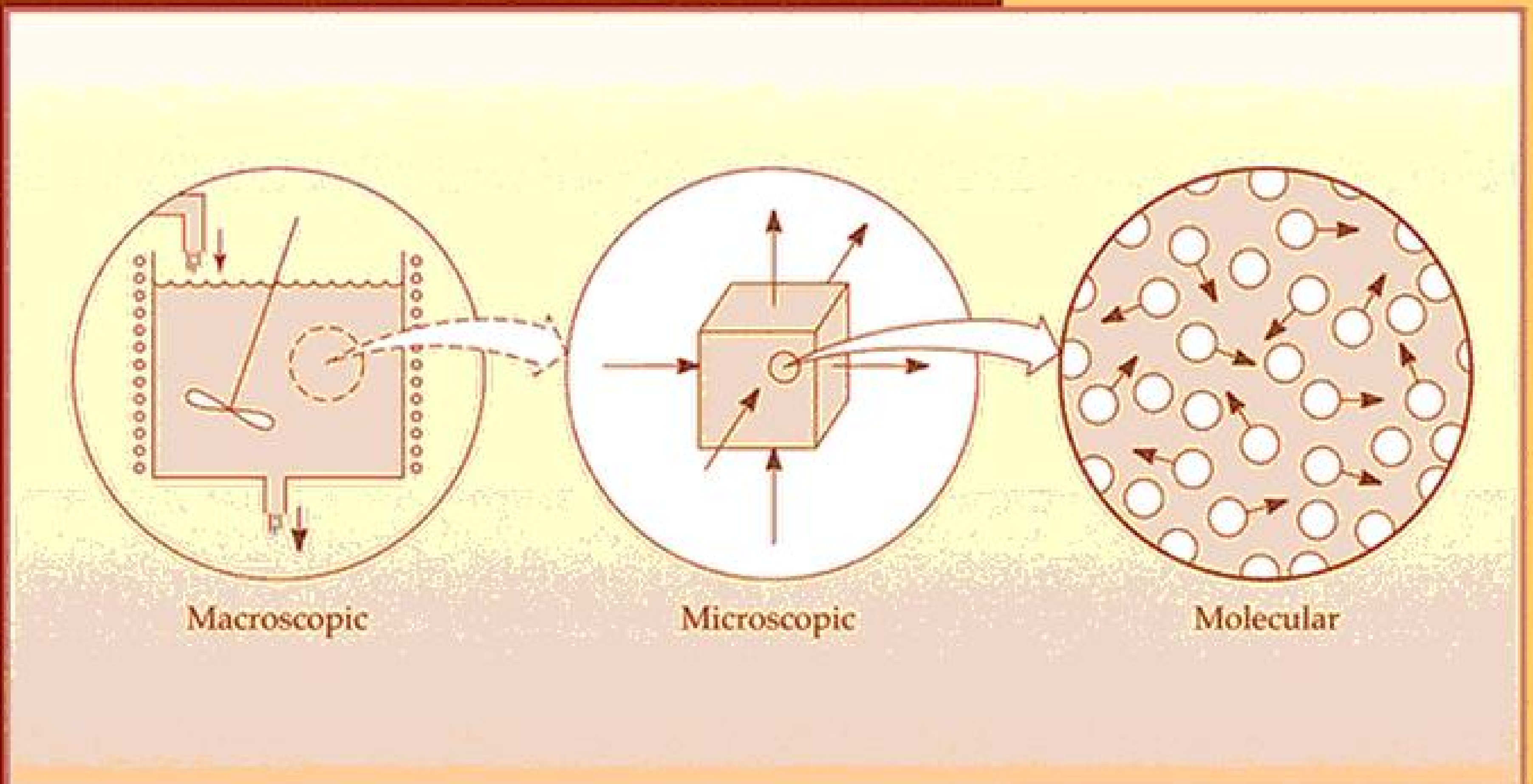
Transport Phenomena

Revised Second Edition



R. Byron Bird • Warren E. Stewart
Edwin N. Lightfoot

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Revised Second Edition

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- Fourier analysis of turbulent transport at high Pr or Sc
- more on heat and mass transfer coefficients
- enlarged discussions of dimensional analysis and scaling
- matrix methods for multicomponent mass transfer
- ionic systems, membrane separations, and porous media
- the relation between the Boltzmann equation and the continuum equations
- use of the "Q+W" convention in energy discussions, in conformity with the leading textbooks in physics and physical chemistry

However, it is always the youngest generation of professionals who see the future most clearly, and who must build on their imperfect inheritance.

Much remains to be done, but the utility of transport phenomena can be expected to increase rather than diminish. Each of the exciting new technologies blossoming around us is governed, at the detailed level of interest, by the conservation laws and flux expressions, together with information on the transport coefficients. Adapting the problem formulations and solution techniques for these new areas will undoubtedly keep engineers busy for a long time, and we can only hope that we have provided a useful base from which to start.

Each new book depends for its success on many more individuals than those whose names appear on the title page. The most obvious debt is certainly to the hard-working and gifted students who have collectively taught us much more than we have taught them. In addition, the professors who reviewed the manuscript deserve special thanks for their numerous corrections and insightful comments: Yu-Ling Cheng (University of Toronto), Michael D. Graham (University of Wisconsin), Susan J. Muller (University of California-Berkeley), William B. Russel (Princeton University), Jay D. Schieber (Illinois Institute of Technology), and John F. Wendt (Von Kármán Institute for Fluid Dynamics). However, at a deeper level, we have benefited from the departmental structure and traditions provided by our elders here in Madison. Foremost among these was Olaf Andreas Hougen, and it is to his memory that this edition is dedicated.

Madison, Wisconsin

R. B. B.
W. E. S.
E. N. L.

Comments on the Revised Second Edition:

Since the appearance of the second edition in 2002, the authors have found a number of errors—some major and some minor—and we have endeavored to correct these. In addition, well over a hundred readers have joined in suggesting how improvements could be made or how errors could be corrected. Some letters came from students, some from teachers, and some from practitioners. We have appreciated receiving their comments and have enjoyed corresponding with them. We regret any problems or confusion caused by our mistakes and have taken seriously the job of correcting the text for this revised second edition, to the extent that it was feasible. The publisher has been very generous in allowing us to do this, and we thank all concerned for their patience. We wish to give special thanks to Professor Carlos A. Ramírez at the University of Puerto Rico (Mayagüez) for his diligence and effort in corresponding with us. His many comments have been particularly valuable, and as a result the book has been greatly improved.

September 2006

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Table 0.2-1 Organization of the Topics in This Book

Type of transport	Momentum	Energy	Mass
Transport by molecular motion	**1** Viscosity and the stress (momentum flux) tensor	**9** Thermal conductivity and the heat-flux vector	**17** Diffusivity and the mass-flux vectors
Transport in one dimension (shell-balance methods)	**2** Shell momentum balances and velocity distributions	**10** Shell energy balances and temperature distributions	**18** Shell mass balances and concentration distributions
Transport in arbitrary continua (use of general transport equations)	**3** Equations of change and their use [isothermal]	**11** Equations of change and their use [nonisothermal]	**19** Equations of change and their use [mixtures]
Transport with two independent variables (special methods)	**4** Momentum transport with two independent variables	**12** Energy transport with two independent variables	**20** Mass transport with two independent variables
Transport in turbulent flow, and eddy transport properties	**5** Turbulent momentum transport; eddy viscosity	**13** Turbulent energy transport; eddy thermal conductivity	**21** Turbulent mass transport; eddy diffusivity
Transport across phase boundaries	**6** Friction factors; use of empirical correlations	**14** Heat-transfer coefficients; use of empirical correlations	**22** Mass-transfer coefficients; use of empirical correlations
Transport in large systems, such as pieces of equipment or parts thereof	**7** Macroscopic balances [isothermal]	**15** Macroscopic balances [nonisothermal]	**23** Macroscopic balances [mixtures]
Transport by other mechanisms	**8** Momentum transport in polymeric liquids	**16** Energy transport by radiation	**24** Mass transport in multi-component systems; cross effects

§0.3 THE CONSERVATION LAWS: AN EXAMPLE

The system we consider is that of two colliding diatomic molecules. For simplicity we assume that the molecules do not interact chemically and that each molecule is homonuclear—that is, that its atomic nuclei are identical. The molecules are in a low-density gas, so that we need not consider interactions with other molecules in the neighborhood. In Fig. 0.3-1 we show the collision between the two homonuclear diatomic molecules, A and B, and in Fig. 0.3-2 we show the notation for specifying the locations of the two atoms of one molecule by means of position vectors drawn from an arbitrary origin.

Actually the description of events at the atomic and molecular level should be made by using quantum mechanics. However, except for the lightest molecules (H_2 and He) at

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QUESTIONS FOR DISCUSSION

1. What are the definitions of momentum, angular momentum, and kinetic energy for a single particle? What are the dimensions of these quantities?
2. What are the dimensions of velocity, angular velocity, pressure, density, force, work, and torque? What are some common units used for these quantities?
3. Verify that it is possible to go from Eq. 0.3-3 to Eq. 0.3-4.
4. Go through all the details needed to get Eq. 0.3-6 from Eq. 0.3-5.
5. Suppose that the origin of coordinates is shifted to a new position. What effect would that have on Eq. 0.3-7? Is the equation changed?
6. Compare and contrast angular velocity and angular momentum.
7. What is meant by internal energy? Potential energy?
8. Is the law of conservation of mass always valid? What are the limitations?

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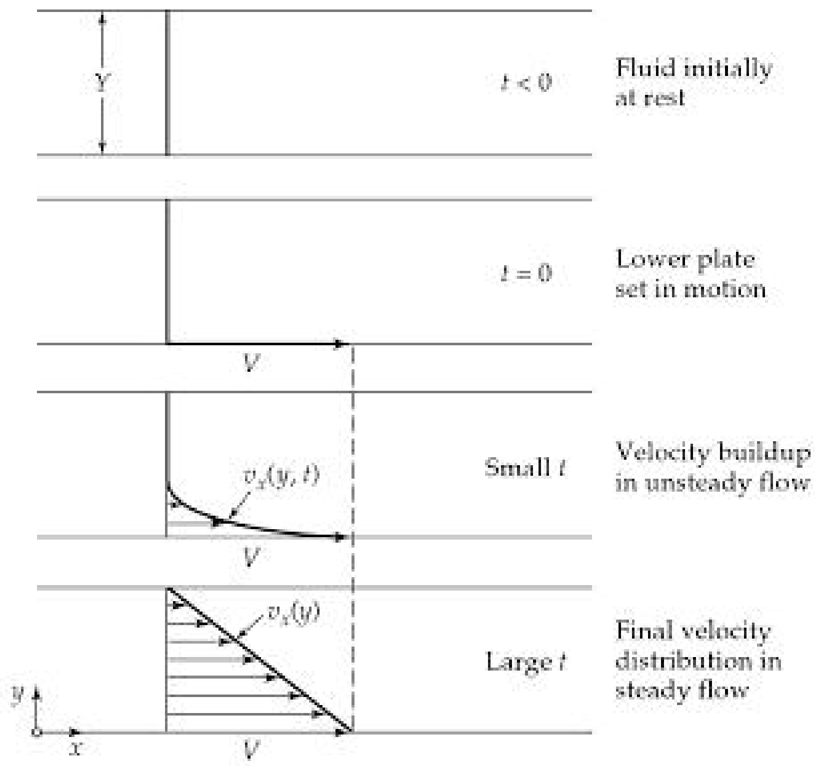

Fig. 1.1-1 The buildup to the steady, laminar velocity profile for a fluid contained between two plates. The flow is called "laminar" because the adjacent layers of fluid ("laminae") slide past one another in an orderly fashion.

has been attained, a constant force F is required to maintain the motion of the lower plate. Common sense suggests that this force may be expressed as follows:

$$\frac{F}{A} = \mu \frac{V}{Y} \tag{1.1-1}$$

That is, the force should be proportional to the area and to the velocity, and inversely proportional to the distance between the plates. The constant of proportionality μ is a property of the fluid, defined to be the *viscosity*.

We now switch to the notation that will be used throughout the book. First we replace F/A by the symbol τ_{yx}, which is the force in the x direction on a unit area perpendicular to the y direction. It is understood that this is the force exerted by the fluid of lesser y on the fluid of greater y. Furthermore, we replace V/Y by $-dv_x/dy$. Then, in terms of these symbols, Eq. 1.1-1 becomes

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \qquad (1.1\text{-}2)^1$$

This equation, which states that the shearing force per unit area is proportional to the negative of the velocity gradient, is often called *Newton's law of viscosity*.[2] Actually we

---

[1] Some authors write Eq. 1.1-2 in the form

$$g_c \tau_{yx} = -\mu \frac{dv_x}{dy} \tag{1.1-2a}$$

in which τ_{yx} [=] $\mathrm{lb}_f/\mathrm{ft}^2$, v_x [=] ft/s, y [=] ft, and μ [=] $\mathrm{lb}_m/\mathrm{ft}\cdot\mathrm{s}$; the quantity g_c is the "gravitational conversion factor" with the value of 32.174 poundals/lb_f. In this book we will always use Eq. 1.1-2 rather than Eq. 1.1-2a.

[2] Sir **Isaac Newton** (1643–1727), a professor at Cambridge University and later Master of the Mint, was the founder of classical mechanics and contributed to other fields of physics as well. Actually Eq. 1.1-2 does not appear in Sir Isaac Newton's *Philosophiae Naturalis Principia Mathematica* (1687), but the germ of the idea is there. For illuminating comments, see D. J. Acheson, *Elementary Fluid Dynamics*, Oxford University Press, 1990, §6.1.

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§1.2 GENERALIZATION OF NEWTON'S LAW OF VISCOSITY

In the previous section the viscosity was defined by Eq. 1.1-2, in terms of a simple steady-state shearing flow in which v_x is a function of y alone, and v_y and v_z are zero. Usually we are interested in more complicated flows in which the three velocity components may depend on all three coordinates and possibly on time. Therefore we must have an expression more general than Eq. 1.1-2, but it must simplify to Eq. 1.1-2 for steady-state shearing flow.

This generalization is not simple; in fact, it took mathematicians about a century and a half to do this. It is not appropriate for us to give all the details of this development here, since they can be found in many fluid dynamics books.[1] Instead we explain briefly the main ideas that led to the discovery of the required generalization of Newton's law of viscosity.

To do this we consider a very general flow pattern, in which the fluid velocity may be in various directions at various places and may depend on the time t. The velocity components are then given by

$$v_x = v_x(x, y, z, t); \qquad v_y = v_y(x, y, z, t); \qquad v_z = v_z(x, y, z, t) \tag{1.2-1}$$

In such a situation, there will be nine stress components τ_{ij} (where i and j may take on the designations x, y, and z), instead of the component τ_{yx} that appears in Eq. 1.1-2. We therefore must begin by defining these stress components.

