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2011

DYNAMICS OF MACHINERY

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KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

INSTITUTE OF DISTANCE LEARNING

(BSC MECHANICAL ENGINEERING, YEAR III)

ME 361**: DYNAMICS OF MACHINERY**

Credit: 3

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Icon #1 Learning Objective	Icon #2 Learning Activity	Icon #3 Unit Assignments	Icon #4 Review	Icon #5 Summary
Icon #6 Time For Activity	Icon #7 Self Assessment	Icon #8 Group Discussion	Icon #9 Read	Icon #10 New Terms
Icon #11 Answer Tips	Icon #12 Note/Learning Tip	Icon #13 Pause	Icon #14 Interactive CD	Icon #15 Online

2. Guidelines for making use of learning support (virtual classroom, etc.)

This course material is also available online at the virtual classroom (v-classroom) Learning Management System. You may access it at www.kvcit.org

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COURSE INTRODUCTION

There are many problems in engineering whose solutions require the application of the principles dynamics. Structural design of any vehicle, such automobile, aeroplane or ship requires consideration of the motion it is subjected to. This is also true for many mechanical devices such as motors, pumps, turbines, vibrators and industrial machines. In addition, common household items such switches, watches, calculators, blenders, microwaves require dynamic analysis. Further, investigations of vehicular accidents and predictions of the motions of artificial satellites, projectiles, and space craft are based on principles of dynamics.

This course introduces students to theories and applications of three-dimensional (3D) engineering rigid-body dynamics. The fundamentals learned in ME 161/162 Basic Mechanics and ME 262 Theory of Machines are extended to three-dimensional rigid bodies.

COURSE OVERVIEW

The course is organised into six units. Each unit is designed to be self-explanatory though the contents of the units are inter-related.

Unit 1 covers 3D kinematics of rigid bodies. Coordinate frames, rotation sequence, and angular velocity and acceleration are determined using the transport theorem. Relative motion equations are derived and applied different moving frames.

In Unit 2, we discuss the intrinsic properties of a rigid body that are related to dynamics. These include mass, mass moment and product of inertia.

Unit 3 covers the 3D kinetics of rigid bodies. We begin the unit with the Newton's second law of motion, force and moment balances. Equations of motion are derived using both rectangular coordinates and Euler's angles. We cover the gyroscopic motion and its effect in machinery.

In Unit 4, we analyse planer linkages to determine forces and moments acting links. The analyses cover both static and dynamic conditions. Unit 5 is solely devoted to balancing of forces in machinery. It begins with the theory and ends with application to multicylinder engines. In Unit 6, we cover flywheels and governors.

COURSE OBJECTIVES

On completion of the course, you should be able to

1. Perform 3D kinematic analysis of a rigid body with a moving frame.

2. Determine centre of mass, and moment and product of inertia of a rigid body about an axis
3. Perform 3D kinetic analysis of a rigid body.
4. Explain and determine gyroscopic effect on a body and machinery
5. Perform static and dynamic analysis of a planer linkage
6. Statically and dynamically balance a planer linkage including multicylinder engines
7. Analyse and select flywheel and governor for a particular application

COURSE OUTLINE

- Unit 1: Rigid bodies motions and applications
- Unit 2: Gyroscopic motion and its effect in gyroscopic effects in machinery
- Unit 3: Geometric properties of rigid bodies
- Unit 3: Static and inertia forces and torques in mechanisms.
- Unit 5: Balancing of multi-cylinder engines including radial and V-engines.
- Unit 6: Fluctuation of energy and speed in machines, crank-effort and turning moment diagrams, flywheels and governors

COURSE STUDY GUIDE

This provides a monthly/weekly schedule of progress of your learning.

Week #	Unit/Session	FFFS/Practical/Exam/Quiz
1	Unit 1	FFFS/Practical/Assignment
2	Unit 2	FFFS/Practical/Assignment/Quiz
3	Unit 3	FFFS/Practical/Assignment
4	Unit 4	FFFS/Practical/Assignment/Exams
5	Unit 5	FFFS/Practical/Assignment
6	Unit 6	FFFS/Practical/Assignment//Exams

GRADING

Continuous assessment: 30%

End of semester examination: 70%

RESOURCES

You will require a calculator with matrix (up to 9 x 9) capability or a laptop computer installed with MathCAD or Matlab software for this course.

REFERENCES

1. R. S. Khumi and J. K. Gupta. *Theory of Machines*, Eurasia Publishing House (PVT.) Ltd, New Delhi. 2005.
2. R. C. Hibbeler, *Principles of Dynamics*, Pearson Prentice Hall, Upper Saddle River, New Jersey, 2006.
3. A. Bedford and W. Fowler, *Engineering Mechanics: Dynamics*, Upper Saddle River, New Jersey, 2005.
4. A. Higdon and W. B. Stiles, *Engineering Mechanics, Vol. II Dynamics*, Prentice-Hall Inc. Englewood Cliffs, New Jersey.
5. F. P Beer, E. R. Johnston, Jr. and W. E. Clausen, *Vector Mechanics for Engineers: Dynamics*, McGraw Hill Higher Education, 2004.

6. James H. Williams, Jr., *Fundamentals of Applied Dynamics*, John Wiley & Sons, Inc., New York 1996.

7. C. E. Wilson and J. P. Sadler, *Kinematics and Dynamics of Machinery*, 2nd Edition, Harper Collins College Publishers, 1993.

8. J.E. Shigley and J. J. Uicker, Jr., *Theory of Machines and Mechanisms*, McGraw-Hill Book Company, 1981.

9. Haim Baruh, *Analytical Dynamics*, McGraw-Hill Companies, 1999.

10. B. J. Hamrock and S. R. Schmid and B. O. Jacobson. *Fundamentals of Machine Elements*, 2nd Edition, McGraw Hill Higher Education, 2005.

11. R. L. Norton. *Design of Machinery: An Indtroduction to Synthesis and Analysis of Mechanisms and Machines.* McGraw Hill, 1992.

READING LIST / RECOMMENDED TEXTBOOKS / WEBSITES / CD ROM

1. 1. R. S. Khumi and J. K. Gupta. *Theory of Machines*, Eurasia Publishing House (PVT.) Ltd, New Delhi. 2005. – **Units 3, 4 and 6**

2. R. C. Hibbeler, *Principles of Dynamics*, Pearson Prentice Hall, Upper Saddle River, New Jersey, 2006. – **Units 1 and 2**

3. A. Bedford and W. Fowler, *Engineering Mechanics: Dynamics*, Upper Saddle River, New Jersey, 2005. – **Units 1 and 2**

4. A. Higdon and W. B. Stiles, *Engineering Mechanics, Vol. II Dynamics*, Prentice-Hall Inc. Englewood Cliffs, New Jersey. – **Units 1 and 2**

5. F. P Beer, E. R. Johnston, Jr. and W. E. Clausen, *Vector Mechanics for Engineers: Dynamics*, McGraw Hill Higher Education, 2004. – **Units 1 and 2**

6. C. E. Wilson and J. P. Sadler, *Kinematics and Dynamics of Machinery*, 2nd Edition, Harper Collins College Publishers, 1993. – **Units 4 and 5**

7. R. L. Norton. *Design of Machinery: An Indtroduction to Synthesis and Analysis of Mechanisms and Machines.* McGraw Hill, 1992. – **Unit 4 and 5**

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UNIT 1

THREE-DIMENSIONAL KINEMATICS OF RIGID BODIES

Introduction

In this unit, we will continue the study of dynamics of rigid bodies by extending the concept of moving frame equations developed in ***ME 262 Mechanisms Synthesis and Analysis I***. We will attach moving coordinate systems to bodies and express components of motions using the moving both fixed and moving coordinate systems. Then, displacement, velocity and accelerations of moving frames will be computed using the ***transport theorem***. This unit ends with general motion analysis. General motion is complex motion, which involves moving coordinate systems, and is a combination of translational and rotational motions.

The subject of dynamics has two parts: *kinematics*, which is the study of the geometry of motion, and *kinetics*, which is the analysis of the forces that cause the motions. Kinematics is concerned with the time behaviour of position, velocity and acceleration of systems and components of the systems without reference to the forces that cause the motion or forces generated by the motion. However, dynamic force analysis or kinetics requires knowledge of kinematics. Thus, a thorough understanding of kinematics is critical to dynamic force analysis, which is important input to machine design process.

The field of kinematics, like any engineering field, consist of two major components: analysis and synthesis (or design). The focus of this unit is on kinematic analysis of three-dimensional mechanisms. Kinematic synthesis is the design or creation of mechanism (interconnect bodies) to perform a desired motion. This field of kinematics is a specialty field, which is beyond the scope of this course.

Learning Objectives

After reading this unit you should be able to:

1. Perform kinematic analysis of a body subjected to rotation and general motion in three-dimension
2. Describe and predict a relative position, velocity and acceleration of a rigid body using translating and rotating axes.

Unit content

Session 1-1: Types of Rigid Body Motion

1-1.1 Pure Translation

1-1.2 Pure Rotation

1-1.3 General Motion

Session 1-2: General Rotational Motion

1-2.1 Euler's Theorem of Rotation
1-2.2 Finite Rotations
1-2.3 Infinitesimal Rotation
1-2.4 Angular Velocity
1-2.5 Linear Velocity
1-2.6 Angular Acceleration
1-2.7 Linear Acceleration
1-2.8 Transport Theorem

Session 1-3: General Motion

1-3.1 Position and Displacement
1-3.2 Velocity
1-3.3 Acceleration

SESSION 1-1: Types of Rigid Body Motion

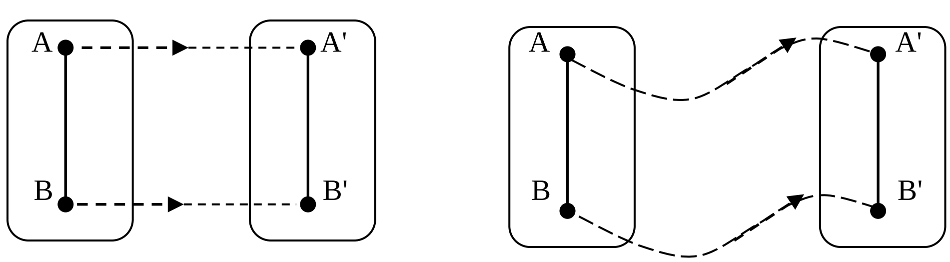
Motion of a rigid body may be classify into three categories:

- Pure Translation
- Pure Rotation
- General Motion

1-1.1 Pure Translation

A rigid body exhibits a pure translational motion when the body moves such that the body remains parallel to its past and future positions at all times. In rectilinear translation, all paths of points on the body are parallel and in straight lines as illustrated in Figure 1(a). A rigid body may translate along a curved path without any rotation, as illustrated in Figure 1(b). The motion exhibited by an aircraft during land is a typical example of pure translational motion along a curved path. Such a motion is called curvilinear translation.

> In translation, there is no rotation of any line in the body at any time though the path of the motion may be a curve.



(a) **(b)**

Figure 1: (a) Rectilinear, and (b) Curvilinear Translational Motions

1-1.2 Pure Rotation

It is the angular motion about a fixed point or line. Rotational motion is described using only rotational parameters such angular displacement, angular velocity and angular acceleration. Rotation may be grouped into two categories:

- Rotation about a fixed axis
- Rotation about a fixed point

When the angular velocity vector lies on a fixed axis or the rotation is about a single axis, as shown in Figure 2(a), then, the rotation is about a fixed axis. All two-dimensional (2D) rotations exhibit fixed axis rotation.

A body exhibits rotation about fixed point when the angular velocity vector of the body lies on more than one axis, as shown in Figure 2(b). In the figure, the shaft spins with an angular velocity ω_1 about the z axis. The sleeve at A rotates about axis m with angular velocity ω_3 while the arm OB rotates about point O with a rate of ω_2. Though the body is rotating about three axis, all the rotations are about the fixed point O.

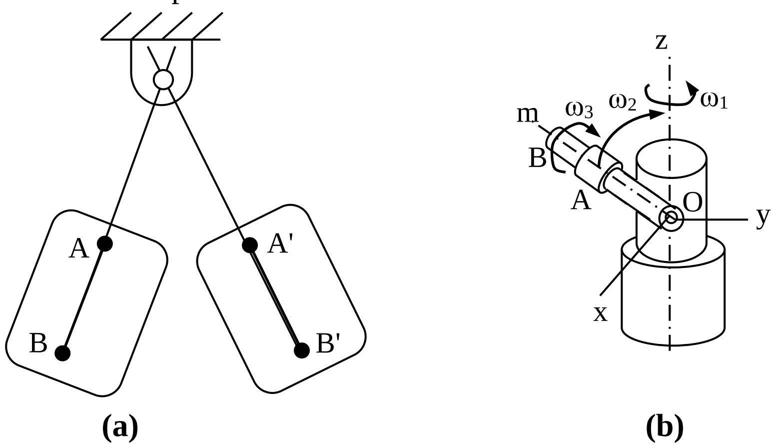


Figure 2: Rotation about a (a) Fixed Axis, and (b) Fixed Point

1-1.3 General Motion

General motion is involves combination of translational and rotational motions. It is a combined translation and rotation, as illustrated in Figure 3. In the figure the paths of A and B are not parallel. In addition, the angular displacement for points A and B are not the same. From the initial position to the final position, the body translates and rotates at the same time. An unrestricted three-dimensional motion of a rigid body has six degrees of freedom, which are translation and rotation about x, y and z axis.

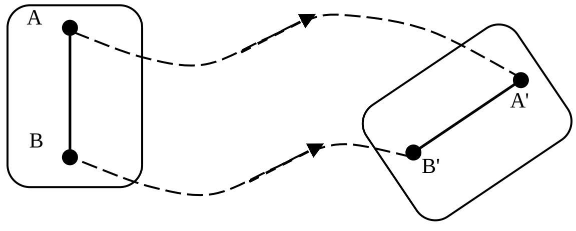


Figure 3: General Motion of a Rigid Body

SESSION 1-2: General Rotational Motion

1-2.1 Euler's Theorem of Rotation

The theorem states that two components of rotations about a different axes passing through a point are equivalent to a single rotation about an axis passing through the point.

1-2.2 Finite Rotations

Consider two finite rotations θ_1 and θ_2 applied to the block shown in Figure 4. Each rotation has a magnitude of 90^o and direction defined by the right-hand rule. The resultant orientation of the block is shown on the right. When these two rotations are applied, the resultant orientation of the order $\theta_1 + \theta_2$, shown in Figure 4(a) is not the same as that of the order $\theta_2 + \theta_1$, shown in Figure 4(b). Thus, finite rotations do not obey the commutative law of addition. That is,

$$\theta_1 + \theta_2 \neq \theta_2 + \theta_1$$

> Finite rotates are not classified as vectors and they do not obey Euler's law of rotation. Also, finite rotations are not commutative.

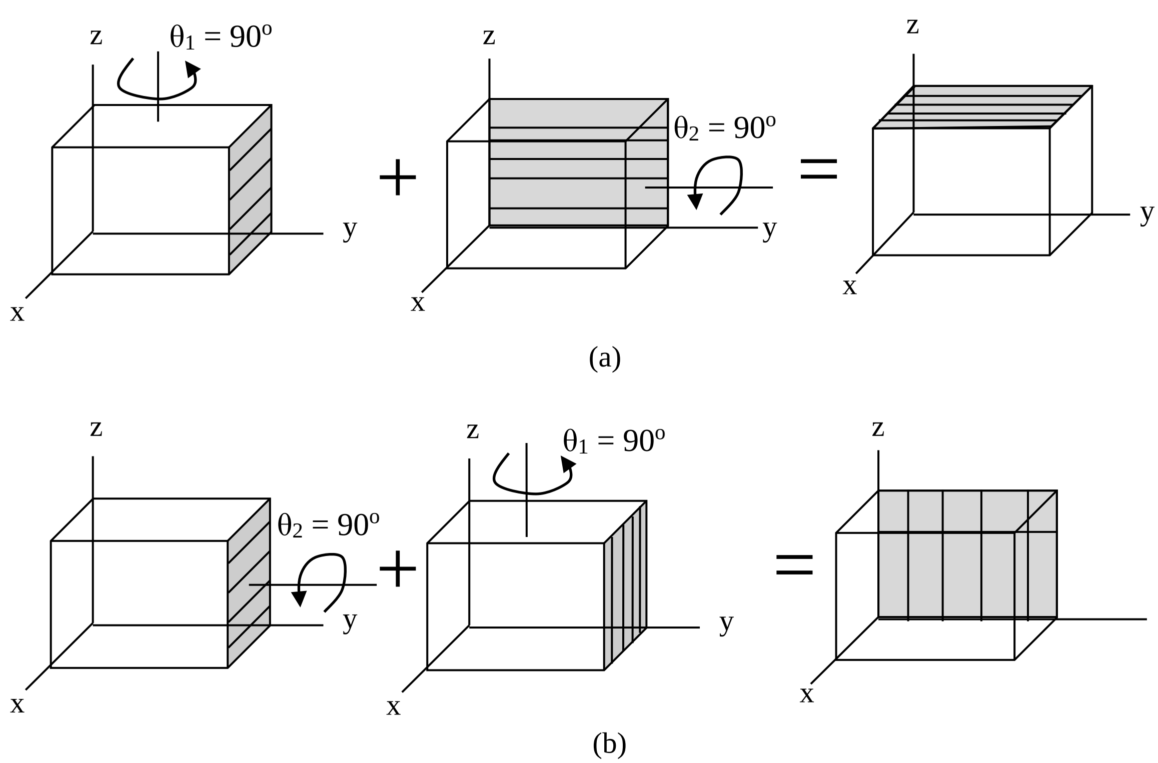


Figure 4: Non-Commutative of Finite Rotation

Learning Activity

Perform the following activities:

1. Orientate a book to the horizontal plane with the binding side pointing toward the left.
2. Rotate the book 90^o in the anti-clockwise direction about the vertical axis
3. Rotate the book 90^o in the anti-clockwise direction about the horizontal axis parallel to a line through your shoulders.
   If you did it correctly, the binding should be parallel to a line through the shoulders and points downward.
4. Orientate the same book to the horizontal plane with the binding pointing toward the left as before.
5. Rotate the book 90^o in the anti-clockwise direction about the horizontal axis parallel to a line through your shoulders.
6. Rotate the book 90^o in the anti-clockwise direction about the vertical axis
   If you did it correctly, the binding should be parallel to the vertical axis and facing you. The two activities give different results, which indicate that finite rotation is not commutative.

1-2.3 Infinitesimal Rotations

In defining angular motions of a body subjected to three-dimensional (3D) motion, only small rotations are considered. Such rotations may be classified as vectors with the same direction as the axis of rotation. In addition, they obey Euler's law of rotation. The resultant rotation of two or more infinitesimal rotations is given as

$$d\theta = d\theta_1 + d\theta_2 + .. + d\theta_n \qquad \textbf{Equation 1}$$

1-2.4 Angular Velocity

Angular velocity of the body is defined as

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \qquad \textbf{Equation 2}$$

Angular velocity ω is vector quantity and has the same directional vector as that of the axis of rotation. If a body is subjected to more than one rotational motion with angular velocities $\omega_1 = \dot{\theta}_1$, $\omega_2 = \dot{\theta}_2$,..., and $\omega_n = \dot{\theta}_n$ then, the angular velocity of the body is the vector sum of all the rotational velocities, given by

$$\omega = \omega_1 + \omega_2 + ..\omega_n \qquad \textbf{Equation 3}$$

The above equation is derived by taking derivate of Equation 1 with respect to time.

1-2.5 Linear Velocity

The linear velocity of any point P on a body rotating about a fixed point is given by the cross product as

$$v = \omega \text{ x } r$$

where r is the position vector of P measured from the fixed point of rotation.

1-2.6 Angular Acceleration

The angular velocity of a body is derived from derivative of the angular velocity ω, as

$$\alpha = \frac{dw}{dt} = \dot{\omega}.$$

1-2.7 Linear Acceleration

The linear acceleration of any point P on a body rotating about a fixed point is given by

$$a = \alpha \text{ x } r + \omega \text{ x } (\omega \text{ x } r) \qquad \textbf{Equation 4}$$

The linear acceleration accounts for both changes in magnitude and direction. The first and second terms on the right of the equation are respectively the *tangential* and *normal components* of the acceleration. Normal acceleration is directed toward the point of rotation, whereas the tangential is tangential to rotation.

1-2.8 Transport Theorem

Consider a vector u observed from a moving coordinate system xyz, which is rotating with angular velocity Ω, as illustrated in Figure 5. The vector u may be expressed as

$$u = u_x i + u_y j + u_z k \qquad \textbf{(a)}$$

Differentiating the above equation with respect to time yields

$$\frac{du}{dt} = \left(\frac{du_x}{dt} i + \frac{du_y}{dt} j + \frac{du_z}{dt} k \right) + \left(u_x \frac{di}{dt} + u_y \frac{dj}{dt} + u_z \frac{dk}{dt} \right) \qquad \textbf{(b)}$$

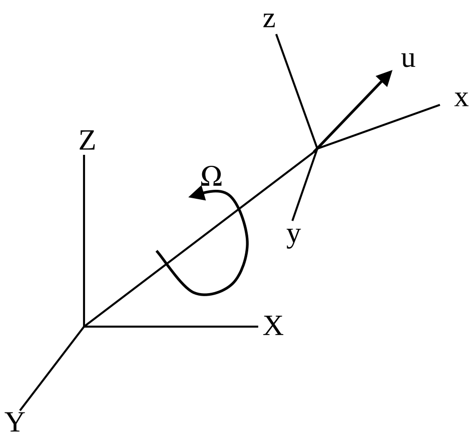


Figure 5: A Moving Frame xyz Observed from a Fixed Frame XYZ

The first bracket on the right side of Equation (b) denotes the change in u as viewed by an observer on the moving frame and it is due to change in magnitudes of the components of u. Let denote this local derivative term by

$$\left(\frac{du}{dt}\right) = \left(\dot{u}\right)_{rel} = \left(\frac{du_x}{dt} i + \frac{du_y}{dt} j + \frac{du_z}{dt} k\right) \quad \textbf{(c)}$$

The next bracket on the right side of Equation (b) is the change in u due to rotation of the coordinate system. The derivative of each unit vector represents only a change in the direction of the vector, which is expressed as

$$u_x \frac{di}{dt} + u_y \frac{dj}{dt} + u_z \frac{dk}{dt} = u_x \Omega \text{ x } i + u_y \Omega \text{ x } j + u_z \Omega \text{ x } k = \Omega \text{ x } u \quad \textbf{(d)}$$

Substituting Equation (c) and (d) into Equation (b) and simplifying, we have

$$\dot{u} = \left(\dot{u}\right)_{rel} + \Omega \text{ x u} \qquad \textbf{Equation 5}$$

The relation given by Equation 5 is known as *transport theorem*. This equation may be written in matrix form as

$$\frac{d}{dt}\{u\} = \left(\dot{u}\right)_{rel} + [\Omega] \text{ x } \{\text{u}\} \qquad \textbf{Equation 6}$$

Example 1-1

Find the angular velocity and angular acceleration of sleeve A of Figure E1-1

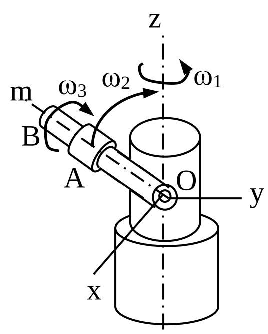


Figure E1-1

Solution

The angular velocity of the sleeve is the sum of the angular velocities, which is

$$\underline{\underline{\omega = \omega_1 + \omega_2 + .\omega_3}}$$

Note that the shaft is not carried by any part, whereas the arm is carried by the shaft and the sleeve is carried by both the shaft and the arm. The relative angular accelerations are determined separately, and then added. Using the transport theorem given by Equation 5, the angular velocity of the shaft is

$$\alpha_1 = (\dot{\omega}_1) + 0 \text{ x} (\omega_1) \qquad\qquad \alpha_1 = \dot{\omega}_1$$

Similarly, angular acceleration of the arm OB which is being transported by the shaft is

$$\alpha_2 = (\dot{\omega}_2) + \omega_1 \text{ x} (\omega_2)$$

Angular acceleration of the sleeve B which is being transported by both the shaft and the arm is

$$\alpha_3 = (\dot{\omega}_3) + \omega_1 \text{ x} (\omega_3) + \omega_2 \text{ x} (\omega_3)$$

The angular acceleration of the sleeve is sum of the angular accelerations, that is

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 \qquad\qquad \underline{\underline{\alpha = (\dot{\omega}_1) + (\dot{\omega}_2 + \omega_1 \text{ x } \omega_2 .) + (\dot{\omega}_3 + \omega_1 \text{ x } \omega_3 + \omega_2 \text{ x } \omega_3)}}$$

Example 1-2

The horizontal shaft BC in Figure E1-2 is rigidly attached to the vertical shaft AB. The vertical shaft AB rotates at a constant angular velocity ω_1 = 4 rad/s about the vertical axis while the disc mounted at the end of BC is rotates about the horizontal axis at ω_2 = 1 rad/s and α_2 =2 rad/s^2. Determine the angular velocity and acceleration of the disc.

Solution

The angular velocity of frame point B is simply the vector addition of angular velocities of shaft BC about AB and disc about point C. Thus,

$$\omega = \omega_1 + \omega_2 = (4k) + (1j) \qquad \underline{\underline{\omega = j + 4k \text{ (rad/s)}}}$$

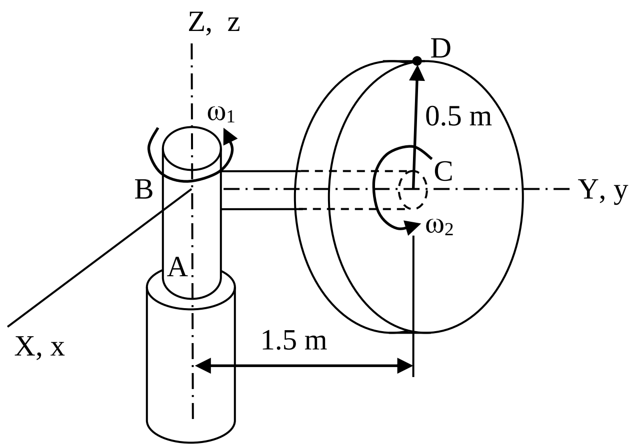


Figure E1-2

The horizontal shaft BC is transporting the disc, while both the disc and the shaft are rotating independently. Using the transport theorem, the angular acceleration of the disk is

$$\alpha = (\dot{\omega}_1 + \dot{\omega}_2)_{x,y,z} + \omega_1 \text{ x } \omega_2 = (0 + 2j) + 4k \text{ x } j \qquad \underline{\underline{\alpha = -4i + 2j \text{ (rad/s}^2)}}$$

Example 1-3

In Figure E1-3, the bent pipe ABC is rigidly attached to a vertical shaft, which rotates with angular velocity ω_1 = 3 rad/s and angular acceleration α_1 = 1 rad/s^2 relative to the ground. The circular disc mounted at the point C rotates with constant angular velocity ω_2 = 5 rad/s relative to the bar. Knowing that $\theta = 30^o$, determine the

(a) angular velocity of the disc relative to the ground
(b) angular acceleration of the disc relative to the ground

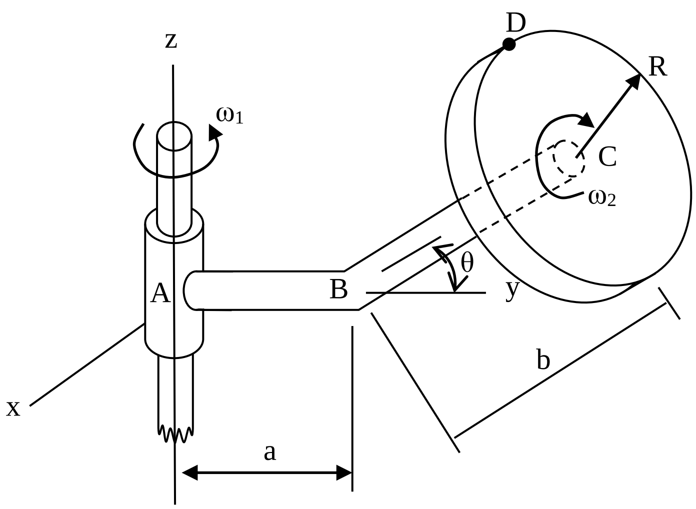


Figure E1-3

Solution

The angular velocity with reference to the fixed frame point B is simply the vector addition of angular velocities of AB about the vertical axis and disc about point C. The angular velocity of the disc with respect to point C is resolved into vertical and horizontal components. Notice that both the vertical and horizontal components of ω_2 are negative. The vertical component is negative because it is as a result of rotation direction which gives j x i = -k. Similarly, the negative sign on the horizontal component come from the result of i x k = -j. Therefore

$$\omega = \omega_1 - \omega_2(\cos\theta\ \mathrm{j} + \sin\theta\ \mathrm{k}) = (3k) - 5(\cos 30^o\ \mathrm{j} + \sin 30^o\ \mathrm{k}) \quad \underline{\underline{\omega = 4.33\mathrm{j} + 0.5\mathrm{k}}}$$

(a) The angular acceleration of the AB is

$$\alpha_1 = (\dot{\omega}_1)_{x,y,z} + 0 \text{ x } \omega_1 = 1 \qquad \alpha_1 = 1\,\mathrm{k}\ \text{rad/s}$$

Similarly, the angular acceleration of the disc with respect to the fixed frame is

$$\alpha_2 = (\dot{\omega}_2)_{x,y,z} + \omega_1 \text{ x } \omega_2 = (0) + (3k) \text{ x } \left[-5(\cos 30^o\ j + \sin 30^o\ k)\right] \qquad \alpha_2 = 12.99i$$

$$\alpha = \alpha_1 + \alpha_2 \qquad \underline{\underline{\alpha = 12.99i + k\ \ (\text{rad/s}^2)}}$$

SESSION 1-3: General Motion

1-3.1 Position and Displacement

Consider two points A and B shown in Figure 6. The location of each point is specified with reference to the fixed frame X, Y and Z. Point A coincides with the origin of the moving frame x, y and z. By vector addition, the position of B is

$$r_B = r_A + r_{B/A} \qquad \textbf{Equation 7}$$

where r_B is the position of point B with reference to the origin of the fixed frame X, Y and Z

r_A is the position of point A with reference to the origin of the fixed frame X, Y and Z

$r_{B/A}$ is the displacement of point B with reference to point A, the origin of the moving frame x, y and z.

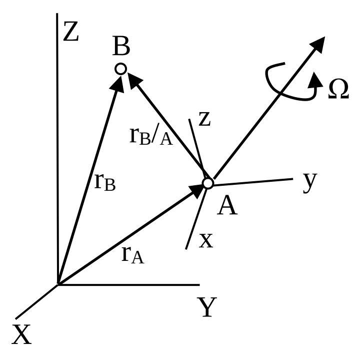


Figure 6: Primary and Secondary Coordinate Systems

1-3.2 Velocity

The velocity of point B with reference to the fixed frame is obtained by differentiating the position vector given by Equation 7 with time, we have

$$\frac{d}{dt}(r_B) = \frac{d}{dt}(r_A) + \frac{d}{dt}(r_{B/A}) \qquad \textbf{(1)}$$

The terms $\frac{d}{dt}(r_B)$ and $\frac{d}{dt}(r_A)$ represent the velocities v_B and v_A, respectively, of points B and A relative to the fixed frame. The term $\frac{d}{dt}(r_{B/A})$ represents the displacement between the two points which are having both translational and rotational motions. Hence, the time derivation of $r_{B/A}$ has two components: translation component due to change magnitude and rotation component due to change in direction. Using the transport theorem given by Equation 5, the time derivative of $r_{B/A}$ is given by

$$\frac{d}{dt}(r_{B/A}) = \left(\frac{d}{dt}(r_{B/A})\right)_{xyz} + \Omega \text{ x } \mathrm{r}_{\mathrm{B/A}} \,,$$

which may written in terms of velocities v_B and v_A as

$$v_{B/A} = (v_{B/A})_{xyz} + \Omega \text{ x } \mathrm{r}_{\mathrm{B/A}} \qquad \textbf{(2)}$$

where $(v_{B/A})_{xyz}$ velocity of point B relative to point A and Ω is the angular velocity of the moving x,y z frame of reference. Substituting equation (2) into equation (1) and replacing $\dot{r}_B$ and $\dot{r}_A$ with v_B and v_A, respectively, we have

$$v_B = v_A + (v_{B/A})_{xyz} + \Omega \text{ x } \mathrm{r}_{\mathrm{B/A}} \qquad \textbf{Equation 8}$$

1-3.3 Acceleration

The acceleration of point B with reference to the fixed frame is obtained by differentiating Equation 8 with time, which gives

$$\frac{d}{dt}(v_B) = \frac{d}{dt}(v_A) + \frac{d}{dt}(v_{B/A})_{xyz} + \frac{d}{dt}(\Omega \text{ x } \mathrm{r}_{\mathrm{B/A}}) \qquad \textbf{(1)}$$

The immediate terms on the left and right of the equation sign of the above equation respectively represent accelerations a_B and a_A of points B and A relative to the fixed XYZ frame. Using the transport theorem, the time derivative of the second term on the right of the above equation is given by

$$\frac{d}{dt}(v_{B/A})_{xyz} = (\dot{v}_{B/A})_{xyz} + \Omega \text{ x } (\mathrm{v}_{\mathrm{B/A}})_{xyz}$$

$$\frac{d}{dt}(v_{B/A})_{xyz} = (a_{B/A})_{xyz} + \Omega \text{ x } (\mathrm{v}_{\mathrm{B/A}})_{xyz} \qquad \textbf{(2)}$$

In addition, the time derivative of the last term of the right of equation (1) gives

$$\frac{d}{dt}(\Omega \text{ x } \mathrm{r}_{\mathrm{B/A}}) = \frac{d}{dt}(\Omega)\text{x } \mathrm{r}_{\mathrm{B/A}} + \Omega \text{ x } \frac{d}{dt}(\mathrm{r}_{\mathrm{B/A}}) = \frac{d}{dt}(\Omega)\text{x } \mathrm{r}_{\mathrm{B/A}} + \Omega \text{ x } (\Omega \text{ x } \mathrm{r}_{\mathrm{B/A}})$$

$$\frac{d}{dt}(\Omega \text{ x } \mathrm{r}_{\mathrm{B/A}}) = \dot{\Omega} \text{ x } \mathrm{r}_{\mathrm{B/A}} + \Omega \text{ x } (\mathrm{v}_{\mathrm{B/A}})_{xyz} + \Omega \text{ x } (\Omega \text{ x } \mathrm{r}_{\mathrm{B/A}}) \qquad \textbf{(3)}$$

where $\frac{d}{dt}(\Omega) = \dot{\Omega}$ = angular acceleration

Substituting equations (2) and (3) into (1) and simplifying, we have

$$a_B = a_A + (\mathrm{a}_{B/A})_{xyz} + \dot{\Omega} \times \mathrm{r}_{B/A} + 2\Omega \times (\mathrm{v}_{B/A})_{xyz} + \Omega \times (\Omega \times \mathrm{r}_{B/A}) \qquad \textbf{Equation 9}$$

For plane motions, the above equation reduces to

$$a_B = a_A + (\mathrm{a}_{B/A})_{xyz} + \dot{\Omega} \times \mathrm{r}_{B/A} + 2\Omega \times (\mathrm{v}_{B/A})_{xyz} - \Omega^2\, \mathrm{r}_{B/A}$$

Equation 9 is applicable to both plane (2D) and spatial (3D) motions.

Example 1-4

Determine the velocity and acceleration of point D at the instant shown in Figure E1-4.

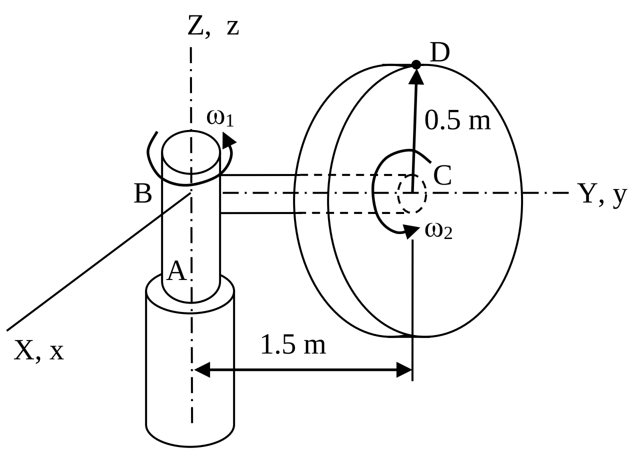


Figure E1-4

Solution

From *Example 1-2*, absolute angular velocity and acceleration are

$$\omega = j + 4k \text{ (rad/s)}$$

$$\alpha = -4i + 2j \text{ (rad/s}^2\text{)}$$

The velocity of point D becomes

$$v_D = v_B + (v_{D/B})_{xyz} + \omega \times \mathrm{r}_{D/B} = 0 + 0 + (j + 4k) \times (1.5\mathrm{j} + 0.5\mathrm{k}) \qquad \underline{\underline{v_D = 5.5i \text{ (m/s)}}}$$

The acceleration of point D is calculated as

$$a_D = a_B + (\mathrm{a}_{\mathrm{D/B}})_{xyz} + \alpha \text{ x } \mathrm{r}_{\mathrm{D/B}} + 2\omega \text{ x } (\mathrm{v}_{\mathrm{D/B}})_{xyz} + \omega \text{ x } (\omega \text{ x } \mathrm{r}_{\mathrm{D/B}})$$

$$a_D = 0 + 0 + (-4i + 2j) \text{ x } (1.5\mathrm{j} + 0.5\mathrm{k}) + 2(j + 4k) \text{ x } (0) + (j + 4k) \text{ x}[(j + 4k) \text{ x } (1.5\mathrm{j} + 0.5\mathrm{k})]$$

$$\underline{\underline{a_D = i + 24j - 11.5k \ (\mathrm{m/s^2})}}$$

Example 1-5

The assembly shown in Figure E1-5 consists of two rods AB and AD, and a plate BCDE, which are welded together. If the assembly is rotating about AB with angular velocity is 15 rad/s anticlockwise, determined the velocity and acceleration of corner E.

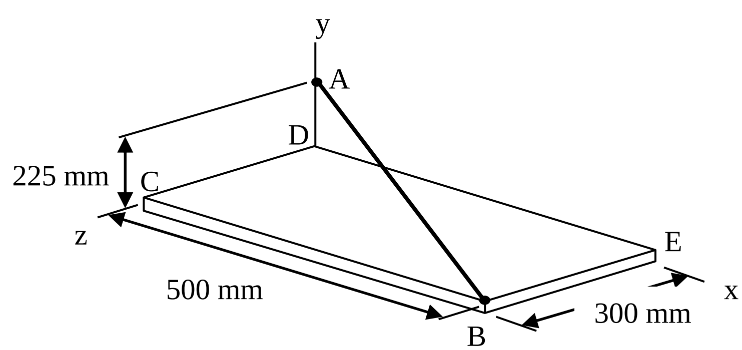


Figure E5

Solution

$$r_{B/A} = 0.5i - 0.225j + 0.3k$$

$$\omega = |\omega| \frac{r_{B/A}}{|r_{B/A}|} = (10) \frac{(0.5i - 0.255j + 0.3k)}{|0.5i - 0.255j + 0.3k|} \qquad \omega = 8.0i - 3.6j + 4.8k \text{ rad/s}$$

$$v_E = v_B + \omega \text{ x } \mathrm{r}_{\mathrm{E/B}} = 0 + \begin{vmatrix} i & j & k \\ 8.0 & -3.6 & 4.8 \\ 0 & 0 & -0.3 \end{vmatrix} \qquad \underline{\underline{v_E = 1.08i + 2.4j \text{ m/s}}}$$

$$a_E = a_B + \alpha \text{ x } \mathrm{r}_{\mathrm{E/B}} + \omega \text{ x } (\omega \text{ x } \mathrm{r}_{\mathrm{E/B}}) = a_B + \alpha \text{ x } \mathrm{r}_{\mathrm{E/B}} + \omega \text{ x } \mathrm{v}_{\mathrm{E}}$$

$$a_E = 0 + 0 + \begin{vmatrix} i & j & k \\ 8.0 & -3.6 & 4.8 \\ 1.08 & 2.4 & 0 \end{vmatrix} \qquad \underline{\underline{a_E = -11.52i + 5.18j + 23.09k \ \text{ m/s}^2}}$$

Example 1-6

In Figure E1-6, rod AB is connect by ball-and-socket joints to collar A and to the 160-mm-radius disc C. Collar A slides on a vertical rod parallel to the y axis. If the disc rotates anticlockwise at a constant rate ω = 3 rad/s in the vertical plane, determine the (a) angular velocity of AB and linear velocity of collar A, and (b) angular acceleration of AB and linear acceleration of collar A for the position shown.

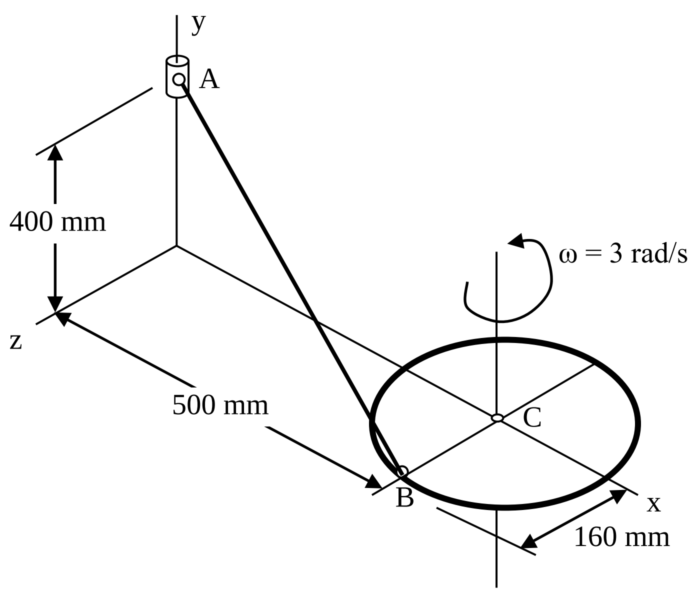


Figure E1-6

Solution

$$v_B = v_C + \omega_{B/C} \text{ x } \mathrm{r_{B/C}} = 0 + (3j \text{ x } 0.16\mathrm{k}) \qquad v_B = 0.48\mathrm{i} \text{ m/s}$$

Denoting the angular velocity of A relative to B by ω_{BA}, we have

$$v_A = v_B + \omega_{BA} \text{ x } \mathrm{r_{A/B}} = 0.48i + \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ -0.5 & 0.4 & -0.16 \end{vmatrix}$$

$$v_A = (0.48 - 0.16\omega_y - 0.48\omega_z)\mathrm{i} + (0.16\omega_x - 0.50\omega_z)\mathrm{j} + (0.40\omega_x + 0.50\omega_y)\mathrm{k} \qquad \text{(a)}$$

Note that point A is attached to the collar and can only move in a direction parallel to the y axis. As a result, we have

$$v_A = V_A j \tag{b}$$

Equating the coefficients of the unit vectors of equations (a) and (b), we have

$$0.16\omega_y + 0.48\omega_z = 0.48 \tag{c}$$

$$0.16\omega_x - 0.50\omega_z = V_A \tag{d}$$

$$0.40\omega_x + 0.50\omega_y = 0 \tag{e}$$

The above three equations cannot be solved completely for the four unknowns. An additional equation is obtained by considering the constraint at point A. The connection at A allows rotation of BA about the vertical rod parallel to the y axis and about axis perpendicular to the plane containing the vertical rod and AB. It prevents rotation of AB about the axis AE illustrated in Figure S1-6, which is perpendicular to vertical axis CD and lies in the plane of the vertical rod and AB. That is, the projection of angular velocity ω on $r_{E/A}$ must be zero, which yields

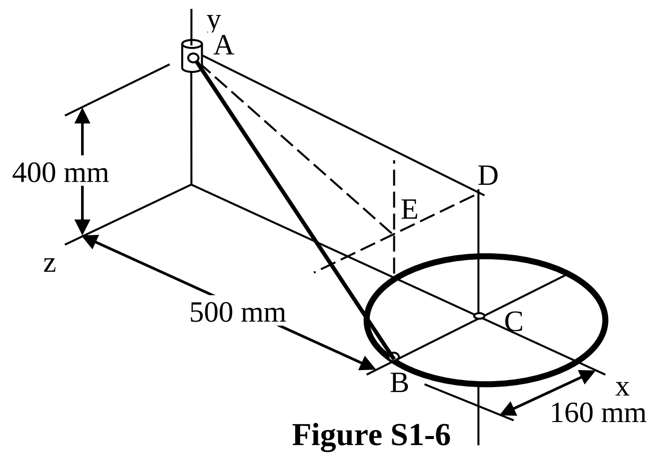


Figure S1-6

$$(\omega_x i + \omega_y j + \omega_z k).(0.5i + 0.16k) = 0 \qquad 0.5\omega_x + 0.16\omega_z = 0 \tag{f}$$

Solving equations (c) to (f) simultaneously, we have

$$\underline{\underline{\omega = (-0.348i + 0.279j + 1.089k)\,\text{rad/s}}}$$

$$\underline{\underline{v_A = -0.6j \text{ m/s}}}$$

Note that

$$v_{A/B} = v_A - v_B = -0.6j - 0.48\text{i}$$

The acceleration of point B is

$$a_B = a_C + \alpha \text{ x } r_{B/C} + \omega \text{ x } v_{B/C} = 0 + 0 + 3j \text{ x } 0.48\text{i} \qquad a_B = 1.44k \text{ m/s}^2$$

Denoting the angular acceleration of BA by α, we have

$$a_A = a_B + \alpha_{BA} \text{ x } \mathrm{r_{A/B}} + \omega_{BA} \text{ x } \mathrm{v_{A/B}} = 1.44k + \begin{vmatrix} i & j & k \\ \alpha_x & \alpha_y & \alpha_z \\ -0.5 & 0.4 & -0.16 \end{vmatrix} + \begin{vmatrix} i & j & k \\ -0.348 & 0.279 & 1.089 \\ -0.48 & -0.6 & 0 \end{vmatrix}$$

$$a_A = \left(0.653 - 0.16\alpha_y - 0.48\alpha_z\right)\mathrm{i} + \left(-0.522 + 0.16\alpha_x - 0.50\alpha_z\right)\mathrm{j} + \left(1.783 + 0.40\alpha_x + 0.50\alpha_y\right)\mathrm{k}$$

Similar to the velocity, the point A can only move in a direction parallel to the y axis. As a result, we have

$$a_A = Aj \tag{g}$$

Equating the coefficients of the unit vectors of the above two equations, we have

$$0.653 - 0.16\alpha_y - 0.48\alpha_z = 0 \tag{h}$$

$$-0.522 + 0.16\alpha_x - 0.50\alpha_z = A \tag{i}$$

$$1.783 + 0.40\alpha_x + 0.50\alpha_y = 0 \tag{j}$$

Similar to the velocity, the projection of the angular acceleration on $\mathrm{r_{E/A}}$ must be zero, which yields

$$\left(\alpha_x i + \alpha_y j + \alpha_z k\right).\left(0.5i + 0.16k\right) = 0 \qquad 0.5\alpha_x + 0.16\alpha_z = 0 \tag{k}$$

Solving equations (g) to (j) simultaneously, we have

$$\underline{\underline{\alpha = \left(-0.752i - 2.965j + 2.349k\right)\mathrm{rad/s^2}}}$$

$$\underline{\underline{a_A = -1.817j \text{ m/s}^2}}$$

Example 1- 7

In Figure E1-7, the pendulum shown consists of two rods AB and BC. AB is pin supported at A, which allows AB to rotate only in the Y-Z plane. BC spins about the AB, and the sleeve at C slides on BC. Determine the velocity and acceleration of the sleeve at the instant shown.

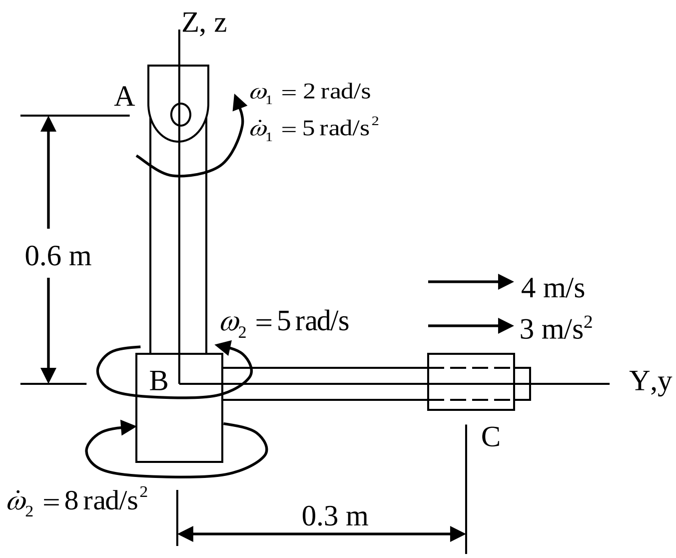


Figure E1-7

Solution

The angular velocity of C is

$$\omega_C = \omega_1 + \omega_2 = (2i) + (5j) \qquad\qquad \omega_C = 2i + 5j$$

Using Transport theorem, the angular acceleration of C is

$$\alpha_C = \dot{\omega}_1 + \dot{\omega}_2 + \omega_1 \text{ x } \omega_2 = (5i) + (-8j) + 2i \text{ x } 5\text{j} \qquad\qquad \alpha_C = 5i - 8j + 10k$$

The velocity of point C is

$$v_C = v_A + (v_{C/A})_{xyz} + \omega \text{ x } \text{r}_{\text{C/A}} = 0 + (4j) + (2i + 5j) \text{ x } (0.3\text{j} - 0.6\text{k}) \qquad \underline{\underline{v_C = -3i + 1.2j + 0.6k \text{(m/s)}}}$$

The acceleration of point C is

$$a_C = a_A + (\text{a}_{\text{C/A}})_{xyz} + \alpha \text{ x } \text{r}_{\text{C/A}} + 2\omega \text{ x } (\text{v}_{\text{C/A}})_{xyz} + \omega \text{ x } (\,\omega \text{ x } \text{r}_{\text{C/A}})$$

$$a_D = 0 + 3j + (5i - 8j + 10k) \text{ x } (0.3\text{j} - 0.6\text{k}) + 2(2i + 5j) \text{ x } (4\text{j}) + (2i + 5j) \text{ x } [(2i + 5j) \text{ x } (0.3\text{j} - 0.6\text{k})]$$

$$\underline{\underline{a_C = 1.8i + 6j + 30k \text{ (m/s}^2\text{)}}}$$

Unit Summary

Rotation about a Fixed Axis

1. A finite rotation is not a vector quantity, while infinitesimal rotation is a vector quantity.

2. Three-dimensional kinematics of rigid bodies can be analysed with Equations 3, 9, 10, 13 and 14 using the following steps:

3. The angular velocity of a body having infinitesimal rotations is the vector sum of all the rotational velocities.

$$\omega = \omega_1 + \omega_2 + \ldots \omega_n$$

4. The angular acceleration must account for both magnitude and directional changes. This is accomplished by the use of transport theorem.

$$\dot{u} = (\dot{u})_{xyz} + \Omega \text{ x } u$$

5. Three-dimensional kinematics of rigid bodies can be analysed with Equations 3, 9, 10, 13 and 14 using the following steps:

6. Select the location and orientation of the XYZ and xyz coordinate system such that

   (a) the origins coincide (b) the axes are collinear, and (c) the axis are parallel.

7. For more than one components of angular velocity, select the secondary coordinates for each component of angular velocity such that only component of angular velocity is observed from one frame.
8. For angular velocity and acceleration, use the Equation 3 and 9 or 10, which are:

$$\omega = \sum_{i=1}^{n} \omega_i$$

$$\alpha = (\dot{\omega})_{rel} + \Omega \text{ x } \omega$$

9. For general motion velocity and acceleration, use the Equation 13 and 14, which are:

$$v_B = v_A + (\mathrm{v}_{\mathrm{B/A}})_{xyz} + \Omega \text{ x } \mathrm{r}_{\mathrm{B/A}}$$

$$a_B = a_A + (\mathrm{a}_{\mathrm{B/A}})_{xyz} + \dot{\Omega} \text{ x } \mathrm{r}_{\mathrm{B/A}} + 2\Omega \text{ x } (\mathrm{v}_{\mathrm{B/A}})_{xyz} + \Omega \text{ x } (\Omega \text{ x } \mathrm{r}_{\mathrm{B/A}})$$

10. For more than one component, consider two points at a time, with oine point as the origin and the other as “new’ point. Work your way from the fixed coordinate or known to unknown. Use relative angular velocities and accelerations instead of the absolute ones.
11. Alternatively, you may use the transport theorem to compute the absolute angular velocity and acceleration of the last point and compute the linear velocity and acceleration using the relations:

$$v = \omega \text{ x } r \qquad\qquad a = \alpha \text{ x } r$$

☑ Key terms/ New Words in Unit

1. Finite rotation

2. Infinitesimal rotation

3. Moving coordinate system

4. Transport theorem

5. Euler's theorem of rotation

Self Assessments 1

1-1. At the instant $\theta = 50^o$, the top in Figure P1.1 has three components of angular motion directed as shown and having magnitudes defined as :

ω_s = 10 rad/s, increasing at the rate of 3 rad/s^2

ω_n = 4 rad/s, increasing at the rate of 2 rad/s^2

ω_p = 5 rad/s, increasing at the rate of 3 rad/s^2

Determine the angular velocity and angular acceleration of the top.

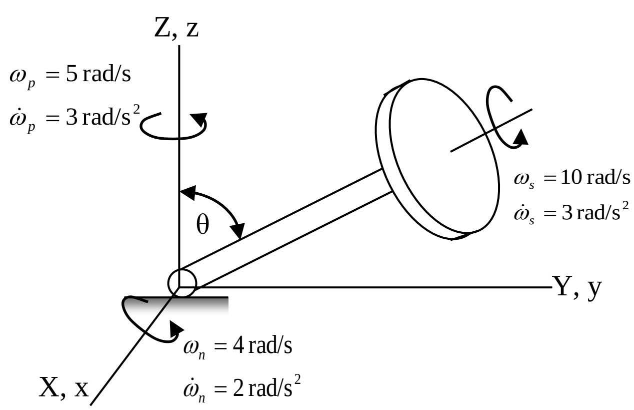


Figure P 1.1

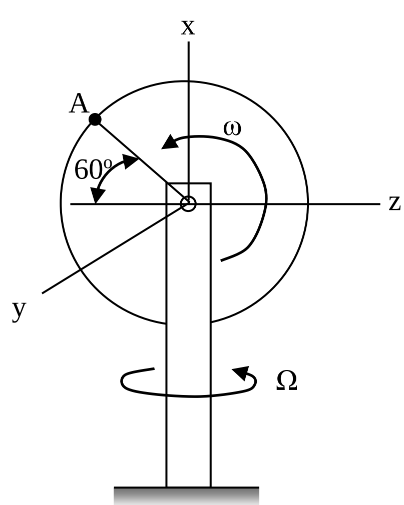


Figure P1.2

1-2. A disc of radius 0.2 m is supported by a vertical shaft as shown in Figure P1.2. The shaft rotates about its vertical axis with a constant angular velocity Ω = 4 rad/s. The disc rotates with a constant angular velocity ω = 6 rad/s relative to the shaft. Determine the angular velocity and acceleration of the disc.

1-3. Find the angular velocity and angular acceleration of disc B shown in Figure P1.3, which is spinning at the constant rate of $\omega_2 = 90/\pi$ rpm. The disc is attached to collar A, which is rotating at the angular speed of $\omega_1 = 5/\pi$ rpm, with the angular speed increasing at $0.5/\pi$ rpm/sec. Rod AB which connects the disc to the collar is pinned to the collar at A. The rod makes an angle of $\theta = 30^o$ with the vertical, which is increasing at a constant rate of $20/\pi^o$/sec. Express the angular velocity and acceleration of the disc in terms of a reference frame attached to the collar.

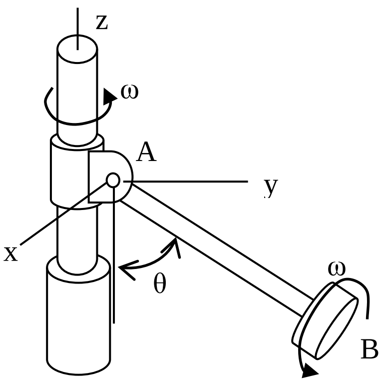


Figure P1.3

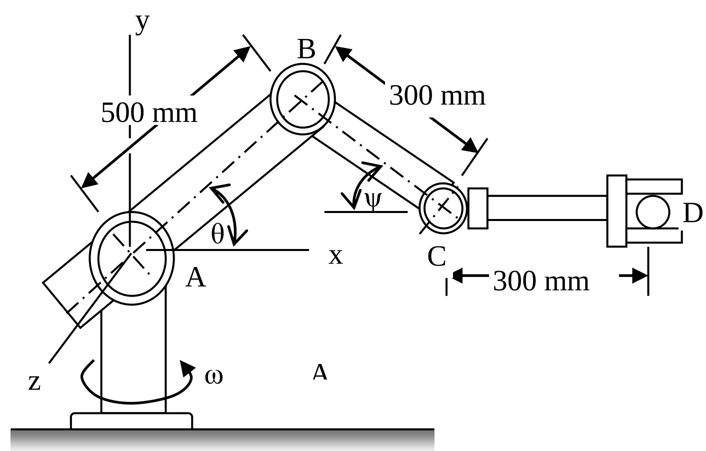


Figure P1.4

1-4. The manipulator shown in Figure P1.4 rotates about the vertical axis with constant angular velocity $\omega = 0.2$ rad/s. At the instant shown, CD is horizontal, the arm AB is rising up at $d\theta/dt = 0.5$ rad/s increase at 0.2 rad/s^2 while BC is rotating about point B with a constant angular velocity of 0.4 rad/s clockwise . Find the angular velocity and acceleration of arm CD about C such that CD is always horizontal.

1-5. In Figure P1.5, the angular velocity vector of the rigid body cube relative to a secondary frame at the centre G is $\omega = 10\mathbf{i}+8\mathbf{j}+6\mathbf{k}$ (rad/s). The velocity of the centre G of the cube relative to the primary reference frame (not shown) is $v_G = 2\mathbf{i}-1.2\mathbf{j}+2\mathbf{k}$ (m/s). Determine the velocity of point A of the cube relative to the primary reference frame.

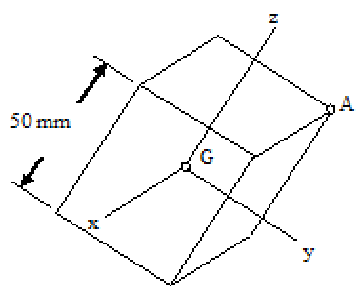


Figure P1.5

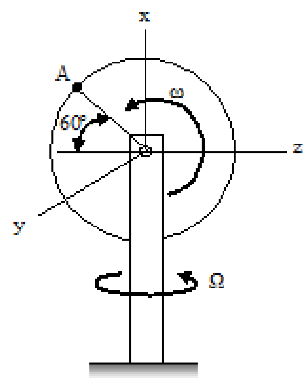


Figure P1.6

1-6. A disc of radius 0.2 m is supported by a vertical shaft as shown in Figure P1.6. The shaft rotates about its vertical axis with constant angular velocity $\Omega = 4$ rad/s. The disc rotates with a constant angular velocity $\omega = 6$ rad/s relative to the shaft. At the instant shown, determine the linear velocity and acceleration of point A.

1-7. In Figure P1.7, Rod AB of length 130 mm is connected by ball-and-socket joints to collars A and B, which slide along the two rods BC and OA. If the collar B moves toward C at a constant speed of 360 mm/s, determine the velocity of collar A.

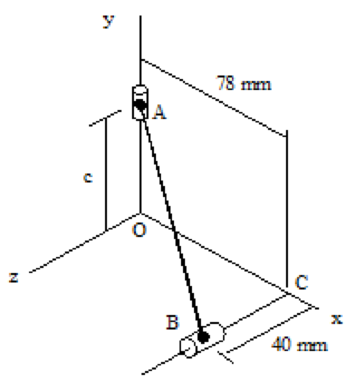


Figure P1.7

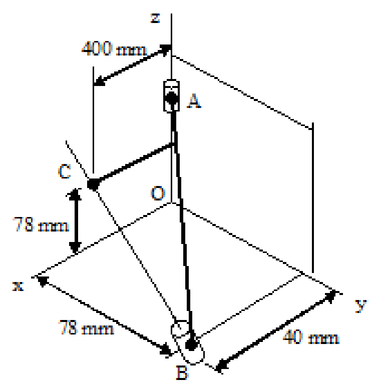


Figure P1.8

1-8. In Figure P1.8, Rod AB of length 500 mm is connected by ball-and-socket joints to collars A and B, which slide along the two rods BC and OA. If the collar B moves away from C at a constant speed of 200 mm/s, determine the velocity of collar A.

1-9. Collars A and B are attached by a rod of length 30 cm, as shown in Figure P1.9. The joint at A is a ball-and-socket and at B a pin joint. Collar A moves in the z direction, while the guide bar for collar B is on xy plane. At the instant shown, collar A is at a height of 24 cm and collar B is moving with a speed of 3 m/s towards the x axis. Find the (a) angular velocity of the rod, and (b) the sliding of collar B.

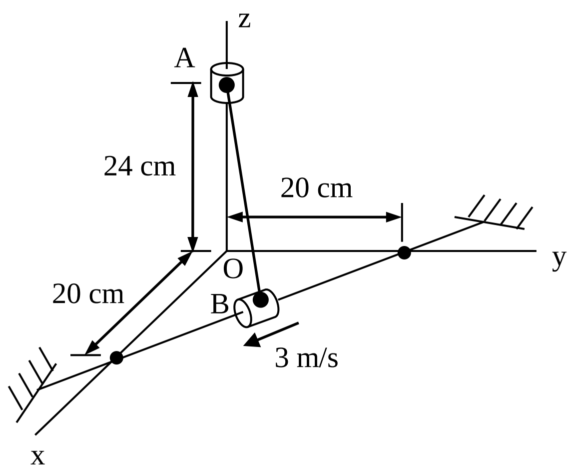


Figure P1.9

1-10. In Figure P1-10, hydraulic actuator CD is contracting a constant rate of 0.5 m/s. Determine the angular velocity of the (a) plate , and (b) actuator

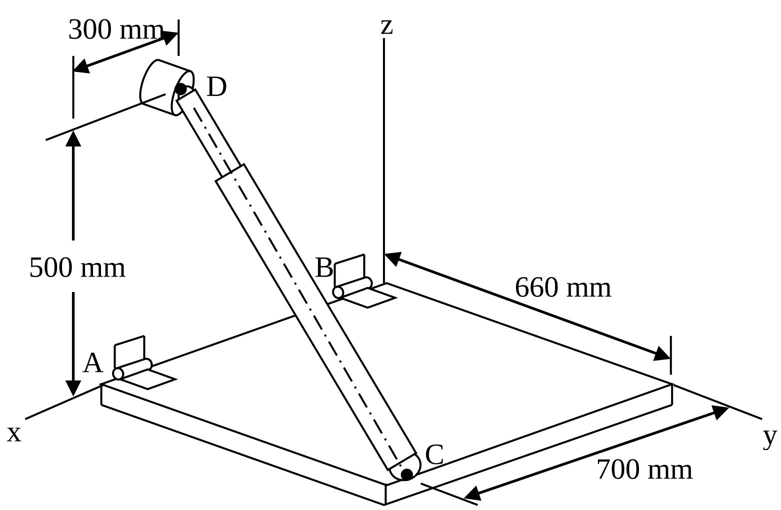


Figure P1.10

UNIT 2

RIGID BODY GEOMETRY

Introduction

In this unit, the geometrical properties of rigid bodies are considered. The term rigid body is a mathematical assumption which makes complex dynamic problems simpler. The type of motion a body exhibits depends on the location of the certain of mass, and constraints involved.

Rigid bodies have physical dimensions, which differentiate them from particles that have no size. These definitions of rigid bodies and particles are mathematical idealization, as all bodies have physical dimensions. In rigid bodies, it is assumed that the distance between any two points that constitute the body remain unchanged. Dynamic analyses of rigid motions consider both translational and rotational motion. Mass of a body represents its resistance to translational motion, and the distribution of the mass about a certain axis represents the body’s resistance to rotational motion about that axis. The mass and distribution of mass of a rigid body has significant effect on the dynamic of that body.

Learning Objectives

After reading this unit you should be able to:

1. Determine centre of mass of a rigid using integration and summation methods.
2. Determine moment and product of inertia of a body.
3. Find inertia matrix of composite body using parallel axis theorem.
4. Find inertia matrix of a rigid body after a set of rotation.
5. Know when to use integration, parallel axis and rotation methods.
6. Find moment of inertia about any arbitrary axis.

Unit content

Session 2-1: Centre of Mass

- 2-1.1 Summation Method
- 1-2.2 Integration Method
- 1-2.3 Velocity and Acceleration of Centre of Mass

Session 2-2: Mass Moments and Products of Inertia

- 2-2.1 Moments of Inertia
- 2-2.2 Products of Inertia
- 2-2.3 Inertia Matrix

Session 2-3: Transformation of Mass Moments of Inertia

2-2.1 Moments of Inertia
2-2.2 Products of Inertia
2-2.3 Inertia Matrix

SESSION 2-1: Centre of Mass

2-1.1 Summation Method

Consider a rigid consisting of N particles, as shown in Figure 7. Let m_i and r_i be the mass and its displacement from the origin O, as shown in Figure 2.1(a), of the *ith* particle (i = 1, 2... n). The location of the centre of mass G is defined as

$$r_G = \frac{1}{m}\sum_{i=1}^{n} m_i r_i \qquad \textbf{Equation 10}$$

where

$$m = \sum_{i=1}^{n} m_i \qquad \textbf{Equation 11}$$

Note that r_G is a three-dimensional vector representing the x, y and z coordinates of the centre of mass. That is, $r_G = x_G i + y_G i + z_G i$

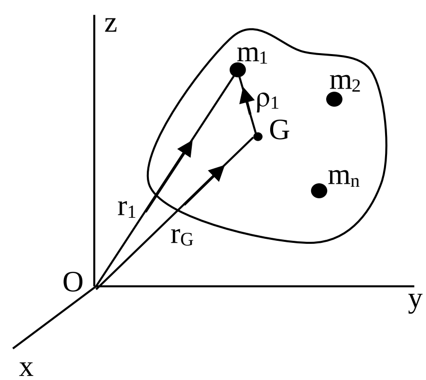


Figure 7: A System of Particles in a Rigid Body

From the figure,

$$r_i = r_G + \rho_i \qquad \text{(a)}$$

Substituting the above equation into Equation 10, we have

$$r_G = \frac{1}{m}\sum_{i=1}^{n} m_i\left(r_G + \rho_i\right)$$

$$r_G = \frac{1}{m}\sum_{i=1}^{n} m_i r_G + \frac{1}{m}\sum_{i=1}^{n} m_i \rho_i$$

The first term of the above equation gives $r_G = \frac{1}{m}\sum_{i=1}^{n} m_i r_G$, which leads to the conclusion that

$$\frac{1}{m}\sum_{i=1}^{n} m_i \rho_i = 0 \qquad \textbf{Equation 12}$$

The above equation indicates the weighted average of the displacement vector about the centre of mass is zero.

2-1.2 Integration Method

Equations 10 and 11 are useful when the rigid body involved is constituted by discrete parts. As n→∞, each particle of the rigid body can be treated as a infinitesimal mass element as shown in Figure 8. Then, the location of centre of mass G may be defined as

$$r_G = \frac{1}{m}\int_{body} r\,dm \qquad \textbf{Equation 13}$$

where

$$m = \int_{body} dm \qquad \textbf{Equation 14}$$

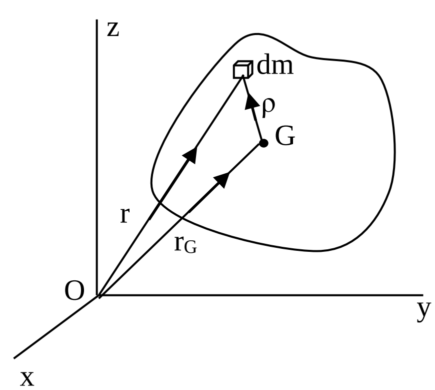


Figure 8: Relative Position of Infinitesimal Element

From the above figure,

$$r = r_G + \rho \qquad \textbf{Equation 15}$$

Substituting the above equation into Equation 11, we have

$$r_G = \frac{1}{m}\int_{body}(r_G + \rho)dm$$

$$r_G = \frac{1}{m}\int_{body} r_G dm + \frac{1}{m}\int_{body} \rho dm$$

The first term of the above equation gives $r_G = \frac{1}{m}\int_{body} r_G dm$, which leads to the conclusion that

$$\frac{1}{m}\int_{body} \rho dm = 0$$

2-1.3 Velocity and Acceleration of Centre of Mass

The first and second derivatives of Equation 10 with respective to time give

$$v_G = \frac{1}{m}\sum_{i=1}^{n} m_i \dot{r}_i$$

$$a_G = \frac{1}{m}\sum_{i=1}^{n} m_i \ddot{r}_i$$

where v_G and a_G are velocity and acceleration of the centre of mass of the body. Similarly, the first and second derivatives of Equation 12 give

$$v_G = \frac{1}{m}\int_{body} \dot{r} dm$$

$$a_G = \frac{1}{m}\int_{body} \ddot{r} dm$$

Using Equation 15, the velocity and acceleration of the centre of mass may be expressed in terms of the centre of mass as

$$v = v_G + \dot{\rho} \qquad \textbf{Equation 16}$$

$$a = a_G + \ddot{\rho} \qquad \textbf{Equation 17}$$

Example 2- 1

Find the location of the centre of mass of the 0.4 tapered aircraft wing shown in Figure E2-1, leaving the answer in terms of L and S.

