Chapter Two
Application of Network Theorems to AC Networks

1. Introduction

When voltages, currents and impedances are treated as complex numbers or phasors, the solution of ac circuits becomes the same as that of dc circuits. In this chapter, we will be working with impedances and phasors instead of just resistors and real numbers.

2. Superposition Theorem

Example 1: Using the superposition theorem, find the current I through the 4-Ω reactance X_{L2} of Fig. 1

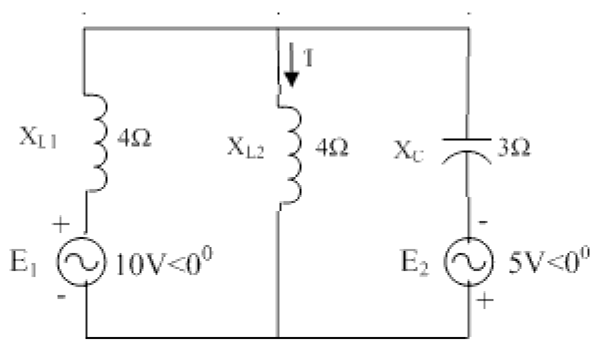


Figure 1 Network for Example 1

Solution:

<u>Voltage source E_1 acting alone:</u>

Effective impedance of parallel branch,

$$X_{L2} \,//\, X_C = \frac{(j4)(-j3)}{j4 - j3} = -j12$$

Total impedance = j4 – j12 = -j8 Ω = $8\Omega\angle -90^{\circ}$

Total current = $10\angle 0^{o} / 8\angle -90^{\circ} = 1.25\ \text{A}\angle 90^{\circ}$ and

I^{1} = voltage across parallel branch / $X_{L2} = \left(12\angle -90^{o} \times 1.25\angle 90^{o}\right)/\,4\angle 90^{o} = 3.76\,A\angle -90^{o}$

<u>Voltage source E_2 acting alone:</u>

Impedance of parallel branch, $X_{L1}//X_{L2}$ = j4//j4 = j2

Total Impedance = j2 – j3 = -j1 = $1\angle -90^{\circ}$

Total Current = $5/1\angle -90^{\circ} = 5\angle 90^{\circ}$

$I^{11} = 5\angle 90^{\circ}/2 = 2.5\angle 90^{\circ}$ because $X_{L1} = X_{L2}$

Actual Current, $I = I^{1} - I^{11} = 3.75\text{A}\angle -90 - 2.5\text{A}\angle 90^{0}$ = -j3.75 – j2.5 = -j6.25 = $6.25\text{A}\angle -90^{0}$

Example 2: Using superposition, find the current through the 6- Ω resistor of Fig. 2

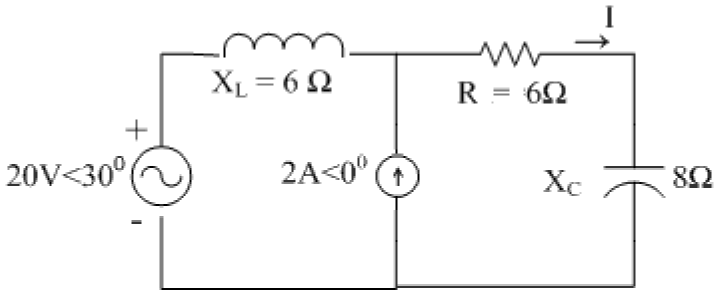


Figure 2 Network for Example 2

Solution: With the current source acting alone, the current through the resistor by current divider rule is

$$I' = \frac{j6}{6 + j(6-8)} \times 2 = \frac{j12}{6 - j2} = \frac{12\angle 90^\circ}{6.32\angle -18.43^\circ} = 1.9\angle 108.43^\circ$$

With the voltage source acting alone,

$$I'' = \frac{20\angle 30^\circ}{6 + j(6-8)} = \frac{20\angle 30^\circ}{6 - j2} = \frac{20\angle 30^\circ}{6.32\angle -18.43^\circ} = 3.16\angle 48.43^\circ$$

Actual Current, $I = I' + I'' = 1.9\angle 108.43^o + 3.16\angle 48.43^\circ = (-0.60 + j1.80) + (2.10 + j2.36)$

$$= 1.50 + j4.16 = 4.42\,\text{A}\angle 70.2^\circ$$

Example 3: One of the most frequent applications of the superposition theorem is to electronic systems in which the dc and ac analysis are treated separately and the total solution is the sum of the two.

For the network of Fig. 3, determine the sinusoidal expression for the voltage v_3 using superposition.

