 henry, and an initial current of 6 amperes. Find the current in the circuit at any time t

Solution:

Here, $E=3\sin 2t$, $R=10$, and $L=0.5$; hence (2.41) becomes: $\frac{dI}{dt}+20I=6\sin 2t$

General solution is: $I=ce^{-20t}+\frac{30}{101}\sin 2t-\frac{3}{101}\cos 2t$

At $t=0$, $I=6$; $6=ce^{-20(0)}+\frac{30}{101}\sin 2(0)-\frac{3}{101}\cos 2(0)\Rightarrow c=\frac{609}{101}$

The current at any time t is: $I=\frac{609}{101}e^{-20t}+\frac{30}{101}\sin 2t-\frac{3}{101}\cos 2t$

The steady state current: $I=\frac{30}{101}\sin 2t-\frac{3}{101}\cos 2t$

4. The RC circuit has an emf given (in volts) by $400\sin t$, a resistance of 100 ohms, an capacitance of 10^{-2} farad. Initially there is no charge on the capacitor. Find the current in the circuit at any time t

Solution:

We first fine the charge q and then use (2.40) to obtain the current. Here, $E=400\cos 2t$, $R=100$, and $C=10^{-2}$; hence (2.39) becomes:

$$\frac{dq}{dt}+q=4\cos 2t$$

This equation is linear, and its solution is (two integrations by parts are required)

$$q=ce^{-t}+\tfrac{8}{5}\sin 2t+\tfrac{4}{5}\cos 2t\,.$$

When there is no charge, that means, $q=0$, so at $t=0, q=0$; hence

$$0=ce^{-(0)}+\tfrac{8}{5}\sin 2(0)+\tfrac{4}{5}\cos(0)$$

$$\Rightarrow c=-\tfrac{4}{5}$$

Thus, $q=-\tfrac{4}{5}e^{-t}+\tfrac{8}{5}\sin 2t+\tfrac{4}{5}\cos 2t$ and using (2.40), we obtain:

$$I=\frac{dq}{dt}=\tfrac{4}{5}e^{-t}+\tfrac{16}{5}\sin 2t-\tfrac{8}{5}\cos 2t$$

Summary

- General first – order differential equation: $F(x,y,y')=0$
- Standard form: $y'=f(x,y)$
- Differential form: $M(x,y)dx+N(x,y)dy=0$
- First – order differential equation is said to be separable, if the differential form have $M(x,y)=m(x)p(y)$, which the product of a function x and a function y and also $N(x,y)=n(x)q(y)$, also a function of x and a function, then we can write the differential form as:

  $$m(x)p(y)dx+n(x)q(y)dy=0$$

  We then divide throughout by $p(y)n(x)$, we get:

  $$\Rightarrow\frac{m(x)}{n(x)}dx+\frac{p(y)}{q(y)}dy=0$$

  now it is separated in each variable x and y.

  Integrating both sides:

  $$\Rightarrow\int\frac{m(x)}{n(x)}dx+\int\frac{p(y)}{q(y)}dy=c$$

  where c is arbitrary constant.

- ***Separation of Variables.*** To solve the initial – value problem

  $$y'=f(x,y);y(x_0)=y_0$$

(a) Separate the variables, so that all of the x-dependence is on one side of the equation, and all of the y-dependence is on the other side of the equation.

(b) Integrate both sides of the equation.

(c) Substitute the initial condition to solve for the constant of integration

- Reduction to Separable
  - A formal definition of homogeneity is: *A function of two variables,* $f(x,y)$*, is said to be* ***homogeneous of degree*** n *if there is constant* n *such that*:
    $f(\lambda x,\lambda y)=\lambda^n f(x,y)$

  - A differential equation of the form (differential form):
    $M(x,y)dx+N(x,y)dy=0$ is **homogeneous** if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the **same degree**.

    Or

    If the equation in standard form: $y'=f(x,y)$ depends only on the ratio $\frac{y}{x}$ or $\frac{x}{y}$, then it is said to be **homogenous**.

    To solve homogeneous equations, turn them into separable ones using the substitution: $y=x\upsilon$, where υ is a function of x; also:

    $$\frac{dy}{dx}=\upsilon+x\frac{d\upsilon}{dx} \text{ or } dy=xd\upsilon+\upsilon dx$$

  - **A Transformation (Equations in which $M(x,y)$ and $N(x,y)$ are linear but not homogeneous)**

Another type of equations that can be reduced to a more basic type by means of a suitable transformation is an equation of the form:
$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$ where
a_1,b_1,c_1,a_2,b_2,c_2 are constants where
$M(x,y)=a_1x+b_1y+c_1$ and $N(x,y)=a_2x+b_2y+c_2$
written as $M(x,y)+N(x,y)y'=0$, which are not homogeneous.
However the differential equation can be made homogeneous and solved accordingly provided;

Case I

If $\dfrac{a_2}{a_1} \neq \dfrac{b_2}{b_1}$, then the transformation:

$x = \mathbf{X} + h$ and $y = \mathbf{Y} + k$, where (h, k) is the solution of the system $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, that is $x = h$ and $y = k$, which reduces to a homogeneous equation:

$$(a_1 \mathbf{X} + b_1 \mathbf{Y}) d\mathbf{X} + (a_2 \mathbf{X} + b_2 \mathbf{Y}) d\mathbf{Y} = 0$$

in the variables $\mathbf{X}$ and $\mathbf{Y}$.

Case II

If $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = k$, then the transformation $z = a_1 x + b_1 y$ reduces the equation to a separable equation in the variables x and z:

$$\frac{dz}{dx} = a_1 + b_1 \frac{dy}{dx}$$

- **Exact Equations**

Given a function $f(x, y)$, its **total differentials**, df, is defined as:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

This shows that the family of curves (or general solution) $f(x, y) = c$ satisfies the differential equation $df = 0$.

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

So if there exists a function $f(x, y)$ such that:

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}$$

then $M(x, y)dx + N(x, y)dy$ is called an **exact differential**, and the equation:

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an **exact** equation, whose solution is the family $f(x, y) = c$.

- **Integrating Factors – Non-exact**

If $M(x, y)dx + N(x, y)dy = 0$ is not exact, sometimes we can turn it into an exact differential equation by multiplying the whole equation by an appropriate factor, called an **integrating factor**, let say $\mu = \mu(x, y)$, which is a function of x, y.

For which we obtain: $\mu M(x,y)dx + \mu N(x,y)dy = 0$ and expect it to be exact.

For it to be exact then: $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x}$

Applying product rule we get: $\mu\dfrac{\partial M}{\partial y} + M\dfrac{\partial \mu}{\partial y} = \mu\dfrac{\partial N}{\partial x} + N\dfrac{\partial \mu}{\partial x}$

$$\Rightarrow M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Simply put: $M\mu_y - N\mu_x = \mu(N_x - M_y)$

$$\Rightarrow N\mu_x - M\mu_y = \mu(M_y - N_x)$$

- **Linear First – Order Equations**

  A **first – order linear** in the standard form is written as

  $y' + p(x)y = q(x)$ ……………………………(2.16)

  We recognise (2.16) as an inexact differential equation which requires an integrating factor $\mu(x)$, that is: $\dfrac{d\mu(x)}{dy} = 0$

  Multiplying through (2.16) by $\mu(x)$ to obtain the following:

  $\mu(x)y'(x) + \mu[p(x)y(x) - q(x)] = 0$ ……………………..(2.17)

  $M(x,y) = \mu(x)[p(x)y - q(x)]$ and $N(x,y) = \mu(x)$

  If (2.17) is exact, then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

  Thus, $\dfrac{\partial}{\partial y}[\mu p y - \mu q] = \dfrac{\partial}{\partial x}\mu(x) = \dfrac{d\mu(x)}{dx}$

  $\Rightarrow \mu p = \dfrac{d\mu}{dx}$ ………….(2.18)

  Separating variables: $\displaystyle\int\frac{d\mu}{\mu} = \int p\,dx \quad \Rightarrow \ln\mu = \int p\,dx$

  $\mu = e^{\int p\,dx}$ ……………………..(2.19)

  Without loose of operating the constant of integration of (2.19) is unity.

  Consider (2.16) and multiply through by: $\mu(x) = e^{\int p(x)dx}$,

  then we have the exact equation:

  $\mu y' + \mu p y = \mu q$

  $\mu y' + \mu' y = \mu q$ by (2.18) that is $\mu p = \mu'$

$$\frac{d}{dx}(\mu p) = \mu p$$

Integrating both sides:

$$\mu y = \int \mu q dx + c$$

$$\Rightarrow y = \mu^{-1} \int \mu q dx + \mu^{-1} c$$

$$\Rightarrow y = e^{-\int p(x)dx} \int e^{\int p(x)dx} q(x) dx + ce^{-\int p(x)dx} \quad \ldots\ldots(2.20)$$

The solution is in two parts in the R.H.S. The term $ce^{-\int p(x)dx}$ = $\mu^{-1}c$, called the **complementary function** is the solution of the reduced equation:

$$y' + p(x)y = 0$$

obtained by setting $q(x) = 0$ in (2.16). The differential equation which has $q(x) = 0$ is called **Homogeneous** differential equation.

The other term on the R.H.S of the solution is dependent on $q(x)$, that $\mu^{-1} \int \mu q dx$ is called the particular integral and it satisfies (with the complementary function) the differential equation:

$$y' + p(x)y = q(x),$$

called the **Non – Homogeneous** differential equation.
Notice that the particular integral contains no constant.

- **Bernoulli's Equation**

A Bernoulli's equation is a first order differential of the form:

$$y' + p(x)y = q(x)y^n \quad \ldots\ldots\ldots\ldots\ldots(2.22)$$

in which n is any real number. Here there is a multiply of y^n, $n \neq 0, n \neq 1$, at the right hand side(RHS)

Then divide (2.22) through by y^n and substitute $z = y^{1-n}$.

$$y'y^{-n} + p(x)y^{1-n} = q(x)$$

Let $\quad z = y^{1-n}$

$$\Rightarrow z' = (1-n)y^{-n}y'$$

$$\Rightarrow \frac{z'}{1-n} = y^n y'$$

The equation becomes:

$$z' + (1-n)p(x)z = (1-n)q(x)$$

The solution is obtained using first order linear equation in first degree solution techniques.

- **Riccati's Equation**

The equation $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$

Consider the new function z defined by:

$$z = \frac{1}{y - y_1}$$

where y_1 is the particular solution.

Then easy calculations give: $\frac{dz}{dx} = -[Q(x) + 2y_1R(x)]z - R(x)$

Which is a linear equation satisfied by new function z. Once it is solved, we go back to via the relation: $y = y_1 + \frac{1}{z}$.

UNIT 3

LINEAR DIFFERENTIAL EQUATION OF HIGHER ORDER

Introduction

So far, all equations we have studied have been first order, which means that they contained only the first derivative, say y'. Now we will turn to higher – order equation. We will start with one of the most common (and, fortunately, easily solved) types of higher – order equation – the linear equations with constant coefficients then we proceed with other forms.

Learning Objectives

After going through this unit, you would be able to:

- solve most linear differential equation of higher order with constant coefficient.
- explain the method of attack to a particular linear differential equation and which method is most efficient
- recognise that solutions of higher order differential equation are linearly independent and can use the Wronskian to determine it.

Session 1-3 Linear Differential Equation of Higher Order

A general nth – order linear differential equation has the form:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=R(x) \quad ..... (3.1)$$

where $a_n(x)\neq 0$ and the coefficients $a_i(x)(i=0,1,...n-1,n)$,and $f(x)$ are given functions of x or are constants.

This equation (3.1) is **homogeneous** if $f(x)=0$ and **non-homogeneous** if $f(x)\neq 0$.

The solution of the **homogeneous** differential equation is called the complimentary function (conventionally y_c) and the solution of the **non-homogeneous** part is known as the particular integral (conventionally y_p).

Then the general solution = complimentary function + particular integral (that is $y = y_c + y_p$).

1-3.1 Homogeneous Equation with Constant Coefficients

A general homogeneous differential equation with constant coefficients is:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \ldots\ldots(3.2)$$

We start problem solving with second order differential equation of the form:

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.3)$$

We consider a trial solution $y = e^{rx}$, exponential function, suitably chosen r to be determined for the constant coefficient linear differential equation. Then (3.3) becomes:

$$\left(a_2 r^2 + a_1 r + a_0\right) e^{rx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.4)$$

clearly $e^{rx} \neq 0$, hence; $a_2 r^2 + a_1 r + a_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$

or for convenience, $ar^2 + br + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$

Hence if r is root of this quadratic equation (3.5), often called the ***auxiliary*** or ***characteristic equations***, e^{rx} is a solution of (3.3).

The roots of (3.5) are: $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

There are three cases to consider:

Case 1: If the roots are real and distinct $\left(b^2 - 4ac > 0\right)$, that is, $r = r_1$, $r = r_2$ and $r_1 \neq r_2$ then the general solution of the differential equation is:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: If the roots are real and identical $\left(b^2 - 4ac = 0\right)$, that is, $r = r_1 = r_2$, if we denote the double root by r, then the general solution of the differential equation is:

$y = c_1 e^{rx} + c_2 x e^{rx}$

Case 3: If the roots are not real. In this case, the roots are complex conjugates $\left(b^2 - 4ac < 0\right)$, that is $r = \alpha \pm i\beta$, then the general solution of the differential equation is $y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$ where $\alpha = \dfrac{b}{2a}$; $\beta = \dfrac{\sqrt{b^2 - 4ac}}{2a}$

Example 3.1

1. Solve the differential equation $y'' + 2y' + y = 0$

Solution:

Let $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$:

$$r^2 e^{rx} + 2re^{rx} + e^{rx} = 0 \quad r^2 + 2r + 1 = 0 : \; r = -1$$

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

2. Solve the differential equation $y'' + 2y' + 5y = 0$.

Solution:

Aux: $r^2 + 2r + 5 = 0 \quad r = -1 \pm 2i$

$$y = e^{-x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

3. Solve $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$

Solution:

Aux: $r^2 + r - 6 = 0 \Rightarrow r_1 = 2,\ r_2 = -3$

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

1-3.2 Homogeneous with constant coefficient of order 3 and higher

Theorem I: Distinct Real Roots

In case 1; for an n th order linear differential equation with constant coefficients, if $r_1, r_2, \ldots r_n$ are distinct roots of the characteristic equation resulting from the differential equation, then the general solution is of the form: $y = c_1 e^{r_1} + c_2 e^{r_2} + \ldots. + c_n e^{r_n}$

Theorem II: Repeated Roots

In case 2; if the characteristic equation has a repeated root r of multiplicity k , then the part of a general solution of the differential equation of the form:

$$y = \left(c_1 + c_2 x + c_3 x^2 + \ldots + c_k x^{k-1}\right)e^{rx}$$

Example 3.2

1. Find a general solution of $y^{(3)} - y'' - 6y' = 0$

Solution:

Aux: $r^3 - r^2 - 6r = 0 \;\; \Rightarrow r\left(r-3\right)\left(r+2\right) = 0\,; r = 0, r = 3, r = -2$

Hence, $y = c_1 + c_2 e^{3x} + c_3 e^{-2x}$

2. Find a general solution of $y^{(4)} + 3y^{(3)} + 3y'' + y' = 0$

Solution:

Aux: $r^4 + 3r^3 + 3r^2 + r = 0$

$r_1 = 0,$ and it has also the triple $(k = 3)$ root $r_2 = -1$

$$y = c_1 + \left(c_2 + c_3 x + c_4 x^2\right)e^{-x}$$

Exercise 3.1

Solve the following linear differential equation.

		Ans.
1.	$y'' + 4y = 0$	$y = c_1 \cos 2x + c_2 \sin 2x$
2.	$4y'' + 4y' - 3y = 0$	$y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{3}{2}x}$
3.	$2y'' - 9y' = 0$	$y = c_1 + c_2 e^{\frac{9}{2}x}$
4.	$9y''' + 12y'' + 4y' = 0$	$y = c_1 + c_2 e^{-(2x/3)} + c_3 x e^{-(2x/3)}$
5.	$y''' + y'' - y' - y = 0$	$y = c_1 e^{x} + c_2 e^{-x} + c_3 x e^{-x}$
6.	$y'' + 4y' + 4y = 0; y(0) = 1, y'(0) = 1$	$y = (1 + 3x)e^{-2x}$
7.	$y'' + 2.2y' + 1.17y = 0; y(0) = 2, y'(0) = -2.6$	$y = 0.3e^{-x/4} + 0.5e^{x/2}$
8.	$y'' - 6y' + 25y = 0; y(0) = 3, y'(0) = 1$	$y = e^{3x}\left(3\cos 4x - 2\sin 4x\right)$
9.	$y^{(4)} + 4y'' = 0$	$y = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x$
10.	$y''' + 10y'' + 25y' = 0; y(0) = 3, y'(0) = 4, y''(0) = 5$	

1-3.3 Theory of Solutions of Linear Differential Equations

Consider a Homogeneous second – order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad \text{..............................(3.6)}$$

Principle of Superposition

Let y_1 and y_2 be two solutions of the homogeneous linear equation (3.6) on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2 \quad \text{.........................(3.7)}$$

is also a solution of (3.6) on I. *Prove as exercise.*

Definition:

Linear Independence of Two Functions and the Wronskian

Two functions defined on an open interval are called *linearly independent* provided that neither is a constant multiple of the other. Two functions are said to be linearly dependent if they are not linearly independent; that is, one is a constant multiple of the other.

In general, a set of functions (solutions to differential equation) $\{y_1(x), y_2(x), ..., y_n(x)\}$ is *linearly dependent* on $a \le x \le b$ if there exist constants, $c_1, c_2, ..., c_n$, not all zero, such that

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n \equiv 0 \quad \text{.........} \ (3.8)$$

on $a \le x \le b$.

Example 3.2

$y_1 = \cos x$ and $y_2 = \sin x$ are two solutions of the equation $y'' + y = 0$. By principle of superposition $y = 3y_1 - 2y_2 = 3\cos x - 2\sin x$ is also a solution.

We can always determine whether two given functions f and g are linearly dependent on an interval I by noting at a glance whether either of the two quotients f/g and g/f as a constant on I.

The following pairs are linearly independent on the entire real line:

$$(\sin x, \cos x),\ (e^x, e^{-2x}),\ (e^x, xe^x),\ [(x+1), x^2], \text{ and } (x, |x|).$$

Given two functions f and g, the **Wronskian** of f and g is the determinant :

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

For example, $W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \ne 0$

while $W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \ne 0 \forall x$

Given two functions $f(x)$ and $g(x)$ that are differentiable on some interval I :

1. If $W(f,g)(x) \ne 0$ for some x_0 in I, then $f(x)$ and $g(x)$ are linearly independent on the interval F.
2. If $f(x)$ and $g(x)$ are linearly dependent on I then $W(f,g)(x) = 0$ for all x is the interval.

Note:
It is possible for two linearly independent functions to have a zero Wronskian.

Suppose that y_1 and y_2 are two solutions of homogeneous second order linear equation $y'' + p(x)y' + q(x)y = 0$ on open interval I on which $p(x)$ and $q(x)$ are continuous. If $W(y_1, y_2) \neq 0$, then the two solutions are called a **fundamental set of solutions** and the general solution to (3.6) is $y = c_1 y_1 + c_2 y_2$.

Theorem 3.1
The n^{th} - order linear *homogeneous* differential equation (3.2) always has n linearly independent solutions. If $y_1(x), y_2(x), ..., y_n(x)$ represent these solution, then the general solution (3.2) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \;\ldots\ldots \quad (3.9)$$

where $c_1, c_2, ..., c_n$ denote arbitrary constants.

Exercise 3.2

Determine if the following sets of functions are linearly dependent or linearly dependent

a) $f(x) = 9\cos 2x,\; g(x) = 2\cos^2 x - 2\sin x$

b) $f(x) = 2x^2,\; g(x) = x^4$

1-3.4 Reduction of Order

Consider the non – constant coefficient, second order differential equations of the form

$$y'' + p(x)y' + q(x)y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (3.6)$$

In general, finding solutions to these kinds of differential equations can be much more difficult than finding solutions to constant coefficient differential equation.

We discuss here the method of **reduction of order**, which enables us to use one known solution, say, y_1 to find a second linearly independent solution y_2.

Solution of Reduction Order:

Consider $\quad y'' + p(x)y' + q(x)y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.6)$

Let $\quad y_2(x) = v(x)y_1(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \; (3.10)$

We begin by substituting the expression in (3.10) into (3.6), using the derivatives:

$$y_2' = vy_1' + v'y_1 \Rightarrow y_2'' = vy_1'' + 2v'y_1' + v''y_1$$

We get: $[vy_1'' + 2v'y_1' + v''y_1] + p[vy_1' + v'y_1] + qvy_1 = 0$ and rearrangement gives

$$v[y_1'' + py_1' + qy_1] + v''y_1 + 2v'y_1' + pv'y_1 = 0$$

but $y_1'' + py_1' + qy_1 = 0$ is y_1 is a solution of (3.6).

This leaves the equation $y_1v'' + (2y_1' + py_1)v' = 0$ ………… (3.11)

The key to the success of this method is that (3.11) is linear in v'. Thus the substitution of (3.10) has reduced the second order linear equation in (3.6) to the first order (in v') in linear equation in (3.11)

If we write $u = v'$ and assume that y_1 never vanishes on I, then (3.10) yields:

$$u' + \left(\frac{2y_1'}{y_1} + p(x)\right)u = 0 \text{ ……………………………… (3.12)}$$

An integrating factor for (3.9) is:

$$\mu = \exp\left\{\int\left[2\frac{y_1'}{y_1} + p(x)\right]dx\right\} = \exp\left[2\ln|y_1| + \int p(x)dx\right]$$

Thus $\mu = y_1^2 e^{\int p(x)dx}$

We now integrate the equation in (3.6) to obtain: $uy_1^2 e^{\int p(x)dx} = c$

So $v' = u = \dfrac{c}{y_1^2} e^{-\int p(x)dx}$

Another integration now gives: $\dfrac{y_2}{y_1} = v = c\displaystyle\int \frac{e^{-\int p(x)dx}}{y_1^2}dx + k$

With the particular choices $c = 1$ and $k = 0$, we get:

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}dx \text{ ………………(3.13)}$$

Example 3.3

1. Find the general solution to $2x^2y'' + xy' - 3y = 0$ given that $y_1(x) = x^{-1}$ is a solution.

Solution:

$$2x^2y'' + xy' - 3y = 0 \Rightarrow y'' + \frac{1}{2x}y' + \frac{3}{2x^2}y = 0 \text{ where } p(x) = \frac{1}{2x}.$$

$$y_2 = y_1\int \frac{e^{-\int p(x)dx}}{y_1^2}dx \Rightarrow y_2 = x^{-1}\int \frac{e^{-\int \frac{1}{2x}dx}}{\left(x^{-1}\right)^2}dx = x^{-1}\int \frac{x^{-\frac{1}{2}}}{x^{-2}}dx$$

$$\Rightarrow y_2 = x^{-1}\int x^{\frac{3}{2}}dx = x^{-1}\left(\tfrac{2}{5}x^{\frac{5}{2}} + c\right) \text{ let } c = 0 \Rightarrow y_2 = \tfrac{2}{5}x^{\frac{3}{2}}$$

Thus a general solution is: $y = c_1x^{-1} + c_2x^{\frac{3}{2}}$

4. Given the solution $y_1 = x$, find a general solution of

$$\left(x^2 - 1\right)y'' - 2xy' + 2y = 0 \ (x^2 < 1)$$

Solution:

First divide each term by $x^2 - 1$ to obtain: $y'' - \dfrac{2x}{x^2-1}y' + \dfrac{2}{x^2-1}y = 0$

With $p(x) = -2x/\left(x^2 - 1\right)$.

Hence $e^{-\int p(x)dx} = \exp\left(\int \dfrac{2x}{x^2-1}dx\right) = \exp\left(\ln\left|x^2 - 1\right|\right) = 1 - x^2$ since $x^2 < 1$.

Therefore, the formula in (37) yields: $y_2 = x\displaystyle\int \frac{1-x^2}{x^2}dx = x\int \left(x^{-2} - 1\right)dx = -1 - x^2$

Thus a general solution is: $y = c_1x + c_2\left(1 + x^2\right)$

Exercise 3.3

Given one solution y_1 for each question, find a second linearly independent solution y_2 of the following differential equation:

1. $y'' - 4y' + 4y = 0;\quad y_1 = e^{2x}$
2. $x^2y'' + 3xy' + y = 0;\quad y_1 = \frac{1}{x}$
3. $x^2y'' - 4xy' + \left(x^2 + 6\right)y = 0\ ;\ y_1 = x^2 \sin x$
4. $x^2y'' + xy' - 9y = 0;\quad y_1 = x^3$
5. $(x+1)y'' - (x+2)y' + y = 0;\quad y_1 = x^3$
6. $4y'' - 4y' + y = 0\ ;\quad y_1 = e^{\frac{x}{2}}$
7. $xy'' - 3y' = 0\ ;\quad y_1 = 1$

1-3.5 Linear Non – Homogeneous Equation with Constant Coefficients

A general linear non – homogeneous constant coefficient equation is:

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = f(x) \text{.....................} \quad (3.14)$$

$f(x)$ is a given function of x, and $a_0, a_1, ..., a_{n-1}, a_n$ also given constants.

The general solution of (3.11) is given as:

General solution = Complimentary Function $y_c(x)$ + Particular Integral $y_p(x)$

There are several different methods for finding solutions of non-homogeneous linear equations. Here we study three methods to find our Particular Integral, since the complimentary is from the homogeneous part, which we have done already;

(I) Method of Undetermined Coefficients
(II) Method of Variation of Parameter
(III) Method of the Linear Differential Operator

1-3.6 Method of Undetermined Coefficients

The main focus of this section is the structure of solutions of non-homogeneous linear differential equations rather than solution methods. Knowledge of the structure of solutions makes it easier to find solutions.

The method of undetermined coefficients is a method for finding particular solutions for some differential equations of the form (3.14). The method consists of three steps:

a. We assume a trial solution function $y_p(x)$, a family of functions that is guaranteed to include a correct particular solution;
b. Find the image of the trial solution whose form depends on $f(x)$ of (3.14).
c. $y_p(x)$ is then substituted into (3.14) and the constants take values which satisfy (3.14) completely

(a) For $f(x) = ae^{bx}$ a, b are constants, if the auxiliary equation of (3.14) has $r = b$ as a root of multiplicity k, then the trial function is of the form $y_p(x) = Ax^k e^{bx}$, where A constant to be determined. If $r = b$ is **Not** a root, take $k = 0$.

Example 3.4

Find a particular integral of $y'' - 3y' + 2y = e^{2x}$.

Solution:

The auxiliary equation is: $r^2 - 3r + 2 = 0$, $\Rightarrow (r-2)(r-1) = 0 \therefore r = 2,1$

Hence $f(x)=e^{2x}$ that is $b=2,$ a root.

The trial function is: $y_p(x)=Axe^{2x}$

Substituting this into the differential equation:

$$y_p(x)=Axe^{2x}, y_p'(x)=Ae^{2x}+2xAe^{2x}$$

$$y_p''(x)=2Ae^{2x}+2Ae^{2x}+4Axe^{2x}$$

$$\Rightarrow 4Ae^{2x}+4Axe^{2x}-3Ae^{2x}-6Axe^{2x}+2Axe^{2x}=e^{2x}$$

$$\Rightarrow Ae^{2x}=e^{2x}\Rightarrow A=1$$

$$\therefore y_p(x)=xe^{2x}$$

General solution: $y=y_c+y_p=c_1e^{2x}+c_2e^{x}+xe^{2x}$

(b) For $f(x)=a\sin bx$ or $\cos bx$ or both, a,b constants, if r^2+b^2 is a factor of the auxiliary equation of multiplicity k, then: $y_p(x)=x^k(A\sin bx+B\cos bx)$, where A,B are constant to be determined. If r^2+b^2 is **Not** a factor, then $k=0$.

Example 3.4

Solve the differential equation $y''+4y=3\sin 2x$.

Solution:

$$r^2+2^2=0,\ b=2,\ k=1$$

$$y_p=Ax\sin 2x+Bx\cos 2x \text{ --------------- (I)}$$

$$y_p'=A(\sin 2x+2x\cos 2x)+B(\cos 2x-2x\sin 2x) \text{ ------ (II)}$$

$$\begin{aligned} y_p''&=A[2\cos 2x+2(\cos 2x-2x\sin 2x)]+\\ &\quad B[-2\sin 2x-2(\sin 2x+2x\cos 2x)] \end{aligned} \text{ --------------- (III)}$$

Substitute back into the equation gives:

$$\begin{aligned} \Rightarrow & A[2\cos 2x+2\cos 2x-4x\sin 2x]+\\ & B[-2\sin 2x-2\sin 2x-4x\cos 2x]+\\ & 4[Ax\sin 2x+Bx\cos 2x]=3\sin 2x \end{aligned}$$

$$\Rightarrow 4A\cos 2x-4B\sin 2x=3\sin 2x$$

Comparing the coefficients:

$$A=0, B=-\frac{3}{4}$$

$$\therefore y_p(x)=-\frac{3}{4}x\cos 2x$$

General solution: $y(x)=c_1\cos 2x+c_2\sin 2x-\frac{3}{4}x\cos 2x$.

(c) For $f(x)=ax^s$, a,s (integer) constant, if the auxiliary equation has $r=0$ as a root of multiplicity k, then trial function: $y_p(x)=x^k\left(Ax^s+Bx^{s-1}+---+Px+Q\right)$ where $A,B,---,P,Q$ are constants to be determined.

If $r=0$ is **Not** a root, then $k=0$

Example 3.5

Find a particular solution of $y''+4y=3x^3$.

Solution:

$$y_p=Ax^3+Bx^2+Cx+D$$

$$y_p'=3Ax^2+2Bx+C$$

$$y_p''=6Ax+2B$$

Substituting in the differential equation, we get:

$$y''+4y_p=(6Ax+2B)+4\left(Ax^3+Bx^2+Cx+D\right)$$

$$=4Ax^3+4Bx^2+(6A+4C)x+(2B+D)=3x^3.$$

We equate coefficients of like powers of x to get the equations:

$$4A=3,\qquad 4B=0$$

$$6A+4C=0,\quad 2B+D=0$$

with solution $A=\frac{3}{4}, B=0, C=-\frac{9}{8}$, and $D=0$. Hence a particular solution of the differential equation is: $y_p=\frac{3}{4}x^3-\frac{9}{8}x$.

(d) If $f(x)$ is the <u>*sum*</u> of any two or all these <u>*special*</u> functions, the particular integral is the appropriate sum of individual particular integral. Hence $f(x)$ takes the trial solution:

$$y_p=x^k\left[\left(A_0+A_1x+\ldots+A_nx^n\right)e^{mx}\cos bx+\left(B_0+B_1x+\ldots+B_nx^n\right)e^{mx}\sin bx\right]$$

Example 3.6

Determine the appropriate form of for a particular solution of

$$y''+6y'+13y=e^{-3x}\cos 2x$$

Solution:

The characteristic equation $r^2+6r+13=0$ has roots: $r=-3\pm 2i$,

So the complementary function is: $y_c=e^{-3x}(A\cos 2x+B\sin 2x)$

This is the same form as a first attempt $e^{-3x}\left(A\cos 2x + B\sin 2x\right)$ at a particular solution, so we must multiply by x to eliminate duplication.

Hence we would take $y_p = e^{-3x}\left(Ax\cos 2x + Bx\sin 2x\right)$

Exercise 3.4

Solve the following linear differential equation.

1. $y'' - y' - 2y = 4x^2$ $\qquad y = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$
2. $y'' - y' - 2y = e^{3x}$ $\qquad y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$
3. $y'' - y' - 2y = \sin 2x$ $\qquad y = c_1e^{-x} + c_2e^{2x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$
4. $y'' - 4y = 2e^{2x}$ $\qquad y = c_1e^{-2x} + c_2e^{2x} + \frac{1}{2}xe^{2x}$
5. $y'' - 6y' + 10y = 20 - e^{2x}$
6. $y'' + 4y = 3x\cos 2x$
7. $y'' - 6y' + 13y = xe^{3x}\sin 2x$
8. $y'' + 3y' + 2y = x\left(e^{-x} - e^{-2x}\right)$
9. $y'' - 2y' - 3y = 4e^{3x} + 5$
10. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 3e^x + 4x^2$
11. $y'' + 6y' + 8y = \cosh x$ given $y(0) = 0$ $\qquad y'(0) = 1$
12. $y'' + y' = 3 + 2e^{-x}$

1-3.7 Method of Variation of Parameters

In method (1-3.6), we presented the method of undetermined coefficients which, when it is applicable, is usually the simplest method of finding a particular solution of a **non-homogeneous** linear differential equation with constant coefficients. This method (1-3.6) cannot succeed with an equation such as $y''+y=\tan x$, because the function $f(x)=\tan x$ has infinitely many linearly independent derivatives.

Consider a non-homogeneous linear second order differential equation of the form:

$$y''+p(x)y'+q(x)y=f(x) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.15)$$

Suppose y_1 and y_2 are the two solutions of the homogeneous equation

$$y''+p(x)y'+q(x)y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.16)$$

Then the complimentary function is:

$$y(x)=c_1y_1(x)+c_2y_2(x) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.17)$$

where c_1 and c_2 are arbitrary constant.

To find the particular integral, this method requires the trial function

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.18)$$

where we replace the constants, or ***parameters*** c_1, c_2 in the complementary function in (3.17) by variables: functions u_1 and u_2 of x. (3.18) is substituted into (3.15) to determine $u_1(x)$ and $u_2(x)$.

To determine $u_1(x)$ and $u_2(x)$ the following conditions should be fulfilled:

(I) (3.18) must satisfy (3.15)

(II) A condition imposed arbitrarily to facilitate the calculation of the function $u_1(x)$ and $u_2(x)$.

$$y_p(x)=u_1y_1+u_2y_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.19)$$

$$y'_p(x)=u'_1y_1+u_1y'_1+u'_2y_2+u_2y'_2$$

Let $u'_1y_1+u'_2y_2=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.19)$: ***condition imposed***

$$\Rightarrow y'_p(x)=u_1y'_1+u_2y'_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.20)$$

$$\Rightarrow y''_p(x)=u'_1y'_1+u_1y''_1+u'_2y'_2+u_2y''_2 \;\ldots\; (3.21)$$

Substituting (3.21), (3.20) and (3.18) into (3.15), we have:

$$\Rightarrow u'_1y'_1+u_1y''_1+u'_2y'_2+u_2y''_2+pu_1y'_1+pu_2y'_2+qu_1y_1+qu_2y_2=f(x)$$

$$\Rightarrow u_1(y''_1+py'_1+qy_1)+u_2(y''_2+py'_2+qy_2)+u'_1y'_1+u'_2y'_2=f(x)$$

But both y_1 and y_2 satisfy the homogeneous equation (3.16) associated with the non-homogeneous equation (3.15).

So $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$

Equation (3.15) finally becomes:

$$u_1'y_1' + u_2'y_2' = f(x) \text{………………….. (3.22)}$$

Using (3.19) and (3.22) u_1 and u_2 can be determined and thus we have a system:

$$\left.\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned}\right. \text{…………………. (3.23)}$$

The Cramer's Rule for the solution of a linear system of equation yields:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$\Rightarrow u_1' = \dfrac{-y_2 f(x)}{W(y_1, y_2)}$, $\Rightarrow u_2' = \dfrac{y_1 f(x)}{W(y_1, y_2)}$; since the denominator is the Wronskian.

$$u_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)}dx, \qquad u_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)}dx$$

Then (3.18) becomes:

$$y_p = -y_1(x)\int \frac{y_2(x) f(x)}{W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x) f(x)}{W(y_1, y_2)}dx \text{……..(3.24)}$$

Example 3.7

Find the general solution of the differential equation

$$y'' + y = \sec x \qquad 0 < x < \tfrac{\pi}{2}$$

Solution:

The two solutions of the homogeneous equation are:

$y_1 = \cos x$ and $y_2 = \sin x$.

