MATH 353

PROBABILITY AND STATISTICS

UNIT 2

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UNIT 2: OUTLINE

1. Introducing Probability
2. Combining Events
3. Exhaustive Probability
4. Independent Probability
5. Conditional Probability

UNIT OBJECTIVES:

After completing this unit, you should be able to:

- Understand the meaning of an **experiment, an outcome, an event, and a sample space.**
- Describe the classical method of assigning probability.
- Calculate probabilities of **exhaustive, independent and conditional events.**

INTRODUCTION TO PROBABILITY

- **Definitions:**
  - **Random Experiment**: Is a mechanism that produces a definite outcome that ca not be predicted with certainty.
  - **Sample Space**: Is the set of all possible outcome of a random experiment.
  - **Event**: Is a subset of a sample space.
  - **Probability**: Is an outcome b in a sample space S, is a number P between 0 and 1 that measures the likelihood that b will occur on a single trial of the corresponding random experiment.

INTRODUCTION TO PROBABILITY cont'd.

- **Definitions:**
  - The value P=0 corresponds to the outcome b being impossible and
  - The value P=1 correspond to the outcome b being certain.
  - **Probability of an Event**: probability of an event A is the sum of probabilities of the individual outcome of which it is composed. It is denoted *P(A)* and read as "$P\ of\ A$".
  - $\boldsymbol{P(A) = \dfrac{\textit{Event of A}}{\textit{Sample space}}}$
  - However, $\boldsymbol{P(A) \geq 0}$ and $\boldsymbol{0 \leq P(A) \leq 1}$

Example 2.1

- Given the table below:

E_i	E_1	E_2	E_3	E_4	E_5
$P(E_i)$	0.05	0.25	0.3	0.15	0.25

Find the following:

(a) Calculate $P(A)$ where $A=\{E_i, i\geq 2\}$

(b) Find $P(B)$ where $B=\{E_i, 1\leq i<5\}$

(c) Find $P(C)$ where $C=\{E_i, i\leq 3\}$

Solution 2.1

- (a) $P(A)=P(E_2)+P(E_3)+P(E_4)+P(E_5)$

  $= 0.25 + 0.3 + 0.15 + 0.25$

  $= 0.95$
- (b) $P(B)=P(E_1)+P(E_2)+P(E_3)+P(E_4)$

  $= 0.05 + 0.25 + 0.3 + 0.15$

  $= 0.75$
- (c) $P(C)=P(E_3)+P(E_4)+P(E_5)$

  $= 0.3 + 0.15 + 0.25$

  $= 0.7$

NOTE:

- Steps in calculating the probability of an event

  (1) Define the experiment

  (2) List all simple events

  (3) Assign probabilities to simple events

  (4) Determine the simple events that constitute an event

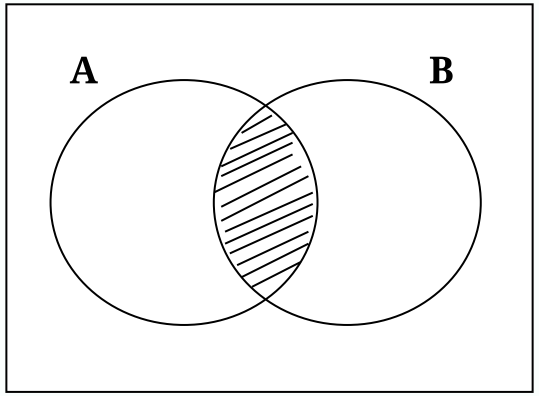
  (5) Add up the simple event probabilities to obtain the probability of the event

THE COMPLEMENTARY EVENT A'

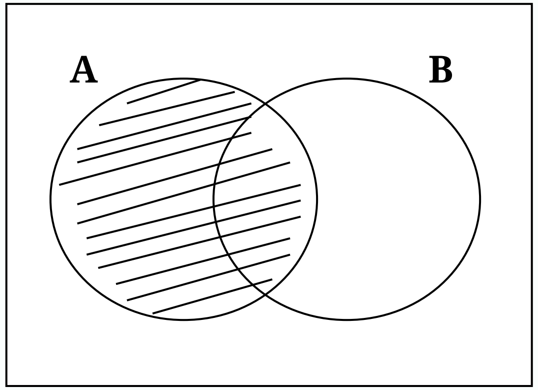
- A' Denotes the event A does not occur..
- $P(A') = 1 - P(A)$
- $=> P(A) + P(A') = 1$
- **NB:**
- NB $\overline{A}$ is written for complementary event instead of A'

PROBABILITY RULE FOR COMBINED EVENTS

- $(A \; union \; B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(B \cap A)$

