ME 261 Dynamics of Solid Mechanics

UNIT 3

KINETICS OF RIGID BODIES

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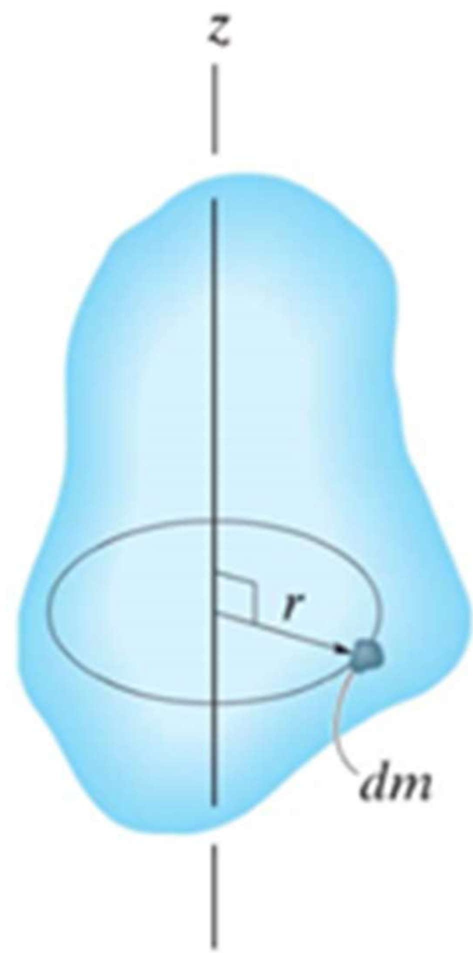
Mass Moment of Inertia

The mass moment of inertia of a body is a measure of how the mass of the body is distributed about a point in the body; it also describes the resistance of a body to a rotational motion. The moment of inertia is always positive since it is the sum of product of mass and square of distances, which are always positive.

Mass moment of inertia

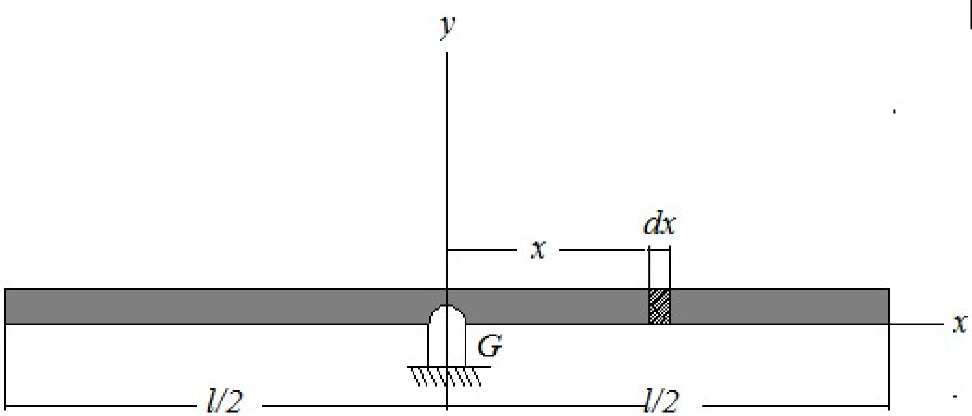
Consider a differential element of mass *dm* about the z-axis, shown in Figure 3-1, the moment of inertia of this elemental mass about point z axis is

$$dI = r^2\,dm \quad or \quad I = \int r^2\,dm$$



Example 3-1: Mass Moment of Inertia

Find the moments and products of inertia of a slender rod of mass *m* and length *l* about an through the centre (G).



Solution

If ρ is its mass per unit length, the mass of the element of length *dl* is *dm = ρdl.* Taking a differential element dx at a distance x from the centre

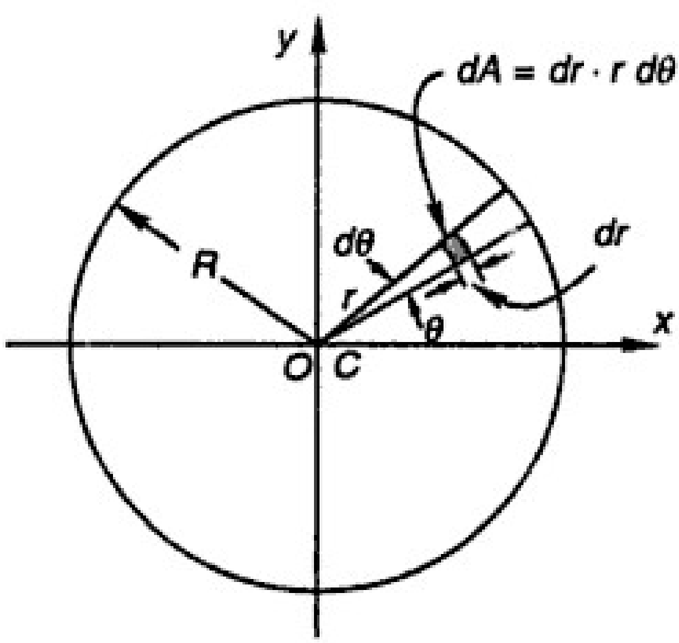
$$I = \int_{-l/2}^{l/2} x^2 dm$$

Since $dm/dl = m/l = \rho$

$$I = \frac{m}{l}\int_{-l/2}^{l/2} x^2 dx = \frac{ml^2}{12}$$

Example 3-2: Mass moment of inertia

Determine the mass moment of inertia of a disc about an axis through its centre(C).