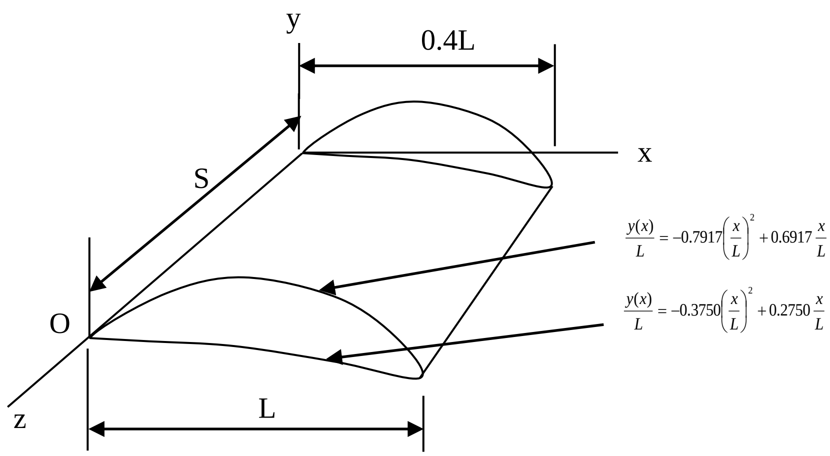


Figure E2-1

Solution

The mass of a differential element is given by $dm = \rho dydxdz$, where ρ is the mass per unit volume (density) of the wing. In this case, ρ is assumed to be constant. The length of the airfoil has a linear relationship with z. At z = 0 and z = S, the lengths of the airfoil are L and 0.4L, respectively. Based on this, the length may be written as

$$x(z) = L\left(0.4 + 0.6\frac{z}{S}\right)$$

Using equation (2.7), the total mass of the wing is given by

$$m = \rho \int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} dydxdz = \rho \int_{z=0}^{S}\int_{x=0.4L}^{L(0.4+0.6z/S)}\int_{y(x)=\left[-0.3750\left(\frac{x}{L}\right)^2+0.2750\frac{x}{L}\right]}^{\left[-0.7917\left(\frac{x}{L}\right)^2+0.6917\frac{x}{L}\right]L} dydxdz \qquad m = 0.0175\mathrm{L}^2\mathrm{S}\rho$$

The location of the centre of mass is defined by

$$r_G = \frac{1}{m}\int_{z_1}^{z_2}\int_{x_1}^{x_2 2}\int_{y_1}^{y_2} r\rho dydxdz = \frac{1}{m}\rho\int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(xi + yj + zk)dydxdz$$

$$r_G = \frac{\rho}{m}\int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} r\,dydxdz = \frac{\rho}{m}\int_{z=0}^{S}\int_{x=0.4L}^{L(0.4+0.6z/S)}\int_{y(x)=\left[-0.3750\left(\frac{x}{L}\right)^2+0.2750\frac{x}{L}\right]}^{\left[-0.7917\left(\frac{x}{L}\right)^2+0.6917\frac{x}{L}\right]L} (x \quad y \quad z)\,dydxdz$$

where the location is given in vector form in terms of unit vector as $r = (x \quad y \quad z)$. Integrating the above equation yields

$$\underline{\underline{r_G = (0.72571\text{L} \quad 0.03343\text{L} \quad 0.27143\text{S})}}$$

Example 2- 2

Find the centre of mass of the body shown in Figure E2-2.

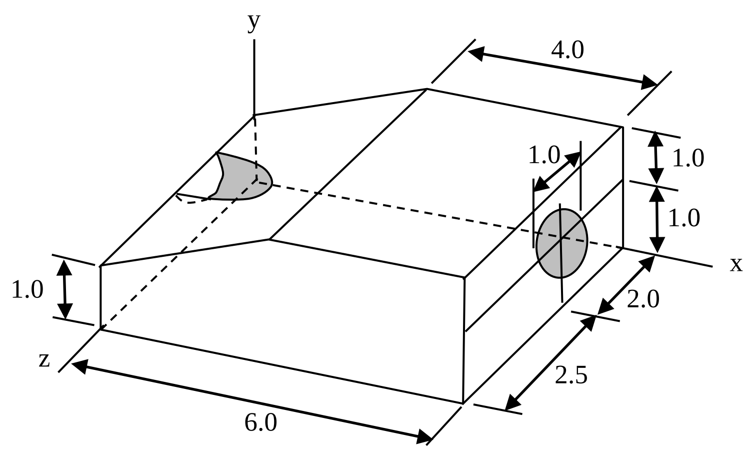


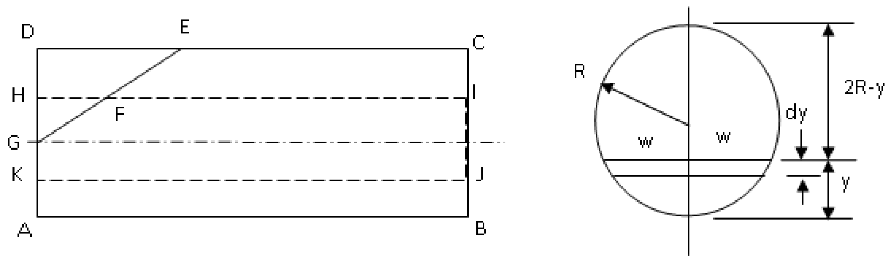
Figure E2-2

Solution

It is assumed that the body is homogeneous and hence the centre of mass is at the centre of its volume. Looking at the body from plane zx, the body may be divided into four primitive parts as shown in Figure S2-2(a). In the figure, ABCD is a solid box, IJKH is a cylindrical hole and GDE is a right triangle prism hole. Note that GFH is part of GED and IJKH, which means that it is deducted from the ABCD twice. Hence, we add FGH to account for it being deducted twice from the box. Then

Volume of ABCD = (6.0)(1.0+1.0)(2.5+2.0) = 54

Volume of GED = $\frac{1}{2}(1)(2)(2.5+2.0) = 4.5$



(a) (b)

Figure S2-2

$$\text{Volume of IJKH} = \frac{\pi}{4}(1)^2(6) = 4.71239$$

Using similar triangles,

$$\frac{HF}{HG} = \frac{DE}{DG} \qquad \frac{HF}{0.5} = \frac{2}{1} \qquad HF = 1$$

From Figure S2-2(b),

$$w.w = y(2R - y) \qquad w = \sqrt{y(2R-y)}$$

The area of the differential section is given by

$$dA = 2wdy = 2\sqrt{y(2R-y)}dy$$

At G (x=0), y = GH = 0.5 and at F , y =0. The volume of GFH is give by

$$V = \int_0^1 \int_{r=0.5}^{r(1-x)} 2\sqrt{y(2R-y)}dydx \qquad V = 0.22603$$

The z coordinate of the centre of mass of GFH is 2 and those of the x and y are determined by integration given as

$$r_x = \frac{1}{V}\int_0^1 \int_{r=0.5}^{r(1-x)} 2\sqrt{y(2R-y)}xdydx \qquad r_x = 0.65151$$

$$r_y = 1 + \frac{1}{V}\int_0^1 \int_{r=0.5}^{r(1-x)} 2\sqrt{y(2R-y)}ydydx \qquad r_y = 1.34849$$

The calculated volumes and centres of mass are fed into Table S2-2. From the table and using Equations 11 and 12, we have

m = 45.01ρ, where ρ = mass per unit volume

Table S2-2

Part	Volume	x	y	z	Vx	Vy	Vz
ABCD	54	3	1	2.25	162	54	121.5
IJKH	-4.71239	3	1	2	-14.1372	-4.71239	-9.42478
GED	-4.5	1.3333	1.666667	2.25	-6	-7.5	-10.125
GFH	0.22603	0.6515	1.34849	2	0.147261	0.304799	0.45206
Total	45.01364	-	-	-	142.0101	42.09241	102.4023

$$r_x = \frac{1}{m}\sum_{i=1}^{n} m_i x_i = \frac{1}{45.01\rho}142.01\rho \qquad \underline{\underline{r_x = 3.155}}$$

$$r_y = \frac{1}{m}\sum_{i=1}^{n} m_i y_i = \frac{1}{45.01\rho}42.09241\rho \qquad \underline{\underline{r_y = 0.935}}$$

$$r_y = \frac{1}{m}\sum_{i=1}^{n} m_i z_i = \frac{1}{45.01\rho}102.4023\rho \qquad \underline{\underline{r_y = 2.275}}$$

SESSION 2-2: Mass Moments and Products of Inertia

2-2.1 Moments of Inertia

The moment of inertia for a differential element about an axis is defined as the product of the mass of the element and square of the shortest perpendicular distance from the axis to the element. Consider the rigid body shown in Figure 9. The moment of inertia of differential element of mass dm about the x axis is

$$dI_{xx} = r_x{}^2 dm = \left(y^2 + z^2\right)dm$$

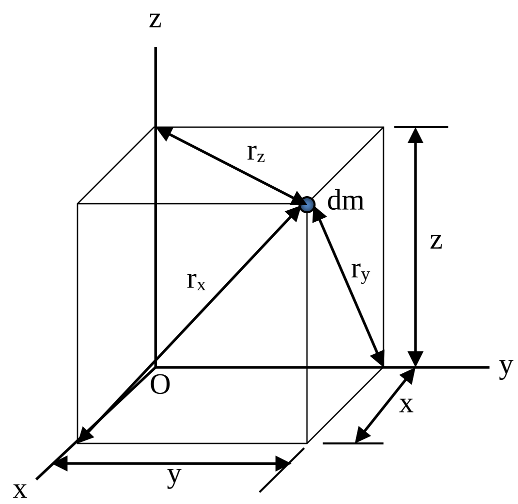


Figure 9: Geometric Dimensions of an Infinitesimal Mass Element

A moment of inertia about an axis is obtained by integrating the expression for the moment of the differential element about the axis over the entire mass of the body. The moments of inertia about the x, y and z axes may be written as

$$I_{xx} = \int_m r_x{}^2 dm = \left(y^2 + z^2\right) dm \qquad \textbf{Equation 18}$$

$$I_{yy} = \int_m r_y{}^2 dm = \left(x^2 + z^2\right) dm \qquad \textbf{Equation 19}$$

$$I_{zz} = \int_m r_{zz}{}^2 dm = \left(x^2 + y^2\right) dm \qquad \textbf{Equation 20}$$

2-2.2 Products of Inertia

The moment of inertia is always positive since it is the sum of product of mass and square of distances, which are always positive. The product of inertia for a differential element with respect a set of two perpendicular planes is defined as the product of the mass of the element and shortest perpendicular distances from the planes to the element. In Figure 9, the product of inertia for differential element of mass dm with respect to the planes y-z and x-z is

$$dI_{xy} = xydm$$

where x is the distance from plane y-z to the element and y is the distance from plane x-z to the element. Note that $dI_{yy} = dI_{xy}$ since multiplication is commutative. A product of inertia with respect to two perpendicular planes is obtained by integrating the expression for the product of inertia for the differential element respect the two perpendicular planes over the entire mass of the body. The products of inertia for each combination of planes may be written as

$$I_{xy} = \int_m xydm \qquad I_{yz} = \int_m yzdm \qquad I_{xz} = \int_m xzdm \qquad \textbf{Equation 21}$$

The moment of inertia may be positive, negative or zero, depending on the signs of the two defining coordinates.

Example 2- 3

Find the moments and products of inertia of a slender of mass m and length l about the x, y and z axes through the centre of mass.

Solution

Consider a uniform slender bar with its length along the y axis as shown in Figure S2-3. If λ is its mass per unit length, the mass of the element of length dy is dm = λdy. Also, if the lateral dimensions are neglected, then

$$I_{yy} = \int_m r_y{}^2 dm = \int_m \left(x^2 + z^2\right) dm = 0\ .$$

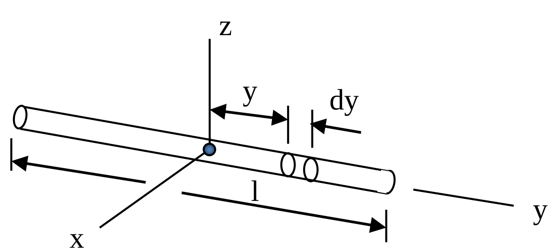


Figure S2-3

The moment of inertia about the x axis, using Equation 21, is

$$I_{xx} = \int_m r_x{}^2 dm = \int_m \left(y^2 + z^2\right) dm = \int_{-l/2}^{l/2} y^2 dm \qquad I_{xx} = \frac{1}{12}\lambda l^3$$

Substituting m = λl into the above express, we have

$$I_{xx} = \frac{1}{12} ml^2$$

The moment of inertia about z axis is equal to the moment of inertia about x axis, i.e.

$$I_{zz} = \frac{1}{12} ml^2$$

Since x and z are zero, the product of inertias are zero, i.e. $I_{xy} = I_{yz} = I_{xz} = 0$

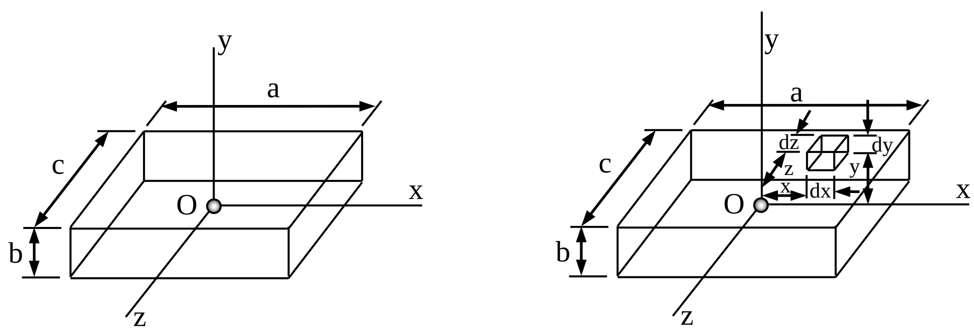
Example 2- 4

Find the moments and products of inertia of a rectangular prism of mass m and dimensions shown in Figure E2-4(a) about the x, y and z axes through the centre of mass.

Solution

Consider a uniform rectangular prism dimensions and orientation shown in Figure E2-4 (b). If λ is its density (mass per unit volume), then the mass of the element of size dx by dy by dz is dm = λdxdydy. The moment of inertia of the infinitesimal element about the x axis is

$$I_{xx} = \int_m \left(y^2 + z^2\right) dm = \int_m \left(y^2 + z^2\right) \lambda dxdydz$$



(a) **(b)**

Figure E2-4

Using triple integral for the above integration, we have

$$I_{xx} = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left(y^2 + z^2\right) \lambda dxdydz = \lambda \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \left[\left(y^2 + z^2\right)x\right]_{-a/2}^{a/2} dydz = \lambda \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \left(y^2 + z^2\right) adydz$$

$$I_{xx} = \lambda \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \left(y^2 + z^2\right) adydz = \lambda a \int_{-c/2}^{c/2} \left[\frac{y^3}{3} + yz^2\right]_{-b/2}^{b/2} dz = \lambda a \int_{-c/2}^{c/2} \left(\frac{b^3}{12} + bz^2\right) dz$$

$$I_{xx} = \lambda a \int_{-c/2}^{c/2} \left(\frac{b^3}{12} + bz^2 \right) dz = \lambda a \left[\frac{b^3}{12} z + b \frac{z^3}{3} \right]_{-c/2}^{c/2} = \lambda a \left[\frac{b^3}{12} c + b \frac{c^3}{12} \right]$$

$$I_{xx} = \lambda \frac{abc}{12} \left[b^2 + c^2 \right] \qquad \text{(a)}$$

The mass of the rectangular prism is the product of density and volume, which is

$$m = \lambda abc \qquad \text{(b)}$$

Substituting (a) into (b) yields

$$I_{xx} = \frac{m}{12} \left[b^2 + c^2 \right]$$

Following similar steps as above, the moments of inertia about the y and z axes are

$$I_{yy} = \frac{m}{12} \left[a^2 + c^2 \right]$$

$$I_{zz} = \frac{m}{12} \left[a^2 + b^2 \right]$$

Now, the product of inertia I_{xy} is given by

$$I_{xy} = \int_m xydm = \int_m xy\lambda dxdydz$$

Using triple integral for the above integration, we have

$$I_{xy} = \int_{z=-c/2}^{c/2} \int_{y=-b/2}^{b/2} \int_{x=-a/2}^{a/2} xy\lambda dxdydz = \lambda \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \left[\frac{x^2}{2} \right]_{-a/2}^{a/2} ydydz = \lambda \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \frac{1}{2} \left[\left(\frac{a}{2} \right)^2 - \left(-\frac{a}{2} \right)^2 \right]_{-a/2}^{a/2} ydydz$$

$$I_{xy} = 0$$

Following similar steps as above, we have

$$I_{yz} = I_{yz} = 0$$

Example 2- 5

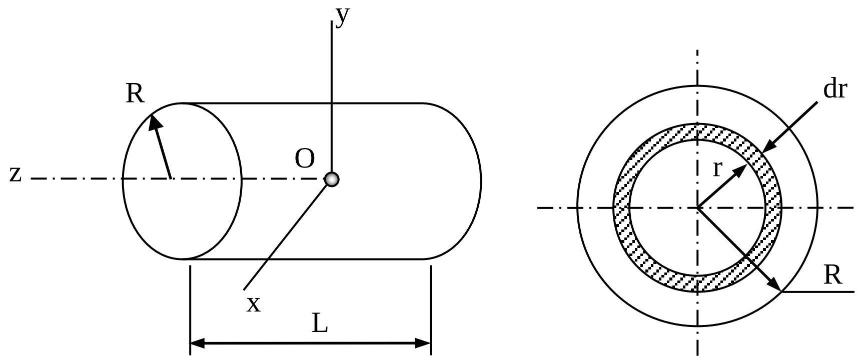
Find the three moments and the three products of inertia of a circular cylinder of mass m, length L and radius R about the x, y and z axes through the centre of mass.

Solution

Consider a uniform circular cylinder of dimensions and orientation shown in Figure S2-5(a). In addition, consider an infinitesimal ring of width *dr* at r radius from the centre of the cylinder as shown in Figure S2-5(b). If λ is the density (mass per unit volume) of the cylinder, then the mass of the infinitesimal ring is $dm = \lambda . 2\pi r dr L$. The moment of inertia of the infinitesimal ring about z axis is

$$I_{zz} = \int_m \left(x^2 + y^2\right) dm = \int_m r^2 dm$$

where is the radius, given by $r^2 = x^2 + y^2$. The length of the cylinder is constant in this case.



(a) (b)

Figure S2-5

Integrating from r = 0 to r = R, we have

$$I_{zz} = \int_0^R r^2 \left(2\lambda\pi r L dr\right) = 2\lambda\pi L \left[\frac{r^4}{4}\right]_0^R = \lambda\pi L \frac{R^4}{2} \qquad \text{(a)}$$

Integrating the mass differential equation gives

$$m = \int_0^R \lambda . 2\pi r dr L = \lambda\pi R^2 L \qquad \text{(b)}$$

Substituting (b) into (a) and simplifying yields

$$I_{zz} = \frac{1}{2} m R^2$$

The moment of inertia about x axis is given as

$$I_{xx} = \int_m \left(y^2 + z^2\right) dm = \int_m y^2 dm + \int_m z^2 dm$$

Let

$$I_{xx,\mathrm{y}} = \int_m y^2 dm$$

$$I_{xx,\mathrm{z}} = \int_m z^2 dm$$

The moment of inertia of the infinitesimal section shown in Figure 2.5 is given by

$$I_{xx,\mathrm{y}} = \int_m y^2 dm = \int_m y^2 \lambda L(2w)dy \qquad \text{(c)}$$

where the mass of the infinitesimal section $dm=\lambda L(2w)dy$.

From geometry and Figure E12-1, $w.w = (R+y)(R-y) \quad \Rightarrow w = \sqrt{(R^2 - y^2)}$ (d)

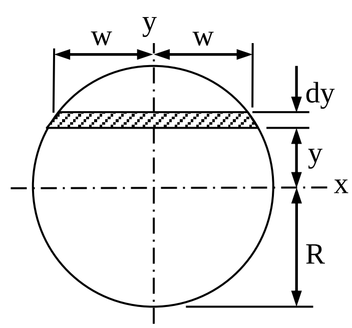


Figure E12-1

Substituting equation (d) into (c)

$$I_{xx,\mathrm{y}} = 2\lambda L \int_{-R}^{R} y^2 \sqrt{(R^2 - y^2)}dy = 2\lambda L\left[-\frac{y}{4}\sqrt{(R^2 - y^2)^3} + \frac{R^2}{8}\left(y\sqrt{(R^2 - y^2)} + R^2 \sin^{-1}\frac{y}{R} \right) \right]_{-R}^{R}$$

$$I_{xx,\mathrm{y}} = \lambda L \frac{\pi R^4}{4} = \frac{mR^2}{4}$$ where m = πR2Lλ.

From Example 2-3,

$$I_{xx,z} = \frac{1}{12} mL^2$$

The moment of inertia I_{xx} is given by:

$$I_{xx} = I_{xx,y} + I_{xx,z} = \frac{mR^2}{4} + \frac{mL^2}{12} = \frac{m}{12}\left(3R^2 + L^2\right) \qquad I_{xx} = I_{yy} = \frac{m}{12}\left(3R^2 + L^2\right)$$

The products of inertia are all zeros.

Example 2- 6

Find the moments and products of inertia of a uniform rigid sphere of mass m and radius R about the x, y and z axes through the centre of mass.

Solution

Consider an infinitesimal element shown in Figure S2-6(a) and the expanded view shown in Figure S2-6(b). The location of the elements in Cartesian coordinate system is

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\phi \tag{a}$$

The volume of the element is given by

$$dV = (rd\phi)(r\sin\phi d\theta)(dr) \tag{b}$$

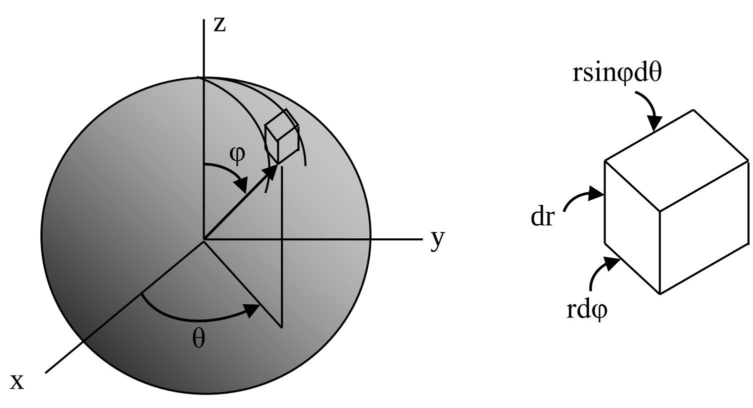
Integrating the above equation gives the total volume of the sphere, which is

$$V = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}(rd\phi)(r\sin\phi d\theta)(dr) \qquad V = \frac{4}{3}\pi R^2 \tag{c}$$

The moment of inertia of the element about the x-axis defined in terms of density λ as

$$I_{xx} = \int_m (y^2 + z^2)dm = \int_m (y^2 + z^2)\lambda dV \tag{d}$$

where λ is the density or mass per unit volume of the sphere.



(a) (b)

Figure S2-6

Substituting equations (a) and (b) into the above equation (d), and rewriting the integral in triple-integral form, we have

$$I_{xx} = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left[(r\sin\theta\sin\phi)^2 + (r\cos\phi)^2\right]\lambda(rd\phi)(r\sin\phi d\theta)(dr)$$

$$I_{xx} = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\lambda r^4\left[\sin^2\theta\sin^3\phi + \cos^2\phi\sin\phi\right]d\phi d\theta dr \qquad \text{(e)}$$

Note that

$$\int_{\phi=0}^{\pi}\sin^3\phi d\phi = \frac{4}{3} \qquad \int_{\phi=0}^{\pi}\cos^2\phi\sin\phi d\phi = \frac{2}{3} \qquad \int_{\theta=0}^{2\pi}\sin^2\theta d\theta = \pi$$

Hence, equation (e) becomes

$$I_{xx} = \lambda\int_{r=0}^{R}\int_{\theta=0}^{2\pi} r^4\left[\sin^2\theta\left(\frac{4}{3}\right) + \left(\frac{2}{3}\right)\right]d\theta dr = \lambda\int_{r=0}^{R} r^4\left[(\pi)\left(\frac{4}{3}\right) + \left(\frac{2}{3}\right)(2\pi)\right]dr = \left(\frac{4}{3}\pi\lambda R^3\right)\frac{2}{5}R^2$$

$$I_{xx} = \frac{2}{5}mR^2$$

where m is the total mass of sphere given by

$$m = \frac{4}{3}\pi\lambda R^3$$

By symmetry,

$$\underline{\underline{I_{yy} = I_{zz} = I_{xx} = \frac{2}{5}mR^2}}$$

Now,

$$I_{xy} = \int_m xydm = \int_V xy\lambda dV \qquad I_{xy} = \int_V (r\cos\theta\sin\phi)(r\sin\theta\sin\phi)\lambda(rd\phi)(r\sin\phi d\theta)(dr)$$

$$I_{xx} = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}(r\cos\theta\sin\phi)(r\sin\theta\sin\phi)\lambda(rd\phi)(r\sin\phi d\theta)(dr)$$

$$I_{xx} = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left(r^3\cos\theta\sin\theta\sin^3\phi\right)\lambda(d\phi)(d\theta)(dr)$$

The presence cosθsinθ makes the above integration from 0 to nπ, where n = 1, 2, .. p, zero. Since the other two products of inertia possess cosθsinθ, all the products of inertia are zero, i.e.

$$\underline{\underline{I_{xy} = I_{xz} = I_{yz} = 0}}$$

2-2.3 Inertia Matrix

The inertia properties of a rigid body are completely defined by nine parameters, which form a 3 x 3 symmetric matrix. Six of the nine parameters are independent of one another. The inertia matrix, which is also called inertia tensor, is defined as

$$[I]=\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \qquad \textbf{Equation 22}$$

Example 2- 7

Find the inertia matrix of Example 2-3.

Solution

From Example 2-3,

$$I_{xx}=I_{zz}=\frac{1}{12}ml^2 \qquad \text{and} \qquad I_{yy}=I_{xy}=I_{xz}=I_{yz}=0$$

$$[I]=\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}=\begin{bmatrix} \frac{1}{12}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} \qquad [I]=\frac{1}{12}ml^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SESSION 2-3: Transformation of Mass Moments of Inertia

Sometimes, it is necessary to relate the inertia properties of a body about two different points and orientations. It can be achieved by transformation of known inertia properties about a point /orientation into that of another point /orientation.

2-3.1 Translation of Coordinate

Consider the two coordinate systems xyz and x'y'z' with origins at O and A, respectively, shown in Figure 10. The coordinates systems are parallel such that x'= x- d_x, y'= y- d_y and z'= z+ d_z. The moment of inertia about x' is given by

$$I_{xx}=\int_m \left(y'^2+z'^2\right)dm=\int_m \left[\left(y-d_y\right)^2+\left(z+d_z\right)^2\right]dm$$

$$I_{x'x'}=\int_m \left(y^2+z^2\right)dm+\int_m \left(d_y{}^2+d_z{}^2\right)dm-2d_y\int_m ydm+2d_z\int_m zdm$$

The first term of the above equation is equal to I_{xx}, and the second term reduces to

$$m\left(d_y^{\,2} + d_z^{\,2}\right)$$

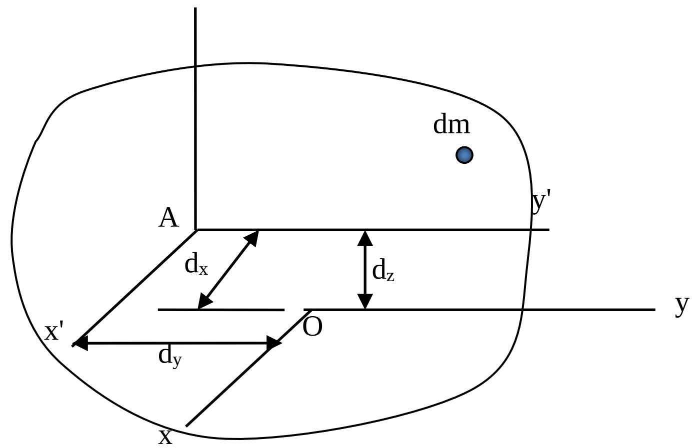


Figure 10: Translation of Coordinate System

The third and fourth terms reduce to zero. The above equation becomes

$$I_{x'x'} = I_{xx} + m\left(d_y^{\,2} + d_z^{\,2}\right) \qquad \textbf{Equation 23}$$

Similarly,

$$I_{y'y'} = I_{yy} + m\left(d_x^{\,2} + d_z^{\,2}\right) \qquad \textbf{Equation 24}$$

$$I_{z'z'} = I_{zz} + m\left(d_x^{\,2} + d_y^{\,2}\right) \qquad \textbf{Equation 25}$$

The product of inertia is given by

$$I_{x'y'} = \int_m x'y'dm \qquad I_{x'y'} = I_{xy} + \int_m d_x d_y dm$$

$$I_{x'y'} = I_{xy} + md_x d_y \qquad \textbf{Equation 26}$$

The parallel-axis may be generalized in matrix form as

$$\left[I_A\right] = \left[I_O\right] + m\left[K\right] \qquad \textbf{Equation 27}$$

where

$$[K]=\begin{bmatrix} d_y^2+d_z^2 & -d_xd_y & -d_xd_z \\ -d_xd_y & d_x^2+d_z^2 & -d_yd_z \\ -d_xd_z & -d_yd_z & d_x^2+d_y^2 \end{bmatrix}$$ **Equation 28**

The transformations given by Equations 23 to 27 are known as parallel-axis theorem.

Equation 28 allows one to determine the moments and products of inertia about axes knowing those of another parallel axis. A complex body may be divided into smaller parts whose centroidal moments and products of inertia can be determined easily or from a table. Then, the total moments and products of inertia are the sum of the individual moments and products of inertia about the same axis.

2-3.2 Radius of Gyration

The radius of gyration of a body is defined as the radius at which the equivalent lumped mass model of the body is located such that the resulting model has the same moment of inertia as the original body. Suppose the radius of gyration of a body of mass m and moment of inertia about axis zz is k_{zz}. From the parallel axis theorem, a concentrated mass m at a radius k_{zz} will have a moment of inertia of

$I_{ZZ} = mk_{zz}{}^2$ from which the radius of gyration becomes

$$k_{zz}=\sqrt{\frac{I_{ZZ}}{m}}$$ **Equation 29**

2-3.3 Rotation of Coordinate Axis

Consider the two coordinate systems xyz and x'y'z' , where x'y'z' is obtained by applying a set of rotation to xyz system. The origins of the two coordinate systems coincide. If [R] is the rotation matrix obtained from the set of rotation and [c] is the direction cosine, then the inertia matrix of x'y'z' is given by

$$[I']=[R][I][R]^T$$ **Equation 30**

$$[I']=[c]^T[I][c]$$ **Equation 31**

It is not necessary to begin with a centroidal set of axes. The following transformation matrices may be used for rotation:

Rotation about x axis $$[R_x]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$ **Equation 32**

Rotation about y axis $$[R_y]=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$ **Equation 33**

Rotation about z axis $$[R_z]=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$ **Equation 34**

2-3.4 Moment of Inertia about an Arbitrary Axis

Suppose that the moments and products of inertia of a rigid body about a given axes are known and one wants to find the moment of inertia of the body about an arbitrary axis AB shown in Figure 11. The moment of inertia of the body about that axis is defined in terms of that of known axes as

$$I_{AB}=I_{xx}u_x^{\ 2}+I_{yy}u_y^{\ 2}+I_{zz}u_z^{\ 2}-2I_{xy}u_xu_y-2I_{xz}u_xu_z-2I_{yz}u_yu_z$$ **Equation 35**

where u_x, u_y and u_z are unit directional vector of AB .

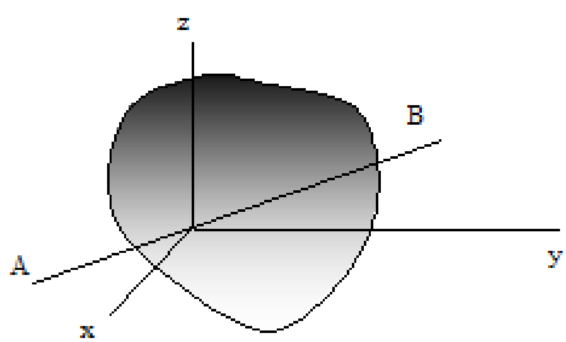


Figure 11: Moment of Inertia of an Arbitrary Axis

Example 2- 8

Determine the moment of inertia of the bent bar shown in Figure E2-8 about the axis AB. The mass and length of each segment is shown on the figure.

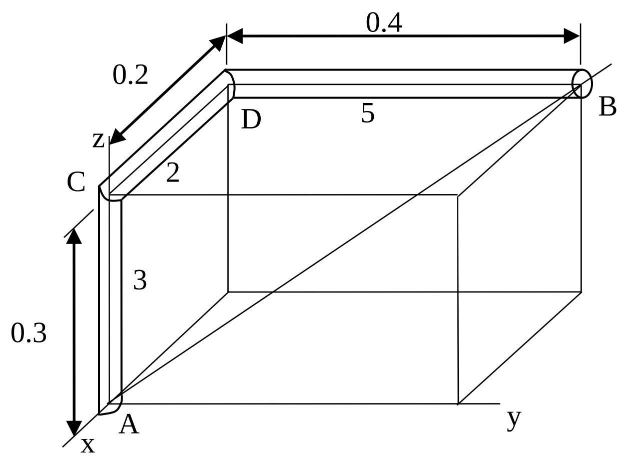


Figure E2-8

Solution

Using the parallel axis theorem, the moment of inertias of the sections of the bar about the origin are:

$$I_{xx,AC} = I_{xx,G} + m\left(d_y^{\,2} + d_z^{\,2}\right) = \frac{1}{12}3(0.3)^2 + 3\left(0^2 + 0.15^2\right) \qquad I_{xx,AC} = 0.09$$

$$I_{yy,AC} = I_{yy,G} + m\left(d_x^{\,2} + d_z^{\,2}\right) = \frac{1}{12}3(0.3)^2 + 3\left(0^2 + 0.15^2\right) \qquad I_{yy,AC} = 0.09$$

$$I_{zz,AC} = I_{zz,G} + m\left(d_x^{\,2} + d_y^{\,2}\right) = 0 + 0 \qquad I_{zz,AC} = 0.0$$

$$I_{xx,CD} = I_{xx,G} + m\left(d_y^{\,2} + d_z^{\,2}\right) = 0 + 2\left(0^2 + 0.3^2\right) \qquad I_{xx,CD} = 0.18 \text{ kg.m}^2$$

$$I_{yy,CD} = I_{yy,G} + m\left(d_x^{\,2} + d_z^{\,2}\right) = \frac{1}{12}2(0.2)^2 + 2\left((-0.1)^2 + 0.2^2\right) \qquad I_{yy,AC} = 0.21 \text{ kg.m}^2$$

$$I_{zz,CD} = I_{zz,G} + m\left(d_x^{\,2} + d_y^{\,2}\right) = \frac{1}{12}2(0.2)^2 + 2\left((-0.1)^2 + 0\right) \qquad I_{zz,CD} = 0.03 \text{ kg.m}^2$$

$$I_{xx,DB} = I_{xx,G} + m\left(d_y^{\,2} + d_z^{\,2}\right) = \frac{1}{12}5(0.4)^2 + 5\left(0.2^2 + 0.3^2\right) \qquad I_{xx,DB} = 0.72 \text{ kg.m}^2$$

$$I_{yy,DB} = I_{yy,G} + m\left(d_x^{\,2} + d_z^{\,2}\right) = 0 + 5\left((-0.2)^2 + 0.3^2\right) \qquad I_{yy,DB} = 0.65 \text{ kg.m}^2$$

$$I_{zz,CD} = 0.47 \text{ kg.m}^2$$

$$I_{xx} = I_{XX,AC} + I_{XX,CD} + I_{XX,DB} \qquad I_{xx} = 0.99 \text{ kg.m}^2$$

$$I_{yy} = I_{yy,AC} + I_{yy,CD} + I_{yy,DB} \qquad I_{yy} = 0.95 \text{ kg.m}^2$$

$$I_{zz} = I_{zz,AC} + I_{zz,CD} + I_{zz,DB} \qquad I_{zz} = 0.49 \text{ kg.m}^2$$

$$I_{xy} = I_{xy,AC} + I_{xy,CD} + I_{xy,DB} = [0+0]+[0+0]+[0+5(-0.2)(0.2)] \qquad I_{xy} = \text{-}0.2 \text{ kg.m}^2$$

$$I_{xz} = [0+0]+[0+2(-0.1)(0.2)]+[0+5(-0.2)(0.2)] \qquad I_{xz} = \text{-}0.24 \text{ kg.m}^2$$

$$I_{yz} = [0+0]+[0+0]+[0+5(0.2)(0.2)] \qquad I_{yz} = 0.20 \text{ kg.m}^2$$

The unit directional vector of AB is

$$u = \frac{\vec{AB}}{|AB|} = \frac{-0.2i+0.4j+0.3k}{\sqrt{\left[(-0.2)^2+0.4^2+0.3^2\right]}} = \text{-}0.3714\text{i}+0.7428\text{j}+0.5571\text{k}$$

$$u_x = \text{-}0.3714,\ u_y = 0.7428,\ u_z = 0.5571$$

Using Equation 35, the moment inertia about the given axis is

$$I_{AB} = 0.99(-0.3714)^2 + 0.95(0.7428)^2 + 0.49(0.5571)^2$$
$$-2(-0.20)(-0.3714)(0.7428) - 2(-0.24)(-0.3714)(0.5571) - 2(0.20)(0.7428)(0.5571)$$

$$\underline{\underline{I_{AB} = 0.988}} \text{ kg.m}^2$$

2-3.5 Principal Moments of Inertia

When all the products of inertia vanish (or equal to zero), the moment of inertia I_{xx}, I_{yy} and I_{zz} are called *principal moments of inertia* and the corresponding axes x, y and z are called *principal axes*. The principal moments of inertia make the equations for rotational motion simpler to handle.

The principal moments of inertia may be obtained by inspection to select axes such that they lie on planes of symmetry. An alternative method for find the principal moments of inertia is by finding the eigenvalue associated with the inertia matrix. The eigenvalues and vector is determine from

$$[I]\{u\} = \lambda\{u\} \qquad \textbf{Equation 36}$$

where λ is the eigenvalues and $\{u\}$ is the eigenvectors. The above equation may be written simplified form as

$$\det([I]-\lambda[1]) = 0 \qquad \textbf{Equation 37}$$

The above equation, which is a third-order polynomial equation in λ, is called *characteristics equation*. Note that the solution of Equation 36 yields three eigenvalues, which are equal to the principal moments of inertia and associated eigenvector. The eigenvectors are perpendicular to each other, which implies that

$$\{u_i\}^T\{u_j\}=0 \qquad \{u_i\}^T[I]\{u_j\}=0 \qquad i\neq j; i,i=1,2,3$$

Example 2- 9

Find the principal moments of inertia of a body whose inertia matrix is given by

$$[I]=\begin{bmatrix}400 & 0 & 150\\ 0 & 500 & 20\\ 150 & 20 & 350\end{bmatrix}$$

Solution

Using Equation 37, we have

$$\det([I]-\lambda[1])=\det\left[\begin{bmatrix}400 & 0 & 150\\ 0 & 500 & 20\\ 150 & 20 & 350\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right]=\det\begin{bmatrix}400-\lambda & 0 & 150\\ 0 & 500-\lambda & 20\\ 150 & 20 & 350-\lambda\end{bmatrix}=0$$

$\lambda_1 = I_{xx} = 222.09$, $\lambda_2 = I_{yy} = 495.5$, $\lambda_3 = I_{zz} = 532.3$

2-3.6 Centre of Percussion

The centre of percussion is a point on a body which when struck with a force there will be a zero reaction and associated with another point, called the *centre of rotation*. Consider a bar of mass m and radius of gyration k about the centre of mass, CG, as illustrated in Figure 12. A sudden force F is applied at point P, which at L_p from the centre of mass. From Newton's second law and the figure, the translational acceleration of the centre of mass of the bar in the direction of y-axis is given by

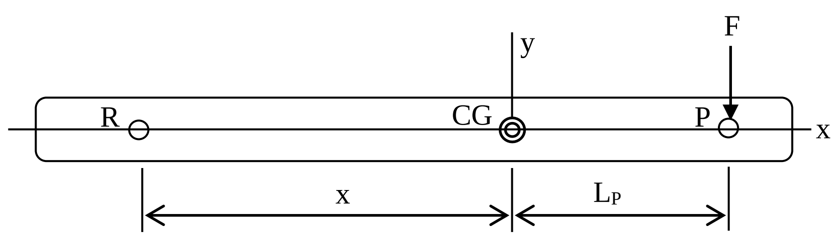


Figure 12: Centre of Percussion

$$a_{Gy}=-\frac{F}{m} \tag{1}$$

The angular acceleration is

$$\alpha = \frac{T}{I_G} = \frac{-FL_p}{I_G} \qquad (2)$$

where IG is the mass moment of inertia about the z-axis through the centre of mass CG.
Using Equation 4, the total acceleration of any distance r from the centre of mass of the bar is given by

$$a_{ytotal} = a_{Gy} + r\alpha \qquad a_{ytotal} = -\frac{F}{m} + r\left(\frac{-FL_p}{I_G}\right) \qquad (3)$$

At the centre of percussion, the reaction force is zero, which means that the total linear acceleration is zero. Equating (3) to zero find the point of percussion x = r, we have

$$x = -\frac{I_G}{mL_p} = -\frac{k^2}{L_p} \qquad \textbf{Equation 38}$$

Unit Summary

Centre of Mass

1. The centre of mass is a very important quantity, as it its use simplifies the kinetic analysis of bodies significantly.

2. For one-component use either single, double or triple integration for locate the centre of mass.

3. For a body consisting of many parts with known formula for the centres of masses, use the relation

$$r_G = \frac{1}{m}\sum_{i=1}^{n} m_i r_i$$

$$m = \sum_{i=1}^{n} m_i$$

Mass Moments of Inertia

4. The mass moment of inertia is a measure of the way the mass is distributed on the body.

5. For a complex shape, use single, double or triple integration to determine moment and product of inertia.

6. If there is symmetry with respect to the yz plane, then $I_{xy} = I_{xz} = 0$.

7. If there is symmetry with respect to the xz plane, then $I_{xy} = I_{yz} = 0$.

8. If there is symmetry with respect to the xy plane, then $I_{yz} = I_{xz} = 0$.

9. The moments and products of inertia form an inertia symmetric matrix

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

10. Use the parallel-axes theorem to determine the moments and products of inertia of axes parallel to the centroidal axes.

11. Use Equation 28 to obtain the inertia matrix of a rigid body about an axis obtained from the set of rotation.

12. The moment of inertia of any arbitrary axis AB with unit direction vector $u = u_x\mathbf{i} + u_j\mathbf{j} + u_z\mathbf{k}$ is given by

$$I_{AB} = I_{xx}u_x^{\ 2} + I_{yy}u_y^{\ 2} + I_{zz}u_z^{\ 2} - 2I_{xy}u_xu_y - 2I_{xz}u_xu_z - 2I_{yz}u_yu_z$$

☑ Key terms/ New Words in Unit

1. Centre of mass, centre of gravity
2. Moment of inertia
3. Product of inertia
4. Inertia matrix / Inertia tensor
5. Parallel axis theorem
6. Centre of rotation
7. Radius of gyration
8. Centre of Percussion
9. Principal moments of inertia

Self Assessment 2

2-1. The homogeneous right circular cone shown in Figure P 2-1 has an altitude h, radius r and mass m. Find (a) the location of centre of gravity from the origin O, (b) moments of inertia I_{xx}, I_{yy} and I_{zz} with reference to the origin O, and (c) moments of inertia I_{xx}, I_{yy} and I_{zz} with reference to the centre of mass G.

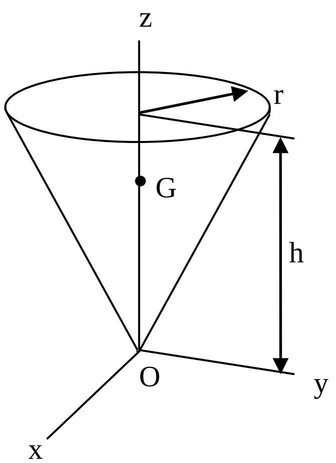


Figure P 2-1

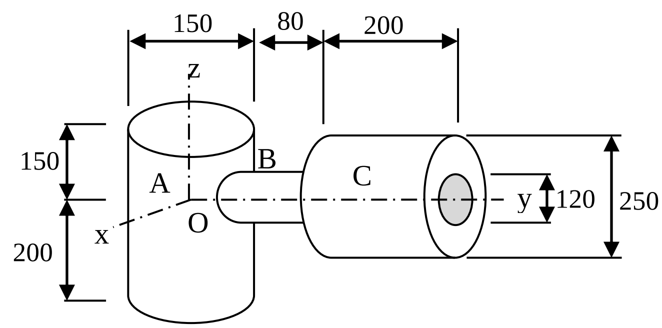


Figure P 2-2

2-2. Figure P 2-2 shows a machine part consisting of a titanium alloy (ρ_t = 4630 kg/m^3) disc C press-fitted on a bronze (ρ_b = 8860 kg/m^3) shafts B. The shaft B is rigidly attached to the vertical steel (ρ_s = 7870 kg/m^3) shaft A. Determine the (a) location of centre of the part, and (b) moments of inertia of the part with respect to the x, y and z axes through the origin O. All dimensions are in mm.

2-3. The shaded area shown in Figure P 2-3 is revolved about the axis to form a homogeneous solid of revolution of mass m. Determine by integration the moment of inertia of the solid with respect to the x axis.

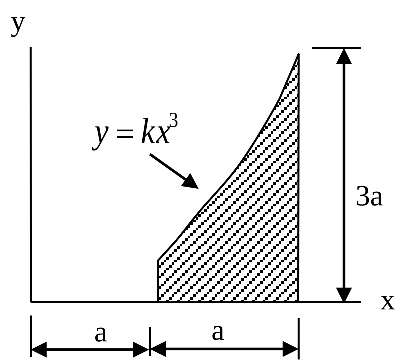


Figure P2-3

2-4. A cylinder of radius R and height h has a square through hole along its longitudinal axis, as shown in Figure P2-4. The sides of the square hole are 0.25R and mass of the cylinder with the hole is M. Find the inertia matrix of the cylinder about its centre of mass C.

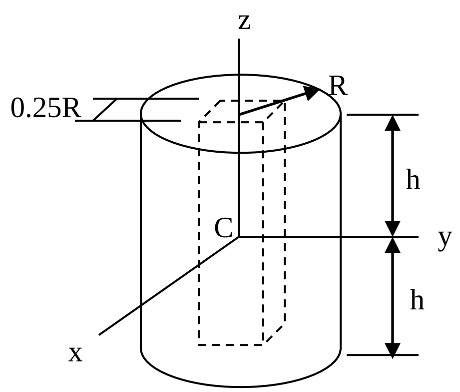


Figure P 2-4

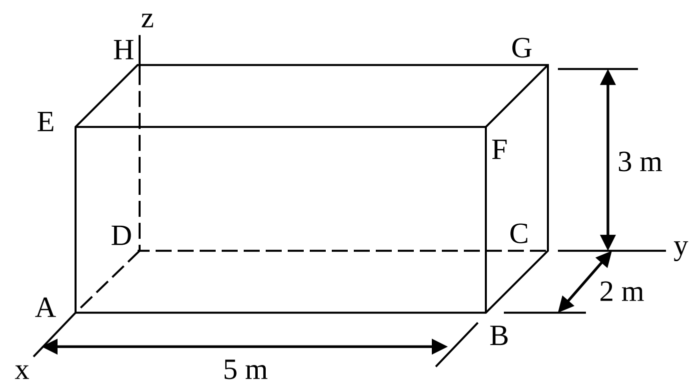


Figure P 2-5

2-5. The rectangular prism shown in Figure P 2-5 has a mass of 5 kg and sides 2 m, 3 m and 5 m. Determine the (a) inertia matrix of the prism with respect to the coordinate axes shown, (b) its moment of inertia with respect t to the diagonal DF, (c) its moment of inertia with respect to the diagonal AC

2-6. The homogeneous object shown in Figure P 2-6 has the shape of a truncated cone and consists of bronze with mass density of 8200 kg/m^3. Determine the moments of inertia of the object about the (a) y axis, (b) x axis

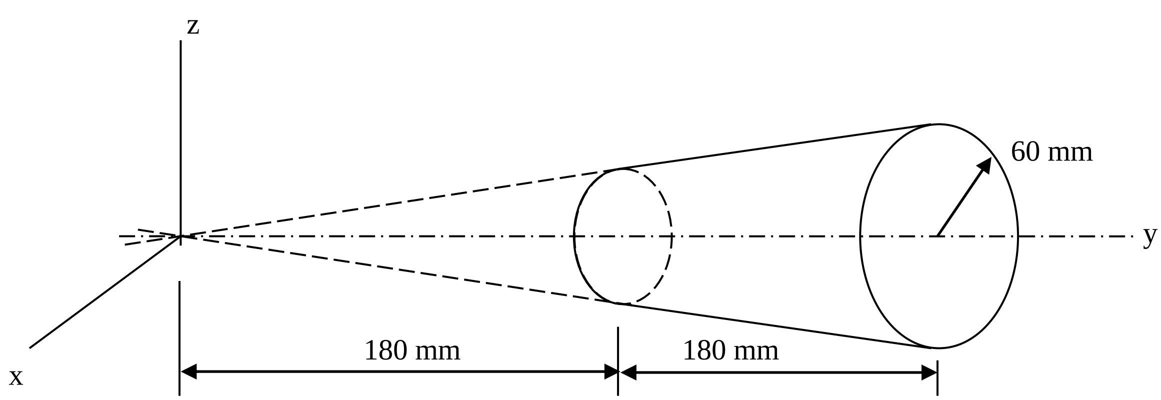


Figure P2-6

2-7. Determine by direct integration the product of inertia I_{yz} and I_{xy} for the homogeneous prism shown in Figure 2-7. Express the result in terms of the mass m and length a of the prism.

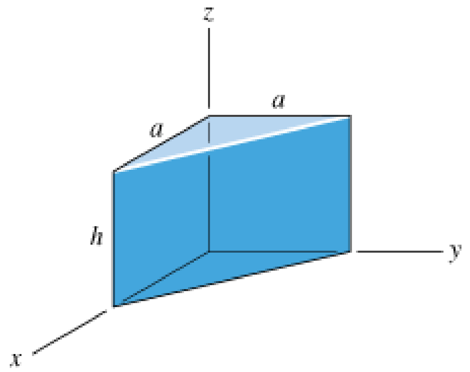


Figure 2-7

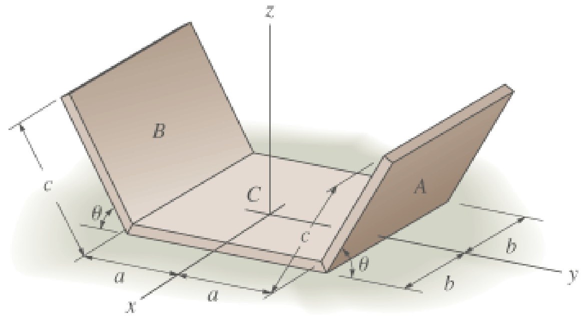


Figure 2-7

2-8. The assembly in Figure 2-8 consists of two square plates A and B which have a mass M_A = 3 kg each and a rectangular plate C which has a mass M_C = 4.5 kg. Knowing that a = 0.3 m, b = 0.2 m, c = 0.4 m and θ= 60°, determine the moments of inertia I_{xx}, I_{yy} and I_{zz}.

2-9. Determine the moment of Inertia about the z axis of the assembly in Figure P2-9, which consists of the rod CD of mass 1.5 kg and length 200 mm, and disc of mass 7 kg and radius 100 mm.

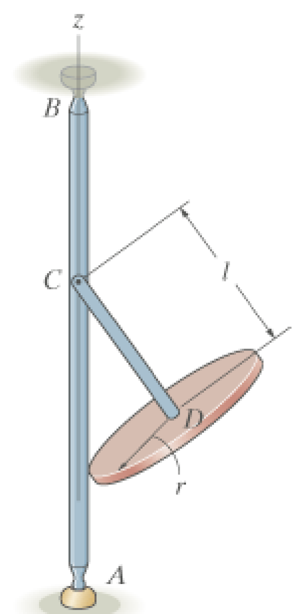


Figure 2-9

Unit Assignments 2

1. Show that the sum of the moments of inertia of a body, $I_{xx}+I_{yy}+I_{zz}$ is constant i.e independent of the orientation of the *x*, *y*, *z* axes and thus depends only on the location of the origin.

UNIT 3

THREE-DIMENSIONAL KINETICS OF RIGID BODIES

Introduction

Three-dimensional dynamics are essential to the design and analysis of vehicles, airplanes, ships. Basically, kinetics is study of forces and their relation with the motion. However, dynamic force kinetics analysis requires prior kinematics analysis.

The field of kinematics, like any engineering field, consist of two major components: analysis and synthesis (or design). This unit focuses on kinematic analysis. Kinematic synthesis is the design or creation of mechanism (interconnect bodies) to perform a desired motion. This field of kinematics is a specialty field, which is beyond the scope of this course notes.

Learning Objectives

After reading this unit you should be able to:

1. To relate moments of inertia to motion
2. To develop and apply equations of motion in three dimensions
3. Apply the equation of motion to gyroscopic and torque-free motions.
4. Able to use Euler's angles to specify the orientation of a rigid body in three dimensions
5. Express equation of motion in Euler's angles
6. Determine gyroscopic effect on machines

Unit content

Session 3-1: Equations of Motion

3-1.1 Equations of Translational Motion

3-1.2 Equations of Rotational Motion

3-1.2.1 Steps for Using Euler's Equations

3-1.2.2 Selection of Secondary Coordinate System

Session 3-2: Gyroscopic Motions

3-2.1 Euler Angles

3-2.2 General Gyroscopic Effects

Session 3-3: Gyroscopic Effects in Machines

3-3.1 Gyroscopic Effects on Airplanes and Naval Ships during Steering

3-3.2 Gyroscopic Effects on Stability of Four-Wheel Drives

3-3.3 Gyroscopic Effects on Stability of Two-Wheel Drives

SESSION 3-1: Equations of Motion

The three-dimensional equation of motion for rigid bodies called Euler's equations. They consist of equations of translational motion based on Newton's second law and equations of angular motion based on equivalent Newton's second law applied to rotational motion.

3-1.1 Equations of Translational Motion

Newton second law states that the sum of external forces acting on a rigid body equals time rate of change of linear momentum. From this law, it can be deduced that the sum of external forces acting on a rigid body equals to the product of the mass of the body and the acceleration of its centre of mass. This law may be stated in vector mathematical form as

$$\sum F = ma_G \qquad \textbf{Equation 39}$$

or in scalar form as

$$\begin{aligned} \sum F_x &= m(a_G)_x \\ \sum F_y &= m(a_G)_y \\ \sum F_z &= m(a_G)_z \end{aligned} \qquad \textbf{Equation 40}$$

where $\sum F = \sum F_x i + \sum F_y j + \sum F_z k$

$a_G = (a_G)_x i + (a_G)_y j + (a_G)_z k$

3-1.2 Equations of Rotational Motion

Newton second law may be used to deduce the equations of rotational motion by taken moment about any point. That is, the sum of moments M about a fixed point of the external forces acting on body is equal to the time rate of change of the total angular momentum H of the body about the point. In vector mathematical form, the above statement is

$$\sum M = \frac{dH}{dt} \qquad \textbf{Equation 41}$$

The total angular momentum may be written as

$$H = \sum_i \left(r_i \text{ x } m_i \frac{dr_i}{dt} \right) \qquad \textbf{Equation 42}$$

Let r_i be the position of *i*th particle relative to the fixed axes XYZ shown in Figure 13. In the figure, xyx is a secondary coordinate system with origin at the centre ofmass of the body. Let the angular velocity of the primary and secondary coordinate systems xyz and XYZ, respectively, be denoted by Ω and ω. The angular velocities and the position vector r may be in terms of components as

$\Omega = \Omega_x i + \Omega_y j + \Omega_z k$ $\qquad \omega = \omega_x i + \omega_y j + \omega_z k$; $\qquad r_i = x_i i + y_i j + z_i k$.

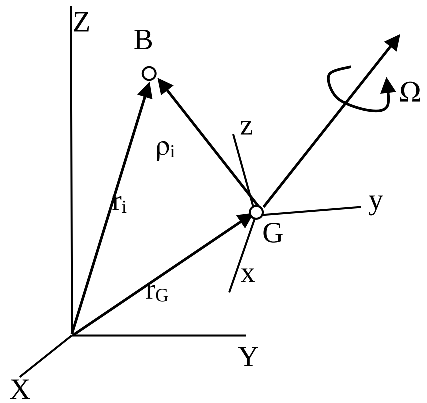


Figure 13: Secondary Axes in Motion

The angular momentums about the origin in discrete and integral forms are

$$H = \sum_i [r_i \text{ x} (\omega \text{ x } r_i)] m_i$$

$$H = \int_m r \text{ x} (\omega \text{ x r}) dm$$

Integrating the above equation, the angular moment of a rigid body is given by

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \textbf{Equation 43}$$

In scalar form, the components of Equation 43 are

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \qquad \textbf{Equation 44}$$

Substituting ω and r into the equation 42 and noting that *dω/dt = ω x r*, the rotational equations of motion are given in matrix form as

$$\begin{pmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{pmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{pmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \textbf{Equation 45}$$

Equation 43 may be written scalar form as

$$\sum M_x = I_{xx}\frac{d\omega_x}{dt} - I_{xy}\frac{d\omega_y}{dt} - I_{xz}\frac{d\omega_z}{dt} - \Omega_z\left(-I_{xy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z\right) + \Omega_y\left(-I_{yx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z\right) \qquad \textbf{Equation 46}$$

$$\sum M_y = -I_{xy}\frac{d\omega_x}{dt} + I_{yy}\frac{d\omega_y}{dt} - I_{yz}\frac{d\omega_z}{dt} + \Omega_z\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xy}\omega_z\right) - \Omega_x\left(-I_{xz}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z\right) \qquad \textbf{Equation 47}$$

$$\sum M_z = -I_{xz}\frac{d\omega_x}{dt} - I_{yz}\frac{d\omega_y}{dt} + I_{zz}\frac{d\omega_z}{dt} - \Omega_y\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right) + \Omega_x\left(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z\right) \qquad \textbf{Equation 48}$$

Note that the moments and products of inertia are calculated with respect to the point or axis of moment.

3-1.3 Steps for Using Euler's Equations

Equation 45 or Equations 46 to 48 are the Euler's equations for three-dimensional motions of rigid bodies. Equation 39 (or 40) and 45 (or 46 to 48) forms equations of rigid bodies in three-dimensional motions and their use involve the following steps:

(a) *Draw the free-body diagram of the body*. Isolate a particular body or a group of bodies under consideration from its support or other bodies. Draw a schematic of the body. In the diagram, indicate clearly the magnitudes if known and symbols if unknown, directions, and locations of all external forces and moments or couples. These include weight, applied forces, reactions, dimensions and angles necessary for sum of forces and moments.

(b) *Choose a coordinate system*. If a body rotates about a fixed point, it is preferable and necessary to use secondary coordinate system with its origin at the fixed point. Otherwise, use a coordinate system with its origin at the centre of mass of the body.

(c) Use Euler's equations of motion given by Equations 39 to 40 and 43 or 46 to relate forces and couples to the motion. Placing the origin of the coordinate system at the centre of mass simplifies the determination of the moments and products of inertia.

3-1.4 Selection of Secondary Coordinate System

A secondary coordinate system is normally defined to simplify the determination of moment matrix and solution of the set of equations of motion. Solution of the Euler's equation is simplified by placing the origin of the secondary coordinate system at the centre of mass. There are three ways in which the origin of a secondary coordinate system may be defined.

Axes Having Motion $\Omega = 0$

The origin of the secondary coordinate system may be placed at the centre of mass of the body such that the secondary axes only translate with respect to the main axes XYZ. Doing so G would simplify the Euler's equation since $\Omega = 0$. However, the body may have rotation ω about primary axes such that the moments and products of inertia of the body would vary with time (or orientation), and need to be expressed as a function of time. Selection of this option may complicate finding solutions to the Euler's equations.

Axes Having Motion $\Omega = \omega$

The origin of the secondary coordinate system may be fixed in the body and move with the body such that $\Omega = \omega$. The moments and products of inertia of the body would be constant during the motion. Using Equation 5, it should be noted that $\dot{\Omega} = (\dot{\Omega})_{x,y,z} + \omega \times \Omega$. Since Ω = ω, ω x Ω = ω x ω = 0, $\dot{\Omega} = (\dot{\Omega})_{x,y,z}$

When Ω = ω, Equation 43 reduces to

$$\begin{pmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{pmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{pmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \textbf{Equation 49}$$

In scalar form, Equation 47 becomes

$$\sum M_x = I_{xx}\frac{d\omega_x}{dt} - I_{xy}\frac{d\omega_y}{dt} - I_{xz}\frac{d\omega_z}{dt} - \omega_z\left(-I_{xy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z\right) + \omega_y\left(-I_{yx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z\right) \qquad \textbf{Equation 50}$$

$$\sum M_y = -I_{xy}\frac{d\omega_x}{dt} + I_{yy}\frac{d\omega_y}{dt} - I_{yz}\frac{d\omega_z}{dt} + \omega_z\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xy}\omega_z\right) - \omega_x\left(-I_{xz}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z\right) \qquad \textbf{Equation 51}$$

$$\sum M_z = -I_{xz}\frac{d\omega_x}{dt} - I_{yz}\frac{d\omega_y}{dt} + I_{zz}\frac{d\omega_z}{dt} - \omega_y\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right) + \omega_x\left(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z\right) \qquad \textbf{Equation 52}$$

Axes Having Motion Ω ≠ ω

It may be necessary to choose the axes xyz such Ω ≠ ω, which is suitable for gyroscopes and spinning tops.

Example 3- 1

A student holds a 5-kg hot rectangular plate with a tong (not shown) at point O in Figure E3-1. Find the couple exerted on the plate by the tong at the instant the plate is horizontal and its moving at angular velocity and acceleration of 5i -3j + 4 k (rad/s) and -5 i +5j – 2k, respectively.

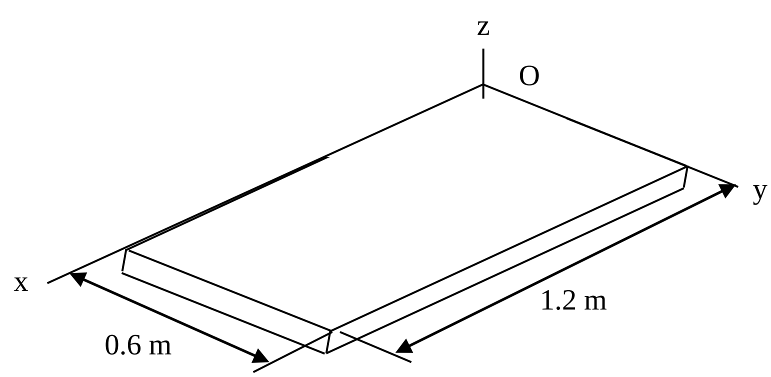


Figure E3-1

Solution

The moments and products of inertia about point O are

$$I_{zz} = \frac{1}{3}m\left(x^2 + y^2\right) = \frac{1}{3}5\left(1.2^2 + 0.6^2\right) \qquad I_{zz} = 3 \text{ kg-m}^2$$

$$I_{xx} = \frac{1}{3}m\left(z^2 + y^2\right) = \frac{1}{3}5\left(0^2 + 0.6^2\right) \qquad I_{xx} = 0.6 \text{ kg-m}^2$$

$$I_{yy} = \frac{1}{3}m\left(x^2 + z^2\right) = \frac{1}{3}5\left(1.2^2 + 0^2\right) \qquad I_{yy} = 2.4 \text{ kg-m}^2$$

$$I_{xz} = I_{yz} = 0$$

$$I_{xy} = I_{xy,G} + m\left(\frac{x}{2}\right)\left(\frac{x}{2}\right) = 0 + 5\left(\frac{1.2}{2}\right)\left(\frac{0.6}{2}\right) \qquad I_{xy} = 0.9 \text{ kg-m}^2$$

Then, the inertia matrix becomes

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \qquad [I] = \begin{bmatrix} 0.6 & -0.9 & 0 \\ -0.9 & 2.4 & 0 \\ 0 & 0 & 3.0 \end{bmatrix}$$

Let the force and couple acting on the plate at the point of contact with the manipulator respectively be F and C. The total moment about point O of the free-body shown in Figure S3-1 is

$$\sum M_O = C + \left(0.6i + 0.3j\right) \text{x} \left[(-5)(9.81)k\right] \qquad \sum M_O = C + -14.715i + 29.43j \quad \text{(a)}$$

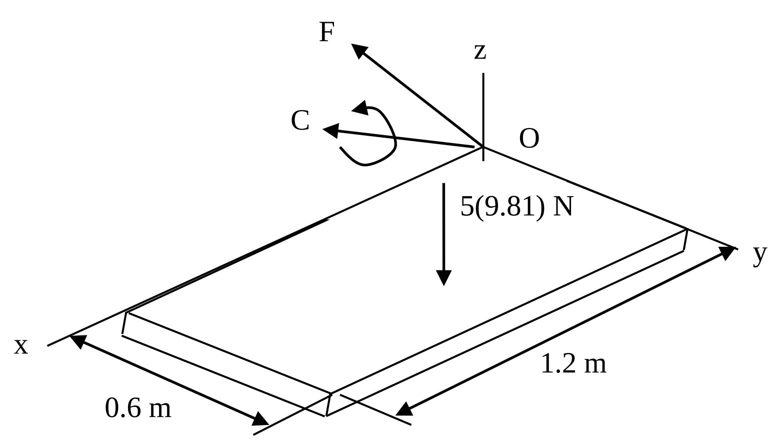


Figure S3-1

Using Equation 45 and note that the fixed and moving coordinates coincide, ie $\Omega = \omega$, we have

$$\begin{pmatrix}\sum M_{Ox}\\ \sum M_{Oy}\\ \sum M_{Oz}\end{pmatrix}=\begin{bmatrix}0.6 & -0.9 & 0\\ -0.9 & 2.4 & 0\\ 0 & 0 & 3.0\end{bmatrix}\begin{pmatrix}-5\\ 5\\ -2\end{pmatrix}+\begin{bmatrix}0 & -(4) & (-3)\\ 4 & 0 & -(5)\\ -(-3) & 5 & 0\end{bmatrix}\begin{bmatrix}0.6 & -0.9 & 0\\ -0.9 & 2.4 & 0\\ 0 & 0 & 3.0\end{bmatrix}\begin{pmatrix}5\\ -3\\ 4\end{pmatrix}$$

$$\begin{pmatrix}\sum M_{Ox}\\ \sum M_{Oy}\\ \sum M_{Oz}\end{pmatrix}=\begin{pmatrix}3.3\\ -20.7\\ -47.4\end{pmatrix} \qquad \text{(b)}$$

$$\sum M_O = C + -14.715i + 29.43j$$

Substituting Equation (b) into Equation (a), and solving for C yields

$$(3.3\mathrm{i}-20.7\mathrm{j}-47.4\mathrm{k}) = C + -14.715\mathrm{i} + 29.43\mathrm{j} \qquad \underline{\underline{C = 18.01\mathrm{i}-50.13\mathrm{j}-47.4\mathrm{k}\ (\mathrm{N\text{-}m})}}$$

Example 3- 2

The inertia matrix of an airplane shown in Figure E3-2 with reference to the coordinate system fixed to centre of mass is

$$[I]=\begin{bmatrix}I_{xx} & -I_{xy} & -I_{xz}\\ -I_{xy} & I_{yy} & -I_{yz}\\ -I_{xz} & -I_{yz} & I_{zz}\end{bmatrix}=\begin{bmatrix}40000 & 0 & 0\\ 0 & 120000 & 0\\ 0 & 0 & 135000\end{bmatrix}\mathrm{kg\text{-}m^2}$$

The angular velocity of the airplane is zero when the pilot actuates the elevators and ailerons, subjecting the airplane to a couple of 7i + 13j –k (kN.m). Determine the angular acceleration of the airplane at this instant.

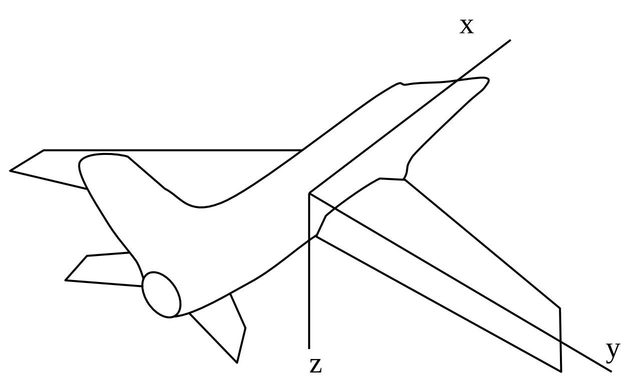


Figure E3-2

Solution

Using Equation 45, we have

$$\begin{pmatrix} 7 \\ 13 \\ -1 \end{pmatrix} = \begin{bmatrix} 40000 & 0 & 0 \\ 0 & 120000 & 0 \\ 0 & 0 & 135000 \end{bmatrix} \begin{pmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\begin{pmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{pmatrix} = \begin{pmatrix} 0.175 \\ 0.108 \\ -0.007 \end{pmatrix} \text{rad/s}^2$$

Example 3- 3

In Figure E3-3, the homogeneous disc of mass 3 kg rotates at a constant speed of ω_1= 16 rad/s with respect with respect to arm ABC, which is welded to shaft DCE rotating at the constant speed of ω_2=8 rad/s. Determine the dynamic reactions at D and E.

