14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

Emmanuel de-Graft Johnson Owusu-Ansah(PhD)

Department of Mathematics

Faculty of Physical and Computational Sciences

College of Science

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values.

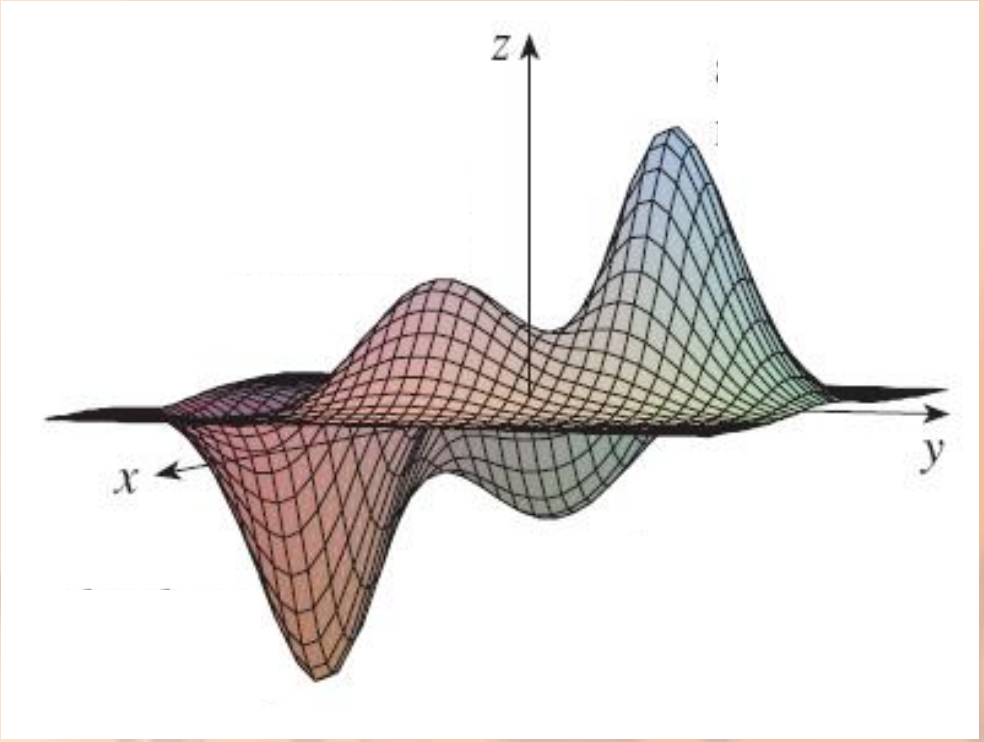
14.7

Maximum and Minimum Values

In this section, we will learn how to:

Use partial derivatives to locate maxima and minima of functions of two variables.

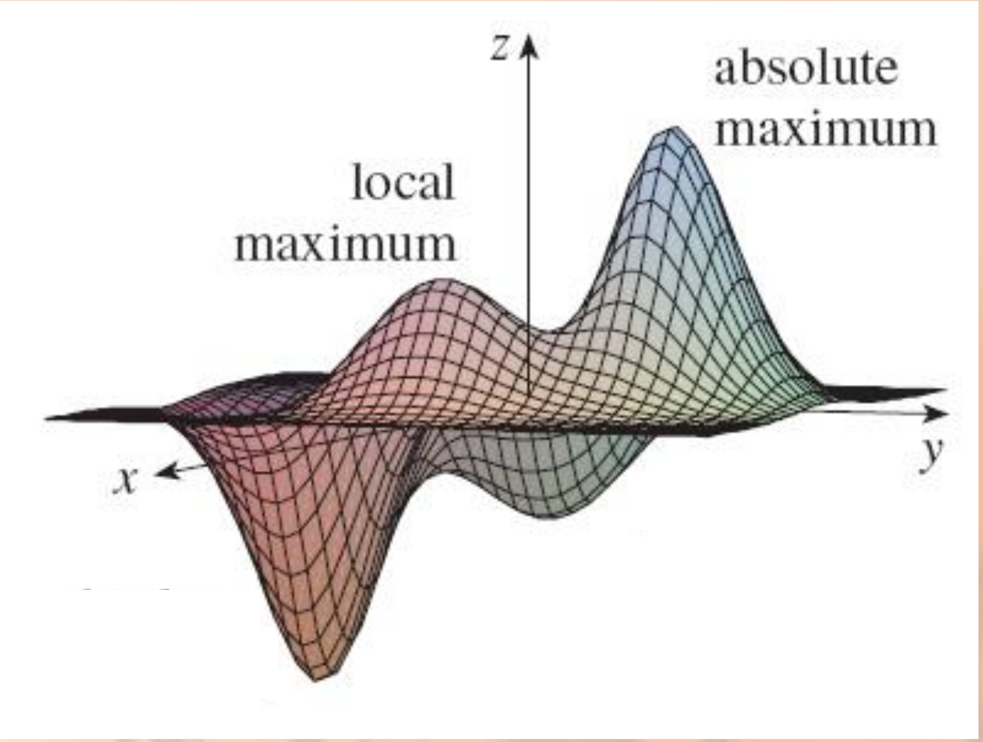
Look at the hills and valleys in the graph of *f* shown here.



ABSOLUTE MAXIMUM

There are two points (a, b) where f has a local maximum—that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

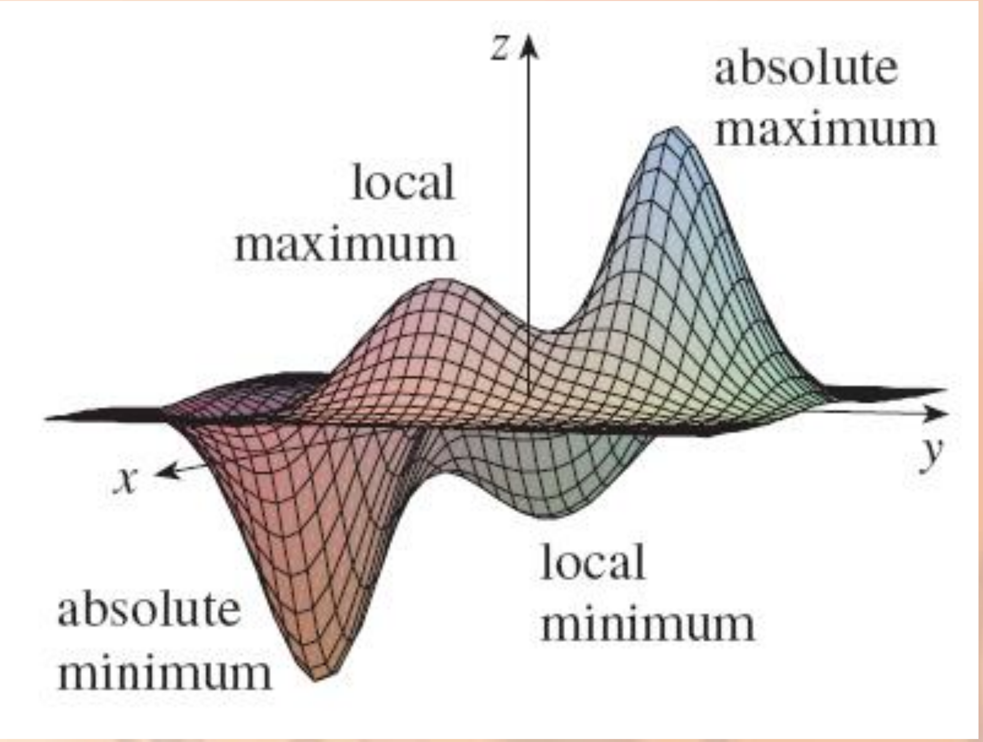
- The larger of these two values is the absolute maximum.



ABSOLUTE MINIMUM

Likewise, *f* has two local minima—where *f*(*a*, *b*) is smaller than nearby values.

- The smaller of these two values is the absolute minimum.



LOCAL MAX. & LOCAL MAX. VAL. **Definition 1**

A function of two variables has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b).

This means that $f(x, y) \le f(a, b)$ for all points (x, y) in some disk with center (a, b).

- The number $f(a, b)$ is called a local maximum value.

LOCAL MIN. & LOCAL MIN. VALUE **Definition 1**

If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b).

$f(a, b)$ is a local minimum value.

ABSOLUTE MAXIMUM & MINIMUM

If the inequalities in Definition 1 hold for all points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).

Theorem 2

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

Proof

Let $g(x) = f(x, b)$.

- If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a.
- So, $g'(a) = 0$ by Fermat's Theorem.

However, $g'(a) = f_x(a, b)$

- See Equation 1 in Section 14.3
- So, $f_x(a, b) = 0$.

Proof

Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain:

$$f_y(a, b) = 0$$

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 2 in Section 14.4), we get:

$$z = z_0$$

THEOREM 2—GEOMETRIC INTERPRETATION

Thus, the geometric interpretation of Theorem 2 is:

- If the graph of *f* has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

CRITICAL POINT

A point (a, b) is called a critical point (or stationary point) of f if either:

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$
- One of these partial derivatives does not exist.

CRITICAL POINTS

Theorem 2 says that, if *f* has a local maximum or minimum at (a, b), then (a, b) is a critical point of *f*.

CRITICAL POINTS

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

- At a critical point, a function could have a local maximum or a local minimum or neither.

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

Then,

$$f_x(x, y) = 2x - 2$$

$$f_y(x, y) = 2y - 6$$

- These partial derivatives are equal to 0 when $x = 1$ and $y = 3$.
- So, the only critical point is (1, 3).

