UNIT ONE

INTRODUCTION TO VECTORS

In the study of engineering, quantities are either described as vectors or scalars. Vectors represent quantities that have both magnitude and direction, while scalars represent quantities that have only magnitude and no direction. Vectors are commonly used in electromagnetic fields and quantities such as the electric field intensity, the electric flux density, the magnetic field intensity and other such quantities are vectors.

Since these quantities may have an arbitrary orientation in space at any given time, we need to define a set of three reference directions at every point by which we can describe vectors drawn to represent these quantities. This unit thus introduces you to the orthogonal coordinate system and their applications to vector fields. It also presents basic vector algebra, a means of manipulations of the field quantities that are represented by the vectors. The final part of this unit presents basic calculus related to fields such as gradient, divergence and curl.

Objectives:

By the end of this chapter, you should be able to:

- Do vector algebra
- Be familiar with the orthogonal coordinate systems
- Carry out vector calculus(Gradient, divergence and curl operations)

SESSION ONE

1.1 VECTOR MULTIPLICATION AND THE ORTHOGONAL COORDINATE SYSTEMS

1.1.1 SCALAR OR DOT PRODUCT

The scalar or dot product of two vectors **A** and **B** is denoted by **A•B** (**A** dot **B**). The result of the dot product of two vectors is a scalar. It is equal to the product of the magnitude of **A** and **B** and the cosine of the angle between them. Thus

$$\mathbf{A}\bullet\mathbf{B} = A\,B\cos\theta_{AB}$$

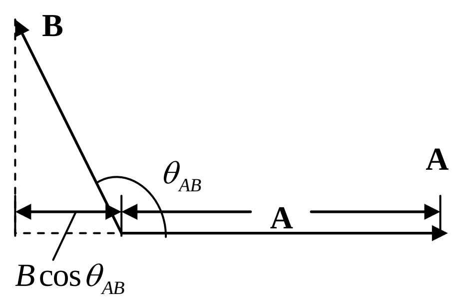


Fig.1 Illustrating the dot product of vectors **A** and **B**

The dot product is commutative. Thus the order of the vectors in a dot product is not important.

$$\mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A}$$

or

$$\mathbf{A}\bullet\mathbf{A} = A^2$$

Example.

Use vectors to prove the law of cosines for a triangle.

Solution

The law of cosine is a scalar relationship that expresses the length of a side of a triangle in terms of the lengths of the two other sides and the angle between them. For the figure below, the law states that

$C = \sqrt{A^2 + b^2 - 2AB\cos\alpha}$

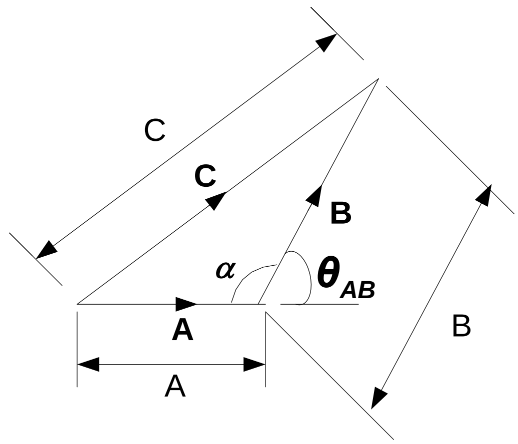


Fig.2 Illustrating the law of cosine for triangles

C^2 =**C•C = (A+B)• (A+B)**

= A•A + B•B + 2 A•B

$= A^2 + B^2 + 2AB\cos\theta_{AB}$

Since θ_{AB} is, by definition, the small angle between **A** and **B** and is equal $(180^\circ - \alpha)$ we know that $\cos\theta_{AB} = \cos(180^\circ - \alpha) = -\cos\alpha$. Therefore,

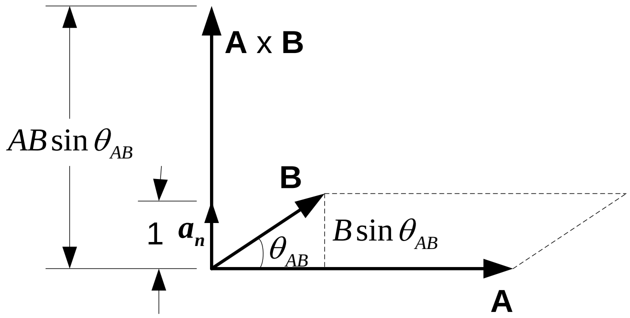
$C^2 = A^2 + B^2 - 2AB\cos\alpha$

1.1.2 VECTOR OR CROSS PRODUCT

The cross product of two vectors **A** and **B** denoted by $\mathbf{A}\times\mathbf{B}$ ("A cross **B**"), is another vector defined by

$\mathbf{A}\times\mathbf{B} = \boldsymbol{a}_n AB\sin\theta_{AB}$

Where θ_{AB} is the small angle between the vectors **A** and **B** $(\leq\pi)$, and $\boldsymbol{a}_n$ is a unit vector normal (perpendicular) to the plane containing **A** and **B**. The direction of $\boldsymbol{a}_n$ follows that of the thumb of a right hand when the fingers rotate from **A** to **B** through the angle θ_{AB} (the right hand rule).



$\mathbf{A}\times\mathbf{B} = \boldsymbol{a}_n AB\sin\theta_{AB}$

Fig. 3Illustration of Vector or cross product

From the definition of vector product and following the right hand rule,

$\mathbf{B}\times\mathbf{A} = -\,\mathbf{A}\times\mathbf{B}$

Hence the cross product is not commutative; and reversing the order of two vectors in a cross product changes the sign of the product.

1.1.3 PRODUCTS OF THREE VECTORS

There are two kinds of products of three vectors:

(1) Scalar triple product

(2) Vector triple product

1. ***Scalar triple product***: this is the dot product of one vector with the result of the cross product of two vectors. A typical form of this is **A•(B ×C**)

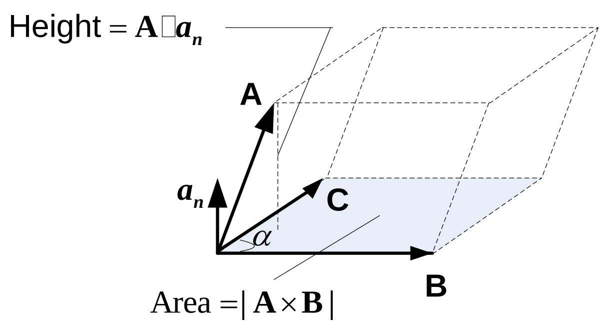


Fig.4 illustrating scalar triple products

2. ***Vector triple product***: This is the cross product of one vector with the result of the cross product of two other vectors. A typical form of this is $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$

1.1.4 ORTHOGONAL COORDINATE SYSTEMS

There are many orthogonal coordinate systems, but the most common and useful are three. These are:

1. Cartesian (or rectangular) coordinates
2. Cylindrical coordinates
3. Spherical coordinates

Cartesian (or Rectangular) Coordinates

A point $P(x_1, y_1, z_1)$ in Cartesian coordinated is the intersection of three planes specified by $x = x_1, y = y_1, z = z_1$.

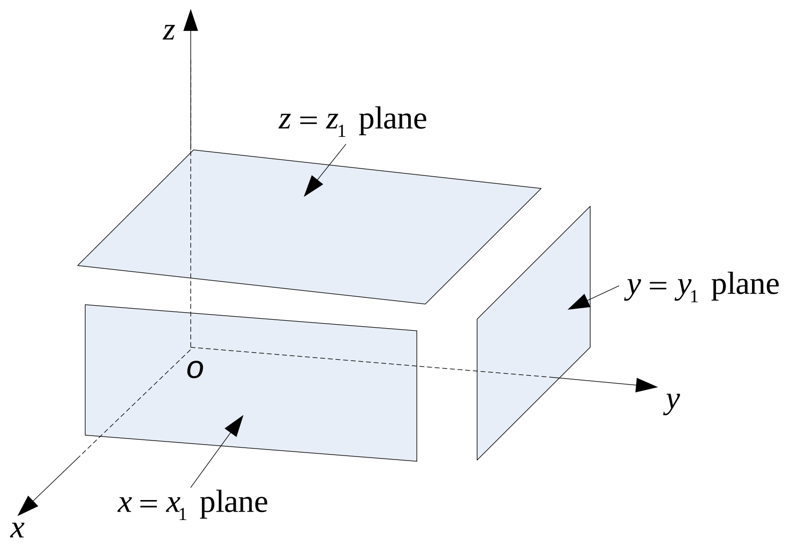


Fig. 5 Three mutually perpendicular planes

The three mutually perpendicular unit vectors, a_x, a_y and a_z, in the three coordinate directions are called the base vectors. For a right-handed system we have the following cyclic properties:

$a_x \times a_y = a_z$

$a_y \times a_z = a_x$

$a_z \times a_x = a_y$

The following relations follow directly

$a_x \bullet a_y = a_y \bullet a_z = a_x \bullet a_z = 0$

and

$a_x \bullet a_x = a_y \bullet a_y = a_z \bullet a_z = 1$

A vector A in Cartesian coordinates

A vector **A** in Cartesian coordinates with components A_x, A_y and A_z can be written as

$$\mathbf{A} = \boldsymbol{a_x} A_x + \boldsymbol{a_y} A_y + \boldsymbol{a_z} A_z$$

Vector differential length in Cartesian coordinates

The expression for a vector differential length is

$$dl = \boldsymbol{a}_x dx + \boldsymbol{a}_y dy + \boldsymbol{a}_z dz$$

Differential volume in Cartesian coordinates

A differential volume is the product of the differential length changes in the three coordinate directions

$$dv = dxdydz$$

Scalar product in Cartesian coordinates

The dot product of vector $\mathbf{A} = \boldsymbol{a_x} A_x + \boldsymbol{a_y} A_y + \boldsymbol{a_z} A_z$ and another vector

$\mathbf{B} = \boldsymbol{a_x} B_x + \boldsymbol{a_y} B_y + \boldsymbol{a_z} B_z$ is

$$\mathbf{A} \bullet \mathbf{B} = (\boldsymbol{a}_x A_x + \boldsymbol{a}_y A_y + \boldsymbol{a}_z A_z) \bullet (\boldsymbol{a}_x B_x + \boldsymbol{a}_y B_y + \boldsymbol{a}_z B_z)$$

or

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Vector product in Cartesian coordinates

The cross product of **A** and **B**

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (\boldsymbol{a_x} A_x + \boldsymbol{a_y} A_y + \boldsymbol{a_z} A_z) \times (\boldsymbol{a_x} B_x + \boldsymbol{a_y} B_y + \boldsymbol{a_z} B_z) \\ &= \boldsymbol{a_x}(A_y B_z - A_z B_y) + \boldsymbol{a_y}(A_z B_x - A_x B_z) + \boldsymbol{a_z}(A_x B_y - A_y B_x)\end{aligned}$$

or

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cylindrical Coordinates

In cylindrical coordinates a point $P(r_1, \phi_1, z_1)$ is the intersection of a circular cylindrical surface $r = r_1$, a half-plane with the *z*-axis as an edge and making an angle $\phi = \phi_1$ with the *xy*-plane, and a parallel to the *xy*-plane at $z = z_1$

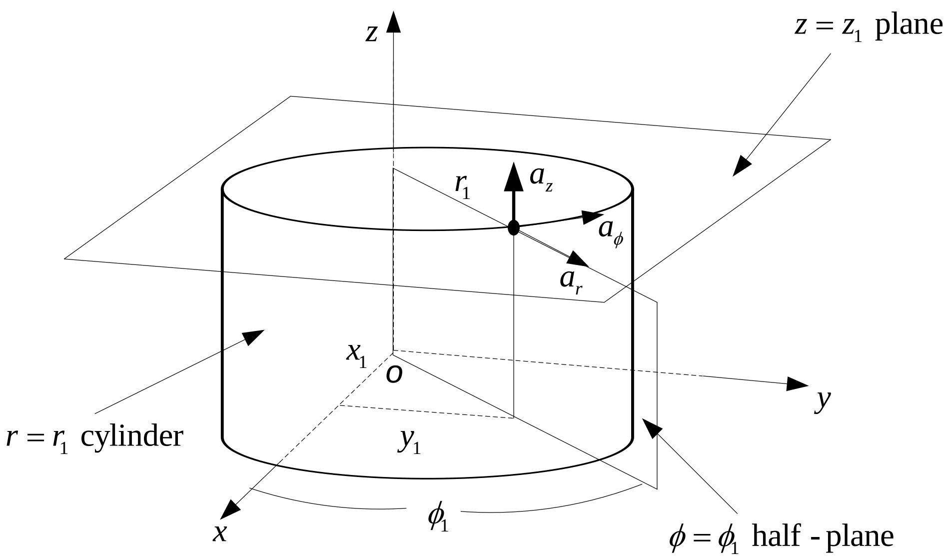


Fig.6 Cylindrical Coordinates

$$a_r \times a_\phi = a_z$$
$$a_\phi \times a_z = a_r$$
$$a_z \times a_r = a_\phi$$