Then $y_p = u_1(x)\cos x + u_2(x)\sin x$

$\Rightarrow y_p' = u_1'\cos x - u_1 \sin x + u_2' \sin x + u_2 \cos x$

Imposition: $u_1' \cos x + u_2' \sin x = 0$ …………….(I)

$y_p' = -u_1(x)\sin x + u_2(x)\cos x$

$y_p'' = -u_1' \sin x - u_1 \cos x + u_2' \cos x - u_2 \sin x$

Then differential equation becomes:

$$-u_1'\sin x - u_1\cos x + u_2'\cos x - u_2\sin x + u_1\cos x + u_2\sin x = \sec x$$

$$\Rightarrow -u_1'\sin x + u_2'\cos x = \sec x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(\text{II})$$

Using (I) and II, we have:

$$u_1'\cos x + u_2'\sin x = 0$$
$$-u_1'\sin x + u_2'\cos x = \sec x$$

The Cramer's Rule for the solution of a linear system of equation yields:

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}, \qquad u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}$$

$$\Rightarrow u_1' = \tan x, \qquad \Rightarrow u_2' = 1$$
$$\Rightarrow u_1 = \ln\cos x, \qquad \Rightarrow u_2 = x$$

Therefore, $y_p = (\ln\cos x)\cos x + x\sin x$.

Hence the General solution:

$$y(x) = c_1\cos x + c_2\sin x + (\ln\cos x)\cos x + x\sin x$$

Exercise 3.5

Determine a particular solution of the following using the method of variation of parameter.

1. $y'' - 5y' + 6y = e^{2x}$
2. $y'' - y' - 2y = 2e^{-x}$
3. $y'' + 2y' + y = 3e^{-x}$
4. $y'' + y = \tan x \quad 0 < x < \frac{\pi}{2}$
5. $y'' + 9y = 9\sec^2 3x \quad 0 < x < \frac{\pi}{6}$
6. $y'' + 4y = 3\operatorname{cosec} 2x \quad 0 < x < \frac{\pi}{2}$
7. $y'' - 3y' + 2y = e^{x}$
8. $y'' - 2y' + y = e^{x}$
9. $y'' - 3y' + 2y = e^{x} + e^{2x}$
10. $y'' + y = \cos x$
11. $y'' + y = \cot x$
12. $y'' - 5y' + 6y = e^{2x} + e^{3x}$
13. $y'' + y = \sin x$

Session 2-3 Linear Differential Operator Method (D-Operator)

A differential operator is a rule D that uses derivatives to assign a function $D(y)$ to any sufficiently differentiable function y. The function $D(y)$ is called the **image** of y under D. This method is very efficient for the solution of a general linear equation of n th order and with constant coefficients. It is most efficient because it can be manipulated algebraically.

Let D denote differentiation with respect to x, D^2 differentiation twice with respect to x, and so on; that is, for positive integral k, $D^k y = \frac{d^k y}{dx^k}$, that is $D = \frac{d}{dx}$ $\quad D^2 = \frac{d^2}{dx^2}$ etc

Then $D(x^2) = 2x \quad D^2(e^{3x}) = 9e^{3x} \quad D^2(x^3 - 4x^2 + 1) = 6x - 8$

The expression $f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ is called a differential operator of order n. It may be defined as that operator which, when applied to any function y, yields the result

$$f(D)y = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y$$

where the function y is assumed to possess as many derivatives as may be encountered in whatever operation take place. Let $f(D)$ be a polynomial in D.

2-3.1 Fundamental Properties of the D–operator

The following properties are satisfied by the D-operator.

$D^n(u+v) = D^n u + D^n v$ (Distributive property)

$D^n(ku) = kD^n u$, k is a constant

$D^m(D^n u) = D^{m+n}(u)$

2-3.2 Factorability

Any constant coefficient n th order linear differential equation (LDE) can be written as: $f(D)[y] = R(x)$, where $f(D)$ is a polynomial function of the operator D.

Example: $y'' - 3y' + 2y = D^2 y - 3Dy + 2y = e^{2x}$

$$\Rightarrow (D^2 - 3D + 2)y = e^{2x} \quad \Rightarrow f(D) = D^2 - 3D + 2$$

This polynomial can be factorized if required. In this case

$$f(D) = D^2 - 3D + 2 = (D-2)(D-1)$$

We recognize that $(D-2)(D-1)=(D-1)(D-2)$ which is valid ONLY for constant coefficient linear equations.

2-3.3 Linear Equations of First Order

Recall the familiar first-order linear equation with constant coefficient: $\frac{dy}{dx}-ay=R(x)$, a is a constant, which is equivalent to:

$$(D-a)y=R(x) \Rightarrow y=\frac{1}{D-a}R(x)------*$$

The integrating factor is $I=e^{-\int adx}=e^{-ax}$. Thus multiplying both sides by the integrating factor we obtain

$$e^{-ax}\frac{dy}{dx}-ae^{-ax}y=e^{-ax}R(x)$$

$$\Rightarrow \frac{d}{dx}\left(ye^{-ax}\right)=e^{-ax}R(x)$$

$$\Rightarrow ye^{-ax}=\int e^{-ax}R(x)dx$$

$$\Rightarrow y=e^{ax}\int e^{-ax}R(x)dx--------(1.1)$$

From $(*)$ $\quad y=\frac{1}{D-a}R(x) \Rightarrow \frac{1}{D-a}R(x)=e^{ax}\int e^{-ax}R(x)dx$

Therefore if we have $(D-2)y=e^{2x} \Rightarrow y=\frac{1}{D-2}e^{2x}$

then $y=e^{2x}\int e^{-2x}\cdot e^{2x}dx \Rightarrow y=e^{2x}\int dx=e^{2x}(x+c)=xe^{2x}+ce^{2x}$

2-3.4 Linear Equation of Higher Order

Here we discuss second order linear equations, $\left(aD^2+bD+c\right)y=f(x)$

Example 1: Solve $\left(D^2-3D+2\right)y=e^{2x}$

Solution

$$\left(D^2-3D+2\right)y=(D-2)(D-1)y=e^{2x}$$

Let $(D-1)y=z$, then $(D-2)z=e^{2x}$

$$z=\frac{1}{D-2}e^{2x}=e^{2x}\int e^{-2x}e^{2x}dx=e^{2x}\left(x+c_1\right)$$

Thus $(D-1)y=e^{2x}\left(x+c_1\right)$

$$\therefore y=\frac{1}{D-1}\left[e^{2x}\left(x+c_1\right)\right]=e^{2x}\int e^{-x}\left[e^{2x}(x+c_1)\right]dx$$

$y=xe^{2x}+Ae^{2x}+Be^{x}$, A and B are arbitrary constants.

Alternatively;

$$(D-1)(D-2)y = e^{2x} \Rightarrow y = \frac{1}{(D-1)(D-2)}e^{2x}$$

Using partial fraction:

$$y = \left(\frac{1}{D-2} - \frac{1}{D-1}\right)e^{2x} = \frac{1}{D-2}e^{2x} - \frac{1}{D-1}e^{2x}$$

$$\Rightarrow y = e^{2x}\int e^{-2x}e^{2x}dx - e^{x}\int e^{-x}e^{2x}dx = xe^{2x} + c_1e^{2x} - e^{2x} + c_2e^{x}$$

$$\Rightarrow y = xe^{2x} + (c_1 - 1)e^{2x} + c_2e^{x} = xe^{2x} + Ae^{2x} + B_2e^{x}$$

The efficiency of the operator method becomes very clear where the auxiliary equation has repeated roots.

Example 2: Solve $(D^2 - 2D + 1)y = 2xe^x$

Solution:

$$(D-1)^2 y = 2xe^x$$

Let $z = (D-1)$

Then $(D-1)z = 2xe^x \Rightarrow z = \frac{1}{D-1}2xe^x = e^x\int e^{-x}2xe^x dx = e^x\left(x^2 + c_1\right)$

$$\therefore (D-1)y = e^x(x^2 + c_1)$$

$$\Rightarrow y = \frac{1}{D-1}\left[e^x(x^2 + c_1)\right] = e^x\int e^{-x}\cdot e^x\left(x^2 + c_1\right)dx$$

$$\Rightarrow y = e^x\int (x^2 + c_1)dx = e^x\left(\tfrac{1}{3}x^3 + c_1x + c_2\right) = \frac{x^3e^x}{3} + c_1xe^x + c_2e^x$$

Exercise 3.6

1. Demonstrate Commutation of the factor in the operator $(D-2)(D-3)$ applied onto:

   i) $\ln x - x^3$ ii) $e^{2x} + x^3$

2. Use equation 1.1 to solve the following;

   i) $(D-2)y = e^x$ ii) $(D-1)y = e^{-x}$

   iii) $(D+2)y = x^2 - e^{-2x}$ iv) $(D-2)y = x^2 + e^{2x}$

   v) $(2D-3)y = \sin x$ vi) $(2D+3)y = \cos x$

   vii) $(3D+2)y = \cos\frac{2}{3}x$ viii) $(3D-2)y = \sin\frac{2}{3}x$

Solve

i)	$(D^2-9)y=e^{3x}$	ii)	$(D^2-9)y=e^{-3x}$
iii)	$(D^2-5D+6)y=e^{3x}$	iv)	$(D^2-5D+6)y=e^{2x}$
v)	$y''-7y'+12y=x$	vi)	$y''-9y'+2y=x$
vii)	$(D-3)^2y=e^{-3x}$	viii)	$(D+3)^2y=e^{3x}$
ix)	$(D-3)^2y=e^{3x}$	x)	$(D+3)^2y=e^{-3x}$
xi)	$(D-2)^2y=xe^{3x}$	xii)	$(D+3)^2y=xe^{-3x}$

2-3.5 $f(D)$ Polynomials with complex roots

Consider a differential equation $f(D)y=R(x)$ for which the auxiliary equation $f(m)=0$ has real coefficients. From elementary algebra we know that if the auxiliary equation has any imaginary roots, those roots must occur in conjugate pairs. Thus, if $m_1=\alpha+i\beta$ is a root of the homogeneous equation $f(m)=0$, with a and b real and $b\neq 0$, then $m_2=\alpha-i\beta$ is also a root of $f(m)=0$.

For $(aD^2+bD+c)y=R(x)$ if $f(m)=0$ have complex roots then we have:

$$(D-\alpha-i\beta)(D-\alpha+i\beta)y=R(x)$$

Let $z=(D-\alpha-i\beta)y$ then $(D-\alpha+i\beta)z=R(x)$.

Example 3: Solve $(D^2+4)y=3$

Solution

$$(D+2i)(D-2i)y=3$$

Let $(D-2i)y=z$

$$\therefore z(D-2i)=3$$

$$\Rightarrow z=\frac{1}{D+2i}3=e^{-2ix}\int 3e^{2ix}dx=\frac{3}{2i}e^{-2ix}\left(e^{2ix}+c\right)$$

$$\Rightarrow z=\frac{3}{2i}+\frac{3}{2i}ce^{-2ix}=\frac{3}{2i}+c_1e^{-2ix},\quad \text{where } c_1=\frac{3}{2i}c$$

$$\therefore y=\frac{1}{D-2i}\left[\frac{3}{2i}+c_1e^{-2ix}\right]=e^{2ix}\int e^{-2ix}\left(\frac{3}{2i}+c_1e^{-2ix}\right)dx$$

$$\Rightarrow y = e^{2ix}\left(\frac{3}{4}e^{-2ix} + \frac{c_1}{4i}e^{-4ix} + c_2\right) = \frac{3}{4} + \frac{c_1}{4i}e^{-2ix} + c_2 e^{2ix}$$

$$\Rightarrow y = \frac{3}{4} + c_3 e^{-2ix} + c_2 e^{-2ix} \quad \text{where} \, c_3 = \frac{c_1}{4i}$$

$$\Rightarrow y = \frac{3}{4} + A\cos 2x + B\sin 2x$$

Exercise 3.7

1. $D(D-1)y = 5$
2. $(D^2+4)y = 0$
3. $(D^2+D+1)y = 2$
4. $(D^2+4)y = \cos x$
5. $(D^2+4)y = e^{2x}$
6. $(D^2+1)y = e^{2x}$
7. $(D^2+9)y = \sin 2x$
8. $(D^2+2D+1)y = e^{-x}\sin x$
9. $(D^2+2D+1)y = e^{-x}\cos x$

A general nth – order linear differential equation has the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \quad \ldots\ldots (3.1)$$

where $a_n(x) \neq 0$ and the coefficients $a_i(x)(i = 0, 1, \ldots n-1, n)$,and $f(x)$ are given functions of x or are constants.

This equation (3.1) is **homogeneous** if $f(x) = 0$ and **non – homogeneous** if $f(x) \neq 0$.

The solution of the **homogeneous** differential equation is called the complimentary function (conventionally y_c) and the solution of the **non – homogeneous** part is known as the particular integral (conventionally y_p).

Then the general solution = complimentary function + particular integral (that is $y = y_c + y_p$).

- **Homogeneous Equation of Second – Order with constant coefficient**

$$a_2\frac{d^2 y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0$$

A trial solution $y = e^{rx}$ substituted gives:

$$\left(a_2 r^2 + a_1 r + a_0\right)e^{rx} = 0$$

clearly $e^{rx} \neq 0$, hence; $a_2r^2 + a_1r + a_0 = 0$
or for convenience,

$$ar^2 + br + c = 0, \text{ where } a = a_2; b = a_1; c = a_0$$

whose roots:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

There are three cases to consider:

Case 1: If the roots are real and distinct $\left(b^2 - 4ac > 0\right)$, that is,

$$r = r_1,\ r = r_2 \text{ and } r_1 \neq r_2$$

then the general solution of the differential equation is:

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

Case 2: If the roots are real and identical $\left(b^2 - 4ac = 0\right)$, that is,

$r = r_1 = r_2$, if we denote the double root by r,

then the general solution of the differential equation is:

$$y = c_1e^{rx} + c_2xe^{rx}$$

Case 3: If the roots are not real. In this case, the roots are complex conjugates $\left(b^2 - 4ac < 0\right)$, that is $r = \alpha \pm i\beta$, then the general solution of the differential equation is:

$$y = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right)$$

where $\alpha = \frac{b}{2a}$; $\beta = \frac{\sqrt{b^2 - 4ac}}{2a}$

- **Wronskian**

Given two functions f and g, the **Wronskian** of f and g is the determinant :

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

Given two functions $f(x)$ and $g(x)$ that are differentiable on some interval I :

i. If $W(f,g)(x) \neq 0$ for some x_0 in I, then $f(x)$ and $g(x)$ are linearly independent on the interval F.
ii. If $f(x)$ and $g(x)$ are linearly dependent on I then $W(f,g)(x) = 0$ for all x is the interval.

- **Reduction Order**

In **reduction of order**, it enables us to use one known solution , say, y_1 to find a second linearly independent solution y_2.

<u>Solution of Reduction Order:</u>

Consider $y'' + p(x)y' + q(x)y = 0$

Then $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$

- **Second – Order Non – Homogeneous Equation**

A general linear non – homogeneous constant coefficient equation is:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

We have three Methods of attack. Each depend on the external function, $f(x)$. The methods are:

(IV) Method of Undetermined Coefficients
(V) Method of Variation of Parameter
(VI) Method of the Linear Differential Operator

Method of Undetermined Coefficients

(e) For $f(x) = ae^{bx}$ a, b are constants, if the auxiliary equation of has $r = b$ as a root of multiplicity k , then the trial function is of the form $y_p(x) = Ax^k e^{bx}$, where A constant to be determined. If $r = b$ is **Not** is a root, take $k = 0$.

(b) For $f(x) = a \sin bx$ or $\cos bx$ or both, a, b constants, if $r^2 + b^2$ is a factor of the auxiliary equation of multiplicity k , then:

$y_p(x) = x^k (A \sin bx + B \cos bx)$

where A, B are constant to be determined.

If $r^2 + b^2$ is **Not** a factor, then $k = 0$.

(c) For $f(x) = ax^s$, a, s (integer) constant, if the auxiliary equation has $r = 0$ as a root of multiplicity k , then trial function:

$y_p(x) = x^k \left(Ax^s + Bx^{s-1} + --- + Px + Q \right)$

where $A, B, ---, P, Q$ are constants to be determined.

If $r = 0$ is **Not** a root, then $k = 0$

(d) If $f(x)$ is the <u>sum</u> of any two or all these <u>special</u> functions, the particular integral is the appropriate sum of individual particular integral. Hence $f(x)$ takes the trial solution:

$$y_p = x^k\left[\left(A_0 + A_1x + \ldots + A_nx^n\right)e^{mx}\cos bx + \left(B_0 + B_1x + \ldots + B_nx^n\right)e^{mx}\sin bx\right]$$

Method of Variation of Parameter

The method of undetermined coefficient does not succeed if the external function, $f(x)$ is not analytic everywhere. The parameters of the homogeneous solution are then varied (that is change to variables).

Given a differential equation: $y'' + py' + qy = f(x)$,

The homogeneous part, $y'' + py' + qy = 0$, gives the solution $y_c = c_1y_1 + c_2y_2$. If the parameters c_1 and c_2 is varied to $u_1(x)$ and $u_2(x)$ respectively, we have:

$$y_p = u_1y_1 + u_2y_2$$

We then determined u_1 and u_2 to get:

$$y_p = -y_1(x)\int\frac{y_2(x)f(x)}{W(y_1,y_2)}dx + y_2(x)\int\frac{y_1(x)f(x)}{W(y_1,y_2)}dx$$

Method Linear Differential Operator

Let $\frac{dy}{dx} - ay = R(x)$, a is a constant,

which is equivalent to: $(D-a)y = R(x)$

$$\Rightarrow y = \frac{1}{D-a}R(x) ------*$$

The integrating factor $I = e^{-\int adx} = e^{-ax}$

Then $\quad y = e^{ax}\int e^{-ax}R(x)dx$

from * $\quad y = \frac{1}{D-a}R(x)$

$$\Rightarrow \frac{1}{D-a}R(x) = e^{ax}\int e^{-ax}R(x)dx$$

UNIT FOUR

LAPLACE TRANSFORM

4.0 INTRODUCTION

In this section we will be looking at how to use Laplace transforms to solve differential equations. Laplace transforms reduce a differential equation to an algebra problem.

We use Laplace transforms when the force function becomes complicated, that is, discontinuous – for instance, when the voltage supplied to an electrical circuit is periodically turned off and on.

Learning Objectives

After going through this section session, you would be able to:

- Know the basic definition of Laplace Transform.
- Solve improper integral with an infinite limit.
- Transform continuous functions like polynomials, exponentials, trigonometric (only cosine and sine ones) to Laplace forms
- Find the inverse a Laplace transform back to polynomial or Laplace Transform.

4.1 DEFINITION: LAPLACE TRANSFORM

Given a function $f(t)$ defined for all $t \geq 0$, the **Laplace transform** of f is the function F of s defined as follows:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \text{ .... } (1.1)$$

for all values of s for which the improper integral converges.

The **improper integral** over an infinite interval is defined as a limit of integrals over finite intervals, that is:

$$\int_a^\infty f(t)dt = \lim_{m\to\infty} \int_a^m f(t)dt \text{............................}(1.2)$$

If the limit in (1.2) exists, then we say that improper integral converges. Otherwise, it diverges or fails to exist. Note that the integrand of the improper integral in (1.1) contains the parameter s in addition to the variable of integration t. Therefore, when the integral (1.1) converges, it converges not merely to a number, but to a function F of s.

Example 4.1

Compute the Laplace transforms of the function

(i) 1, (ii) t, (iii) e^{at}, (iv) $\cos t$, for $t \geq 0$.

Solution:

(i) $$L[1] = \int_0^\infty 1 \cdot e^{-st}\,dt = \lim_{m\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^m = \frac{1}{s} \text{ for every } s > 0$$

(ii) $$L[t] = \int_0^\infty t \cdot e^{-st}\,dt = \lim_{m\to\infty}\left[-\frac{t}{s}e^{-st}\right]_0^m - \int_0^\infty\left(-\frac{1}{s}\right)e^{-st}\,dt$$

$$= \lim_{m\to\infty}\left[-\frac{1}{s^2}e^{-st}\right]_0^m = \frac{1}{s^2} \text{ for every } s > 0$$

(iii) $$L\left[e^{at}\right] = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a},\ s > a$$

(iv) $$L[\cos at] = \int_0^\infty e^{-st}\cos at\,dt$$

$$= \lim_{m\to\infty}\left[\frac{1}{a}e^{-st}\sin at\right]_0^m - \int_0^\infty \frac{1}{a}(-s)e^{-st}\sin at\,dt \qquad = \frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt$$

$$= \lim_{m\to\infty}\left[\frac{1}{a}e^{-st}\sin at\right]_0^m - \int_0^\infty \frac{1}{a}(-s)e^{-st}\sin at\,dt$$

$$= \frac{s}{a^2} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\cos at\,dt = \frac{s}{a^2} - \frac{s^2}{a^2}L[\cos at]$$

$$\Rightarrow L[\cos at] = \frac{s}{a^2} - \frac{s^2}{a^2}L[\cos at] = \frac{s}{s^2+a^2}$$

Find the following: $L\left[t^2\right]$, $L\left[t^3\right]$, $L\left[t^4\right]$, and $L\left[t^n\right]$

4.2 ELEMENTARY PROPERTIES OF LAPLACE TRANSFORM

Theorem 4.1: (Linear Property)

Let f and g be functions whose Laplace transforms exist, and let c_1, c_2 be any two real numbers, then:

$$L[c_1 f(t) + c_2 g(t)] = c_1 L[f(t)] + c_2 L[g(t)]$$

Prove as exercise.

Example 4.2

Find the Laplace transforms of the given functions:

$$f(t) = 6e^{-5t} + e^{3t} + 5t^2 - 9$$

$$g(t) = 4\cos 4t - 9\sin 4t + \cos 10t$$

Solution:

1. $$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$\Rightarrow F(s) = \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

2. $$G(s) = 4\frac{s}{s^2+4^2} - 9\frac{4}{s^2+4^2} + 2\frac{s}{s^2+10^2}$$

$$\Rightarrow G(s) = \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

Example 4.3

1. Let $f(t) = \cosh at$. Find $L[f(t)]$.

Solution:

$$L[\cosh at] = L\left[\tfrac{1}{2}\left(e^{at} + e^{-at}\right)\right] = \tfrac{1}{2}L\left[e^{at}\right] + \tfrac{1}{2}L\left[e^{-at}\right] = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}$$

$$\Rightarrow L[\cosh at] = \frac{s}{s^2 - a^2}, \; s > a \ge 0$$

2. Let $f(t)=e^{iwt}$. Find $L[f(t)]$.

Solution:

$$L\left[e^{iwt}\right]=\frac{1}{s-iw}=\frac{s+iw}{(s-iw)(s+iw)}=\frac{s+iw}{s^2+w^2}=\frac{s}{s^2+w^2}+\frac{iw}{s^2+w^2}$$

By Euler formula: $e^{iwt}=\cos wt+i\sin wt$

$$L\left[e^{iwt}\right]=L[\cos wt+i\sin wt]=L[\cos wt]+iL[\sin wt]$$

Equating the real and imaginary parts of these two equations, we obtain the transforms of cosine and sine.

$$L[\cos wt]=\frac{s}{s^2+w^2},\ L[\sin wt]=\frac{w}{s^2+w^2}.$$

Theorem 4.2: (First Shifting Theorem)

Replacement of s by $s-a$ in the function.

If $f(t)$ has the transformation $F(s)$ (where $s>k$), then $e^{at}f(t)$ has the transform $F(s-a)$(where $s-a>k$). In formula:

$$L\left[e^{at}f(t)\right]=F(s-a)$$

or, if we take the inverse on both sides

$$e^{at}f(t)=L^{-1}[F(s-a)].$$

Example 4.4

1. $L\left[e^{at}\cos wt\right]=\dfrac{s-a}{(s-a)^2+w^2}$

2. $L\left[e^{at}\sin wt\right]=\dfrac{w}{(s-a)^2+w^2}$

Theorem 4.3: (Multiplying by t and t^n)

If $L[f(t)] = F(s)$ then $L[tf(t)] = -F'(s)$. Because:

$$L\left[tf(t)\right] = \int_0^\infty tf(t)e^{-st}dt = \int_0^\infty f(t)\left(-\frac{de^{-st}}{ds}\right)dt$$

$$= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = -F'(s)$$

For example; $L[\sin 2t] = \dfrac{2}{s^2+4}$

Therefore, $L\left[t\sin 2t\right] = -\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right) = \dfrac{4s}{\left(s^2+4\right)^2}$.

Since $\dfrac{d^n}{ds^n}\left(e^{-st}\right) = (-1)^n t^n e^{-st}$, we have:

$$F^{(n)}(s) = \int_0^\infty \frac{d^n}{ds^n}\left[e^{-st} f(t)\right]dt = (-1)^n \int_0^\infty e^{-st} t^n f(t)dt$$

$$= (-1)^n L\left[t^n f(t)\right]$$

Hence $L\left[t^n f(t)\right] = (-1)^n \dfrac{d^n}{ds^n}\left[F(s)\right] = (-1)^n F^{(n)}(s)$

Theorem 4.4:

Let f be a piecewise – continuous function on $[0,\infty)$ exponential order.

Define a function g by:

$$g(t) = \begin{cases} 0 & 0 \le t \le a \\ f(t-a) & a < t \end{cases}$$

where a positive number,

then $L[g] = e^{-as}L[f(t)]$

Example 4.5

1. Let $g(t)=\begin{cases}0 & 0\le t\le a\\ 1 & a<t\end{cases}$

   Then $L[g]=\frac{1}{s}e^{-as}$

2. Let $g(t)=\begin{cases}0 & 0\le t\le \pi\\ \cos(t-\pi) & \pi<t\end{cases}$

   Then $e^{-s\pi}\,\frac{s}{s^2+1}$

Theorem 4.5:

If $L\left[f(t)\right]=F(s)$ and if $\lim\limits_{\substack{t\to 0\\ t>0}}\frac{f(t)}{t}$ exists, then

$$L\left[\frac{1}{t}f(t)\right]=\int_s^\infty F(x)dx$$

Theorem 4.6:

If $L\left[f(t)\right]=F(s)$, then

$$L\left[\int_0^t f(x)dx\right]=\frac{1}{s}F(s)$$

Theorem 4.7:

If $f(t)$ is periodic with period T, that is, $f(t+\mathrm{T})=f(t)$, then

$$L\left[f(t)\right]=\frac{\int_0^\infty e^{-st}f(t)dt}{1-e^{-\mathrm{T}s}}$$

Example 4.5 Further examples

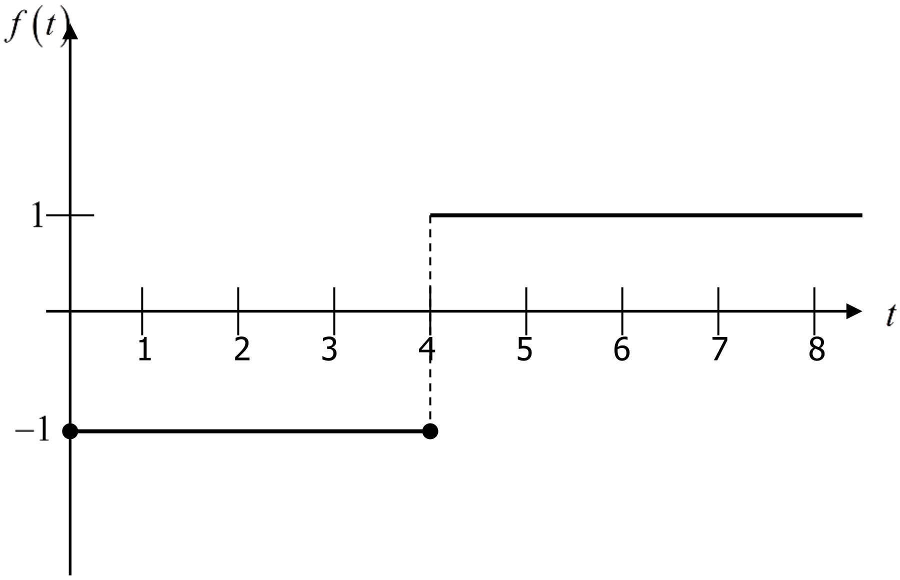
1. Find the Laplace transform of $f(t)=\begin{cases}e^t & t\le 2\\ 3 & t>2\end{cases}$

Solution:

$$L\left[f(t)\right]=\int_0^{\infty}e^{-st}f(t)dt=\int_0^{2}e^{-st}e^{t}dt+\int_2^{\infty}e^{-st}(3)dt$$

$$\Rightarrow L\left[f(t)\right]=\frac{1-e^{-2(s-1)}}{s-1}+\frac{3}{s}e^{-2s}\,(s>0)$$

2. Find the Laplace transform of the function graph below:



Solution:

The function is $f(t)=\begin{cases}-1 & x\le 4\\ 1 & x>4\end{cases}$

Therefore,

$$L\left[f(t)\right]=\int_0^{\infty}e^{-st}f(t)dt=\int_0^{4}e^{-st}(-1)dt+\int_4^{\infty}e^{-st}(1)dt$$

$$\Rightarrow L\left[f(t)\right]=\frac{2e^{-4s}}{s}-\frac{1}{s}(s>0)$$

3. Find $L\left[\frac{\sin 3t}{t}\right]$.

Using Theorem 1.5, let $f(t)=\sin 3t$, then $F(s)=\frac{3}{s^2+9}$, which also

can be written as: $F(x)=\frac{3}{x^2+9}$

Therefore,

$$L\left[\frac{\sin 3t}{t}\right]=\int_s^{\infty}\frac{3}{x^3+9}dx=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{3}\right)$$

4. Find $L\left[\int_0^t \sinh 2x dx\right]$

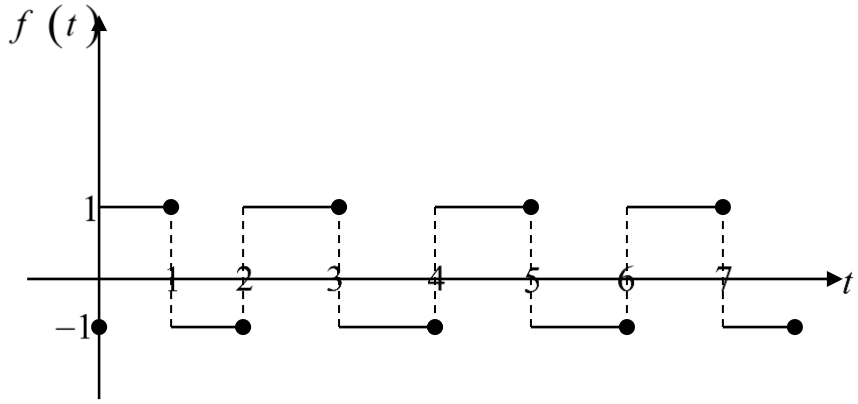
Solution:

Taking $f(x)=\sinh 2x$, we have $f(t)=\sinh 2t$

$F(s)=\dfrac{2}{\left(s^2-4\right)}$, and then, from Theorem 1.6 that

$$L\left[\int_0^t \sinh 2x dx\right]=\frac{1}{s}\left(\frac{2}{s^2-4}\right)=\frac{2}{s\left(s^2-4\right)}$$

5. Find the Laplace for the square wave shown below



Solution:

Note that $f(t)$ is periodic with $\mathrm{T}=2$, and in the interval $0<t\leq 2$ it can be defined analytically by:

$$f(t)=\begin{cases}1 & 0<t\leq 1\\ -1 & 0<t\leq 2\end{cases}$$

From Theorem 1.7, we have:

$$L\left[f(t)\right]=\frac{\int_0^2 e^{-st}f(t)dt}{1-e^{-2s}}$$

Since $\int_0^2 e^{-st} f(t)dt = \int_0^1 e^{-st}(1)\,dt + \int_1^2 e^{-st}(-1)dt$

$$= \frac{1}{s}\left(e^{-2s} - 2e^{-s} + 1\right) = \frac{1}{s}\left(e^{-s} - 1\right)^2$$

It follows that,

$$F(s) = \frac{\left(e^{-s}-1\right)^2}{s\left(1-e^{-2s}\right)} = \frac{\left(1-e^{-s}\right)^2}{s\left(1-e^{-s}\right)\left(1+e^{-s}\right)} = \frac{1-e^{-s}}{s\left(1+e^{-s}\right)}$$

$$\left[\frac{e^{s/2}}{e^{s/2}}\right]\left[\frac{1-e^{-s}}{s\left(1+e^{-s}\right)}\right] = \frac{e^{s/2}-e^{-s/2}}{s\left(e^{s/2}+e^{-s/2}\right)} = \frac{1}{s}\tanh\frac{s}{2}$$

Exercise4.1

Find the Laplace Transforms of the following functions

1. (i) $f(t) = \sin \omega t$ $\qquad F(s) = \dfrac{\omega}{s^2+\omega^2}$

   (ii) $f(t) = \sinh t$ $\qquad F(s) = \dfrac{1}{s^2-1}$

   (iii) $f(t) = 2x^2 - 3x + 4$ $\qquad F(s) = \frac{4}{s^3} - \frac{3}{s^2} + \frac{4}{s}$

   (iv) $f(t) = \sum_{k=1}^{\infty} \dfrac{t^k}{k!}$.

2. $f(t) = e^{3t}\cos 6t$ $\qquad F(s) = \dfrac{s-3}{(s-3)^2+36}$

3. $f(t) = 3\sinh 2t + 3\sin 2t$

4. $f(t) = e^{3t} + \cos 6t - e^{3t}\cos 6t$

5. $f(t) = t^2 \sin 2t$ $\qquad F(s) = \dfrac{12s^2-16}{(s+4)^3}$

6. $f(t) = t^{\frac{3}{2}}$ $\qquad F(s) = \dfrac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}$

7. $f(t) = tg'(t)$ $\qquad F(s) = -G(s) - sG'(s)$

8. $f(t)=\sin 3t\cos 3t$

9. $f(t)=(1+t)^3$

10. $f(t)=\begin{cases}1 & 0\le t\le 1\\ 0 & 1<t\end{cases}$

11. $f(t)=\begin{cases}t & 0\le t\le 1\\ 0 & 1<t\end{cases}$

12. Find the Laplace transform of the period function

$$f(t)=\begin{cases}t & 0\le t\le \pi\\ 2\pi-t & \pi\le t\le 2\pi\end{cases} \qquad \frac{1}{s^2}\tanh\frac{\pi s}{2}$$

13. Find $L\left[e^{4t}t\int_0^t \frac{1}{x}e^{-4x}\sin 3x dx\right]$

$$\frac{\pi}{2(s-4)^2}-\frac{1}{(s-4)^2}\tan^{-1}\left(\frac{s}{3}\right)+\frac{3}{(s-4)(s^2+9)}$$

4.3 INVERSE LAPLACE

The inverse Laplace is given by:

$$L^{-1}[F(s)] = f(t).$$

Linearity of inverse Laplace transforms:

$L^{-1}[aF(s)+bG(s)] = aL^{-1}[F(s)]+bL^{-1}[G(s)]$ for any constant a,b.

For example:

$L[1]=1/s$, $L[t]=1/s^2$, $L\left[e^{at}\right]=1/(s-a)$, and $L[\cos at]=s/\left(s^2+a^2\right)$.

Hence $L^{-1}[1/s]=1$, $L^{-1}\left[1/s^2\right]=t$, $L^{-1}[1/(s-a)]=e^{at}$.

