Linear Algebra I 2006 Exam Solutions

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SECTION A

A1. (a) $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

$B^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$ 4

(b) $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$

$AB - BA = \begin{pmatrix} bg-cf, & f(a-d)+b(h-e) \\ c(e-h)+g(d-a), & cf-bg \end{pmatrix}$ 3

A2. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, i.e. $M^2 = 2M$. 2

Hence $M^3 = 2M^2 = 4M$

and $M^4 = 4M^2 = 8M$. 2

This pattern implies that $M^n = 2^{n-1}M$

for $n \in \mathbb{Z}$, $n \geq 1$ (over $\mathbb{Q}$ or $\mathbb{R}$). 2

Over $\mathbb{F}_2$, $M^n = 0$ for $n \geq 2$. 1

A3. $D_1 = 1 \times \begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{vmatrix} + 2 \times \begin{vmatrix} 0 & 2 & -1 \\ 2 & 0 & 0 \\ 0 & -1 & 4 \end{vmatrix}$

by Laplace expansion about the first row

$= 1 \times 3 \times (8-1) + 2 \times (-2) \times (8-1)$

by Laplace expansion about the second rows

$= 3 \times 7 - 4 \times 7 = -7$ 3

$D_2 = D_1$ with the middle two rows swapped.
Hence $D_2 = -D_1 = +7$ 2

D_3 is such that the last row is the sum of the other rows. If any row of a determinant is a linear combination of other rows then the determinant vanishes. Hence $D_3 = 0$. 2

A4. $u = \sum_{i=1}^{n} k_i v_i = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n, \quad k_i \in \mathbb{K}$. 2

(a) Any vector $u = a(1,0,1) + b(0,1,0) \in U \ \forall a, b \in \mathbb{R}$ since it has the form stated above, e.g.
$u = (0,0,0)$ by taking $a = b = 0$ or
$u = (1,1,1)$ by taking $a = b = 1$ or ... 2

(b) Any vector not of the above form is not in U, eg $v = (1,0,2)$.
Proof: Does $(1,0,2) = a(1,0,1) + b(0,1,0)$ for some $a, b \in \mathbb{R}$? Try to solve
$1 = a, \ 0 = b, \ 2 = a$. No solution since $1 \neq 2$. 3

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A5. A set of vectors $\{v_1, v_2, \dots v_n\}$ in a vector space V over a field $\mathbb{K}$ is linearly independent if the only solution of $k_1 v_1 + \dots + k_n v_n = 0$ for $k_1, \dots k_n \in \mathbb{K}$ is $k_1 = \dots = k_n = 0$. (2)

A set of vectors $\{v_1, v_2, \dots v_n\}$ in a vector space V over a field $\mathbb{K}$ is a spanning set for V if every $v \in V$ has the form $v = k_1 v_1 + \dots + k_n v_n$ for some $k_1, \dots k_n \in \mathbb{K}$. (2)

A basis for V is a linearly independent spanning set. (1)

The dimension of V is the number of vectors in any basis. (1)

The basis for V is not unique. As a counterexample, $\{(1,0), (0,1)\}$ and $\{(1,1), (1,-1)\}$ are both bases for $\mathbb{R}^2$. (1)

A6. A linear map $\alpha : U \to V$, where U, V are vector spaces over the same field $\mathbb{K}$, is a map such that

$$\alpha(au + bv) = a\alpha(u) + b\alpha(v)$$

$$\forall\ u, v \in U,\ a, b \in \mathbb{K}.$$ (2)

α is not linear. As a counterexample, take $u = (1,1,1) \in \mathbb{R}^3$, $a = 2 \in \mathbb{R}$, $v = 0$.

$$\alpha(au) = \alpha(2,2,2) = (4,4,4) \neq$$

$$a\alpha(u) = 2(1,1,1) = (2,2,2).$$ (2)

β is linear. Proof.

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$$\beta(au + bv) = \beta(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$
$$= (ay_1 + by_2 + az_1 + bz_2, az_1 + bz_2 + ax_1 + bx_2, ax_1 + bx_2 + ay_1 + by_2)$$
$$= a(y_1 + z_1, z_1 + x_1, x_1 + y_1) + b(y_2 + z_2, z_2 + x_2, x_2 + y_2)$$
$$= a\beta(u) + b\beta(v).$$

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A7. An eigenvalue and eigenvector for an $n \times n$ matrix A over $\mathbb{C}$ are a scalar $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.