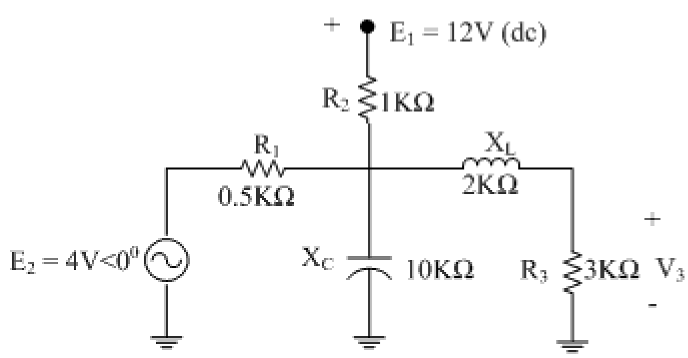
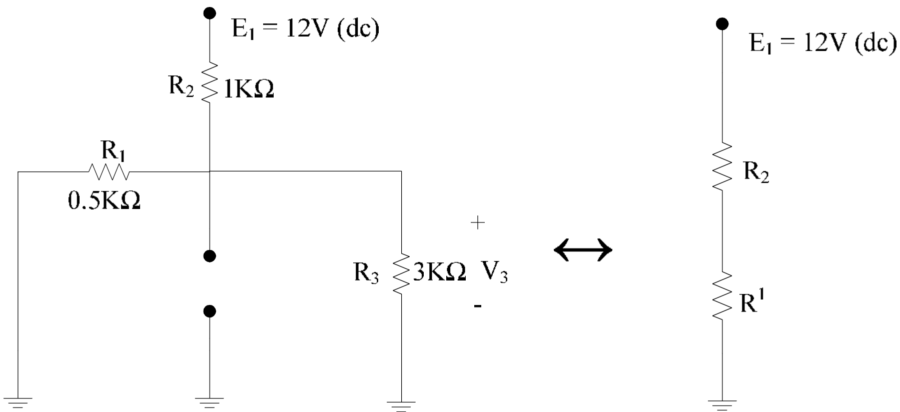


Figure 3 Network for Example 3

Solution: For the dc analysis, in steady state, the capacitor is open-circuited and the inductor short-circuited. The result is the network below



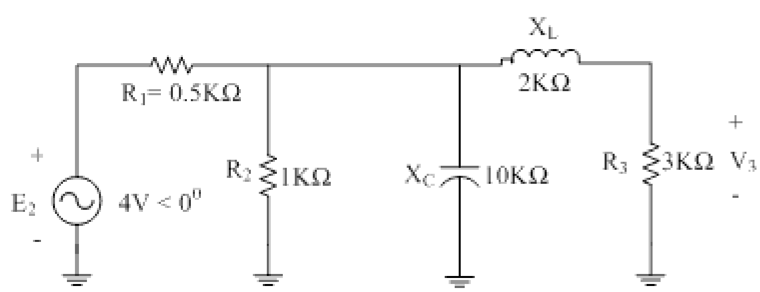
Resistance of parallel branch is given by

$$R' = \frac{05 \times 3}{05 + 3} = 0.429\,\text{k}\Omega$$

and the voltage across R_3 = voltage across the parallel branch is

$$V' = \frac{R'}{R' + R_2} \times E_1 = \frac{0.429}{0.429 + 1} \times 12 = \frac{5.148}{1.429} = 3.6 \text{ Vdc}$$

For the ac source acting alone, the network is redrawn as shown below:



The impedance of the parallel branch is given by

$$\frac{1}{Z_p} = \frac{1}{R_2} + \frac{1}{-jX_c} + \frac{1}{R_3 + jX_L} = \frac{1}{1} + \frac{1}{-j10} + \frac{1}{3 + j2} = 1 + j0.1 + \frac{1}{3.61\angle 33.69^\circ}$$

$$= 1 + j0.1 + 0.231 - j0.154$$

$$= 1.231 - j0.054 = 1.231\angle -2.51^\circ$$

$$Z_p = \frac{1}{1.231\angle -2.51^\circ} = 0.812 + j0.036$$

Total impedance $Z_T = Z_1 + Z_p = 0.5 + 0.812 + j0.036 = 1.312 + j0.036 = 1.312\angle -1.57^\circ$

$$\text{Total or source current } = \frac{4\angle 0^o}{1.312\angle -1.57^\circ} = 3.05\,\text{mA}\angle\text{-}1.57^\circ$$

$$\text{Current in } R_3 = \frac{\text{Voltage across parallel branch}}{R_3 + jX_L}$$

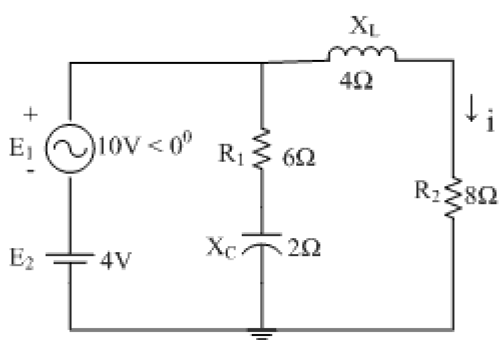
$$= \left[\frac{1}{1.231\angle -2.51^\circ} \times 3.05\,\text{mA}\angle\text{-}1.57^\circ\right] \times \frac{1}{3.61\angle 33.69^\circ} = 0.686\,\text{mA}\angle -32.75^\circ$$

$$V_3 = \text{voltage across } R_3 = 0.686\,\text{mA}\angle -32.75^\circ \times 3\angle 0^\circ = 2.06\,\text{V}\angle -32.75^\circ$$

