14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

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14.2
Limits and Continuity

In this section, we will learn about:

Limits and continuity of

various types of functions.

Let’s compare the behavior of the functions

$$f(x,y)=\frac{\sin(x^2+y^2)}{x^2+y^2} \quad \text{and} \quad g(x,y)=\frac{x^2-y^2}{x^2+y^2}$$

as *x* and *y* both approach 0
(and thus the point (*x*, *y*) approaches
the origin).

The following tables show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin.

Table 1

This table shows values of $f(x, y)$.

TABLE 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2

This table shows values of *g*(*x*, *y*).

TABLE 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Notice that neither function is defined at the origin.

- It appears that, as (x, y) approaches (0, 0), the values of $f(x, y)$ are approaching 1, whereas the values of $g(x, y)$ aren't approaching any number.

It turns out that these guesses based on numerical evidence are correct.

Thus, we write:

- $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$
- $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$ does not exist.

In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to indicate that:

- The values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.

In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).

- A more precise definition follows.

LIMIT OF A FUNCTION

Definition 1

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b).

Then, we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L.

Definition 1

We write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if:

- For every number $\varepsilon > 0$, there is a corresponding number $\boldsymbol{\delta} > 0$ such that,

  if $(x,y)\in D$ and $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$

  then $|f(x,y)-L|<\varepsilon$

LIMIT OF A FUNCTION

Other notations for the limit in Definition 1 are:

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$$

$$f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Notice that:

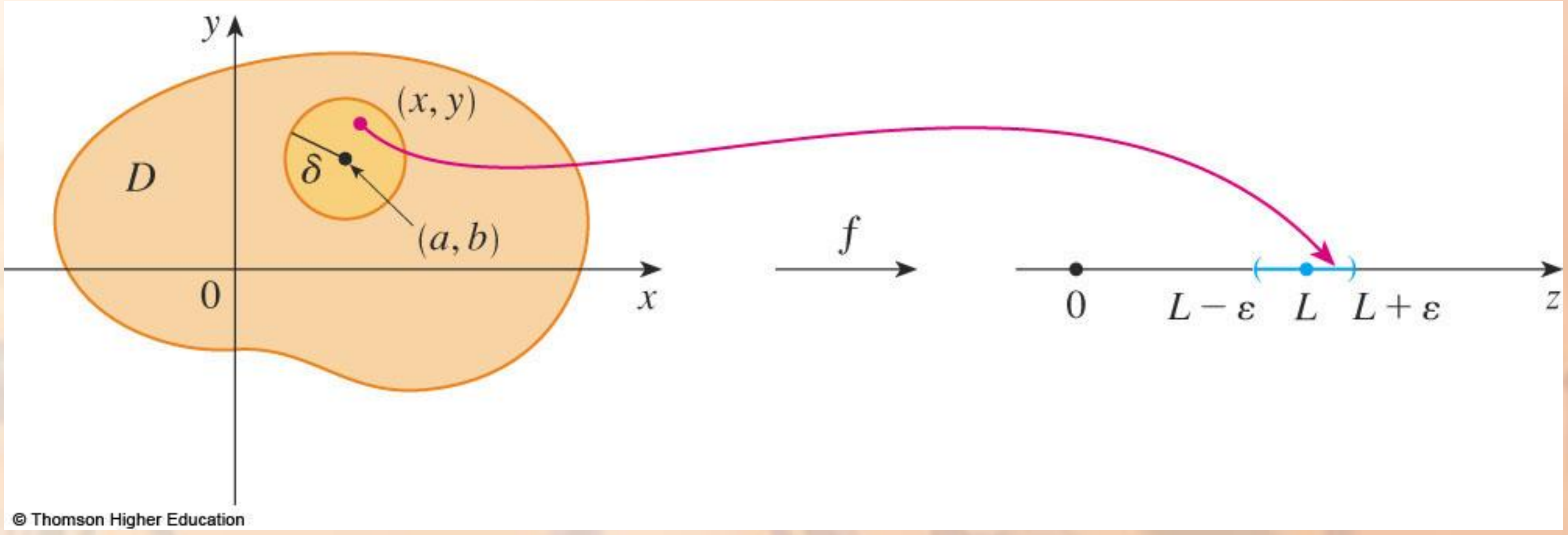
- $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L
- $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b).