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Two matrices A, B are similar if $\exists$ an invertible matrix P such that $B = P^{-1}AP$. 2

A is similar to a diagonal matrix if (a) it has n distinct eigenvalues or (b) it is real and symmetric.

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(a) Take the eigenvectors as the columns of P.

(b) Ditto, after ensuring that any eigenvectors corresponding to the same eigenvalue are chosen to be orthogonal.

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[Only one of (a) and (b) above is required.]

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A8. A symmetric matrix S satisfies $S^T = S$.
An antisymmetric matrix A satisfies $A^T = -A$. (2)

$S = \frac{1}{2}(M + M^T)$ is symmetric,
$A = \frac{1}{2}(M - M^T)$ is antisymmetric,
and $M = S + A$. (2)

The symmetric part of $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is

$$S = \frac{1}{2}\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix}.$$ (1)

Let $q = x^T A x$. Transposing gives
$q = x^T A^T x = -x^T A x = -q$.
Hence $2q = 0 \Rightarrow q = 0$. (2)

SECTION B

B1. (a) Closure: $u+v \in V \quad \forall\, u,v \in V$.
Commutativity: $u+v = v+u \quad \forall\, u,v \in V$.
Associativity: $u+(v+w) = (u+v)+w \;\forall\, u,v,w \in V$.
Identity: $\exists\, 0 \in V$ such that $v+0 = v \;\forall\, v \in V$.
Inverse: $\forall\, v \in V \;\exists\, -v \in V$ such that $v+(-v)=0$ 5

(b) $\mathbb{K}^n$ is the set of all n-tuples of elements of $\mathbb{K}$, i.e. $\mathbb{K}^n = \{(k_1, k_2, \dots, k_n) \mid k_i \in \mathbb{K}\}$, such that if $x,y \in \mathbb{K}^n$ then $x+y = (x_1+y_1, \dots, x_n+y_n)$ and if $k \in \mathbb{K}$ then $kx = (kx_1, \dots, kx_n)$. 5

(c) Either $U \neq \emptyset$ or $0_V \in U$.
$k_1u_1 + k_2u_2 \in U \quad \forall\, u_1,u_2 \in U$ and $\forall\, k_1,k_2 \in \mathbb{K}$. 3

(d) (i) U is a vector space (since the conditions are linear).
$(0,0,0,0) \in U$ since $0=0$.
Let $u_1 = (w_1,x_1,y_1,z_1)$, $u_2 = (w_2,x_2,y_2,z_2)$.
$u_1,u_2 \in U \Rightarrow w_1=y_1,\, x_1=z_1$ and $w_2=y_2,\, x_2=z_2$

Let $a,b \in \mathbb{R}$. Then $au_1+bu_2 =$
$(aw_1+bw_2,\, ax_1+bx_2,\, ay_1+by_2,\, az_1+bz_2)$.

$w_1=y_1$ and $w_2=y_2 \Rightarrow aw_1+bw_2 = ay_1+by_2$
$x_1=z_1$ and $x_2=z_2 \Rightarrow ax_1+bx_2 = az_1+bz_2$
Hence $au_1+bu_2 \in U$ 6

(ii) U is not a vector space (since the conditions are non linear).
$u=(1,1,1,1) \in U$ since $1=1^2$
But $2u = (2,2,2,2) \notin U$ since $2 \neq 2^2$. 3

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B2. (a) $\ker(\alpha) = \{ u \in U \mid \alpha(u) = 0 \}$ ①
$\operatorname{im}(\alpha) = \{ \alpha(u) \mid u \in U \}$ ①

(b) $\ker(\alpha) = \{ (x,y,z) \mid x+y=0,\ 2x+5y+z=0,\ 3y+z=0 \}$
Solving the equations: $x = -y$, $z = -3y$ and
$-2y + 5y - 3y = 0 \ \forall y$.
Hence $\ker(\alpha) = \{ y(-1,1,-3) \mid y \in \mathbb{R} \}$
so $\{(-1,1,-3)\}$ is a basis for $\ker(\alpha)$.
Proof: spanning by construction and linearly independent since only one vector.
Take as ordered basis for the domain
$\mathcal{B} = (1,0,0), (0,1,0), (1,-1,3)$. ④

