14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

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One of the most important ideas in single-variable calculus is:

- As we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line.
- We can then approximate the function by a linear function.

Here, we develop similar ideas in three dimensions.

- As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane).
- We can then approximate the function by a linear function of two variables.

We also extend the idea of a differential to functions of two or more variables.

14.4

Tangent Planes and Linear Approximations

In this section, we will learn how to:

Approximate functions using tangent planes and linear functions.

TANGENT PLANES

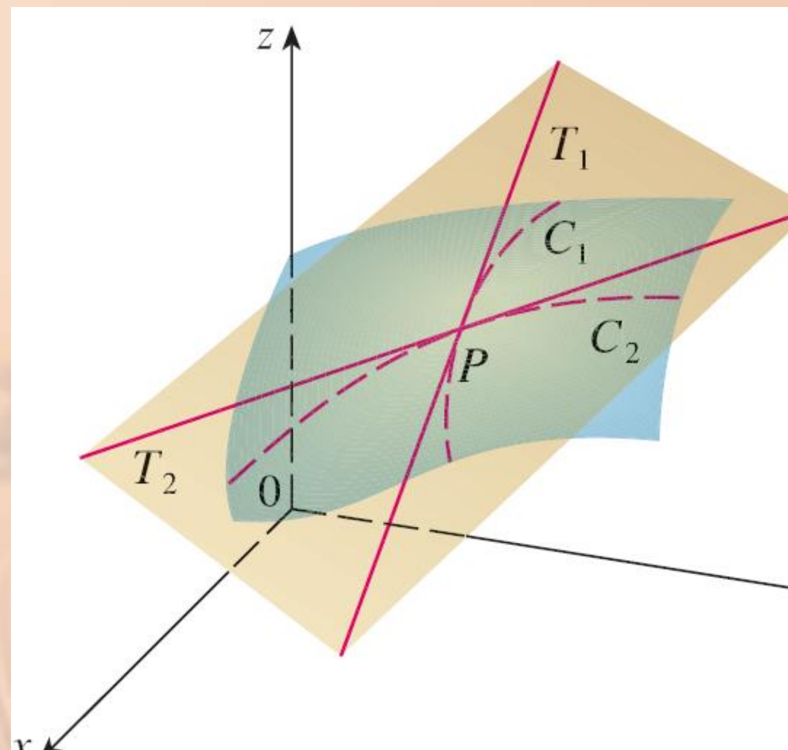
Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives.

Let $P(x_0, y_0, z_0)$ be a point on S.

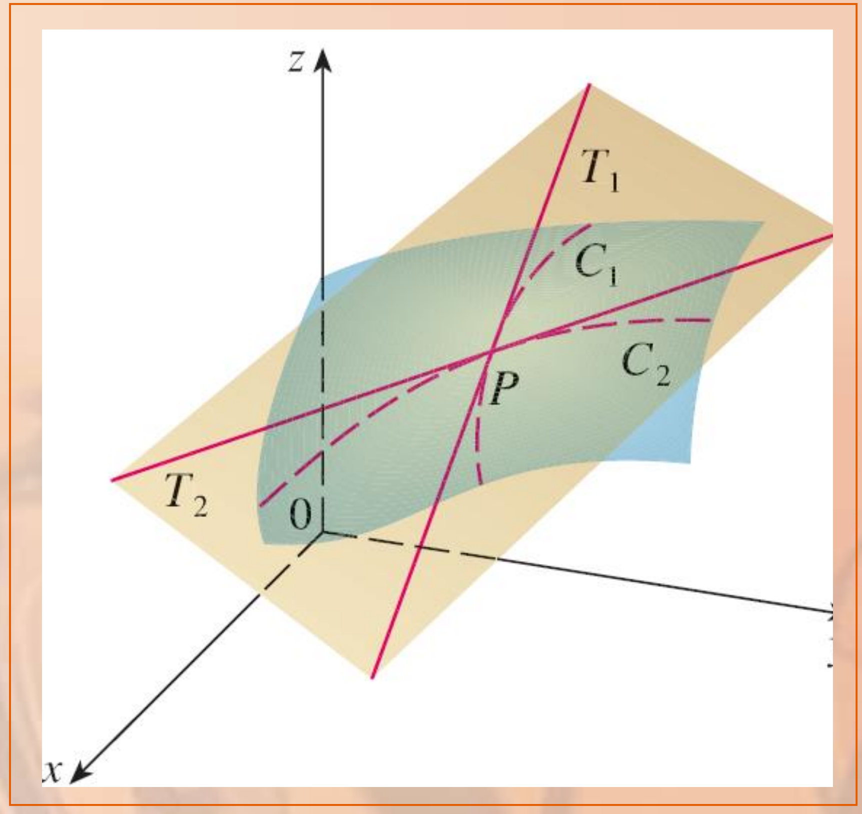
TANGENT PLANES

As in Section 14.3, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.

- Then, the point P lies on both C_1 and C_2.

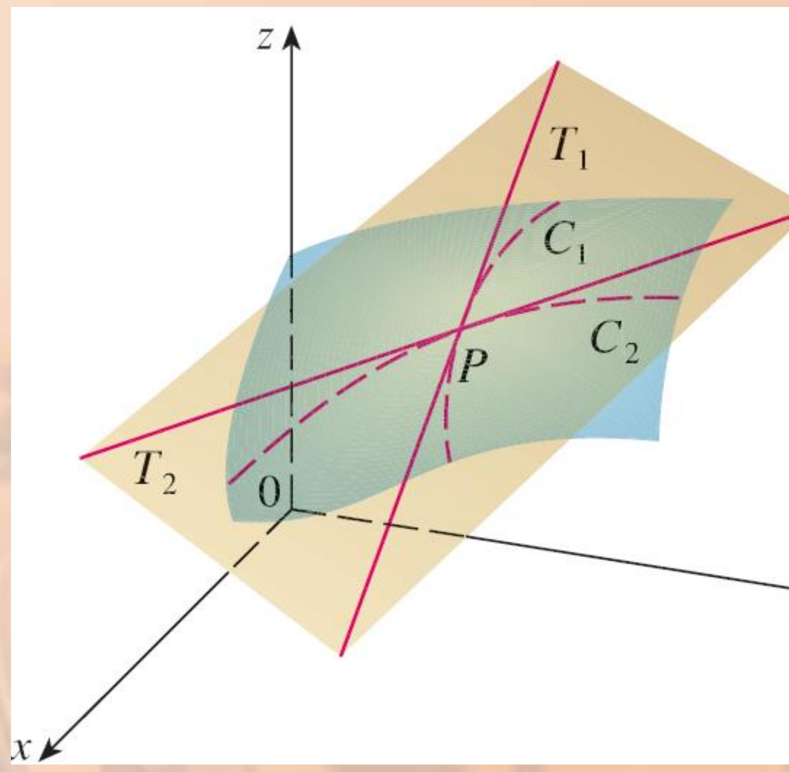


Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.



