18.06 Quiz 2 Solution November 10, 1999 Closed Book

1 (25 pts.) (a) Find equations (**do not solve**) for the coefficients C, D, E in $b = C + Dt + Et^2$, the parabola which best fits the four points $(t, b) = (0,0), (1,1), (1,3)$ and $(2,2)$.

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ has no solution. We need to look for its least square solution and solve the system

$$A^T A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = A^T \mathbf{b}, \qquad \text{i.e.,} \qquad \begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

(b) In solving this problem you are projecting the vector $\underline{\mathbf{b} = (0,1,3,2)}$ onto the subspace spanned by <u>the column vectors of A</u>. The projection in terms of C, D, E is

$$P = A\hat{x} = A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} C \\ C + D + E \\ C + D + E \\ C + 2D + 4E \end{bmatrix}$$

2 (28 pts.) Let

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

(a) Find the eigenvalues of the singular matrix A.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 & 6 \\ 0 & 1-\lambda & 0 \\ -1 & -2 & -2-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)\lambda,$$

so the eigenvalues of A are $0, 1, 1$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{has special solution} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$(A - I)\mathbf{x} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{has special solutions} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So one such basis is

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

(c) By expressing $(1, 1, 1)$ as a combination of eigenvectors or by diagonalizing $A = S\Lambda S^{-1}$, compute

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

First method:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A^{99}(6v_1) + A^{99}(v_2) + A^{99}(-5v_3) = 0 + v_2 - 5v_3 = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$

Second method:

$$A = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1},$$

$$A^{99} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{99} S^{-1} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1} = A.$$

So

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$

3 (25 pts.) Start with two vectors (the columns of A):

$$a_1 = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} \qquad \text{and} \qquad a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) With $q_1 = a_1$ find an orthonormal basis q_1, q_2 for the space spanned by a_1 and a_2 (column space of A).

$$b_2 = a_2 - a_2 \cdot q_1\, q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \cos\theta \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 1-\cos^2\theta \\ 0 \\ -\cos\theta\sin\theta \end{bmatrix},$$

$$q_2 = \frac{b_2}{|b_2|} = \begin{bmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{bmatrix}.$$

(b) What shape is the matrix R in $A = QR$ and why is $R = Q^T A$? Here Q has columns q_1 and q_2. Compute the matrix R.

R is a 2×2 upper triangular matrix.

$$A = QR \quad \Rightarrow \quad Q^T A = Q^T QR \quad \Rightarrow \quad Q^T A = IR = R$$

$$R = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}$$

(c) Find the projection matrices P_A and P_Q onto the column spaces of A and Q.

$$\text{Since } C(A) = C(Q), \quad P_A = P_Q = QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If you notice that the second entry of both a_1 and a_2 are zero, then you know you are looking for the projection matrix onto the xz-plane. You can obtain the answer without doing any matrix multiplication.

4 (22 pts.) (a) If Q is an orthogonal matrix (square with orthonormal columns), show that $\det Q = 1$ or -1.

$$
\begin{aligned}
& Q^T Q = I \\
\Rightarrow \quad & |Q^T Q| = |I| \\
\Rightarrow \quad & |Q^T||Q| = 1 \\
\Rightarrow \quad & |Q||Q| = 1 \quad \text{because } |A^T| = |A| \\
\Rightarrow \quad & |Q| = \pm 1.
\end{aligned}
$$

(b) How many of the 24 terms in $\det A$ are nonzero, and what is $\det A$?

$$
A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
$$

There are four nonzero terms in $\det A$:

$$
\begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ 1 & 0 & \mathbf{-1} & 0 \\ 0 & -1 & 0 & \mathbf{1} \end{bmatrix},
\begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ 1 & 0 & \mathbf{-1} & 0 \\ 0 & \mathbf{-1} & 0 & 1 \end{bmatrix},
\begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ \mathbf{1} & 0 & -1 & 0 \\ 0 & -1 & 0 & \mathbf{1} \end{bmatrix},
\begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{-1} & 0 & 1 \end{bmatrix}
$$

Each of the four terms is equal to -1, so det A = -4.