In Fig. 1.2-1 is shown a small cube-shaped volume element within the flow field, each face having unit area. The center of the volume element is at the position x, y, z. At

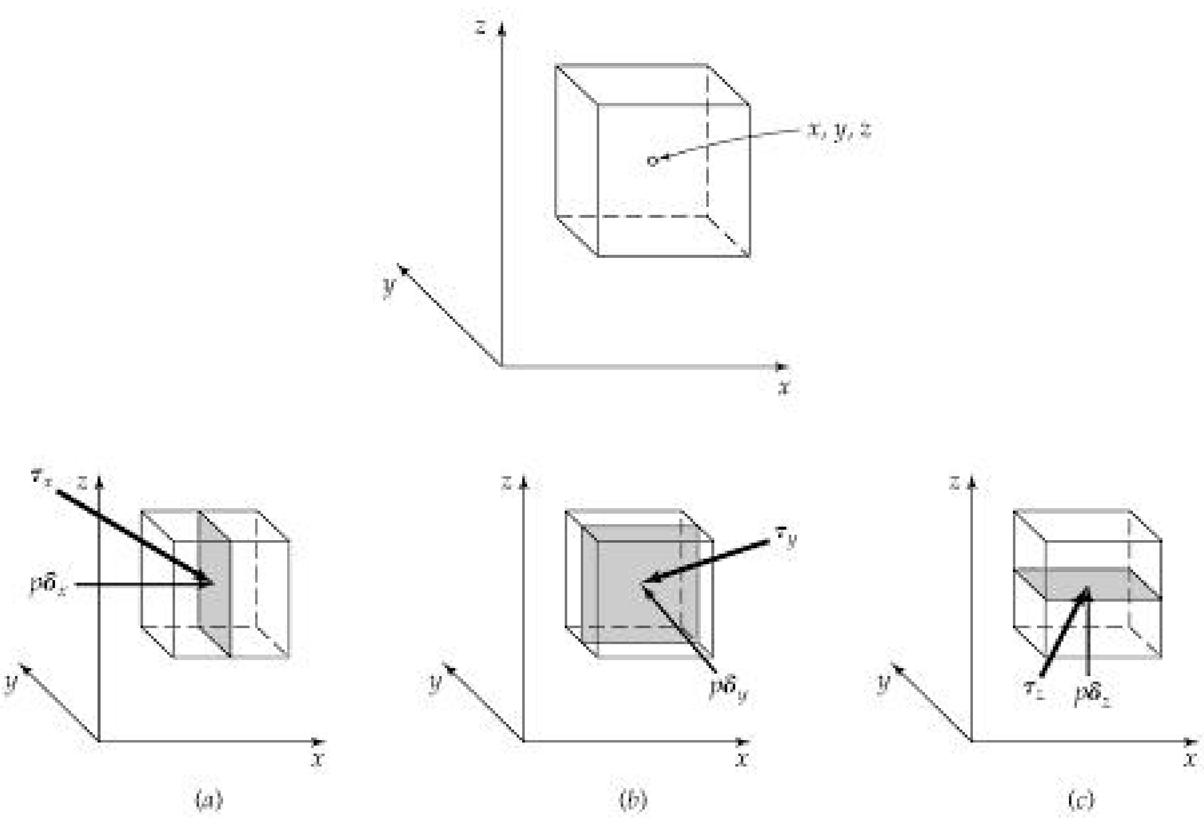

Fig. 1.2-1 Pressure and viscous forces acting on planes in the fluid perpendicular to the three coordinate directions. The shaded planes have unit area.

---

[1] W. Prager, *Introduction to Mechanics of Continua*, Ginn, Boston (1961), pp. 89–91; R. Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Prentice-Hall, Englewood Cliffs, N.J. (1962), pp. 30–34, 99–112; L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, London, 2nd edition (1987), pp. 44–45. **Lev Davydovich Landau** (1908–1968) received the Nobel prize in 1962 for his work on liquid helium and superfluid dynamics.

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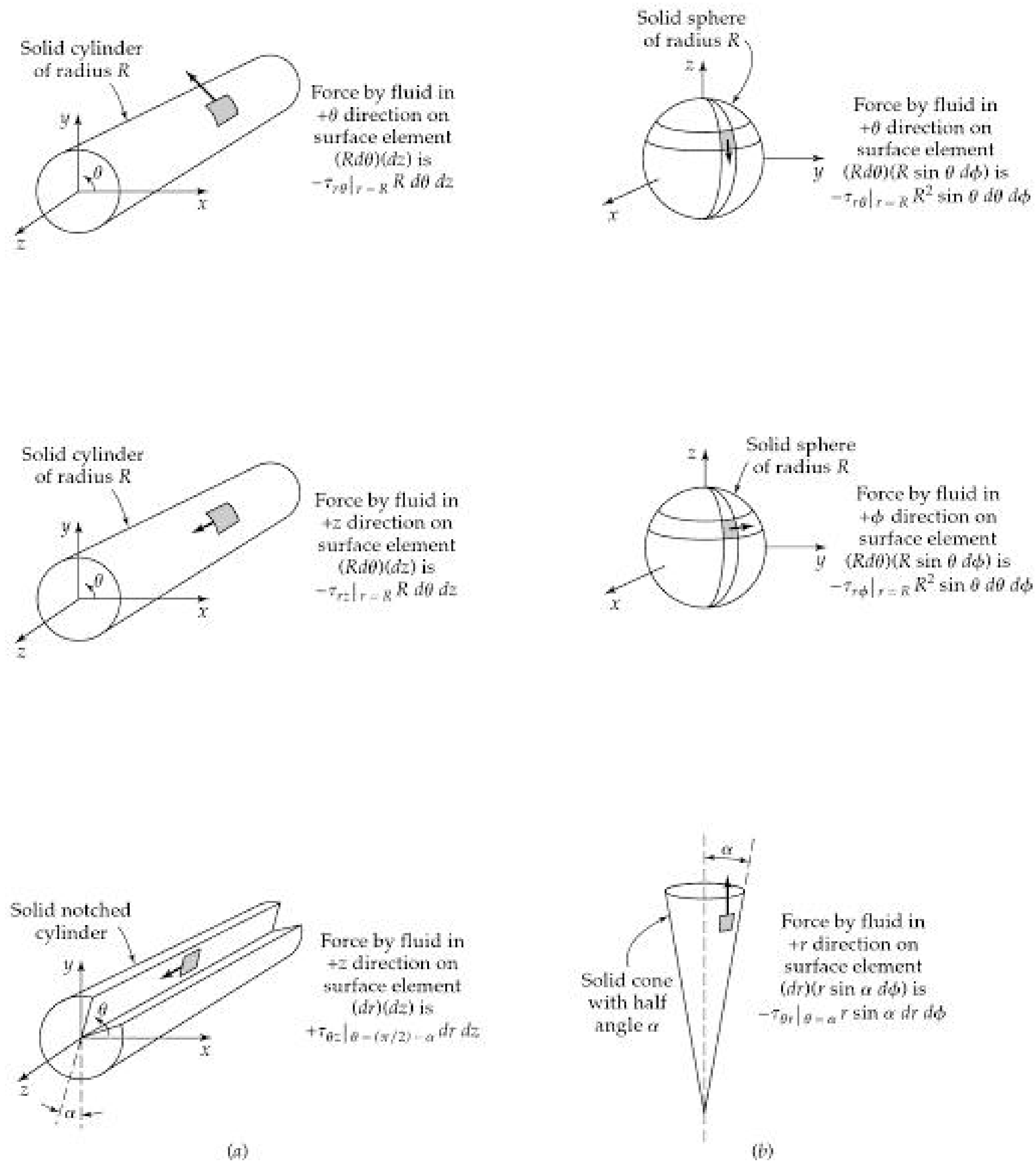

Fig. 1.2-2 (*a*) Some typical surface elements and shear stresses in the cylindrical coordinate system. (*b*) Some typical surface elements and shear stresses in the spherical coordinate system.

§1.7 and Table 19.2-2); (c) in Eq. 1.2-2, the terms $p\delta_{ij}$ and τ_{ij} have the same sign affixed, and the terms p and τ_{ij} are both positive in compression (in accordance with common usage in thermodynamics); (d) all terms in the entropy production in Eq. 24.1-5 have the same sign. Clearly the sign convention in Eqs. 1.1-2 and 1.2-6 is arbitrary, and either sign convention can be used, provided that the physical meaning of the sign convention is clearly understood.

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The average distance traveled by a molecule between successive collisions is the *mean free path* λ, given by

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n} \tag{1.4-3}$$

On the average, the molecules reaching a plane will have experienced their last collision at a distance a from the plane, where a is given very roughly by

$$a = \tfrac{2}{3}\lambda \tag{1.4-4}$$

The concept of the mean free path is intuitively appealing, but it is meaningful only when λ is large compared to the range of intermolecular forces. The concept is appropriate for the rigid-sphere molecular model considered here.

To determine the viscosity of a gas in terms of the molecular model parameters, we consider the behavior of the gas when it flows parallel to the xz-plane with a velocity gradient dv_x/dy (see Fig. 1.4-1). We assume that Eqs. 1.4-1 to 4 remain valid in this nonequilibrium situation, provided that all molecular velocities are calculated relative to the average velocity $\mathbf{v}$ in the region in which the given molecule had its last collision. The flux of x-momentum across any plane of constant y is found by summing the x-momenta of the molecules that cross in the positive y direction and subtracting the x-momenta of those that cross in the opposite direction, as follows:

$$\tau_{yx} = Zmv_x|_{y-a} - Zmv_x|_{y+a} \tag{1.4-5}$$

In writing this equation, we have assumed that all molecules have velocities representative of the region in which they last collided and that the velocity profile $v_x(y)$ is essentially linear for a distance of several mean free paths. In view of the latter assumption, we may further write

$$v_x|_{y\pm a} = v_x|_y \pm \tfrac{2}{3}\lambda \frac{dv_x}{dy} \tag{1.4-6}$$

By combining Eqs. 1.4-2, 5, and 6 we get for the net flux of x-momentum in the positive y direction

$$\tau_{yx} = -\tfrac{1}{3}nm\bar{u}\lambda \frac{dv_x}{dy} \tag{1.4-7}$$

This has the same form as Newton's law of viscosity given in Eq. 1.1-2. Comparing the two equations gives an equation for the viscosity

$$\mu = \tfrac{1}{3}nm\bar{u}\lambda = \tfrac{1}{3}\rho\bar{u}\lambda \tag{1.4-8}$$

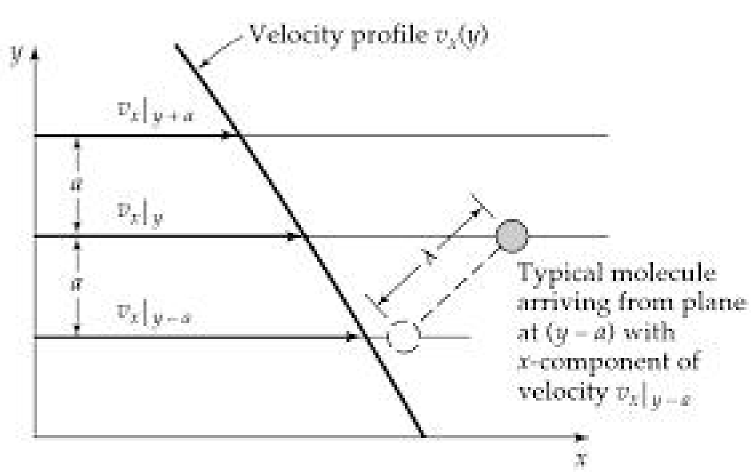

Fig. 1.4-1 Molecular transport of x-momentum from the plane at $(y-a)$ to the plane at y.

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EXAMPLE 1.4-1

Computation of the Viscosity of a Pure Gas at Low Density

Compute the viscosity of CO_2 at 200, 300, and 800K and 1 atm.

SOLUTION

Use Eq. 1.4-14. From Table E.1, we find the Lennard-Jones parameters for CO_2 to be ε/κ = 190 K and σ = 3.996 Å. The molecular weight of CO_2 is 44.01. Substitution of M and σ into Eq. 1.4-14 gives

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{44.01T}}{(3.996)^2\Omega_\mu} = 1.109 \times 10^{-5} \frac{\sqrt{T}}{\Omega_\mu} \tag{1.4-17}$$

in which μ [=] g/cm · s and T [=] K. The remaining calculations may be displayed in a table.

				Viscosity (g/cm · s)	
T (K)	$\kappa T/\varepsilon$	Ω_μ	$\sqrt{T}$	Predicted	Observed[11]
200	1.053	1.548	14.14	1.013×10^{-4}	1.015×10^{-4}
300	1.58	1.286	17.32	1.494×10^{-4}	1.495×10^{-4}
800	4.21	0.9595	28.28	3.269×10^{-4}	···

Experimental data are shown in the last column for comparison. The good agreement is to be expected, since the Lennard-Jones parameters of Table E.1 were derived from viscosity data.

EXAMPLE 1.4-2

Prediction of the Viscosity of a Gas Mixture at Low Density

Estimate the viscosity of the following gas mixture at 1 atm and 293K from the given data on the pure components at the same pressure and temperature:

Species α	Mole fraction, x_α	Molecular weight, M_α	Viscosity, μ_α (g/cm · s)
1. CO_2	0.133	44.01	1462×10^{-7}
2. O_2	0.039	32.00	2031×10^{-7}
3. N_2	0.828	28.02	1754×10^{-7}

SOLUTION

Use Eqs. 1.4-16 and 15 (in that order). The calculations can be systematized in tabular form, thus:

α	β	M_α/M_β	μ_α/μ_β	$\Phi_{\alpha\beta}$	$\sum_{\beta=1}^{3} x_\beta\Phi_{\alpha\beta}$
1.	1	1.000	1.000	1.000	
	2	1.375	0.720	0.730	0.763
	3	1.571	0.834	0.727	
2.	1	0.727	1.389	1.394	
	2	1.000	1.000	1.000	1.057
	3	1.142	1.158	1.006	
3.	1	0.637	1.200	1.370	
	2	0.876	0.864	0.993	1.049
	3	1.000	1.000	1.000	

[11] H. L. Johnston and K. E. McCloskey, *J. Phys. Chem.*, **44**, 1038–1058 (1940).

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Newton's law of viscosity (Eq. 1.1-2 or 1.2-7) with two modifications: (i) the viscosity μ is replaced by an *effective viscosity* μ_{eff}, and (ii) the velocity and stress components are then redefined (with no change of symbol) as the analogous quantities averaged over a volume large with respect to the interparticle distances and small with respect to the dimensions of the flow system. This kind of theory is satisfactory as long as the flow involved is steady; in time-dependent flows, it has been shown that Newton's law of viscosity is inappropriate, and the two-phase systems have to be regarded as viscoelastic materials.[1]

The first major contribution to the theory of the *viscosity of suspensions of spheres* was that of Einstein.[2] He considered a suspension of rigid spheres, so dilute that the movement of one sphere does not influence the fluid flow in the neighborhood of any other sphere. Then it suffices to analyze only the motion of the fluid around a single sphere, and the effects of the individual spheres are additive. The *Einstein equation* is

$$\frac{\mu_{\text{eff}}}{\mu_0} = 1 + \frac{5}{2}\phi \qquad (1.6\text{-}1)$$

in which μ_0 is the viscosity of the suspending medium, and ϕ is the volume fraction of the spheres. Einstein's pioneering result has been modified in many ways, a few of which we now describe.