The total solution $V_3 = V_3\,(\text{dc}) + V_3\,(\text{ac}) = 3.6\,\text{V} + 2.06\,\text{V}\angle -32.75^\circ$

From which $v_3(t) = 3.6 + 2.06\sqrt{2}\sin(\omega t - 32.75^\circ) = 3.6 + 2.91\sin(\omega t - 32.75^\circ)$ volts

Exercise 1: Using superposition, find the sinusoidal expression for the current i shown in the circuit below:



Ans: $i = 0.5 + 1.581 \sin(\omega t - 26.565^o)$ amps

3. Thevenin's Theorem

The only change is replacement of the term resistance with impedance. Unlike the superposition, it is applicable to only one frequency since reactance is frequency dependent. (Note: it has to be one frequency so that we can have one Thevenin's impedance)

Example 4: Find the Thevenin equivalent circuit for the network external to resistor R in Fig. 4.

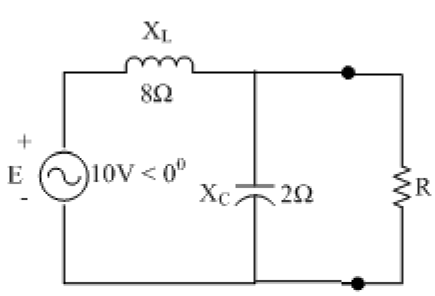
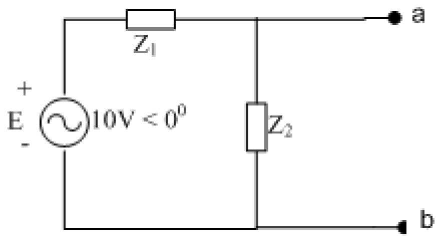


Figure 4 Network for Example 4

Solution: With R disconnected, the circuit becomes

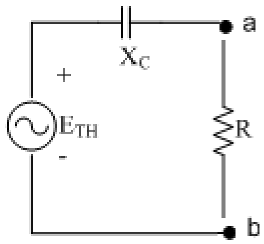


where

$Z_1 = j8$ and $Z_2 = -j2$

$$Z_{TH} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(j8)(-j2)}{(j8 - j2)} = \frac{16}{j6} = \frac{16}{6\angle 90^\circ} = 2.67\,\Omega\angle -90^\circ$$

By voltage divider rule, $E_{TH} = V_{ab} = \dfrac{Z_2}{Z_1 + Z_2} \times E = \dfrac{-j2}{j6} \times 10\angle 0^\circ = \dfrac{-j20}{j6} = 3.33\,\text{V}\angle -180^\circ$



$E_{TH} = 3.33\,\text{V}\angle -180^\circ$

$X_C = Z_{TH} = 2.67\,\Omega\angle -90^\circ$

Example 5: Find the Thevenin equivalent circuit seen at terminals a and b in Fig. 5.

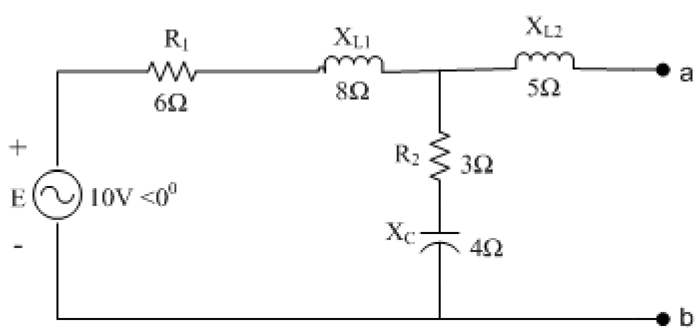


Figure 5 Network for Example 5

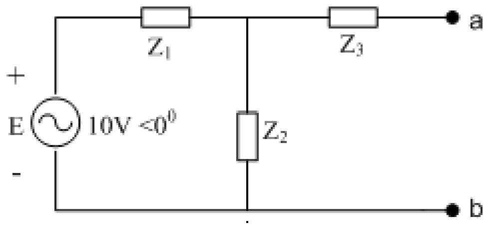
Solution: Let

$Z_1 = R_1 + jX_{L1} = 6 + j8 = 10\,\Omega\angle 53.13^\circ$

$Z_2 = R_2 - jX_C = 3 - j4 = 5\,\Omega\angle -53.13^\circ$

$Z_3 = jX_{L2} = j5$

With these impedances, the circuit becomes as shown below



$$Z_{TH} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2} = j5 + \frac{10\Omega\angle 53.13^\circ \times 5\Omega\angle -53.13^\circ}{(6+j8)+(3-j4)} = j5 + \frac{50\angle 0^\circ}{9.85\angle 23.96^\circ} = 5.49\Omega\angle 32.36^\circ$$

$$E_{TH} = \frac{Z_2}{Z_1 + Z_2} \times E = \frac{5\Omega\angle -53.13^\circ \times 10\angle 0^\circ}{9.85\angle 23.96^\circ} = 5.08\angle -77.09^\circ$$