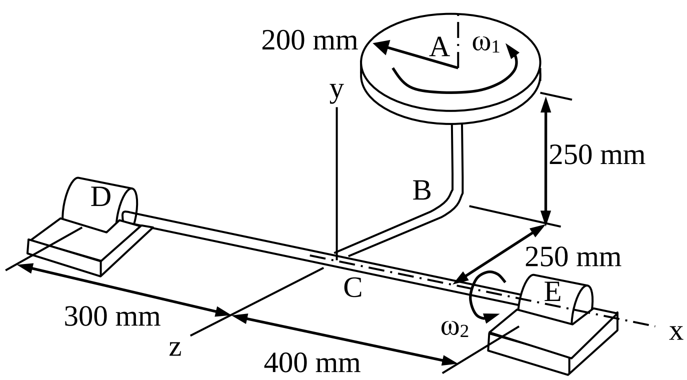


Figure E3-3

Solution

Using the parallel axis theorem, the moments of inertias of the disc about point O are:

$$I_{xx} = \frac{1}{4}mr^2 + md^2 = \frac{1}{4}3(0.2)^2 + 3\left(0.225^2 + 0.225^2\right) \qquad I_{xx} = 0.33375\ \text{kg-m}^2$$

$$I_{yy} = \frac{1}{2}mr^2 + md^2 = \frac{1}{2}3(0.2)^2 + 3\left(0.2250^2\right) \qquad I_{yy} = 0.211875\ \text{kg-m}^2$$

$$I_{zz} = \frac{1}{4}mr^2 + md^2 = \frac{1}{4}3(0.2)^2 + 3\left(0.225^2\right) \qquad I_{zz} = 0.181875\ \text{kg-m}^2$$

All the products of inertias are zero. Let the primary and secondary frames coincide at C and set $\Omega = \omega$. Then, $\Omega = \omega = \omega_2 \mathrm{i} + \omega_1 \mathrm{j} = 8i + 16j$

The acceleration is defined using the transport theorem given by Equation 5 as

$$\frac{d\omega}{dt} = \dot{\omega} = (\dot{\omega})_{xyz} + \omega_2 \mathrm{i} \text{ x } (\omega_2 \mathrm{i} + \omega_1 \mathrm{j}) = 0 + 8\mathrm{i} \text{ x } (8i + 16j) \qquad \frac{d\omega}{dt} = 128\,\mathrm{k}$$

Components of velocities and acceleration are

$$\Omega_x = \omega_x = 8 \qquad \Omega_y = \omega_y = 16 \qquad \Omega_z = \omega_z = 0$$

$$\frac{d\omega_x}{dt} = 0 \qquad \frac{d\omega_y}{dt} = 0 \qquad \frac{d\omega_z}{dt} = 128$$

The free-body of the bar and disc are shown in Figure S3-3 and taking moment about point C using 49, we have

$$\begin{pmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{pmatrix} = \begin{bmatrix} 0.33375 & 0 & 0 \\ 0 & 0.211875 & 0 \\ 0 & 0 & 0.181875 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 128 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 16 \\ 0 & 0 & -8 \\ -16 & 8 & 0 \end{bmatrix} \begin{bmatrix} 0.33375 & 0 & 0 \\ 0 & 0.211875 & 0 \\ 0 & 0 & 0.181875 \end{bmatrix} \begin{pmatrix} 8 \\ 16 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7.68 \end{pmatrix}$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 7.68$$

Taking moment about C along the z and y axes

$$\sum M_z = 0.3E_y - 0.3D_y = 7.68 \qquad E_y - D_y = 25.6 \qquad (1)$$

$$\sum M_y = 0.3E_z - 0.3D_z = 0 \qquad E_z - D_z = 0 \qquad (2)$$

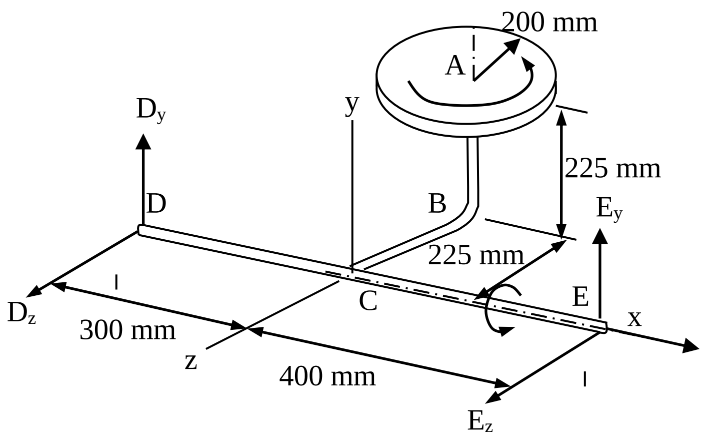


Figure S3-3

From $\sum F = ma_G$**,** and noting that there is no force along the x direction, we have

$$\sum F_y = m(a_G)_y \qquad \sum F_y = E_y + D_y - m\omega_2^{\,2} \sqrt{(0.225^2 + 0.225^2)} \sin\theta = 0$$

The angle θ is the angle between the horizontal plane xz and line CA

$$\theta = \tan^{-1}\left(\frac{225}{225}\right) \qquad \theta = 45^o$$

$$E_y + D_y - (3)(8)^2\sqrt{(0.225^2 + 0.225^2)}\sin 45 = 0 \qquad E_y + D_y = 43.2 \qquad (3)$$

Solving Equations (1) and (3) simultaneously, $E_y = 34.4\text{ N}$ and $D_y = 8.8\text{ N}$

Similarly,

$$\sum F_z = m(a_G)_z \qquad \sum F_z = E_z + D_z - m\omega_2{}^2\sqrt{(0.225^2 + 0.225^2)}\cos 45 = 0$$

$$E_z + D_z - (3)(8)^2\sqrt{(0.225^2 + 0.225^2)}\cos 45 = 0 \qquad E_z + D_z = 43.2 \qquad (4)$$

Solving Equations (2) and (4) simultaneously, $E_z = 21.6\text{ N}$ and $D_z = 21.6\text{ N}$

The reactions at D and E are: $\underline{\underline{E = 34.4\text{j} + 21.6\text{k N}}}$

$\underline{\underline{D = 8.8\text{j} + 21.6\text{k N}}}$

Example 3- 4

The 10-kg cylinder shown in Figure E3-4 rotates about the shaft at a constant speed of ω_s = 8 rad/s. At the same time, the shaft is rotating about the vertical axis about an axis through A with an angular velocity ω_p = 4 rad/s. If A is thrust bearing and B is a journal bearing, determine the components of force reaction at each bearing due to the motion.

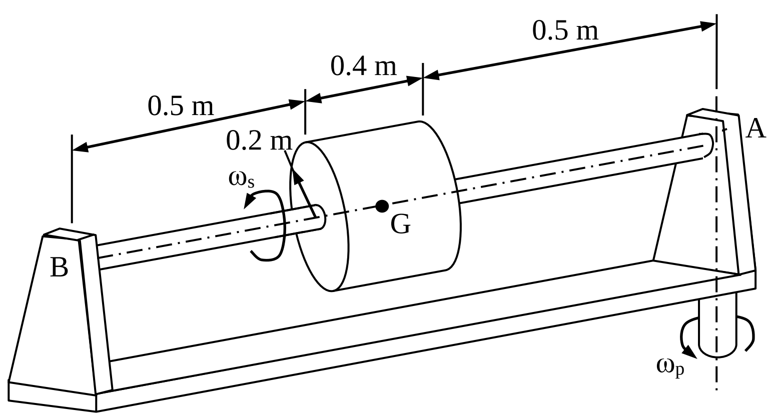


Figure E3-4

Solution

Free-body diagram of the cylinder and the shaft is shown in Figure S3-4. In the figure the origin of the coordinate systems coincide at the centre of mass G of the cylinder. Thus, the products of inertia are equal to zero.

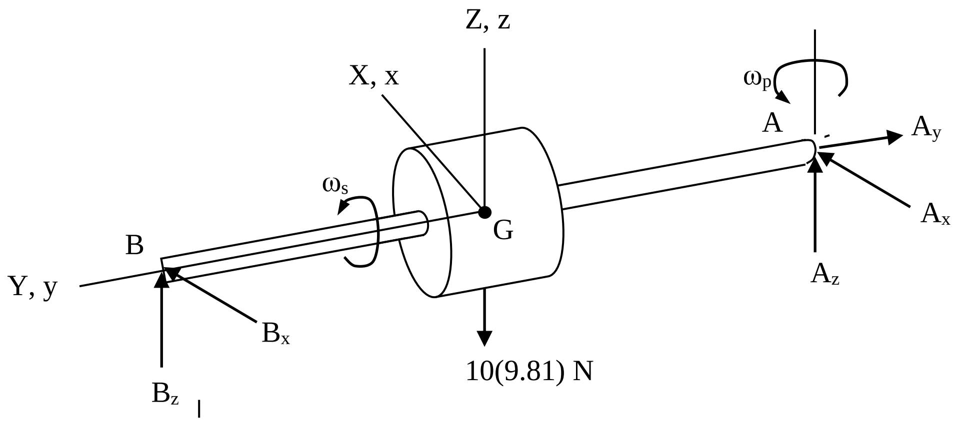


Figure S3-4

Although the cylinder spins relative to the axis, the moments of inertia remain constant. The moments of inertia are:

$$I_{xx} = I_{zz} = \frac{1}{12}m\left(3r^2 + h^2\right) = \frac{1}{12}10\left(3(0.2)^2 + 0.4^2\right) \qquad I_{xx} = I_{zz} = 0.233\ \text{kg - m}^2$$

$$I_{yy} = \frac{1}{2}mr^2 = \frac{1}{2}10(0.2)^2 \qquad I_{yy} = 0.2\ \text{kg - m}^2$$

Alternative I

The angular velocities and accelerations are

$$\Omega = \omega_p = 4\,\text{k},\ \dot{\Omega} = 0 \qquad \text{and} \qquad \omega = \omega_s = 8\,\text{j} \qquad \dot{\omega} = 0$$

The components of angular velocities are

$$\Omega_x = \Omega_y = 0,\ \Omega_z = 4 \qquad \text{and} \qquad \omega_y = 8,\ \omega_x = \omega_z = 0$$

Using equation 43, we have

$$\begin{pmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{pmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{bmatrix} 0 & -4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.233 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.233 \end{bmatrix} \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{pmatrix} = \begin{pmatrix} -6.4 \\ 0 \\ 0 \end{pmatrix}$$

The moments can also be determined by letting $\Omega = \omega = 8\text{j} + 4\,\text{k}$. Then, using the transport theorem the acceleration becomes

$$\frac{d\omega}{dt} = (\dot{\omega})_{xyz} + \omega_p\ \text{x}\ \omega = 0 + 4\,\text{k x}\left(8\text{j} + 4\,\text{k}\right) \qquad \frac{d\omega}{dt} = -32\,\text{i}$$

$$\begin{pmatrix}\sum M_x\\ \sum M_y\\ \sum M_z\end{pmatrix}=\begin{bmatrix}I_{xx} & -I_{xy} & -I_{xz}\\ -I_{xy} & I_{yy} & -I_{yz}\\ -I_{xz} & -I_{yz} & I_{zz}\end{bmatrix}\begin{pmatrix}d\omega_x/dt\\ d\omega_y/dt\\ d\omega_z/dt\end{pmatrix}+\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}\begin{bmatrix}I_{xx} & -I_{xy} & -I_{xz}\\ -I_{xy} & I_{yy} & -I_{yz}\\ -I_{xz} & -I_{yz} & I_{zz}\end{bmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

$$\begin{pmatrix}\sum M_x\\ \sum M_y\\ \sum M_z\end{pmatrix}=\begin{bmatrix}0.233 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.233\end{bmatrix}\begin{pmatrix}-32\\ 0\\ 0\end{pmatrix}+\begin{bmatrix}0 & -4 & 8\\ 4 & 0 & 0\\ -8 & 0 & 0\end{bmatrix}\begin{bmatrix}0.233 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.233\end{bmatrix}\begin{pmatrix}0\\ 8\\ 4\end{pmatrix} \qquad \begin{pmatrix}\sum M_x\\ \sum M_y\\ \sum M_z\end{pmatrix}=\begin{pmatrix}-6.4\\ 0\\ 0\end{pmatrix}$$

From the figure, the moments about the centre of mass G are

$$\sum M_x = -A_z(0.5+0.2)+B_z(0.5+0.2) = -6.4 \qquad -A_z + B_z = -9.1429 \qquad (1)$$

$$\sum M_z = A_x(0.5+0.2)-B_x(0.5+0.2) = 0 \qquad A_x = B_x \qquad (2)$$

Applying Equation 40, we have

$$\sum F_x = m(a_G)_x \qquad A_x + B_x = 0 \qquad \underline{\underline{A_x = B_x = 0}}$$

$$\sum F_y = m(a_G)_y \qquad A_y = m\Omega_z^2|AG|$$ (Centrifugal force due to rotation about the vertical axis)

$$A_y = 10(4)^2(0.5+0.2) \qquad \underline{\underline{A_y = 112 \text{ N}}}$$

$$\sum F_z = A_z + B_z - mg = 0 \qquad A_z + B_z = 10 \text{ x } 9.81 \qquad (3)$$

Solving Equations (1) and (3) simultaneously, we have

$\underline{\underline{A_z = 53.62 \text{ N}}}$ and $\underline{\underline{B_z = 44.48 \text{ N}}}$

SESSION 3-2: Gyroscopic Motion

3-2.1 Euler Angles

Unlike planer motion which requires only one angle, describing the orientation of a rigid body in spatial motion requires three angles measured from three non-parallel axes. The three angles used for describing the orientation of a rigid body or for transforming one coordinate set into another one are commonly referred to as *Euler angles*.

3-2.2 General Gyroscopic Effects

Consider the motion of a rigid body which is symmetrical with respect to an axis and moving about a fixed point O lying on the axes X, Y and Z as shown in Figure 14 (a). In the figure, the rigid body which is referred to as top, is attached to point O the origin of the fixed

XYZ coordinate and moving xyz coordinate. Suppose that the top is given the following angular displacements:

(a) Rotate the top and the xyz coordinate through φ ($0 \leq \phi \leq 2\pi$) about the Z (or z) axis, as illustrated in Figure 14(b).
(b) Rotate the top and the xyz coordinate through θ ($0 \leq \theta \leq \pi$) about the x, as illustrated in Figure 14(c).
(c) Rotate the top and the xyz coordinate through ψ ($0 \leq \psi \leq 2\pi$) about the z axis to obtain the final position, as illustrated in Figure 14 (d).

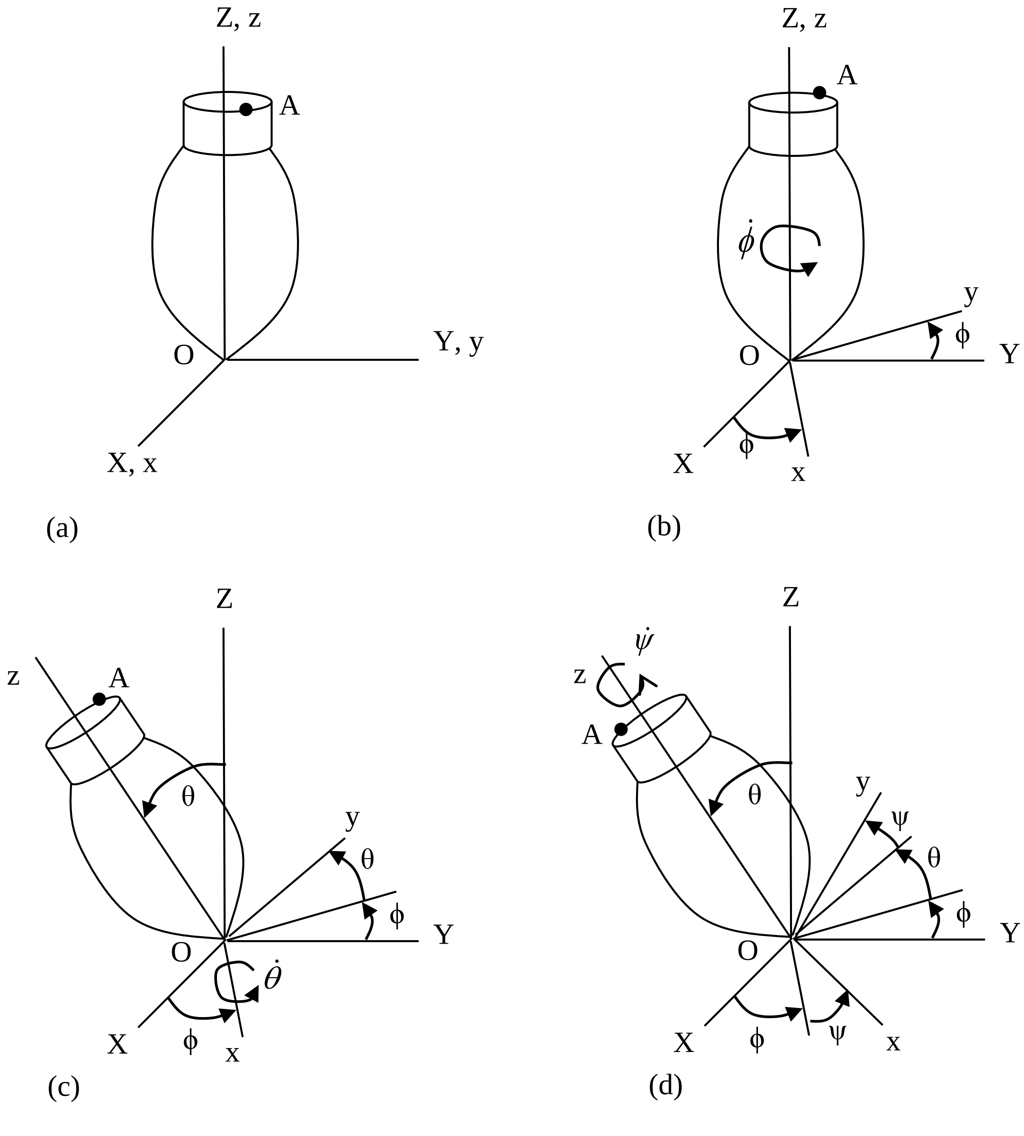


Figure 14: Euler's Angles of Spinning Top

Note that the finite rotations ϕ, θ and ψ are not vector quantities. However, the differential rotations dϕ, dθ and dψ are vectors, and hence the angular velocities and accelerations relating to these rotations are vectors. The angular displacements, velocities and accelerations relating to the angles ϕ, θ and ψ are referred to as *precession*, *nutation* and *spin*, respectively. These rotational vectors are not perpendicular to one another, as illustrated in Figure 15. The angular velocity of the top can expressed in terms of these angular velocities. If the top is considered to be orientated such that ψ = 0, then angular velocity of the xyz coordinate is $\Omega = \omega_p + \omega_n$. The angular velocity of the body has three components along x, y and z axes given as $\omega = \omega_x i + \omega_y j + \omega_z k$.

$$\omega = \dot{\theta}\,\mathrm{i} + \left(\dot{\phi}\sin\theta\right)\mathrm{j} + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\mathrm{k} \qquad \textbf{Equation 53}$$

Since the motion of the xyz coordinate is not affected by the spin motion, its angular velocity is given by

$$\Omega = \Omega_x \mathrm{i} + \Omega_y \mathrm{j} + \Omega_z \mathrm{k}.$$

The xyz axes represent the principal axes of inertia of the top for any spin, and hence, the products of inertia are:

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

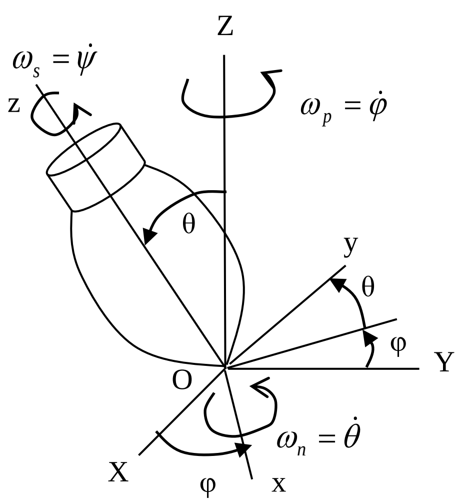


Figure 15: Angular Vectors of Rotating Top

Hence, the angular equations of motion for top are

$$\sum M_x = I_{xx}\ddot{\theta} + \left(I_{zz} - I_{xx}\right)\dot{\phi}^2 \sin\theta\cos\theta + I_{zz}\dot{\psi}\dot{\phi}\sin\theta \qquad \textbf{Equation 54}$$

$$\sum M_y = I_{xx}\left(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta\right) - I_{zz}\dot{\theta}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)$$ **Equation 55**

$$\sum M_z = I_{zz}\left(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta\right)$$ **Equation 56**

Note that each moment of inertia of Equations 54 to 56 is calculated with reference to axes through point O or the centre of mass of the body. Equations 54 to 56 are a set of second order non-homogeneous nonlinear differential equations; hence closed-form solutions may not be obtainable. Instead a numerical or computer technique may be used to obtain the Euler angles as functions of time. However, special cases exist, which simplify the solutions. These special cases are:

3-2.2.1 Steady Precession

Steady precession motion occurs when the spin rate $\dot{\psi}$ relative to the xyz coordinate, nutation angle θ and the precession rate $\dot{\phi}$ are (assumed to be) constant. This type of motion is commonly observed in tops and gyroscopes. Based on these assumptions equations 54 to 56 reduce to:

$$\sum M_x = \left(I_{zz} - I_{xx}\right)\dot{\phi}^2 \sin\theta\cos\theta + I_{zz}\dot{\psi}\dot{\phi}\sin\theta$$ **Equation 57**

$$\sum M_y = 0 \qquad \sum M_z = 0$$

3-2.2.2 Precession of a Top

This motion is peculiar to tops. When a top is set into a vertical rotational motion, its spin axis may initially remain vertical for some time, and exhibits a form of motion called *sleeping*. As friction reduces the spin rate, the spin axis begins to tip to one side as shown in Figure 16. This phase of the motion is approximately a steady precession. However, the spin rate continuously decreases due to friction. The moments of the weight of the top about the origin O are given by

$$\sum M_x = mgh\sin\theta \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

Substituting the above equations into Equations 54 to 56 assuming a steady precession, we have

$$mgh = \left(I_{zz} - I_{xx}\right)\dot{\phi}^2\cos\theta + I_{zz}\dot{\psi}\dot{\phi}$$ **Equation 58**

When θ = 90º, the dynamic behaviour of top reduces to spinning rotor as shown in Figure 17. As θ = 90º, Equation 46 or 57 reduces to

$$\sum M_x = I_{zz}\dot{\psi}\dot{\phi} = I_{zz}\omega_z\Omega_y$$ **Equation 59**

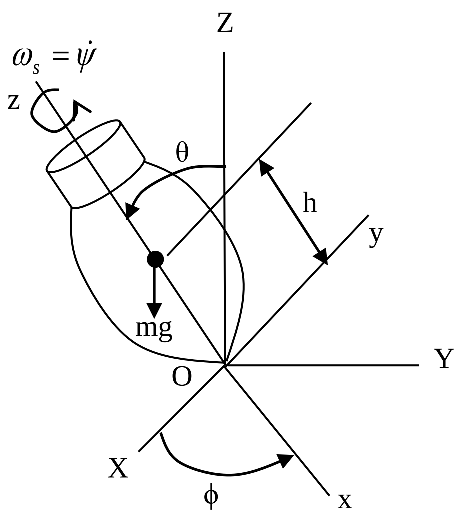


Figure 16: Free-Body Diagram of Rotating Top

It is seen from the figure that vectors $\sum M_x$, Ω_y and ω_z all act along their respective positive axes and are mutually perpendicular. Instinctively, one would expect the top to fall down under the influence of gravity. However, this is not case if the right hand side of the above equation is enough to counterbalance the moment of the rotor's weight about point O. This unusual phenomenon of rigid body motion is often referred to as the *gyroscopic effect*.

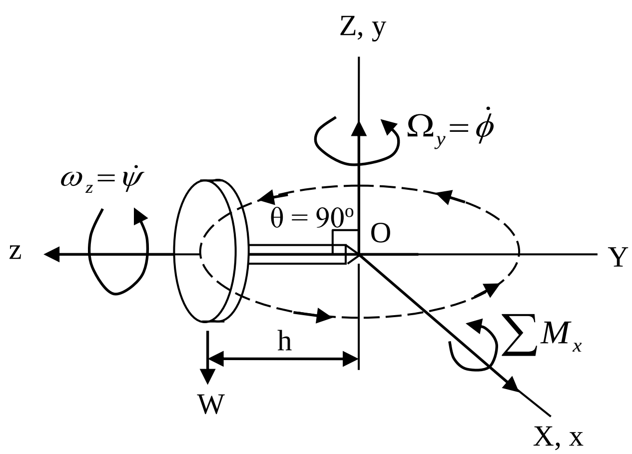


Figure 17: Spinning Rotor

3-2.2.3 Moment -Free Steady Precession

An axisymmetric object which is free of all external moments (or torques) and force except gravitational force and exhibiting steady processional motion is referred to as moment-free (or torque-free) steady precession. An axisymmetric satellite in orbit exhibits this motion.

This motion is also exhibited by American football and cocoa pod when they are thrown in a spirally and wobbly motion. Equating the moment in Equation 57 to zero and simplifying, we have

$$\left(I_{zz} - I_{xx}\right)\dot{\phi}\cos\theta + I_{zz}\dot{\psi} = 0 \qquad \text{or} \qquad \dot{\psi} = \left(\frac{I_{xx}}{I_{zz}} - 1\right)\dot{\phi}\cos\theta \qquad \textbf{Equation 60}$$

The z axis represents the axis of symmetry, which implies that $I_{xx} = I_{yy} = I$ for the body. Since the gravitational force is the only force acting on the body, the sum of moments about the centre of mass is zero. From Equation 41, the angular momentum about the centre of mass is constant, that is

$$H_G = constant$$

The angular momentum H_G is directed along the positive Z axis and y lies in the plane formed by z and Z axes. From Figure 16, the Euler angle between Z and z is θ. With this choice of axes and the above argument, the angular momentum H_G may be expresses as

$$H_G = H_G \sin\theta \,\mathrm{j} + H_G \cos\theta \,\mathrm{k} \qquad \text{(a)}$$

The angular momentum H_G may be expresses in terms of components of angular velocity as

$$H_G = I\omega_x i + I\omega_y j + I_{zz}\omega_z k \qquad \text{(b)}$$

Equating the respective i, j and k components of equations (a) and (b), we have

$$\omega_x = 0 \qquad \omega_y = \frac{H_G \sin\theta}{I} \qquad \omega_z = \frac{H_G \cos\theta}{I_{zz}} \qquad \textbf{Equation 61}$$

Similarly, equating the respective i, j and k components of Equation 53 to Equation 57, we have

$$\dot{\theta} = 0 \qquad \Rightarrow \dot{\theta} = \text{constant}$$

$$\dot{\phi}\sin\theta = \frac{H_G \sin\theta}{I} \qquad \dot{\phi} = \frac{H_G}{I} \qquad \textbf{Equation 62}$$

$$\dot{\phi}\cos\theta + \dot{\psi} = \frac{H_G \cos\theta}{I_{zz}} \qquad \dot{\psi} = \left(\frac{I - I_{zz}}{I_{zz}}\right) H_G \cos\theta \qquad \textbf{Equation 63}$$

Example 3- 5

A top of mass of 0.5 kg is precessing about the vertical axis at a constant angle of 80°, as shown in Figure E3-5. If it spins at ω_s = 100 rad/s, determine the precessional velocity ω_p.

Assume that the axial and transverse moments of inertia of the top are 0.45×10^{-3} kg.m^2 and 1.20×10^{-3} kg.m^2, respectively, measured with respect to the fixed point O.

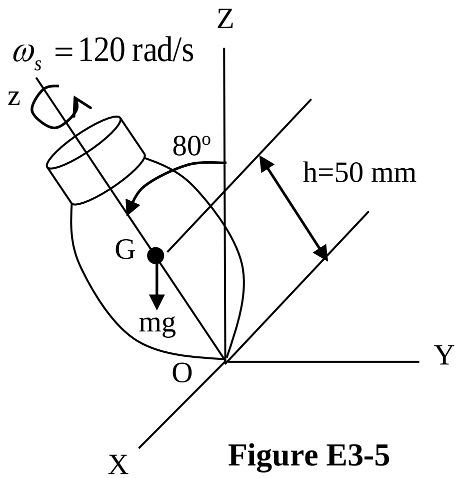


Figure E3-5

Solution

From Figure E3-5, the coordinate system is established such that the vertical points toward the positive Z direction, which is the axis of precession. Based on the coordinated system, the following given data are

$I_{xx} = 1.20 \times 10^{-3}$ kg.m^2 $I_{zz} = 0.45 \times 10^{-3}$ kg.m^2 m = 0.5 kg h = 50 mm = 0.05 m $\theta = 80^o$

Assuming that the motion is a steady precession and using Equation 54, the sum moments about point O is

$$mgh\sin\theta = \left(I_{zz} - I_{xx}\right)\dot{\phi}^2 \sin\theta\cos\theta + I_{zz}\dot{\psi}\dot{\phi}\sin\theta$$

$$(0.5)(9.81)(0.05)\sin 80^o = \left(0.45 \times 10^{-3} - 1.20 \times 10^{-3}\right)\dot{\phi}^2\left(\sin 80^o\right)\left(\cos 80^o\right) + \left(0.45 \times 10^{-3}\right)(100)\dot{\phi}\sin 80^o$$

$$0.375\,\dot{\phi}^2 - 45\dot{\phi} + 245.25 = 0$$

Solving the above quadratic equation yields

$$\dot{\phi} = 114.3 \text{ rad/s}$$

$$\dot{\phi} = 5.7 \text{ rad/s}$$

In reality, the low precession angular velocity would occur since that requires lower kinetic energy. Hence,

$$\underline{\underline{\dot{\phi} = 5.7 \text{ rad/s}}}$$

Example 3- 6

The wheel in Figure E3-6 has mass 3 kg and radius of gyration of 120 mm about the spin axis. In the figure the wheel rotates at end A of the shaft with angular velocity of 10 rad/s in the direction shown. A non-rotating body of mass 4 kg is p laced on the shaft a distance h from the point of rotation of the shaft. Neglecting the mass of the shaft, determine the angular velocity of precession ω_p when (a) h= 200 mm, and (b) h = 300 mm.

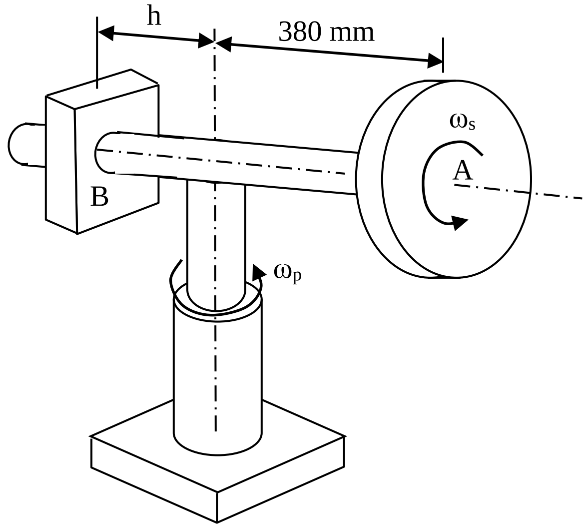


Figure E3-6

Solution

The free-body diagram of the wheel-shaft assembly is shown in Figure S3-6, where R_x, R_y and R_z represent the reaction at point D along x, y and z directions, respectively. The coordinate system is established with the vertical pointing to the positive y direction. The z axis is along the axis of spin, so that $\theta = 90^o$. Assuming a steady precession, and equating $\theta = 90^o$ we have

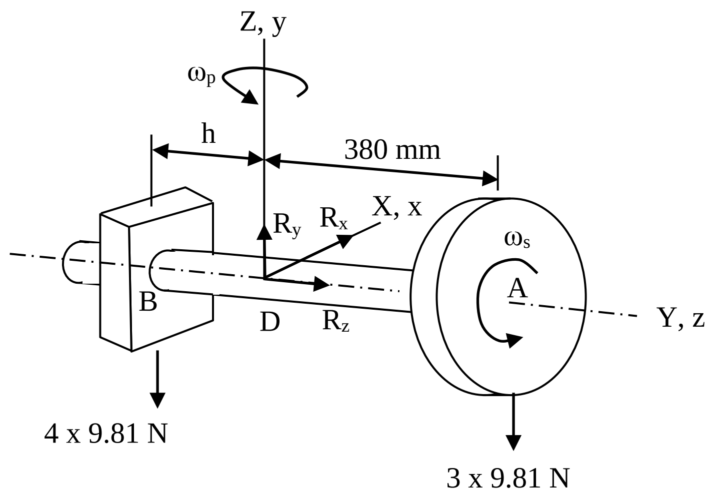


Figure S3-6

$$\sum M_x = I_{zz}\dot{\psi}\dot{\phi} = I_{zz}\omega_p\omega_s$$

Summing moment about point D on the axis of precession, we have

$$\sum M_x = (3)(9.81)(0.38) - (4)(9.81)(h) = [(3)(0.12^2)](10)\omega_p$$

(a) h = 0.2 m $\quad (3)(9.81)(0.38) - (4)(9.81)(0.2) = [(3)(0.12^2)](10)\omega_p \quad \underline{\underline{\omega_p = 7.72 \text{ rad/s}}}$

(b) h = 0.3 m $\quad (3)(9.81)(0.38) - (4)(9.81)(0.3) = [(3)(0.12^2)](10)\omega_p \quad \underline{\underline{\omega_p = -1.36 \text{ rad/s}}}$

Example 3- 7

A young man threw a cocoa pod and the motion of it is observed to be directed 30° from the horizontal, while it is precessing about the vertical axis at 4 rad/s, as shown in Figure E3-7. Neglecting the effect of air resistance and knowing that the ratio of the axial to transverse moments of inertia of the cocoa pod is 7/20 measured from the centre of mass, determine the magnitude of the spin of the pod and its angular velocity.

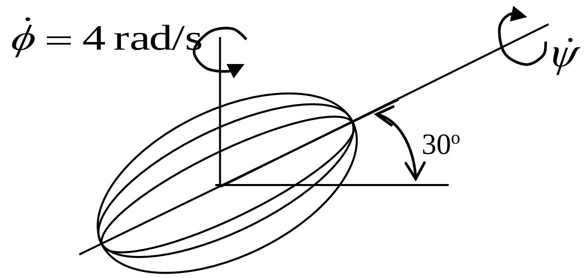


Figure E3-7

Solution

Since the weight of the cocoa pod is the only force acting on it, it is exhibiting a moment-free motion. The Z-axis is established along the vertical (which is the precession axis) and the spin axis is set as the z –axis, as shown in Figure S3-7.

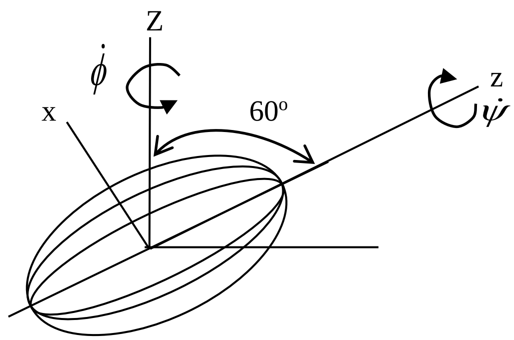


Figure S3-7

Using equation 3.19, we have

$$\dot{\psi} = \left(\frac{I_{xx}}{I_{zz}} - 1\right)\dot{\phi}\cos\theta = \left(\frac{\frac{20}{7} I_{zz}}{I_{zz}} - 1\right) 4\cos 60^o$$

$$\underline{\underline{\dot{\psi} = 3.71\,\text{rad/s}}}$$

Using equation 56 and for symmetry along the z axis, $H_G = \dot{\phi} I_{yy} = \dot{\phi} I_{xx} = (4)(I)$

Noting that the angular velocity along the x axis is zero, we have

$\omega_x = 0$ $\qquad \omega_y = \dfrac{[4I]\sin 60}{I}$ $\qquad \omega_y = 3.46\,\text{rad/s}$

$$\omega_z = \frac{[4I]\cos 60}{I_{zz}} = \frac{[4I]\cos 60}{\left(\frac{7}{20}\right) I} \qquad \omega_z = 5.71\,\text{rad/s}$$

$$\omega = \sqrt{\left[(\omega_x)^2 + (\omega_y)^2 + (\omega_z)^2\right]} = \sqrt{\left[0^2 + (3.46)^2 + (5.71)^2\right]} \qquad \underline{\underline{\omega = 6.68\,\text{rad/s}}}$$

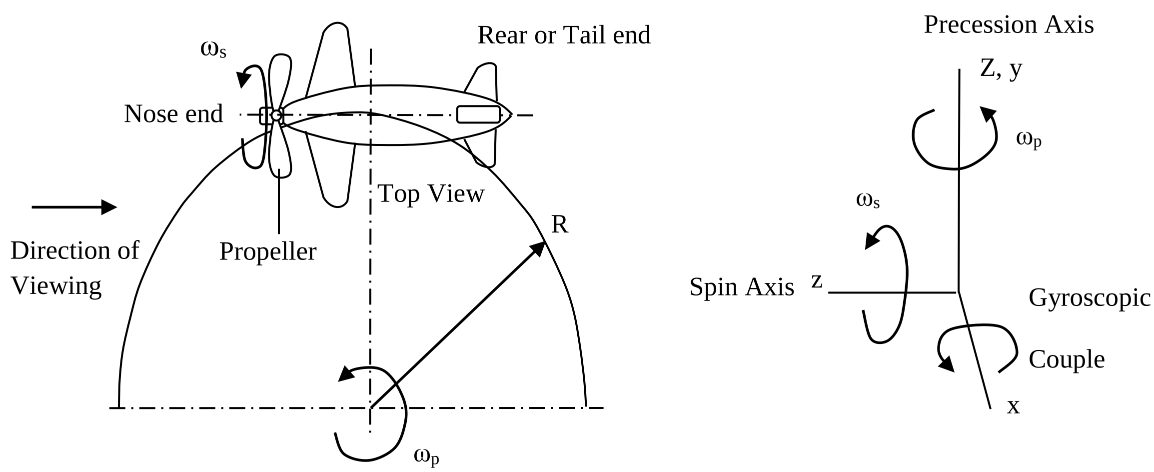
SESSION 3-3: Gyroscopic Effects in Machines

Gyroscopic effect is felt or utilized in many machines that have more than two non-parallel axis of rotation, including rotation of the whole machine. These machines include automobiles, aeroplanes, ships, submarines, hovercrafts, projectiles, missiles, gyrocompass, etc. Free gyro mechanism is utilized in gyrocompass where the spin axis is directed north. This device is useful for local navigation of ships and aircrafts. The gyroscopic effect influences steering, pitching and rolling of aeroplanes and ships, and stability of projectiles, missiles, four- and two-wheel drives. Gyroscopic effect must also be taken into consideration in the designs of shafts, bearings and supports of bearing of rotors which are subjected to forced precessions.

3-3.1 Gyroscopic Effects on Airplanes and Naval Ships during Steering

Consider an airplane making a turn about a curve of radius of curvature R and moving at a constant horizontal speed v as shown in Figure 17 (a). In the figure, viewing from the nose end, the propeller is spinning at a constant rate ω_s in the anti-clockwise direction. The airplane is also precessing steadily at ω_p about the vertical axis, which corresponds to the Z axis on the coordinate systems shown in Figure 19(b). The angular velocity of precession is derived from:

$$\dot{\phi} = \omega_p = \frac{v}{R}$$



(a) (b)

Figure 18: (a)Airplane Making a Turn, (b) Direction of Rotation and Couple

The gyroscopic couple acting on the plane is determined using Equation 51 and setting $\theta = 90^o$. The moment of inertia I_{zz} is the moment of inertia of the engine moving parts and the propeller(s) about the spin axis. Note that if the airplane is not moving horizontally, then $\theta \neq 90^o$.

The gyroscopic effect is a couple about the x axis as illustrated in Figure 17(b). In the figure, if ω_s and ω_p are both positive, the gyroscopic effect would turn nose downward, as viewed from the nose direction. Depending on the direction of precession and spin, the gyroscopic effect may turn the nose upwards and the tail downwards or vice versa. Similarly, gyroscopic couple affect steering of naval ships.

Example 3- 8

An airplane makes a right turn, as view from the nose end, about a semi circular path of radius 120 m when cruising horizontally at 200 km/h. The rotating parts of the engine and the propeller have a combined mass of 500 kg and a radius of gyration of 0.3. If the engine and the propeller rotate at 2500 rpm clockwise as viewed from nose end, determine the gyroscopic couple on the airplane and state its effect on it.

Solution

The angular velocity of precession is

$$\omega_p = \frac{\left(200 \text{ x } \frac{1000}{60^2}\right)}{120} \qquad \omega_p = 0.463 \text{ rad/s}$$

The spin rate of the engine and propeller is

$$\omega_s = \frac{-2500 \text{ x } 2\pi}{60} \qquad \omega_s = -261.8 \text{ rad/s}$$

The Z-axis is established along the vertical (which is the precession axis) and the spin axis is set as the z –axis, as shown in Figure 17(a). In the figure, the vertical axis is normal to the page, and the figure is a two-dimensional sketch illustrating the position of the airplane in relation to the axis of precession. A three-dimensional version of the figure is shown in Figure 17(b). The gyroscopic couple on the airplane is

$$C = I\omega_s\omega_p = \left(mk^2\right)\omega_s\omega_p = \left[(500)\left(0.3^2\right)\right](-261.8)(0.463) \qquad \underline{\underline{C = -5.45 \text{ kN - m}}}$$

The gyroscopic effect is opposite to the gyroscopic couple direction indicated in Figure 19(b). The couple would raise the nose upwards and tail downwards.

3-3.2 Gyroscopic Effects on Stability of Four-Wheel Drives

Neglecting air and aerodynamic effects, three factors affect the stability of a four-wheel drive. The three effects are gyroscopic couples, centrifugal force and weight of the vehicle. The reaction due to each effect would be calculated separately. The method of superposition would be used to calculate the overall reaction on each wheel due to the effects.

Consider a four-wheel vehicle making a turn about a curve of radius of curvature R and moving at a constant speed v. Suppose that road is banked at β to the horizontal. From method of superposition, the total effect on each wheel is the sum of individual effect. The individual effects are :

Gyroscopic Couple

Most vehicles are designed to minimise or eliminate the differences in rotational speeds of inner and outer wheels during curving. This is accomplished by the use of drive differentials. Using Figure 18(a) and assuming that inner and outer wheels have the same precession rate, the angular velocities of precession of the wheels are

$$\omega_p = \dot{\phi}_i = \frac{v}{R}$$

Using equation 51, adapting the coordinate system shown in Figure 18 (b), and noting that the nutation angle is $\theta = 90^o + \beta$, the total gyroscopic couples due to all wheels may be calculated as

$$C_W = 4\left[\left(I_{zz} - I_{xx}\right)\dot{\phi}^2 \sin\left(90^o + \beta\right)\cos\left(90^o + \beta\right) + I_{zz}\dot{\psi}\dot{\phi}\sin\left(90^o + \beta\right)\right] \qquad \textbf{Equation 64}$$

where I_{zz} and I_{xx} are the moments of inertia about axis normal and perpendicular , respectively, to the spin axis and through the centre of gravity. If there is no banking ie β = 0, then Equation 58 reduces to Equation 59.

$$C_W = 4I_{zz}\dot{\psi}\dot{\phi} \qquad \textbf{Equation 65}$$

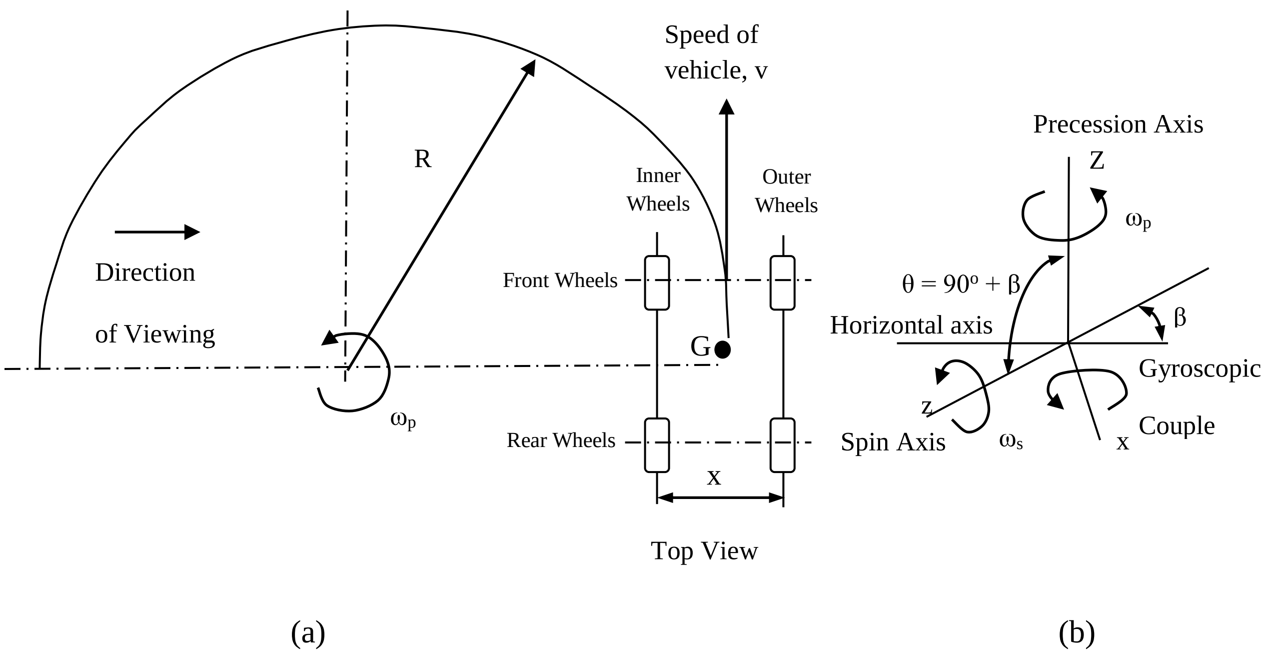


Figure 19: Four-Wheel Drives Making a Turn, (b) Direction of Rotation and Couple

Rotations of engine parts also induce gyroscopic couple on vehicles. If an engine's crankshaft rotates about the same axis as that of the wheels, the banking affects the gyroscopic couple to due to the rotating engine parts. Equation 55 is used by accounting for the banking angle. However, if the rotation of the engine is normal to the front of the vehicle, the gyroscopic couple due to the rotating parts of the engine is not affected by banking. Hence, the gyroscopic couple of the engines parts are determined by substituting θ = 90° into Equation 57. The net gyroscopic effect on each wheel is given by

$$C = C_W \pm C_E$$ **Equation 66**

where the subscripts W and E respectively represent the gyroscopic couples due to wheels and rotating engine parts. In the above equation, a positive sign (+) is used if the two couples have the same sense of direction, otherwise a negative sign is used.

To determine the reactions on the inner and outer wheels due to the combined gyroscopic couple, a free-body diagram showing the couple is drawn as shown in Figure 19. Taking moment about point A, the reactions on inner and outer wheels are

$$C = R_{o,c} x = R_{i,c} x$$

The above equation shows that the reactions are equal. From the above equation, the reaction on each wheel due to the gyroscopic couples are given by

$$R_c = \frac{C}{2x}$$ **Equation 67**

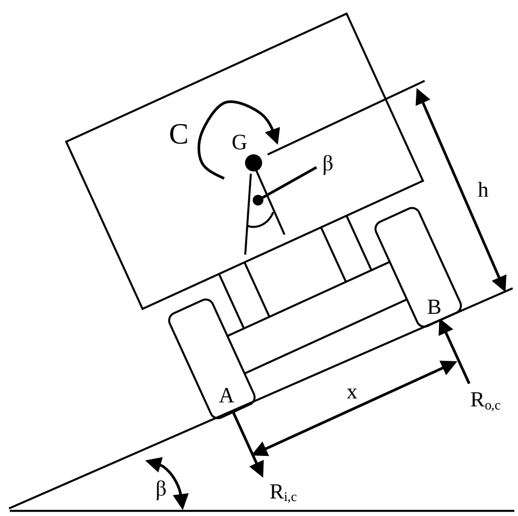


Figure 20: Reaction on Wheels due to Gyroscopic Couple

Centrifugal Force and Weight

Since the vehicle moves in a circular path, centrifugal force acts outwardly at the centre of gravity of the vehicle. The centrifugal force on a vehicle of mass m is give by

$$F_c = \frac{mv^2}{R} \qquad \textbf{Equation 68}$$

The centrifugal force acts to overturn the vehicle as shown in Figure 20 (a). Summing moments about point A, we have

$$R_B = \frac{mv^2}{Rx}\left[h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left[-h\sin\beta + \frac{x}{2}\cos\beta\right] \qquad \textbf{Equation 69}$$

Similarly, summing moments about point B, we have

$$R_A = -\frac{mv^2}{Rx}\left[h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left[h\sin\beta - \frac{x}{2}\cos\beta\right] \qquad \textbf{Equation 70}$$

In Equations 69 and 70, R_A and R_B are reactions on the two inner and two outer wheels respectively. The first square brackets in each of the above two equations represents the reaction due to centrifugal force. To determine the reactions on the front and rear wheels due to the weight of the vehicle, the free-body diagram in Figure 21(b) is used. Summing moments about the front wheel, we have

$$R_r = W\frac{y_1}{y_1 + y_2} \qquad \textbf{Equation 71}$$

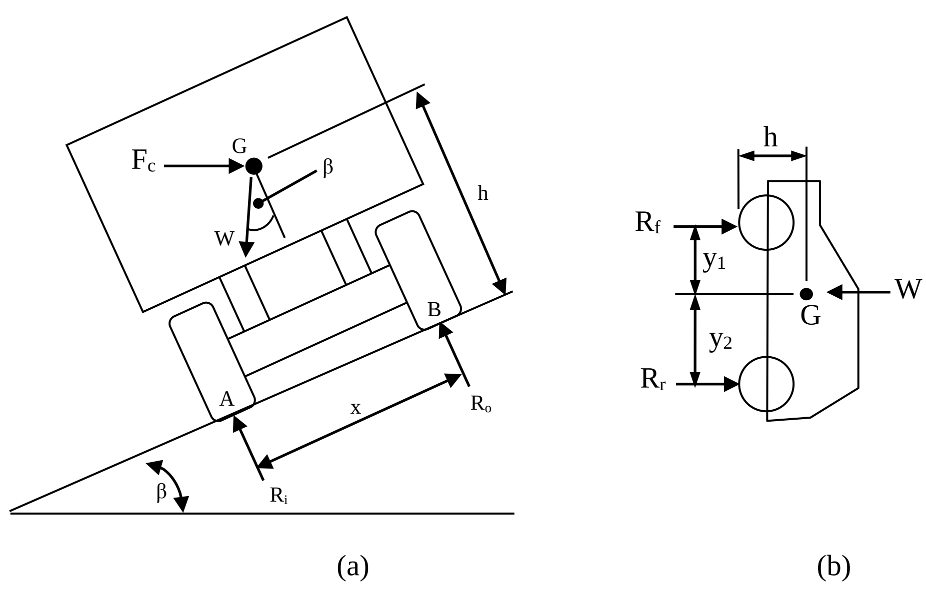


Figure 21: Reaction on Wheels due to Centrifugal Force

Similarly, summing moments about the rear wheel, we have

$$R_f = W\frac{y_2}{y_1 + y_2} \qquad \textbf{Equation 72}$$

Superposition of Equations 69 to 72, the reactions on the individual wheels due to weight and centrifugal forces are

$$R_{o,f} = \frac{mv^2}{2Rx}\left[h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left(\frac{y_2}{y_1 + y_2}\right)\left[-h\sin\beta + \frac{x}{2}\cos\beta\right] \qquad \textbf{Equation 73}$$

$$R_{o,r} = \frac{mv^2}{2Rx}\left[h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left(\frac{y_1}{y_1 + y_2}\right)\left[-h\sin\beta + \frac{x}{2}\cos\beta\right] \qquad \textbf{Equation 74}$$

$$R_{i,f} = \frac{mv^2}{2Rx}\left[-h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left(\frac{y_2}{y_1 + y_2}\right)\left[h\sin\beta + \frac{x}{2}\cos\beta\right] \qquad \textbf{Equation 75}$$

$$R_{i,r} = +\frac{mv^2}{2Rx}\left[-h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left(\frac{y_1}{y_1 + y_2}\right)\left[h\sin\beta + \frac{x}{2}\cos\beta\right] \qquad \textbf{Equation 76}$$

In the above equations, the subscripts i, o, f and r denote inner, outer, front and rear wheels, respectively. Adding all the reactions on the wheels by superposition, the reaction on a wheel is given by

$$P_{j,l} = \mathrm{sgn}\frac{C}{2x} + R_{j,l} \qquad \textbf{Equation 77}$$

Where

j = inner or outer,

l = rear or front,

sgn = + when j = inner and sgn = - when j = outer.

For an inner wheel to make contact with the road surface,

$$-\frac{C}{2x} + R_{i,l} \geq 0 \qquad \textbf{Equation 78}$$

Example 3- 9

A four wheel automobile is travelling along the Suame circle in Kumasi, which has a mean radius of 40 m. Each wheel has a moment of inertia of 2.0 kg-m^2 and effective radius of

0.25 m. The rotating parts of the engine have a moment of inertia of 1.0 kg-m^2. The axis of rotation of the engine is parallel to the rear axle and the crankshaft rotates in the same sense as the wheels. The ratio of engine speed to the back axle speed is 3:1. The automobile has a mass of 1500 kg, its centre of gravity is 0.5 m above the road level and the distance between the inner and outer wheels is 1.1 m. The centre of gravity of the automobile lies at middle of the back axle and at 1/3d from the front wheel, where d is distance between the front and rear wheels. Determine the maximum speed the automobile can travel around the circle for the four wheels to maintain contact with the ground.

Solution

$$I_{zz,wheel} = 2.0\,\mathrm{kg\text{-}m^2}$$

If v is the linear velocity if the automobile, then spin velocity of each wheel is

$$\omega_W = \frac{v}{r_w} = \frac{v}{0.25} \qquad\qquad \omega_W = 4v$$

The angular velocity of precession is

$$\omega_p = \frac{v}{R} = \frac{v}{40} \qquad\qquad \omega_p = 0.025v$$

From the velocity ratio, we have

$$\omega_E = 3\omega_W = 3(4v) \qquad\qquad \omega_E = 12v$$

Since there is no banking the gyroscopic couple due to the four wheels and engine parts are

$$C_w = 4I_{zz,w}\omega_w.\omega_p = 4(2.0)(4v).(0.025v) \qquad\qquad C_w = 0.8v^2$$

$$C_E = I_{zz,E}\omega_E.\omega_p = (1.0)(12v).(0.025v) \qquad\qquad C_E = 0.3v^2$$

Since the wheels and engine crankshaft rotate in the same direction, the total gyroscopic couple is

$$C = C_W + C_E = 0.8v^2 + 0.3v^2 \qquad\qquad C = 1.1v^2$$

Using Equation 67, the magnitude of vertical reaction on each wheel is

$$R_c = \frac{C}{2x} = \frac{1.1v^2}{2(1.1)} \qquad\qquad R_c = 0.5v^2$$

The reaction would be vertically upward on the outer wheels and vertically downward on the inner wheels. The outer wheels would always make contact with the road surface. The reactions due to centrifugal force and weight of the automobile on the inner wheels are

$$R_{i,f} = \frac{mv^2}{2Rx}\left[-h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left(\frac{y_2}{y_1+y_2}\right)\left[h\sin\beta + \frac{x}{2}\cos\beta\right]$$

$$R_{i,f} = \frac{(1500)v^2}{2(40)(1.1)}\left[-(0.5)\cos(0) + \frac{1.1}{2}\sin(0)\right] + \frac{(1500 \text{ x } 9.81)}{1.1}\left(\frac{\frac{2}{3}d}{d}\right)\left[(0.5)\sin(0) + \frac{1.1}{2}\cos(0)\right]$$

$$R_{i,f} = -8.523v^2 + 4905$$

$$R_{i,f} = \frac{mv^2}{2Rx}\left[-h\cos\beta + \frac{x}{2}\sin\beta\right] + \frac{W}{x}\left(\frac{y_1}{y_1+y_2}\right)\left[h\sin\beta + \frac{x}{2}\cos\beta\right]$$

$$R_{i,f} = \frac{(1500)v^2}{2(40)(1.1)}\left[-(0.5)\cos(0) + \frac{1.1}{2}\sin(0)\right] + \frac{(1500 \text{ x } 9.81)}{1.1}\left(\frac{\frac{1}{3}d}{d}\right)\left[(0.5)\sin(0) + \frac{1.1}{2}\cos(0)\right]$$

$$R_{i,r} = -8.523v^2 + 2452.5$$

The total reaction on each of the inner front wheel is

$$P_{i,f} = -R_c + R_{i,f} = -0.5v^2 - 8.523v^2 + 4905 \qquad P_{i,f} = -9.023v^2 + 4905$$

Similarly,

$$P_{i,r} = -R_c + R_{i,r} = -0.5v^2 - 8.523v^2 + 2452.5 \qquad P_{i,f} = -9.023v^2 + 4905$$

For the inner front wheel to make contact with the road surface,

$$P_{i,f} \geq= 0 \qquad -9.023v^2 + 4905 \geq 0 \qquad 9.023v^2 \leq 4905$$

$$v \leq 23.3 \text{ m/s} \qquad v \leq 83.94 \text{ km/h}$$

For the inner rear wheel to make contact with the road surface,

$$P_{i,r} \geq= 0 \qquad -9.023v^2 + 2452.5 \geq 0 \qquad 9.023v^2 \leq 2452.5$$

$$v \leq 16.48 \text{ m/s} \qquad v \leq 59.35 \text{ km/h}$$

Comparing the speeds, if the automobile travels beyond 59.35 km, the inner rear wheel would not make contact with the surface of the road. Hence, the answer is the minimum of the two velocities. That is

$$\underline{\underline{v \leq 59.35 \text{ km/h}}}$$

3-3.3 Gyroscopic Effects on Stability of Two-Wheel Drives

Similar to a four-wheel drive, three main factors contribute to the stability of a two-wheel drive. Again, the three effects are gyroscopic couples, centrifugal force and weight of the vehicle. The reaction due to each effect are calculated separately and added together using the method of superposition.

Gyroscopic Couple

The spin rate of the wheels are related to the linear speed by

$$\omega_w = \frac{v}{r_w}$$

where r_w is radius of the wheel. The spin rate of the rotating parts of the engine is given by

$$\omega_E = G\omega_w \qquad\qquad \omega_E = G\frac{v}{r_w}$$

where G is gear ratio $G = \omega_E/\omega_W$. The velocity of precession is given as

$$\omega_p = \frac{v}{R}$$

When two-wheel drive makes a turn, the vehicle is always inclined at an angle λ to to the vertical, as illustrated in Figure 22. This angle is known as *angle of heel*. The precession is vertical through the vehicle moves in a circular path in at the angle of heel to the vertical. The gyroscopic couples of both the wheels and rotating parts of the engine may calculate using Equation 55. Alternatively, by neglecting the bracket in Equation 57 the total gyroscopic couple may be approximate as

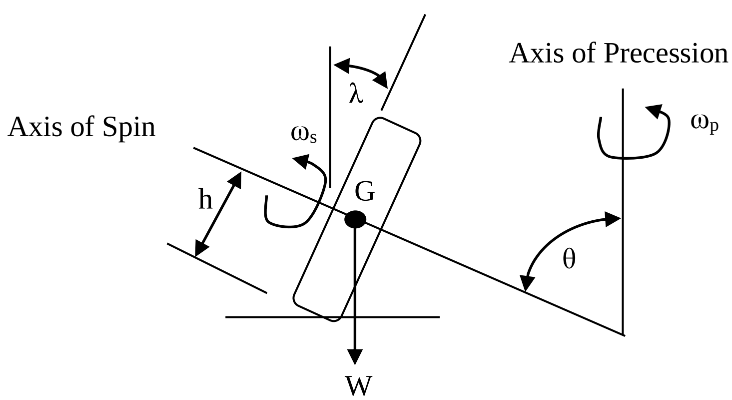


Figure 22: Gyroscopic on Two-Wheel Vehicle

$$C = \left(2I_W \omega_W \omega_p \pm I_E \omega_E \omega_p\right)\sin\theta \qquad \textbf{Equation 79}$$

The angle of heel λ is related to the nutation angle θ by $\theta = 90^o - \lambda$ or $\sin\theta = \cos\lambda$
Substituting for the angle of heel, spin rates ω_E and ω_W, and precession rate ω_p, the above equation reduces to

$$C_G = \frac{v^2}{Rr_W}\left(2I_W \pm GI_E\right)\cos\lambda \qquad \textbf{Equation 80}$$

Centrifugal Force

The centrifugal force on a vehicle of mass m is give by

$$F_c = \frac{mv^2}{R} \qquad \textbf{Equation 81}$$

The centrifugal couple is

$$C_C = F_c.h\cos\lambda \qquad\qquad C_C = \frac{mv^2}{R}h\cos\lambda$$

Both gyroscopic couple and centrifugal couples act to overturn the vehicle. For the vehicle to be stable, the overturning couple must be balance by the moment of the weight. The total overturning moment is

$$C = C_G + C_C = \frac{v^2}{Rr_W}\left(2I_W \pm GI_E\right)\cos\lambda + \frac{mv^2}{R}h\cos\lambda$$

Summing moments at the point where the wheel makes contact with the ground, we have

$$\frac{v^2}{R}\left[\frac{1}{r_W}\left(2I_W \pm GI_E\right) + mh\right]\cos\lambda = mgh\sin\lambda \qquad \textbf{Equation 82}$$

Example 3- 10

A motor bike and its rider have mass 250 kg goes around the circular at the entrance to the Kwame Nkrumah University of Science and Technology, Kumasi, at speed of 100 km/h. The combined centre of gravity of the rider and the bike is 0.6 m above the road surface when the vehicle is vertical. Each wheel has a radius of 0.25 m and moment of inertia of 1.2 kg-m^2, and the engine's flywheel has moment of inertia of 0.3 kg-m^2 and rotating at 5 times the speed of the wheels and in the same direction. Find the (a) relationship between the radius of turn and maximum angle of inclination of the wheels to the vertical, and (b) the maximum angle of turn if the radius of turn is 40 m.

Solution

(a) Using equation 82, we have

$$\frac{v^2}{R}\left[\frac{1}{r_W}\left(2I_W \pm GI_E\right) + mh\right]\cos\lambda = mgh\sin\lambda$$

$$\frac{\left(100 \text{ x } \frac{1000}{60^2}\right)^2}{R}\left[\frac{1}{(0.25)}\left(2(1.2) + (5)(0.3)\right) + (250)(0.6)\right]\cos\lambda = (250)(9.81)(0.6)\sin\lambda$$

$$\tan\lambda = \frac{86.835}{R} \qquad \text{or} \qquad \text{R}\tan\lambda = 86.835$$

A plot of radius of turn on horizontal verses angle of heel of the bike is shown in Figure 23. The figure shows that as the radius increases, the angle of heel decrease. It may be explained the gyroscopic couple and centrifugal force are inversely proportional to the radius of turn. As a result, increase in radius results in decrease in moment of weight required to balance gyroscopic couple and centrifugal force.

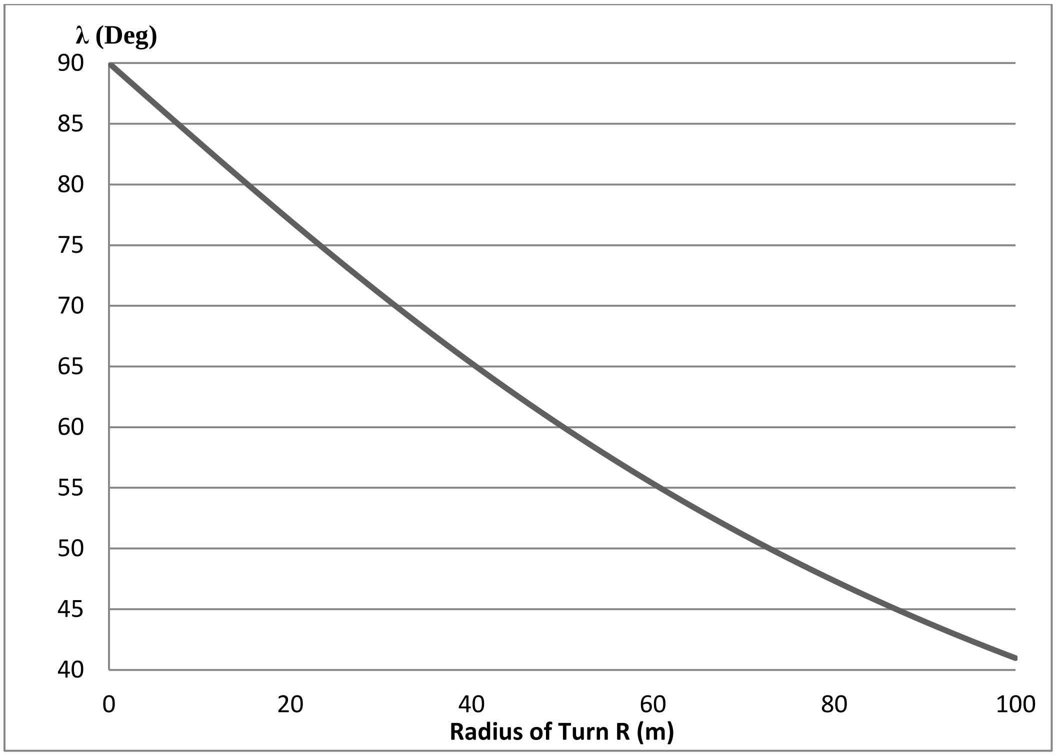


Figure E23

(b) From the above relation,

$$\tan\lambda = \frac{86.835}{(40)} \qquad \underline{\underline{\lambda = 65.3^o}}$$

Unit Summary

Equations of Motion

The scale equations of translational motion for a rigid body that moves in three dimensions are:

$$\sum F_x = m(a_G)_x$$
$$\sum F_y = m(a_G)_y$$
$$\sum F_z = m(a_G)_z$$

The scale equations of rotation motion for a rigid body that moves in three dimensions are:

$$\sum M_x = I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z - \Omega_z\left(-I_{xy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z\right) + \Omega_y\left(-I_{yx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z\right)$$
$$\sum M_y = -I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y - I_{yz}\dot{\omega}_z + \Omega_z\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xy}\omega_z\right) - \Omega_x\left(-I_{xz}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z\right)$$
$$\sum M_z = -I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z - \Omega_y\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right) + \Omega_x\left(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z\right)$$

If the axes are oriented such that the axes are principal axes of inertia, then all products of inertia are zero. The scale equations of rotation motion for a rigid body that moves in three dimensions reduce to

$$\sum M_x = I_{xx}\dot{\omega}_x - I_{yy}\omega_y\Omega_z + I_{zz}\omega_z\Omega_y$$
$$\sum M_y = I_{yy}\dot{\omega}_y + I_{xx}\omega_x\Omega_z - I_{zz}\omega_z\Omega_x$$
$$\sum M_z = I_{zz}\dot{\omega}_z - I_{xx}\omega_x\Omega_y + I_{yy}\omega_y\Omega_x$$

If the axes are fixed in and move with the rotation of the body i.e. $\Omega = \omega$, then the above equations reduce to

$$\sum M_x = I_{xx}\dot{\omega}_x - I_{yy}\omega_y\omega_z + I_{zz}\omega_z\omega_y$$
$$\sum M_y = I_{yy}\dot{\omega}_y + I_{xx}\omega_x\omega_z - I_{zz}\omega_z\omega_x$$
$$\sum M_z = I_{zz}\dot{\omega}_z - I_{xx}\omega_x\omega_y + I_{yy}\omega_y\omega_x$$

Gyroscopic Motion

The angular motion of gyroscope is described using changes in three angles called Euler angles. These angles are the precession ϕ, the nutation θ and the spin ψ. If $\dot{\theta} = 0$,and $\dot{\phi}$ and $\dot{\psi}$ are constant, the equations of rotation motion becomes

$$\sum M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right)$$
$$\sum M_y = 0$$
$$\sum M_z = 0$$

A body is subjected to only gravitational force has no moment it about its centre of mass. The motion exhibited by a body under such condition is described as torque-free (or moment-free) motion.

Gyroscopic Effect in Machines

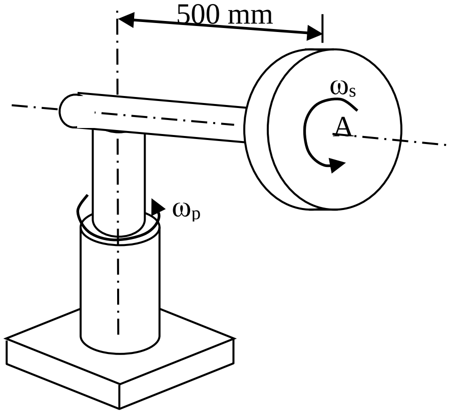
Gyroscopic effect is affect that have more than two non-parallel axis of rotation, including rotation of the whole machine. The gyroscopic effect influences steering, stability, pitching and rolling of aeroplanes, ships, missiles, four-and two-wheel drives.

☑ Key terms/ New Words in Unit

1. Angle of heel
2. Precession
3. Spin
4. Nutation
5. Euler angle
6. Steady precession
7. Moment-free steady precession
8. Euler equations
9. Gyroscopic effect
10. Centrifugal force
11. Angular momentum
12. Sleeping

Self Assessment 3

3-1. A flywheel of radius of gyration 150 mm is spinning at ω_s = 250 rad/s about a shaft of negligible mass, as shown in Figure P3-1. If the shaft is to be horizontal and the flywheel is at find the rotational speed ω_p of the vertical shaft.



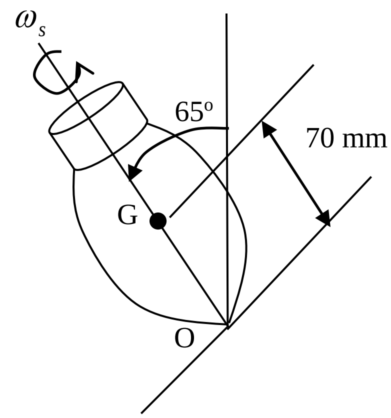


Figure P3-1 **Figure P3-2**

3-2. A top of mass of 2 kg is at a constant speed of 1.2 rad/s about the vertical axis, as shown in Figure P3-2. The centre of gravity is at point G, which is 70 mm from a plane that goes point O and perpendicular line GO. The axial and transverse radius of gyrations of the top are 30 mm and 50 mm, respectively, measured with respect to the fixed point O. At the instant shown, determine the spin rate ω_s of the top.

3-3. A young man threw a cocoa pod and the motion of it is observed to be directed 30° from the horizontal, while it is precessing about the vertical axis at 4 rad/s, as shown in Figure P3-3. Neglecting the effect of air resistance and knowing that the ratio of the axial to transverse moments of inertia of the cocoa pod is 7/20 measured from the centre of mass, determine the magnitude of the pod's (a) spin, and (b) angular velocity.