Solution

$$dm = \rho dA = \rho\, r dr d\theta$$

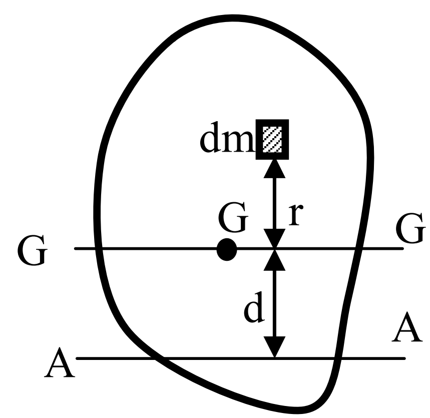
$$I = \int r^2 dm = \rho \int_0^{2\pi}\int_0^{r} r^2 \left(r dr d\theta\right) = \frac{mr^2}{2}$$

Parallel-axis theorem

- The parallel-axis theorem for the mass moment of inertia states that the mass moment of inertia with respect to any axis is equal to the moment of inertia with respect to a parallel axis through the centre of mass plus the product of the mass and the square of the perpendicular distance between the axes. Mathematically

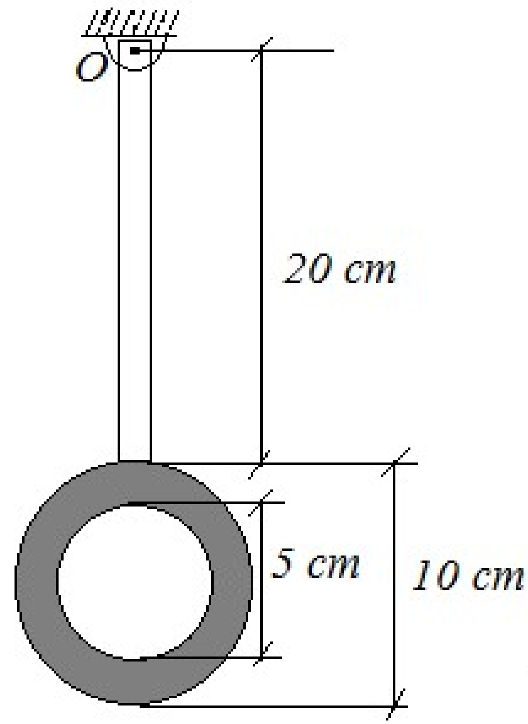
$$I_{AA} = I_{GG} + md^2$$

- Where I_{AA} is the moment of inertia about an axis AA and I_{GG} is the moment of inertia about an axis parallel to AA and passing through the centre of mass G. The two axes are at distance d apart and the mass of the body is m.



Example 3-3: Parallel-Axis theorem and composite bodies

A clock pendulum consists of a slender rod and a circular disc with a hole in it as shown. The rod has a density of 7000 kg/m^3 and cross sectional area of 50 mm^2 . The disc has a density of 8000 kg/m^3 and a thickness of 5 mm. Compute the moment of inertia of the pendulum about O.



Solution

Solution

Let us consider the pendulum to consist of three composite parts, a slender rod 20 cm long (A), a solid disc 10 cm diameter (B), a solid disc 5 cm diameter(C).0.01333=0.0833

$$\therefore\ I_O = I_{AO} + I_{OB} - I_{OC}$$

For A

Using the parallel-axis theorem

$$I_{AO} = I_{AOG} + md^2$$

$$I_{AO} = \frac{ml^2}{12} + md^2 = \rho V \frac{l^2}{12} + md^2$$

$$= \left(7000 \times 0.2 \times 50 \times 10^{-6} \times \frac{0.2^2}{12}\right) + \left(7000 \times 0.2 \times 50 \times 10^{-6} \times 0.1^2\right)$$

$$I_{AO} = 0.0833$$

Solution

For B

$$I_{BO} = I_{BOG} + md^2$$

$$I_{BO} = \frac{mr^2}{2} = \rho V \frac{r^2}{2}$$

$$I_{BO} = (8000 \times \pi \times 0.05^2 \times 5 \times 10^{-3} \times \frac{0.5^2}{2}) + (8000 \times \pi \times 0.05^2 \times 5 \times 10^{-3} \times 0.25^2)$$

$$I_{BO} = 7.893$$

For C

$$I_{CO} = I_{COG} + md^2$$

$$I_{CO} = \left(8000 \times \pi \times 0.025^2 \times 5 \times 10^{-3} \times \frac{0.025^2}{2}\right) + (8000\pi \times 0.05^2 \times 5 \times 10^{-3} \times 0.25^2)$$

$$I_{CO} = 0.0196$$

$$I_O = 0.0833+7.893-0.0196$$

$$I_O = 7.957\ kgm^2$$

Radius of Gyration

- The radius of gyration of a body is defined as the radius at which the equivalent lumped mass model of the body is located such that the resulting model has the same moment of inertia as the original body.

$$I = mk^2$$

- Sometimes it is convenient to express the moment of inertia of a body about a given axis in the form

$$k = \sqrt{\frac{I}{m}}$$

- where m is the mass of the body and k is the radius of gyration about the axis under consideration.

Equations of Motion for Planar Rigid Bodies

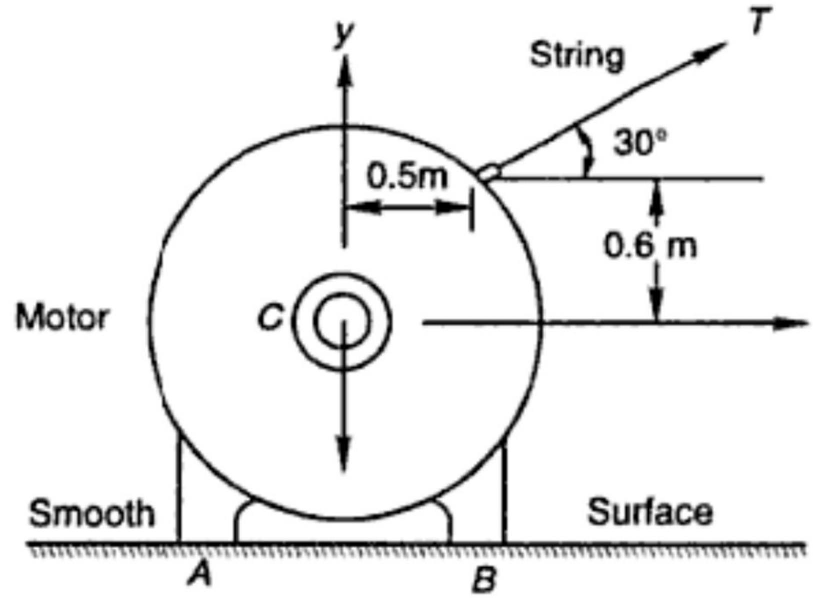
Recalling the fact that the acceleration and velocity of each element on a rigid body must be the same in a pure translation, the translational motion must be governed by the Newton's law, and given by

$$\sum F = ma_G \quad \text{and} \quad \sum M_G = 0$$

where $\sum F$ is the net external force acting on the body and $\boldsymbol{a_G}$ is the acceleration of any point on the body

Example 3 4: Equation of Motion for Rigid Bodies

A motor of mass 8000 kg resting on two supports A and B is pulled along a smooth horizontal surface by a string passing through a hook as shown in Figure E3-1. Calculate the acceleration of the motor and the reactions at the supports for a tension T applied in the string. Calculate the maximum tension in the string for the sliding motion.



