ME 355
STRENGTH OF MATERIALS II

UNIT 1

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Introduction

DEFLECTION OF BEAMS

LEARNING OBJECTIVES

- This unit discusses the various methods used to determine the deflection of beams, which include the strain energy method, the method of calculus and the Macaulay's method

After reading this unit you should be able to:

- Define deflection of a beam
- Derive the various equations used to determine the deflection of beams
- Select the appropriate boundary conditions for each type of situation
- Determine the deflection of beam using the three methods

Differential Equation of the Deflection Curve

Curvature of the Bending Beam

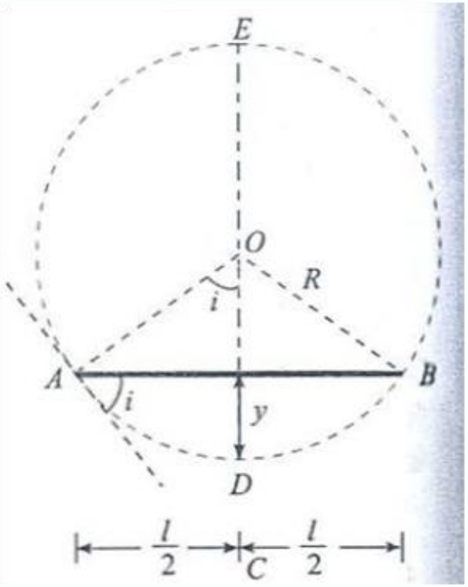


Fig. 1: : Curvature of the beam

Let

- *l* Length of the beam *AB*
- *M* Bending moment
- *R* Radius of curvature of the bent up beam
- *I* Moment of inertia of the beam section
- *E* Modulus of elasticity of beam material
- *y* Deflection of the beam *(i.e., CD)* and
- *i* Slope of the beam

- From the geometry of a circle, and neglecting y^2

$$AC\ X\ CB = EC\ X\ CD \Rightarrow {}^{l}\!/_{2}\, x\, {}^{l}\!/_{2} = (2R - y)y$$

- Therefore

$$y = {}^{l^2}\!/_{8R} \quad ....(\text{i})$$

- From ME 255, for a loaded beam,

$$\frac{M}{I} = \frac{E}{R} \Rightarrow R = \frac{EI}{M}$$

- Now substituting this value of R in equation *(i)*,

$$y == \frac{Ml^2}{8EI}$$

- From the geometry of the figure, the slope of the beam i at *A* or *B* and angle *i* being small

$$i = \frac{AC}{OA} = \frac{l}{2R} \ldots (ii)$$

- Again substituting the value of *R* in equation (ii),

$$i == \frac{Ml}{2EI} \quad radians \cdots (iii)$$

Relationship between Slope, Deflection and Radius of Curvature

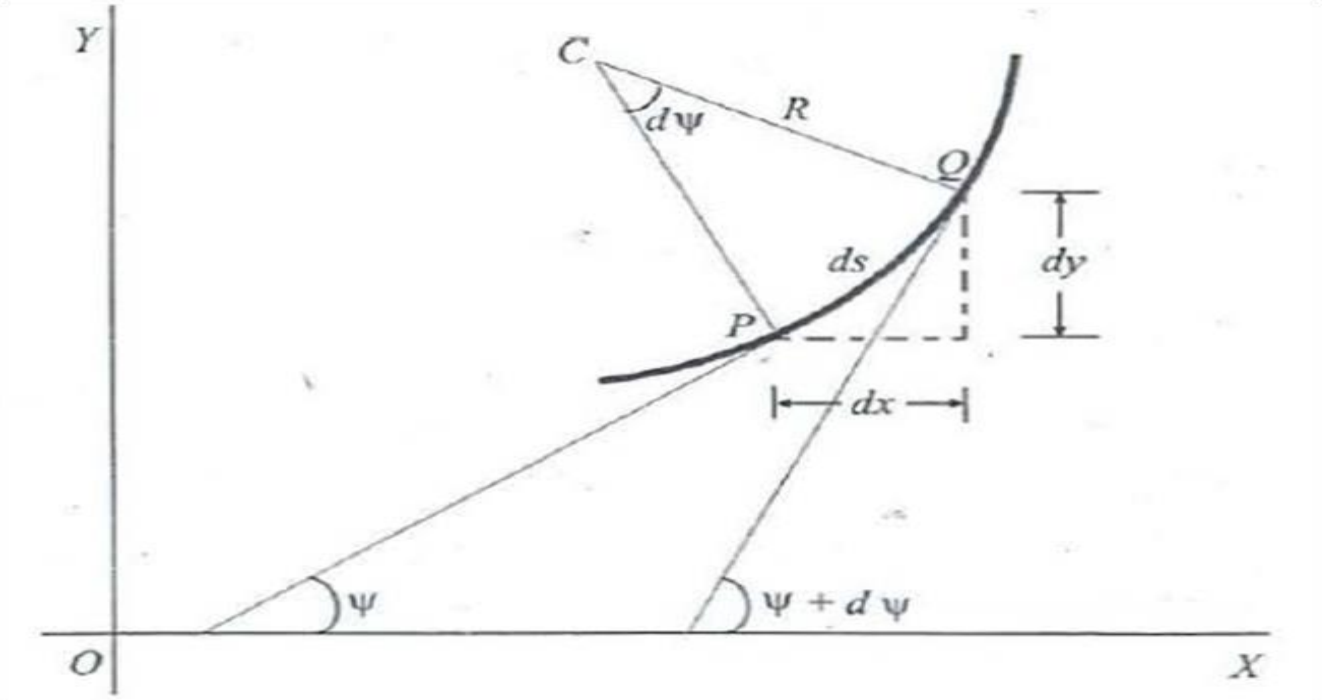


Fig. 2, Beam bent into an arc

Let

- *ds* Length of the beam *PQ*
- *R* Radius of the arc, into which the beam bas been bent
- C Centre of the arc
- Ψ Angle, which the tangent at *P* makes with x-x axis and
- Ψ+dΨ Angle which the tangent at *Q* makes with x-x axis.

- From the geometry of the figure, we find that

$$\angle PCQ = d\psi \quad and \quad ds = R.d\psi$$

- Considering *ds* = *dx*

$$R = \frac{ds}{d\psi} = \frac{dx}{d\psi} \, or \, \frac{1}{R} = \frac{d\psi}{dx}$$

- At point P and for very small angle

$$\psi = \frac{dy}{dx}$$

- Therefore,

$$\frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

- We also know that $$\frac{M}{I} = \frac{E}{R} \Rightarrow M = \frac{EI}{R}$$
- Therefore, $$M = EI \frac{d^2 y}{dx^2}$$
- **NOTE:** The above equation is also based only on the bending moment. The effect of shear force, being very small as compared to the bending moment, is neglected.

Self Assessment

- What is deflection of beams and state the relation between deflection and bending moment.

