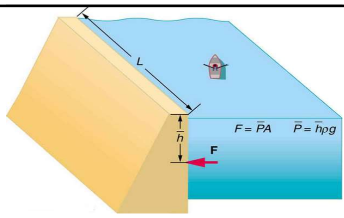

ME 251 Introduction to Fluid Mechanics

Fluid Statics

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1

Areas to Cover

- Fluid Statics
  - Hydrostatic forces on submerged surfaces
  - Buoyancy and flotation
  - Fluids in rigid-body motion

2

FLUID STATICS

- Fluid statics deals with problems associated with fluids at rest.
- Fluid statics is generally referred to as
  - Hydrostatics when the fluid is a liquid and
  - Aerostatics when the fluid is a gas.
- No relative motion between adjacent fluid layers, and thus there are no shear (tangential) stresses in the fluid trying to deform it.
- The only stress to deal with is the normal stress, which is the pressure,
  - the variation of pressure is due only to the **weight** of the fluid.

3

HYDROSTATIC FORCES ON SUBMERGED PLANE SURFACES

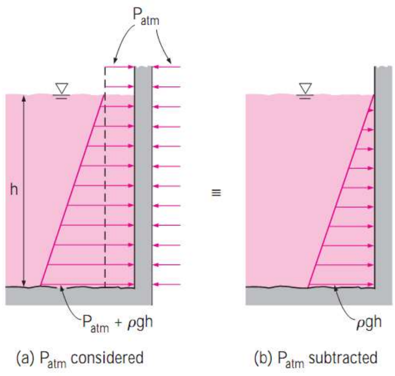
- A plate exposed to a liquid is subjected to fluid pressure distributed over its surface.
- On a plane surface, the hydrostatic forces form a system of parallel forces.
  - the magnitude of the force and
  - its point of application (center of pressure).

4

HYDROSTATIC FORCES ON SUBMERGED PLANE SURFACES

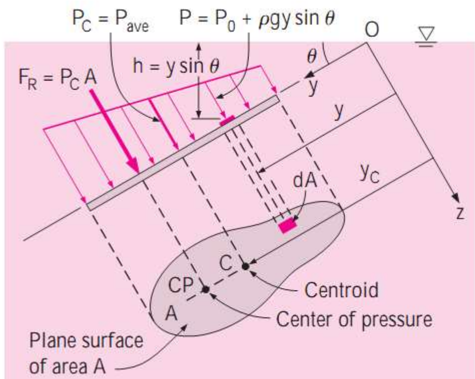
- The other side of the plate may be open to the atmosphere (such as the dry side of a gate),
- Atmospheric pressure acts on both sides of the plate, yielding a zero resultant.
- It is convenient to work with the gauge pressure only .
- At the bottom of the lake

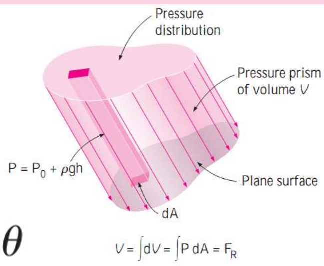
$$P_{gauge} = \rho gh.$$



5

- A flat plate of arbitrary shape is completely submerged in a liquid.
- The plane of the surface intersects the horizontal free surface with an angle θ.
- The absolute pressure above the liquid is P_0, which is the local atmospheric pressure P_{atm}.
- Then the absolute pressure at any point on the plate is

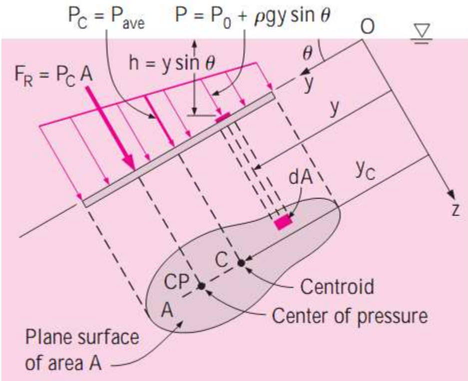




$$P = P_0 + \rho gh = P_0 + \rho gy \sin\theta$$

6

- The resultant hydrostatic force F_R acting on the surface is determined by integrating the force **PdA** acting on a differential area **dA** over the entire surface area,



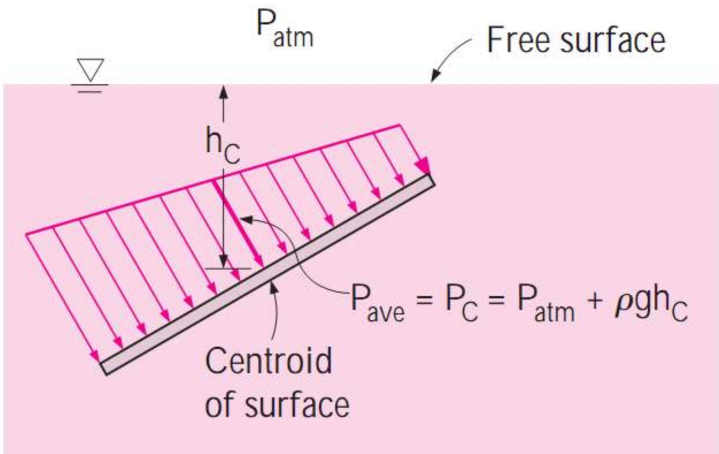
$$F_R = \int_A P\,dA = \int_A (P_0 + \rho gy \sin\theta)\,dA = P_0A + \rho g \sin\theta \int y\,dA$$

first moment of area $\int_A y\,dA$ $\qquad y_C = \frac{1}{A}\int_A y\,dA$

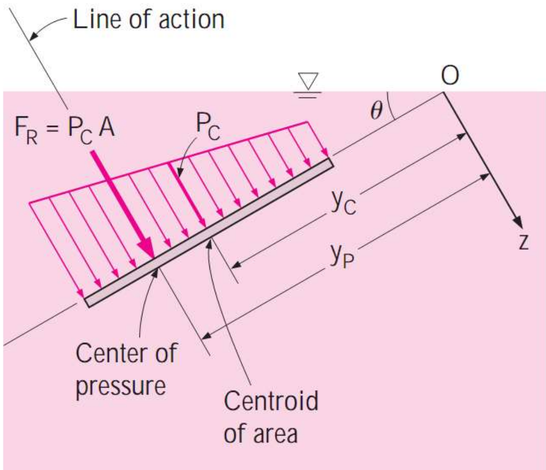
$$F_R = (P_0 + \rho gy_C \sin\theta)A = (P_0 + \rho gh_C)A = P_C A = P_{ave} A$$

where $P_C = P_0 + \rho gh_C$ is the pressure at the centroid of the surface, which is equivalent to the average pressure on the surface, and $h_C = y_C \sin\theta$ is the vertical distance of the centroid from the free surface of the liquid.

$$F_R = (P_0 + \rho gy_C \sin\theta)A = (P_0 + \rho gh_C)A = P_C A = P_{ave} A$$



The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid is equal to the product of the pressure P_C at the centroid of the surface and the area **A** of the surface



- Two parallel force systems are equivalent if they have the same magnitude and the same moment about any point.
- The line of action of the resultant hydrostatic force, in general, does not pass through the centroid of the surface
- It lies underneath where the pressure is higher.
- The point of intersection of the line of action of the resultant force and the surface is the **center of pressure.**
- The vertical location of the line of action is determined by equating the moment of the resultant force to the moment of the distributed pressure force about the x-axis.

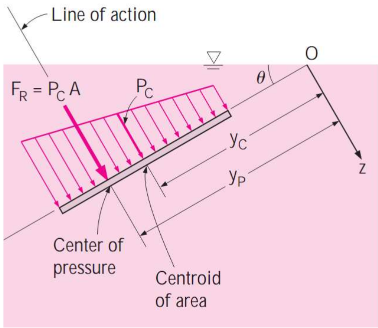
$$y_P F_R = \int_A yP\,dA = \int_A y(P_0 + \rho g y \sin\theta)\,dA = P_0 \int_A y\,dA + \rho g \sin\theta \int_A y^2\,dA$$

$$y_P F_R = P_0 y_C A + \rho g \sin\theta\, I_{xx,\,O}$$

Second moment of area (area moment of inertia) about the x-axis

$$I_{xx,\,O} = \int_A y^2\,dA$$

- The second moments of area about two parallel axes are related to each other by the parallel axis theorem, which in this case is expressed as



$$I_{xx,\,O} = I_{xx,\,C} + y_C^2 A$$

where $I_{xx,\,C}$ is the second moment of area about the x-axis passing through the centroid of the area and y_C (the y-coordinate of the centroid) is the distance between the two parallel axes.