TANGENT PLANE

Then, the tangent plane to the surface *S* at the point *P* is defined to be the plane that contains both tangent lines T_1 and T_2.



TANGENT PLANES

We will see in Section 14.6 that, if *C* is any other curve that lies on the surface *S* and passes through *P*, then its tangent line at *P* also lies in the tangent plane.

TANGENT PLANES

Therefore, you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P.

- The tangent plane at P is the plane that most closely approximates the surface S near the point P.

TANGENT PLANES

We know from Equation 7 in Section 12.5 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Equation 1

By dividing that equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

TANGENT PLANES

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1.

Setting $y = y_0$ in Equation 1 gives:

$$z - z_0 = a(x - x_0)$$
$$y = y_0$$

- We recognize these as the equations (in point-slope form) of a line with slope a.

However, from Section 14.3, we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

- Therefore, $a = f_x(x_0, y_0)$.

TANGENT PLANES

Similarly, putting $x = x_0$ in Equation 1, we get:

$$z - z_0 = b(y - y_0)$$

This must represent the tangent line T_2.

- Thus, $b = f_y(x_0, y_0)$.

Equation 2

Suppose f has continuous partial derivatives.

An equation of the tangent plane to the surface z = $f(x, y)$ at the point $P(x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

- Let $f(x, y) = 2x^2 + y^2$.
- Then,

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

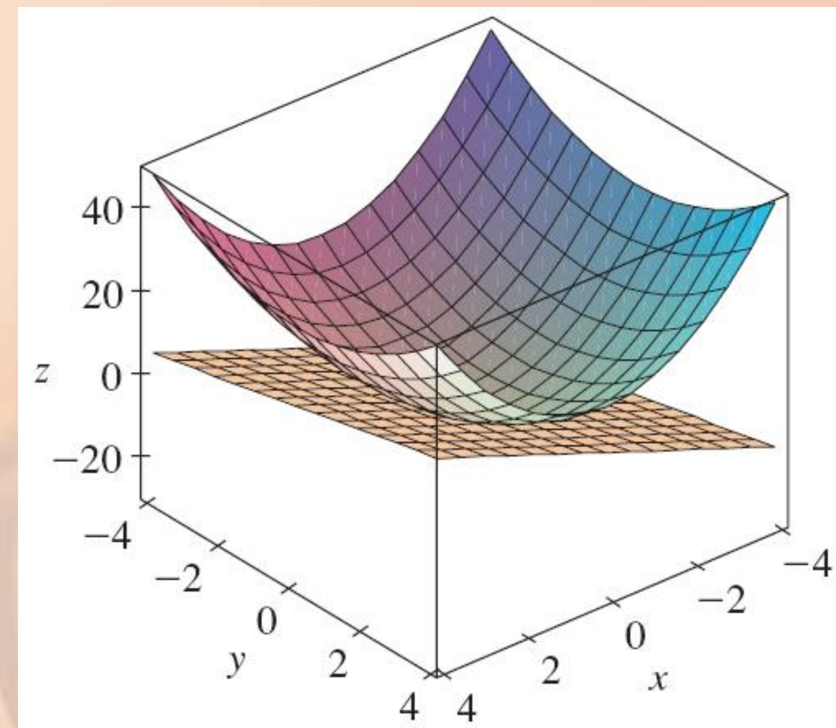
- So, Equation 2 gives the equation of the tangent plane at (1, 1, 3) as:

$$z - 3 = 4(x - 1) + 2(y - 1)$$

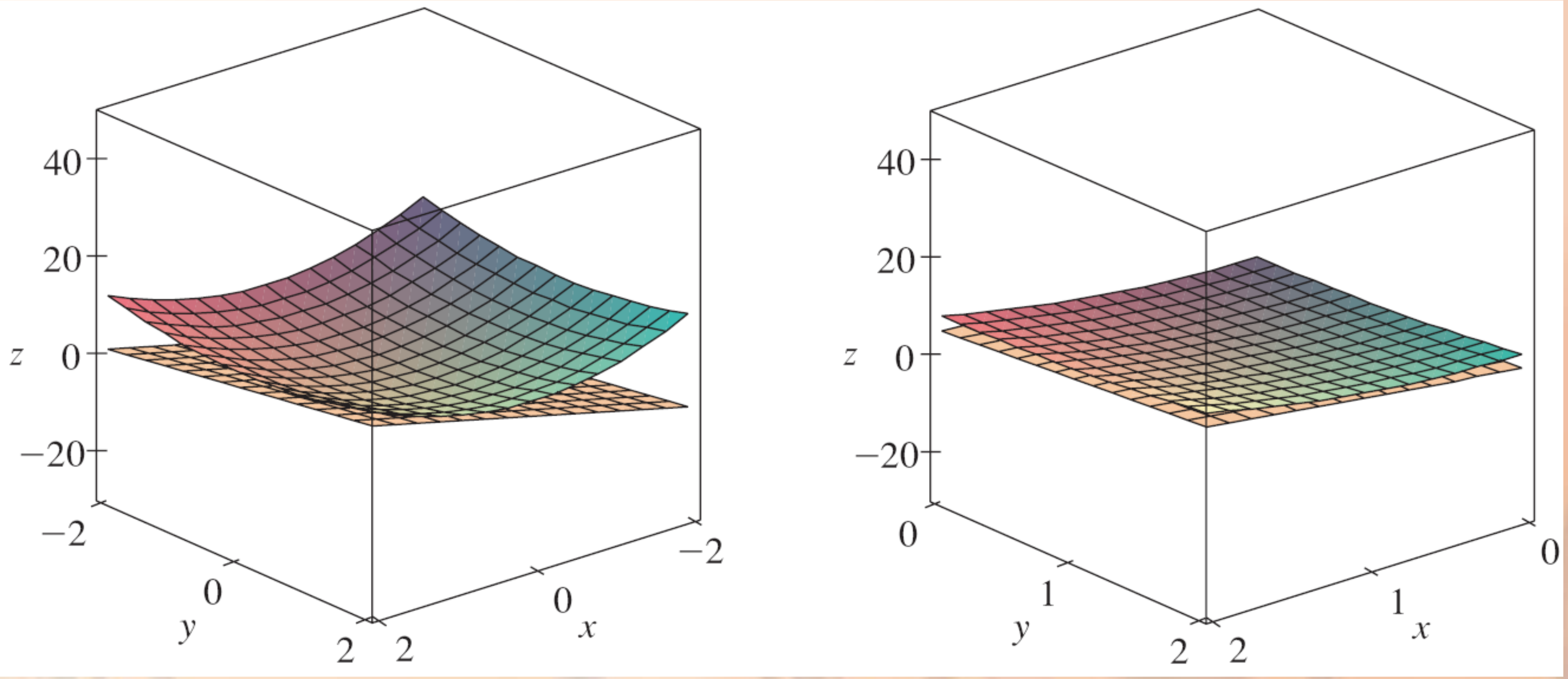
or

$$z = 4x + 2y - 3$$

The figure shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1.

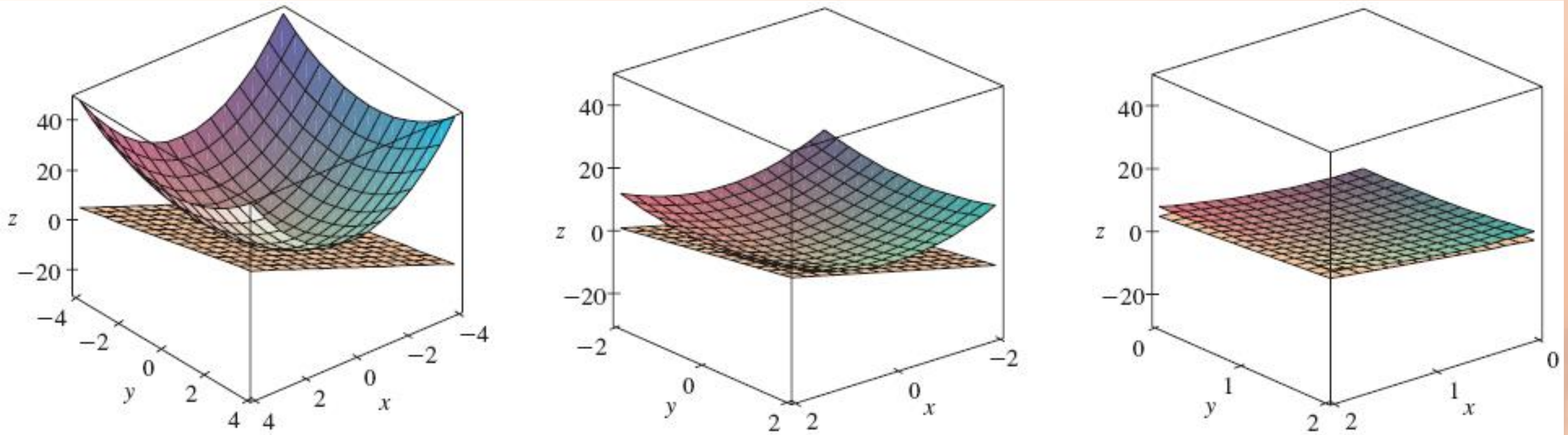


Here, we zoom in toward the point by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.



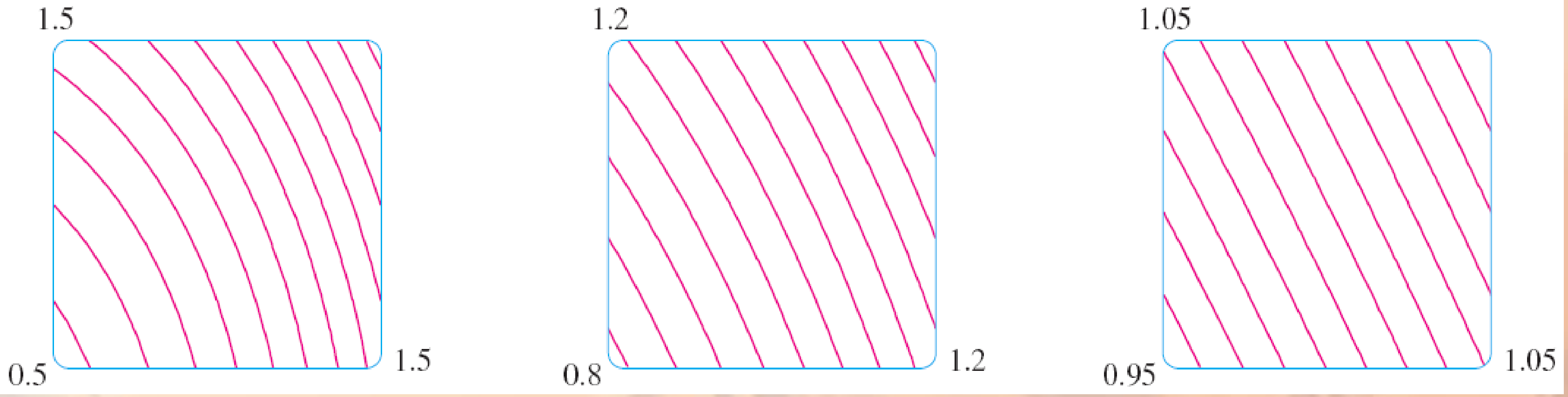
Notice that, the more we zoom in,

- The flatter the graph appears.
- The more it resembles its tangent plane.