For *dilute suspensions of particles of various shapes* the constant $\frac{5}{2}$ has to be replaced by a different coefficient depending on the particular shape. Suspensions of elongated or flexible particles exhibit non-Newtonian viscosity.[3,4,5,6]

For *concentrated suspensions of spheres* (that is, ϕ greater than about 0.05) particle interactions become appreciable. Numerous semiempirical expressions have been developed, one of the simplest of which is the *Mooney equation*[7]

$$\frac{\mu_{\text{eff}}}{\mu_0} = \exp\left(\frac{\frac{5}{2}\phi}{1 - (\phi/\phi_0)}\right) \qquad (1.6\text{-}2)$$

in which ϕ_0 is an empirical constant between about 0.74 and 0.52, these values corresponding to the values of ϕ for closest packing and cubic packing, respectively.

---

[1] For dilute suspensions of rigid spheres, the linear viscoelastic behavior has been studied by H. Fröhlich and R. Sack, *Proc. Roy. Soc.*, **A185**, 415–430 (1946), and for dilute emulsions, the analogous derivation has been given by J. G. Oldroyd, *Proc. Roy. Soc.*, **A218**, 122–132 (1953). In both of these publications the fluid is described by the Jeffreys model (see Eq. 8.4-4), and the authors found the relations between the three parameters in the Jeffreys model and the constants describing the structure of the two-phase system (the volume fraction of suspended material and the viscosities of the two phases). For further comments concerning suspensions and rheology, see R. B. Bird and J. M. Wiest, Chapter 3 in *Handbook of Fluid Dynamics and Fluid Machinery*, J. A. Schetz and A. E. Fuhs (eds.), Wiley, New York (1996).

[2] **Albert Einstein** (1879–1955) received the Nobel prize for his explanation of the photoelectric effect, not for his development of the theory of special relativity. His seminal work on suspensions appeared in A. Einstein, *Ann. Phys.* (*Leipzig*), **19**, 289–306 (1906); erratum, *ibid.*, **34**, 591–592 (1911). In the original publication, Einstein made an error in the derivation and got ϕ instead of $\frac{5}{2}\phi$. After experiments showed that his equation did not agree with the experimental data, he recalculated the coefficient. Einstein's original derivation is quite lengthy; for a more compact development, see L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, Oxford, 2nd edition (1987), pp. 73–75. The mathematical formulation of multiphase fluid behavior can be found in D. A. Drew and S. L. Passman, *Theory of Multicomponent Fluids*, Springer, Berlin (1999).

[3] H. L. Frisch and R. Simha, Chapter 14 in *Rheology*, Vol. 1, (F. R. Eirich, ed.), Academic Press, New York (1956), Sections II and III.

[4] E. W. Merrill, Chapter 4 in *Modern Chemical Engineering*, Vol. 1, (A. Acrivos, ed.), Reinhold, New York (1963), p. 165.

[5] E. J. Hinch and L. G. Leal, *J. Fluid Mech.*, **52**, 683–712 (1972); **76**, 187–208 (1976).

[6] W. R. Schowalter, *Mechanics of Non-Newtonian Fluids*, Pergamon, Oxford (1978), Chapter 13.

[7] M. Mooney, *J. Coll. Sci.*, **6**, 162–170 (1951).

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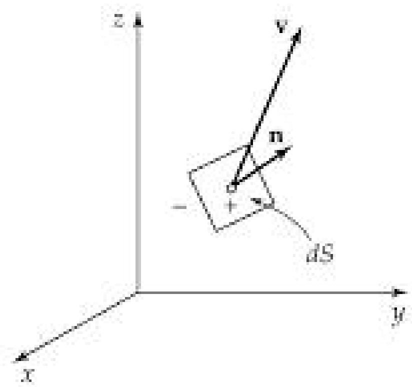

Fig. 1.7-2 The convective momentum flux through a plane of arbitrary orientation $\mathbf{n}$ is $(\mathbf{n} \cdot \mathbf{v})\rho\mathbf{v} = [\mathbf{n} \cdot \rho\mathbf{v}\mathbf{v}]$.

Next we ask what the convective momentum flux would be through a surface element whose orientation is given by a unit normal vector $\mathbf{n}$ (see Fig. 1.7-2). If a fluid is flowing through the surface dS with a velocity $\mathbf{v}$, then the volume rate of flow through the surface, from the minus side to the plus side, is $(\mathbf{n} \cdot \mathbf{v})dS$. Hence the rate of flow of momentum across the surface is $(\mathbf{n} \cdot \mathbf{v})\rho\mathbf{v}dS$, and the convective momentum flux is $(\mathbf{n} \cdot \mathbf{v})\rho\mathbf{v}$. According to the rules for vector-tensor notation given in Appendix A, this can also be written as $[\mathbf{n} \cdot \rho\mathbf{v}\mathbf{v}]$—that is, the dot product of the unit normal vector $\mathbf{n}$ with the convective momentum flux tensor $\rho\mathbf{v}\mathbf{v}$. If we let $\mathbf{n}$ be successively the unit vectors pointing in the x, y, and z directions (i.e., $\boldsymbol{\delta}_x$, $\boldsymbol{\delta}_y$, and $\boldsymbol{\delta}_z$), we obtain the entries in the second column of Table 1.7-1.

Similarly, the *total molecular momentum flux* through a surface of orientation $\mathbf{n}$ is given by $[\mathbf{n} \cdot \boldsymbol{\pi}] = p\mathbf{n} + [\mathbf{n} \cdot \boldsymbol{\tau}]$. It is understood that this is the flux from the minus side to the plus side of the surface. This quantity can also be interpreted as the force per unit area exerted by the minus material on the plus material across the surface. A geometric interpretation of $[\mathbf{n} \cdot \boldsymbol{\pi}]$ is given in Problem 1D.2.

In this chapter we defined the *molecular transport* of momentum in §1.2, and in this section we have described the *convective transport* of momentum. In setting up shell momentum balances in Chapter 2 and in setting up the general momentum balance in Chapter 3, we shall find it useful to define the *combined momentum flux*, which is the sum of the molecular momentum flux and the convective momentum flux:

$$\boldsymbol{\phi} = \boldsymbol{\pi} + \rho\mathbf{v}\mathbf{v} = p\boldsymbol{\delta} + \boldsymbol{\tau} + \rho\mathbf{v}\mathbf{v} \tag{1.7-2}$$

Keep in mind that the contribution $p\boldsymbol{\delta}$ contains no velocity, only the pressure; the combination $\rho\mathbf{v}\mathbf{v}$ contains the density and products of the velocity components; and the contribution $\boldsymbol{\tau}$ contains the viscosity and, for a Newtonian fluid, is linear in the velocity gradients. All these quantities are second-order tensors.

Most of the time we will be dealing with components of these quantities. For example the components of $\boldsymbol{\phi}$ are

$$\phi_{xx} = \pi_{xx} + \rho v_x v_x = p + \tau_{xx} + \rho v_x v_x \tag{1.7-3a}$$

$$\phi_{xy} = \pi_{xy} + \rho v_x v_y = \tau_{xy} + \rho v_x v_y \tag{1.7-3b}$$

and so on, paralleling the entries in Tables 1.2-1 and 1.7-1. The important thing to remember is that

ϕ_{xy} = the combined flux of y-momentum across a surface perpendicular to the x direction by molecular and convective mechanisms.

The second index gives the component of momentum being transported and the first index gives the direction of transport.

The various symbols and nomenclature that are used for momentum fluxes are given in Table 1.7-2. The same sign convention is used for all fluxes.

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Chapter 2

Shell Momentum Balances and Velocity Distributions in Laminar Flow

In this chapter we show how to obtain the velocity profiles for laminar flows of fluids in simple flow systems. These derivations make use of the definition of viscosity, the expressions for the molecular and convective momentum fluxes, and the concept of a momentum balance. Once the velocity profiles have been obtained, we can then get other quantities such as the maximum velocity, the average velocity, or the shear stress at a surface. Often it is these latter quantities that are of interest in engineering problems.

In the first section we make a few general remarks about how to set up differential momentum balances. In the sections that follow we work out in detail several classical examples of viscous flow patterns. These examples should be thoroughly understood, since we shall have frequent occasions to refer to them in subsequent chapters. Although these problems are rather simple and involve idealized systems, they are nonetheless often used in solving practical problems.

The systems studied in this chapter are so arranged that the reader is gradually introduced to a variety of factors that arise in the solution of viscous flow problems. In §2.2 the falling film problem illustrates the role of gravity forces and the use of Cartesian coordinates; it also shows how to solve the problem when viscosity may be a function of position. In §2.3 the flow in a circular tube illustrates the role of pressure and gravity forces and the use of cylindrical coordinates; an approximate extension to compressible flow is given. In §2.4 the flow in a cylindrical annulus emphasizes the role played by the boundary conditions. Then in §2.5 the question of boundary conditions is pursued further in the discussion of the flow of two adjacent immiscible liquids. Finally, in §2.6 the flow around a sphere is discussed briefly to illustrate a problem in spherical coordinates and also to point out how both tangential and normal forces are handled.

The methods and problems in this chapter apply only to *steady rectilinear flow*. By "steady" we mean that the pressure, density, and velocity components at each point in the stream do not change with time. We give the equations for unsteady flow in Chapter 3.

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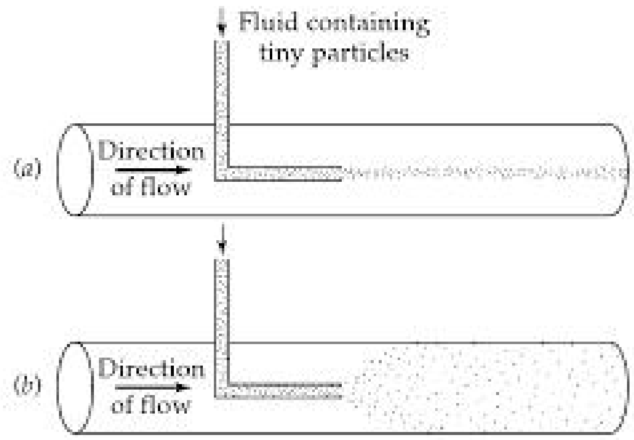

Fig. 2.0-1 (*a*) Laminar flow, in which fluid layers move smoothly over one another in the direction of flow, and (*b*) turbulent flow, in which the flow pattern is complex and time-dependent, with considerable motion perpendicular to the principal flow direction.

This chapter is concerned only with *laminar flow*. "Laminar flow" is the orderly flow that is observed, for example, in tube flow at velocities sufficiently low that tiny particles injected into the tube move along in a thin line. This is in sharp contrast with the wildly chaotic "turbulent flow" at sufficiently high velocities that the particles are flung apart and dispersed throughout the entire cross section of the tube. Turbulent flow is the subject of Chapter 5. The sketches in Fig. 2.0-1 illustrate the difference between the two flow regimes.

§2.1 SHELL MOMENTUM BALANCES AND BOUNDARY CONDITIONS

The problems discussed in §2.2 through §2.5 are approached by setting up momentum balances over a thin "shell" of the fluid. For *steady flow*, the momentum balance is

$$\begin{Bmatrix}\text{rate of}\\ \text{momentum in}\\ \text{by convective}\\ \text{transport}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{momentum out}\\ \text{by convective}\\ \text{transport}\end{Bmatrix} + \begin{Bmatrix}\text{rate of}\\ \text{momentum in}\\ \text{by molecular}\\ \text{transport}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{momentum out}\\ \text{by molecular}\\ \text{transport}\end{Bmatrix} + \begin{Bmatrix}\text{force of gravity}\\ \text{acting on system}\end{Bmatrix} = 0 \quad (2.1\text{-}1)$$

This is a restricted statement of the law of conservation of momentum. In this chapter we apply this statement only to one component of the momentum—namely, the component in the direction of flow. To write the momentum balance we need the expressions for the convective momentum fluxes given in Table 1.7-1 and the molecular momentum fluxes given in Table 1.2-1; keep in mind that the molecular momentum flux includes both the pressure and the viscous contributions.

In this chapter the momentum balance is applied only to systems in which there is just one velocity component, which depends on only one spatial variable; in addition, the flow must be rectilinear. In the next chapter the momentum balance concept is extended to unsteady-state systems with curvilinear motion and more than one velocity component.

The procedure in this chapter for setting up and solving viscous flow problems is as follows:

- Identify the nonvanishing velocity component and the spatial variable on which it depends.
- Write a momentum balance of the form of Eq. 2.1-1 over a thin shell perpendicular to the relevant spatial variable.
- Let the thickness of the shell approach zero and make use of the definition of the first derivative to obtain the corresponding differential equation for the momentum flux.

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- Integrate this equation to get the momentum-flux distribution.
- Insert Newton's law of viscosity and obtain a differential equation for the velocity.
- Integrate this equation to get the velocity distribution.
- Use the velocity distribution to get other quantities, such as the maximum velocity, average velocity, or force on solid surfaces.

In the integrations mentioned above, several constants of integration appear, and these are evaluated by using "boundary conditions"—that is, statements about the velocity or stress at the boundaries of the system. The most commonly used boundary conditions are as follows:

a. At *solid–fluid* interfaces the fluid velocity equals the velocity with which the solid surface is moving; this statement is applied to both the tangential and the normal component of the velocity vector. The equality of the tangential components is referred to as the "no-slip condition." See Problem 2D.3 for further ideas.

b. At a *liquid–liquid* interfacial plane of constant x, the tangential velocity components v_y and v_z are continuous through the interface (the "no-slip condition") as are also the molecular stress-tensor components $p + \tau_{xx}$, τ_{xy}, and τ_{xz}.

c. At a *liquid–gas* interfacial plane of constant x, the stress-tensor components τ_{xy} and τ_{xz} are taken to be zero, provided that the gas-side velocity gradient is not too large. This is reasonable, since the viscosities of gases are much less than those of liquids.

In all of these boundary conditions it is presumed that there is no material passing through the interface; that is, there is no adsorption, absorption, dissolution, evaporation, melting, or chemical reaction at the surface between the two phases. Boundary conditions incorporating such phenomena appear in Problems 3C.5 and 11C.6, and §18.1.

In this section we have presented some guidelines for solving simple viscous flow problems. For some problems slight variations on these guidelines may prove to be appropriate.

§2.2 FLOW OF A FALLING FILM

The first example we discuss is that of the flow of a liquid down an inclined flat plate of length L and width W, as shown in Fig. 2.2-1. Such films have been studied in connection with wetted-wall towers, evaporation and gas-absorption experiments, and applications of coatings. We consider the viscosity and density of the fluid to be constant.