Vector differential length in Cylindrical Coordinates

The expression for a vector differential length is

$dl = \boldsymbol{a}_r\, dr + \boldsymbol{a}_Ø r\, dØ + \boldsymbol{a}_z\, dz$

Differential volume in Cylindrical Coordinates

A differential volume is the product of the differential length changes in the three coordinate directions

$dv = r dr d\phi dz$

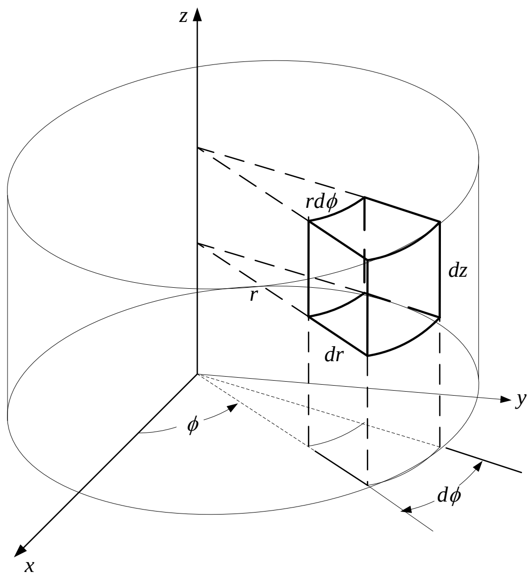


Fig.7 A differential volume element in cylindrical coordinates

A vector A in Cylindrical coordinates

A vector **A** in cylindrical coordinates with components A_r, A_ϕ and A_z can be written as

$$\mathbf{A} = \boldsymbol{a_r} A_r + \boldsymbol{a_\phi} A_\phi + \boldsymbol{a_z} A_z$$

Transformation of Vector Components in Cylindrical Coordinates to Cartesian Coordinates

The relation between the components of a vector in Cartesian and Cylindrical Coordinates is written in matrix form as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Transformation of the location of a Point in Cylindrical Coordinates to Cartesian Coordinates

The coordinates of a point in cylindrical coordinates (r, ϕ, z) can be transformed into those in Cartesian coordinates (x, y, z)

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

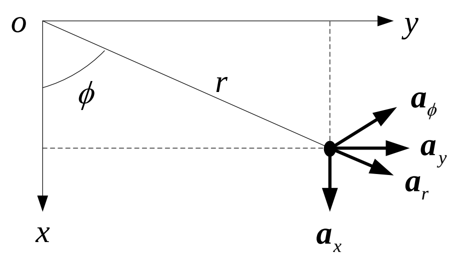


Fig.8 Relations among the base vectors

Spherical Coordinates

A point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates is specified as the intersection of the following three surfaces: a spherical surface centered at the origin with a radius $R = R_1$; a right circular cone with its apex at the origin, its axis coinciding with the + z-axis and having a half-angle $\theta = \theta_1$; and a half-plane with the z-axis as an edge and making an angle $\phi = \phi_1$ with the xz-plane.

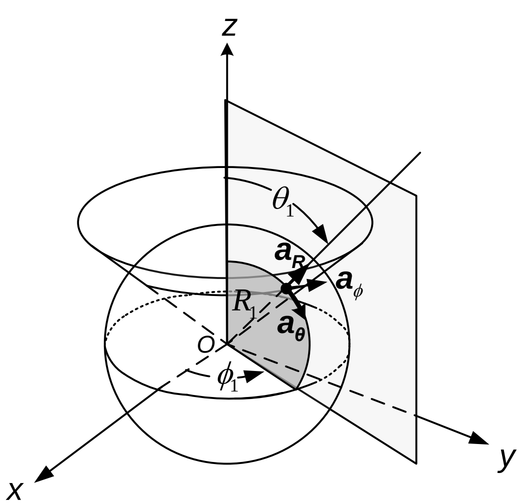


Fig.9Spherical Coordinates

$$a_R \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_R$$
$$a_\phi \times a_R = a_\theta$$

A vector in Spherical coordinates

A vector **A** in Spherical coordinates with components A_R, A_θ and A_ϕ can be written as

$$\mathbf{A} = \boldsymbol{a_R} A_R + \boldsymbol{a}_\theta A_\theta + \boldsymbol{a}_\phi A_\phi$$

Vector differential length in Spherical Coordinates

The expression for a vector differential length is

$$dl = \boldsymbol{a_R} dR + \boldsymbol{a}_\theta R d\theta + \boldsymbol{a}_\phi R \sin\theta d\phi$$

Differential volume in Spherical Coordinates

A differential volume is the product of the differential length changes in the three coordinate directions

$$dv = R^2 \sin\theta dR d\theta d\phi$$

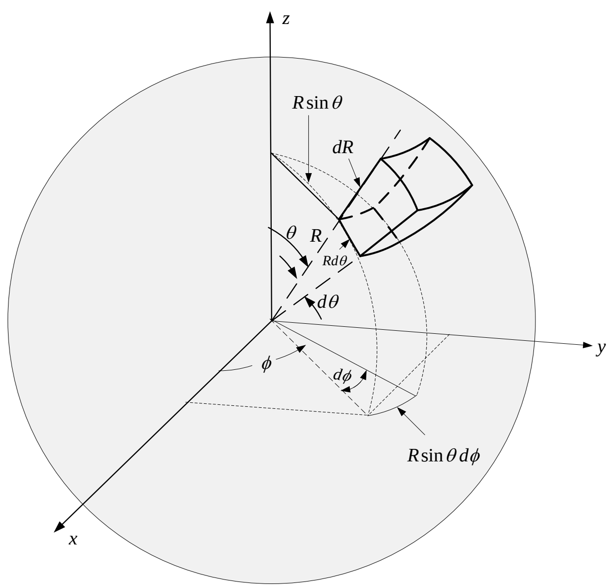


Fig.10 A differential volume element in spherical coordinates

Transformation of the location of a Point in Spherical Coordinates to Cartesian Coordinates

The coordinates of a point in spherical coordinates (R, θ, ϕ) can be transformed into those in Cartesian coordinates (x, y, z)

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

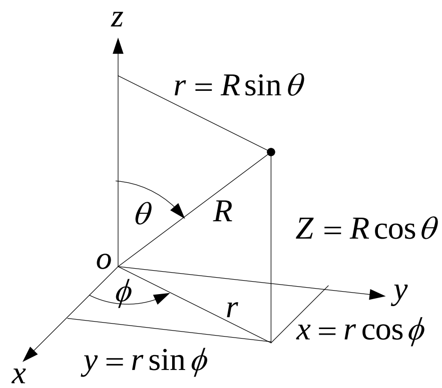


Fig.11 Showing relationship of space variables $(x, y, z), (r, \phi, z),$ and (R, θ, ϕ)

Table:1 Three Basic Orthogonal Coordinate Systems

		Cartesian Coordinates (x, y, z)	**Cylindrical Coordinates** (r, ϕ, z)	**Spherical Coordinates** (R, θ, ϕ)
Base Vectors	$\boldsymbol{a}_{u_1}$	$\boldsymbol{a}_x$	$\boldsymbol{a}_r$	$\boldsymbol{a}_R$
	$\boldsymbol{a}_{u_2}$	$\boldsymbol{a}_y$	$\boldsymbol{a}_\phi$	$\boldsymbol{a}_\theta$
	$\boldsymbol{a}_{u_3}$	$\boldsymbol{a}_z$	$\boldsymbol{a}_z$	$\boldsymbol{a}_\phi$
Metric Coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R\sin\theta$
Differential Volume	dv	$dxdydz$	$rdrd\phi dz$	$R^2\sin\theta dRd\theta d\phi$

SESSION TWO

1.2 GRADIENT OF A SCALAR FIELD AND DIVERGENCE OF A VECTOR FIELD

1.2.1 GRADIENT OF A SCALAR FIELD

The vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar is defined as the gradient of that scalar. It is represented by the del operator, ∇ or **grad** V.

$$\textbf{grad}\ V = \boldsymbol{a}_n \frac{dV}{dn}$$

$$\nabla V = \boldsymbol{a}_n \frac{dV}{dn}$$

The space rate of increase of V in terms of ∇V is given as

$$dV = (\nabla V) \bullet dl$$

∇V in Cartesian Coordinates is given by:

$$\nabla V = \boldsymbol{a}_x \frac{\partial V}{\partial x} + \boldsymbol{a}_y \frac{\partial V}{\partial y} + \boldsymbol{a}_z \frac{\partial V}{\partial z},$$

or

$$\nabla V = \left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z} \right) V$$

From the above equation, ∇ in Cartesian coordinates as a vector differential operator is given by

$$\nabla \equiv \boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}$$

In general, orthogonal coordinates (u_1, u_2, u_3) with metric coefficients (h_1, h_2, h_3) can define ∇ as:

$$\nabla \equiv \left(\boldsymbol{a}_{u_1} \frac{\partial}{h_1 \partial x} + \boldsymbol{a}_{u_2} \frac{\partial}{h_2 \partial y} + \boldsymbol{a}_{u_3} \frac{\partial}{h_3 \partial z} \right)$$

1.2.2 DIVERGENCE OF A VECTOR FIELD

We define the divergence of a vector field ***A*** at a point, abbreviated *div* ***A***, as the net outward flux flow of ***A*** per unit volume as the volume about the point tends to zero.

$$div\ A = \lim_{\Delta v \to 0} \frac{\oint_S A \cdot ds}{\Delta v}$$

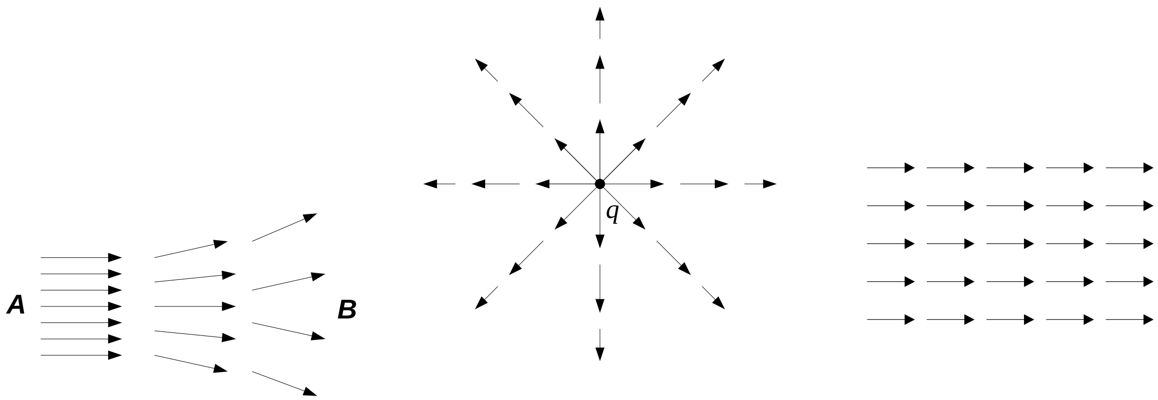


Fig.12 Flux lines of vector fields

$\nabla \bullet \boldsymbol{A}$ in Cartesian Coordinates is given by

$$div\ \boldsymbol{A} = \boldsymbol{a}_x \frac{\partial A_x}{\partial x} + \boldsymbol{a}_y \frac{\partial A_y}{\partial y} + \boldsymbol{a}_z \frac{\partial A_z}{\partial z}$$

$div\ \boldsymbol{A} \equiv \nabla \bullet \boldsymbol{A}$

In general, orthogonal coordinates (u_1, u_2, u_3) with metric coefficients (h_1, h_2, h_3) can define $\nabla \bullet \boldsymbol{A}$ as:

$$\nabla \bullet \boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

$\nabla \bullet \boldsymbol{A}$ in spherical coordinates is written as:

$$\nabla \bullet \boldsymbol{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R \sin\theta}\frac{\partial}{\partial \emptyset}(A_\emptyset)$$

$\nabla \bullet \boldsymbol{A}$ in cylindrical coordinates is written as:

$$\nabla \bullet \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\emptyset}{\partial \emptyset} + \frac{\partial A_z}{\partial z}$$

1.2.3 DIVERGENCE THEOREM

This states that the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume

$$\int_s \nabla \bullet \boldsymbol{A}\, dv = \oint_s \boldsymbol{A} \bullet ds$$

1.2.4 CURL OF A VECTOR

The curl of a vector field **A**, denoted by *curl* ***A*** or $\nabla \times \boldsymbol{A}$, is a vector whose magnitude is the maximum net circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum

$$curl\ \boldsymbol{A} \equiv \nabla \times \boldsymbol{A}$$

$$\nabla \times \boldsymbol{A} = \boldsymbol{a}_u \bullet (\nabla \times \boldsymbol{A}) = \lim_{\Delta s_u \to 0} \frac{1}{\Delta s_u}\left(\oint_{c_u} A \bullet dl\right)$$