LIMIT OF A FUNCTION

Thus, Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

LIMIT OF A FUNCTION

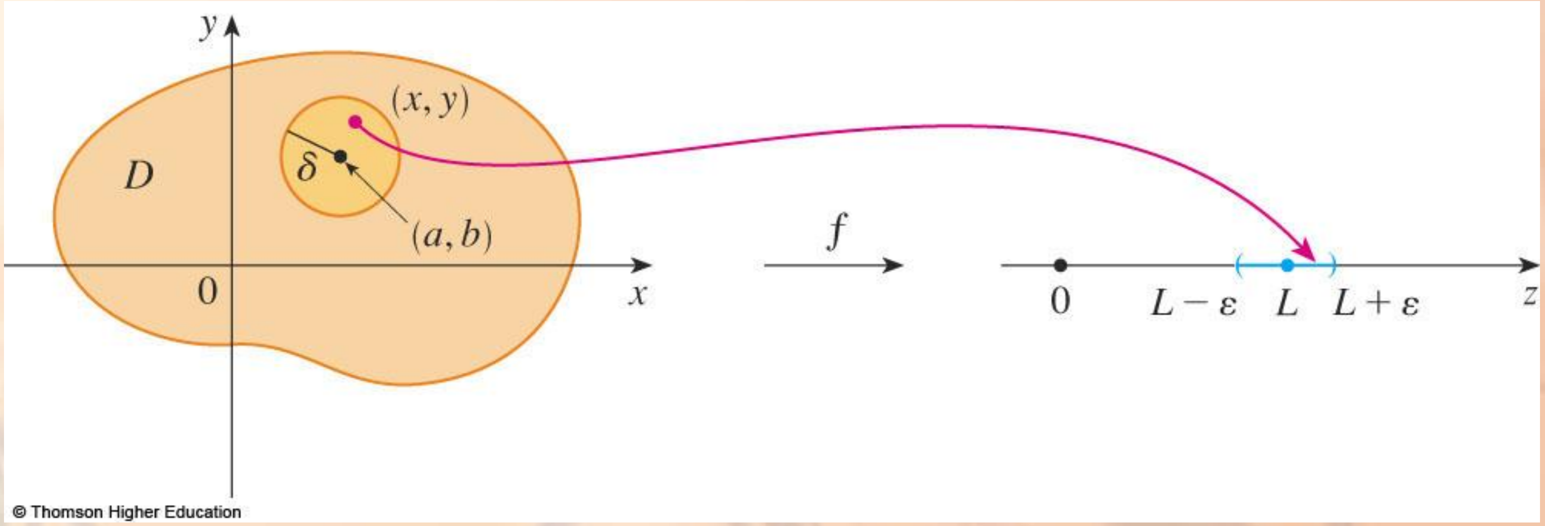
The figure illustrates Definition 1 by means of an arrow diagram.



LIMIT OF A FUNCTION

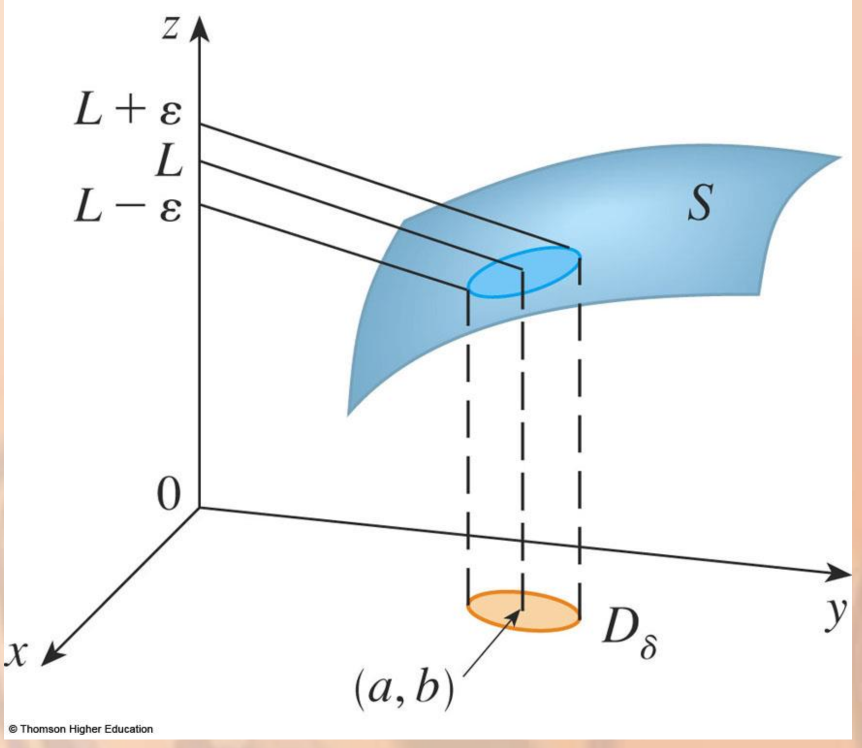
If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that:

- f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.



LIMIT OF A FUNCTION

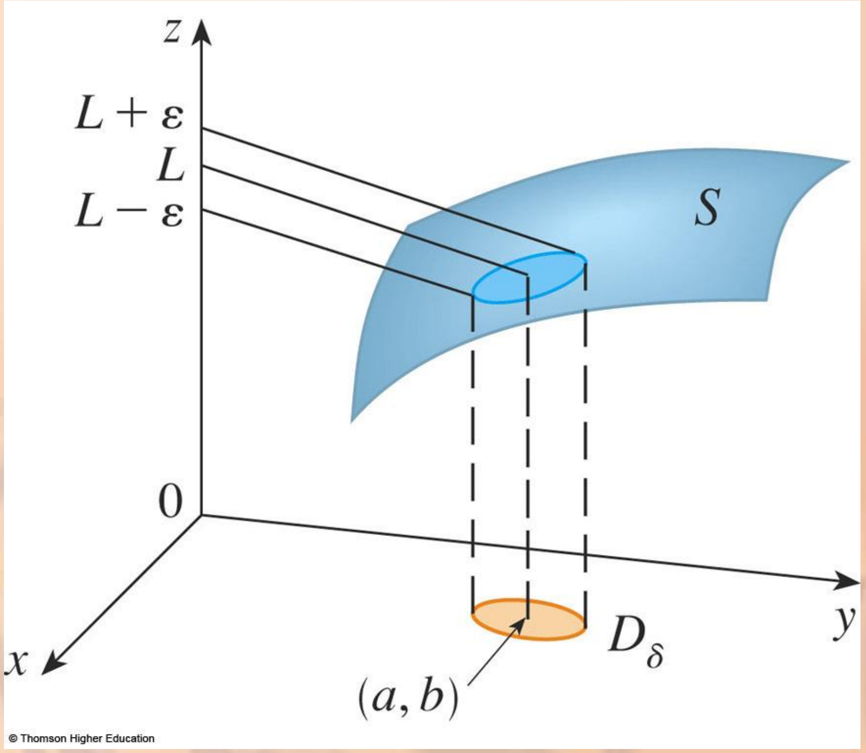
Another illustration of Definition 1 is given here, where the surface *S* is the graph of *f*.



LIMIT OF A FUNCTION

If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that, if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then

- The corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.



SINGLE VARIABLE FUNCTIONS

For functions of a single variable, when we let *x* approach *a*, there are only two possible directions of approach, from the left or from the right.