By completing the square, we find:

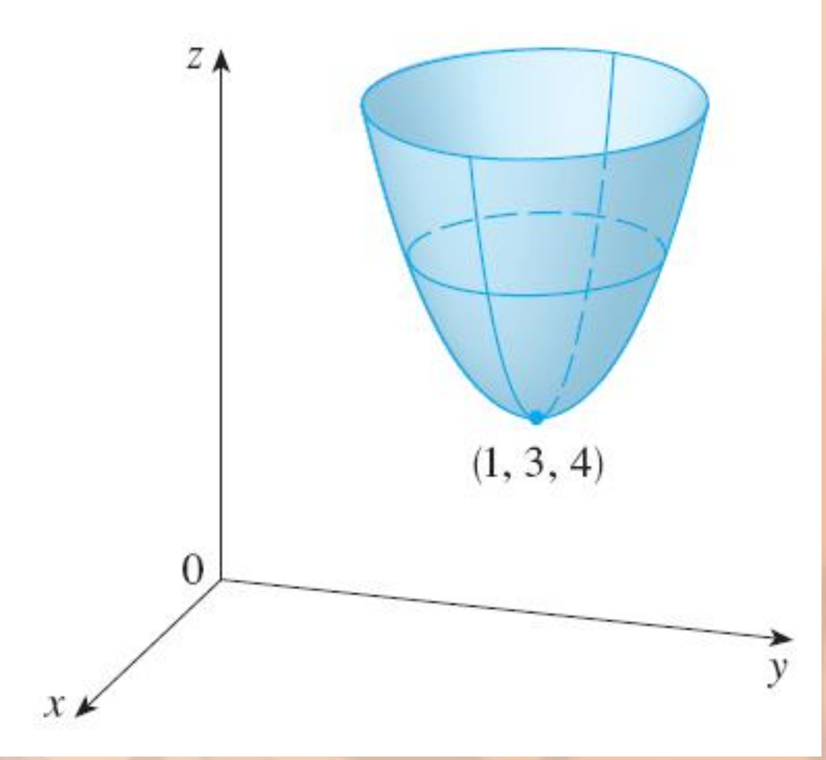
$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

- Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y.
- So, $f(1, 3) = 4$ is a local minimum.
- In fact, it is the absolute minimum of f.

LOCAL MINIMUM

Example 1

This can be confirmed geometrically from the graph of *f*, which is the elliptic paraboloid with vertex (1, 3, 4).



Find the extreme values of

$$f(x, y) = y^2 - x^2$$

- Since $f_x = -2x$ and $f_y = -2y$, the only critical point is (0, 0).

Notice that, for points on the *x*-axis, we have $y = 0$.

- So, $f(x, y) = -x^2 < 0$ (if $x \neq 0$).

For points on the *y*-axis, we have $x = 0$.

- So, $f(x, y) = y^2 > 0$ (if $y \neq 0$).

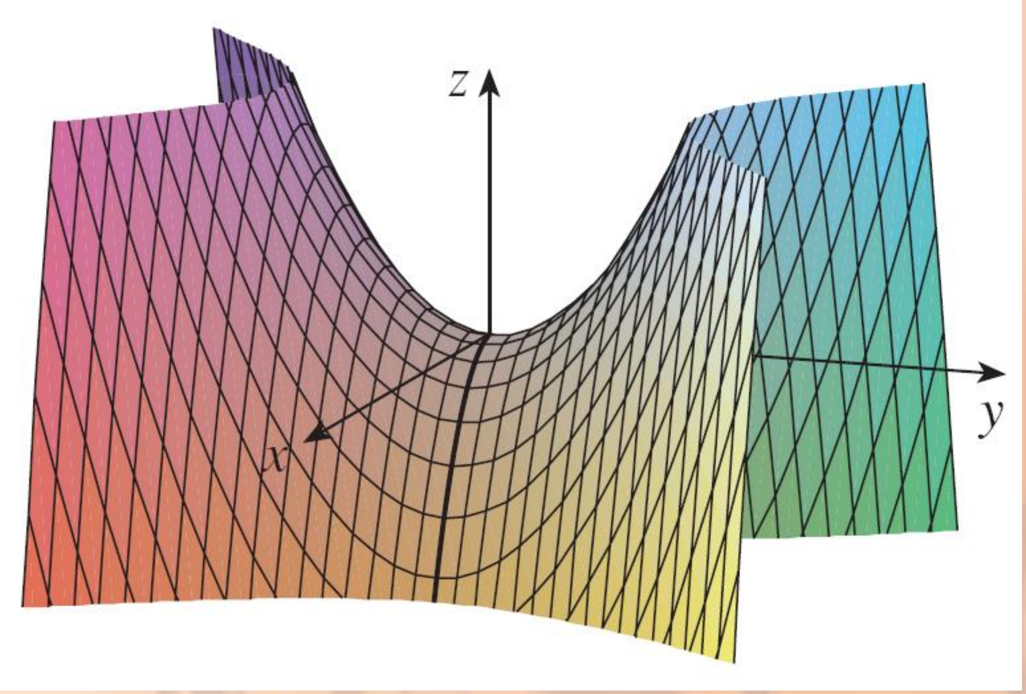
Thus, every disk with center (0, 0) contains points where *f* takes positive values as well as points where *f* takes negative values.

- So, $f(0, 0) = 0$ can't be an extreme value for *f*.
- Hence, *f* has no extreme value.

Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point.

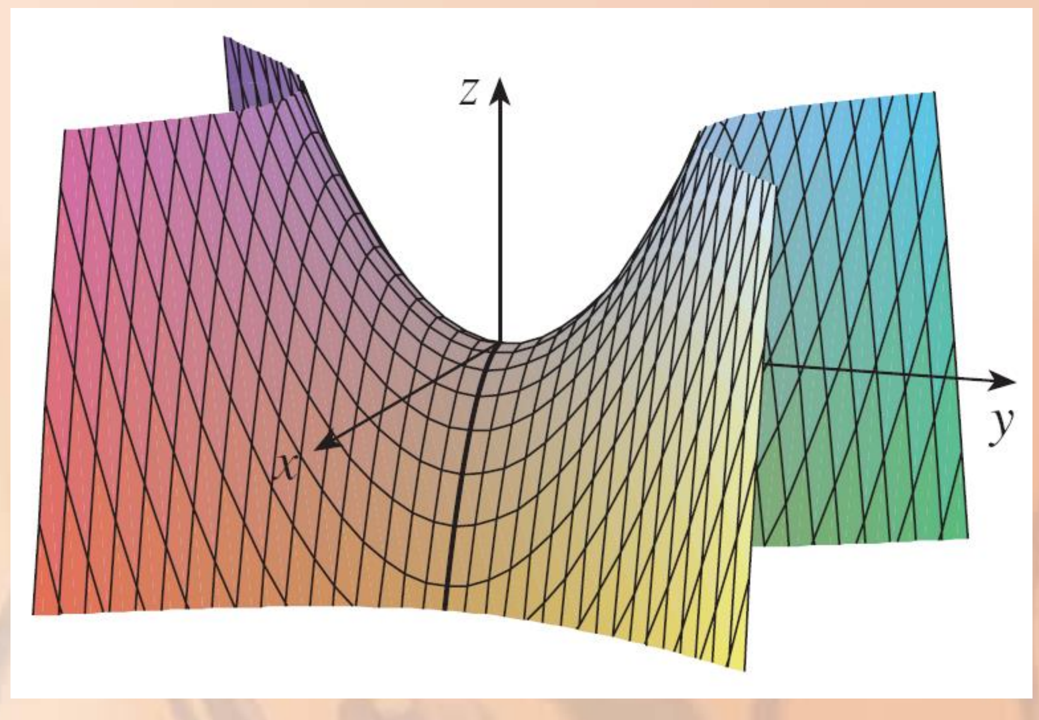
The figure shows how this is possible.

- The graph of *f* is the hyperbolic paraboloid $z = y^2 - x^2$.
- It has a horizontal tangent plane ($z = 0$) at the origin.



You can see that $f(0, 0) = 0$ is:

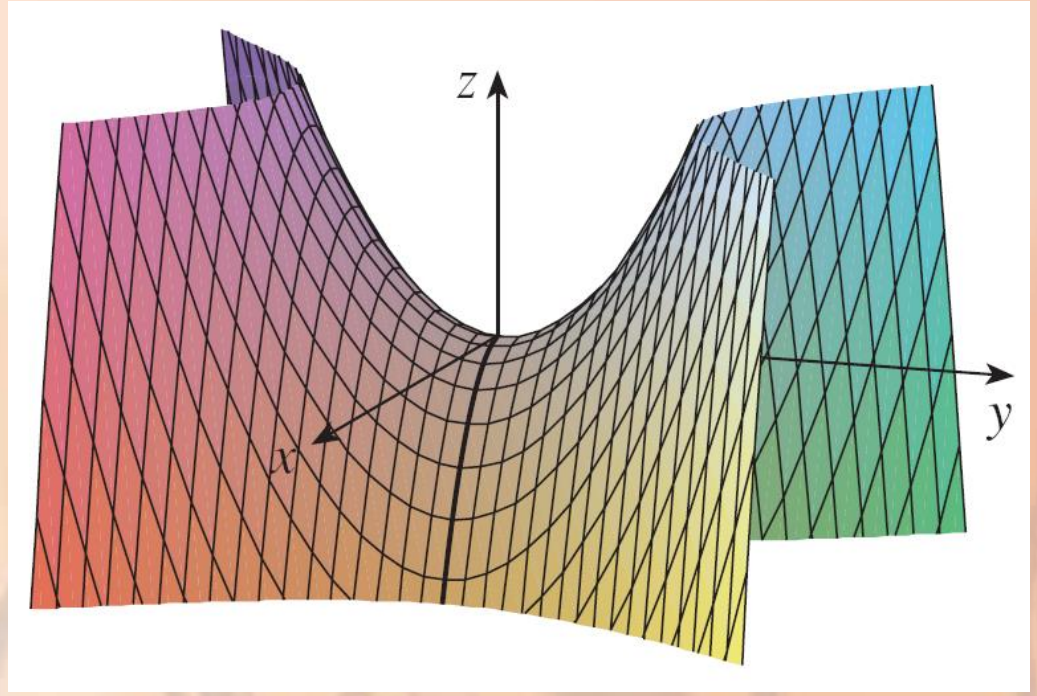
- A maximum in the direction of the *x*-axis.
- A minimum in the direction of the *y*-axis.