A complete description of the liquid flow is difficult because of the disturbances at the edges of the system ($z = 0$, $z = L$, $y = 0$, $y = W$). An adequate description can often be

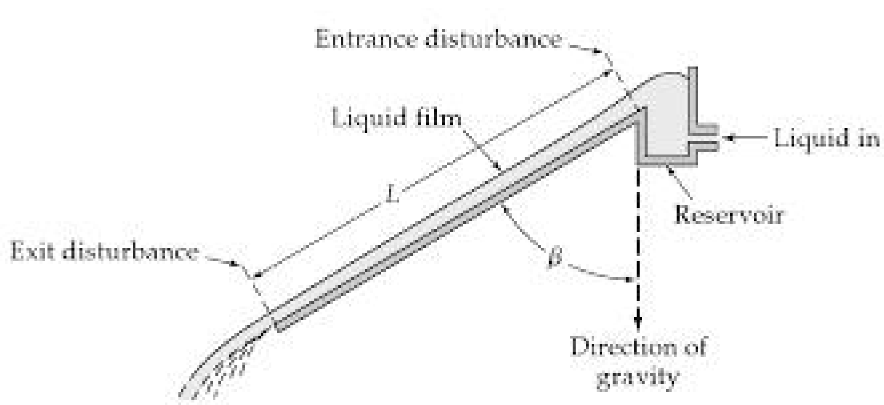

Fig. 2.2-1 Schematic diagram of the falling film experiment, showing end effects.

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obtained by neglecting such disturbances, particularly if W and L are large compared to the film thickness δ. For small flow rates we expect that the viscous forces will prevent continued acceleration of the liquid down the wall, so that v_z will become independent of z in a short distance down the plate. Therefore it seems reasonable to *postulate* that $v_z = v_z(x)$, $v_x = 0$, and $v_y = 0$, and further that $p = p(x)$. From Table B.1 it is seen that the only nonvanishing components of $\boldsymbol{\tau}$ are then $\tau_{xz} = \tau_{zx} = -\mu(dv_z/dx)$.

We now select as the "system" a thin shell perpendicular to the x direction (see Fig. 2.2-2). Then we set up a z-momentum balance over this shell, which is a region of thickness Δx, bounded by the planes $z = 0$ and $z = L$, and extending a distance W in the y direction. The various contributions to the momentum balance are then obtained with the help of the quantities in the "z-component" columns of Tables 1.2-1 and 1.7-1. By using the components of the "combined momentum-flux tensor" $\boldsymbol{\phi}$ defined in 1.7-1 to 3, we can include all the possible mechanisms for momentum transport at once:

rate of z-momentum in across surface at $z = 0$	$(W\Delta x)\phi_{zz}\big\|_{z=0}$	(2.2-1)
rate of z-momentum out across surface at $z = L$	$(W\Delta x)\phi_{zz}\big\|_{z=L}$	(2.2-2)
rate of z-momentum in across surface at x	$(LW)(\phi_{xz})\big\|_{x}$	(2.2-3)
rate of z-momentum out across surface at $x + \Delta x$	$(LW)(\phi_{xz})\big\|_{x+\Delta x}$	(2.2-4)
gravity force acting on fluid in the z direction	$(LW\,\Delta x)(\rho g \cos\beta)$	(2.2-5)

By using the quantities ϕ_{xz} and ϕ_{zz} we account for the z-momentum transport by all mechanisms, convective and molecular. Note that we take the "in" and "out" directions in the direction of the positive x- and z-axes (in this problem these happen to coincide with the directions of z-momentum transport). The notation $|_{x+\Delta x}$ means "evaluated at $x + \Delta x$," and g is the gravitational acceleration.

When these terms are substituted into the z-momentum balance of Eq. 2.1-1, we get

$$LW(\phi_{xz}|_x - \phi_{xz}|_{x+\Delta x}) + W\Delta x(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}) + (LW\,\Delta x)(\rho g \cos\beta) = 0 \qquad (2.2\text{-}6)$$

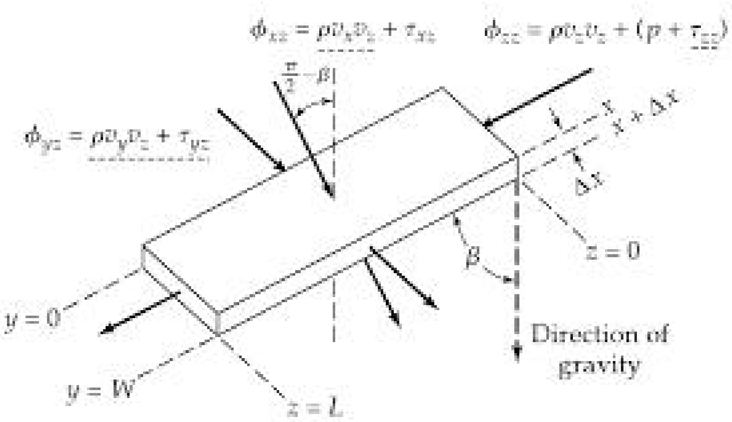

Fig. 2.2-2 Shell of thickness Δx over which a z-momentum balance is made. Arrows show the momentum fluxes associated with the surfaces of the shell. Since v_x and v_y are both zero, $\rho v_x v_z$ and $\rho v_y v_z$ are zero. Since v_z does not depend on y and z, it follows from Table B.1 that $\tau_{yz} = 0$ and $\tau_{zz} = 0$. Therefore, the dashed-underlined fluxes do not need to be considered. Both p and $\rho v_z v_z$ are the same at $z = 0$ and $z = L$, and therefore do not appear in the final equation for the balance of z-momentum, Eq. 2.2-10.

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When this equation is divided by $LW\,\Delta x$, and the limit taken as Δx approaches zero, we get

$$\lim_{\Delta x \to 0}\left(\frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_x}{\Delta x}\right) - \frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} = \rho g \cos \beta \tag{2.2-7}$$

The first term on the left side is exactly the definition of the derivative of ϕ_{xz} with respect to x. Therefore Eq. 2.2-7 becomes

$$\frac{\partial \phi_{xz}}{\partial x} - \frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} = \rho g \cos \beta \tag{2.2-8}$$

At this point we have to write out explicitly what the components ϕ_{xz} and ϕ_{zz} are, making use of the definition of $\boldsymbol{\phi}$ in Eqs. 1.7-1 to 3 and the expressions for τ_{xz} and τ_{zz} in Appendix B.1. This ensures that we do not miss out on any of the forms of momentum transport. Hence we get

$$\phi_{xz} = \tau_{xz} + \rho v_x v_z = -\mu \frac{\partial v_z}{\partial x} + \rho v_x v_z \tag{2.2-9a}$$

$$\phi_{zz} = p + \tau_{zz} + \rho v_z v_z = p - 2\mu \frac{\partial v_z}{\partial z} + \rho v_z v_z \tag{2.2-9b}$$

In accordance with the postulates that $v_z = v_z(x)$, $v_x = 0$, $v_y = 0$, and $p = p(x)$, we see that (i) since $v_x = 0$, the $\rho v_x v_z$ term in Eq. 2.2-9a is zero; (ii) since $v_z = v_z(x)$, the term $-2\mu(\partial v_z/\partial z)$ in Eq. 2.2-9b is zero; (iii) since $v_z = v_z(x)$, the term $\rho v_z v_z$ is the same at $z = 0$ and $z = L$; and (iv) since $p = p(x)$, the contribution p is the same at $z = 0$ and $z = L$. Hence τ_{xz} depends only on x, and Eq. 2.2-8 simplifies to

$$\boxed{\frac{d\tau_{xz}}{dx} = \rho g \cos \beta} \tag{2.2-10}$$

This is the differential equation for the momentum flux τ_{xz}. It may be integrated to give

$$\tau_{xz} = (\rho g \cos \beta)x + C_1 \tag{2.2-11}$$

The constant of integration may be evaluated by using the boundary condition at the gas–liquid interface (see §2.1):

B.C. 1: $\quad$ at $x = 0$, $\quad \tau_{xz} = 0 \qquad$ (2.2-12)

Substitution of this boundary condition into Eq. 2.2-11 shows that $C_1 = 0$. Therefore the momentum-flux distribution is

$$\tau_{xz} = (\rho g \cos \beta)x \tag{2.2-13}$$

as shown in Fig. 2.2-3.

Next we substitute Newton's law of viscosity

$$\tau_{xz} = -\mu \frac{dv_z}{dx} \tag{2.2-14}$$

into the left side of Eq. 2.2-13 to obtain

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos \beta}{\mu}\right)x \tag{2.2-15}$$

which is the differential equation for the velocity distribution. It can be integrated to give

$$v_z = -\left(\frac{\rho g \cos \beta}{2\mu}\right)x^2 + C_2 \tag{2.2-16}$$

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Reynolds number should be used to delineate the flow regimes. We shall have more to say about this in §3.7.

This discussion illustrates a very important point: theoretical analysis of flow systems is limited by the postulates that are made in setting up the problem. It is absolutely necessary to do experiments in order to establish the flow regimes so as to know when instabilities (spontaneous oscillations) occur and when the flow becomes turbulent. Some information about the onset of instability and the demarcation of the flow regimes can be obtained by theoretical analysis, but this is an extraordinarily difficult subject. This is a result of the inherent nonlinear nature of the governing equations of fluid dynamics, as will be explained in Chapter 3. Suffice it to say at this point that experiments play a *very* important role in the field of fluid dynamics.

EXAMPLE 2.2-1

Calculation of Film Velocity

An oil has a kinematic viscosity of $2 \times 10^{-4}\ \mathrm{m^2/s}$ and a density of $0.8 \times 10^3\ \mathrm{kg/m^3}$. If we want to have a falling film of thickness of 2.5 mm on a vertical wall, what should the mass rate of flow of the liquid be?

SOLUTION

According to Eq. 2.2-21, the mass rate of flow in kg/s is

$$w = \frac{\rho g \delta^3 W}{3\nu} = \frac{(0.8 \times 10^3)(9.80)(2.5 \times 10^{-3})^3 W}{3(2 \times 10^{-4})} = 0.204W \tag{2.2-24}$$

To get the mass rate of flow one then needs to insert a value for the width of the wall in meters. This is the desired result provided that the flow is laminar and nonrippling. To determine the flow regime we calculate the Reynolds number, making use of Eqs. 2.2-21 and 24

$$\mathrm{Re} = \frac{4\delta \langle v_z \rangle \rho}{\mu} = \frac{4w/W}{\nu\rho} = \frac{4(0.204)}{(2 \times 10^{-4})(0.8 \times 10^3)} = 5.1 \tag{2.2-25}$$

This Reynolds number is sufficiently low that rippling will not be pronounced, and therefore the expression for the mass rate of flow in Eq. 2.2-24 is reasonable.

EXAMPLE 2.2-2

Falling Film with Variable Viscosity

Rework the falling film problem for a position-dependent viscosity $\mu = \mu_0 e^{-\alpha x/\delta}$, which arises when the film is nonisothermal, as in the condensation of a vapor on a wall. Here μ_0 is the viscosity at the surface of the film and α is a constant that describes how rapidly μ decreases as x increases. Such a variation could arise in the flow of a condensate down a wall with a linear temperature gradient through the film.

SOLUTION

The development proceeds as before up to Eq. 2.2-13. Then substituting Newton's law with variable viscosity into Eq. 2.2-13 gives

$$-\mu_0 e^{-\alpha x/\delta} \frac{dv_z}{dx} = \rho g x \cos\beta \tag{2.2-26}$$

This equation can be integrated, and using the boundary conditions in Eq. 2.2-17 enables us to evaluate the integration constant. The velocity profile is then

$$v_z = \frac{\rho g \delta^2 \cos\beta}{\mu_0}\left[e^{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) - e^{\alpha x/\delta}\left(\frac{x}{\alpha\delta} - \frac{1}{\alpha^2}\right)\right] \tag{2.2-27}$$

As a check we evaluate the velocity distribution for the constant-viscosity problem (that is, when α is zero). However, setting $\alpha = 0$ gives $\infty - \infty$ in the two expressions within parentheses.

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This difficulty can be overcome if we expand the two exponentials in Taylor series (see §C.2), as follows:

$$
\begin{aligned}
(v_z)_{\alpha=0} &= \frac{\rho g \delta^2 \cos\beta}{\mu_0} \lim_{\alpha \to 0} \left[\left(1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots \right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) \right. \\
&\quad \left. - \left(1 + \frac{\alpha x}{\delta} + \frac{\alpha^2 x^2}{2!\delta^2} + \frac{\alpha^3 x^3}{3!\delta^3} + \cdots \right)\left(\frac{x}{\alpha\delta} - \frac{1}{\alpha^2}\right)\right] \\
&= \frac{\rho g \delta^2 \cos\beta}{\mu_0} \lim_{\alpha \to 0} \left[\left(\frac{1}{2} + \frac{1}{3}\alpha + \cdots\right) - \left(\frac{1}{2}\frac{x^2}{\delta^2} - \frac{1}{3}\frac{x^3}{\delta^3}\alpha + \cdots\right)\right] \\
&= \frac{\rho g \delta^2 \cos\beta}{2\mu_0}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \qquad (2.2\text{-}28)
\end{aligned}
$$

which is in agreement with Eq. 2.2-18.

From Eq. 2.2-27 it may be shown that the average velocity is

$$\langle v_z \rangle = \frac{\rho g \delta^2 \cos\beta}{\mu_0}\left[e^{\alpha}\left(\frac{1}{\alpha} - \frac{2}{\alpha^2} + \frac{2}{\alpha^3}\right) - \frac{2}{\alpha^3}\right] \qquad (2.2\text{-}29)$$

The reader may verify that this result simplifies to Eq. 2.2-20 when α goes to zero.

§2.3 FLOW THROUGH A CIRCULAR TUBE

The flow of fluids in circular tubes is encountered frequently in physics, chemistry, biology, and engineering. The laminar flow of fluids in circular tubes may be analyzed by means of the momentum balance described in §2.1. The only new feature introduced here is the use of cylindrical coordinates, which are the natural coordinates for describing positions in a pipe of circular cross section.

We consider then the steady-state, laminar flow of a fluid of constant density ρ and viscosity μ in a vertical tube of length L and radius R. The liquid flows downward under the influence of a pressure difference and gravity; the coordinate system is that shown in Fig. 2.3-1. We specify that the tube length be very large with respect to the tube radius, so that "end effects" will be unimportant throughout most of the tube; that is, we can ignore the fact that at the tube entrance and exit the flow will not necessarily be parallel to the tube wall.

We postulate that $v_z = v_z(r)$, $v_r = 0$, $v_\theta = 0$, and $p = p(z)$. With these postulates it may be seen from Table B.1 that the only nonvanishing components of $\boldsymbol{\tau}$ are $\tau_{rz} = \tau_{zr} = -\mu(dv_z/dr)$.