Example 4.6

1. $$L^{-1}=\left[\frac{2}{s}-\frac{3}{s^2}\right]=2L^{-1}\left[\frac{1}{s}\right]-3L^{-1}\left[\frac{1}{s^2}\right]=2-3t$$

2. $$L^{-1}\left[\frac{5}{s-3}+\frac{s}{s^2+4}\right]=5L^{-1}\left[\frac{1}{s-3}\right]+L^{-1}\left[\frac{s}{s^2+4}\right]=5e^{3t}+\cos 2t$$

Example 4.7

Find the Laplace inverse of the function

$$F(s)=\frac{2s+3}{(s-5)(s+4)}$$

Solution:

By partial fraction:

$$\frac{2s+3}{(s-5)(s+4)}=\frac{13}{9(s-5)}+\frac{5}{9(s+4)}$$

$$L^{-1}\left[\frac{2s+3}{(s-5)(s+4)}\right]=L^{-1}\left[\frac{13}{9(s-5)}+\frac{5}{9(s+4)}\right]$$

$$=\frac{13}{9}L^{-1}\left[\frac{1}{(s-5)}\right]+\frac{5}{9}L^{-1}\left[\frac{1}{s+4}\right]=\frac{13}{9}e^{5t}+\frac{5}{9}e^{-4t}$$

Exercise 4.2

Find the Laplace inverse of the following;

1. $\dfrac{s}{s^2+6}$ $\qquad\qquad$ $\cos\sqrt{6}t$

2. $\dfrac{5s}{\left(s^2+1\right)^2}$ $\qquad\qquad$ $\dfrac{5}{2}t\sin t$

3. $\dfrac{s+1}{s^2-9}$ $\qquad\qquad$ $\cosh 3t+\dfrac{1}{3}\sinh 3t$

4. $\dfrac{s+4}{s^2+4s+8}$ $\qquad\qquad$ $e^{-2t}\cos 2t+e^{-2t}\sin 2t$

5. $\dfrac{1}{(s+1)\left(s^2+1\right)}$ $\qquad\qquad$ $\dfrac{1}{2}e^{-t}-\dfrac{1}{2}\cos t+\dfrac{1}{2}\sin t$

6. $\dfrac{s^2+3s+1}{(s+2)\left(s^2+16\right)}$

7. $\dfrac{e^{as}}{s}$

8. $\dfrac{s+5}{s^2+25}$

9. $\dfrac{1}{s}\left(e^{-2s}-e^{-s}\right)$

10. $\dfrac{e^{-\pi s}}{s\left(s^2+4\right)}$

11. $\dfrac{s^2}{(s+2)^3}$

4.4 CONVOLUTION

Definition: Let $f(t)$ and $g(t)$ be piecewise – continuous functions on $[0,\infty)$ of exponential order. The function called the **convolution** of $f(t)$ and $g(t)$, denoted by $f*g$, is defined by:

$$(f*g)(t)=\int_0^t f(x)g(t-x)dx$$

Note: $f*g=g*f$

Theorem: Let f and g be piecewise continuous on $[0,\infty)$ of exponential order. Then:

$$L[f*g]=L[f]L[g] \text{ or, equivalently, } L^{-1}[L[f]L[g]]=f*g$$

Example 4.8

1. $L^{-1}\left[\dfrac{1}{s(s+3)}\right]=1*e^{-3t}=\int_0^t 1\cdot e^{-3(t-x)}dx=\dfrac{1}{3}e^{-3(t-x)}\Big|_0^t=\dfrac{1}{3}\left(1-e^{-3t}\right)$

2. Using convolution, find the inverse $f(t)$ of;

$$F(s)=\frac{1}{\left(s^2+1\right)^2}=\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}$$

Solution:

$$f(t)=L^{-1}[F(s)]=\sin t*\sin t=\int_0^t \sin\tau\sin(t-\tau)d\tau$$

$$=\frac{1}{2}\int_0^t -\cos t\,d\tau+\frac{1}{2}\int_0^t \cos(2\tau-t)d\tau=-\frac{1}{2}t\cos t+\frac{1}{2}\sin t$$

3. Let $F(s)=\dfrac{1}{s^2(s-a)}$. Find $f(t)$

Solution:

$$L^{-1}\left[\frac{1}{s^2}\right]=t \qquad L^{-1}\left[\frac{1}{s-a}\right]=e^{-at}$$

$$f(t)=t*e^{at}=\int_0^t xe^{a(t-x)}dx=e^{at}\int_0^t xe^{-ax}dx=\frac{1}{a^2}\left(e^{at}-at-1\right)$$

4.5 SOLUTION OF LINEAR DIFFERENTIAL EQUATION

The use of Laplace Transforms enables us to reduce the problem of finding a solution y to a linear differential equation with constant coefficients to the problems of solving a linear algebraic equation for $L[y]$ and then determining y.

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$\Rightarrow L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f] - sf(0) - f'(0)$$

In general;

$$L\left[f^{(n)}\right] = s^n L(f) - s^{n-1} f(0) - s^{n+2} f'(0) - \cdots - f^{(n-1)}(0)$$

Example 4.9

1. Let $f(t) = t^2$. Derive $L[f]$ for $L(1)$.

Solution:

Since $f(0) = 0$, $f'(0) = 0$. $f''(t) = 2$, and $L[2] = 2L[1] = 2/s$,

we obtain $L[f''] = L[2] = 2/s = s^2 L[f]$

Hence $L\left[t^2\right] = 2/s^3$

2. Let $f(t) = \sin^2 t$. Find $L[f]$.

Solution:

We have $f(0) = 0$, $f'(t) = 2\sin t\cos t = \sin 2t$

$$L[\sin 2t] = \frac{2}{s^2+4} = sL[f] \text{ or}$$

$$L\left[\sin^2 t\right] = \frac{2}{s\left(s^2+4\right)}$$

Example 4.10

1. Solve the differential equation (IVP)

$$y' + 3y = 1 \qquad y(0) = 2$$

Solution:

$$L[y' + 3y = 1] = L[y'] + 3L[y] = L[1]$$

But $\quad L[y'] = sL[y] - y(0) = sL[y] - 2$

Hence $\{sL[y] - 2\} + 3L[y] = \frac{1}{s}$

$$\Rightarrow (s+3)L[y] = 2 + \frac{1}{s}$$

$$\Rightarrow L[y] = \frac{2}{s+3} + \frac{1}{s(s+3)}$$

$$\Rightarrow y = L^{-1}\left[\frac{2}{s+3} + \frac{1}{s(s+3)}\right] = 2L^{-1}\left[\frac{1}{s+3}\right] + L^{-1}\left[\frac{1}{s(s+3)}\right]$$

$$\Rightarrow y = 2e^{-3t} + \frac{1}{3} - \frac{1}{3}e^{-3t} = \frac{5}{3}e^{-3t} + \frac{1}{3}$$

2. Solve the differential equation (IVP)

$$y'' - 3y' + 2y = e^{-t} \qquad y(0) = 3, y'(0) = 4$$

Solution:

$$L[y''] - 3L[y'] + 2L[y] = L\left[e^{-t}\right] \text{..........}\mathbf{(I)}$$

but $\quad L[y''] = s^2L[y] - sy(0) - y'(0)$

$$\Rightarrow L[y''] = s^2L[y] - 3s - 4 \text{...............}\mathbf{(II)}$$

and $\quad L[y'] = sL[y] - y(0) = sL[y] - 3 \text{.........}\mathbf{(III)}$

Put **(III)** and **(II)** into **(I)**

$$\left(s^2L[y] - 3s - 4\right) - 3(sL[y] - 3) + 2L[y] = \frac{1}{s+1}$$

$$\Rightarrow \left(s^2 - 3s + 2\right)L[y] = 3s - 5 + \frac{1}{s+1}$$

$$\Rightarrow L[y] = \frac{3s-5}{(s-2)(s-1)} + \frac{1}{(s-2)(s-1)(s+1)}$$

$$\Rightarrow y = L^{-1}\left[\frac{3s-5}{(s-2)(s-1)} + \frac{1}{(s-2)(s-1)(s+1)}\right]$$

$$\Rightarrow y = L^{-1}\left[\frac{3s-5}{(s-2)(s-1)}\right] + L^{-1}\left[\frac{1}{(s-2)(s-1)(s+1)}\right]$$

by partial fraction, we have:

$$y = L^{-1}\left[\frac{3s-5}{(s-2)(s-1)}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{S-2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right]$$

$$\Rightarrow y = e^{2t} + 2e^{t} + \tfrac{1}{3}e^{2t} - \tfrac{1}{2}e^{t} + \tfrac{1}{6}e^{-t}$$

$$\Rightarrow y = \tfrac{4}{3}e^{2t} + \tfrac{3}{2}e^{t} + \tfrac{1}{6}e^{-t}$$

Exercise 4.3

Work out the following (on convolution)

1. $1 * 1$ 2. $1 * \sin\omega t$ 3. $t * e^{t}$ 4. $\sin\omega t * \cos\omega t$ 5. $e^{it} * e^{-it}$

Inverse Transforms by Convolution. *Find* $f(t)$ *by "the convolution theorem"*

1. $\frac{6}{s(s+3)}$ 2. $\frac{1}{s^2(s-1)}$ 3. $\frac{1}{s(s^2+4)}$ 4. $\frac{s^2}{(s^2+\pi^2)^2}$ 5. $\frac{e^{-as}}{s(s-2)}$

Solve the following differential equations using Laplace Transforms

1. $y'' + 2y' + 5y = 5 \quad y(0) = 0,\, y'(0) = 0$
2. $y'' - 5y' - 6y = 2\cos 2t + e^{t} \quad y(0) = 0,\ y'(0) = \tfrac{1}{10}$
3. $y'' + y = F(t) \quad y(0) = 0,\, y'(0) = 0$; where $F(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & 1 < t \end{cases}$
4. $y'' + \beta^2 y = A\sin\omega t \quad y(0) = 1,\, y'(0) = 0$
5. $y'' + y = F(t) \quad y(0) = 0,\, y'(0) = 0$ where $F(t) = \begin{cases} 4 & 0 \le t \le 2 \\ t+2 & t > 2 \end{cases}$

4.6 DIRAC'S DELTA FUNCTION

Definition: $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$ and $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$.

Also $\int_{n}^{m} f(t)\delta(t-a)dt = f(a)$ and $\int_{n}^{m} \delta(t-a)dt$,

provided in each case, that $n < a < m$.

In the Laplace transform form:

$L[f(t)\delta(t-a)] = f(a)e^{-as}$ and $L[\delta(t-a)] = e^{-as}$

Note: $\delta(t-a)$ is not a function in the ordinary sense.

Example 4.11

1. $\int_{2}^{5}(2t^2+3)\delta(t-2)dt = f(2) = 2(2^2)+3 = 11$
2. $\int_{0}^{\pi} \sin 5t \cdot \delta\left(t-\frac{\pi}{2}\right)dt = f\left(\frac{\pi}{2}\right) = \sin\frac{5}{2}\pi = 1$
3. $\int_{0}^{\infty}(2t^3+2t^2-5t-6)\delta(t-1)dt = -7$
4. $\int_{2}^{5} e^{-2t}\delta(t-3)dt = f(3) = e^{-2\times 3} = e^{-6}$
5. $L[\delta(t-3)] = e^{-3s}$
6. $L[2\delta(t-3)] = f(3)e^{-3s} = 2e^{-3s}$
7. $L[(2t^2-t+1)\delta(t-2)] = f(2)e^{-2s} = [2(2^2)-2+1]e^{-2s} = 7e^{-2s}$
8. $L\left[\sin 3t \cdot \delta\left(t-\frac{\pi}{2}\right)\right] = \left(\sin\frac{3}{2}\pi\right)e^{-\frac{\pi}{2}s} = -e^{\frac{\pi}{2}s}$
9. $L[\cosh 2t \cdot \delta(t)] = \cosh 2(0)\cdot e^{0} = \cosh(0)\cdot 1 = 1\cdot 1 = 1$

Exercise 4.4

Evaluate the following;

1. $\int_1^5 \left(3+2t-4t^2\right)\cdot \delta(t-3)dt$

2. $\int_0^\infty \sinh 2t \cdot \delta(t)dt$

3. $\int_0^5 e^{-2t}\delta(t-3)dt$

4. $\int_0^\infty \cos 5t \cdot \delta\left(t-\frac{\pi}{2}\right)dt$

Determine the following;

5. $L[2\cdot\delta(t-1)]$

6. $L\left[e^{-3t}\cdot\delta(t-2)\right]$

7. $L[\sin 3t\cdot\delta(t-\pi)]$

Solve the following differential equation;

8. $y''+y=\delta(t-\pi)-\delta(t-2\pi)$ $\quad y(0)=0, y'(0)=1$

9. $y''+4y'+5y=\delta(t-1)$ $\quad y(0)=0, y'(0)=3$

10. $y''+2y'-3y=8e^{-t}+\delta\left(t-\frac{1}{2}\right)$ $\quad y(0)=3, y'(0)=-5$

UNIT 5

SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

Introduction

In the preceding course (MATH 251) we have discussed methods for solving Ordinary Differential Equations with constant coefficient that involves only one dependent variable. Many applications, however, require the use of two or more dependent variables, hence in this unit, we shall deal with more than one dependent variable and more than one equation.

Learning Objectives

After going through this unit, you would be able to:

- put systems of differential equations in normal form.
- reduce a higher order linear differential equation into first order system of differential equations.
- put systems of equations in matrix form and solve.

Session 1-1 Introduction and Motivation

In real life situations quantities and their rate of change depend on more than one variable. For example, the rabbit population, though it may be represented by a single number, depends on the size of predator populations and the availability of food. In order to represent and study such complicated problems we need to use more than one dependent variable and more than one equation.

Such problems lead naturally to a ***system*** of simultaneous Ordinary Differential Equations. Systems of differential equations are the tools to use. The differential equations are very much helpful in many areas of science. But most of interesting real life problems involves more than one unknown function. Therefore, the use of system of differential equations is very useful.

As a motivation let us consider an island with two types of species: Rabbits and Fox. Clearly one plays the role of **predator** while the other one the role of a **prey**. If we are interested to model the populations' growths of both species, then we have to keep in mind that if, for example, the population of the Fox increases, then the Rabbit population will be affected. So the rate of change of the population of one type will depend on the actual population of the other type. For example, in the absence of the Rabbit population, the Fox population will decrease (and fast) to face a certain extinction, something that most of us would like to avoid. A model for this Predator-Prey problem was developed by Lotka (in 1925) and Volterra (in 1926) and is known as the Lotka-Volterra system

$$\begin{cases} \dfrac{dR}{dt} = aR - \alpha RF \\ \dfrac{dy}{dt} = -bF + \beta RF \end{cases}$$

where $R(t)$ measures the Rabbit population, $F(t)$ measures the Fox population, and all the involved constant (a, b, α, γ) are positive numbers. Note that a and b are the growth rate of the prey, and the death rate of the predator. α and β are measures of the effect of the interaction between the Rabbits and The Fox. *Can we check **Iraq** population with system of differential equation?*

Note that in the Lotka-Volterra system, the variable *t* is missing. This kind of system is called **autonomous system** and is written

$$\begin{cases} \dfrac{dy}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

We will usually denote the independent variable by t and the dependent variables (the unknown functions of t) by $x_1, x_2, x_3, \cdots$, or $x, y, z, \cdots$.

For instance, a system of two first order differential equations in the dependent variables x and y has the general form:

$$\left.\begin{aligned} f\left(t,x,y,\frac{dx}{dt},\frac{dy}{dt}\right)=0 \\ g\left(t,x,y,\frac{dx}{dt},\frac{dy}{dt}\right)=0 \end{aligned}\right\} \text{---------}(\kappa)$$

where the functions f and g are given.

A solution of this system is a *pair* ($\big(x(t),y(t)\big)$ of functions of t that satisfy both equations identically over some real interval of values of t.

1-1.1 Types of Linear Systems

Definition 1:

The general linear systems of **two** *first-order* differential equations in **two** unknown functions x and y are of the form:

$$\left.\begin{aligned} a_1(t)x'+a_2(t)y'+a_3(t)x+a_4(t)y= f_1(t) \\ b_1(t)x'+b_2(t)y'+b_3(t)x+b_4(t)y= f_2(t) \end{aligned}\right\} \text{............ (4.1)}$$

where the *primes* denote derivatives with respect to the independent variable t.

i.e. $\quad x'=\dfrac{dx}{dt} \qquad y'=\dfrac{dy}{dt}$

Note: We shall be concerned with systems of this type that have constant coefficients and the number of equations

Example 1.1

1. $\begin{aligned} x'+3y'+2x+y=t^2 \\ x'-2y'-3x-5y=e^t \end{aligned}$

2. $\begin{aligned} x'-2y'+x-y=t^2-1 \\ x'+y'-x=t+2 \end{aligned}$

3. $\begin{aligned} 2x'+y-2x-y=e^t \\ x'+3x+y=\cos t \end{aligned}$

4. $\begin{aligned} (D+2)x-(D-1)y=1 \\ (D+1)x-(D+2)y=t-1 \end{aligned}$

We shall say that a *solution* of system (1.1) is an ordered pair real functions (f, g) such that $x = f(t)$, $y = g(t)$ simultaneously satisfy both equations of the system (4.1) on some real interval $a \le t \le b$.

Definition 2:

The general linear systems of three first-order differential equations in three unknown functions x, y and z is of the form:

$$\left.\begin{aligned} a_1(t)x' + a_2(t)y' + a_3(t)z' + a_4(t)x + a_5(t)y + a_6(t)z &= f_1(t) \\ b_1(t)x' + b_2(t)y' + b_3(t)z' + b_4(t)x + b_5(t)y + b_6(t)z &= f_2(t) \\ c_1(t)x' + c_2(t)y' + c_3(t)z' + c_4(t)x + c_5(t)y + c_6(t)z &= f_3(t) \end{aligned}\right\} \text{..... (4.2)}$$

Example 1.2

1. $$\begin{aligned} x' + y' - 2z' + 2x - 3y + z &= t \\ 2x' - y' + 3z' + x + 4y - 5z &= \sin t \\ x' + 2y' + z' - 3x + 2y - z &= \cos t \end{aligned}$$

2. $$\begin{aligned} x' + z' - 2x - 5y + z &= 1 - t \\ y' + 3x - 5y - z &= \tan t \\ x' - y' + z' - 2z &= e^t \end{aligned}$$

3. $$\begin{aligned} (D-1)x_1 - x_2 + x_3 &= 0 \\ -2x_1 - (D-3)x_2 + 4x_3 &= 0 \\ -4x_1 - x_2 + (D+4)x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + 2x_2 + 2x_3 \\ \frac{dx_2}{dt} &= x_1 + 4x_2 + x_3 \\ \frac{dx_3}{dt} &= -2x_1 - 4x_2 - 2x_3 \end{aligned}$$

We shall say that a solution of system (4.2) is an ordered triple of real functions (f, g, h) such that: $x = f(t), y = g(t), z = h(t)$ simultaneously satisfy all three equations of the system (4.2) on some interval $a \le t \le b$.

Systems of the form (4.1) and (4.2) contained only first derivative, and we now consider the basic linear system involving higher derivatives. This is the general linear system of two second-order differential equations in two unknown functions x and y, and is a system of the form:

$$\left.\begin{aligned} a_1(t)x'' + a_2(t)y'' + a_3(t)x' + a_4(t)y' + a_5(t)x + a_6(t)y &= f_1(t) \\ b_1(t)x'' + b_2(t)y'' + b_3(t)x' + b_4(t)y' + b_5(t)x + b_6(t)y &= f_2(t) \end{aligned}\right\} \text{..... (4.3)}$$

An example is provided below with constant coefficient:

$$\begin{aligned} 2x'' + 5y'' + 7x' + 3y' + 2y &= 3t + 1 \\ 3x'' + 2y'' - 2y' + 4x + y &= 0 \end{aligned}$$

For given fixed positive integers m and n, we could proceed in like manner, to exhibit other general linear systems of n m^{th} - order differential equations in n unknown functions and give examples each of such type of system (I hope you can do that on your own). Instead we proceed to introduce the *standard type* of linear system.

We now introduce standard type as a special case of the system (4.1) of two first-order differential equations in two unknown functions x and y. We consider the special type of linear system (4.1), which is of the form:

$$\left.\begin{aligned} x' &= a_{11}(t)x + a_{12}(t)y + f_1(t) \\ y' &= a_{21}(t)x + a_{22}(t) + f_2(t) \end{aligned}\right\} \quad \text{................} \quad (4.4)$$

This is called the **normal form** in the case of two linear differential equations in two unknown functions. The characteristic feature of such a system is apparent from the manner in which the derivatives appear in it. An example of such a system with variable coefficients is

$$\begin{aligned} x' &= t^2 x + (t+1)y + t^3 \\ y' &= te^t x + t^3 y - e^t \end{aligned},$$

while one with constant coefficients is

$$\begin{aligned} x' &= 5x + 7y + t^2 \\ y' &= 2x - 3y - 2t \end{aligned}$$

Note that example (4) of equation (4.2) is in the *normal* form.

The *normal* form in the case of a linear system of three differential equations in three unknown functions x, y and z is:

$$\begin{aligned} x' &= a_{11}(t)x + a_{12}(t)y + a_{13}(t)z + f_1(t) \\ y' &= a_{21}(t)x + a_{22}(t)y + a_{23}(t)z + f_2(t) \\ z' &= a_{31}(t)x + a_{32}(t)y + a_{33}(t)z + f_3(t) \end{aligned}$$

An example of such a system is the constant coefficient system:

$$
\begin{aligned}
x' &= 3x + 3y + z + t \\
y' &= 2x - 4y + 5z - t^2 \\
z' &= 4x + y - 3z + 2t + 1
\end{aligned}
$$

The **normal form** in the general case of a linear system of n differential equations in n unknown functions $x_1, x_2, \cdots, x_n$ is:

$$
\left.\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}\right\} \text{............ } (4.5)
$$

An important fundamental property of a normal linear system (4.5) is its relationship to a single nth – order linear differential equation in one unknown function. Specifically, consider the so-called normalized (meaning, the coefficient of the highest derivative is one) nth – order linear differential equation

$$
x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_{n-1}(t)x' + a_n(t)x = f(t) \text{\_\_\_\_\_\_ } (4.6)
$$

in the one unknown function x.

Let

$$
\left.\begin{aligned}
x_1 &= x, \\
x_2 &= x' \\
x_3 &= x'' \\
&\vdots \\
x_{n-1} &= x^{(n-2)} \\
x_n &= x^{(n-1)}
\end{aligned}\right\} \text{ .... } (4.7)
$$

From (4.7), we get:

$$
x' = x_1',\; x'' = x_2', \cdots, x^{(n-1)} = x_{n-1}',\;\; x^{(n)} = x_n' \text{ .... } (4.8)
$$

Then using both (4.7) and (4.8), the single nth – order equation (4.6) can be transformed into:

$$\left.\begin{aligned} &x_1' = x_2 \\ &x_2' = x_3 \\ &\vdots \\ &x_{n-1}' = x_n \\ &x_n' = -a_n(t)x_1 - a_{n-1}(t)x_2 - \cdots - a_1(t)x_n + f(t) \end{aligned}\right\} \text{.......... (4.9)}$$

which is a special case of the normal linear system (4.5) of n equations in n unknown functions. Thus we see that a single nth – order linear differential equation of the form (4.6) in one unknown function is indeed **intimately related** to a normal linear system (4.5) of n first – order differential equation in n unknown functions.

Example 1.3

Reduced the following to normal form:

(a) $y'''(t) + a_1 y''(t) + a_2 y'(t) + a_3 y(t) = f(t)$

(b) $\dfrac{d^2x}{dt^2} + e^t \dfrac{dx}{dt} + (t-1)x = 0$

(c) $\dfrac{d^2x}{dt^2} + \sinh x = 3x$

Solution (a):

$$y'''(t) + a_1 y''(t) + a_2 y'(t) + a_3 y(t) = f(t) \text{ .... } (\alpha)$$

Let $y(t) = x_0(t)$

$$y'(t) = x_0'(t) = x_1(t)$$

$$y''(t) = x_0''(t) = x_1'(t) = x_2(t)$$

$$y'''(t) = x_0'''(t) = x_1''(t) = x_2'(t)$$

then from (α), we have

$$y'''(t) = f(t) - a_1 y''(t) - a_2 y'(t) - a_3 y(t) \text{ .... } (\beta)$$

from above

$$\left.\begin{aligned} &x_0'(t) = x_1(t) \text{ .... } (\gamma) \\ &x_1'(t) = x_2(t) \text{ .... } (\delta) \\ &x_2'(t) = f(t) - a_1x_2(t) - a_2x_1(t) - a_3x_0(t) \text{ .... } (\varepsilon) \end{aligned}\right\} \text{system}$$

so equation (γ), (δ) and (ε) are the 1st – order linear differential equation equivalent to (α). These three (3) equations can be solved to obtain the solution to (α).

Solution (b):

Let $$x = x_0 \Rightarrow \frac{dx}{dt} = \frac{dx_0}{dt} = x_1 \Rightarrow \frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} = \frac{dx_1}{dt} = -e^t \frac{dx}{dt} - (t-1)x$$

$\therefore$ The normal form is:

$$\left.\begin{aligned} \frac{dx_0}{dt} &= x_1 \\ \frac{dx_1}{dt} &= -e^t x_1 - (t-1)x_0 \end{aligned}\right\}$$

Exercise 1.2

In each of exercises 1 – 4, transform the single linear differential equation of the form (1.6) into a system of first – order differential equations of the form (1.9).

1. $\dfrac{dx^2}{dt^2} - 3\dfrac{dx}{dt} + 2x = t^2$ 2. $\dfrac{d^3x}{dt^3} - 2\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 2x = e^{3t}$

3. $\dfrac{d^3x}{dt^3} + t\dfrac{d^2x}{dt^2} + 2t^3\dfrac{dx}{dt} - 5t^4 = 0$ 4. $\dfrac{d^4x}{dt^4} - t^2\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - 2tx = \cos t$

Example 1.5

Reduce the following to normal form;

1. $$2\frac{dx}{dt} + \frac{dy}{dt} - 2x + 5y = 2t$$
$$\frac{dx}{dt} - 2\frac{dy}{dt} + x + 3t = 0$$

2. $$\frac{dx}{dt} + 2\frac{dy}{dt} = 2x + y$$
$$2\frac{dx}{dt} - \frac{dy}{dt} = -x + y$$

Solution:

$$2\frac{dx}{dt} + \frac{dy}{dt} = 2x - 5y + 2t \_\_\_\_\_\_\_\_\_\_(1)$$

$$\frac{dx}{dt}-2\frac{dy}{dt}=-x-3t \_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)$$

Solving for $\frac{dx}{dt}$ by the method of elimination (i.e. eliminating $\frac{dy}{dt}$ first) then $(1)\times 2+(2)$

$$\therefore 5\frac{dx}{dt}=3x-5y+t \Rightarrow \frac{dx}{dt}=3x-y+\frac{1}{5}t$$

Also solving for $\frac{dy}{dt}$ then $(1)-(2)\times 2$

$$\therefore 5\frac{dx}{dt}=4x-5y+8t \Rightarrow \frac{dy}{dt}=\frac{4}{5}x-y+\frac{8}{5}t$$

Therefore we have the normal linear system as:

$$\left.\begin{aligned}\frac{dx}{dt}&=\frac{3}{5}x-y+\frac{1}{5}t\\ \frac{dy}{dt}&=\frac{4}{5}x-y+\frac{8}{5}t\end{aligned}\right\}$$

Exercise 1.3

Find the first order system of the following differential equations

1. $\frac{d^2y}{dt^2}-4y=0$

2. $9\frac{d^3y}{dt^3}+12\frac{d^2y}{dt^2}+4\frac{dy}{dt}=\cos t$

3. $m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$, where $f(t)$ is an external force.

4. $3\frac{d^3x}{dt^3}-2\frac{d^2x}{dt^2}+12\frac{dx}{dt}-8x=e^{-t}\sin t$

5. $\frac{d^4x}{dt^4}-5\frac{d^2x}{dt^2}+4x=e^{t}-te^{2t}$

6. $\frac{d^5x}{dt^5}+2\frac{d^3x}{dt^3}+2\frac{d^2x}{dt^2}=3t^2-1$

Reduce each of these to the normal form.

1. $$\frac{dy}{dt}+2\frac{dx}{dt}=e^t$$
$$\frac{dy}{dt}-\frac{dx}{dt}=x-y+\sin t$$

2. $$2\frac{dx}{dt}+\frac{dy}{dt}-x=e^t$$
$$\frac{dx}{dt}-3\frac{dy}{dt}-2x+4y=\sinh t$$

3. $$\frac{dy}{dt}+\frac{dx}{dt}=x+2y+\sin t$$
$$\frac{dy}{dt}-\frac{dx}{dt}=4e^t-x$$

4. $$\frac{d^2x}{dt^2}-6\frac{dx}{dt}+3\frac{dy}{dt}=0$$
$$\frac{d^2y}{dt^2}+\frac{dx}{dt}+x-y=4t$$

1-1.2 Solution to a Linear System of Differential Equations

The solution to the system:

$$\left.\begin{aligned}\frac{dx_1}{dt} &= f_1(t,x_1,x_2,x_3,\cdots x_n)\\ \frac{dx_2}{dt} &= f_2(t,x_1,x_2,x_3,\cdots x_n)\\ &\vdots\\ \frac{dx_n}{dt} &= f_n(t,x_1,x_2,x_3,\cdots x_n)\end{aligned}\right\} \text{......................} \quad (1.10)$$

is the parametric vector $\mathbf{x}=\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}$ on open real interval $a<t<b$ for which the system is true and the vector $\mathbf{x}$ must be continuous on the open real interval.

Session 2-1 Methods of Solving SLDE

The most elementary approach to **S**ystems of **L**inear **D**ifferential **E**quations (**SLDE**) with constant coefficients involves the elimination of dependent variables by appropriately combining pairs of equations.

The main methods we will consider in solving such systems are:

A) The differential operator method,

B) The matrix method.

The differential operator method for linear differential system is quite similar to the solution of linear algebraic systems by elimination of variables until only one remains. It is most convenient in the case of manageably small systems: those consisting of no more than four equations. For large systems of differential equations, the matrix method is preferable.

2-1.1 The Differential Operator Method

Recall that $D = \frac{d}{dt} \Rightarrow D^2 = \frac{d^2}{dt^2}$ and also $D^n = \frac{d^n}{dt^n}$

System of 1st Order Equations: The differential operator method for linear differential system is quite similar to the solution of linear algebraic systems by elimination of variables until only one remains. It is most convenient in the case of manageably small systems: those consisting of no more than four equations. For large systems of differential equations, the matrix method is preferable.

Example 1.6

Solve the system

$$\left.\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x + y \end{aligned}\right\} \Rightarrow \frac{dx}{x+y} = \frac{dy}{4x+y}$$

Note: *this system is called* ***Autonomous system*** *since time is not explicitly expressed in the equation*.

Solution:

We use matrix/determinant techniques. We rewrite the problem as follows:

$$\begin{aligned} (D-1)x - y &= 0 \\ -4x + (D-1)y &= 0 \end{aligned} \quad \Rightarrow \begin{pmatrix} D-1 & -1 \\ -4 & D-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Cramer's Rule:

$$\begin{vmatrix} D-1 & -1 \\ -4 & D-1 \end{vmatrix} x = \begin{vmatrix} 0 & -1 \\ 0 & D-1 \end{vmatrix}$$

$$\left[(D-1)^2 - 4\right]x = 0 \Rightarrow \left(D^2 - 2D - 3\right)x = 0 \Rightarrow x \neq 0 \quad D = 3, -1$$

(Just like finding the roots of a characteristic equation in linear differential equation we did last semester)

$$\therefore x = Ae^{3t} + Be^{-t}$$

Then go back to the system to pick one equation to determine y.

$$y = \frac{dx}{dt} - x = 3Ae^{3t} - Be^{-t} - \left(Ae^{3t} + Be^{-t}\right) = 2Ae^{3t} - 2Be^{-t}$$

Then the solution is written as:

$$x(t) = Ae^{3t} + Be^{-t}$$
$$y(t) = 2Ae^{3t} - 2Be^{-t}$$

Example 1.7

1. Solve the system $\dfrac{dx}{3x-y} = \dfrac{dy}{x+y}$, when $x(0) = 1$, $y(0) = 2$

Solution:

$$\begin{aligned} \frac{dx}{dt} &= 3x - y \Rightarrow (D-3)x + y = 0 \\ \frac{dy}{dt} &= x + y \Rightarrow -x + (D-1)y = 0 \end{aligned} \Rightarrow \begin{pmatrix} D-3 & 1 \\ -1 & D-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Cramer's Rule:

$$\begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix} x = \begin{vmatrix} 0 & 1 \\ 0 & D-1 \end{vmatrix} \Rightarrow \left[(D-3)(D-1) + 1\right]x = 0$$

$$\Rightarrow \left[D^2 - D - 3D + 4\right]x = 0$$

$$\Rightarrow \left[D^2 - 4D + 4\right]x = 0 \Rightarrow \left[(D-2)^2\right]x = 0 \text{ but } x \neq 0$$

$$\therefore x = (At + B)e^{2t} \text{\_\_\_\_\_\_\_\_\_\_}(*)$$

Solving for y, we have: $y = 3x - \dfrac{dx}{dt}$

$$\Rightarrow y = 3e^{2t}(At + B) - [A + 2(At + B)]e^{2t}$$

$$\Rightarrow y = 3e^{2t}(At+B) - [2At + (2B+A)]e^{2t}$$

$$\Rightarrow y = Ate^{2t} + (B-A)e^{2t} \_\_\_\_\_ (**)$$

with the initial conditions:

$$(*) \qquad \Rightarrow B = 1$$

$$(**) \qquad \Rightarrow 2 = B - A \Rightarrow A = -1$$

$$\therefore \qquad x = (1-t)e^{2t}$$

$$y = -te^{2t} + 2e^{2t}$$

2. Solve the system:

$$(D-1)x - Dy = 2t+1$$

$$(2D+1)x + 2Dy = t$$

Solution:

$$(D-1)x - Dy = 2t+1 \text{........(I)}$$

$$(2D+1)x + 2Dy = t \text{..........(II)}$$

Subtracting twice (I) from (II), we have: $3x = -3t - 2$.

Substituting $x = -t - 2/3$ in (I), we obtain:

$$Dy = 2t + 1 - (D-1)x = t + \tfrac{4}{3}$$

$$y = \tfrac{1}{2}t^2 + \tfrac{4}{3}t + c$$

The complete solution: $x = -t - \frac{2}{3}$, $y = \frac{1}{2}t^2 + \frac{4}{3}t + c$.

Exercise 4.4

1. $\dfrac{dx}{dt} = y$, $\dfrac{dy}{dt} = x$

2. $\dfrac{dx}{dt} = -y$, $\dfrac{dy}{dt} = x$

3. $\dfrac{dx}{dt} = 6x + 8y$, $\dfrac{dy}{dt} = 6x - 2y$

4. $\dfrac{dx}{dt} = -2x + y$, $\dfrac{dy}{dt} = x - y$

5. $\dfrac{dx}{dt} = 3x + 2y + t$, $\dfrac{dy}{dt} = 2x + 3y + 3$ non-homogeneous

6. $$\frac{dx}{dt} = 6x + 8y$$
$$\frac{dy}{dt} = 6x - 2y + e^{-t}$$

7. $$\frac{dx}{dt} = 3x + 2y + 2z$$
$$\frac{dy}{dt} = x + 4y + z$$
$$\frac{dz}{dt} = -2x - 4y - z$$

8. $$\frac{dx_1}{dt} = 3x_1 + x_2$$
$$\frac{dx_2}{dt} = 3x_1 - 2x_2$$
$$\frac{dx_3}{dt} = x_1 + x_2 + x_3$$

9. $$\frac{dx_1}{dt} = x_1 - x_2 - x_3 + e^t$$
$$\frac{dx_2}{dt} = x_1 - x_2 + t$$
$$\frac{dx_3}{dt} = x_1 - x_3 + 1$$

10. Find the solution of the system of differential equation Ex. 1 such that $x(0) = 1$, $y(0) = 2$.

11. Find the solution of the system of differential equation is Ex.5 such that $x(0) = 3$, $y(0) = 1$.

12. Find the solution of the system of differential equations in Ex.7 such that $x(0) = 2$, $y(0) = 1$, $z(0) = 0$.

13. Find the solution of the system of differential equations in Ex.9 such that $x_1(0) = 1$, $x_2(0) = -1$, $x_3(0) = 0$.