Expression of $\nabla \times \boldsymbol{A}$ ***in Cartesian coordinates***

$$\nabla \times \boldsymbol{A} = \boldsymbol{a}_x\left(\frac{\partial \boldsymbol{A}_z}{\partial y} - \frac{\partial \boldsymbol{A}_y}{\partial z}\right) + \boldsymbol{a}_y\left(\frac{\partial \boldsymbol{A}_x}{\partial z} - \frac{\partial \boldsymbol{A}_z}{\partial x}\right) + \boldsymbol{a}_z\left(\frac{\partial \boldsymbol{A}_y}{\partial x} - \frac{\partial \boldsymbol{A}_x}{\partial y}\right)$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \boldsymbol{A}_x & \boldsymbol{A}_y & \boldsymbol{A}_z \end{vmatrix}$$

Expression of $\nabla \times \boldsymbol{A}$ ***in Spherical coordinates***

$$\nabla \times \boldsymbol{A} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \boldsymbol{a}_R & \boldsymbol{a}_\theta R & \boldsymbol{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \boldsymbol{A}_R & R\boldsymbol{A}_\theta & (R\sin\theta)\boldsymbol{A}_z \end{vmatrix}$$

Expression $\nabla \times \boldsymbol{A}$ ***in general orthogonal coordinates system***

$$\nabla \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \boldsymbol{a}_x h_1 & \boldsymbol{a}_y h_2 & \boldsymbol{a}_z h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1\boldsymbol{A}_x & h_2\boldsymbol{A}_y & h_3\boldsymbol{A}_z \end{vmatrix}$$

1.2.5 STOKES THEOREM

It states that the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

$$\int_s (\nabla \times \boldsymbol{A}) \bullet ds = \oint_c \boldsymbol{A} \bullet dl$$

1.2.6 TWO NULL IDENTITIES

1. The curl of the gradient of any scalar field is identically zero.

   $\nabla \times (\nabla V) \equiv 0$

2. The divergence of the curl of any vector field is identically zero.

   $\nabla \bullet (\nabla \times \boldsymbol{A}) \equiv 0$

SOME USEFUL VECTOR IDENTITIES

$\boldsymbol{A} \bullet \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{B} \bullet \boldsymbol{C} \times \boldsymbol{A} = \boldsymbol{C} \bullet \boldsymbol{A} \times \boldsymbol{B}$

$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \bullet \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \bullet \boldsymbol{B})$

$\nabla(\psi \boldsymbol{V}) = \psi \nabla \boldsymbol{V} + \boldsymbol{V} \nabla \psi$

$\nabla \bullet (\psi A) = \psi \nabla \bullet \boldsymbol{A} + \boldsymbol{A} \bullet \nabla \psi$

$\nabla \times (\psi A) = \psi \nabla \times \boldsymbol{A} + \nabla \psi \times \boldsymbol{A}$

$\nabla \bullet (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \bullet (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \bullet (\nabla \times \boldsymbol{B})$

a$\nabla \bullet \nabla V = \nabla^2 V$

$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \bullet \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

SOLVED EXAMPLES

1.1) Given a vector **A**= -a_x+ $a_y 2$ - $a_z 2$ in Cartesian coordinates find

   a) Its magnitude $A=|\mathbf{A}|$

   b) The expression of its unit vector a_A in the direction of **A**

   c) The angle **A** makes with the z-axis

Solution

a)$A=+\sqrt{\mathbf{A}.\mathbf{A}}$

$\mathbf{A}.\mathbf{A} = (-1)(-1) + (2)(2) + (-2)(-2) = 9$

$A=\sqrt{9} = 3$

b) $a_A = \frac{\mathbf{A}}{A} = \frac{1}{3}\left(-a_x + a_y 2 - a_z 2\right) = \left(-\frac{1}{3}a_x + a_y\frac{2}{3} - a_z\frac{2}{3}\right)$

c) $\mathbf{A} \bullet \boldsymbol{a_z} = A\cos\theta_z$

$(-a_x + a_y 2 - a_z 2) \bullet \boldsymbol{a_z} = -2 = 3\cos\theta_z$

$\theta_z = \cos^{-1}\left(\frac{-2}{3}\right) = 131.8°$

1.2) Express the unit vector $\boldsymbol{a}_z$ *in spherical coordinates*

Solution

$\boldsymbol{a}_z \bullet \boldsymbol{a}_R = \cos\theta$

$\boldsymbol{a}_z \bullet \boldsymbol{a}_\theta = -\sin\theta$

$\boldsymbol{a}_z \bullet \boldsymbol{a}_\varnothing = 0$

Thus,

$\boldsymbol{a}_z = \boldsymbol{a}_R \cos\theta - \boldsymbol{a}_\theta \sin\theta$

1.3) The electrostatic field intensity **E** is the negative gradient of a scalar electric *potential, that is* $\mathbf{E} = -\nabla V$. *Determine* **E** *at point (1,1,0) if*

a) $V = V_o e^{-x}\sin\frac{\pi y}{4}$

b) $V = E_o R\cos\theta$ $(Try - Spherical\ coordinates)$

Solution

a) $\mathbf{E} = -\left[a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right] V_o e^{-x}\sin\frac{\pi y}{4}$

$= \left(a_x \sin\frac{\pi y}{4} - a_y\frac{\pi}{4}\cos\frac{\pi y}{4}\right) V_o e^{-x}$

at $\mathbf{E}(1,1,0) = \left(a_x - a_y\frac{\pi}{4}\right)\frac{V_o}{\sqrt{2}}$

ASSIGNMENTS

1) Show that $\nabla \times \mathbf{A} = 0$ if

   a) $\mathbf{A} = \boldsymbol{a}_{\emptyset}(k/r)$ in cylindrical coordinates where k is a constant, or

   b) $\mathbf{A} = \boldsymbol{a}_R f(R)$ in spherical coordinates where $f(R)$ is any function of the radial distance R.

2) What is the difference between a scalar quantity and a scalar field and between a vector quantity quantity and a vector field.

3) Find the result of the following

   a)$\boldsymbol{a}_{\emptyset} \bullet \boldsymbol{a}_x$

   b) $\boldsymbol{a}_R \bullet \boldsymbol{a}_y$

   c) $\boldsymbol{a}_z \bullet \boldsymbol{a}_R$

   d) $\boldsymbol{a}_{\emptyset} \times \boldsymbol{a}_x$

   e) $\boldsymbol{a}_r \times \boldsymbol{a}_R$

   f) $\boldsymbol{a}_{\theta} \times \boldsymbol{a}_z$

4) Given the scalar field $V = 2xy - yz + xz$

   a)find the vector representing the direction and magnitude of the maximum rate of increase of V at point P(2,-1,0)

   b)find the rate of increase of V at point P in the direction toward the point Q(0,2,6)

UNIT TWO

STATIC ELECTRIC FIELDS (ELECTROSTATICS)

Electrostatics is the study of electric charges at rest. As far back as 600 B.C. there was evidence of the existence of the knowledge of static electricity. The word electricity was derived from the Greek word for amber. It was common knowledge that, when amber was rubbed against fur, it was able to attract small pieces of papers. This was the beginning of the field of static electricity and shortly after 1600, Colonel Charles Coulomb performed experiments to demonstrate the force that exist between two objects each having static electricity. He published the results of his work, which came to be known as Coulomb's law.

This unit begins with the fundamental postulate of static electric field and develops by deductive approach, Coulombs law. Quantities relevant to the study of static electric fields are introduced. The interaction between material media and these field quantities is presented and extended into the determination of boundary conditions at the interface of two media. A related law, Gauss law is introduced in relation to its application on static fields; which leads to a discussion on capacitors and capacitance. Finally, the applications of Poisson and Laplace equations on fields are also presented

Objectives:

By the end of this chapter, you should be able to:

- Know the fundamental postulates of electrostatics in free space.
- Apply Coulomb's law to a system of discrete charges and a continuous distribution of charges
- Apply Gauss law of electrostatics to a surface charge density
- Calculate Electric potential, electric flux density and electrostatic energy
- Know about the different kinds of material media in electrostatics and about dielectric constants
- Determine the boundary conditions for electrostatic fields
- Calculate the Capacitance of a capacitor of a given configuration
- Apply Poisson and Laplace's equations to fields

SESSION ONE

2.1 FUNDAMENTAL POSTULATES OF ELECTROSTATICS IN FREE SPACE

Electrostatics is the study of electric charges at rest.

Electric field intensity is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists.

$$E = \lim_{q \to 0} \frac{F}{q} \quad \ldots\ldots. (1) \qquad \text{Units in N/C or V/m}$$

From this equation, $E \,\alpha\, F$ (E is in the same direction as F)

From (1) above, the force on the electron is $\vec{F} = q\vec{E}$ (N)

The two fundamental postulates of electrostatics in free space specify the divergence and curl of E as:

$$\text{Divergence:} \qquad \nabla \cdot E = \frac{\rho v}{\varepsilon_o} \quad \ldots\ldots (2)$$

ρv is the volume charge density given by $\rho v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v}$ and ε_o is the permittivity of free space, a universal constant = 8.854 x 10^{-12}

and Curl: $\nabla \times \vec{E} = 0$ ……. (3)

The curl-free nature of the electric field indicates that, the electric field is irrotational.

Equation (2) indicates that the static electric field is not divergenless or solenoidal unless $\rho v = 0$

Equations (2) and (3) are the differential forms of the postulates of electrostatics.

We are more interested in the total field in practice and that can be obtained by taking the integral volume of both sides of equation (2)

$$\int_V \nabla \cdot \vec{E} dv = \frac{1}{\varepsilon_o} \int_v \rho v dv \quad \ldots\ldots..(4)$$

From divergence theorem, $\int_v \nabla \cdot E dv = \oint_s E \cdot ds$ and $\int_v \rho v dv = Q$

i.e. $\oint_s \vec{E} \cdot ds = \dfrac{Q}{\varepsilon_O}$ …….(5) where Q is the total charge contained in ***v***

For the curl relations,

$\int_s \nabla \times \vec{E} = 0$

From stokes theorem

$\int_s \nabla \times \vec{E} = \oint_c E \cdot dl$ i.e. $\oint_c \vec{E} \cdot dl = 0$ ……… (6) Line integral performed over an arbitrary closed contour, c.

Equation (6) means, the scalar line integral of the static electric field intensity around a closed path is zero..

Observe $E\left({}^{v}\!/\!_{m}\right) \cdot dl(m) = v$ (volts, units for voltage) i.e. *E.dl* integrated over any path is the voltage along that path. i.e. equation (6) is an expression of Kirchoff's voltage law in circuit theory: which is, the algebraic sum of voltage drops around any closed circuit is zero.

Also put: the scalar line integral of the irrotational *E* field from one point (say P_1) to any other point (say P_2) along any path (say P_3) is cancelled by that from P_2 to P_1 along any other path.

Finally, the line integral of *E* from point P_1 to P_2 represents the work done by *E* in moving a unit charge from P_1 to P_2.

($E = \dfrac{F}{q} = \dfrac{F}{1} = F \quad \int E \cdot dl = E \int dl = E \cdot l = F \cdot l = work$)

In other words, the work done in moving a unit charge around a closed path in an electrostatic field is zero. This is the principle of conservation of work (energy) in an electrostatic field.

2.1.1 <u>COULOMBS LAW</u>

For a point charge situated in free space, it radiates equally in all directions and a sphere of an arbitrary radius can be described round the charge (center).

$$\oint_s E \cdot ds = \oint_s (a_R E_R) \cdot a_R ds = \frac{q}{\varepsilon_O} \qquad a_R.a_R = 0$$

$$E_R \oint_s ds = E_R \left(4\pi R^2\right) = \frac{q}{\varepsilon_o}$$

i.e. $E = a_R E_R = a_R \dfrac{q}{4\pi\varepsilon_o R^2}$ ……. (7)

If q is situated at a distance R^1 from the origin, then the distance between the charge and the point P, R distance away from the origin where *E* needs to be determined is $R-R^1$.

The unit vector in that direction is $a_R = \dfrac{(R-R^1)}{|R-R^1|}$ and

$$E = \frac{q(R-R^1)}{4\pi\varepsilon_O |R-R^1|^3} \quad \text{……. (8)}$$

Let a point charge q_2 be placed in the field of another point charge q_1, a force F_{12} is experienced by q_2 due to E_{12} at q_2 and is given by

$$F_{12} = q_2 E_{12} = a_{12} \frac{q_1 q_2}{4\pi\varepsilon_o R_{12}^2} \quad \text{………(9)} \quad \text{Coulomb's law}$$

Coulomb's law states that the force between two point charges is proportional to the product of the charges and inversely proportional to the square of the distance of separation.