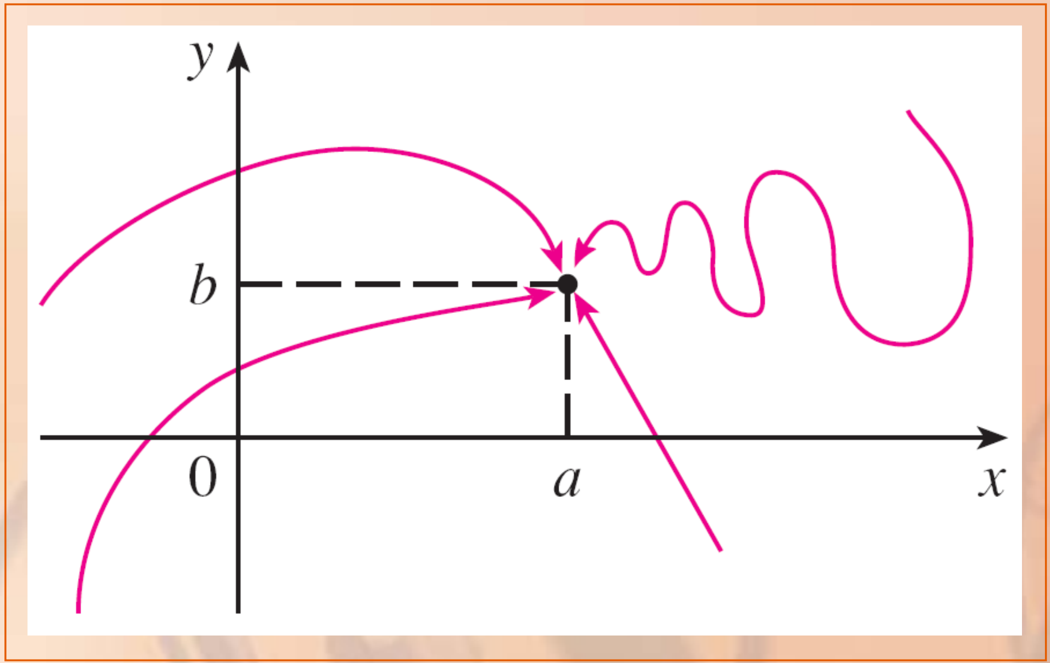
- We recall from Chapter 2 that, if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

DOUBLE VARIABLE FUNCTIONS

For functions of two variables, the situation is not as simple.

DOUBLE VARIABLE FUNCTIONS

This is because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever as long as (x, y) stays within the domain of f.



Definition 1 refers only to the distance between (x, y) and (a, b).

- It does not refer to the direction of approach.

LIMIT OF A FUNCTION

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).

LIMIT OF A FUNCTION

Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that

$$\lim_{(x,y)\to(a,b)} f(x, y)$$ does not exist.

LIMIT OF A FUNCTION

If

$f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1

and

$f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2,

where $L_1 \neq L_2$,

then

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.

Example 1

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

- Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

Example 1

First, let's approach (0, 0) along the x-axis.

- Then, $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$.
- So, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

Example 1

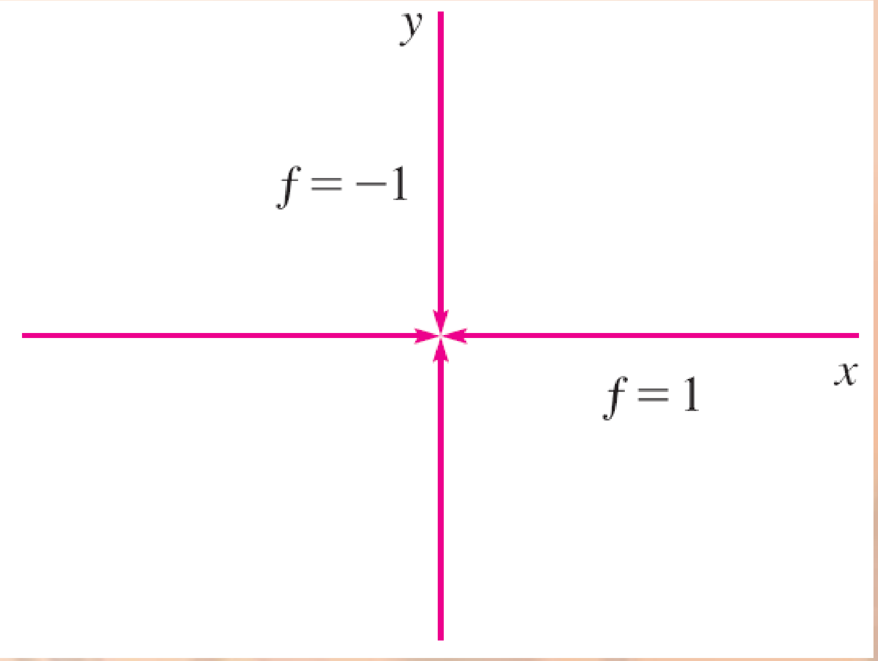
We now approach along the y-axis by putting $x = 0$.

- Then, $f(0, y) = -y^2/y^2 = -1$ for all $y \neq 0$.
- So, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

Example 1

Since *f* has two different limits along two different lines, the given limit does not exist.

- This confirms the conjecture we made on the basis of numerical evidence at the beginning of the section.



Example 2

If

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

does

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exist?

Example 2

If $y = 0$, then $f(x, 0) = 0/x^2 = 0$.