2-1.2 Reduction of n^{th} – Order Equation to a System of 1st Order Equations

The number of integration constants in an n^{th} order differential equation, in the same vein, is the same as the number of integration constant in a system of n 1st order differential equation.

Example 1.8

1. Convert $y'' - 3y' + 2y = e^t$ into a system of equations and solve completely.

Solution:

Let
$$y(t) = x_0(t)$$
$$y'(t) = x_0'(t) = x_1(t)$$
$$y''(t) = x_1'(t)$$

The system becomes:

$$\left.\begin{aligned} x_0'(t) &= x_1(t) \\ x_1'(t) &= e^t + 3x_1(t) - 2x_0(t) \end{aligned}\right\} \textbf{non} - \textbf{homogeneous} \text{ example}$$

Then the solution of the system:

$$\begin{pmatrix} D & -1 \\ 2 & D-3 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

Using Cramer's Rule:

$$\begin{vmatrix} D & -1 \\ 2 & D-3 \end{vmatrix} x_1 = \begin{vmatrix} 0 & -1 \\ e^t & D-3 \end{vmatrix}$$

$$\Rightarrow [D(D-3)+2]x_1 = e^t$$

$$\Rightarrow [D^2 - 3D + 2]x_1 = e^t$$

$$\Rightarrow (D-2)(D-1)x_1 = e^t$$

Let $(D-2)x_1 = z$

$$\therefore (D-1)z = e^t \Rightarrow z = \frac{1}{D-1}e^t$$

$$\Longrightarrow \boldsymbol{z = e^t \int e^{-t} e^t\, dt} \quad \Rightarrow z = te^t + k_1 e^t$$

Then

$$(D-2)x_1 = te^t + k_1 e^t \quad \Rightarrow x_1 = e^{2t}\int \left(te^t + k_1 e^t\right)e^{-2t}\, dt$$

$$\Rightarrow x_1 = e^{2t}\int\left(te^{-t}+k_1e^{-t}\right)dt \qquad \Rightarrow x_1 = e^{2t}\left[-te^{-t}+e^{-t}-k_1e^{-t}+k_2\right]$$

$$\Rightarrow x_1 = e^{t}(1-t)-k_1e^{-t}+k_2e^{2t}$$

Since we are solving for $y(t)$, then we have

$$y = Ae^{-t}+Be^{2t}+e^{t}(1-t)$$

Exercise 1.4

1. $y''-7y'+12y=t$
2. $(D-3)^2y=te^{3t}$
3. $\left(D^2-5D+6\right)y=e^{2t}$
4. $D(D-1)y=5$
5. $\left(D^2+2D\right)y=e^{-x}\cos x$
6. $\left(D^2+D\right)y=x$
7. $\left(D^2+1\right)y=\tan t^2, -\pi/2<t<\pi/2$
8. $\left(D^2-D\right)y=2^t$

2-1.3 The Matrix Method

Although the simple differential operator method suffice for the solution of small linear systems containing only two or three equations with constant coefficients, the general properties of linear systems – as well as solution methods suitable for large systems – are most easily and concisely described in matrix notation.

We discuss here the general system of n first order linear equations being the form: (***note*** equation 4.5)

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1+a_{12}(t)x_2+\cdots+a_{1n}(t)x_n+f_1(t)\\
\frac{dx_2}{dt} &= a_{21}(t)x_1+a_{22}(t)x_2+\cdots+a_{2n}(t)x_n+f_2(t)\\
&\vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1+a_{n2}(t)x_2+\cdots+a_{nn}(t)x_n+f_n(t)
\end{aligned} \qquad \text{______ (4.5)}$$

where $f_1(t), f_2(t), f_3(t),\cdots, f_n(t)$ are continuous on a real interval $a\le t\le b$.

The Matrix Notation form of (4.5) is given as:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

This can simply be written as: $\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{F}(t)$ …………............... .(**I**)

which is a **NON-HOMOGENEOUS FORM**.

where $\mathbf{F}(t)$ is the **non-homogeneous term** or the **force term**.

If $\mathbf{F}(t) = \mathbf{0}$, that is $f_1(t) = f_2(t) = f_3(t) = \cdots = f_n(t) = 0$ for the real interval $a \le t \le b$ then

we have $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ ……………….............. (**II**)

Equation (**II**) is known as the **HOMOGENEOUS FORM**.

Examples of non – homogeneous systems of linear differential equation

1. $$\frac{dx}{dt} = x + y + e^{-t}$$

$$\frac{dy}{dx} = x - y + 6\sin t$$

The matrix notation form is given as:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 6\sin t \end{pmatrix}$$ where $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and

$$\mathbf{F}(t) = \begin{pmatrix} e^{-t} \\ 6\sin t \end{pmatrix}$$

$$
2. \quad \begin{aligned}
\frac{dx_1}{dt} &= 3x_1 + x_2 - 2x_3 + t^2 \\
\frac{dx_2}{dt} &= 4x_1 - 2x_2 + x_3 - e^t \quad , \\
\frac{dx_3}{dt} &= \quad -2x_2 + 5x_3 - \sin t
\end{aligned}
$$

The matrix notation form is given as:

$$
\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 \\ 4 & -2 & 1 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} t^2 \\ -e^t \\ -\sin t \end{pmatrix} \text{ where } \frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 3 & 1 & -2 \\ 4 & -2 & 1 \\ 0 & -2 & 5 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

and $\mathbf{F}(t) = \begin{pmatrix} t^2 \\ -e^t \\ -\sin t \end{pmatrix}$

Note that $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{F}(t)$ can be simply be written as $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{F}(t)$.

<u>Examples of Homogeneous systems of linear differential equation</u>

$$
1. \quad \begin{aligned}
\frac{dx}{dt} &= 5x - 3y \\
\frac{dy}{dt} &= -2x + 4y
\end{aligned}
$$

The matrix notation form is given as : $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ where $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$,

$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$
2. \quad \begin{aligned}
\frac{dx_1}{dt} &= x_1 + x_2 - x_3 \\
\frac{dx_2}{dt} &= 2x_1 \quad - 4x_3 \\
\frac{dx_3}{dt} &= \quad -2x_1 + 3x_3
\end{aligned}
$$

The matrix notation form is given as:

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -4 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ where } \frac{d\mathbf{x}}{dt} = \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -4 \\ 0 & -2 & 3 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Note that $\dfrac{d\mathbf{x}}{dt} = \mathbf{Ax}$ can simply be written as $\mathbf{x}' = \mathbf{Ax}$

Definition

Let $\mathbf{A}$ and $\mathbf{F}$ be defined on an open interval *I*. A vector function $\mathbf{u}$ is a solution of equation (**I**) on *I* if:

$$\mathbf{u}'(t) = \mathbf{Au}(t) + \mathbf{F}(t)$$

Example 1.10

Consider the differential equation

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix} \mathbf{x} \text{ -------------- (1i)}$$

The vector function: $\mathbf{u}(t) = \begin{pmatrix} 3e^{7t} \\ -2e^{7t} \end{pmatrix}$ is a solution of equation (1i) on

$(-\infty, \infty)$ since $\mathbf{u}'(t) = \begin{pmatrix} 21e^{7t} \\ -14e^{7t} \end{pmatrix}$ and $\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 3e^{7t} \\ -2e^{7t} \end{pmatrix} = \begin{pmatrix} 21e^{7t} \\ -14e^{7t} \end{pmatrix}$

Similarly, $\mathbf{v}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ is a solution of equation (1) on $(-\infty, \infty)$ since: $\mathbf{v}'(t) = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$ and

$$\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Example 1.11

Consider the equation:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \mathbf{x} \text{ ----------- (1ii)}$$

The vector functions: $\mathbf{u}(t) = \begin{pmatrix} e^{-3t} \\ 7e^{-3t} \\ 11e^{-3t} \end{pmatrix}$ and $\mathbf{v}(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ e^{2t} \end{pmatrix}$ are solutions on $(-\infty, \infty)$ of equation (1ii) since: $\mathbf{u}'(t) = \begin{pmatrix} -3e^{-3t} \\ -21e^{-3t} \\ -33e^{-3t} \end{pmatrix}$ $\mathbf{v}'(t) = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 2e^{2t} \end{pmatrix}$ and

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ 7e^{-3t} \\ 11e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -21e^{-3t} \\ -33e^{-3t} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 2e^{2t} \end{pmatrix}$$

It is now left for you to verify that : $\mathbf{w}(t) = \begin{pmatrix} e^{t} \\ e^{t} \\ e^{t} \end{pmatrix}$ is also a solution on $(-\infty, \infty)$ of equation (1ii)

Theorem 1.1

If $\mathbf{u}$ and $\mathbf{v}$ are solutions on an open interval I of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and c_1, c_2 are any two constants, then $\mathbf{w}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$ is also a solution on I .

Example 1.12

In example 1.10 we found that $\mathbf{u}(t) = \begin{pmatrix} 3e^{7t} \\ -2e^{7t} \end{pmatrix}$ and $\mathbf{v}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ are solutions of $\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix} \mathbf{x}$

By **Theorem 1.1,** $c_1\begin{pmatrix} 3e^{7t} \\ -2e^{7t} \end{pmatrix}+c_2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}=\begin{pmatrix} 3c_1e^{7t}+c_2e^{2t} \\ -2c_1e^{7t}+c_2e^{2t} \end{pmatrix}$ is also a solution for any choice of the constants c_1 and c_2.

The vector – valued functions $\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n$ are **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \cdots, c_n$ *not all zero* such that:

$c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t)+\cdots+c_n\mathbf{x}_n(t)=0$ for all t in I . Otherwise they are **linearly independent**.

Equivalently, they are linearly independent provided that no one of them is a linear combination of the others.

Exercise 1.5

a. Write the given system in the form: $\mathbf{x}'(t) = \mathbf{A} + \mathbf{F}(t)$

1. $\begin{aligned} x' &= 3x - 2y \\ y' &= 2x + y \end{aligned}$

2. $\begin{aligned} x' &= x - 2y + \cos t \\ y' &= x + y - \sin t \end{aligned}$

3. $\begin{aligned} x' &= 2x + 4y + 3e^{t} \\ y' &= 5x - y - t^{2} \end{aligned}$

4. $\begin{aligned} x' &= 3x - 4y + z + t \\ y' &= x - 3z + t^{2} \\ z' &= 6y - 7z + t^{3} \end{aligned}$

5. $\begin{aligned} x' &= 4x - y + z \\ y' &= 2x + y - z \\ z' &= x + 3y + z \end{aligned}$

6. State whether homogeneous or non – homogeneous.

b. Verify that the given vectors are solutions to the given system, and then use the Wronskian to show that they are linearly independent. Finally write the general solution of the system.

1. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\mathbf{x};\ \mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix},\ \mathbf{x}_2 = \begin{pmatrix} 2e^{-2t} \\ e^{-t} \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}\mathbf{x};\ \mathbf{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mathbf{x}_2 = e^{-2t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\mathbf{x};\ \mathbf{x}_1 = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \mathbf{x}_2 = e^{2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x};\ \mathbf{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \mathbf{x}_3 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix}\mathbf{x};\ \mathbf{x}_1 = e^{t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},\ \mathbf{x}_2 = e^{3t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{x}_3 = e^{5t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Theorem 1.2: Principle of Superposition

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n$ be n solutions of homogeneous linear equation (**II**) i.e.: $\dfrac{d\mathbf{x}}{dt} = \mathbf{Ax}$ on the open interval $(-\infty, \infty)$. If $c_1, c_2, \cdots, c_n$ are constants, then the linear combination,
$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$ is also a solution of (**II**). ***Prove as exercise***.

2-1.4 The Eigenvalue Method of Homogeneous Linear Systems

In this section we will present a method for finding a general solution of the differential equation: $\dfrac{d\mathbf{x}}{dt} = \mathbf{Ax}$ ………. (**III**)

where **A** is an $n \times n$ matrix.

Recall that ordinary differential equation: $Dy = ay$ or $(D^2 + aD + b)y = 0$, has solutions of the form: $y(t) = ce^{rt}$, where c is an arbitrary constant, r is the root(s) of the characteristic. We will show that equation (**III**) has analogous function for solutions in system of differential equation. That is, we will show that equation (**III**) has solutions of the form: $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ where λ is a constant (possibly a complex number) and $\mathbf{v}$ is a constant.

To begin, we note that if: $\mathbf{v} = \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \\ \vdots \\ \upsilon_v \end{pmatrix}$ then:

$$\frac{d}{dt}\left(e^{\lambda t}\mathbf{v}\right) = \frac{d}{dt}\begin{pmatrix} e^{\lambda t}\upsilon_1 \\ e^{\lambda t}\upsilon_2 \\ \vdots \\ e^{\lambda t}\upsilon_n \end{pmatrix} = \begin{pmatrix} \lambda e^{\lambda t}\upsilon_1 \\ \lambda e^{\lambda t}\upsilon_2 \\ \vdots \\ \lambda e^{\lambda t}\upsilon_n \end{pmatrix} = \lambda e^{\lambda t}\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \\ \vdots \\ \upsilon_n \end{pmatrix} = \lambda e^{\lambda t}\mathbf{v}$$

Thus $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{x}' = \mathbf{Ax}$ if and only if $\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{Av}$ or, equivalently, if and only if $\mathbf{Av} = \lambda\mathbf{v}$

2-1.5 Definition: Eigenvalue and Eigenvector

The number λ (either zero or non – zero) is called an eigenvalue of $n \times n$ matrix provided:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| = \mathbf{0}$$

An eigenvector associated with the eigenvalue λ is a non-zero vector $\mathbf{v}$ such that $\mathbf{Av} = \lambda\mathbf{v}$, so that: $(\mathbf{A} - \lambda\mathbf{v}) = \mathbf{0}$

Lemma: If $\mathbf{x}(t) = \mathbf{v}_k e^{\lambda_k t}$ is a solution to (**III**) for the k th eigenvalue of $\mathbf{A}$ of order $n \times n$ then: $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$ is also a solution, where c_i is a constant for $i = 1,2,\cdots,n$

In outline, the eigenvalue method for solving system $\mathbf{x}' = \mathbf{Ax}$ goes as follows:

(I) Solve the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| = \mathbf{0}$ for the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of matrix $\mathbf{A}$.

(II) We attempt to find n linearly independent eigenvector $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ (so that $\mathbf{v}_k \neq 0$ for each k) associated with these eigenvalues. That is $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_k = 0$ for each λ_k. This will not always be possible, but when it is, we get n linearly independent solutions:
$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda t}, \cdots, \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda t}$

(III) The general solution is then a linear combination of these n solutions :
$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2\mathbf{v}_2 e^{\lambda t} + \cdots + c_n\mathbf{v}_n e^{\lambda t}$

We will discuss the various cases that can occur separately, depending upon whether the eigenvalues are distinct or repeated, real or complex.

Case 1: Distinct Real Eigenvalues

If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real and distinct, then we substitute each of them in turn in $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$ and solve for the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

Example 4.13

Find a general solution of the system

$$\begin{aligned} x_1' &= 4x_1 + 2x_2 \\ x_2' &= 3x_1 - x_2 \end{aligned}$$

Solution:

The matrix form of the system is: $\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\mathbf{x}$

The characteristic equation of the coefficient matrix is:

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(-1-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda+2)(\lambda-5) = 0$$

So we have the distinct real eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = 5$

For the coefficient matrix $\mathbf{A}$ the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$ takes the form:

$$\begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{……….(δ)}$$

When we substitute the first eigenvalue $\lambda_1 = -2$, we find that the two scalar equations that follow are: $6\upsilon_1 + 2\upsilon_2 = 0$ and $3\upsilon_1 + \upsilon_2 = 0$.

These equations are equivalent i.e. they are redundant, so only one is needed; we can choose υ_1 arbitrary (but non-zero) and solve for υ_2 . The simplest choice is $\upsilon_1 = 1$, which yields $\upsilon_2 = -3$, and thus: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector associated with $\lambda_1 = -2$ (as is any zero constant multiple of υ_1).

Remark: If, instead of the “simplest” choice $\upsilon_1 = 1$, $\upsilon_2 = -3$, we had made another choice, $\upsilon_1 = c$, $\upsilon_2 = -3c$, we would have obtained the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} c \\ -3c \end{pmatrix} = c\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Because this is a constant multiple of our previous result, any choice we make leads to the same solution: $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t}$

Next we substitute in (δ) the second eigenvalue $\lambda_2 = 5$ and get the equivalent scalar equations $-\upsilon_1 + 2\upsilon_2 = 0$ and $3\upsilon_1 - 6\upsilon_2 = 0$.

With $\upsilon_2 = 1$ we obtain $\upsilon_1 = 2$, so: $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector associated with $\lambda_2 = 5$. A different choice $\upsilon_1 = 2c$, $\upsilon_2 = c$ would merely give a (constant) multiple of $\mathbf{v}_2$.

These two eigenvalues and associated eigenvectors yield the two solutions $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t}$ and $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$.

They are linearly independent because their Wronskian $\begin{vmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{vmatrix} = 7e^{3t} \neq 0$, since $e^{3t} \neq 0$ is nonzero. Hence a general solution of the system is:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t};$$

in scalar form: $x_1 = c_1e^{-2t} + 2c_2e^{5t}$, $\quad x_2 = -3c_1e^{-2t} + c_2e^{5t}$.

Case 2: Repeated Real Eigenvalues

Suppose that the $n \times n$ matrix $\mathbf{A}$ has $m < n$ distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then at least one of these eigenvalues is a repeated root of the characteristic equation: $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An eigenvalue is of **multiplicity** k if it is a $k-$fold root.

Let λ_1 and λ_2 be eigenvalues of the system matrix such that $\lambda_1 = \lambda_2 = \lambda$ then the system has two linearly independent solutions of the form: $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, where $\mathbf{x}_1 = \mathbf{v}e^{\lambda t}$ and $\mathbf{x}_2 = \mathbf{v}te^{\lambda t} + \mathbf{w}e^{\lambda t}$. And we use the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$ for the second part that is $\mathbf{x}_2$.

Example 1.15

Find the general solution of the system

$$x_1' = -3x_1 + x_2$$
$$x_2' = -x_1 - x_2$$

Solution:

The characteristic equation is:

$$\begin{vmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda+2)^2 = 0 \Rightarrow \lambda = \lambda_1 = \lambda_2 = 2$$

The eigenvector will be solution of: $\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix}\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For $\lambda = 2$, $\Rightarrow \begin{cases} -\upsilon_1 + \upsilon_2 = 0 \\ -\upsilon_1 + \upsilon_2 = 0 \end{cases}$ then $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{x}_1 = \mathbf{v}_1 e^{-2t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

With this eigenvector $\mathbf{v_1}$ and eigenvalue $\lambda = -2$ we use the equation: $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}_1$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow -\omega_1 + \omega_2 = 1$$

Let $\omega_1 = 0$ then $\omega_2 = 1$

Therefore $\mathbf{x}_2 = \mathbf{v}_1 t e^{-2t} + \mathbf{w} e^{-2t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$

Hence $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} t \\ t+1 \end{pmatrix} e^{-2t}$.

Use WRONSKIAN to determine whether their linearly independent.

<u>Case 3</u>: Complex Eigenvalues

To find the general solution of a two – dimensional system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where the eigenvalues of $\mathbf{A}$ are nonreal (i.e. complex conjugates), $\lambda = \alpha \pm i\beta$:

I. Find a complex eigenvector $\mathbf{v} = \mathbf{v_1} + i\mathbf{v_2}$ corresponding to an eigenvalue λ.

II. Construct a complex – valued solution of the system as:

$\mathbf{x} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}$, $\lambda = \alpha + i\beta$.

III. Separate the solution into a real part $\mathbf{x_1}$ and an imaginary $\mathbf{x_2}$.

IV. The general solution of the system is $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$

Example 1.14

1. Find a general solution of the system

$$x_1' = -x_1 + 4x_2$$
$$x_2' = -2x_1 + 3x_2$$

Solution:

The characteristic equation of the coefficient matrix is given as:

$$\begin{vmatrix} -1-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) - 8 = \lambda^2 - 2\lambda + 5 = 0$$

Thus, $\lambda = 1 \pm 2i$, ***Note: the complex roots are conjugates.***

Next we choose $\lambda = 1 + 2i$ and find an eigenvector.

The eigenvectors will be solutions of:

$$\begin{pmatrix} -1-(1+2i) & 4 \\ -2 & 3-(1+2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2-2i & 4 \\ -2 & 2-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{cases} (-2-2i)v_1 + 4v_2 = 0 \\ -2v_1 + (2-2i)v_2 = 0 \end{cases}$$

As before, the equation corresponding to the tow row are redundant, so only one is needed. Using the top row, we obtain the equation: $(-2-2i)v_1 + 4v_2 = 0$ or $v_2 = [(1+i)/2]v_1$ we may avoid fractions by choosing $v_1 = 2$; then $v_2 = 1 + i$. Hence, we have the eigenvector:

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

A complex – valued solution is constructed from the formula $\mathbf{x} = e^{\lambda t}\mathbf{v}$ and Euler's formula:

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix} e^t \left(\cos 2t + i \sin 2t\right) = e^t \begin{pmatrix} 2\cos t + 2i \sin 2t \\ (1+i)\cos 2t + (-1+i)\sin 2t \end{pmatrix}$$

The real part of this solution is: $\mathbf{x}_1 = e^t \begin{pmatrix} 2\cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}$ and the imaginary part is:

$$\mathbf{x}_2 = e^t \begin{pmatrix} 2\sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

Both of these real-value functions are solutions of the system, and with them we get a general solution: $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$

$$\mathbf{x} = e^t \left[c_1 \begin{pmatrix} 2\cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin 2t \\ \cos 2t + \sin 2t \end{pmatrix} \right]$$

<u>Case 4:</u> Systems with Zero as an Eigenvalue

We discussed the case of system with two distinct real eigenvalues, repeated (nonzero) eigenvalue, and complex eigenvalues. But we did not discuss the case when one of the eigenvalues is zero.

Example 1.15:

Find the general solution to $\begin{aligned} x_1' &= 2x_1 - x_2 \\ x_2' &= -2x_1 + x_2 \end{aligned}$

Solution:

The procedure for solving a system with zero eigenvalue is the same as system with distinct real eigenvalue.

$$\begin{vmatrix} 2-\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

The characteristic polynomial of this system is: $\lambda^2 - (2+1)\lambda + 0 = 0$, which reduces to $\lambda^2 - 3\lambda = 0$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$.

Let us find the associated eigenvectors;

For $\lambda_1 = 0$, $(\mathbf{A} - \mathbf{I}\lambda_1)\mathbf{v}_1 = 0$

$$\begin{bmatrix} 2-0 & -1 \\ -2 & 1-0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation $(\mathbf{A} - \mathbf{I}\lambda_1)\mathbf{v}_1 = \mathbf{0}$ translates into: $\begin{cases} 2v_1 - v_2 = 0 \\ -2v_1 + v_2 = 0 \end{cases}$

The two equations are the same. So we have $2v_1 = v_2$. Hence an eigenvector is: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda_2 = 3$, set: $\mathbf{v}_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

The equation $(\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \mathbf{0}$ translates into: $\begin{bmatrix} 2-3 & -1 \\ -2 & 1-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or

$$\begin{cases} 2v_1 - v_2 = 3v_1 \\ -2v_1 + v_2 = 3v_2 \end{cases} \Rightarrow \begin{matrix} -v_1 - v_2 = 0 \\ -2v_1 - 2v_2 = 0 \end{matrix}$$

The two equations are the same (as $-v_1 - v_2 = 0$). So we have $v_2 = -v_1$. Hence an eigenvector is: $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore the general solution is: $\mathbf{x} = c_1 e^{0t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercise 1.6

a. Apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find the corresponding particular solution.

1. $\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$

2. $\begin{aligned} x_1' &= 3x_1 + 4x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}$; $x_1(0) = x_2(0) = 1$

3. $\begin{aligned} x_1' &= 6x_1 - 7x_2 \\ x_2' &= x_1 - 2x_2 \end{aligned}$

4. $\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$

5. $\begin{aligned} x_1' &= 4x_1 + x_2 \\ x_2' &= 6x_1 - x_2 \end{aligned}$

6. $\begin{aligned} x_1' &= 9x_1 + 5x_2 \\ x_2' &= -6x_1 - 2x_2 \end{aligned}$;

b. Find general solutions of the systems in these exercises.

1. $\begin{aligned} x_1' &= 5x_1 \phantom{{}-x_2} - 6x_3 \\ x_2' &= 2x_1 - x_2 - 2x_3 \\ x_3' &= 4x_1 - 2x_2 - 4x_3 \end{aligned}$

3. $\begin{aligned} x_1' &= 3x_1 + 2x_2 + 2x_3 \\ x_2' &= -5x_1 - 4x_2 - 2x_3 \\ x_3' &= 5x_1 + 5x_2 + 3x_3 \end{aligned}$

2. $$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= -5x_1 - 3x_2 - x_3 \\ x_3' &= 5x_1 + 5x_2 + 3x_3 \end{aligned}$$

4. $$\begin{aligned} x_1' &= 2x_1 + x_2 - x_3 \\ x_2' &= -4x_1 - 3x_2 - x_3 \\ x_3' &= 4x_1 + 4x_2 + 2x_3 \end{aligned}$$

c. Show that the coefficient matrix of the system $\begin{aligned} x_1' &= 2x_1 + 3x_2 + 3x_3 \\ x_2' &= \quad\; - x_2 - 3x_3 \\ x_3' &= \qquad\qquad 2x_3 \end{aligned}$ has the eigenvalue $\lambda = 2$ of multiplicity 2. Find two linearly independent eigenvectors associated with $\lambda = 2$, and then find a general solution of the system.

d. i) Show that the coefficient matrix of the system

$$\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= -x_1 \quad - x_3 \\ x_3' &= x_1 + 2x_2 + 3x_3 \end{aligned}$$

has the single eigenvalue $\lambda = 2$ of multiplicity 3, but with only one linearly independent eigenvector $\mathbf{u}$ associated with it. Then one solution is $\mathbf{x}_1(t) = \mathbf{u}e^{2t}$.

ii) Find a second solution of the form $\mathbf{x}_2(t) = \mathbf{u}te^{2t} + \mathbf{v}e^{2t}$ where $\mathbf{v}$ is a nontrivial solution of $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{u}$.

iii) Find a third solution of the form $\mathbf{x}_3(t) = \frac{1}{2}\mathbf{u}t^2e^{2t} + \mathbf{v}te^{2t} + \mathbf{w}e^{2t}$ where $\mathbf{w}$ is a nontrivial solution of $(\mathbf{A} - 2\mathbf{I})\mathbf{w} = \mathbf{v}$.

e. Solve for the system $\begin{aligned} x' + y' &= 2x + 4y \\ 2x' + 3y' &= 2x + 6y \end{aligned}$

Hint: the system above is not in the normal form, so reduce it to normal form first before solving it.

2-1.6 Non–Homogeneous Linear Systems

Here we discuss two ways (methods) of solving non – homogeneous linear systems. The methods are;

I. Undetermined coefficients

II. Variation of parameters

These are analogous to finding a particular solution of a single non-homogeneous n th order linear differential equation.

Given the non-homogeneous first order linear system:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x} + \mathbf{F}(t) \quad \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

The general solution of (1) has the form:

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) \quad \ldots\ldots\ldots\ldots\ldots\text{(2)}$$

where $\mathbf{x}_c$ denotes a general solution of the associated homogeneous system: $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}$ and $\mathbf{x}_p$ is a single particular solution of the equation (1)

I. Undetermined Coefficient

The method is essentially the same for a system as for single differential equations – we make an intelligent guess as to the general form of a particular solution $\mathbf{x}_p$ and then attempt to determine the coefficients in $\mathbf{x}_p$ by substitution in $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} + \mathbf{F}(t)$.

Example 1.16

Find a general solution of the system

$$\begin{aligned} x' &= 4x + 2y - 8t \\ y' &= 3x - y + 2t + 3 \end{aligned} \quad \ldots\ldots\ldots\ldots\text{(3)}$$

Solution:

We found the general solution

$$\mathbf{x}_c = \begin{pmatrix} x_c \\ y_c \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{5t} \quad \ldots\ldots\ldots\ldots\text{(4)}$$

of the associated homogeneous system. Because there is no duplication between the terms $\mathbf{x}_c$ and the non – homogeneous terms, we assume a trial solution of the form:

$$\mathbf{x}_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \mathbf{a}t + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{.............. (5)}$$

Upon substitution of (5) into (3) we get:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} + \begin{pmatrix} -8t \\ 2t + 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4a_1 + 2a_2 - 8 \\ 3a_1 - a_2 + 2 \end{pmatrix} t + \begin{pmatrix} 4b_1 + 2b_2 \\ 3b_1 - b_2 + 3 \end{pmatrix}$$

When we equate the coefficients of t and constant terms, we get the equations:

$$\begin{aligned} 4a_1 + 2a_2 &= 8, \\ 3a_1 - \ a_2 &= -2, \\ 4b_1 + 2b_2 &= a_1, \\ 3b_1 - \ b_2 &= a_2 - 3 \end{aligned}$$

hence $a_1 = \frac{2}{5}, a_2 = \frac{16}{5}, b_1 = \frac{2}{25}$ and $b_2 = \frac{1}{25}$.

Thus our particular solution is: $\mathbf{x}_p = \begin{pmatrix} \frac{2}{5}t + \frac{2}{25} \\ \frac{16}{5}t + \frac{1}{25} \end{pmatrix}$ and the general solution of the system is given

as: $\begin{aligned} x &= c_1 e^{-2t} + 2c_2 e^{5t} + \tfrac{2}{5}t + \tfrac{2}{25} \\ y &= -3c_1 e^{-2t} + c_2 e^{5t} + \tfrac{16}{5}t + \tfrac{1}{25} \end{aligned}$

Example 1.17

Find a general solution of the system

$$\begin{aligned} x' &= 4x + 2y, \\ y' &= 3x - y + e^{-2t} \end{aligned} \text{.................(6)}$$

Solution:

The complimentary solution is given as:

$$\mathbf{x}_c = \begin{pmatrix} x_c \\ y_c \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} \text{........ (7)}$$

Because of duplication of the term e^{-2t} in non-homogeneous system, we assume a trial solution of the form: $\mathbf{x}_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \mathbf{a}te^{-2t} + \mathbf{b}e^{-2t} = \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} e^{-2t}$ ......(8)

(rather than $\mathbf{a}te^{-2t}$ alone). Then $\mathbf{x}'_p = (\mathbf{a} - 2\mathbf{b})e^{-2t} - 2\mathbf{a}te^{-2t}$ .... (9)

Substitution of (8) and (9) into (6), and cancellation of e^{-2t} throughout, yields:

$$\begin{pmatrix} a_1 - 2b_1 - 2a_1 t \\ a_2 - 2b_2 - 2a_2 t \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If we equate the coefficients of t and the constant terms, we get the equations:

$$\begin{aligned} 6a_1 + 2a_2 &= 0, \\ 3a_1 + \ a_2 &= 0, \\ 6b_1 + 2b_2 &= a_1, \\ 3b_1 + \ b_2 &= a_2 - 1 \end{aligned}$$

The first two equations imply that $a_2 = -3a_1$. In order for the last two equations to be consistent, we must have $a_1 = 2(a_2 - 1) = 2(-3a_1 - 1) = -6a_1 - 2$.

It follows that $a_1 = -\frac{2}{7}$, and so $a_2 = \frac{6}{7}$ and $b_2 = -3b_1 - \frac{1}{7}$

Hence the general solution of the system is given by:

$$\begin{aligned} x &= c_1 e^{-2t} + 2c_2 e^{5t} - \tfrac{2}{7} t e^{-2t} \\ y &= -3c_1 e^{-2t} + c_2 e^{5t} + \tfrac{6}{7} t e^{-2t} - \tfrac{1}{7} e^{-2t} \end{aligned}$$

Fundamental Matrices

Suppose that $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_n(t)$ are n linearly independent solutions on some open interval of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of n linear equations. Then the $n \times n$ matrix

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

having as column vectors the solution vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ is called a **fundamental matrix** for the system. Because its column vectors are linearly independent, it follows that the matrix $\mathbf{\Phi}(t)$ is non-singular and therefore has an inverse matrix $\mathbf{\Phi}^{-1}(t)$. In terms of the fundamental matrix $\mathbf{\Phi}(t)$, the general solution: $\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$ can be written in the form: $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}$ ………….(α)

where $\mathbf{c} = \left[c_1, c_2, ..., c_n\right]^{\mathrm{T}}$

In order that the solution $\mathbf{x}(t)$ satisfy the initial condition: $\mathbf{x}(t_0)=\mathbf{b}$ where $b_1, b_2, \cdots b_n$ are given, it therefore will suffice for the coefficient vector $\mathbf{c}$ in $\mathbf{x}(t)=\mathbf{\Phi}(t)\mathbf{c}$ to satisfy $\mathbf{\Phi}(t_0)\mathbf{c}=\mathbf{b}$; that is: $\mathbf{c}=\mathbf{\Phi}^{-1}(t_0)\mathbf{b}$ ....(β)

When we combine (α) and (β), it becomes clear that the solution of the initial value problem $\mathbf{x}'=\mathbf{A}\mathbf{x}$, $\mathbf{x}(t_0)=\mathbf{b}$ is given by: $\mathbf{x}(t)=\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{b}$ ....(γ)

Example 1.18

Find the fundamental matrix for the system: $\begin{array}{l} x'=4x+2y \\ y'=3x-y \end{array}$ and use it to find the solution of the system which satisfies the initial conditions $x(0)=1, y(0)=-1$.

Solution:

The linearly independent solutions: $\mathbf{x}_1(t)=\begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix}$ and $\mathbf{x}(t)=\begin{pmatrix} 2e^{5t} \\ e^{5t} \end{pmatrix}$ yield the fundamental matrix: $\mathbf{\Phi}(t)=\begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix}$

Then $\mathbf{\Phi}(0)=\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$ and the inverse matrix is given as: $\mathbf{\Phi}^{-1}(0)=\frac{1}{7}\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$

Hence the formula (γ) gives the solution:

$$\mathbf{x}(t)=\begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix}\frac{1}{7}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}=\frac{1}{7}\begin{pmatrix} 3e^{-2t}+4e^{5t} \\ -9e^{-2t}+2e^{5t} \end{pmatrix}$$

Thus the solution of the original initial value problem is given by

$$x=\tfrac{3}{7}e^{-2t}+\tfrac{4}{7}e^{5t}$$
$$y=-\tfrac{9}{7}e^{-2t}+\tfrac{2}{7}e^{5t}$$

II. Variation of Parameters

Theorem

If $\mathbf{\Phi}(t)$ is the fundamental matrix of $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$. Let $\mathbf{x}_p$ be a solution to $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{F}(t)$, then we have the following; $\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt \text{ …………..} \ (\sigma)$$

Then also with definite integration we have $\mathbf{x}_p = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}(s)\mathbf{F}(s)ds$

If we add this particular solution and the homogeneous solution, we get the solution:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{b} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)ds \text{ ……….} (\varepsilon)$$

of the initial value problem $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x} + \mathbf{F}(t), \mathbf{x}(t_0) = \mathbf{b}$.

Example 1.19

Solve the initial problem $\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 15 \\ 4 \end{pmatrix} te^{-2t}, x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution:

If we were to use the method of undetermined coefficients, our trial solution would be of the form $\mathbf{x}_p(t) = \mathbf{a}t^2e^{-2t} + \mathbf{b}te^{-2t} + \mathbf{c}e^{-2t}$, and we would have six scalar coefficients to determine. Here it will be simpler to use the method of variation of parameters.

