Math 210, Answer to Exam 2, 12/7/04

All answers should be supported by valid arguments or calculations. Clear presentation is part of the answer.

(1) (15 pts) The base of an aquarium (i.e. a box without the top lid) with a given volume V is made of slate and the sides are made of glass. If slate costs 5 times as much (per unit area) as glass, find the dimensions of the aquuarium that minimize the cost of the materials.

(You don't need to use the second derivative test to show the answer is a local minimum.)

Answer: Let x and y be the dimensions of base of the aquarium and z be its height. Hence $V = xyz$. Let C be the cost of the material. Therefore we like to maximize

$$C = 5xy + 2xz + 2yz$$

subject to the constraint $V = xyz$. Putting $z = V/(xy)$ into the cost function,

$$C = 5xy + 2V\frac{x+y}{xy} .$$

Thus

$$\frac{\partial C}{\partial x} = 5y - 2\frac{V}{x^2} , \quad \frac{\partial C}{\partial y} = 5x - 2\frac{V}{y^2} .$$

Setting both partial derivatives to be zero to find the critical point of C, we have

$$\begin{cases} 5y &= 2\frac{V}{x^2} \\ 5x &= 2\frac{V}{y^2} \end{cases} \tag{1}$$

Hence $5yx^2 = 5xy^2$. Since both x and y are not zero, we have $x = y$. Thus putting back into the first equation of (1), we obtain

$$5x^3 = 2V$$

Therefore $x = \left(\frac{2V}{5}\right)^{1/3}$, which leads to $y = \left(\frac{2V}{5}\right)^{1/3}$ and $z = V/(xy) = (\frac{5}{2})^{2/3}V^{1/3}$.

(2) (15 pts) Evaluate $\int\int_D xy\, dA$ where D is the region bounded by $y = 8x$ and $y = x^4$.

Answer: First we find the point of intersection of the two curves. Setting $x^4 = 8x$, we have $x = 0$ or $x = 2$. Hence the points of intersection are $(0, 0)$ and $(2, 16)$. Now,

$$\int\int_D xy\, dA = \int_{x=0}^{2} dx \int_{y=x^4}^{8x} xy\, dy = \frac{1}{2}\int_{x=0}^{2} x[(8x)^2 - x^8]\, dx = 384/5$$

(3) (15 pts) Evaluate $\int\int\int_E \sqrt{x^2+y^2}\, dV$, where E is the region lying above the xy-plane and below the cone $z = 4 - \sqrt{x^2+y^2}$.

Answer: Let the shadow of the cone be D, which is the circle of radius 4 centered at the origin. Hence

$$\begin{aligned}
\int\int\int_E \sqrt{x^2+y^2}\, dV &= \int\int_D dx\, dy \int_{z=0}^{4-\sqrt{x^2+y^2}} \sqrt{x^2+y^2} dz \\
&= \int\int_D dx\, dy \sqrt{x^2+y^2}(4-\sqrt{x^2+y^2}) \\
&= \int\int_D r dr\, d\theta\, r(4-r) \\
&= \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{4} r^2(4-r) dr \\
&= 128\pi/3
\end{aligned}$$

(4) (15 pts) Evaluate $\int\int\int_E z^3\sqrt{x^2+y^2+z^2}\, dV$, where E is the solid hemisphere that lies above the xy-plane and centers at the origin with radius 1.

Answer: Use spherical coordinates. Since $z = \rho\cos\phi$ and $\sqrt{x^2+y^2+z^2} = \rho$, and $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, thus

$$\begin{aligned}
\int\int\int_E z^3\sqrt{x^2+y^2+z^2}\, dV &= \int\int\int_E \rho^6 \sin\phi\ \cos^3\phi\, d\rho\, d\phi\, d\theta \\
&= \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi/2} \sin\phi\ \cos^3\phi\ d\phi \int_{\rho=0}^{1} \rho^6 d\rho \\
&= \frac{2\pi}{7} \int_{\phi=0}^{\pi/2} \sin\phi\ \cos^3\phi\ d\phi \\
&= \frac{\pi}{14}
\end{aligned}$$

by making the substitution $u = \cos\phi$.