SADDLE POINT

Near the origin, the graph has the shape of a saddle.

- So, (0, 0) is called a saddle point of *f*.



EXTREME VALUE AT CRITICAL POINT

We need to be able to determine whether or not a function has an extreme value at a critical point.

The following test is analogous to the Second Derivative Test for functions of one variable.

Suppose that:

- The second partial derivatives of f are continuous on a disk with center (a, b).
- $f_x(a, b) = 0$ and $f_y(a, b) = 0$
[that is, (a, b) is a critical point of f].

SECOND DERIVATIVES TEST

Theorem 3

Let $D = D(a, b)$

$$= f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

a) If $D > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum.

b) If $D > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum.

c) If $D < 0$, $f(a, b)$ is not a local maximum or minimum.

In case c,

- The point (a, b) is called a saddle point of f .
- The graph of f crosses its tangent plane at (a, b).

SECOND DERIVATIVES TEST Note 2

If $D = 0$, the test gives no information:

- f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

To remember the formula for *D*,

it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

SECOND DERIVATIVES TEST **Example 3**

Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

We first locate the critical points:

$$f_x = 4x^3 - 4y$$

$$f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain:

$$x^3 - y = 0$$

$$y^3 - x = 0$$

- To solve these equations, we substitute $y = x^3$ from the first equation into the second one.

This gives:

$$\begin{aligned} 0 &= x^9 - x \\ &= x(x^8 - 1) \\ &= x(x^4 - 1)(x^4 + 1) \\ &= x(x^2 - 1)(x^2 + 1)(x^4 + 1) \end{aligned}$$

So, there are three real roots:

$$x = 0, 1, -1$$

- The three critical points are:

$$(0, 0), (1, 1), (-1, -1)$$

SECOND DERIVATIVES TEST — Example 3

Next, we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - (f_{xy})^2 \\ &= 144x^2y^2 - 16 \end{aligned}$$

SECOND DERIVATIVES TEST

Example 3

As $D(0, 0) = -16 < 0$, it follows from case c of the Second Derivatives Test that the origin is a saddle point.

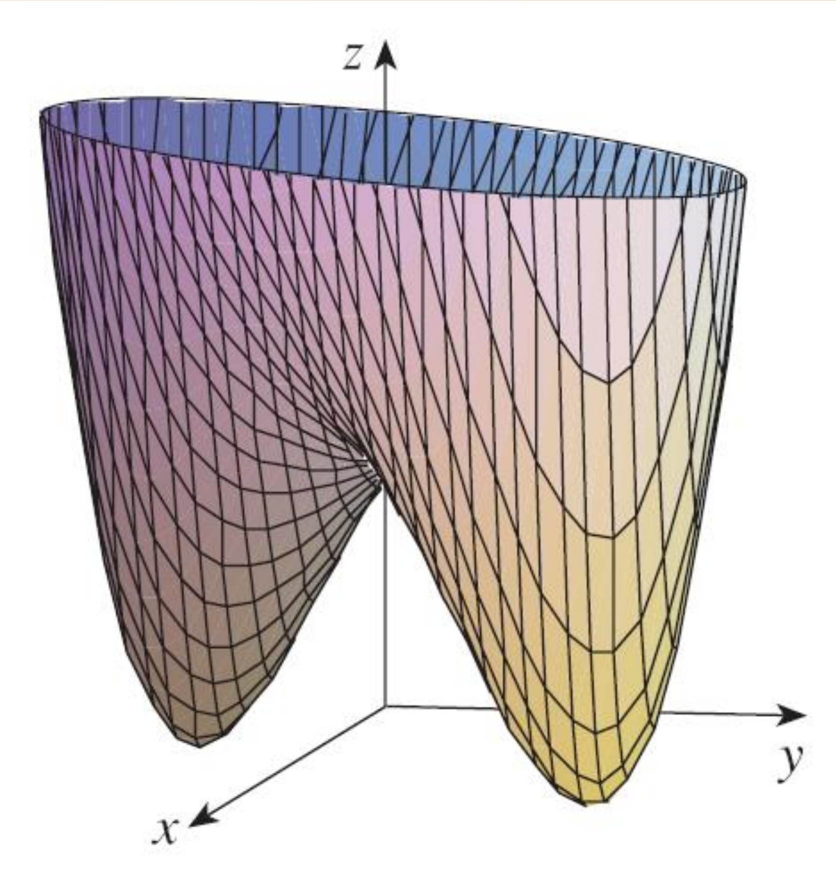
- That is, f has no local maximum or minimum at (0, 0).

As $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case a of the test that $f(1, 1) = –1$ is a local minimum.

Similarly, we have $D(–1, –1) = 128 > 0$ and $f_{xx}(–1, –1) = 12 > 0$.

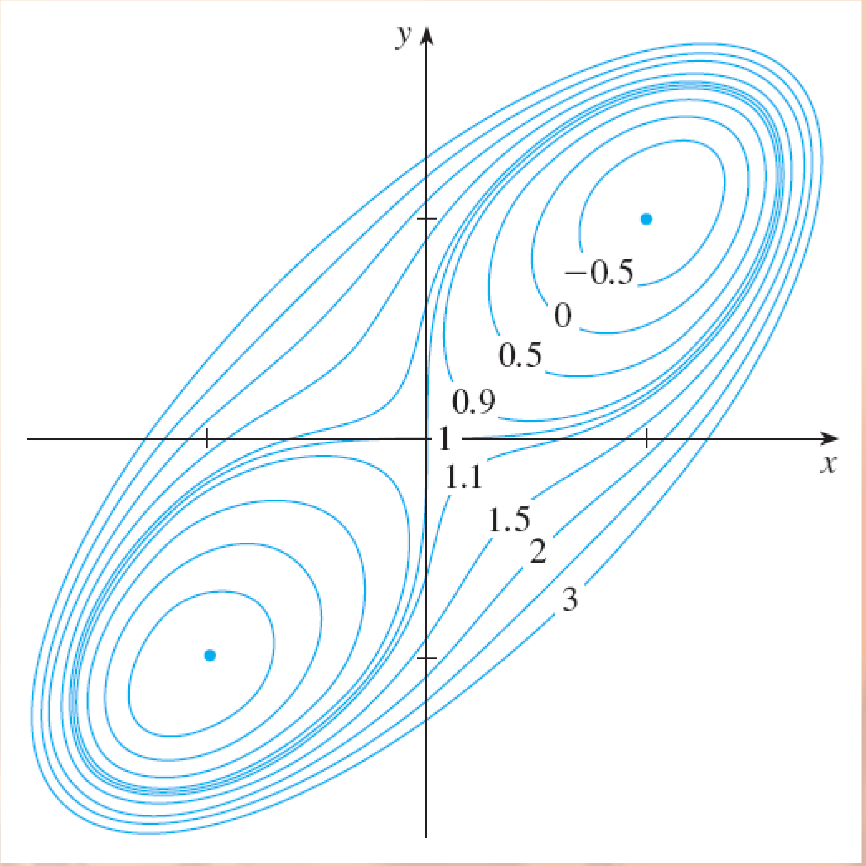
- So $f(–1, –1) = –1$ is also a local minimum.

The graph of *f* is shown here.



CONTOUR MAP

A contour map of the function in Example 3 is shown here.

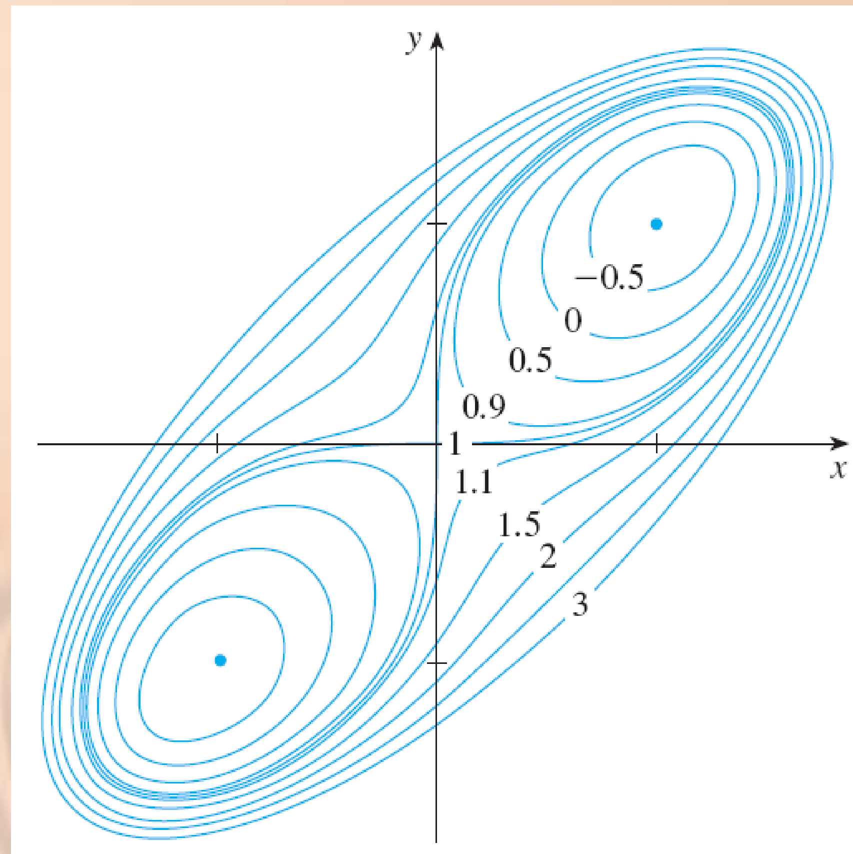


CONTOUR MAP

The level curves near (1, 1) and (–1, –1) are oval in shape.

