ASSIGNMENT 8 SOLUTION

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1. Stewart 15.4.26

[5 pts] *Find the volume of the solid region bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.*

Solution:

We work in polar coordinates. First we locate the bounds on (r, θ) in the xy-plane. The curve of intersection of the two surfaces is cut out by the two equations $z = 3$ and $x^2 + y^2 = 1$. Therefore the x- and y- coordinates in this solid region must lie in the disk of radius one, i.e., where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. To find the volume between the surfaces, we subtract the equation of the lower surface, $3x^2 + 3y^2 = 3r^2$, from that of the higher one, $4 - x^2 - y^2 = 4 - r^2$, and integrate over these values of r, θ. Thus our integral becomes

$$\int_0^{2\pi} \int_0^1 (4 - 4r^2) r \, dr d\theta = \int_0^{2\pi} \left[4r - 4r^3/3 \right]_0^1 d\theta = (8/3)2\pi = 16\pi/3$$

2. Stewart 15.4.36a,b,c

[5pts]

(a) *We defne the improper inetgral (over $\mathbb{R}^2$)*

$$I = \iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx dy = \lim_{a \to \infty} \iint\limits_{D_a} e^{-(x^2+y^2)} \, dA,$$

where D_a is the disk of radius a at the origin. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx dy = \pi$$

Solution: We are given that the double integral on the left is equal to the limit of an integral whose domain of integration is D_a. So we should just calculate that limit, for arbitrary a, and then let $a \to \infty$. Since the disk of radius a is given in polar coordinates by $0 \le r \le \theta$, $0 \le \theta \le 2\pi$, we have

$$\iint\limits_{D_a} e^{-(x^2+y^2)} \, dA = \int_0^{2\pi} \int_0^a e^{-r^2} r \, dr d\theta = -\frac{1}{2} \int_0^{2\pi} (e^{-a^2} - 1) \, d\theta = \pi(1 - e^{-a^2})$$

But when $a \to \infty$, this expression goes to π, so our integral I defined by this limit is π.

(b) *An equivalent definition of the improper integral in part (a) is*

$$\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dA = \lim_{a \to \infty} \iint\limits_{S_a} e^{-(x^2+y^2)} \, dA,$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy = \pi$$

Solution:

For any value of a, we have

$$\iint\limits_{S_a} e^{-(x^2+y^2)} \, dA = \int_{-a}^{a} \int_{a}^{a} e^{-x^2} e^{-y^2} \, dx dy = \int_{-a}^{a} e^{-x^2} \, dx \int_{-a}^{a} e^{-y^2} \, dy$$

Therefore taking the limits of both sides as $a \to \infty$ gives

$$\lim_{a \to \infty} \iint\limits_{S_a} e^{-(x^2+y^2)} \, dA = \lim_{a \to \infty} \int_{-a}^{a} e^{-x^2} \, dx \int_{-a}^{a} e^{-y^2} \, dy$$

Applying the defintion given here in (b) and the result of part (a) to the left hand side shows that the right hand side must be equal to π.

3. Stewart 15.8.18

[5 pts] *Sketch the solid whose volume is given by the integral and evaluate the integral:*

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho d\phi d\theta$$

Solution: The given region lies between two hemispheres, of radius 1 and 2, and below the xy-plane. Since the volume enclosed by a sphere of radius R is $\frac{4}{3}\pi R^3$, the given volume should be $\frac{1}{2}(\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3) = 14\pi/3$. Let's now calculate this by evaluating the integral:

$$\begin{aligned}
\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho d\phi d\theta &= \int_0^{2\pi}\int_{\pi/2}^{\pi}\left[\frac{\rho^3}{3}\right]_1^2 \sin\phi \, d\phi d\theta \\
&= \frac{7}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_{\pi/2}^{\pi} \, d\theta \\
&= \frac{7}{3}\int_0^{2\pi} d\theta = 14\pi/3
\end{aligned}$$