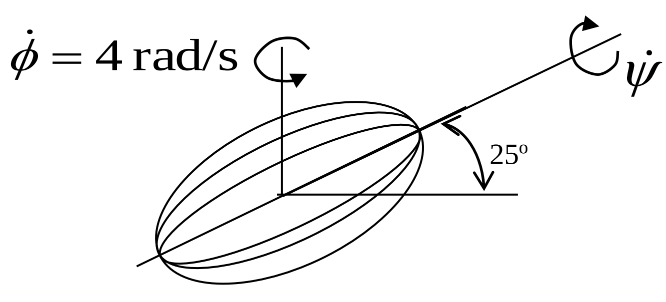


Figure P3-3

3-4. A racing motor cyclist travels at 140 km/h round a curve of 120 m radium measured horizontally. The cycle and rider have mass of 150 kg and their centre of gravity lies at 0.55 m above the ground level when the motor cycle is vertical. Each wheel has

0.70 m diameter and 1.5 kg-m^2 moment of inertia about its axis of rotation. The rotating parts of the engine are equivalent to a flywheel of moment of inertia 0.25 kg-m^2 and rotational speed five times that of the wheel in the same direction. For no side sliding, find the inclination of the cycle and rider to the vertical.

3-5. The spin axis of an axisymmetric turbine is horizontal and aligned with the longitudinal axis of the ship. The turbine has moment of inertia of 3000 kg-m^2 about its axial axis and rotates at 10 000 rpm. If the ship turns at a constant rate of 60 degrees per minute, what is the magnitude of the moment exerted on the ship by the turbine?

3-6. An airplane makes a right turn, as view from the nose end, about a semi circular path of radius 120 m when cruising horizontally at 200 km/h. The rotating parts of the engine and the propeller have a combined mass of 500 kg and a radius of gyration of 0.3. If the engine and the propeller rotate at 2500 rpm clockwise as viewed nose end, determine the gyroscopic couple on the airplane and state its effect on it.

3-7. The propeller on a single-engine airplane has a mass 20 kg and a centroidal radius of gyration 0.4 computed about the axis of spin. When viewed from the front of the airplane, the propeller is turning clockwise at 1200 rpm about the spin axis. If the airplane enters a vertical curve having a radius 100 m and is travelling at speed 200 km/h, determine the gyroscopic bending moment which the propeller exerts on the bearings of the engine when the airplane is in its lowest position.

3-8. A car is traveling at velocity of 120 km/h around the horizontal curve having a radius of 100 m. The distance between the inner and outer wheels is 1.4 m. If each wheel has mass 15 kg, radius of gyration 300 mm about its spinning axis, and radius 400 mm, determine the difference between the normal forces of the rear wheels, caused by the gyroscopic effect.

3-9. An airplane flying at 300 km/h makes a turn at a radius of 60 m toward the left as viewed from the front. The rotating parts of the engine and the propeller of the plane have a total mass of 300 kg and radius of gyration of 0.3 m. The engine rotates at 2500 rpm clockwise as viewed from the front. Find the gyroscopic couple on the airplane and state its effect on it.

3-10. A four-wheel electric train car has a total mass of 3.6 tonnes and two axles. Each axle together with its wheels and gearings has a total moment of inertia of 25 kg-m^2. The centre distance between the two wheels on axle is 1.5 m and radius of each wheel is 400 mm. Each axle is driven a motor with a speed reduction of 1:4. Each motor runs in a direction opposite to that of its axle. The centre of gravity of the car is 0.85 m above the rails. Assuming that no can is provided for the outer rail, determine the maximum speed at which the car can round the curve of 200 m radius such that no wheel leaves the rail.

3-11. Each wheel of a motor bicycle has moment of inertia of 1.2 kg-m^2 and diameter of 0.5 m. The rotating parts of the engine of the bicycle have a total moment of inertia of 0.3 kg-m^2. The rotational speed of the engine shaft is four times that of the wheels. The mass of the bicycle with its rider is 250 kg and their centre of gravity is 0.5 m above the ground level. Knowing that the bicycle is making a turn of radius 30 m at a speed of 60 km/h, determine the angle of heel if the engine and the wheels are rotating in (a) the same directions, and (b) opposite directions. Assume that there is no side sliding.

UNIT 4

FORCE ANALYSIS OF MECHANISMS AND MACHINES

Introduction

This unit focuses on forces in machines. Two types of forces act on a machine part in motion. They are static and dynamic forces. Static forces are associated with bodies in equilibrium while dynamic forces are associated with accelerating masses. Since machines contain both non-accelerating and accelerating parts, static and dynamic forces are always present in machines. These dynamic forces and torques are called *inertia force* and *inertia torque*, respectively.

In *Unit 1*, Kinematic analysis was used to define set of motions for two-dimensional and three-dimensional systems without referring to the forces that cause the motion. In *Unit 2*, the geometric properties of bodies were defined. These properties are related to dynamic forces and torques. *Unit 3* focused on forces associated with single body in three-dimensional motion. In this unit, an *inverse dynamic* or *kinetostatic* analysis is used to determine dynamic forces and torques required to give a particular motion to a linkage. Basically, this unit focuses on solving for forces and torques that result from, and/or required to drive a kinematic system in such a way as to provide the desired acceleration.

Learning Objectives

After reading this unit you should be able to:

1. Identify static and dynamic forces in linkages
2. Compute static and dynamic reactions on joints
3. Determine the force or torque required to drive a linkage
4. Computes forces and moments which cause vibration in machine supports or frames.

Unit content

Session 4-1: Force Analysis Methods

4-1.1 Principle of Superposition
4-1.2 Free-Body Diagram
4-1.3 Equilibrium Conditions
4-1.3 Newton’s Second Law of Motion and D'Alembert’s Principle

SESSION 4-1: Force Analysis Methods

There are several methods used for analyses of forces and torques in mechanisms. For a mechanism, a method that may be employed depends on the degree of complexity, which is determined by the input information, assumptions made, and output desired, as listed in Table 1. These methods are either based on the Newton's laws of motion or the energy method. Some of the methods include principle of superposition, method of virtual work and Newtonian solution methods.

Of all the methods, drawing of free-body diagrams and application of Newton's second law of motion is the most elaborative and give most information about the forces internal to a mechanism. As such, in this text, drawing of free-body diagrams and application of Newton's second law of motion using linear algebra and matrix are explored to find forces/torques in linkages.

4-1.1 Principle of Superposition

It states that, for linear system, the net effect of two or more loads on a system is equal to vector sum of the effects of the individual loads considered separately. This tool is for analysing systems with two or more loads. Each load is applied to the linkage separately and its effect on each joint and/ or member are determined. Then, for each joint/member, the net effect of the loads is the sum of the individual separate effects.

4-1.2 Free-Body Diagrams

Free-body diagrams are extremely important and useful tool for force analysis. It is a sketch of part or group of parts, isolated from its supports and other bodies. Free-body diagrams of some support and contacts as shown in Figure 23. Steps for drawing of free-body diagrams are as follows:

Table 1: Methods of Analysis of Forces and Torques in Mechanisms

Input information, Assumptions & Masses	Zero	Specified	Specified
Loading	Specified or given parametric form, as in input/output ratio	Specified or can be determined at each position	Specified in terms of position, velocity, and/or time
Motion	Positions specified	Position, velocity and acceleration specified	Unknown
Output Information (sought)	Required input force to balance the load, mechanical advantage at each position, and/or pin reactions	Required input force/torque to sustain assumed/desired motion, pin reactions	Position, velocity and acceleration of each member as a function of time: i.e. the actual motion
Required analytical tools	Statics, linear algebra	D'Alembert's principle, statics, linear algebra	Writing differential equations of motion, solution by numerical solution using programming

1. Decide which body (or group of bodies) is under consideration, imagine it to be isolated from all other bodies, and sketch the outlined shape of the body (or group of bodies).
2. Identify all the forces acting on the body or group of bodies including known and unknown forces, weight, tension in ropes and contact reactions.
3. Using arrows, indicate at points of application all external forces and moments acting on the body. They should include (a) weight of the body, if required, (b) all externally applied loads on the body (c) reactions at supports and other contacts with other bodies.
4. Indicate directions of known external forces and assumed direction for all unknown external forces.
5. Indicate on the diagram dimensions and angles necessary for calculating and resolving forces and moments.
6. The weight of a body represents the gravitational attraction on the earth, and always acts vertically downward through the *centre of gravity* of the body. Unless otherwise stated

weights of links, cables, ropes, pulleys, springs and supports should be neglected. Support reactions represent the constraining forces that are exerted on a body by the supports and connected bodies.

7. Newton's third law of motion must be observed carefully when showing direction of a reaction.

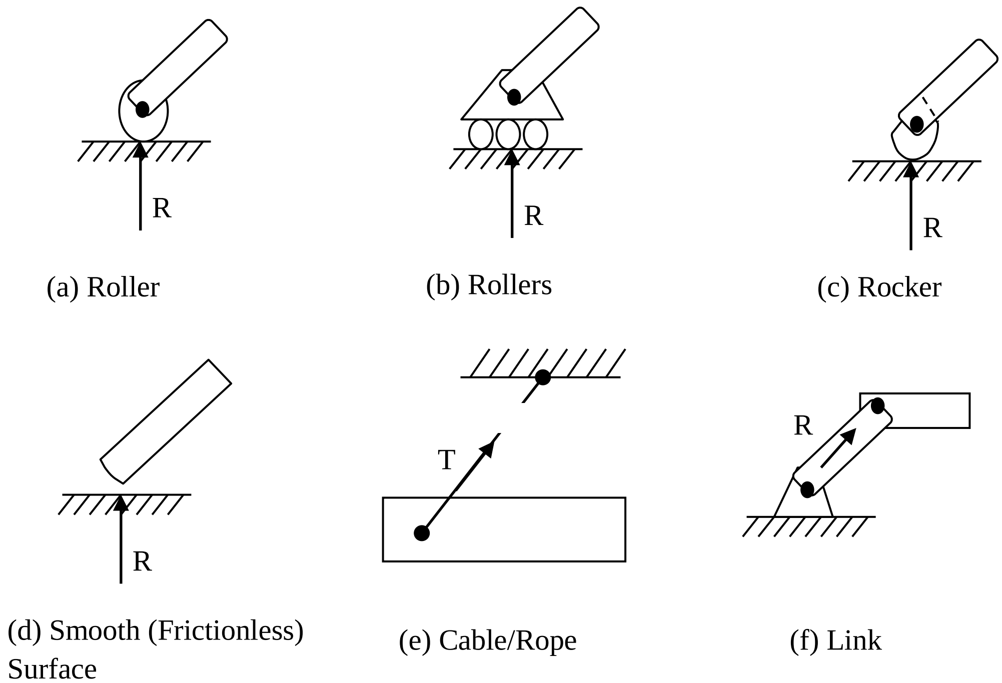


Figure 23: Free-Body Diagrams of 2D Supports

4-1.3 Equilibrium Conditions

For a body or a group of bodies to be in a static equilibrium the vector sum of all forces acting on it must be equal to zero and the vector sum of moments about any arbitrary point must be equal to zero. In mathematical form, the equilibrium conditions are

$$\sum F = 0 \qquad \sum M = 0 \qquad \textbf{Equation 83}$$

The above equation may be reduced to their respective scalar form by summation along the x, y and z axes. For plane mechanism, the above equation reduce to

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0 \qquad \textbf{Equation 84}$$

4-1.4 Newton's Second Law of Motion and D'Alembert's Principle

4-1.4.1 Newton's Second Law of Motion

In mathematical form, it states that:

$$\sum F = ma \qquad \sum M = I_G \alpha \qquad \textbf{Equation 85}$$

The above equation may be reduced to their respective scalar by summation along the x, y and z axes. For plane mechanism, the above equation reduce to

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum M = I_G \alpha \qquad \textbf{Equation 86}$$

4-1.4.2 D'Alembert's Principle

Consider a rigid body of mass m with centre of mass at G and acted upon by external forces F_1, F_2 and F_3, as shown in Figure 24. From Newton's second law,

$$\sum F = ma_G \qquad \textbf{Equation 87}$$

$$\sum M = I_G \alpha \qquad \textbf{Equation 88}$$

Where a $\sum F$ and $\sum M$ are respectively the sum of the external forces and moments acting on the body. The terms a_G, α and I are acceleration of centre of mass, angular acceleration and mass moment of inertia about the centre of mass. In general, the line-of-action of this result will not pass through the centre of mass of the body, but be displaced by some distance, say h, as shown in the figure. To balanced the

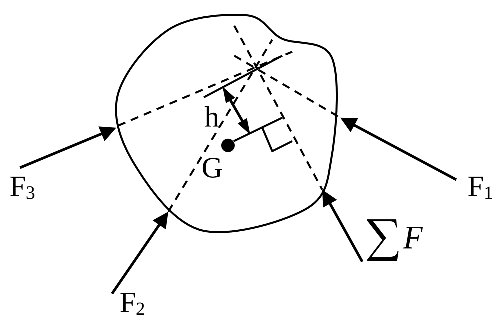


Figure 24: Forces Acting on a Moving Body

The Equations 87 and 88 may be written as

$$\sum F - ma_G = 0 \qquad \textbf{Equation 89}$$

$$\sum M - I_G \alpha = 0 \qquad \textbf{Equation 90}$$

The perpendicular distance of the line-of-action of the resultant force from the centre of mass is

$$h = \frac{|I_G \alpha|}{|ma_G|} = \frac{\left|\sum M\right|}{\left|\sum F\right|} \qquad \textbf{Equation 91}$$

Equation 89 simply states that the sum of all the external forces acting on a body plus fictitious force $-ma_G$ is zero. This fictitious force is called *inertia force* and its line-of-action is same as that of the acceleration a_G , but in opposite direction. Similarly, Equation 88 simply states that the sum of all the external moments acting on a body plus fictitious torque $-I\alpha$ is zero. This fictitious torque is called *inertia torque* and it acts in opposite direction to the angular acceleration α. Equations 89 and 90 are known as D'Alembert's principles. It states that the *reverse forces and torques and the external forces and torques on a body give statical equilibrium.* The D'Alembert's principles are used to convert a dynamic problem into static one.

SESSION 4-2: Review of Position Analysis of Two-Dimensional Linkages

Position analysis is required for velocity, acceleration and forces analysis. Usually, the lengths of members of a linkage and the position(s) of input(s) are given during position analysis. However, in machine design it is required to find lengths of members of a linkage that give a desired motion. Complete motion analysis include virtually moving the linkage through its intended path. Doing so requires examination of the linkage at various positions. If a mechanism is to examined for only one or two positions, a graphical methods are convenient. However, for multiple position analysis, analytical methods are more convenient that graphical methods.

4-2.1 Crank-Slider Linkages

Consider an offset crank-slider linkage, which has crank of length r and at angle θ to the positive x-axis, and connecting rod of length l and sliding along the horizontal direction as shown in Figure 25. The crank length r, angular position θ and connecting rod length l are normally the inputs for determine the position of the slider. The vertical position of point B is

$$r \sin\theta = e + l \sin\phi$$

which leads to

$$\phi = \sin^{-1}\left(\frac{r\sin\theta - e}{l}\right) \qquad \textbf{Equation 92}$$

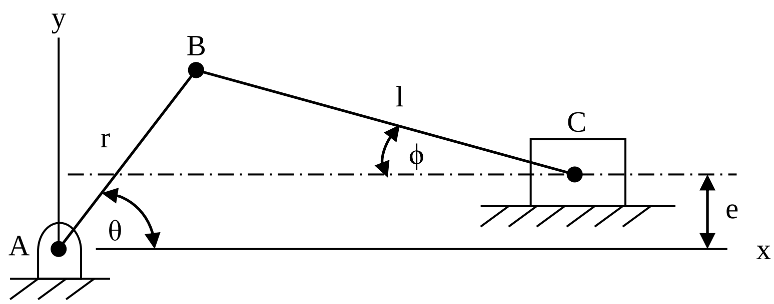


Figure 25: Positions of a Crank-Slider Linkage

Then, the position of the slider is given as

$$x = r\cos\theta + l\cos\phi \qquad \textbf{Equation 93}$$

Differentiating the above equation once and twice with respect to time respectively give velocity and acceleration of the slider.

Example 4- 1

Find the horizontal position x_0 of the slider C and the inclination of the connecting rod BC to the horizontal axis of the crank-slider mechanism shown in Figure E4-1, which has the following lengths:

Crank: $r_1 = 30$ mm

Connecting rod BC: $r_2 = 100$ mm

Vertical offset of slider from horizontal axis: $y_o = 10$ mm

Orientation of crank to horizontal axis: $\theta_1 = 45^o$

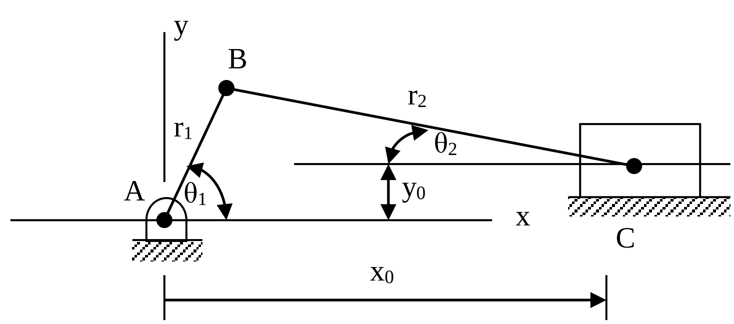


Figure E4-1

Solution

Using Equation 3-2,

$$\theta_2 = \sin^{-1}\left(\frac{r_1 \sin\theta_1 - y_o}{r_2}\right) = \sin^{-1}\left(\frac{30\sin 45 - 10}{100}\right) \qquad \theta_2 = 6.44^o$$

Using Equation 3-1,

$$x_0 = r_1\cos\theta_1 + r_2\cos\theta_2 = (30)\cos 45 + 100\cos(6.44) \qquad \underline{\underline{x_0 = 120.58\,\text{mm}}}$$

4-2.2 Four-Bar Linkages

Consider a four-bar linkage shown in Figure 26. The vector of diagonal BD is given by the vector loop equation

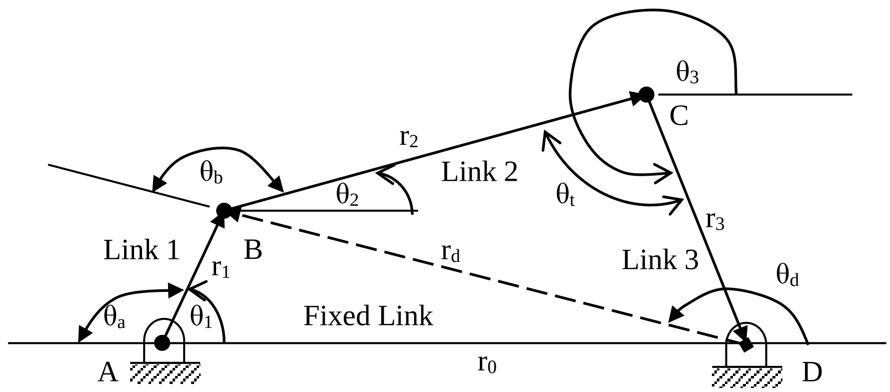


Figure 26: Positions of a Four-Bar Linkage

$$r_d = r_0 + r_1 \tag{1}$$

Taking the dot product of each side of the above equation, we have

$$r_d . r_d = (r_0 + r_1).(r_0 + r_1) = r_0^{\;2} + r_1^{\;2} + 2r_0 r_1 \cos\theta_a$$

From which we have

$$r_d^{\;2} = r_0^{\;2} + r_1^{\;2} + 2r_0 r_1 \cos(180 - \theta_1) \qquad \textbf{Equation 94}$$

or
$$r_d^{\;2} = r_0^{\;2} + r_1^{\;2} - 2r_0 r_1 \cos\theta_1$$

which is basically the cosine law. The direction of the diagonal is given in terms of the x and y components of r_0 and r_1 as

$$\sin\theta_d = \frac{r_{0y} + r_{1y}}{r_d} \qquad \textbf{Equation 95}$$

$$\cos\theta_d = \frac{r_{0x} + r_{1x}}{r_d}$$ **Equation 96**

where $r_{ix} = r_i \cos\theta_i$ and $r_{iy} = r_i \sin\theta_i$ and subscript i stands for link i. It can be shown that

$$\tan\left(\frac{\theta_d}{2}\right) = \frac{1 - \cos\theta_d}{\sin\theta_d}$$ **Equation 97**

Alternatively, the orientation of the diagonal vector may be determined from Table 2.

Considering the loop $B\vec{C} + C\vec{D} + D\vec{B} = 0$, then

$$-r_3 = r_d + r_2 \qquad (2)$$

Taking the dot product of each side of the above equation, we have

$$r_3.r_3 = (r_d + r_2).(r_d + r_2)$$

Table 2: Quadrant and Corresponding Value of θ_d

cos θ_d	*sin θ_d*	θ_d
positive	positive	$\theta_d = \tan^{-1}\left(\frac{\sin\theta_d}{\cos\theta_d}\right)$
negative	positive	$\theta_d = 180 - \tan^{-1}\left(\left\lvert\frac{\sin\theta_d}{\cos\theta_d}\right\rvert\right)$
negative	negative	$\theta_d = 180 + \tan^{-1}\left(\left\lvert\frac{\sin\theta_d}{\cos\theta_d}\right\rvert\right)$
positive	negative	$\theta_d = 360 - \tan^{-1}\left(\left\lvert\frac{\sin\theta_d}{\cos\theta_d}\right\rvert\right)$

From which we have

$$\cos\theta_b = \frac{r_3^{\ 2} - r_d^{\ 2} - r_2^{\ 2}}{2 r_d r_2}$$ **Equation 98**

Then,

$$\theta_2 = \theta_d \mp \theta_b$$ **Equation 99**

where θ_b is negative when the vector loop $r_2r_3r_d$ is clockwise, otherwise positive. The x and y components of r_3 are determined from the loop equation for the entire linkage. Substituting equation (1) into (2), we have

$$r_3 = -(r_0 + r_1 + r_2) \qquad \textbf{Equation 100}$$

The above equation may be written in terms of the vertical and horizontal components as

$$r_{3x} = -(r_{0x} + r_1\cos\theta_1 + r_2\cos\theta_2) \qquad \textbf{Equation 101}$$

$$r_{3y} = -(r_{0y} + r_1\sin\theta_1 + r_2\sin\theta_2) \qquad \textbf{Equation 102}$$

As a check, the calculated length of link 3, r_3, must be equal to the given length.

$$r_3 = \sqrt{(r_{3x})^2 + (r_{3y})^2}$$

If the fixed link aligned to the horizontal, then

$$r_{0y} = 0$$

$$r_{0x} = r_0$$

If the fixed link is inclined at angle θ_0 to the horizontal axis, then

$$r_{0y} = r_0\sin\theta_0$$

$$r_{0x} = r_0\cos\theta_0$$

The orientation of link 3 is determined from horizontal and vertical lengths of link 3 as

$$\sin\theta_3 = \frac{r_{3y}}{r_3} \qquad \textbf{Equation 103}$$

$$\cos\theta_3 = \frac{r_{3x}}{r_3} \qquad \textbf{Equation 104}$$

Then, θ_3 is determined from the equation

$$\tan\left(\frac{\theta_3}{2}\right) = \frac{1-\cos\theta_3}{\sin\theta_3} \qquad \textbf{Equation 105}$$

Alternatively, θ_3 may be determined using Table 2.

Example 4- 2

A four-bar linkage has the following lengths:

Fixed link:	r_0 = 45 mm
Crank:	r_1 = 10 mm
Coupler:	r_2 = 50 mm
Rocker:	r_3 = 20 mm

Find the positions of links 2 and 3 when the crank is at $\theta_1 = 45^o$ to the x-axis. Assume that the fixed link is aligned to the x-axis.

Solution

Using Equation 94,

$$r_d^{\;2} = r_0^{\;2} + r_1^{\;2} + 2r_0 r_1 \cos(180-\theta_1) = (45)^2 + (10)^2 + 2(45)(10)\cos(180-45) \qquad r_d = 38.58\,\text{mm}$$

Using Equations 95 and 96

$$\sin\theta_d = \frac{r_{0y} + r_{1y}}{r_d}$$

$$\cos\theta_d = \frac{r_{0x} + r_{1x}}{r_d}$$

The orientation of the diagonal vector is determined from

$$\sin\theta_d = \frac{r_{0y} + r_{1y}}{r_d} = \frac{0 + 10\sin 45}{38.58} \qquad \sin\theta_d = 0.1833$$

$$\cos\theta_d = \frac{r_{0x} + r_{1x}}{r_d} = \frac{-45 + 10\cos 45}{38.58} \qquad \cos\theta_d = -0.9831$$

Using Equation 97

$$\tan\left(\frac{\theta_d}{2}\right) = \frac{1-(-0.9831)}{(0.1833)} \qquad \theta_d = 169.4^o$$

Alternatively, since sin θ_d is positive and cos θ_d is negative, the diagonal vector lies in second quadrant. Hence, using Table 2, θ_d is given by

$$\theta_d = 180 - \tan^{-1}\left(\left|\frac{0.1833}{-0.9831}\right|\right) \qquad \theta_d = 169.4^o$$

Using Equation 98,

$$\cos\theta_b = \frac{r_3^{\;2} - r_d^{\;2} - r_2^{\;2}}{2 r_d r_2} = \frac{(20)^2 - (38.58)^2 - (50)^2}{2(38.58)(50)} \qquad \theta_b = 158.45^o$$

For assembly mode 1 and using Equation 99,

$$\theta_2 = \theta_d - \theta_b = 169.4 - 158.45 \qquad \underline{\underline{\theta_2 = 10.95^o}}$$

For assembly mode 2,

$$\theta_2 = \theta_d - \theta_b = 169.4 + 158.45 \qquad \underline{\underline{\theta_2 = 327.85^o}}$$

The horizontal and vertical components of r_3 are

$$r_{3x} = -(r_{0x} + r_1\cos\theta_1 + r_2\cos\theta_2) = -(-45 + 10\cos 45 + 50\cos 10.95) \qquad r_{3x} = -11.16\,\text{mm}$$

$$r_{3y} = -(r_{0y} + r_1\sin\theta_1 + r_2\sin\theta_2) = -(0 + 10\sin 45 + 50\sin 10.95) \qquad r_{3y} = -16.57\,\text{mm}$$

Then

$$r_3 = \sqrt{r_{3x}^2 + r_{3y}^2} = \sqrt{(-11.16)^2 + (-16.57)^2} \qquad r_3 = 19.978\,\text{mm}$$

which is approximately equal to r_3

$$\sin\theta_3 = \frac{r_{3y}}{r_3} = \frac{-16.57}{20} \qquad \sin\theta_3 = -0.8285$$

$$\cos\theta_3 = \frac{r_{3x}}{r_3} = \frac{-11.16}{20} \qquad \cos\theta_3 = -0.558$$

$$\tan\left(\frac{\theta_3}{2}\right) = \frac{1-\cos\theta_3}{\sin\theta_3} = \frac{1-(-0.558)}{(-0.8285)} \qquad \underline{\underline{\theta_3 = -124^o(236^o)}}$$

Alternatively, both sine and cosine are negative, meaning we are in the third quadrant. Hence, θ_3 is given by

$$\theta_3 = 180 + \tan^{-1}\left(\left|\frac{-0.8285}{-0.558}\right|\right) \qquad \underline{\underline{\theta_3 = 236^o}}$$

The alternative mode, the horizontal and vertical components of r_3 are

$$r_{3x} = -(r_{0x} + r_1\cos\theta_1 + r_2\cos\theta_2) = -(-45 + 10\cos 45 + 50\cos 327.85) \qquad r_{3x} = -4.4\,\text{mm}$$

$$r_{3y} = -(r_{0y} + r_1\sin\theta_1 + r_2\sin\theta_2) = -(0 + 10\sin 45 + 50\sin 327.85) \qquad r_{3y} = 19.54\,\text{mm}$$

Check

$$r_3 = \sqrt{r_{3x}^2 + r_{3y}^2} = \sqrt{(-4.4)^2 + (19.54)^2} \qquad r_3 = 20.0\,\text{mm}$$

which is approximately equal to r_3. The orientation of link 3 is determined from

$$\sin\theta_3 = \frac{19.54}{20} \qquad \sin\theta_3 = 0.977$$

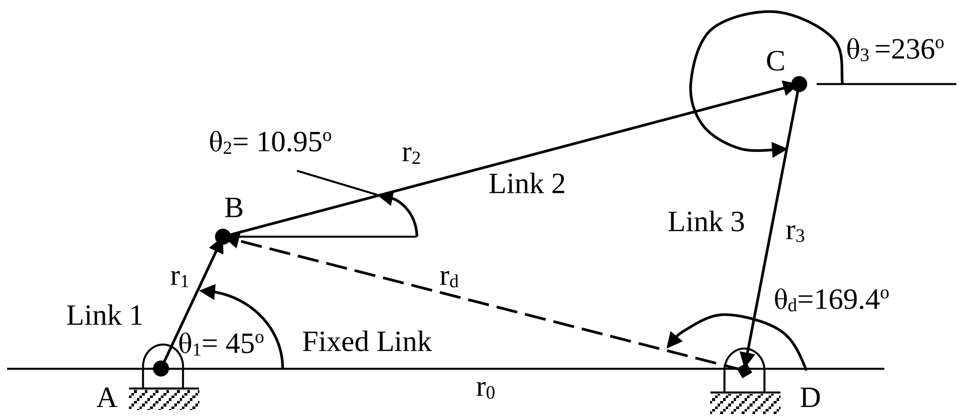
$$\cos\theta_3 = \frac{-4.4}{20} \qquad \cos\theta_3 = -0.22$$

Then

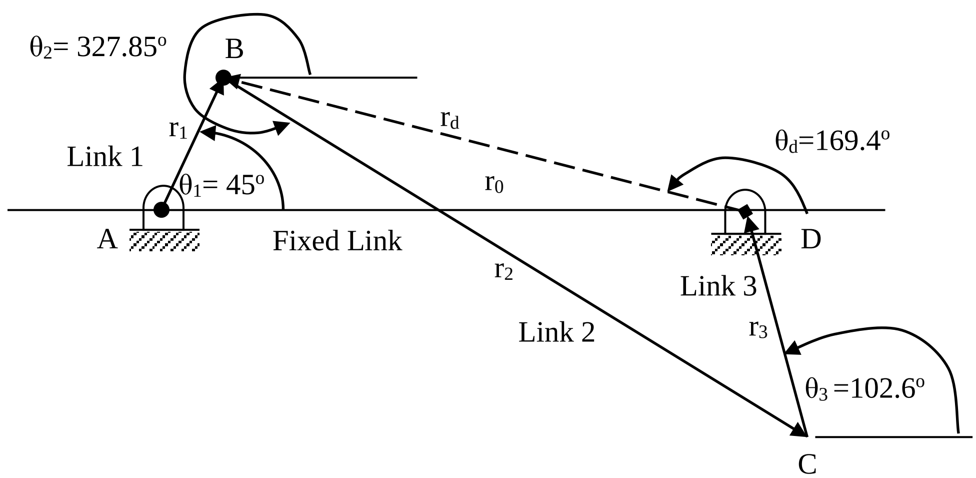
$$\tan\left(\frac{\theta_3}{2}\right) = \frac{1-(-0.22)}{(0.977)} \qquad \underline{\underline{\theta_3 = 102.6^o}}$$

Alternatively, using Table 2, θ_3 is given by

$$\theta_3 = 180 - \tan^{-1}\left(\left|\frac{0.977}{-0.22}\right|\right) \qquad \underline{\underline{\theta_3 = 102.6^o}}$$



(a) Assembly Mode 1



(b) Assembly Mode 2

Figure E4-2: Modes of Assembly of Example 4-2

SESSION 4-3: Static Forces in Mechanisms

4-3.1 Static Force Analysis of Crank-Slider and Inverted Crank-Slider Linkages

Consider the slider-crank linkage shown in Figure 27. This linkage may represent a machine such as reciprocating compressor, internal combustion engine, press, etc. Suppose it is desired to find the torque T required to overcome load P on the slider at C. First, the free-body diagrams of the components of the mechanism are shown in Figure 28. On the diagram, gravitational forces are assumed to be negligible.

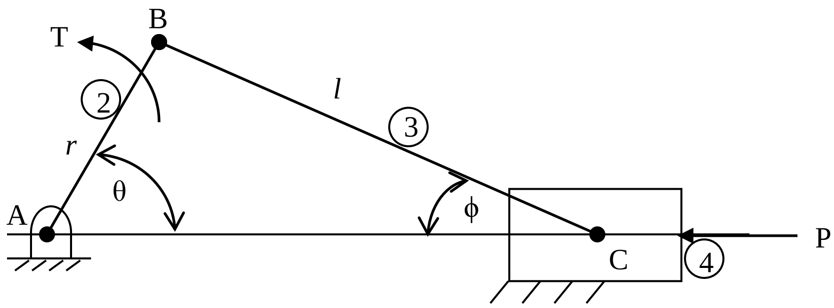


Figure 27: Drive Torque and External Force on Crank-Slider Linkage

Slider C

The slider is a three-force member. Applying equilibrium equation along the horizontal axis, we have

$$\sum F_x = F_C \cos\phi - P = 0 \qquad F_C = \frac{P}{\cos\phi} \tag{1}$$

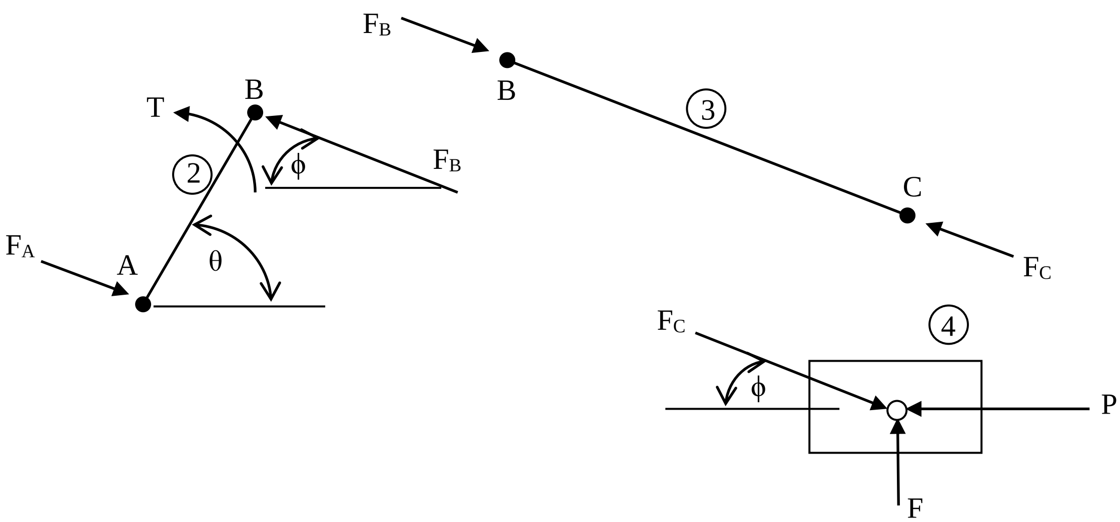


Figure 28: Free-Body Diagrams of Components of Crank-Slider Linkage

Connecting Rod BC

The connecting rod is a two-force member, which means that the forces are collinear. Thus,

$$F_B = F_C \qquad\qquad F_B = \frac{P}{\cos\phi} \tag{2}$$

Crank AB

Since the pins at A and B are assumed to be frictionless, forces F_A and F_B have the same magnitude, but opposite directions. Summing moments about point A, we have

$$\sum M_A = (F_B \cos\phi)(r\sin\theta) + (F_B \sin\phi)(r\cos\theta) + T = 0 \qquad T = -F_B r[\sin\theta\cos\phi + \sin\phi\cos\theta]$$

Substituting for F_B from equation (2) into the above equation, the torque require to overcome load P is

$$T = -\frac{P}{\cos\phi} r[\sin\theta\cos\phi + \sin\phi\cos\theta] \tag{3}$$

Using Equation 92 with e = 0, we have

$$\sin\phi = \frac{r\sin\theta}{l}$$

Substituting the above equation into Equation (3) and simplifying, we have

$$T = -\Pr\sin\theta\left[1 + \frac{r\cos\theta}{\sqrt{l^2 - (r\sin\theta)^2}}\right] \qquad \textbf{Equation 106}$$

4-3.2 Static Force Analysis of Four-Bar Linkages

Consider the slider-crank linkage shown in Figure 29. This linkage may represent any part of a machine such as part of steering rag, four-bar vehicle suspension without the springs and dumpers (also called shock absorber), pick-and-place mechanism. The external loads are denoted F_i and T_i, where F is the applied force and T is the applied torque, and the subscript i stand for the link on which the loads are applied. Thus, loads F_1, F_2 and F_3 and torques T_1, T_2, and T_3 respectively are external loads on links AB, BC and CD. External loads may include weight of the link or reaction due to attachment to other linkages. Suppose it is desired to determine the torque T required to overcome loads and couples on all the links. The free-body diagram of each link is shown in Figure 30. On the figure, frictional forces are neglected, and R_{ji} denotes the reaction force from link j on link i. From the Newton's law of motion,

$$F_{ij} = -F_{ji} \qquad \textbf{Equation 107}$$

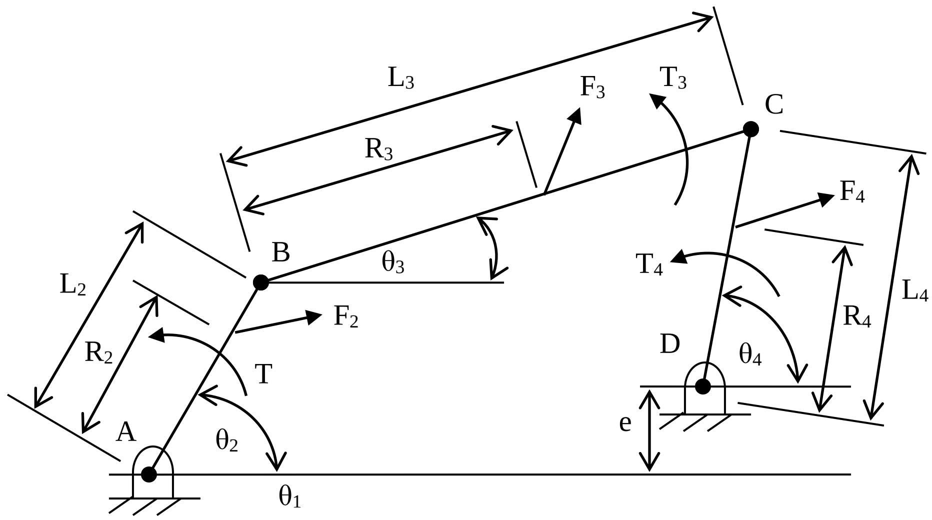


Figure 29: A Schematic of Four-Bar Linkage

In addition, each applied force F_i and reaction F_{ji} at the joints are resolved into x and y components. A more advanced four-bar analysis may include frictional forces and couples at the joints A, B, C and C. Such analyses are only useful if the coefficients of friction are known and reliable.

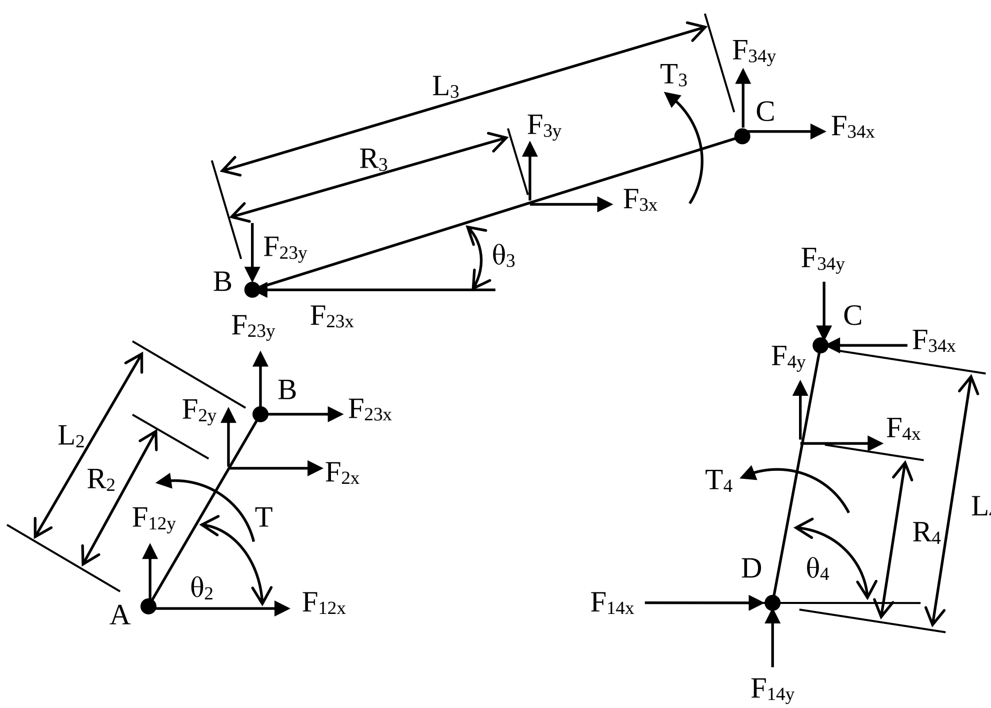


Figure 30: Static Free-Body Diagram of Four-Bar Linkage

Link CD

Consider the free body diagram of CD. Summing up forces in the x and y axes, and moments about point D, we have

$$F_{41x} - F_{34x} + F_{4x} = 0 \tag{1}$$

$$F_{41y} - F_{34y} + F_{4y} = 0 \tag{2}$$

$$-\left(F_{34y}L_{4x} - F_{34x}L_{4y}\right) + \left(F_{4y}R_{4x} - F_{4x}R_{4y}\right) + T_4 = 0 \tag{3}$$

Where

$$L_{ix} = L_i \cos\theta_i \quad , \quad L_{iy} = L_i \sin\theta_i \qquad \textbf{Equation 108}$$

$$R_{ix} = R_i \cos\theta_i \quad , \quad R_{iy} = R_i \sin\theta_i \qquad \textbf{Equation 109}$$

Equations (1) to (3) have four unknowns, meaning that they cannot be solved completely.

Link BC

Summing up forces in the x and y axes, and moments about point B using the free body diagram of link CD, we have

$$-F_{23x} + F_{3x} + F_{34x} = 0 \tag{4}$$

$$-F_{23y} + F_{3y} + F_{34y} = 0 \tag{5}$$

$$\left(F_{34y}L_{3x} - F_{34x}L_{3y}\right) + \left(F_{3y}R_{3x} - F_{3x}R_{3y}\right) + T_3 = 0 \tag{6}$$

Equations (1) and (6) have six unknowns meaning that they can be solved completely. Note that equations (3) and (6) have only two unknowns: F_{34x} and F_{34y}. Rearranging equations (3) and (6) yields

$$a_{11}F_{34x} + a_{12}F_{34y} = b_1 \tag{3a}$$

$$a_{21}F_{34x} + a_{22}F_{34y} = b_2 \tag{6a}$$

The above equations may be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} F_{34x} \\ F_{34y} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{7}$$

where

$$a_{11} = L_{4y} = L_4 \sin\theta_4, \qquad a_{12} = -L_{4x} = -L_4 \cos\theta_4,$$

$$a_{21} = -L_{3y} = -L_3 \sin\theta_3, \qquad a_{22} = L_{3x} = L_3 \cos\theta_3,$$

$$b_1 = -\left(F_{4y}R_{4x} - F_{4x}R_{4y}\right) - T_4 \qquad b_2 = -\left(F_{3y}R_{3x} - F_{3x}R_{3y}\right) - T_3$$

Solving equation (7) using Cramer's rule, we have

$$F_{34x} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad F_{34x} = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}}$$ **Equation 110**

$$F_{34y} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad F_{34y} = \frac{a_{11}b_2 - a_{21}b_1}{a_{22}a_{11} - a_{12}a_{21}}$$ **Equation 111**

From equations (4) and (5), we have

$$F_{23x} = F_{3x} + F_{34x}$$ **Equation 112**

$$F_{23y} = F_{2y} + F_{34y}$$ **Equation 113**

Similarly, from equations (1) and (2), we have

$$F_{41x} = F_{34x} - F_{4x}$$ **Equation 114**

$$F_{41y} = F_{34y} - F_{4y}$$ **Equation 115**

Link AB

Consider the free body diagram of AB. Summing up forces in the x and y axes, and moments about point A, we have

$$F_{12x} = -F_{2x} - F_{23x}$$ **Equation 116**

$$F_{12y} = -F_{2y} - F_{23y}$$ **Equation 117**

$$T_2 = -\left(F_{23y}L_{2x} - F_{23x}L_{2y}\right) - \left(F_{2y}R_{2x} - F_{2x}R_{2y}\right) \qquad \textbf{Equation 118}$$

Equations (1) to (9) may written in matrix form as

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_{2y} & L_{2x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L_{3y} & L_{3x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -L_{4y} & L_{4x} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{23x} \\ F_{23y} \\ F_{34x} \\ F_{34y} \\ F_{41x} \\ F_{41y} \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -F_{2x} \\ -F_{2y} \\ -\left(F_{2y}R_{2x} - F_{2x}R_{2y}\right) \\ -F_{3x} \\ -F_{3y} \\ -\left(F_{3y}R_{3x} - F_{3x}R_{3y}\right) - T_3 \\ -F_{4x} \\ -F_{4y} \\ -\left(F_{4y}R_{4x} - F_{4x}R_{4y}\right) - T_4 \end{Bmatrix} \qquad \textbf{Equation 119}$$

Let

$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_{2y} & L_{2x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L_{3y} & L_{3x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & L_{4y} & -L_{4x} & 0 & 0 & 0 \end{bmatrix}, \{X\} = \begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{23x} \\ F_{23y} \\ F_{34x} \\ F_{34y} \\ F_{41x} \\ F_{41y} \\ T_2 \end{Bmatrix} \text{ and}$$

$$\{B\} = \begin{Bmatrix} -F_{2x} \\ -F_{2y} \\ -\left(F_{2y}R_{2x} - F_{2x}R_{2y}\right) \\ -F_{3x} \\ -F_{3y} \\ -\left(F_{3y}R_{3x} - F_{3x}R_{3y}\right) - T_3 \\ -F_{4x} \\ -F_{4y} \\ -\left(F_{4y}R_{4x} - F_{4x}R_{4y}\right) - T_4 \end{Bmatrix}$$

Then, the solution column matrix becomes

$$\{X\} = [A]^{-1}\{B\} \qquad \textbf{Equation 120}$$

The above equation may be solved using any commercial software such as Matlab and MathCAD or calculator with matrix algebra capabilities.

Example 4- 3

The four-bar linkage shown in Figure E4-3 is subject loads and torques indicated in on the links. In the position shown, determine the shaft torque T on input link AB and bearing reactions on all the joints for static equilibrium.

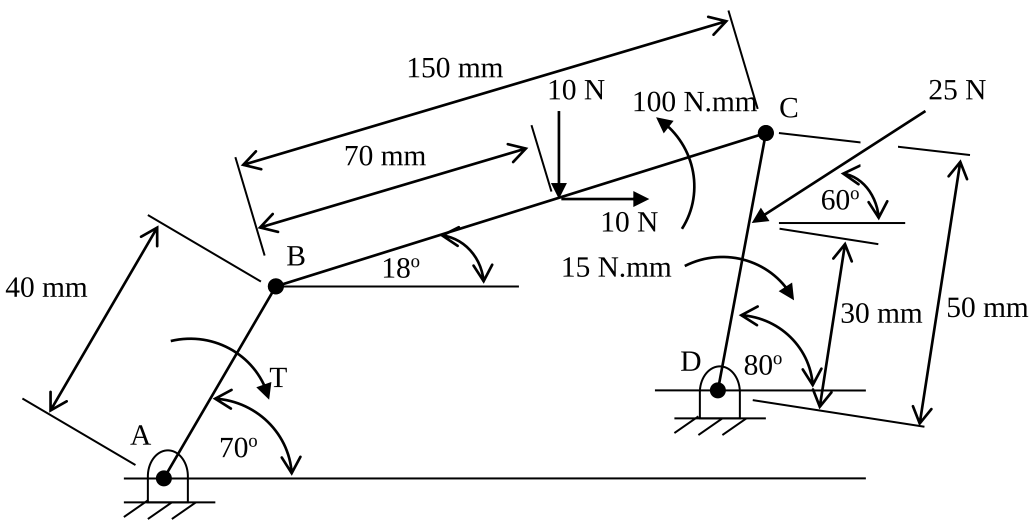


Figure E4-3

Solution

Comparing the diagrams in Figure E4-3 to that of Figure 30,

$\theta_2 = 70^o$, $\theta_3 = 18^o$, $\theta_4 = 80^o$

$F_{2x} = F_{2y} = 0$ $\quad F_{3x} = 10\text{ N}$, $\quad F_{3y} = -10\text{ N}$

$F_{4x} = -25\cos 60^o = -12.5\text{ N}$ $\quad F_{4y} = -25\sin 60^o = -21.65\text{ N}$

T_3 =30 N.mm, T_4 = -15 N.mm

The parameters of Equations 110 and 111 are calculated as follows:

$$a_{11} = L_{4y} = L_4 \sin\theta_4 = 50\sin 80^o, \qquad a_{11} = L_{4y} = 49.24\,\text{mm}$$

$$a_{12} = L_{4x} = -L_4 \cos\theta_4 = 50\cos 80^o, \qquad a_{12} = L_{4x} = -8.68\,\text{mm}$$

$$a_{21} = -L_3 \sin\theta_3 = -150\sin 18^o, \qquad a_{21} = L_{3y} = -46.35\,\text{mm}$$

$$a_{22} = L_{3x} = L_3 \cos\theta_3 = 150\cos 18^o, \qquad a_{22} = L_{3x} = 142.66\,\text{mm}$$

$$R_{4y} = R_4 \sin\theta_4 = 30\sin 80^o, \qquad R_{4y} = 29.54\,\text{mm}$$

$$R_{4x} = R_4 \cos\theta_4 = 30\cos 80^o, \qquad R_{4x} = 5.21\,\text{mm}$$

$$R_{3y} = R_3 \sin\theta_3 = 70\sin 18^o, \qquad R_{3y} = 21.63\,\text{mm}$$

$$R_{3x} = R_3 \cos\theta_3 = 70\cos 18^o, \qquad R_{3x} = 66.57\,\text{mm}$$

$$b_1 = -\left(F_{4y}R_{4x} - F_{4x}R_{4y}\right) - T_4 = -\left[(-21.65)(5.21) - (-12.5)(29.54)\right] - (-15)$$

$$b_1 = -241.515\,\text{N-mm}$$

$$b_2 = -\left(F_{3y}R_{3x} - F_{3x}R_{3y}\right) - T_3 = -\left[(-10)(66.57) - (10)(21.63)\right] - 30$$

$$b_2 = 852\,\text{N-mm}$$

Using Equations 110 and 111, we have

$$F_{34x} = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}} = \frac{(142.66)(-241.515) - (-8.68)(852)}{(142.66)(49.24) - (-8.68)(-46.35)} \qquad \underline{\underline{F_{34x} = -4.086\,\text{N}}}$$

$$F_{34y} = \frac{a_{11}b_2 - a_{21}b_1}{a_{22}a_{11} - a_{12}a_{21}} = \frac{(49.24)(852) - (-46.35)(-241.515)}{(142.66)(49.24) - (-8.68)(-46.35)} \qquad \underline{\underline{F_{34y} = 4.645\,\text{N}}}$$

From Equations 112 and 113

$$F_{23x} = F_{3x} + F_{34x} = (10) + (-4.086) \qquad \underline{\underline{F_{23x} = 5.914\,\text{N}}}$$

$$F_{23y} = F_{3y} + F_{34y} = (-10) + (4.645) \qquad \underline{\underline{F_{23y} = -5.355\,\text{N}}}$$

Using Equations 114 and 115

$$F_{41x} = F_{34x} - F_{4x} = (-4.086) - (-12.5) \qquad \underline{\underline{F_{41x} = 8.414\ \text{N}}}$$

$$F_{41y} = F_{34y} - F_{4y} = (4.645) - (-21.65) \qquad \underline{\underline{F_{41y} = 26.296\ \text{N}}}$$

Using Equations 116 to 118, we have

$$F_{12x} = -F_{2x} - F_{23x} = -(0) - (5.914) \qquad \underline{\underline{F_{12x} = -5.914\ \text{N}}}$$

$$F_{12y} = -F_{2y} - F_{23y} = (0) - (-5.355) \qquad \underline{\underline{F_{12y} = 5.355\ \text{N}}}$$

$$L_{2y} = L_2 \sin\theta_1 = 40 \sin 70^o, \qquad L_{2y} = 37.59\ \text{mm}$$

$$L_{2x} = L_2 \cos\theta_2 = 40 \cos 70^o, \qquad L_{2x} = 13.68\ \text{mm}$$

$$T_2 = -(F_{23y} L_{2x} - F_{23x} L_{2y}) - (F_{2y} R_{2x} - F_{2x} R_{2y})$$

$$T_2 = -[(-5.355)(13.68) - (5.914)(37.59)] - ((0)R_{2x} - (0)R_{2y})$$

$$\underline{\underline{T_2 = 295.56\ \text{N-mm}}}$$

Alternatively, we may use the matrix form. Using MathCAD, the matrix A and vector B are

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -37.59 & 13.68 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -46.35 & 142.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 49.24 & -8.68 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ 10 \\ 852.05 \\ -10 \\ 10 \\ -241.52 \end{bmatrix}$$

The inverse of matrix A is

$$
\text{A\_inverse} = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -1.311\times10^{-3} & 0 & 0 & -0.022 \\
0 & 1 & 0 & 0 & 1 & -7.436\times10^{-3} & 0 & 0 & -7\times10^{-3} \\
0 & 0 & 0 & -1 & 0 & 1.311\times10^{-3} & 0 & 0 & 0.022 \\
0 & 0 & 0 & 0 & -1 & 7.436\times10^{-3} & 0 & 0 & 7\times10^{-3} \\
0 & 0 & 0 & 0 & 0 & 1.311\times10^{-3} & 0 & 0 & 0.022 \\
0 & 0 & 0 & 0 & 0 & 7.436\times10^{-3} & 0 & 0 & 7\times10^{-3} \\
0 & 0 & 0 & 0 & 0 & 1.311\times10^{-3} & 1 & 0 & 0.022 \\
0 & 0 & 0 & 0 & 0 & 7.436\times10^{-3} & 0 & 1 & 7\times10^{-3} \\
0 & 0 & 1 & -37.588 & 13.681 & -0.052 & 0 & 0 & 0.714
\end{pmatrix}
$$

Then, the solution column vector is given by

$$\{X\} = [A]^{-1}\{B\} \qquad X := \text{A\_inverse}\cdot\text{B}$$

$$X = \begin{bmatrix} F_{12x} & F_{12y} & F_{23x} & F_{23y} & F_{34x} & F_{34y} & F_{41x} & F_{41y} & T_2 \end{bmatrix}^T$$

$$X = \begin{bmatrix} -5.914 & 5.355 & 5.914 & -5.355 & -4.086 & 4.645 & -14.086 & 14.645 & 295.562 \end{bmatrix}^T$$

Example 4- 4

Determine the input torque T_2 required for static equilibrium of the mechanism shown in Figure E4-4. The force F on the slider has a magnitude of 2.5 kN.

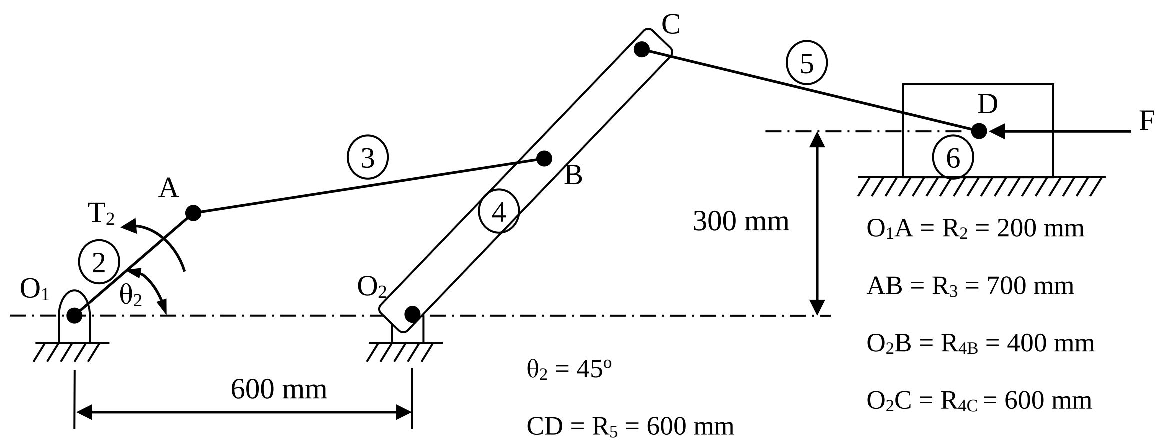


Figure E4-4

Solution

Position Analysis

$$|AO_2|^2 = r_d{}^2 = r_1{}^2 + r_2{}^2 - 2r_2r_1\cos\theta_2 = (0.6)^2 + (0.2)^2 - 2(0.6)(0.2)\cos 45 \qquad r_d = 0.4799\,\text{m}$$

Using Equations 95 and 96

$$\sin\theta_d = \frac{r_{1y} + r_{2y}}{r_d} \quad \text{and} \quad \cos\theta_d = \frac{r_{1x} + r_{2x}}{r_d}$$

The orientation of the diagonal vector is determined from

$$\sin\theta_d = \frac{r_{1y} + r_{2y}}{r_d} = \frac{0 + 0.2\sin 45}{0.4799} \qquad \sin\theta_d = 0.2947$$

$$\cos\theta_d = \frac{r_{1x} + r_{2x}}{r_d} = \frac{-0.6 + 0.2\cos 45}{0.4799} \qquad \cos\theta_d = -0.9556$$

Using Equation 97

$$\tan\left(\frac{\theta_d}{2}\right) = \frac{1 - \cos\theta_d}{\sin\theta_d} = \frac{1 - (-0.9556)}{(0.2947)} \qquad \theta_d = 162.9^o$$

Alternatively, since sin θ_d is positive and cos θ_d is negative, the diagonal vector lies in second quadrant. Hence, using Table 2, θ_d is given by

$$\theta_d = 180 - \tan^{-1}\left(\left|\frac{0.2947}{-0.9556}\right|\right) \qquad \theta_d = 162.9^o$$

Using Equation 98, we have

$$\cos\theta_b = \frac{r_4{}^2 - r_d{}^2 - r_3{}^2}{2r_dr_3} = \frac{|O_2B|^2 - r_d{}^2 - |AB|^2}{2r_d|AB|} = \frac{(0.4)^2 - (0.4799)^2 - (0.7)^2}{2(0.4799)(0.7)}$$

$$\theta_b = 146.5^o$$

Figure E4-4 shows that the linkage is mode 1 configuration. Hence,

$$\theta_3 = \theta_d - \theta_b = 162.9 - 146.5 \qquad \underline{\underline{\theta_3 = 16.4^o}}$$

The horizontal and vertical components of r_4 are

$$r_{4x} = -(r_1\cos\theta_1 + r_2\cos\theta_2 + r_3\cos\theta_3) = -(-0.6\cos 0^o + 0.2\cos 45^o + 0.7\cos 16.4^o)$$

$$r_{4x} = -0.2129\ \text{m}$$

$$r_{4y} = -(r_1\sin\theta_1 + r_2\sin\theta_2 + r_3\sin\theta_3) = -(-0.6\sin 0^o + 0.2\sin 45^o + 0.7\sin 16.4^o)$$

$$r_{4y} = -0.3391\ \text{m}$$

Then

$$r_4 = \sqrt{r_{4x}^2 + r_{4y}^2} = \sqrt{(-0.2129)^2 + (-0.3391)^2} \qquad r_4 = 0.40\ \text{m}$$

which is approximately equal to r_4

$$\sin\theta_4 = \frac{r_{4y}}{r_4} = \frac{-0.2129}{0.4} \qquad \sin\theta_4 = -0.53225$$

$$\cos\theta_4 = \frac{r_{4x}}{r_4} = \frac{-0.3391}{0.40} \qquad \cos\theta_4 = -0.84775$$

$$\tan\left(\frac{\theta_4}{2}\right) = \frac{1-\cos\theta_4}{\sin\theta_4} = \frac{1-(-0.53225)}{(-0.84775)} \qquad \underline{\underline{\theta_4 = 212.1^o}}$$

The calculated angles are indicated on the linkage diagram shown in Figure S4-4(a).

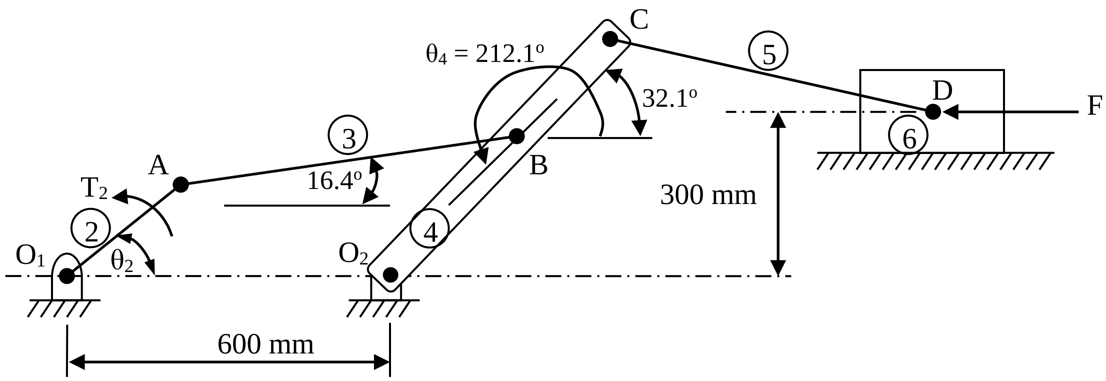


Figure S4-4(a)

From the above figure, the vertical displacement of point C from O_2 is given by, $r_{4y} = |O_2C|\sin 32.1^o = 0.3 + |CD|\sin\theta_5$

$$r_{4y} = |0.6|\sin 32.1^o = 0.3 + |0.6|\sin\theta_5 \qquad \sin\theta_5 = 1.8^o$$

Static Force Analysis

The free body diagrams of elements of the linkage are shown in Figure S4-4(b).

Slider

$$\sum F_x = 0 \qquad F_D \cos\theta_5 = F \qquad F_D = \frac{F}{\cos\theta_5} = \frac{1000}{\cos 1.8} \qquad F_D = 1000.5\ \text{N}$$

Link 5

$$\sum F_{along\ \text{axis of CD}} = 0 \qquad F_C = F_D = 1000.5\ \text{N}$$

(2)

Link 4

$$\sum M_{O_2} = 0$$

$$F_B \cos\theta_3 . R_{4B} \sin\theta_4 - F_B \sin\theta_3 . R_{4B} \cos\theta_4 - F_C \cos\theta_5 . R_{4C} \sin\theta_4 + F_C \sin\theta_5 . R_{4C} \cos\theta_4 = 0 \quad (3)$$

$$F_B = F_C \frac{R_{4C}}{R_{4B}} \left[\frac{\cos\theta_5 \sin\theta_4 - \sin\theta_5 . \cos\theta_4}{\cos\theta_3 \sin\theta_4 - \sin\theta_3 . \cos\theta_4} \right] = (1000.5)\left(\frac{0.6}{0.4}\right)\left[\frac{\cos 1.8 \sin 32.1 - \sin 1.8 . \cos 32.1}{\cos 16.4 \sin 32.1 - \sin 16.4 . \cos 32.1}\right]$$

$$F_B = 2798.1\ \text{N}$$

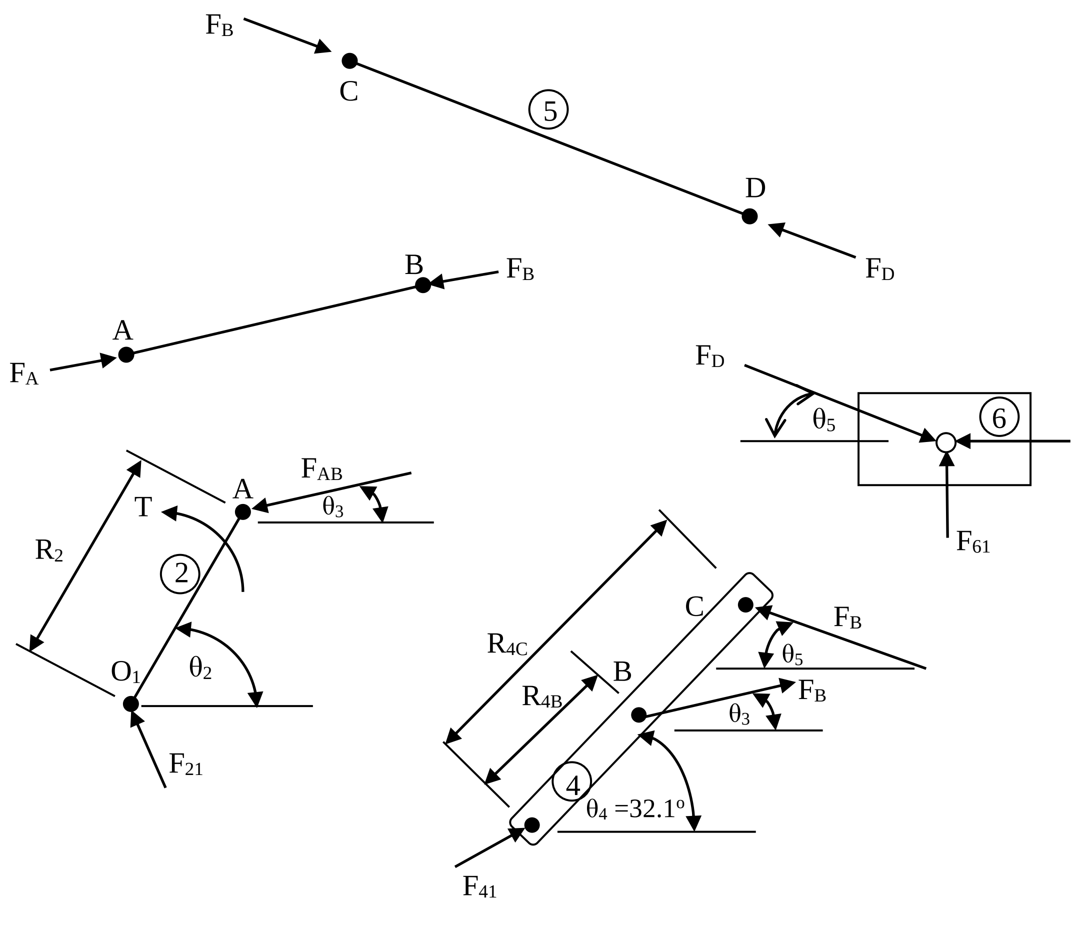


Figure S4-4(a)

Link 3

$\sum F_{along \text{ axis of AB}} = 0$ $\quad F_A = F_B \quad F_A = 2798.1\,\text{N}$

Link 2

$\sum M_{O_1} = 0$

$\sum M_{O_2} = 0 \quad T_2 + F_A \cos\theta_3 . R_2 \sin\theta_2 - F_A \sin\theta_3 . R_2 \cos\theta_2 = 0$

$$T_2 = F_A R_2 (\sin\theta_3 \cos\theta_2 - \cos\theta_3 \sin\theta_2) = (2798.1)(0.2)(\sin 32.1 \cos 45 - \cos 32.1 \sin 45)$$

$$\underline{\underline{T_2 = -124.9\,\text{N-m}}}$$

SESSION 4-4: Inertia Forces and Torques in Mechanisms

4-4.1 Dynamic Force Analysis of a Single Link in Pure Rotational Motion

Consider a single link in a pure rotational motion as shown in Figure 29(a). The corresponding free-body diagram is shown in Figure 29(b). In the figure, CG, T and force F_p denote the centre of mass, external torque and force, respectively, acting on the link. The displacements of the applied force and the point of rotation from the CG are R_p and R_{12}.