Example 6: Fig. 6 shows ac equivalent network for a transistor amplifier. Find Thevenin equivalent circuit for the network external to the resistor R_L and then determine V_L.

Solution: With R_L disconnected,

$$E_{TH} = V_{ab} = \text{-}(100 I_1) R_C = -\text{-}(100 \times \frac{E_1}{2.8}) \times 2 = -\frac{200 E_i}{2.8} = -71.42 E_i$$

The R_B which is far greater than 2.3 kΩ has been ignored.

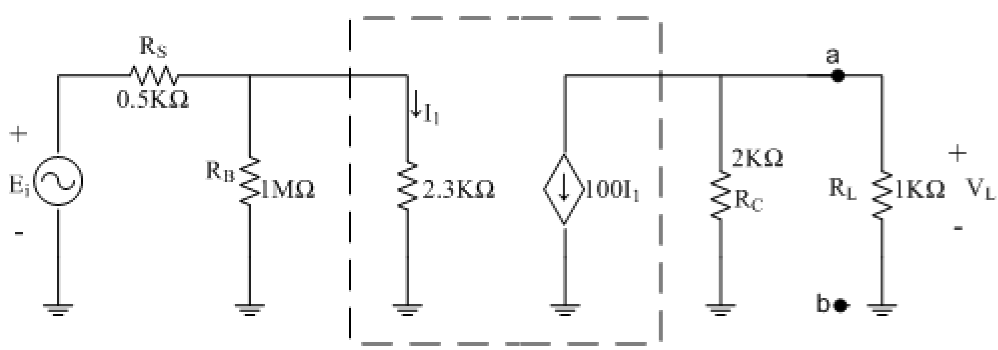


Figure 6 Network for Example 6

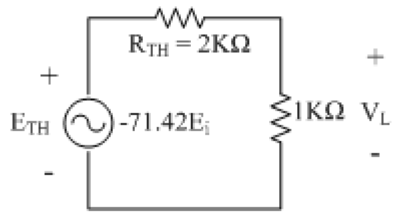
To calculate R_{TH}, the terminals *a* and *b* are short-circuited, and the I_{SC} determined. When the terminals *a* and *b* are short-circuited,

$$I_{sc} = \text{-}100 I_i \text{ or } I_{sc} = \text{-}100 \times \frac{E_i}{2.8} \text{ mA}$$

Also

$$E_{TH} = -\frac{200E_i}{2.8}$$

Therefore $R_{TH} = \frac{E_{TH}}{I_{SC}} = -\frac{200E_i}{2.8} \times -\frac{2.8}{100E_i} = 2\,\text{k}\Omega$



$$V_L = -\frac{1}{2+1} \times 71.42E_i = -23.8E_i$$

Example 7: Find the Thevenin equivalent circuit between the indicated terminals a and b of the network in Fig. 7

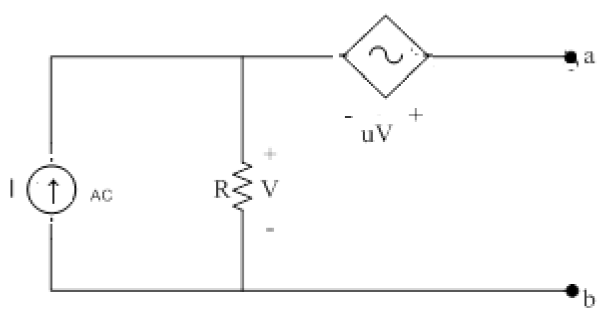


Figure 7 Network for Example 7

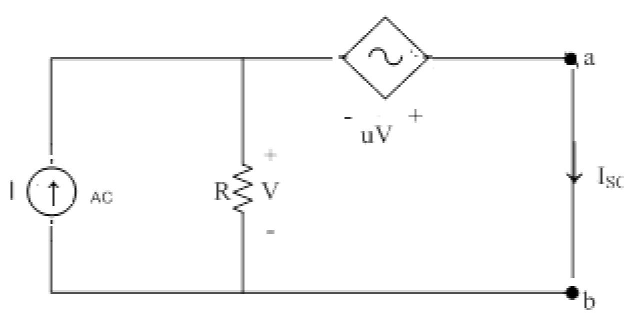
Solution

<u>Calculation of E_{TH}</u>:

$E_{TH} = V_{ab} = uV + V = (u+1)\ V = (u+1)IR$

<u>Calculation of R_{TH}</u>

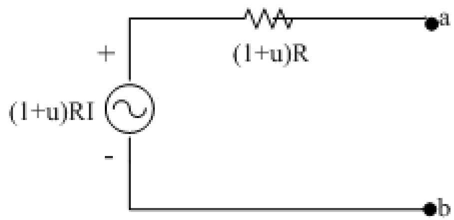
Short-circuit terminals and calculate I_{SC}:



From KVL

$V + uV = 0$ or $(1+u)V = 0$. This implies that $V = 0$

Therefore current in R is zero, $I_{SC} = I$ and $Z_{TH} = \dfrac{E_{TH}}{I_{SC}} = \dfrac{(1+\mu)IR}{I} = (1+\mu)R$



Example 8: Fig. 8 shows the equivalent circuit of a transistor used most frequently today. Determine the Thevenin equivalent circuit at terminal a and b:

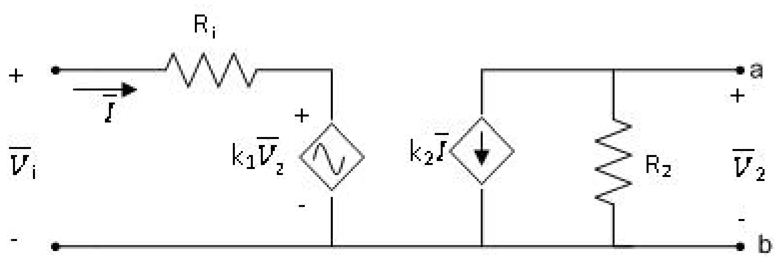


Figure 8 Network for Example 8

Solution:

<u>Calculation of $\overline{E}_{TH}$:</u>

$$\overline{E}_{TH} = \overline{V}_2 = -k_2 R_2 \overline{I}$$

$$\overline{I} = \frac{\overline{V}_i - k_1\overline{V}_2}{R_1}$$

Substituting this for $\overline{I}$ in the first equation, we obtain

$$\overline{V}_2 = k_2 R_2\left[\frac{k_1\overline{V}_2 - \overline{V}_i}{R_1}\right] = k_1 k_2\left(\frac{R_2}{R_1}\right)\overline{V}_2 - k_2\left(\frac{R_2}{R_1}\right)\overline{V}_i$$

$$\left[1 - k_1 k_2\frac{R_2}{R_1}\right]\overline{V}_2 = -k_2\frac{R_2}{R_1}\overline{V}_i$$

$$\overline{V}_2 = \frac{k_2 R_2}{[k_1 k_2 R_2 - R_1]}\overline{V}_i$$

Hence $\overline{E}_{TH} = \dfrac{k_2 R_2}{[k_1 k_2 R_2 - R_1]}\overline{V}_i$

Calculation of $\bar{I}_{SC}$:

When the terminals *a* and *b* are short-circuited $\bar{V}_2 = 0$. Hence

$$\bar{I}_{SC} = -k_2 \bar{I} \text{ and } \bar{I} = \frac{\bar{V}_i}{R_1}$$

Therefore $\bar{I}_{SC} = -k_2 \times \dfrac{\bar{V}_i}{R_1}$

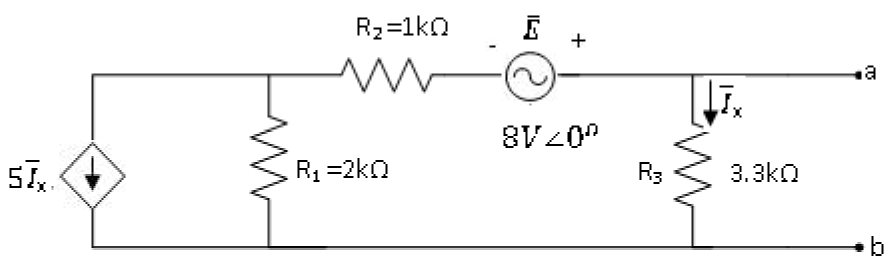
Calculation of Z_{TH}:

$$Z_{TH} = \frac{\bar{E}_{TH}}{\bar{I}_{SC}} = \frac{k_2 R_2}{[k_1 k_2 R_2 - R_1]} \bar{V}_i \times (-1) \frac{R_1}{k_2 \bar{V}_i} = \frac{R_1 R_2}{R_1 - k_1 k_2 R_2} = \frac{R_2}{1 - k_1 k_2 \frac{R_2}{R_1}}$$

Frequently, the approximation $k_1 \approx 0$ is applied. Then we obtain

$$\bar{E}_{TH} = -\frac{k_2 R_2}{R_1} \bar{V}_i \text{ and } Z_{TH} = R_2$$

Exercise 2: Find the Thevenin equivalent circuit for the network given below to the left of terminals a and b. I_x is in mA.