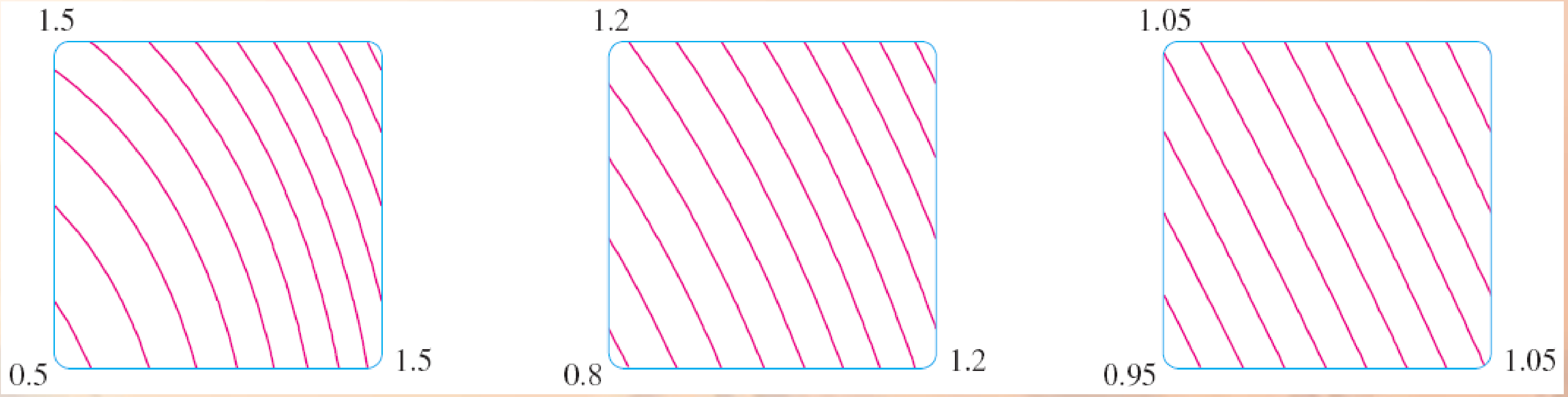
TANGENT PLANES

Here, we corroborate that impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$.



TANGENT PLANES

Notice that, the more we zoom in, the more the level curves look like equally spaced parallel lines—characteristic of a plane.



In Example 1, we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is:

$$z = 4x + 2y - 3$$

LINEAR APPROXIMATIONS

Thus, in view of the visual evidence in the previous two figures, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1).

LINEARIZATION & LINEAR APPROXIMATION

The function L is called the linearization of f at (1, 1).

The approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the linear approximation or tangent plane approximation of f at (1, 1).

LINEAR APPROXIMATIONS

For instance, at the point (1.1, 0.95), the linear approximation gives:

$$f(1.1, 0.95)$$
$$\approx 4(1.1) + 2(0.95) - 3$$
$$= 3.3$$

- This is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$

However, if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

- In fact, $L(2, 3) = 11$, whereas $f(2, 3) = 17$.

In general, we know from Equation 2 that an equation of the tangent plane to the graph of a function *f* of two variables at the point $(a, b, f(a, b))$ is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

LINEARIZATION

Equation 3

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b).

LINEAR APPROXIMATION **Equation 4**

The approximation

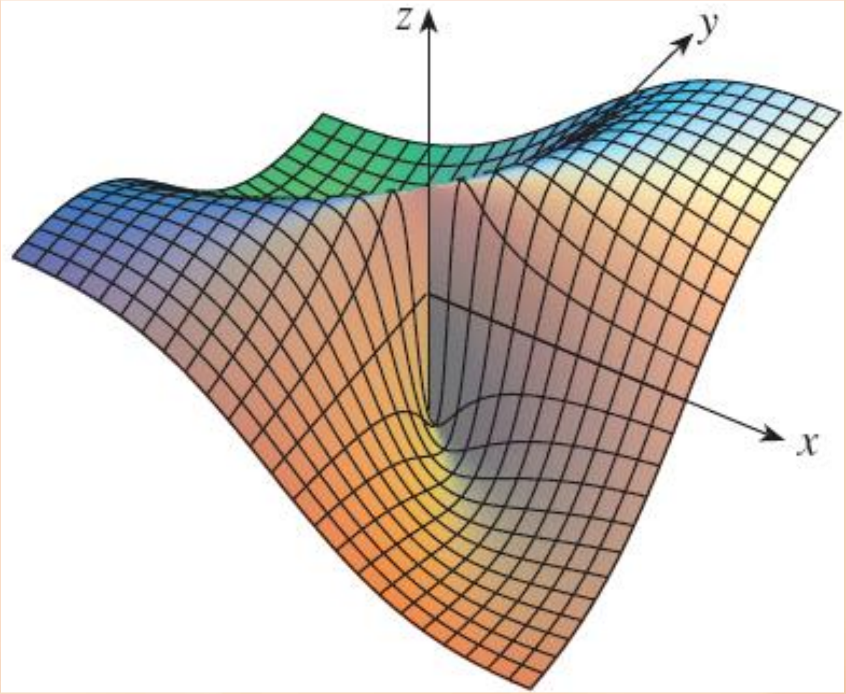
$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent plane approximation of f at (a, b).

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives.

- What happens if f_x and f_y are not continuous?

The figure pictures such a function.



- Its equation is:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

You can verify (see Exercise 46) that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$.

However, f_x and f_y are not continuous.

Thus, the linear approximation would be $f(x, y) \approx 0$.

- However, $f(x, y) = ½$ at all points on the line $y = x$.

Thus, a function of two variables can behave badly even though both of its partial derivatives exist.

- To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that, for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as:

$$\Delta y = f(a + \Delta x) - f(a)$$

Equation 5

In Chapter 3 we showed that,

if f is differentiable at a, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$

LINEAR APPROXIMATIONS

Now, consider a function of two variables, $z = f(x, y)$.

Suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta x$.

Equation 6

Then, the corresponding increment of z is:

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus, the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

- By analogy with Equation 5, we define the differentiability of a function of two variables as follows.

Definition 7

If z = $f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation in Equation 4 is a good approximation when (x, y) is near (a, b).

- That is, the tangent plane approximates the graph of f well near the point of tangency.

It’s sometimes hard to use Definition 7 directly to check the differentiability of a function.

- However, the next theorem provides a convenient sufficient condition for differentiability.

Theorem 8

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there.

Then, use it to approximate $f(1.1, –0.1)$.