We select as our system a cylindrical shell of thickness Δr and length L and we begin by listing the various contributions to the z-momentum balance:

rate of z-momentum in across annular surface at $z = 0$	$(2\pi r\Delta r)(\phi_{zz})\|_{z=0}$	(2.3-1)
rate of z-momentum out across annular surface at $z = L$	$(2\pi r\Delta r)(\phi_{zz})\|_{z=L}$	(2.3-2)
rate of z-momentum in across cylindrical surface at r	$(2\pi rL)(\phi_{rz})\|_r = (2\pi rL\phi_{rz})\|_r$	(2.3-3)
rate of z-momentum out across cylindrical surface at $r + \Delta r$	$(2\pi(r+\Delta r)L)(\phi_{rz})\|_{r+\Delta r} = (2\pi rL\phi_{rz})\|_{r+\Delta r}$	(2.3-4)
gravity force acting in z direction on cylindrical shell	$(2\pi r\Delta rL)\rho g$	(2.3-5)

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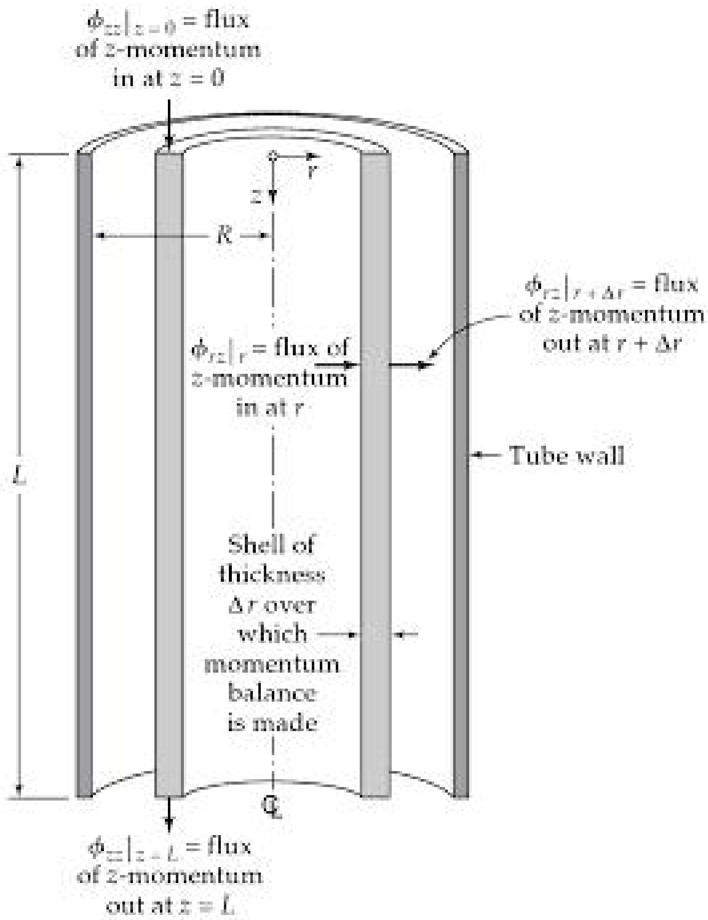

Fig. 2.3-1 Cylindrical shell of fluid over which the z-momentum balance is made for axial flow in a circular tube (see Eqs. 2.3-1 to 5). The z-momentum fluxes ϕ_{rz} and ϕ_{zz} are given in full in Eqs. 2.3-9a and 9b.

The quantities ϕ_{zz} and ϕ_{rz} account for the momentum transport by all possible mechanisms, convective and molecular. In Eq. 2.3-4, $(r + \Delta r)$ and $(r)|_{r+\Delta r}$ are two ways of writing the same thing. Note that we take "in" and "out" to be in the positive directions of the r- and z-axes.

We now add up the contributions to the momentum balance:

$$(2\pi r L \phi_{rz})|_r - (2\pi r L \phi_{rz})|_{r+\Delta r} + (2\pi r \Delta r)(\phi_{zz})|_{z=0} - (2\pi r \Delta r)(\phi_{zz})|_{z=L} + (2\pi r \Delta r L)\rho g = 0 \quad (2.3\text{-}6)$$

When we divide Eq. 2.3-6 by $2\pi L \Delta r$ and take the limit as $\Delta r \to 0$, we get

$$\lim_{\Delta r \to 0}\left(\frac{(r\phi_{rz})|_{r+\Delta r} - (r\phi_{rz})|_r}{\Delta r}\right) = \left(\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} + \rho g\right)r \quad (2.3\text{-}7)$$

The expression on the left side is the definition of the first derivative of $r\phi_{rz}$ with respect to r. Hence Eq. 2.3-7 may be written as

$$\frac{\partial}{\partial r}(r\phi_{rz}) = \left(\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} + \rho g\right)r \quad (2.3\text{-}8)$$

Now we have to evaluate the components ϕ_{rz} and ϕ_{zz} from Eq. 1.7-1 and Appendix B.1:

$$\phi_{rz} = \tau_{rz} + \rho v_r v_z = -\mu \frac{\partial v_z}{\partial r} + \rho v_r v_z \quad (2.3\text{-}9a)$$

$$\phi_{zz} = p + \tau_{zz} + \rho v_z v_z = p - 2\mu \frac{\partial v_z}{\partial z} + \rho v_z v_z \quad (2.3\text{-}9b)$$

Next we take into account the postulates made at the beginning of the problem—namely, that $v_z = v_z(r)$, $v_r = 0$, $v_\theta = 0$, and $p = p(z)$. Then we make the following simplifications:

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(i) because $v_r = 0$, we can drop the term $\rho v_r v_z$ in Eq. 2.3-9a; (ii) because $v_z = v_z(r)$, the term $\rho v_z v_z$ will be the same at both ends of the tube; and (iii) because $v_z = v_z(r)$, the term $-2\mu \partial v_z/\partial z$ will be zero. Hence Eq. 2.3-8 simplifies to

$$\frac{d}{dr}(r\tau_{rz}) = \left(\frac{(p_0 - \rho g 0) - (p_L - \rho g L)}{L}\right) r = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right) r \tag{2.3-10}$$

in which $\mathscr{P} = p - \rho g z$ is a convenient abbreviation for the sum of the pressure and gravitational terms.[1] Equation 2.3-10 may be integrated to give

$$\tau_{rz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) r + \frac{C_1}{r} \tag{2.3-11}$$

The constant C_1 is evaluated by using the boundary condition

B.C. 1: $\qquad$ at $r = 0$, $\quad \tau_{rz} = \text{finite}$ $\qquad$ (2.3-12)

Consequently C_1 must be zero, for otherwise the momentum flux would be infinite at the axis of the tube. Therefore the momentum flux distribution is

$$\boxed{\tau_{rz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) r} \tag{2.3-13}$$

This distribution is shown in Fig. 2.3-2.

Newton's law of viscosity for this situation is obtained from Appendix B.1 as follows:

$$\tau_{rz} = -\mu \frac{dv_z}{dr} \tag{2.3-14}$$

Substitution of this expression into Eq. 2.3-13 then gives the following differential equation for the velocity:

$$\frac{dv_z}{dr} = -\left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}\right) r \tag{2.3-15}$$

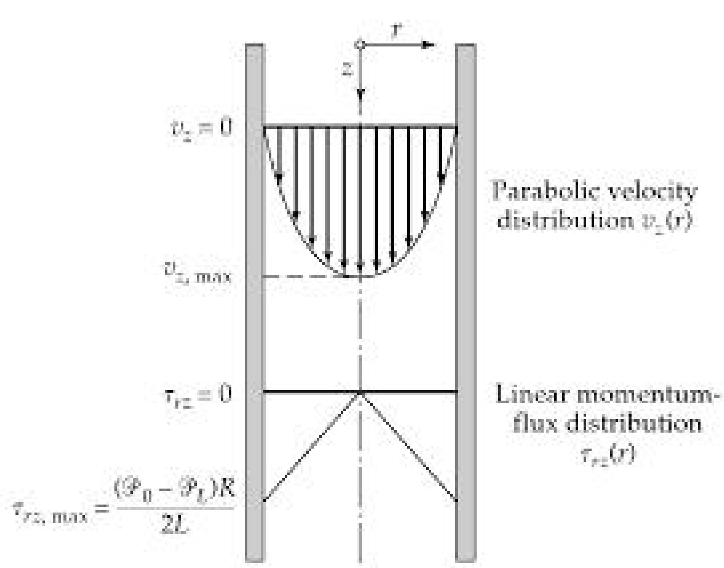

Fig. 2.3-2 The momentum-flux distribution and velocity distribution for the downward flow in a circular tube.

[1] The quantity designated by $\mathscr{P}$ is called the *modified pressure*. In general it is defined by $\mathscr{P} = p + \rho g h$, where h is the distance "upward"—that is, in the direction opposed to gravity from some preselected reference plane. Hence in this problem $h = -z$.

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This first-order separable differential equation may be integrated to give

$$v_z = -\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu L}\right)r^2 + C_2 \tag{2.3-16}$$

The constant C_2 is evaluated from the boundary condition

B.C. 2: $$\text{at } r = R, \qquad v_z = 0 \tag{2.3-17}$$

From this C_2 is found to be $(\mathcal{P}_0 - \mathcal{P}_L)R^2/4\mu L$. Hence the velocity distribution is

$$\boxed{v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{2.3-18}$$

We see that the velocity distribution for laminar, incompressible flow of a Newtonian fluid in a long tube is parabolic (see Fig. 2.3-2).

Once the velocity profile has been established, various derived quantities can be obtained:

(i) The *maximum velocity* $v_{z,\max}$ occurs at $r = 0$ and is

$$v_{z,\max} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L} \tag{2.3-19}$$

(ii) The *average velocity* $\langle v_z \rangle$ is obtained by dividing the total volumetric flow rate by the cross-sectional area

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L} = \tfrac{1}{2}v_{z,\max} \tag{2.3-20}$$

(iii) The *mass rate of flow* w is the product of the cross-sectional area πR^2, the density ρ, and the average velocity $\langle v_z \rangle$

$$w = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4\rho}{8\mu L} \tag{2.3-21}$$

This rather famous result is called the *Hagen–Poiseuille*[2] *equation*. It is used, along with experimental data for the rate of flow and the modified pressure difference, to determine the viscosity of fluids (see Example 2.3-1) in a "capillary viscometer."

(iv) The z-component of the *force*, F_z, *of the fluid on the wetted surface* of the pipe is just the shear stress τ_{rz} integrated over the wetted area

$$\begin{aligned} F_z &= (2\pi RL)\left(-\mu\frac{dv_z}{dr}\right)\Bigg|_{r=R} = \pi R^2(\mathcal{P}_0 - \mathcal{P}_L) \\ &= \pi R^2(p_0 - p_L) + \pi R^2 L\rho g \end{aligned} \tag{2.3-22}$$

This result states that the viscous force F_z is counterbalanced by the net pressure force and the gravitational force. This is exactly what one would obtain from making a force balance over the fluid in the tube.

[2] G. Hagen, *Ann. Phys. Chem.*, **46**, 423–442 (1839); J. L. Poiseuille, *Comptes Rendus*, **11**, 961 and 1041 (1841). **Jean Louis Poiseuille** (1799–1869) (pronounced "Pwa-zø'-yuh," with ø being roughly the "oo" in book) was a physician interested in the flow of blood. Although Hagen and Poiseuille established the dependence of the flow rate on the fourth power of the tube radius, Eq. 2.3-21 was first derived by E. Hagenbach, *Pogg. Annalen der Physik u. Chemie*, **108**, 385–426 (1860).

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The results of this section are only as good as the postulates introduced at the beginning of the section—namely, that $v_z = v_z(r)$ and $p = p(z)$. Experiments have shown that these postulates are in fact realized for Reynolds numbers up to about 2100; above that value, the flow will be turbulent if there are any appreciable disturbances in the system—that is, wall roughness or vibrations.[3] For circular tubes the Reynolds number is defined by $\text{Re} = D\langle v_z\rangle\rho/\mu$, where $D = 2R$ is the tube diameter.

We now summarize all the assumptions that were made in obtaining the Hagen–Poiseuille equation.

(a) The flow is laminar; that is, Re must be less than about 2100.

(b) The density is constant ("incompressible flow").

(c) The flow is "steady" (i.e., it does not change with time).

(d) The fluid is Newtonian (Eq. 2.3-14 is valid).

(e) End effects are neglected. Actually an "entrance length," after the tube entrance, of the order of $L_e = 0.035D\,\text{Re}$, is needed for the buildup to the parabolic profile. If the section of pipe of interest includes the entrance region, a correction must be applied.[4] The fractional correction in the pressure difference or mass rate of flow never exceeds L_e/L if $L > L_e$.

(f) The fluid behaves as a continuum—this assumption is valid, except for very dilute gases or very narrow capillary tubes, in which the molecular mean free path is comparable to the tube diameter (the "slip flow region") or much greater than the tube diameter (the "Knudsen flow" or "free molecule flow" regime).[5]

(g) There is no slip at the wall, so that B.C. 2 is valid; this is an excellent assumption for pure fluids under the conditions assumed in (f). See Problem 2B.9 for a discussion of wall slip.

EXAMPLE 2.3-1

Determination of Viscosity from Capillary Flow Data

Glycerine ($CH_2OH \cdot CHOH \cdot CH_2OH$) at 26.5°C is flowing through a horizontal tube 1 ft long and with 0.1 in. inside diameter. For a pressure drop of 40 psi, the volume flow rate w/ρ is 0.00398 ft³/min. The density of glycerine at 26.5°C is 1.261 g/cm³. From the flow data, find the viscosity of glycerine in centipoises and in Pa · s.

SOLUTION

From the Hagen–Poiseuille equation (Eq. 2.3-21), we find

$$\begin{aligned}\mu &= \frac{\pi(p_0 - p_L)R^4}{8(w/\rho)L}\\ &= \frac{\pi\left(40\,\dfrac{\text{lb}_f}{\text{in.}^2}\right)\left(6.8947\times 10^4\,\dfrac{\text{dyn/cm}^2}{\text{lb}_f/\text{in.}^2}\right)\left(0.05\text{ in.}\times\dfrac{1}{12}\,\dfrac{\text{ft}}{\text{in.}}\right)^4}{8\left(0.00398\,\dfrac{\text{ft}^3}{\text{min}}\times\dfrac{1}{60}\,\dfrac{\text{min}}{\text{s}}\right)(1\text{ ft})}\\ &= 4.92\text{ g/cm}\cdot\text{s} = 492\text{ cp} = 0.492\text{ Pa}\cdot\text{s}\end{aligned} \tag{2.3-23}$$

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[3] A. A. Draad [Doctoral Dissertation, Technical University of Delft (1996)] in a carefully controlled experiment, attained laminar flow up to Re = 60,000. He also studied the nonparabolic velocity profile induced by the earth's rotation (through the Coriolis effect). See also A. A. Draad and F. T. M. Nieuwstadt, *J. Fluid. Mech.*, **361**, 207–308 (1998).

[4] J. H. Perry, *Chemical Engineers Handbook*, McGraw-Hill, New York, 3rd edition (1950), pp. 388–389; W. M. Kays and A. L. London, *Compact Heat Exchangers*, McGraw-Hill, New York (1958), p. 49.

[5] **Martin Hans Christian Knudsen** (1871–1949), professor of physics at the University of Copenhagen, did key experiments on the behavior of very dilute gases. The lectures he gave at the University of Glasgow were published as M. Knudsen, *The Kinetic Theory of Gases*, Methuen, London (1934); G. N. Patterson, *Molecular Flow of Gases*, Wiley, New York (1956). See also J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases*, North-Holland, Amsterdam (1972), Chapter 15.