We already know the fundamental matrix: $\mathbf{\Phi}(t) = \begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix}$ of the associated homogeneous system. The determinant , $\Delta\mathbf{\Phi}(t) = 7e^{3t}$, so the inverse matrix:

$$\mathbf{\Phi}^{-1}(t) = \tfrac{1}{7}e^{-3t}\begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix}$$

From the theorem gives the particular solution:

$$\mathbf{x}_p = \mathbf{\Phi}(t)\int_0^t \tfrac{1}{7}e^{-3s}\begin{pmatrix} e^{5s} & -2e^{5s} \\ 3e^{-2s} & e^{-2s} \end{pmatrix}\begin{pmatrix} -15se^{-2s} \\ -4se^{-2s} \end{pmatrix}ds$$

$$= \mathbf{\Phi}(t)\int_0^t \tfrac{1}{7}e^{-3s}\begin{pmatrix} -7se^{3s} \\ -49se^{-4s} \end{pmatrix}ds$$

$$= \mathbf{\Phi}(t)\int_0^t \begin{pmatrix} -s \\ -7se^{-7s} \end{pmatrix} ds$$

$$= \mathbf{\Phi}(t)\left[\begin{pmatrix} -\frac{1}{2}s^2 \\ se^{-7s} + \frac{1}{7}e^{-7s} \end{pmatrix}\right]_{s=0}^{s=t}$$

$$= \begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix}\begin{pmatrix} -\frac{1}{2}t^2 \\ te^{-7t} + \frac{1}{7}e^{-7t} - \frac{1}{7} \end{pmatrix}$$

Therefore:

$$\mathbf{x}_P(t) = \tfrac{1}{14}e^{-2t}\begin{pmatrix} 4 + 28t - 7t^2 \\ 2 + 14t + 21t^2 \end{pmatrix} - \tfrac{1}{7}e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This is a particular solution such that $\mathbf{x}_p(0) = \mathbf{0}$.

We already know (from previous examples) the solution: $\mathbf{x}_c(t) = \frac{1}{7}\begin{pmatrix} 3e^{-2t} + 4e^{5t} \\ -9e^{-2t} + 2e^{5t} \end{pmatrix}$ of the associated homogeneous systems such that: $\mathbf{x}_c(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Hence the solution of the initial value problem is given by:

$$x = \tfrac{1}{14}\left(10 + 28t - 7t^2\right)e^{-2t} + \tfrac{2}{7}e^{5t}$$
$$y = \tfrac{1}{14}\left(-16 + 14t - 7t^2\right)e^{-2t} + \tfrac{1}{7}e^{5t}$$

Summary: The Method of Solving Non-Homogeneous systems

1. Solve for the homogeneous part $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$.
2. Form your fundamental matrix $\mathbf{\Phi}(t)$
3. Find $\mathbf{\Phi}^{-1}(s)$
4. Evaluate $\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)ds$
5. Find $\mathbf{x}_p = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}(s)\mathbf{F}(s)ds$.
6. Your final step $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{b} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)ds$.

Exercise 1.7

a. Apply the method of undetermined coefficients to find a particular solution of each of the system below. If initial conditions are given, find the particular solution that satisfies these conditions.

1. $\begin{aligned} x' &= x + 2y + 3 \\ y' &= 2x + y - 2 \end{aligned}$

2. $\begin{aligned} x' &= 3x + 4y + 3 \\ y' &= 3x + 2y + t^2 \end{aligned}$ $x(0) = y(0) = 0$

3. $\begin{aligned} x' &= 6x - 7y + 10 \\ y' &= x - 2y - 2e^{-t} \end{aligned}$

4. $\begin{aligned} x' &= -3x + 4y + \sin t \\ y' &= 6x - 5y \end{aligned}$ $x(0) = 1, y(0) = 0$

5. $\begin{aligned} x' &= x - 5y + \cos 2t \\ y' &= x - y \end{aligned}$

6. $\begin{aligned} x' &= 2x + 4y + 2 \\ y' &= x + 2y + 3 \end{aligned}$ $x(0) = 1, y(0) = -1$

7. $\begin{aligned} x' &= 2x + 3y + 5 \\ y' &= 2x + y - 2t \end{aligned}$

8. $\begin{aligned} x' &= 4x + y + e^t \\ y' &= 6x - y - e^t \end{aligned}$ $x(0) = y(0) = 1$

9. $\begin{aligned} x' &= 9x + y + 2e^t \\ y' &= -8x - 2y + te^t \end{aligned}$

10. $\begin{aligned} x' &= x - 5y + 2\sin t \\ y' &= x - y - 3\cos t \end{aligned}$

11. $\begin{aligned} x' &= x - 2y \\ y' &= 2x - y + e^t \sin t \end{aligned}$

12. $\begin{aligned} x' &= x + y + 2t \\ y' &= x + y - 2t \end{aligned}$

b. Find the fundamental matrix of each of the systems below and then find a solution satisfying the given initial conditions.

1. $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

6. $\mathbf{x}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

7. $\mathbf{x}' = \begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ -5 & 5 & 3 \end{pmatrix}\mathbf{x}, \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

c. In each of the problem below apply the method of variation of parameters to find a particular solution of the given system.

1. $\mathbf{x}' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2t \\ 3 \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -4 \\ 5 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \sec t \\ \tan t \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ e^{3t}\cos 2t \end{pmatrix}$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3e^{2t} \\ 5t \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{1}{t^2} \\ \frac{1}{t^3} \end{pmatrix}$

9. $\mathbf{x}' = \begin{pmatrix} 3 & -5 \\ 5 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \cos 4t \\ \sin 4t \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{2t}\tan t \\ 0 \end{pmatrix}$

11. $\mathbf{x}' = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \ln t \\ t \end{pmatrix}$

6. Find the particular solution for each of the following above such that $x_1(1) = x_2(1) = 0$.

Summary

- **Normal Form**

The **normal form** in the general case of a linear system of n differential equations in n unknown functions $x_1, x_2, \cdots, x_n$ is:

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

real

- **Solution To A Linear System Of Differential Equations**

The solution to the system:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, x_2, x_3, \cdots x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, x_2, x_3, \cdots x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, x_2, x_3, \cdots x_n)
\end{aligned}\right\}$$

is the parametric vector

$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ on open real interval $a < t < b$ for which the

system is true and the vector $\mathbf{x}$ must be continuous on the open

- **Matrix Form**

The Matrix Notation form of the normal is given as:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

can simply be written as: $\dfrac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{F}(t)$ ………….**(I)** which is a **NON – HOMOGENEOUS FORM**.

where $\mathbf{F}(t)$ is the **non – homogeneous term** or the **force term**.

If $\mathbf{F}(t)=\mathbf{0}$, that is $f_1(t)=f_2(t)=f_3(t)=\cdots=f_n(t)=0$ for the real interval $a\le t\le b$ then we have

$$\frac{d\mathbf{x}}{dt}=\mathbf{Ax}\ \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{II})$$

Equation (**II**) is known as the **HOMOGENEOUS FORM**.

- **Solution of Homogeneous Equation**

In outline, the eigenvalue method for solving system $\mathbf{x}'=\mathbf{Ax}$ goes as follows:

(IV) Solve the characteristic equation

$$\det(\mathbf{A}-\lambda\mathbf{I})=|\mathbf{A}-\lambda\mathbf{I}|=\mathbf{0}$$

for the eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ of matrix $\mathbf{A}$.

(V) We attempt to find n linearly independent eigenvector $\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n$ (so that $\mathbf{v}_k\neq 0$ for each k) associated with these eigenvalues.

That is $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_k=0$ for each λ_k. This will not always be possible, but when it is, we get n linearly independent solutions:

$$\mathbf{x}_1(t)=\mathbf{v}_1e^{\lambda t},\mathbf{x}_2(t)=\mathbf{v}_2e^{\lambda t},\cdots,\mathbf{x}_n(t)=\mathbf{v}_ne^{\lambda t}$$

(VI) The general solution is then a linear combination of these n solutions :

$$\mathbf{x}(t)=c_1\mathbf{v}_1e^{\lambda t}+c_2\mathbf{v}_2e^{\lambda t}+\cdots+c_n\mathbf{v}_ne^{\lambda t}$$

- **Non – Homogenous**

The Method of Solving Non – Homogeneous systems

7. Solve for the homogeneous part

$$\frac{d\mathbf{x}}{dt}=\mathbf{Ax}\ .$$

8. Form your fundamental matrix $\mathbf{\Phi}(t)$

9. Find $\mathbf{\Phi}^{-1}(s)$

10. Evaluate $\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)ds$

11. Find $\mathbf{x}_p = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}(s)\mathbf{F}(s)ds$.

Your final step $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{b} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)ds$.

UNIT SIX
SOLUTION SERIES

2.0 INTRODUCTION

We have seen that a linear differential equation with constant coefficients can be reduced to the algebraic problem of finding the roots of its characteristic equation. There is no similar procedure for solving linear differential equations with ***variable coefficients***, (at least not routinely and in finitely many steps) with the exception of special types, and the occasional equation that can be solved by inspections (perhaps followed by reduction of order).

For linear ordinary differential equations with variable coefficients and of order greater than one, probably the most generally effective method of attack is that based upon the use of ***power series***.

To simplify our work and statements, the equations treated (that is the ones to be solved completely) here will by restricted to those with polynomial coefficients. The difficulties to be encountered, the methods of attack, and the results accomplished all remain essentially unchanged when the coefficients are permitted to be functions that have ***power series*** expansion valid about some point (such function are called ***analytic functions***).

2.1 SERIES SOLUTIONS TO LINEAR DIFFERENTIAL EQUATION

There are two standard methods of solving linear differential equation in series form:

1. **Power Series Method**
2. **Frobenius Method** (which is an extension of Power Series Method).

Recall the theory of power series:

We first remember that a power series (in powers of $x - x_0$) is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_k(x-x_0)^k + \cdots \quad (2.1$$

where $a_0, a_1, a_2, \cdots$ are constants, called the coefficients of the series, x_0 is a constant, called the centre of the series, and x is a variable.

If in particular $x_0 = 0$, we obtain a power series in powers of x as:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots \_\_\_\_\_ \quad (2.2)$$

we assume that all variables and constant are real.

Recall typical Maclaurin's series:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \ldots\ldots \text{(II)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \ldots\ldots\ldots \text{(III)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(IV)}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \ldots\ldots\ldots \text{(V)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n!} \quad \ldots\ldots\ldots\ldots \text{(VI)}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots x^k + \cdots = \sum_{n=0}^{\infty} x^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VII)}$$

Power series such as those listed above are often derived as ***Taylor Series***.
The Taylor series with centre $x = x_0$ of the function f is the power series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

in powers of $x-x_0$, under the hypothesis that f is infinitely differentiable at x_0 (so that the coefficients of the power series are all defined). If the Taylor series of f converges to $f(x)$ for all x in some open interval containing x_0, then we say that the function is *analytic* at $x=x_0$ or we say converges to $f(x)$ in some neighbourhood of x_0.

Polynomials, $\sin x$, $\cos x$ and e^x are analytic everywhere; so too are sums, difference and products of these functions. Quotients of any two of these functions are analytic all powers where the denominator is not zero.

Note: $0!=1$

2.2 SHIFT OF INDEX OF SUMMATION OF INFINITE SERIES

1. The index of summation of an infinite series as a dummy parameter that is:

$$\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}=\sum_{k=0}^{\infty}\frac{2^k x^k}{k!}=\sum_{p=0}^{\infty}\frac{2^p x^p}{p!}$$

2. Indices used to maintain a particular generic term in an infinite series.

Example 2.1

$$\sum_{n=2}^{\infty}a_n x^n=a_2x^2+a_3x^3+a_4x^4+\cdots=a_{0+2}x^{0+2}+a_{1+2}x^{1+2}+a_{2+2}x^{2+2}+\cdots$$

$$=\sum_{r=0}^{\infty}a_{r+2}x^{r+2}=\sum_{n=0}^{\infty}a_{n+2}x^{n+2}$$

(ii) Consider $\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}$ and maintain the generic term $(x-x_0)^n$ in the infinite series:

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}=(4)(3)a_2(x-x_0)^0+$$
$$(5)(4)a_3(x-x_0)+(6)(5)a_4(x-x_0)^2+\cdots$$

$$=(4+0)(3+0)a_{2+0}(x-x_0)^0+(4+1)(3+1)a_{2+1}(x-x_0)+$$
$$(4+2)(3+2)a_{2+2}(x-x_0)^2+\cdots$$

$$= \sum_{n=0}^{\infty} (4+n)(3+n) a_{2+n} (x - x_0)^n$$

(iii) $x^2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$ and maintain the generic term x^{n+r}.

$$x^2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = \sum_{n=0}^{\infty} (n+r) x^{n+r+1}$$

$$= (0+r) a_0 x^{0+r+1} + (1+r) a_1 x^{1+r+1} + (2+r) a_2 x^{2+r+1} + \cdots$$

$$= \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r}$$

(iiii) Solve the expression

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

Solution:

we shift index of summation of the first series, then we get:

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n .$$

$\Rightarrow \sum_{n=0}^{\infty} [(n+1) a_{n+1} - a_n] x^n = 0$, which is true for all x.

Then $(n+1) a_{n+1} - a_n = 0 \quad n = 1,2,3,\cdots$

$$\Rightarrow a_{n+1} = \frac{a_n}{n+1}$$

Hence for $a_0 \neq 0$

$$a_1 = \frac{a_0}{1} = \frac{a_0}{1!}$$

$$a_2 = \frac{a_1}{2} = \frac{a_0}{1 \cdot 2} = \frac{a_0}{2!}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{2! \cdot 3} = \frac{a_0}{3!}$$

$$\boxed{a_k = \frac{a_0}{k!}}$$

$$\overline{\qquad\qquad\qquad} \quad (*)$$

Thus $(*)$ allows the determination of:

$$a_n = \frac{a_0}{n!} \qquad \forall n.$$

If asked to find $\sum_{n=0}^{\infty} a_n x^n$, with a_n so determined we have:

$$\sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

Exercise 2.1

With (iv) determine a_n so that

1. $\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$ 2. $\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$

Theorem 2.1: IDENTITY PRINCIPLE

If $\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n$

for every point x in some open interval I,

Then $a_n = b_n$ for all $n \geq 0$.

2.3 THE POWER SERIES METHOD

Consider the general inhomogeneous linear differential equation of n^{th} order as:

$$b_n(x)\frac{d^n y}{dx^n} + b_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + b_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + b_1(x)\frac{dy}{dx} + b_0(x)y = f(x)$$

Then the second – order inhomogeneous equation would be:

$$b_2(x)\frac{d^2 y}{dx^2} + b_1(x)\frac{dy}{dx} + b_0(x)y = f(x) \text{.............} \quad (2.3)$$

or $b_2(x)y'' + b_1(x)y' + b_0(x)y = f(x)$..................... (2.3)

The idea of the power series method for solving differential equations is simple natural.

For (2.3) $b_2(x)y'' + b_1(x)y' + b_0(x)y = f(x)$

$$\Rightarrow y'' + \frac{b_1(x)}{b_2(x)} y' + \frac{b_0(x)}{b_2(x)} y = \frac{f(x)}{b_2(x)}$$

This can be simplified as:

$$y'' + p(x)y' + q(x)y = g(x) \ldots\ldots\ldots\ldots\ldots\ldots \text{(2.4)}$$

where $p(x) = \frac{b_1(x)}{b_2(x)}$, $q(x) = \frac{b_0(x)}{b_2(x)}$ and $g(x) = \frac{f(x)}{b_2(x)}$.

For a given homogeneous linear equation of the second – order of (2.4)

That is $y'' + p(x)y' + q(x)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2.5)

we first represents $p(x)$ and $q(x)$ by power series in powers of x (or $x - x_0$, if solutions in power of $x - x_0$ is what we needed).

Let

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \cdots = \sum_{n=0}^{\infty} a_nx^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(a)}$$

$$\Rightarrow y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + ka_kx^{k-1} + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots \text{(b)}$$

$$\Rightarrow y'' = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots + k(k-1)a_kx^{k-2} + \cdots = \sum_{n=1}^{\infty} n(n-1)a_nx^{n-2} \text{ (c)}$$

Then we substitute (a), (b) and (c) into (2.5).

The coefficient $a_0, a_1, a_2, \cdots$ of an analytic solution $y(x) = \sum_{n=0}^{\infty} a_nx^n$ (i.e. (a)) of equation (2.5) can be computed as in the following steps;

Step 1: Insert $y(x) = \sum_{n=0}^{\infty} a_nx^n$ into the differential equation [or (a), (b), (c) into (2.5)].

Step 2: Rearrange the left – hand side of the equation so that the equation has the form $\sum_{n=0}^{\infty} c_nx^n = 0$ where c_n is an equation in terms of a_j terms.

Step 3: By theorem (2.1) $c_n = 0$ for every n. Solve this equation for the a_j term having the largest subscript. The resulting equation is known as the **Recurrence formula** for the given differential equation.

Step 4: Use the **Recurrence formula** to sequentially determine $a_j\ (j = 2,3,\cdots)$ in terms of, a_0 and a_1 if it is second order differential equation, and a_0 if it is first – order.

Example 2.2

1. Solve the linear differential equation $y' - y = 0$.

Solution:

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \cdots = \sum_{n=1}^{\infty} a_nx^n$.

$$\Rightarrow y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + ka_kx^{k-1} + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$$

Substituting these expressions into the equation, we get:

$$\sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n = 0$$

we shift index in the summation of first sum, i.e. replacing $n+1$ by n in the summation to maintain the **generic term** x^n, we have

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n]x^n = 0$$

by Identity principle:

$$(n+1)a_{n+1} - a_n = 0.$$

$$\Rightarrow a_n = \frac{a_n}{n+1},$$

which is a recursion formula that gives as the following:

$$a_1 = \frac{a_0}{1},\ a_2 = \frac{a_1}{2} = \frac{a_0}{1\cdot 2} = \frac{a_0}{2!},\quad a_3 = \frac{a_2}{3} = \frac{a_0}{3\cdot 2!} = \frac{a_0}{3!},\ldots,$$

Continuing the processes, then: $a_n = \dfrac{a_0}{n!}$

Therefore, $y = \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} \dfrac{a_0}{n!}x^n$

$$\Rightarrow y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The solution is given as:

$y = a_0 e^x$, where a_0 is an arbitrary constant.

Alternatively,

$$y' - y = 0$$

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = 0$$

Then we collect like powers of x, finding

$$(a_1 - a_0) + (2a_0 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have:

$$a_1 - a_0 = 0,\ 2a_0 - a_1 = 0,\ 3a_3 - a_2 = 0$$

Solving these equations, we may express $a_1, a_2, a_3, \cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0,\ a_2 = \frac{a_1}{2} = \frac{a_0}{2!},\ a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \ldots$$

with these coefficients the power series:

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$$

becomes: $$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 \cdots + \frac{a_0}{k!}x^k + \cdots$$

and we see that we have obtained the familiar general solution:

$y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$, where a_0 is arbitrary.

2. Solve $y' - 2xy = 0$.

Solution:

$$y' = 2xy$$

$$\Rightarrow a_1 + 2a_1 x + 3a_3 x^2 + \cdots = 2x\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$\Rightarrow a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + \cdots$$

equating coefficients, we see that:

$$a_1 = 0,\ 2a_2 = 2a_0,\ 3a_3 = 2a_1,\ 4a_4 = 2a_2,\ 5a_5 = 2a_3,\ \ldots$$

hence coefficients with odd subscript, $a_3 = 0$, $a_5 = 0$,... and for the coefficients with even subscripts:

$$a_2 = a_0,\ a_4 = \frac{a_2}{2} = \frac{a_0}{2!},\ a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \cdots$$

a_0 remains arbitrary. With these coefficients the power series gives the following solution, which you should conform by the method of separating variables:

$$y = a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = a_0 e^{x^2}$$

Alternatively,

Solve $y' - 2xy = 0$

Solution:

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

By substitution:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

For the same generic term, we shift the left index by $+2$, to maintain $n+1$:

$$\Rightarrow \sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since the two summations do not start from the same point (i.e either $n = 0$ or $n = -1$). We let them start from the same point. We then find the term $n = -1$(that is the left summation, which the start with $n = 0$):

$$\Rightarrow (-1+2) a_{(-1+2)} x^{-1+1} + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Group like terms:

$$\Rightarrow (1)a_1x^0 + \sum_{n=0}^{\infty}\left[(n+2)a_{n+2} - 2a_n\right]x^{n+1} = 0$$

Comparing coefficients, we get:

$a_1 = 0$, and $(n+2)a_{n+2} - 2a_n = 0$

$$\Rightarrow a_{n+2} = \frac{2a_n}{(n+2)}$$

Which is the recursion formula for $n = 0,1,2,\cdots$

$$a_1 = 0 \qquad a_5 = \frac{2a_3}{(3+2)} = \frac{2(0)}{5} = 0$$

$$a_2 = \frac{2a_0}{(0+2)} = \frac{2a_0}{2} = a_0 \qquad a_6 = \frac{2a_4}{(4+2)} = \frac{2a_0}{2!\cdot 6} = \frac{a_0}{3!}$$

$$a_3 = \frac{2a_1}{(1+2)} = \frac{2(0)}{3} = 0 \qquad a_7 = \frac{2a_5}{(5+2)} = \frac{2(0)}{5} = 0$$

$$a_4 = \frac{2a_2}{(2+2)} = \frac{2a_0}{4} = \frac{a_0}{2!} \qquad a_8 = \frac{2a_6}{(6+2)} = \frac{2a_0}{3!\cdot 8} = \frac{a_0}{4!}$$

then we have:

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{a_0}{\left(\frac{n}{2}\right)!} & \text{if } n \text{ is even} \end{cases} \qquad \text{or } a_{2n} = \frac{a_0}{n!}$$

$$a_2 = a_0,\ a_4 = \frac{a_2}{2} = \frac{a_0}{2!},\ a_6 = \frac{a_4}{3} = \frac{a_0}{3!},\cdots$$

Hence by substitution:

$$y = a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = a_0e^{x^2}$$

3. Solve $y'' + y = 0$.

Solution:

Let $y = \sum_{n=0}^{\infty} a_nx^n$

Then we get: $\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n = 0$

Shifting index in the first summation, and maintaining the generic term x^n.

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

Now, by theorem 2.1: $(n+2)(n+1)a_{n+2}+a_n=0$

$$\Rightarrow a_{n+2}=-\frac{a_n}{(n+2)(n+1)}$$

Which is the recursion formula for $n=0,1,2,\cdots$

$$a_2=\frac{-a_0}{2!} \qquad a_3=\frac{-a_1}{3\cdot 2}=\frac{-a_1}{3!}$$

$$a_4=\frac{-a_2}{4\cdot 3}=\frac{-(-a_0)}{4\cdot 3\cdot 2!}=\frac{a_0}{4!} \qquad a_5=\frac{-a_3}{5\cdot 4}=\frac{-(-a_1)}{5\cdot 4\cdot 3!}=\frac{a_1}{5!}$$

$$a_6=\frac{-a_4}{6\cdot 5.}=\frac{-a_0}{6\cdot 5\cdot 4!}=\frac{-a_0}{6!} \qquad a_7=\frac{-a_5}{7\cdot 6}=\frac{-a_1}{7\cdot 6\cdot 5!}=\frac{-a_1}{7!}$$

Since, $y=a_0+a_1x+a_2x^2+\cdots+a_kx^k+\cdots$

Thus,

$$y=a_0+a_1x+\left(\frac{-a_0}{2!}\right)x^2+\left(\frac{-a_1}{3!}\right)x^3+\left(\frac{a_0}{4!}\right)x^4+\left(\frac{a_1}{5!}\right)x^5+\cdots$$

$$y=a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)$$

$$y=a_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$y=a_0\cos x+a_1\sin x$$

Exercise 2.2

In each of 1-16, find a power series solution of the given differential equation. Determine the radius of convergence of the resulting series.

1. $y' + 2y = 0$
2. $y' = y$
3. $2y' + 3y = 0$
4. $y' + 2xy = 0$
5. $y' = x^2 y$
6. $(x-2)y' = y$
7. $y' = 4y$
8. $(2x-1)y' + 2y = 0$
9. $2(x+1)y' = y$
10. $(x-1)y' + 2y = 0$
11. $2(x-1)y' = 3y$
12. $(x-3)y' + 2y = 0$
13. $xy' - 3y = 0$
14. $(x-2)y' = 2xy$
15. $(1-x^2)y' = 2xy$
16. $xy' = (x^2 - 1)y$

Exercise 2.3

Fine two linearly independent power series solutions of to given differential equation. Determine the radius of convergence of each series, and identify the general solution in terms of familiar elementary functions.

1. $y'' = y$
2. $y'' + 9y = 0$
3. $y'' + 4y = 0$
4. $y'' - 4xy + (4x^2 - 2)y = 0$
5. $y'' - 3y' + 2y = 0$
6. $(1-x^2)y'' - 2xy' + 2y = 0$.
7. $y'' = 4y$
8. $y'' + y = x$

2.4 SOLUTIONS ABOUT NEIGHBOURHOOD POINTS

The power series method is used to solve linear differential equations such as

$y''+p(x)y'+q(x)y=g(x)$ .... (2.4)

where $p(x)$ and $q(x)$ are given function as if $(x-x_0)$ solutions obtained are power series solution in the neighbourhood of $x=x_0$.

To obtain this solution $p(x)$ and $q(x)$ need to satisfy the following;

(1) $x=x_0$ is an **ordinary point** if both of the following limits exist and are finite: $\lim\limits_{x\to x_0} p(x)$, $\lim\limits_{x\to x_0} q(x)$

i.e.: $\lim\limits_{x\to x_0} |p(x)|<\infty$, $\lim\limits_{x\to x_0} |q(x)|<\infty$.

(2) $x=x_0$ is **singular point** if either

$$\lim_{x\to x_0} |p(x)|=\infty, \ \lim_{x\to x_0} |q(x)|=\infty$$

i.e. they are not finite (infinite)

a) $x=x_0$ is a ***regular point*** if it is a **singular point** and if both of the following limits exist and are finite:

$$\lim_{x\to x_0} |(x-x_0)\,p(x)|, \ \lim_{x\to x_0} \left|(x-x_0)^2\,q(x)\right|$$

i.e. $\lim\limits_{x\to x_0} |(x-x_0)\,p(x)|<\infty$, $\lim\limits_{x\to x_0} \left|(x-x_0)^2\,q(x)\right|<\infty$

b) $x=x_0$ is a ***irregular singular point*** if either (or both) of limits do not exist.

$$\lim_{x\to x_0} |(x-x_0)\,p(x)|=\infty, \ \lim_{x\to x_0} \left|(x-x_0)^2\,q(x)\right|=\infty$$

i.e. they are not finite (infinite).

NOTE:

The solutions we have obtained from the examples of Power series is about the point $x = 0$ (i.e $x_0 = 0$) $p(0)$ and $q(0)$ are ordinary point.

Example 2.3

1.

☞ The differential equation

$$\left(1-x^2\right)y'' - 6xy' - 4y = 0$$

has $x = 1$ and $x = -1$ as its only **singular points** in the finite plane.

☞ The equation

$$y'' + 2xy' + y = 0$$

has **no singular points** in the finite plane.

☞ The equation

$$xy'' + y' + xy = 0$$

has the origin $x = 0$ as the only **singular poin**t in the finite plane.

2. Determine whether $x = 0$ is an ordinary point of the differential equation

$$y'' - xy' + 2y = 0.$$

Here $p(x) = -x$ and $q(x) = 2$ are both polynomials; hence they are analytic everywhere. Therefore, every value of x, in particular $x = 0$, is an ordinary point.

3. Determine whether $x = 0$ is an ordinary point of the differential equation

$$y'' + y = 0.$$

Here $p(x) = 0$ and $q(x) = 1$ are both constants; hence they are analytic everywhere. Therefore, every value of x, in particular $x = 0$, is an ordinary point.

4. Determine whether $x = 0$ is an ordinary point of the differential equation

$$x^2y'' + 2y' + xy = 0.$$

Here $p(x)=\frac{2}{x^2}$ and $q(x)=\frac{1}{x}$. Neither of these functions is analytic at $x=0$, so $x=0$ is not an ordinary point, rather, a singular point.

5. Determine whether $x=0$ or $x=1$ is an ordinary point of the differential equation (called Legendre's equation)

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

for any positive integer n.

Hence $p(x)=\frac{-2x}{1-x^2}$ and $q(x)=\frac{n(n+1)}{1-x^2}$;

If $x=0$, both are analytic there and $x=0$ is an ordinary point.

If $x=1$, neither function is analytic there. Consequently, $x=1$ is a singular point.

6. Classify the singular points, in the finite plane, of the equation

$$x(1-x)^2(x+2)y''+x^2y'-(x^3+2x-1)y=0$$

Hence $p(x)=\frac{x}{(x-1)^2(x+2)}$ and $q(x)=\frac{-\left(x^3+2x-1\right)}{x(x-1)^2(x+2)}$;

The singular points in the finite plane are $x=0,1,-2$.

Consider $x=0$:

The factor x is absent from the denominator of $p(x)$ and it appears to the first power in the denominator of $q(x)$. Hence $x=0$ is a **regular singular point.**

Consider $x=1$:

The factor $(x-1)$ appears to the second power in the denominator of $p(x)$. That is a higher power than is permitted in the definition of a regular singular point. Hence it does not matter how $(x-1)$ appears in $q(x)$; the point $x=1$ is an **irregular singular point.**

Consider $x=-2$

The factor $(x+2)$ appears to the first power in the denominator of $p(x)$, just as high as is permitted, and to the first power also in the denominator of $q(x)$, so $x=-2$ is a **regular singular point**.

In summary, the equation has in finite plane the following singular points:

~ Regular singular points at $x=0, x=-2$.

~ Irregular singular points at $x=1$.

7. Classify the singular points in the finite plane for the equation

$$x^4\left(x^2+1\right)(x-1)^2 y''+4x^3(x-1)y'+(x+1)y=0$$

Hence $p(x)=\dfrac{4}{x\left(x^2+1\right)(x-1)}=\dfrac{4}{x(x-i)(x+i)(x-1)}$ and

$$q(x)=\frac{x+1}{x^4(x+i)(x-i)(x-1)^2}$$

Therefore the desired classification is;

Regular Singular Point (R.S.P) at $x=i,-i,1$ and

Irregular Singular Point (I.S.P) at $x=0$

8. In the equation

$$(\sin 2x)y''-x(x-1)y'+x^2y=0$$

there are infinite number of singular points, namely,

$$x=\pm\frac{\pi}{2},\pm\pi,\pm3\frac{\pi}{2},\cdots$$

Note that $x=0$ is not a singular point, since $\lim_{x\to 0}\left|\frac{-x(x-1)}{\sin 2x}\right| = \frac{1}{2} < \infty$ and $\lim_{x\to 0}\left|\frac{x^2}{\sin 2x}\right| = 0 < \infty$

Exercise 2.4

For each equation, list all of the singular points in the finite plane.

1. $\left(x^2+4\right)y''-6xy'+3y=0$
2. $x(3-x)y''-(3-x)y'+4xy=0$
3. $4y''+3xy'+2y=0$
4. $x(x-1)^2 y''+3xy'+(x-1)y=0$
5. $x^2y''+xy'+\left(1-x^2\right)y=0$
6. $x^4+y=0$
7. $\left(1+x^2\right)y''-2xy'+6y=0$
8. $\left(x^2-4x+3\right)y''+x^2y'-4y=0$
9. $x^2(1-x)^3 y''+(1+2x)y=0$
10. $6xy''+\left(1-x^2\right)y'+2y=0$
11. $4y''+y=0$
12. $x^2\left(x^2-9\right)y''+3xy'-y=0$
13. $x^2\left(1+4x^2\right)y''-4xy'+y=0$
14. $4xy''+y=0$
15. $(2x+1)(x-3)y''-y'+(2x+1)y=0$
16. $x^3\left(x^2-4\right)^2 y''+2\left(x^2-4\right)y'-xy=0$
17. $x\left(x^2+1\right)y''-y=0$
18. $\left(x^2+6x+8\right)y''+3y=0$
19. $(4x+1)y''+3xy'+y=0$

Exercise 2.5

For each equation, locate and classify all its singular points in the finite plane.

1. $x^3(x-1)y''+(x-1)y'+4xy=0$
2. $x^2\left(x^2-4\right)y''+2x^3y'+3y=0$
3. $y''+xy=0$
4. $x^2y''+y=0$
5. $x^4y''+y=0$
10. $x^2(x+2)y''+(x+2)y'+4y=0$
11. $x(x+3)y''+y'-y=0$
12. $(x-1)(x+2)y''+5(x+2)y'+x^2y=0$
13. $(x-1)^2(x+4)^2 y''+(x+4)y'+7y=0$
14. $\left(1+4x^2\right)y''+6x\left(1+4x^2\right)y'-9y=0$

6. $\left(x^2+1\right)(x-4)^3 y''+(x-4)^2 y'+y=0$

7. $x^2(x-2)y''+3(x-2)y'+y=0$

8. $x^2(x-4)^2 y''+3xy'-(x-4)y=0$

9. $\left(1+4x^2\right)^2 y''+6xy'-9y=0$

15. $x^3y''+4y=0$

16. $\left(1+4x^2\right)y''+6xy'-9y=0$

17. $x^4y''+2x^3y'+4y=0$

18. $(2x+1)^4 y''+(2x+1)y'-8y=0$

<u>Exercise 2.6</u>

Determine and classify the singular points (if any) of each of the following equation.

1. $\left(x^3+7x+8\right)y''-(x+3)y'+xy=0$
2. $\left(x^4-16\right)y''+(x-2)y'+(x+2)y=0$
3. $\left(x^4-4x^2+4\right)y''+xy'=0$
4. $\sin 2xy''-(x-3)y'+e^x y=0$
5. $3x^2(x+2)^2 y''-2x(x-1)y'+\left(x^2+2\right)y=-xe^x$
6. $\left(x^2+x+1\right)y''+(\sin 2x)y'-4x^2y=\frac{1}{2}-\sin 2x$
7. $\left(1-\cos^2 x\right)y''-3xy'+4y=0$
8. $\left(e^x-1\right)y''+14y'-12e^x(\cos x)y=0$
9. $\left(\sin 3x^2\right)y''-x(x-2)y'-5xe^x y=2e^{2x}$
10. $x(\sinh x)y''-3(\sin x)y'+14e^x y=0$

<u>Examples of Solution about a neighbourhood point</u> $\underline{x=x_0}$

1. Find the general solution near $x=0$ (i.e. $x_0=0$) of $y''-xy'+2y=0$.

Solution:

Here $p(x)=-x$ and $q(x)=2$.

$x=0$ is an ordinary point since $p(x)$ and $q(x)$ are polynomials that means it is analytic everywhere.

Let $\quad y=\sum_{n=0}^{\infty}a_nx^n \quad \Rightarrow y'=\sum_{n=1}^{\infty}na_nx^{n-1} \quad \Rightarrow y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$

Therefore,

$$\Big[2a_2+6a_3x+12a_4x^2+\cdots+n(n-1)a_nx^{n-2}+(n+1)na_{n+1}x^{n-1}+$$
$$(n+2)(n+1)a_{n+2}x^n+\cdots\Big]+-x\Big[a_1+2a_2x+3a_3x^2+4a_4x^3+\cdots+na_nx^{n-1}+$$
$$(n+1)a_{n+1}x^n+(n+2)a_{n+2}x^{n+1}+\cdots\Big]+2\Big[a_0+a_1x+a_2x^2+a_3x^3+\cdots+$$
$$a_nx^n+a_{n+1}x^{n+1}+\cdots\Big]=0$$

Combining terms that contain like powers of x, we have:

$$(2a_2+2a_0)+x(6a_3+a_1)+x^2(12a_4)+x^3(20a_5-a_3)+\cdots+$$
$$x^n\big[(n+2)(n+1)a_{n+2}-na_n+2a_n\big]+\cdots=0+0x+0x^2+\cdots+0x^k+$$

The last equation holds if and only if each coefficient in the left – hand sides is zero. Thus:

$$2a_2+a_0=0, 6a_3+a_1=0, 12a_4=0, 20a_5-a_3=0$$

In general: $(n+2)(n+1)a_{n+2}-(n-2)a_n=0$, or

$$a_{n+2}=\frac{(n-2)}{(n+2)(n+1)}a_n$$

which is the recurrence formula for this problem.