They indicate that:

- As we move away from (1, 1) or (–1, –1) in any direction, the values of *f* are increasing.

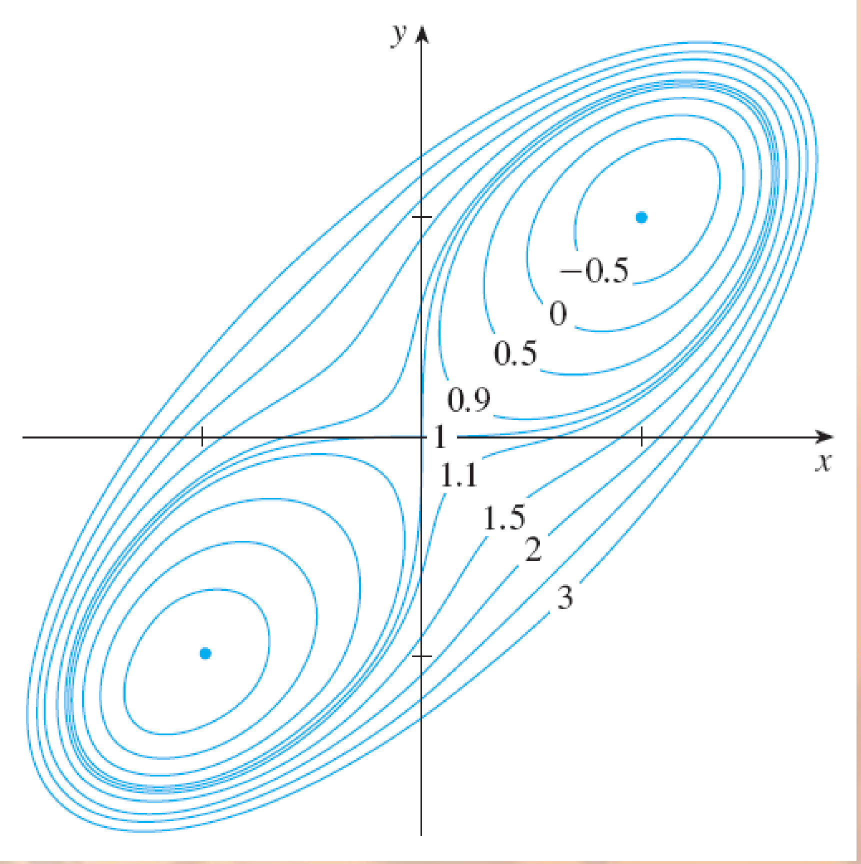


CONTOUR MAP

The level curves near (0, 0) resemble hyperbolas.

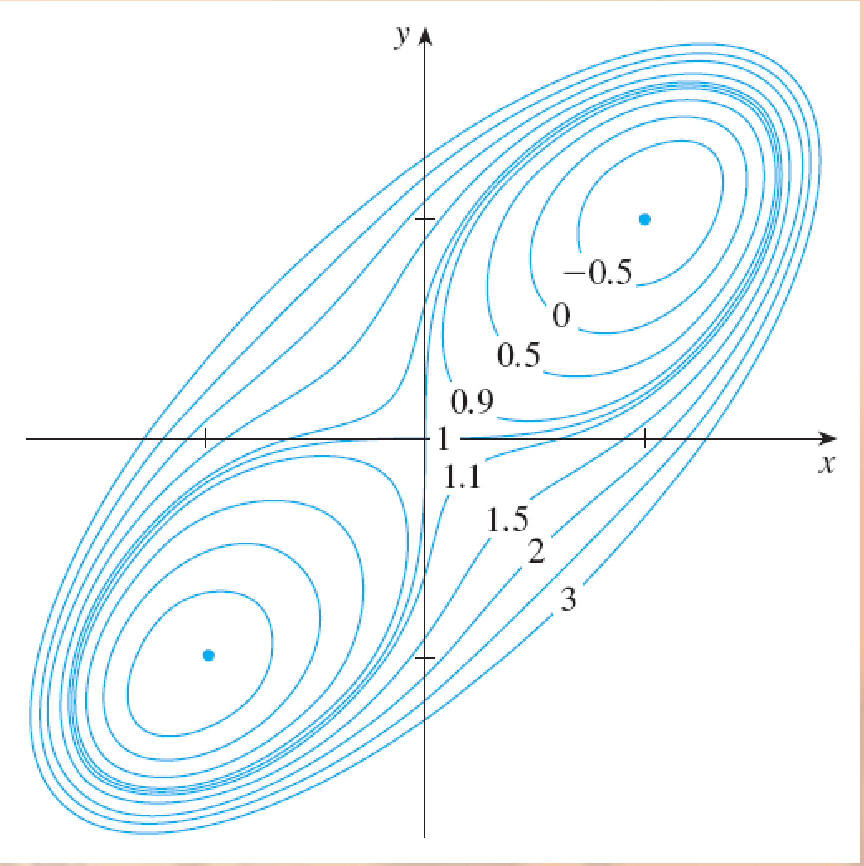
They reveal that:

- As we move away from the origin (where the value of *f* is 1), the values of *f* decrease in some directions but increase in other directions.



CONTOUR MAP

Thus, the map suggests the presence of the minima and saddle point that we found in Example 3.



Example 4

Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also, find the highest point on the graph of *f*.

Example 4

The first-order partial derivatives are:

$$f_x = 20xy - 10x - 4x^3$$

$$f_y = 10x^2 - 8y - 8y^3$$

So, to find the critical points, we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0$$

$$5x^2 - 4y - 4y^3 = 0$$

From Equation 4, we see that either:

- $x = 0$
- $10y - 5 - 2x^2 = 0$

Example 4

In the first case ($x = 0$), Equation 5 becomes:

$$-4y(1 + y^2) = 0$$

So, $y = 0$, and we have the critical point (0, 0).

E. g. 4—Equation 6

In the second case ($10y - 5 - 2x^2 = 0$), we get:

$$\boxed{x^2 = 5y - 2.5}$$

- Putting this in Equation 5, we have:

$$25y - 12.5 - 4y - 4y^3 = 0$$

So, we have to solve the cubic equation

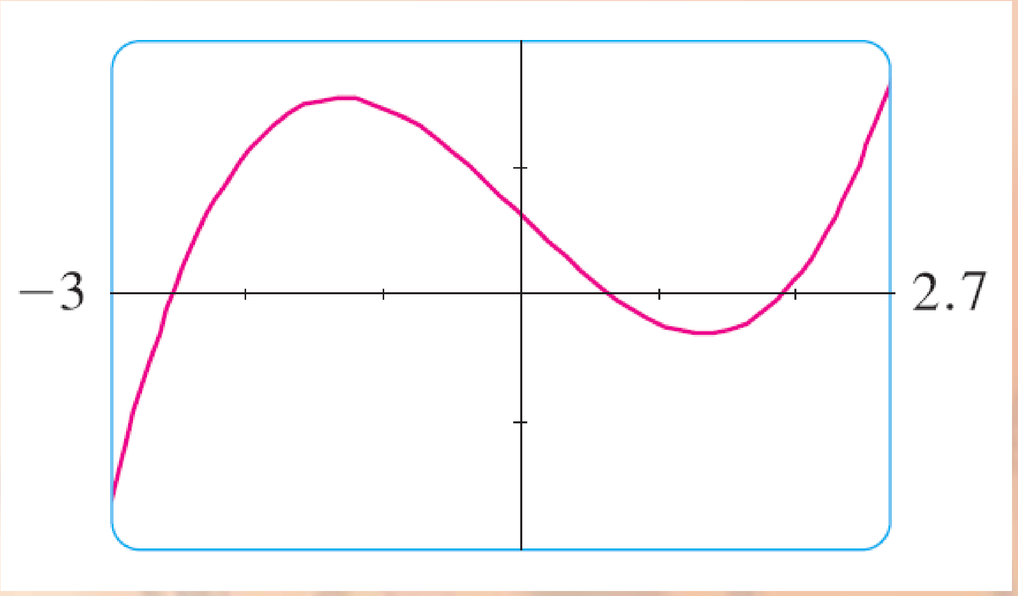
$$4y^3 - 21y + 12.5 = 0$$

 Example 4

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 – 21y + 12.5$$

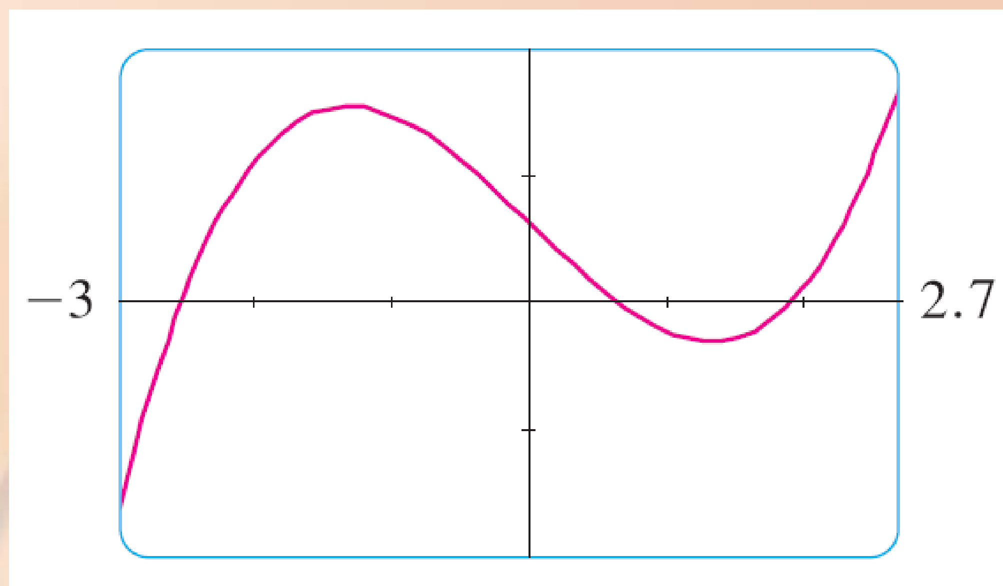
we see Equation 7 has three real roots.