Electric Field Due To A System Of Discrete Charges

The electric field contribution due to charges q_1, q_2 …..q_n at distances $R_1', R_2' ....R_n'$ from point of interest (situated R away) respectively, then,

$$E = \frac{1}{4\pi\varepsilon_O}\left[\frac{q_1(R-R_1')}{|R-R_1'|^3} + \frac{q_2(R-R_2')}{|R-R_2'|^3} + ... + \frac{q_n(R-R_n')}{|R-R_n'|^3}\right]$$

$$E = \frac{1}{4\pi\varepsilon_O}\sum_{k=1}^{n}\frac{q_k(R-R')}{|R-R'|^3} \quad \text{…….(10)}$$

Electric Field Due To A Continuous Distribution Of Charge

Volume Charge Density

The contribution of a charge $\rho v dv$ in a differential volume element dv' to the electric field intensity at any point P is,

$$E = \frac{1}{4\pi\varepsilon_O}\int_{v'} a_R \frac{\rho v}{R^2} dv' \quad \text{(V/m) …….(11)}$$

<u>Note</u>

$\int_v \rho v dv = q$, $\int_s \rho s ds = q$, $\int_l \rho l dl = q$

Surface Charge Density

If the charge is distributed on a surface charge density $\rho s\left(c/_{m^2}\right)$,

$$E = \frac{1}{4\pi\varepsilon_O}\int_{S'} a_R \frac{\rho s}{R^2} ds' \quad \text{………..(12)}$$

For a line charge, $E = \frac{1}{4\pi\varepsilon_o}\int_{l'} a_R \frac{\rho l}{R^2} dl'$ ………(13) where ρl = line charge density

2.1.2 <u>GAUSS LAW</u>

Gauss law states that the total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_o, i.e.

$$\oint_s E \cdot ds = \frac{Q}{\varepsilon_o}$$

The essence of applying Gauss's law lies first in the recognition of symmetry conditions; and also in the suitable choice of a surface over which the normal component of E resulting from a given charge distribution is constant. Such a surface is called Gaussian surface. E.g. Surface of a sphere centered at the point charge.

2.1.3 <u>ELECTRIC POTENTIAL</u>

A scalar Electric Potential is defined so that the electric field is given by

$$E = -\nabla V$$

The electrical potential is related to the work done in carrying a charge from one point to another. The negative sign must be included to account for the convention that, in going against the field, the electric potential increases.

In moving a unit charge from point P_1 to P_2 in an electric field, work is done against the field.

The elemental work along the path P_1P_2 is $dW = -F \cdot dl$

The total work $\int dW = -\int F \cdot dl$

$$F = qE \Rightarrow \int dW = -\int_{P1}^{P2} qE \cdot dl \Rightarrow \frac{W}{q} = -\int_{P1}^{P2} E \cdot dl$$

$$\frac{W}{q} = -\int_{P1}^{P2} E \cdot dl$$

The work done per unit charge moving from point P_1 to P_2 is the electric potential difference between P_1 and P_2 (i.e. $V_2 - V_1$)

$$\therefore V_2 - V_1 = -\int_{P1}^{P2} E \cdot dl$$

Let P_1 be R_1 from the source and P_2 be R_2 from the source then,

$$V_2 - V_1 = -\int_{R1}^{R2} E \cdot dl \quad \text{where} \quad E = \frac{q}{4\pi\varepsilon_o r^2}$$

Let a_r be a unit vector in the direction of R.

$$dl = a_r \cdot dr$$

$$E = a_r \cdot Er$$

then, $V_2 - V_1 = \int_{R1}^{R2} a_R E_R \cdot a_R dr \Rightarrow V_2 - V_1 = -\frac{q}{4\pi\varepsilon_o}\int_{R1}^{R2} \frac{1}{R^2} dr$

after integration, the negative sign vanishes.

$$V_2 - V_1 = \int_{R1}^{R2} E_R \cdot dR = \int_{R1}^{R2} \frac{q}{4\pi\varepsilon_o R^2} dR = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{R_2} - \frac{1}{R_1}\right]$$

$$V_2 - V_1 = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{R_2} - \frac{1}{R_1}\right]$$

If we want to determine the potential at a point A, then we assume the point P_1 is at infinity. Then,

$$V_A = \int_{\infty}^{R} \frac{q}{4\pi\varepsilon_o R^2} dr \Rightarrow V_A = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{R} - \frac{1}{\infty}\right] = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{R}\right]$$

$$V_A = \frac{q}{4\pi\varepsilon_o R}$$

The electric potential due to a continuous distribution of charges confined in a given region is obtained by integrating the contribution of an element of a charge over the charged region.

Thus for a volume charge distribution, $V = \frac{1}{4\pi\varepsilon_o}\int_v \frac{\rho v}{R} dV$

For a surface charge distribution, $V = \frac{1}{4\pi\varepsilon_o}\int_s \frac{\rho s}{R} dS$

For a line charge distribution, $V = \frac{1}{4\pi\varepsilon_o}\int_l \frac{\rho l}{R} dl$

SESSION TWO

2.2 MATERIAL MEDIA IN STATIC ELECTRIC FIELD AND BOUNDARY CONDITIONS

Materials exist in three types:

- Conductors
- Semiconductors
- Insulators (dielectrics)

2.2.1 CONDUCTORS IN STATIC ELECTRIC FIELD

If a conducting body is placed in electrostatic field, and time is allowed for the drift motion of changes under the influence of the field to stop, the electric field of induced charges will exactly cancel out the external field and the total electric field at all points of a conductor will be zero. i.e in electrostatic field, E = 0 inside conductors.

Applying Gauss's law to a surface that is completely inside the conductor; because E is zero at all points in S, the total charge enclosed by S will also be zero. In other words, if there are excess charges, they must be distributed over the surface of the conductor. In electrostatics, a conductor has charges only on its surface.

Because there is no field inside the conductors, the tangential component of the electric field strength on the very surface of the conductor is also zero (otherwise, it will produce organized motion of charge on its surface.)

Because the tangential component of E is zero on conductor surfaces, the potential difference between any 2 points on a conductor is zero($V = \int E.dl = 0$ as $\mathrm{E} = 0$). i.e. the surface of a conductor in an electrostatic field is equipotential. And because the field inside the conductor is zero also, it follows that, all points of the conductor have the same potential.

Now consider a small cylindrical surface ΔS on the surface of the conductor: which has a height Δh —> 0; portion of the cylinder in air and in conductor.

The only field present is the normal component given by $En\Delta S$ and from Gauss law; this should equal the charge in the surface divided by ε_o $\frac{\rho s \Delta s}{\varepsilon_o}$

$$E_n \Delta s = \frac{\rho s}{\varepsilon_o} \Delta s$$

$$E_n = \frac{\rho s}{\varepsilon_o}$$

2.2.2 IMAGE METHOD

For a point charge Q at a certain distance d above a perfectly conducting plane, it is not easy to determine the electric field intensity using the methods we have discussed so far. The image method is most helpful in finding E above a perfectly conducting plane.

The image theory states that: any given charge configuration above an infinite, perfectly conducting plane is electrically equivalent to the combination of the given charge configuration and its image configuration, with the conducting plane removed. The method is represented by the charge Q at a distance d and its image –Q at a distance d below. That is, the distance between two charges will be 2d.

2.2.3 DIELECTICS IN STATIC ELECTRIC FIELD

When a dielectric material is placed in an electric field, the field exerts a force on each charge causing the charges (both positive and negative) to be displaced in opposite directions by small amount. This causes the polarization of the dielectric materials and creates electric dipoles. The sum of all the dipole moments per differential volume gives the polarization vector P.

In other words, the polarization vector is the volume density of electric dipole moment and it is related to another quantity (polarization volume charge density) by:

$P_{\rho v} = -\nabla \cdot P$; which is the net total charge flowing out of V as the result of polarization.

2.2.4 ELECTRIC FLUX DENSITY AND DIELECTRIC CONSTANT

Since a polarized dielectric gives rise to an equivalent volume charge density $P_{\rho v}$, the electric field intensity to a given source distribution in a dielectric will be different from that in free space. i.e.

$\nabla \cdot E = \frac{\rho v}{\varepsilon_o}$ has to be modified to $\nabla \cdot E = \frac{1}{\varepsilon_o}\left(\rho_v + \rho_{\rho v}\right)$ but $p_{\rho v} = -\nabla \cdot P$

$$\nabla \cdot \varepsilon_o E = \rho v - \nabla \cdot P$$
$$\nabla \cdot \varepsilon_o E + \nabla \cdot P = \rho v$$
$$\Rightarrow \nabla \cdot (\varepsilon_o E + P) = \rho v$$

The electric flux density or electric displacement D is defined such that,

$D = \varepsilon_o E + P$ C/m^2

Thus $\nabla \cdot D = \rho v$ ………..(X)

$\nabla \cdot D = \rho v$ and $\nabla \cdot E = 0$ are the two governing fundamental differential equations for electrostatics in any medium. ρv is the volume density of free charges

From equation (X),

$$\int_v \nabla \cdot D dv = \int_v \rho v dv = 0$$

or using Gauss' law

$\oint_s D \cdot ds = Q$ …… Generalized Gauss' law

Gauss law states that the total outward flux of the electric displacement (or simply, the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

For a linear and isotropic dielectric medium,

$P \infty E \Rightarrow P = \varepsilon_o X_e E$ where X_e is the electric susceptibility.

A dielectric medium is linear if *Xe* is independent of E and is homogenous if *Xe* is independent of space coordinate.

But $D = \varepsilon_o E + P$

$$= \varepsilon_o E + \varepsilon_o \chi_e E = \varepsilon_o (1 + \chi_e) E$$

$= \varepsilon_o \varepsilon_r E = \varepsilon E$ where $\varepsilon_r = 1 + \chi_e$ is the relative permittivity of a dielectric constant of a medium, and $\varepsilon = \varepsilon_o \varepsilon_r$ is the absolute permittivity.

The maximum electric field intensity that a dielectric material air withstands without breakdown is the dielectric strength of the material.

2.2.5 BOUNDARY CONDITIONS FOR ELECTROSTATIC FIELDS

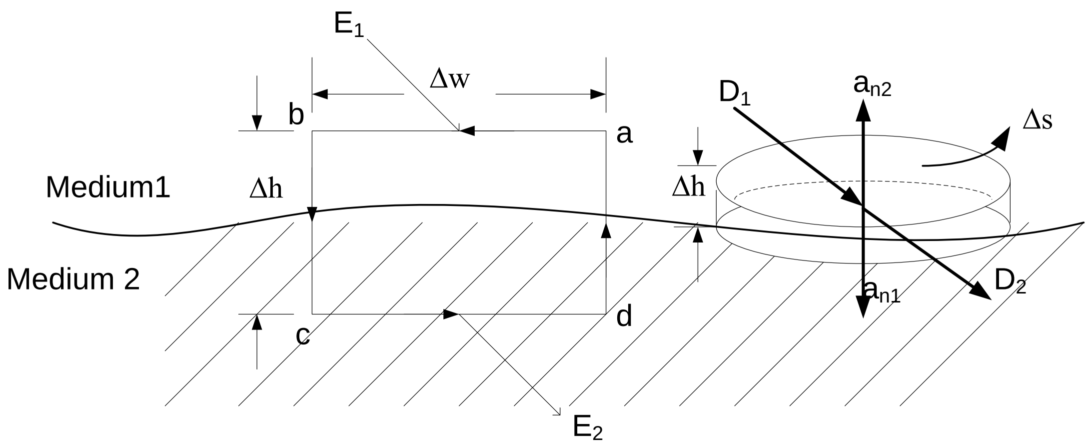


Fig.13Electrostatic Fields at the boundary of dielectric media

$$bc = ad = \Delta h \rightarrow 0$$
$$ba = cd = \Delta w$$

Contribution of the line integral of E along abcda is

$$\oint_{abcda} E \cdot dl = E_1 \cdot \Delta w + E_2 \cdot (-\Delta w)$$

As $\Delta h \rightarrow 0$ i.e. E_1 and E_2 at equipotential

$$E_{1t}\Delta w - E_{2t}\Delta w = 0$$

$\therefore E_{1t} = E_{2t}$: The tangential component of an E-field is continuous across an interface

$$D = \varepsilon E \text{ or } E = \frac{D}{\varepsilon}$$

$$\therefore \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

As $\Delta h \to 0$, $\oint_v D \cdot dv = \rho v dv$ as $\oint_s D \cdot ds = \rho s \Delta s$

$$\oint_s D \cdot ds = \left(D_1 \cdot a_{n2} + D_2 \cdot a_{n1}\right)\Delta s \qquad \text{but } a_{n2} = -a_{n1}$$

$$\begin{aligned} &= \left(D_1 a_{n2} - D_2 a_{n2}\right)\Delta s = a_{n2} \cdot \left(D_1 - D_2\right)\Delta s \\ &= \rho s \Delta s \\ &\Rightarrow a_{n2} \cdot \left(D_1 - D_2\right) = \rho s \end{aligned}$$

Or $D_{1n} - D_{2n} = \rho s$ i.e. the normal component of D field is discontinuous across an interface where a surface charge exists: the amount of discontinuity being equal to the surface charge density.

If there are no free charges at the interface, $\rho s = 0$ as $D_{1n} = D_{2n}$ or $E_1 E_{1n} = E_2 E_{2n}$

2.2.6 <u>CAPACITANCES AND CAPACITORS</u>

The charge of a conducting body Q, is directly proportional to its potential V.

i.e. Q ∞ V $\Rightarrow Q = CV$ or $C = \dfrac{Q}{V}(F)$ where C is the capacitance of the isolated conducting body .