The partial derivatives are:

$f_x(x, y) = e^{xy} + xye^{xy}$ $f_y(x, y) = x^2e^{xy}$

$f_x(1, 0) = 1$ $f_y(1, 0) = 1$

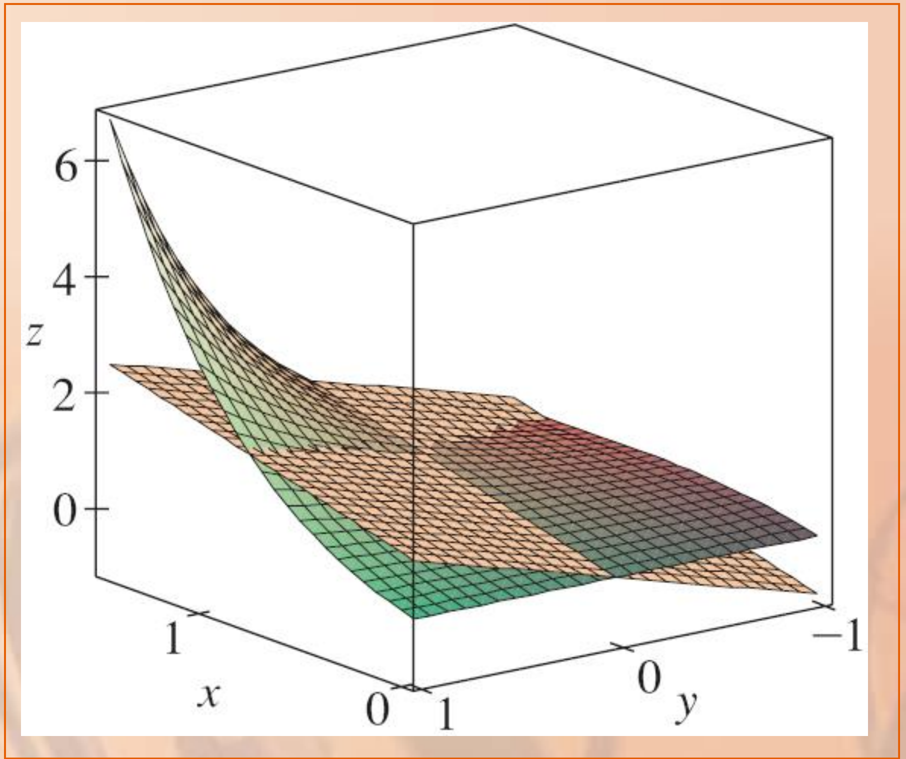
- Both f_x and f_y are continuous functions.
- So, f is differentiable by Theorem 8.

The linearization is:

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y$$

$$= x + y$$



The corresponding linear approximation is:

$$xe^{xy} \approx x + y$$

So,

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

- Compare this with the actual value of

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$$

At the beginning of Section 14.3, we discussed the heat index (perceived temperature) I as a function of:

- The actual temperature T
- The relative humidity H

We gave this table of values from the National Weather Service.

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96°F and H is near 70%.

Use it to estimate the heat index when the temperature is 97°F and the relative humidity is 72%.

 Example 3

We read that $f(96, 70) = 125$.

- In Section 14.3, we used the tabular values to estimate that: $f_T(96, 70) \approx 3.75$ and $f_H(96, 70) \approx 0.9$

Relative humidity (%)

Actual temperature (°F)	T \ H	50	55	60	65	70	75	80	85	90
	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

So, the linear approximation is:

$$f(T, H) \approx f(96, 70) + f_T(96, 70)(T - 96) + f_H(96, 70)(H - 70)$$

$$\approx 125 + 3.75(T - 96) + 0.9(H - 70)$$

In particular,

$$f(92, 72) \approx 125 + 3.75(1) + 0.9(2)$$
$$= 130.55$$

- Thus, when T = 97°F and H = 72%, the heat index is:

$$I \approx 131°\text{F}$$

DIFFERENTIALS

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable.

- That is, dx can be given the value of any real number.

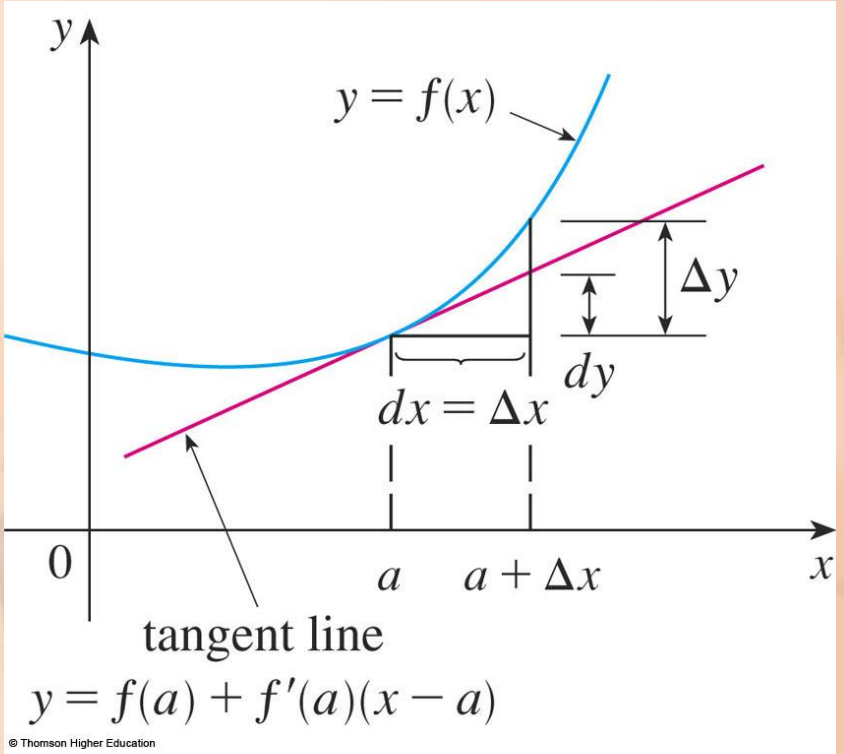
Then, the differential of y is defined as:

$$dy = f'(x)\, dx$$

- See Section 3.10

DIFFERENTIALS

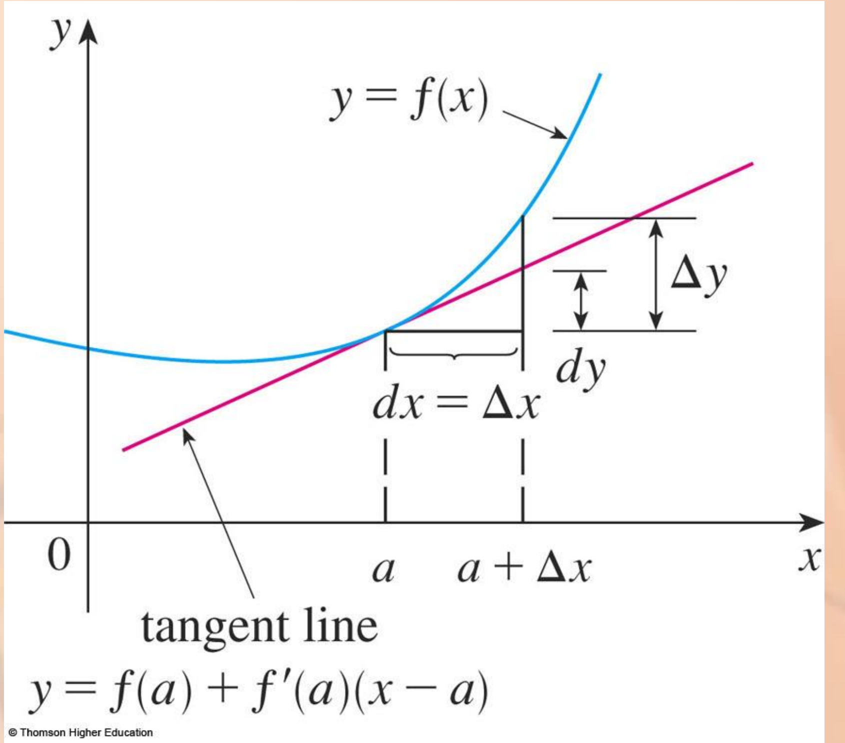
The figure shows the relationship between the increment Δy and the differential *dy*.



DIFFERENTIALS

Δy represents the change in height of the curve $y = f(x)$.

dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.



DIFFERENTIALS

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables.

- That is, they can be given any values.

TOTAL DIFFERENTIAL

Equation 10

Then the differential *dz*, also called the total differential, is defined by:

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

- Compare with Equation 9.
- Sometimes, the notation *df* is used in place of *dz*.

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is:

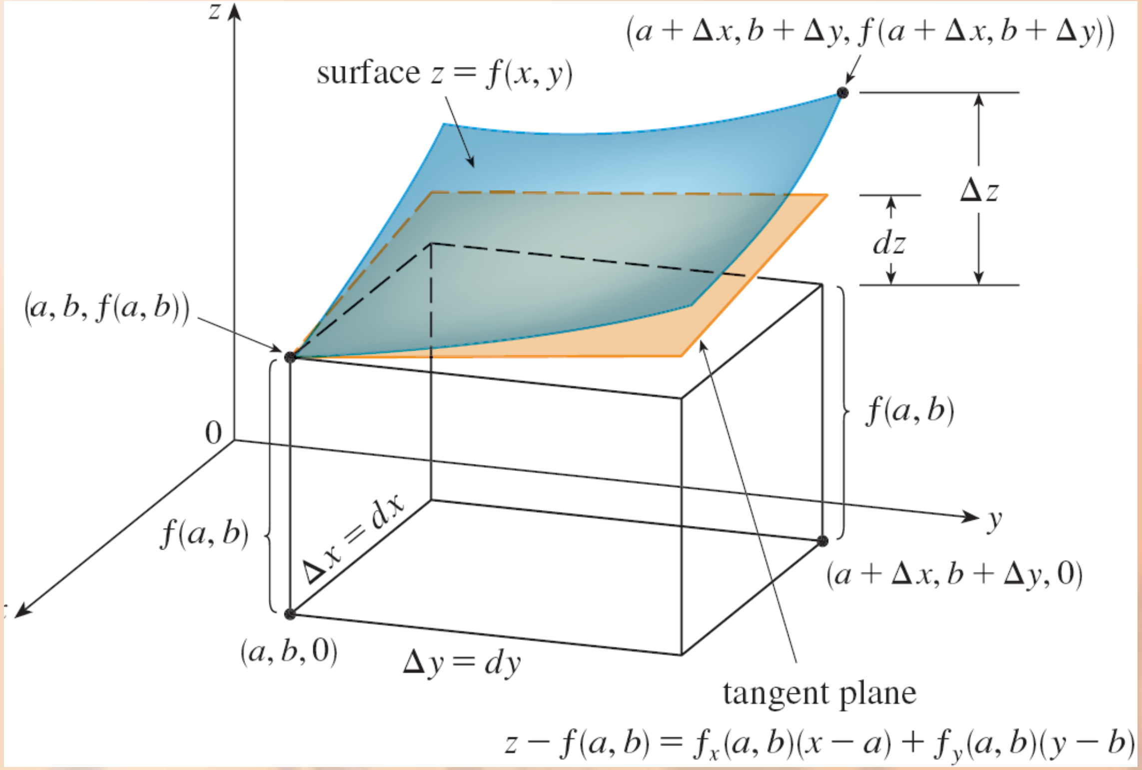
$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- So, in the notation of differentials, the linear approximation in Equation 4 can be written as:

$$f(x, y) \approx f(a, b) + dz$$

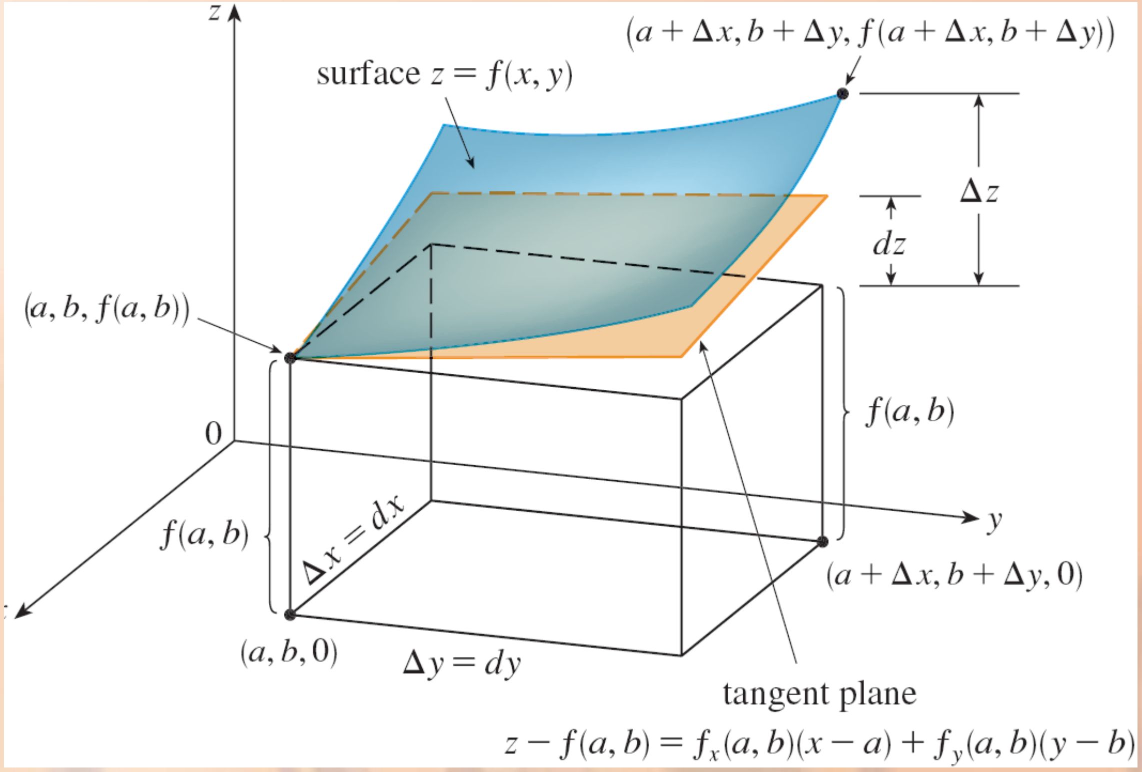
DIFFERENTIALS

The figure is the three-dimensional counterpart of the previous figure.



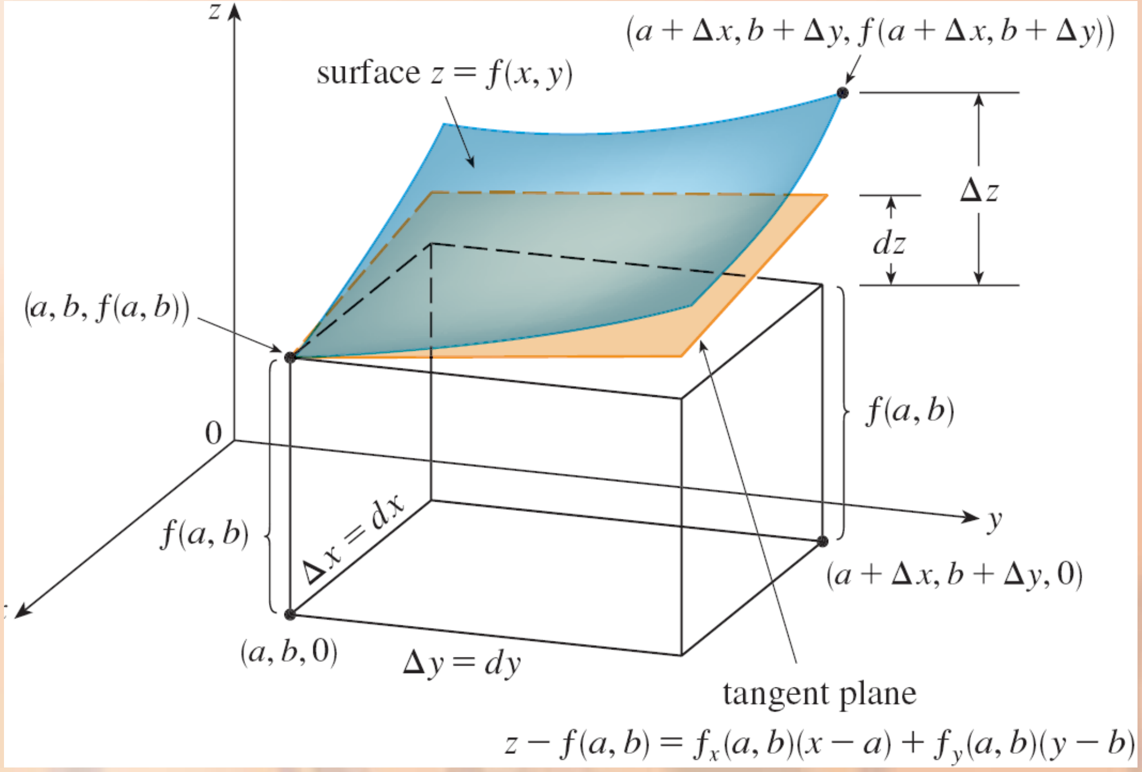
DIFFERENTIALS

It shows the geometric interpretation of the differential *dz* and the increment Δ*z*.