11

- Substituting the F_R relation and the $I_{xx,\,O}$ relation and solving for y_P

$$F_R = (P_0 + \rho g y_C \sin\theta)A = (P_0 + \rho g h_C)A = P_C A = P_{ave} A$$

$$I_{xx,\,O} = I_{xx,\,C} + y_C^2 A$$

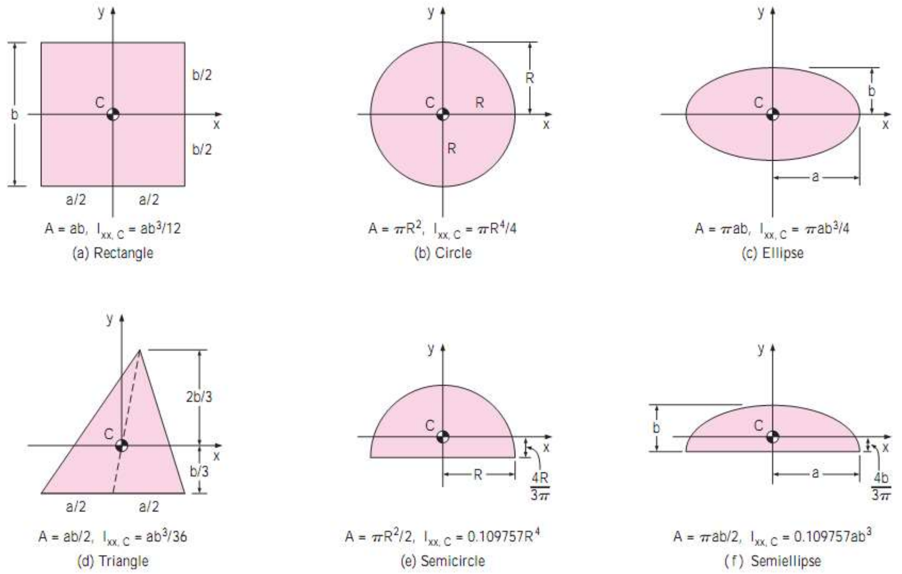
$$y_P F_R = P_0 y_C A + \rho g \sin\theta \, I_{xx,\,O}$$

$$y_P = y_C + \frac{I_{xx,\,C}}{[y_C + P_0/(\rho g \sin\theta)]A} \qquad y_P = y_C + \frac{I_{xx,\,C}}{y_C A}$$

Knowing y_P, the vertical distance of the center of pressure from the free surface is determined from $h_P = y_P \sin\theta$.

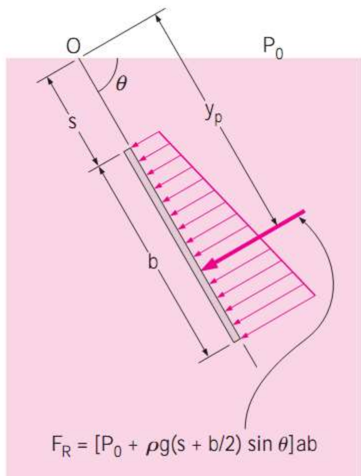
12

The centroid and the centroidal moments of inertia for some common geometries



Special Case: Submerged Rectangular Plate

- Consider a completely submerged rectangular flat plate of height b and width a tilted at an angle θ from the horizontal and whose top edge is horizontal and is at a distance s from the free surface along the plane of the plate,
- The resultant hydrostatic force on the upper surface is equal to the average pressure, which is the pressure at the midpoint of the surface, times the surface area A. That is,



(a) Tilted plate

$$F_R = P_C A = [P_0 + \rho g(s + b/2)\sin\theta]ab$$

Submerged Rectangular Plate

- The force acts at a vertical distance of $h_P = y_P \sin\theta$ from the free surface directly beneath the centroid of the plate where,

$$y_P = s + \frac{b}{2} + \frac{ab^3/12}{[s + b/2 + P_0/(\rho g \sin\theta)]ab}$$
$$= s + \frac{b}{2} + \frac{b^2}{12[s + b/2 + P_0/(\rho g \sin\theta)]}$$

When the upper edge of the plate is at the free surface and thus s = 0, then

$$F_R = [P_0 + \rho g (b \sin\theta)/2]ab$$

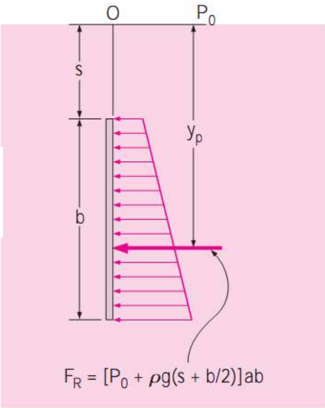
15

Submerged Rectangular Plate

- For a completely submerged vertical plate (θ = 90°) whose top edge is horizontal, the hydrostatic force can be obtained by setting sinθ = 1

$$F_R = [P_0 + \rho g(s + b/2)]ab$$

$$F_R = (P_0 + \rho g b/2)ab$$



(b) Vertical plate

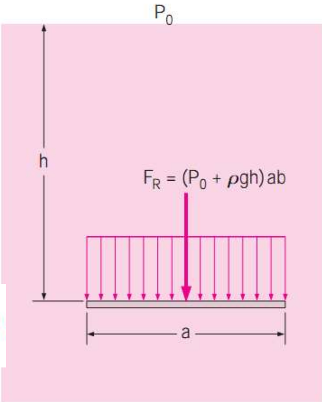
16

Submerged Rectangular Plate

- The pressure distribution on a submerged horizontal surface is uniform,
- and its magnitude is $P = P_0+\rho gh$, where h is the distance of the surface from the free surface.
- Therefore, the hydrostatic force acting on a horizontal rectangular surface is

$$F_R = (P_0 + \rho gh)ab$$

- and it acts through the midpoint of the plate.



(c) Horizontal plate

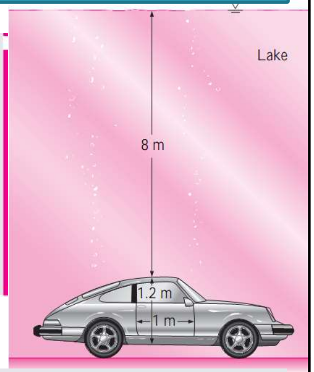
17

A heavy car plunges into a lake during an accident and lands at the bottom of the lake on its wheels. The door is 1.2 m high and 1 m wide, and the top edge of the door is 8 m below the free surface of the water. Determine the hydrostatic force on the door and the location of the pressure center, and discuss if the driver can open the door.

SOLUTION A car is submerged in water. The hydrostatic force on the door is to be determined, and the likelihood of the driver opening the door is to be assessed.

Assumptions **1** The bottom surface of the lake is horizontal. **2** The passenger cabin is well-sealed so that no water leaks inside. **3** The door can be approximated as a vertical rectangular plate. **4** The pressure in the passenger cabin remains at atmospheric value since there is no water leaking in, and thus no compression of the air inside. Therefore, atmospheric pressure cancels out in the calculations since it acts on both sides of the door. **5** The weight of the car is larger than the buoyant force acting on it.

Properties We take the density of lake water to be 1000 kg/m³ throughout.