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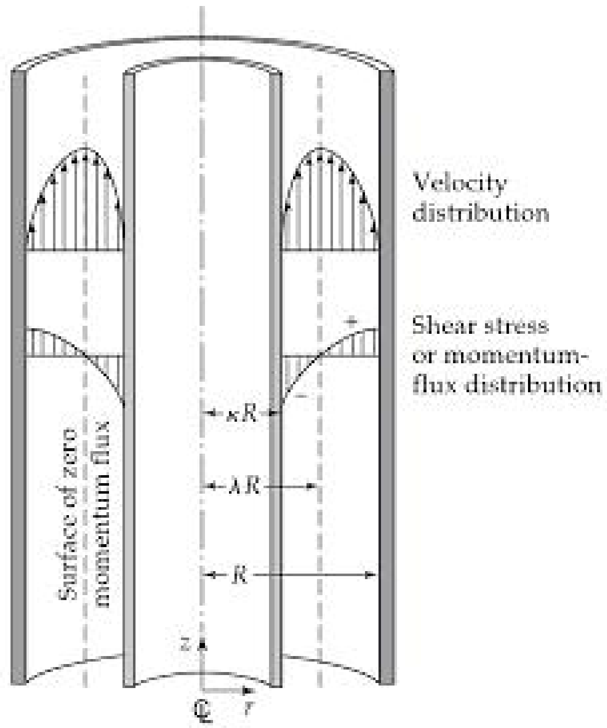

Fig. 2.4-1 The momentum-flux distribution and velocity distribution for the upward flow in a cylindrical annulus. Note that the momentum flux changes sign at the same value of r for which the velocity has a maximum.

the tube—that is, in the direction opposed to gravity. We make the same postulates as in §2.3: $v_z = v_z(r)$, $v_\theta = 0$, $v_r = 0$, and $p = p(z)$. Then when we make a momentum balance over a thin cylindrical shell of liquid, we arrive at the following differential equation:

$$\frac{d}{dr}(r\tau_{rz}) = \left(\frac{(p_0 + \rho g 0) - (p_L + \rho g L)}{L}\right)r = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right)r \tag{2.4-1}$$

This differs from Eq. 2.3-10 only in that $\mathcal{P} = p + \rho g z$ here, since the coordinate z is in the direction opposed to gravity (i.e., z is the same as the h of footnote 1 in §2.3). Integration of Eq. 2.4-1 gives

$$\tau_{rz} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2L}\right)r + \frac{C_1}{r} \tag{2.4-2}$$

just as in Eq. 2.3-11.

The constant C_1 cannot be determined immediately, since we have no information about the momentum flux at the fixed surfaces $r = \kappa R$ and $r = R$. All we know is that there will be a maximum in the velocity curve at some (as yet unknown) surface $r = \lambda R$ at which the momentum flux will be zero. That is,

$$0 = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2L}\right)\lambda R + \frac{C_1}{\lambda R} \tag{2.4-3}$$

When we solve this equation for C_1 and substitute it into Eq. 2.4-2, we get

$$\tau_{rz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L}\left[\left(\frac{r}{R}\right) - \lambda^2\left(\frac{R}{r}\right)\right] \tag{2.4-4}$$

The only difference between this equation and Eq. 2.4-2 is that the constant of integration C_1 has been eliminated in favor of a different constant λ. The advantage of this is that we know the geometrical significance of λ.

We now substitute Newton's law of viscosity, $\tau_{rz} = -\mu(dv_z/dr)$, into Eq. 2.4-4 to obtain a differential equation for v_z

$$\frac{dv_z}{dr} = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2\mu L}\left[\left(\frac{r}{R}\right) - \lambda^2\left(\frac{R}{r}\right)\right] \tag{2.4-5}$$

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Integration of this first-order separable differential equation then gives

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L}\left[\left(\frac{r}{R}\right)^2 - 2\lambda^2 \ln\left(\frac{r}{R}\right) + C_2\right] \tag{2.4-6}$$

We now evaluate the two constants of integration, λ and C_2, by using the no-slip condition on each solid boundary:

B.C. 1: at $r = \kappa R$, $v_z = 0$ (2.4-7)

B.C. 2: at $r = R$, $v_z = 0$ (2.4-8)

Substitution of these boundary conditions into Eq. 2.4-6 then gives two simultaneous equations:

$$0 = \kappa^2 - 2\lambda^2 \ln \kappa + C_2; \qquad 0 = 1 + C_2 \tag{2.4-9, 10}$$

From these the two integration constants λ and C_2 are found to be

$$C_2 = -1; \qquad 2\lambda^2 = \frac{1 - \kappa^2}{\ln(1/\kappa)} \tag{2.4-11, 12}$$

These expressions can be inserted into Eqs. 2.4-4 and 2.4-6 to give the momentum-flux distribution and the velocity distribution[1] as follows:

$$\boxed{\tau_{rz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L}\left[\left(\frac{r}{R}\right) - \frac{1 - \kappa^2}{2\ln(1/\kappa)}\left(\frac{R}{r}\right)\right]} \tag{2.4-13}$$

$$\boxed{v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2 - \frac{1 - \kappa^2}{\ln(1/\kappa)}\ln\left(\frac{R}{r}\right)\right]} \tag{2.4-14}$$

Note that when the annulus becomes very thin (i.e., κ only slightly less than unity), these results simplify to those for a plane slit (see Problem 2B.5). It is always a good idea to check "limiting cases" such as these whenever the opportunity presents itself.

The lower limit of $\kappa \to 0$ is not so simple, because the ratio $\ln(R/r)/\ln(1/\kappa)$ will always be important in a region close to the inner boundary. Hence Eq. 2.4-14 does not simplify to the parabolic distribution. However, Eq. 2.4-17 for the mass rate of flow does simplify to the Hagen–Poiseuille equation.

Once we have the momentum-flux and velocity distributions, it is straightforward to get other results of interest:

(i) The *maximum velocity* is

$$v_{z,\max} = v_z|_{r=\lambda R} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L}[1 - \lambda^2(1 - \ln \lambda^2)] \tag{2.4-15}$$

where λ^2 is given in Eq. 2.4-12.

(ii) The *average velocity* is given by

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_{\kappa R}^{R} v_z r dr d\theta}{\int_0^{2\pi}\int_{\kappa R}^{R} r dr d\theta} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}\left[\frac{1 - \kappa^4}{1 - \kappa^2} - \frac{1 - \kappa^2}{\ln(1/\kappa)}\right] \tag{2.4-16}$$

(iii) The *mass rate of flow* is $w = \pi R^2(1 - \kappa^2)\rho\langle v_z \rangle$, or

$$w = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4\rho}{8\mu L}\left[(1 - \kappa^4) - \frac{(1 - \kappa^2)^2}{\ln(1/\kappa)}\right] \tag{2.4-17}$$

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[1] H. Lamb, *Hydrodynamics*, Cambridge University Press, 2nd edition (1895), p. 522.

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(iv) The *force exerted by the fluid on the solid surfaces* is obtained by summing the forces acting on the inner and outer cylinders, as follows:

$$\begin{aligned} F_z &= (2\pi\kappa RL)(-\tau_{rz}|_{r=\kappa R}) + (2\pi RL)(+\tau_{rz}|_{r=R}) \\ &= \pi R^2(1-\kappa^2)(\mathcal{P}_0 - \mathcal{P}_L) \end{aligned} \tag{2.4-18}$$

The reader should explain the choice of signs in front of the shear stresses above and also give an interpretation of the final result.

The equations derived above are valid only for laminar flow. The laminar–turbulent transition occurs in the neighborhood of Re = 2000, with the Reynolds number defined as $\text{Re} = 2R(1-\kappa)\langle v_z\rangle\rho/\mu$.

§2.5 FLOW OF TWO ADJACENT IMMISCIBLE FLUIDS[1]

Thus far we have considered flow situations with solid–fluid and liquid–gas boundaries. We now give one example of a flow problem with a liquid–liquid interface (see Fig. 2.5-1).

Two immiscible, incompressible liquids are flowing in the z direction in a horizontal thin slit of length L and width W under the influence of a horizontal pressure gradient $(p_0 - p_L)/L$. The fluid flow rates are adjusted so that the slit is half filled with fluid I (the more dense phase) and half filled with fluid II (the less dense phase). The fluids are flowing sufficiently slowly that no instabilities occur—that is, that the interface remains exactly planar. It is desired to find the momentum-flux and velocity distributions.

A differential momentum balance leads to the following differential equation for the momentum flux:

$$\frac{d\tau_{xz}}{dx} = \frac{p_0 - p_L}{L} \tag{2.5-1}$$

This equation is obtained for both phase I and phase II. Integration of Eq. 2.5-1 for the two regions gives

$$\tau_{xz}^{\text{I}} = \left(\frac{p_0 - p_L}{L}\right)x + C_1^{\text{I}} \tag{2.5-2}$$

$$\tau_{xz}^{\text{II}} = \left(\frac{p_0 - p_L}{L}\right)x + C_1^{\text{II}} \tag{2.5-3}$$

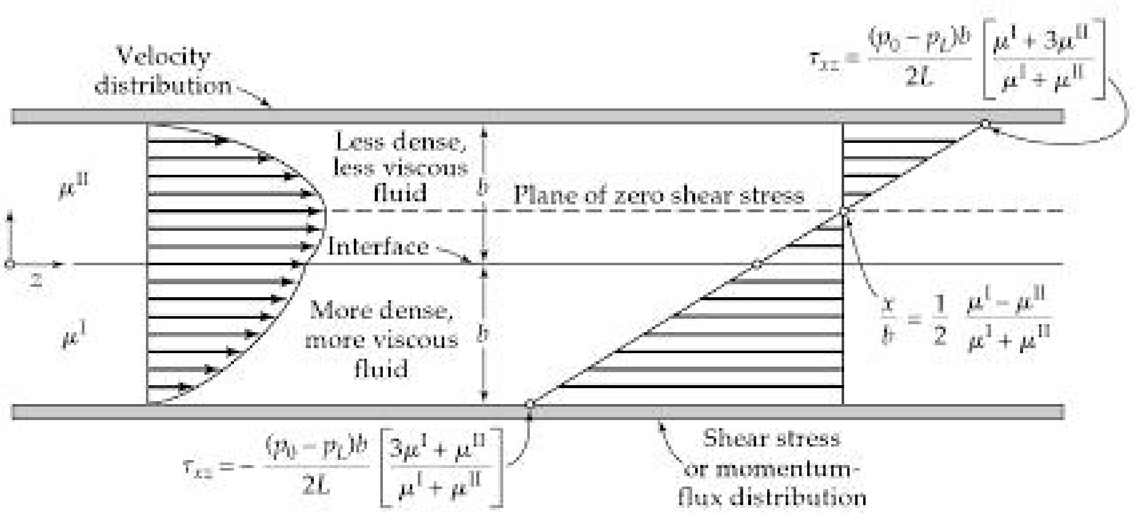

Fig. 2.5-1 Flow of two immiscible fluids between a pair of horizontal plates under the influence of a pressure gradient.

[1] The adjacent flow of gases and liquids in conduits has been reviewed by A. E. Dukler and M. Wicks, III, in Chapter 8 of *Modern Chemical Engineering*, Vol. 1, "Physical Operations," A. Acrivos (ed.), Reinhold, New York (1963).

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We may immediately make use of one of the boundary conditions—namely, that the momentum flux τ_{xz} is continuous through the fluid–fluid interface:

B.C. 1: at $x = 0$, $\tau^{\mathrm{I}}_{xz} = \tau^{\mathrm{II}}_{xz}$ (2.5-4)

This tells us that $C^{\mathrm{I}}_1 = C^{\mathrm{II}}_1$; hence we drop the superscript and call both integration constants C_1.

When Newton's law of viscosity is substituted into Eqs. 2.5-2 and 2.5-3, we get

$$-\mu^{\mathrm{I}}\frac{dv^{\mathrm{I}}_z}{dx} = \left(\frac{p_0 - p_L}{L}\right)x + C_1 \tag{2.5-5}$$

$$-\mu^{\mathrm{II}}\frac{dv^{\mathrm{II}}_z}{dx} = \left(\frac{p_0 - p_L}{L}\right)x + C_1 \tag{2.5-6}$$

These two equations can be integrated to give

$$v^{\mathrm{I}}_z = -\left(\frac{p_0 - p_L}{2\mu^{\mathrm{I}}L}\right)x^2 - \frac{C_1}{\mu^{\mathrm{I}}}x + C^{\mathrm{I}}_2 \tag{2.5-7}$$

$$v^{\mathrm{II}}_z = -\left(\frac{p_0 - p_L}{2\mu^{\mathrm{II}}L}\right)x^2 - \frac{C_1}{\mu^{\mathrm{II}}}x + C^{\mathrm{II}}_2 \tag{2.5-8}$$

The three integration constants can be determined from the following no-slip boundary conditions:

B.C. 2: at $x = 0$, $v^{\mathrm{I}}_z = v^{\mathrm{II}}_z$ (2.5-9)

B.C. 3: at $x = -b$, $v^{\mathrm{I}}_z = 0$ (2.5-10)

B.C. 4: at $x = +b$, $v^{\mathrm{II}}_z = 0$ (2.5-11)

When these three boundary conditions are applied, we get three simultaneous equations for the integration constants:

from B.C. 2: $C^{\mathrm{I}}_2 = C^{\mathrm{II}}_2$ (2.5-12)

from B.C. 3: $0 = -\left(\frac{p_0 - p_L}{2\mu^{\mathrm{I}}L}\right)b^2 + \frac{C_1}{\mu^{\mathrm{I}}}b + C^{\mathrm{I}}_2$ (2.5-13)

from B.C. 4: $0 = -\left(\frac{p_0 - p_L}{2\mu^{\mathrm{II}}L}\right)b^2 - \frac{C_1}{\mu^{\mathrm{II}}}b + C^{\mathrm{II}}_2$ (2.5-14)

From these three equations we get

$$C_1 = -\frac{(p_0 - p_L)b}{2L}\left(\frac{\mu^{\mathrm{I}} - \mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right) \tag{2.5-15}$$

$$C^{\mathrm{I}}_2 = +\frac{(p_0 - p_L)b^2}{2\mu^{\mathrm{I}}L}\left(\frac{2\mu^{\mathrm{I}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right) = C^{\mathrm{II}}_2 \tag{2.5-16}$$

The resulting momentum-flux and velocity profiles are

$$\boxed{\tau_{xz} = \frac{(p_0 - p_L)b}{L}\left[\left(\frac{x}{b}\right) - \frac{1}{2}\left(\frac{\mu^{\mathrm{I}} - \mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right)\right]} \tag{2.5-17}$$

$$\boxed{v^{\mathrm{I}}_z = \frac{(p_0 - p_L)b^2}{2\mu^{\mathrm{I}}L}\left[\left(\frac{2\mu^{\mathrm{I}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right) + \left(\frac{\mu^{\mathrm{I}} - \mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right]} \tag{2.5-18}$$

$$\boxed{v^{\mathrm{II}}_z = \frac{(p_0 - p_L)b^2}{2\mu^{\mathrm{II}}L}\left[\left(\frac{2\mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right) + \left(\frac{\mu^{\mathrm{I}} - \mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right]} \tag{2.5-19}$$

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These distributions are shown in Fig. 2.5-1. If both viscosities are the same, then the velocity distribution is parabolic, as one would expect for a pure fluid flowing between parallel plates (see Eq. 2B.3-2).