Solution

The free-body diagram of the motor is shown below.

By the equations of motion for a rigid body in translation

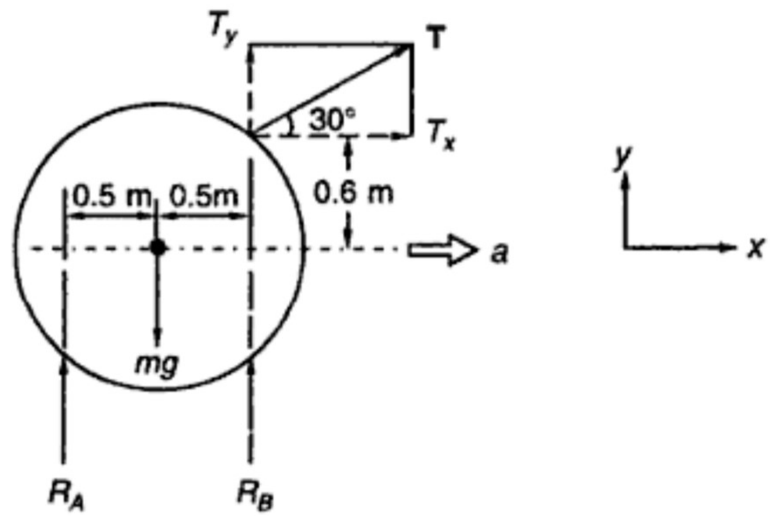
$$F_x = ma_x \qquad F_y = ma_y = 0 \qquad M_{Gz} = 0$$

From these equations

$$-T\sin 30° + 8000 \times 9.81 + R_A + R_B = 0 \dots\dots\dots 2$$

$$T\cos 30° = 8000\, a \dots\dots\dots 1$$

$$0.6T\cos 30° - 0.5T\sin 30° + 0.5(R_A - R_B) = 0 \dots\dots\dots 3$$



Solution

- Solving the three equations simultaneously gives

$$a = 0.000\ 108\ T$$

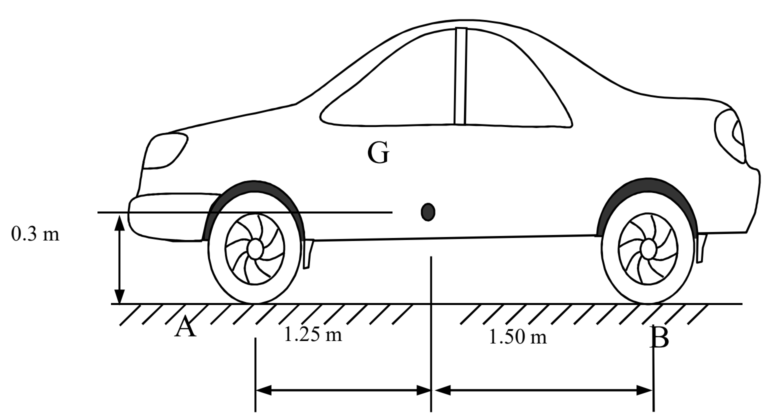
$$R_A = 39\ 250 - 0.52T$$

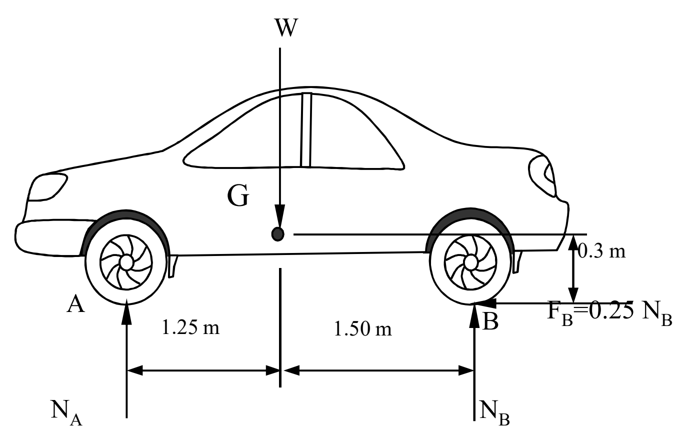
$$R_B = 39\ 250 + 0.02\ T$$

- It can be observed that the reaction R_A decreases with the tension T increasing. In the limiting case of sliding, $R_A = 39\ 250 - 0.52T = 0 \quad which\ gives\ T = 75.48\ kN$
- beyond this value of *T*, point A will not be restrained to move along the surface. The motor may then overturn forward.

Example 3-5: Planar Translational Motion

The car shown has a mass of 2.5 Mg and a centre of mass at G. Determine the acceleration of the car if the rear driving wheels are always slipping, whereas the front wheels are free to rotate. Neglect the mass of the wheels. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.25$.





Solution

Free-body is shown in Figure E3-5(a). The equations of motion are

$$+\leftarrow \sum F_x = m(a_G)_x \qquad 0.25N_B = 2500(a_G)_x \qquad (1)$$

$$+\uparrow \sum F_y = m(a_G)_y \qquad N_A + N_B - 2500 \text{ x } 9.81 = 0 \qquad (2)$$

$$+\circlearrowleft \sum M_G = 0 \qquad N_A(1.25) - N_B(1.5) + 0.25N_B(0.3) = 0 \qquad (3)$$

Solving the above equations simultaneously yields

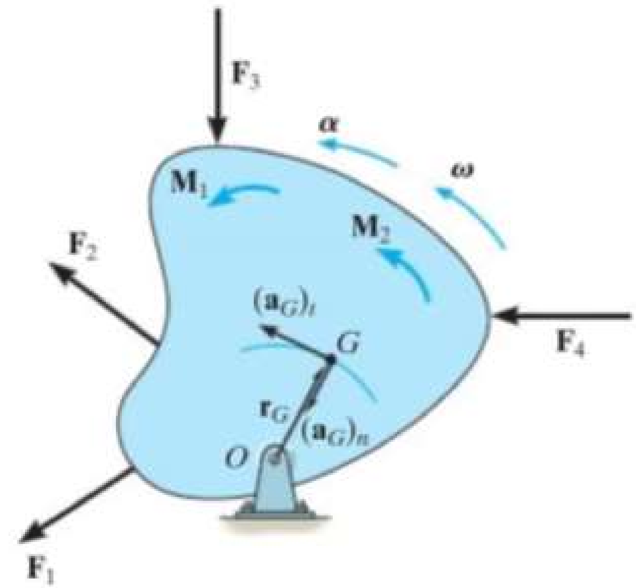
$$\underline{(a_G)_x = 1.15 \text{ m/s}^2} \qquad \underline{\underline{N_A = 13.06 \text{ kN}}} \qquad \underline{\underline{N_B = 11.46 \text{ kN}}}$$

Equations of Motion: Rotation about a Fixed Axis

- Consider the rigid body which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at O.