For two parallel plate capacitor, V_{12} being the potential difference between the two conductors, $C = \dfrac{Q}{V_{12}}(F)$

To determine the capacitance C between two conductors,

1. Choose an appropriate coordinate system for the given geometry.
2. Assume charges +Q and –Q on the conductors
3. Find E from Q by Gauss law or other relations $E = \dfrac{\rho s}{\varepsilon_o}$, $\rho s = \dfrac{Q}{S} \Rightarrow E = \dfrac{Q}{\varepsilon_o S}$
4. Find V_{12} by evaluating $V_{12} = -\int_2^1 E \cdot dR$
5. Find C from the ratio $\dfrac{Q}{V_{12}}$

2.2.7 <u>POISSON AND LAPLACE'S EQUATION</u>

The two fundamental differential equations for electrostatics

$$\nabla \cdot D = \rho v$$
$$\nabla \times E = 0$$

Now $E = -\nabla V$ and in a linear isotropic medium, $D = \varepsilon E$

$\therefore \nabla \cdot D = \rho v \Rightarrow \nabla \cdot \varepsilon E = \rho v$

Or $\nabla \cdot (\varepsilon \nabla v) = -\rho v$ for a simple medium, ε is constant

$$\therefore \nabla \cdot \nabla V = -\frac{\rho v}{\varepsilon}$$

Or $\nabla^2 V = -\frac{\rho v}{\varepsilon}$ ……….. Laplacian equation. Where ∇^2 (del square) is the Laplacian operator

In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right) \cdot \left(a_x \frac{\partial V}{\partial x} + a_y \frac{\partial V}{\partial y} + a_z \frac{\partial V}{\partial z} \right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho v}{\varepsilon}$$

In cylindrical,

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In spherical,

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi}$$

Now, at a point in a simple medium where $\rho v = 0$,

$\nabla^2 V = 0$ …… Laplace equation

In Cartesian,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

2.2.8 ELECTROSTATIC ENERGY

The general expression for the electric potential energy of a group of N discrete point charges at rest is given by

$$W_e = \frac{1}{2}\sum_{k-1}^{N} Q_k V_k$$ where V_k , the electric potential at Q_k is caused by all the other charges.

Considering the situation of a simple charge Q_1,

$$W_e = \frac{1}{2} Q_1 V_1 \ldots\ldots\ldots \text{Joules} \quad \text{but } Q_1 = C_1 V_1 \quad \therefore W_e = \frac{1}{2} C V^2$$

In terms of field quantities,

$$W_e = \frac{1}{2}\int_{v'} D \cdot E dv', \text{ as } \quad D = \varepsilon E$$

$$W_w = \frac{1}{2}\int_{v'} \varepsilon E^{2v} dv'$$

SOLVED QUESTIONS

2.1) Determine the electric field intensity at P(-0.2,0,-2.3) due to a point charge of +5 nC at Q(0.2,0.1,-2.5) in air. All dimensions are meters.

Solution

The position vector for the field point P

$$R = \overrightarrow{OP} = -a_x 0.2 - a_z 2.3$$

$$R' = \overrightarrow{OQ} = a_x 0.2 + a_y 0.1 - a_z 2.5$$

The difference

$$R - R' = a_x 0.4 - a_y 0.1 + a_z 0.2$$

$$\left|R - R'\right| = \left(0.4^2 + 0.1^2 + 0.2^2\right)^{1/2} = 0.458\ m$$

$$\overrightarrow{E_p} = \frac{Q}{4\pi\varepsilon_o}\frac{R - R'}{\left|R - R'\right|^3} = 5x10^{-9}\,x\ \frac{9x10^9}{0.458^3}\,x\ \left(a_x 0.4 - a_y 0.1 + a_z 0.2\right)$$

$$\overrightarrow{E_p} = 214.5\left(-a_x 0.873 - a_y 0.218 + a_z 0.437\right)\ V/m$$

$$\left|E_p\right| = 214.5\ V/m \qquad a_{QP} = \frac{R - R'}{\left|R - R'\right|^3} = \left(-a_x 0.873 - a_y 0.218 + a_z 0.437\right)$$

2.2) Two point charges $-q$ and $+\frac{1}{2}q$ are situated at the origin and at the point $(a,0,0)$ respectively. At what point along the axis does electric field vanish?

Solution

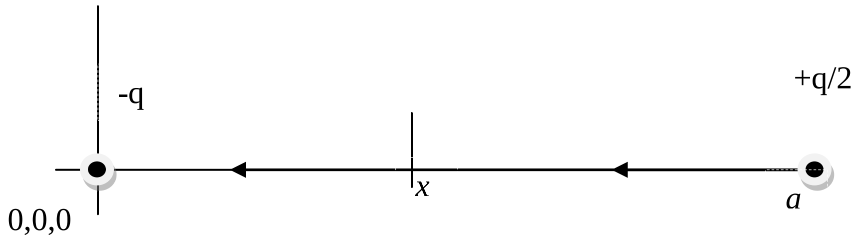


Fig 14.Illustration of Question 2.2

Let *x* be thr vanishing point

$$E = \frac{Q}{4\pi\varepsilon_\circ R^2}$$ \

$E_R = E_1 + E_2 = 0$

$$E_1 = \frac{Q_1}{4\pi\varepsilon_\circ {R_1}^2} = \frac{1}{4\pi\varepsilon_\circ}\frac{(-q)}{x^2}$$

$$E_2 = \frac{Q_2}{4\pi\varepsilon_\circ {R_2}^2} = \frac{1}{4\pi\varepsilon_\circ}\frac{\left(\frac{1}{2}q\right)}{(a-x)^2}$$

$$E_R = \frac{1}{4\pi\varepsilon_\circ}\left[\frac{-q}{x^2} + \frac{\frac{1}{2}q}{(a-x)^2}\right] = 0$$

$$\frac{q}{4\pi\varepsilon_\circ}\left[\frac{-1}{x^2} + \frac{1}{2(a-x)^2}\right] = 0$$

$$\frac{x^2 - 2(a-x)^2}{2x^2(a-x)^2} = 0$$

$x^2 - 2(a-x)^2 = 0$

$x^2 - 4ax + 2a^2 = 0$

$x = \frac{4a \pm \sqrt{16a^2 - 4(2a^2)}}{2}$

$x = 2a \pm a\sqrt{2}$

$x = a(2 \pm \sqrt{2})$

2.3) a) Calculate the capacitance of a square parallel plate capacitor having two dielectrics ε_{r1}=1.5 and ε_{r2}=3.5 each comprising one half of the area between the plates.The area of a plate is $4.0x10^4$ cm^2 and the two plates separated by a distance of 20mm.

b) If a 12V volts electric potential is applied across the terminals of this capacitor, find the electric potential energy stored in the capacitor.

Solution

a)

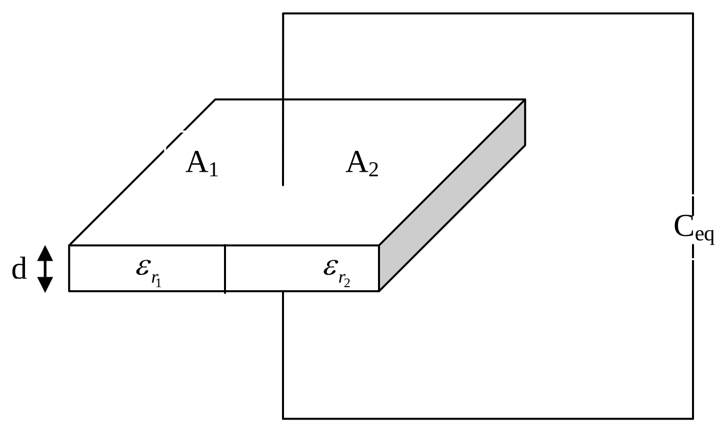


Fig.15Illustration of Question 2.3

ε_{r1}=1.5

ε_{r2}=3.5

A=4.0$x10^4$ cm^2 = 4 m^2

$A_1 = A_2 = \frac{1}{2}A = 2m^2$

d = 20mm = 20x10^{-3}m = 0.02m

$C_1 = \frac{\varepsilon_{r1}\varepsilon_o A_1}{d} = \frac{1.5\times 8.854\times 10^{-12}\times 2}{0.02} = 1.3281 \times 10^{-9}F = 1.3281nF$

$C_2 = \frac{\varepsilon_{r2}\varepsilon_o A_2}{d} = \frac{3.5\times 8.854\times 10^{-12}\times 2}{0.02} = 3.0989 \times 10^{-9}F = 3.0989nF$

$C_{eq} = C_1 + C_2$, since they are in parallel

$C_{eq} = 1.3281nF + 3.0989nF = 4.427nF$

b) $W_e = \frac{1}{2}CV^2 = \frac{1}{2} \times 4.427nF \times (12)^2 = 3.187 \times 10^{-7}J$

2.4) Assuming that a cloud of electrons confined in a region between two spheres of radii 2 and 5 cm has a charge density of

$$\frac{-3\times 10^{-8}}{R^4} cos^2\emptyset \quad (c/m^3)$$

find the total charge contained in the region.

Solution

$$\rho_v = \frac{-3\times 10^{-8}}{R^4} cos^2\emptyset$$

$$Q = \int \rho_v dv$$

The given conditions of the problem show the use of spherical coordinates. Using the expression for *dv* in spherical coordinates,

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{0.02}^{0.05} \rho_v R^2 \sin\theta \, dR \, d\theta \, d\emptyset = -1.8\pi$$

ASSIGNMENTS

1) A parallel plate capacitor has a surface charge on the lower side of the upper plate of $+\rho_s\ C/m^2$. The upper surface of the lower plate contains $\rho_s\ C/m^2$. Find the D and E in the region between the plates using Gauss Law and neglecting fringing.

$$Ans:\ D = \rho_s\ a_n$$
$$E = \rho_s/\varepsilon_o\ \ a_n$$

2) Derive an expression for the electric field intensity E due to charge uniformly distributed over an infinite plane with surface charge density ρ_s. Assume charge in $Z = 0$ - plane and use cylindrical coordinate.

$$Ans:\ \frac{\rho_s}{2\varepsilon_o} a_n\ \ \mathrm{V/m}$$

3) For the potential function $V = 4x + 8y$ volts, in free space, calculate the energy stored in a cubic volume centred at the origin.

$$Ans:\ W = 354.0\ pJ/m^3$$

4) Two infinite radial planes are inclined to each other at an angle α. There is an infinite small insulating gap at $r = 0$. Solve the one dimensional Laplace's equations in cylindrical coordinates to obtain the potential V as a function of φ. Boundary coordinates are given as $V = 0$ at $\varphi = 0$ and $V = V_o$ at $\varphi = \alpha$

$$Ans:\ V = \frac{V_o}{\alpha} \cdot \varphi \qquad \text{and}$$
$$E = -\frac{1}{r}\frac{V_o}{\alpha}\ a_\varphi$$

UNIT THREE

STATIC MAGNETIC FIELDS (MAGNETOSTATICS)

.

The phenomenon of magnetism was first discovered when pieces of magnetic lodestone were found to exhibit magical attractive power. Since the lodestone was found near the ancient Greek city called Magnesia, the terms magnet, magnetism, magnetization and magnetron have come to be used. We study magnetism by introducing magnetic field. A magnetic field can be caused by a permanent magnet (like lodestone), by moving charges or by a current flow.

Objectives:

By the end of this chapter, you should be able to:

- Know about the different types of magnetic materials
- Calculate Vector magnetic potential, magnetic field intensity and magnetic energy
- Know about boundary conditions for magnetostatic fields
- Know about Inductances and inductors

SESSION ONE

3.1 MAGNETOSTATICS

In a region of space where there is magnetic field, which is characterized by magnetic flux density B, a charge in motion or a current element experiences magnetic force F_m. The force according to experiment is given by

$F_m = qu \times B$ where q is the charge and u its drift velocity. B is measured in Wb/m^2 or Tesla (T).

If the charge is moving in the magnetic field B under the influence of an Electric field E, then the total force on the charge is given by $F = F_e + F_m$, where F_e is the force due to the electric field E.

Noting that $F_e = qE$, then

$$\begin{aligned} F &= qE + qu \times B \\ F &= q(E + u \times B) \end{aligned} \qquad (1)$$

Equation (1) is called the Lorentz's force equation.

Charges in motion produce current that in turn creates a magnetic field. Steady currents are accompanied by static magnetic field.

The fundamental postulates of magnetostatics which specifies the divergence and curl of B are:

$$\nabla \cdot B = 0 \qquad \text{divergenless field or solenoidal} \qquad (2)$$

$$\nabla \times B = \mu_0 J \quad \text{(in non magnetic media)} \qquad (3)$$

In integral form

$$\oint_S B \cdot dS = 0 \qquad (4)$$

Note the difference of equation (4) from $\oint_S E \cdot dS = \frac{Q}{\varepsilon_0}$.

While the latter shows that the electric field arises from the charge, the former shows that there is no magnetic analogue of the electric charge. In other words, you can not isolate a single magnetic pole as the flow source. No matter how small you break up a magnetic material, you still have a north and a south pole. Further, equation (4) states that, the total magnetic flux through any closed surface (S) is zero (for static fields) which is the law of conservation of magnetic flux.