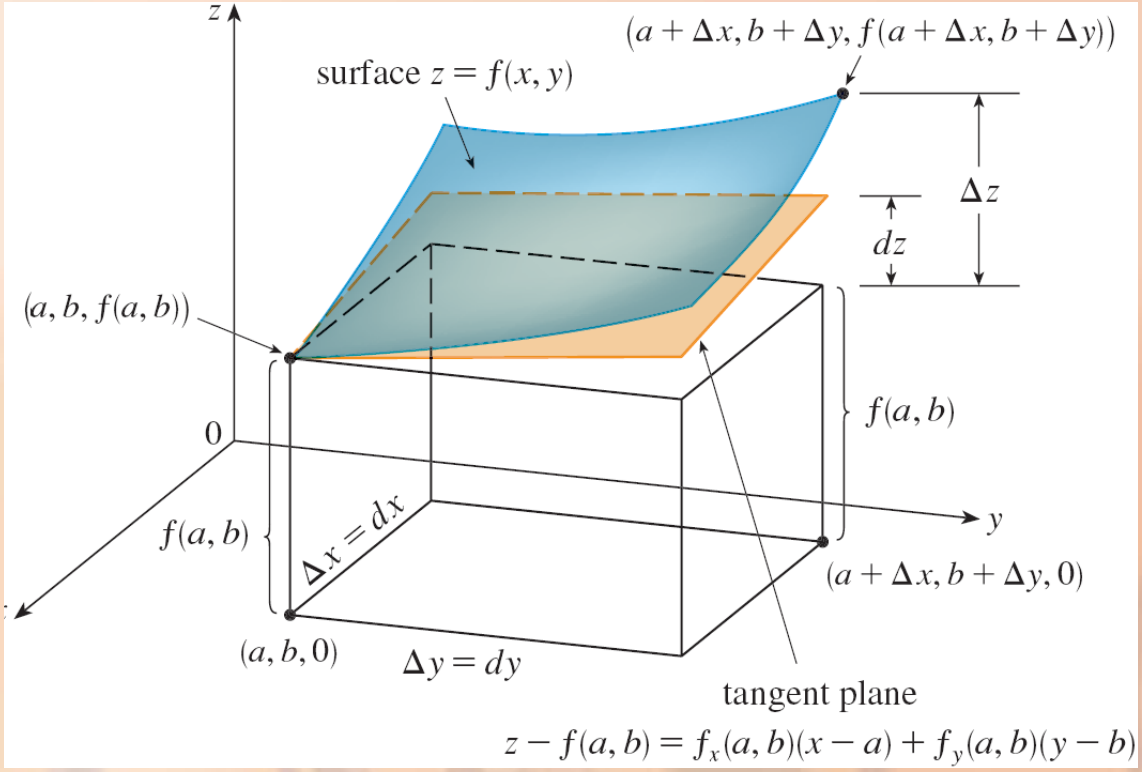


DIFFERENTIALS

dz is the change in height of the tangent plane.



Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.



Example 4

a. If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

b. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare Δz and dz.

Definition 10 gives:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= (2x + 3y)\, dx + (3x - 2y)\, dy$$

Putting

$x = 2$, $dx = \Delta x = 0.05$, $y = 3$, $dy = \Delta y = -0.04$,

we get:

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04)$$

$$= 0.65$$

The increment of z is:

$$\Delta z = f(2.05, 2.96) - f(2, 3)$$

$$= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2]$$

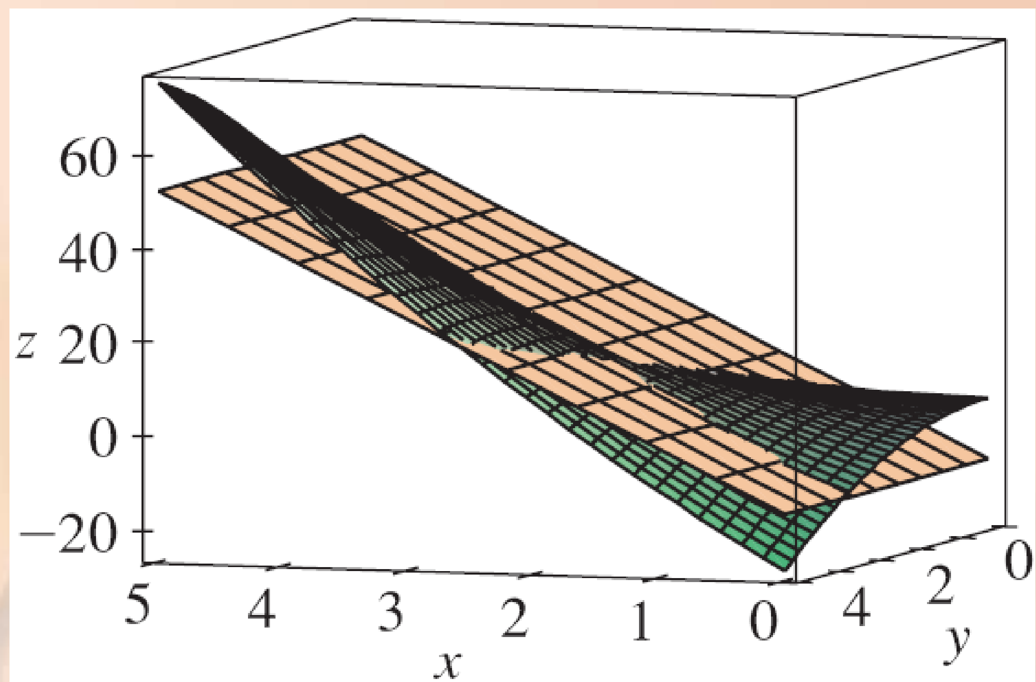
$$- [2^2 + 3(2)(3) - 3^2]$$

$$= 0.6449$$

- Notice that $\Delta z \approx dz$, but dz is easier to compute.

DIFFERENTIALS

In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near (2, 3, 13).



Example 5

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each.

- Use differentials to estimate the maximum error in the calculated volume of the cone.

The volume V of a cone with base radius r and height h is $V = \pi r^2 h/3$.

So, the differential of V is:

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh$$

Each error is at most 0.1 cm.

So, we have:

$$|\Delta r| \le 0.1$$

$$|\Delta h| \le 0.1$$

To find the largest error in the volume, we take the largest error in the measurement of r and of h.

- Therefore, we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$.

That gives:

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1)$$

$$= 20\pi$$

- So, the maximum error in the calculated volume is about 20π cm^3 ≈ 63 cm^3.

FUNCTIONS OF THREE OR MORE VARIABLES

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables.

A differentiable function is defined by an expression similar to the one in Definition 7.

- The linear approximation is:

  $$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

- The linearization $L(x, y, z)$ is the right side of this expression.

FUNCTIONS OF THREE OR MORE VARIABLES

If $w = f(x, y, z)$, the increment of w is:

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The differential *dw* is defined in terms of the differentials *dx*, *dy*, and *dz* of the independent variables by:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Example 6

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm.

- Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

If the dimensions of the box are x, y, and z, its volume is $V = xyz$.

Thus,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

We are given that

$$|\Delta x| \le 0.2,\ |\Delta y| \le 0.2,\ |\Delta z| \le 0.2$$

- To find the largest error in the volume, we use

$$dx = 0.2,\ dy = 0.2,\ dz = 0.2$$

together with

$$x = 75,\ y = 60,\ z = 40$$

Thus,

$$\begin{aligned}\Delta V \approx dV &= (60)(40)(0.2) + (75)(40)(0.2) \\ &\quad + (75)(60)(0.2) \\ &= 1980\end{aligned}$$

So, an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume.

- This may seem like a large error.
- However, it's only about 1% of the volume of the box.