- Because the body's centre of mass G moves around a circular path, the acceleration of this point is represented by its tangential and normal components.

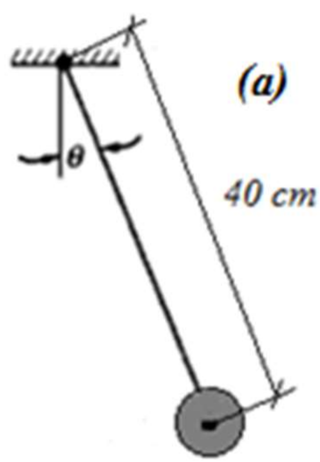
- The equations of motion which apply to the body can be written in the form

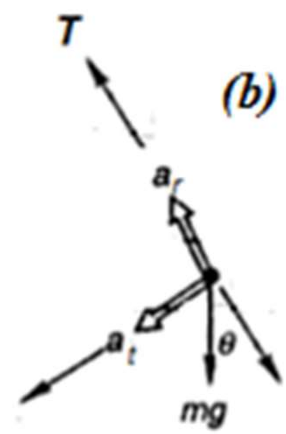


$$\sum F_n = m(a_G)_n = m\omega^2 r_G \qquad \sum F_t = m(a_G)_t = m\alpha r_g \qquad \sum M_G = I_G\alpha$$

Example 3-6: Equations of Motion: Rotation about a Fixed Axis

A Simple Pendulum of mass 100 kg and length 40 cm is supported by a string from released from rest when θ = 60° as shown. For a plane curvilinear motion of the bar, determine the tension in the string at the instant when it is released from rest. Also find the moment about the fixed point.





Solution

$$\boldsymbol{M_Z} = I_Z \alpha \qquad T - mg\cos\theta = ma_r \qquad (1)$$

$$\boldsymbol{F_r} = m\boldsymbol{a_r} \qquad mg\sin\theta = ma_t \qquad (2)$$

$$\boldsymbol{F_t} = m\boldsymbol{a_t} \qquad mgl\sin\theta = I_o\alpha = \boldsymbol{M_o} \qquad (3)$$

The acceleration of any point on the body is made up of two components (As discussed in Unit 1); the radially inward component, i.e., $a_r = r\omega^2 = 0.4\omega^2$

Substituting the radial and tangential components of accelerations in to equations (1) and (2);

$$a_t = r\alpha = 0.4\alpha \qquad T - 981\cos 60° = 40\omega^2 \qquad 981\sin 60° = 40\alpha$$

At the instant of release

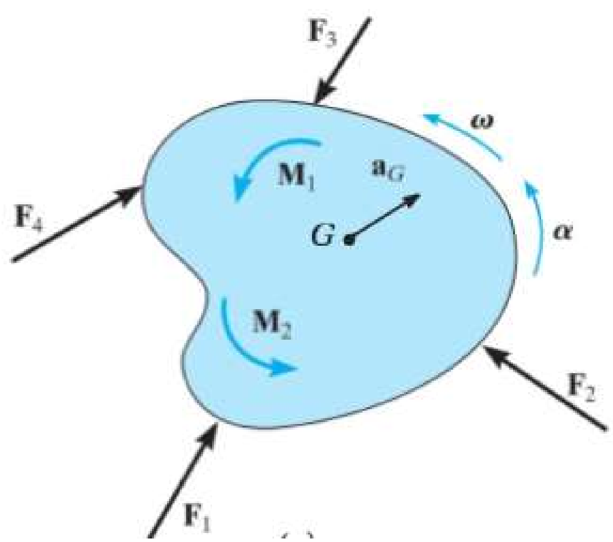
$$\theta = 60°, \omega = 0 \qquad \therefore T = 490.5\ N \ \ and \ \ \alpha = 21.24\ rad/s^2$$

$$\boldsymbol{M_o} = mgl\sin\theta = 849.57\ N - m$$

Equations of Motion: General Plane Motion

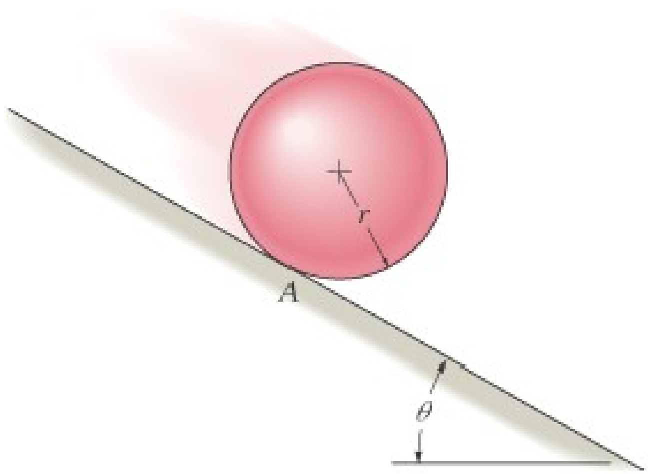
The rigid body shown in Figure 3-5 is subjected to general plane motion caused by the externally applied force and couple-moment system. If an x and y inertial coordinate system is established as shown, the three equations of motion are

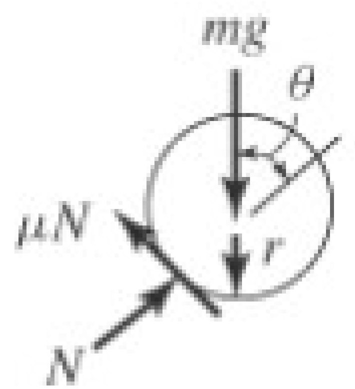
$$\sum F_x = m(a_G)_x \qquad \sum F_y = m(a_G)_y \qquad \sum M_G = I_G \alpha$$

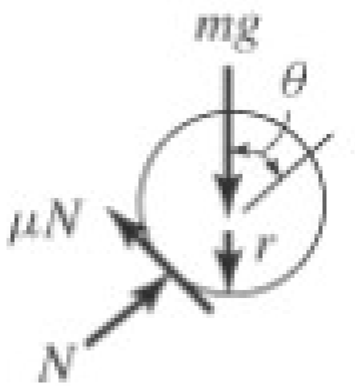


Example 3-9: General Plane Motion

A ball of mass m and radius r rolls along plane inclined for which the coefficient of static friction is μ. If the ball is released from rest, determine the maximum angle θ for the incline so that it rolls without slipping at A.







