MULTIVARIABLE CALCULUS

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Overview

Definition

A vector function $r(t)$ is a functio whose domain is a set of real numbers and whose range is a set of vectors.

In other words,

$$r(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

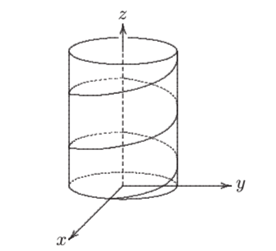
f, g, h, h are called the component function of $\mathbf{r}$.

Example 1

Sketch the curve whose vector equation is $\mathbf{r}(t) = \langle \cos t,\ \sin t,\ t \rangle$.

Solution

The parametric equations of the curve are $x = \cos t$, $y = \sin t$, $z = t$. Consider a point $P(x, y, z)$ on this curve. Since x, y and z coordinates of P satisfy the relation $x^2 + y^2 = 1$,it lies on the cylinder $x^2 + y^2 = 1$.



Solution Cont'd

Moreover, P lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$. Since $z = t$, the curve spirals upward around the cylindr as t increases. The curve is a Helix.

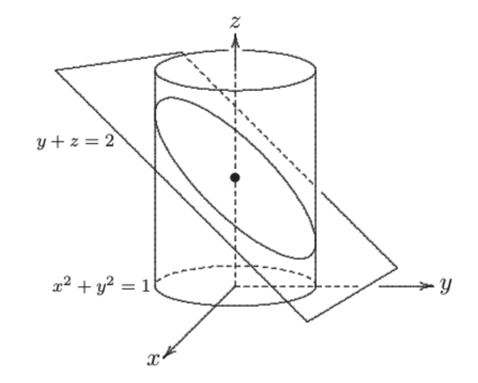
Example 2

Find the vector function that represents the curve of intercesection C of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution

Since C lies on the cylinder which projects onto the circle $x^2 + y^2 = 1$ on the $xy-$plane, we can write $x = \cos t$, $y = \sin t$ with $0 \leq t \leq 2\pi$. Since C also lies on the plane, its x, y, z coordinates should satisfy the equation of the plane. Thus, $z = 2 - y = 2 - \sin t$. Consequently, the vector equation of C is $r(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$.

Solution Contd



The curve C is an elipse with centre (0, 0, 2) and it inclines at an angle 45° to the horizontal plane.

Solution Contd

Let $\mathbf{r} = \langle f(t), g(t), h(t) \rangle$. The limit of $\mathbf{r}(t)$ as t tends to a is defined by:

$$\lim_{t \mapsto a} \mathbf{r}(t) = \langle \lim_{t \mapsto a} f(t), \lim_{t \mapsto a} g(t), \lim_{t \mapsto a} h(t) \rangle.$$

Given a vector function $\mathbf{r}(t)$. Its derivative is defined by:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \mapsto 0} \frac{\mathbf{r}(t+h) - \mathbf{r}}{h}$$

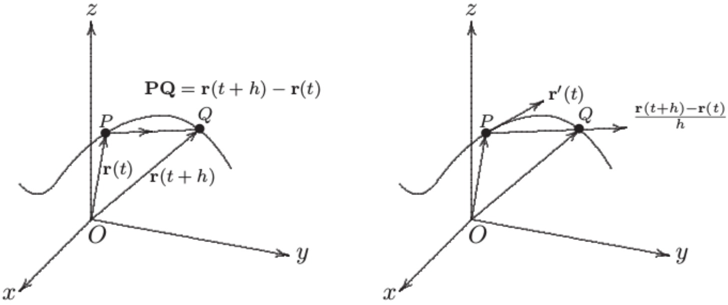


Figure : Derivative of a vector function

Derivative of a vector function

If $\mathbf{r}'(t)$ exists and is nonzero, we call it a tangent vector to the curve defined by $\mathbf{r}(t)$ at the point P. See figure 9. In this case, $\mathbf{T}(t) = \mathbf{r}'(t)/\mathbf{r}'(t)$ is called the unit tangent vector.

Theorem 1

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, h are differentiable functions of t. Then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Example 3

Let $\mathbf{r}(t) = \langle 1 + t^3, 2t, 1 \rangle$. Find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ at the point where $t = 0$.

Solution

First we have $\mathbf{r}'(t) = \langle 3t^2, 2, 0 \rangle$. Thus, $\mathbf{r}'(0) = \langle 0, 2, 0 \rangle = 2\mathbf{j}$. Therefore, $\mathbf{T}(0) = \dfrac{2\mathbf{j}}{2} = \mathbf{j}$.

Example 4

Find the parametric equations for the tangent line l to the helix with parametric equations $x = 2\cos t$, $y = \sin t$, $z = t$ at $t = \dfrac{\pi}{2}$.

Solution

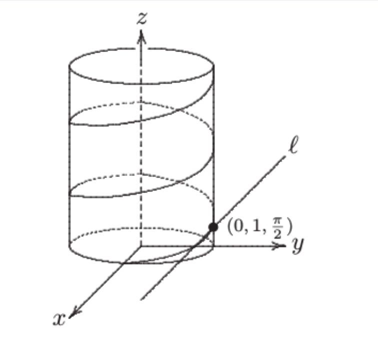


Figure : The tangent to the helix

Solution Contd

The vector equation of bthe helix is $\mathbf{r}(t) = \langle -2\sin t, \cos t, 1\rangle$ and $\mathbf{r}'(\frac{\pi}{2}) = \langle -2, 2, 1\rangle$ is a tangent vector to the helix at $t = \frac{\pi}{2}$. Therefore, the parametric equations of the tangent line l are given by:
$x = 0 + (-2)t$, $y = 1 + (0)t$, $z = \frac{\pi}{2} + (1)t$. That is
$x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$.
Given a vector function $\mathbf{r}(t)$, we may compute successively $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, $\mathbf{r}'''(t)$ etc, provided they exist.