Methods for Slope and Deflection at a Section

- Strain Energy Method (Castigliano's Theorem)
- Double Integration Method
- Singularity Function (Macaulay's Method)

Strain Energy Method

- Consider a short length of beam δc ,under the action of a bending moment M.
- The strain energy of the length δx is given by

$$\delta U = \int \left(\sigma^2 / 2E\right) dV$$

- But $dV = dA.\delta x x$ and $\sigma^2 = \left(My / I\right)^2$

$$\Rightarrow \delta U = \left(M^2 \delta x / 2EI^2\right) \int y^2 dA$$

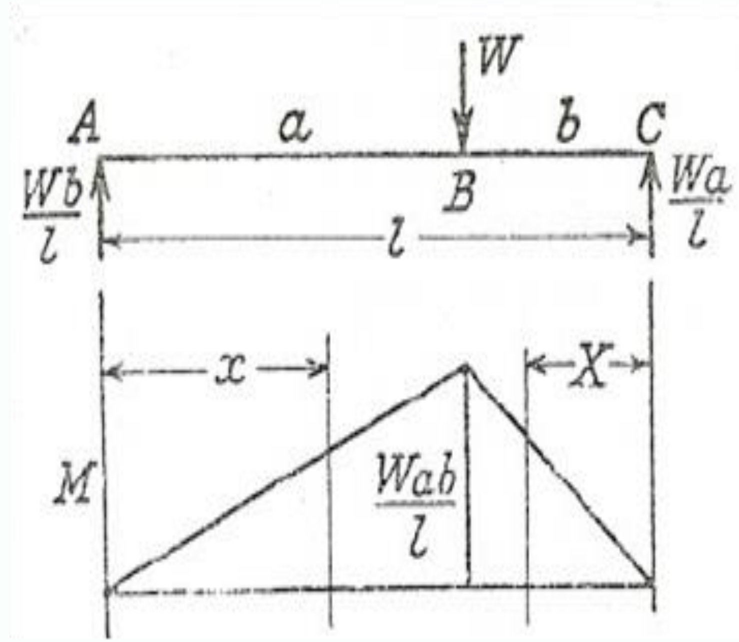
- But $\int y^2 dA = I$

- hence $\delta U = \left(M^2 / 2EI \right) \delta x$

- For the whole beam: $U = \int \left(M^2 / 2EI \right) dx$

- The product ***EI*** is called the ***Flexural Rigidity*** of the beam

- *Example* 1-1: *A simply supported beam of length l carries a concentrated load W at distances of a and b from the two ends. Find expressions for the total strain energy of the beam and the deflection under the load.*



Solution

- The integration for strain energy can only be applied over a length of beam for which a continuous expression for *M* can be obtained.
- This usually implies a separate integration for each section between two concentrated loads or reactions.
- Referring to Figure 3, for the section AB, $M = (Wb/l)x$
- Taking a variable x measured from A

$$U_a = \int \left(M^2 / 2EI \right) dx = \int \frac{W^2 b^2}{2l^2 EI} x^2 dx = \frac{W^2 b^2}{2l^2 EI} \left[\frac{x^3}{3} \right]_0^a = W^2 a^3 b^2 / 6EIl^2$$

- Similarly, by taking a variable *X* measured from C

$$U_b = \int \left(M^2 / 2EI \right) dx = \int \frac{W^2 a^2}{2l^2 EI} X^2 dX = \frac{W^2 a^2}{2l^2 EI} \left[\frac{X^3}{3} \right]_0^b = W^2 a^2 b^3 / 6EIl^2$$

- The total strain energy is

$$U = U_a + U_b$$

$$\therefore U = \left(\frac{W^2a^2b^2}{6EIl^2}\right)(a+b) = \frac{W^2a^2b^2}{6EIl}$$

- But, if δ is the deflection under the load, the strain energy must equal the work done by the load (gradually applied),

i.e.
$$\frac{1}{2}W\delta = \frac{W^2a^2b^2}{6EIl} \Rightarrow \delta = \frac{Wa^2b^2}{3EIl}$$

- For a central load, *a* =*b* =*1*/2, and

$$\delta = \frac{W\left(\frac{l}{2}\right)^2\left(\frac{l}{2}\right)^2}{3EIl} = \frac{Wl^3}{48EI}$$

Example 1-2: Compare the strain energy of a beam, simply supported at its ends and loaded with a uniformly distributed load, with that of the same beam centrally loaded and having the same value of maximum bending stress.

Solution

- If *l* is the span and *EI* the flexural rigidity, then for a uniformly distributed load w, the end reactions are *wl/2*, and at a distance x from one end,

$$M = \left({wl}/{2} \right) x - {wx^2}/{2} = \left({wx}/{2} \right)(l - x)$$

$$U_1 = \int \left({M^2}/{2EI} \right) dx = \int_0^l \frac{w^2 x^2 (l-x)^2}{8EI} dx = \frac{w^2}{8EI} \int_0^l \left(l^2 x^2 - 2lx^3 + x^4 \right) dx = \frac{w^2 l^5}{240EI}$$

- For central load of *W*, (Example 1-1),

$$U_2 = \tfrac{1}{2} W\delta = \frac{1}{2} W\left(\frac{Wl^3}{48EI}\right) = \frac{W^2 l^3}{96EI}$$

- Maximum bending stress=M/Z.
- Since the section modulus is the same then for the two bending stresses to be the same, the maximum bending moment must be the same.
- Hence equating maximum bending moments, we have

$$wl^2/8 = Wl/4 \Rightarrow W = \tfrac{1}{2} wl$$

- Hence $U_2 = \frac{W^2 l^3}{96EI} = \frac{\left(\frac{1}{2} wl\right)^2 l^3}{96EI} = \frac{wl^5}{384EI}$
- Ratio $\frac{U_1}{U_2} = \left(\frac{wl^5}{240EI}\right)\left(\frac{384EI}{wl^5}\right) = 1.6$

Double Integration Method

- Bending moment at a point is given by

$$M = EI\frac{d^2y}{dx^2}$$

- The **value of slope at any point**

$$EI\frac{dy}{dx} = \int M$$

- The **value of deflection at any point,**

$$EI.y = \iint M$$

Case 1: Simply Supported Beam with a Central Point Load

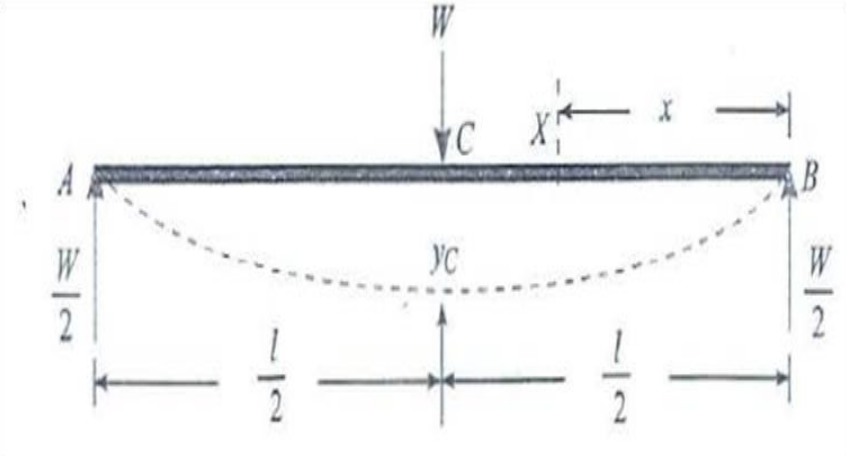


Fig. 4: Simply supported beam with a central point load

Reactions at *A and B are*

$$R_A = R_B = \frac{W}{2}$$

The bending moment at this section

$$M_x = R_B x = \frac{Wx}{2}$$

Therefore $$EI\frac{d^2y}{dx^2} = \frac{Wx}{2}$$

Integrating

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1$$

Using Boundary Condition

$$x = \frac{l}{2} \qquad \frac{d^2 y}{dx} = 0 \qquad C_1 = -\frac{Wl^2}{16}$$

Hence,

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}$$

Integrating the above equation

$$EI.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16} + C_2$$

Using Boundary Condition

$x = 0$, $y = 0$, $C_2 = 0$

Hence,

$$EI.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16}$$

Example 1-3: A simply supported beam of span 3 m is subjected to a central load of 10 kN. Find the maximum slope and deflection of the beam. Take I= 12 X 10^6 mm^4 and E = 200 GPa

