Section 13.7: Surface Area

In this section, we will use double integrals to compute the area of a surface defined by the equation $z = f(x, y)$.

Theorem: (Surface Area)
Suppose that f has continuous first-order partial derivatives f_x and f_y. Then the area of the surface defined by $z = f(x, y)$, $(x, y) \in D$ is

$$A = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA.$$

Example: Find the area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$.

The triangular region T can be described by

$$T = \{(x, y) | 0 \leq y \leq 1, 0 \leq x \leq y\}.$$

Using the previous theorem with $f(x, y) = x + y^2$ gives

$$\begin{aligned} A &= \iint_T \sqrt{(1) + (2y)^2 + 1} dA \\ &= \int_0^1 \int_0^y \sqrt{4y^2 + 2} dx dy \\ &= \int_0^1 y\sqrt{4y^2 + 2} dy \\ &= \frac{1}{8}\left[\frac{2}{3}(4y^2 + 2)^{3/2}\Big|_0^1\right] \\ &= \frac{1}{12}(6\sqrt{6} - 2\sqrt{2}). \end{aligned}$$

Example: Find the area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies above the annular region $1 \leq x^2 + y^2 \leq 4$.

The annular region D can be described in polar coordinates by

$$D = \{(r, \theta) | 0 \leq \theta \leq 2\pi, 1 \leq r \leq 2\}.$$

Using the previous theorem with $f(x, y) = y^2 - x^2$ gives

$$\begin{aligned} A &= \iint_D \sqrt{(-2x)^2 + (2y)^2 + 1} dA \\ &= \int_0^{2\pi} \int_1^2 \sqrt{4r^2 + 1} r dt d\theta \\ &= 2\pi \int_1^2 r\sqrt{4r^2 + 1} dr \\ &= \frac{\pi}{4} \left[\frac{2}{3}(4r^2 + 1)^{3/2} \Big|_0^1 \right] \\ &= \frac{\pi}{6}(5\sqrt{5} - 1). \end{aligned}$$