Zooming in, we can find the roots to four decimal places:

$y \approx -2.5452$ $y \approx 0.6468$ $y \approx 1.8984$

- Alternatively, we could have used Newton's method or a rootfinder to locate these roots.



From Equation 6, the corresponding x-values are given by:

$$x = \pm\sqrt{5y - 2.5}$$

- If $y \approx -2.5452$, x has no corresponding real values.
- If $y \approx 0.6468$, $x \approx \pm\, 0.8567$
- If $y \approx 1.8984$, $x \approx \pm\, 2.6442$

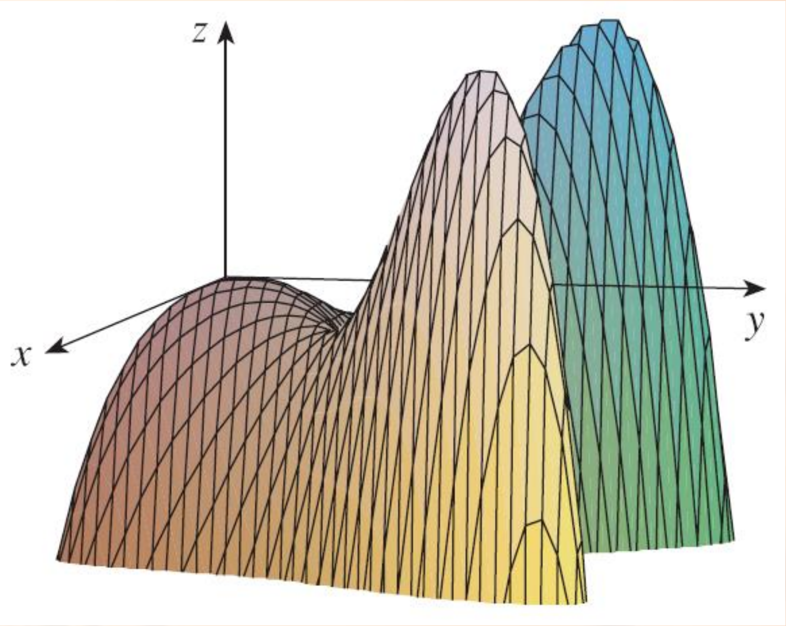
So, we have a total of five critical points, which are analyzed in the chart.

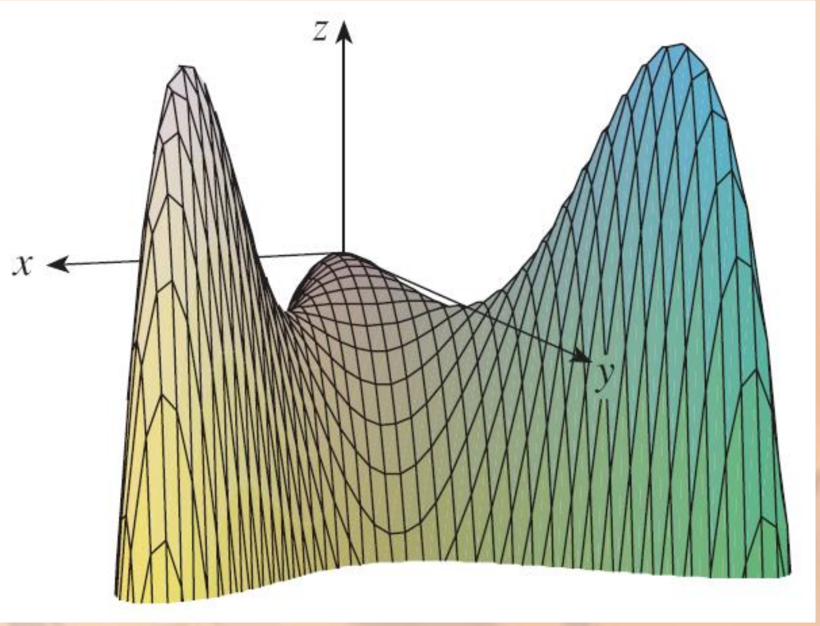
- All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	-10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	-55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	-1.48	-5.87	-187.64	saddle point

These figures give two views of the graph of *f*.

- We see that the surface opens downward.

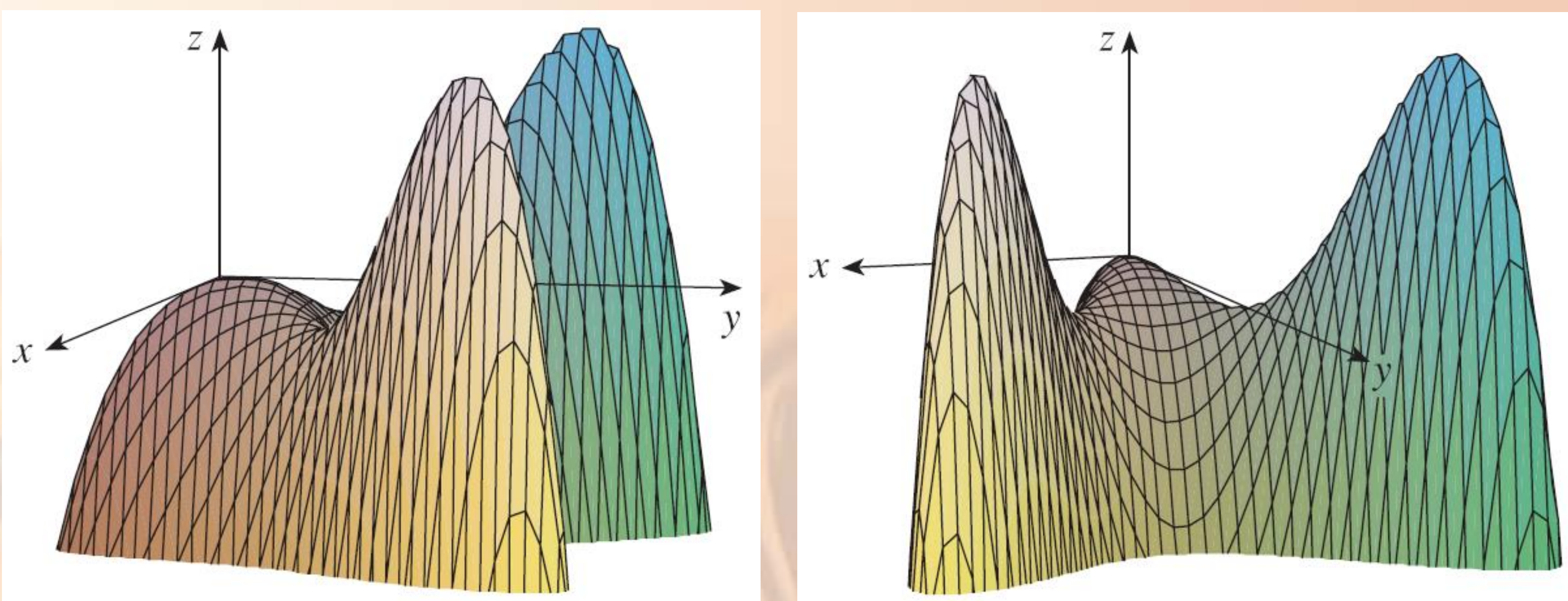




 Example 4

This can also be seen from the expression for $f(x, y)$:

- The dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.