- Therefore,

  $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis.

If $x = 0$, then $f(0, y) = 0/y^2 = 0$.

- So,

  $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the y-axis.

Although we have obtained identical limits along the axes, that does not show that the given limit is 0.

Let’s now approach (0, 0) along another line, say $y = x$.

- For all $x \neq 0$,

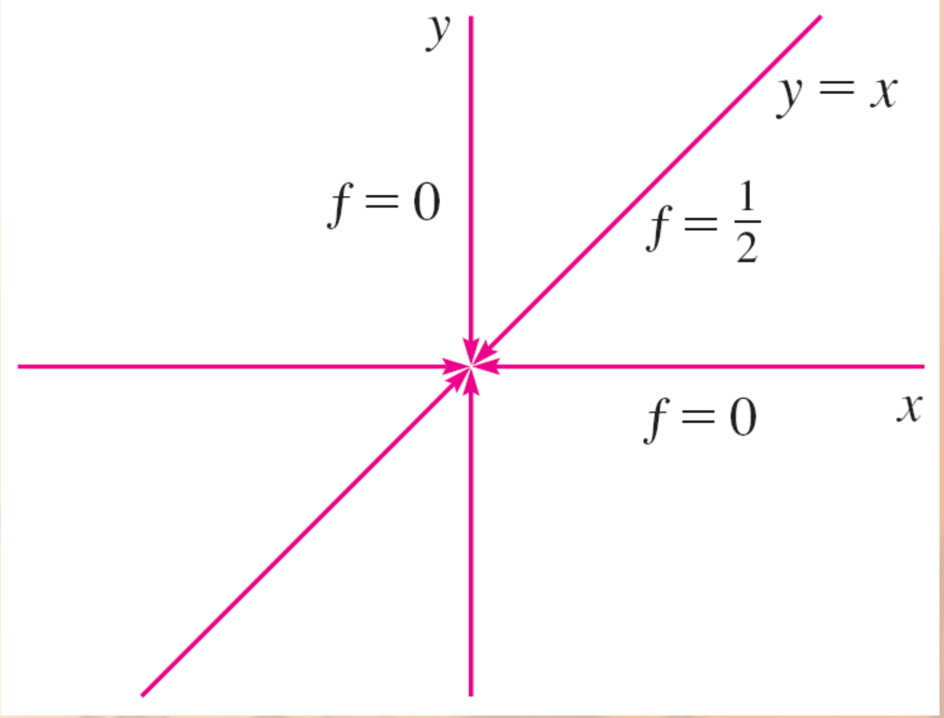
$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

- Therefore,

$$f(x, y) \to \tfrac{1}{2} \text{ as } (x, y) \to (0, 0) \text{ along } y = x$$

Example 2

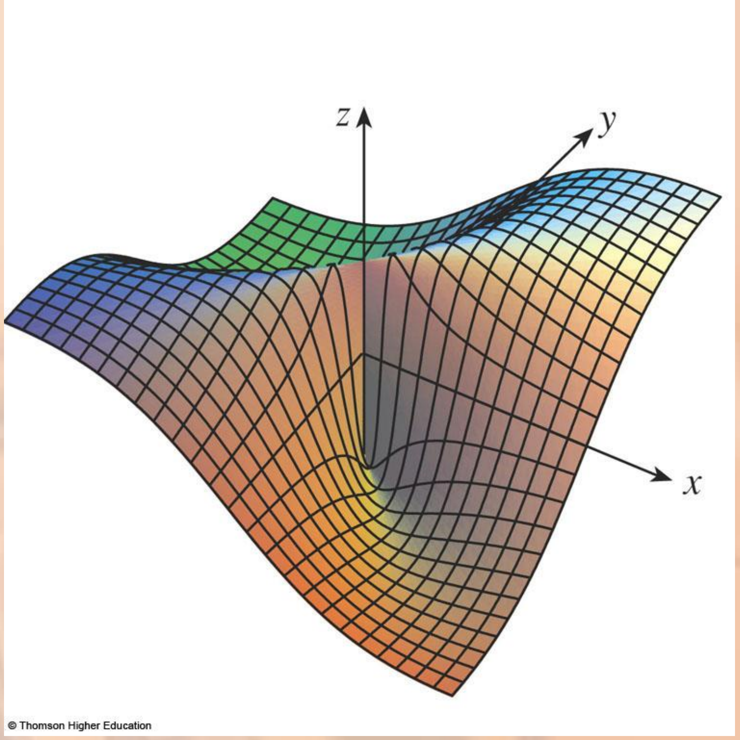
Since we have obtained different limits along different paths, the given limit does not exist.