Solution

Given: Span (*l*) = 3 m = 3 x 10^3 mm; Central load *(W)* = 10 kN = 10 x 10^3 N; Moment of inertia (*I*)= 12 x 10^6 mm^4 and Modulus of elasticity *(E)*= 200 GPa = 200 x 10^3 N/mm^2,

@ x= 0

Maximum slope of the beam

$$EI\,i_{max} = \frac{W(0)^2}{4} - \frac{Wl^2}{16}$$

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}$$

$$i_{max} = -\frac{Wl^2}{16EI}$$

$$i_{max} = -\frac{Wl^2}{16EI} = -\frac{(10x10^3)(3x10^3)^2}{16(200x10^3)(12x10^6)} = -0.0023\,rads$$

$$EI.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16}$$

@ x= l/2

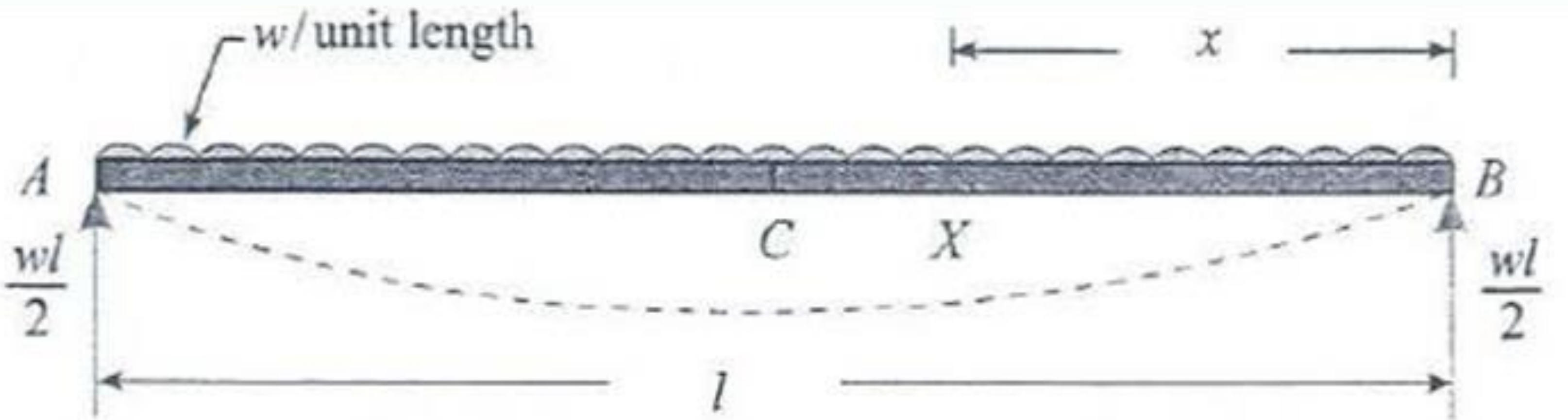
Maximum deflection of the beam

$$EI.y_{\max} = \frac{W(l/2)^3}{12} - \frac{Wl^2(l/2)}{16}$$

$$y_{\max} = \frac{Wl^3}{48EI}$$

$$y_{\max} = -\frac{Wl^3}{48EI} = \frac{(10000)(3000)^3}{48(200x10^3)(12x10^6)} = 2.34\,mm$$

Case 2: Simply Supported Beam with a Uniformly Distributed Load



The bending moment at this section

$$M_x = R_B x = \frac{wlx}{2} - \frac{wx^2}{2}$$

- Reactions at *A and B,*

$$R_A = R_B = \frac{wl}{2}$$

Therefore

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}$$

- Integrating the above equation,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1$$

Using Boundary Condition

$$x = \frac{l}{2} \quad \frac{dy}{dx} = 0 \quad C_1 = -\frac{wl^3}{24}$$

Hence,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}$$

Integrating the above equation

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2$$

Using Boundary Condition

$x = 0$, $y = 0$, $C_2 = 0$

Hence,

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}$$

- *Example 1-4: A simply supported beam of span 4 m is carrying a uniformly distributed load of 2 kN/m over the entire span. Find the maximum slope and deflection of the beam. Take EI for the beam as 80 x 10^9 N-mm^2*

Solution

Given: Span *(l)* = 4 m = 4 *x* 10^3 mm; Uniformly distributed load *(w)* = 2 kN/m = *2NImm* and flexural rigidity (*EI*)= 80 *x* 10^9 N-mm^2

Maximum slope of the beam ***@ x= 0***

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}$$

$$EIi_{\max} = \frac{wl(0)^2}{4} - \frac{w(0)^3}{6} - \frac{wl^3}{24}$$

$$i_{\max} = -\frac{wl^3}{24EI}$$

$$i_{\max} = -\frac{wl^3}{24EI} = -\frac{2(4000)^3}{24(80x10^9)} = 0.067\ rad$$

- ***Maximum deflection of the beam***

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}$$

@ x= l/2

$$EI.y_{\max} = \frac{wl(l/2)^3}{12} - \frac{w(l/2)^4}{24} - \frac{wl^3(l/2)}{24}$$

$$y_{\max} = \frac{wl^4}{384EI}$$

$$y_C = \frac{5wl^4}{384EI} = \frac{5(2)(4000)^4}{384(80x10^9)} = 83.3\,mm$$

- *Example 1-5: A simply supported beam of span 6 m is subjected to a uniformly distributed load over the entire span. If the deflection at the centre of the beam is not to exceed 4 mm, find the value of the load. Take E= 200 GPa and I = 300 x 10^6 mm^4.*