Theorem

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, c a scalar and f a real-valued function. Then we have the followings:

1. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$.
2. $\frac{d}{dt}(c\mathbf{u}(t)) = c\mathbf{u}'(t)$.
3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{v}'(t)$.
4. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.
5. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.
6. (Chain Rule) $\frac{d}{dt}(\mathbf{u}(f(t))) = f'(t)\mathbf{u}'(f(t))$.

Example 5

Suppose $|\mathbf{r}(t)| = c$, where c is a positive constant. Show that $\mathbf{r}(t)$ is orthogonal to $\diagdown'(t)$ for all t.

Solution

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a continous vector function. The definite integral of $\mathbf{r}(t)$ from $t = 1$ to $t = b$ is defined as:

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)d(t)\right)\mathbf{i} + \left(\int_a^b g(t)d(t)\right)\mathbf{j} + \left(\int_a^b h(t)d(t)\right)\mathbf{k}.$$

Example 6

Let $\mathbf{r}(t)dt = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$. Find $\int_0^{\frac{\pi}{2}} \mathbf{r}(t)dt$.

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \mathbf{r}(t)dt &= [2\sin t]_0^{\frac{\pi}{2}}\mathbf{i} - [\cos t]_0^{\frac{\pi}{2}}\mathbf{j} + [t^2]_0^{\frac{\pi}{2}}\mathbf{k} \\ &= 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}.\end{aligned}$$

Definition

A function of 2 variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. Here D is called the domain of f. The set of variables that f takes on is called the range of f. That is Range of $f = \{f(x, y)|(x, y) \in D\}$.

Functions of 2 variables.

We usually write $z = f(x, y)$ to indicate that z is a function of x and y. Moreover, x, y are called the independent variables and z is called the dependent variable.

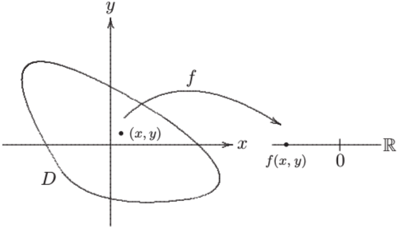


Figure : $f : D \mapsto \mathbb{R}$

Example 7

Find the domain of $f(x, y) = x\ln(y^2 - x)$.

Solution

The expression $x\ln(y^2 - x)$ is defined only when $y^2 - x > 0$.That is $y^2 > x$. The curve $y^2 = x$ separates the plane into two regions, one satisfying the inequality $y^2 > x$, the other satisfying $y^2 < x$. To find out which region is determined by the inequality $y^2 > x$. Pick any point in one of the regions and test whether it satisfies the inequality. If it does, then by 'connectivity',that whole region is the one satisfying $y^2 > x$, otherwise, it must be the other region.

Solution Cont'd

For example, pick the point (3, 2). Since $2^3 > 3$, the region satisfying $y^2 > x$ is the one containing $(3, 2)$. Thus, domain of f is $\{(x, y) \in \mathbb{R}^2 | y^2 > x\}$.

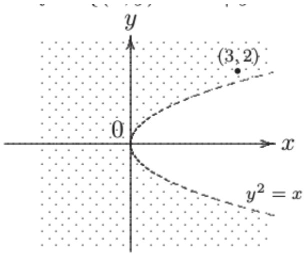


Figure : Domain of $x \ln(y^2 - x)$

Example 8

Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution

The domain of g is $\{(x, y) \in \mathbb{R}^2 | 9 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 3^2\}$ which is a circular disk of radius 3. Since $0 \leq g(x, y) = \sqrt{9 - x^2 - y^2} \leq 3$, the range of g lies in $[0, 3]$. Clearly every number in $[0, 3]$ can be expressed as $g(x, y)$ for certain (x, y). Therefore the range of g is the interval $[0, 3]$.

Definition

Let f be a function of 2 variables with domain D. The graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$.

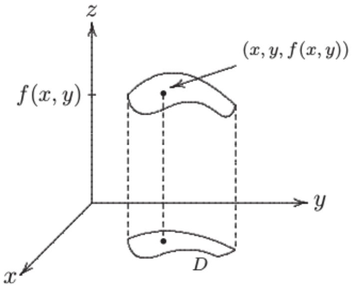


Figure : The graph of f

In general, the graph of $f(x, y)$ is a surface in $\mathbb{R}^3$.

Example 9

The graph of $f(x, y) = 6 - 3x - 2y$ is a plane. See figure 23.

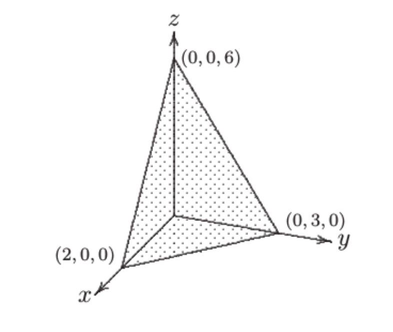


Figure : $z = 6 - 3x - 2y$

Example 10

The graph of $h(x, y) = 4x^2 + y^2$ is an elliptic paraboloid.

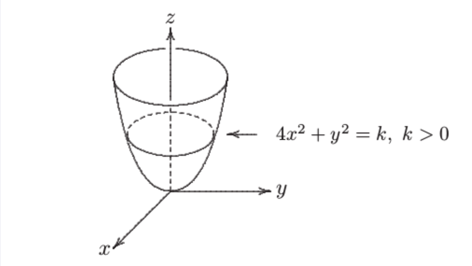


Figure : $z = 4x^2 + y^2$

The domain of h is $\mathbb{R}^2$. Since $4x^2 + y^2 \geq 0$, the range of h is $[0, \infty)$. Each horizontal trace is an ellipse with equation given by $4x^2 + y^2 = k$, where $k > 0$.

Level Curves.