And for the curl equation (3), the integral form is:

$$\int_S \nabla \times B \cdot dS = \mu_0 \int_S J \cdot dS \quad \text{as J} \cdot \text{dS} = \text{I, and using stokes theorem,}$$
$$\oint_C B \cdot dl = \mu_0 I \qquad (5)$$

Equation (5) allows you to find B due to a current I where there is a closed path C around I in which B is constant.

3.1.1 <u>TYPES OF MAGNETIC MATERIALS</u>

Magnetic materials can basically be grouped into three categories which are:

- Diamagnetic material: It is a type of material for which *the relatively permeability is less than 1 by a very small amount*. In other words, *the magnetic susceptibility is a very small negative number* (from equation (17).
- Paramagnetic material: It is a type of material for which the *relatively permeability* is *greater than 1 by a very small amount*. In other words, the *magnetic susceptibility is a very small positive* number (from equation (17).
- Ferromagntic Material: It is a type of material for which *the relatively permeability is far greater than 1*. In other words, *the magnetic susceptibility is a very large positive number* (from equation (17).

3.1.2 <u>THE VECTOR MAGNETIC POTENTIAL</u>

From the vector identity $\nabla \cdot (\nabla \times A) = 0,$ and from $\nabla \cdot B = 0,$ then we can write:

$$B = \nabla \times A$$

i.e. any divergenless or solenoidal field (as in $\nabla \cdot B = 0$), can be expressed as the curl of another vector (say A, to be determined). A as it is defined is called the vector magnetic potential, similar to electric scalar potential, V (for which $E = -\nabla V$).

Now, $B = \nabla \times A$

Also, $\nabla \times B = \mu_0 J$

That is, $\nabla \times B = \nabla \times \nabla \times A = \mu_0 J$

Recall the identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$

Then, $\nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 J$ (6)

It can be verified that, setting $\nabla \cdot A = 0$ will still satisfy the vector identities while simplifying equation (6) above. This is called the Coulomb's condition for divergence of A.

Now, setting $\nabla \cdot A = 0$

Then $-\nabla^2 A = \mu_0 J$

That is $\nabla^2 A = -\mu_0 J$ (7)

Equation (7) is called the vector Poisson's equation.

The scalar electric potential $V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_v}{R} \cdot dV$ is actually the solution for the Poisson's equation $\nabla^2 V = -\frac{\rho_v}{\varepsilon_0}$. Which is similar to equation (7). Hence equation (7) has a similar solution given by:

$$A = \frac{\mu_0}{4\pi} \int_V \frac{J}{R} dV \tag{8}$$

Now, noting that $J \cdot dV = JdSdl$ and $JdS = I$

The $JdV = Idl$ and Hence:

$$A = \frac{\mu_0 I}{4\pi} \oint_C \frac{dl}{R} \tag{9}$$

Now we have said, $B = \nabla \times A$

$$B = \nabla \times \left[\frac{\mu_0 I}{4\pi} \oint_C \frac{dl}{R} \right]$$

$$= \frac{\mu_0 I}{4\pi} \oint_C \nabla \times \left(\frac{dl}{R} \right)$$, which simplifies to:

$$B = \frac{\mu_0 I}{4\pi} \oint_C \frac{dl \times a_R}{R^2} \tag{10}$$

Equation (10) is called the Bito-Savart law and it is a formula for determining the magnetic flux density B due a current I in a closed path.

Now, writing $B = \oint_C dB$,

Then $dB = \frac{\mu_0 I}{4\pi}\left(\frac{dl \times a_R}{R^2}\right)$ or

$$dB = \frac{\mu_0 I}{4\pi} \cdot \frac{dl \sin\theta}{R^2} \qquad (11)$$

a_R is a unit vector in direction of R and θ is the angle between *dl* and a_R

3.1.3 <u>MAGNETIC FIELD INTENSITY, H</u>

Consider the atomic model which shows electrons orbiting round the nucleus of the atom. As the electrons orbit round the nucleus, they also spin on their axes. The orbiting electrons constitute *circulating currents* which forms microscopic *magnetic dipoles*. The spinning electrons do so with a certain *magnetic dipole moments*. In the absence of an external magnetic field, the magnetic dipoles are in random directions. However, once there is an external magnetic field, then the magnetic dipoles are aligned and the sum of the magnetic dipole moment per unit volume, called the *magnetizing vector M*, will give rise to a *magnetization volume current density* J_{mv}.

It has been proven analytically that, the magnetizing vector *M* is related to the magnetization volume current density J_{mv} by:

$$J_{mv} = \nabla \times M$$

Thus, for a magnetic material in a external magnetic field, equation (3) has to be modified to:

$$\nabla \times B = \mu_0 (J + J_{mv})$$

$$= \mu_0 (J + \nabla \times M)$$

$$\frac{1}{\mu_0}\nabla \times B - \nabla \times M = J$$

or

$$\nabla \times \left(\frac{B}{\mu_0} - M\right) = J \qquad (12)$$

Thus, another fundamental field quantity, the *magnetic field intensity H*, is defined as:

$$H = \frac{B}{\mu_0} - M \qquad (13)$$

Then from (12), $\nabla \times H = J$ (14)

In integral form, $\int_S (\nabla \times H) \cdot dS = \int_S J \cdot dS$ or using Stokes Theorem,

$$\oint_C H \cdot dl = I \qquad (14)$$

C is the contour bounding the surface, and I is the total current passing through S. Equation (14) is called Ampere's Circuital Law and it states: The circulation of the magnetic field intensity around any closed path is equal to the free current flowing through the surface bounded by the path.

In a linear, isotropic medium, the *magnetizing vector (M)* is related to the *magnetic field intensity (H)* by:

$$M = \chi_m H \qquad (15)$$

Where χ_m is the magnetic susceptibility.

Recall equation (13), $H = \frac{B}{\mu_0} - M$, then becomes:

$$H = \frac{B}{\mu_0} - \chi_m H \text{ or}$$

$$B = \mu_0 H + \mu_0 \chi_m H \text{ which simplifies to:}$$

$$B = \mu_0 (1 + \chi_m) H \qquad (16)$$

We define another quantity μ_r called the relative permeability of the material given by:

$$\mu_r = 1 + \chi_m \qquad (17)$$

Thus (16) becomes:

$$B = \mu_0 \mu_r H \text{ , which is written as:}$$

$$B = \mu H \qquad (18)$$

Where μ is the absolute permeability of the material given by:

$$\mu = \mu_0 \mu_r \qquad (19)$$

SESSION TWO

3.2.1 BOUNDARY CONDITIONS FOR MAGNETOSTATIC FIELD.

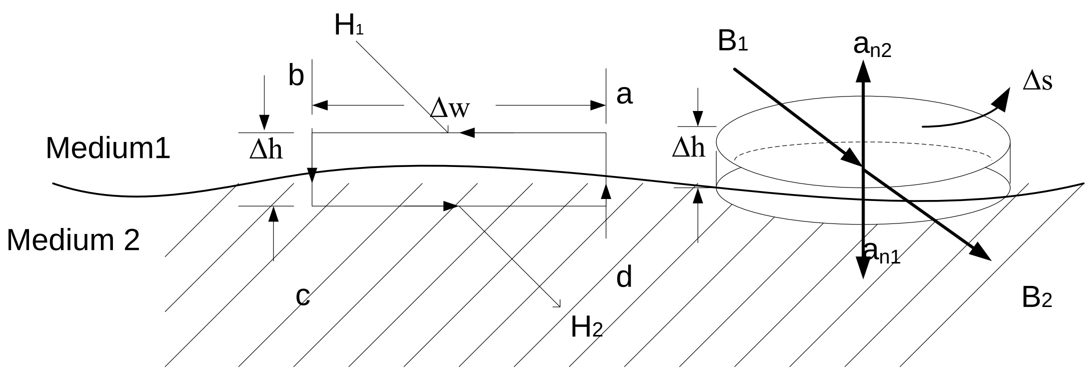


Fig.16Magnetostatic Fields at the boundary of dielectric media

$$bc = ad = \Delta h \to 0$$
$$ba = cd = \Delta w$$

Recall, in electrostatics, applying Gauss law ($\oint_S D.dS = \int_S \rho_s \cdot dS$) gave the boundary condition $D_{1n} - D_{2n} = \rho s$. Similarly, applying Gauss law of magnetostatics($\int_S B \cdot dS = 0$) gives the boundary condition $B_{1n} - B_{2n} = 0$ (20)

That is, the normal component of B is continuous across the boundary between two different media.

$$B_1 = \mu_1 H_1 \qquad \text{and} \qquad B_2 = \mu_2 H_2$$

$$\therefore \mu_1 H_{1n} = \mu_2 H_{2n} \tag{21}$$

Again, in electrostatics, $\oint_C E.dl = 0$ gives the boundary condition: $E_{1t} = E_{2t}$. However, in Magnetostatics, $\oint_C H \cdot dl = \int_S J_S \cdot dS$

Now, as *Δh* approaches 0, the surface of the loop approaches a thin line of length *Δl*. Thus: $H_{1t}\Delta l + H_{2t}(-\Delta l) = J_S \Delta l$ or

$$H_{1t} - H_{2t} = J_S \qquad (22)$$

That means, the tangential component of the magnetic field intensity is discontinuous across the boundary between two media; and the amount of discontinuity is equal to the surface current density. However, surface current can only exist on the surface of perfect conductors and superconductors. Hence at the interface between two media with finite conductivities, *Js* = 0.

$$H_{1t} - H_{2t} = 0 \text{ or } H_{1t} = H_{2t} \qquad (23)$$

3.2.2 <u>INDUCTANCES AND INDUCTORS</u>

Consider two coils C_1 with surface area S_1 and C_2 with surface area S_2 as shown in the figure below:

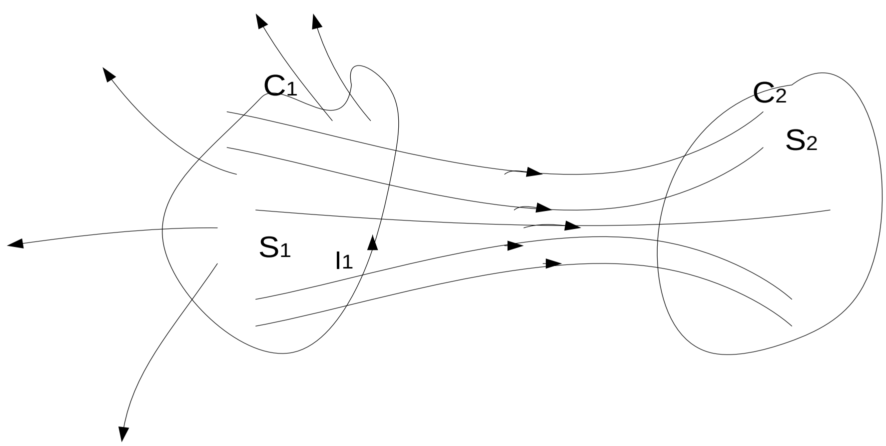


Fig.17Magnetic Flux Linkage in Coils

C_1 is carrying a current I_1 which will create a magnetic flux characterized by magnetic flux density B_1. The flux created by C_1 will then link C_2. Thus the mutual flux from C_1 to C_2, Φ_{12} is given by:

$$\Phi_{12} = \int_{S_2} B_1 \cdot dS \qquad (24)$$

From Biot-Savart law, $B = \frac{\mu_0 I}{4\pi} \oint_C \frac{dl \times a_R}{R^2}$, the flux density B is directly proportional to the current I. Thus $\Phi_{12} \propto I_1$, and the constant of proportionality is L_{12}, called the mutual inductance between the coils C_1 and C_2. Thus $\Phi_{12} = L_{12} I_1$

The coils may be made up of several turns in which case the flux linkage Λ_{12} is given by:

$\Lambda_{12} = N\Phi_{12}$

The mutual inductance in that case is given by:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} B_1 \cdot dS_2$$

Some of the magnetic flux produced by I_1 only links with C_1 itself and the total flux linkage with C_1 is $\Lambda_{11} = N\Phi_{11}$

The self inductance of C_1, L_{11} is then defined as the magnetic flux linkage per unit current in the loop itself; that is:

$$L_{11} = \frac{\Lambda_{11}}{I_1} = \frac{N_1}{I_1} \int_{S_1} B_1 \cdot dS_1$$

Determining the self-inductance of an inductor

- Choose an appropriate coordinate system for the given geometry
- Assume a current I in the conducting wire.
- Find B from I by Ampere's Circuital law or Biot-Savart law
- Find the flux Φ from B by integration: $\Phi = \int_S B \cdot dS$
- Find the flux linkage $\Lambda = N\Phi$
- Find L by taking the ratio $L = \Lambda / I$

3.2.3 MAGNETIC ENERGY

If a current I flows through a single inductor with inductance L, then the stored magnetic energy W_m is given by:

$$W_m = \frac{1}{2} L I^2$$

In terms of field quantities:

$$W_m = \frac{1}{2}\int_v H \cdot B dv$$

$$W_m = \frac{1}{2}\int_v \frac{B^2}{\mu} dv$$

SOLVED QUESTIONS

3.1) Obtain the magnetic field intensity, *H* due to an infinitely long straight filament of current *I* using Amperes law.

$$H_\varphi = \frac{I}{2\pi r}\ a_\varphi$$

Solution

Applying Ampere's law integrating over a circular contour of radius r,

Ampere's law

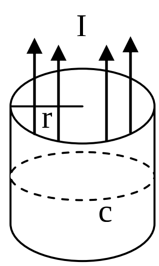


Fig.18Illustration of Question 3.1

$$\oint_c H \cdot dl = I$$

$$H_\varphi \cdot 2\pi r = I$$

$$H_\varphi = \frac{I}{2\pi r}$$

3.2) The magnitude of H at a radius of 1 m from a long linear conductor is 1 A/m. Calculate the current in the wire.

$$I = 2\pi$$

Solution

$$H_\varphi \cdot 2\pi r = I$$
$$I = 1 \; x \; 2\pi \; x \; 1$$
$$I = 2\pi$$

3.3) Determine H for a solid cylindrical conductor of radius a where the current I is uniformly distributed over the cross-section.