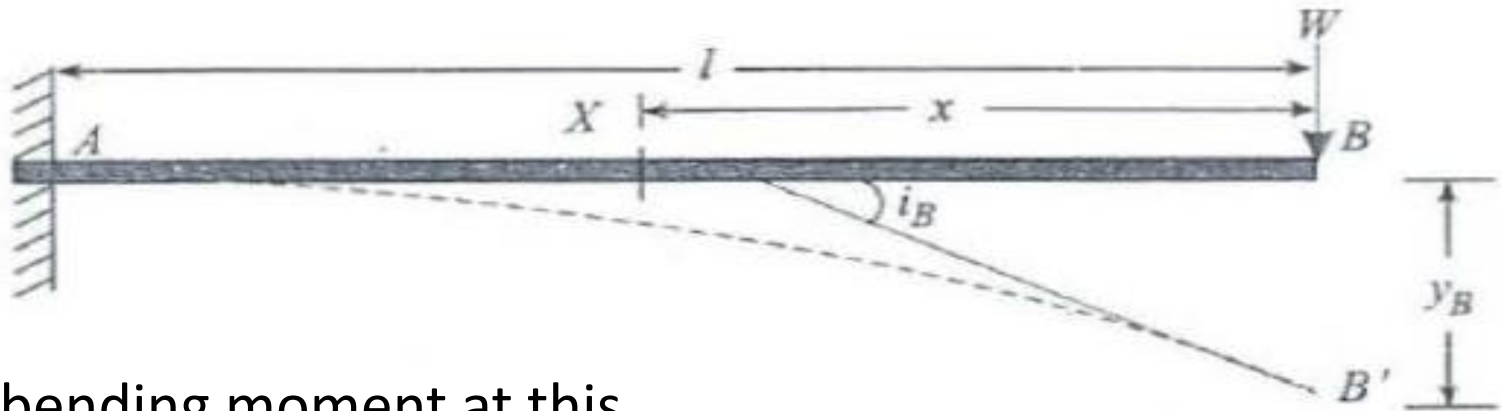
Solution

Given: Span (*l*)= -6 m = 6 x 10^3 mm ; Deflection at the centre (Y_c) = 4 mm ; modulus of elasticity(*E*)= 200 GPa = 200 x 10^3 N/mm^2 and moment of inertia(*I*)= 300 x 10^6 mm^4

Let *w* = Value of uniformly distributed load in N/mm or kN/m.

$$4=\frac{5wl^4}{384EI}=\frac{5w(6000)^4}{384(200x10^3)(300x10^6)}=0.281w\Rightarrow w=\frac{4}{0.281}=14.2\ N/mm$$

Case 3: Cantilever with a Point Load at the Free End



The bending moment at this section

$$M_x = -Wx$$

Therefore

$$EI\frac{d^2y}{dx^2} = -Wx$$

Integrating the above equation

$$EI\frac{dy}{dx} = -\frac{Wx}{2} + C_1$$

Using Boundary Condition

$$x = l$$

$$\frac{dy}{dx} = 0$$

$$C_1 = \frac{Wl^2}{2}$$

Hence,

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{Wl^2}{2}$$

Integrating the above equation

$$EI.y = -\frac{Wx^3}{6} + \frac{Wl^2x}{2} + C_2$$

Using Boundary Condition

$x = l$ and $y = 0$

$$C_2 = -\frac{Wl^3}{3}$$

Hence,

$$EI.y = -\frac{Wx^3}{6} + \frac{Wl^2x}{2} - \frac{Wl^3}{3}$$

- *Example 1-7: A cantilever beam 120 nun wide and 150 mm deep is 1.8 m long. Determine the slope and deflection ac the free end of the beam, when it carries a point load of20 kN at its free end. Take E for the cantilever beam as 200 GPa*

Solution

Given: Width *(b)*= 120 mm; Depth (d)= 150 mm; Span(*l*)= 1.8 m = 1.8 x 10^3 mm; Point load *(W)* = 20 kN = 20 x 10^3 N and modulus of elasticity (*E*) = 200 GPa = 200 x 10^3 N/mm^2

$$I = \frac{bd^3}{12} = \frac{(120)(150)^3}{12} = 33.75x10^6\ mm^4$$

Slope at the free end

$$EIi_B = -\frac{W(0)^2}{2} + \frac{Wl^2}{2}$$

$$i_B = \frac{Wl^2}{2EI} = \frac{(20000)(1800)^2}{2(200x10^3)(33.75x10^6)} = 0.0048\ rad$$

Deflection at the free end

$$EI.y_B = -\frac{W(0)^3}{6} + \frac{Wl^2(0)}{2} - \frac{Wl^3}{3}$$

$$y_B = -\frac{Wl^3}{3EI} = -\frac{(20000)(18000)^3}{3(200x10^3)(33.75x10^6)} = -5.76\ mm$$

- *Example 1-8: A cantile1-er beam of 160 mm width and 240 mm depth is 1.75 m long. What load can be placed at the free end of the cantilever if its deflection under the load is not to exceed 4.5 mm? Take E for the beam material as 180 GPa*

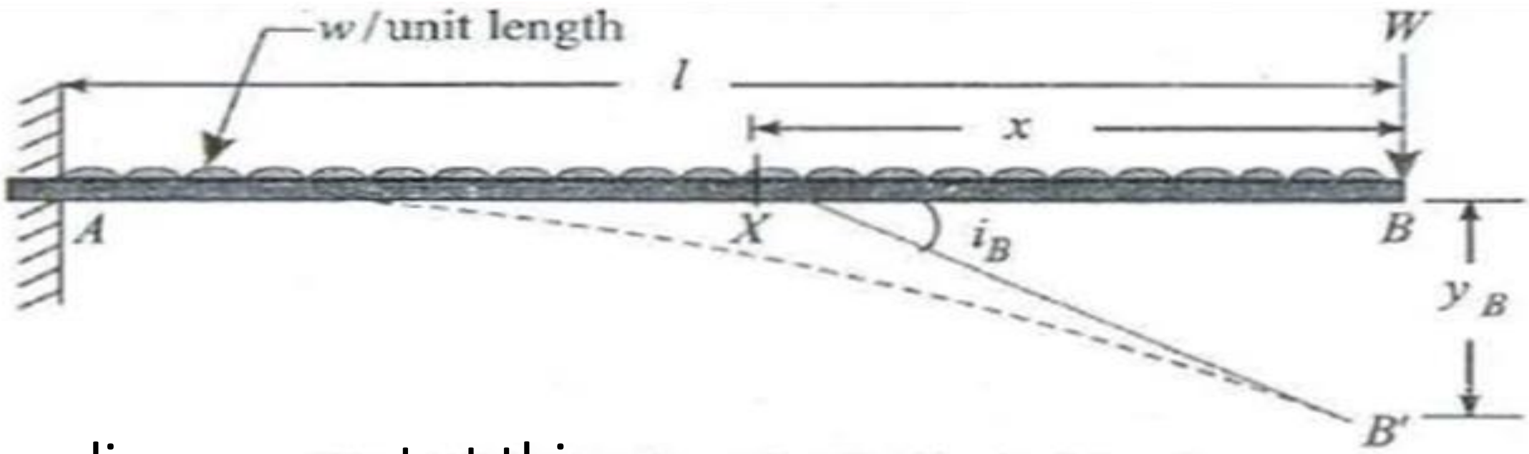
Solution

Given: Width *(b)*= 160 mm: Depth (d)= 240 mm; Span(/) = 1.75 m = 1.75 x 10^3 mm ; Deflection under the load *(y_B)* = 4.5 mm and modulus of elasticity *(E)* = 180 GPa = 180 x 10^3 N/mm^2

$$I = \frac{bd^3}{12} = \frac{(160)(240)^3}{12} = 184.32x10^6 \ mm^4$$

$$4.5 = \frac{Wl^3}{3EI} = \frac{W(18000)^3}{3(801x10^3)(1824.32x10^6)} \Rightarrow W = 83.57\,kN$$

Case 4: Cantilever with a Uniformly Distributed Load.