Applying the Newton's second law, we have

$$\sum F = F_p + F_{12} = m_2 a_G \qquad (1)$$

$$\sum M_{CG} = T + (R_p \text{x} F_p) + (\text{R}_{12} \text{x} F_{12}) = I_G \alpha \qquad (2)$$

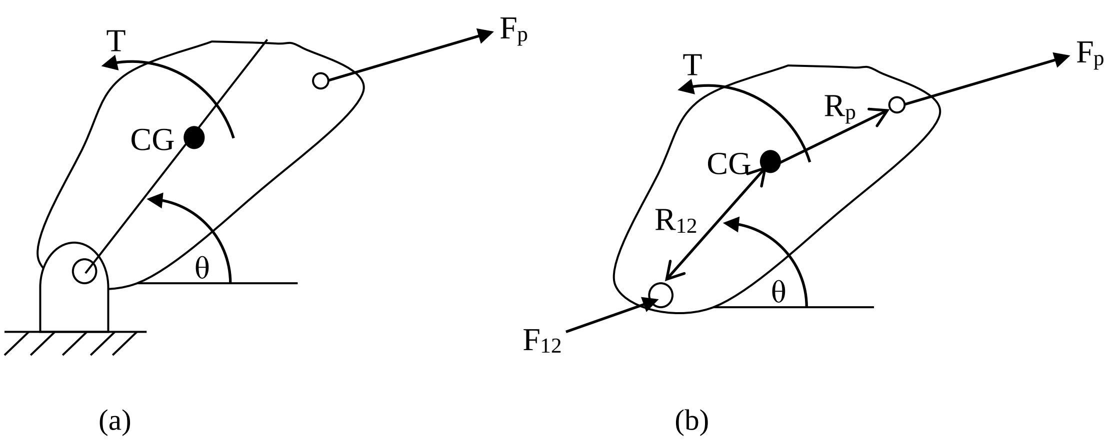


Figure 31: (a) Single Link in Pure Rotational Motion (b) Free-body Diagram of the link

Note that the moment is sum of moments about the centre of mass CG. Equation (1) can be split into x and y components and equation (2) expanded as

$$F_{px} + F_{12x} = m_2 a_{Gx} \tag{3}$$

$$F_{py} + F_{12y} = m_2 a_{Gy} \tag{4}$$

$$T + \left(R_{px} F_{py} - R_{py} F_{px}\right) + \left(R_{12x} F_{12y} - R_{12y} F_{12x}\right) = I_G \alpha \tag{5}$$

The unknowns F_{12x}, F_{12y} and T are determined by solving the equations (3) to (5) simultaneously. Alternatively, equations (3) to (5) may be put into matrix form as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12y} & R_{12x} & 1 \end{bmatrix} \begin{Bmatrix} F_{12x} \\ F_{12y} \\ T \end{Bmatrix} = \begin{Bmatrix} m_2 a_{Gx} - F_{px} \\ m_2 a_{Gy} - F_{py} \\ I_G \alpha - \left(R_{Px} F_{Py} - R_{Py} F_{Px}\right) \end{Bmatrix} \tag{6}$$

The above equation may be written as

$$[A]\{X\} = \{B\}$$

where $[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12y} & R_{12x} & 1 \end{bmatrix}$, $\{X\} = \begin{Bmatrix} F_{12x} \\ F_{12y} \\ T \end{Bmatrix}$ and $\{B\} = \begin{Bmatrix} m_2 a_{Gx} - F_{px} \\ m_2 a_{Gy} - F_{py} \\ I_G \alpha - \left(R_{Px} F_{Py} - R_{Py} F_{Px}\right) \end{Bmatrix}$

The matrix [A] contains all the geometric information and {B} contains all the dynamic information about the system. The solution column matrix then determined from

$$\{X\} = [A]^{-1}\{B\}$$ **Equation 121**

Example 4- 5

A link of length 250 mm has a mass of 2.5 kg and moment of inertia about its centre of mass of 0.007 kg-m^2. The centre of mass is located at 80 mm from the point of rotation. An external force of 200 N is applied horizontally at the other end of the link. Find the reaction force F_{12} at the pin-joint and the driving torque required to maintain the motion with the following instant data

$\theta = 30^o$, $\alpha = 20$ rad/s^2, $a_G = 25\angle 200^o$

Solution

The free-body diagram of the body showing the displacement vectors R_P and R_{12} is shown in Figure S4-5. From the figure, the displacement vectors are

$$R_{12} = 0.08\angle 210^o \qquad R_{12x} = 0.08\cos 210^o \qquad R_{12x} = -0.0693$$

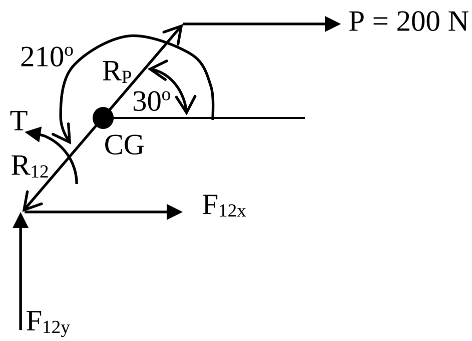


Figure S4-5

$$R_{12y} = 0.08\sin 210^o \qquad R_{12y} = -0.04$$

$$R_P = 0.170\angle 30^o \qquad R_{Px} = 0.170\cos 30^o \qquad R_{Px} = 0.14723$$

$$R_{Py} = 0.170\sin 30^o \qquad R_{Py} = 0.085$$

$$a_G = 25\angle 200^o \qquad a_{Gx} = 25\cos 200^o \qquad a_{Gx} = -23.492 \text{ m/s}^2$$

$$a_{Gy} = 25\sin 200^o \qquad a_{Gy} = -8.551 \text{ m/s}^2$$

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(0.04) & -0.0693 & 1 \end{bmatrix} \text{ and } \{B\} = \begin{Bmatrix} (2.5)(-23.492) - 200 \\ (2.5)(-8.551) - 0 \\ (0.007)(20) - (R_{Px}(0) - (0.085)(200)) \end{Bmatrix}$$

$$\begin{Bmatrix} F_{12x} \\ F_{12y} \\ T \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(0.04) & -0.0693 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} (2.5)(-23.492) - 200 \\ (2.5)(-8.551) - 0 \\ (0.007)(20) - (R_{Px}(0) - (0.085)(200)) \end{Bmatrix}$$

$$\begin{Bmatrix} F_{12x} \\ F_{12y} \\ T \end{Bmatrix} = \begin{Bmatrix} -258.73 \\ -21.378 \\ 26.01 \end{Bmatrix}$$

4-4.2 Dynamic Force Analysis of Four-Bar linkages

Consider the four-bar linkage shown in Figure 30. It is assumed that the all dimensions of link lengths, link positions, applied external force positions, locations of centre of mass CG, linear and angular accelerations are known or can be determined. Let the subscript i denotes the link, and variables T, F and R respectively denote torque, force and displacement from the centre of mass. Then, an external force F_{Pi} and torque T_i act on link i, and the displacement of the external force from the centre of mass CG_i is R_{Pi}. It is desired to determine all the reactions at the joints and the torque required to drive the linkage.

Figure 31 shows the free-body diagrams of the all the links of Figure 30and the various dimensions are indicated on the diagrams. In the figure, F_{ji} denotes reaction force from link j on link i at the joint between links i and j, and the corresponding displacement from the centre of mass is R_{ji}. From Newton's third law

$$F_{ij} = -F_{ji}$$ **Equation 122**

Summing up forces and moments acting on link 2, we have

$$F_{12x} + F_{32x} + F_{p2x} = m_2 a_{G2x}$$ **Equation 123**

$$F_{12y} + F_{32y} + F_{p2y} = m_2 a_{G2y}$$ **Equation 124**

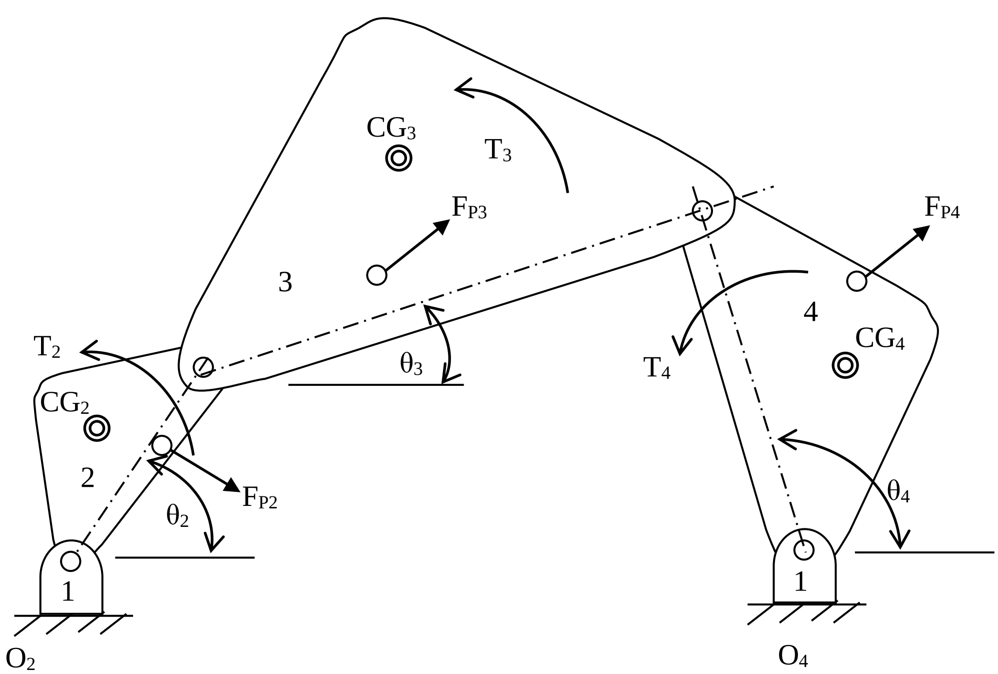


Figure 32: A Schematic of a Four-Bar Linkage

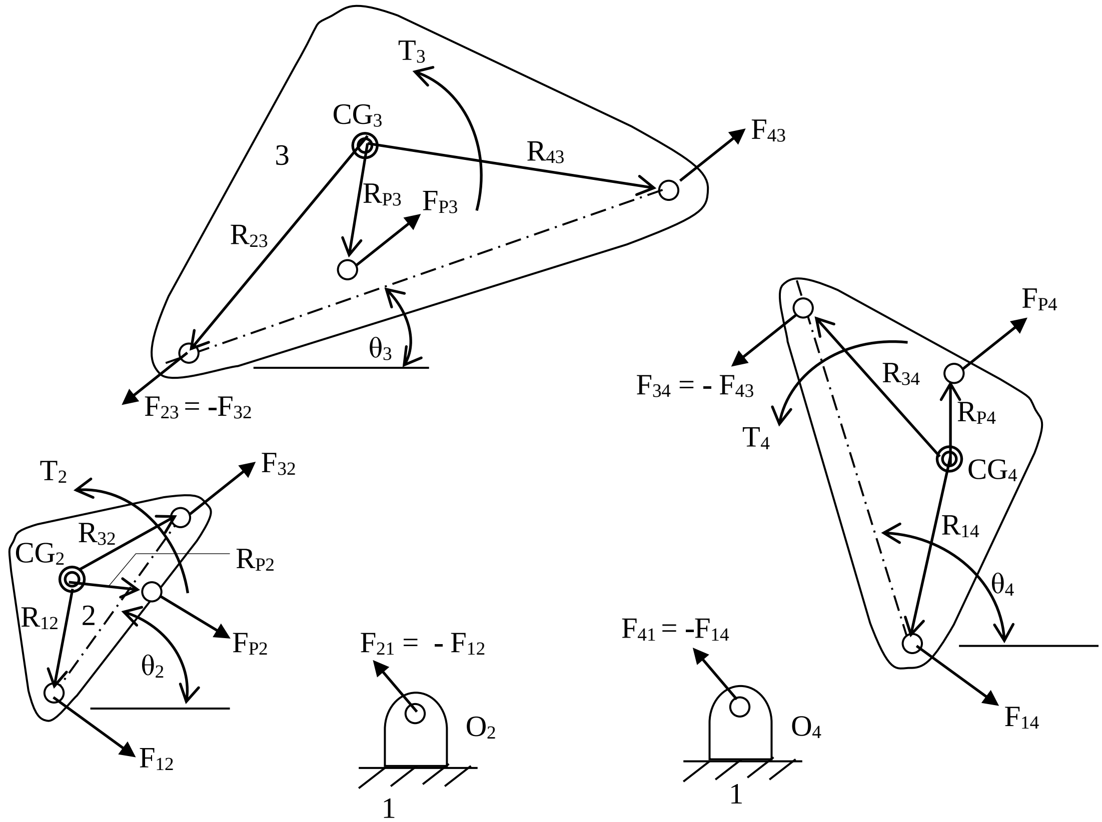


Figure 33: The Free-Body Diagrams of Links of Four-Bar Linkage

$$T_2 + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) - \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{p2x}F_{p2y} - R_{p2y}F_{p2x}\right) = I_{G2}\alpha_2 \quad \textbf{Equation 125}$$

Applying Newton's second law to link 3 and substituting $-F_{32} = F_{23}$, we have

$$F_{43x} - F_{32x} + F_{p3x} = m_3 a_{G3x} \quad \textbf{Equation 126}$$

$$F_{43y} - F_{32y} + F_{p3y} = m_3 a_{G3y} \quad \textbf{Equation 127}$$

$$T_3 + \left(R_{43x}F_{43y} - R_{43y}F_{43x}\right) - \left(R_{23x}F_{32y} - R_{23y}F_{32x}\right) + \left(R_{p3x}F_{p3y} - R_{p3y}F_{p3x}\right) = I_{G3}\alpha_3 \quad \textbf{Equation 128}$$

Applying Newton's second law to link 4 and substituting $-F_{43} = F_{34}$, we have

$$F_{14x} - F_{43x} + F_{p4x} = m_3 a_{G4x} \quad \textbf{Equation 129}$$

$$F_{14y} - F_{43y} + F_{p4y} = m_3 a_{G4y} \quad \textbf{Equation 130}$$

$$T_4 + \left(R_{14x}F_{14y} - R_{14y}F_{14x}\right) - \left(R_{34x}F_{43y} - R_{34y}F_{43x}\right) + \left(R_{p4x}F_{p4y} - R_{p4y}F_{p4x}\right) = I_{G4}\alpha_4 \quad \textbf{Equation 131}$$

The input torque T_2 is the torque required to drive the linkage, and it is unknown. The other unknowns are F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14x} and F_{14y}. The nine equations (110 to 118) are then solved simultaneously. To apply matrix methods, the terms are arranged to put the unknowns on the left and knowns on the right of the equation, which result in Equation 119

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12y} & R_{12x} & -R_{32y} & R_{32x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23y} & -R_{23x} & -R_{43y} & R_{43x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R_{34y} & -R_{34x} & -R_{14y} & R_{14x} & 0 \end{bmatrix} \begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ T_2 \end{Bmatrix} =$$

$$\begin{Bmatrix} m_2 a_{G2x} - F_{P2x} \\ m_2 a_{G2y} - F_{P2y} \\ I_{G2}\alpha_2 - \left(R_{P2x}F_{P2y} - R_{P2y}F_{P2x}\right) \\ m_3 a_{G3x} - F_{P3x} \\ m_3 a_{G3y} - F_{P3y} \\ I_{G3}\alpha_3 - \left(R_{P3x}F_{P3y} - R_{P3y}F_{P3x}\right) - T_3 \\ m_4 a_{G4x} - F_{P4x} \\ m_4 a_{G4y} - F_{P4y} \\ I_{G4}\alpha_4 - \left(R_{P4x}F_{P4y} - R_{P4y}F_{P4x}\right) - T_4 \end{Bmatrix}$$

Equation 132

$$\text{Let } [A] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12y} & R_{12x} & -R_{32y} & R_{32x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23y} & -R_{23x} & -R_{43y} & R_{43x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R_{34y} & -R_{34x} & -R_{14y} & R_{14x} & 0 \end{bmatrix}$$

Equation 133

$$\{X\} = \begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ T_2 \end{Bmatrix} \quad \textbf{Equation 134}$$

$$\{B\} = \begin{Bmatrix} m_2 a_{G2x} - F_{P2x} \\ m_2 a_{G2y} - F_{P2y} \\ I_{G2}\alpha_2 - \left(R_{P2x}F_{P2y} - R_{P2y}F_{P2x}\right) \\ m_3 a_{G3x} - F_{P3x} \\ m_3 a_{G3y} - F_{P3y} \\ I_{G3}\alpha_3 - \left(R_{P3x}F_{P3y} - R_{P3y}F_{P3x}\right) - T_3 \\ m_4 a_{G4x} - F_{P4x} \\ m_4 a_{G4y} - F_{P4y} \\ I_{G4}\alpha_4 - \left(R_{P4x}F_{P4y} - R_{P4y}F_{P4x}\right) - T_4 \end{Bmatrix} \quad \textbf{Equation 135}$$

Then $$[A]\{X\} = \{B\}$$

From which the solution column matrix is given by

$$\{X\} = [A]^{-1}\{B\} \quad \textbf{Equation 136}$$

The above equation may be solved using any commercial software or calculator with matrix algebra capabilities.

Example 4- 6

In Figure E4-6, the link AB has an anticlockwise angular velocity of 12 rad/s and a clockwise angular acceleration of 250 rad/s^2. The mass properties of the links are given in Table E4-6. Determine the instantaneous of drive torque required to produce the assumed motion. Neglect gravity and friction, and assumed uniform cross section for all the links.

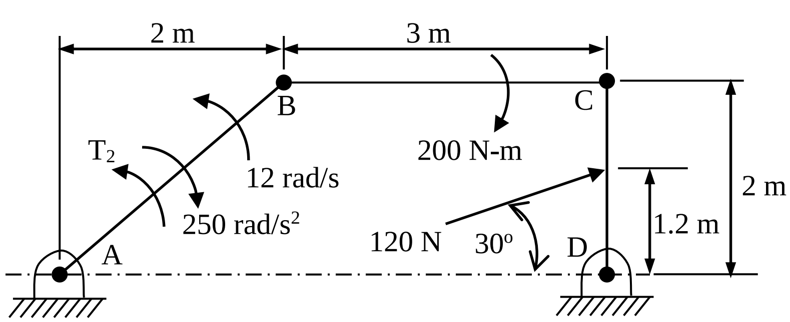


Figure E4-6

Table E4-6

Link	Mass, m (kg)	Moment of inertia, I_G (kg.m^2)
AB	0.5	0.001
BC	1.0	0.004
CD	1.2	0.002

Solution

First, a kinematics analysis using either graphical or analytical method is performed to determine accelerations of the centres of mass of all the links.

Kinematics Analysis

> *Students are to refer to ME 262 Mechanisms Synthesis and Analysis I lecture notes or any text book on how to perform velocity and acceleration analysis of planar mechanism.*

Since the angular acceleration formula has angular velocity component, first the velocities of the links BC and CD are determined. Then, the determined velocities are used to determine accelerations of the links. The velocity of point B

$$v_B = v_A + \omega_{AB} \text{ x } \mathrm{r}_{B/A} \qquad v_B = 0 + 12\text{ k x }(2\mathrm{i} + 2\mathrm{j}) \qquad v_B = -24i + 24j \text{ (m/s)}$$

Let ω_{BC} and ω_{CD} be the angular velocities of links BC and CD, respectively. The velocity of point C in terms of the velocity of point B is

$$v_C = v_B + \omega_{BC} \text{ x } \mathrm{r}_{C/B}$$

$$v_C = (-24i + 24j) + (\omega_{BC}k) \text{ x } (3\mathrm{i}) \qquad v_C = -24i + (3\omega_{BC} + 24)j \tag{1}$$

Similarly, the velocity of point C in terms of the velocity of point D is

$$v_C = v_D + \omega_{CD} \text{ x } \mathrm{r}_{C/D} \qquad v_C = 0 + (\omega_{CD}k) \text{ x } (2\mathrm{j}) = -2\omega_{CD}i \tag{2}$$

Equating the two expressions for the velocity of point C, i.e. equations (1) and (2), yields

$$-24i + (3\omega_{BC} + 24)j = -2\omega_{CD}i$$

Equation the i and the j components of the above equation and solving for the angular velocities, we have

for i component: $-2\omega_{CD} = -24$, $\omega_{CD} = 12$ rad/s

for j component: $(3\omega_{BC} + 24) = 0$, $\omega_{BC} = -8$ rad/s

Figure S4-6(a) shows the centres of mass and the combinations of inertia forces and couples on the links. The acceleration of the centre of mass, G1, of AB is given in terms that of point A as

$$a_{G2} = a_A + \alpha \text{ x } \mathrm{r_{G1/A}} - \omega^2 r_{G1/A} = 0 - 250\,\mathrm{k} \text{ x} \left(\frac{2}{2}\mathrm{i} + \frac{2}{2}\mathrm{j}\right) - (12^2)\left(\frac{2}{2}i + \frac{2}{2}j\right)$$

$$a_{G2} = 106i - 394\,j \text{ (m/s}^2\text{)}$$

The acceleration point B is given in terms that of point A is as

$$a_B = a_A + \alpha \text{ x } \mathrm{r_{B/A}} - \omega^2 r_{B/A} = 0 - 250\,\mathrm{k} \text{ x } (2\mathrm{i} + 2\mathrm{j}) - (12^2)(2i + 2j)$$

$$a_B = 212i - 788\,j \text{ (m/s).}$$

Let α_{BC} and α_{CD} be the angular accelerations of links BC and CD, respectively. The acceleration of point C in terms of that of point B is

$$a_C = a_B + \alpha_{BC} \text{ x } \mathrm{r_{C/B}} - \omega_{BC}^2 \mathrm{r_{C/B}}$$

$$a_C = (212i - 788\,j) + \alpha_{BC}\mathrm{k} \text{ x } 3\mathrm{i} - (-8)^2(3i)$$

$$a_C = 20i + (3\alpha_{BC} - 788)j \tag{3}$$

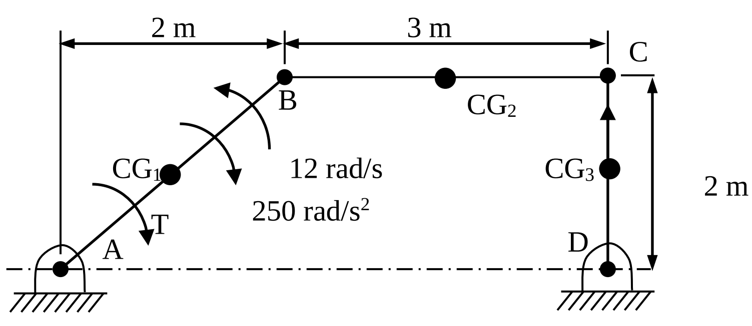


Figure S4-6(a)

Similarly, the acceleration of point C from point D is

$$a_C = a_D + \alpha_{CD} \text{ x } \mathrm{r_{D/C}} - \omega_{CD}^2 \mathrm{r_{C/D}}$$

$$a_C = 0 + \alpha_{CD}\mathrm{k} \text{ x } 2\mathrm{j} - (12)^2(2\,j)$$

$$a_C = -2\alpha_{CD}\mathrm{i} - 288j \tag{4}$$

Equating the two expressions for the acceleration of point C (i.e. equations (3) and (4)) gives

$$20i + (3\alpha_{BC} - 788)j = -2\alpha_{CD}\mathrm{i} - 288j$$

Equating i and j components of the above equation and solving for the angular accelerations, we have

i component: $-2\alpha_{CD} = 20$, $\alpha_{CD} = \alpha_4 = -10$ rad/s²

j component: $3\alpha_{BC} - 788 = -288$, $\alpha_{BC} = \alpha_3 = 166.7$ rad/s²

Then, the acceleration of the centre of mass of BC is

$$a_{G3} = a_B + \alpha_{BC} \text{ x } \mathrm{r}_{\mathrm{G2/B}} - \omega_{BC}^2 \mathrm{r}_{\mathrm{G2/B}} = (212i - 788j) + 166.7\mathrm{k} \text{ x } \frac{3}{2}\mathrm{i} - (-8)^2\left(\frac{3}{2}i\right)$$

$$a_{G3} = 116i - 538j \text{ (m/s)}$$

The acceleration of the centre of mass of CD is

$$a_{G4} = a_D + \alpha_{CD} \text{ x } \mathrm{r}_{\mathrm{G3/C}} - \omega_{CD}^2 \mathrm{r}_{\mathrm{G3/D}} = 0 + (-10)\mathrm{k} \text{ x } \frac{2}{2}\mathrm{j} - (12)^2\left(\frac{2}{2}j\right)$$

$$a_{G4} = 10\mathrm{i} - 144j \text{ (m/s)}$$

Kinetics Analysis

$F_{P2x} = F_{P2y} = F_{P3x} = F_{P3y} = 0$ $T_4 = 0$ $T_3 = -200$ N - m

$F_{P4x} = 120\cos 30^o$ $F_{P4x} = 103.9$ N

$F_{P4y} = 120\sin 30^o$ $F_{P4y} = 60.0$ N

$R_{P4x} = 0$ $R_{P4y} = 0.2$ m

The free-body diagrams of the links of the linkage are shown in Figure S4-6(b). In the figure the distances from the centres of mass to the joints and external forces, if any, are indicated. The geometric and mass properties are substituted in Equation 101

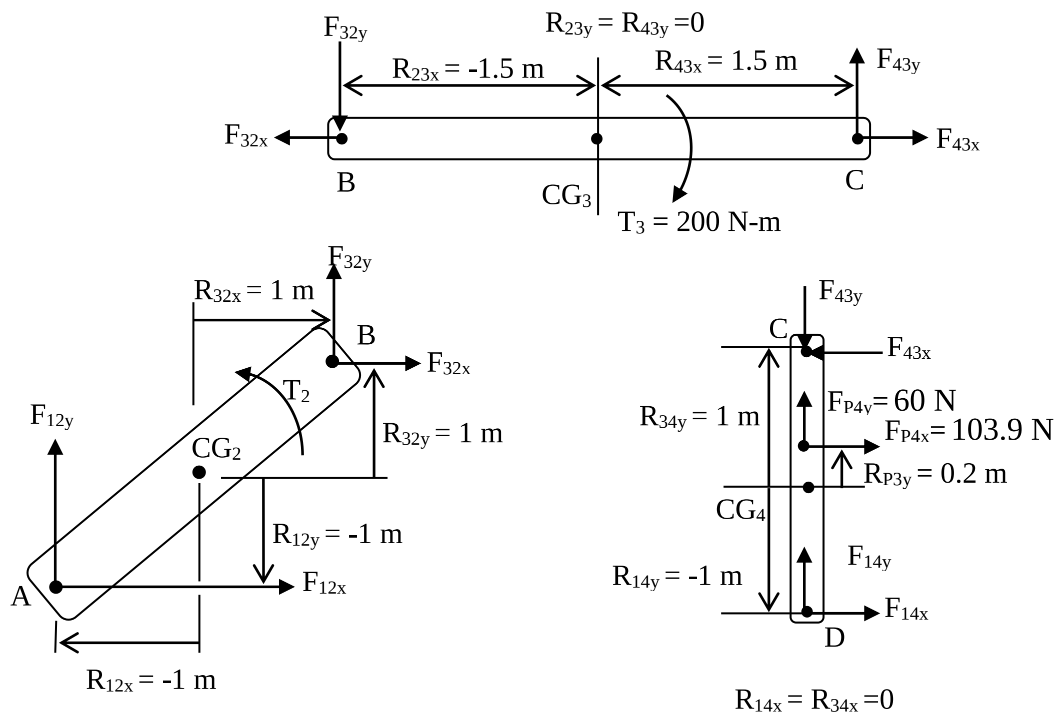


Figure S4-6(b)

$$A=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -(-1) & (-1) & -(1) & (1) & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & (0) & -(-1.5) & -(0) & (1.5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & (1) & -(0) & -(-1) & (0) & 0 \end{bmatrix} \quad B=\begin{Bmatrix} (0.5)(106)-0 \\ (0.5)(-394)-0 \\ (0.001)(-250)-(0) \\ (1.0)(10)-0 \\ (1.0)(-144)-0 \\ (0.004)(166.7)-(0)-(-200) \\ (1.2)(10)-(103.9) \\ (1.2)(-144)-(60.0) \\ (0.002)(-10)-[(0)(60)-(0.2)(103.9)]-(0) \end{Bmatrix}$$

Using MatCAD, we have

$$A=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad B=\begin{bmatrix} 53 \\ -197 \\ -0.25 \\ 10 \\ -144 \\ 200.67 \\ -91.9 \\ -232.8 \\ 20.76 \end{bmatrix}$$

$$A\_inverse := A^{-1}$$

$$A\_inverse = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0.5 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 0.5 & -0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.333 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 1 & -0.667 & -1 & 0 & 1 \end{pmatrix} \quad X := A\_inverse \cdot B \quad X = \begin{pmatrix} 6.67 \\ -335.889 \\ 46.33 \\ 138.889 \\ 56.33 \\ -5.111 \\ -35.57 \\ -237.911 \\ -435.368 \end{pmatrix}$$

4-4.3 Dynamic Force Analysis of Four-Bar Crank-Slider linkages

The approach used for the four-bar pin-jointed linkage is equally valid for a four-bar crank-slider linkage. Consider a four-bar crank-slider linkage shown in Figure 32. An external force acts on the link 4 and it is desired to determine the reactions at the joints and the driving torque required on the crank to provide the specified accelerations. First, a kinematic analysis needs to be performed to determine the positions, velocities and accelerations. In this case, it assumed that the kinematic parameters are known.

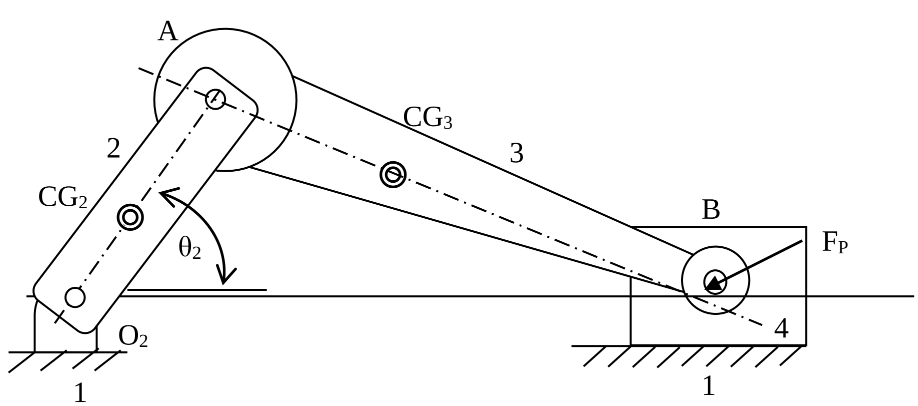


Figure 34: A Schematic of a Crank-Slider Linkage

The free-body diagrams of the links are shown in Figure 33. In the figure, no external forces are indicated on links 2 and 3 though it is possible to apply external forces on them. This implies that Equations 108 to 113 are applicable to links 2 and 3 of Figure 33. Removing external components of Equations 108 to 113, the equations of motion for links 2 and 3 are:

Link 2

$$F_{12x} + F_{32x} = m_2 a_{G2x} \qquad \textbf{Equation 137}$$

$$F_{12y} + F_{32y} = m_2 a_{G2y} \qquad \textbf{Equation 138}$$

$$T_2 + \left(R_{12x} F_{12y} - R_{12y} F_{12x}\right) + \left(R_{32x} F_{32y} - R_{32y} F_{32x}\right) = I_{G2} \alpha_2 \qquad \textbf{Equation 139}$$

Link 3

$$F_{43x} - F_{32x} = m_3 a_{G3x} \qquad \textbf{Equation 140}$$

$$F_{43y} - F_{32y} = m_3 a_{G3y} \qquad \textbf{Equation 141}$$

$$T_3 + \left(R_{43x} F_{43y} - R_{43y} F_{43x}\right) - \left(R_{23x} F_{32y} - R_{23y} F_{32x}\right) = I_{G3} \alpha_3 \qquad \textbf{Equation 142}$$

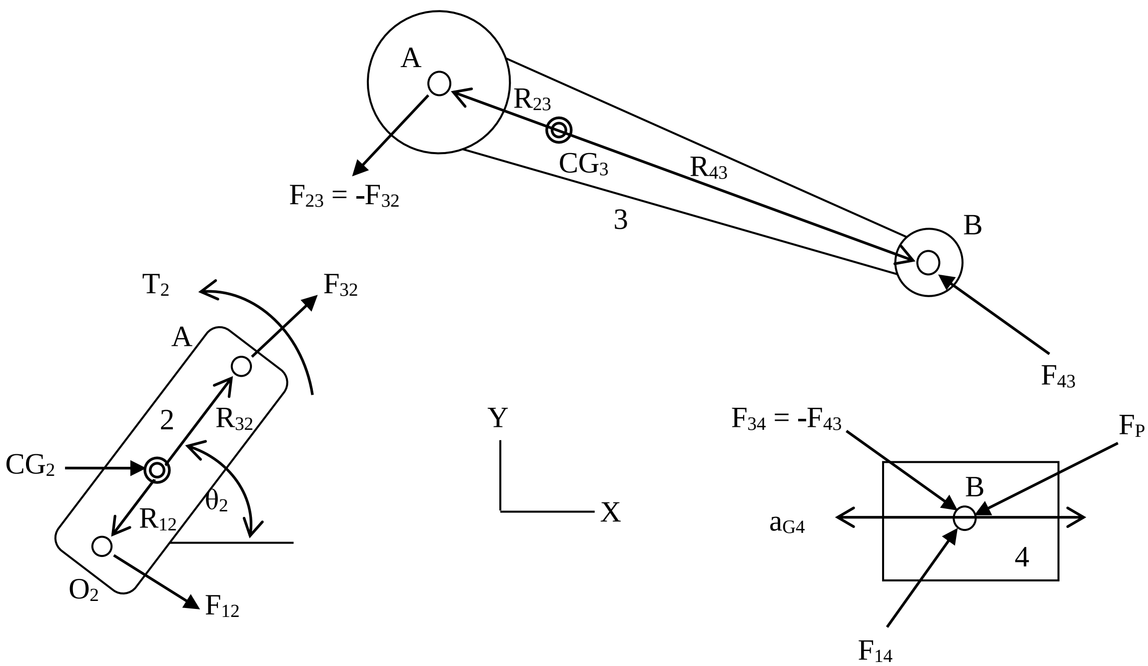


Figure 35: The Free-Body Diagrams of Links of a Crank-Slider Linkage

For cases whereby there are external forces and torques on links 2 and 3, then use Equations 108 to 111.

The slider (link 4) is in pure translational motion against the stationary ground; thus it has no angular acceleration or angular velocity. In addition, there is no motion along the vertical direction, which implies that the vertical acceleration is zero. The sum of moments about the centre of mass of the link is zero since we have concurrent force system and all the forces acting on the link 4 pass through its centre of mass. Thus,

$$a_{G4y} = 0 \qquad \alpha_4 = 0$$

From the above arguments, the equations of motion for link 4 are

$$F_{14x} - F_{43x} + F_{Px} = m_3 a_{G3x}$$ **Equation 143**

$$F_{14y} - F_{43y} + F_{Py} = 0$$ **Equation 144**

The only force directed along the x axis that exists between link 4 and the ground is frictional force. Suppose there is Coulomb (dry) friction between the slider and ground (link 1), then the x component may be expressed in terms of the normal force N of the force at interface as

$$F = \operatorname{sgn} \mu N$$

where μ is coefficient of friction and the sgn indicates the friction forces opposes motion.

$$\operatorname{sgn} = -\frac{v_{4x}}{|v_{4x}|}$$ **Equation 145**

The above equation makes the sign of μ opposite of the sign of the velocity of link 4. Thus

$$F_{14x} = \operatorname{sgn} \mu F_{14y}$$ **Equation 146**

Substituting the above equation into Equation 121, we have

$$\operatorname{sgn} F_{14y} - F_{43x} + F_{Px} = m_3 a_{G3x}$$ **Equation 147**

This last substitution reduces the number of unknowns to eight, which are F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14y} and T_2. These eight unknowns are solved for simultaneously. Alternatively, Equations 117 to 122, 124 and 127 are used to assemble the matrices for solution.

$$[A]=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -R_{12y} & R_{12x} & -R_{32y} & R_{32x} & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & R_{23y} & -R_{32x} & -R_{43y} & R_{43x} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \operatorname{sgn}\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$ **Equation 148**

$$\{X\}=\begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14y} \\ T_2 \end{Bmatrix}$$ **Equation 149**

$$\{B\}=\begin{Bmatrix} m_2 a_{G2x} \\ m_2 a_{G2y} \\ I_{G2}\alpha_2 \\ m_3 a_{G3x} \\ m_3 a_{G3y} \\ I_{G3}\alpha_3 \\ m_4 a_{G4x} - F_{Px} \\ -F_{Py} \end{Bmatrix}$$ **Equation 150**

where $[A]\{X\}=\{B\}$

The solution of the above equation will yield complete dynamic force information for the four-bar crank-slider linkage. Thus,

$$\{X\}=[A]^{-1}\{B\}$$ **Equation 151**

Once again, the above equation may be solved using any commercial software or calculator with matrix algebra capabilities.

4-4.4 Dynamic Force Analysis of Four-Bar Inverted Crank-Slider Linkages

The approach used for the four-bar pin-jointed linkage is equally valid for a four-bar inverted crank-slider linkage. Consider a four-bar inverted crank-slider linkage shown in Figure 34. Again, we wish to determine the reactions at the joints and the driving torque required on the crank, link 2, to provide the specified accelerations. First, a kinematic analysis needs to be performed to determine the positions, velocities and accelerations. Again, it assumed that the kinematic parameters, including instantaneous positions, velocities and accelerations are known. Note that there is no Coriolis component of acceleration. The free-body diagrams of members of the four-bar inverted crank-slider are illustrated in Figure 35. Equations 109 to 118 are applicable to this linkage, except Equation 115. An equivalent equation is written for Figure 35 as in Equation 139, where there is no external torque on link 3.

$$\left(R_{43x}F_{43y}-R_{43y}F_{43x}\right)-\left(R_{23x}F_{32y}-R_{23y}F_{32x}\right)+\left(R_{p3x}F_{p3y}-R_{p3y}F_{p3x}\right)=I_{G3}\alpha_3$$ **Equation 152**

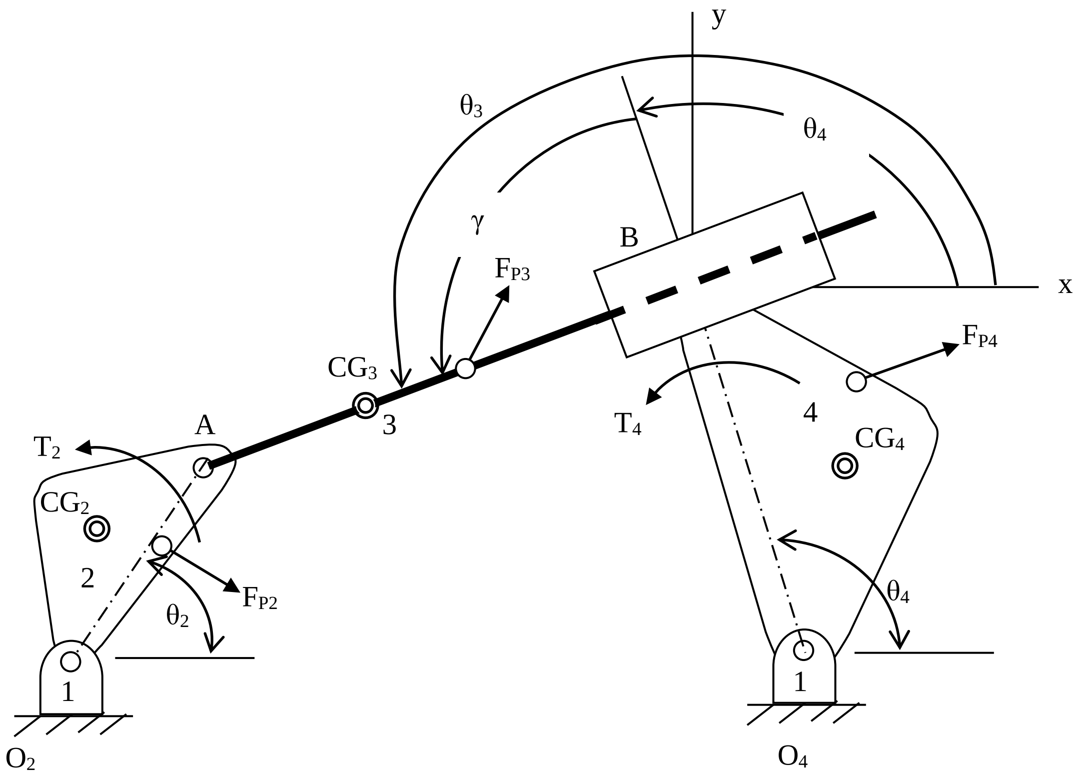


Figure 36: A Schematic of a Four-Bar Inverted Crank-Slider Linkage

The sliding joint between links 3 and 4 can only transmit force from link 3 to 4 or vice versa along a line perpendicular to the axis of sliding. This line is called the *axis of transmission*. In the absence of friction force F_{34} or F_{43} is always perpendicular to the axis of sliding. Then, cross product of the unit direction vector of sliding and the force F_{43} is zeros. Thus

$$\hat{u}.F_{43}=0$$ **Equation 153**

which expands to

$$u_x F_{43x}+u_y F_{43y}=0$$ **Equation 154**

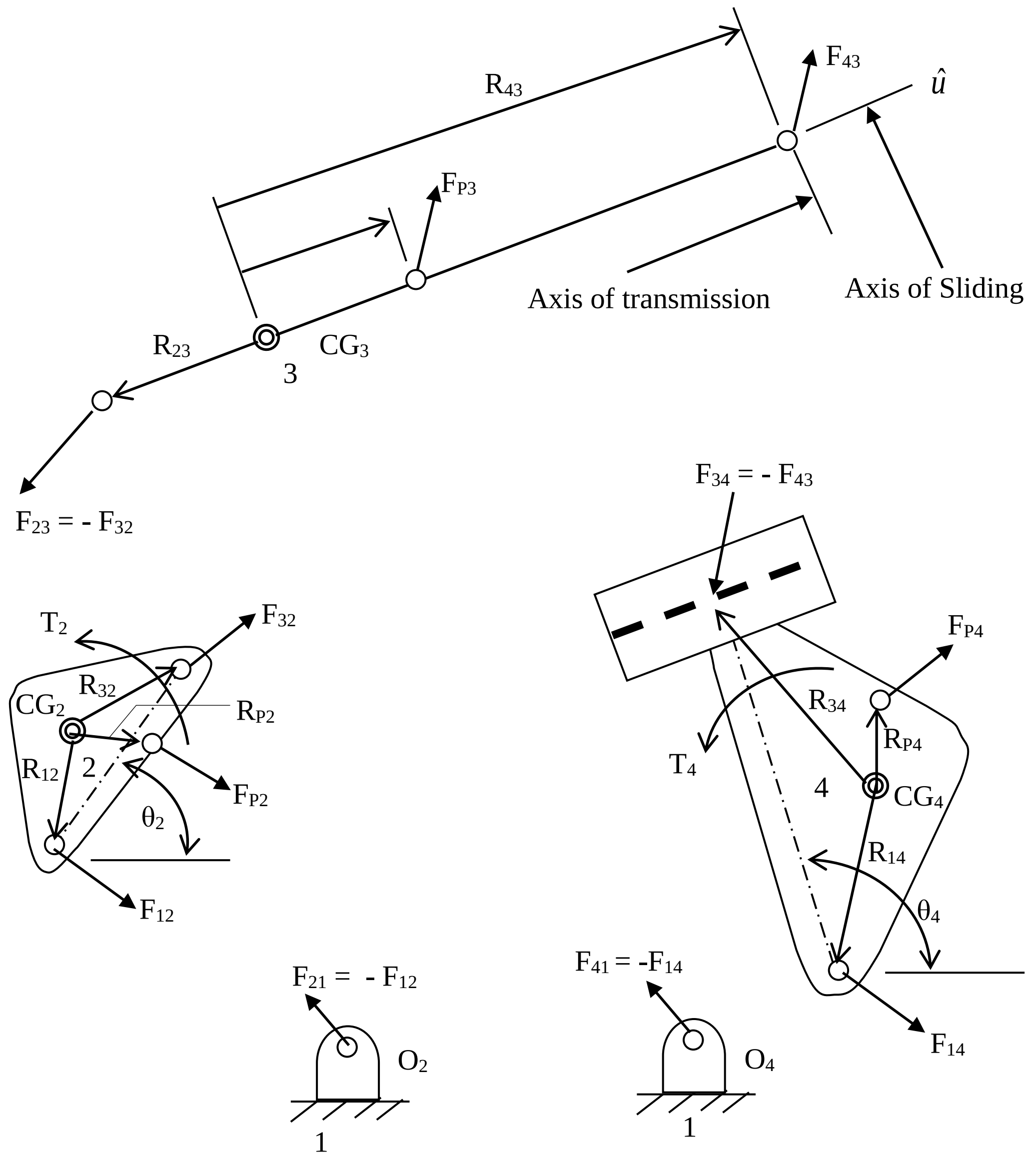


Figure 37: Free-Body Diagrams of Links of the Schematic Inverted Crank-Slider Linkage

The unit direction vector of sliding is defined in terms of the sliding direction as

$$u_x = \cos\theta_3 \qquad\qquad u_y = \sin\theta_3 \qquad \textbf{Equation 155}$$

The angular velocity of link 3 is the same as that of link 4. The moment equations for links 3 and 4 are

$$\left(R_{43x}F_{43y} - R_{43y}F_{43x}\right) - \left(R_{23x}F_{32y} - R_{23y}F_{32x}\right) + \left(R_{p3x}F_{p3y} - R_{p3y}F_{p3x}\right) = I_{G3}\alpha_4 \qquad \textbf{Equation 156}$$

$$T_4 + \left(R_{14x}F_{14y} - R_{14y}F_{14x}\right) - \left(R_{34x}F_{43y} - R_{34y}F_{43x}\right) + \left(R_{p4x}F_{p4y} - R_{p4y}F_{p4x}\right) = I_{G4}\alpha_4 \qquad \textbf{Equation 157}$$

Adding the above two equations gives

$$\left[\left(R_{43x}-R_{34x}\right)F_{43y}-\left(R_{43y}-R_{34y}\right)F_{43x}\right]-\left(R_{23x}F_{32y}-R_{23y}F_{32x}\right)+\left(R_{p3x}F_{p3y}-R_{p3y}F_{p3x}\right)+\left(R_{p4x}F_{p4y}-R_{p4y}F_{p4x}\right)=\left(I_{G3}+I_{G4}\right)\alpha_4-T_4$$ **Equation 158**

Equations 108 to 112, 143, 114, 115 and 140 give nine equations with nine unknowns, which can be solved simultaneously or put in a matrix form for a solution.

By letting

$$[A]=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12y} & R_{12x} & -R_{32y} & R_{32x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23y} & -R_{23x} & -\left(R_{43y}-R_{34y}\right) & \left(R_{43x}-R_{34x}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & u_x & u_y & 0 & 0 & 0 \end{bmatrix}$$ **Equation 159**

$$\{X\}=\begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ T_2 \end{Bmatrix}$$ **Equation 160**

$$\{B\} = \left\{ \begin{array}{c} m_2 a_{G2x} - F_{P2x} \\ m_2 a_{G2y} - F_{P2y} \\ I_{G2}\alpha_2 - \left(R_{P2x}F_{P2y} - R_{P2y}F_{P2x}\right) \\ m_3 a_{G3x} - F_{P3x} \\ m_3 a_{G3y} - F_{P3y} \\ \left(I_{G3} + I_{G4}\right)\alpha_4 - \left(R_{P3x}F_{P3y} - R_{P3y}F_{P3x}\right) - \left(R_{P4x}F_{P4y} - R_{P4y}F_{P4x}\right) - T_4 \\ m_4 a_{G4x} - F_{P4x} \\ m_4 a_{G4y} - F_{P4y} \\ 0 \end{array} \right\} \quad \textbf{Equation 161}$$

Then $[A]\{X\} = \{B\}$

From which the solution column matrix is given by

$$\{X\} = [A]^{-1}\{B\}$$

4-4.5 Force Analysis of Linkages with More than Four Links

For more than four links, start with a set of at most four links and the result fed to the next four set of links as external forces. Figure 36 is a schematic of a typical linkage with more than links. Link 2 is the input link and we wish to determine the reactions at all the pin joints and the driving torque required on link 2. To do so, the assembly is subdivided into three sub-assemblies as

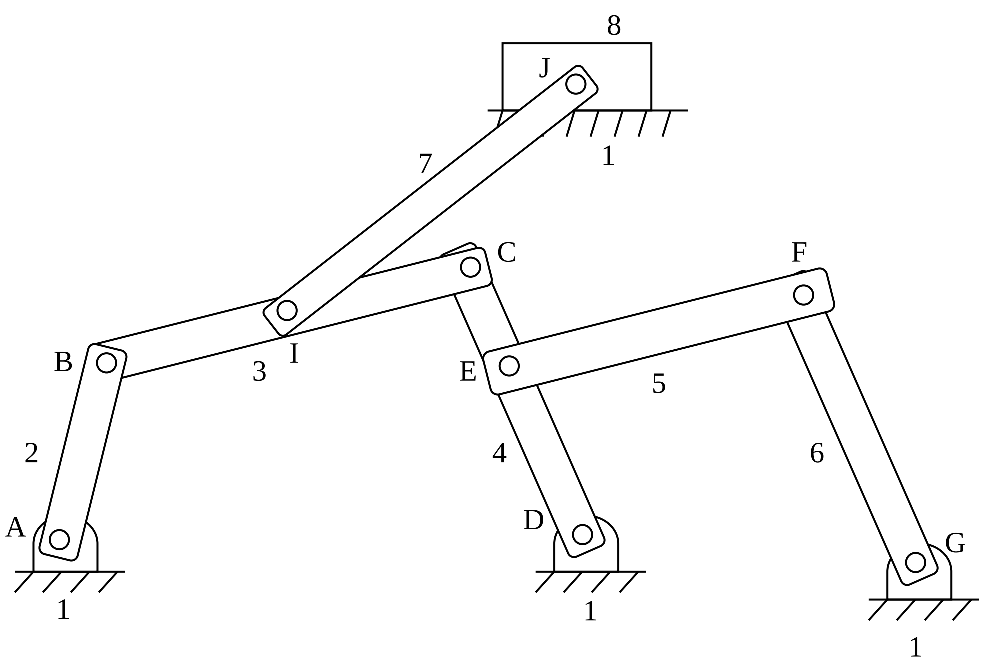


Figure 38: A schematic of a Linkage with More than Four Links

Assembly 1: Links 2, 3, 4 and 1

Assembly 2: Links 7, 8 and 1

Assembly 3: Links 5, 6 and 1

The assembly 2 is analysed as crank-slider mechanism and the vertical and horizontal components of the force at point I required to give it the desired position, velocity and acceleration become the vertical and horizontal components of the external force at point I on the link 3 of the Assembly 1. Similarly, the assembly 3 is analysed as three-bar linkage and the vertical and horizontal components of the force at point E required to give it the desired position, velocity and acceleration become the vertical and horizontal components of the external force at point E on link 4 of the Assembly 1. Note that in the analysis of the assemblies, external torques are not included at the interface joints I and E because pin-joints cannot transmit any torque.

Alternatively, the matrix method may be extended to linkage with more than four links. Let j, i and k denote any link, previous link and next link in a chain. Thus,

i = j – 1, k = j + i, for j = 1, 2 ... n = number of links.

The vector form of equations of motion for link j become

$$F_{ij} + F_{jk} + \sum F_{Pjn} = m_j a_{Gj}$$ **Equation 162**

$$(R_{ij} \text{x} F_{ij}) + (R_{jk} \text{x} F_{jk}) + \sum (R_{Pn} \text{ x } F_{Pn}) + \sum T_j = I_j \alpha_j$$ **Equation 163**

$$F_{ji} = -F_{ij}; \qquad F_{kj} = -F_{jk}$$ **Equation 164**

where j = 2,3, ... n; i = j-1;

k = j + 1 for $j \neq n$

k = 1 for j = n,

In Equations 162 and 163, the subscripts p and n denote external force and index of an external force on link j. The sum of forces equation given by Equation 162 for each link may be split into its x and y components. The sum of moments given by Equation 163 is taken about the centre of mass of each link replacing the forces by their x and y components. Any link may have external force and/or external torques applied to it. When sliding joints are present, it is necessary to add constraints on the allowable directions of forces at those joints as was in the cases of crank-slider and inverted crank-slider linkages. The allowable directions in a case of crank-slider are sliding and normal to sliding.

Example 4- 7

In Figure E4-7, the arm ABC of a model excavator has a mass of 1200 kg and moment of 3600 kg-m^2, calculated about its centre of mass G. At the instant shown, the hydraulic cylinder BD is vertical, and the angular velocity and acceleration of the arm are 3 rad/s and 1.2 rad/s^2, both counter clockwise. Determine the (a) force exerted by vertical hydraulic cylinder BD on the arm at B, and (b) the reaction at A.

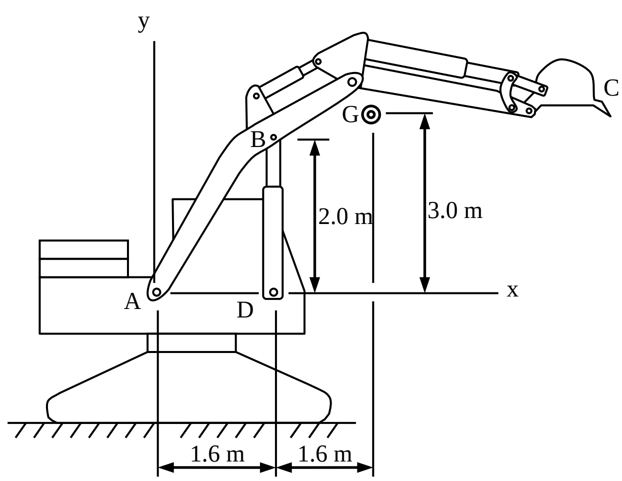


Figure E4-7

Solution

Using the parallel axis theorem, the moment of inertia of the arm about point A is

$$I_A = I_G + md^2 = 3600 + 1200\left[(1.6+1.6)^2 + (3.0)^2\right] \qquad I_A = 26688\ \text{kg - m}^2$$

Let the reaction at A be A_x and A_y acting along the horizontal and vertical axes, respectively. The free-body diagram of the arm is shown in Figure S4-7. Taking moment about A, we have

$$\sum M_A = I_A\alpha_{AB} \qquad F(1.6) - (1200\ \text{x}\ 9.81)(1.6+1.6) = (26688)(1.2)$$

$$\underline{\underline{F = 43.56\ \text{kN}}}$$

The linear acceleration of the centre of mass G is expressed in terms of the acceleration of point A as:

$$a_G = a_A + \alpha_{AB}\ \text{x}\ \text{r}_{G/A} - \omega_{AB}^{\ 2}\text{r}_{G/A} = 0 + 1.2\text{k}\ \text{x}\ (3.2\text{i} + 3.0\text{j}) - 3^2(3.2\text{i} + 3.0\text{j})$$

$$a_G = -32.4\text{i} - 23.16\,\text{j}$$

$$\sum F = ma_G \qquad A_x \mathrm{i} + A_y \mathrm{j} + (43560)j - (1200\text{x}9.81)\mathrm{j} = 1200(-32.4i - 23.16j))$$

$$\underline{\underline{A_x = -38.88\,\text{kN}}}$$

$$\underline{\underline{A_y = -59.58\,\text{kN}}}$$

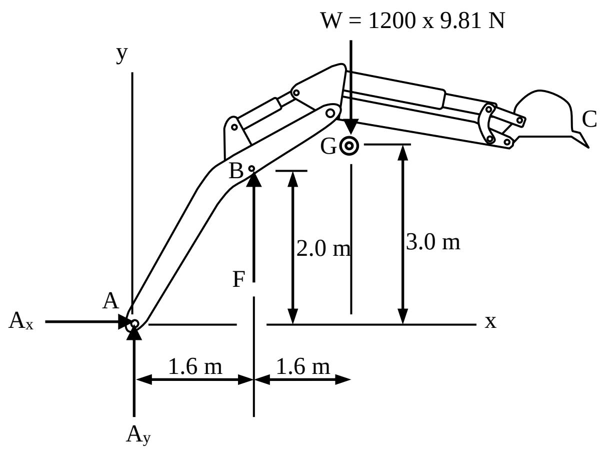


Figure S4-7

4-4.6 Shaking Forces and Shaking Torques

Shaking effects are net dynamic forces and torques felt on the ground plane. These effects may set up vibrations in the support structure of a machine. The sum of all the reaction forces on the ground plane is called the *Shaking Forces (F_s)*. It is defined in mathematical form as

$$F_s = \sum F_{j1} \qquad \textbf{Equation 165}$$

In all the examples given Sessions 4-3, the shaking force is equal to

$$F_s = F_{21} + F_{41}$$

The sum of all the reaction torques felt by the ground plane is called the *Shaking Torque (T_s)*. It is defined in mathematical form as

$$T_s = \sum -T_j \qquad \textbf{Equation 166}$$

where T_j is the torque of links connected to the ground. In all the given examples in Sessions 4-3, the shaking torques are equal to

$$T_s = -T_2 - T_4$$

4-4.7 Linkage Force Analysis Using Energy Method

The Newtonian methods of force analysis described so far have considerable advantage of providing complete information about all internal forces at pin joints as well as the external forces and torques on a system. However, the application of the methods is relatively complex as it requires dividing mechanisms into free bodies and simultaneous solution of large systems of equations. The method of virtual work, which is a form the Energy Method, is considerably easier to implement but gives less information. It leads to direct determination of input and output forces and/or torques without the intermediate computation of joint forces. The main advantage is this method is its use as a quick check on correctness of Newtonian solution for input forces and/or torque. The method is applicable to both static-force and dynamic-force analysis.

4-4.7.1 Dynamic-Force Analysis

The energy method is based on the principle of conservation of energy. That is, in the absence of losses, the total energy of a system remains constant. This implies that the rate of change of energy in a system at any instant must be equal to the rate at which energy is externally passing into or out of the system. This statement can be expressed in a mathematical form as summation of the energies (or power) due to each moving element (or link) in the system. Thus,

$$\sum_{k=2}^{n} F_k . v_k + \sum_{k=2}^{n} T_k . \omega_k = \sum_{k=2}^{n} m_k a_k . v_k + \sum_{k=2}^{n} I_k \alpha_k . \omega_k \qquad \textbf{Equation 167}$$

The subscript k represents a moving or movable link or element in the system, and n represents the number of links in the system. To use the above equation, the angular and linear velocities and accelerations must be calculated first by performing kinematics analysis, and masses and moments of inertias known. Expanding Equation 154 to create a scalar equation, we have

$$\sum_{k=2}^{n} \left(F_{kx} . v_{kx} + F_{ky} . v_{ky} \right) + \sum_{k=2}^{n} T_k . \omega_k = \sum_{k=2}^{n} m_k \left(a_{kx} . v_{kx} + a_{ky} . v_{ky} \right) + \sum_{k=2}^{n} I_k \alpha_k . \omega_k \qquad \textbf{Equation 168}$$

4-4.7.2 Static-Force Analysis

For a static-force analysis, there is no motion. Equation 154 and 155 are power (or energy per unit time) are modified to energy form and removing all energy terms due to motion, using the method of virtual work. The term virtual work comes from the concept of each force causing infinitesimal (or virtual) displacement in a static system over infinitesimal change in time. For static condition, equation 154 reduces to

$$\sum_{k=2}^{n} F_k.\partial R_k + \sum_{k=2}^{n} T_k.\partial \theta_k = 0 \qquad \textbf{Equation 169}$$

where δR and δθ are respectively virtual linear and angular displacement vectors. Expanding the above equation to x and y components of the forces, we have

$$\sum_{k=2}^{n} \left(F_{kx}.\partial R_{kx} + F_{ky}.\partial R_{ky}\right) + \sum_{k=2}^{n} T_k.\partial \theta_k = 0 \qquad \textbf{Equation 170}$$

Example 4- 8

Use the method of virtual work to determine the torque required to overcome load P on the slider at C of Session 4-2.1.

Solution

The crank-slider linkage is analysed with the sketched shown in Figure S4-8 .The quantities δθ and δx are virtual displacements. Using Equation 160, we have

$$\left(F_{2x}.\partial R_{2x} + F_{2y}.\partial R_{2y}\right) + \left(F_{3x}.\partial R_{3x} + F_{3y}.\partial R_{3y}\right) + \left(F_{4x}.\partial R_{4x} + F_{4y}.\partial R_{4y}\right) +$$
$$T_2.\partial \theta_2 + T_3.\partial \theta_3 + T_4.\partial \theta_4 = 0$$

$$(0) + (0) + ((-P).(-\partial x) + 0) + (T)(.\partial \theta) + 0 + 0 = 0$$

$$P\partial x + T\partial \theta = 0 \qquad T = -P\frac{\partial x}{\partial \theta} \qquad (1)$$

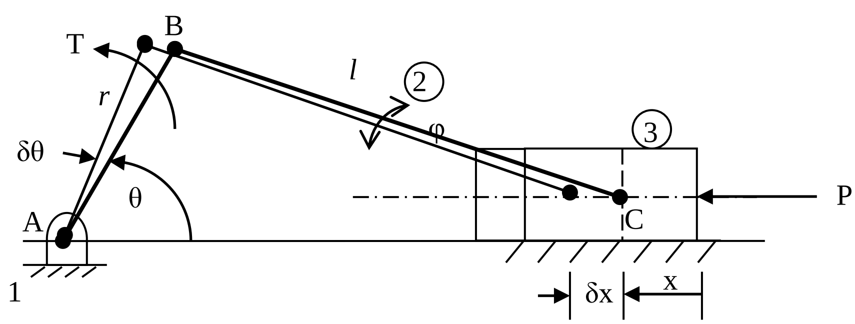


Figure S4-8

The displacement x from the original position when θ = 0 is

$$x = R + L - (R\cos\theta + L\cos\phi) \qquad x = R(1-\cos\theta) + L(1-\cos\phi) \tag{2}$$

Using sine rule

$$\frac{\sin\phi}{r} = \frac{\sin\theta}{l} \qquad \sin\phi = \frac{r\sin\theta}{l} \tag{3}$$

Substituting (3) into (2), we have

$$x = r(1-\cos\theta) + l\left[1 - \sqrt{1 - \left(\frac{r}{l}\sin\theta\right)^2}\right]$$ **Equation 171**

Partially differentiating the equation with respect to θ, we have

$$\frac{\partial x}{\partial\theta} = r\sin\theta\left[1 + \frac{r\cos\theta}{\sqrt{l^2 - (r\sin\theta)^2}}\right] \tag{4}$$

Substituting the above expression into equation (1) gives

$$T = -\Pr\sin\theta\left[1 + \frac{r\cos\theta}{\sqrt{l^2 - (r\sin\theta)^2}}\right]$$

Example 4- 9

An instant kinematic and mass data of a four-bar linkage shown in Figure E4-9 is listed in Table E4-9. There is external force of 120 N at 60^o applied on link 3 at E where the linear velocity is 6.5 m/s at 145^o. Also, there is external torque of 200 N-m on link 4. Determine the driving torque required on the crank to maintain the motion with the instantaneous position of the linkage. Use the method of virtual work.

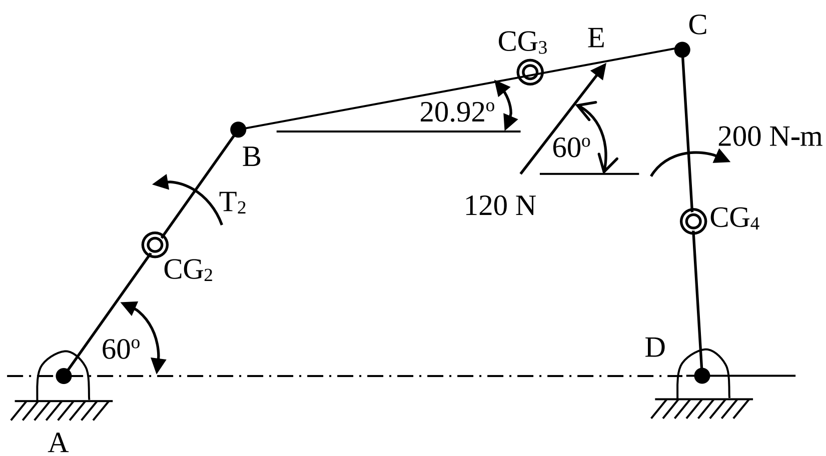


Figure E4-9

Table E4-9

Link	M (kg)	I_G (kg-m^2)	ω (rad/s)	α (rad/s^2)	v_G (m/s)	a_G (m/s^2)
Crank, link 2	0.5	0.4	20	-40	7.5 at 180^o	68.6 at 271^o
Coupler, link 3	2.5	1.2	-6.2	125	7.2 at 150^o	87.7 at 254^o
Rocker, link 4	2.0	0.8	8.4	280	4.0 at 190^o	36 at 207^o

Solution

The x and y components of the external forces are:

$F_{P3x} = 120\cos 60^o \qquad F_{P3x} = 60 \text{ N} \qquad F_{P3y} = 120 \sin 60^o \qquad F_{P3y} = 103.9 \text{ N}$

$F_{P2x} = F_{P2y} = F_{P4x} = F_{P4y} = 0$

The external torques are:

$T_3 = 0 \qquad T_4 = -200 \text{ N-m}$

The x and y components of the linear velocities are:

$v_{G2} = 7.5\angle 180^o \qquad v_{G2x} = 7.5\cos 180^o \qquad v_{G2x} = -7.5 \text{ m/s}$

$v_{G2y} = 7.5\sin 180^o \qquad v_{G2y} = 0$

$v_{G3} = 7.2\angle 150^o \qquad v_{G3x} = 7.2\cos 150^o \qquad v_{G3x} = -6.23 \text{ m/s}$

$v_{G3y} = 7.2\sin 150^o \qquad v_{G3y} = 3.6 \text{ m/s}$

$v_{G4} = 4.0\angle 190^o \qquad v_{G4x} = 4.0\cos 190^o \qquad v_{G4x} = -3.94 \text{ m/s}$

$v_{G4y} = 4.0\sin 190^o \qquad v_{G4y} = -0.69 \text{ m/s}$

$v_{P3} = 6.5\angle 145^o \qquad v_{P3x} = 6.5\cos 145^o \qquad v_{P3x} = -5.32 \text{ m/s}$

$$v_{P3y} = 3.73\sin 145^o \qquad v_{P3y} = 3.73 \text{ m/s}$$

The x and y components of the linear accelerations are

$$a_{G2} = 68.6\angle 271^o \qquad a_{G2x} = 68.6\cos 271^o \qquad a_{G2x} = 1.20 \text{ m/s}^2$$

$$a_{G2y} = 68.6\sin 271^o \qquad a_{G2y} = -68.59 \text{ m/s}^2$$

$$a_{G3} = 87.7\angle 254^o \qquad a_{G3x} = 87.7\cos 254^o \qquad a_{G3x} = -24.17 \text{ m/s}^2$$

$$a_{G3y} = 87.7\sin 254^o \qquad a_{G3y} = -84.3 \text{ m/s}^2$$

$$a_{G4} = 36.0\angle 207^o \qquad a_{G4x} = 36.0\cos 207^o \qquad a_{G4x} = -32.07 \text{ m/s}^2$$

$$a_{G4y} = 36.0\sin 207^o \qquad a_{G4y} = -16.34 \text{ m/s}^2$$

Applying the energy method to the four-bar linkage, we have

$$\begin{aligned}
&\left[\left(F_{P2x}.v_{P2x} + F_{P2y}.v_{P2y}\right) + \left(F_{P3x}.v_{P3x} + F_{P3y}.v_{P3y}\right) + \left(F_{P4x}.v_{P4x} + F_{P4y}.v_{P4y}\right)\right] + \left[T_2.\omega_2 + T_3.\omega_3 + T_4.\omega_4\right] \\
&= \left[m_2\left(a_{G2x}.v_{G2x} + a_{G2y}.v_{G2y}\right) + m_3\left(a_{G3x}.v_{G3x} + a_{G3y}.v_{G3y}\right) + m_4\left(a_{G4x}.v_{G4x} + a_{G4y}.v_{G4y}\right)\right] + \\
&\left[I_{G2}\alpha_2.\omega_2 + I_{G3}\alpha_3.\omega_3 + I_{G4}\alpha_4.\omega_4\right]
\end{aligned}$$

Substituting the accelerations, velocities, external forces and external torques into the above equation, we have

$$\begin{aligned}
&\left[(0) + \{(60).(-5.32) + (103.9).(3.73)\} + (0)\right] + \left[T_2.(20) + (0).\omega_3 + (-200).(8.4)\right] \\
&= \begin{bmatrix} (0.5)\{(1.20).(-7.5) + (-68.59).(0)\} + (2.5)\{(-24.17).(-6.23) + (-84.3).(3.6)\} + \\ (2.0)\{(-32.07).(-3.94) + (-16.34).(-0.69)\} \end{bmatrix} + \\
&\left[(0.4)(-40).(20) + (1.2)(125)(.-6.2) + (2.0)(280).(8.4)\right]
\end{aligned}$$

$$T_2 = \frac{4954.13}{20} \qquad \underline{\underline{T_2 = 247.7 \text{ N - m}}}$$

Unit Summary

Methods

There are several methods that may be employed for forces analysis, depending on the degree of complexity, input information, assumptions made, and output desired. These methods are either based on the Newton's laws of motion or the energy method.

Free-Body Diagram

Drawing of free-body diagrams are required for determine both internal and external forces using Newton's laws of motion. It is a sketch of part or group of parts, isolated from its supports and other bodies

Principle of Superposition

Use this tool for analysis systems with two or more loads. Each load is applied to the linkage separately and its effect on each joint and/ or member are determined. Then, for each joint/member, the net effect of all the loads is the sum of the individual separate effects.

Equilibrium Conditions

Equilibrium conditions are applicable to static loading: Velocity equal to zero or very slow motion, or constant velocity. For a body or a group of bodies to be in a static equilibrium the vector sum of all forces acting on it (along any direction) must be equal to zero and the vector sum of moments about any arbitrary point must be equal to zero. In mathematical form, the equilibrium conditions are

$$\sum F = 0 \qquad\qquad \sum M = 0$$

Newton's Second Law of Motion (or Non-Equilibrium Condition)

For a body or a group of bodies to be in a static equilibrium the vector sum of all forces acting on it (along any direction) must be equal to the product of mass of the body and acceleration of the centre of mass along that direction zero and the vector sum of moments about any arbitrary point must be equal to the product moment of inertia about the point of moment and angular acceleration.In mathematical form, it states that:

$$\sum F = ma \qquad\qquad \sum M = I_G \alpha$$

Static Force Analysis

Decide the method to be used: Newtonian method with free-body diagram or energy method

For energy method, assume virtual movement and sum of virtual work equals zero. Solve for the unknown forces or/and torques. This method cannot be used to determine internal loads and it is recommended problems with more than two unknowns or machine design where internal loads are to be determined.

To use Newtonian method, first draw free-body diagrams and apply equilibrium conditions to determine the unknowns. For a complicated linear problem the method superposition be employed.

Dynamic Force Analysis

Perform kinematic analysis to determine linear and angular accelerations of parts. Then, draw free-body diagrams and apply Newton's second law of motion to individual or group of bodies to determine the unknowns. If a linkage has more than four links, break it to a group of related sub-linkages.

☑ Key terms/ New Words in Unit

1. Equilibrium Condition
2. D'Alembert's principle
3. Newton's second law of motion
4. Assembly mode
5. Crank
6. Fixed link/ground link
7. Connecting Rod
8. Coupler

Self Assessment 4

4-1. For static equilibrium of the crank-slider linkage shown in Figure P4-1, determine the required input torque T if (a) $\theta = 60^o$ and e = 0, (b) $\theta = 60^o$ and e = 10 mm, and (c) $\theta = 60^o$, e = 10 mm and coefficient of static friction between block C and ground = 0.3.