The *average velocity* in each layer can be obtained and the results are

$$\langle v_z^{\mathrm{I}} \rangle = \frac{1}{b}\int_{-b}^{0} v_z^{\mathrm{I}} dx = \frac{(\mathcal{P}_0 - \mathcal{P}_L)b^2}{12\mu^{\mathrm{I}}L}\left(\frac{7\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right) \tag{2.5-20}$$

$$\langle v_z^{\mathrm{II}} \rangle = \frac{1}{b}\int_{0}^{b} v_z^{\mathrm{II}} dx = \frac{(\mathcal{P}_0 - \mathcal{P}_L)b^2}{12\mu^{\mathrm{II}}L}\left(\frac{\mu^{\mathrm{I}} + 7\mu^{\mathrm{II}}}{\mu^{\mathrm{I}} + \mu^{\mathrm{II}}}\right) \tag{2.5-21}$$

From the velocity and momentum-flux distributions given above, one can also calculate the maximum velocity, the velocity at the interface, the plane of zero shear stress, and the drag on the walls of the slit.

§2.6 CREEPING FLOW AROUND A SPHERE[1,2,3,4]

In the preceding sections several elementary viscous flow problems have been solved. These have all dealt with rectilinear flows with only one nonvanishing velocity component. Since the flow around a sphere involves two nonvanishing velocity components, v_r and v_θ, it cannot be readily analyzed by the techniques explained at the beginning of this chapter. Nonetheless, a brief discussion of flow around a sphere is warranted here because of the importance of flow around submerged objects. In Chapter 4 we show how to obtain the velocity and pressure distributions. Here we only cite the results and show how they can be used to derive some important relations that we need in later discussions. The problem treated here, and also in Chapter 4, is concerned with "creeping flow"—that is, very slow flow. This type of flow is also referred to as "Stokes flow."

We consider here the flow of an incompressible fluid about a solid sphere of radius R and diameter D as shown in Fig. 2.6-1. The fluid, with density ρ and viscosity μ, ap-

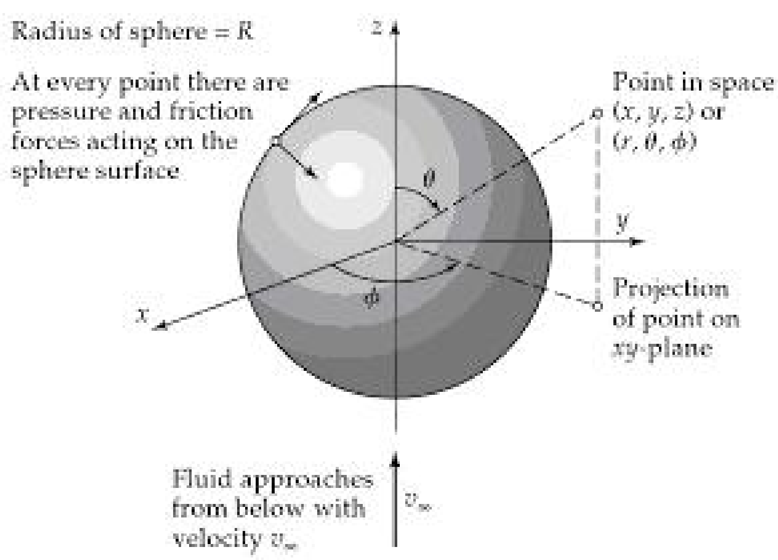

Fig. 2.6-1 Sphere of radius R around which a fluid is flowing. The coordinates r, θ, and ϕ are shown. For more information on spherical coordinates, see Fig. A.8-2.

[1] G. G. Stokes, *Trans. Cambridge Phil. Soc.*, **9**, 8–106 (1851). For creeping flow around an object of arbitrary shape, see H. Brenner, *Chem. Engr. Sci.*, **19**, 703–727 (1964).

[2] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd edition, Pergamon, London (1987), §20.

[3] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press (1967), §4.9.

[4] S. Kim and S. J. Karrila, *Microhydrodynamics: Principles and Selected Applications*, Butterworth-Heinemann, Boston (1991), §4.2.3; this book contains a thorough discussion of "creeping flow" problems.

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is $-(p + \tau_{rr})|_{r=R}(\cos\theta)$. We now multiply this by a differential element of surface $R^2 \sin\theta\, d\theta\, d\phi$ to get the force on the surface element (see Fig. A.8-2). Then we integrate over the surface of the sphere to get the resultant normal force in the z direction:

$$F^{(n)} = \int_0^{2\pi}\int_0^{\pi} (-(p + \tau_{rr})|_{r=R} \cos\theta) R^2 \sin\theta\, d\theta\, d\phi \tag{2.6-7}$$

According to Eq. 2.6-5, the normal stress τ_{rr} is zero[5] at $r = R$ and can be omitted in the integral in Eq. 2.6-7. The pressure distribution at the surface of the sphere is, according to Eq. 2.6-4,

$$p|_{r=R} = p_0 - \rho g R \cos\theta - \frac{3}{2}\frac{\mu v_\infty}{R}\cos\theta \tag{2.6-8}$$

When this is substituted into Eq. 2.6-7 and the integration performed, the term containing p_0 gives zero, the term containing the gravitational acceleration g gives the buoyant force, and the term containing the approach velocity v_∞ gives the "form drag" as shown below:

$$F^{(n)} = \tfrac{4}{3}\pi R^3 \rho g + 2\pi\mu R v_\infty \tag{2.6-9}$$

The buoyant force is the mass of displaced fluid ($\tfrac{4}{3}\pi R^3\rho$) times the gravitational acceleration (g).

Integration of the Tangential Force

At each point on the solid surface there is also a shear stress acting tangentially. The force per unit area exerted in the $-\theta$ direction by the fluid (region of greater r) on the solid (region of lesser r) is $+\tau_{r\theta}|_{r=R}$. The z-component of this force per unit area is $(\tau_{r\theta}|_{r=R})\sin\theta$. We now multiply this by the surface element $R^2 \sin\theta\, d\theta d\phi$ and integrate over the entire spherical surface. This gives the resultant force in the z direction:

$$F^{(t)} = \int_0^{2\pi}\int_0^{\pi} (\tau_{r\theta}|_{r=R} \sin\theta) R^2 \sin\theta\, d\theta\, d\phi \tag{2.6-10}$$

The shear stress distribution on the sphere surface, from Eq. 2.6-6, is

$$\tau_{r\theta}|_{r=R} = \frac{3}{2}\frac{\mu v_\infty}{R}\sin\theta \tag{2.6-11}$$

Substitution of this expression into the integral in Eq. 2.6-10 gives the "friction drag"

$$F^{(t)} = 4\pi\mu R v_\infty \tag{2.6-12}$$

Hence the total force F of the fluid on the sphere is given by the sum of Eqs. 2.6-9 and 2.6-12:

$$F = \underbrace{\tfrac{4}{3}\pi R^3 \rho g}_{\substack{\text{buoyant}\\ \text{force}}} + \underbrace{2\pi\mu R v_\infty}_{\substack{\text{form}\\ \text{drag}}} + \underbrace{4\pi\mu R v_\infty}_{\substack{\text{friction}\\ \text{drag}}} \tag{2.6-13}$$

or

$$F = F_b + F_k = \underbrace{\tfrac{4}{3}\pi R^3 \rho g}_{\substack{\text{buoyant}\\ \text{force}}} + \underbrace{6\pi\mu R v_\infty}_{\substack{\text{kinetic}\\ \text{force}}} \tag{2.6-14}$$

[5] In Example 3.1-1 we show that, for incompressible, Newtonian fluids, all three of the normal stresses are zero at fixed solid surfaces in all flows.

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The first term is the *buoyant force*, which would be present in a fluid at rest; it is the mass of the displaced fluid multiplied by the gravitational acceleration. The second term, the *kinetic force*, results from the motion of the fluid. The relation

$$F_k = 6\pi\mu R v_\infty \tag{2.6-15}$$

is known as *Stokes' law*.[1] It is used in describing the motion of colloidal particles under an electric field, in the theory of sedimentation, and in the study of the motion of aerosol particles. Stokes' law is useful only up to a Reynolds number $\text{Re} = Dv_\infty\rho/\mu$ of about 0.1. At $\text{Re} = 1$, Stokes' law predicts a force that is about 10% too low. The flow behavior for larger Reynolds numbers is discussed in Chapter 6.

This problem, which could not be solved by the shell balance method, emphasizes the need for a more general method for coping with flow problems in which the streamlines are not rectilinear. That is the subject of the following chapter.

EXAMPLE 2.6-1

Determination of Viscosity from the Terminal Velocity of a Falling Sphere

Derive a relation that enables one to get the viscosity of a fluid by measuring the terminal velocity v_t of a small sphere of radius R in the fluid.

SOLUTION

If a small sphere is allowed to fall from rest in a viscous fluid, it will accelerate until it reaches a constant velocity—the *terminal velocity*. When this steady-state condition has been reached the sum of all the forces acting on the sphere must be zero. The force of gravity on the solid acts in the direction of fall, and the buoyant and kinetic forces act in the opposite direction:

$$\tfrac{4}{3}\pi R^3 \rho_s g = \tfrac{4}{3}\pi R^3 \rho g + 6\pi\mu R v_t \tag{2.6-16}$$

Here ρ_s and ρ are the densities of the solid sphere and the fluid. Solving this equation for the viscosity gives

$$\mu = \tfrac{2}{9} R^2 (\rho_s - \rho) g / v_t \tag{2.6-17}$$

This result may be used only if the Reynolds number is less than about 0.1.

This experiment provides an apparently simple method for determining viscosity. However, it is difficult to keep a homogeneous sphere from rotating during its descent, and if it does rotate, then Eq. 2.6-17 cannot be used. Sometimes weighted spheres are used in order to preclude rotation; then the left side of Eq. 2.6-16 has to be replaced by m, the mass of the sphere, times the gravitational acceleration.

QUESTIONS FOR DISCUSSION

1. Summarize the procedure used in the solution of viscous flow problems by the shell balance method. What kinds of problems can and cannot be solved by this method? How is the definition of the first derivative used in the method?
2. Which of the flow systems in this chapter can be used as a viscometer? List the difficulties that might be encountered in each.
3. How are the Reynolds numbers defined for films, tubes, and spheres? What are the dimensions of Re?
4. How can one modify the film thickness formula in §2.2 to describe a thin film falling down the interior wall of a cylinder? What restrictions might have to be placed on this modified formula?
5. How can the results in §2.3 be used to estimate the time required for a liquid to drain out of a vertical tube that is open at both ends?
6. Contrast the radial dependence of the shear stress for the laminar flow of a Newtonian liquid in a tube and in an annulus. In the latter, why does the function change sign?

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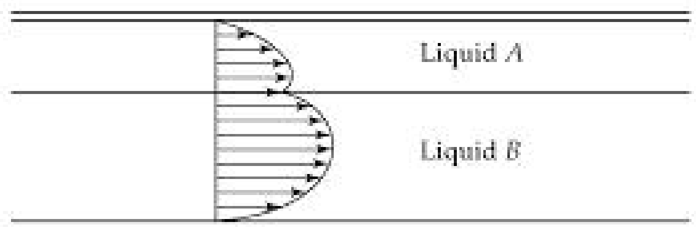
7. Show that the Hagen–Poiseuille formula is dimensionally consistent.
8. What differences are there between the flow in a circular tube of radius R and the flow in the same tube with a thin wire placed along the axis?
9. Under what conditions would you expect the analysis in §2.5 to be inapplicable?
10. Is Stokes' law valid for droplets of oil falling in water? For air bubbles rising in benzene? For tiny particles falling in air, if the particle diameters are of the order of the mean free path of the molecules in the air?
11. Two immiscible liquids, A and B, are flowing in laminar flow between two parallel plates. Is it possible that the velocity profiles would be of the following form? Explain.



12. What is the terminal velocity of a spherical colloidal particle having an electric charge e in an electric field of strength $\mathscr{E}$? How is this used in the Millikan oil-drop experiment?

PROBLEMS

2A.1 Thickness of a falling film. Water at 20°C is flowing down a vertical wall with Re = 10. Calculate **(a)** the flow rate, in gallons per hour per foot of wall width, and **(b)** the film thickness in inches.

Answers: **(a)** 0.727 gal/hr · ft; **(b)** 0.00361 in.

2A.2 Determination of capillary radius by flow measurement. One method for determining the radius of a capillary tube is by measuring the rate of flow of a Newtonian liquid through the tube. Find the radius of a capillary from the following flow data:

Length of capillary tube	50.02 cm
Kinematic viscosity of liquid	4.03×10^{-5} m²/s
Density of liquid	0.9552×10^3 kg/m³
Pressure drop in the horizontal tube	4.829×10^5 Pa
Mass rate of flow through tube	2.997×10^{-3} kg/s

What difficulties may be encountered in this method? Suggest some other methods for determining the radii of capillary tubes.

2A.3 Volume flow rate through an annulus. A horizontal annulus, 27 ft in length, has an inner radius of 0.495 in. and an outer radius of 1.1 in. A 60% aqueous solution of sucrose ($C_{12}H_{22}O_{11}$) is to be pumped through the annulus at 20°C. At this temperature the solution density is 80.3 lb_m/ft^3 and the viscosity is 136.8 $lb_m/ft \cdot hr$. What is the volume flow rate when the impressed pressure difference is 5.39 psi?

Answer: 0.110 ft³/s

2A.4 Loss of catalyst particles in stack gas.

(a) Estimate the maximum diameter of microspherical catalyst particles that could be lost in the stack gas of a fluid cracking unit under the following conditions:

Gas velocity at axis of stack	= 1.0 ft/s (vertically upward)
Gas viscosity	= 0.026 cp
Gas density	= 0.045 lb_m/ft^3
Density of a catalyst particle	= 1.2 g/cm³

Express the result in microns (1 micron = 10^{-6}m = 1μm).

(b) Is it permissible to use Stokes' law in (a)?

Answers: **(a)** 110 μm; Re = 0.93

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2B.1 Different choice of coordinates for the falling film problem. Rederive the velocity profile and the average velocity in §2.2, by replacing x by a coordinate $\bar{x}$ measured away from the wall; that is, $\bar{x} = 0$ is the wall surface, and $\bar{x} = \delta$ is the liquid–gas interface. Show that the velocity distribution is then given by

$$v_z = (\rho g \delta^2/\mu)[(\bar{x}/\delta) - \tfrac{1}{2}(\bar{x}/\delta)^2] \cos\beta \tag{2B.1-1}$$

and then use this to get the average velocity. Show how one can get Eq. 2B.1-1 from Eq. 2.2-18 by making a change of variable.