Definition

The level curves of a function of 2 variables are the curves in the $xy-$plane with equation $f(x, y) = K$, where K is a constant. (K is in the range of f)

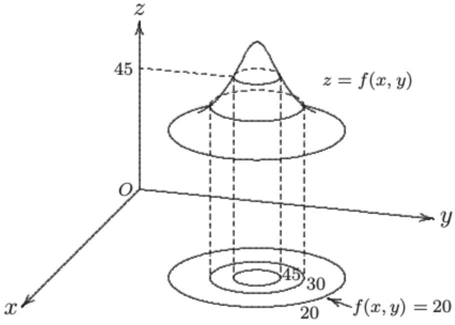


Figure : Level curves

Example 11

Sketch the level curves of $f(x, y) = 6 - 3x - 2y$ for $K = -6,\ 0,\ 6,\ 12$.

Solution

The level curves are $6 - 3x - 2y = K$ which are straight lines.

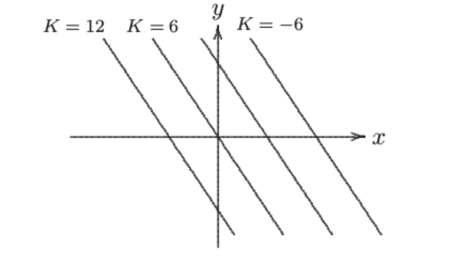


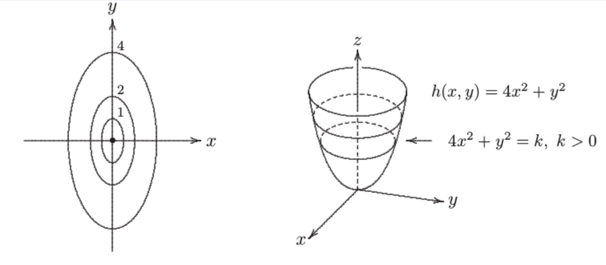
Figure : Level curves of $f(x, y) = 6 - 3x - 2y$

Example 12

Sketch some level curves of $h(x, y) = 4x^2 + y^2$.

Solution

If $k < 0$, then $4x^2 + y^2 = K$ has no solution in (x, y). Therefore, there is no level curves for $K < 0$. If $K = 0$, then $4x^2 + y^2 = 0$ has only one solution $(0, 0)$. Thus, the level curve consists of one single point at $(0, 0)$.



Solution Cont'd

If $K > 0$, the, $4x^2 + y^2 = K$ is an ellipse. We may write this equation in the standard form:

$$\frac{x^2}{(\frac{\sqrt{K}}{2})^2} + \frac{y^2}{(\sqrt{K})^2} = 1.$$

Thus a larger K gives rise to an ellipse with longer major and minor axes.

Let $f : D \subseteq \mathbb{R}^3 \mapsto R$ be a function of three variables. We can describe f by examining the level surfaces of f. These are surfaces in $\mathbb{R}^3$ given by the equations $f(x, y, z) = K$, where $K \in \mathbb{R}$.

Example 13

Let $f(x, y, z) = x^2 + y^2 + z^2$. The level of surfaces of f are concentric spheres with equatioms of the form $x^2 + y^2 + z^2 = K$ for $K > 0$. If $K > 0$, then the level surface reduces to a point at the origin of $\mathbb{R}^3$. For $K < 0$, there is no level surface for f.

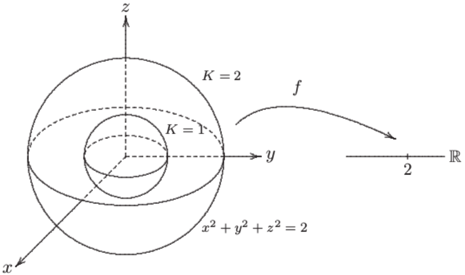


Figure : Level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$

Definition

Let f be a function of two variables whose domain D includes points arbitrarily to (a, b). We say that the limit $f(x, y)$ as (x, y) approaches (a, b) is L and we write $\lim\limits_{(x,y)\mapsto(a,b)} f(x, y) = L$ if for any positive number ϵ, there is a corresponding positive number δ such that

$$(x, y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Longrightarrow |f(x, y) - L|| < \epsilon.$$

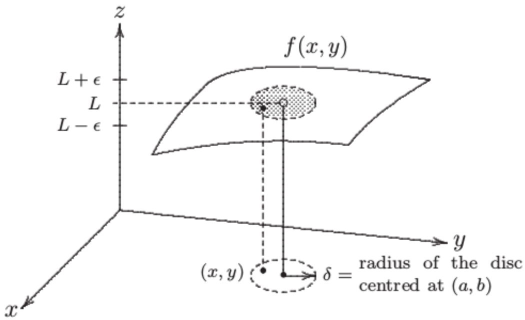


Figure : $\lim_{(x,y)\mapsto(a,b)} f(x,y) = L$

Note that f is not required to be defined at (a, b). The idea is that as (x, y) approaches (a, b), $f(x, y)$ approaches L as we wish by requiring (x, y) sufficiently close to (a, b). This is the meaning of the above definition.

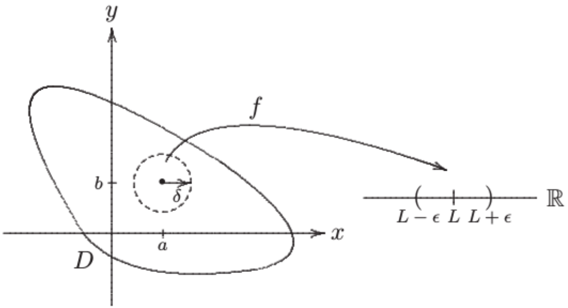


Figure : $\lim_{(x,y)\mapsto(a,b)} f(x, y) = L$

The implication in the definition says that all points (x, y) which are inside the disc centred at (a, b) with radius δ are mapped by f into the interval $(L - \epsilon,\ L + \epsilon)$. See figure 33.