The bending moment at this section

$$M_x = -\frac{wx^2}{2}$$

Therefore

$$EI\frac{d^2y}{dx^2} = -\frac{w.x^2}{2}$$

Integrating the above equation

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$$

Using Boundary Condition

$$x = l$$

$$\frac{dy}{dx} = 0 \qquad C_1 = \frac{wl^3}{6}$$

Hence,

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6}$$

Integrating the above equation

$$EI.y = -\frac{wx^4}{24} + \frac{wl^3x}{6} + C_2$$

Using Boundary Condition

$x = l$ and y = 0 $\qquad C_2 = -\frac{wl^4}{8}$

Hence,

$$EI.y = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8}$$

- *Example 1-9: A cantilever beam 2 m long is subjected to a uniformly distributed load of 5 kN/m over its entire length. Find the slope and deflection of the cantilever beam at its free end. Take EI= 2.5 x 10^{12} mm^2*

Solution

Given: Span (*l*) = 2 m = 2 x 10^3 mm ; Uniformly distributed load (w) = 5 kN/m =*5N/mm* and flexural rigidity (*EI*)= 2.5 x 10^{12} N-mm^2

Slope of the cantilever beam at its free end

$$EIi_B = -\frac{w(0)^3}{6} + \frac{wl^3}{6}$$

$$i_B = -\frac{wl^3}{6EI} = \frac{(5)(2000)^3}{6(2.5x10^{12})} = 0.0027\ rad$$

Deflection of the cantilever beam at its free end

$$EI.y_B = -\frac{w(0)^4}{24} + \frac{wl^3(0)}{6} - \frac{wl^4}{8}$$

$$y_B = \frac{wl^4}{8EI} = \frac{(5)(2000)^4}{8(2.5x10^{12})} = 4\ mm$$

- Example 1-10: *A cantilever beam 100 mm wide and 180 mm deep is projecting 2 m from a wall. Calculate the uniformly distributed load, which the beam should carry, if the deflection of the free end should not exceed 3.5 mm. Take E as 200 GPa*

Solution

Given: Width *(b)*= 100 mm; Depth (d)= 180 mm; Span(*l*)= 2m= 2 x 10^3 mm; Deflection at the free end *(y_B)* = 3.5 mm and modulus of elasticity (*E*)= 200 GPa = 200 x 10^3 N/mm^2

The moment of inertia of the beam reaction,

$$I = \frac{bd^3}{12} = \frac{(100)(180)^3}{12} = 48.6x10^6 \quad mm^4$$

$$3.5 = \frac{wl^4}{8EI} = \frac{w(2000)^4}{8(200x10^3)(48.6x10^6)} = 0.206w \Rightarrow w = \frac{3.5}{0.206} = 17\ N/mm$$

- *Example 1-11: A cantilever beam of length 3 m is carrying a uniformly distributed load of w kN/m. Assuming rectangular cross-section with depth (d) equal to twice the width (b), determine the dimensions of the beam, so that vertical deflection at the free end does not exceed 8 mm. Take maximum bending stress = 100 MPa and E = 200 GPa*

Solution

Given: Span (*l*)= 3m= 3 x 10^3 mm; Uniformly distributed load= *w* kN/m = *w N/mm;* Depth *(d)* = 2*b;* Deflection at the free end *(y_B)* = 8 mm ; Maximum bending stress= 100 MPa = 100 N/mm^2 and modulus of elasticity (*E*) = 200 GPa = 200 x 10^3 N/mm^2.

Moment of inertia:

$$I = \frac{bd^3}{12} = \frac{(b)(2b)^3}{12} = \tfrac{2}{3} b^4 mm^4$$

Deflection at the free end of the cantilever

$$8 = \frac{wl^4}{8EI} = \frac{w(2000)^4}{8(200x10^3)\left(\tfrac{2}{3}b^4\right)} \Rightarrow b^4 = 9.5x10^6 w..........(i)$$

Moment at the fixed end of the cantilever

$$M = \frac{wl^2}{2} = 4.5x10^6 w$$

Bending Stress

$$100 = \frac{My}{I} = \frac{\left(4.5x10^6 w\right)}{\frac{2}{3}b^4}(b) = 4.5x10^6 w \Rightarrow b^3 = 67.5x10^3 w....(ii)$$

Dividing equation (i) by (ii), we have

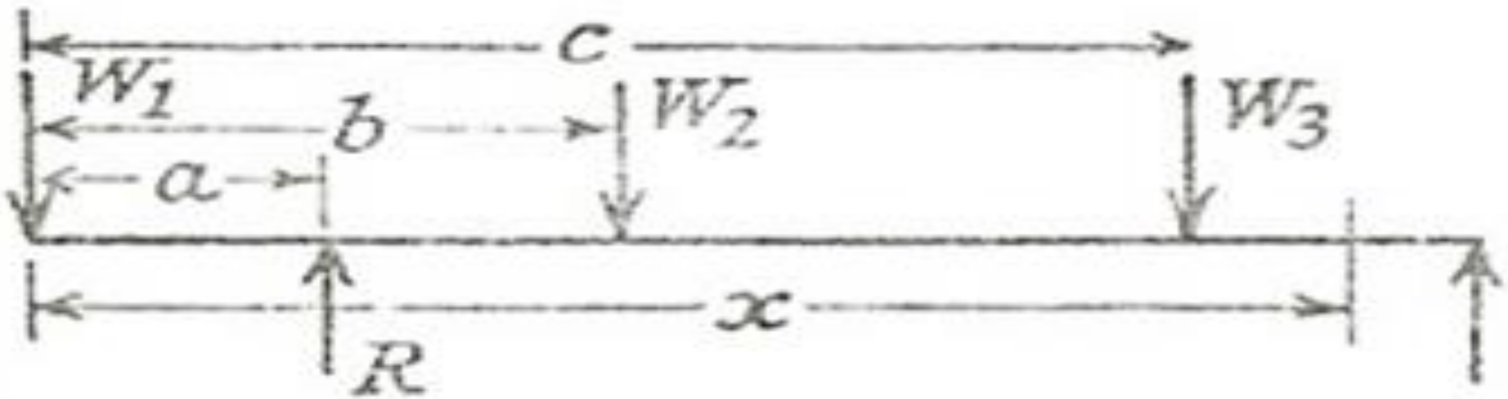
$$b = \frac{9.5x10^6}{67.5x10^3} = 141mm \Rightarrow d = 2b = 2(141mm) = 282\,mm$$

Singularity Function (Macualay's Method)

- Macaulay's method enables
  - one continuous expression for bending moment to be obtained, and
  - the same constants of integration for all sections of the beam.

- For the purpose of illustration, it is advisable to deal with the different types of loading separately.