18

Analysis The average pressure on the door is the pressure value at the centroid (midpoint) of the door and is determined to be

$$P_{ave} = P_C = \rho g h_C = \rho g(s + b/2)$$
$$= (1000\ kg/m^3)(9.81\ m/s^2)(8 + 1.2/2\ m)\left(\frac{1\ kN}{1000\ kg \cdot m/s^2}\right)$$
$$= 84.4\ kN/m^2$$

Then the resultant hydrostatic force on the door becomes

$$F_R = P_{ave}A = (84.4\ kN/m^2)(1\ m \times 1.2\ m) = 101.3\ kN$$

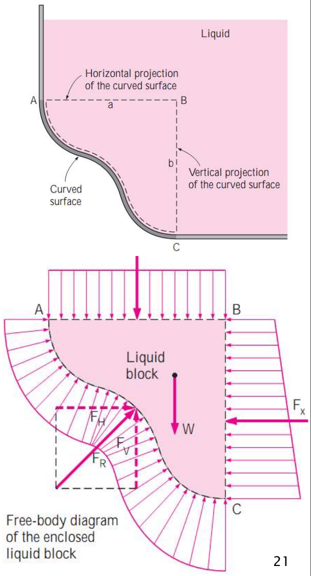
The pressure center is directly under the midpoint of the door, and its distance from the surface of the lake is determined from Eq. 3–24 by setting $P_0 = 0$ to be

$$y_P = s + \frac{b}{2} + \frac{b^2}{12(s + b/2)} = 8 + \frac{1.2}{2} + \frac{1.2^2}{12(8 + 1.2/2)} = 8.61\ m$$

Discussion A strong person can lift 100 kg, whose weight is 981 N or about 1 kN. Also, the person can apply the force at a point farthest from the hinges (1 m farther) for maximum effect and generate a moment of 1 kN · m. The resultant hydrostatic force acts under the midpoint of the door, and thus a distance of 0.5 m from the hinges. This generates a moment of 50.6 kN · m, which is about 50 times the moment the driver can possibly generate. Therefore, it is impossible for the driver to open the door of the car. The driver's best bet is to let some water in (by rolling the window down a little, for example) and to keep his or her head close to the ceiling. The driver should be able to open the door shortly before the car is filled with water since at that point the pressures on both sides of the door are nearly the same and opening the door in water is almost as easy as opening it in air.

HYDROSTATIC FORCES ON SUBMERGED CURVED SURFACES

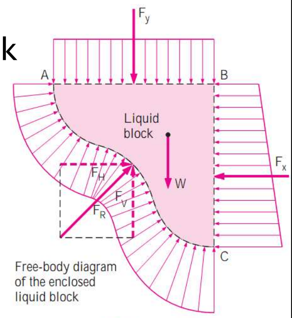
- The resultant hydrostatic force F_R is of the horizontal F_H and vertical F_V components.
- The vertical surface of the liquid block considered is simply the projection of the curved surface on a vertical plane, and
- The horizontal surface is the projection of the curved surface on a horizontal plane.
- The resultant force acting on the curved solid surface is then equal and opposite to the force acting on the curved liquid surface (Newton's third law).



21

SUBMERGED CURVED SURFACES

- The weight of the enclosed liquid block of volume V is simply $W = \rho gV$,
- It acts downward through the centroid of this volume.
- Noting that the fluid block is in static equilibrium, the force balances in the horizontal and vertical directions give



Horizontal force component on curved surface: $F_H = F_x$

Vertical force component on curved surface: $F_V = F_y + W$

where the summation $F_y + W$ is a vector addition (i.e., add magnitudes if both act in the same direction and subtract if they act in opposite directions).

22

SUBMERGED CURVED SURFACES

- 1. The horizontal component of the hydrostatic force acting on a curved surface is equal (in both magnitude and the line of action) to the hydrostatic force acting on the vertical projection of the curved surface.
- 2. The vertical component of the hydrostatic force acting on a curved surface is equal to the hydrostatic force acting on the horizontal projection of the curved surface, plus (minus, if acting in the opposite direction) the weight of the fluid block.

23

SUBMERGED CURVED SURFACES

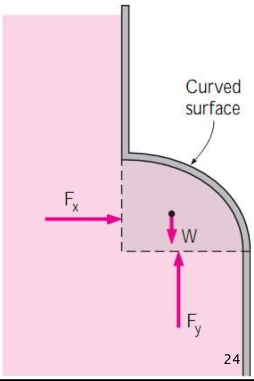
- The magnitude of the resultant hydrostatic force acting on the curved surface is

$$F_R = \sqrt{F_H^2 + F_V^2}$$

- and the tangent of the angle it makes with the horizontal is

$$\tan \alpha = F_V / F_H$$

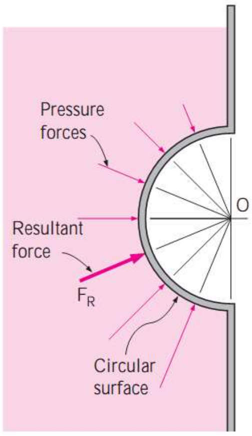
- The exact location of the line of action of the resultant force (e.g., its distance from one of the end points of the curved surface) can be determined by taking a moment about an appropriate point.
- NB: a curved surface above a liquid; the weight of the liquid is subtracted from the vertical component



24

SUBMERGED CURVED SURFACES

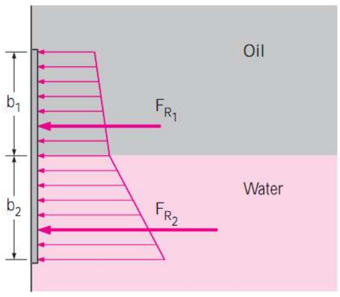
- When the curved surface is a circular arc (full circle or any part of it),
- The resultant hydrostatic force acting on the surface always passes through the center of the circle.
  - This is because the pressure forces are normal to the surface, and all lines norma to the surface of a circle pass through the center of the circle.
- Thus, the pressure forces form a concurrent force system at the center, which can be reduced to a single equivalent force at that point.



25

SUBMERGED CURVED SURFACES

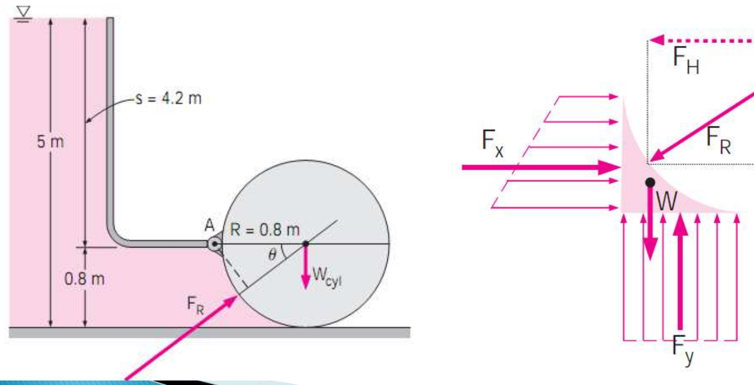
- Hydrostatic forces acting on a plane or curved surface submerged in a multilayered fluid of different densities can be determined by
  - considering different parts of surfaces in different fluids as different surfaces,
  - finding the force on each part, and
- then adding them using vector addition.
- For a plane surface, it can be expressed as



$$F_R = \sum F_{R,i} = \sum P_{C,i} A_i \qquad P_{C,i} = P_0 + \rho_i g h_{C,i}$$

26

A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate. When the water level reaches 5 m, the gate opens by turning about the hinge at point A. Determine (a) the hydrostatic force acting on the cylinder and its line of action when the gate opens and (b) the weight of the cylinder per m length of the cylinder



27

SOLUTION The height of a water reservoir is controlled by a cylindrical gate hinged to the reservoir. The hydrostatic force on the cylinder and the weight of the cylinder per m length are to be determined.

Assumptions **1** Friction at the hinge is negligible. **2** Atmospheric pressure acts on both sides of the gate, and thus it cancels out.

Properties We take the density of water to be 1000 kg/m³ throughout.

Analysis (*a*) We consider the free-body diagram of the liquid block enclosed by the circular surface of the cylinder and its vertical and horizontal projections. The hydrostatic forces acting on the vertical and horizontal plane surfaces as well as the weight of the liquid block are determined as

Horizontal force on vertical surface:

$$F_H = F_x = P_{ave}A = \rho g h_C A = \rho g(s + R/2)A$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(4.2 + 0.8/2\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg} \cdot \text{m/s}^2}\right)$$

$$= 36.1\ \text{kN}$$

Vertical force on horizontal surface (upward):

$$F_y = P_{ave}A = \rho g h_C A = \rho g h_{bottom} A$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg} \cdot \text{m/s}^2}\right)$$

$$= 39.2\ \text{kN}$$

28

Weight of fluid block per m length (downward):

$$W = mg = \rho g V = \rho g(R^2 - \pi R^2/4)(1\ \text{m})$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})^2(1 - \pi/4)(1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 1.3\ \text{kN}$$

Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\ \text{kN}$$

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = 52.3\ \text{kN}$$

$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4°$$

Therefore, the magnitude of the hydrostatic force acting on the cylinder is 52.3 kN per m length of the cylinder, and its line of action passes through the center of the cylinder making an angle 46.4° with the horizontal.