Solution

$$+\rightarrow \sum F_n = m(a_G)_n \qquad N - mg\cos\theta = 0 \qquad N = mg\cos\theta \ldots 1$$

$$+\downarrow \sum F_t = m(a_G)_t \qquad mg\sin\theta - \mu N = ma_G \qquad mg\sin\theta - \mu mg\cos\theta = ma_G \ldots 2$$

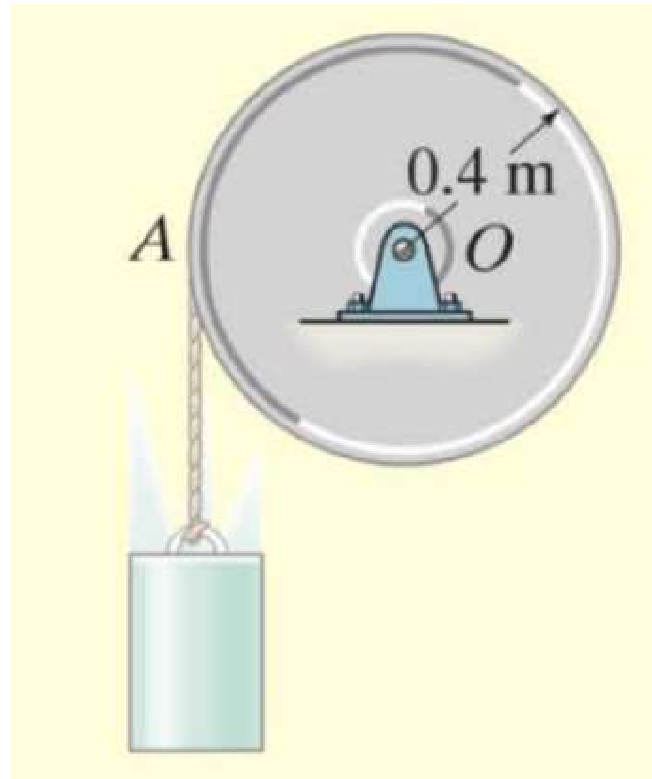
$$M_G = I_G\alpha \qquad (\mu N)r = I_G\alpha \qquad (\mu mg\cos\theta)r = \left(\frac{2}{5}mr^2\right)\alpha \ldots 3$$

Substituting (1) into (2) and (3), and solving

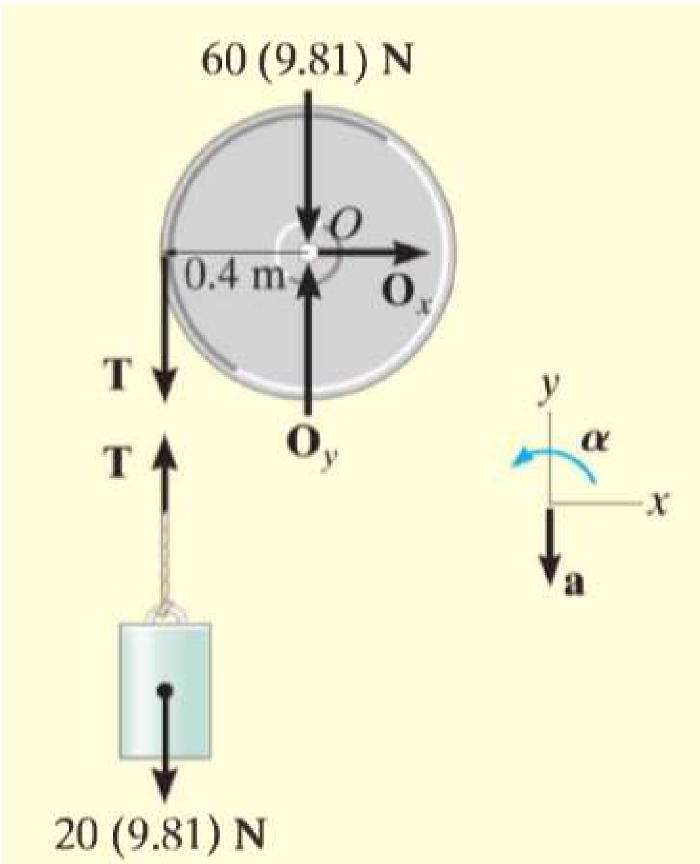
$$\theta = \tan^{-1}\left(\frac{7\mu}{2}\right)$$

Example 3-7: Equations of Motion: Rotation about a Fixed Axis

The drum shown has a mass of 60 kg and a radius of gyration k_O = 0.25 m. A cord of negligible mass is wrapped around the periphery of the drum and attached to a block having a mass of 20 kg. If the block is released, determine the drum's angular acceleration.