Calculate

$a_2=-a_0$ $\qquad a_3=-\frac{1}{6}a_1$

$a_4=0$ $\qquad a_5=\frac{1}{20}a_3=\frac{1}{20}\left(-\frac{1}{6}a_1\right)=-\frac{1}{120}a_1$

$a_6=\frac{2}{30}a_4=\frac{1}{15}(0)$ $\qquad a_7=\frac{1}{14}\left(-\frac{1}{120}\right)a_1=-\frac{1}{1680}a_1$

$a_8=\frac{4}{56}a_6=\frac{1}{14}(0)=0$ $\qquad a_9=.....$

Note that since $a_4=0$, it follows from the recurrence formula that all the even coefficients beyond a_4 are also zero.

Substituting we have

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{6} a_1 x^3 + 0x^4 - \frac{1}{120} a_1 x^5 + 0x^6 - \frac{1}{1680} a_1 x^7 + \cdots$$

$$\Rightarrow y = a_0\left(1 - x^2\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \frac{1}{1680}x^7 - \cdots\right)$$

2. Solve the equation $y'' + (x-1)^2 y' - 4(x-1)y = 0$ about the ordinary point $x = 1$.

Solution:

To solve an equation "about the point $x = x_0$" means to obtain solution valid in a region surrounding the point, solutions expressed in powers of $x = x_0$.

We first translate the axes, putting $x - 1 = v$.

The equation then becomes

$$\frac{d^2 y}{dv^2} + v^2 \frac{dy}{dv} - 4vy = 0$$

always in a pure translation, $x - x_0 = v$,

we have $\frac{dy}{dx} = \frac{dy}{dv}$, and so on.

As usual we put : $\quad y = \sum_{n=0}^{\infty} a_n v^n$

and we obtain: $\quad \sum_{n=2}^{\infty} n(n-1) a_n v^{n-2} + \sum_{n=1}^{\infty} n a_n v^{n+1} - 4\sum_{n=0}^{\infty} a_n v^{n+1} = 0$

collecting like terms yields:

$$\sum_{n=2}^{\infty} n(n-1) a_n v^{n-2} + \sum_{n=0}^{\infty} (n-4) a_n v^{n+1} = 0$$

which, with a shift of index from n to $(n-3)$ in the second series, gives:

$$\sum_{n=0}^{\infty} n(n-1) a_n v^{n-2} + \sum_{n=3}^{\infty} (n-7) a_{n-3} v^{n-2} = 0$$

Therefore a_0 and a_1 are arbitrary and for the remainder we have

$n = 2: \; 2a_2 = 0,$

$n \geq 3: \; n(n-1) + (n-7) a_{n-3} = 0$

$$a_n = -\frac{n-7}{n(n-1)} a_{n-3}$$

$$a_3 = -\frac{-4}{3\cdot 2} a_0 \qquad a_4 = -\frac{-3}{4\cdot 3} a_1 \qquad a_5 = -\frac{-2}{5\cdot 4} a_2 = 0$$

$$a_6 = -\frac{-1}{6\cdot 5} a_3 \qquad a_7 = -\frac{0}{7\cdot 6} a_4 = 0 \qquad a_7 = (\ ) a_5 = 0$$

$$a_9 = -\frac{2}{9\cdot 8} a_6 \qquad a_{10} = -\frac{3}{10\cdot 9} a_7 = 0 \qquad a_{11} = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{3n} = -\frac{3n-7}{3n(3n-1)} a_{3n-3} \qquad a_{3n+1} = 0, n \geq 2 \qquad a_{3n+2} = 0, n \geq 1$$

with the usual multiplication scheme, the first column yields:

$$n \geq 1: \quad a_{3n} = \frac{(-1)^n \left[(-4)\cdot(-1)\cdot 2 \cdot \cdots \cdot (3n-7)\right] a_0}{\left[3\cdot 6\cdot 9 \cdot \cdots \cdot (3n)\right]\left[2\cdot 5\cdot 8 \cdot \cdots \cdot (3n-1)\right]}$$

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \left[(-4)(-1)\cdot 2 \cdot \cdots \cdot (3n-7)\right] v^{3n}}{\left[3\cdot 6\cdot 9 \cdot \cdots \cdot (3n)\right]\left[2\cdot 5\cdot 8 \cdot \cdots \cdot (3n-1)\right]}\right] + a_1 \left(v + \tfrac{1}{4} v^4\right)$$

since $v = x-1$, the solution appears as:

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \left[(-4)(-1)\cdot 2 \cdot \cdots \cdot (3n-7)\right] (x-1)^{3n}}{\left[3\cdot 6\cdot 9 \cdot \cdots \cdot (3n)\right]\left[2\cdot 5\cdot 8 \cdot \cdots \cdot (3n-1)\right]}\right] + a_1 \left((x-1) + \tfrac{1}{4}(x-1)^4\right)$$

but $3\cdot 6\cdot 9 \cdot \cdots \cdot (3n) = 3^n (1\cdot 2\cdot 3 \cdots n) = 3^n n!$

Furthermore, all but the first two factors inside the square bracket in the numerator also appear in the denominator, and from $n > 2$, they cancelled out, it can be shown that :

$$y = a_0 \sum_{n=0}^{\infty} \frac{4(-1)^n (x-1)^{3n}}{3^n (3n-1)(3n-4) n!} + a_1 \left[(x-1) + \tfrac{1}{4}(x-1)^4\right]$$

Exercise 2.7

1. Find the general solution in powers of x of $\left(x^2-4\right)y''+3xy'+y=0$.

   Then find the particular solution with $y(0)=4, y'(0)=1$

2. Determine the general series solution of the equation

   $\left(x^2+1\right)y''+xy'-4y=0$ about the point $x=0$.

3. Find the general solution near $x=2$ of $y''-(x-2)y'+2y=0$.

4. $y''-2(x+3)y'-3y=0$. Solve about $x=-3$.

5. $y''+(x-2)y=0$. Solve about $x=2$

6. Find the general solution of the Legendre's equation

   $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

2.5 THE FROBENIUS METHOD

The standard form of *linear homogeneous differential equation of second order of equation (2.5) is*:

$$y''+p(x)y'+q(x)y=0 \text{............(2.5)}$$

We shall confine our discussion to a solution of (2.5) in the neighbourhood of $x=0$. The Frobenius method applicable when $x=0$ is *ordinary or regular point* of (2.5). Any differential equation of the form:

$$y''+\frac{p(x)}{x}y'+\frac{q(x)}{x^2}y=0 \text{.........(2.6)}$$

where the functions $p(x)$ and $q(x)$ are analytic at $x=0$, we use a more general method to assume a trial solution of the form:

$$y=x^r\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots\right)=x^r\sum_{n=0}^{\infty}a_nx^n \text{ ...(2.7)}$$

where a_0 is the first coefficient that is not zero (i.e. $a_0\neq 0$).

Indicial Equation

Assume a series solution of the form:

$$y = x^r \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = x^r \sum_{n=0}^{\infty} a_n x^n \text{ ....(2.7) } a_0 \neq 0$$

of the differential equation

$$x^2 y'' + xp(x)y' + q(x)y = 0 \text{ ....(2.6a)}$$

we first expand $p(x)$ and $q(x)$ in power series:

$$p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \qquad q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots$$

Termwise differentiation of trial solution in (2.7)

$$y'(x) = x^{r-1}\left[ra_0 + (r+1)a_1 x + \cdots\right] = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$\Rightarrow y''(x) = x^{r-2}\left[r(r-1)a_0 + r(r+1)a_1 x + \cdots\right] = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

by inserting all these series into (2.6a) we readily obtain:

$$\begin{aligned} &x^r\left[r(r-1)a_0 + \cdots\right] + \left(q_0 + q_1 x + \cdots\right) x^r \left[ra_0 + (r+1)a_1 + \cdots\right] + \\ &\left(q_0 + q_1 x + \cdots\right) x^r \left(a_0 + a_1 x + \cdots\right) = 0 \end{aligned} \text{ ..(2.8)}$$

we now equate the sum of the coefficients of each power $x^r, x^{r+1}, x^{r+2}, \cdots$ to zero. This yields a system of equations involving these unknown coefficients a_n. The equation corresponding to the power x^r is

$$\left[r(r+1) + p_0 r + q_0\right] a_0 = 0 \text{ ....(2.9)}$$

Since by assumption $a_0 \neq 0$, the expression in the brackets must by zero. This gives

$$r(r+1) + p_0 r + q_0 = 0 \text{ ....(2.10)}$$

This important quadratic equation is called ***indicial equation*** of the differential equation (2.7).

This is the coefficient of the lowest power of x gives the indicial equation from which values of r are obtained, $r = r_1$ and $r = r_2$, the roots of the equation.

Three Cases Arises

Case 1: Distinct roots not differing by an Integer.

A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1x + a_2x^2 + \cdots\right) \text{...........}\textbf{(I)}$$

and $$y_2(x) = x^{r_2}\left(c_0 + c_1x + c_2x^2 + \cdots\right) \text{...........}\textbf{(II)}$$

with coefficients obtained successively from (2.8) with $r = r_1$ and $r = r_2$, respectively.

Case 2: Double root $r = r_1 = r_2$.

A basis is

$$y_1(x) = x^{r}\left(a_0 + a_1x + a_2x^2 + \cdots\right) \text{..........}\textbf{(III)}$$

$$\left[r = \tfrac{1}{2}(1 - p_0)\right]$$

and $$y_2(x) = y_1(x)\ln x + x^{r}\left(c_1x + c_2x^2 + \cdots\right) \text{.....}\textbf{(IV)} \qquad x > 0$$

Case 3: Roots differing by an integer.

A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1x + a_2x^2 + \cdots\right) \text{..........}\textbf{(V)}$$

and $$y_2(x) = ky_1(x)\ln x + x^{r_2}\left(c_0 + c_1x + c_2x^2 + \cdots\right) \text{.......}\textbf{(VI)}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

Example 2.5

Solve the equation $4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$.

Solution:

$$y''+\frac{1}{2x}y'+\frac{1}{4x}y=0$$

clearly $p(x)=\frac{1}{2x}$, $q(x)=\frac{1}{4x}$.

Therefore the problem makes the point $x=0$ a regular singular point.

Let $y=\sum_{n=0}^{\infty}a_nx^{n+r}$

$$\Rightarrow y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$\Rightarrow y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$$

substituting the above into the differential equation we get:

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+2\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$2\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)\right]a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$2r(2r-1)a_0x^{r-1}+2\sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$2r(2r-1)a_0x^{r-1}+2\sum_{n=0}^{\infty}(n+r+1)(2n+2r+1)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$2r(2r-1)a_0x^{r-1}+\sum_{n=0}^{\infty}\left[2(n+r+1)(2n+2r+1)a_{n+1}+a_n\right]x^{n+r}=0$$

This last expression is true for all x, therefore the coefficients of all powers of x must vanish.

The coefficient of the lowest power of x,

$$2r(2r-1)a_0$$

is called the indicial term and it determines the values of the index r.

In this case i.e. **Case 1**, assuming $a_0 \neq 0$ then $r = 0, \frac{1}{2}$.

The other coefficients of x^{n+r} for $n = 0,1,2,3\cdots$ lead to the recurrence relationship:

$$2(n+r+1)(2n+2r+1)a_{n+1} + a_n = 0$$

$$a_{n+1} = \frac{-a_n}{2(n+r+1)(2n+2r+1)}$$

$2r(2r-1)a_0 = 0$........ indicial equation

If $r = 0$, $a_0 \neq 0$, then the recurrence relation of coefficient becomes:

$$a_{n+1} = \frac{-a_n}{2(n+1)(2n+1)}$$

if $n = 0$, $a_1 = \frac{-a_0}{2(1)(1)} = \frac{-a_0}{2!}$

if $n = 1$, $a_2 = \frac{-a_1}{2(2)(3)} = \frac{-a_1}{4(3)} = \frac{a_0}{4!}$

if $n = 2$, $a_3 = \frac{-a_2}{2(3)(5)} = \frac{-a_2}{6(5)} = \frac{-a_0}{6!}$

if $n = 3$, $a_4 = \frac{-a_3}{2(4)(7)} = \frac{-a_3}{8(7)} = \frac{-a_0}{8!}$

therefore

$$a_n = \frac{(-1)^n a_0}{(2n)!}$$

Recall that **(I)**, $y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$.

$$= x^0\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots\right)$$

$$= a_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^k x^k}{(2k)!} + \cdots\right)$$

$$= a_0\left[1 - \frac{\left(\sqrt{x}\right)^2}{2!} + \frac{\left(\sqrt{x}\right)^4}{4!} - \frac{\left(\sqrt{x}\right)^6}{6!} + \cdots + \frac{(-1)^k\left(\sqrt{x}\right)^{2k}}{2!} + \cdots\right]$$

$\Rightarrow y = a_0 \cos\sqrt{x}$, a_0 is arbitrary constant

The other value obtained for $r=\frac{1}{2}$.

Recall the recurrence relation:

$$c_{n+1}=\frac{-c_n}{2\left(n+\frac{3}{2}\right)(2n+2)} \quad c_0 \text{ is arbitrary,}$$

if $n=0$, $c_1=\dfrac{-c_0}{(3)(2)}=\dfrac{-c_0}{3!}$

if $n=1$, $c_2=\dfrac{-c_1}{2\left(\frac{5}{2}\right)(4)}=\dfrac{-c_1}{5(4)}=\dfrac{c_0}{5!}$

if $n=2$, $c_3=\dfrac{-c_2}{2\left(\frac{7}{2}\right)(6)}=\dfrac{-c_2}{7(6)}=\dfrac{-c_0}{7!}$

Therefore, $\quad c_n=\dfrac{(-1)^n c_0}{(2n+1)!}$

Recall that **(II),** $y_2=x^{r_2}\sum_{n=0}^{\infty} c_n x^n$

$$=x^{\frac{1}{2}}\left(c_0+c_1x+c_2x^2+\cdots+c_kx^k+\cdots\right)$$

$$=x^{\frac{1}{2}}c_0\left(1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots+\frac{(-1)^k x^k}{(2k+1)!}+\cdots\right)$$

$$=c_0\left[\sqrt{x}-\frac{\left(\sqrt{x}\right)^3}{3!}+\frac{\left(\sqrt{x}\right)^5}{5!}-\frac{\left(\sqrt{x}\right)^7}{7!}+\cdots+\frac{(-1)^k\left(\sqrt{x}\right)^k}{k!}+\cdots\right]$$

$y_2=c_0\sin\sqrt{x}$, $\qquad c_0$ is arbitrary

Hence for $y=y_1+y_2$

The general solution is given as:

$y=a_0\cos\sqrt{x}+c_0\sin\sqrt{x}$ or.

Example 2.6

Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y''+3x(1+x)^3y'-(1-x^2)y=0$$

Solution:

We divide each term by $2x^2(1+x)$ to recast the differential equation in the form

$$y''+\frac{\left(\frac{3}{2}\right)\left(1+2x+x^2\right)}{x}y'-\frac{\left(\frac{1}{2}\right)(1-x)}{x^2}y=0$$

and thus see that $p_0=\frac{3}{2}$ and $q_0=-\frac{1}{2}$

Hence the indicial equation is:

$$r(r-1)+\tfrac{3}{2}r-\tfrac{1}{2}=r^2+\tfrac{1}{2}r-\tfrac{1}{2}=(r+1)\left(r-\tfrac{1}{2}\right)=0$$

with roots $r_1=\frac{1}{2}$ and $r_2=-1$.

The two possible Frobenius series solutions are then of the forms

$$y_1=x^{\frac{1}{2}}\sum_{n=0}^{\infty}a_nx^n \text{ and } y_2=x^{-1}\sum_{n=0}^{\infty}c_nx^n .$$

Example 2.7

Find a Frobenius series solution of Bessel's equation of order zero:

$$x^2y''+xy'+x^2y=0$$

Solution:

$$y''+\frac{1}{x}y'+\frac{x^2}{x^2}y=0$$

Hence $x=0$ is a regular singular point with $p(x)=1$ and $q(x)=x^2$, so our series will converge for all $x>0$.

Because $p_0=1$ and $q_0=0$, the indicial equation is

$$r(r-1)+r=r^2=0$$

Thus we obtain only the single exponent $r=0$ and so there is only one Frobenius series solution

$$y = x^0 \sum_{n=0}^{\infty} a_n x^n$$

substitute in the differential equation; the result is:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

we combine the first two sums and shift the index of summation in the third by -2 to obtain:

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Therefore, the recursion formula is:

$$a_n = -\frac{a_{n-2}}{n^2} \quad \text{for } n \geq 2$$

$$a_2 = -\frac{a_0}{2^2}, \qquad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 4^2} \text{ and } a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 4^2 6^2}$$

$$a_{2n} = \frac{(-1)^n a_0}{2^2 \cdot 4^2 \cdots (2n)^2} = \frac{(-1)^n a_0}{2^{2n} (n!)^2}$$

The choice $a_0 = 1$ gives us one of the most important special functions in mathematics, the **Bessel function of order zero of the first kind**, denoted by $J_0(x)$.

Thus $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$

In this example we have not been able to find a second linearly independent solution of Bessel's equation of order zero.

Exercise 2.9

In each exercise, determine whether $x=0$ is an ordinary point, a regular singular point, or an irregular singular point.

If it is a regular point, find the exponent of the differential equation at $x=0$.

1. $xy''+\left(x-x^3\right)y'+(\sin x)y=0$ 1a. $xy''+x^2y'+\left(e^x-1\right)y=0$

2. $x^2y''+(\cos x)y'+xy=0$ 2a. $x^2y''+(6\sin x)y'+6y=0$

3. $3x^3y''+2x^2y'+\left(1-x^2\right)y=0$

4. Apply the method of Frobenius of Bessel's equation of order $\frac{1}{2}$

   $x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$, to derive its general solution for $x>0$,

   $y=a_0\dfrac{\cos x}{\sqrt{x}}+a_1\dfrac{\sin x}{\sqrt{x}}$

5. Show that Bessel's equation of order 1,

   $x^2y''+xy'+\left(x^2-1\right)y=0$

   has exponents $r_1=1$ and $r_2=-1$ at $x=0$, and that the Frobenius series solution corresponding to $r_1=1$ is $J_1(x)=\dfrac{x}{2}\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{n!(n+1)!2^{2n}}$

6. Obtain the solution

   $y''+y=0 \qquad -\infty<x<\infty$

   by Frobenius method.

7. Find the series solutions in powers of Airy's equation

   $y''=xy \qquad -\infty<x<\infty$

8. Find the general series solution of the following

   a) $2xy''+(3-2x)y'+4y=0$ b) $2\left(x-x^2\right)y''+(1-9x)y'+3y=0$

   c) $x^2y''+xy'+\left(x^2-1\right)y=0$

UNIT SEVEN
FOURIER SERIES

Definition: Periodic Function

If the value of each ordinate $f(t)$ repeats itself at equal intervals in abscissa, t, the $f(t)$ is said to be ***periodic***. Thus if $f(t) = f(t+\mathbf{T}) = f(t+2\mathbf{T}) = \ldots$, then $\mathbf{T}$ is called the least period (or period) of the function $f(t)$.

Example 3.1

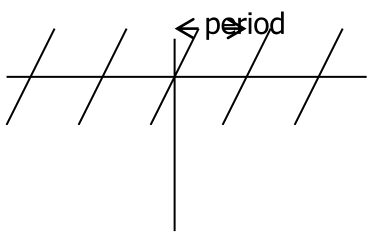
1: The function $\sin x$ has periods $2\pi, 4\pi, 6\pi$ since $\sin(x+2\pi)$, $\sin(x+4\pi)$, $\sin(x+6\pi)$ all equal $\sin x$. So the *least period* is 2π.

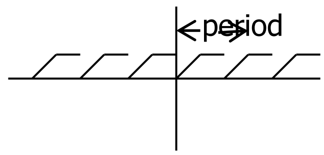
2: The period of $\sin nx$ or $\cos nx$, where n a positive integer, is $2\pi/n$.

3: The period of $\tan x$ is π.

4: A constant has any positive number as period.

Other examples of periodic functions are shown in graphs below:





Piecewise Smooth Functions

Definition: A function $f(x)$ is *piecewise continuous on the open interval* $a < x < b$ if:

1. $f(x)$ is continuous everywhere in $a < x < b$ with the possible exception of at most a *finite* number of points $x_1, x_2, \ldots, x_n$ and

2. at these points of discontinuity, the right- and left – hand limits of $f(x)$, respectively $\lim\limits_{\substack{x \to x_j \\ x > x_j}} f(x)$ and $\lim\limits_{\substack{x \to x_j \\ x < x_j}} f(x)$, exist ($j = 1, 2, \ldots, n$)

(Note that a continuous function is piecewise continuous).

Definition: A function $f(x)$ is *piecewise continuous on the closed interval* $a \le x \le b$ if:

1. it is piecewise continuous on the open interval $a < x < b$,
2. the right – hand limit of $f(x)$ exists at $x = a$, and
3. the left – hand limit of $f(x)$ exists at $x = b$.

Definition: A function $f(x)$ is *piecewise smooth* on $[a,b]$ if both $f(x)$ and $f'(x)$ are piecewise continuous on $[a,b]$.

Examples 3.2

1. Determine whether

$$f(x) = \begin{cases} x^2+1 & x \ge 0 \\ 1/x & x < 0 \end{cases} \text{ is piecewise continuous on } [-1,1]$$

Solution:

The given function is continuous everywhere on $[-1,1]$ except at $x = 0$. Therefore, if the right- and left- hand limits exist at $x = 0$, $f(x)$ will be piecewise continuous on $[-1,1]$. We have:

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = \lim_{\substack{x\to 0\\ x>0}} \left(x^2+1\right) = 1 \qquad \lim_{\substack{x\to 0\\ x<0}} f(x) = \lim_{\substack{x\to 0\\ x<0}} \frac{1}{x} = -\infty$$

Since the left – hand limit does not exist, $f(x)$ is not piecewise continuous on $[-1,1]$.

2. Is $f(x) = \begin{cases} \sin \pi x & x > 1 \\ 0 & 0 \le x \le 1 \\ e^x & -1 < x < 0 \\ x^3 & x \le -1 \end{cases}$ piecewise continuous on $[-2,5]$?

Solution:

The given function is continuous on $[-2,5]$ except at the two points $x_1 = 0$ and $x_2 = -1$.(Note that $f(x)$ continuous at $x = 1$). At the two points of discontinuity, we find that:

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = \lim_{x\to 0} 0 = 0 \qquad \lim_{\substack{x\to 0\\ x>0}} f(x) = \lim_{x\to 0} e^x = 1$$

and $$\lim_{\substack{x\to 1\\ x>-1}} f(x) = \lim_{x\to -1} e^x = e^{-1} \qquad \lim_{\substack{x\to 1\\ x>-1}} f(x) = \lim_{x\to -1} x^3 = -1$$

since all require limits exist, $f(x)$ is piecewise continuous on $[-2,5]$

3. Is $f(x) = \begin{cases} x^2+1 & x<0 \\ 1 & 0\le x\le 1 \\ 2x+1 & x>1 \end{cases}$ piecewise smooth on $[-2,2]$?

Solution:

The function is continuous everywhere on $[-2,2]$ except at $x_1 = 1$. Since the require limits exist at x_1, $f(x)$ is piecewise continuous.

Differentiating $f(x)$, we obtain

$$f'(x) = \begin{cases} 2x & x<0 \\ 0 & 0\le x\le 1 \\ 2 & x>1 \end{cases}$$

The derivative does not exist at $x_1 = 1$ but is continuous at all other points in $[-2,2]$. At x_1 the required limits exist; hence $f'(x)$ is piecewise continuous. It follows that $f(x)$ is piecewise smooth on $[-2,2]$.

4. Is $f(x) = \begin{cases} 1 & x<0 \\ \sqrt{x} & 0\le x\le 1 \\ x^3 & x>1 \end{cases}$ piecewise smooth on $[-1,3]$?

Solution:

The function is continuous everywhere on $[-1,3]$ except at $x_1 = 0$. Since the require limits exist at x_1, $f(x)$ is piecewise continuous. Differentiating $f(x)$, we obtain:

$$f'(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{2\sqrt{x}} & 0 \le x \le 1 \\ 3x^2 & x > 1 \end{cases}$$

which is continuous everywhere on $[-1,3]$ except at the two points $x_1 = 0$ and $x_2 = 1$where the derivative does not exist.

At $x_1 = 0$,

$$\lim_{\substack{x\to x_1 \\ x>x_1}} f'(x) = \lim_{\substack{x\to 0 \\ x>0}} \frac{1}{2\sqrt{x}} = \infty$$

Hence , one of the required limits does not exist. If follows that $f'(x)$ is not piecewise continuous, and therefore that $f(x)$ is not piecewise smooth on $[-1,3]$.

Definition: Fourier Series

A non – sinusoidal period function $f(x)$ (which is integrable in $(-\pi,\pi)$) can be expressed in a converging series of the form:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \text{ ....(3.1)}$$

called the ***Fourier Series***, where $a_0,a_1,a_2,...,b_1,b_2,b_3,...$ are constants that can easily be determined, and

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx, \qquad n=1,2,... \qquad \text{......... (3.2)}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx \qquad n=1,2,...$$

where $a_0, a_n,$ and b_n are called **Fourier coefficients**.

The function $\cos x$ and $\sin x$ are called fundamentals for $n>1$ an integral $\cos nx$ and $\sin nx$ are called harmonics, which are multiples of the fundamental frequencies.

Orthogonality Conditions For The Sine And Cosine Functions

The following integrals are very useful in the Fourier expansions:

For m,n integers $m\neq n$:

1. $\int_0^{2\pi}\sin nx dx=0$
2. $\int_0^{2\pi}\cos nx dx=0$
3. $\int_0^{2\pi}\sin^2 nx dx=\pi$
4. $\int_0^{2\pi}\cos^2 nx dx=\pi$
5. $\int_0^{2\pi}\sin nx\sin mx dx=0$
6. $\int_0^{2\pi}\cos nx\sin mx dx=0$
7. $\int_0^{2\pi}\cos nx\cos mx dx=0$
8. $\int_0^{2\pi}\sin nx\cos mx dx=0$

The iterated integration by parts is also very useful:

$$\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$$

where $u' = \dfrac{du}{dx}$, $u'' = \dfrac{d^2u}{dx^2}$, $u''' = \dfrac{d^3u}{dx^3}, \ldots$

$$v_1 = \int v dx,\ v_2 = \int v_1 dx,\ v_3 = \int v_2 dx,\ v_4 = \int v_3 dx, \ldots$$

Also for N which is an integer $n \in N$,

$$\sin n\pi = 0,\ \cos n\pi = (-1)^n$$

Determination of Fourier Coefficients

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ with period 2π .

a) Then to find a_0: Integrate both sides of (3.1) w.r.t. x for $0 < x < 2\pi$

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} a_0 dx + \int_0^{2\pi} \sum_{n=1}^{\infty} a_n \cos nx dx + \int_0^{2\pi} \sum_{n=1}^{\infty} b_n \sin nx dx$$

$\Rightarrow \int_0^{2\pi} f(x)dx = a_0 \int_0^{2\pi} dx = a_0 2\pi$, other term vanishes by orthogonal properties.

$$\therefore a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx \ldots\ldots\ldots(\mathrm{I})$$

b) To find a_n: Multiply each side of equation (3.1) by $\cos nx$ and integrate throughout with respect to x, $0 < x < 2\pi$

$$\int_0^{2\pi} f(x)\cos nx dx = \int_0^{2\pi} a_0 \cos nx dx + \int_0^{2\pi} \sum_{n=1}^{\infty} a_n \cos^2 nx dx + \int_0^{2\pi} \sum_{n=1}^{\infty} b_n \cos nx \sin nx dx$$

$$\Rightarrow \int_0^{2\pi} f(x)\cos nx dx = a_n \int_0^{2\pi} \cos nx \cos nx dx = a_n \pi$$

$$\therefore a_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\cos nx dx \ldots\ldots\ldots(\mathrm{II})$$

c) To find b_n: Multiply each side of equation (3.1) by $\sin nx$ and integrate throughout with respect to x, $0 < x < 2\pi$

$$\int_0^{2\pi} f(x)\cos nxdx = \int_0^{2\pi} a_0 \sin nxdx + \int_0^{2\pi} \sum_{n=1}^{\infty} a_n \cos nx \sin xdx + \int_0^{2\pi} \sum_{n=1}^{\infty} b_n \sin^2 nxdx$$

$$\Rightarrow \int_0^{2\pi} f(x)\sin nxdx = b_n \int_0^{2\pi} \sin nx \sin nxdx = b_n \pi$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \text{ ....(III)}$$

Example and Solution

If the following function are defined over the interval $-\pi < x < \pi$ and $f(x+2\pi)$, state whether or not each function can be represented by a Fourier series.

1. $f(x) = x^3$ - Yes
2. $f(x) = 4x - 5$ - Yes
3. $f(x) = \dfrac{2}{x}$ - No, infinite discontinuity at $x = 0$
4. $f(x) = \dfrac{1}{x-4}$ - Yes
5. $f(x) = \tan x$ - No: infinite discontinuity at $x = \pi/2$.

$f(x) = y$, where $x^2 + y^2 = 9$ - No; two valued

Dirichlet Conditions

Suppose that;

1. $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$.
2. $f(x)$ is periodic outside $(-L, L)$ with period $2L$.
3. $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series with Fourier coefficients converges to

a) $f(x)$ if x is a point of continuity

b) if x is point of discontinuity.

Here $f(x+0)$ and $f(x-0)$ are the right – and left – hand limits of $f(x)$ at x represents $\lim_{\varepsilon\to 0^+} f(x+\varepsilon)$ and $\lim_{\varepsilon\to 0^-} f(x-\varepsilon)$ respectively.

The conditions 1, 2 and 3 imposed on $f(x)$ are sufficient but not necessary, and are generally satisfied in practice.

Example 3.1:

Find the Fourier series representing the periodic function $f(x)=x$ and $0<x<2\pi$ and sketch its graph from $x=-4\pi$ to $x=4\pi$.

Solution:

Clearly $f(x)=x$ is periodic then,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Recall:

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx,\ a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx,\ b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx$$

Thus;

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}xdx=\frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=\pi,$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x\cos nxdx=\frac{1}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\sin nxdx\right]=0$$

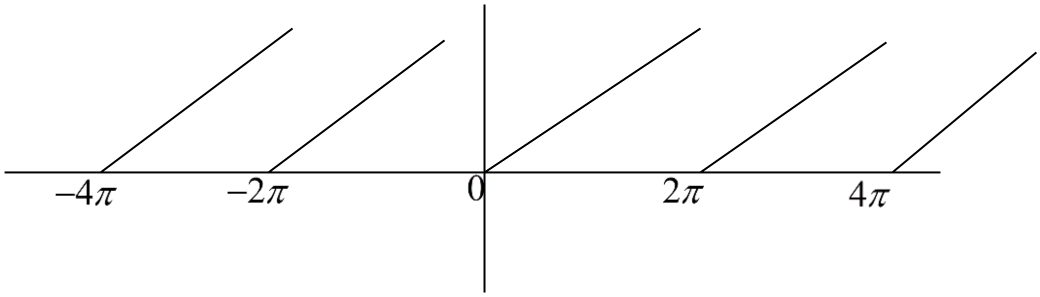
$$b_n=\frac{1}{\pi}\int_0^{2\pi}x\cos nxdx=\frac{1}{\pi}\left[\frac{-x}{n}\cos nx\Big|_0^{2\pi}+\frac{1}{n}\int_0^{2\pi}\cos nxdx\right]=-\frac{2}{n}$$

Put the values into:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$\therefore f(x)=\pi-2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{k}\sin kx+\ldots\right]$$

Sketch



Function Defines in two or more sub – interval

Example

1. Find the Fourier coefficients corresponding to the function

$$f(x)=\begin{cases}0 & -5<x<0\\ 3 & 0<x<5\end{cases} \qquad period = 10$$

2. Write the corresponding Fourier series

3. How should $f(x)$ be defined at $x=-5$, Fourier series will converge to $f(x)$ for $-5\le x\le 5$?

Solution:

1. Period $2L=10$ and $L=5$.

Choose the interval c to $c+2L$ as -5 to 5, so that $c=-5$.

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)dx=\frac{1}{2(5)}\left[\int_{-5}^{0}0dx+\int_{0}^{5}3dx\right]$$

$$\Rightarrow a_0=0+\frac{1}{2(5)}\left[3x\Big|_0^5\right]=\frac{3}{2}$$

Then $a_n=\dfrac{1}{L}\displaystyle\int_{c}^{c+2L} f(x)\cos\frac{n\pi x}{L}=\frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}dx$

$$=\frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}dx+\int_{0}^{5}(3)\cos\frac{n\pi x}{5}\right\}=\frac{3}{5}\int_{0}^{5}\cos\frac{n\pi x}{5}dx$$

$$=\frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5=0 \text{ if } n\neq 0$$

$$b_n=\frac{1}{L}\int_{c}^{c+2L} f(x)\sin\frac{n\pi x}{L}=\frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}dx$$

$$=\frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}dx+\int_{0}^{5}(3)\sin\frac{n\pi x}{5}\right\}=\frac{3}{5}\int_{0}^{5}\sin\frac{n\pi x}{5}dx$$

$$=\frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Big|_{0}^{5}=\frac{3(1-\cos n\pi)}{n\pi}$$

2. The corresponding Fourier series is:

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)$$

$$=\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$=\frac{3}{2}+\frac{6}{\pi}\left(\sin\frac{\pi x}{5}+\frac{1}{3}\sin\frac{3\pi x}{5}+\frac{1}{5}\sin\frac{5\pi x}{5}+\ldots\right)$$

3. Since $f(x)$ satisfying the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to t at points of discontinuity. At $x=-5,0$ and 5, which are points of discontinuity, the series converges to $(3+0)/2=3/2$.

If we redefine $f(x)$ as follows:

$$f(x)=\begin{cases}3/2 & x=-5\\ 0 & -5<x<0\\ 3/2 & x=0\\ 0 & 0<x<5\\ 3/2 & x=5\end{cases}$$

Then the series will converges to $f(x)$ for $-5\le x\le 5$.

Example 3.5

Find the coefficients of the period function

$$f(x)=\begin{cases}-k & \text{if } -\pi<x<0\\ k & \text{if } 0<x<\pi\end{cases} \quad \text{and } f(x+2\pi)=f(x)$$

Solution:

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}-kdx+\int_{0}^{\pi}kdx\right]=0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx + \int_{0}^{\pi}(k)\cos nx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\Bigg|_{0}^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi, 0,$ and π for all $n = 1, 2, ...$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx + \int_{0}^{\pi}(k)\sin nx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Bigg|_{-\pi}^{0} + -k\frac{\cos nx}{n}\Bigg|_{0}^{\pi}\right]$$

since $\cos(-\alpha) = \cos\alpha; \cos 0 = 1$, this yields.