Case 1: Concentrated loads

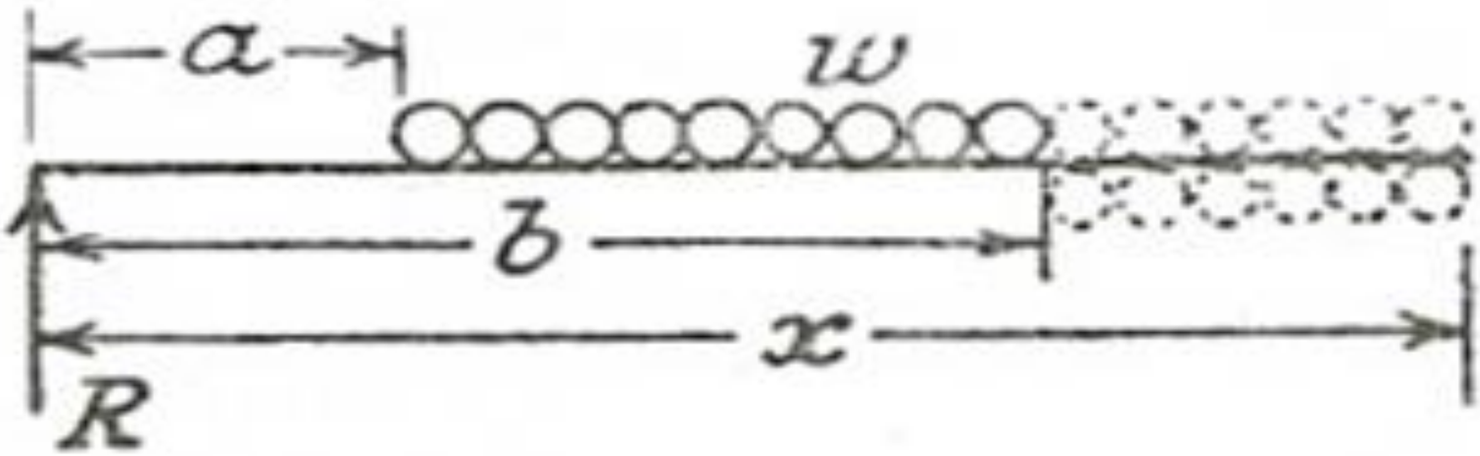


$$EI\,{d^2y}/{dx^2} = M = \left(-W_1x + R[x-a] - W_2[x-b] - W_3[x-c]\right)$$

$$EI\,{dy}/{dx} = \frac{1}{2}\left(-W_1x^2 + R[x-a]^2 - W_2[x-b]^2 - W_3[x-c]^2\right) + A$$

$$EIy = \frac{1}{6}\left(-W_1x^3 + R[x-a]^3 - W_2[x-b]^3 - W_3[x-c]^3\right) + Ax + B$$

Case 2: Uniformly distributed loads

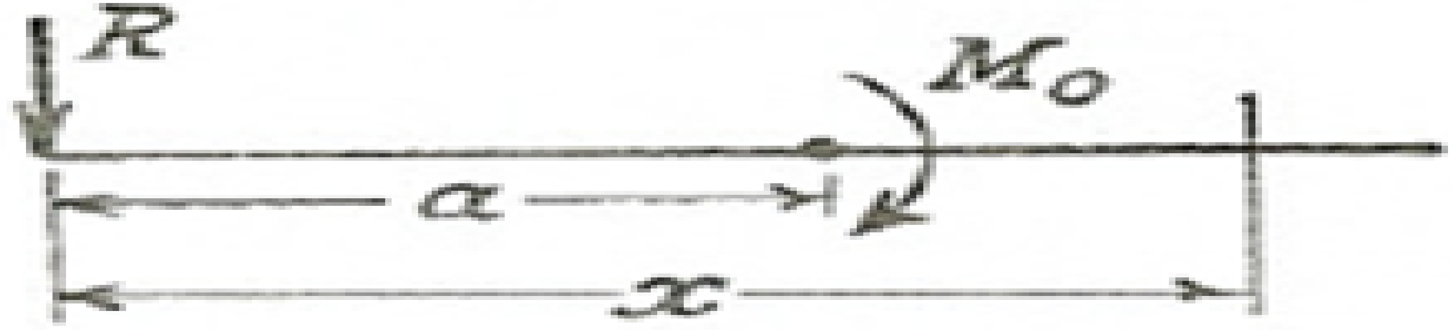


$$EI\,{d^2y}\big/{dx^2} = M = Rx - w[x-a]^2 + w[x-b]^2$$

$$EI\,{dy}\big/{dx} = M = \frac{R}{2}x^2 - \frac{w}{3}[x-a]^3 + \frac{w}{3}[x-b]^3 + A$$

$$EI.y = M = \frac{R}{6}x^3 - \frac{w}{12}[x-a]^4 + \frac{w}{12}[x-b]^4 + Ax + B$$