Solution

$$for \; r < a$$

$$\oint_c H \cdot dl = I$$

$$H_\varphi \cdot 2\pi r = I \cdot \frac{\pi r^2}{\pi a^2}$$

$$H_\varphi = \frac{Ir}{2\pi a^2}$$

$$H = \frac{Ir}{2\pi a^2} a_\varphi$$

$$for \; r > a$$

$$\oint_c H \cdot dl = I$$

$$H_\varphi \cdot 2\pi r = I$$

$$H = \frac{I}{2\pi r} a_\varphi$$

3.4) A thin linear conductor of length l carrying a current I is coincident with the y-axis. The medium surrounding the conductor is air. One end of the conductor is at a distance y_1 and the other at a distance y_2 as shown in fig. below. Show that the flux density due to the conductor at a point on the x-axis at a distance X_1 from the origin is

$$B = \frac{\mu_o I}{4\pi x_1}\left[\frac{y_2}{\sqrt{x_1^{\;2} + y_2^{\;2}}} - \frac{y_1}{\sqrt{x_1^{\;2} + y_1^{\;2}}}\right]$$

Solution

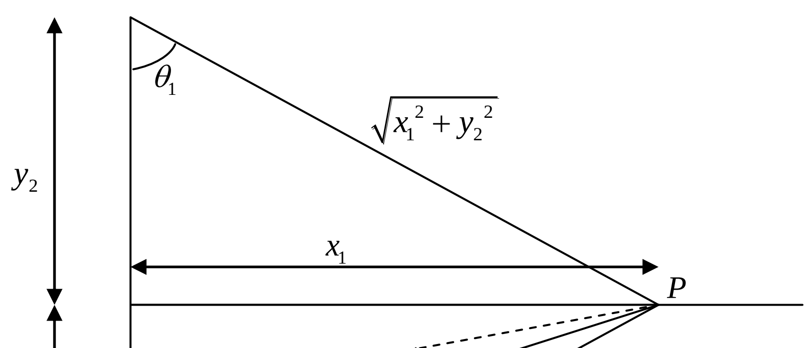


Fig.19Illustration of Question 3.4

Let P be the point where ***B*** is to be determined. Consider an element dl at a distance x from P. Then by Biot-Savart Law, the flux density B at P due to element is given by

$$\sin\theta = \frac{DE}{dl}$$

$$\Rightarrow DE = dl\sin\theta$$

$$\frac{DE}{x} = d\theta \quad \text{or} \quad DE = xd\theta$$

$$\therefore dl\sin\theta = xd\theta$$

$$x_1 = x\sin\theta$$

$$\therefore \frac{1}{x} = \frac{\sin\theta}{x_1}$$

from Biot-Savart law,

$$dB = \frac{\mu_0 I}{4\pi} \cdot \frac{dl\sin\theta}{x^2}$$

$$dB = \frac{\mu_0 I}{4\pi} \cdot \frac{xd\theta}{x^2}$$

$$dB = \frac{\mu_0 I}{4\pi} \cdot \frac{d\theta}{x}$$

$$dB = \frac{\mu_0 I}{4\pi} \frac{\sin\theta d\theta}{x_1}$$

$$B = \frac{\mu_0 I}{4\pi x_1} \int_{\theta_1}^{\theta_2} \sin\theta d\theta$$

$$B = \frac{\mu_0 I}{4\pi x_1} \left[-\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$B = \frac{\mu_0 I}{4\pi x_1} \left[\cos\theta_1 - \cos\theta_2\right]$$

$$\cos\theta_1 = \frac{y_2}{\sqrt{x_1^2 + y_2^2}} \text{ and } \cos\theta_2 = \frac{y_1}{\sqrt{x_1^2 + y_1^2}}$$

$$\therefore B = \frac{\mu_0 I}{4\pi x_1} \left[\frac{y_2}{\sqrt{x_1^2 + y_2^2}} - \frac{y_1}{\sqrt{x_1^2 + y_1^2}} \right]$$

ASSIGNMENTS

1) Using the cylindrical coordinate, calculate the magnetic field in a coaxial line of inner conductor of radius R_1 and outer conductor of inner radius R_2 and thickness, t for the regions

   $(a)\ r \le R_1\ (b)\ R_1 \le r \le R_2\ (c)\ R_2 \le r \le R_2 + t$

   $Ans:$ For the region $r < R_{2,}$

   $$(a)\ B_\varphi = \frac{\mu_o I r}{2\pi R_1^2} \qquad r \le R_{1,}$$

   $$(b)\ B_\varphi = \frac{\mu_o I}{2\pi r} \qquad \mathrm{R}_1 \le r \le R_2$$

   (c) For $R_2 \le r \le R_2 + t$

   $$B_\varphi = \frac{\mu_o I}{2\pi r}\left[1 - \frac{r - R_2^{\ 2}}{\left(R_2 + t\right)^2 - R_2^{\ 2}}\right]$$

   (d) and for $r \ge R_2 + t,\ \ B_\varphi = 0$

2) A current loop is formed by joining the ends of the two parallel long wires of radius r separated by a distance d between their axes. Neglecting the ends effects

and the magnetic flux within the wires, show that the self-inductance of length l of the parallel wire is given by

$$L = \frac{\mu_o l}{\pi} \log_e \frac{d-r}{r}$$

3)

i. If the electric field strength of a plane wave is 1 V/m, what is the strength of a magnetic field H in free space?

ii. If the magnitude of H in a plane wave is 1 A/m, what is the magnitude of E for a plane wave in free space?

Ans: (a) 2.6 mA/m
(b) 376.8 V/m

4) (a) Determine the expression for the magnetic field intensity H for a solid cylindrical conductor of radius a, where the current I is uniformly distributed over the cross-section, for the regions $r < a$ and $r > a$.

Find thereby conversely J (the current density) from H.

Ans

$$H = \frac{I \cdot r}{2\pi a^2}\ a_\varphi \quad r < a$$

$$H = \frac{I}{2\pi r}\ a_\varphi \quad r > a$$

$$J = \frac{I}{2\pi r} \quad r < a$$

$$J = 0 \qquad r > a$$

(b) Show that the equation $J = \sigma E$ is Ohm's law in point form. (J is current density, σ is conductivity of the material and E is the electric field intensity).

UNIT FOUR
MAXWELL'S EQUATIONS AND WAVE PROPAGATION

So far we have only dealt with fields that do not change with time. We observe that **E** and **D** in electrostatic model do not relate to **B** and **H** in magnetostatic model. In a conducting medium, static electric and static magnetic fields may both exist to form electromagnetostatics. Static fields do not give rise to wave that propagate and carry energy and information. Static models are also not adequate for explaining time-varying electromagnetic phenomena. In this chapter we see that a changing magnetic field induces an electric field, and vice versa. Under time varying conditions it is necessary to construct an electromagnetic model to properly represent and relate the electric field vectors **E** and **D** and magnetic field vectors **B** and **H**.

Objectives:

By the end of this chapter, you should be able to:

- Know the four Maxwell's Equations and their relations to other laws
- Solve for boundary conditions at a dielectric interface and at the interface of perfect conductors
- Work with time harmonic electromagnetics
- Derive expressions for the velocity of propagation of electromagnetic wave
- Know about plane waves in lossless media

SESSION ONE

4.1 BASIS FOR MAXWELL'S EQUATIONS AND BOUNDARY CONDITIONS

The work of James Clerk Maxwell, published in 1873, called Maxwell's equations, describes electric and magnetic phenomena at the macroscopic level. From purely theoretical point of view, Maxwell hypothesized the existence of the electric displacement current, which led to the discovery of electromagnetic wave propagation by Marconi and Hertz. Maxwell's work was based on the theoretical and empirical knowledge developed by Faraday, Ampere, Gauss and others. The basis for Maxwell's hypothesis are discussed in this unit.

Recall:

Electric field

$E \propto D$

$D = \varepsilon E$

$\nabla \times E = 0$

$\nabla \bullet D = \rho v$

Magnetic field

$B \propto H$

$H = \frac{1}{\mu} B$

$\nabla \bullet B = 0$

$\nabla \times H = J$

Now E and D are not related to B and H for static field. However, under time varying conditions, the electric field is always coupled with the magnetic field. Thus a model is required for which the electric field vectors (E and D) will be properly related to the magnetic field vectors B and H. i.e. the electromagnetic model.

4.1.1 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Faraday discovered that a current was induced in a conducting loop when the magnetic flux linking the loop changed. To investigate the relationship between the induced emf and the rate of change of flux linkage (which is Faraday's Law) we state yet another postulate for time varying electromagnetic field.

The fundamental postulate for electromagnetic induction is

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\int_S \nabla \times E.ds = -\int_S \frac{\partial B}{\partial t}.ds$$ (time varying magnetic field gives rise to an electric field.)

or

$$\oint_C E.dl = -\int_S \frac{\partial B}{\partial t}.ds$$

For a stationery circuit, with contour C, and surface, S $\oint_C E.dl$ is the voltage (emf) induced in the circuit and

$$\phi = \int_s B.ds$$

$$\therefore \oint_C E.dl = -\int_S \frac{\partial B}{\partial t}.ds$$

$$V = -\frac{d}{dt}\int_S B.ds = -\frac{d\phi}{dt}$$

Faradays law of electromagnetic induction: Faraday's law thus states *the electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking circuit.*

The negative sign is a result of Lenz's law "The induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change it in the linking magnetic flux. The induced emf is called transformer emf.

4.1.2 MAXWELL'S EQUATIONS

For time varying field the curl equation of H has to be modified.

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times H = J$$

$$\nabla \bullet D = \rho v$$

$$\nabla \bullet B = 0$$

From the principle of conservation of charge, eqn of continuity

$$\nabla \bullet J = -\frac{\partial \rho v}{\partial t}$$

Observe that

$$\nabla \bullet \nabla \times H = \nabla \bullet J$$

$$\nabla \bullet J = 0$$

But ie inconsistent

$$\nabla \bullet J = -\frac{\partial \rho v}{\partial t}$$

Thus (2) also needs to be modified so that

$$\nabla \bullet \nabla \times H = 0 = \nabla \bullet J + \frac{\partial \rho v}{\partial t}$$

$$\nabla \bullet J = -\frac{\partial \rho v}{\partial t}$$

thus

$$\nabla \bullet \nabla \times H = \nabla \bullet J + \frac{\partial \rho v}{\partial t}$$

$$\nabla \bullet \nabla \times H = \nabla \bullet J + \frac{\partial \rho v}{\partial t}$$

$$\nabla \bullet (\nabla \times H) = \nabla \bullet (J + \frac{\partial}{\partial t} D)$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

The term $\frac{\partial D}{\partial t}$ is the displacement current density

Thus, the Maxwell's equations are:

Differential Form

$\nabla \times E = -\frac{\partial B}{\partial t}$	Faraday's Law
$\nabla \times H = J + \frac{\partial B}{\partial t}$	Ampere's Circuital Law (corrected)
$\nabla \bullet D = \rho v$	Gauss Law
$\nabla \bullet B = 0$	No isolated Magnetic change

Integral Form

$\oint_C E.dl = -\int_S \frac{\partial B}{\partial t}.ds$	Faraday's Law
$\oint_C H.dl = \int_S (J + \frac{\partial D}{\partial t}).ds$	Ampere's Circuital Law (corrected)
$\int_S D.ds = \int_V \rho v dv$	Gauss Law
$\int_S B.ds = 0$	No isolated Magnetic change

Since electric current is the flow of charges it can be said that, the electric charge density ρv, is the ultimate source of the electric field.