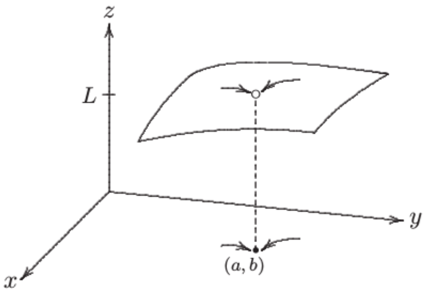


Figure : $f(x, y)$ approaches L along different paths

It can be proved from the definition that $\lim\limits_{(x,y)\mapsto(a,b)} f(x,y) = L$ exists, then

(i) its value L is unique, and

(ii) L is independent of the choice of any path approaching (a,b).

Proposition

Let $x = \alpha(t)$, $y = \beta(t)$ be the parametric equations of a path in $\mathbb{R}^2$ such that $(\alpha(t),\, \beta(t))$ lies in the domain of $f(x, y)$ for all t in a certain open interval containing t_0 and $\lim\limits_{t \mapsto t_0} \alpha(t) = a$ and $\lim\limits_{t \mapsto t_0} \beta(t) = b$. Suppose $\lim\limits_{(x,y) \mapsto (a,b)} f(x, y) = L$. Then $\lim\limits_{t \mapsto t_0} f(\alpha(t),\, \beta(t)) = L$

Example 14

Show that $\lim\limits_{(x,y)\mapsto(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution

Let $f(x) = \dfrac{x^2 - y^2}{x^2 + y^2}$. First let's approach $(0,0)$ along the x-axis.

$$\begin{aligned} \underbrace{\lim_{(x,y)\mapsto(0,0)}}_{\text{along } y=0} f(x,y) &= \lim_{x\mapsto 0} f(x,0) \\ &= \lim_{x\mapsto 0} \frac{x^2 - 0^2}{x^2 + 0^2} = \lim_{x\mapsto 0} 1 = 1 \end{aligned}$$

Example 14

Solution contd

Next let's approach $(0,0)$ along the y-axis.

$$\begin{aligned}
\underbrace{\lim_{(x,y)\mapsto(0,0)}}_{\text{along } x=0} f(x,y) &= \lim_{x\mapsto 0} f(0,y) \\
&= \lim_{y\mapsto 0} \frac{0^2 - y^2}{0^2 + y^2} = \lim_{x\mapsto 0} -1 = -1
\end{aligned}$$

Since f has two different limits along 2 differents paths, the given limit does not exist.

Example 15

Show that $\lim\limits_{(x,y)\mapsto(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Solution

Let $f(x,y) = \lim\limits_{(x,y)\mapsto(0,0)} \dfrac{xy}{x^2+y^2}$. First let's approach $(0,0)$ along the x-axis.

$$\underbrace{\lim_{(x,y)\mapsto(0,0)}}_{\text{along } y=0} f(x,y) = \lim_{x\mapsto 0} f(x,0) = \lim_{x\mapsto 0} \frac{x\cdot 0}{x^2+0^2} = 0$$

Example 15

Solution Cont'd

Now let's approach $(0,0)$ along the y-axis.

$$\underbrace{\lim_{(x,y)\mapsto(0,0)}}_{\text{along } x=0} f(x,y) = \lim_{y\mapsto 0} f(0,y) = \lim_{y\mapsto 0} \frac{0\cdot y}{0^2+y^2} = 0.$$

At this point, we cannot conclude anything as the limit may not exist. Now let's approach $(0,0)$ along the path $y = x$

$$\underbrace{\lim_{(x,y)\mapsto(0,0)}}_{\text{along } x=y} f(x,y) = \lim_{x\mapsto 0} f(x,x) = \lim_{x\mapsto 0} \frac{x^2}{x^2+y^2} = \frac{1}{2}$$

Since f has two different limits along 2 different paths, the given limit does not exist.

Definition

A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\mapsto(a,b)} f(x, y) = f(a, b).$$

f is said to be continuous on $D \subseteq \mathbb{R}^2$ if f is continuous at each point (a, b) in D.

Example 16

Every polynomial in x, y is continuous on $\mathbb{R}^2$. Each rational function is continuous in its domain. For instance, the rational function $f(x,y) = \dfrac{x^2 + x^3 y}{x+y}$ is continuous of

$$D = \{(x,y) \in \mathbb{R}^2 | x + y \neq 0\}$$

.

Example 17

Find $\displaystyle\lim_{(x,y)\mapsto(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution

We shall change to polar coordinates.

$$\begin{aligned}
\lim_{(x,y)\mapsto(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{(x,y)\mapsto(0,0)} r^2\ln(r^2) \\
&= \lim_{(x,y)\mapsto(0,0)} \frac{2\ln r}{r^{-2}} \quad \text{using L'Hopital's rule} \\
&= \lim_{(x,y)\mapsto(0,0)} \frac{2(1/r)}{(-2)(1/r^3)} = \lim_{(x,y)\mapsto(0,0)} -r^2
\end{aligned}$$

Definition

$\lim_{(x,y,z)\mapsto(a,b,c)} f(x,y,z) = L$ if for any $\epsilon > 0$, there is a corresponding $\delta > 0$ such that
$(x,y,z) \in D$ and
$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta \implies |f(x,y,y) - L| < \epsilon$

Definition

A function f is continuous at (a,b,c) if

$$\lim_{(x,y,z)\mapsto(a,b,c)} f(x,y,z) = f(a,b,c)$$

Partial Derivatives.

Definition

Let f be a function of two variables. The partial derivative of f with respect to x at (a, b) is

$$f_x(a, b) = \lim_{h \mapsto 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

The partial derivative of f with respect to y at (a, b) is

$$f_y(a, b) = \lim_{h \mapsto 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

Partial Derivatives.