LIMIT OF A FUNCTION

This figure sheds some light on Example 2.

- The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = ½$ for all points (x, y) on that line except the origin.



Example 3

If

$$f(x,y)=\frac{xy^2}{x^2+y^4}$$

does

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exist?

Example 3

With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any nonvertical line through the origin.

Then, $y = mx$, where m is the slope,

and

$$f(x,y) = f(x,mx)$$

$$= \frac{x(mx)^2}{x^2 + (mx)^4}$$

$$= \frac{m^2x^3}{x^2 + m^4x^4}$$

$$= \frac{m^2x}{1 + m^4x^2}$$

Example 3

Therefore,

$$f(x, y) \to 0 \text{ as } (x, y) \to (0, 0) \text{ along } y = mx$$

- Thus, *f* has the same limiting value along every nonvertical line through the origin.

Example 3

However, that does not show that the given limit is 0.

- This is because, if we now let
  $(x, y) \to (0, 0)$ along the parabola $x = y^2$
  we have:

$$f(x,y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

- So,
  $f(x, y) \to ½$ as $(x, y) \to (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist.

Now, let’s look at limits that do exist.

LIMIT OF A FUNCTION

Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

LIMIT OF A FUNCTION

The Limit Laws listed in Section 2.3 can be extended to functions of two variables.

For instance,

- The limit of a sum is the sum of the limits.
- The limit of a product is the product of the limits.

Equations 2

In particular, the following equations are true.

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\lim_{(x,y)\to(a,b)} c = c$$

Equations 2

The Squeeze Theorem

also holds.

Example 4

Find

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$$

if it exists.

As in Example 3, we could show that the limit along any line through the origin is 0.

- However, this doesn't prove that the given limit is 0.

Example 4

However, the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0.

- So, we begin to suspect that the limit does exist and is equal to 0.

Example 4

Let $\varepsilon > 0$.

We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{that is, if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \frac{3\mathrm{x}^2 |y|}{x^2 + y^2} < \varepsilon$$

Example 4

However,

$$x^2 \leq x^2 = y^2 \text{ since } y^2 \geq 0$$

- Thus,

$$x^2/(x^2 + y^2) \leq 1$$

Therefore,

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$$

Thus, if we choose $\delta = \varepsilon/3$

and let $0 < \sqrt{x^2 + y^2} < \delta$

then

$$\left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3} \right)$$

$$= \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$$

CONTINUITY OF SINGLE VARIABLE FUNCTIONS

Recall that evaluating limits of continuous functions of a single variable is easy.

- It can be accomplished by direct substitution.
- This is because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$

CONTINUITY OF DOUBLE VARIABLE FUNCTIONS

Continuous functions of two variables are also defined by the direct substitution property.

Definition 4

A function *f* of two variables is called continuous at (*a*, *b*) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say *f* is continuous on *D* if *f* is continuous at every point (*a*, *b*) in *D*.

CONTINUITY

The intuitive meaning of continuity is that, if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

- This means that a surface that is the graph of a continuous function has no hole or break.

CONTINUITY

Using the properties of limits, you can see that sums, differences, products, quotients of continuous functions are continuous on their domains.

- Let’s use this fact to give examples of continuous functions.

POLYNOMIAL

A polynomial function of two variables (polynomial, for short) is a sum of terms of the form $cx^m y^n$,

where:

- c is a constant.
- m and n are nonnegative integers.

RATIONAL FUNCTION

A rational function is

a ratio of polynomials.

RATIONAL FUNCTION VS. POLYNOMIAL

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial.

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in Equations 2 show that the functions

$$f(x, y) = x, \; g(x, y) = y, \; h(x, y) = c$$

are continuous.

CONTINUOUS POLYNOMIALS

Any polynomial can be built up out of the simple functions *f*, *g*, and *h* by multiplication and addition.

- It follows that all polynomials are continuous on $\mathbb{R}^2$.

CONTINUOUS RATIONAL FUNCTIONS

Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

Evaluate

$$\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$$

- $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial.
- Thus, it is continuous everywhere.