 Example 4

Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is:

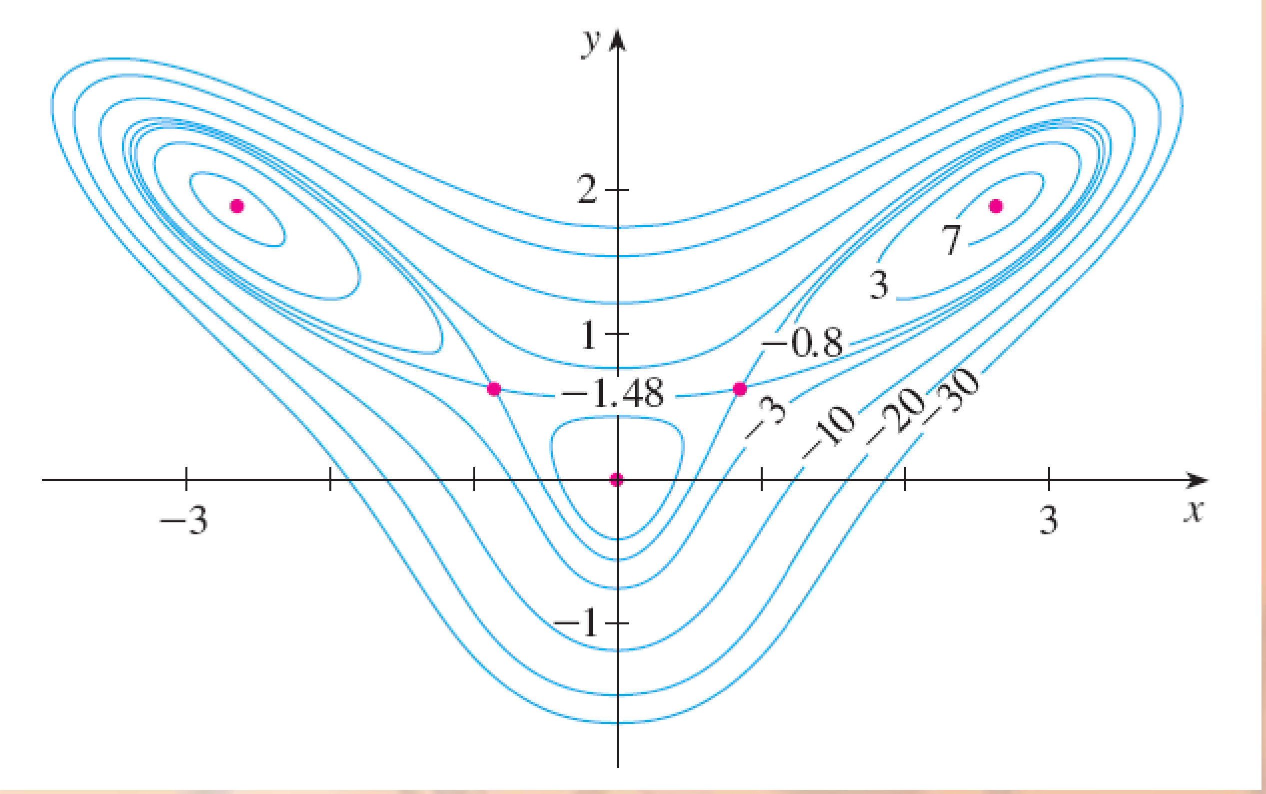
$$f(\pm 2.64, 1.90) \approx 8.50$$

- That is, the highest points on the graph of f are:

$$(\pm 2.64, 1.90, 8.50)$$

 Example 4

The five critical points of the function *f* in Example 4 are shown in red in this contour map of *f*.



Find the shortest distance from the point (1, 0, –2) to the plane $x + 2y + z = 4$.

- The distance from any point (x, y, z) to the point (1, 0, –2) is:

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

However, if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 – x – 2y$.

- Thus, we have:

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

 Example 5

We can minimize *d* by minimizing the simpler expression

$$\begin{aligned} d^2 &= f(x, y) \\ &= (x-1)^2 + y^2 + (6 - x - 2y)^2 \end{aligned}$$

By solving the equations

$$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$.

Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$,

we have:

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0 \text{ and } f_{xx} > 0$$

- So, by the Second Derivatives Test, f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$.

 Example 5

Intuitively, we can see that this local minimum is actually an absolute minimum:

- There must be a point on the given plane that is closest to (1, 0, –2).

If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$
$$= \sqrt{\left(\tfrac{5}{6}\right)^2 + \left(\tfrac{5}{3}\right)^2 + \left(\tfrac{5}{6}\right)^2} = \tfrac{5}{6}\sqrt{6}$$

- The shortest distance from (1, 0, –2) to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

Example 6

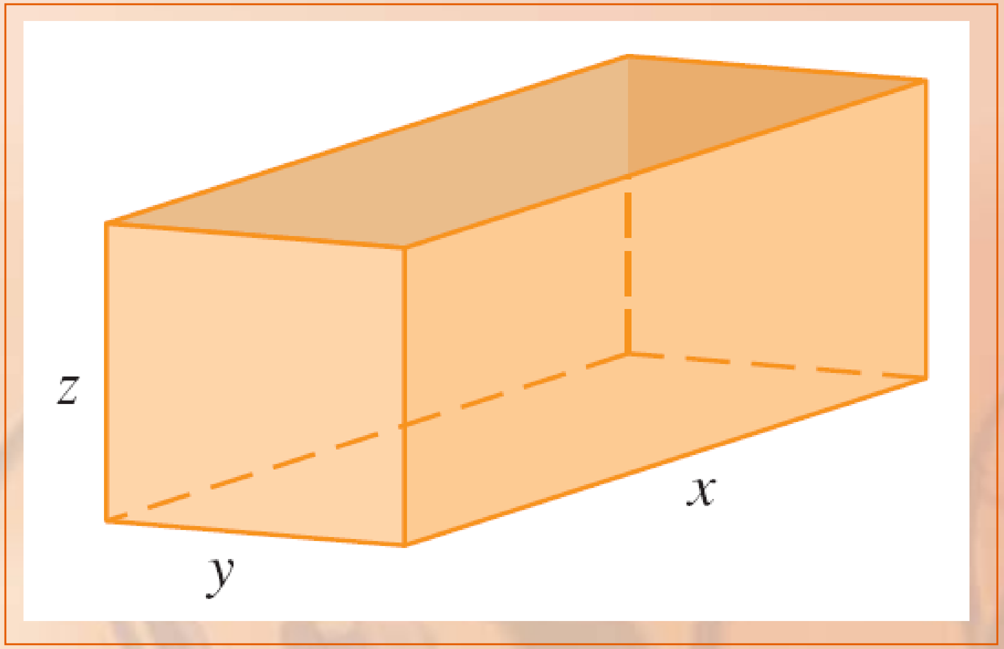
A rectangular box without a lid is to be made from 12 m^2 of cardboard.

Find the maximum volume of such a box.

 Example 6

Let the length, width, and height of the box (in meters) be x, y, and z.

- Then, its volume is: $V = xyz$



We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is:

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get:

$$z = (12 - xy)/[2(x + y)]$$

- So, the expression for V becomes:

$$V = xy\frac{12 - xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2}$$

$$\frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$$

Example 6

If V is a maximum, then

$$\partial V/\partial x = \partial V/\partial y = 0$$

- However, $x = 0$ or $y = 0$ gives $V = 0$.

So, we must solve:

$$12 - 2xy - x^2 = 0$$

$$12 - 2xy - y^2 = 0$$

- These imply that $x^2 = y^2$ and so $x = y$.
- Note that x and y must both be positive in this problem.

If we put $x = y$ in either equation, we get:

$$12 - 3x^2 = 0$$

- This gives:

$$x = 2$$
$$y = 2$$
$$z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$$

 Example 6

We could use the Second Derivatives Test to show that this gives a local maximum of V.

 Example 6

Alternatively, we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of *V*.

- So, it must occur when $x = 2$, $y = 2$, $z = 1$.

 Example 6

Then,

$$V = 2 \cdot 2 \cdot 1$$
$$= 4$$

- Thus, the maximum volume of the box is 4 m^3.

For a function f of one variable,

the Extreme Value Theorem says that:

- If f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value.

ABSOLUTE MAXIMUM & MINIMUM VALUES

According to the Closed Interval Method in Section 4.1, we found these by evaluating *f* at both:

- The critical numbers
- The endpoints *a* and *b*

CLOSED SET

There is a similar situation for functions of two variables.

Just as a closed interval contains its endpoints, a closed set in $\mathbb{R}^2$ is one that contains all its boundary points.

BOUNDARY POINT

A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.

For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$ is a closed set because:

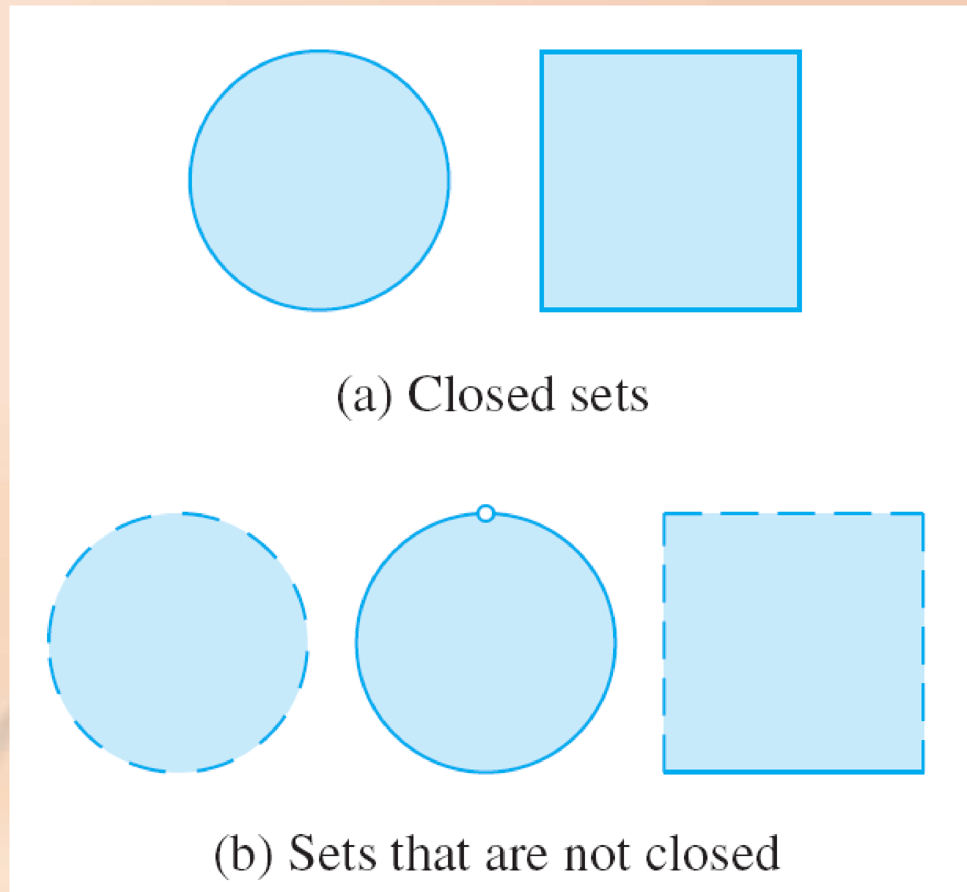
- It contains all of its boundary points (the points on the circle $x^2 + y^2 = 1$).