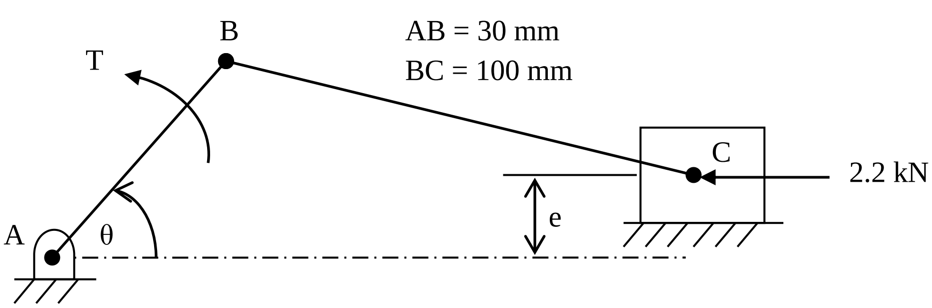


Figure P4-1

4-2. For a static equilibrium of the crank-slider linkage shown in Figure P4-2, determine the required input torque T_1 at the crank AB at $\theta = 60^o$.

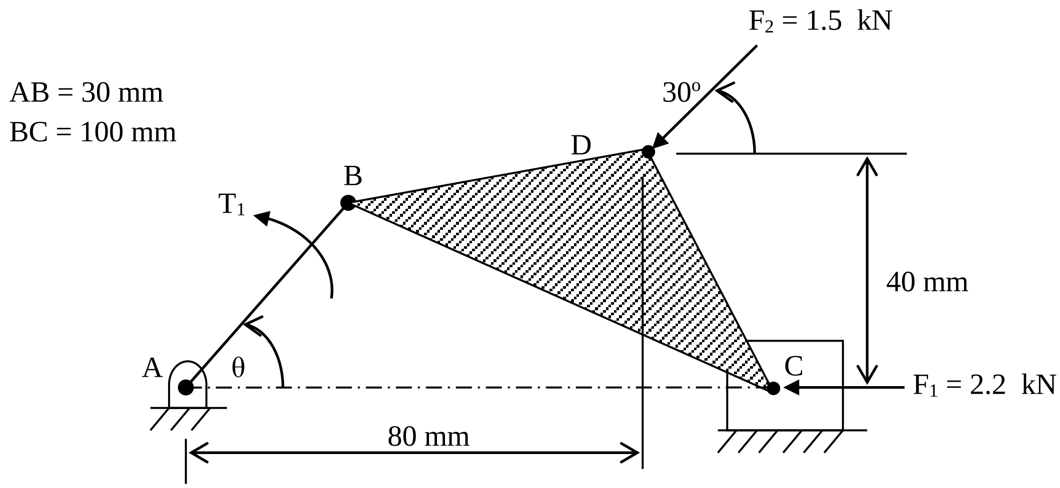


Figure 4-2

4-3. For a static equilibrium, determine the force required on the piston of the hydraulic actuator EF of the linkage shown in Figure P4-3.

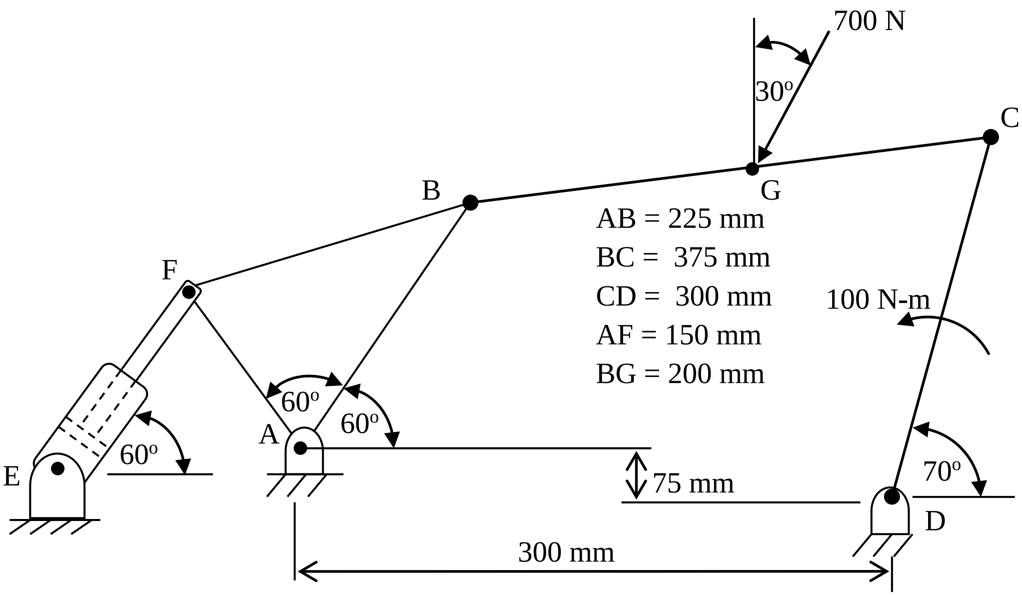


Figure P4-3

4-4. In Figure E4-4, the arm ABC of a model excavator has a mass of 1200 kg and moment of 3600 kg-m^2, calculated about its centre of mass G. At the instant shown, the hydraulic cylinder BD is vertical, and the angular velocity and acceleration of the arm are 3 rad/s and 1.2 rad/s^2, both counter clockwise. Determine the (a) force exerted by vertical hydraulic cylinder BD on the arm at B, and (b) the reaction at A.

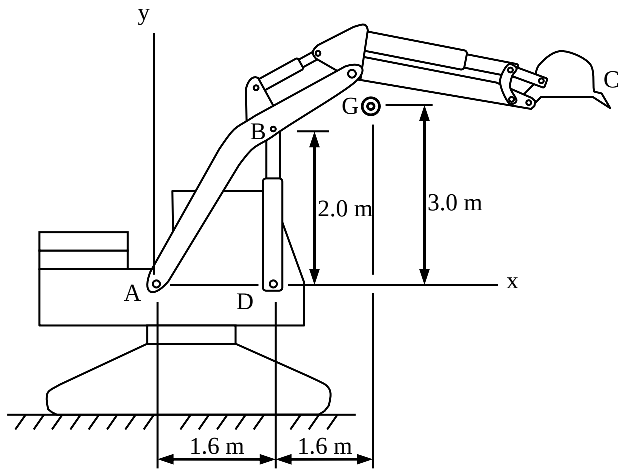


Figure E4-4

4-5. The slider-crank mechanism shown in Figure E4-5 has a constant angular velocity of 50 rad/s anti-clockwise. The crank has a mass of 0.5 kg and moment of inerta about its centre of mass of 3.0 x 10^{-4} kg-m^2. The connecting rod has a mass of 1.0 kg, with a mass-moment of inertia of about its centre of mass of 1.02 x 10^{-3} kg-m^2. The crank has a uniform cross section and the centre of mass of the connecting rod is at 1/3 of the length of the connecting rod from C. The piston has a weight of 0.7 kg. Determine all bearing forces and the required input torque T on crank AB for the position shown. The coefficient of dynamic friction between the piston and the ground is 0.25. Neglect friction at all pin joints.

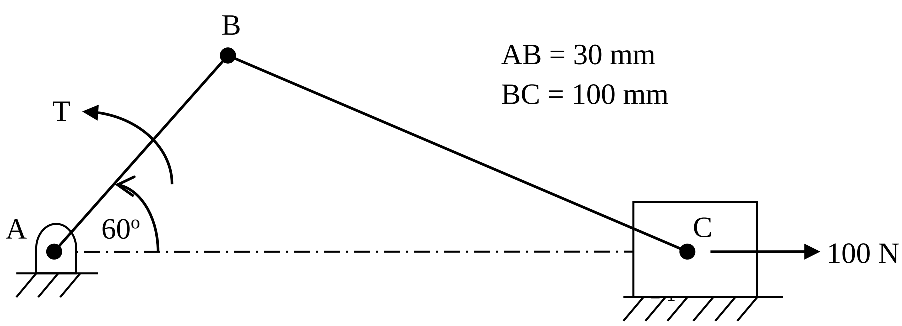


Figure E4-5

4-6. The four-bar linkage shown in Figure 4-6 has a input constant angular velocity of 200 rad/s clockwise. This results in the following accelerations: $a_{G3} = 863<298^o$ m/s^2, $\alpha_3 =$ 2670 rad/s^2 ccw, $a_{G4} = 365<294^o$ m/s^2, $\alpha_4 = 6940$ rad/s^2 cw. In the figure, the G refers to the centre of mass and the mechanism has the following mass properties:

$m_2 = 0.5$, $I_{G2} = 3.39 \times 10^{-3}$ kg-m^2
$m_3 = 1.2$, $I_{G3} = 6.78 \times 10^{-2}$ kg-m^2
$m_4 = 1.8$, $I_{G4} = 4.02 \times 10^{-3}$ kg-m^2

Determine the instantaneous value of drive torque T required to produce the assumed motion.

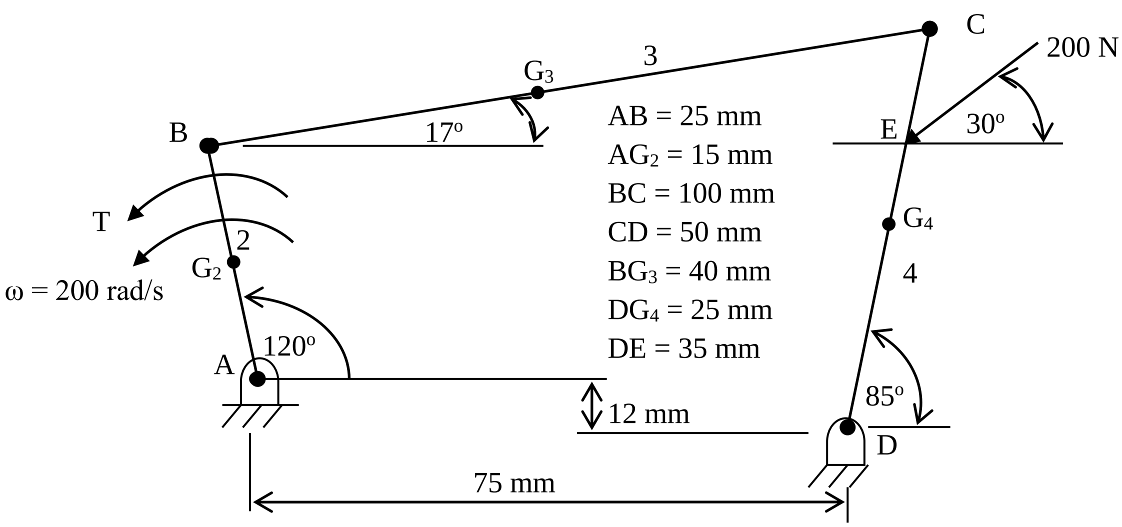


Figure E4-6

4-7. The crank and connecting rod of an internal combustion engine are 40 mm and 180 mm, respectively. The piston has a diameter of 80 mm and reciprocating mass of 1.2 kg. When the piston has moved 10 mm from the inner dead centre during the power stroke, the pressure in the combustion chamber is 0.8 N/mm^2. If the engine is running at 2000 rpm, determine the (a) net load on the gudgeon pin (i.e the pin connecting the piston and the connecting rod), (b) the trust in the connecting rod, (c) the normal reaction between the piston and the cylinder, (d) the speed at which the above values become zero.

4-8. Determine the Torque required to maintain a constant crank speed of 1000 rpm ccw for the two-cylinder engine shown in Figure E4-8. The individual pistons and connecting rods have masses of 1.0 kg and 0.8 kg, respectively. Consider the case where the V angle $\psi = 90^o$ and the crank angle is $\phi = 20^o$. The dimensions of mechanism are: AB = 60 mm BC = BD = 200 mm, and $BG_2 = BG_3 = 100$ mm. [Wilson & Sadler]

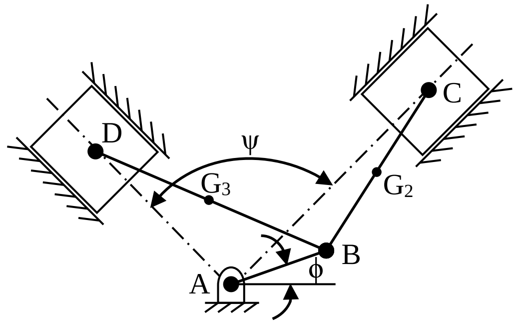


Figure E4-8

UNIT 5

BALANCING OF MACHINERY

Introduction

Unbalance linkages produce shaking forces and torques, as was learnt in Unit 4. In this unit, we learn about design linkages to eliminate shaking effects, which cause vibration in machine supports. The process of eliminating or minimizing shaking effects is called balancing of linkage. Balancing of linkages is more difficulty than balancing of rotors, and in many cases complete balance cannot be achieved by practical means.

Many methods have been developed to balance linkages. Some of the methods achieve a complete balance of one dynamic factor, such as shaking force at the expense of shaking torque, or vice versa. This typically involves adding counterweights or/and redistributing the masses of links to change the locations of the centres of mass. This unit covers balancing of four-bar linkage to balancing of multi-cylinder engines.

Complete balance of any mechanism may be obtained by connecting a mirror image mechanism to cancel the dynamic forces and moments. Certain configurations of multi-cylinder internal combustion (IC) engines and reciprocating compressors utilises this principle. As a result most multi-cylinder IC engines have even number of cylinders, such as four cylinders, six cylinders, eight cylinders, which are arranged such that shaking effect of cylinder cancels that of another cylinder.

Increase in pin force and weight also result in an increase in shaking torque compared to the value on unbalance linkage. Balancing by addition of weight will increase mass and mass moment of inertia. As a result, the torque required to drive the linkage will be greater.

Learning Objectives

After reading this unit you should be able to:

1. Determine shaking forces and shaking torques of linkages
2. Forces balancing of four-bar linkages
3. Determine Equivalent Lumped mass dynamic models of cranks and connecting rods
4. Reduce or balance single- and multi-cylinder Internal Combustion engines

Unit content

SESSION 5-1: Balancing of Linkages

5-1.1 Complete Force-Balance of Linkages

A rotating link of a linkage may be balanced using similar technique as the one for balancing rotating rotor. This requires making the centre of mass of the linkage stationary. This method works for any linkage having revolute (pin) and prismatic (slider) joints, provided that the prismatic joint is not connected to the ground. This means that this method is applicable to crank-slider machines such as internal combustion engines and reciprocating compressors.

Consider a four-bar linkage shown in Figure 37. In the figure CG and R, respectively, denotes centre of mass, and displacement of the centre of mass from the fixed revolute joint O_2. The subscript 1 to 4 and t denotes links 1 to 4 and the linkage. The total mass of the linkage is simply the sum of the masses of the individual links.

$$m_t = m_2 + m_3 + m_4 \qquad \textbf{Equation 172}$$

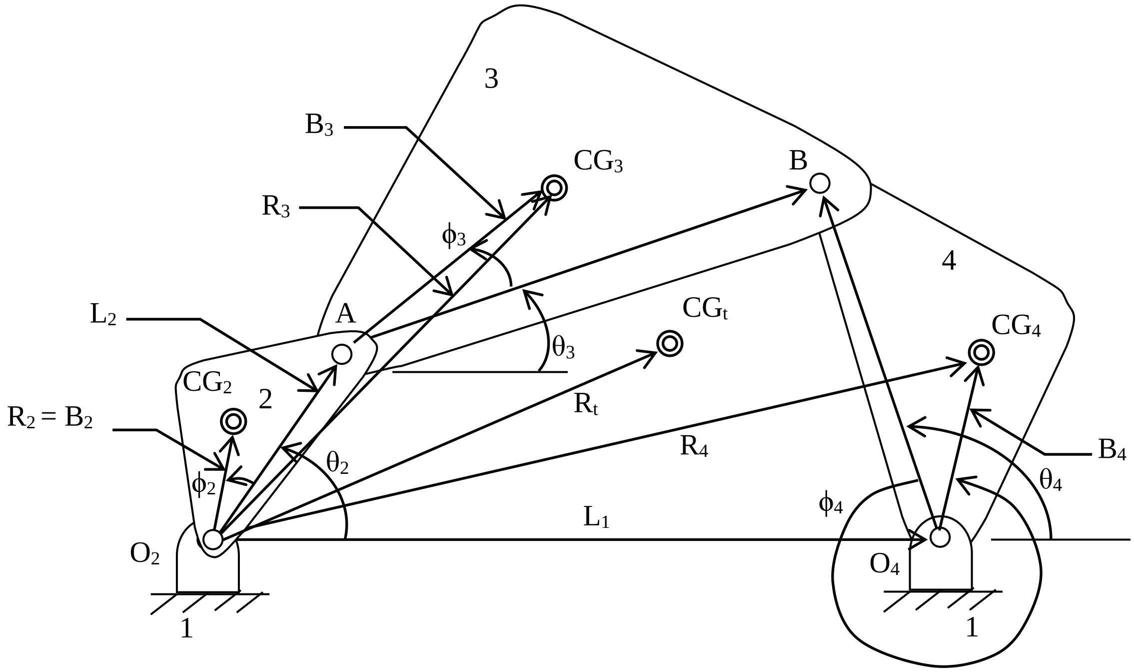


Figure 39: Force Balancing of Four-Bar Linkage

The total mass moment of the linkage about point O_2 is simply the sum of mass moments of masses of individual links about the same point.

$$m_t R_t = m_2 R_2 + m_3 R_3 + m_4 R_4 \qquad \textbf{Equation 173}$$

or

$$R_t = \frac{m_2 R_2 + m_3 R_3 + m_4 R_4}{m_t}$$

From the geometry in Figure 37 and using complex algebra, we have

$$R_2 = B_2 e^{j(\theta_2 + \phi_2)} = B_2 e^{j\theta_2} e^{j\phi_2} \qquad (1)$$

$$R_3 = L_2 e^{j\theta_2} + B_3 e^{j(\theta_3 + \phi_3)} = L_2 e^{j\theta_2} + B_3 e^{j\theta_3} e^{j\phi_3} \qquad (2)$$

$$R_4 = L_1 e^{j\theta_1} + B_4 e^{j(\theta_4 + \phi_4)} = L_1 e^{j\theta_1} + B_4 e^{j\theta_4} e^{j\phi_4} \qquad (3)$$

Substituting Equations (1) to (3) into Equation 160 yields

$$m_t R_t = m_2\left(B_2 e^{j\theta_2} e^{j\phi_2}\right) + m_3\left(L_2 e^{j\theta_2} + B_3 e^{j\theta_3} e^{j\phi_3}\right) + m_4\left(L_1 e^{j\theta_1} + B_4 e^{j\theta_4} e^{j\phi_4}\right) \qquad (4)$$

Using the vector loop equation for the linkage

$$L_2 e^{j\theta_2} + L_3 e^{j\theta_3} - L_4 e^{j\theta_4} - L_1 e^{j\theta_1} = 0$$

Links 2 and 4 are in pure rotational motion, while link 3 is in complex motion. The angular displacement of link 3 is derived from the above equation in terms parameters of links 2 and 4 as

$$e^{j\theta_3} = \frac{L_4 e^{j\theta_4} + L_1 e^{j\theta_1} - L_2 e^{j\theta_2}}{L_3} \tag{5}$$

Substituting Equation (5) into Equation (4), simplifying and isolating θ's, we have

$$\begin{aligned} m_t R_t = &\left(m_2 B_2 e^{j\phi_2} + m_3 L_2 - m_3 B_3 \frac{L_2}{L_3} e^{j\phi_3} \right) e^{j\theta_2} + \\ &\left(m_4 B_4 e^{j\phi_4} + m_3 B_3 \frac{L_4}{L_3} e^{j\phi_3} \right) e^{j\theta_4} + \\ &\left(m_4 L_1 + m_3 B_3 \frac{L_1}{L_3} e^{j\phi_3} \right) e^{j\theta_1} \end{aligned} \tag{6}$$

The terms in parenthesis and θ_1 of the above equation are constants ie time-independent, whereas the variables θ_2, θ_3 and θ_4 time-dependent. For the linkage to be force-balance, R_t must be constant, i.e. time-independent. For R_t to be constant, first and second terms in parenthesis of the above equation must be zero. That is

$$m_2 B_2 e^{j\phi_2} + m_3 L_2 - m_3 B_3 \frac{L_2}{L_3} e^{j\phi_3} = 0 \tag{7}$$

$$m_4 B_4 e^{j\phi_4} + m_3 B_3 \frac{L_4}{L_3} e^{j\phi_3} = 0 \tag{8}$$

From the above two equations, we have

$$m_2 B_2 e^{j\phi_2} = m_3 \left(B_3 \frac{L_2}{L_3} e^{j\phi_3} - L_2 \right) \tag{9}$$

$$m_4 B_4 e^{j\phi_4} = -m_3 B_3 \frac{L_4}{L_3} e^{j\phi_3} \tag{10}$$

Equations (9) and (10) involve three links and are in vector form. To change the centre of mass of the linkage it is easy to add counterweights to links 2 and 4, which are pure rotation. The above two equations may be written scalar form as

$$\left(m_2 B_2 \right)_x = m_3 \left(B_3 \frac{L_2}{L_3} \cos\phi_3 - L_2 \right)$$ **Equation 174**

$$(m_2 B_2)_y = m_3\left(B_3 \frac{L_2}{L_3}\sin\phi_3\right) \qquad \textbf{Equation 175}$$

$$(m_4 B_4)_x = -m_3 B_3 \frac{L_4}{L_3}\cos\phi_3 \qquad \textbf{Equation 176}$$

$$(m_4 B_4)_y = -m_3 B_3 \frac{L_4}{L_3}\sin\phi_3 \qquad \textbf{Equation 177}$$

Equations 160 to 164 are used to force balance a four-bar linkage without prismatic joint to the ground. Normally, links 3 is designed first, and then the above equations are used to size links 2 and 4. If links 2 and 4 have been designed, a product of mR is added or subtracted in order to satisfy the above four equations.

Example 5- 1

The masses of uniform cross section links 2, 3 and 4 shown Figure E5-1 are 0.5 kg, 1.0 kg and 1.2 kg, respectively. The angles between lines of centres and centres of mass of links 2, 3 and 4 are 7^o, 10^o and -3^o, respectively. For complete force balance, determine the masses that must attached to link 2 at radius 0.6 m from the fixed joints O_2 and link 4 at radius 1.0 m from O_4, respectively. Also, determine the orientation of the centres of mass of the masses from their respective lines of centres of joints.

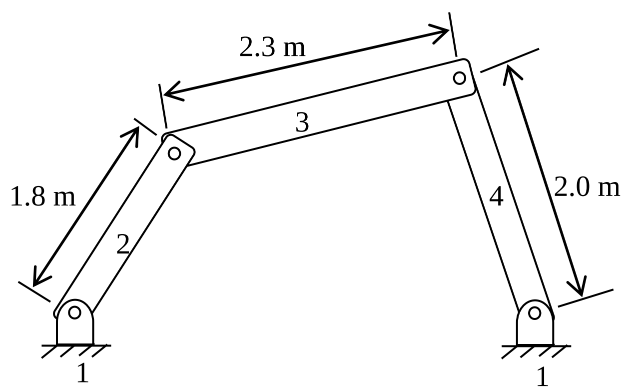


Figure E5-1

Solution

The givens are:

$L_2 = 1.8\,\text{m}$ $\qquad$ $L_3 = 2.3\,\text{m}$ $\qquad$ $L_4 = 2.0\,\text{m}$

$B_2 = \frac{L_2}{2} = 0.9\,\text{m}$ $\quad B_3 = \frac{L_3}{2} = 1.15\,\text{m}$ $\quad B_4 = \frac{L_4}{2} = 1.0\,\text{m}$

$m_2 = 0.5\,\text{kg}$ $\quad m_3 = 1.0\,\text{kg}$ $\quad m_4 = 1.2\,\text{kg}$

$\phi_2 = 7^o,$ $\quad \phi_3 = 10^o,$ $\quad \phi_4 = 3^o$

Using Equations 160 to 164, we have

$$(m_2 B_2)_x = m_3\left(B_3 \frac{L_2}{L_3}\cos\phi_3 - L_2\right) = (1.0)\left[(1.15)\frac{(1.8)}{(2.3)}\cos(10) - 1.8\right] \qquad (m_2 B_2)_x = -0.9137\,\text{kg - m}$$

$$(m_2 B_2)_y = m_3\left(B_3 \frac{L_2}{L_3}\sin\phi_3\right) = (1.0)\left[(1.15)\frac{(1.8)}{(2.3)}\sin(10)\right] \qquad (m_2 B_2)_y = 0.1563\,\text{kg - m}$$

$$(m_4 B_4)_x = -m_3 B_3 \frac{L_4}{L_3}\cos\phi_3 = -(1.0)(1.15)\frac{(2.0)}{(2.3)}\cos(10) \qquad (m_4 B_4)_x = -0.9848\,\text{kg - m}$$

$$(m_4 B_4)_y = -m_3 B_3 \frac{L_4}{L_3}\sin\phi_3 = -(1.0)(1.15)\frac{(2.0)}{(2.3)}\sin(10) \qquad (m_4 B_4)_y = -0.1736\,\text{kg - m}$$

Let m_{2a} and m_{4a} , and m_{2c} and m_{4c} be masses that should be attached to and original masses of links 2 and 4, respectively. Also, let ϕ_{2a} and ϕ_{4a} be orientation of the attached masses from the centre lines of links 2 and 4, respectively. Then

$$(m_{2a}B_{2a})\cos\phi_{2a} + (m_{2c}B_{2c})\cos\phi_{2c} = (m_2 B_2)_x$$

$$(m_{2a}B_{2a})\cos\phi_{2a} + [(0.5)(0.9)\cos(7)] = -0.9137 \qquad (m_{2a}B_{2a})\cos\phi_{2a} = -1.3603$$

$$(m_{2a}B_{2a})\sin\phi_{2a} + (m_{2c}B_{2c})\sin\phi_{2c} = (m_2 B_2)_y$$

$$(m_{2a}B_{2a})\sin\phi_{2a} + [(0.5)(0.9)\sin(7)] = 0.1565 \qquad (m_{2a}B_{2a})\sin\phi_{2a} = 0.10166$$

$$m_{2a}B_{2a} = \sqrt{(m_{2a}B_{2a}\cos\phi_{2a})^2 + (m_{2a}B_{2a}\sin\phi_{2a})^2} = \sqrt{(-1.3603)^2 + (0.10166)^2}$$

$$m_{2a}B_{2a} = m_{2a}(0.6) = 1.3641\,\text{kg - m} \qquad \underline{\underline{m_{2a} = 2.2735\,\text{kg}}}$$

Since $(m_{2a}B_{2a})\cos\phi_{2a}$ is negative and $(m_{2a}B_{2a})\sin\phi_{2a}$, ϕ_{2a} lies within $90 < \phi_{2a} < 180$

$$\phi_{2a} = 180 - \tan^{-1}\left(\left|\frac{(m_{2a}B_{2a})\sin\phi_{2a}}{(m_{2a}B_{2a})\cos\phi_{2a}}\right|\right) = 180 - \tan^{-1}\left(\left|\frac{0.10166}{-1.3603}\right|\right)$$

$\underline{\underline{\phi_{2a} = 175.7^o}}$ to the line centres of link 2

Similarly for link 4, we have

$$\left(m_{4a}B_{4a}\right)\cos\phi_{4a}+\left(m_{4c}B_{4c}\right)\cos\phi_{4c}=\left(m_4B_4\right)_x$$

$$\left(m_{4a}B_{4a}\right)\cos\phi_{4a}+\left[(1.2)(1.0)\cos(-3)\right]=-0.9848 \qquad \left(m_{4a}B_{4a}\right)\cos\phi_{4a}=-2.1832$$

$$\left(m_{4a}B_{4a}\right)\sin\phi_{4a}+\left(m_{4c}B_{4c}\right)\sin\phi_{4c}=\left(m_4B_4\right)_y$$

$$\left(m_{4a}B_{4a}\right)\sin\phi_{4a}+\left[(1.2)(1.0)\sin(-3)\right]=-0.1736 \qquad \left(m_{4a}B_{4a}\right)\sin\phi_{2a}=-0.1108$$

$$m_{4a}B_{4a}=\sqrt{\left(m_{4a}B_{4a}\cos\phi_{4a}\right)^2+\left(m_{4a}B_{4a}\sin\phi_{4a}\right)^2}=\sqrt{(-2.1832)^2+(0.1108)^2}$$

$$m_{4a}B_{4a}=m_{4a}(1.0)=2.186\ \text{kg - m} \qquad m_{4a}=2.186\ \text{kg}$$

Since both $\left(m_{4a}B_{4a}\right)\cos\phi_{4a}$ and $\left(m_{4a}B_{4a}\right)\sin\phi_{4a}$ are negative, ϕ_{4a} lies within $180<\phi_{2a}<270$

$$\phi_{2a}=180+\tan^{-1}\left(\left|\frac{\left(m_{4a}B_{4a}\right)\sin\phi_{4a}}{\left(m_{4a}B_{4a}\right)\cos\phi_{4a}}\right|\right)=180+\tan^{-1}\left(\left|\frac{-0.1108}{-2.1832}\right|\right)$$

$\underline{\underline{\phi_{2a}=182.9^o}}$ to the line centres of link 4

The location of masses to be attached is illustrated in Figure S5-1

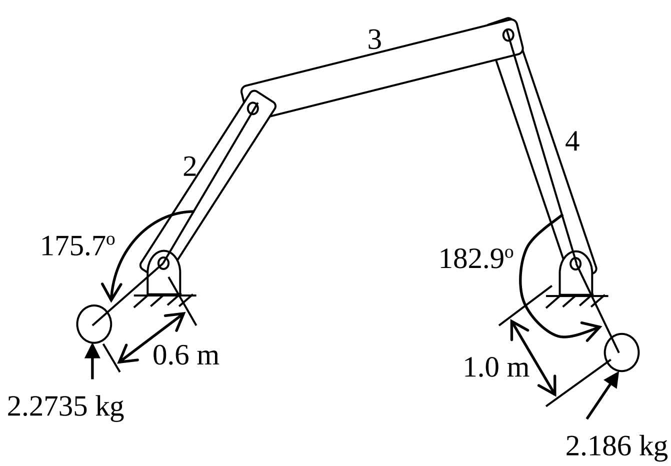


Figure S5-1

5-1.2 Effects of Force Balance on Shaking and Pin Forces, and Input Torque

Force balance essential reduces shaking force to zero. However, the pin forces and shaking torque do not disappear as result of force balance. These pin forces can be larger due to addition of counterweights to the crank and the rocker. Increase in pin forces and weight of linkages also result in an increase shaking torque compared to the value on unbalance linkage. Force balancing by addition of weight increases the total mass and total mass moment of inertia of a linkage. As a result, the required driving torque generally increases.

5-1.3 Balancing Shaking Moment in Linkages

The shaking moment in a linkage is the sum of moment of the shaking forces about a plane of the cylinders and is given as

$$M_s = \sum_{i=1}^{n} z_i F_{s_i} \qquad \textbf{Equation 178}$$

Where z is the perpendicular axial distance from the plane of reference to the cylinder, F_s is shaking force and the subscript i denote cylinder number.

Many techniques have been developed that utilise optimization to find mass configuration that would give minimum shaking moment. Most of these techniques required significant numerical analysis, which is beyond the scope of this course.

SESSION 5-2: Lumped-Mass Modeling of Crank-Slider Mechanisms

In a crank-slider mechanism, the crank exhibits pure rotational motion while the slider performs pure translational motion. Pure rotational and translational motions are relatively simple motions to analyse. However, the connecting rod exhibits complex motion, except the ends. Figure 38 shows a typical connecting rod. In the figure, B is the crankpin that connects crankshaft, and A is wrist pin that is connected to a slider. The end A exhibits translational motion and the end B performs rotational motion.

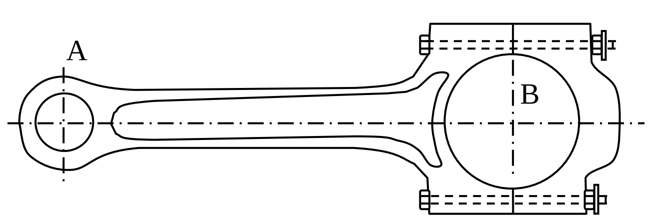


Figure 40: A Sketch of Connecting Rod

5-2.1 Lumped-Mass Dynamic Model of Connecting Rod

The motion of the connecting rod could be simplified as pure rotational or translational motions. This can be accomplished by modelling the mass of the connecting rod as two lumped point masses concentrated at crank pin and wrist pin, as shown in Figure 39. The lumped mass at the crank pin, point B, would perform rotational motion while the lumped mass at wrist pin, point A, would perform translational motion. It is assumed that these point masses have no size and are connected with a massless rigid rod. The model and actual connecting rods would by dynamically equivalent if the following three conditions are satisfied:

1. The centres of mass must be at the same location, point CG_2.
2. The total mass must be equal.
3. The moment of inertia with respect to the centre of mass must be equal.

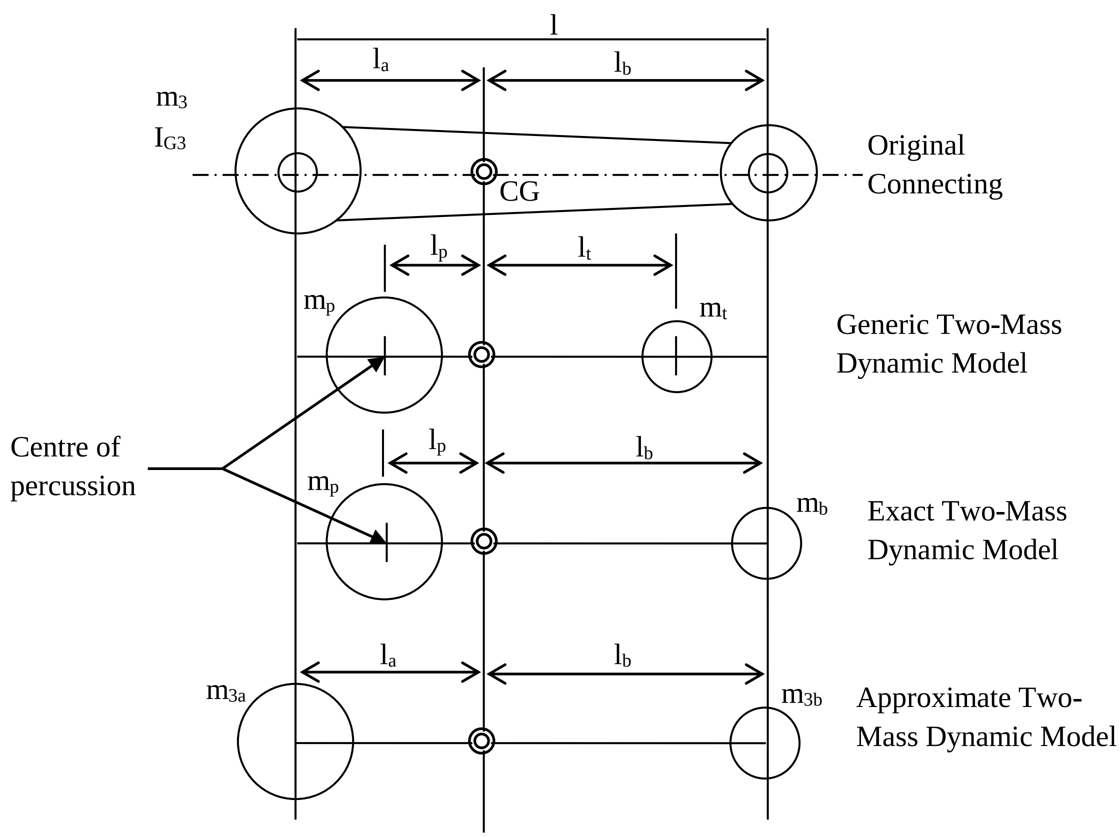


Figure 41: Lumped-Mass Dynamic Model of a Connecting Rod

Based on the above requirements, we have

$$m_p l_p = m_t l_t \qquad \textbf{Equation 179}$$

$$m_p + m_t = m_3 \qquad \textbf{Equation 180}$$

$$m_p l_p^{\,2} + m_t l_t^{\,2} = I_{G3} \qquad \textbf{Equation 181}$$

Equations 166 to 168 have four unknowns, which can be solved completely if a value for any one of the variables is chosen. By letting $l_t = l_b$, the solution of Equations 166 and 168 are

$$m_p = m_3 \frac{l_b}{l_p + l_b} \qquad \textbf{Equation 182}$$

$$m_b = m_3 \frac{l_p}{l_p + l_b} \qquad \textbf{Equation 183}$$

Substituting Equations 169 and 170 into Equation 168, we have

$$\left(m_3 \frac{l_b}{l_p + l_b} \right) l_p^{\,2} + \left(m_3 \frac{l_b}{l_p + l_b} \right) l_b^{\,2} = I_{G3}$$

which simplifies to

$$l_p = \frac{I_{G3}}{m_3 l_b} \qquad \textbf{Equation 184}$$

Or

$$l_p = \frac{k_{G3}^{\,2}}{l_b}$$

The above equation also defines the centre of percussion of the connecting rod, and its geometric relationship to the corresponding centre of rotation. Thus, the second lumped mass m_p must be placed at the centre of percussion P **(*refer to Subsection 2-3.6*)** using point B as its centre of rotation to obtain the exact two-mass dynamic equivalent. A typical connecting rod, as shown in Figure 38, is large at crank pin B and small at the wrist pin end, A. This put the centre of mass close to the crank pin. For this reason, the second lumped mass can be placed at end A with relatively small error in accuracy of the dynamic model. That is, $l_p = l_a$. Equations 169 and 170 become

$$m_{3a} = m_3 \frac{l_b}{l_a + l_b} \qquad \textbf{Equation 185}$$

$$m_{3b} = m_3 \frac{l_p}{l_a + l_b} \qquad \textbf{Equation 186}$$

These two equations define the amounts of the total lumped masses to place at each end of the massless rod to approximately and dynamically equal to the actual connecting rod. In the absence of any data on the geometry of the connecting rod at the beginning of a design, the rule of thumb is to place two-thirds of the mass of the connecting rod at the crank pin and one-third at the wrist pin.

Example 5- 2

The distance between the connecting centres of a connecting rod of a single-cylinder two-stroke internal combustion engine of mass 8 kg is 240 mm. The mass moment of inertia of the connecting rod about an axis through the centre of mass is 6500 kg-mm^2, and its centre of mass is 90 mm from its large end centre. If one of the masses is to be located at the small end centre, determine the dynamical equivalent of the two lumped masses of the connecting rod.

Solution

$$m_{3a} = m_3 \frac{l_b}{l_a + l_b} = (8)\frac{(240-90)}{240} \qquad \underline{\underline{m_{3a} = 5\,\text{kg}}}$$

$$m_{3b} = m_3 \frac{l_a}{l_a + l_b} = (8)\frac{90}{240} \qquad \underline{\underline{m_{3b} = 3\,\text{kg}}}$$

The original connecting rod and the two-mass model of this example are illustrated in Figure S5-2.

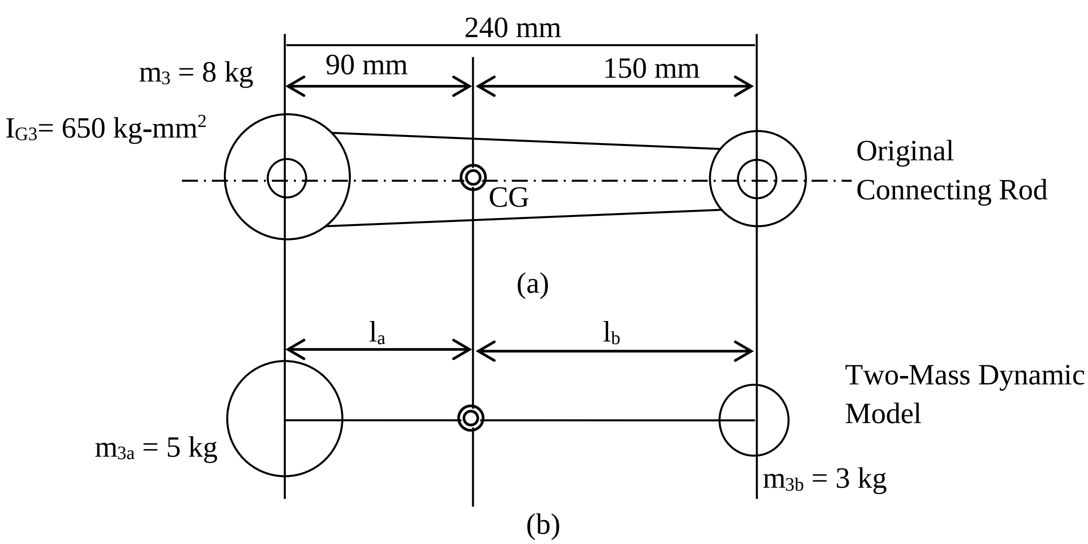


Figure S5-2

5-2.2 Lumped-Mass Dynamic Model of Crank

A similar lumped mass model can be developed for cranks. The principal concern is with a steady state analysis where its angular velocity is constant. Thus, the dynamic analysis of such crank reduces to static analysis. For static equivalent, the following requirements must be satisfied:

1. The mass of the model must equal to that of the original crank.
2. The centre of gravity must be the same location as that of the original crank

Consider a crank of mass m_2, centre-to-centre distance r and centre of mass at r_{G2} from the fixed point O_2, as shown in Figure 40. Based on the requirements listed above, we have

$$m_{2a} + m_{2O2} = m_2 \quad \textbf{Equation 187}$$

$$m_{2a} r = m_{202} r_{G2} \quad \text{or} \quad m_{2a} = \frac{m_{202} r_{G2}}{r} \quad \textbf{Equation 188}$$

Substituting Equation 175 into 174 and making m_{2O2} the subject yields

$$m_{2O2} = m_2 \left(1 - \frac{r_{G2}}{r} \right) \quad \textbf{Equation 189}$$

The lumped masses m_{2a} and m_{2O2} are placed at points A and O_2, respectively, to represent the unbalanced crank. From the models of the connecting rod and the crank, the mass at points A, B and O_2 are the sum of that of the crank and the connecting rod at those points, which are:

$$m_A = m_{2a} + m_{3a} \quad \textbf{Equation 190}$$

$$m_B = m_{3b} + m_4 \quad \textbf{Equation 191}$$

$$m_{O2} = m_{2O2} \quad \textbf{Equation 192}$$

In Equation 178, m_4 is the mass of slider (or link 4).

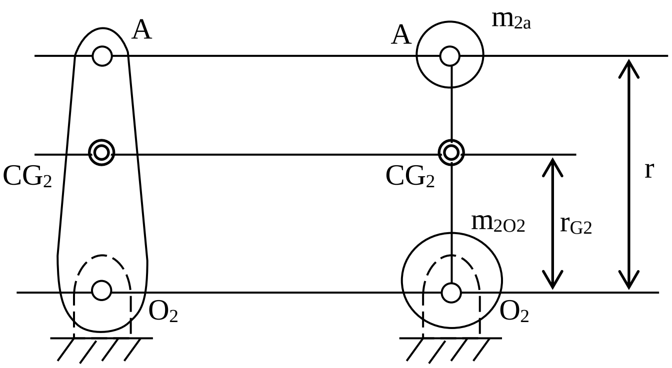


Figure 42: Lumped Mass Dynamic Model of Crank

5-2.3 Shaking Force and Torques in Reciprocating Machines

The lumped mass models developed for crank and connecting rod in Sections 5-1.6 can be used to develop expressions for the inertia forces and inertia torques, which were developed in Section 4-3.3. The lumped mass model of crank-slider mechanism and its corresponding free-body diagrams of the lumped mass crank and connecting rod are shown in Figure 41 and 42, respectively. The position vector of point A is

$$r_A = r\cos\theta\ \mathrm{i} + r\sin\theta\ \mathrm{j} = r\cos\omega t\ \mathrm{i} + r\sin\omega t\ \mathrm{j}$$

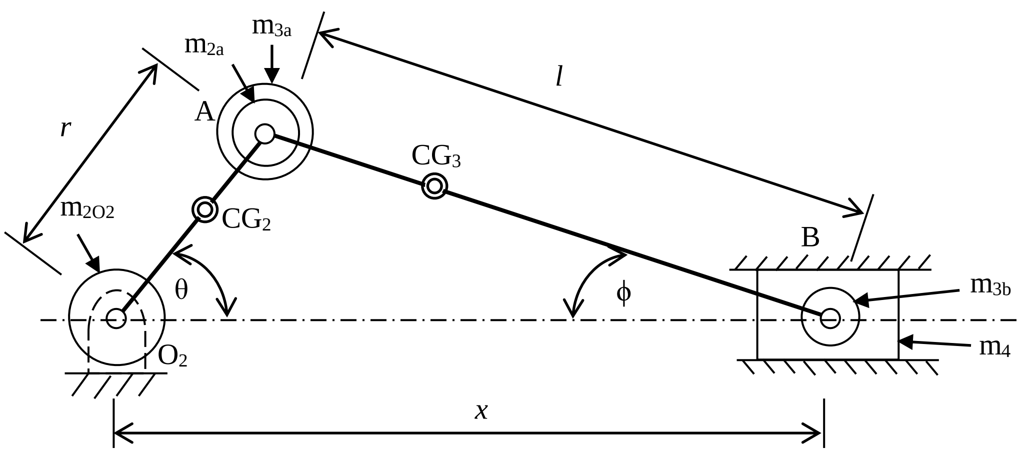


Figure 43: Lumped-Mass Dynamic Model of Crank-Slider Mechanism

Differentiating the above position vector of point A twice with respect to time gives the acceleration of point A. Assuming a constant crankshaft angular velocity ω, the angular acceleration of point A is

$$a_A = -\omega^2 r\left(\cos\omega t\ \mathrm{i} + \sin\omega t\ \mathrm{j}\right) \qquad \textbf{Equation 193}$$

From the geometry of the crank-slider mechanism shown in Figure 40, the position vector of the slider (i.e. point B) is

$$x = r\cos\omega t + l\sqrt{1-\left(\frac{r}{l}\sin\omega t\right)^2} \qquad \textbf{Equation 194}$$

Differentiating the above equation twice with respect to time and using binomial expansion, the acceleration of the slider is approximately equal to

$$\ddot{x} = a_B \cong -\omega^2 r\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) \qquad \textbf{Equation 195}$$

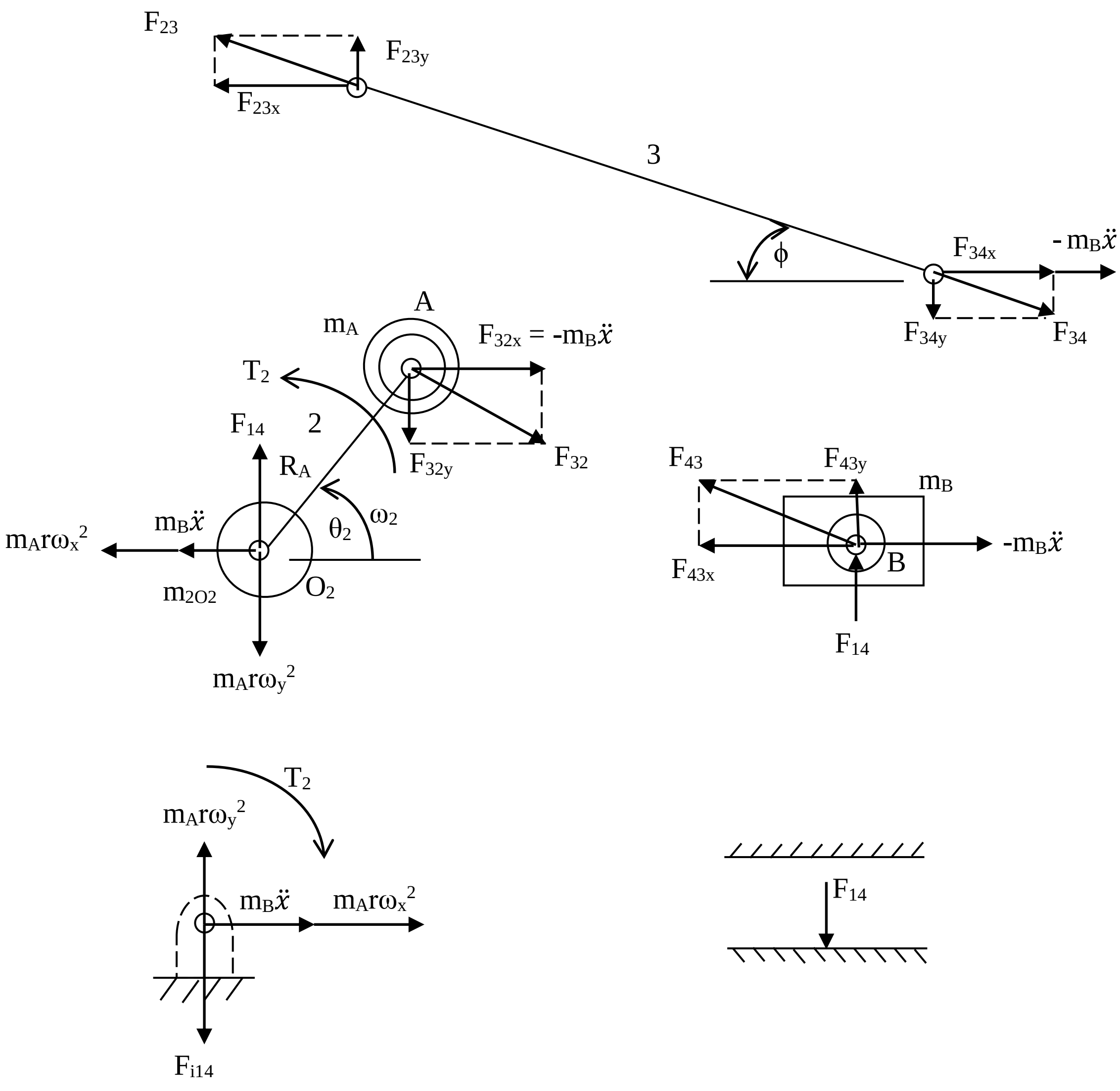


Figure 44: Free-Body Diagrams of Components of Lumped-Mass Model of Crank-Slider Mechanism

The shaking force, which is the sum of all the inertia forces acting on the ground, is given as

$$F_s = -m_A a_A - m_B a_B$$

$$F_s = m_A \omega^2 r(\cos\omega t\,\mathrm{i} + \sin\omega t\,\mathrm{j}) + m_B\,\omega^2 r\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right)\mathrm{i} \qquad \textbf{Equation 196}$$

In a scalar form, the x and y components of the shaking force are

$$F_{sx} = m_A \omega^2 r\cos\omega t + m_B\,\omega^2 r\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) \qquad \textbf{Equation 197}$$

$$F_{sy} = m_A \omega^2 r\sin\omega t \qquad \textbf{Equation 198}$$

Note that the gas force on the slider due to combustion in the case of internal combustion engines and compression in the case of reciprocating compressor does not contribute to the shaking force. Only the inertia and external forces contribute to the shaking force.

5-2.4 Shaking Torque

The shaking torque is equal to inertia torque from a linkage to the ground, which It is given by

$$T_s = \frac{1}{2} m_B r^2 \omega^2 \left(\frac{r}{2l}\sin\omega t - \sin 2\omega t - \frac{3r}{2l}\sin\omega t\right)\mathrm{k} \qquad \textbf{Equation 199}$$

5-2.5 Balancing of Single-Cylinder Machines

One method that is commonly used for partial force balancing of single-cylinder machines is addition of a mass at some radius to the crank. The counterweight m_c placed at radius r_c counteract the rotating unbalance due to the masses of the crank and rotating part of the connecting rod. Figure 43 shows a dynamic model of a crank-slider mechanism with counterweight. The mass-radius product $m_c r_c$ is determined such that the centre of mass of the lumped mass at A and mass of the counterweight m_c is at O_2. In IC engines and reciprocating machines, it is sensible to place the counterweight at a small radius to reduce space. The counterweight has to be kept close to the centreline to clear the piston at bottom-dead-centre (BDC).

Addition of counterweight reduces the equations of the horizontal and vertical components of the shaking force to

$$F_{sx} = m_A\omega^2 r\cos\omega t + m_B\,\omega^2 r\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) - m_c\omega^2 r_c\cos\omega t \qquad \textbf{Equation 200}$$

$$F_{sy} = m_A\omega^2 r\sin\omega t - m_c\omega^2 r_c\sin\omega t \qquad \textbf{Equation 201}$$

If $m_c r_c = m_A r$, the y component of the shaking force is completely eliminated. However, the x component is not completely eliminated, but reduces to

$$F_{sx} = m_B\,\omega^2 r\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) \qquad \textbf{Equation 202}$$

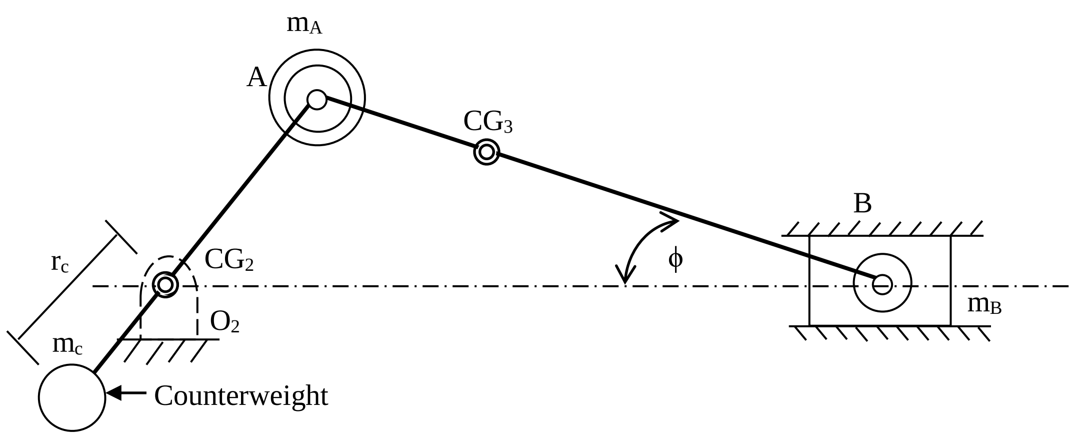


Figure 45: Lumped-Mass Dynamic Model of Single-Cylinder Crank-Slider Mechanism with Counterweight

The first and second terms of Equation 189 are the primary and secondary forces (or harmonics). Note this expression is an approximate equation, which excludes the third and higher harmonics. Clearly, the counterweight cannot eliminate the shaking force completely.

SESSION 5-3: Design Trade-Offs and Ratios

In the design of crank-slider machines, there are conflicting requirements, demands and desires, which must be traded-off to achieve optimum results. Some factors that affect the performance of crank-slider machines include crank/connecting rod ratio, bore to stroke ratio, materials and manufacturing processes.

5-3.1 Crank-Connecting Rod Ratio

The crank-connecting rod length ratio r/l appears in the equations of position (or displacement), velocity, acceleration, force and torque. Generally, smaller r/l ratio gives smooth acceleration function and all other functions related to acceleration. For acceptable transmission angle in a crank-slider linkage, the connecting rod length must be at least twice that of the crank. The ideal value of l/r from kinematic point of view is infinity, which would result in pure harmonic acceleration function of the piston (or slider). In a design of the ratio l/r must falls in a range that satisfies both transmission angle and smooth acceleration and its related functions. In most engine designs, the acceptable value of l/r ratio ranges from 3 to 5.

5-3.2 Bore-Stroke Ratio

In crank-slider machine, the bore diameter of is slightly bigger than the piston to allow piston movement. The stroke is the distance travelled by the piston, which is equal to the distance from Top Dead Centre (TDC) to the bottom Dead Centre (BDC). The stroke equals twice the crank length (i.e. S = 2r). The constant stroke volume is given by

$$V = \frac{\pi}{4} \mathrm{B}^2 S$$

A large bore and a small stroke would result in a large gas pressure on the piston (i.e. force = pressure x area) and piston pin. In contrast, a small bore and a large stroke would result in high inertia forces and torques that would adversely affect the piston pins. An B/S ratio is traded-off in order to minimizes gas pressure and inertia forces and torques. The bore-to-stroke ratio of most Internal Combustion (IC) engines ranges from 0.75 to 1.5.

5-3.3 Materials and Manufacturing Processes

The forces and torques in crank-slider machinery can be quite high. These forces and torques are due to explosion in IC engines and gas compression in compressors, and inertia of moving elements. Masses of moving elements must be kept as low as possible in order to reduce the inertia forces. Reducing masses require use of minimal amount of materials. However, the elements must enough strength to withstand the forces. As a result, materials with high strength/weight ratio are preferred. Pistons are usually made of cast or forged aluminium alloys, and connecting rods often made from cast iron or forged steel. However, connecting rods of very small engines such as lawn mowers, trimmers, chain saw and motorcycle are most often made aluminium alloys, whereas high-performance engines have titanium connecting rods. Crankshafts are usually cast iron or forged steel. Wrist and Piston pins are of hardened steel tubing or rod, and plain bearings of a special soft, nonferrous metal alloy called Babbitt. Four-stroke engine blocks are cast iron or cast aluminium alloy. The cylinder bore of most aluminium blocks are fitted with cast liners to reduce wear due to movement of chrome-plated steel piston rings.

SESSION 5-4: Balancing of Multicylinder Machines

Many applications of crank-slider mechanism such as IC engines, reciprocating compressors and pumps involve use of multiple crank-slider mechanisms that are synchronized to give large and smoother flow of fluid or power than can be accomplished by just a single cylinder machine. The multicylinder crank-slider machines are designed with a wide variety of arrangements from simple inline to Vee, opposed and radial configurations. For all configurations, cranks and connecting rods are same. The cranks are formed together with the common shaft, which is referred to as *crankshaft*. Figure 44 shows a schematic of a multicylinder Engine. Each crank is referred to as *crank throw*. The angle between a crank throw and a reference plane is known as *phase angle*, and the angle between two crank throws is called *delta phase angle*.

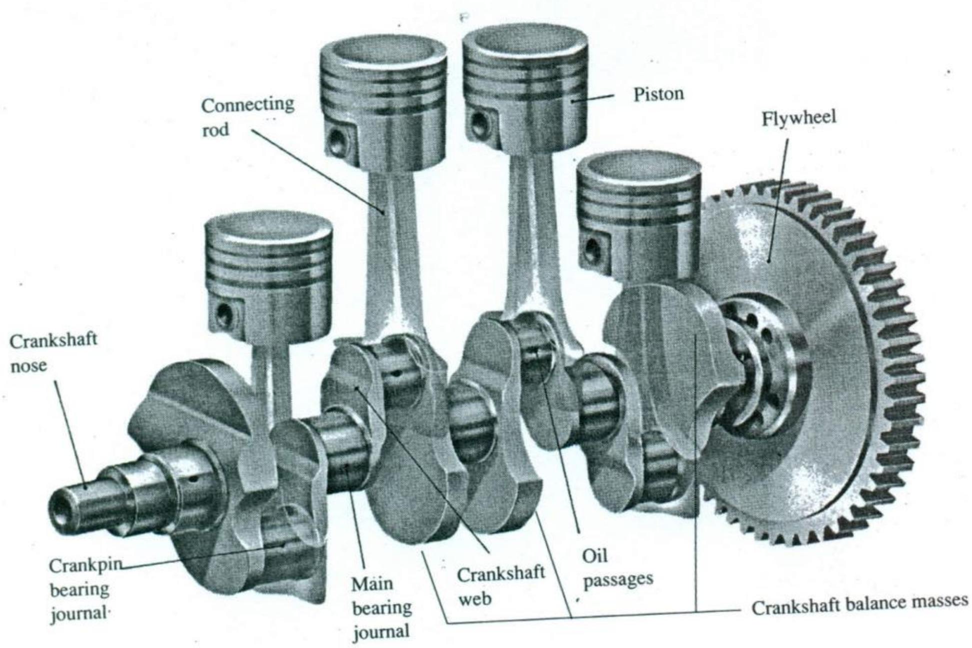


Figure 46: Multicylinder Engine Showing Dynamic Features

5-4.1 Shaking Force in Multicylinder Machines

Consider a general multicylinder configuration with a total number of N cylinders. For simplicity a three-cylinder configuration is illustrated in Figure 45. It is assumed that all the crank and connecting are the same in terms of length, mass, mass moment of inertia and location of centre of mass. The cylinder orientations are defined by angles ϕ_n, n = 1, 2 ….N, which are measured from the x-axis. The angular position of each crank throw is defined by θ_n, n = 1, 2, ….N, which are measured from the axis of the respective cylinder. Suppose that each crank throw is counter balance by product $m_A r_A$, which reduces the shaking force on each cylinder given by Equation 183 and rewritten for cylinder n as Equation 184. The line-of-action of each shaking force is along the centre line of the cylinder.

$$F_{sn} = m_B \omega^2 r \left[\cos(\omega t + \phi_n) + \frac{r}{l} \cos 2(\omega t + \phi_n) \right] \mathrm{i}$$ **Equation 203**

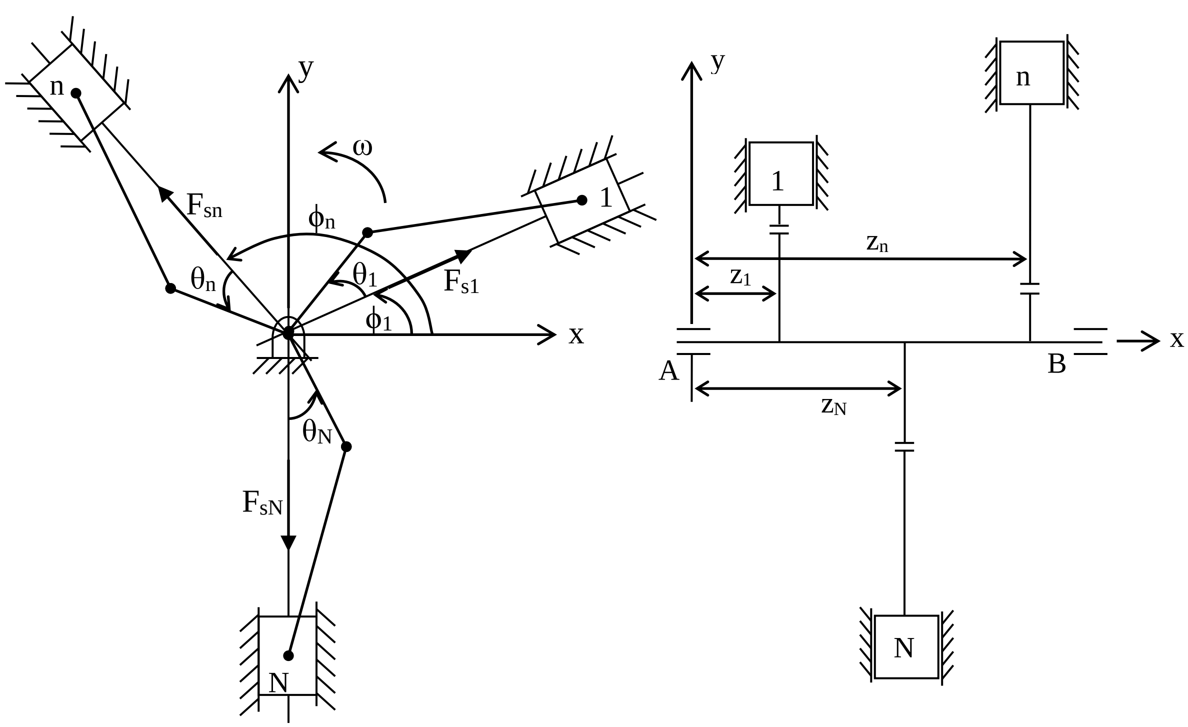
Expanding the above equation gives

$$F_{sn} = m_B \omega^2 r \left[(\cos\omega t \sin\phi_n - \sin\omega t \cos\phi_n) + \frac{r}{l} (\cos 2\omega t \sin 2\phi_n - \sin 2\omega t \cos 2\phi_n) \right] \mathrm{i}$$

The total shaking force F_s for the multicylinder machine is the sum of the individual shaking forces. That is

$$F_s = m_B \omega^2 r \left[\begin{array}{l} \left(\cos\omega t \sum_{n=1}^{N} \sin\phi_n - \sin\omega t \sum_{n=1}^{N} \cos\phi_n \right) + \\ \frac{r}{l} \left(\cos 2\omega t \sum_{n=1}^{N} \sin 2\phi_n - \sin 2\omega t \sum_{n=1}^{N} \cos 2\phi_n \right) \end{array} \right] \mathrm{i}$$ **Equation 204**

Since product ωt cannot be zero when the machine is running, the total shaking force can only be zero if the coefficients of the time-dependent sine and cosine functions are all zero. Based on this argument, the conditions for force balance are



(a) (b)

Figure 47: Schematic of Multicylinder Engine

$$\sum_{n=1}^{N} \sin\phi_n = 0 \qquad \sum_{n=1}^{N} cos\phi_n = 0 \qquad \sum_{n=1}^{N} \sin 2\phi_n = 0 \qquad \sum_{n=1}^{N} cos 2\phi_n = 0$$ **Equation 205**

The first two sub-equations of Equation 192 are conditions for balancing of primary forces, and the last two sub-equations are conditions for balancing of secondary forces.

5-4.2 Shaking Moment in Multicylinder Machines

Figure 45 (b) shows schematic of a general multicylinder machine showing location of cranks along the axis of the crank. Summing moments in the planes of the cylinders about any convenient point A, which is at one end of the crankshaft, gives

$$\sum M_A = \sum_{n=1}^{N} \mathrm{z_n} k \text{ x } F_{sn} i$$

$$M_s = m_B \omega^2 r \left[\begin{array}{l} \left(\cos\omega t \sum_{n=1}^{N} \mathrm{z_n} \sin\phi_n - \sin\omega t \sum_{n=1}^{N} \mathrm{z_n} \cos\phi_n \right) + \\ \frac{r}{l} \left(\cos 2\omega t \sum_{n=1}^{N} \mathrm{z_n} \sin 2\phi_n - \sin 2\omega t \sum_{n=1}^{N} \mathrm{z_n} \cos 2\phi_n \right) \end{array} \right] \mathrm{j}$$ **Equation 206**

The shaking moment can be only zero for all values of ωt if

$$\sum_{n=1}^{N} \mathrm{z_n} \sin\phi_n = 0 \qquad \sum_{n=1}^{N} \mathrm{z_n} cos\phi_n = 0 \qquad \sum_{n=1}^{N} \mathrm{z_n} \sin 2\phi_n = 0 \qquad \sum_{n=1}^{N} \mathrm{z_n} cos 2\phi_n = 0$$ **Equation 207**

Equation 188 will not guarantee zero shaking moments through the second harmonics.

5-4.3 Inline Configuration

The inline configuration is the most common and simplest configuration of multicylinder machine. Shown in Figure 46 is schematic of a six-cylinder inline IC engine configuration.

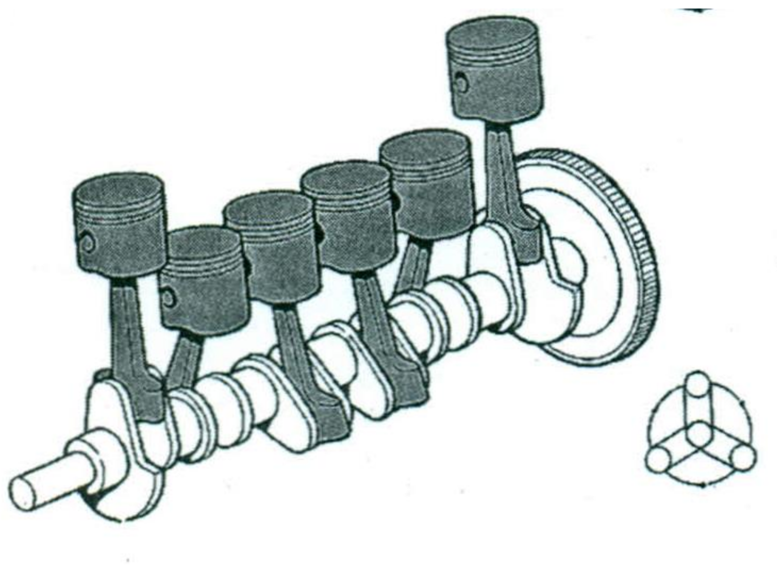


Figure 48: Inline Six-Cylinder Engine Configuration

Inline machines generally have even number of cylinders and the crank phase are arranged such that one cylinder is at 180^o to another one, as illustrated in Figure 47. As result, shaking force of one cylinder counterbalances that of the opposite cylinder. However, the shaking moments may not be balanced completely. From the figure, the arrangements of four-cylinder machines are arranged at 0^o, 90^o, 180^o and 270^o, which give delta phase angle of 90^o. These phase angles satisfy Equation 192 leading to a complete force-balance. However, this arrangement will not guarantee zero shaking moments through the second harmonics.