- Hence, we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$$

$$= 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2$$

$$= 11$$

Example 6

Where is the function

$$f(x,y)=\frac{x^2-y^2}{x^2+y^2}$$

continuous?

Example 6

The function f is discontinuous at (0, 0) because it is not defined there.

Since f is a rational function, it is continuous on its domain, which is the set

$$D = \{(x, y) \mid (x, y) \neq (0, 0)\}$$

Let

$$g(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

- Here, *g* is defined at (0, 0).
- However, it is still discontinuous there because

$$\lim_{(x,y)\to(0,0)} g(x,y)$$

does not exist (see Example 1).

Example 8

Let

$$f(x,y)=\begin{cases}\dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)\neq(0,0)\end{cases}$$

Example 8

We know *f* is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there.

Also, from Example 4, we have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$$

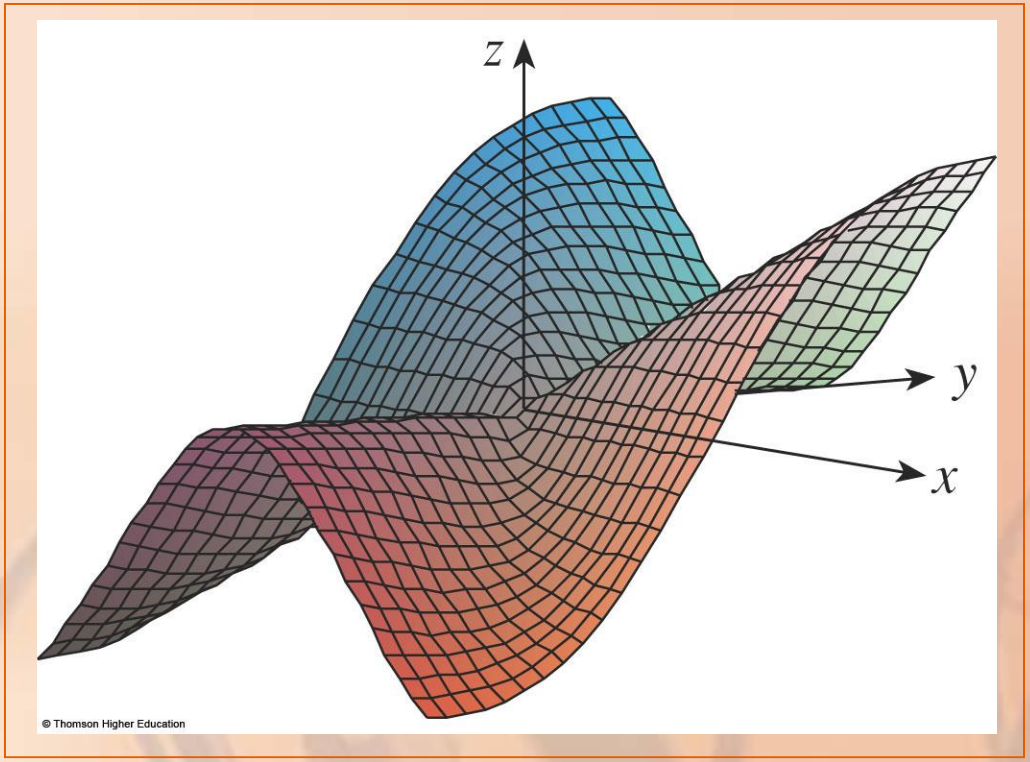
$$= 0 = f(0,0)$$

Thus, *f* is continuous at (0, 0).

- So, it is continuous on P^2.

CONTINUITY

This figure shows the graph of the continuous function in Example 8.



COMPOSITE FUNCTIONS

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

COMPOSITE FUNCTIONS

In fact, it can be shown that, if f is a continuous function of two variables and g is a continuous function of a single variable defined on the range of f, then

- The composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

Example 9

Where is the function $h(x, y) = \arctan(y/x)$ continuous?

- The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$.
- The function $g(t) = \arctan t$ is continuous everywhere.