There are different notations for the partial derivative of a function. If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x},$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y}.$$

In other words, in order to find f_x, we may simply regard y as constant and differentiate $f(x, y)$ with respect to x. Similarly, to find f_y, one can simply regard x as constant and differentiate $f(x, y)$ with respect to y. That is

$$f_x(a,b) = \frac{d}{dx} f(x,b)\Big|_{x=a} \text{ and } f_y(a,b) = \frac{d}{dy} f(a,y)\Big|_{y=b}$$

.

Example 18

Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Then $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$. Thus for example, $f_x(1, 1) = 5$ and $f_y(1, 1) = -1$. Geometrically, $f_x(a, b)$ measures the rate of change of f in the direction of $\mathbf{i}$ at the point (a, b). If we consider the line $y = b$ on the xy-plane to the x-axis and passing through the point (a, b), the image of this line under f is a curve C_1 on the surface $z = f(x, y)$.

Example 18 Cont'd

Then $f_x(a, b)$ is just the gradient of the tangent line to C_1 at (a, b). Similarly, $f_y(a, b)$ is just the derivative at (a, b) of the curve C_2 traced out as the image of the line $x = a$ under f.

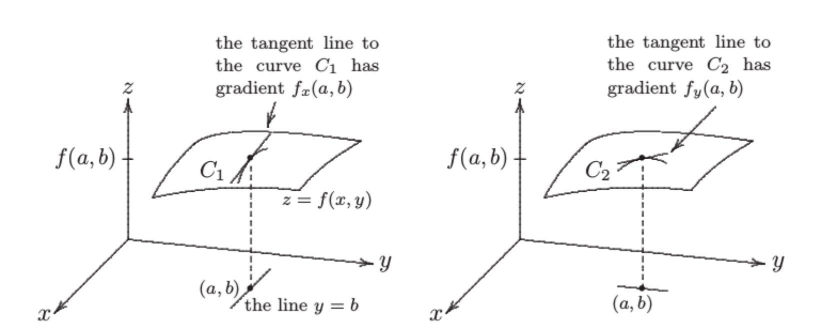


Figure : Partial derivatives

Example 19

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined as a function of x and y by

$$x^3 + y^3 + z^3 + 6xyz = 1$$

.

Solution

The partial derivative with respect to x on both sides:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) = 0$$

. Solving for $\frac{\partial z}{\partial x}$, we have

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Partial Derivatives.
Example 19

Solution Cont'd

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

. For functions of more than two variables, as such $w = f(x, y, z)$, we can similarly define

$$f_x, f_y, f_z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ or } \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$$

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Partial Derivatives.
Tangent Plane.

Let f be a function of two variables. The graph of f is a surface in $\mathbb{R}^3$ with equation $z = f(x, y)$. Let $P(x_0, y_0, z_0)$ be a point on this surface. Thus, $z_0 = f(x_0, y_0)$. Assuming a tangent plane to the surface exists, we shall find its equation.

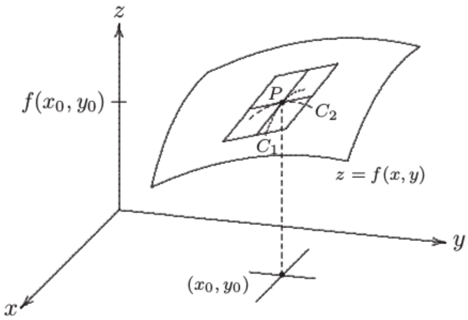


Figure : The tangent plane

Partial Derivatives.
Tangent Plane.

Recall that the equation of a plane passing through $P(x_0, y_0, z_0)$ is of the form $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. Assuming the plane is not vertical, we have C is not zero. Thus we may write the equation of the plane as

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

. The tangent line to C_1 at P is obtained by taking $y = y_0$ in the above equation. That is $z - z_0 = a(x - x_0)$. Since $f_x(x_0, y_0)$ is the gradient of the tangent line C_1 at P, we have $a = f_x(x_0, y_0)$. Similarly, $b = f_y(x_0, y_0)$. Consequently, the equation of the tangent plane to the surface $z = f(x, y)$ at P is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

.

Tangent Plane.
Example 20

Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$. Then $f_x(x, y) = 4x$ and $f_y(x, y) = 2y$ so that $f_x(1, 1) = 4$ and $f_y(1, 1) = 2$. Hence, the equation of the tangent plane at $(1, 1, 3)$ is given by $z = 3 + 4(x - 1) + 2(y - 1)$. That is $z = 4x_2y - 3$.

Linear Approximation.

Since the tangent plane to the surface $z = f(x, y)$ at P is very close to the surface at least when it is near P, we may use the function defining the tangent plane as alinear approximation to f. Recall that the equation of the tangent plane to the graph of $f(x, y)$ at $P(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

.

Definition

The linear function L whose graph is this tangent is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

. L is called the linearization of f at (a, b). The approximation

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or tangent plane approximation of f at (a, b)

Linear Approximation.
Example 21

Let $f(x, y) = xe^{xy}$. Find the linearization of f at $(1, 0)$. Use it to approximate $f(1.1, -0.1)$.

Solution

First we have $f_x(x, y) = e^{xy} + xye^{xy}$ and $f_y(x, y) = x^2 e^{xy}$. Thus $f_x(1, 0) = 1$ and $f_y(1, 0) = 1$. Then, $L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = x + y$. The corresponding linear approximation is $xe^{xy} \approx x + y$. Therefore, $f(1.1, -0.1) \approx 1.1 + (-0.1) = 1$. The actual value of $f(1.1, -0.1)$ is 0.98542 up to 5 decimal places.

The differential.

Let $z = f(x, y)$. As in the case of functions of one variable, we take the differentials dx and dy to be independent variables.

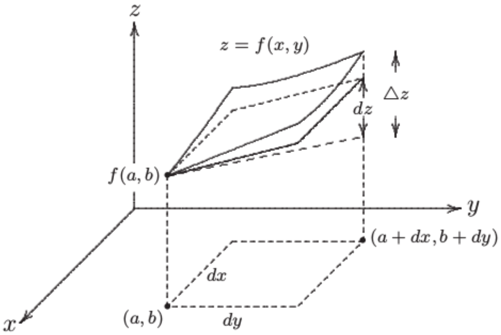


Figure : The differential

The differential.