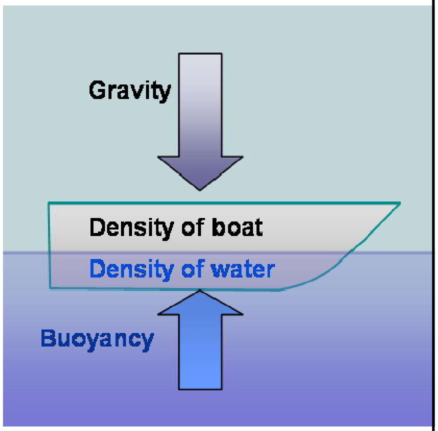
(*b*) When the water level is 5 m high, the gate is about to open and thus the reaction force at the bottom of the cylinder is zero. Then the forces other than those at the hinge acting on the cylinder are its weight, acting through the center, and the hydrostatic force exerted by water. Taking a moment about point *A* at the location of the hinge and equating it to zero gives

$$F_R R \sin\theta - W_{cyl} R = 0 \rightarrow W_{cyl} = F_R \sin\theta = (52.3\ \text{kN}) \sin 46.4° = 37.9\ \text{kN}$$

Discussion The weight of the cylinder per m length is determined to be 37.9 kN. It can be shown that this corresponds to a mass of 3863 kg per m length and to a density of 1921 kg/m³ for the material of the cylinder.

BUOYANCY AND STABILITY

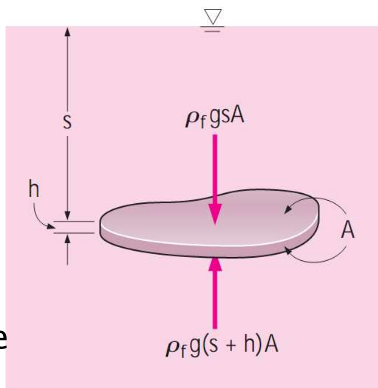
- An object feels lighter and weighs less in a liquid than it does in air.
- An objects made of wood or other light materials float on water.
- A fluid exerts an upward force on a body immersed in it called the **buoyant** force.
- The buoyant force is caused by the increase of pressure in a fluid with depth.



31

BUOYANCY AND STABILITY

- The pressures at the top and bottom surfaces of the plate are $\rho_f gs$ and $\rho_f g(s+h)$, respectively.
- Force $F_{top}=\rho_f gsA$ acts downward on the top surface
- Force $F_{bottom}=\rho_f g(s+h)A$ acts upward on the bottom surface
- The difference between these two forces is a net upward force (the buoyant force),

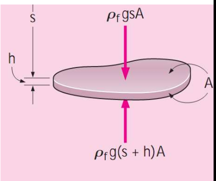


$$F_B = F_{bottom} - F_{top} = \rho_f g(s + h)A - \rho_f gsA = \rho_f ghA = \rho_f gV$$

32

BUOYANCY AND STABILITY

$$F_B = \rho_f g V$$



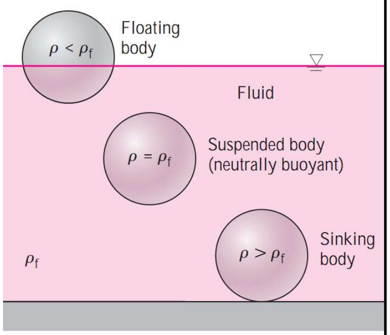
- $V = hA$ is the volume of the plate.
- But the relation $\rho_f gV$ is the weight of the liquid whose volume is equal to the volume of the plate.

  Buoyant force = weight of the liquid displaced
- the buoyant force is independent of
  - the distance of the body from the free surface.
  - the density of the solid body.
- The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.

33

BUOYANCY AND STABILITY

- For floating bodies,
  - the weight of the entire body must be equal to the buoyant force, (weight of the fluid whose volume is the volume of the submerged portion of the floating body.

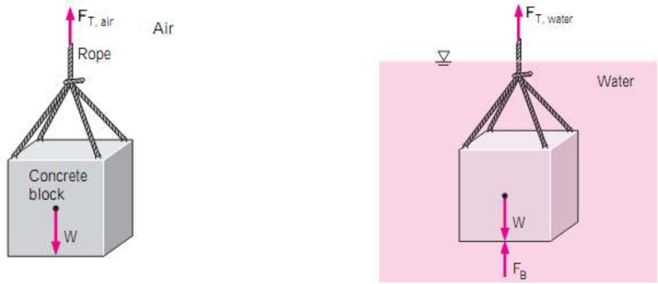


$$F_B = W \rightarrow \rho_f g V_{sub} = \rho_{ave,\ body} g V_{total} \rightarrow \frac{V_{sub}}{V_{total}} = \frac{\rho_{ave,\ body}}{\rho_f}$$

The submerged volume fraction of a floating body is equal to the ratio of the average density of the body to the density of the fluid. When the density ratio is equal to or greater than one, the floating body becomes completely submerged.

34

A crane is used to lower weights into the sea (density = 1025 kg/m3) for an underwater construction project. Determine the tension in the rope of the crane due to a rectangular 0.4-m x 0.4-m x 3-m concrete block (density = 2300 kg/m3) when it is
(a) suspended in the air and
(b) completely immersed in water.



SOLUTION A concrete block is lowered into the sea. The tension in the rope is to be determined before and after the block is in water.

Assumptions **1** The buoyancy of air is negligible. **2** The weight of the ropes is negligible.

Properties The densities are given to be 1025 kg/m³ for seawater and 2300 kg/m³ for concrete.

35

Analysis (*a*) Consider the free-body diagram of the concrete block. The forces acting on the concrete block in air are its weight and the upward pull action (tension) by the rope. These two forces must balance each other, and thus the tension in the rope must be equal to the weight of the block:

$$V = (0.4\text{ m})(0.4\text{ m})(3\text{ m}) = 0.48\text{ m}^3$$

$$F_{T,\,air} = W = \rho_{concrete}\, gV$$

$$= (2300\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.48\text{ m}^3)\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) = 10.8\text{ kN}$$

(*b*) When the block is immersed in water, there is the additional force of buoyancy acting upward. The force balance in this case gives

$$F_B = \rho_f\, gV = (1025\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.48\text{ m}^3)\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) = 4.8\text{ kN}$$

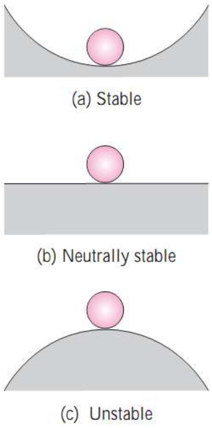
$$F_{T,\,water} = W - F_B = 10.8 - 4.8 = 6.0\text{ kN}$$

Discussion Note that the weight of the concrete block, and thus the tension of the rope, decreases by (10.8 − 6.0)/10.8 = 55 percent in water.

36

Stability of Immersed and Floating Bodies

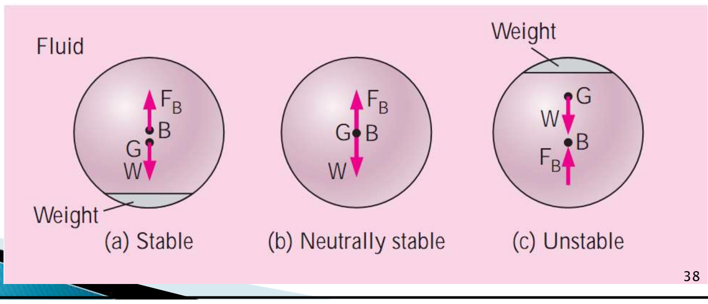
- An important application of the buoyancy concept is the assessment of the stability of immersed and floating bodies with no external attachments.
- Case (a) is **stable** since any small disturbance generates a restoring force (due to gravity) that returns it to its initial position.
- Case (b) is **neutrally stable** because if someone moves the ball to the right or left, it would stay put at its new location. It has no tendency to move back to its original location, nor does it continue to move away.
- Case (c) is a situation in which the ball may be at rest at the moment, but any disturbance, even an infinitesimal one, causes the ball to roll off the hill—it does not return to its original position; rather it diverges from it (**unstable**).