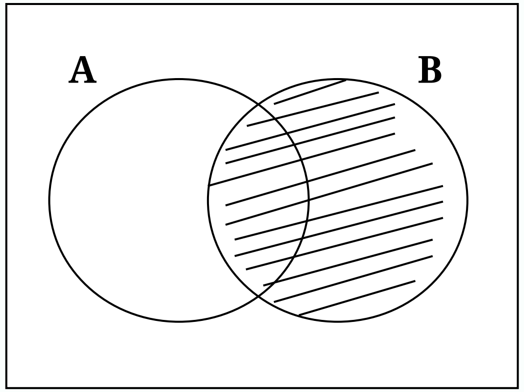


- $P(A) = P(A \cap B) + P(A \cap B')$

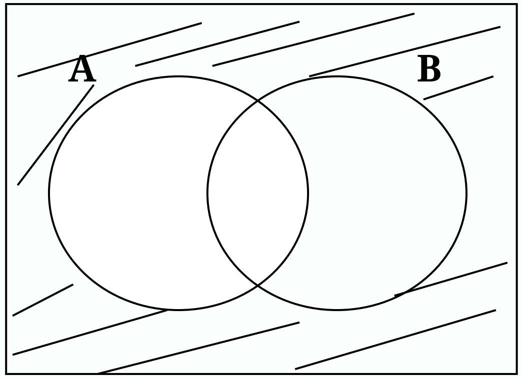


PROBABILITY RULE FOR COMBINED EVENTS

- $P(B) = P(A \cap B) + P(B \cap A')$



- $P(A' \cap B') = P([A \cup B]') = 1 - P(A \cup B)$



Example 2.2

- Event A and B are such that $P(A)=0.4$, $P(B)=0.5$ and $P(A \cap B)=0.2$
- Find
- $P(A \cup B)$ (b) $P(A \cap B')$ (c) $P(A' \cap B)$ (d) $P(A' \cap B')$

- **NB:**

$$P\left(A' \cup B'\right) = 1 - P(A \cap B)$$

Solution 2.2

- (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  $= 0.4 + 0.5 - 0.2$

  $= 0.7$
- (b) $P(A) = P\left(A \cap B'\right) + P(A \cap B)$

  $P\left(A \cap B'\right) = P(A) - P(A \cap B)$

  $= 0.4 - 0.2 = 0.2$
- (c) $P(B) = P\left(A' \cap B\right) + P(A \cap B)$

  $P\left(A' \cap B\right) = P(B) - P(A \cap B)$

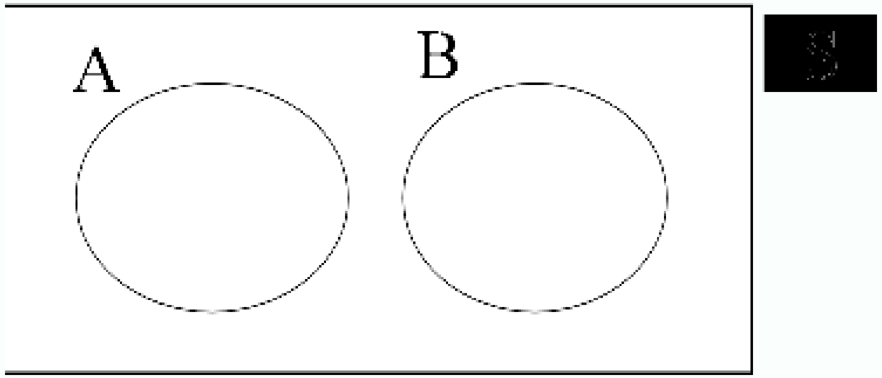
  $= 0.5 - 0.2 = 0.3$
- (d) $P\left(A' \cap B'\right) = 1 - P(A \cup B)$

  $= 1 - 0.7 = 0.3$

EXCLUSIVE (OR MUTUALLY EXCLUSIVE EVENT)

- If A and B are exclusive, then $P(A \cap B)=0$ since $A \cap B$ is an impossible event.
- Exclusive events for combined events
- $P(A \cup B)=P(A)+P(B)$

  OR
- $P(A \text{ or } B)=P(A)+P(B)$



- Extending this results to an exclusive events
- $P(A_1 \text{ or } A_2 \text{ or } A_3 \ldots A_n)=P(A_1)+P(A_2)+P(A_3)+\ldots+P(A_n)$

Example 2.3

- Given that $P(A)=0.6$ and $P(B)=0.25$ where A and B are mutually exclusive.

  (a) Find $P(A \cup B)$.

  (b) Calculate $P(A \cap B)$.

  (c) Find $P\left(A \cap B'\right)$.

  (d) Find $P(A' \cap B)$.