Case 3: Concentrated bending moment

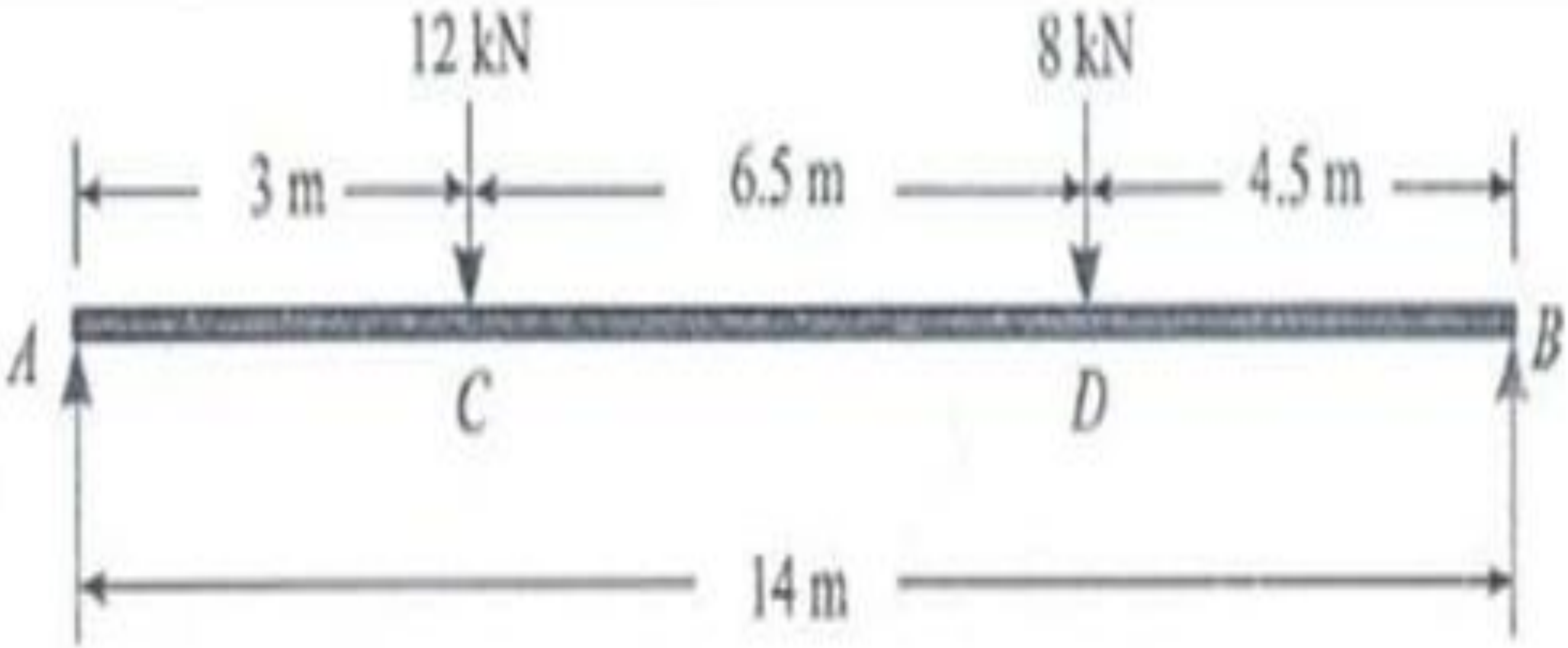


$$EI\,{d^2y}/{dx^2} = M = \left(Rx - M_0[x-a]^0\right)$$

$$EI\,{dy}/{dx} = M = \left(Rx^2/2 - M_0[x-a]\right) + A$$

$$EIy = M = \left(Rx^3/6 - M_0[x-a]^2\right) + Ax + B$$

- *Example 1-12: A horizontal steel girder having uniform cross-section is 14m long and is simply supported at its ends. It carries two concentrated loads as shown in Fig. 17. Calculate the deflections of the beam under the loads C and D. Take E = 200 GPa and 1 = 160x10^6 mm^4*



Solution

Given: Span (l)= 14m= 14 x 10^3 mm; Load at C (W_1) = 12 kN = 12 x 10^3 N; Load at D (W_2) = 8 kN = 8 x 10^3 N; Modulus of elasticity (E)= 200 GPa = 200 x I 0^3 N/mm^2 and moment of inertia(I)= 160 x 10^6 mm^4

Taking moments about A and equating the same

$$R_A = 12 + 8 - 8 = 12000\,N$$

$$14R_B = 12(3) + 8(9.5) = 112 \Rightarrow R_B = 8000\,N$$

Now taking A as the origin and using Macaulay's method, the bending moment at any section X at a distance x from A,

$$EI\frac{d^2y}{dx^2} = 12000x - 12000[x - 3000] - 8000[x - 9500]$$

Integrating the above equation

$$EI\,\frac{dy}{dx} = 6000x^2 - 6000[x-3000]^2 - 4000[x-9500]^2 + C_1$$

Integrating the above equation once again

$$EI.y = 2000x^3 - 2000[x-3000]^3 - 1333[x-9500]^3 + C_1 x + C_2$$

Using Boundary Condition

x= 0 and *y = 0*, then $C_2 = 0$

x= 14000 mm and *y* = 0, then $C_1 = 193.2x10^9$

Hence

$$EI.y = 2000x^3 - 2000[x-3000]^3 - 1333[x-9500]^3 - 193.2x10^6\,x..(i)$$

- For 12 kN load; $x = 3$ m (or 3×10^3 mm)

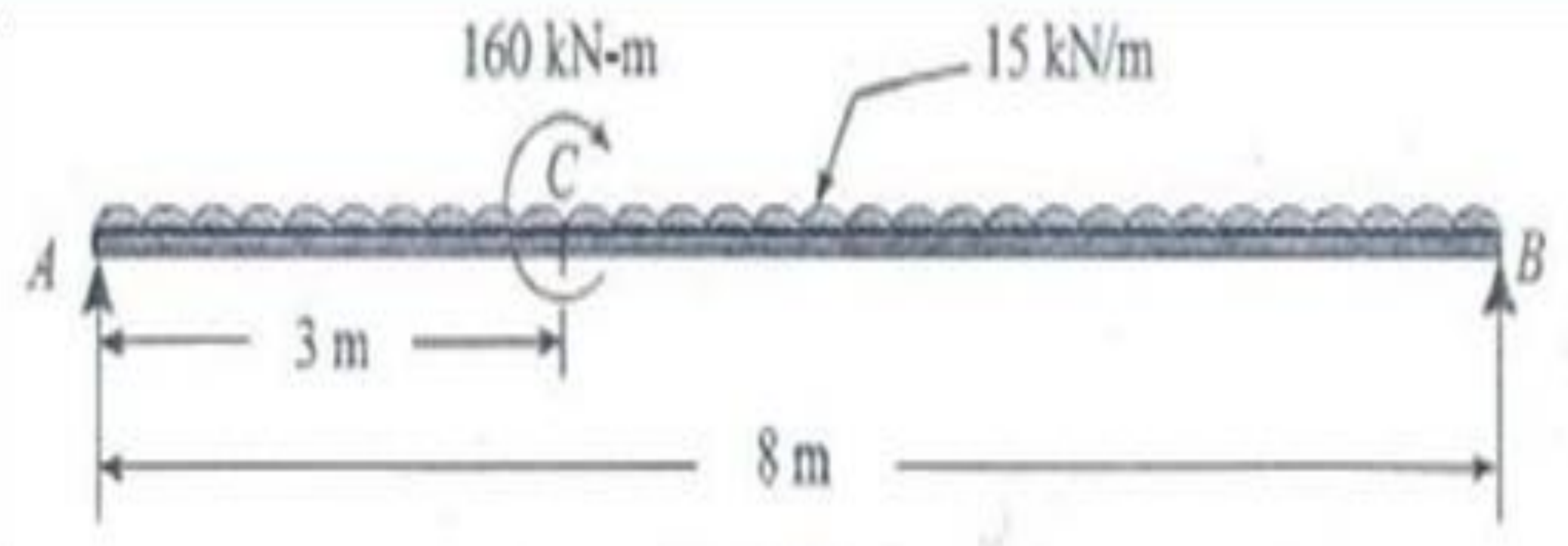
$$EIy_C = 2000(3000)^3 - 193.2x10^9(3000) = -525.6x10^{12}$$

$$\Rightarrow y_C = \frac{-525.6x10^{12}}{(200x10^3)(160x10^6)} = 16.4\,mm$$

$$EIy_D = 2000(9500)^3 - 193.2x10^9(9500) - 2000[6500]^3 = -669.9x10^{12}$$

$$\Rightarrow y_D = \frac{-669.6x10^{12}}{(200x10^3)(160x10^6)} = -20.9\,mm$$

- *Example 1-13: A horizontal beam AB is freely supported at A and B 8 m apart and carries a uniformly distributed load of 15 kN/m run (including its own weight). A clockwise moment of 160 kN-m is applied to the beam at a point C, 3m from the left hand support A. Calculate the slope of the beam at C, if EI = 40 x* 10^3 kN-m^2.