37

Rotational Stability

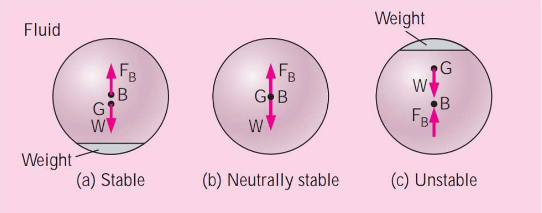
- The rotational stability of an immersed body depends on the relative locations of the **center of gravity, G** of the body and the **center of buoyancy, B,** which is the centroid of the displaced volume.
- An immersed body is stable if the body is bottom-heavy and thus **point G is directly below point B.**

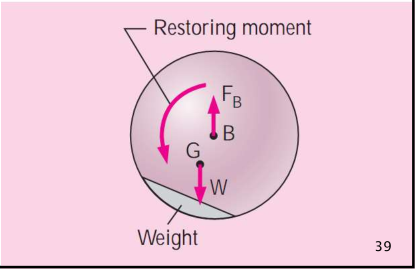


38

Rotational Stability

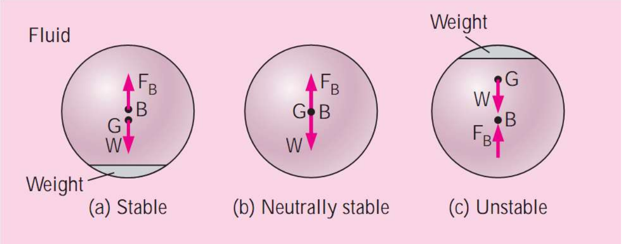
- A rotational disturbance of a stable body produces a restoring moment to return the body to its original stable position.
  - A stable design for a submarine calls for the engines and the cabins for the crew to be located at the lower half in order to shift the weight to the bottom as much as possible.
  - Hot-air or helium balloons (which can be viewed as being immersed in air) are also stable since the cage that carries the load is at the bottom.





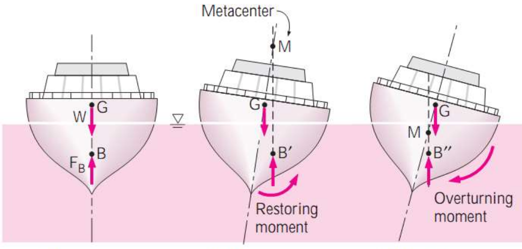
Rotational Stability

- An immersed body whose center of gravity G is directly above point B is unstable, and any disturbance will cause this body to turn upside down.
- A body for which G and B coincide is neutrally stable.
  - For neutrally stable bodies, there is no tendency to overturn or right themselves.



Stability of Floating Bodies

- If a floating body is bottom-heavy and thus the center of gravity G is directly below the center of buoyancy B, the body is **always stable.**
- But unlike immersed bodies, a floating body may still be stable when G is directly above B.

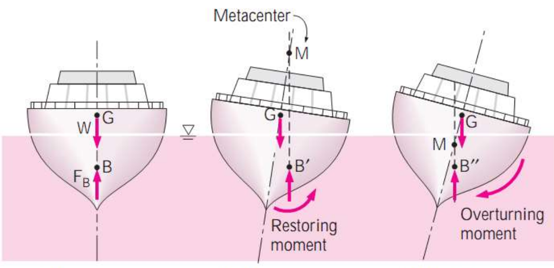


(a) Stable (b) Stable (c) Unstable

41

Stability of Floating Bodies

- The centroid of the displaced volume shifts to the side to a point B/ during a rotational disturbance while the center of gravity G of the body remains unchanged.
- If point B/ is sufficiently far, these two forces create a restoring moment and return the body to the original position.

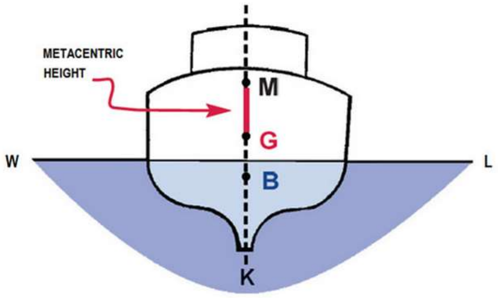


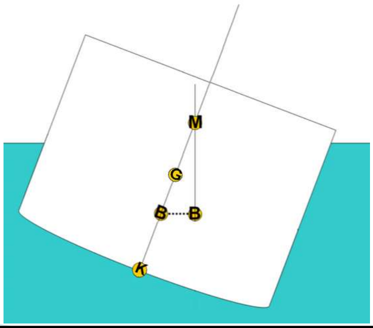
(a) Stable (b) Stable (c) Unstable

42

Stability of Floating Bodies

- **Metacentric height, GM**, is the distance between the center of gravity G and the metacenter M (the intersection point of the lines of action of the buoyant force through the body before and after rotation).

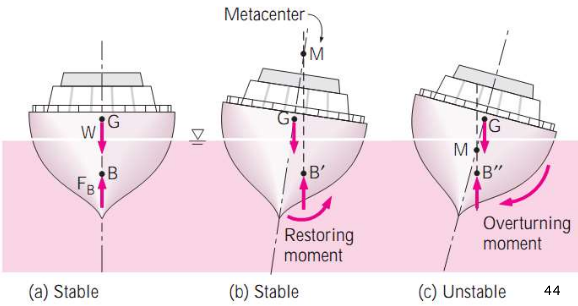




43

Stability of Floating Bodies

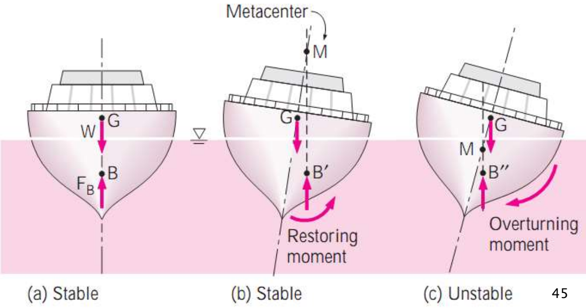
A floating body is
- stable if point M is above point G, (GM is positive)
- unstable if point M is below point G, (GM is negative).
  - the weight and the buoyant force acting on the tilted body generate an overturning moment instead of a restoring moment, causing the body to capsize.



44

Stability of Floating Bodies

- The length of the metacentric height GM above G is a measure of the stability
- Larger GM is, the more stable is the floating body

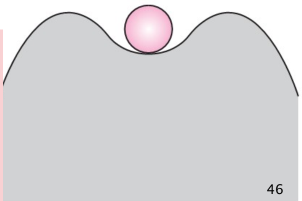


45

Stability of Floating Bodies

- A boat can tilt to some maximum angle without capsizing, but beyond that angle it overturns (and sinks).
- The ball returns to its stable equilibrium position after being perturbed—up to a limit.
- If the perturbation amplitude is too great, the ball rolls down the opposite side of the hill and does not return to its equilibrium position.
- Stable up to some limiting level of disturbance, but unstable beyond.

The metacenter may be considered to be a fixed point for most hull shapes for small rolling angles up to about 20°.



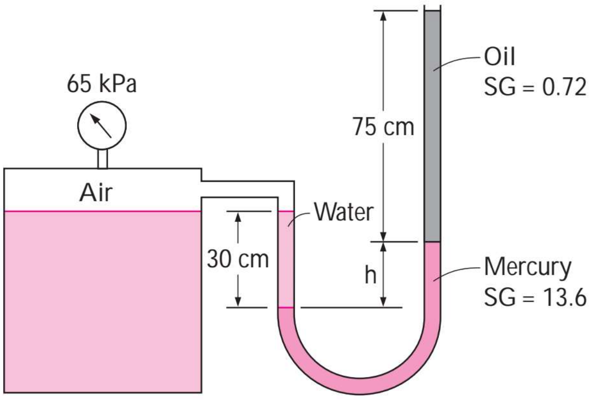
46

Floating Body Experiment

- 1.
- The pressure in an automobile tire depends on the temperature of the air in the tire. When the air temperature is 25°C, the pressure guage reads 210 kPa. If the volume of the tire is 0.025 m^3, determine the pressure rise in the tire when the air temperature in the tire rises to 50°C. Assume the atmospheric pressure to be 100 kPa.