NON-CLOSED SETS

However, if even one point on the boundary curve were omitted,

the set would not

be closed.



BOUNDED SET

A bounded set in Υ^2 is one that is contained within some disk.

- In other words, it is finite in extent.

CLOSED & BOUNDED SETS

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem (EVT) for functions of two variables in two dimensions.

EVT (TWO-VARIABLE FUNCTIONS) **Theorem 8**

If f is continuous on a closed, bounded set D in Υ^2, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

EVT (TWO-VARIABLE FUNCTIONS)

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1), then (x_1, y_1) is either:

- A critical point of f
- A boundary point of D

EVT (TWO-VARIABLE FUNCTIONS)

Thus, we have the following extension of the Closed Interval Method.

To find the absolute maximum and minimum values of a continuous function *f* on a closed, bounded set *D*:

1. Find the values of *f* at the critical points of *f* in *D*.
2. Find the extreme values of *f* on the boundary of *D*.
3. The largest value from steps 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle

$$D = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$$

Example 7

As f is a polynomial, it is continuous on the closed, bounded rectangle D.

- So, Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

Example 7

According to step 1 in Method 9, we first find the critical points.

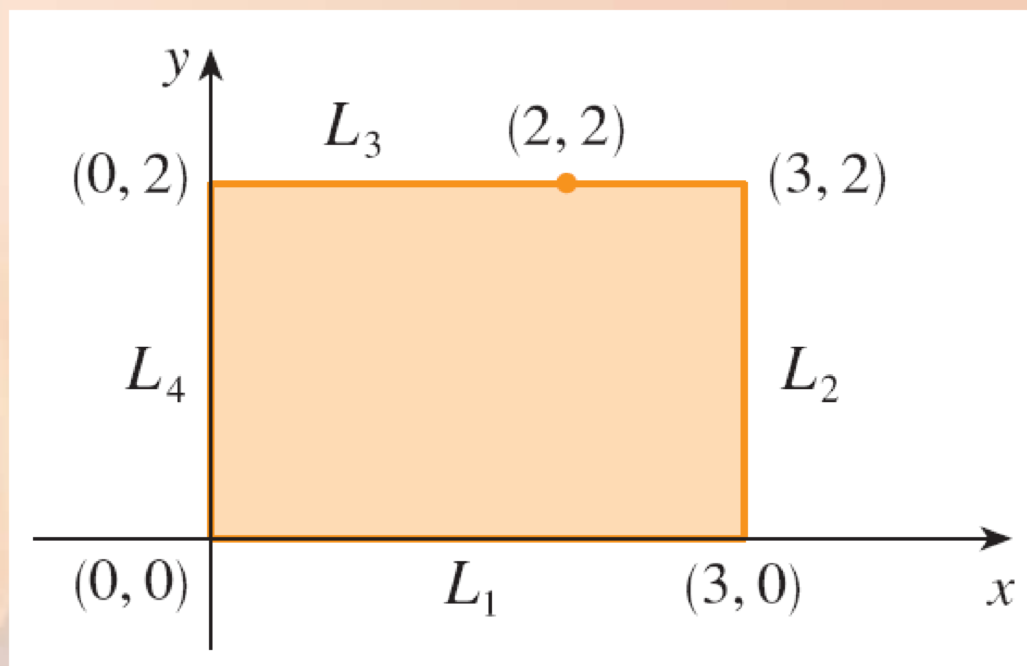
- These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

- So, the only critical point is (1, 1).
- The value of f there is $f(1, 1) = 1$.

In step 2, we look at the values of *f* on the boundary of *D*.

- This consists of the four line segments L_1, L_2, L_3, L_4



 Example 7

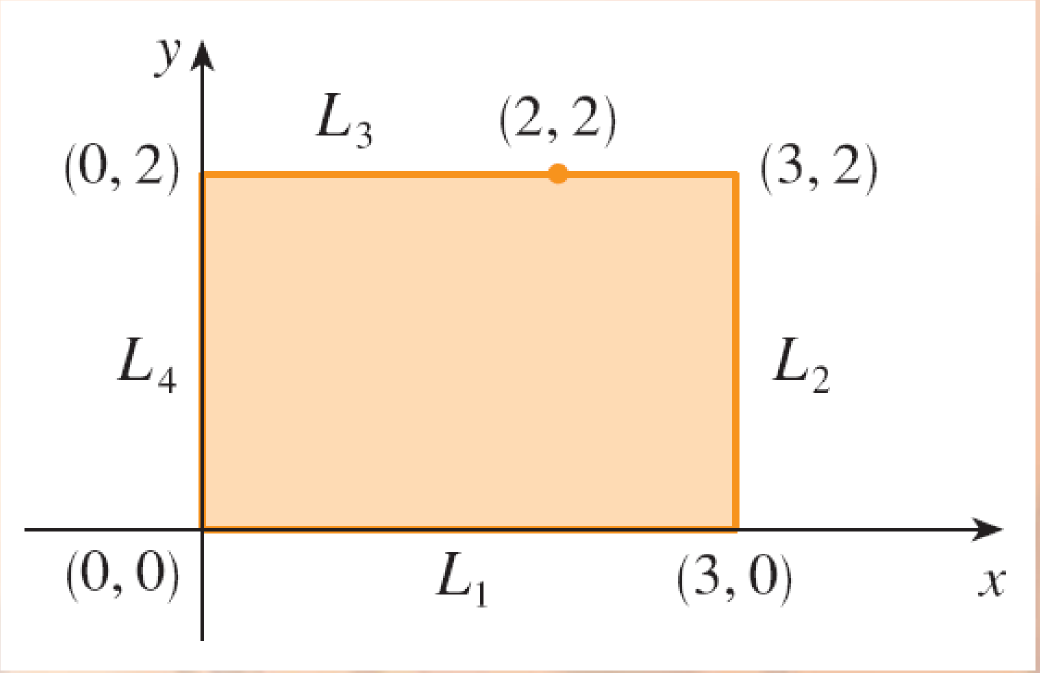
On L_1, we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leq x \leq 3$$

This is an increasing function of x.

So,

- Its minimum value is:
  $f(0, 0) = 0$
- Its maximum value is:
  $f(3, 0) = 9$



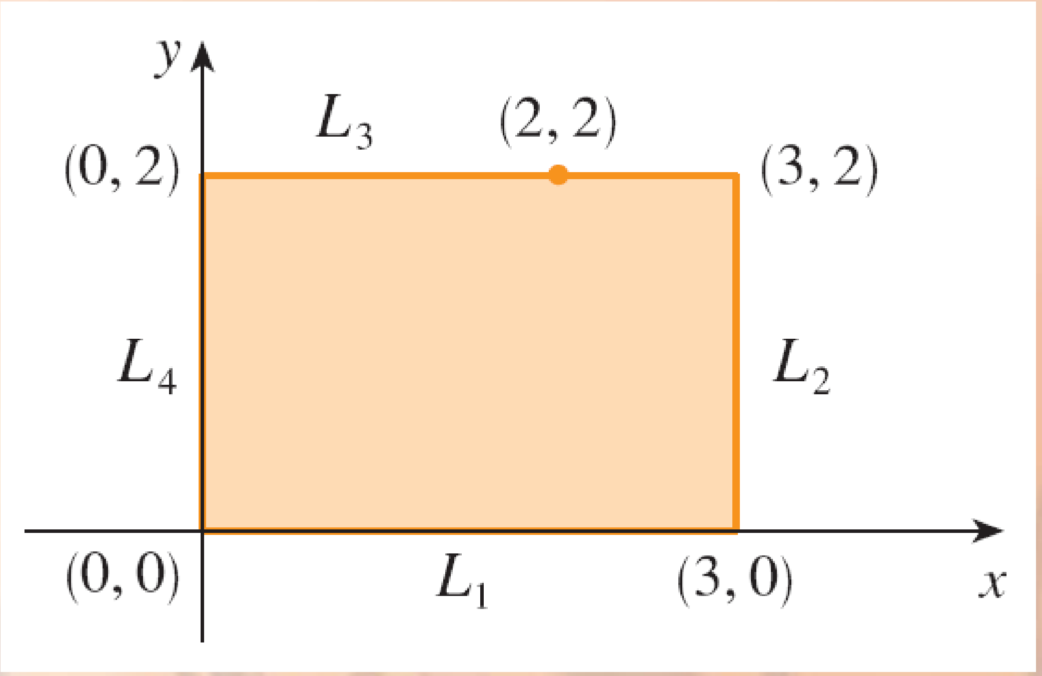
On L_2, we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \le y \le 2$$

This is a decreasing function of y.