Solution 2.3

- Mutually exclusive event
- (a) $P(A \cup B) = P(A) + P(B) = 0.6 + 0.25 = 0.85$
- (b) $P(A \cap B) = 0$ for mutually exclusive event
- (c) $P\left(A \cap B'\right) = P(A) - P(A \cap B) = 0.6 - 0 = 0.6$
- (d) $P\left(A' \cap B\right) = P(B) - P(A \cap B) = 0.25 - 0 = 0.25$

EXHAUSTIVE EVENTS

- If two events A and B are such that between they make up the whole of the possible space, S,
- Then A and B are said to be exhaustive events.
- That is P(AuB)=1.

Example 2.4

- A and B are two events such that $P(A)=0.8$ and $P(B)=0.4$ and $P(A \cap B)=0.2$. Are A and B Exhaustive event?

Solution 2.4

- Exhaustive Event: $P(A \cup B)=1$
- $P(A \cup B)=P(A)+P(B)-P(A \cap B) = 0.8+0.4-0.2=1$
- Hence A and B are exhaustive events

INDEPENDENT EVENTS

- Two events such that the occurrence of one has no effect on the occurrence of the other.
- ***Probability of Independent Event:***
- If A and B are independent events then the probability that both A and B will occur is
- $P(\mathrm{A\ and\ B}) = P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}).\, P(\mathrm{B})$

Example 2.5

- The events A and B are independent and are such that

  $P(A) = x$, $P(B) = x + 0.2$ and $P(A \cap B) = 0.15$.

(a) Find x.

(b) Find $P(A \cup B)$.

Solution 2.5

- (a) Using $P(A \cap B) = P(A).P(B)$

$$0.15 = x(x + 0.2)$$
$$0.15 = x^2 + 0.2x$$
$$x^2 + 0.2x - 0.15 = 0$$
$$(x + 0.5)(x - 0.3)$$
$$x + 0.5 = 0 \quad \text{and} \quad x - 0.3 = 0$$
$$x = -0.5 \qquad\qquad x = 0.3$$

- Negative value is impossible for probability $\therefore x = 0.3$

$\Rightarrow P(A) = 0.3$ and $P(B) = 0.3 + 0.2 = 0.5$

Solution 2.5 cont'd.

- (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.3 + 0.5 - 0.15$$

$$= 0.65$$

Example 2.6

- The probability that a certain type of machine will break down in the first month of operation is 0.1. If a firm has two of such machines which are installed in the same time. Find the probability that at the end of the first month, just one has broken down. Assume that the performance of the two machines are independent.

Solution 2.6

- M_1: machine 1 breaks down $P(M_1)= 0.1$, $P(M'_1)=0.9$
- M_2: machine 2 breaks down $P(M_2) = 0.1$, $P(M'_2) =0.9$
- Now M_1 and M'_2 are independent as M'_1 and M_2
- So $P(M_1 \cap M_2') + P(M_2 \cap M_1') = P(M_1').P(M_2) + P(M_1).P(M_2')$
- $0.1 \times 0.9 + 0.9 \times 0.1 = 0.18$
- The probability that after one month just one machine has broken down is 0.18

CONDITIONAL PROBABILITY

- Probability that event B will occur depending on whether event A has occurred.
- This is called the **conditional probability** of B given A and is written as: $\mathbf{P(B|A)}$.
- **Probability of Dependent Events**
- If A and B are dependent events, then the probability that both A and B occur is $\mathrm{P(A\ and\ B)} = \mathrm{P(A \cap B)} = \mathrm{P(A).P(B|A)}$.

CONDITIONAL PROBABILITY (RULES)

-

1) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

2) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$

3) $P(A \cap B) = P(B|A) \cdot P(B)$

4) $P(A \cap B) = P(A|B) \cdot P(A)$

5) $P(A|B) \cdot P(A) = P(B|A) \cdot P(B)$

Example 2.7

- If $P(A) = 0.3$, $P(B) = 0.4$ and $P(B|A) = 0.5$

(a) Find $P(A \cap B)$

(b) Find $P(A \cup B)$

(c) Find $P(B|A')$

Solution 2.7

- (a) $P(A \cap B) = P(B|A).P(A)$

$$= 0.5 \times 0.3 = 0.15$$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.3 + 0.4 - 0.15 = 0.55$$

(c) $P(B|A') = \frac{P(B \cap A')}{P(A')}$

$$= \frac{P(B) - P(A \cap B)}{1 - P(A)}$$

$$= \frac{0.4 - 0.15}{1 - 0.3} = \frac{0.25}{0.7}$$

$$= 0.36$$

Example 2.8

- The probability that regularly scheduled flight depart on time is $P(D) = 0.83$, the probability that it arrives on time is $P(A) = 0.92$ and the probability that it both departs and arrives on time $P(A \cap D) = 0.78$

a) Find the probability that a plane arrives on time given that it departed on time.

b) Find the probability a plane did not depart on time given that it fails to arrive on time.

End of Slides

Thank You