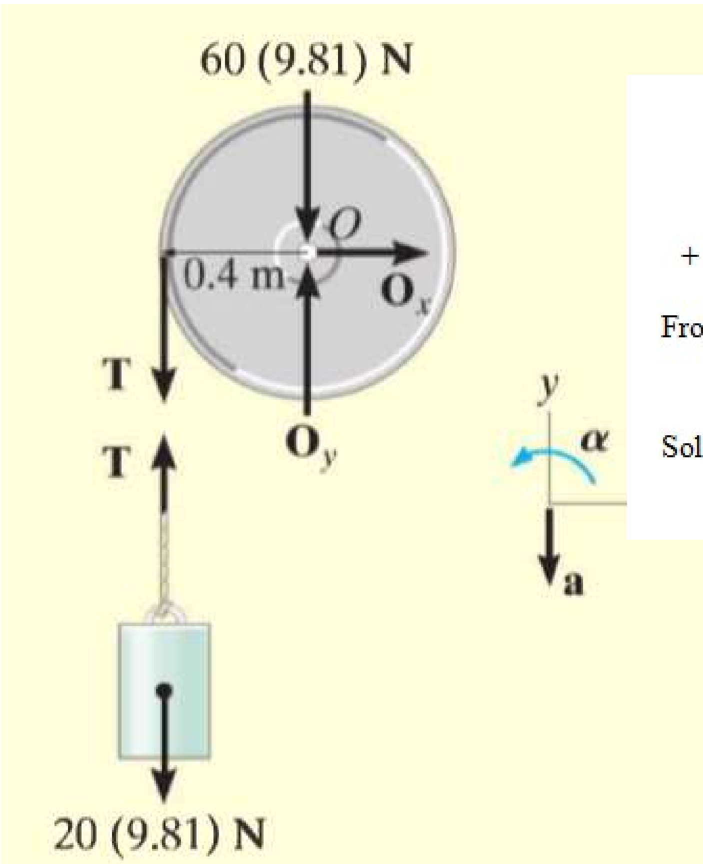
Example 3-7: Equations of Motion: Rotation about a Fixed Axis



Here we will consider the drum and block separately, as shown. The moment of inertia of the drum is

$$I_O = mk_O^{\ 2} = (60)(0.25)^2 \qquad I_O = 3.75 \text{ kg - m}^2$$

Example 3-7: Equations of Motion: Rotation about a Fixed Axis



$$\sum M_O = I_O \alpha \qquad T(0.4) = (3.75)\alpha \qquad (1)$$

$$+\uparrow \sum F_y = m(a_G)_y \qquad T - 20(9.81) = -20a \qquad (2)$$

From kinematics

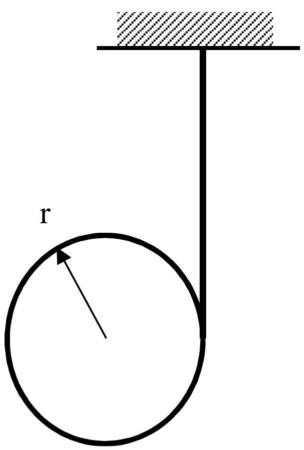
$$a = \alpha r \qquad a = \alpha(0.4) \qquad (3)$$

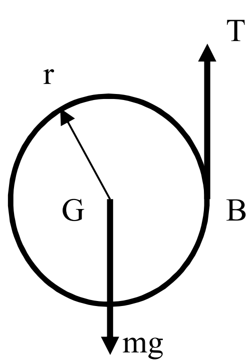
Solving equations 1 to 3 simultaneously yields

$$\underline{\underline{T = 106\ \text{N}}} \qquad \underline{\underline{a = 4.52\ \text{m/s}^2}} \qquad \underline{\underline{\alpha = 11.3\ \text{rad/s}^2}}$$

Example 3-10: General Plane Motion

A wire of negligible mass is wrapped around the outer surface of the disk of mass m and radius r. If the disk is released from rest, determine its angular velocity at time t.





$$\sum M_B = I_B \alpha$$

$$mgr = \left(I_G + mr^2\right)\alpha = \left(\frac{1}{2}mr^2 + mr^2\right)\alpha$$

$$\Rightarrow \alpha = \frac{2}{3}\frac{g}{r}$$

$$\omega_f = \omega_i + \alpha t = 0 + \left(\frac{2}{3}\frac{g}{r}\right)t \qquad \omega_f = \frac{2}{3}\frac{g}{r}t$$

Planar Kinetics of Rigid Bodies: Work and Energy

Work-Energy Formulation for Plane Motion

The work done due to a force F acting at an arbitrary point P on a body equals the dot product of the force with the displacement of the point of application of the force $dW = \boldsymbol{F} dr$

For a finite displacement of the point of application of force i.e., from r_1 to r_2 the work done is given by

$$dW = \int_{r_1}^{r_2} \boldsymbol{F}\ dr$$

If, instead, it is sought to evaluate the integral $dW' = \int_{r_1}^{r_2} \boldsymbol{F}\ dr_c$

The action of a moment $\boldsymbol{M}$ due to a couple on a rigid body results in an angular displacement $d\theta$ of the body. The work of the moment is expressed by

$$dW = \boldsymbol{M} d\theta$$

Kinetic Energy and Potential Energy

- The kinetic energy of a rigid body in a general plane motion, may be thought of as the sum of: The translational and rotational kinetic energy

$$KE = KE_{trans} + KE_{rot}$$

$$KE = \frac{1}{2} m v_G{}^2 + \frac{1}{2} I_G \omega^2$$

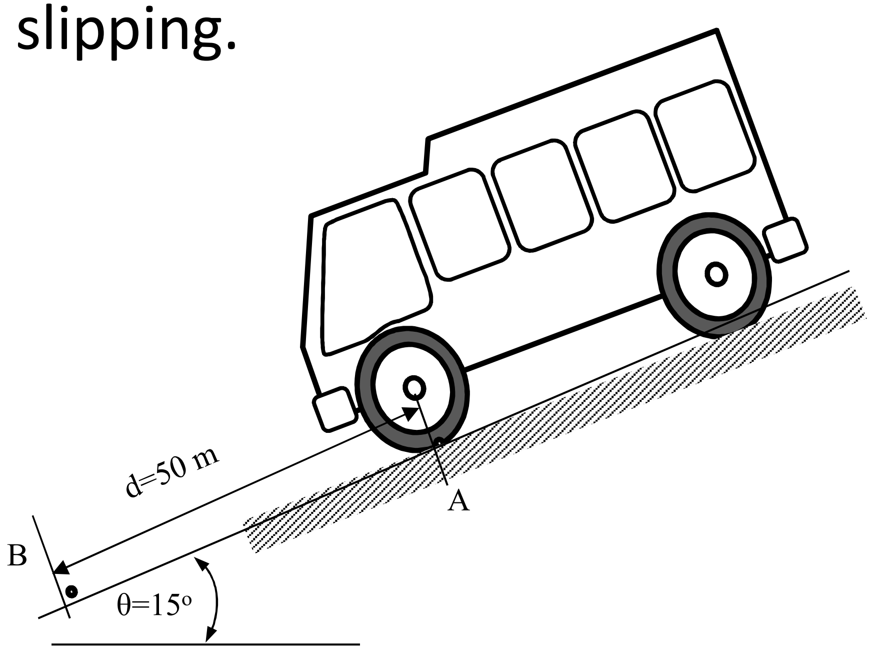
- The linear spring and potential energies are the same as derived for particles. Torsional spring energy is given by

$$PEe_{torsion} = \frac{1}{2} k_t \theta^2$$

where k_t is torsional stiffness (N.m/rad) and θ is angular displacement in rad

Example 3-11: Conservation of Energy for planar rigid bodies

The minibus shown in Figure E3-11 has mass m_B=4400 kg including the passengers but excluding its four wheels. Each wheel has mass m_w = 40 kg, radius r = 0.35 m, and radius of gyration k = 0.30 m, computed about an axis passing through the wheel's axle. Determine the speed of the minibus after it has travelled a distance d =50 m along a plane of inclination θ = 15º starting from rest. Use the energy method and assume that the wheels roll without slipping.