2.

The gauge pressure of the air in the tank shown in Fig. 1 is measured to be 65 kPa. Determine the differential height *h* of the mercury column.



FLUIDS IN RIGID-BODY MOTION

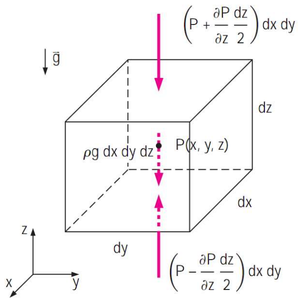
- Many fluids such as milk and gasoline are transported in tankers.
- In an accelerating tanker, the fluid rushes to the back, and some initial splashing occurs.
- But then a new free surface (usually nonhorizontal) is formed, each fluid particle assumes the same acceleration, and the entire fluid moves like a rigid body.

❖No shear stresses develop within the fluid body since there is no deformation and thus no change in shape.

❖Rigid-body motion of a fluid also occurs when the fluid is contained in a tank that rotates about an axis.

FLUIDS IN RIGID-BODY MOTION

- Consider a differential rectangular fluid element of side lengths dx, dy, and dz in the x-, y-, and z-directions, respectively, with the z-axis being upward in the vertical direction.
- If the differential fluid element behaves like a rigid body, Newton's second law of motion can be applied.



$$\delta\vec{F} = \delta m \cdot \vec{a}$$

where $dm = \rho dV = \rho dxdydz$ is the mass of the fluid element.

51

FLUIDS IN RIGID-BODY MOTION

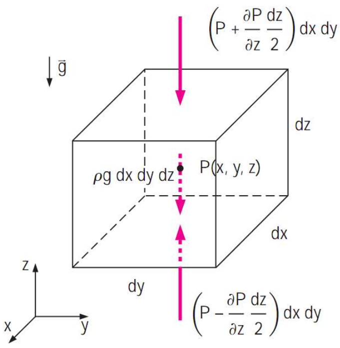
The forces acting on the fluid element consist of

- **Body forces** - Gravity that act throughout the entire body of the element and are proportional to the volume of the body (and also electrical and magnetic forces, which will not be considered),
- **Surface forces** - Pressure forces that act on the surface of the element and are proportional to the surface area (shear stresses are also surface forces, but they do not apply in this case since the relative positions of fluid elements remain unchanged).
- NB: Pressure represents the compressive force applied on the fluid element by the surrounding fluid and is always directed to the surface.

52

FLUIDS IN RIGID-BODY MOTION

- Taking the pressure at the center of the element to be P, the pressures at the top and bottom surfaces of the element can be expressed as **P+(δP/δz.dz/2)** and **P-(δP/δz.dz/2)**, respectively.
- The net surface force acting on the element in the z-direction is the difference between the pressure forces acting on the bottom and top faces,



$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z} dx\, dy\, dz$$

53

FLUIDS IN RIGID-BODY MOTION

Similarly, the net surface forces in the x- and y-directions are

$$\delta F_{S,x} = -\frac{\partial P}{\partial x} dx\, dy\, dz \quad \text{and} \quad \delta F_{S,y} = -\frac{\partial P}{\partial y} dx\, dy\, dz$$

- The surface force (the pressure force) acting on the entire element is expressed in vector form as

$$\delta \vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$

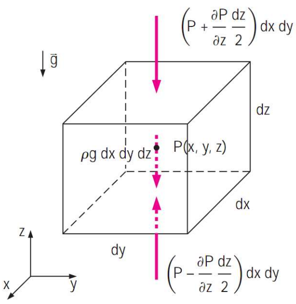
$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\, dy\, dz = -\vec{\nabla} P\, dx\, dy\, dz$$

$$\vec{\nabla} P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$

54

FLUIDS IN RIGID-BODY MOTION

- The only body force acting on the fluid element is the weight of the element acting in the negative z-direction, and it is expressed as



$$\delta F_{B,\,z} = -g\delta m = -\rho g dxdydz$$

- or in vector form as

$$\delta\vec{F}_{B,\,z} = -g\delta m\vec{k} = -\rho g\, dx\, dy\, dz\vec{k}$$

Then the total force acting on the element becomes

$$\delta\vec{F} = \delta\vec{F}_S + \delta\vec{F}_B = -(\vec{\nabla}P + \rho g\vec{k})\, dx\, dy\, dz$$

55

FLUIDS IN RIGID-BODY MOTION

- The general equation of motion for a fluid that acts as a rigid body (no shear stresses) is determined to be

Rigid-body motion of fluids $\qquad \vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}$

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g\vec{k} = -\rho(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

Accelerating fluids:

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

56

Special Case 1: Fluids at Rest

- For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are zero.

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

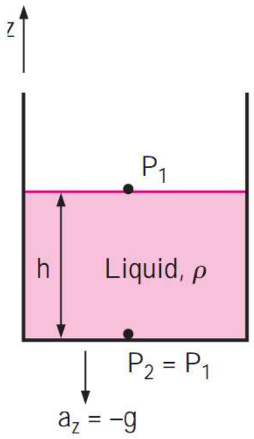
Which confirm that, in fluids at rest, the pressure remains constant in any horizontal direction (P is independent of x and y) and varies only in the vertical direction as a result of gravity [and thus P=P(z)]. These relations are applicable for both compressible and incompressible fluids.

57

Special Case 2: Free Fall of a Fluid Body

- A freely falling body accelerates under the influence of gravity.
- When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is zero.

  $a_x = a_y = 0$ and $a_z = -g$
- Then the equations of motion for accelerating fluids reduce to



$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \quad \rightarrow \quad P = \text{constant}$$

58

Special Case 2: Free Fall of a Fluid Body

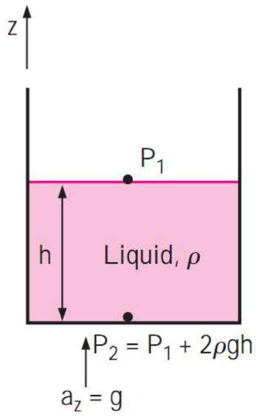
- When the direction of motion is reversed and the fluid is forced to accelerate vertically with

$$a_z = +g$$

by placing the fluid container in an elevator or a space vehicle propelled upward by a rocket engine, the pressure gradient in the z-direction is

$$\delta P/\delta z = -2\rho g$$

- The pressure difference across a fluid layer now doubles relative to the stationary fluid.

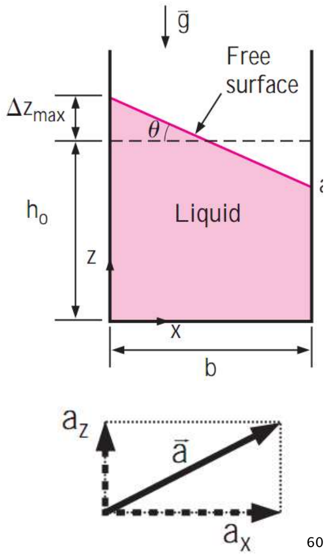


59

Acceleration on a Straight Path

Considering a container partially filled with a liquid.

- Moving on a straight path with a constant acceleration.
- Taking the projection of the path of motion on the horizontal plane to be the x-axis, and the projection on the vertical plane to be the z-axis.
- The x- and z-components of acceleration are a_x and a_z.