For four-stroke four-cylinder IC engine, there is power pulse in each cylinder every two revolutions. If the firing is even, then the delta phase angle of the crank throws is

$$\Delta\phi_{four\_stroke} = \frac{2(360^o)}{n} = \frac{2(360^o)}{4} \qquad \Delta\phi_{four\_stroke} = 180^o$$

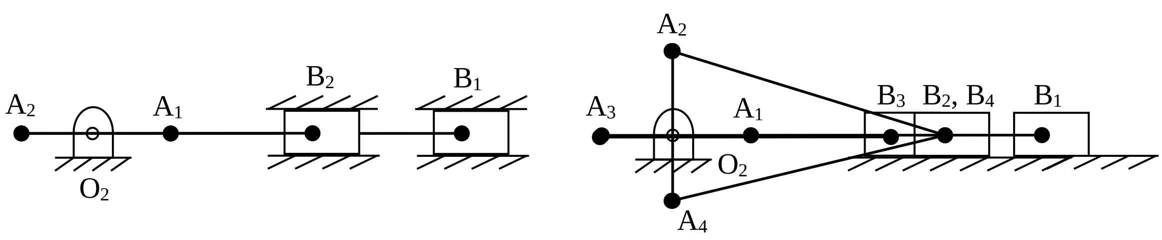


Figure 49: Geometry of (a) Two-Cylinder Inline, and (b) Four-Cylinder Inline

The primary and secondary harmonics of a multicylinder inline machine may be balanced completely using either the analytical and/or graphical methods**.** Both methods are based on the following two conditions:

1. The vector sum of primary forces must be equal to zero. For graphical method, the force polygon must be closed.
2. The vector sum of primary couples must be equal to zero. For graphical method, the couple polygon must be closed

Consider the axial and angular positions of an inline engine shown in Figure 48. In the figure, the plane A is the reference plane, and the angular positions are measured from the positive x-axis. Summing up forces along the x-and y-axes, we have

$$\sum_{i=11}^{n} m_{Bi} r_i \omega^2 \cos\theta_i = 0 \qquad \sum_{i=11}^{n} m_{Bi} r_i \omega^2 \sin\theta_i = 0$$

Dividing the above equations by the angular speed ω^2 yields

$$\sum_{i=11}^{n} m_{Bi} r_i \cos\theta_i = 0 \qquad \sum_{i=11}^{n} m_{Bi} r_i \sin\theta_i = 0 \qquad \textbf{Equation 208}$$

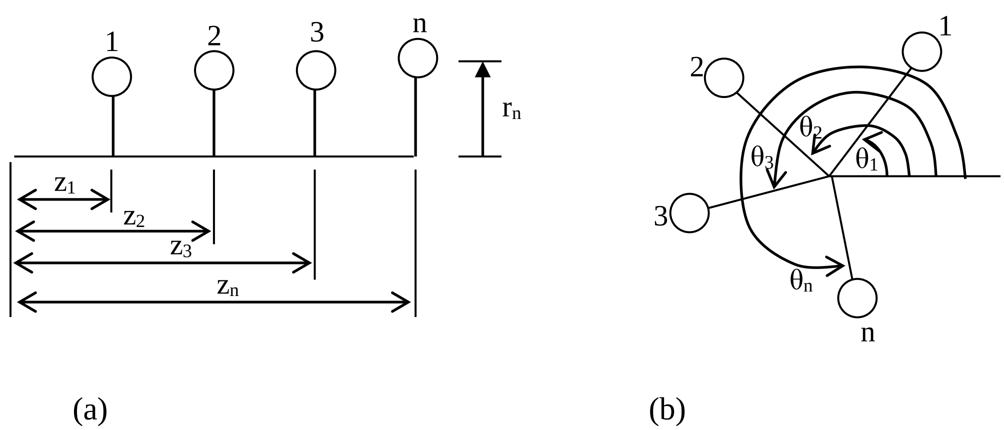


Figure 50: (a) Axial and (b) Angular Positions of Crank Throws

Summing up moments along the x-and y-axes about plane A, we have

$$\sum_{i=11}^{n} m_{Bi} r_i z_i \omega^2 \cos\theta_i = 0 \qquad \sum_{i=11}^{n} m_{Bi} r_i z_i \omega^2 \sin\theta_i = 0$$

Dividing the above equations by the angular speed ω^2 yields

$$\sum_{i=11}^{n} m_{Bi} r_i z_i \cos\theta_i = 0 \qquad \sum_{i=11}^{n} m_{Bi} r_i z_i \sin\theta_i = 0 \qquad \textbf{Equation 209}$$

Equations 195 and 196 are conditions for static equilibrium. For a graphical method, the force polygon which is equivalent to Equation 195 and the couple polygon which is equivalent to

Equation 196 must be closed. These equations are applicable to all inline machines even if the cranks are not the same. For the second harmonics, crank angles θ are replaced with 2θ in Equations 195 and 196.

Example 5- 3

A four-cylinder inline engine has cranks 150 mm long. The planes of rotation of the cranks are equally spaced at 200 mm interval staring from the first to the fourth. The reciprocating masses of the first, second and fourth cranks are 50 kg, 60 kg and 55 kg respectively. Find the mass of the reciprocating parts of the third cylinder and the angular positions of the cranks with respect to the first crank in order that engine may be in complete primary balance.

Analytical Solution

Shown in Figure S5-3 are axial positions and angular positions. The angular positions are measured from crank 1. Summing up moments about the plane of crank 3, we have

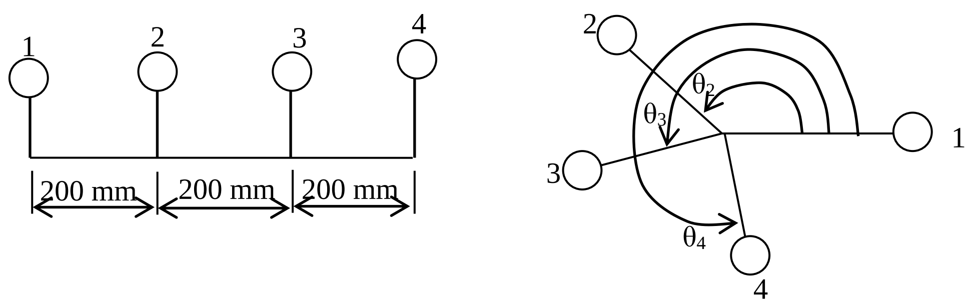


Figure E5-3: (a) Axial and (b) Angular Positions of Cranks

$$\sum_{i=11}^{n} m_{Bi} r_i z_i \cos\theta_i = 0$$

$$(50)(0.15)(-0.4)\cos 0 + (60)(0.15)(-0.2)\cos\theta_2 + (m_3)(0.15)(0)\cos\theta_3 + (55)(0.15)(0.2)\cos\theta_4 = 0$$

$$-12\cos\theta_2 + 11\cos\theta_4 = 20 \tag{1}$$

$$\sum_{i=11}^{n} m_{Bi} r_i z_i \sin\theta_i = 0$$

$$(50)(0.15)(-0.4)\sin 0 + (60)(0.15)(-0.2)\sin\theta_2 + (m_3)(0.15)(0)\sin\theta_3 + (55)(0.15)(0.2)\sin\theta_4 = 0$$

$$-12\sin\theta_2 + 11\sin\theta_4 = 0 \tag{2}$$

Solving equations (1) and (2) simultaneously,

$$\underline{\underline{\theta_2 = 151.9^o}} \qquad \underline{\underline{\theta_4 = 30.9^o}}$$

Summing up forces along the x and y axes, we have

$$\sum_{i=11}^{n} m_{Bi} r_i \cos\theta_i = 0$$

$$(50)(0.15)\cos 0 + (60)(0.15)\cos 151.9^o + (m_3)(0.15)\cos\theta_3 + (55)(0.15)\cos 30.9 = 0$$

$$m_3 \cos\theta_3 = -44.256 \tag{3}$$

$$\sum_{i=11}^{n} m_{Bi} r_i \sin\theta_i = 0$$

$$(50)(0.15)\sin 0 + (60)(0.15)\sin 151.9^o + (m_3)(0.15)\sin\theta_3 + (55)(0.15)\sin 30.9 = 0$$

$$m_3 \sin\theta_3 = -56.539 \tag{4}$$

Since both equations (3) and (4) are negatives, θ_3 lies in the third quadrant. Hence , the angular position of crank is

$$\underline{\underline{\theta_3 = 231.95^o}}$$

The reciprocating mass at crank 3 is

$$m_3 = \sqrt{(-44.256)^2 + (-56.539)^2} \qquad \underline{\underline{m_3 = 71.8\,\text{kg}}}$$

Graphical Solution

The method for balancing rotor is employed. Table S5-3 is created from the given data. Based on the table, the couple polygon shown in Figure S5-3 (a) is constructed with the data in mrz column. First, the moment of crank 1 given by -3 is constructed at 0^o. Because the moment has a negative value, the arrow points towards the negative x-direction.

A compass is placed at on ruler and end to measure 1.8 units taking into consideration the chosen scale. Then, the compass is placed at one end of the 3-unit moment to describe arcs above and below the 3-unit moment line. Similar arcs are described with the length equivalent to 1.65 units to intercept the 1.8-unit arcs. Since the moment of crank 2 has negative magnitude, the interception below the 3-unit crank arrow is accepted. From the couple polygon shown below,

$$\theta_2 = 151.9^o \qquad \theta_4 = 30.9^o$$

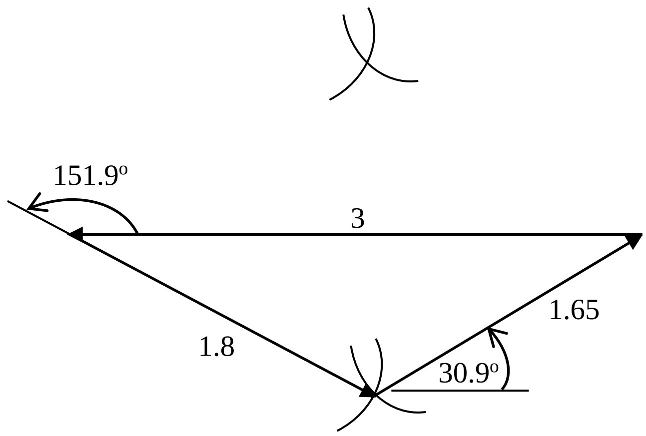


Figure S5-3(a): Couple Polygon

The force polygon is constructed with the data in mr column as shown in Figure S5-3(b). The closing side of the polygon, indicated with dotted line corresponds to the mr of crank 3. From the force polygon,

$$\underline{\underline{\theta_3 = 232^o}}$$

$$0.15m_3 = 10.68 \qquad \underline{\underline{m_3 = 71.2\,\text{kg}}}$$

Table S5-3

Crank	Reciprocating mass (m) kg	Crank Radius (r) m	mr	Axial Distance from crank 3	mrz
1	50	0.15	7.5	-0.4	-3
2	60	0.15	9.0	-0.2	-1.8
3	m_3	0.15	0.15 m_3	0	0
4	55	0.15	8.25	0.2	1.65

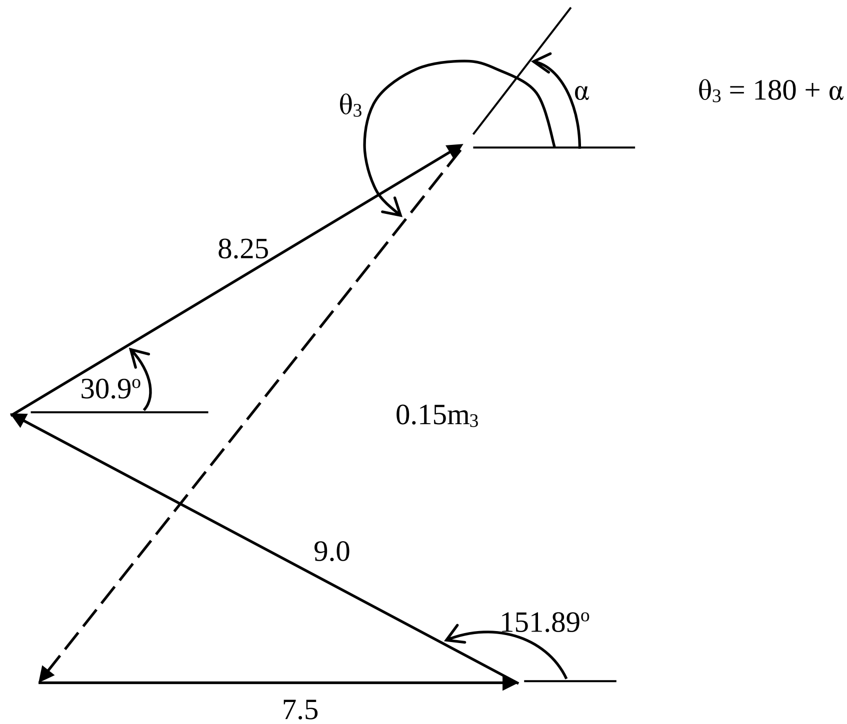


Figure S5-3(b): Force Polygons

Example 5- 4

The two outer cranks of a four-cylinder inline engine is set at 120° to each other and their reciprocating masses are each 20 kg. The distances between the planes of rotation of the cranks starting from the left are 250 mm, 550 mm and 450 mm. If the engine is to be complete primary balance, find the required reciprocating mass and relative angular position for each of the inner cranks.

If the length of each crank is 200 mm and that of each connecting rod is 1000 mm, find the maximum secondary shaking force when the speed of rotation is 300 rpm.

Solution

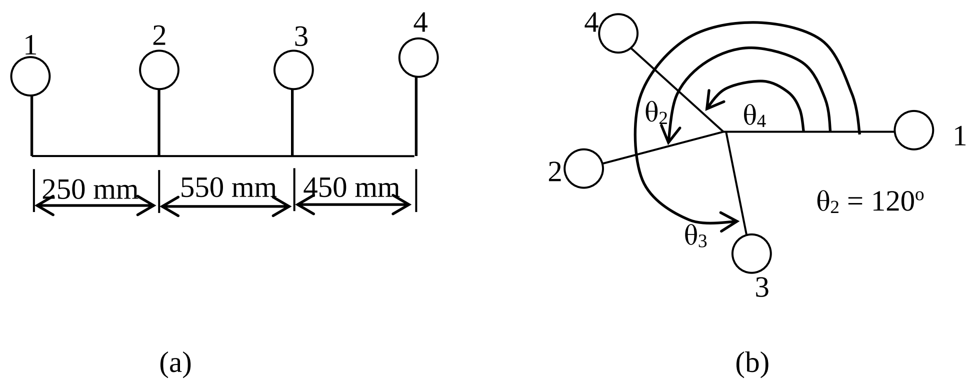


Figure S5-4: (a) Axial and (b) Angular Positions of Cranks

Taking moment about crank 2, we have

$$\sum_{i=11}^{n} m_{Bi} r_i z_i \cos\theta_i = 0$$

$$(20)(r)(-0.25)\cos 0^o + (m_2)(r)(0)\cos\theta_2 + (m_3)(r)(0.55)\cos\theta_3 + (20)(r)(1.0)\cos 120^o = 0$$

$$0.55 m_3 \cos\theta_3 = 5 \tag{1}$$

$$\sum_{i=11}^{n} m_{Bi} r_i z_i \sin\theta_i = 0$$

$$(20)(r)(-0.25)\sin 0^o + (m_2)(r)(0)\sin\theta_2 + (m_3)(r)(0.55)\sin\theta_3 + (20)(r)(1.0)\sin 120^o = 0$$

$$0.55 m_3 \sin\theta_3 = -17.32 \tag{2}$$

Squaring and adding equation (1) and (2) yield s

$$(0.55 m_3)^2 \left[(\sin\theta_3)^2 + (\cos\theta_3)^2\right] = (0.55 m_3)^2 = \left[(5)^2 + (-17.32)^2\right]$$

$$m_3 = \frac{1}{0.55}\sqrt{\left[(5)^2 + (-17.32)^2\right]} \qquad \underline{\underline{m_3 = 18.03\,\text{kg}}}$$

Equations (1) and (2) have negative and positive values, respectively. Hence the crank 3 is in the fourth quadrant with respect to crank 1. Hence

$$\theta_3 = 360^o - \tan^{-1}\left(\left|\frac{0.55\sin\theta_3}{0.55\cos\theta_3}\right|\right) \qquad \theta_3 = 360^o - \tan^{-1}\left(\left|\frac{-17.32}{5}\right|\right) \qquad \underline{\underline{\theta_3 = 286.1^o}}$$

The reciprocating mass of crank 2 and its angular position may be determined either by summing up force along the horizontal and vertical axes or summing up moments about crank 3. Summing up forces, we have

$$\sum_{i=11}^{n} m_{Bi} r_i \cos\theta_i = 0$$

$$(20)(r)\cos 0^o + (m_2)(r)\cos\theta_2 + (18.03)(r)\cos 286.1^o + (20)(r)\cos 120^o = 0$$

$$m_2 \cos\theta_2 = -15.0 \tag{3}$$

$$\sum_{i=11}^{n} m_{Bi} r_i \sin\theta_i = 0$$

$$(20)(r)\sin 0^o + (m_2)(r)\sin\theta_2 + (18.03)(r)\sin 286.1^o + (20)(r)\sin 120^o = 0$$

$$m_2 \sin\theta_2 \approx 0 \tag{4}$$

Since equation (3) is negative, and equation (4) equal to zero, θ_2 is equal to 180^o . That is

$$\theta_2 = 180^o \qquad\qquad m_2 = 15.0\,\text{kg}$$

The secondary shaking force is given by

$$F_s = \sum_{i=11}^{n} m_{Bi} r_i \omega^2 \left(\frac{r}{l}\right)\cos 2\theta_i = r\omega^2\left(\frac{r}{l}\right)\sum_{i=11}^{n} m_{Bi}\cos 2\theta_i$$

$$F_s = (0.2)\left(\frac{0.2}{1.0}\right)\left(300 \text{ x } \frac{2\pi}{60}\right)^2 \left[(20)\cos 2(0^o) + (15)\cos 2(180) + (18.03)\cos 2(286.1^o) + (20)\cos 2(120^o)\right]$$

$$F_s = 384.64\ \text{N}$$

Example 5- 5

The cranks and connecting rods of a five-cylinder two-stroke inline engine are 60 mm and 240 mm, respectively, and the pitch distance between the cylinder lines are 100 mm, 150 mm, 150 mm and 100 mm respectively. The reciprocating mass of each cylinder is 2 kg. The cylinders are numbered 1 to 5 in sequence from the left end, and the crank intervals are equal. If the firing order is 1-3-5-2-4 and crankshaft is rotating at 1800 rpm, determine the (a) unbalance primary and secondary forces, (b) unbalance primary and secondary couples with reference to the central plane of the engine.

Solution

For a two-stroke five-cylinder inline engine with equal firing order, the angle between cranks is given by

$$\theta = \frac{360^o}{n} = \frac{360^o}{5} \qquad\qquad \theta = 72^o$$

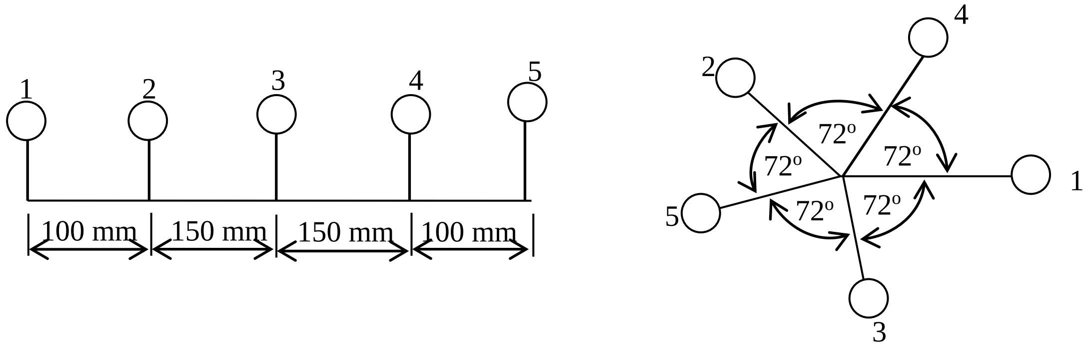


Figure S5-5: (a) Axial and (b) Angular Positions of Cranks

(b) The primary shaking force is given by

$$F_p = \sum_{i=11}^{n} m_{Bi} r_i \omega^2 \cos\theta_i = m_B r \omega^2 \sum_{i=11}^{n} \cos\theta_i$$

$$F_p = (2)(0.06)\left(1800 \text{ x } \frac{2\pi}{60}\right)^2 \left[\cos(0^o) + \cos(72) + \cos(2\text{x}72) + \cos(3\text{x}72) + \cos(4\text{x}72)\right] \qquad \underline{\underline{F_p = 0}}$$

The secondary shaking force is given by

$$F_s = \sum_{i=11}^{n} m_{Bi} r_i \omega^2 \frac{r}{l} \cos 2\theta_i = m_B r \omega^2 \frac{r}{l} \sum_{i=11}^{n} \cos 2\theta_i$$

$$F_s = (2)(0.06)\left(1800 \text{ x } \frac{2\pi}{60}\right)^2 \left(\frac{0.06}{0.24}\right)\left[\cos 2(0^o) + \cos 2(72) + \cos 2(2\text{x}72) + \cos 2(3\text{x}72) + \cos 2(4\text{x}72)\right]$$

$$\underline{\underline{F_s = 0}}$$

(c) The primary shaking couple is given by

$$M_p = \sum_{i=11}^{n} m_{Bi} r_i \omega^2 z_i \cos\theta_i = m_B r \omega^2 \sum_{i=11}^{n} z_i \cos\theta_i$$

$$M_p = (2)(0.06)\left(1800 \text{ x } \frac{2\pi}{60}\right)^2 \left[-0.25\cos(0^o) - 0.15\cos(72) + 0.0\cos(2\text{x}72) + 0.15\cos(3\text{x}72) + 0.25\cos(4\text{x}72)\right]$$

$$\underline{\underline{M_p = -1.452 \text{ kN - m}}}$$

The secondary shaking Couple is given by

$$M_s = \sum_{i=11}^{n} m_{Bi} r_i \omega^2 \frac{r}{l} z_i \cos 2\theta_i = m_B r \omega^2 \frac{r}{l} \sum_{i=11}^{n} z_i \cos 2\theta_i$$

$$M_s = (2)(0.06)\left(1800 \text{ x } \frac{2\pi}{60}\right)^2 \left(\frac{0.06}{0.24}\right) \begin{bmatrix} -0.25\cos 2(0^o) - 0.15\cos 2(72) + \\ 0.0\cos 2(2\text{x}72) + 0.15\cos 2(3\text{x}72) + 0.25\cos 2(4\text{x}72) \end{bmatrix}$$

$$\underline{\underline{M_s = -303.3 \text{ N - m}}}$$

5-4.4 V Configurations

The design principles of inline configuration are applicable to the V (or Vee) configuration. A V configuration is made up of two inline configurations with one on each bank, as shown in Figure 49. A V-6 engine is a six-cylinder engine which essentially consists of two three-cylinder inline engines on a common crankshaft. Similarly, a V-8 engine consists of two-cylinder inline engines on a common crankshaft.

In V configurations, the bank angle introduces additional phase shift of the forces. Consider a V engine of V angle 2γ, which corresponds to a bank angle of γ, as shown in Figure 50. Consider first two-cylinder engine with one cylinder in each bank and with both sharing a common crank throw. By letting the y-axis be the central axis of the banks, the shaking force for a single cylinder in the direction piston motion with angle θ measured from the piston axis is expressed as

$$F_s = m_B r \omega^2 \left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$$

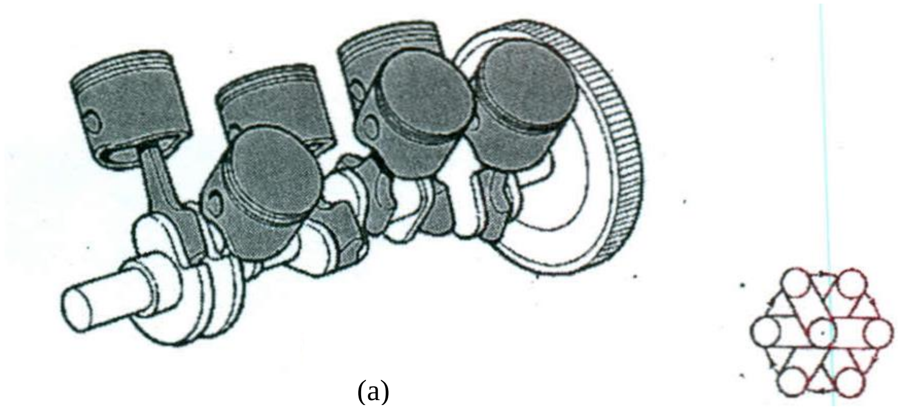
The total shaking force is the vector sum of the individual shaking forces, which is

$$F_s = F_{sL} + F_{sR} \qquad \textbf{Equation 210}$$

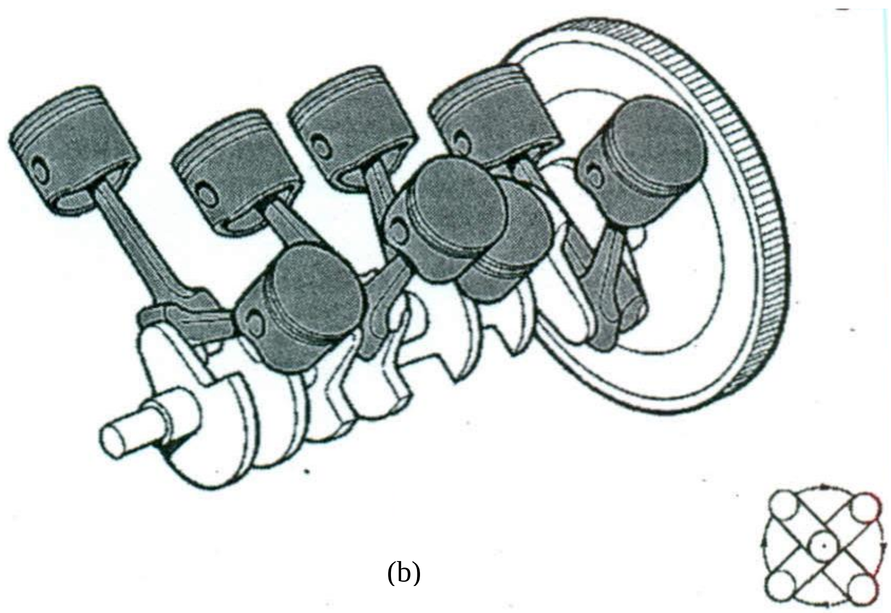
where the shaking forces of the right (R) and left (L) banks in plane of the respective banks can be expressed as

$$F_{sR} = m_B r \omega^2 \left[\cos(\theta - \gamma) + \frac{r}{l}\cos 2(\theta - \gamma)\right]\hat{r} \qquad \textbf{Equation 211}$$

$$F_{sL} = m_B r \omega^2 \left[\cos(\theta + \gamma) + \frac{r}{l}\cos 2(\theta + \gamma)\right]\hat{l} \qquad \textbf{Equation 212}$$



(a)



(b)

Figure 51: (a) Six- and (b) Eight-Cylinder Vee Engine Configurations

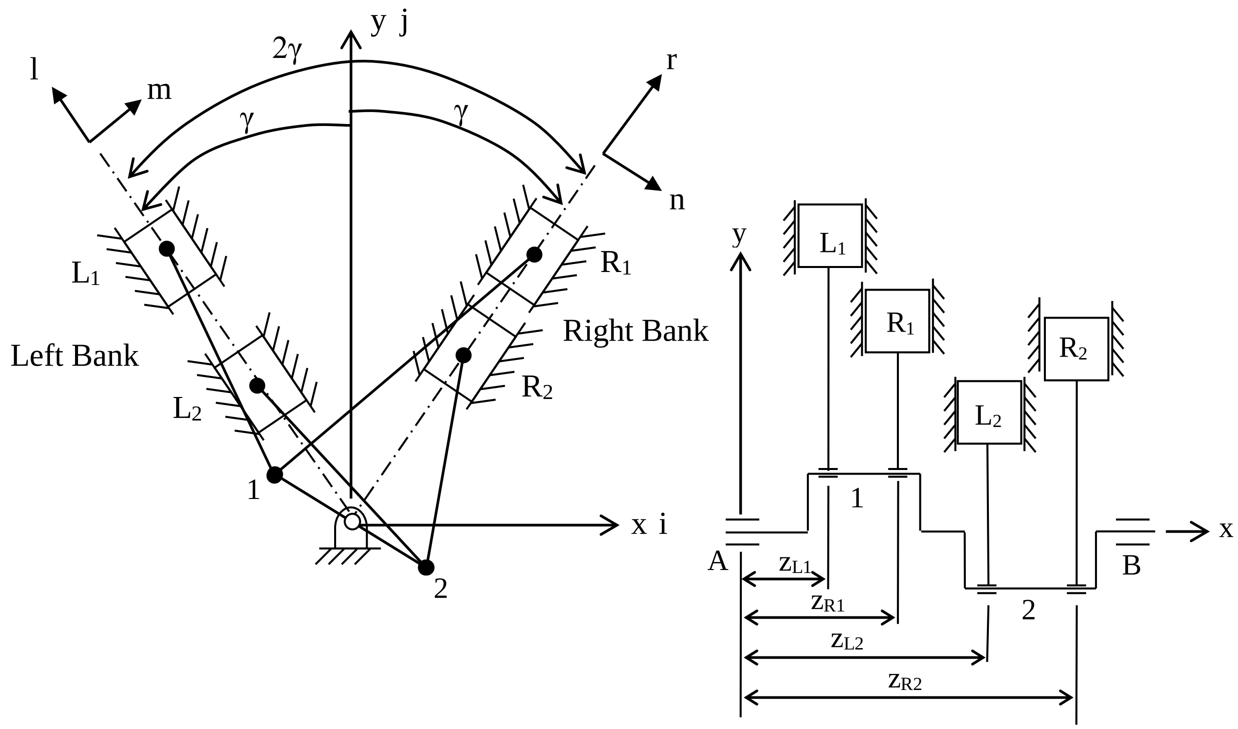


Figure 52: A Schematic of V Engine

Substituting the trigonometry identities given by Equations (1) and (2) into Equations 198 and 199 gives Equation 200 and 201

$$\cos(\theta+\gamma)=\cos\theta\cos\gamma-\sin\theta\sin\gamma \tag{1}$$

$$\cos(\theta-\gamma)=\cos\theta\cos\gamma+\sin\theta\sin\gamma \tag{2}$$

$$F_{sR}=m_B r\omega^2\left[\left(\cos\theta\cos\gamma+\sin\theta\sin\gamma\right)+\frac{r}{l}\left(\cos 2\theta\cos 2\gamma+\sin 2\theta\sin 2\gamma\right)\right]\hat{r}$$ **Equation 213**

$$F_{sL}=m_B r\omega^2\left[\left(\cos\theta\cos\gamma-\sin\theta\sin\gamma\right)+\frac{r}{l}\left(\cos 2\theta\cos 2\gamma-\sin 2\theta\sin 2\gamma\right)\right]\hat{l}$$ **Equation 214**

To account for multiple cylinders and phase angle ϕ, the crank position is replaced with $\theta = \omega t + \phi$. Using the trigonometry identities given by Equations (3) and (4), and introducing summation sign for multiple cylinders, the total shaking forces on banks are given by Equations 202 and 203.

$$\cos(\omega t+\phi)=\cos\omega t\cos\phi-\sin\omega t\sin\phi \tag{3}$$

$$\sin(\omega t-\phi)=\sin\omega t\cos\phi-\cos\omega t\sin\phi \tag{4}$$

$$F_{sR}=m_B r\omega^2\begin{bmatrix}(\cos\omega t\cos\gamma+\sin\omega t\sin\gamma)\sum_{n=1}^{N/2}\cos\phi_n+\\(\cos\omega t\sin\gamma-\sin\omega t\cos\gamma)\sum_{n=1}^{N/2}\sin\phi_n+\\\frac{r}{l}(\cos 2\omega t\cos 2\gamma+\sin 2\omega t 2\sin\gamma)\sum_{n=1}^{N/2}\cos 2\phi_n+\\\frac{r}{l}(\cos 2\omega t\sin 2\gamma-\sin 2\omega t\cos 2\gamma)\sum_{n=1}^{N/2}\sin 2\phi_n\end{bmatrix}\hat{r} \qquad \textbf{Equation 215}$$

$$F_{sL}=m_B r\omega^2\begin{bmatrix}(\cos\omega t\cos\gamma-\sin\omega t\sin\gamma)\sum_{n=1}^{N/2}\cos\phi_n+\\-(\cos\omega t\sin\gamma+\sin\omega t\cos\gamma)\sum_{n=1}^{N/2}\sin\phi_n+\\\frac{r}{l}(\cos 2\omega t\cos 2\gamma-\sin 2\omega t 2\sin\gamma)\sum_{n=1}^{N/2}\cos 2\phi_n+\\-\frac{r}{l}(\cos 2\omega t\sin 2\gamma+\sin 2\omega t\cos 2\gamma)\sum_{n=1}^{N/2}\sin 2\phi_n\end{bmatrix}\hat{l} \qquad \textbf{Equation 216}$$

The total shaking force can be resolved into x and y components as

$$F_{sx}=(F_{sR}-F_{sL})\sin\gamma \qquad \textbf{Equation 217}$$

$$F_{sy}=(F_{sR}+F_{sL})\cos\gamma \qquad \textbf{Equation 218}$$

Equations 204 to 205 provide additional means for eliminating shaking forces beside the choice of phase angles and axial locations. The shaking moment is equal to moment of the shaking force given by Equations 202 and 203 about plane yz in Figure 50 as

$$M_{sR}=m_B r\omega^2\begin{bmatrix}(\cos\omega t\cos\gamma+\sin\omega t\sin\gamma)\sum_{n=1}^{N/2}z_n\cos\phi_n+\\(\cos\omega t\sin\gamma-\sin\omega t\cos\gamma)\sum_{n=1}^{N/2}z_n\sin\phi_n+\\\frac{r}{l}(\cos 2\omega t\cos 2\gamma+\sin 2\omega t 2\sin\gamma)\sum_{n=1}^{N/2}z_n\cos 2\phi_n+\\\frac{r}{l}(\cos 2\omega t\sin 2\gamma-\sin 2\omega t\cos 2\gamma)\sum_{n=1}^{N/2}z_n\sin 2\phi_n\end{bmatrix}\hat{n} \qquad \textbf{Equation 219}$$

$$M_{sL} = m_B r\omega^2 \begin{bmatrix} (\cos\omega t\cos\gamma - \sin\omega t\sin\gamma)\sum_{n=1}^{N/2} z_n \cos\phi_n + \\ -(\cos\omega t\sin\gamma + \sin\omega t\cos\gamma)\sum_{n=1}^{N/2} z_n \sin\phi_n + \\ \frac{r}{l}(\cos 2\omega t\cos 2\gamma - \sin 2\omega t 2\sin\gamma)\sum_{n=1}^{N/2} z_n \cos 2\phi_n + \\ -\frac{r}{l}(\cos 2\omega t\sin 2\gamma + \sin 2\omega t\cos 2\gamma)\sum_{n=1}^{N/2} z_n \sin 2\phi_n \end{bmatrix}\hat{m} \qquad \textbf{Equation 220}$$

The total shaking moment can be resolved into x and y components as

$$M_{sx} = (M_{sL} + M_{sR})\cos\gamma \qquad \textbf{Equation 221}$$

$$M_{sy} = (M_{sL} - M_{sR})\sin\gamma \qquad \textbf{Equation 222}$$

Similar to shaking force, Equations 208 to 209 provide additional means for eliminating shaking moment beside the choice of phase angles and axial locations.

5-4.5 Opposed Configuration

An opposed configuration is basically a vee configuration with V angle of 180^o, as shown in Figure 51. Using Equation 205, F_{sy} = 0 and if F_{sL}= F_{sR}, then the shaking force is completely eliminated. Similarly, M_{sy} = 0 and if M_{sL}= M_{sR}, then the shaking moment is completely eliminated. If $M_{sL} \neq M_{sR}$, there will be significant shaking couple (both primary and secondary) due to the staggering of the crank throws. Clearly, the smaller the spacing z is, the better will be the sign from the point view of balancing. One method for eliminating the shaking couple is to reduce z to zero by using double connecting rods for one of the cylinders, as shown in Figure 50.

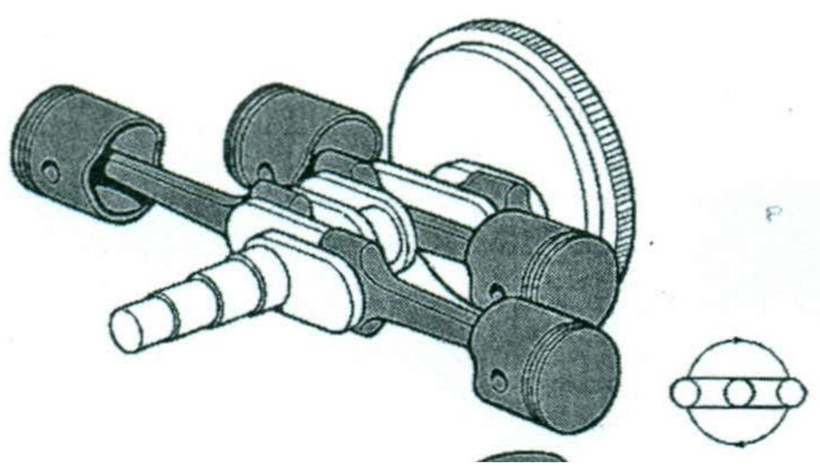


Figure 53: Four-Cylinder Opposed Engine Configuration

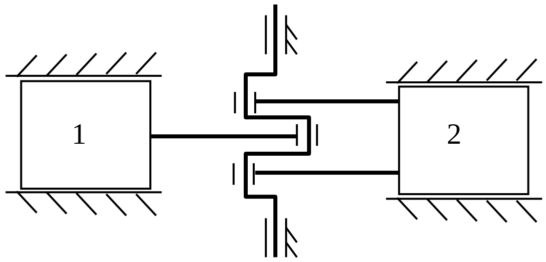


Figure 54: Double Connecting Rod for Cylinder 2 to Reduce or Eliminate Shaking Couple

Example 5- 6

Figure E5-6 shows an inline two-cylinder engine configuration in which the cranks are spaced 90°. For each cylinder, $m_B r\omega^2$ = 8.5 kN and r/l = 0.25, where m_B is the mass of reciprocating mass, r is crank length and l is connecting rod length. The cylinders are spaced between two bearings as shown in Figure E5-6. Determine the total shaking force and shaking moment in terms ωt.

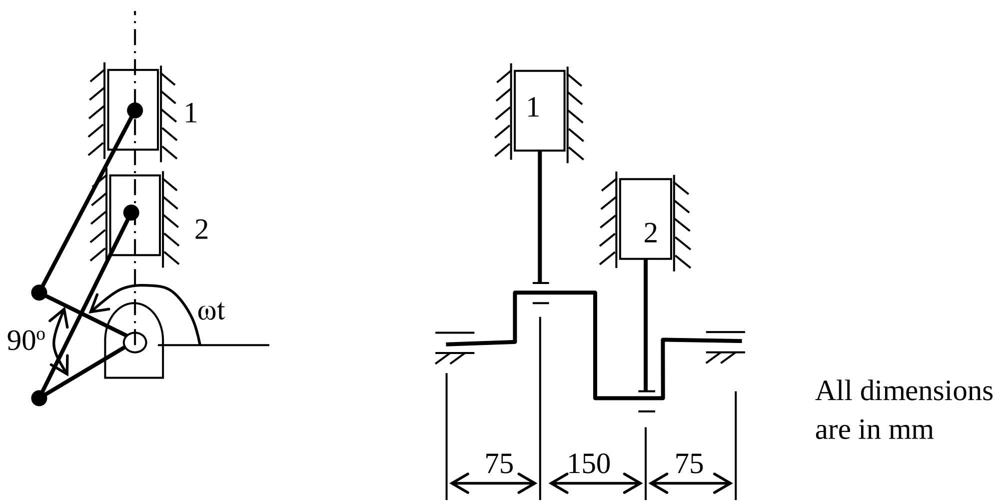


Figure E5-6

Solution

Using Equation 191, we have

$$F_s = 8500\begin{bmatrix} (\cos\omega t\,(\sin 0 + \sin 90) - \sin\omega t\,(\cos 0 + \cos 90)) + \\ 0.25(\cos 2\omega t\,(\sin 2(0) + \sin 2(90)) - \sin 2\omega t\,(\cos 2(0) + \cos 2(90))) \end{bmatrix}$$

$$\underline{\underline{F_s = 8500(\cos\omega t - \sin\omega t\ )\ (\mathrm{N})}}$$

From the figure z_1=0.075 m, z_2=(0.075 + 0.15) = 0.225 mm

Substituting the above given into Equation 193, the shaking moment is

$$M_s = 8500\begin{bmatrix} (\cos\omega t\ (0.075\sin 0 + 0.225\sin 90) - \sin\omega t\ (0.075\cos 0 + 0.225\cos 90)) + \\ 0.25(\cos 2\omega t\ (0.075\sin 2(0) + 0.225\sin 2(90)) - \sin 2\omega t\ (0.075\cos 2(0) + 0.225\cos 2(90))) \end{bmatrix}$$

$$\underline{\underline{M_s = 8500[0.225\cos\omega t - 0.075\sin\omega t + 0.0375\sin 2\omega t]\mathrm{j}\ (\mathrm{N\text{-}m})}}$$

☑ Key terms/ New Words in Unit

1. Lumped-Mass model
2. Centre of percussion
3. Force balance
4. Primary balance
5. Secondary balance
6. Firing order/sequence
7. Crank throw
8. crankshaft

Self Assessment 5

5-1. A connecting rod of mass 30 kg is suspended from 20 mm above the centre of the small end, and 600 mm above the centre of gravity. When the connecting rod is is given a small displacement and released, the period of oscillation is found to be 1.64 s. Find (a) the radius of gyration of the rod (Hint: Refer to Simple Harmonic Motion of ME 161/162 Basic Mechanics), (b) the distance between the two end centres, and (c) the dynamic equivalent masses of a two-lumped mass model placed at the end centres.

5-2. Figure E5-2 shows a two-cylinder inline engine configuration in which the cranks are spaced at 90°. Determine the expression for the net shaking force F_s and shaking moments in terms of ωt. For each cylinder, $mr\omega^2 = 8.8$ kN and crank length to connecting rod ratio, r/l = 0.25, where m is the reciprocating mass.

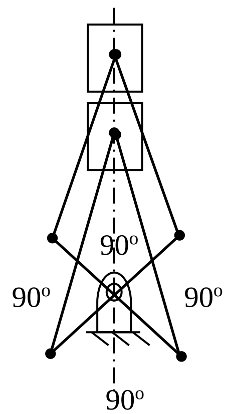


Figure E5-2

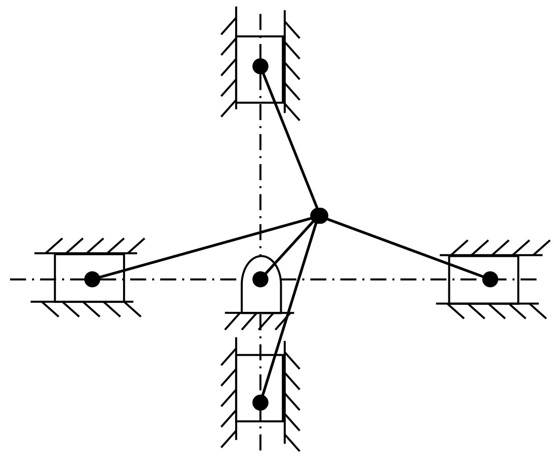


Figure E5-3

5-3. The four-cylinder radial engine depicted in Figure E5-3 is an excellent engine from the point of view of dynamic balance. Show that the engine can be balanced by means of a single rotating counterweight mounted on the crankshaft, and determine the magnitude and location and of such a counterweight. The crank length is r, the connecting rod lengths all equal l, and the reciprocating masses all equal m. The rotating masses are balanced, and all four cylinders lie in a single transverse plane [Charles E. Wilson & J. Peter Sadler].

UNIT 6

FLYWHEELS AND GOVERNORS

Introduction

Large variation in accelerations within mechanisms can cause significant variations in inertia and pin forces, supports reactions and torque required drive it at a constant or near constant speed. To stabilize the back-and-forth flow kinetic energy of rotating equipments, flywheels and governors are often attached to shafts. The intermittent nature of power delivery in reciprocating machines makes the use of flywheels and sometimes governors mandatory.

Flywheels are used in machines as an energy reservoir. They store energy during periods when the energy is more than required, and release energy to the system when the energy requirement is more than supplied. Thus, flywheels buffer the speed fluctuation in machines.

Governors are also used to control the mean speed of machines by regulating the flow working fluid, which power the machines. They rely on centrifugal force to function and control variation of speeds caused by load variation.

This unit focuses on dynamic analysis of flywheels and governors.

Learning Objectives

After reading this unit you should be able to:

1. Perform Dynamic Analysis of Flywheels and Governors
2. Select Suitable Flywheel for a given Energy and speed fluctuations
3. Select Suitable Governor for a given Load and speed variations

Unit content

SESSION 6-1: Turning Moment

6-1.1 Turning Moment

Turning moment is equal to the moment of crank pin force about the centre of crankshaft. It is equal to the cross product of position vector of crank pin and the crank pin force. Turning moment is also known as crank effort or turning drive torque. Mathematically, it is defined as

$$T_2 = R_A \text{ x } \mathrm{F}_{32}$$

For IC engines, the inertial forces and gas forces contribute to the turning moments. The turning moment due to gas force on a single cylinder is approximately equal to

$$T_{2g} \cong F_g r \sin \omega t \left(1 + \frac{r}{l} \cos \omega t \right) \qquad \textbf{Equation 223}$$

where F_g = gas force, r = crank radius, l = length of connecting rod, ω = angular speed and t is time. The gas force may be derived from the gas pressure P_g and bore diameter B as

$$F_g = \frac{\pi}{4} P_g B^2 \qquad \textbf{Equation 224}$$

The inertia torque on the crank is given by

$$T_{2i} \cong \frac{1}{2} m_B r^2 \omega^2 \left(\frac{r}{2l} \sin \omega t - \sin 2\omega t - \frac{3r}{2l} \sin 3\omega t \right) \qquad \textbf{Equation 225}$$

The above equation has a third harmonic and the second term is dominant as it has the largest coefficient of one. The total engine torque is the sum of the gas and initial torques on the crankpin given as

$$T_2 = T_{2g} + T_{2i} \qquad \textbf{Equation 226}$$

6-1.2 Turning Moment Diagram

Turning moment diagram (also known as crank effort diagram) is a plot of turning moment, T_2, on the vertical axis against the crank angle on the horizontal axis. A typical turning moment diagram of a one-cylinder four-stroke IC engine is shown in Figure 53. In the figure, the negative pressure inside the cylinder during suction forms negative loop. At the compression stage the loop is negative as work is done on the gas. During the power stroke, fuel burns and expands, thereby creating a large positive loop.

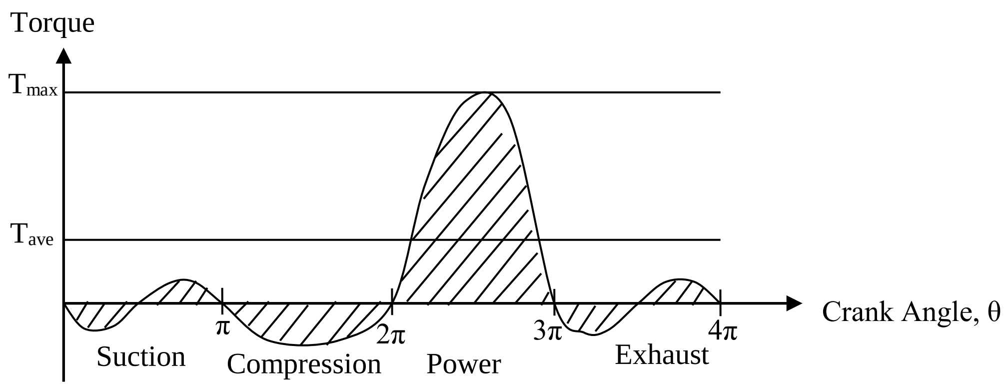


Figure 55: Turning Moment Diagram of One-Cylinder Four-Stroke IC Engine

Since the work done by or on crank is the product of the turning moment and the angle turned, the area under turning moment diagram represents work done. When the turning moment diagram is positive, work is done by the expansion fluid in the cylinder; otherwise work is done on the fluid as it contracts in the case of reciprocating compressors. The crankshaft accelerates when the turning moment is greater than the mean torque and retards when the torque is less than mean torque.

6-1.3 Energy Fluctuation

Energy fluctuation may be determined from the turning moment diagram. Consider the turning moment diagram shown in Figure 54. On the figure the horizontal line AG represents the mean torque. Let a_1, a_2, a_3, a_4, a_5 and a_6 be the mean area between the turning moment curve and the average torque line AG. The energy at any point is the sum of energy from the origin. Let the energy at point E equal to E. Then, the energy at various points on the curve are:

Energy at B = E + a_1

Energy at C = E + a_1-a_2

Energy at D = E + a_1-a_2+a_3

Energy at E = E + a_1-a_2+a_3-a_4

Energy at F = E + a_1-a_2+a_3-a_4 + a_5

Energy at F = E + a_1-a_2+a_3-a_4 + a_5-a_6

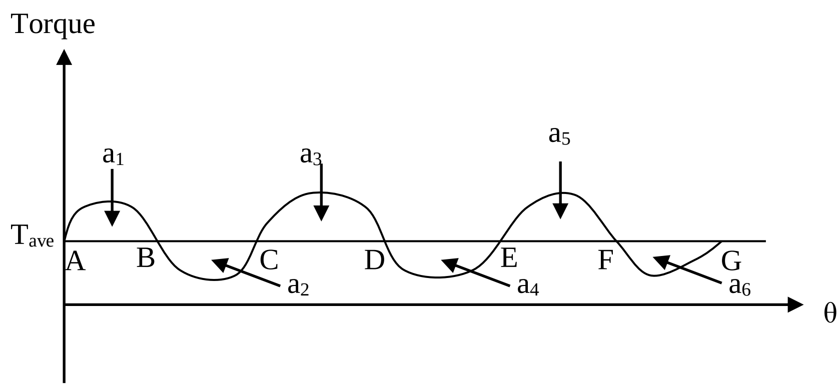


Figure 56: Turning Moment Diagram

The maximum energy fluctuation is defined as

ΔE = maximum of energy from A to G - maximum of energy from A to G **Equation 227**

Coefficient of energy fluctuation is defined as

$$C_E = \frac{\Delta E}{\text{Work done per Cycle}} = \frac{\Delta E}{\mathrm{T_{ave}}\theta}$$ **Equation 228**

where θ is the angle of turn per cycle. The angles of turn θ for two-stroke and four-stroke engines are 2π and 4π. Average torque may be defined in terms of power transmitted and angular speed as

$$\mathrm{T_{ave}} = \frac{P}{\omega} = \frac{60P}{2\pi N}$$ **Equation 229**

where P = power in Watts, N = angular speed in rpm and ω = angular speed in rad/s. Work done per cycle can be obtained from power P as

$$\text{Work done per Cycle} = \frac{60P}{n}, \qquad \textbf{Equation 230}$$

where n = number strokes per minute

n = N in case of steam engines and two-stroke engines

n = N/2 in case of four-stroke engines

Example 6- 1

The turning moment diagram for a two-stroke one-cylinder engine is shown in Figure E6-1. The area between the turning moment curve. If the mean torque is 77 N-m, find (a) the maximum energy fluctuation (b) coefficient of energy fluctuation.

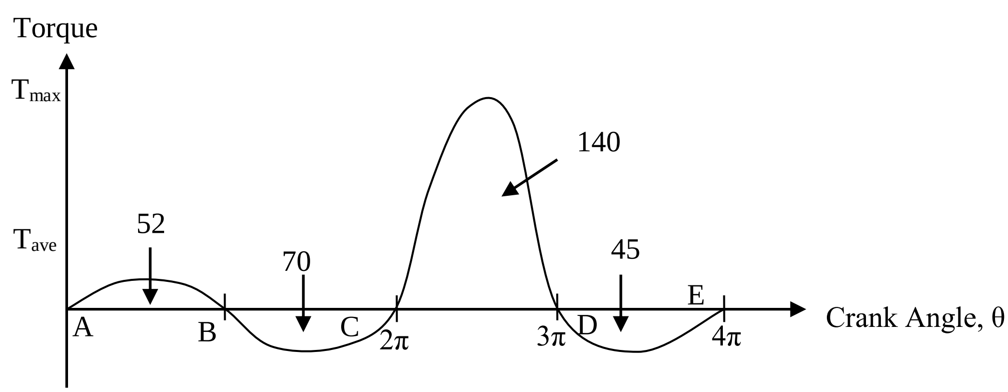


Figure E6-1

Solution

Let the energy at A be E. Then, the energy at various points are:

Energy at B = E +52

Energy at C = E + 52-70 = E-18

Energy at D = E + 52-70+140 =E +122

Energy at E = E + 52-70+140-45 =E + 77

The maximum energy fluctuation is

ΔE = maximum of energy from A to E - maximum of energy from A to E

$\Delta E = (E + 122) - (E - 18) = 140$

Using Equation 215, the coefficient of Energy fluctuation is given by

$$C_E = \frac{\Delta E}{\text{Work done per Cycle}} = \frac{\Delta E}{\mathrm{T_m}\theta} = \frac{140}{(77)(2\pi)} \qquad \underline{\underline{C_E = 0.289}}$$

SESSION 6-2: Flywheels

Flywheels have the following functions:

1. Reduce amplitude of speed fluctuation.
2. Reduce maximum torque required.
3. Stores and release energy when needed during cycling.

6-2.1 Speed Fluctuation

The maximum speed fluctuation is defined as

$$\Delta N = N_{\max} - N_{\min} \qquad \textbf{Equation 231}$$

$$\Delta\omega = \omega_{\max} - \omega_{\min} \qquad \textbf{Equation 232}$$

where N and ω speeds in rpm and rad/s, respectively, and the subscripts max and min denote maximum and minimum. The mean speed in rpm and rad/s are defined as

$$N_{ave} = \frac{N_{\max} + N_{\min}}{2} \qquad \textbf{Equation 233}$$

$$\omega_{ave} = \frac{\omega_{\max} - \omega_{\min}}{2} \qquad \textbf{Equation 234}$$

Coefficient of speed fluctuation is defined as

$$C_S = \frac{\Delta N}{\mathrm{N_{ave}}} = \frac{2(N_{\max} - N_{\min})}{(N_{\max} + N_{\min})} = \qquad \textbf{Equation 235}$$

$$C_S = \frac{\Delta\omega}{\omega_{\mathrm{ave}}} = \frac{2(\omega_{\max} - \omega_{\min})}{(\omega_{\max} + \omega_{\min})}$$

$$C_S = \frac{\Delta v}{v_{\mathrm{ave}}} = \frac{2(v_{\max} - v_{\min})}{(v_{\max} + v_{\min})} \qquad \text{(in terms of linear speed)}$$

The coefficient of speed fluctuation is a design factor that depends upon the nature of service conditions. Some service conditions and corresponding coefficient of speed fluctuation are listed in Table 2. An another terminology which is sometimes used to describe speed fluctuation is coefficient of steadiness, defined as

$$C_{St} = \frac{1}{C_s} \qquad \textbf{Equation 236}$$

Table 3: Coefficient of Speed Fluctuation for Various Machines

Type of Equipment	Coefficient of Fluctuation
Crushing Machinery	0.0200
Electrical Machinery	0.003
Direct Driven Electrical Machinery	0.002
Engine with belt transmission	0.030
Flour Milling Machinery	0.020
Gear Wheel Transmission	0.02
Hammering machine	0.200
Machine Tools	0.030
Paper-making Machinery	0.025
Pumping Machinery	0.03-0.050
Shearing Machinery	0.03-0.050
Spinning Machinery	0.010-0.020
Textile Machinery	0.025

6-2.2 Energy Stored in Flywheels

Consider a flywheel of mass m, moment of inertia I and radius of gyration K. Suppose that the velocity of the flywheel varies from ω_{min} to ω_{max}. The mean kinetic energy of the flywheel is

$$E_{ave} = \frac{1}{2} I\omega_{ave}^{2}$$ **Equation 237**

The maximum energy fluctuation is then

$$\Delta E = \frac{1}{2} I\omega_{\max}^{2} - \frac{1}{2} I\omega_{\min}^{2} = \frac{1}{2} I(\omega_{\max} + \omega_{\min})(\omega_{\max} - \omega_{\min})$$

Substituting Equation 217 into the above equation, we have

$$\Delta E = I\omega_{ave}(\omega_{\max} - \omega_{\min})$$ **Equation 238**

Substituting Equation 218 into the above equation, we have

$$\Delta E = IC_s\omega_{ave}^{2}$$ **Equation 239**

Substituting $\omega_{ave}^{2} = 2\frac{E_{ave}}{I}$ from Equation 220 into the above equation yields

$$\Delta E = 2C_s E$$ **Equation 240**

Suppose mean torque and load torque on the flywheel are Tm and T_l, respectively, as shown in Figure 55. Summing up moments about the axis of rotation of the flywheel, we have

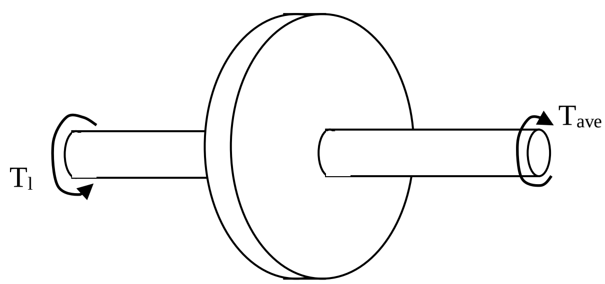


Figure 57: Flywheel

$$\sum M = (-T_L + T_m) = I\frac{d\omega}{dt} = I\frac{d\omega}{dt}\frac{d\theta}{d\theta} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta} \qquad \int_{\theta_{\min}}^{\theta_{\max}}(-T_L + T_m)d\theta = \int_{\omega_{\min}}^{\omega_{\max}} I\omega d\omega$$

$$\int_{\theta_{\min}}^{\theta_{\max}}(-T_L + T_m)d\theta = \frac{1}{2} I(\omega_{\max}^{2} - \omega_{\min}^{2})$$ **Equation 241**

The right hand side Equation 228 represents the change in kinetic energy between maximum and minimum speeds and is equal to the area under the turning moment (torque-angle) diagram between the extreme values of speed.

6-2.3 Flywheel Sizing

Sizing of flywheel is done to determine the dynamic parameters (mass, I, radius of gyration) required to stabilize energy fluctuation to an acceptable level in speed fluctuation. By knowing the desired C_s for a specific application and obtaining the kinetic energy from torque-angle curve, I can be determined. Efficient design of flywheels required optimization of mass and shape to achieve low mass and high moment of inertia. The following steps may be taken to perform dynamic design and/or selection of a flywheel:

1. Plot the load torque T_l verse θ for one cycle
2. Determine $Tl,_{\text{ave over}}$ one cyclye
3. Find the location of θmax, θmax and θmin
4. Determine the mind + the flywheel
5. Determine ω_{ave} Find the diameter of the flywheel

Example 6- 2

The mass and radius of gyration of the crankshaft of an engine are 100 kg and 1.2 m. If the engine runs at a mean speed of 120 rpm and maximum energy fluctuation is 3.79 kJ, determine the maximum and minimum speeds of the engine.

Solution

$$I = mk^2 = (100)(1.2)^2 \qquad\qquad I = 144\,\text{kg - m}^2$$

$$\Delta E = 3790 = I\omega_{ave}(\omega_{\max} - \omega_{\min}) = (144)\left(120 \text{ x } \frac{2\pi}{60}\right)(N_{\max} - N_{\min})\left(\frac{2\pi}{60}\right)$$

$$(N_{\max} - N_{\min}) = 20 \qquad (1)$$

$$N_{ave} = \frac{1}{2}(N_{\max} + N_{\min}) = 120$$

$$N_{\max} + N_{\min} = 240 \qquad (2)$$

Solving equations (1) and (2) simultaneously, we have

$$\underline{\underline{N_{max} = 130 \text{ rpm}}} \qquad \text{and} \qquad \underline{\underline{N_{min} = 110 \text{ rpm}}}$$

Example 6- 3

The total mass and radius of gyration of a flywheel and crankshaft of an engine are 25 kg and 1.2 m. The engine operates between speeds 100 rpm and 120 rpm, and starting torque of 150 N-m. When the engine starts from rest, it must reach the minimum and maximum operating speeds before 0.5 s and 0.8 s, respectively. Determine (a) minimum constant angular acceleration at which the engine must be ran in order to reach the operating speeds in the required time, (b) the minimum torque required to drive the engine at that constant acceleration, (c) the kinetic energy of the flywheel and crankshaft 0.7 s from the start of the engine.

Solution

$$I = mk^2 = (25)(1.2)^2 \qquad I = 36 \text{ kg - m}^2$$

(a) From the start of the engine to the minimum operating speed

$$\omega_f = \omega_i + \alpha t \qquad \left(100 \text{ x} \frac{2\pi}{60}\right) = (0) + \alpha(0.5) \qquad \alpha = 20.94 \text{ rad/s}^2$$

From the start of the engine to the maximum operating speed

$$\omega_f = \omega_i + \alpha t \qquad \left(120 \text{ x} \frac{2\pi}{60}\right) = (0) + \alpha(0.8) \qquad \alpha = 15.71 \text{ rad/s}^2$$

From the minimum speed to the maximum speed

$$\omega_f = \omega_i + \alpha t \qquad \left(120 \text{ x} \frac{2\pi}{60}\right) = \left(120 \text{ x} \frac{2\pi}{60}\right) + \alpha(0.8 - 0.5)$$

$$\alpha = 7.0 \text{ rad/s}^2$$

The angular acceleration is the maximum of the three accelerations

$$\underline{\alpha = 20.94 \text{ rad/s}^2}$$

\

(b) $\sum M = I\alpha$

$$T - T_{starting} = I\alpha \qquad T = T_{starting} + I\alpha = 150 + 36(20.94) \qquad \underline{\underline{T = 903.8 \text{ N - m}}}$$

(c) $\omega_f = \omega_i + \alpha t = 0 + 20.94(0.7)$ $\qquad \omega_f = 14.658 \text{ rad/s}$

$$KE = \frac{1}{2} I\omega_f^{\ 2} = \frac{1}{2}(36)(14.658\ )^2 \qquad \underline{\underline{KE = 3.87\ \text{kJ}}}$$

Example 6- 4

A four-stroke engine is to deliver 450 kW at a mean speed of 100 rpm. The coefficient of energy fluctuation is 0.2 and operational speeds must be kept within ±4% of the mean speed. Determine the radius of gyration of the flywheel if the mass of the flywheel must not exceed 5000 kg.

Solution

$$\omega_{\max} = \left(1 + \frac{4}{100}\right)\omega_{ave} \qquad \omega_{\max} = 1.04\omega_{ave}$$

$$\omega_{\min} = \left(1 - \frac{4}{100}\right)\omega_{ave} \qquad \omega_{\max} = 0.96\omega_{ave}$$

$$C_s = \frac{\omega_{\max} - \omega_{\min}}{\omega_{ave}} = \frac{1.04\omega_{ave} - 0.96\omega_{ave}}{\omega_{ave}} \qquad C_s = 0.08$$

$$\text{Work done per Cycle} = \frac{60P}{n} = \frac{60P}{(N/2)} = \frac{60\left(450\ \text{x}\ 10^3\right)}{(100/2)} \qquad \text{Work done per Cycle} = 900\ \text{kJ/cycle}$$

$$C_E = \frac{\Delta E}{\text{work done per cycle}} = \frac{\Delta E}{900\ \text{x}\ 10^3} = 0.2 \qquad \Delta E = 180000$$

Using Equation 222, we have

$$\Delta E = 180000 = IC_s\omega_{ave}^{\ 2} = I(0.08)\left(100\ \text{x}\ \frac{2\pi}{60}\right)^2 \qquad I = 20517.53\ \text{kg - m}^2$$

$$I = mk^2 \qquad k = \sqrt{\frac{I}{m}} = \sqrt{\frac{20517.5}{5000}} \qquad \underline{\underline{k = 2.03\ \text{m}}}$$

SESSION 6-3: Governors

Governors are broadly classified into centrifugal and inertia governors. Centrifugal governors are based on balancing of centrifugal force and moments of a rotating mass by an equal force and moments provided by either by action of a weight or springs.

In gravity controlled governors, the balancing force is provided by weight. Figure 56 shows a schematic of a typical gravity controlled governor. It consists of balls, called governor or fly balls that are supported by arms and revolve with the spindle. The spindle is driven by a pair of bevel gears or a universal joint. As the governor rotates, the centrifugal forces on the fly balls cause them to move outward, which lift the sleeve. The upward and downward movements of the sleeve drive a separate mechanism (not shown), which control a throttle valve that feeds working fluid into a machine.

The controlling force in a spring controlled governor is provided by springs. Figure 57 shows a schematic of a typical spring controlled governor. It consists of fly balls that are supported by L-shaped arms and revolve with the spindle. As the governor rotates, the centrifugal forces on the fly balls cause them to move outward. The centrifugal forces on the balls are balanced by spring(s). As in all governors, the outward movement of the fly balls lifts the sleeve, which drives a separate throttle valve mechanism (not shown).

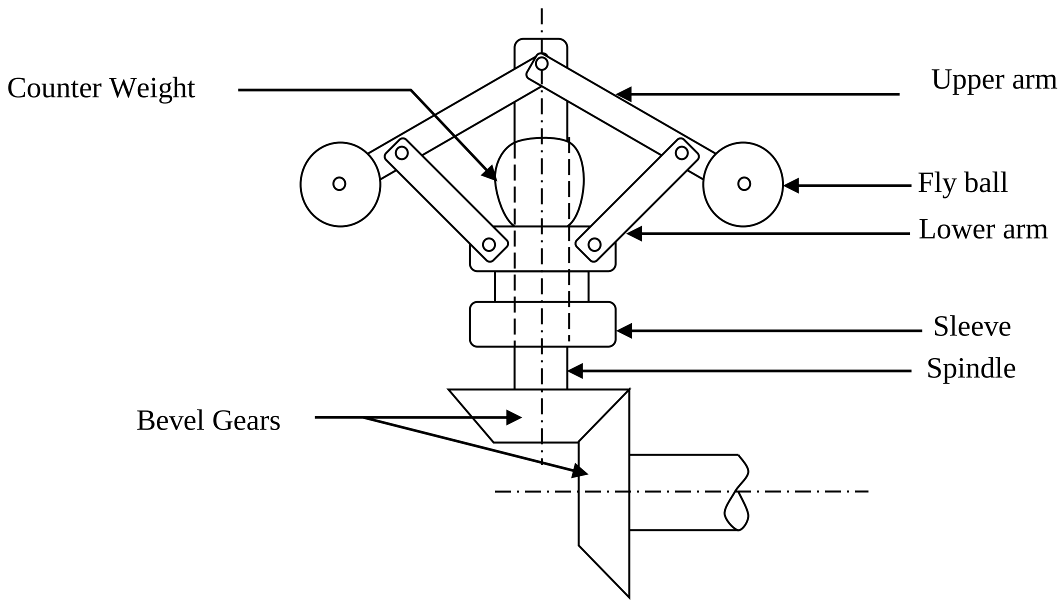


Figure 58: A Schematic of Typical Governor

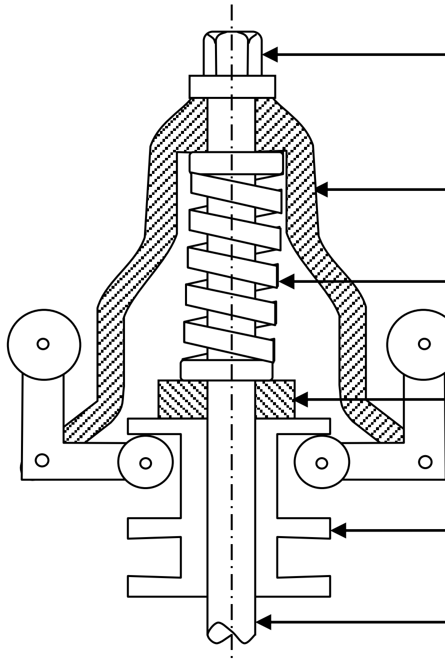


Figure 59: A Schematic of a Spring Controlled Governor

6-3.1 Terminologies of Governor

1. *Height of a governor* is the vertical distance between two extreme positions.
2. *Equilibrium speed* is the speed at which the centrifugal force acting on each fly ball is in equilibrium with the restoring force.
3. *Mean equilibrium speed* is the speed at the mean equilibrium position.
4. *Maximum and Minimum speeds are* the speed at which maximum and minimum positions, respectively, of a governor.
5. Control force is the force that opposes and balances the centrifugal force on a fly ball.
6. Sleeve lift is the minimum vertical distance that sleeve travels to change the equilibrium speed.

6-3.2 Watt Governor

Watt governor is the simplest centrifugal governor. It consists of a pendulum attached to light weight links, as shown in Figure 58. In the figure, the fly balls are suspended on arms. The symbols w and W in the figure denotes weights of each fly ball and sleeve, respectively, and α and β denotes the angles the upper and lower arms makes with the vertical axis, respectively.

It is assumed that the masses of arms and sleeve are negligible. Summing up vertical forces on the sleeve, we have

$$\sum F_y = 0 \qquad W = nT_2 \cos\beta \tag{1}$$

where n is number of fly balls, W is weight of sleeve. Since the mass of the sleeve is negligible, $T_2 = 0$

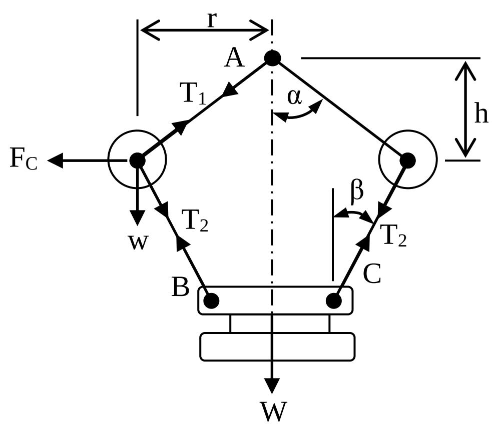


Figure 60: Forces on Fly Ball of Watt Governor

Summing up the vertical and horizontal forces on each fly ball, we have

$$\sum F_x = 0 \qquad F_C = T_1 \sin\alpha + T_2 \sin\beta \qquad F_C = T_1 \sin\alpha \tag{2}$$

$$\sum F_y = 0 \qquad w + T_2 \cos\beta = T_1 \cos\alpha \qquad w = T_1 \cos\alpha \tag{3}$$

Substituting equation (3) into (2), we have

$$F_C = m\omega^2 r = w\frac{\sin\alpha}{\cos\alpha} = mg\tan\alpha \qquad \omega^2 r = g\tan\alpha \tag{4}$$

From the geometry of the governor,

$$\tan\alpha = \frac{r}{h} \tag{5}$$

Substituting equation (5) into (4) and simplifying yield

$$\omega^2 = \frac{g}{h}$$ **Equation 242**

Note that friction force between the sleeve and the spindle was not accounted for in the derivation of Equation 229. As the governor rotates the centrifugal forces move the fly balls outward, which lift up the sleeve. The above equation indicates that the speed of a governor is inversely proportional to the square root of the height h. As a result a small change in speed ω corresponds to a small change in lift height h. This change in h may not be sufficient to enable the governor to drive a throttle mechanism to give the necessary change in fuel supply. Hence Watt governors work satisfactory at relatively low speeds between 60 to 80 rpm.

Example 6- 5

Find the vertical height of a Watt governor rotating at 65 rpm. Also determine the change in height corresponding to 1 rpm increase in the speed.

Solution

Using Equation 225, $$h = \frac{g}{\omega^2} = \frac{9.81}{\left(65 \text{ x } \frac{2\pi}{60}\right)^2}$$ h = 212 mm

Partially differentiating $h = \frac{g}{\omega^2}$ yields $dh = -\frac{2g}{\omega^3} d\omega$

Substituting *ω = 65 rpm* and *dω = 1 rpm* into the above equation gives

$$dh = -\frac{2g}{\omega^3} d\omega = -\frac{2(9.81)}{\left(65 \text{ x } \frac{2\pi}{60}\right)^3}\left(1 \text{ x } \frac{2\pi}{60}\right)$$ *dh = -6.515 mm*

6-3.3 Porter Governor

The Porter governor is a modification of the Watt governor with an addition of load to sleeve to provide additional controlling force, as shown 59. The addition of the load increases the speed required to raise the sleeve to appreciable level.