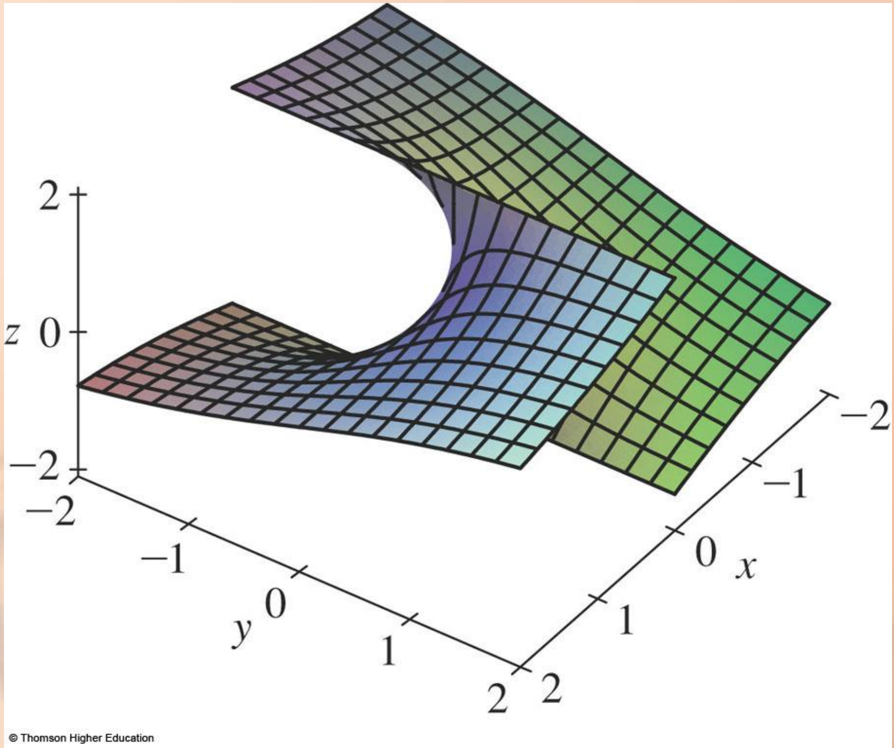
- So, the composite function

$$g(f(x, y)) = \arctan(y, x) = h(x, y)$$

is continuous except where $x = 0$.

Example 9

The figure shows the break in the graph of *h* above the *y*-axis.



FUNCTIONS OF THREE OR MORE VARIABLES

Everything that we have done in this section can be extended to functions of three or more variables.

The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that:

- The values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f.

MULTIPLE VARIABLE FUNCTIONS

The distance between two points (x, y, z) and (a, b, c) in P^3 is given by:

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

- Thus, we can write the precise definition as follows.

For every number $\varepsilon > 0$, there is

a corresponding number $\delta > 0$ such that,

if (x, y, z) is in the domain of f

and $0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$

then

$$|f(x, y, z) - L| < \varepsilon$$

The function *f* is continuous at (*a*, *b*, *c*) if:

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$$

MULTIPLE VARIABLE FUNCTIONS

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables.

- So, it is continuous at every point in P^3 except where $x^2 + y^2 + z^2 = 1$.

In other words, it is discontinuous on the sphere with center the origin and radius 1.

MULTIPLE VARIABLE FUNCTIONS

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

MULTIPLE VARIABLE FUNCTIONS Equation 5

If *f* is defined on a subset *D* of P^n,

then $\lim_{x \to a} f(\mathbf{x}) = L$ means that, for every number *ε* > 0, there is a corresponding number *δ* > 0 such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta$$

$$\text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

MULTIPLE VARIABLE FUNCTIONS

If $n = 1$, then

$$\mathbf{x} = x \quad \text{and} \quad \mathbf{a} = a$$

- So, Equation 5 is just the definition of a limit for functions of a single variable.

MULTIPLE VARIABLE FUNCTIONS

If n = 2, we have

$$\mathbf{x} = \langle x, y \rangle$$

$$\mathbf{a} = \langle a, b \rangle$$

$$|\mathbf{x} - \mathbf{a}| = \sqrt{(x-a)^2 + (y-b)^2}$$

- So, Equation 5 becomes Definition 1.

MULTIPLE VARIABLE FUNCTIONS

If $n = 3$, then

$$\mathbf{x} = \langle x, y, z \rangle \quad \text{and} \quad \mathbf{a} = \langle a, b, c \rangle$$

- So, Equation 5 becomes the definition of a limit of a function of three variables.

In each case, the definition of continuity can be written as:

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$