Solution

Applying the energy conservation equation given by Equation 2-13, we have

$$KE_A + PE_A + (PEe)_A = KE_B + PE_B + (PEe)_B + E_{loss}$$

Since there is no sliding (i.e. frictional force neglect), $E_{loss} = 0$. In additional there is no spring element means that $(PEe)_A = (PEe)_B = 0$. Setting the datum at point B makes $PE_B = 0$ and

$$PE_A = (m_B + 4m_w) gd \sin\theta \tag{1}$$

The kinetic energy at A is zero since the vehicle started from rest, i.e. $KE_A = 0$. The kinetic energy at B is given by

$$KE_B = KE_{rot} + KE_{trans} = 4\left(\frac{1}{2} I_w \omega^2\right) + \frac{1}{2}(m_B + 4m_W) v^2$$

Solution

where ω and v are angular velocity of each wheel and linear velocity of each wheel and the body, and I_W is mass moment of inertia of each wheel. I_W is given by

$$I_w = m_w k^2 \qquad (3)$$

For slipping, the angular velocity of the wheels is related to the linear velocity as

$$\omega = \frac{v}{r} \qquad (4),$$

where r is the radius of each wheel. Substituting the initial conditions and (2) to (4) into (1), we have

$$0 + (m_B + 4m_w)gd\sin\theta + 0 = 4\left[\frac{1}{2}(m_w k^2)\left(\frac{v}{r}\right)^2\right] + \frac{1}{2}(m_B + 4m_W)v^2 + 0 + 0 + 0$$

$$v = \sqrt{\frac{2(m_B + 4m_W)gd\sin\theta}{4m_W\left(\frac{k^2}{r^2}+1\right) + m_B}} = \sqrt{\frac{2[4400 + 4(40)](9.81)(50)\sin 15}{4(40)\left(\frac{0.30^2}{0.35^2}+1\right) + 4400}} \qquad \underline{\underline{v = 15.7 \text{ m/s}}}$$

Planar Kinetics of Rigid Bodies: Impulse and Momentum

Linear and Angular Momentum

Linear Momentum

$$L = mv_G$$

Angular Momentum

The angular momentum of this particle about point P is equal to the moment of the particle's linear momentum about point P

$$(H_P)_i = r \text{ x } m_i v_i \qquad H_G = I_G \omega$$

Translation

$$L = mv_G \quad \text{and} \quad H_G = 0$$

Rotation about a Fixed Axis

$$L = mv_G \qquad H_G = I_G \omega$$

General Motion

When a rigid body is subjected to general plane motion the total momentum about G, is the effect of both translational and rotational momentum. i.e.,

$$H_O = I_G \omega + d(mv_G)$$

where d is the moment arm about G.

Principle of Impulse and Momentum for Planar Rigid Bodies

- ***Principle of Linear Impulse and Momentum***

$$\sum \int_{t_1}^{t_2} F dt = m(v_G)_2 - m(v_G)_1$$

- ***Principle of Angular Impulse and Momentum***

The equation of general plane motion for a rigid of mass may be written as

$$\sum M_G = \frac{d}{dt}(I_G \omega)$$

Rearranging the terms and integrating between the limits ω = ω_1 at t =t_1 and ω = ω_2 at t =t_2 we have

$$\sum \int_{t_1}^{t_2} M_G dt = I_G \omega_2 - I_G \omega_1$$

- ***General Principle of Impulse and Momentum for a Rigid Body***

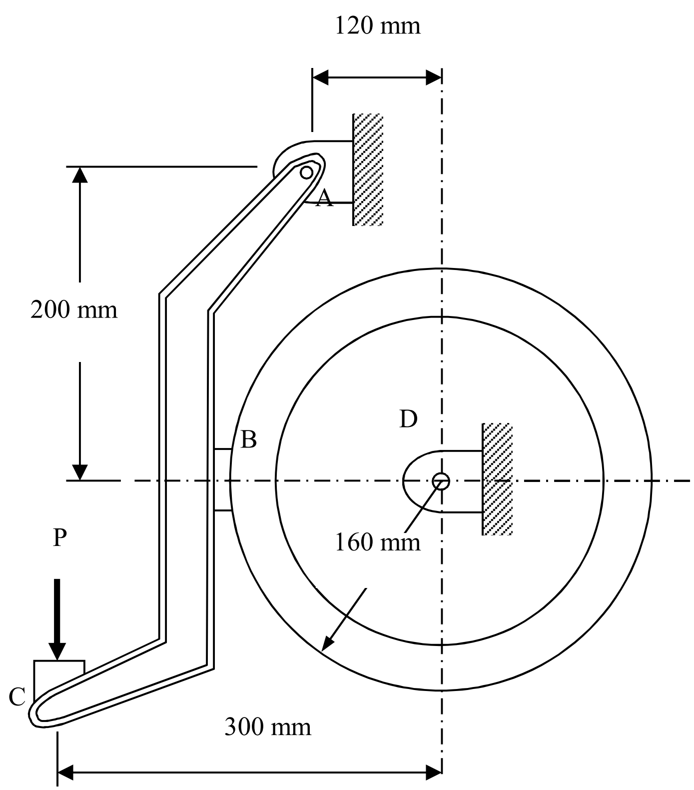
$$m(v_{Gx})_1 + \sum \int_{t_1}^{t_2} F_x dt = m(v_{Gx})_2$$

$$m(v_{Gy})_1 + \sum \int_{t_1}^{t_2} F_y dt = m(v_{Gy})_2$$

$$I_G \omega_1 + \sum \int_{t_1}^{t_2} M_G dt = I_G \omega_2$$

Example 3-14: Principle of Impulse and Momentum

The 160-mm-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the drum and the flywheel is 18 kg.m^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the angular velocity of the fly wheel is 360 rpm anti-clockwise when a force P of magnitude 300 N is applied to the pedal C, determine the time take for flywheel to come to a stop.