Definition

The fifferential dz or the total differential, is defined to be

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

.

Consider the differential of f at the point (a, b). The tangent plane approximation of f at (a, b) implies that for a small change dx of a and a small change dy of b, the actual change Δz of z is approximately equal to dz. In other words,

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy$$

.

The differential.
Example 22

Let $f(x, y) = x^2 + 3xy - y^2$. Find dz. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution

$dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy = (2x + 3y)dx + (3x - 2y)dy$. At the point $(2, 3)$, $dz = ((2)(2) + 3(3))dx + ((3)(2) - (2)(3))dy$. That is $dz = 13dx$. Now we take $dx = 2.05 - 2 = 0.05$ and $dy = 2.96 - 3 = -0.04$. Thus, $dz = 13(0.05) = 0.65$. For the actual change in z, we have $\Delta z = f(2.05, 2.96) - f(2, 3) = 0.6449$.

The differential.

Definition

Let $z = f(x, y)$.f is said to be differentiable at (a, b) if

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

, where $\lim_{(\Delta x, \Delta y) \mapsto (0,0)} \epsilon_1 = 0$ and $\lim_{(\Delta x, \Delta y) \mapsto (0,0)} \epsilon_2 = 0$

Example 23

Prove that $f(x, y)$ is differentiable at (a, b).

Solution

First $f_x(x, y) = y$ and $f_x(x, y) = x$. At the point (a, b), we have $f_x(a, b) = b$ and $f_x(a, b) = a$.

$$\begin{aligned}
\Delta x &= f(a + \Delta x, b + \Delta y) - f(a, b) \\
&= (a + \Delta x)(b + \Delta y) - ab \\
&= b\Delta x + a\Delta y + \Delta x\Delta y \\
&= f_x(a, b)\Delta x + f_y(a, b)\Delta y + \Delta x\Delta y.
\end{aligned}$$

Here $\epsilon_1 = 0$ and $\epsilon_2 = \Delta x$. Clearly, $\lim\limits_{(\Delta x,\Delta y)\mapsto(0,0)} \epsilon_1 = 0$ and $\lim\limits_{(\Delta x,\Delta y)\mapsto(0,0)} \epsilon_2 = 0$. Thus, $f(x, y) = xy$ is differentiable at (a, b).

Theorem

Suppose $f_x(x, y)$ and $f_y(x, y)$ exist in an open disk containing (a, b) and are continuous at (a, b). Then f is differentiable at (a, b).

Theorem (The chain rule, case 1)

Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$, $y = h(t)$ are both differentiable functions of t. Then z is a differentiable functions of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

.

Theorem (The chain rule, case 2)

Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$, $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable functions of$s$ and t and

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

.

Implicit Differentiation.

Suppose $F(x, y) = 0$ defines y implicitly as a function of x. That is $y = f(x)$. Then $F(x, f(x)) = 0$. Now we use the chain rule(case 1) to differentiate F with respect to x. Thus

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dy} = 0$$

. Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

.

Example 24

Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Solution

Let $F(x, y) = x^3 + y^3 - 6xy$. The given equation is simply $F(x, y) = 0$. Therefore,

$$\frac{dy}{dy} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

. Next, suppose z is given implicitly as afunction of x and y by an equation $F(x, y, z) = 0$. In other words, one may solve z locally in terms of x and y. We thus obtain:

$$F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_x \frac{\partial z}{\partial x} = 0$$

.

Example 24

Solution Cont'd

Note that $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$. Thus $F_x + F_z\dfrac{\partial z}{\partial x} = 0$. Hence,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

. Similarly,

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

.

Definition

Let f be a function of x and y. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \mapsto 0} \frac{f(x_0 + ha,\, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

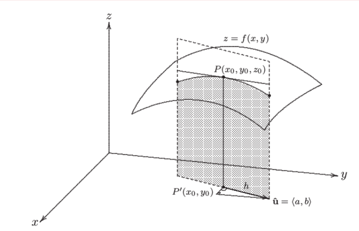


Figure : Directional Derivative

Note that $D_i f(x_0, y_0) = f_x(x_0, y_0)$ and $D_j f(x_0, y_0) = f_y(x_0, y_0)$, where **i** and **j** are the standard basis vectors in $\mathbb{R}^2$.

Theorem

Let f be a differentiable function of x and y. Then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y),\ f_y(x, y) \rangle \cdot \mathbf{u}$$

.

Definition

Let f be a differentiable function of x and y. The gradient of f is the vector function

$$\bigtriangledown f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

. Thus we have the following formula for the directional derivative in terms of the gradient of f.

$$D_{\mathbf{u}} f = \bigtriangledown f \cdot \mathbf{u}, \qquad \text{where } \mathbf{u} \text{ is a unit vector.}$$

Example 25

Let $f(x,y) = x^2y^3 - 4y$. Find the directional derivative of f at $(2,-1)$ in the direction $3\mathbf{i} + 4\mathbf{j}$.

Solution

The unit vector along $3\mathbf{i} + 4\mathbf{j}$ is $\mathbf{u} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$. The gradient of f is $\bigtriangledown f = \langle 2xy^3, 3x^2y^2 - 4\rangle$. Thus $\bigtriangledown f(2,-1) = \langle -4, 8\rangle$. Consequently, $D_{\mathbf{u}}f(2,-1) = \bigtriangledown f(2,-1)\cdot\mathbf{u} = \langle -4, 8\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = 4$.

Directional Derivatives and the Gradient Vector.