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}[1 - \cos n\pi]$$

Now, $\cos\pi = -1, \cos 2\pi = 1, \cos 3\pi = -1,$ etc;

In general $\cos n\pi = \begin{cases} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$ and thus

$$1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

Hence the Fourier Coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, b_2 = 0, b_3 = \frac{4k}{\pi}, b_4 = 0, b_5 = \frac{4k}{5\pi}$$

Since the a_n are zero, the Fourier series of $f(x)$ is:

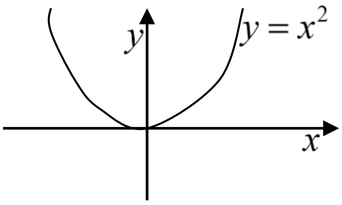
$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

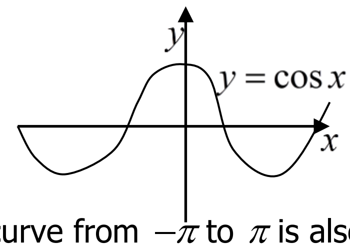
ODD AND EVEN FUNCTION

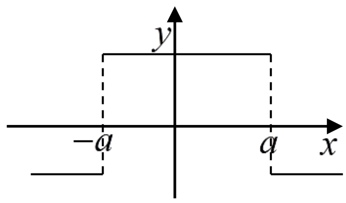
Even Functions

Definition: A function $f(x)$ is said to be even (or symmetric) if $f(-x)=f(x)$

The graph of a symmetric function is symmetrical about the $y-$axis







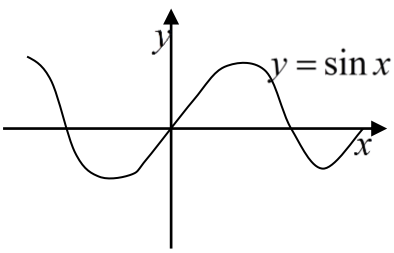
The area under such a curve from $-\pi$ to π is also twice the area from 0 to π that is $\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$.

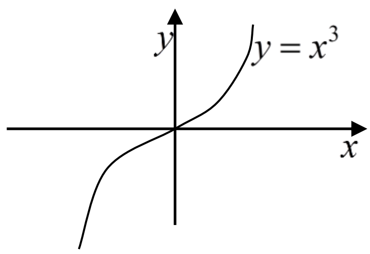
Thus $x^4, 2x^6-4x^2+5, \cos x, e^x+e^{-x}$ are even function

Odd functions

Definition: A function is called odd (or skew symmetric) if $f(-x)=-f(x)$.

Some graphs of odd functions:





The area under the curve from $-\pi$ to π is zero; i.e. $\int_{-\pi}^{\pi} f(x)dx = 0$.

Thus $x^3, x^5-3x^3+2x, \sin x, \tan 3x$ are odd functions.

<u>Products of Odd and Even Functions</u>

$$(even)\times(even)=(even)$$
$$(odd)\times(odd)=(even)$$
$$(odd)\times(even)=(odd)$$

Expansion of the Even Function, $f(x)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0$$

Therefore the series of the even function contains only cosine terms

Expansion of the Odd Function, $f(x)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nxdx$$

Therefore the series of the odd function contains only sine terms.

Remark: Functions which are neither even nor odd contains both sine and cosine terms.

Example 3.5

Consider the function

$$f(x) = \begin{cases} -6 & -\pi < x < 0 \\ 6 & 0 < x < \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

Before we do any evaluation, we can see that this is an odd function and therefore sine terms only; i.e. $a_0 = 0$ $\quad a_n = 0$ $\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$

$f(x)\sin nx$ is a product of odd functions and is therefore even.

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} 6\sin nx dx = \frac{12}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{12}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \dfrac{24}{n\pi} & n \text{ odd} \end{cases}$$

The Fourier series is then:

$$f(x) = \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\}$$

Example 3.6

Obtain a Fourier expansion for $f(x) = x^3$ for $-\pi < x < \pi$.

Solution:

$f(x) = x^3$ is an odd function, therefore $a_0 = a_n = 0 \quad \forall n$

for $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x^3 \sin nx dx = 2(-1)^n\left[\frac{-\pi}{n} + \frac{6}{n^3}\right]$

$$x^3 = 2\left[-\left(\frac{-\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(\frac{-\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(\frac{-\pi^2}{3} + \frac{6}{3^3}\right)\sin x + \ldots\right]$$

Half – Range Series, Periodic for $(0,\pi)$

Let the given function be defined in the interval $(0,\pi)$. To get the series of **cosine** only, the function $f(x)$ is an even function in the interval $(-\pi,\pi)$.

Then,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \text{ and } b_n = 0 \quad \forall n$$

To expand $f(x)$ as series of **sine** only, extend the interval to $(-\pi,\pi)$ and treat $f(x)$ as odd function.

Then,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx, \ a_n = 0 \ \forall n$$

Example 3.7

Find the Fourier sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$

Solution:

Let $e^{ax} = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx dx = \frac{2n}{\left(a^2+n^2\right)\pi}\left[1-(-1)^n e^{a\pi}\right]$$

Thus,

$$b_1 = \frac{2\left(1+e^{a\pi}\right)}{\left(a^2+1^2\right)\pi}, \ b_2 = \frac{2\cdot 2\left(1-e^{a\pi}\right)}{\left(a^2+2^2\right)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\left(\frac{1+e^{a\pi}}{a^2+1^2}\right)\sin x + 2\left(\frac{1-e^{a\pi}}{a^2+2^2}\right)\sin 2x + \ldots\right]$$

Example 3.8

Expand $f(x) = x, 0 < x < 2$ in a half range

a. sine series b. cosine series

Solution:

a) $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}dx$$

$$= \left\{\left[(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right)\right] - \left[(1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right]\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos\pi$$

Then $f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\right)$$

b) Thus $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2}dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{if } n \neq 0.$$

If $n = 0, a_0 = \int_0^2 x dx = 2$

Then $f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$

$$1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right)$$

Change of Integral and Functions of Arbitrary Period

For some problems the period of the function is not 2π but t or $2L$. To conform to the general theory, this period must be converted to its equivalent of 2π, and such will equally affect the argument of the function proportionally.

Let $f(x)$ be defined in the interval $(-L, L)$, to convert this to 2π;

$2L$ is the interval for variable x

1 is the interval for the variable $\dfrac{x}{2L}$

and therefore 2π is the interval for the variable

$$\frac{2\pi x}{2L} = \frac{\pi x}{L}$$

let $z = \dfrac{\pi x}{L}$, then $x = \dfrac{zL}{\pi}$

Thus the function $f(x)$ of period $2L$ is transformed to the function $f\left(\dfrac{zL}{\pi}\right)$ of period 2π. Thus,

$$f\left(\frac{zL}{\pi}\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz$$

where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\left(\frac{zL}{\pi}\right) dz = \frac{1}{2\pi}\int_0^{2L} f(x)\, d\left(\frac{\pi x}{L}\right)$ o

but $x = \dfrac{zL}{\pi}$ and so $a_0 = \dfrac{1}{2L}\displaystyle\int_0^{2L} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{zL}{\pi}\right)\cos nz\,dz = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\, d\left(\frac{\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{zL}{\pi}\right)\sin nz\,dz = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

In general the Fourier series with period $T = 2L$ is stated as:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \quad ....(3.3)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas:**

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos nx dx, \qquad n = 1,2,...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin nx dx \qquad n = 1,2,...$$

Corollary: For half range series in the interval $(0, L)$;

Cosine series: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

where $a_0 = \frac{1}{L}\int_0^L f(x)dx$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$

Sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

Example 3.9

1. Given $f(x) = \int_1^0 \begin{matrix} 0 < x < c \\ c < x < 2c \end{matrix}$

   Expand $f(x)$ in the Fourier series of period $2c$

Solution:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Recall that:

$$a_0 = \frac{1}{2c}\int_0^{2c} f(x)dx = \frac{1}{2c}\int_0^c 0dx + \frac{1}{2c}\int_c^{2c}(1)dx = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{c}\left[\int_0^{c}(0)\cos\frac{n\pi x}{c}dx + \int_c^{2c}(1)\cos\frac{n\pi x}{c}dx\right] = 0$$

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{c}\left[\int_0^{c}(0)\sin\frac{n\pi x}{c}dx + \int_c^{2c}(1)\sin\frac{n\pi x}{c}dx\right]$$

$$\Rightarrow b_n = \begin{cases} \frac{-2}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$\therefore f(x) = \frac{1}{2} - \frac{2}{\pi}\left\{\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \ldots\right\}$$

2. Find the Fourier series of the function $f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

Solution: We determine the Fourier series, but the period $2c = 4$

$$\Rightarrow c = 2$$

$$a_0 = \frac{1}{2c}\int_{-c}^{c} f(x)dx = \frac{1}{2(2)}\int_{-2}^{2} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{4}\left[\int_{-2}^{-1}(0)dx + \int_{-1}^{1}kdx + \int_{1}^{2}(0)dx\right] = \frac{k}{2}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

$$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

$$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}dx = 0$$

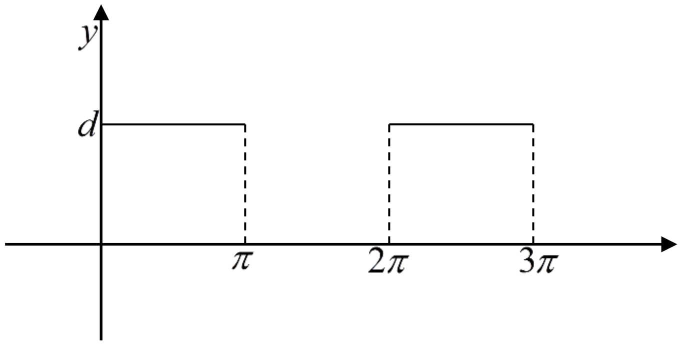
then

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left\{\frac{1}{1}\sin\frac{\pi}{2}\cos\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi}{2}\cos\frac{3\pi}{2}x + \ldots\right\}$$

$$f(x)=\frac{k}{2}+\frac{2k}{\pi}\left\{\cos\frac{\pi x}{2}-\frac{1}{3}\cos\frac{3\pi}{2}x+...\right\}$$

Fourier Series from the Sketches of the Wave Form

Example 1: Derive the Fourier series for the sequence wave form up to the fire four terms. The figure is shown below:



Solution:

Let $y=f(x)$ defined as $f(x)=\begin{cases} d & 0<x<\pi \\ 0 & \pi<x<2\pi \end{cases}$

Then by Fourier series expansion, with period 2π

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

where $L=\pi$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx=\frac{1}{2\pi}\left[\int_0^{\pi}(d)dx+\int_0^{\pi}(0)dx\right]=\frac{d}{2}$$

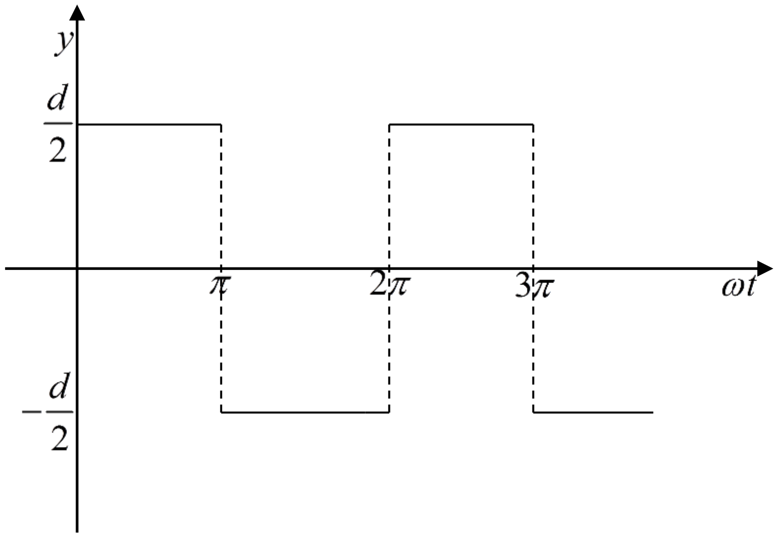
$$a_n=\frac{1}{L}\int_0^{2L}f(x)\cos\frac{n\pi x}{L}dx=\frac{1}{\pi}\left[\int_0^{\pi}(d)\cos xdx+\int_{\pi}^{2\pi}(0)\cos xdx\right]=0$$

$$b_n=\frac{1}{\pi}\left[\int_0^{\pi}(d)\sin xdx+\int_{\pi}^{2\pi}(0)\sin xdx\right]=\begin{cases}\frac{2d}{n\pi} & n\text{ is odd}\\ 0 & n\text{ is even}\end{cases}$$

Then

$$y=f(x)=\frac{d}{2}+\frac{2d}{\pi}\left[\frac{1}{1}\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right]$$

Example 2: Find the first three terms of the Fourier series for the complete square waveform shown in the following figure:



Solution:

Let $y = f(\omega t) = \begin{cases} d/2 & 0 < \omega t < \pi \\ -d/2 & \pi < \omega t < 2\pi \end{cases}$

Notice that the square wave form is skew – symmetric. Then the Fourier series is to comprise of only sine terms. Thus,

$a_0 = 0$, $a_n = 0 \ \forall n$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\omega t)\sin(\omega t)\,d(\omega t)$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_0^{\pi}\left(\frac{d}{2}\right)\sin(\omega t)\,d(\omega t) + \int_{\pi}^{2\pi}\left(-\frac{d}{2}\right)\sin(\omega t)\,d(\omega t)\right]$$

$$b_n = \frac{d}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \dfrac{2d}{n\pi} & \text{odd} \\ 0 & \text{even} \end{cases}$$

$$y = f(\omega t) = \frac{2d}{\pi}\left[\frac{1}{1}\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right]$$

Problems

1. Determine whether the following functions are piecewise continuous on $[-1,5]$: **Ans**

a) $f(x)=\begin{cases} x^2 & x\geq 0 \\ 4 & 0<x<2 \\ x & x\leq 0 \end{cases}$ yes

b) $f(x)=\begin{cases} 1/(x-2)^2 & x>2 \\ 5x^2-1 & x\leq 2 \end{cases}$ no $\left(\lim\limits_{\substack{x\to 2\\ x>2}} f(x)=\infty\right)$

c) $f(x)=\dfrac{1}{x-2}$ no $\left(\lim\limits_{\substack{x\to 2\\ x>2}} f(x)=\infty\right)$

d) $f(x)=\dfrac{1}{x+2}$ yes,

2. Which of the following functions are piecewise smooth on $[-2,3]$?

a) $f(x)=\begin{cases} x^3 & x<0 \\ \sin \pi x & 0\leq x\leq 1 \\ x^2-5x & x>1 \end{cases}$ yes

b) $f(x)=\begin{cases} e^x & x<1 \\ \sqrt{x} & x\geq 1 \end{cases}$ yes

c) $f(x)=\ln|x|$ no $\left(\lim\limits_{\substack{x\to 0\\ x>0}} \ln|x|=-\infty\right)$

d) $f(x)=\begin{cases} (x-1)^2 & x\leq 1 \\ (x-1)^{1/3} & x>1 \end{cases}$ no $\left(\lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{1}{3(x-1)^{2/3}}=\infty\right)$

3. State whether each of the following products is odd, even or neither

(a) $x^2 \sin x$ (f). $(2x+3)\sin 4x$

(b) $x^3 \cos x$ (g). $\sin 3x \cos 3x$

(c) $\cos 2x \cos 3x$ (h). $x^3 e^x$

(d) $x \sin x$ (i). $\left(x^4+4\right)\sin 2x$

(e) $3\sin x \cos 4x$ (j). $\dfrac{1}{x+2}\cosh x$

4. Find the smallest positive period of

(a) $\cos x, \sin x, \cos 2x, \sin 2x, \cos \pi x, \sin \pi x, \cos 2\pi x, \sin 2\pi x$

(b) $\cos nx, \sin nx, \cos \frac{2\pi x}{k}, \sin \frac{2\pi x}{k}, \cos \frac{2\pi nx}{k}, \sin \frac{2\pi nx}{k}$

5. Find the Fourier series of the function $f(x)$, which is a assumed to have the period 2π, and plot a accurate graphs of the first three partial sum.

(a) $f(x) = x; (-\pi < x < \pi)$ (d) $f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$

(b) $f(x) = x^2; (-\pi < x < \pi)$

(e) $f(x) = \begin{cases} -\pi/4 & \text{for } -\pi < x < 0 \\ \pi/4 & \text{for } 0 < x < \pi \end{cases}$ and also $f(-\pi) = f(0) = f(\pi) = 0$

(c) $f(x) = x^3; (-\pi < x < \pi)$ (f) $f(x) = \begin{cases} 1 & \text{for } -\pi < x < 0 \\ -1 & \text{for } 0 < x < \pi \end{cases}$

(g) $f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\pi/2 \\ 0 & \text{for } -\pi/2 < x < \pi/2 \\ 1 & \text{for } \pi/2 < x < \pi \end{cases}$

6. Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression for $f(x)$. Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

7. State whether the given function is even or odd. Find the Fourier series. Sketch the function and some partial sums.

a) $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

b) $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

c) $f(x) = x^2/2 \quad (-\pi < x < \pi)$

8. Find the Fourier cosines as well as the Fourier sine series.

(a) Find a Fourier sine series for $f(x) = 1$ on $(0,1)$.

Ans: $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(-1)^n\right]\sin n\pi x$

(b) Find a Fourier sine series for $f(x) = x$ on $(0,3)$.

Ans: $-\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{3}$

(c) Find a Fourier cosine series for $f(x) = x^2$ on $(0,\pi)$.

Ans: $\frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

(d) Find a Fourier cosine series for $f(x) = \begin{cases} 0 & x \le 2 \\ 2 & x > 2 \end{cases}$ on $(0,3)$

Ans: $\frac{2}{3} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi}{3}\cos\frac{n\pi x}{3}$

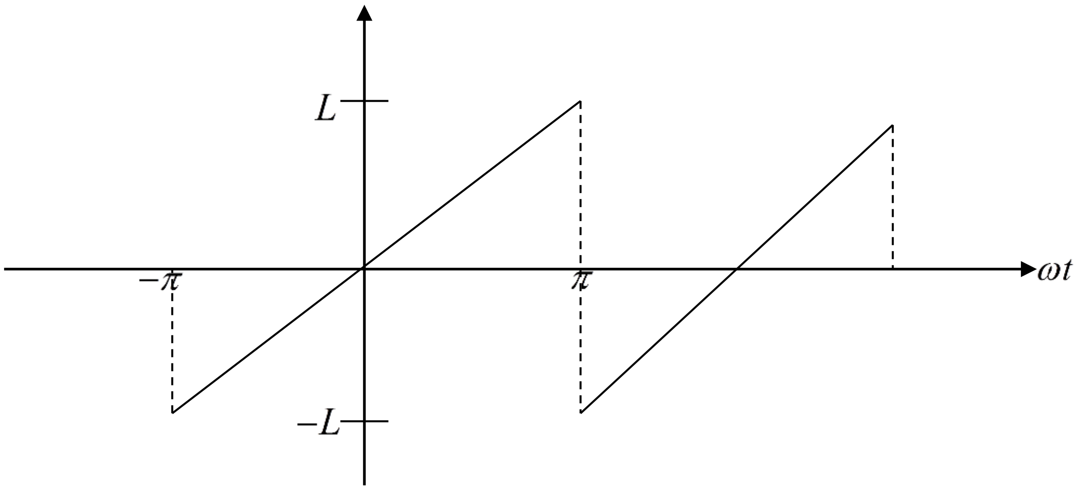
(e) Find a Fourier cosine series for $f(x) = 1$ on $(0,7)$.

Ans: 1

(f) Find a Fourier sine series for $f(x)=\begin{cases} x & x\le 1 \\ 2 & x>1 \end{cases}$ on $(0,2)$

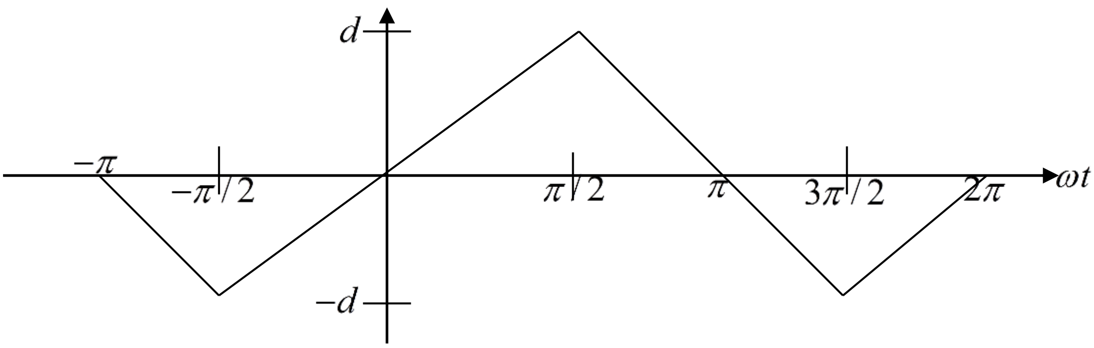
Ans: $\sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{2}{n\pi}\cos\frac{n\pi}{2}-\frac{4}{n\pi}\cos n\pi\right)\sin\frac{n\pi x}{3}$

9. Find the first four terms of the Fourier series for the saw – tooth wave form as shown below:



Ans: $y=f(\omega t)=\frac{2L}{\pi}\left[\frac{1}{1}\sin\omega t-\frac{1}{2}\sin 2\omega t+\frac{1}{3}\sin 3\omega t-\frac{1}{4}\sin 4\omega t+...\right]$

10. Determine the first four terms the Fourier series for the triangular wave form in the following figure;



Hint: $$f(x)=\begin{cases} -\dfrac{2d}{\pi}\omega t & 0<\omega t<\pi/2 \\ -\dfrac{2d}{\pi}\omega t+2d & \pi/2<\omega t<3\pi/2 \\ \dfrac{2d}{\pi}\omega t-4d & 3\pi/2<\omega t<2\pi \end{cases}$$

Ans: $$y=f(\omega t)=\frac{8d}{\pi^2}\left[\frac{1}{1^2}\sin\omega t-\frac{1}{3^2}\sin 3\omega t+\frac{1}{5^2}\sin 5\omega t-\frac{1}{7^2}\sin 7\omega t+\ldots\right]$$

UNIT EIGHT

TOTAL DIFFERENTIAL EQUATIONS

4.0 Recall that a function $f(x,y)$, its **total differentials**, df , is defined as:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \text{---------------------}(\cdot)$$

This shows that the family of curves $f(x,y)=c$ satisfies the differential equation $df = 0$.

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \text{----------------------}(\cdot\,\cdot)$$

So if there exists a function $f(x,y)$ such that:

$$M(x,y) = \frac{\partial y}{\partial x} \text{ and } N(x,y) = \frac{\partial f}{\partial y}$$

then $M(x,y)dx + N(x,y)dy$ is called an **exact differential**, and the equation:

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be an **exact equation**, whose solution is the family $f(x,y)=c$.

The general form of **Total Differential Equation** is given as:

$$P(x_1,x_2,...,x_n)dx_1 + Q(x_1,x_2,...x_n)dx_2 + ... + S(x_1,x_2,..x_n)dx_n = 0 \text{..(4.1)}$$

Examples:

1. $\left(3x^2y^2 - e^x z\right)dx + \left(2x^3y + \sin z\right)dy + \left(y\cos z - e^x\right)dz = 0$
2. $(3xz + 2y)dx + xdy + x^2dz = 0$
3. $ydx + dy + dz = 0$

It may be verified readily that **example 1** is the exact differential of:

$f(x,y,z) = x^3y^2 - e^x z + y\sin z = c$, c is an arbitrary constant. Such an equation is called ***exact.***

Example 2 is not exact, but the use of x as an integrating factor yields:

$$\left(3x^2z + 2xy\right)dx + x^2dy + x^3dz = 0$$

which is the exact differential of $x^3z + x^2y = c$. Equations 1 and 2 are called ***integrable***.

Example 3 is not integrable; that is, no primitive (or general solution).

4.1 CONDITION OF INTEGRABILITY AND EXACTNESS

The condition of integrability of the total differential equation

$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ ...(4.2)

is $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$, identically....(4.3)

And the conditions for exactness are:

$\frac{\partial P}{\partial y} = \frac{\partial Q}{Dx}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ ....(4.4)

Example 4.1

1. The equation $(3xz + 2y)dx + xdy + x^2dz = 0$

   $P = 3xz + 2y$, $\frac{\partial P}{\partial y} = 2$, $\frac{\partial P}{\partial z} = 3x$; $Q = x$, $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial Q}{\partial z} = 0$; and

   $R = x^2$, $\frac{\partial R}{\partial x} = 2x$, $\frac{\partial R}{\partial y} = 0$.Then from (4.3),we have:

   $(3xz + 2y)(0-0) + x(2x - 3x) + x^2(2-1) = 0 - x^2 + x^2 = 0$ then the equation is integrable.

2. The equation $ydx + dy + dz = 0$

   $P = y, \frac{\partial P}{\partial y} = 1, \frac{\partial P}{\partial z} = 0$; $Q = 1, \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial z} = 0$; $R = 1, \frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = 0$.

   Then from (4.3),we have:

   $y(0-0) + 1(0-0) + 1(1-0) = 1 \neq 0$, the equation is not integrable

3. The equation

$$\left(3x^2y^2 - e^x z\right)dx + \left(2x^3y + \sin z\right)dy + \left(y\cos z - e^x\right)dz = 0$$

$P = 3x^2y^2 - e^x z$, $\dfrac{\partial P}{\partial y} = 6x^2 y$, $\dfrac{\partial P}{\partial z} = -e^x$

$Q = 2x^3 y + \sin z$, $\dfrac{\partial Q}{\partial x} = 6x^2 y$, $\dfrac{\partial Q}{\partial z} = \cos z$

$R = y\cos z - e^x$, $\dfrac{\partial R}{\partial x} = -e^x$, $\dfrac{\partial R}{\partial z} = \cos z$

Hence the equation is exact.

4.2 AN APPROACH TO SOLVE AN INTEGRABLE TOTAL DIFFERENTIAL EQUATION (in three variables)

I) If the equation is exact, the solution is evident after, at most, a regrouping of terms.

II) If the equation not exact, it may be possible to find an integrating factor (note: integrating factor would be given to help you solve).

III) If the equation is homogeneous, one variable, say z, can be separated from the others by the transformation $x = uz$, $y = vz$.

IV) If no integrating factor can be found, consider one of the variables, say z, as a constant. Integrate the resulting equation, denoting the arbitrary constant of integration by $\phi(z)$, a function of z. Take the total differential of the integral just obtained and compare the coefficients of its differentials with those of the given differential equation, thus determining $\phi(z)$.

Example 4.2

1. Solve $(x - y)dx - xdy + zdz = 0$

Solution:

Here $P = x - y$, $Q = -x$, $R = z$

Then $\frac{\partial P}{\partial x} = -1 = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = 0 = \frac{\partial P}{\partial z}$, the equation is exact.

Upon regrouping thus $xdx - (xdy + ydx) + zdz = 0$

$$\int xdx - \int (xdy + ydx) + \int zdz = k$$

$$\Rightarrow \tfrac{1}{2}x^2 - xy + \tfrac{1}{2}z^2 = k$$

$$\Rightarrow x^2 - 2xy + z^2 = c$$

2. Solve $y^2dx - zdy + ydz = 0$.

Solution:

Here $P = y^2, \frac{\partial P}{\partial y} = 2y, \frac{\partial P}{\partial z} = 0$; $Q = -z, \frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial z} = -1$,

$R = y, \frac{\partial R}{\partial x} = 0, \frac{\partial R}{\partial y} = 1$:

Then

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial X}\right) =$$
$$y^2(-1-1) - z(0-0) + y(2y-0) = 0$$

and the equation is integrable. The integrable factor $\frac{1}{y^2}$ reduce the equation to $dx + \frac{ydz + -zdy}{y^2} = 0$, whose solution is:

$$x + \frac{z}{y} = c.$$

3. Solve the homogeneous equation

$$2(y+z)dx-(x+z)dy+(2y-x+z)dz=0.$$

Solution:

The equation is integrable since:

$2(y+z)(-1-2)-(x+z)(-1-2)+(2y-x+z)(2+1)=0$ The transformation $x=uz, y=vz$ reduces the given equation to:

$$2z(v+1)(udz+zdu)-z(u+1)(vdz+zdv)+z(2v-u+1)dz=0$$

Dividing by z and rearranging, we have:

$2z(v+1)du-z(u+1)dv+(uv+u+v+1)dz=0$ or, dividing by $z(uv+u+v+1)=z(u+1)(v+1)$,

$$\Rightarrow \frac{2du}{u+1}-\frac{dv}{v+1}+\frac{dz}{z}=0$$

Then $2\ln(u+1)-\ln(v+1)+\ln z=\ln k$

$$\Rightarrow z(u+1)^2=k(v+1)$$

$$\Rightarrow (x+z)^2=k(y+z)$$

$$\Rightarrow y+z=c(x+z)^2.$$

Exercise 4.1 (Solve the following)

1. $(2x^3+1)dx+x^4dy+x^2\tan zdz=0$ I.F. $1/x^2$
2. $(2x^3-z)zdx+2x^2yzdy+x(z+x)dz=0$ I.F. $1/x^2z$
3. $yzdx+z^2dy-xydz=0$
4. $(2y-z)dx+2(x-z)dy-(x+2y)dz=0$

UNIT NINE
PARTIAL DIFFERENTIAL EQUATION

5.0 NOTATION AND TERMINOLOGY

Let u denote a function of several independent variables; say, $u = u(x, y, z, t)$. At (x, y, z, t), the *partial derivative of* u *with respect to* x is defined by

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h, y, z, t) - u(x, y, z, t)}{h} \quad \text{........(5.0)}$$

provided the limits exists. We will use the following subscript notation:

$$\frac{\partial u}{\partial x} \equiv u_x \quad \frac{\partial u}{\partial y} \equiv u_y \quad \frac{\partial^2 u}{\partial x^2} \equiv u_{xx} \quad \frac{\partial^2 u}{\partial y^2} \equiv u_{yy} \quad \frac{\partial^2 u}{\partial x \partial y} \equiv u_{yx} \text{ or } \frac{\partial^2 u}{\partial y \partial x} \equiv u_{xy} \ \ldots$$

5.1 Definition

A *partial differential equation* (abbreviated *PDE*) is an equation involving one or more partial derivatives of an unknown function of several independent variables. The ***order*** of a PDE is the order of the highest derivative that appears in the equation.

The partial differential equation $F(x, y, z, t; u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, \ldots) = 0$ is said to be linear if the function F is algebraically linear in each of the variables $u, u_x, u_y, \ldots,$ and if the coefficients of u and its derivatives are functions only of the independent variables. An equation that is not linear is said to be *nonlinear*; a nonlinear equation is *quasilinear* if it is linear in the highest – order derivatives.

Example 5.1

1. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ is of order one.

2. $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ is of order two.

3. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ One – dimensional wave equation

4. $\dfrac{\partial^2 u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ One – dimensional heat equation

5. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ Two – dimensional Laplace equation

6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$ Two – dimensional Poisson equation

7. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$ Two – dimensional wave equation

8. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ Three – dimensional Laplace equation

Their solution is in the form: $u = f(x, y, z, t, ...)$

5.2 SOLUTION TO A FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

A linear partial differential equation of order one, involving a dependent variable z and two independent variables x and y, is of the form:

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \text{.......................(5.1)}$$

where P, Q, R are functions of x, y, z.

If $P = 0$, or $Q = 0$, equation (5.1) may be solved easily. Thus, the equation $\dfrac{\partial z}{\partial x} = 2x + 3y$ has as solution $z = x^2 + 3xy + \phi(y)$, where ϕ is an arbitrary function.

Lagrange reduced the problem of finding the general solution of 5.1 to that of solving an auxiliary system (called the Lagrange system) of *ordinary* differential equations:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \text{..................(5.2)}$$

by showing that:

$$\phi(u,v)=0 \quad ....(5.3) \quad (\phi,\text{arbitrary})$$

is the general solution of equation (5.1) provided $u=u(x,y,z)=a$ and $v=v(x,y,z)=b$ are two independent solutions of equation (5.2). Here, a and b are arbitrary constants and at least one of u,v must contain z.

Example 5.1

Find the general solution of

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z .$$

Solution:

The auxiliary system is:

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}.$$

From $\dfrac{dx}{x}=\dfrac{dz}{3z}$, we obtain $u=\dfrac{z}{x^3}=a$; and from $\dfrac{dx}{x}=\dfrac{dz}{y}$, we obtain $v=\dfrac{y}{x}=b$.

Thus, the general solution is $\phi\left(\dfrac{z}{x^3},\dfrac{y}{x}\right)=0$, where ϕ is arbitrary.

Of course, from $\dfrac{dy}{y}=\dfrac{dz}{3z}$, we obtain $\dfrac{z}{y^3}=c$, and we may write

$$\psi\left(\frac{z}{x^3},\frac{z}{y^3}\right)=0 \text{ or } \lambda\left(\frac{z}{y^3},\frac{y}{x}\right)=0$$

where ψ and λ are arbitrary. However, these are all equivalent and we shall call any one of them ***the*** general solution

COMPLETE SOLUTIONS

If $u = a$ and $v = b$ are two independent solutions of equations (5.2) and if α, β are arbitrary constants,

$$u = \alpha v + \beta$$

is called a *complete solution* of (5.1).

Thus, for the equation is the last example above ()

$z / x^3 = \alpha(y/x) + \beta$ is a complete solution.

Example 5.2

Find the general solution of

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 1$$

Solution:

The auxiliary system is : $\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1}$

From $\frac{dx}{2} = \frac{dz}{1}$, we have $x - 2z = a$; and from $\frac{dx}{2} = \frac{dy}{3}$, we have $3x - 2y = b$. Thus, the general solution is:

$$\phi(x - 2y, 3x - 2y) = 0.$$

The complete solution $x - 2z = \alpha(3x - 2y) + \beta$ is a two – parameter of family of planes.

5.3 TYPES OF SECOND – ORDER EQUATIONS

In the linear PDE of order two in two variables,

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \text{ ....(5.3)}$$

if u_{xx} is formally replaced by α^2, u_{xy} by $\alpha\beta$, u_{yy} by β^2, u_x by α, and u_y by β, then associated with (5.3) is a polynomial of degree two in α and β:

$$P(\alpha, \beta) \equiv a\alpha^2 + 2b\alpha\beta + c\beta^2 + d\alpha + e\beta + f$$

The mathematical properties of the solutions of (5.3) are largely determined by the algebraic properties of the polynomial $P(\alpha,\beta)$. $P(\alpha,\beta)$ - and along with it, the PDE (5.3) – is classified as *hyperbolic, parabolic, pr elliptic* according as its discriminant, $b^2 - ac$ is *positive*, *zero*, or *negative*. Note that the type of (5.3) is determined solely by its *principal part* (the terms involving the highest – order derivatives of u) and that the type will generally change with position in the $xy-$ plane unless $a, b,$ and c are constants.

Example 5.2

1. The PDE $3u_{xx} + 2u_{xy} + 5u_{yy} + xu_y = 0$ is elliptic, since

$$b^2 - ac = 1^2 - 3(5) = -14 < 0$$

2. The *Tricomi equation* for transonic flow, $u_{xx} + yu_{yy} = 0$, has

$$b^2 - ac = 0^2 - (1)y = -y$$

Thus, the equation is elliptic for $y > 0$, parabolic for $y = 0$, and hyperbolic for $y < 0$.