Consider the forces acting on a fly ball and a sleeve of a Porter governor as shown 60. In the figure, the forces in the upper and lower arms are T_1 and T_2, respectively. Let w and W denote weights of each fly ball and sleeve, respectively. The force required to control the throttle valve mechanism which the governor is being used to drive may be included in W. The symbols α and β denotes the angles the upper and lower arms makes with the vertical axis, respectively.

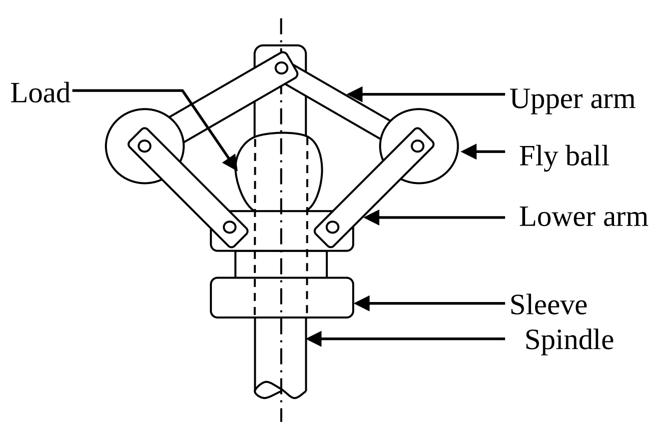


Figure 61: Porter Governor

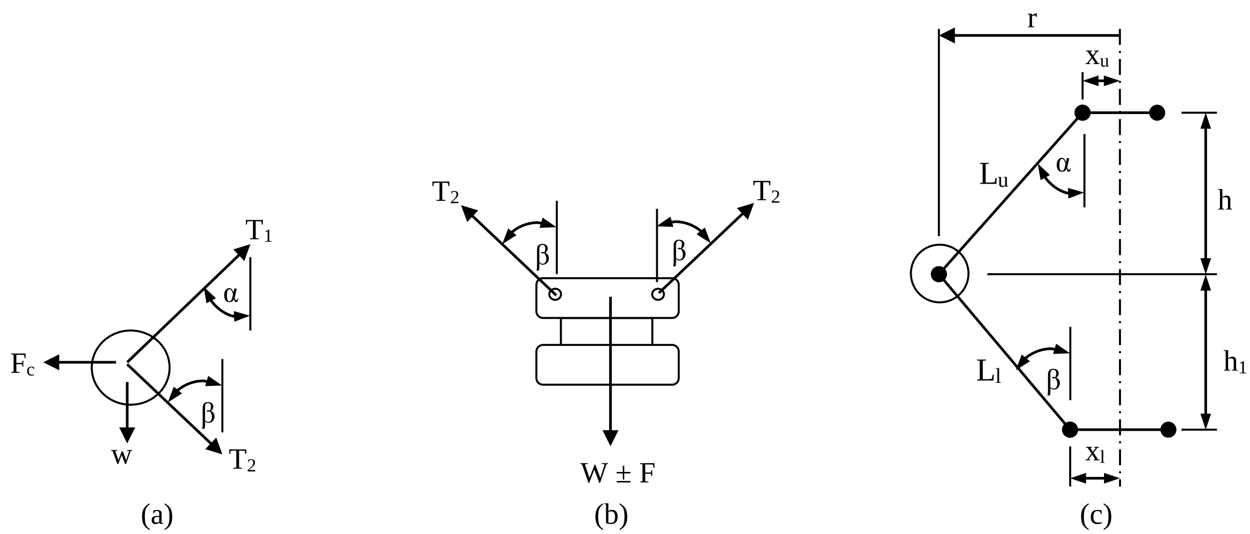


Figure 62: Diagram of (a) Forces acting on Fly Ball, (b) Forces acting on Sleeve and Load and (c) Geometry of Porter Governor

Summing up the vertical and horizontal forces on each fly ball, we have

$$\sum F_x = 0 \qquad F_C = T_1 \sin\alpha + T_2 \sin\beta \tag{1}$$

$$\sum F_y = 0 \qquad w + T_2 \cos\beta = T_1 \cos\alpha \tag{2}$$

Summing up the vertical forces on each fly ball, we have

$$\sum F_y = 0 \qquad nT_2 \cos\beta = W \pm F \qquad \Rightarrow T_2 = \frac{1}{n\cos\beta}(W \pm F) \tag{3}$$

where F = friction force between the sleeve and the spindle. The sign of the above equation is positive (+) when the sleeve is moving upward and negative (-) when the sleeve is moving downward. Substituting equation (3) into equation (2) yields

$$w + \frac{1}{n}(W \pm F) = T_1 \cos\alpha \qquad T_1 = \frac{1}{\cos\alpha}\left[w + \frac{1}{n}(W \pm F)\right] \tag{4}$$

Substituting equation (4) into equation (3), we have

$$F_C = \left[w + \frac{1}{n}(W \pm F)\right]\frac{\sin\alpha}{\cos\alpha} + \frac{1}{n}(W \pm F)\frac{\sin\beta}{\cos\beta}$$

$$\frac{F_c}{\tan\alpha} = \left[w + \frac{1}{n}(W \pm F)\right] + \frac{1}{n}(W \pm F)\frac{\tan\beta}{\tan\alpha} = w + \frac{1}{n}(W \pm F)(1+q)$$

$$\frac{F_c}{\tan\alpha} = w + \frac{1}{n}(W \pm F)(1+q) \qquad \textbf{Equation 243}$$

where $q = \dfrac{\tan\beta}{\tan\alpha}$. From the geometry of a Porter governor shown in Figure 60 (c),

$$\tan\alpha = \frac{r - x_u}{h} \qquad \text{and} \qquad \tan B = \frac{r - x_l}{h_l}$$

Equation 244

which implies that

$$q = \frac{\tan\beta}{\tan\alpha} = \left(\frac{r - x_l}{h_l}\right) \Big/ \left(\frac{r - x_u}{h}\right) \qquad q = \left(\frac{r - x_l}{r - x_u}\right)\left(\frac{h}{h_l}\right) \qquad \textbf{Equation 245}$$

The vertical distances h and h_l are determined from the geometry as

$$h = L_u \cos\alpha \qquad \text{and} \qquad h_l = L_l \cos\beta \qquad \textbf{Equation 246}$$

Substituting $\tan\alpha = (r - x_u)/h$, $F_c = m\omega^2 r$ and $w = mg$ into Equation 226 and simplifying gives

$$\omega^2 = \left[\frac{g}{r} + \frac{1}{nmr}(W \pm F)(1+q)\right]\left(\frac{r - x_u}{h}\right) \qquad \textbf{Equation 247}$$

Example 6- 6

The upper and lower arms of a three-ball Porter governor are 300 mm and 250 mm long, respectively. The upper arms are pivoted on the axis of rotation, and the lower arms pivoted on the sleeve at a distance 40 mm from the axis of rotation. The total mass of the sleeve and the load are 40 kg, and mass of each fly ball is 2 kg. Neglecting friction, determine the speed of rotation of the governor if the radius of each ball is 200 mm.

Solution

$$\sin\alpha = \frac{r}{300} = \frac{200}{300} \qquad \alpha = 41.8^o$$

$$\sin\beta = \frac{r - 40}{250} = \frac{200 - 40}{250} \qquad \beta = 39.8^o$$

$$q = \frac{\tan\alpha}{\tan\beta} = \frac{\tan 41.8^o}{\tan 39.8^o} \qquad q = 0.9318$$

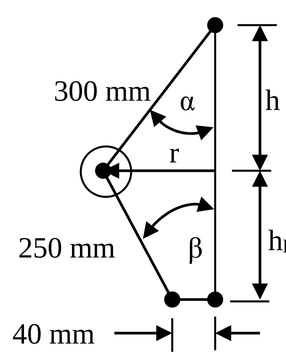


Figure S6-6

$h = 300\cos 41.8^o$ $\qquad h = 223.64\,\text{mm}$

$h_l = 250\cos 39.8^o$ $\qquad h_l = 192.1\,\text{mm}$

$$\omega^2 = \left[\frac{g}{r} + \frac{1}{nmr}(W \pm F)(1+q)\right]\left(\frac{r - x_u}{h}\right)$$

$$\omega^2 = \left[\frac{9.81}{0.20} + \frac{1}{(3)(2)(0.2)}(40 \text{ x } 9.81 + 0)(1 + 0.9318)\right]\left(\frac{0.20 - 0}{0.22364}\right) \qquad \underline{\underline{\omega = 24.67 \text{ rad/s}}}$$

$$\underline{\underline{N = 235.6\,\text{rpm}}}$$

Example 6- 7

The upper and lower arms of a three-fly ball Porter governor are each 350 mm long and pivoted on the axis of rotation. The mass of each fly ball is 3 kg and that of the sleeve is 40 kg. The radius of rotation of the balls ranges from 150 mm to 250 mm. If the friction force between the sleeve and the spindle is 24 N, find the speed range of the governor.

Solution

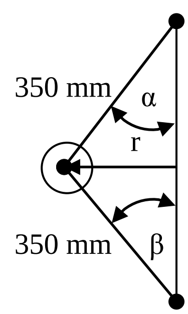


Figure S6-7

Since the arms are pivoted on the axis of rotation, $x_u = x_l = 0$.. From the geometry of the governor shown in Figure S6-7, $\alpha = \beta$ which implies that $q = \tan\beta/\tan\alpha = 1$ and $h = h_l = \sqrt{(350^2 - r^2)}$

At r = 150 mm, $h = h_l = \sqrt{(350^2 - 150^2)}$

$$h = h_l = 316.23\,\text{mm}$$

At r = 250 mm, $h = h_l = \sqrt{(350^2 - 250^2)}$ $\qquad h = h_l = 244.95\,\text{mm}$

Using Equation 230 at radius r = 150 mm, we have

$$\omega^2 = \left[\frac{9.81}{0.15} + \frac{1}{(3)(3)(0.15)}[(40 \text{ x } 9.81) + 24](1+1)\right]\left(\frac{0.15 - 0}{0.31623}\right) \qquad \omega = 17.99\,\text{rad/s}$$

$$N = \frac{2\pi}{60}\omega = \frac{2\pi}{60}17.99 \qquad N = 171.8\,\text{rpm}$$

At radius r = 150 mm, we have

$$\omega^2 = \left[\frac{9.81}{0.25} + \frac{1}{(3)(3)(0.25)}\left[(40 \text{ x } 9.81) + 24\right](1+1)\right]\left(\frac{0.25-0}{0.24495}\right) \qquad \omega = 20.44 \text{ rad/s}$$

$$N = \frac{2\pi}{60}\omega = \frac{2\pi}{60}20.44 \qquad N = 195.2 \text{ rpm}$$

Hence the speed of the governor is within the range: $\underline{\underline{171.8 \le N \le 195.2 \text{ rpm}}}$

6-3.4 Proell Governor

The Proell governor is a modification of the Porter governor. In a Proell governor each lower arm is extended beyond its joint with the upper arm. In addition, the fly balls are fixed at the end of the extensions, as shown in Figure 61.

Consider the forces acting on one fly ball and lower arm together, as shown in Figure 62(a). In the figure, the force in the upper arm is T, and w and W in denotes weights of a fly ball and sleeve, respectively. The vertical and horizontal reactions on the pin at A are Ay and $A_{x,}$ respectively. A line through the two attachment points of the upper arm intercept another line perpendicular to the axis of rotation and through point A, where the lower arm is attached to the sleeve. Also, the free-body diagram of a sleeve showing only two arms is shown in Figure 62(b). In on the diagram W and F are respectively the total weight of sleeve and other loads, and friction force. The other load in W include the forces required control the throttle valve mechanism that the governor is being used to drive. Summing up the vertical forces on the sleeve, we have

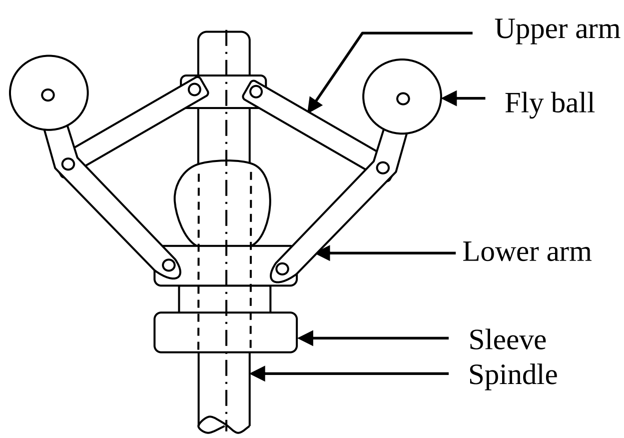


Figure 63: Proell Governor

$$\sum F_y = 0 \qquad nA_y = (W \pm F) \qquad A_y = \frac{1}{n}(W \pm F) \tag{1}$$

Taking moment at point A of Figure 62 (a), we have

$$\sum M_C = 0 \qquad F_C(h_1 + h_2) = w(x_3 - x_1) + A_y(x_3 + x_4)$$

$$F_C = w\left(\frac{x_3 - x_1}{(h_1 + h_2)}\right) + A_y\left(\frac{x_3 + x_4}{(h_1 + h_2)}\right) \tag{2}$$

Substituting equation (1) into (2) yields

$$F_C = m\omega^2 r = w\left(\frac{x_3 - x_1}{(h_1 + h_2)}\right) + \frac{1}{n}(W \pm F)\left(\frac{x_3 + x_4}{(h_1 + h_2)}\right)$$

$$\omega^2 = \frac{g}{r}\left(\frac{x_3 - x_1}{(h_1 + h_2)}\right) + \frac{1}{nmr}(W \pm F)\left(\frac{x_3 + x_4}{h_1 + h_2}\right) \qquad \textbf{Equation 248}$$

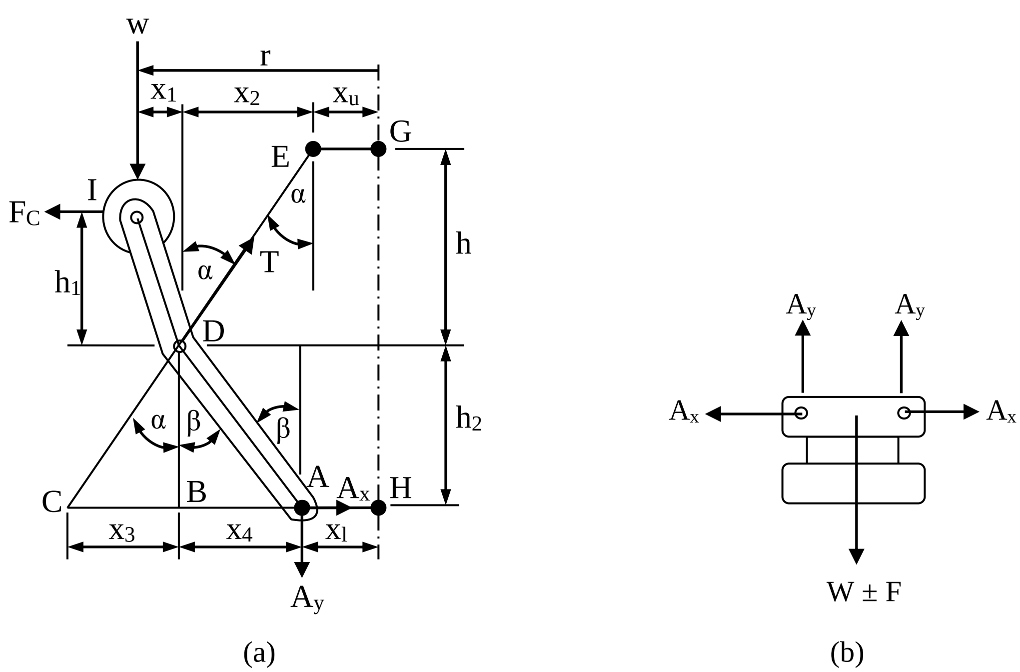


Figure 64: Free Body Diagrams of (a) Lower Arm and Fly Ball, and (b) Sleeve

Let the lengths of DE, AD and DI be denoted by L_1, L_2 and L_3, respectively. Then, from the Figure 59 (a), the vertical extension of the lower arm is given

$$h = L_1 \cos\alpha \qquad \textbf{Equation 249}$$

The horizontal length of the upper is

$$x_2 = L_1 \sin\alpha \qquad \textbf{Equation 250}$$

The horizontal extension of the lower arm is determined from the radius of fly ball r, horizontal length of upper arm x_2 and the horizontal distance of point of pivot of upper from the axis of rotation x_u as

$$x_1 = r - x_2 - x_u \qquad \textbf{Equation 251}$$

Using Pythagoras theorem, the vertical extension of the lower arm is

$$h_1 = \sqrt{L_3^{\ 2} - x_1^{\ 2}} \qquad \textbf{Equation 252}$$

The horizontal distance from the axis of rotation to the joint of the upper and lower arms is given by $x_2 + x_u = x_4 + x_l = L_2 \sin\beta + x_l$

$$\sin\beta = \frac{x_2 + x_u - x_l}{L_2} \qquad \textbf{Equation 253}$$

The lengths h_2, x_3 and x_4 are determined from

$$h_2 = L_2 \cos\beta \qquad \textbf{Equation 254}$$

$$x_3 = h_2 \tan\alpha \qquad \textbf{Equation 255}$$

$$x_4 = h_2 \tan\beta \qquad \textbf{Equation 256}$$

The angular velocity is determined by substituting of Equations 232 to 239 into 231. Comparing the Proell to Porter governors, Proell governors require less load on sleeve than that of Porter governor in order to reach the equilibrium height.

Example 6- 8

A three-ball Proell governor has equal arms of length 250 mm. The upper and lower arms are pivoted 30 mm and 40 mm respectively from the axis of rotation. Each ball has mass of 5 kg and that of the sleeve and load combined is 80 kg. The friction force between sleeve and spindle is 25 N. Determine the speed of the governor knowing that the radius of rotation is 150 mm, and the extension link of the lower arm is 70 mm and vertical.

Solution

$$x_1 = 0 \qquad h_1 = 70\,\text{mm}$$

$$\sin\alpha = \frac{x_2}{|DE|} = \frac{150-30}{250} \qquad \alpha = 28.68^o$$

$$\sin\beta = \frac{x_4}{|DA|} = \frac{150-40}{250} \qquad \beta = 26.10^o$$

$$h_2 = |DA|\cos\beta = 250\cos 26.10^o \qquad h_2 = 224.51\,\text{mm}$$

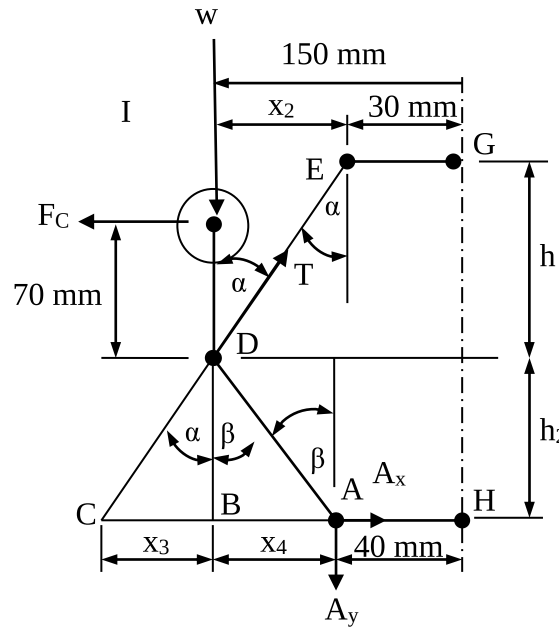
$$x_3 = h_2 \tan\alpha = 224.51\tan 28.68^o$$

$x_3 = 122.81\,\text{mm}$

$$x_4 = h_2 \tan\beta = 224.51\tan 26.10^o$$

$x_4 = 109.99\,\text{mm}$

$$\omega^2 = \frac{g}{r}\left(\frac{x_3 - x_1}{(h_1 + h_2)}\right) + \frac{1}{nmr}(W \pm F)\left(\frac{x_3 + x_4}{(h_1 + h_2)}\right)$$



$$|DE| = |DA| = 250\,\text{mm}$$

Figure E42

$$\omega^2 = \frac{9.81}{0.15}\left(\frac{0.12281-0}{(0.07+0.2245)}\right) + \frac{1}{(3)(5)(0.15)}(80 \text{ x } 9.81 + 24)\left(\frac{0.12281+0.10999}{0.07+0.2245}\right)$$

$$\omega^2 = 325.8 \qquad \omega = 18.05\,\text{rad/s}$$

$$N = \frac{60}{2\pi}\omega = \frac{60}{2\pi}18.05\,\text{rpm} \qquad \underline{\underline{N = 172.4\,\text{rpm}}}$$

Example 6- 9

A Proell governor has four fly balls, each of mass 3.5 kg. The lengths of all the arms are 250 mm and distance of each pivot of arm from the axis of rotation is 30 mm. The length of extension of lower arms on which each fly ball is attached is 80 mm, and inclined at 30º to the lower arm. The mass of the sleeve is 60 kg. Knowing that the arms are inclined at 50° to the axis of rotation, determine the equilibrium speed of the governor.

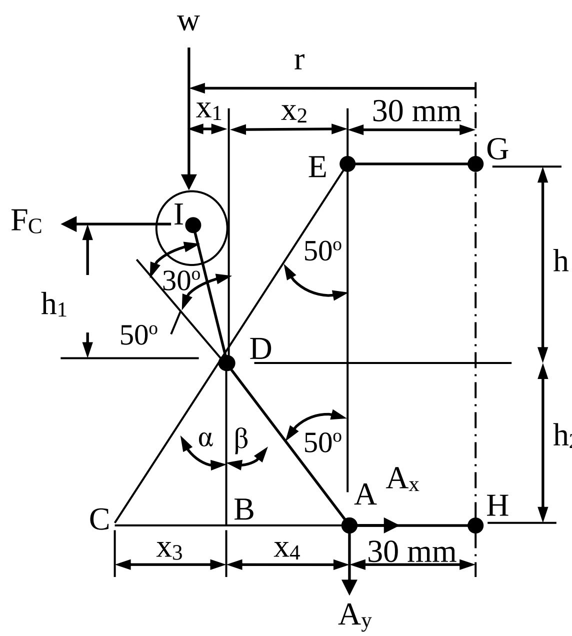
Solution

$$x_2 = |DE|\sin 50^o = 250\sin 50^o \qquad x_2 = 191.5\text{ mm}$$

$$x_1 = |DI|\sin\left(50^o - 30^o\right) = 80\sin 20^o \qquad x_1 = 27.4\text{ mm}$$

$$r = x_1 + x_2 + 30 = 27.4 + 191.5 + 30.0 \qquad r = 248.9\text{ mm}$$

$$h_1 = |DI|\cos\left(50^o - 30^o\right) = 80\cos\left(50^o - 30^o\right) \qquad h_1 = 75.2\text{ mm}$$



$$|DE| = |DA| = 250\text{ mm} \quad \text{and} \quad |DI| = 80\text{ mm}$$

Figure S6-9

$$h_2 = |DA|\cos 50^o = 250\cos 50^o \qquad h_2 = 160.7\text{ mm}$$

$$x_3 = h_2\tan 50^o = 160.7\tan 50^o \qquad x_3 = 191.5\text{ mm}$$

$$x_4 = h_2\tan 50^o = 160.7\tan 50^o \qquad x_4 = 191.5\text{ mm}$$

$$\omega^2 = \frac{g}{r}\left(\frac{x_3 - x_1}{(h_1 + h_2)}\right) + \frac{1}{nmr}(W \pm F)\left(\frac{x_3 + x_4}{(h_1 + h_2)}\right)$$

$$\omega^2 = \frac{9.81}{0.2489}\left(\frac{0.1915-0.0274}{0.0752+0.1607}\right)+\frac{1}{(4)(3.5)(0.2489)}(60 \text{ x } 9.81+0)\left(\frac{0.1915+0.1915}{0.0752+0.1607}\right)$$

$$\omega^2 = 301.66$$

$\omega = 17.37$ rad/s $\qquad N = \frac{60}{2\pi}\omega = \frac{60}{2\pi}17.37$ rpm $\qquad \underline{\underline{N = 165.9 \text{ rpm}}}$

6-3.5 Hartnell Governor

Figure 63 shows a typical Hartnell governor. It consists of at least two crank bell levers pivoted on a frame. Each lever carries a fly ball at the end of the vertical arm and roller at the end of the horizontal arm. A helical compression spring which is seated on the sleeve at the bottom and on the frame at the top provides the controlling force.

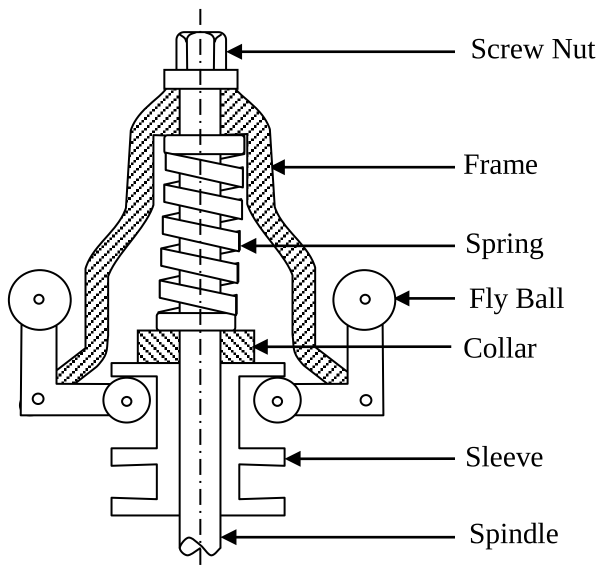


Figure 65: Hartnell Governor

Consider the forces acting on one fly ball and crank bell lever together, as shown in Figure 64. In the figure, the forces acting on the fly ball are its weight w and centrifugal force F_c and they are balanced by the load W due to weight and drive force for a throttle valve mechanism, and spring force S. The symbols y and x denote the lengths of the vertical and horizontal arms of the crank bell lever, and R is the radius of the pivot of crank bells from the axis of rotation. Suppose that the horizontal arm is horizontal at the minimum speed. Also lever is in equilibrium when the horizontal arm is orientated at angle θ to the horizontal, and the governor rotating at a speed of ω at radius r.

Taking moment at point the pivot the lever, we have

$$\sum M_{pivot} = 0 \qquad F_C\, y\cos\theta + wy\sin\theta = \frac{1}{n}(W+S)x\cos\theta$$

$$F_C = m\omega^2 r = \frac{1}{n}(W+S)\frac{x}{y} - w\tan\theta$$

$$\omega^2 = \frac{1}{nmr}(W+S)\frac{x}{y} - \frac{g}{r}\tan\theta \qquad \textbf{Equation 257}$$

From the lever geometry,

$$r = R + y\sin\theta \qquad \text{or} \qquad \theta = \sin^{-1}\left(\frac{r-R}{y}\right) \qquad \textbf{Equation 258}$$

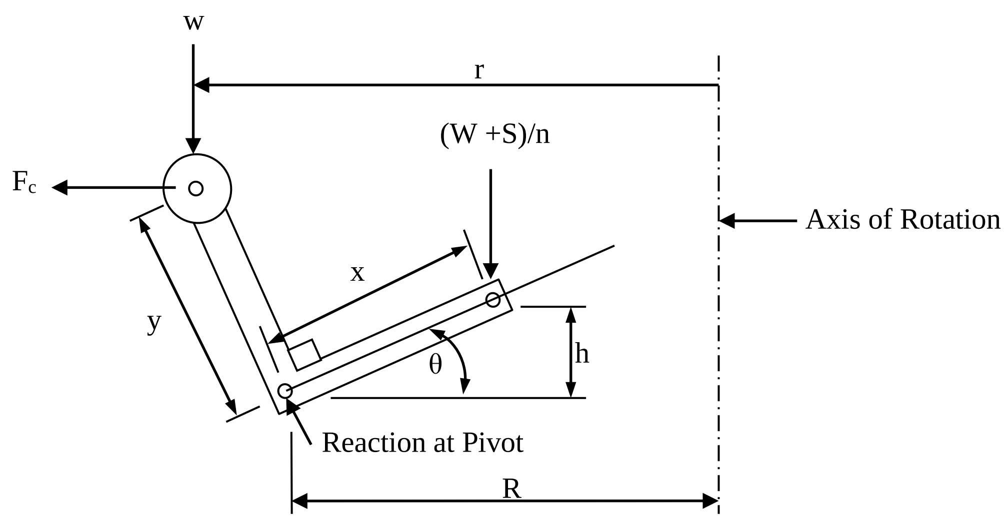


Figure 66: Free Body Diagram of Crank Bell Lever of Hartnell Governor

From Equation 244,

$$W + S = \frac{y}{x}nm\left(\omega^2 r + g\tan\theta\right) \qquad \textbf{Equation 259}$$

Let the stiffness of the spring denote k, and deflection of spring at speed ω be h. Then, the spring force is defined as

$$S = kh$$

At speeds ω_1 and ω_2, the spring forces are

$$W + S_1 = W + kh_1 \qquad (1)$$

$$W + S_2 = W + kh_2 \quad (2)$$

Subtracting equation (1) from (2) gives

$$S_2 - S_1 = k(h_2 - h_1) \quad (3)$$

If h_1 is the lift at minimum speed, then the lift from the minimum speed is

$$S_2 - S_1 = k(h_2 - h_1) = kh \qquad \textbf{Equation 260}$$

Again, from the geometry, $h = x\sin\theta$ **Equation 261**

For small angles, $h \cong x\theta$ (θ is in radian) **Equation 262**

To account for friction force F between the spindle and the sleeve, Equation 246 becomes

$$W + S \pm F = \frac{y}{x} nm(\omega^2 r + g\tan\theta) \qquad \textbf{Equation 263}$$

Example 6- 10

The extreme radii of rotations of fly balls of Hartnell type of governor are 80 mm and 120 mm. The governor has a central sleeve spring and three right-angled bell crank levers, which vertical and horizontal arms 120 mm and 80 mm, respectively. The levers are pivoted 120 mm from the axis of rotation. Each fly ball has a mass of 3 kg. Knowing that the two extreme speeds of the govern are 400 rpm and 450 rpm for a sleeve lift of 18 mm, and that the horizontal arm is perpendicular to the axis of rotation at the minimum speed, determine (a) total load on the sleeve at the lowest and highest speeds, (b) stiffness of the spring, and (c) deflection of the spring at the minimum speed if the mass of the sleeve is 50 kg. Neglect friction at all contacts.

Solution

Figure S6-10 shows illustration of the governor at minimum and maximum positions.

(a)

$$W + S_1 = \frac{y}{x} n(m\omega_1^2 r_1 + mg\tan\theta_1) = \left(\frac{120}{80}\right)(3)\left[(3)\left(400 \text{ x } \frac{2\pi}{60}\right)^2(0.12) + (3)(9.81)\tan 0\right]$$

$$\underline{\underline{W + S_1 = 2842.4 \text{ N}}}$$

The inclination of the horizontal arm to the horizontal axis is given by

$$\sin\theta = \frac{h}{x} = \frac{18}{80} \qquad \theta = 13.0^o$$

$$r_2 = R + y\sin\theta = 120 + 120\sin 13 \qquad r_2 = 146.99\ \text{mm}$$

$$W + S_2 = \frac{x}{y} n\left(m\omega_2{}^2 r_2 + mg\tan\theta_2\right) = \left(\frac{120}{80}\right)(3)\left[(3)\left(450\ \text{x}\ \frac{2\pi}{60}\right)^2(0.14699) + (3)(9.81)\tan 13.0^o\right]$$

$$\underline{\underline{W + S_2 = 4498.3\ \text{N}}}$$

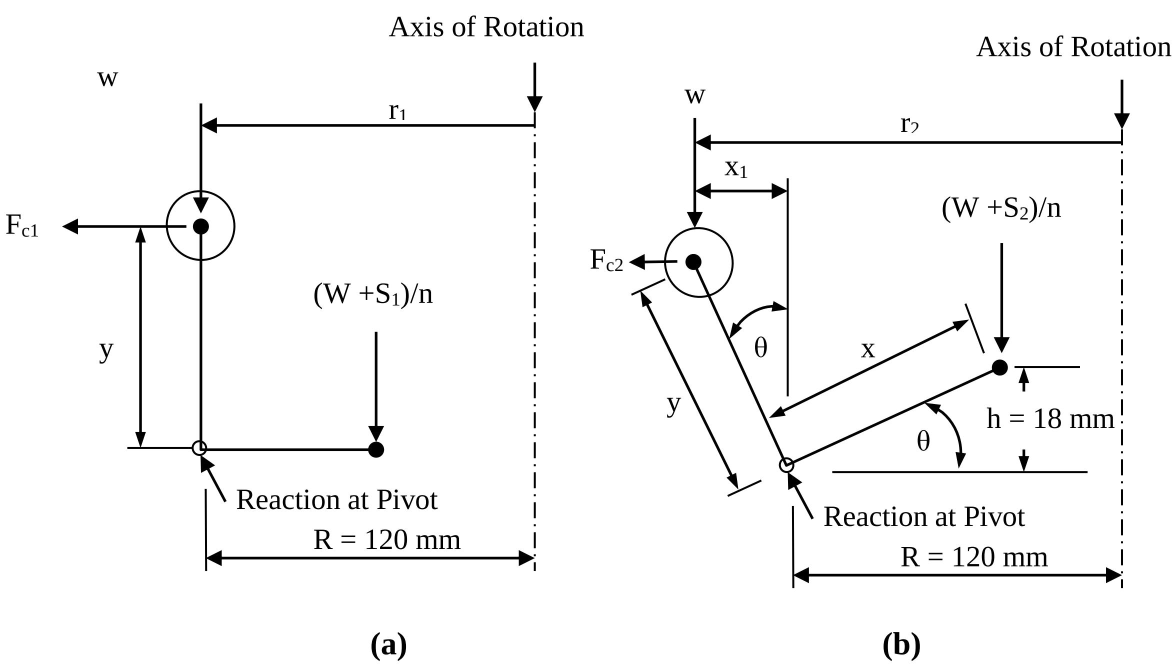


Figure S6-10 (a) Minimum, and (b) Maximum Positions

(b) $(W + S_2) - (W + S_1) = S_2 - S_1 = 4498.3 - 2842.4 \qquad S_2 - S_1 = 1655.9\ \text{N}$

$$S_2 - S_1 = kh = 1655.9 \qquad k = \frac{1655.9}{h} = \frac{1655.9}{18\ \text{x}\ 10^{-3}} \qquad \underline{\underline{k = 92.0\ \text{kN/m}}}$$

(c) $S_1 = 1655.9 - W = 1655.9 - 50\ \text{x}\ 9.81 \qquad S_1 = 1165.4\ \text{N}$

The initial deflection is related to the initial spring force as

$$S_1 = kh_0 \qquad h_0 = \frac{S_1}{k} = \frac{1165.4}{92.0\ \text{x}\ 10^3} \quad h_0 = 0.0196\ \text{m} \qquad \underline{\underline{h_0 = 12.67\ \text{mm}}}$$

6-3.6 Hartung Governor

The Hartung governor is spring-loaded governor similar to Hartnell governors except that the placements of springs are different. In Hartung governors, each vertical arm of the crank bell levers is fitted with compressive a coiled spring that presses against the frame of the governor at the other end, as shown in Figure 65.

Consider the forces acting on one fly ball and crank bell lever together, as shown in Figure 66. In the figure, the forces acting on the fly ball are its weight w and centrifugal force F_c. These forces are balanced by the load W due to weight and drive force for a throttle valve mechanism and spring force S. The symbols y and x denote the lengths of the vertical and horizontal arms of the crank bell lever, y_s is vertical distance from the axis of the coiled spring to the pivot and R is the radius of the pivot of crank bells from the axis of rotation. Suppose that the horizontal arm is horizontal at the minimum speed. Also, the lever is in equilibrium when the horizontal arm is orientated at angle θ to the horizontal, and the governor rotating at a speed of ω at radius r.

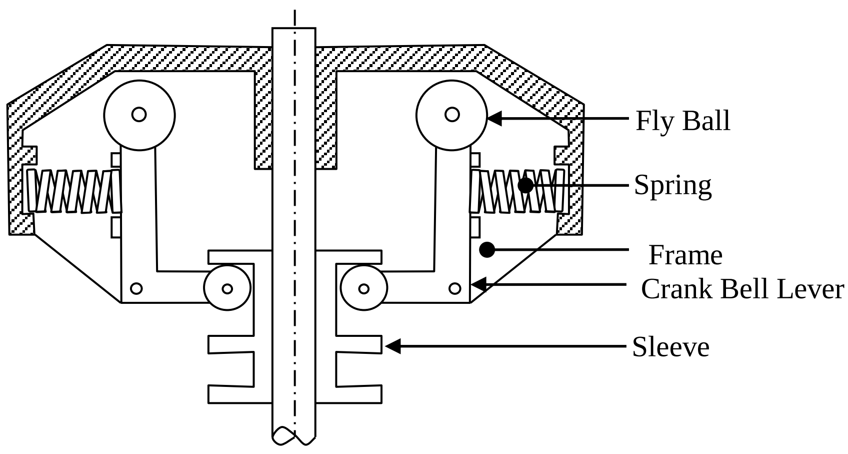


Figure 67: Hartung Governor

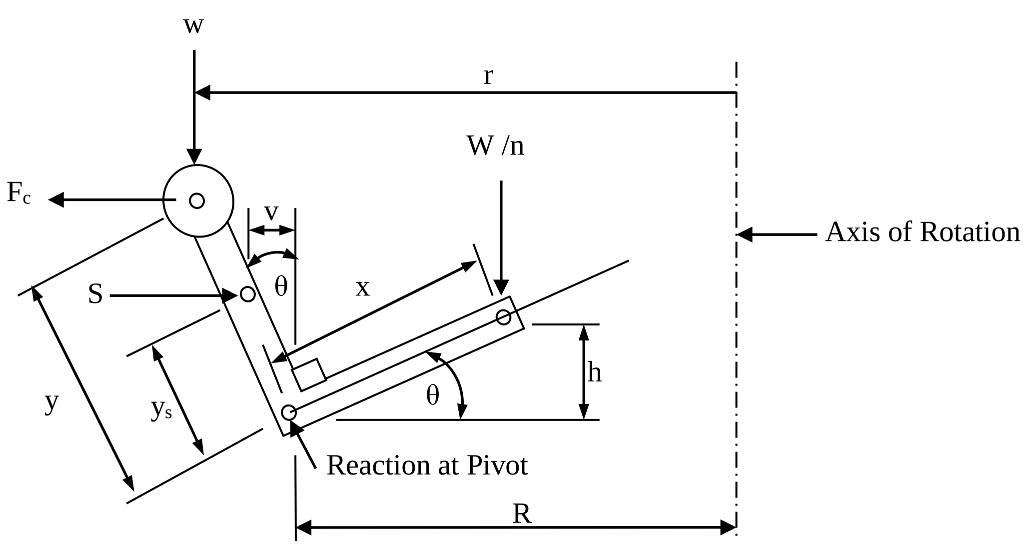


Figure 68: A Free Body Diagram of Crank Bell Lever of Hartung Governor

Taking moment at point the pivot the lever, we have

$$\sum M_{pivot} = 0 \qquad F_C y\cos\theta + wy\sin\theta = \frac{1}{n}Wx\cos\theta + Sy_s\cos\theta$$

$$F_C = m\omega^2 r = \frac{1}{n}W\frac{x}{y} + S\frac{y_s}{y} - w\tan\theta$$

$$\omega^2 = \frac{1}{nmr}W\frac{x}{y} + \frac{S}{mr}\frac{y_s}{y} - \frac{g}{r}\tan\theta \qquad \textbf{Equation 264}$$

From the lever geometry,

$$r = R + y\sin\theta \text{ or } \qquad \theta = \sin^{-1}\left(\frac{r-R}{y}\right) \text{ (Same as Eqn 245)} \qquad \textbf{Equation 265}$$

$$h = x\sin\theta \qquad (1)$$

$$v = y_s\sin\theta \qquad (2)$$

Dividing Equation (2) by (1) and making v the subject, we have

$$v = \frac{y_s}{x}h \qquad \textbf{Equation 266}$$

From Equation 247,

$$S\frac{y_s}{y} + \frac{1}{n}W\frac{x}{y} = m\omega^2 r + w\tan\theta \qquad \textbf{Equation 267}$$

Let the stiffness of the spring denote k, v be deflection of spring for a lift of h. Then the spring force is defined as

$$S = kv \qquad (3)$$

At speeds ω_1 and ω_2, the spring forces are

$$S_1\frac{y_s}{y} + \frac{1}{n}W\frac{x}{y} = m\omega_1^{\,2}r_1 + w\tan\theta_1 \qquad (4)$$

$$S_2\frac{y_s}{y} + \frac{1}{n}W\frac{x}{y} = m\omega_2^{\,2}r_2 + w\tan\theta_2 \qquad (5)$$

Subtracting equation (4) from (5) gives

$$(S_2 - S_1) = \frac{y}{y_s}\left[m\left(\omega_2^{\,2}r_2 - \omega_1^{\,2}r_1\right) + w\left(\tan\theta_2 - \tan\theta_1\right)\right] = k(v_2 - v_1) \qquad \textbf{Equation 268}$$

Substituting Equation 249 into (5), we have

$$\left(S_2 - S_1\right) = k\left(v_2 - v_1\right) = k\frac{y_s}{x}\left(h_2 - h_1\right) \qquad \left(S_2 - S_1\right) = k\frac{y_s}{x}\left(h_2 - h_1\right)$$ **Equation 269**

To account for friction force F between the spindle and the sleeve, Equation 250 becomes

$$S\frac{y_s}{y} + \frac{1}{n}\left(W \pm F\right)\frac{x}{y} = m\omega^2 r + w\tan\theta$$ **Equation 270**

Example 6- 11

The extreme radii of rotations of the three fly balls of Hartung type of governor are 80 mm and 120 mm. The governor has a central sleeve spring and three right-angled bell crank levers, which vertical and horizontal arms are 120 mm and 80 mm, respectively. The levers are pivoted at 120 mm from the axis of rotation. Each fly ball has a mass of 3 kg. A retaining spring is attached to each the vertical arm of each crank bell lever at 100 mm from the pivot of the crank and to the frame of the governor. The axis of each spring is perpendicular to the axis of rotation. Knowing that the two extreme speeds of the govern are 400 rpm and 450 rpm for a sleeve lift of 18 mm, and that the horizontal arm is perpendicular to the axis of rotation at the minimum speed, determine (a) stiffness of the spring, and (b) deflection of the spring at the minimum speed if the mass of the sleeve is 50 kg. Neglect friction at all contacts.

Solution

Figure S6-11 shows illustration of the governor at minimum and maximum positions.

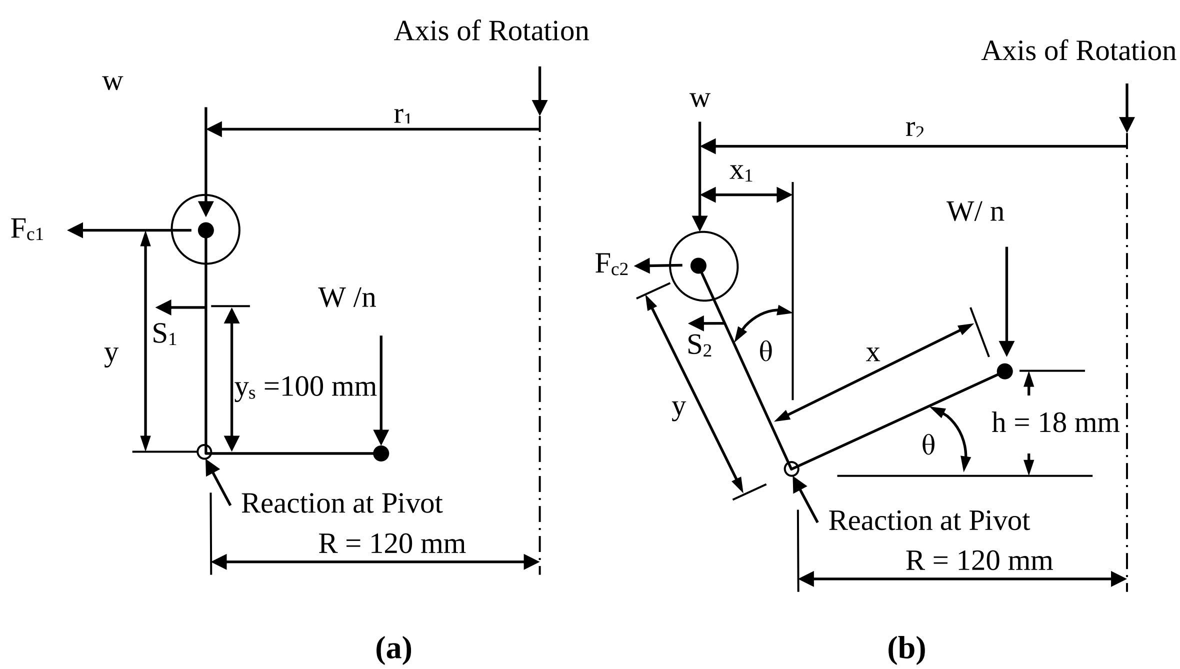


Figure S6-10 (a) Minimum, and (b) Maximum Positions

(a) $y_s = 100\text{ mm}$

$$S_1 \frac{y_s}{y} + \frac{1}{n} W \frac{x}{y} = m\omega_1{}^2 r_1 + w\tan\theta_1$$

$$S_1\left(\frac{100}{120}\right) + \frac{1}{3}W\left(\frac{80}{120}\right) = (3)\left(400 \text{ x } \frac{2\pi}{60}\right)^2 (0.12) + (3 \text{ x } 9.81)\tan 0$$

$$S_1 + 0.26667W = 757.99 \qquad (1)$$

The inclination of the horizontal arm to the horizontal axis is given by

$$\sin\theta = \frac{h}{x} = \frac{18}{80} \qquad \theta = 13.0^o$$

$$r_2 = R + y\sin\theta = 120 + 120\sin 13 \qquad r_2 = 146.99\text{ mm}$$

$$S_2\left(\frac{100}{120}\right) + \frac{1}{3}W\left(\frac{80}{120}\right) = (3)\left(450 \text{ x } \frac{2\pi}{60}\right)^2 (0.14699) + (3 \text{ x } 9.81)\tan 13.0^o$$

$$S_2 + 0.26667W = 1183.2 \qquad (2)$$

Subtracting (1) from (2) gives

$$(S_2 + 0.26667W) - (S_1 + 0.26667W) = 1183.2 - 757.99$$

$$S_2 - S_1 = 425.26\text{ N}$$

(a)

Now $(S_2 - S_1) = k\frac{y_s}{x}(h_2 - h_1) = k\frac{y_s}{x}h$ $\qquad (S_2 - S_1) = 425.26 = k\left(\frac{100}{80}\right)\left(\frac{18}{1000}\right)$

$$\underline{\underline{k = 18.9\text{ kN/m}}}$$

(b) From (1)

$$S_1 + 0.26667W = 757.99$$

$$S_1 = 757.99 - 0.26667W = 757.99 - 0.26667(50 \text{ x } 9.81) \qquad S_1 = 267.49\text{ N}$$

$$S_1 = 267.49 = k\frac{y_s}{x}h_1 = (18.9 \text{ x } 10^3)\left(\frac{100}{80}\right)h_1 \qquad h_1 = 0.0113\text{ m} \qquad \underline{\underline{h_1 = 11.3\text{ mm}}}$$

6-3.7 Pickering Governors

Pickering governors are mostly used for driving gramophones. Figure 67 shows a schematic of a Pickering governor. It consists of three straight leaf springs (only two shown in the figure) arranged at equal angular interval around the spindle. Each spring carries a weight at the centre. As the governor rotates, the centrifugal force causes the attached masses to move radially outward, which results in the bending of the leaf springs. The bending of the springs lifts the sleeve, which is free to slide on the spindle.

Consider a Pickering governor with each governor have a mass equal to m. Suppose that when the governor is at rest, the radial distance from the axis of rotation to the centre of the a governor mass is a. The maximum deflection of a leaf spring with both ends fixed and carrying load P at the centre is given by

$$\delta = \frac{Pl^3}{192EI}$$ **Equation 271**

where l = effective mass of leaf spring, E = modulus of elasticity of the left spring material, and I is moment of inertia of its cross-section about the neutral axis. The area moment of inertia is defined in terms thickness t and width b as

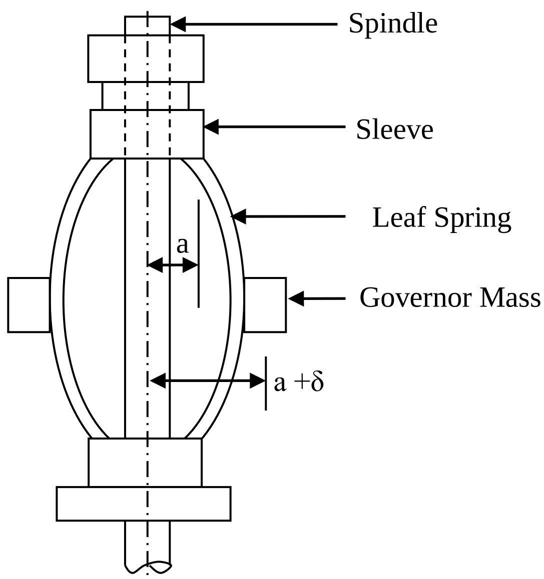


Figure 69: A Schematic of Pickering Governor

$$I = \frac{1}{12}bt^3$$ **Equation 272**

Neglecting the mass of the left spring, the total load on the spring is due to centrifugal force given by

$$P = m\omega^2 (a+\delta)$$ **Equation 273**

Substituting Equation 260 into 258, we have

$$\delta = \frac{m\omega^2 (a+\delta) l^3}{192EI}$$

$$\omega^2 = \frac{192EI\delta}{(a+\delta) l^3 m}$$ **Equation 274**

The lift is given in terms of the deflection δ and the length l by the empirical formula

$$h = \frac{2.4\delta^2}{l}$$ **Equation 275**

Example 6- 12

Each left spring of Pickering governor which drives gramophone is 5 mm wide and 0.12 mm thick. When the governor is at rest, the effective length of each leaf spring is 45 mm and the distance from the axis of rotation to the centre of gravity of each 22-g governor mass attached to each leaf spring is 10 mm. Find the speed of the governor when the lift of the sleeve is 1.0 mm. Take E = 200 GPa.

Solution

$$I = \frac{1}{12} bt^3 = \frac{1}{12}\left(\frac{5}{1000}\right)\left(\frac{0.12}{1000}\right)^3 \qquad I = 7.2 \text{ x} 10^{-16} \text{ m}^4$$

$$h = \frac{2.4\delta^2}{l} \qquad (1) = \frac{2.4\delta^2}{(45-1)} \qquad \delta = 4.282 \text{ mm}$$

$$\omega^2 = \frac{192EI\delta}{(a+\delta) l^3 m} = \frac{192(200 \text{ x} 10^{9})(7.2 \text{ x} 10^{-16})(4.282 \text{ x} 10^{-3})}{(10 \text{ x} 10^{-3} + 4.282 \text{ x} 10^{-3})(44 \text{ x} 10^{-3})^3 (22 \text{ x} 10^{-3})}$$

$$\omega = 2.103 \text{ rad/s} \qquad N = \frac{60}{2\pi}\omega = \frac{60}{2\pi} 2.103 \qquad \underline{\underline{N = 20.1 \text{ rpm}}}$$

6-3.8 Wilson-Hartnell governors and Auxiliary Control Force Elements

Wilson-Hartnell governors use two springs. The two springs are main and auxiliary springs, as shown in Figure 68. The auxiliary spring is attached to the sleeve mechanism via a straight bar or crank bell lever. Each main spring is connected to two fly balls. Note that it is possible to have an auxiliary spring and a connecting lever for other governors if the control force from the main spring or counterweight is not enough to reach the desired equilibrium speed. Auxiliary springs and lever are recommended for high speed governors, where the centrifugal forces are very high.

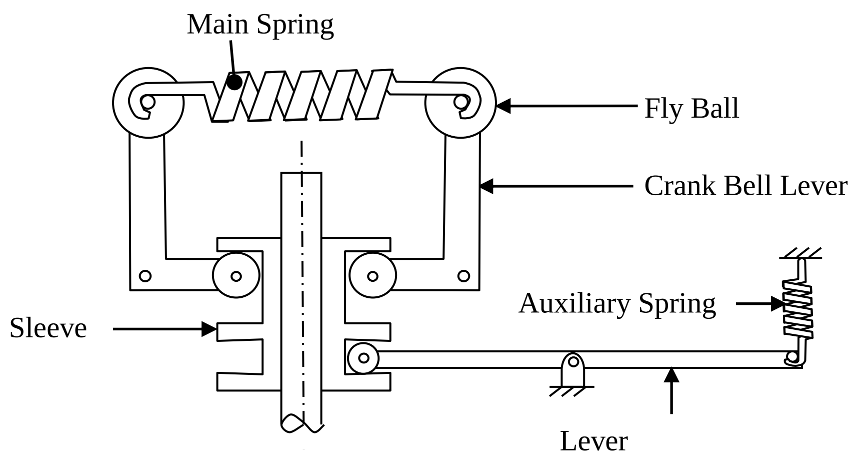


Figure 70: A Schematic of Wilson-Hartnell Governor with Auxiliary Spring

Consider the free-body diagram of the auxiliary lever and spring in Figure 68. Suppose that there is a sleeve lift h and the corresponding auxiliary spring force and displacement are in S_a and h_a, as shown in Figure 69. In the figure, F_a-is the control force from the auxiliary spring. Summing moment about the pivot, we have

$$F_a L_1 - S_a L_2 = 0 \qquad F_a = S_a \frac{L_2}{L_1}$$ **Equation 276**

The force F_a is then added to the restore force on the sleeve. Using similar triangles to Figure 69(b)

$$\frac{h_a}{L_2} = \frac{h}{L_1} \qquad h_a = h\frac{L_2}{L_1}$$ **Equation 277**

The auxiliary spring force is determined from its deflection as

$$S_a = k_a h \frac{L_2}{L_1}$$ **Equation 278**

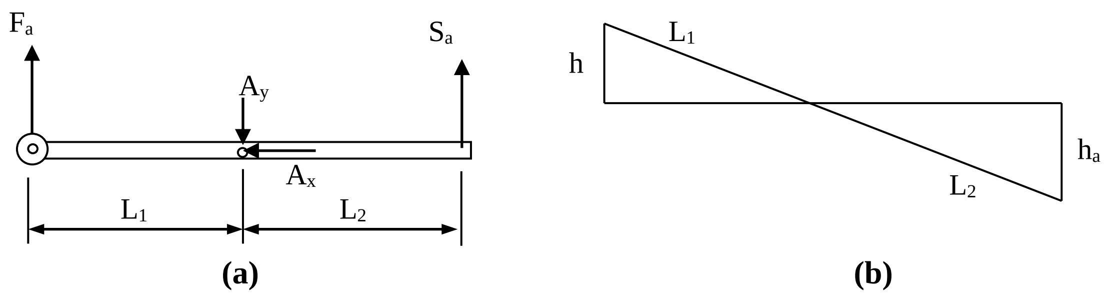


Figure 71: (a) Free-Body Diagram, and (b) Displacement of Auxiliary Lever

SESSION 6-4: Properties of Governors

6-4.1 Sensitiveness

The sensitiveness of a governor is defined as the ratio of the difference between the maximum and minimum equilibrium speeds to the mean speed. That is

$$C_s = \frac{\omega_{\max} - \omega_{\min}}{\frac{1}{2}(\omega_{\max} + \omega_{\min})} \qquad C_s = \frac{2(\omega_{\max} - \omega_{\min})}{(\omega_{\max} + \omega_{\min})}$$ **Equation 279**

$$C_s = \frac{2(N_{\max} - N_{\min})}{(N_{\max} + N_{\min})}$$ **Equation 280**

6-4.2 Hunting

The sensitiveness of a governor is the same as coefficient of speed fluctuation in flywheels. When a governor is too sensitive, the governor cause the speed of the engine or the system is controlling to fluctuate continuously above and below the equilibrium speed. Such a governor is said to be hunting.

6-4.3 Stability

A governor is said to be stable there is only one radius of rotation of the fly balls for a particular speed with the operating range at which the governor is in equilibrium. For stable governor, the radius of rotation increases with the speed increases.

6-4.4 Isochronous

A governor is said to be isochronous when the equilibrium speed is constant for all radii of rotation of fly balls with the working range. Gravity controlled governors such as Porter governor are not isochronous, but spring loaded governors are capable. The isochronous governors have no practical use because the sleeve move to its extreme position immediately the speed deviates from the isochronous speed. It is possible to use this governor as angular speed-dependent toggle position switch.

Self Assessment 6

6-1. The turning moment diagram of a four-stroke engine is shown in Figure E6-1. The engine is supposed to operate between the speeds 110 rpm to 120 rpm. If the resisting torque is constant, determine the moment of inertia of a flywheel that must be attached to crankshaft of the engine to keep the engine running within the acceptable speed range.

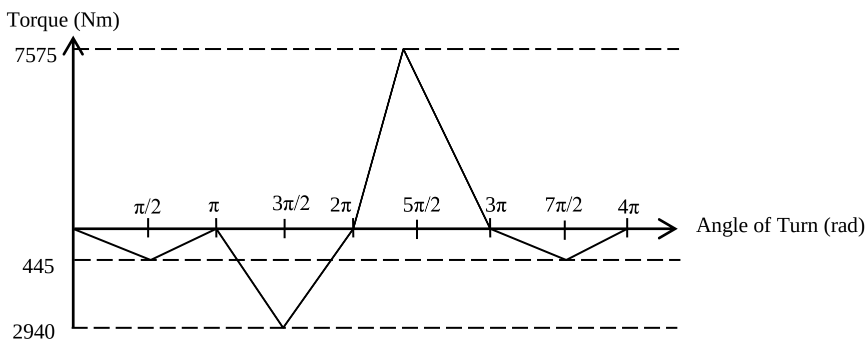


Figure E6-1

6-2. A punching press is fitted with a flywheel that rotates at a maximum speed of 220 rpm. The press punches 750 holes per hour. Each punching operation takes 2 s and requires 15 kJ of energy. If the press is driven by a constant electric motor and the speed of the flywheel is not to fall below 200 rpm, find the moment of inertia of the flywheel.

6-3. The mass and radius of gyration of a flywheel attached to an engine is 7 tonnes and 2.0 m. If the fluctuation of energy of an engine at a mean speed of 120 rpm is 60 kJ, determine the minimum and maximum speeds of the engine.

6-4. An engine develops 200 kW of power at a mean speed of 75 rpm. The coefficient of energy fluctuation is 0.15 and speed fluctuates within ±3% of the mean speed. Find the moment of inertia of the flywheel that must attached to the crankshaft of the engine in order to maintain the speed within the acceptable range.

6-5. Find the rotation speed of a Watt governor rotating if lift is 300 mm.

6-6. The length of the upper arm of a Watt governor is 300 mm and its inclination to the axis of rotation is 35^{o}. Find the percentage increase in speed if the balls rise by 22 mm.

6-7. The upper and lower arms of a three-ball Porter governor are each 350 long. The upper arms are pivoted on the axis of rotation, and the lower arms pivoted on the sleeve at a distance 30 mm from the axis of rotation. The total mass of the sleeve and the load are 50 kg, and mass of each fly ball is 3 kg. Neglecting friction, determine the speed of rotation of the governor if the radius of each ball is 250 mm.

6-8. The upper and lower arms of a three-fly ball Porter governor are each 350 mm long and pivoted on the axis of rotation. The mass of each fly ball is 3 kg and that of the sleeve is 45 kg. The radius of rotation of the balls ranges from 100 mm to 200 mm. If the friction force between the sleeve and the spindle is 25 N, find the speed range of the governor.

6-9. The upper and lower arms of a two-fly ball Porter governor are each 350 mm long and pivoted on the axis of rotation, and the mass of each fly ball is 3 kg. When the radius of rotation of the balls is 250 mm, the speeds of governor are, respectively, 160 rpm when the sleeve is moving upward and 150 rpm when the sleeve is moving downward. Find the mass of the sleeve and the friction force acting between the sleeve and the spindle.

6-10. A two-ball Proell governor has equal arms of length 300 mm. The upper and lower arms are pivoted 35 mm and 40 mm respectively from the axis of rotation. Each ball has mass of 4 kg and that of the sleeve and load combined is 70 kg. The friction force between sleeve and spindle is 30 N. Determine the speed of the governor knowing that the radius of rotation is 200 mm, and the extension link of the lower arm is 60 mm and vertical.

6-11. The extreme radii of rotations of the three fly balls of Hartung type of governor are 120 mm and 160 mm. The governor has a central sleeve spring and three right-angled bell crank levers, which vertical and horizontal arms are 150 mm and 100 mm, respectively. The levers are pivoted at 120 mm from the axis of rotation. Each fly ball has a mass of 3 kg. A retaining spring is attached to each the vertical arm of each crank bell lever at 100 mm from the pivot of the crank and to the frame of the governor. The axis of each spring is perpendicular to the axis of rotation. Knowing that the two extreme speeds of the govern are 400 rpm and 450 rpm for a sleeve lift of 25 mm, and that the horizontal arm is perpendicular to the axis of rotation at the minimum speed, determine (a) total load on the sleeve at the lowest and highest speeds, (b) stiffness of the spring, and (c) deflection of the spring at the minimum speed if the mass of the sleeve is 50 kg. Neglect friction at all contacts.

6-12. A Proell governor has two fly balls, each of mass 2 kg. The lengths of all the arms are 250 mm. The upper and lower arms are pivoted at 30 mm and 40 mm, respectively, from the axis of rotation. The length of extension of lower arms on which each fly ball is attached is 60 mm, and inclined at 25^o above and to the lower arm. The mass of the sleeve is 60 kg and friction force between the sleeve and the spindle is 50 N. Knowing that the arms are inclined at 50^o to the axis of rotation, determine the equilibrium speed of the governor.

6-13. Two 5-mm wide and 0.15 mm think leaf springs are used in a Pickering governor to drive a gramophone. When the governor is at rest, the effective length of each leaf spring is 500 mm and the distance from the axis of rotation to the centre of gravity of the mass attached to each spring is 12 mm. When the speed of the governor is 10 rpm, the sleeve lift is 1.2 mm. Knowing that the mass attached to each left spring is 15 g, find the mass of the fly ball attached to each leaf spring.

☑ Key terms/ New Words in Unit

1. Coefficient of speed fluctuation
2. Coefficient of energy fluctuation
3. Lift
4. Hunting
5. Stability
6. Isochronous
7. Sensitiveness

SELECTED ANSWERS TO UNIT ASSIGNMENTS

Unit 1

1-1. $\omega = -4\mathbf{i} + 7.66\mathbf{j} + 11.43\mathbf{k}$ (rad/s), $\alpha = -40.30\mathbf{i} + 8.01\mathbf{j} - 28.07\mathbf{k}$ (rad/s^2)

1-2. $\omega = 4\mathbf{i} + 6\mathbf{j}$ (rad/s), $\alpha = 24\mathbf{k}$ (rad/s^2)

1-3. $\omega = 0.111\mathbf{i} + 2.598\mathbf{j} - 1.333\mathbf{k}$ (rad/s), $\alpha = -0.433\mathbf{i} + 0.185\mathbf{j} + 0.289\mathbf{k}$ (rad/s^2)

1-4. $\omega = 0.1\mathbf{k}$ (rad/s), $\alpha = 0.2\mathbf{k}$ (rad/s^2)

1-5. $2.05\,\mathbf{i} - 1.6\,\mathbf{j} + 2.45\,\mathbf{k}$ (m/s)

1-6. (a) $\omega = 4\,\mathbf{i} + 6\,\mathbf{j}$ (rad/s) $\alpha = 24\,\mathbf{k}$ (rad/s^2) (b) $v = -1.198\,\mathbf{i} + 0.799\,\mathbf{j}$ - -$1.039\,\mathbf{k}$, (m/s)

$a = 4.157\,\mathbf{j}$ (m/s^2)

1-7. 0.15 j m/s
1-8. -0.333 k m/s
1-9. $8.839\,\mathbf{i} - 8.839\,\mathbf{j}$ (rad/s) (b) $-2.25\,\mathbf{k}$ (m/s)
1-10. (a) $\omega_{plate} = 0.164i$ (b) $\omega_{actuator} = -0.749\,i + 0.341\,j$

Unit 2

2-1. (a) x = y = 0, z = 3h/4, (b) $I_{xx,O} = I_{yy,O} = m\left(\frac{3}{5}h^2 + \frac{3}{20}r^2\right)$, $I_{zz,O} = \frac{3}{10}mr^2$

(c) $I_{xx,G} = I_{yy,G} = m\left(\frac{3}{80}h^2 + \frac{3}{20}r^2\right)$, $I_{zz,G} = \frac{3}{10}mr^{2\;2}$

2-2. (a) $x_G = 0$, $y_G = 182.5$ mm, $z_G = -6.2$ mm (b) $I_{xx} = 4.661$ kg-m^2, $I_{yy} = 0.983$ kg-m^2, $I_{zz} =$ 4.2019 kg-m^2

2-3. $I_{xx} = 4.886ma^2$

2-4. $I_{xx} = I_{yy} = m(0.081R^2 - 0.025h^2)$, $I_{zz} = 0.1618mR^2$, $I_{xy} = I_{yz} = I_{xz} = 0$

2-5. (a) $[I] = \begin{bmatrix} 56.67 & -12.5 & -7.5 \\ -12.5 & 21.67 & -18.75 \\ -7.5 & -18.75 & 48.33 \end{bmatrix}$ kg - m^2 (b) $I_{\mathrm{DF}} = 7.917$ kg - m^2

(c) $I_{\mathrm{AC}} = 35.118$ kg - m^2

2-6. (a) $I_{yy} = 0.012$ kg-m^2 (a) $I_{xx} = 844$ kg-m^2

2-7. $I_{xy} = \frac{1}{12} ma^2 \quad I_{yz} = \frac{1}{6} mah$

2-8. $I_{xx} = 1.36$ kg-m^2 , $I_{yy} = 0.380$ kg-m^2 and $I_{zz} = 1.25$ kg-m^2

2-9. $I_{zz} = 0.0915$ kg-m^2

Unit 3

3-1. 0.872 rad/s
3-2. 636 rad/s
3-3. (a) 3.14 rad/s, (b) 6.04 rad/s
3-4. 5.4^o
3-5. 54.8 kN-m
3-6. 164 kN-m (Nose tilts upward)
3-7. 112 N-m
3-8. 26.8 N
3-9. 58.9 kN-m (Nose tilts downward)
3-10. 108 km/h
3-11. (a) 42^o, (b) 40^o

Unit 4
4-1. (a) 46.7 N-m ccw (b) 43.6 N-m ccw (c) 50.3 N-m ccw
4-2. 0.101 N-m cw
4-3. 643.7 N
4-4. 181.8 N-m ccw
4-5. $A_x = -153$ N, $A_y = -20.6$ N, $B_x = 144$ N, $B_y = 4.4$ N, $C_x = 114$ N, $C_y = -17.3$ N, T = 3.68 N-m
4-6. 363.5 N-m ccw
4-7. (a) 2.24 kN, (b) 2.26 kN, (c) 306 N, (d) 3002 rpm ccw
4-8. 5.94 N-m cw

Unit 5
5-1. (a) 0.203 m (b) 650.7 mm (c) $m_1 = 3.26$ kg and $m_2 = 26.74$ kg
5-2. Fs = 0, Ms = 0
5-3. 2mr at 180^o to the crank

Unit 6
6-1. 830.5 kg-m^2
6-2. 190 kg-m^2
6-3. $N_{max} = 128$ rpm $N_{min} = 112$ rpm
6-4. 6485 kg-m^2
6-5. 54.6 rpm

6-6. -4.19%

6-7. 247 rpm

6-8. $158.5 \leq N \leq 209$ rpm

6-9. M = 16.75 kg and F = 12.5 N

6-10. 212 rpm

6-11. (a) $W+S_1$ = 2.84 kN, $W+S_2$ = 4.75 kN (b) 76.5 kN/m, (c) 18.6 mm

6-12. 267.8 rpm

6-13. 5.86 g

MY PAGE

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Contact: Tel. ____________ Email: ________________ Emergency Name/Phone: __________

Important numbers: Student number ______________ Examination number ________________

Program: ___________ Year: _______ Course code/title: _______________________________

Course objectives: ____

____

Course dates/Semester No (   ): Starts_____________________ Ends ____________________

FFFS schedule/Dates: ____

Quiz dates: ____

Assignments hand in dates: ____

Revision dates: ____

Group discussion/work members (names and contacts): _________________________________

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End of course Self Evaluation:
I have completed all Units & interactive sessions□, mastered all learning objectives, completed all self Assessments, unit summary, key words and terms, discussion questions, review questions, reading activity, web activity, unit assignments, and submitted all CA scoring assignments, learner feedback on this course and submitted my comments and course focus contributory questions to facilitator for discussion.

Self-grading: self assessment questions score ______ %   Unit Assignments scored ______ %

My course conclusion remarks: ____

____

____

____ (may continue on reverse side)

Learner Feedback Form/[insert course code]

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Dear Learner,
While studying the units in the course, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill out and send the following questionnaire, which pertains to this course. If you find the space provided insufficient, kindly use a separate sheet.

1. How many hours did you need for studying the units

Unit no.	1	2	3	4	5	6
No. of hours						

2. Please give your reactions to the following items based on your reading of the course

Items	**Excellent**	**Very good**	**Good**	**Poor**	*Give specific examples, if poor*
Presentation quality	☐	☐	☐	☐	
Language and style	☐	☐	☐	☐	
Illustrations used (diagrams, tables, etc.)	☐	☐	☐	☐	
Conceptual clarity	☐	☐	☐	☐	
Self assessment	☐	☐	☐	☐	
Feedback to SA	☐	☐	☐	☐	

3. Any other comments (may continue on reverse side)

Unit 1: ____________________

Unit 2: ____________________

Unit 3: ____________________

Unit 4: ____________________

Unit 5: ____________________

Unit 6: ____________________