Solution

The free-body diagram is shown in Figure.

Pedal

$$\sum \int_{t_1}^{t_2} M_A dt = I_A\omega_2 - I_A\omega_1 \qquad \left[P(0.14) - R(0.2) + \mu R(0.04)\right]t = 0 - 0$$

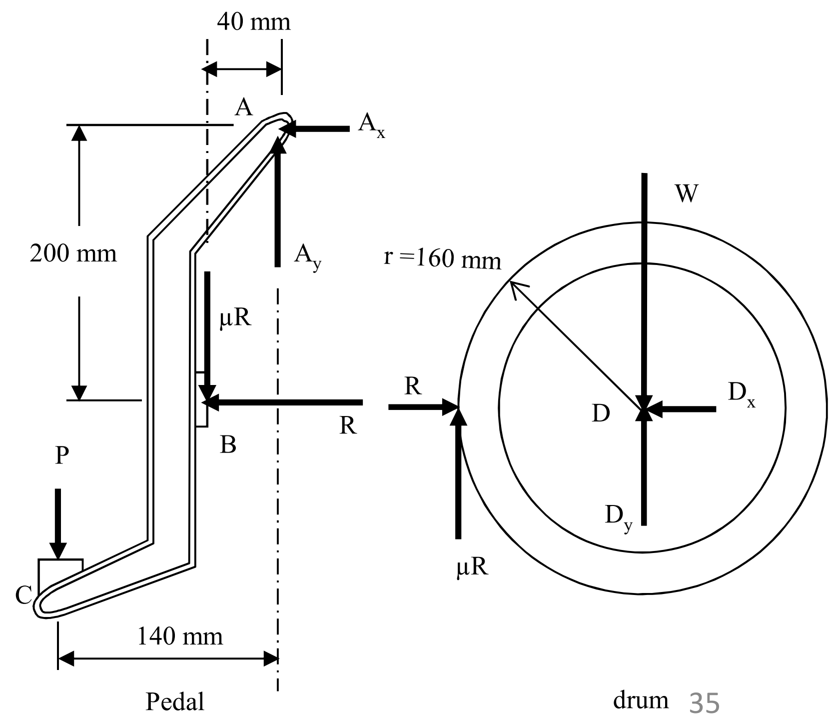
$$\Rightarrow R = \frac{P(0.14)}{0.2 - 0.04\mu} = \frac{(300)(0.14)}{0.2 - 0.04(0.35)} \qquad R = 225.8\ \mathrm{N}$$

Drum

$$\sum \int_{t_1}^{t_2} M_D dt = I_D\omega_2 - I_D\omega_1$$

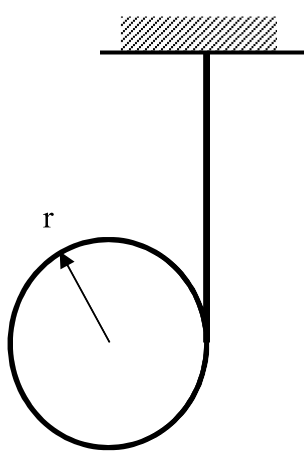
$$\left[-\mu R(0.16)\right]t = I_D(0) - I_D\omega_1$$

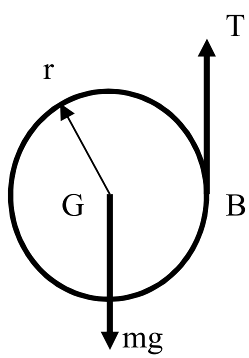
$$t = \frac{I_D\omega_1}{0.16\mu R} = \frac{(18)\left(360\frac{2\pi}{60}\right)}{0.16(0.35)(225.8)} \qquad t = 53.7\ \mathrm{s}$$



Example 3-10: General Plane Motion

A wire of negligible mass is wrapped around the outer surface of the disk of mass m and radius r. If the disk is released from rest, determine its angular velocity at time t.





$$I_B\omega_2 = I_B\omega_1 + \sum\int_{t_1}^{t_2} M_B dt$$

$$\left(I_G + mr^2\right)\omega_f = \left(I_G + mr^2\right)(0) + mgrt$$

$$\left(\frac{1}{2}mr^2 + mr^2\right)\omega_f = mgrt$$

$$\omega_f = \frac{2}{3}\frac{g}{r}t$$

Principle of Conservation of Momentum

Conservation of Linear Momentum

$$\sum m_i (v_i)_1 = \sum m_i (v_i)_2$$

Conservation of Angular Momentum

$$\sum I_G \omega_1 = \sum I_G \omega_2$$

Example 3-15: Conservation of Angular Momentum

Two wheels A and B have masses m_A= 10 kg and m_B = 6 kg and radii of gyration about their central vertical axes of k_A= 0.5 m and k_B= 0.8 m, respectively. If they are freely rotating in the same direction at ω_A = 5 rad/s and ω_B =8 rad/s about the same vertical axis, determine their common angular velocity after they are brought into contact and slipping between them stops.

Solution

Using the principle of conservation of angular momentum, we have

$$I_A(\omega_1)_A + I_B(\omega_1)_B = I_A(\omega_2)_A + I_B(\omega_2)_B$$

$$\left(m_A k_A{}^2\right)(\omega_1)_A + \left(m_B k_B{}^2\right)(\omega_1)_B = \left(m_A k_A{}^2\right)(\omega_2)_A + \left(m_B k_B{}^2\right)(\omega_2)_B \ldots\ldots 1$$

Since the two disc coupled,

$$(\omega_2)_A = (\omega_2)_B = \omega_2$$

Substituting it into (1) and making the subject we have

$$\omega_2 = \frac{\left(m_A k_A{}^2\right)(\omega_1)_A + \left(m_B k_B{}^2\right)(\omega_1)_B}{m_A k_A{}^2 + m_B k_B{}^2} = \frac{\left(10\text{x}0.5^2\right)(5) + \left(6\text{x}0.8^2\right)(8)}{\left(10\text{x}0.5^2\right) + \left(6\text{x}0.8^2\right)}$$

$$\omega_2 = 6.82\ \text{rad/s}$$