Ans: $\bar{Z}_{TH}$ = 607.42 Ω∠0° $\bar{E}_{TH}$= 1.62 V∠0°

4. Norton's Theorem

Here too, resistance is replaced with impedance and it is applicable to only one frequency since reactance is frequency dependent.

Example 9: Determine the Norton equivalent circuit for the network external to the 6-Ω resistor of Fig. 9.

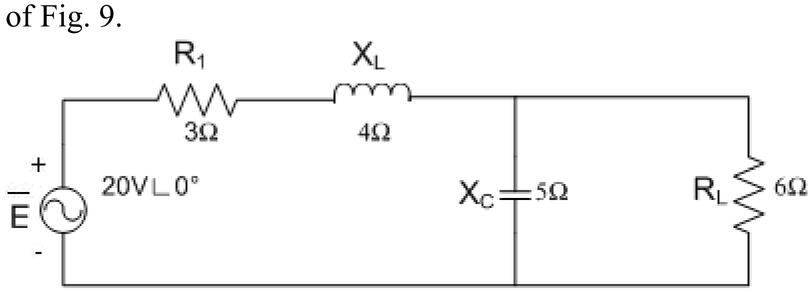


Figure 9 Network for Example 9

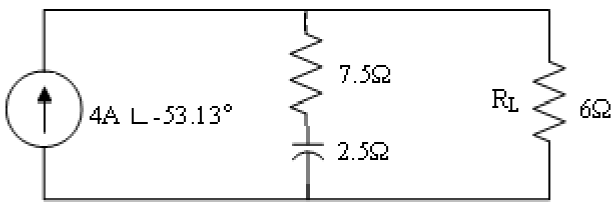
Solution:
Calculate Norton's current: Since X_C is short-circuited

$$\bar{I}_N = \frac{\bar{E}}{R_1 + jX_L} = \frac{20\angle 0^\circ}{3 + j4} = \frac{20\angle 0^\circ}{5\angle 53.13^\circ} = 4\,\text{A}\angle\text{-}53.13^\circ$$

Calculate Z_N:

$$\bar{Z}_N = \frac{(R_1 + jX_L)(-jX_C)}{R_1 + j(X_L - X_C)} = 7.91\Omega\angle -18.44^\circ = 7.50 - j2.50\Omega$$

Equivalent circuit:



Example 10: Find the Norton equivalent circuit for the network external to the 7-Ω capacitive reactance in Fig. 10.

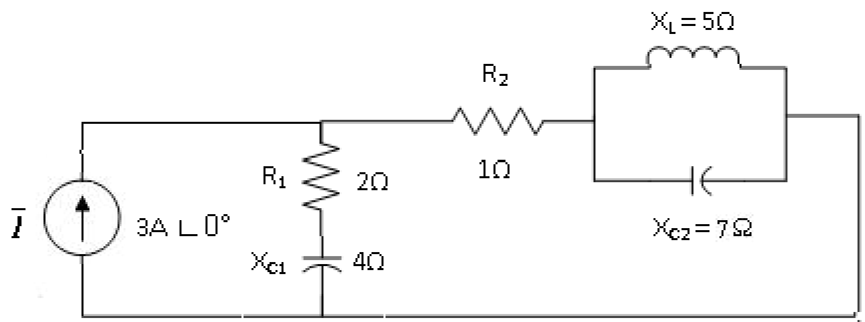
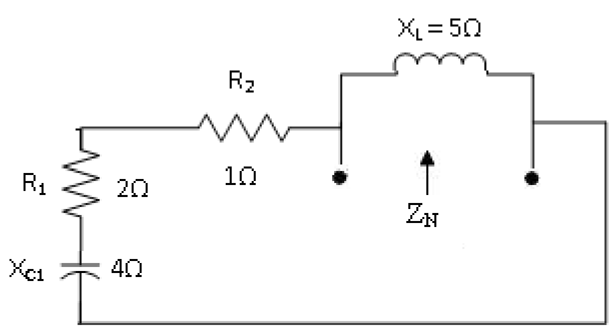


Figure 10 Network for Example 10

Solution:
Calculate Z_N:



$$\bar{Z}_N = jX_L // (R_1 + R_2 - jX_C) = \frac{(j5)(3 - j4)}{3 + j1} = 7.91\Omega\angle 18.44^\circ = 7.50 + j2.50\Omega$$

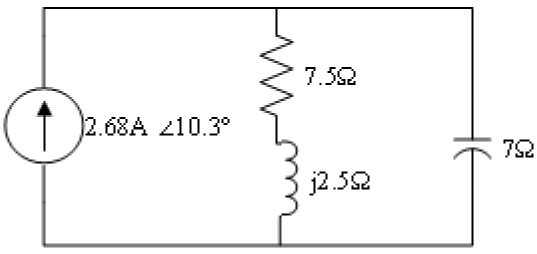
Calculate: $\bar{I}_N$

X_L is short–circuited and the parallel branch consists of (R_1 +jX_{C1}) // R_2.