Definition

Let f be a function of x, y, z. The directional derivatives of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ in $\mathbb{R}^3$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \mapsto 0} \frac{f(x_0 + ha,\ y_0 + hb,\ z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.
Similarly, the gradient of a differentiable function f is defined to be

$$\bigtriangledown f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

. The formula $D_{\mathbf{u}} f = \bigtriangledown f \cdot \mathbf{u}$ is also valid for any function f of more than 2 variables.

Theorem

Let f be a differentiable function of 2 or 3 variables. Let P be a point in the domain of f. The maximum value of $D_{\mathbf{u}}f(P)$ is $|\bigtriangledown f(P)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\bigtriangledown f(P)$.

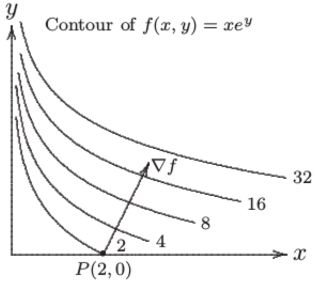


Figure : $\bigtriangledown f$ is the direction of steepest ascend

Tangent Planes to Level surfaces.

Let S be a surface with equation $F(x, y, z) = k$, where k is a constant. That is S is a level surface of F. Let $P(x_0, y_0, z_0)$ be a point in S. Let's find the equation of the tangent plane S at P.

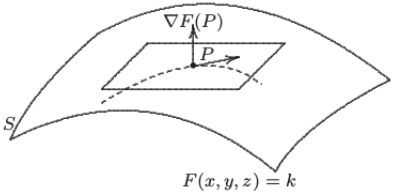


Figure : Tangent plane

Tangent Planes to Level surfaces.

Take any curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ on the surface S such that $\mathbf{r}(0) = (x_0, y_0, z_0)$. It's tangent vector $\mathbf{r}'(0)$ shall lie on the tangent plane to S at P. Now if we use the chain rule to differentiate $F(x(t), y(t), z(t)) = k$ with respect to t, we have

$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0$$

. In other words, $\bigtriangledown F \cdot \mathbf{r}'(t) = 0$. At $t = 0$, we have $\bigtriangledown F(P) \cdot \mathbf{r}'(0) = 0$. Therefore, $\bigtriangledown F(P)$ is perpendicular to the tangent plane.

Equation of tangent plane: $\quad \langle x - x_0, y - y_0, z - z_0\rangle \cdot \bigtriangledown F(x_0, y_0, z_0) = 0$

.

Maximum and Minimum Values.

Definition

$f(x, y)$ has a local maximum (minimum) at (a, b) if $f(x, y) \leq f(a, b)$ $(f(x, y) \geq f(a, b))$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a local maximum value (local minimum value).

Figure 56 A local maximum A local minimum

Maximum and Minimum Values

Theorem

If f has a local maximum or local minimum at (a, b) and $f_x(a, b)$ and $f_y(a, b)$ exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. That is $\bigtriangledown f(a, b) = 0$.

Definition

A point (a, b) is called a critical point of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

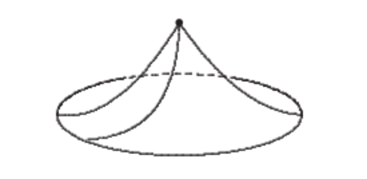


Figure : A critical point

Example 26

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the local maxima and local minima of f.

Solution

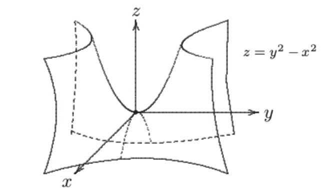
First $f_x = 2x - 2$ and $f_y = 2y - 6$. Thus, $f_x = 0$ and $f_y = 0$ if and only if $(x, y) = (1, 3)$. Therefore f has a critical point at $(1, 3)$. So f has a possible local maximum or local minimum at $(1, 3)$. As $f(x, y) = 4 + (x - 1)^2 + (y - 3)^2 \geq 4$, we see that f has a local (in fact absolute) minimum at $(1, 3)$.

Example 27

Find the local extrema (i.e. local maximum or local minimum) of $f(x, y) = y^2 - x^2$.

Solution

First $f_x = -2x$ and $f_y = 2y$. Therefore, the only critical point is $(0, 0)$. However, f has neither a maximum nor a minimum at $(0, 0)$. To see this, consider the function f along $y = 0$, $f(x, 0) = -x^2 < 0$ for $x \neq 0$. So f has a local maximum along $y = 0$. On the other hand, if we consider f along $x = 0$, we have $f(0, y) = y^2 > 0$ for all $y \neq 0$. Thus f has a local minimum along $x = 0$. Therefore f has neither a maximum nor a minimum at $(0, 0)$. Such a point is called a **saddle point**.



Definition

f is said to have a saddle point at (a, b) if there is a disk centered at (a, b) such that f assumes its maximum value on one diameter of the disk only at (a, b), and assume its minimum value on another diameter of the disk only at (a, b).

Theorem(The Second Derivative Test)

Suppose f_{xx}, f_{xy}, f_{yx} and f_{yy} are continuous on a disk with center (a, b) and suppose $f_x(a, b) = 0$, $f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$$

.

- If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D < 0$, then f has a saddle point at (a, b).

Note that if $D = 0$, then no conclusion can be drawn from it. The point can be a local maximum, a local minimum, a saddle point or neither of these.

Definition

A bounded set in $\mathbb{R}^2$ is one that is contained in some disk. A closed set in $\mathbb{R}^2$ is one that contains all its boundary oints.