Solution

Given: Span (*l*) = 8 m; Uniformly distributed load (*w*) = 15 kN/m; Moment at *C* (μ) = 160 kN-m (clockwise) and flexural rigidity *(EI)* = 40 x 10^3 kN-m^2

Taking moments about *A*,

$$8R_B = 15(8)(4) + 160 = 640 \Rightarrow R_B = 80\,kN$$

$$R_A = 15(8) - 80 = 40\,kN$$

Now taking *A* as the origin and using Macaulay's method, the bending moment at any section *X* at a distance *x* from *A*,

$$EI\frac{d^2y}{dx^2} = 40x - 160[x-3]^0 - 7.5x^2$$

Integrating the above equation

$$EI\frac{dy}{dx} = 20x^2 - 160[x-3] - 2.5x^3 + C_1 \ldots (i)$$

Integrating the above equation once again

$$EI.y = 6.67x^3 - 80[x-3]^2 - 0.625x^4 + C_1x + C_2 ...(ii)$$

Using Boundary Condition

x= 0 and *y = 0*, then $C_2 = 0$

x= 8 m and *y* = 0, then $C_1 = 356.7$

Hence

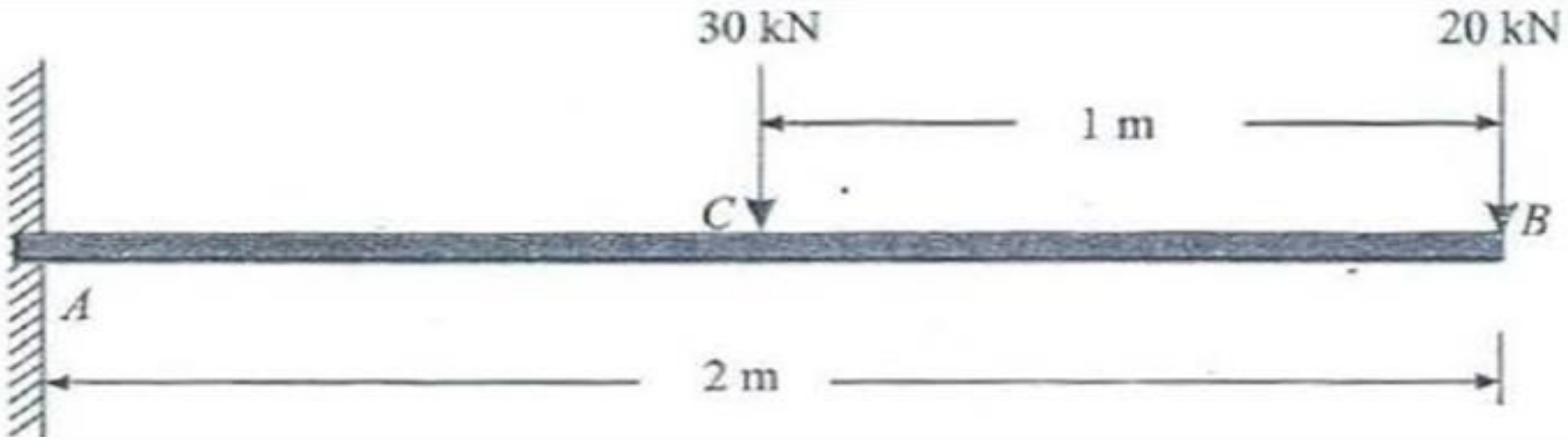
$$EI\frac{dy}{dx} = 20x^2 - 356.7 - 160[x-3] - 2.5x^3 ..(iii)$$

$$EI.y = 6.67x^3 - 356.7x - 80[x-3]^2 - 0.625x^4 ..(iv)$$

slope at C, substituting *x* = 3 m in equation (iii)

$$EIi_C = 20(3)^2 - 356.7 - 2.5(3)^3 = -244.2 \Rightarrow i_C = \frac{-244.2}{40x10^3} = -0.0061rad$$

- *Example 1-14: A cantilever AB 2 m long is carrying a load of 20 kN at free end and 30 kN at a distance 1 m from the free end. Find the slope and deflection at the free end. Take E = 200 GPa and I = 150 x 10^6 mm^4*



Solution

Given: Span *AB* (l)=2m= 2 x 10^3 mm: Load at the free end (\W_1)= 20 kN = 20 x10^3 N: Load at *C* (W_2) = 30 kN = 30 x 10^3 N; Length *AC* (l_1) = 1 m = 1 x 10^3 mm; Modulus of elasticity (E) = 200 GPa = 200 x 10^3 N/mm^2 and moment of inertia (I) = 150 x 10^6 mm^4

Taking moments about *A*

$$14R_B = 12(3) + 8(9.5) = 112 \Rightarrow R_B = 8000\,N$$

$$R_A = 12 + 8 - 8 = 12000\,N$$

Now taking *A* as the origin and using Macaulay's method, the bending moment at any section *X* at a distance *x* from *A*

$$EI\frac{d^2y}{dx^2} = 12000x - 12000[x - 3000] - 8000[x - 9500]$$

Integrating the above equation

$$EI\frac{dy}{dx} = 6000x^2 - 6000[x - 3000]^2 - 4000[x - 9500]^2 + C_1$$

Integrating the above equation once again

$$EI.y = 2000x^3 - 2000[x-3000]^3 - 1333[x-9500]^3 + C_1x + C_2$$

Using Boundary Condition

x= 0 and $y = 0$, then $C_2 = 0$

x= 14000 mm and y = 0, then $C_1 = 193.2x10^9$

Hence

$$EI.y = 2000x^3 - 2000[x-3000]^3 - 1333[x-9500]^3 - 193.2x10^6 x..(i)$$

Deflection under the 12 kN load; substituting *x* = 3 m (or 3 x 10^3 mm)

$$EIy_C = 2000(3000)^3 - 193.2x10^9(3000) = -525.6x10^{12}$$

$$\Rightarrow y_C = \frac{-525.6x10^{12}}{(200x10^3)(160x10^6)} = 16.4\ mm$$

Deflection under the 8 kN load, substituting x = 9.5 m (or 9.5 x 10^3 mm)

$$EIy_D = 2000(9500)^3 - 193.2x10^9(9500) - 2000[6500]^3 = -669.9x10^{12}$$

$$\Rightarrow y_D = \frac{-669.6x10^{12}}{(200x10^3)(160x10^6)} = -20.9\ mm$$

Quiz 1
Time: 30 Minutes

Assignment 1

Problem 6: *A steel girder of uniform section,* 14 *metres long is simply supported at its ends. It carries concentrated loads of 90 kN and* 60 *kN at two points* 3 *metres and* 4.5 *metres from the two ends respectively. Calculate:*

(i) *The deflection of the girder at the points under the two loads.*

(ii) The maximum deflection. Take: $1=64 \times 10^{8}$ mm^{4} *and* $E = 210 \times 10^{3}$ N/mm^{2}.

Problem 7: *A beam AB of* 4 *metres span is simply supported at the ends and is loaded as shown in Fig.* 13. *Determine (i) Deflection at* C,

ii) Maximum deflection and

(iii) Slope at the end A.

Given: $E = 200 \times 10^{6}$ kN/m^{2}, *and* $1 = 20 \times 10^{-6}$ mm^{4}

Problem 8: *A beam AB of span* 8 *metres is simply supported at the ends. It carries a uniformly distributed load of 30 kN/m over its entire length and a concentrated. Load of 60 kN at* 3 *metres from the support A. Determine the maximum deflection in the beam and the location where the deflection occurs. Take:* $1= 80 \times 10^{-4}$ m^{4}; $E = 200 \times 10^{6}$ kN/mm^{2}