The image of this ordered basis is
$$\left.\begin{aligned} \alpha(1,0,0) &= (1,2,0) \\ \alpha(0,1,0) &= (1,5,3) \\ \alpha(1,-1,3) &= (0,0,0) \end{aligned}\right\} \quad (*)$$
so $(1,2,0), (1,5,3)$ is the corresponding ordered basis for $\operatorname{im}(\alpha)$.
Proof: the image of a spanning set is a spanning set and the two vectors are linearly independent since neither vector is a multiple of the other.
Take as ordered basis for the codomain
$\mathcal{C} = (1,2,0), (1,5,3), (0,0,1)$. ④

(c) From $(*)$ above, $A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ①

$$\left.\begin{array}{l} \alpha(1,0,0) = (1,2,0) \\ \alpha(0,1,0) = (1,5,3) \\ \alpha(0,0,1) = (0,1,1) \end{array}\right\} \Rightarrow A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix}.$$ 1

(d) $A' = (\alpha, \mathcal{D}, \mathcal{C}) = PAQ$ and $A = (\alpha, \mathcal{E}_3, \mathcal{E}_3)$.
Hence $Q = (Id, \mathcal{D}, \mathcal{E}_3)$ and $P = (Id, \mathcal{E}_3, \mathcal{C})$

Find Q:

$Id(1,0,0) = (1,0,0) = e_1$

$Id(0,1,0) = (0,1,0) = e_2$

$Id(1,-1,3) = (1,-1,3) = 1e_1 - 1e_2 + 3e_3$

Hence $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ 4

Find P:

$Id(1,0,0) = (1,0,0) = a(1,2,0) + b(1,5,3) + c(0,0,1)$

$$\Rightarrow \left.\begin{array}{l} 1 = a + b \\ 0 = 2a + 5b \end{array}\right\} \Rightarrow -2 = 3b \Rightarrow b = -2/3$$
$$\& \quad a = 1 - b = 5/3$$
$$0 = 3b + c \qquad \text{so} \quad c = 2.$$

Hence $Id(1,0,0) = 5/3(1,2,0) - 2/3(1,5,3) + 2(0,0,1)$.

$Id(0,1,0) = (0,1,0) = a(1,2,0) + b(1,5,3) + c(0,0,1)$

$$\Rightarrow \left.\begin{array}{l} 0 = a + b \\ 1 = 2a + 5b \end{array}\right\} \Rightarrow 1 = 3b \Rightarrow b = 1/3$$
$$\& \quad a = -1/3$$
$$0 = 3b + c \qquad \text{so} \quad c = -1.$$

Hence $Id(0,1,0) = -1/3(1,2,0) + 1/3(1,5,3) - (0,0,1)$.

$Id(0,0,1) = (0,0,1)$

Hence $P = \begin{pmatrix} 5/3 & -1/3 & 0 \\ -2/3 & 1/3 & 0 \\ 2 & -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & -1 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 3 \end{pmatrix}$. 6

Optional check:

$$PAQ = \frac{1}{3}\begin{pmatrix} 5 & -1 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5 & -1 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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B3.(a) The elementary row operations are:

- interchange two rows, which changes the sign of the determinant; ②
- multiply a row by a scalar, which multiplies the determinant by the scalar; ②
- add a multiple of one row to another row, which does not change the determinant. ②

(b) Use elementary row operations to reduce A to row canonical form. If this is the identity matrix then the operations correspond to premultiplication by A^{-1}. Apply the same operations to I, which therefore turns into A^{-1}. Do this by adjoining I to A (on the right). ③

(c)
$$AI = \left(\begin{array}{ccc|ccc} -2 & 0 & 3 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 4 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$ ①

* $R_1 \leftrightarrow R_2$
$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 1 & 0 \\ -2 & 0 & 3 & 1 & 0 & 0 \\ 4 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 4R_1$
$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & 2 & 0 \\ 0 & 6 & 7 & 0 & -4 & 1 \end{array}\right)$$