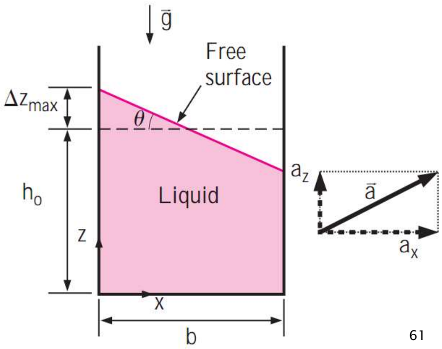
60

Acceleration on a Straight Path

- No movement in the y-direction, thus the acceleration in that direction is zero,
  $a_y = 0$.
- The equations of motion for accelerating fluids reduce to

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = 0,$$

$$\text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$



61

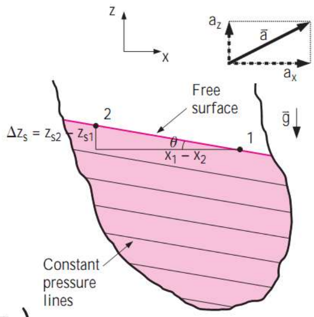
Acceleration on a Straight Path

- Pressure is independent of y
- The total differential of P=P(x, z), which is (δP/δx)dx + (δP/δz)dz, becomes

$$dP = -\rho a_x\, dx - \rho(g + a_z)\, dz$$

For ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integration to be

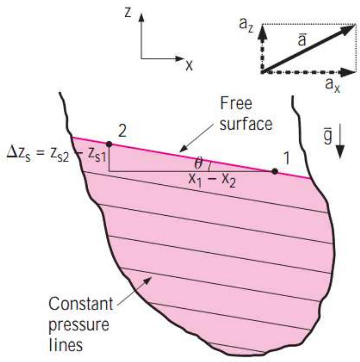
$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$



62

Acceleration on a Straight Path

- Taking point 1 to be the origin (x = 0, z = 0) where the pressure is P_0 and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as



$$P = P_0 - \rho a_x x - \rho(g + a_z)z$$

63

Acceleration on a Straight Path

- The vertical rise (or drop) of the free surface at point 2 relative to point 1 can be determined by choosing both 1 and 2 on the free surface (so that $P_1 = P_2$), and solving for $z_2 - z_1$

$$\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g + a_z}(x_2 - x_1)$$

Z_s is the z-coordinate of the liquid's free surface.

64

Acceleration on a Straight Path

- The equation for surfaces of constant pressure, called isobars, is obtained by setting $dP = 0$ and replacing z by z_{isobar}, which is the z-coordinate (the vertical distance) of the surface as a function of x.

Surfaces of constant pressure: $$\frac{dz_{isobar}}{dx} = -\frac{a_x}{g + a_z} = \text{constant}$$

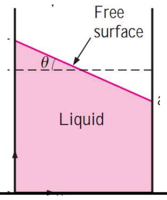
- Thus we conclude that the isobars (including the free surface) in an incompressible fluid with constant acceleration in linear motion are parallel surfaces whose slope in the xz-plane is

Slope of isobars: $$\text{Slope} = \frac{dz_{isobar}}{dx} = -\frac{a_x}{g + a_z} = -\tan\theta$$

65

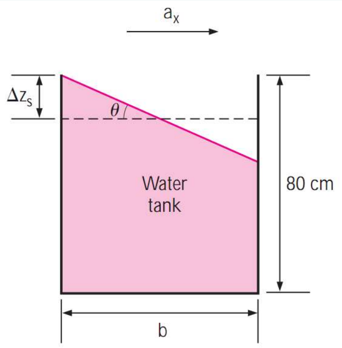
Acceleration on a Straight Path

- Obviously, the free surface of such a fluid is a plane surface, and it is inclined unless $a_x = 0$ (the acceleration is in the vertical direction only).
- Also, the conservation of mass together with the assumption of incompressibility (ρ =constant) requires that the volume of the fluid remain constant before and during acceleration.
- The rise of fluid level on one side must be balanced by a drop of fluid level on the other side.



66

An 80-cm-high fish tank of cross section 2 m x 0.6 m that is initially filled with water is to be transported on the back of a truck. The truck accelerates from 0 to 90 km/h in 10 s.
If it is desired that no water spills during acceleration, determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?



SOLUTION A fish tank is to be transported on a truck. The allowable water height to avoid spill of water during acceleration and the proper orientation are to be determined.

Assumptions **1** The road is horizontal during acceleration so that acceleration has no vertical component ($a_z = 0$). **2** Effects of splashing, braking, driving over bumps, and climbing hills are assumed to be secondary and are not considered. **3** The acceleration remains constant.

Analysis We take the *x*-axis to be the direction of motion, the *z*-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is

$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90 - 0)\ \text{km/h}}{10\ \text{s}}\left(\frac{1\ \text{m/s}}{3.6\ \text{km/h}}\right) = 2.5\ \text{m/s}^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g + a_z} = \frac{2.5}{9.81 + 0} = 0.255 \qquad (\text{and thus } \theta = 14.3°)$$

The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical midplane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the midplane for the two possible orientations becomes

Case 1: The long side is parallel to the direction of motion:

$$\Delta z_{s1} = (b_1/2)\tan\theta = [(2\text{ m})/2] \times 0.255 = 0.255\text{ m} = 25.5\text{ cm}$$

Case 2: The short side is parallel to the direction of motion:

$$\Delta z_{s2} = (b_2/2)\tan\theta = [(0.6\text{ m})/2] \times 0.255 = 0.076\text{ m} = 7.6\text{ cm}$$

Therefore, assuming tipping is not a problem, the tank should definitely be oriented such that its short side is parallel to the direction of motion. Emptying the tank such that its free surface level drops just 7.6 cm in this case will be adequate to avoid spilling during acceleration.

Discussion Note that the orientation of the tank is important in controlling the vertical rise. Also, the analysis is valid for any fluid with constant density, not just water, since we used no information that pertains to water in the solution.

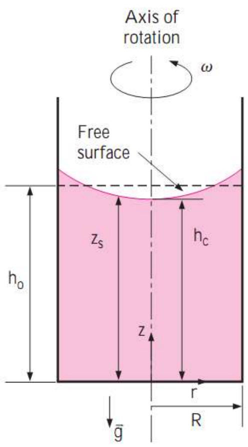
Rotation in a Cylindrical Container

- It is known from experience that when a glass filled with water is rotated about its axis, the fluid is forced outward as a result of the so-called centrifugal force, and the free surface of the liquid becomes concave.
- This is known as the forced vortex motion.
- Consider a vertical cylindrical container partially filled with a liquid. The container is now rotated about its axis at a constant angular velocity of ω
- After initial transients, the liquid will move as a rigid body together with the container.
- There is no deformation, and thus there can be no shear stress, and every fluid particle in the container moves with the same angular velocity

Rotation in a Cylindrical Container

- Best analyzed in cylindrical coordinates (r, θ, z), with z taken along the centerline of the container directed from the bottom toward the free surface,
- since the shape of the container is a cylinder, and the fluid particles undergo a circular motion.
- The centripetal acceleration of a fluid particle rotating with a constant angular velocity of ω at a distance r from the axis of rotation is $r\omega^2$ and is directed radially toward the axis of rotation (negative r-direction). That is,

$$a_r = -r\omega^2.$$

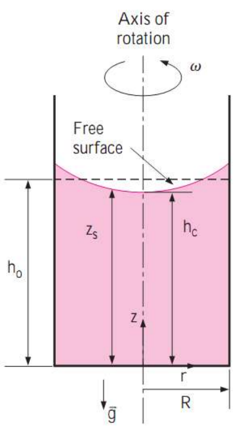


71

Rotation in a Cylindrical Container

- There is symmetry about the z-axis, which is the axis of rotation, and thus there is no θ dependence.
- Then
  $P = P(r, z)$ and $a_\theta = 0$. Also, $a_z = 0$ since there is no motion in the z-direction.
- Equations of motion for rotating fluids

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$



$$\frac{\partial P}{\partial r} = \rho r\omega^2, \quad \frac{\partial P}{\partial \theta} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho g$$

72

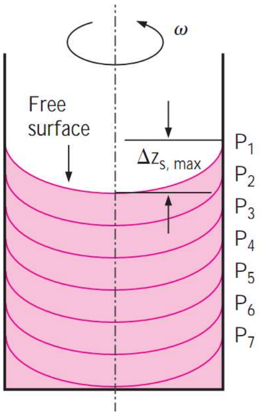
Rotation in a Cylindrical Container

- The total differential of P=P(r, z) which is

  dP=(δP/δr)dr+(δP/δz)dz

becomes

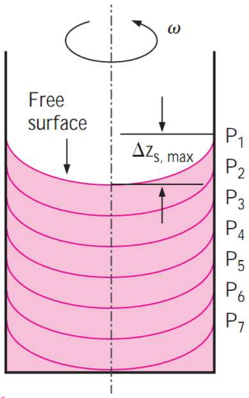
$$dP = \rho r\omega^2\, dr - \rho g\, dz$$



73

Rotation in a Cylindrical Container

- The equation for surfaces of constant pressure is obtained by setting dP = 0 and replacing z by z_{isobar}, which is the z-value of the surface as a function of r.



$$\frac{dz_{isobar}}{dr} = \frac{r\omega^2}{g} \qquad z_{isobar} = \frac{\omega^2}{2g}r^2 + C_1$$

which is the equation of a parabola.
The surfaces of constant pressure, including the free surface, are paraboloids of revolution.