4.1.3 BOUNDARY CONDITIONS

From $\oint_s D.ds = \oint_v \ell_v dv$ in the limit as Δh approaches zero

$$\Delta SD_{2n} - \Delta SD_{1n} = \Delta S\ell_s .$$

or $D_{2n} - D_{1n} = \ell_s$ in vector form.

$$\hat{n}(\bar{D}_2 - \bar{D}_1) = \ell_s$$

Also $\oint_s B.ds = 0$ ensures that

$$\Delta SB_{2n} - \Delta SB_{1n} = 0 .$$

or $B_{2n} = B_{1n}$

in vector form

$$\hat{n}.B_{2n} = \hat{n} B_{1n}$$

$$\oint_c E.dl = \frac{-d}{dt}\int_s B.ds$$

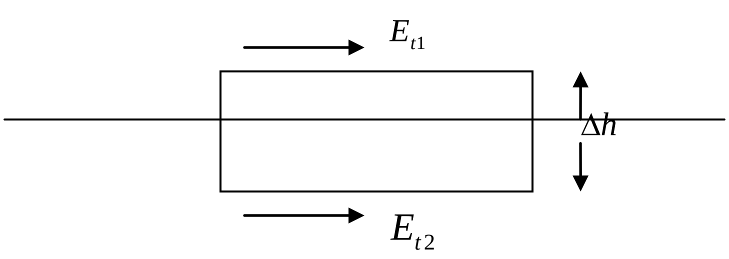


Fig.20Tangential Electric field at the boundary of two media

In the limit as $\Delta h \to 0$. Surface integral of B vanishes and

$$\Delta l E_{t1} - \Delta l E_{t2} = 0$$

or $E_{t1} = E_{t2}$.

$$\oint H.dl = \int_s \left(J + \frac{\partial D}{\partial t} \right).ds$$

In the limit as $h \to 0$ surface integral of D vanishes.

and $ds \to dl$

$$\therefore \Delta l H_{2t} - \Delta l H_{1t} = \Delta l J$$

or $H_{2t} - H_{1t} = J$

and $\hat{n}.(H_2 - H_1) = J_s$

Field at a Dielectric Surface

At an interface b/n 2 lossless dielectric materials there are usually no free charges and no surface current at the interface. $\ell_s = 0$, $J_s = 0$

Then

$$E_{1t} = E_{t2} \to \frac{D_{1t}}{D_{2t}} = \frac{E_1}{E_2}$$

$$H_{1t} = H_{t2} \to \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}.$$

$$D_{1n} = D_{2n} \to E_1 \mathrm{E}_{1n} = E_2 \mathrm{E}_{2n}$$

$$B_{1n} = B_{2n} \to \mu_1 H_{1n} = \mu_2 H_{2n}$$

Field at the interface of a perfect Conductor

In the interior of the conductor, E=0

$$\oint_c E.dl = -\int \frac{dB}{dt}.ds \Rightarrow B = 0$$

$$D = \varepsilon \mathrm{E} \Rightarrow D = 0$$

$$H = \frac{B}{\mu} \Rightarrow B = 0$$

Conductor is medium 2.

$$E_2 = B_2 = H_2 = D_2 = 0$$

Then

$$E_{1t} = 0 \quad , \; E_{2t} = 0$$

$$\hat{n} \times H_1 = J_s \; H_{2t} = 0$$

$$D_{1n} = \ell_s \quad D_{2n} = 0$$

$$B_{1n} = 0 \; B_{2n} = 0$$

SESSION TWO

4.2.1 <u>**TIME HARMONIC ELECTROMAGNETICS**</u>

The field vectors do vary with space coordinate and are sinusoidal functions of time.

Thus the electric field intensity E can be represented as

$$E(x, y, z, t) = \Re e[E(x, y, z)e^{jwt}]$$

Similarly, $H(x, y, z, t) = \Re e[H(x, y, z)e^{jwt}]$

Thus $\frac{\partial E}{\partial t} = jwE \qquad \int E dt = \frac{E}{jw}$

And $\frac{\partial H}{\partial t} = jwH$ $\int Hdt = \frac{H}{jw}$

Maxwell's equations can be written then as

$$\nabla \times E = -jw\mu H \qquad (B = \mu H)$$

$$\nabla \times H = J + jw\mu\varepsilon \mathrm{E} \qquad (D = \varepsilon \mathrm{E})$$

The wave equation and basic wave solution

$$\nabla \times E = -jw\mu H \text{ (in some free field, J=0, } \ell_s = 0\text{)}$$

$$\nabla \times H = jw\mu\varepsilon \mathrm{E}$$

$$\nabla \times \nabla \times E = -jw\mu(\nabla \times H)$$

$$= -jw\mu . jw\varepsilon E$$

$$\nabla \times \nabla \times E = w^2 \mu\varepsilon E$$

In source free field

$$\ell_s = 0, \nabla \bullet D = \ell_s, \nabla \bullet E = \frac{\ell_s}{\varepsilon}, \nabla \bullet E = 0,$$

$$\Rightarrow \nabla(\nabla \bullet E) - \nabla^2 E = w^2 \mu\varepsilon E \quad .$$

$$\therefore -\nabla^2 E = w^2 \mu\varepsilon E \quad \text{or} \quad \nabla^2 E + w^2 \mu\varepsilon E = 0$$

This is called the wave equation. Similarly

$$\nabla^2 H + w^2 \mu\varepsilon H = 0$$

$w^2 \mu\varepsilon$ is at constant.

Let $K^2 = w^2 \mu\varepsilon$

so that

$$K = w\sqrt{\mu\varepsilon}$$

then $\nabla^2 E + k^2 E = 0$, K is called the wave number.

4.2.2 **PLANE WAVES IN LOSSLESS MEDIA**

In lossless media μ and ε are real, so k is real. In Cartesian coordinate, the wave equation can be written as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)E_x=0$$ for the x-component.

If the wave is uniform along the x and y direction

Then $\frac{\partial^2 Ex}{\partial x^2}=\frac{\partial^{2Ex}}{\partial y^2}Ex=0$ so that (wave traveling in the z direction)

$$\frac{\partial^2 Ex}{\partial x^2}+k^2Ex=0$$ ODE.

Solution

$$Ex(z)=E^+x(z)+E^-x(z)$$

$$=E^+e^{-jkz}+E^-e^{-jkz}$$

or $Ex(z)=E^+\cos(wt-kz)+E^-\cos(wt+kz)$

i.e. wave traveling in the z-direction.

Set $wt-kz=$ constant (say A)

$$wt-kz=A$$

or $kz=wt-A$

$$z=\frac{wt-A}{k}$$

$$V_p=\frac{dz}{dt}=\frac{w}{k}=\frac{w}{w\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu\varepsilon}}$$

In free space

$$V_p=\frac{1}{\sqrt{\mu_o\varepsilon_o}}\approx 3\times10^8 m/s=C$$

That is the wave travels with the speed of light. Wavelength, λ is defined so the distance between two successive maxima (minima) or any reference point.

Thus $[(wt-kz)]-[wt-k(z+\lambda)]=2\pi$

Or $\quad k\lambda=2\pi$ or $V_p=\frac{w}{k}$ or $k=\frac{w}{V_p}$

$$\lambda=\frac{2\pi V_p}{w}=\frac{2\pi V_p}{2\pi f}=\frac{V_p}{f}$$

For H,

$$\nabla\times E=-jw\mu H$$

$$\Rightarrow\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E^+x(z) & 0 & 0 \end{vmatrix}=-jw\mu(a_x H^+{}_x+a_y H^+{}_y+a_z H^+{}_z)$$

Which leads

$$H^+{}_x=0 \quad H^+{}_y=\frac{1}{-jw\mu}\frac{\partial H^+{}_x(z)}{\partial z} \qquad H^+{}_z=0$$

$$\text{Ie} \quad H^+{}_y=\frac{1}{-jw\mu}\frac{\partial\left(E^+{}_o e^{-jkz}\right)}{\partial z}=\frac{-jkE^+{}_x(z)}{-jw\mu}$$

$$H=a_y H^+{}_y(z)=a_y\frac{k}{w\mu}E^+{}_x(z)$$

$$H=a_y\frac{1}{\eta}E^+{}_x(z) \qquad \eta=\sqrt{\frac{\mu}{\varepsilon}}$$

in free space

$$\eta=\sqrt{\frac{\mu_o}{\varepsilon_o}}\approx 377\Omega$$

which is the wave impedance.

SOLVED EXAMPLES

4.1) Using cylindrical coordinates, calculate the field from an infinite wire of finite radius R_1 for the regions $r \leq R_1$ and $r \geq R_1$

Solution

Applying Ampere's circuital law and integrating over a circular contour of radius R, we have

$$\oint B_\varphi \cdot dl = \int \mu_o J \cdot ds$$

$$\int_0^{2\pi} B_\varphi \cdot dl = \mu_o \int_0^r \int_0^{2\pi} J \cdot r d\varphi dr$$

$$\int_0^{2\pi} B_\varphi \cdot r d\varphi = \mu_o J \int_0^{2\pi} d\varphi \left[\frac{r^2}{2}\right]_0^r$$

$$B_\varphi r [\varphi]_0^{2\pi} = \mu_o J [\varphi]_0^{2\pi} \cdot \frac{r^2}{2}$$

$$B_\varphi = \mu_o J \cdot \frac{r}{2}$$

But $J = \dfrac{I}{\pi R_1^2}$

$$B_\varphi = \frac{\mu_o I r^2}{2\pi R_1^2} \; Wb/m^2 \quad \text{for} \quad r \leq R_1$$

For $r \geq R_1$, Total current enclosed is I

$$\int_0^{2\pi} B_\varphi \cdot r d\varphi = \mu_o I$$

$$B_\varphi r [\varphi]_0^{2\pi} = \mu_o I$$

$$B_\varphi \cdot 2\pi r = \mu_o I$$

$$B_\varphi = \frac{\mu_o I}{2\pi r} \qquad Wb/m^2$$

4.2) An electric vector E of an electromagnetic wave in free space is given by the expressions $E_x = E_z = 0$, $E_y = Ae^{jw\left(t - \frac{z}{v}\right)}$.

Using Maxwell's equation for free space condition, determine the expression for the components of the magnetic vector H

Solution

$E_x = E_z = 0$

$$E_y = Ae^{jw\left(t-\frac{z}{v}\right)}$$

$$\nabla \times \mathrm{E} = -\mu_0 \frac{\partial \mathrm{H}}{\partial t} = -\mu_0 \frac{\partial}{\partial t}\left[a_x H_x + a_y H_y + a_z H_z\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

$$\nabla \times \mathrm{E} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -a_x \frac{\partial E_y}{\partial z} + a_z \frac{\partial E_y}{\partial x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .2$$

Comparing equations 1 and 2

$H_y = 0$

$$\frac{\partial E_y}{\partial z} = \mu_o \frac{\partial H_x}{\partial t}$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_o}\frac{\partial}{\partial z}\left[Ae^{jw\left(t-\frac{z}{v}\right)}\right] = \frac{A}{\mu_o} e^{jw\left(t-\frac{z}{v}\right)} . \left(\frac{-jw}{v}\right)$$

$$\frac{\partial H_x}{\partial t} = \frac{-Ajw}{\mu_o v} e^{jw\left(t-\frac{z}{v}\right)}$$

$$H_x = \frac{-Ajw}{\mu_o v}\int e^{jw\left(t-\frac{z}{v}\right)} \cdot dt + 0$$

$$H_x = \frac{-A}{\mu_0\sqrt{\frac{1}{\mu_o \varepsilon_o}}} \cdot e^{jw\left(t-\frac{z}{v}\right)} = -\sqrt{\frac{\varepsilon_o}{\mu_o}} \cdot Ae^{jw\left(t-\frac{z}{v}\right)}$$

$$\frac{\partial E_y}{\partial x} = -\mu_o \frac{\partial H_z}{\partial t}$$

$$\frac{\partial E_y}{\partial x} = 0 \therefore H_z = 0$$

4.3) Using Maxwell's equations, discuss the boundary conditions that exist at the interface between two lossless media. Make mention of the variation or otherwise of the tangential and normal components of the vector *E*, *D*, *B* and *H*

Solution

At the interface between two lossless media dielectric material, there are usually no free charges and no surface current at the interface.

$\rho_s = 0, J_s = 0$

Then $E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2}$

$H_{1t} = H_{2t}$

$D_{1t} = D_{2t}$

$B_{1t} = B_{2t}$

ASSIGNMENT

1) Starting from Maxwell's wave equation,

   $$\nabla^2 E = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2},$$

   Deduce that the electromagnetic wave in transverse in nature

2) The electric field intensity of an electromagnetic wave on free space is given by

$E_x = 0, \;\; E_z = 0, \;\; E_y = E_o \cos \omega \left(t - \frac{Z}{V} \right)$

Determine the expression for the components the magnetic field intensity, *H* using the Maxwell's equation.

3) In free space, $E = E_o \sin\left(\omega t - \beta Z \right) \; a_y . a_y$ is a unit vector in the direction of y-axis. Calculate *D*, *B* and *H*. (The constant of integration may be taken as zero for these fields).

4) Using Maxwell's equation, discuss the boundary conditions that exist at the interface between a dielectric and a perfect conductor. Make mention of the variation or otherwise of the tangential and normal components of the vector *E, D, B* and *H*