Using the current divider rule, the Norton's current = current through R_2 is given by

$$\bar{I}_N = \left[\frac{R_1 - jX_{C1}}{R_1 + R_2 - jX_{C1}}\right] \times \bar{I} = 2.68\,\text{A}\angle 10.3^\circ$$

Equivalent circuit:



Example 11: Find the Norton equivalent circuit for the network shown in Fig 11

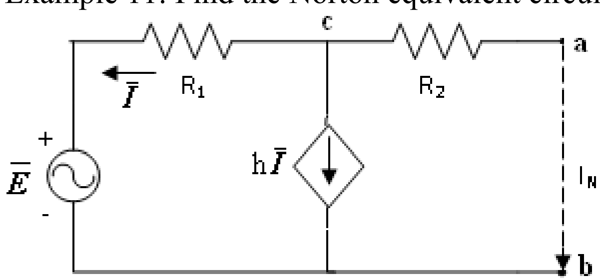


Figure 11 Network for Example 11

Solution:

Calculate $\bar{I}_N$

The Norton's current is the current in a short circuit placed between terminals a and b as shown in Fig. 11. Applying KCL at node c, we obtain

$0 = \bar{I} + h\bar{I} + \bar{I}_N$ or $\bar{I}_N = -(1+h)\bar{I}$

Applying KVL, we obtain

$$\bar{E} = R_2\bar{I}_N - \bar{I}R_1 = -[(1+h)R_2 + R_1]\bar{I} \text{ or } \bar{I} = -\frac{\bar{E}}{R_1 + (1+h)R_2}$$

Therefore $\bar{I}_N = \dfrac{(1+h)\bar{E}}{R_1 + (1+h)R_2}$

Calculate $\bar{E}_{TH}$:

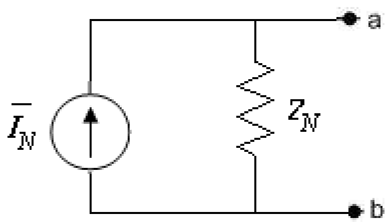
$\bar{E}_{TH} = \bar{V}ab = \bar{E} + \bar{I}R_1$

Applying KCL to node c, we obtain $\bar{I} + h\bar{I} = 0$ or $(1+h)\bar{I} = 0$ or $\bar{I} = 0$. Hence $\bar{E}_{TH} = \bar{E}$

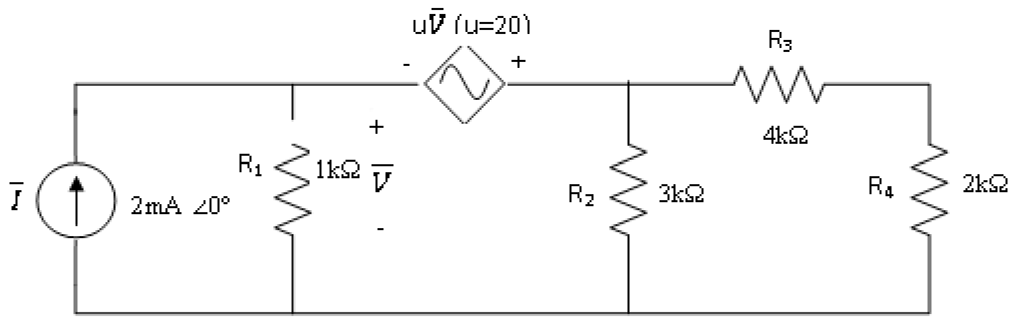
Calculate Z_N:

$$\overline{Z}_N = \frac{\overline{E}_{TH}}{\overline{I}_N} = \frac{\overline{E}}{(1+h)\overline{E}} \times [R_1 + (1+h)R_2] = \frac{R_1 + (1+h)R_2}{(1+h)} = R_2 + \frac{R_1}{1+h}$$

Equivalent Circuit:



Exercise 3: For the network given below, find the Norton equivalent circuit for the circuit external to the 2-kΩ resistor.



Ans: Z_N = 6.63 kΩ ∠ 0° $\overline{I}_N$ = 0.792 mA ∠0°

5. Maximum Power Transfer

Here, maximum power is delivered to the load when its impedance is the conjugate of the Thevenin impedance. This implies that input power factor under maximum power condition is unity.

Example 12: Find the load impedance Z_L in Fig. 12 for maximum power to the load and find the maximum power.