74

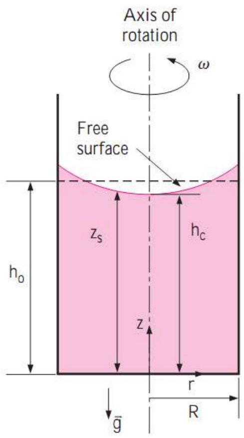
Rotation in a Cylindrical Container

$$z_{isobar} = \frac{\omega^2}{2g} r^2 + C_1$$

- The value of the integration constant C_1 is different for different paraboloids of constant pressure (i.e., for different isobars).
- For the free surface, setting r = 0 gives

$$z_{isobar}(0) = C_1 = h_c,$$

- h_c is the distance of the free surface from the bottom of the container along the axis of rotation.

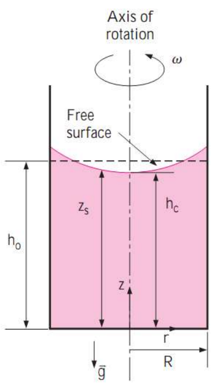


Rotation in a Cylindrical Container

- the equation for the free surface becomes

$$z_s = \frac{\omega^2}{2g} r^2 + h_c$$

where z_s is the distance of the free surface from the bottom of the container at radius r.
The underlying assumption in this analysis is that there is sufficient liquid in the container so that the entire bottom surface remains covered with liquid.



Rotation in a Cylindrical Container

- The volume of a cylindrical shell element of radius r, height z_s,and thickness dr is

$$dV=2\pi r z_s dr$$

- Then the volume of the paraboloid formed by the free surface is

$$V = \int_{r=0}^{R} 2\pi z_s r\, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c\right) r\, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$

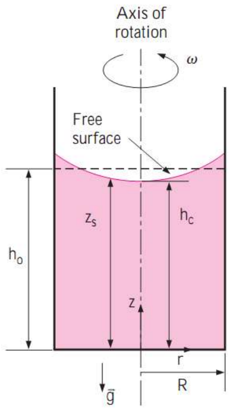
77

Rotation in a Cylindrical Container

- Since mass is conserved and density is constant, this volume must be equal to the original volume of the fluid in the container, which is

$$V = \pi R^2 h_0$$

- where h_0 is the original height of the fluid in the container with no rotation.



78

Rotation in a Cylindrical Container

- Setting these two volumes equal to each other, the height of the fluid along the centerline of the cylindrical container becomes

$$\pi R^2 h_0 = \pi R^2\left(\frac{\omega^2 R^2}{4g} + h_c\right) \Longrightarrow h_c = h_0 - \frac{\omega^2 R^2}{4g}$$

$$z_s = \frac{\omega^2}{2g} r^2 + h_c \Longrightarrow z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

The maximum vertical height occurs at the edge where r = R, and the maximum height difference between the edge and the center of the free surface is determined by evaluating z_s at r = R and also at r = 0, and taking their difference

$$\Delta z_{s,\,max} = z_s(R) - z_s(0) = \frac{\omega^2}{2g} R^2$$

79

Rotation in a Cylindrical Container

- When ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integrating

$$dP = \rho r \omega^2 dr + \rho g dz$$

This yields

$$P_2 - P_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$$

80

Rotation in a Cylindrical Container

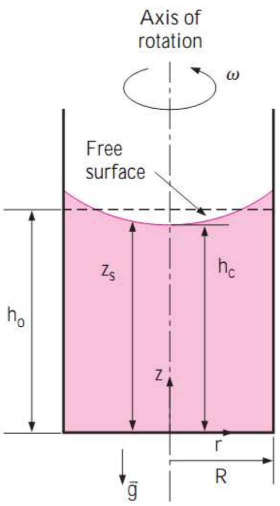
$$P = P_0 + \frac{\rho\omega^2}{2}r^2 - \rho g z$$

- At a fixed radius, the pressure varies hydrostatically in the vertical direction, as in a fluid at rest.
- For a fixed vertical distance z, the pressure varies with the square of the radial distance r, increasing from the centerline toward the outer edge.
- In any horizontal plane, the pressure difference between the center and edge of the container of radius R is $\Delta P = r\omega^2R^2/2$.

81

Rotation in a Cylindrical Container

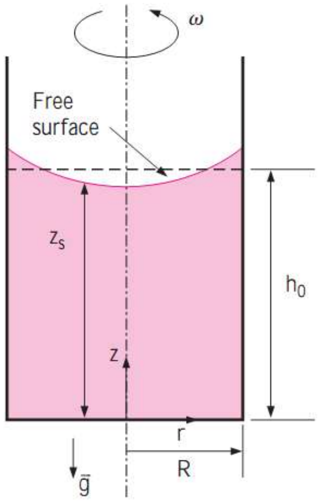
- Taking point 1 to be the origin (r = 0, z = 0) where the pressure is P_0 and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as



$$P = P_0 + \frac{\rho\omega^2}{2}r^2 - \rho g z$$

82

A 20-cm-diameter, 60-cm-high vertical cylindrical container, shown in figure, is partially filled with 50-cm-high liquid whose density is 850 kg/m³. Now the cylinder is rotated at a constant speed. Determine the rotational speed at which the liquid will start spilling from the edges of the container



SOLUTION A vertical cylindrical container partially filled with a liquid is rotated. The angular speed at which the liquid will start spilling is to be determined.

Assumptions **1** The increase in the rotational speed is very slow so that the liquid in the container always acts as a rigid body. **2** The bottom surface of the container remains covered with liquid during rotation (no dry spots).

Analysis Taking the center of the bottom surface of the rotating vertical cylinder as the origin ($r = 0$, $z = 0$), the equation for the free surface of the liquid is given as

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

Then the vertical height of the liquid at the edge of the container where $r = R$ becomes

$$z_s(R) = h_0 + \frac{\omega^2 R^2}{4g}$$

where $h_0 = 0.5$ m is the original height of the liquid before rotation. Just before the liquid starts spilling, the height of the liquid at the edge of the container equals the height of the container, and thus $z_s(R) = 0.6$ m. Solving the last equation for ω and substituting, the maximum rotational speed of the container is determined to be

$$\omega = \sqrt{\frac{4g[z_s(R) - h_0]}{R^2}} = \sqrt{\frac{4(9.81 \text{ m/s}^2)[(0.6 - 0.5) \text{ m}]}{(0.1 \text{ m})^2}} = \mathbf{19.8 \text{ rad/s}}$$

Noting that one complete revolution corresponds to 2π rad, the rotational speed of the container can also be expressed in terms of revolutions per minute (rpm) as

$$\dot{n} = \frac{\omega}{2\pi} = \frac{19.8 \text{ rad/s}}{2\pi \text{ rad/rev}}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = \mathbf{189 \text{ rpm}}$$

Therefore, the rotational speed of this container should be limited to 189 rpm to avoid any spill of liquid as a result of the centrifugal effect.

Discussion Note that the analysis is valid for any liquid since the result is independent of density or any other fluid property. We should also verify that our assumption of no dry spots is valid. The liquid height at the center is

$$z_s(0) = h_0 - \frac{\omega^2 R^2}{4g} = 0.4 \text{ m}$$

Since $z_s(0)$ is positive, our assumption is validated.

85