(5) (10 pts) Find the surface area of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$.

Answer: The shadow of the plane will just be the disk $x^2 + y^2 \leq 9$. Call it D. Hence

$$\text{surface area} = \int\int_D \sqrt{1 + z_x^2 + z_y^2} dA$$

where $z = 10 - 2x - 5y$. Thus

$$\text{surface area} = \int\int_D \sqrt{1+4+25} dA = \sqrt{30}\int\int_D dA = \sqrt{30} * \text{area of D} = 9\pi\sqrt{30}$$

(6a) (8 pts) Let $\mathbf{F} = (y - 2x^2)\mathbf{i} + (2x + y)\mathbf{j}$. Evaluate $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the straight line from $(1, 0)$ to $(0, 1)$.

Answer: The curve C_1 can be parametrized by

$$x = 1 - t\ , \qquad y = t\ , \qquad 0 \le t \le 1\ .$$

Thus $dx/dt = -1$ and $dy/dt = 1$.

$$\begin{aligned}
\textstyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \textstyle\int_{t=0}^{1} \begin{pmatrix} y - 2x^2 \\ 2x + y \end{pmatrix} \cdot \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} dt \\
&= 2 \textstyle\int_{t=0}^{1} (x^2 + x) dt \\
&= 2 \textstyle\int_{t=0}^{1} [(1 - t)^2 + 1 - t]\ dt \\
&= 5/3
\end{aligned}$$

(6b) (7 pts) With the same $\mathbf{F}$ as in part (a), use the Green's theorem (or otherwise) to evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangular path from $(0, 0)$ to $(1, 0)$ to $(0, 1)$ to $(0, 0)$.

Answer: Let $P = y - 2x^2$ and $Q = 2x + y$ and the triangle (including its interior) be denoted by D. Using the Green's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy = \int \int_D (Q_x - P_y)\ dA$$

Since $Q_x - P_y = 2 - 1 = 1$, thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_D\ dA = \text{area of D} = \frac{1}{2}$$

(7) (3 pts) (a) Show that $\mathbf{F}(x,y) = x^3y^4\mathbf{i} + (x^4y^3 + y)\mathbf{j}$ is a conservative vector field.

Answer: Let $P = x^3y^4$ and $Q = x^4y^3 + y$. Since $Q_x = P_y = 4x^3y^3$, and P and Q are "nice" in the whole $\mathbf{R}^2$, which is simply connected. Therefore $\mathbf{F}$ is conservative.

(b) (8 pts) Find a f such that $\nabla f = \mathbf{F}$.

Answer: With

$$\frac{\partial f}{\partial x} = P = x^3y^4 \ ,$$

$$\frac{\partial f}{\partial y} = Q = x^4y^3 + y$$

Integrate the first equation, $f(x,y) = \frac{1}{4}x^4y^4 + g(y)$ for some function g. Putting into the second equation,

$$\frac{\partial}{\partial y}(\frac{1}{4}x^4y^4 + g(y)) = x^4y^3 + y$$

Thus, $g'(y) = y$ which gives $g(y) = \frac{1}{2}y^2 + C$. Take $C = 0$, so

$$f(x,y) = \frac{1}{4}x^4y^4 + \frac{1}{2}y^2$$

(c) (4 pts) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve: $x = \sqrt{t}$ and $y = 1 + t^3$ from $t = 0$ to $t = 1$.

Answer: Since $\mathbf{F}$ is conservative in $\mathbf{R}^2$, the line integral is path independent. When $t = 0$, $(x,y) = (0,1)$. When $t = 1$, $(x,y) = (1,2)$. Hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{1} \nabla f \cdot \frac{d\mathbf{r}}{dt} dt = f(x(1), y(1)) - f(x(0), y(0)) = f(1,2) - f(0,1) = 11/2$$

where f is given in part (b).