The general linear PDE of order two in n variables has the form:

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = d \text{ ....(5.4)}$$

If $u_{x_i x_j} = u_{x_j x_i}$, then the principal part of (5.4) can always be arranged so that $a_{ij} = a_{ji}$; thus, the $n \times n$ matrix $\mathbf{A} = \left[a_{ij}\right]$ can be assumed symmetric. In linear algebra it is shown that every real, symmetric $n \times n$ matrix has n real eigenvalues. These eigenvalues are the (possibly repeated) zeros of an nth – degree polynomial in λ, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where $\mathbf{I}$ is the $n \times n$ identity matrix. Let P denote the number of positive eigenvalues, and Z the number of zero eigenvalues (i.e., the multiplicity of the eigenvalues zero), of the matrix $\mathbf{A}$. Then (5.4) is:

hyperbolic if $Z = 0$ and $P = 1$ or $Z = 0$ and $P = n - 1$

parabolic if $Z > 0$ (equivalently, if $\det \mathbf{A} = 0$)

elliptic if $Z = 0$ and $P = n$ or $Z = 0$ and $P = 0$

ultrahyperbolic if $Z = 0$ and $1 < P < n-1$

if any of the a_{ij} is nonconstant, the type of (5.4) can vary with position.

Example 5.3

For the PDE $3u_{x_1x_1} + u_{x_2x_2} + 4u_{x_2x_3} + 4u_{x_3x_3} = 0$,

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \text{ and } \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{bmatrix} = (3-\lambda)(\lambda)(\lambda-5) = 0$$

Because $\lambda = 0$ is an eigenvalues, the PDE is parabolic (throughout $x_1x_2x_3-$space).

Exercise 5.1

1. Classify according to type:

   a) $u_{xx} + 2yu_{xy} + xu_{yy} - u_x + u = 0$

   b) $2xyu_{xy} + xu_y + yu_x = 0$

   c) $u_{xx} + u_{xy} + 5u_{yx} + u_{yy} + 2u_{yz} + u_{zz} = 0$

2. Describe the regions where the equation is hyperbolic, parabolic, elliptic:

   a) $u_{xx} - u_{xy} - 2u_{yy} = 0$

   b) $u_{xx} + 2u_{xy} + u_{yy} = 0$

   c) $2u_{xx} + 4u_{xy} + 3u_{yy} - 5u = 0$

   d) $u_{xx} + 2xu_{xy} + u_{yy} + (\cos xy)u_x = u$

   e) $yu_{xx} - 2u_{xy} + e^x u_{yy} + u = 3$

   f) $e^{xy}u_{xx} + (\sinh x)u_{yy} + u = 0$

   g) $xu_{xx} + 2xyu_{xy} - yu_{yy} = 0$

   h) $xu_{xx} + 2xyu_{xy} + yu_{yy} = 0$

SOLUTIONS TO SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION.

Example 5.3

Solve $\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that $x = 0$, $u = \cos 2t$ and $\frac{\partial u}{\partial x} = \sin t$.

Solution

$$\frac{\partial u}{\partial x} = \int 12x^2(t+1)\,dx = 4x^3(t+1) + \phi(t)$$

$$u = x^4(t+1) + x\phi(t) + \theta(t)$$

Using initial conditions $x = 0; \frac{\partial u}{\partial x} = \sin t$ at $u = \cos 2t$

$$\Rightarrow \phi(t) = \sin t \qquad \theta(t) = \cos 2t$$

hence $u = x^4(t+1) + x\sin t + \cos 2t$.

Exercise

Solve the equation $\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y)$, given that

$$y = 0, \frac{\partial u}{\partial x} = 1; x = 0, u = (y-1)^2$$

Initial Conditions and Boundary Conditions

As with any differential equation, the arbitrary constant or arbitrary functions in any particular case are determine from initial conditions or, more generally, the boundary conditions since they do not always refer to zero values of independent variables.

5.4 SEPARATION OF VARIABLES

Solving second – order PDE with two independent variables we assume a trial solution of the form $u(x,t)=\mathrm{X}(x)\mathrm{T}(t)$ (or simply $u=\mathrm{XT}$)where;

$\mathrm{X}(x)$ is a function of x only.

$\mathrm{T}(t)$ is a function of t only.

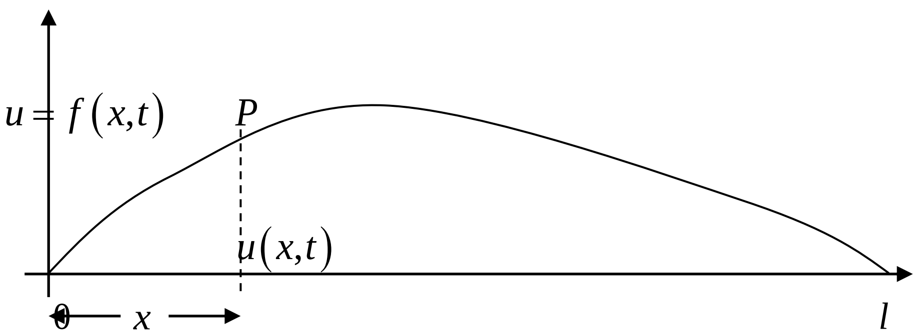
Therefore $\dfrac{\partial u}{\partial x}=\mathrm{X}'\mathrm{T}\Rightarrow\dfrac{\partial^2 u}{\partial x^2}=\mathrm{X}''\mathrm{T};\qquad \dfrac{\partial u}{\partial t}=\mathrm{XT}'\Rightarrow\dfrac{\partial^2 u}{\partial t^2}=\mathrm{XT}''$

Hence; the Wave equation $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$ can be written as $\dfrac{\mathrm{X}''}{\mathrm{X}}=\dfrac{1}{c^2}\dfrac{\mathrm{T}''}{\mathrm{T}}$.

the Heat equation $\dfrac{\partial^2 u}{\partial t}=c^2\dfrac{\partial^2 u}{\partial x^2}$ can be written as $\dfrac{\mathrm{X}''}{\mathrm{X}}=\dfrac{1}{c^2}\dfrac{\mathrm{T}'}{\mathrm{T}}$.

the Laplace equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ can be written as $\dfrac{\mathrm{X}''}{\mathrm{X}}=-\dfrac{\mathrm{Y}''}{\mathrm{Y}}$.

5.5 SOLUTION OF THE WAVE EQUATION



The wave equation $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ has a solution $u(x,t)$.

Boundary conditions:

a) The string is fixed at both ends, i.e. at $x=0$ and $x=l$ for all values of time t. Therefore $u(x,t)$ becomes

$$\left.\begin{aligned}u(0,t)&=0\\u(l,t)&=0\end{aligned}\right\}\text{ for all values of } t\geq 0$$

Initial conditions:

b) If the initial deflection of P at $t=0$ is denoted by $f(x)$, then $u(x,0)=f(x)$.

c) Let the initial velocity of P be $g(x)$, then $\left[\dfrac{\partial u}{\partial t}\right]_{t=0}=g(x)$

The wave equation $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\cdot\dfrac{\partial^2 u}{\partial t^2}$ becomes $\dfrac{\mathrm{X}''}{\mathrm{X}}=\dfrac{1}{c^2}\dfrac{\mathrm{T}''}{\mathrm{T}}$.

Let $\dfrac{\mathrm{X}''}{\mathrm{X}}=k$ also $\dfrac{1}{c^2}\dfrac{\mathrm{T}''}{\mathrm{T}}=k$ $\quad\therefore \mathrm{X}''-k\mathrm{X}=0$ and $\mathrm{T}''-c^2k\mathrm{T}=0$

For oscillatory motion let $k=-p^2$ i.e. $k<0$.

$\therefore \mathrm{X}''+p^2\mathrm{X}=0 \qquad \Rightarrow \mathrm{X}=A\cos px+B\sin px$

and also $\mathrm{T}''+c^2p^2\mathrm{T}=0 \qquad \Rightarrow \mathrm{T}=C\cos cpt+D\sin cpt$

so our suggested solution $u=\mathrm{XT}$ now becomes

$u(x,t)=(A\cos px+B\sin px)(C\cos cpt+D\sin cpt)$

and, if we put $cp=\lambda \ \therefore p=\lambda/c$

$\Rightarrow u(x,t)=\left(A\cos\dfrac{\lambda}{c}x+B\sin\dfrac{\lambda}{c}x\right)(C\cos\lambda t+D\sin\lambda t)$

where A,B,C,D are arbitrary constants.

Now, using the boundary conditions

$u=0$ and $x=0 \ \Rightarrow 0=A(C\cos\lambda t+D\sin\lambda t)\forall t \quad \therefore A=0$

Therefore, $u(x,t)=B\sin\dfrac{\lambda}{c}x(C\cos\lambda t+D\sin\lambda t)$

$u=0 \qquad x=l\,\forall x \quad \Rightarrow 0=B\sin\dfrac{\lambda l}{c}(C\cos\lambda t+D\sin\lambda t)$

Now, $B\neq 0$ or $u(x,t)$ would be identically zero.

$$\therefore \sin\frac{\lambda l}{c}=0 \qquad \Rightarrow \frac{\lambda l}{c}=n\pi \qquad \Rightarrow \lambda=\frac{nc\pi}{l} \text{ for } n=1,2,3,...$$

As we can see, there is an infinite set of values of λ and each separate value gives a particular solution for $u(x,t)$. The values of λ are called the *eigenvalues* and each corresponding solution the *eigenfunction*.

	Eigenvalues	**Eigenfunctions**
n	$\lambda=\frac{nc\pi}{l}$	$u(x,t)=B\sin\frac{\lambda x}{c}\{C\cos\lambda t+D\sin\lambda t\}$
1	$\lambda_1=\frac{c\pi}{l}$	$u_1=\sin\frac{\pi x}{l}\left\{C_1\cos\frac{c\pi t}{l}+D_1\sin\frac{c\pi t}{l}\right\}$
2	$\lambda_2=\frac{2c\pi}{l}$	$u_2=\sin\frac{2\pi x}{l}\left\{C_2\cos\frac{2c\pi t}{l}+D_2\sin\frac{2c\pi t}{l}\right\}$
3	$\lambda_3=\frac{3c\pi}{l}$	$u_3=\sin\frac{3\pi x}{l}\left\{C_3\cos\frac{3c\pi t}{l}+D_3\sin\frac{3c\pi t}{l}\right\}$
$\vdots$		$\vdots$
r	$\lambda_r=\frac{rc\pi}{l}$	$u_r=\sin\frac{r\pi x}{l}\left\{C_r\cos\frac{rc\pi t}{l}+D_r\sin\frac{rc\pi t}{l}\right\}$

Note $BC=C_n$ and $BD=D_n$, where $C_1,C_2,C_3,....$ and $D_1,D_2,D_3,....$ are arbitrary constants.

If $u_1,u_2,u_3,....$ are solutions, then the linear combination is also a solution, which is a more general solution, i.e. $u=u_1+u_2+u_3+\cdots$

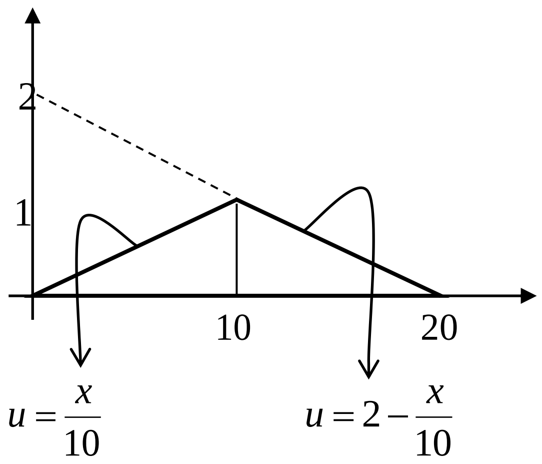
So $u(x,t)=\sum_{r=1}^{\infty}u_r=\sum_{r=1}^{\infty}\left[\sin\frac{r\pi x}{l}\left(C_r\cos\frac{rc\pi t}{l}+D_r\sin\frac{rc\pi t}{l}\right)\right]$

To solve for the arbitraries C_r, D_r we use the initial conditions which we have not yet taken into account.

Example5.2

A stretched string of length $20cm$ and is set oscillating by displacing its midpoint a distance $1cm$ from its rest position and releasing it with zero initial velocity. Solve the wave equation $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t}$ where $c^2=1$ to determine the resulting motion, $u(x,t)$.

Solution:



From data given the boundary conditions are

$u(0,t)=0; \quad u(20,t)=0$ fixed endpoints.

$$u(x,0)=f(x)=\begin{cases}\frac{x}{10} & 0\le x\le 10\\ \frac{20-x}{10} & 10\le x\le 20\end{cases} \qquad \left[\frac{\partial u}{\partial t}\right]_{t=0}$$ (zero initial velocity)

Since $c=1$, then we have $X''T=T''X$

Which implies $X''+p^2X=0$ and $\quad T''+p^2T=0$

The respective solutions are

$$X=A\cos px+B\sin px \text{ and } T=C\cos pt+D\sin pt$$

$$\therefore u(x,t)=XT=(A\cos px+B\sin px)(C\cos pt+D\sin pt)$$

$\Rightarrow u(x,t)=(A\cos\lambda x+B\sin\lambda x)(C\cos\lambda t+D\sin\lambda t)$ where $\lambda=cp$ but $c=1$

$u(0,t)=0, x=0 \;\therefore 0=A(C\cos\lambda t+D\sin\lambda t)\Rightarrow A=0$

Therefore, $(x,t)=B\sin\lambda x(C\cos\lambda t+D\sin\lambda t)$

$u(20,t)=0, x=20 \;\therefore 0=B\sin 20\lambda(C\cos\lambda t+D\sin\lambda t)$

$B\neq 0$ or u would be identically zero $\therefore \sin 20\lambda=0 \quad \therefore 20\lambda=n\pi \qquad \Rightarrow \lambda=\dfrac{n\pi}{20}$

Hence, $u(x,t)=B\sin\dfrac{n\pi}{20}x\left(C\cos\dfrac{n\pi}{20}t+D\sin\dfrac{n\pi}{20}t\right)$

Follow the table below:

	Eigenvalues	**Eigenfunctions**
n	$\lambda=\dfrac{n\pi}{20}$	$u(x,t)=B\sin\lambda x\{C\cos\lambda t+D\sin\lambda t\}$
1	$\lambda_1=\dfrac{\pi}{20}$	$u_1=\sin\dfrac{\pi x}{20}\left\{C_1\cos\dfrac{\pi t}{20}+D_1\sin\dfrac{\pi t}{20}\right\}$
2	$\lambda_2=\dfrac{2\pi}{20}$	$u_2=\sin\dfrac{2\pi x}{20}\left\{C_2\cos\dfrac{2\pi t}{20}+D_2\sin\dfrac{2\pi t}{20}\right\}$
3	$\lambda_3=\dfrac{3\pi}{20}$	$u_3=\sin\dfrac{3\pi x}{20}\left\{C_3\cos\dfrac{3\pi t}{20}+D_3\sin\dfrac{3\pi t}{20}\right\}$
$\vdots$		$\vdots$
r	$\lambda_r=\dfrac{r\pi}{20}$	$u_r=\sin\dfrac{r\pi x}{20}\left\{C_r\cos\dfrac{r\pi t}{20}+D_r\sin\dfrac{r\pi t}{20}\right\}$

Note $BC=C_n$ and $BD=D_n$, where $C_1,C_2,C_3,....$ and $D_1,D_2,D_3,....$ are arbitrary constants.

$$u=u_1+u_2+u_3+\cdots$$

$$\therefore u(x,t)=\sum_{r=1}^{\infty}u_r=\sum_{r=1}^{\infty}\left[\sin\frac{r\pi x}{20}\left(C_r\cos\frac{r\pi t}{20}+D_r\sin\frac{r\pi t}{20}\right)\right]$$

To find C_r and D_r we use the rest of the conditions i.e.;

$$u(x,0) = f(x) = \begin{cases} \dfrac{x}{10} & 0 \le x \le 10 \\ \dfrac{20-x}{10} & 10 \le x \le 20 \end{cases}$$

$$u(x,0) = \sum_{r=1}^{\infty} C_r \sin\frac{r\pi x}{20}$$

Then $Cr = 2 \times$ mean value of $f(x)\sin\dfrac{r\pi x}{20}$ between $x = 0$ and $x = 20$

$$C_r = \frac{2}{20}\int_0^{20} f(x)\sin\frac{r\pi x}{20}dx$$

$$\therefore 10C_r = \int_0^{10}\frac{x}{10}\sin\frac{r\pi x}{20}dx + \int_{10}^{20}\frac{20-x}{10}\sin\frac{r\pi x}{20}dx$$

Then $10C_r = -\dfrac{20}{r\pi}\cos\dfrac{r\pi}{2} + \dfrac{40}{r^2\pi^2}\sin\dfrac{r\pi}{2} + \dfrac{20}{r\pi}\cos\dfrac{r\pi}{2} - \dfrac{40}{r^2\pi^2}\sin r\pi$

$\therefore$ For $\mathrm{r}=1,2,3,\ldots\ C_r = \dfrac{4}{r^2\pi^2}\sin\dfrac{r\pi}{2}$

$$\therefore u(x,t) = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left(\frac{4}{r^2\pi^2}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20} + D_r\sin\frac{r\pi t}{20}\right)$$

and for the condition $\left[\dfrac{\partial u}{\partial t}\right]_{t=0}$ (zero initial velocity), which is $t = 0; \dfrac{\partial u}{\partial t} = 0$

$$\therefore \frac{\partial u(x,t)}{\partial t} = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left[\left(\frac{4}{r^2\pi^2}\sin\frac{r\pi}{2}\right)\left(-\frac{r\pi}{20}\sin\frac{r\pi t}{20}\right) + D_r\frac{r\pi}{20}\cos\frac{r\pi t}{20}\right]$$

Therefore, at $t = 0$, $0 = \displaystyle\sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}D_r\frac{r\pi}{20} \qquad \therefore D_r = 0$

So finally we have $u(x,t) = \dfrac{4}{\pi^2}\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{r\pi x}{20}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20}$

5.5 HEAT AND LAPLACE EQUATION

The Heat and Laplace equations look slightly different from the wave equation, but the method of solution is very much along the same lines.

1. For Heat equations;

$$c^2 u_{xx} = u_t \qquad \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}$$

$\Rightarrow X'' + p^2 X = 0$ given $X = A\cos px + B\sin px$

$\Rightarrow T' + p^2c^2 T = 0$ given $T = Ce^{-p^2c^2t}$

$$\therefore u(x,t) = XT = (A\cos px + B\sin px)Ce^{-p^2c^2t}$$

$$pc = \lambda \qquad u(x,t) = \left(AC\cos\frac{\lambda}{c}x + BC\sin\frac{\lambda}{c}x\right)e^{-\lambda^2 t}$$

2. For Laplace's equation:

$$u_{xx} + u_{yy} = 0 \qquad \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

$\Rightarrow X'' + p^2 X = 0$ given $X = A\cos px + B\sin px$

$\Rightarrow Y'' - p^2 Y = 0$ given $Y = C\cosh py + D\sinh py$, which we can express as: $E\sinh p(y+\phi)$

$$\therefore u(x,y) = (A\cos px + B\sin px)E\sinh p(y+\phi)$$

Exercise 5.2

1. A bar of length $2m$ is fully insulated along its sides. It is initially at a uniform temperature of $10°C$ and at $t = 0$ the ends are plunged into ice and maintained at a temperature of $0°C$. Determine an expression for the temperature at a point P a distance x from one end at any subsequent time t seconds after $t = 0$.

2. Determine a solution $u(x,y)$ of the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial^2 y}=0$

subject to the following boundary conditions:

$u=0$ when $x=0$; $u=0$ when $x=\pi$;

$u\to 0$ when $y\to\infty$; $u=3$ when $y=0$.

SOLUTIONS TO SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION.

Example 5.3

Solve $\frac{\partial^2 u}{\partial x^2}=12x^2(t+1)$ given that $x=0$, $u=\cos 2t$ and $\frac{\partial u}{\partial x}=\sin t$.

Solution

$$\frac{\partial u}{\partial x}=\int 12x^2(t+1)dx=4x^3(t+1)+\phi(t)$$

$$u=x^4(t+1)+x\phi(t)+\theta(t)$$

Using initial conditions $x=0;\frac{\partial u}{\partial x}=\sin t$ at $u=\cos 2t$

$$\Rightarrow \phi(t) = \sin t \qquad \theta(t) = \cos 2t$$

hence $u = x^4(t+1) + x\sin t + \cos 2t$.

Exercise

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial y} = \sin(x+y)$, given that

$$y = 0, \frac{\partial u}{\partial x} = 1; x = 0, u = (y-1)^2$$

Initial Conditions and Boundary Conditions

As with any differential equation, the arbitrary constant or arbitrary functions in any particular case are determine from initial conditions or, more generally, the boundary conditions since they do not always refer to zero values of independent variables.

5.4 SEPARATION OF VARIABLES

Solving second – order PDE with two independent variables we assume a trial solution of the form $u(x,t) = \mathrm{X}(x)\mathrm{T}(t)$ (or simply $u = \mathrm{XT}$)where;

$\mathrm{X}(x)$ is a function of x only.

$\mathrm{T}(t)$ is a function of t only.

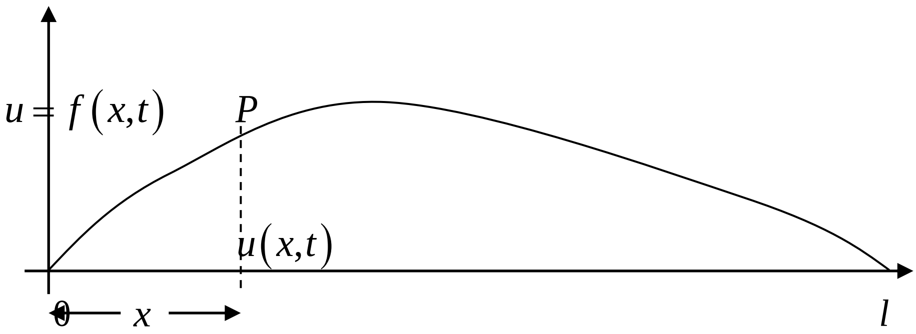
Therefore $\dfrac{\partial u}{\partial x} = \mathrm{X'T} \Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \mathrm{X''T}$; $\qquad \dfrac{\partial u}{\partial t} = \mathrm{XT'} \Rightarrow \dfrac{\partial^2 u}{\partial t^2} = \mathrm{XT''}$

Hence; the Wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ can be written as $\dfrac{\mathrm{X''}}{\mathrm{X}} = \dfrac{1}{c^2}\dfrac{\mathrm{T''}}{\mathrm{T}}$.

the Heat equation $\dfrac{\partial^2 u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ can be written as $\dfrac{\mathrm{X''}}{\mathrm{X}} = \dfrac{1}{c^2}\dfrac{\mathrm{T'}}{\mathrm{T}}$.

the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ can be written as $\dfrac{\mathrm{X''}}{\mathrm{X}} = -\dfrac{\mathrm{Y''}}{\mathrm{Y}}$.

5.5 SOLUTION OF THE WAVE EQUATION



The wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ has a solution $u(x,t)$.

Boundary conditions:

d) The string is fixed at both ends, i.e. at $x = 0$ and $x = l$ for all values of time t. Therefore $u(x,t)$ becomes

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(l,t) &= 0 \end{aligned}\right\} \text{ for all values of } t \geq 0$$

Initial conditions:

e) If the initial deflection of P at $t = 0$ is denoted by $f(x)$, then $u(x,0) = f(x)$.

f) Let the initial velocity of P be $g(x)$, then $\left[\dfrac{\partial u}{\partial t}\right]_{t=0} = g(x)$

The wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ becomes $\dfrac{\mathrm{X}''}{\mathrm{X}} = \dfrac{1}{c^2}\dfrac{\mathrm{T}''}{\mathrm{T}}$.

Let $\dfrac{\mathrm{X}''}{\mathrm{X}} = k$ also $\dfrac{1}{c^2}\dfrac{\mathrm{T}''}{\mathrm{T}} = k$ $\quad \therefore \mathrm{X}'' - k\mathrm{X} = 0$ and $\mathrm{T}'' - c^2 k\mathrm{T} = 0$

For oscillatory motion let $k = -p^2$ i.e. $k < 0$.

$\therefore \mathrm{X}'' + p^2\mathrm{X} = 0 \qquad \Rightarrow \mathrm{X} = A\cos px + B\sin px$

and also $\mathrm{T}'' + c^2 p^2 \mathrm{T} = 0 \qquad \Rightarrow \mathrm{T} = C\cos cpt + D\sin cpt$

so our suggested solution $u = \mathrm{XT}$ now becomes

$$u(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$$

and, if we put $cp = \lambda \;\therefore\; p = \lambda / c$

$$\Rightarrow u(x,t) = \left(A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right)(C\cos\lambda t + D\sin\lambda t)$$

where A, B, C, D are arbitrary constants.

Now, using the boundary conditions

$u = 0$ and $x = 0 \;\Rightarrow 0 = A(C\cos\lambda t + D\sin\lambda t)\forall t \quad \therefore A = 0$

Therefore, $u(x,t) = B\sin\frac{\lambda}{c}x(C\cos\lambda t + D\sin\lambda t)$

$u = 0 \qquad x = l\,\forall x \quad \Rightarrow 0 = B\sin\frac{\lambda l}{c}(C\cos\lambda t + D\sin\lambda t)$

Now, $B \neq 0$ or $u(x,t)$ would be identically zero.

$\therefore \sin\frac{\lambda l}{c} = 0 \qquad \Rightarrow \frac{\lambda l}{c} = n\pi \qquad \Rightarrow \lambda = \frac{nc\pi}{l}$ for $n = 1, 2, 3, \ldots$

As we can see, there is an infinite set of values of λ and each separate value gives a particular solution for $u(x,t)$. The values of λ are called the *eigenvalues* and each corresponding solution the *eigenfunction*.

	Eigenvalues	**Eigenfunctions**
n	$\lambda = \frac{nc\pi}{l}$	$u(x,t) = B\sin\frac{\lambda x}{c}\{C\cos\lambda t + D\sin\lambda t\}$
1	$\lambda_1 = \frac{c\pi}{l}$	$u_1 = \sin\frac{\pi x}{l}\left\{C_1\cos\frac{c\pi t}{l} + D_1\sin\frac{c\pi t}{l}\right\}$
2	$\lambda_2 = \frac{2c\pi}{l}$	$u_2 = \sin\frac{2\pi x}{l}\left\{C_2\cos\frac{2c\pi t}{l} + D_2\sin\frac{2c\pi t}{l}\right\}$
3	$\lambda_3 = \frac{3c\pi}{l}$	$u_3 = \sin\frac{3\pi x}{l}\left\{C_3\cos\frac{3c\pi t}{l} + D_3\sin\frac{3c\pi t}{l}\right\}$

$\vdots$		$\vdots$
r	$\lambda_r = \frac{rc\pi}{l}$	$u_r = \sin\frac{r\pi x}{l}\left\{C_r \cos\frac{rc\pi t}{l} + D_r \sin\frac{rc\pi t}{l}\right\}$

Note $BC = C_n$ and $BD = D_n$, where $C_1, C_2, C_3, \ldots.$ and $D_1, D_2, D_3, \ldots.$ are arbitrary constants.

If $u_1, u_2, u_3, \ldots.$ are solutions, then the linear combination is also a solution, which is a more general solution, i.e. $u = u_1 + u_2 + u_3 + \cdots$

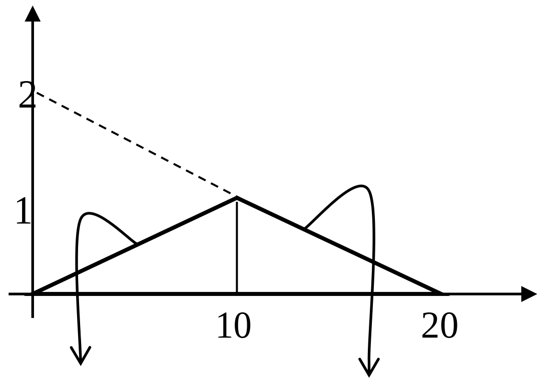
So $u(x,t) = \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty}\left[\sin\frac{r\pi x}{l}\left(C_r \cos\frac{rc\pi t}{l} + D_r \sin\frac{rc\pi t}{l}\right)\right]$

To solve for the arbitraries C_r, D_r we use the initial conditions which we have not yet taken into account.

Example5.2

A stretched string of length $20cm$ and is set oscillating by displacing its midpoint a distance $1cm$ from its rest position and releasing it with zero initial velocity. Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t}$ where $c^2 = 1$ to determine the resulting motion, $u(x,t)$.

Solution:



From data given the boundary conditions are

$u(0,t)=0;\quad u(20,t)=0$ fixed endpoints.

$$u(x,0)=f(x)=\begin{cases}\dfrac{x}{10} & 0\le x\le 10\\ \dfrac{20-x}{10} & 10\le x\le 20\end{cases}\qquad \left[\frac{\partial u}{\partial t}\right]_{t=0}$$ (zero initial velocity)

Since $c=1$, then we have $\mathrm{X}''\mathrm{T}=\mathrm{T}''\mathrm{X}$

Which implies $\mathrm{X}''+p^2\mathrm{X}=0$ and $\quad \mathrm{T}''+p^2\mathrm{T}=0$

The respective solutions are

$\mathrm{X}=A\cos px+B\sin px$ and $\mathrm{T}=C\cos pt+D\sin pt$

$\therefore u(x,t)=\mathrm{XT}=(A\cos px+B\sin px)(C\cos pt+D\sin pt)$

$\Rightarrow u(x,t)=(A\cos\lambda x+B\sin\lambda x)(C\cos\lambda t+D\sin\lambda t)$ where $\lambda=cp$ but $c=1$

$u(0,t)=0, x=0\ \therefore 0=A(C\cos\lambda t+D\sin\lambda t)\Rightarrow A=0$

Therefore, $(x,t)=B\sin\lambda x(C\cos\lambda t+D\sin\lambda t)$

$u(20,t)=0, x=20\ \therefore 0=B\sin 20\lambda(C\cos\lambda t+D\sin\lambda t)$

$B\neq 0$ or u would be identically zero $\therefore \sin 20\lambda=0\quad \therefore 20\lambda=n\pi\qquad \Rightarrow\lambda=\frac{n\pi}{20}$

Hence, $u(x,t)=B\sin\frac{n\pi}{20}x\left(C\cos\frac{n\pi}{20}t+D\sin\frac{n\pi}{20}t\right)$

Follow the table below:

	Eigenvalues	Eigenfunctions
n	$\lambda=\frac{n\pi}{20}$	$u(x,t)=B\sin\lambda x\{C\cos\lambda t+D\sin\lambda t\}$
1	$\lambda_1=\frac{\pi}{20}$	$u_1=\sin\frac{\pi x}{20}\left\{C_1\cos\frac{\pi t}{20}+D_1\sin\frac{\pi t}{20}\right\}$
2	$\lambda_2=\frac{2\pi}{20}$	$u_2=\sin\frac{2\pi x}{20}\left\{C_2\cos\frac{2\pi t}{20}+D_2\sin\frac{2\pi t}{20}\right\}$
3	$\lambda_3=\frac{3\pi}{20}$	$u_3=\sin\frac{3\pi x}{20}\left\{C_3\cos\frac{3\pi t}{20}+D_3\sin\frac{3\pi t}{20}\right\}$
$\vdots$		$\vdots$
r	$\lambda_r=\frac{r\pi}{20}$	$u_r=\sin\frac{r\pi x}{20}\left\{C_r\cos\frac{r\pi t}{20}+D_r\sin\frac{r\pi t}{20}\right\}$

Note $BC=C_n$ and $BD=D_n$, where $C_1,C_2,C_3,\ldots.$ and $D_1,D_2,D_3,\ldots.$ are arbitrary constants.

$$u=u_1+u_2+u_3+\cdots$$

$$\therefore u(x,t)=\sum_{r=1}^{\infty}u_r=\sum_{r=1}^{\infty}\left[\sin\frac{r\pi x}{20}\left(C_r\cos\frac{r\pi t}{20}+D_r\sin\frac{r\pi t}{20}\right)\right]$$

To find C_r and D_r we use the rest of the conditions i.e.;

$$u(x,0) = f(x) = \begin{cases} \dfrac{x}{10} & 0 \le x \le 10 \\ \dfrac{20-x}{10} & 10 \le x \le 20 \end{cases}$$

$$u(x,0) = \sum_{r=1}^{\infty} C_r \sin\frac{r\pi x}{20}$$

Then $Cr = 2 \times$ mean value of $f(x)\sin\dfrac{r\pi x}{20}$ between $x=0$ and $x=20$

$$C_r = \frac{2}{20}\int_0^{20} f(x)\sin\frac{r\pi x}{20}dx$$

$$\therefore 10C_r = \int_0^{10}\frac{x}{10}\sin\frac{r\pi x}{20}dx + \int_{10}^{20}\frac{20-x}{10}\sin\frac{r\pi x}{20}dx$$

Then $10C_r = -\dfrac{20}{r\pi}\cos\dfrac{r\pi}{2} + \dfrac{40}{r^2\pi^2}\sin\dfrac{r\pi}{2} + \dfrac{20}{r\pi}\cos\dfrac{r\pi}{2} - \dfrac{40}{r^2\pi^2}\sin r\pi$

$\therefore$ For r=1,2,3,... $C_r = \dfrac{4}{r^2\pi^2}\sin\dfrac{r\pi}{2}$

$$\therefore u(x,t) = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left(\frac{4}{r^2\pi^2}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20} + D_r\sin\frac{r\pi t}{20}\right)$$

and for the condition $\left[\dfrac{\partial u}{\partial t}\right]_{t=0}$ (zero initial velocity), which is $t=0; \dfrac{\partial u}{\partial t} = 0$

$$\therefore \frac{\partial u(x,t)}{\partial t} = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left[\left(\frac{4}{r^2\pi^2}\sin\frac{r\pi}{2}\right)\left(-\frac{r\pi}{20}\sin\frac{r\pi t}{20}\right) + D_r\frac{r\pi}{20}\cos\frac{r\pi t}{20}\right]$$

Therefore, at $t=0$, $0 = \sum_{r=1}^{\infty}\sin\dfrac{r\pi x}{20}D_r\dfrac{r\pi}{20}$ $\quad\therefore D_r = 0$

So finally we have $u(x,t) = \dfrac{4}{\pi^2}\sum_{r=1}^{\infty}\dfrac{1}{r^2}\sin\dfrac{r\pi x}{20}\sin\dfrac{r\pi}{2}\cos\dfrac{r\pi t}{20}$

5.5 HEAT AND LAPLACE EQUATION

The Heat and Laplace equations look slightly different from the wave equation, but the method of solution is very much along the same lines.

3. For Heat equations;

$$c^2 u_{xx} = u_t \qquad \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}$$

$\Rightarrow X'' + p^2 X = 0$ given $X = A\cos px + B\sin px$

$\Rightarrow T' + p^2 c^2 T = 0$ given $T = Ce^{-p^2c^2t}$

$$\therefore u(x,t) = XT = (A\cos px + B\sin px)Ce^{-p^2c^2t}$$

$$pc = \lambda \quad u(x,t) = \left(AC\cos\frac{\lambda}{c}x + BC\sin\frac{\lambda}{c}x \right)e^{-\lambda^2 t}$$

4. For Laplace's equation:

$$u_{xx} + u_{yy} = 0 \qquad \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

$\Rightarrow X'' + p^2 X = 0$ given $X = A\cos px + B\sin px$

$\Rightarrow Y'' - p^2 Y = 0$ given $Y = C\cosh py + D\sinh py$, which we can express as: $E\sinh p(y+\phi)$

$$\therefore u(x,y) = (A\cos px + B\sin px)E\sinh p(y+\phi)$$

<u>Exercise 5.2</u>

3. A bar of length $2m$ is fully insulated along its sides. It is initially at a uniform temperature of $10°C$ and at $t = 0$ the ends are plunged into ice and maintained at a temperature of $0°C$. Determine an expression for the temperature at a point P a distance x from one end at any subsequent time t seconds after $t = 0$.

4. Determine a solution $u(x,y)$ of the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial^2 y}=0$ subject to the following boundary conditions:

$u=0$ when $x=0$; $u=0$ when $x=\pi$;

$u\to 0$ when $y\to\infty$; $u=3$ when $y=0$.