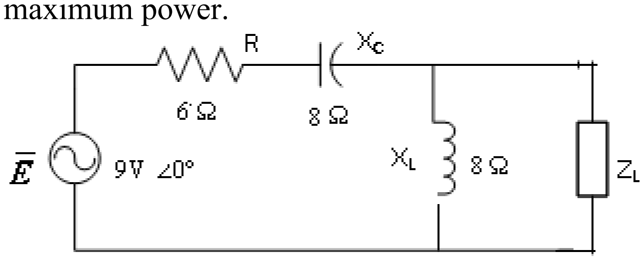


Figure 12 Network for Example 12

Solution:

<u>Calculate Z_{TH} and Z_L:</u>

$$Z_{TH} = (R - jX_C) // jX_L = \frac{(6-j8)(j8)}{6-j8+j8} = \frac{64}{6} + j8 = R_{TH} + jX_{TH}$$

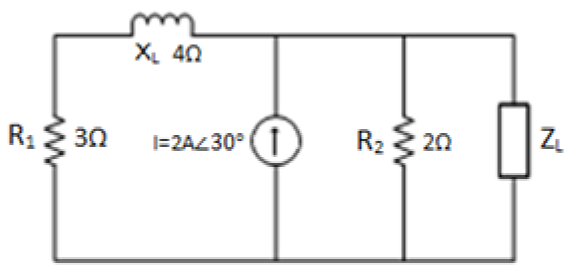
Hence $Z_L = \frac{64}{6} - j8$

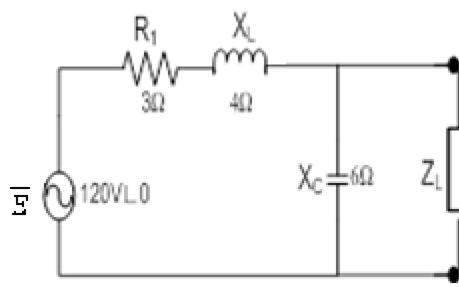
<u>Calculate $\overline{E}_{TH}$:</u>

By voltage divider rule, $\overline{E}_{TH} = \frac{jX_L}{R + jX_L - jX_C}\overline{E} = \frac{j8}{6+j8-j8} \times 9 = 12\,\text{V}\angle 90^\circ$

$$P_{\max} = I_L^2 R_L = I_L^2 R_{TH} = \left(\frac{E_{TH}}{2R_{TH}}\right)^2 \times R_{TH} = \frac{E_{TH}^2}{4R_{TH}} = \frac{144}{4} \times \frac{6}{64} = 3.38\,\text{W}$$

Exercise 4: Find the load impedance Z_L for the networks given below for maximum power to the load.

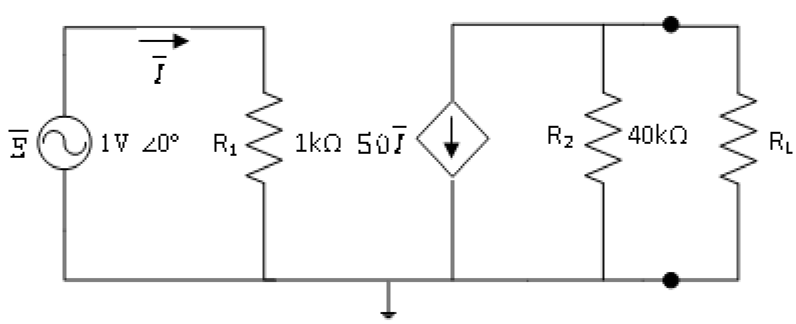




Ans: $Z_L = 8.32\Omega\angle 3.18^\circ$ $P_{max} = 1198.2$W

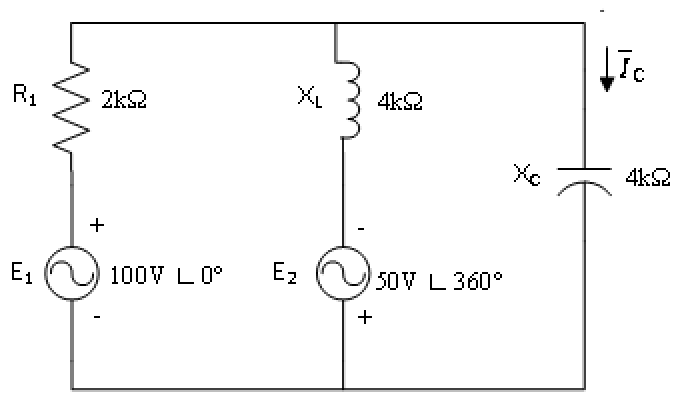
Ans: $Z_L = 1.562\Omega\ \angle{-14.47^\circ}$ $P_{max} = 1.614$W

Exercise 5: Find the load impedance R_L for the network given below for maximum power to the load, and find the maximum power to the load.



Ans: 40kW, 25W

Exercise 6: Using Millman's theorem, determine the current through the 4-kΩ capacitive reactance of the figure given below:



Ans: 25.77mA∠104.4°