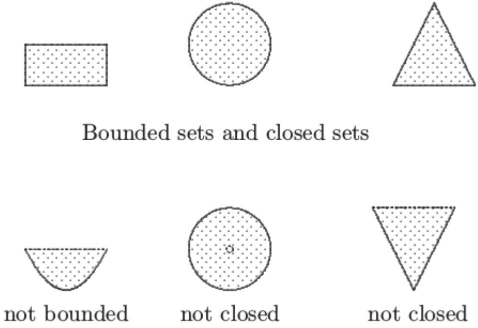


Figure : Sets in $\mathbb{R}^2$

Maximum and Minimum Values

Theorem(Extreme Value Theorem)

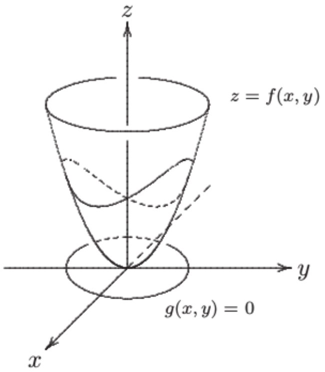
If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

The following is a procedure to find the absolute maximum and the absolute minimum value of a function defined on a closed and bounded set.

1. Find the values of f at the critical points.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from 1. and and 2. is the absolute maximum value and the smallest of the values from 1. and and 2. is the absolute minimum value.

Lagrange Multipliers.

In this section we consider the problem of maximizing or minimizing a function $f(x, y)$ subject to a constraint $g(x, y) = 0$.



If we confine the point (x, y) to lie on the curve $g(x, y) = 0$ on the xy-plane, its image under f gives a curve on the graph of $z = f(x, y)$. We are looking for the highest and the lowest points of this curve. Suppose the extreme value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ is k and is attained at the point (x_0, y_0). By examining at the contour of f, we see that at the extreme point, the curve $g(x, y) = 0$ cuts across the level curve $f(x, y) = k$, one can still move the point along $g(x, y) = 0$ so as to increase or decrease the value of f. In other words, the gradients of f and g must be parallel at the extreme point (x_0, y_0). Consequently, we must have $\bigtriangledown f(x_0, y_0) = \lambda \bigtriangledown g(x_0, y_0)$ if $\bigtriangledown g(x_0, y_0) \neq 0$.

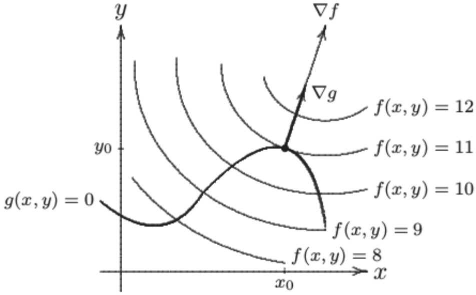


Figure : Lagrange Multiplier

The same principle applied to functions of three variables. Let's state the method of Lagrange Multiplier in this setting. The objective is to find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ (assuming that these extreme values exist).

Below is an outline of the procedure.

- Find all x, y, z and λ such that

$$\bigtriangledown f(x, y, x) = \lambda \bigtriangledown g(x, y, z) \tag{1}$$

  and $g(x, y, z) = 0$. (Assuming at each of these solutions $\bigtriangledown g \neq 0$.)

- Evaluate f at all points (x, y, z) obtained in the first. The largest of these values is the absolute maximum of f; the smallest is the absolute minimum of f.

  The number λ is called the **Lagrange Multiplier**.

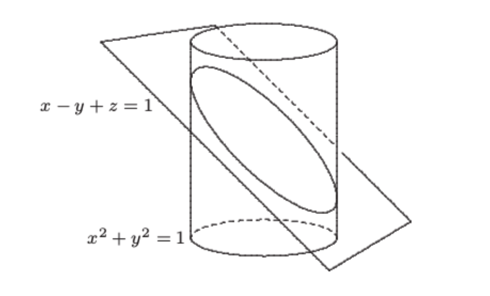
  The equation (1) is equivalent to $f_x = \lambda g_x,\ f_y = \lambda g_y,\ f_z = \lambda g_z$.

Example 28

Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution

We wish to maximize $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z - 1$ and $h(x, y, z) = x^2 + y^2 - 1$.



Example 28

Solution Cont'd

First we have $\bigtriangledown f = \langle 1, 2, 3 \rangle$, $\bigtriangledown g = \langle 1, -1, 1 \rangle$ and $\bigtriangledown h = \langle 2x, 2y, 0 \rangle$. Thus we need to solve the system of equations:
$\bigtriangledown f = \lambda \bigtriangledown g + \mu \bigtriangledown h$, $x - y + z = 1$, $x^2 + y^2 = 1$. That is

$$
\begin{aligned}
1 &= \lambda + 2x\mu && (2)\\
2 &= -\lambda + 2y\mu && (3)\\
3 &= \lambda + 0 && (4)\\
x - y + z &= 1 && (5)\\
x^2 + y^2 &= 1 && (6)
\end{aligned}
$$

Example 28

Solution Cont'd

From (4), $\lambda = 3$. Substituting this into (2) and (3), we get $x = -\dfrac{1}{\mu}$ and $y = \dfrac{5}{2\mu}$. Note that $\mu \neq 0$ by (3) and (4).

From (5), we have $z = 1 - x + y = 1 + \dfrac{1}{\mu} + \dfrac{5}{2\mu} = 1 + \dfrac{7}{2\mu}$.

Using (6), we have $(-\dfrac{1}{\mu})^2 + (\dfrac{1}{\mu})^2 = 1$. From this, we can solve μ, giving $\mu = \pm\dfrac{\sqrt{29}}{2}$.

Thus $x = -\dfrac{2}{\sqrt{29}}$ or $x = \dfrac{2}{29}$. The corresponding values of y are $\dfrac{5}{\sqrt{29}}, -\dfrac{5}{\sqrt{29}}$.

Using (7), the corresponding values of z are $1 + \dfrac{7}{\sqrt{29}}, 1 - \dfrac{7}{\sqrt{29}}$.

Example 28

Solution Cont'd

Therefore, the two possible extreme values are at

$$P_1 = \left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) \text{ and } P_2 = \left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right)$$

As $f(P_1) = 3 + \sqrt{29}$ and $f(P_2) = 3 - \sqrt{29}$, the maximum value is $3 + \sqrt{29}$ and the minimum value is $3 - \sqrt{29}$.

The End