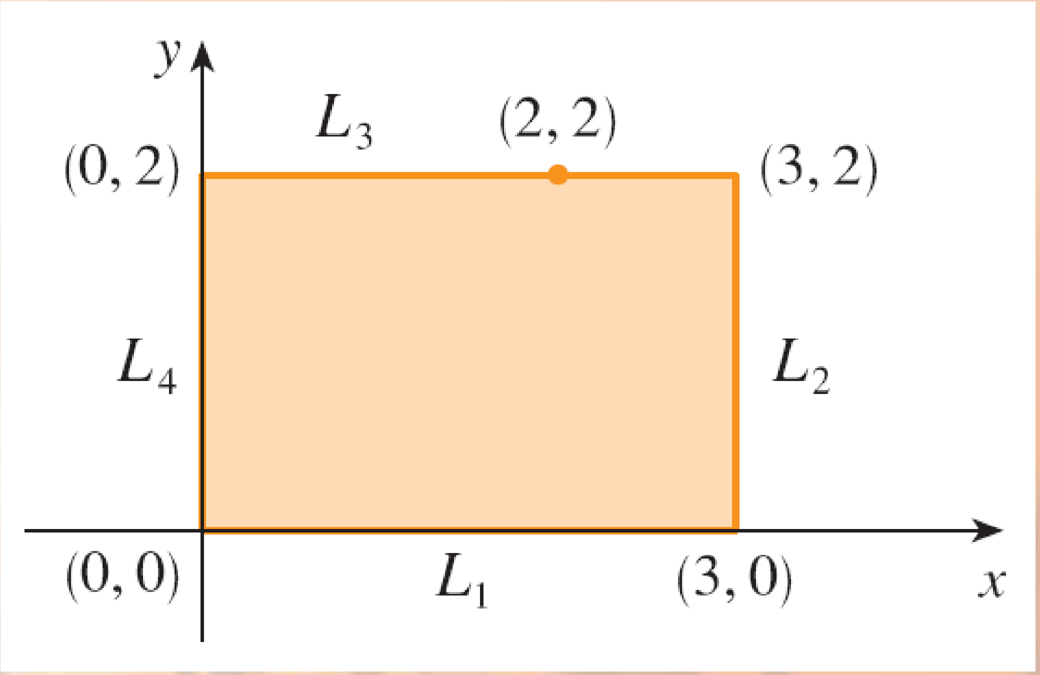
So,

- Its maximum value is:
  $f(3, 0) = 9$
- Its minimum value is:
  $f(3, 2) = 1$



On L_3, we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \le x \le 3$$



By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that:

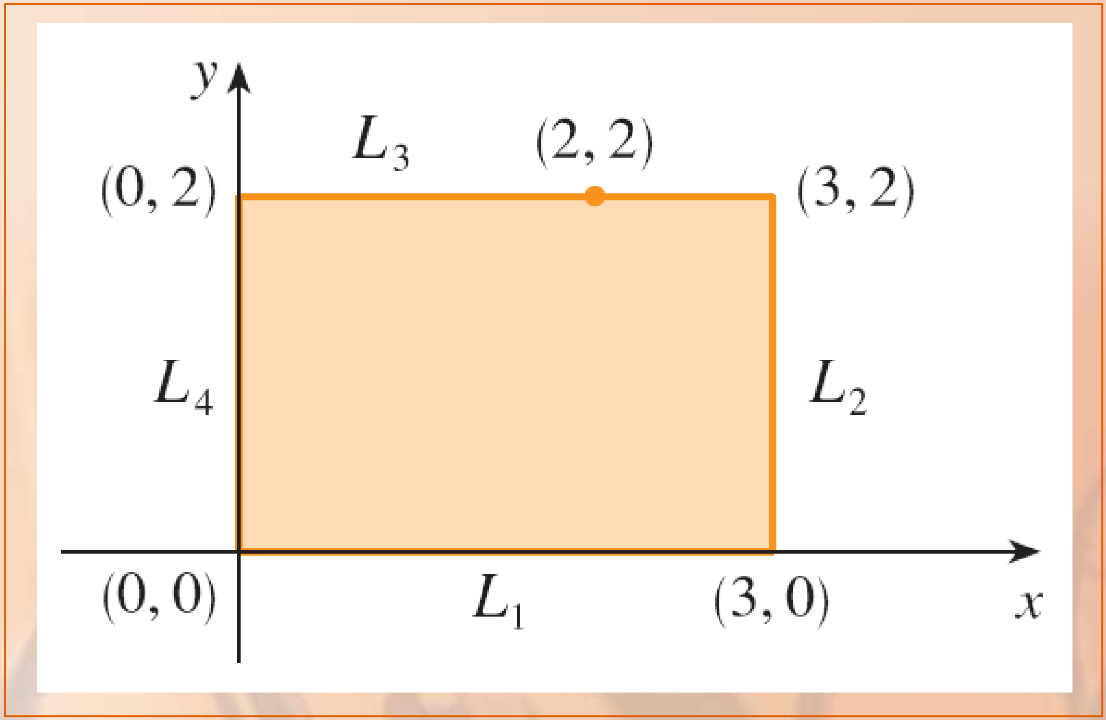
- The minimum value of the function is $f(2, 2) = 0$.
- The maximum value of the function is $f(0, 2) = 4$.

Finally, on L_4, we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \le y \le 2$$

with:

- Maximum value $f(0, 2) = 4$
- Minimum value $f(0, 0) = 0$



Example 7

Thus, on the boundary,

- The minimum value of f is 0.
- The maximum value of f is 9.

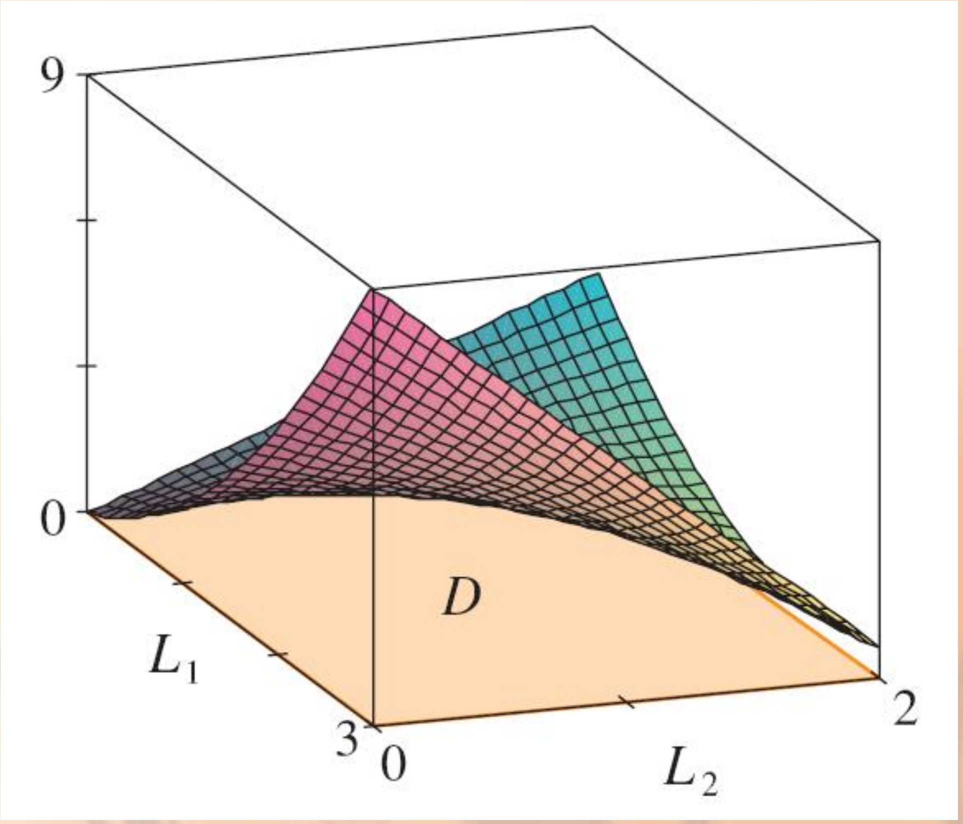
In step 3, we compare these values with the value $f(1, 1) = 1$ at the critical point.

We conclude that:

- The absolute maximum value of f on D is $f(3, 0) = 9$.
- The absolute minimum value of f on D is $f(0, 0) = f(2, 2) = 0$.

Example 7

This figure shows the graph of *f*.



SECOND DERIVATIVES TEST

We close this section by giving a proof of the first part of the Second Derivatives Test.

- The second part has a similar proof.

SECOND DERIVS. TEST (PART A) **Proof**

We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$.

The first-order derivative is given by Theorem 3 in Section 14.6:

$$D_{\mathbf{u}}f = f_x h + f_y k$$

Applying the theorem a second time, we have:

$$
\begin{aligned}
D_{\mathbf{u}}^2 f &= D_{\mathbf{u}}(D_{\mathbf{u}} f) \\
&= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\
&= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\
&= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2
\end{aligned}
$$

(Clairaut's Theorem)

Proof—Equation 10

If we complete the square in that expression, we obtain:

$$D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}^2\right)$$

SECOND DERIVS. TEST (PART A) **Proof**

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$.

However, f_{xx} and $D = f_{xx} f_{yy} - f_{xy}^2$ are continuous functions.

- So, there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B.

Therefore, by looking at Equation 10, we see that

$$D_{\mathbf{u}}^2 f(x, y) > 0$$

whenever (x, y) is in B.

This means that:

- If C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of **u**, then C is concave upward on an interval of length 2δ.

This is true in the direction of every vector **u**.

- So, if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P.

Thus, $f(x, y) \geq f(a, b)$ whenever (x, y) is in B.

- This shows that $f(a, b)$ is a local minimum.