* $R_2 \to -\frac{1}{2}R_2$
$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 & -1 & 0 \\ 0 & 6 & 7 & 0 & -4 & 1 \end{array}\right)$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 - 6R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & -1 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 \end{array}\right)$$

* $R_3 \to \frac{1}{4} R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & -1 & 0 \\ 0 & 0 & 1 & 3/4 & 1/2 & 1/4 \end{array}\right)$$

$R_1 \to R_1 + \frac{3}{2} R_3$
$R_2 \to R_2 - \frac{1}{2} R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/8 & 3/4 & 3/8 \\ 0 & 1 & 0 & -7/8 & -5/4 & -1/8 \\ 0 & 0 & 1 & 3/4 & 1/2 & 1/4 \end{array}\right)$$ 9

Hence $A^{-1} = \frac{1}{8}\begin{pmatrix} 5 & 6 & 3 \\ -7 & -10 & -1 \\ 6 & 4 & 2 \end{pmatrix}$ 1

Optional check: $AA^{-1} =$

$$\begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & -2 \\ 4 & 2 & -1 \end{pmatrix} \frac{1}{8}\begin{pmatrix} 5 & 6 & 3 \\ -7 & -10 & -1 \\ 6 & 4 & 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(d) The elementary row operations flagged * above change determinant, multiplying it by factors $(-1) \times (-\frac{1}{2}) \times \frac{1}{4} = \frac{1}{8}$.
The determinant of the reduced matrix is 1, hence $\underline{\det(A) = 8}$. 2

Optional check: $(-2 + 0 + 6) - (-12 + 8)$
$= 4 - (-4) = 8.$

B4. (a) A set of vectors is <u>orthogonal</u> if any two distinct vectors u, v in the set are orthogonal, i.e. their inner product vanishes, $\langle u, v \rangle = 0$. 2

A set of vectors is <u>orthonormal</u> if it's orthogonal and every vector v in the set has unit norm, i.e. $\|v\| = 1$ (or $\langle v, v \rangle = 1$). 2

(b) If A is an orthogonal matrix then $AA^T = A^TA = I$. 1

Let C_i denote the ith column of A.
Then $(A^TA)_{ij} = C_i^T C_j = I_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$
Hence $\{C_i\}$ is an orthonormal set of vectors. 3

(c) (i)
$$\begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -6 \\ 0 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$
Hence $(1,0,1,0)$, $(1,0,-1,0)$ are eigenvectors with respective eigenvalues $4, 6$. 2

(ii) The eigenvalues satisfy the characteristic equation

$$\begin{vmatrix} 5-\lambda & 0 & -1 & 0 \\ 0 & 1-\lambda & 0 & -1 \\ -1 & 0 & 5-\lambda & 0 \\ 0 & -1 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad ①$$

$$\Rightarrow (5-\lambda)\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 5-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1-\lambda & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)^2\left((1-\lambda)^2-1\right) - \left((1-\lambda)^2-1\right) = 0 \qquad ②$$

$$\Rightarrow \left((5-\lambda)^2-1\right)\left((1-\lambda)^2-1\right) = 0$$

$$\Rightarrow (24-10\lambda+\lambda^2)(-2\lambda+\lambda^2) = 0$$

$$\Rightarrow (\lambda-6)(\lambda-4)\lambda(\lambda-2) = 0$$

$$\Rightarrow \lambda = 0, 2, 4, 6$$

so the other two eigenvalues are $0, 2$. ②

$\underline{\lambda = 0}$:

$$\begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} 5x - z &= 0 \\ y - t &= 0 \\ -x + 5z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x = z &= 0 \\ t &= y \end{aligned}$$

Thus an eigenvector is $(0, 1, 0, 1)$. ②

$\underline{\lambda = 2}$: $\begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$

$$\Rightarrow \left.\begin{aligned} 3x - z &= 0 \\ -y - t &= 0 \\ -x + 3z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x = z &= 0 \\ t &= -y \end{aligned}$$

Thus an eigenvector is $(0, 1, 0, -1)$. 2

(iii) The eigenvectors are orthogonal and all have norm $\sqrt{2}$. Hence

$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$ is orthogonal and

$PAQ = \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$ is diagonal

where $P = Q^T$ 3

<u>Optional check:</u> Q^TAQ

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 4 & 6 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & -6 \\ 0 & -2 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$