2B.2 Alternate procedure for solving flow problems. In this chapter we have used the following procedure: (i) derive an equation for the momentum flux, (ii) integrate this equation, (iii) insert Newton's law to get a first-order differential equation for the velocity, (iv) integrate the latter to get the velocity distribution. Another method is: (i) derive an equation for the momentum flux, (ii) insert Newton's law to get a second-order differential equation for the velocity profile, (iii) integrate the latter to get the velocity distribution. Apply this second method to the falling film problem by substituting Eq. 2.2-14 into Eq. 2.2-10 and continuing as directed until the velocity distribution has been obtained and the integration constants evaluated.

2B.3 Laminar flow in a narrow slit (see Fig. 2B.3).

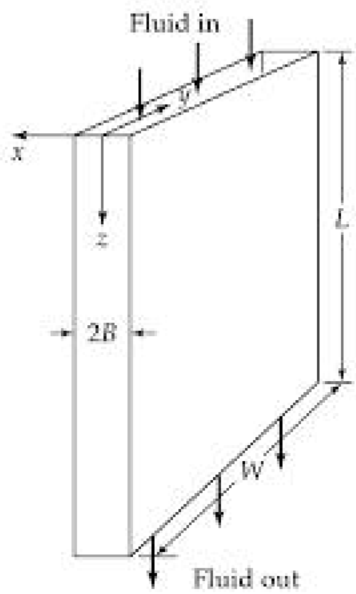

Fig. 2B.3 Flow through a slit, with $B << W << L$.

(a) A Newtonian fluid is in laminar flow in a narrow slit formed by two parallel walls a distance $2B$ apart. It is understood that $B << W$, so that "edge effects" are unimportant. Make a differential momentum balance, and obtain the following expressions for the momentum-flux and velocity distributions:

$$\tau_{xz} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right)x \tag{2B.3-1}$$

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] \tag{2B.3-2}$$

In these expressions $\mathcal{P} = p + \rho g h = p - \rho g z$.

(b) What is the ratio of the average velocity to the maximum velocity for this flow?

(c) Obtain the slit analog of the Hagen–Poiseuille equation.

(d) Draw a meaningful sketch to show why the above analysis is inapplicable if $B = W$.

(e) How can the result in (b) be obtained from the results of §2.5?

Answers: **(b)** $\langle v_z \rangle / v_{z,\max} = \frac{2}{3}$

(c) $w = \dfrac{2}{3}\dfrac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W \rho}{\mu L}$

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2B.4 Laminar slit flow with a moving wall ("plane Couette flow"). Extend Problem 2B.3 by allowing the wall at $x = B$ to move in the positive z direction at a steady speed v_0. Obtain (*a*) the shear-stress distribution and (*b*) the velocity distribution. Draw carefully labeled sketches of these functions.

Answers: $\tau_{xz} = \left(\dfrac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right)x - \dfrac{\mu v_0}{2B}$; $v_z = \dfrac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L}\left[1 - \left(\dfrac{x}{B}\right)^2\right] + \dfrac{v_0}{2}\left(1 + \dfrac{x}{B}\right)$

2B.5 Interrelation of slit and annulus formulas. When an annulus is very thin, it may, to a good approximation, be considered as a thin slit. Then the results of Problem 2B.3 can be taken over with suitable modifications. For example, the mass rate of flow in an annulus with outer wall of radius R and inner wall of radius $(1 - \varepsilon)R$, where ε is small, may be obtained from Problem 2B.3 by replacing $2B$ by εR, and W by $2\pi(1 - \frac{1}{2}\varepsilon)R$. In this way we get for the mass rate of flow:

$$w = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4\varepsilon^3\rho}{6\mu L}(1 - \tfrac{1}{2}\varepsilon) \tag{2B.5-1}$$

Show that this same result may be obtained from Eq. 2.4-17 by setting κ equal to $1 - \varepsilon$ everywhere in the formula and then expanding the expression for w in powers of ε. This requires using the Taylor series (see §C.2)

$$\ln(1 - \varepsilon) = -\varepsilon - \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{3}\varepsilon^3 - \tfrac{1}{4}\varepsilon^4 - \cdots \tag{2B.5-2}$$

and then performing a long division. The first term in the resulting series will be Eq. 2B.5-1. *Caution:* In the derivation it is necessary to use the first *four* terms of the Taylor series in Eq. 2B.5-2.

2B.6 Flow of a film on the outside of a circular tube (see Fig. 2B.6). In a gas absorption experiment a viscous fluid flows upward through a small circular tube and then downward in laminar flow on the outside. Set up a momentum balance over a shell of thickness Δr in the film,

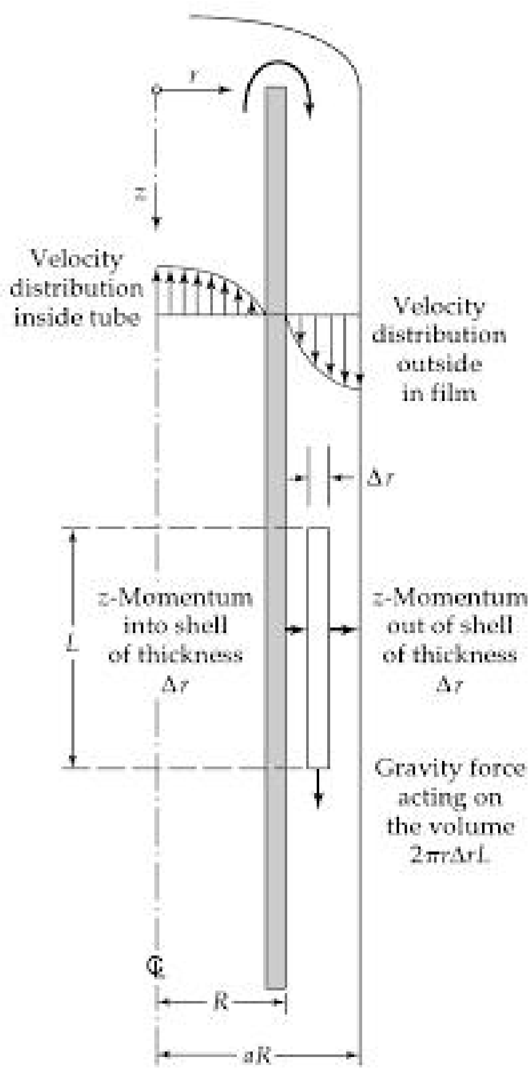

Fig. 2B.6 Velocity distribution and z-momentum balance for the flow of a falling film on the outside of a circular tube.

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as shown in Fig. 2B.6. Note that the "momentum in" and "momentum out" arrows are always taken in the positive coordinate direction, even though in this problem the momentum is flowing through the cylindrical surfaces in the negative r direction.

(a) Show that the velocity distribution in the falling film (neglecting end effects) is

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right] \tag{2B.6-1}$$

(b) Obtain an expression for the mass rate of flow in the film.

(c) Show that the result in (b) simplifies to Eq. 2.2-21 if the film thickness is very small.

2B.7 **Annular flow with inner cylinder moving axially** (see Fig. 2B.7). A cylindrical rod of radius κR moves axially with velocity $v_z = v_0$ along the axis of a cylindrical cavity of radius R as seen in the figure. The pressure at both ends of the cavity is the same, so that the fluid moves through the annular region solely because of the rod motion.

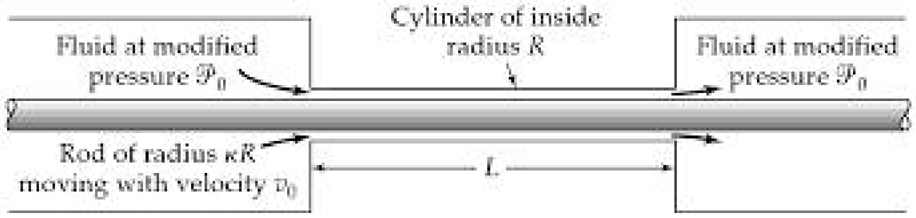

Fig. 2B.7 Annular flow with the inner cylinder moving axially.

(a) Find the velocity distribution in the narrow annular region.

(b) Find the mass rate of flow through the annular region.

(c) Obtain the viscous force acting on the rod over the length L.

(d) Show that the result in (c) can be written as a "plane slit" formula multiplied by a "curvature correction." Problems of this kind arise in studying the performance of wire-coating dies.[1]

Answers: **(a)** $\dfrac{v_z}{v_0} = \dfrac{\ln(r/R)}{\ln \kappa}$

(b) $w = \dfrac{\pi R^2 v_0 \rho}{2}\left[\dfrac{(1-\kappa^2)}{\ln(1/\kappa)} - 2\kappa^2\right]$

(c) $F_z = -2\pi L \mu v_0 / \ln(1/\kappa)$

(d) $F_z = \dfrac{-2\pi L \mu v_0}{\varepsilon}(1 - \frac{1}{2}\varepsilon - \frac{1}{12}\varepsilon^2 + \cdots)$ where $\varepsilon = 1 - \kappa$ (see Problem 2B.5)

2B.8 **Analysis of a capillary flowmeter** (see Fig. 2B.8). Determine the rate of flow (in lb_m/hr) through the capillary flow meter shown in the figure. The fluid flowing in the inclined tube is

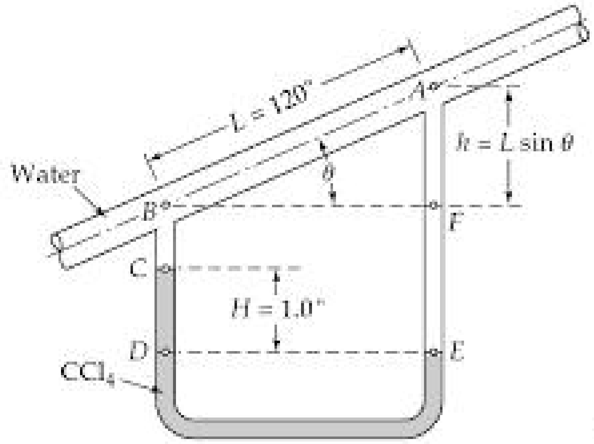

Fig. 2B.8 A capillary flow meter.

[1] J. B. Paton, P. H. Squires, W. H. Darnell, F. M. Cash, and J. F. Carley, *Processing of Thermoplastic Materials*, E. C. Bernhardt (ed.), Reinhold, New York (1959), Chapter 4.

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(a) Assume that locally the velocity distribution in the gap can be very closely approximated by that for flow between parallel plates, the upper one moving with a constant speed. Verify that this leads to the *approximate* velocity distribution (in spherical coordinates)

$$\frac{v_\phi}{r} = \Omega\left(\frac{(\pi/2) - \theta}{\psi_0}\right) \tag{2B.11-1}$$

This approximation should be rather good, because ψ_0 is so small.

(b) From the velocity distribution in Eq. 2B.11-1 and Appendix B.1, show that a reasonable expression for the shear stress is

$$\tau_{\theta\phi} = \mu(\Omega/\psi_0) \tag{2B.11-2}$$

This result shows that the shear stress is uniform throughout the gap. It is this fact that makes the cone-and-plate viscometer quite attractive. The instrument is widely used, particularly in the polymer industry.

(c) Show that the torque required to turn the cone is given by

$$T_z = \tfrac{2}{3}\pi\mu\Omega R^3/\psi_0 \tag{2B.11-3}$$

This is the standard formula for calculating the viscosity from measurements of the torque and angular velocity for a cone–plate assembly with known R and ψ_0.

(d) For a cone-and-plate instrument with radius 10 cm and angle ψ_0 equal to 0.5 degree, what torque (in dyn · cm) is required to turn the cone at an angular velocity of 10 radians per minute if the fluid viscosity is 100 cp?

Answer: **(d)** 40,000 dyn · cm

2B.12 Flow of a fluid in a network of tubes (Fig. 2B.12). A fluid is flowing in laminar flow from A to B through a network of tubes, as depicted in the figure. Obtain an expression for the mass flow rate w of the fluid entering at A (or leaving at B) as a function of the modified pressure drop $\mathcal{P}_A - \mathcal{P}_B$. Neglect the disturbances at the various tube junctions.

Answer: $w = \dfrac{3\pi(\mathcal{P}_A - \mathcal{P}_B)R^4\rho}{20\mu L}$

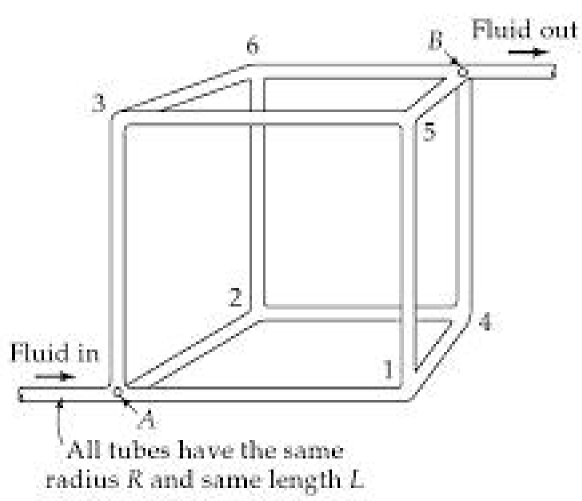

Fig. 2B.12 Flow of a fluid in a network with branching.

2C.1 Performance of an electric dust collector (see Fig. 2C.1)[5].

(a) A dust precipitator consists of a pair of oppositely charged plates between which dust-laden gases flow. It is desired to establish a criterion for the minimum length of the precipitator in terms of the charge on the particle e, the electric field strength $\mathcal{E}$, the pressure difference

[5] The answer given in the first edition of this book was incorrect, as pointed out to us in 1970 by Nau Gab Lee of Seoul National University.

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(a) Begin by analyzing the system without the rotation of the cone. Assume that it is possible to apply the results of Problem 2B.3 locally. That is, adapt the solution for the mass flow rate from that problem by making the following replacements:

replace $(\mathcal{P}_0 - \mathcal{P}_L)/L$ by $\quad -d\mathcal{P}/dz$

replace W by $\quad 2\pi r = 2\pi z \sin\beta$

thereby obtaining

$$w = \frac{2}{3}\left(-\frac{d\mathcal{P}}{dz}\right)\frac{B^3\rho \cdot 2\pi z \sin\beta}{\mu} \tag{2C.6-1}$$

The mass flow rate w is a constant over the range of z. Hence this equation can be integrated to give

$$(\mathcal{P}_1 - \mathcal{P}_2) = \frac{3}{4\pi}\frac{\mu w}{B^3\rho \sin\beta}\ln\frac{L_2}{L_1} \tag{2C.6-2}$$

(b) Next, modify the above result to account for the fact that the cone is rotating with angular velocity Ω. The mean centrifugal force per unit volume acting on the fluid in the slit will have a z-component *approximately* given by

$$(F_{\text{centrif}})_z = K\rho\Omega^2 z \sin^2\beta \tag{2C.6-3}$$

What is the value of K? Incorporate this as an additional force tending to drive the fluid through the channel. Show that this leads to the following expression for the mass rate of flow:

$$w = \frac{4\pi B^3\rho \sin\beta}{3\mu}\left[\frac{(\mathcal{P}_1 - \mathcal{P}_2) + (\frac{1}{2}K\rho\Omega^2 \sin^2\beta)(L_2^2 - L_1^2)}{\ln(L_2/L_1)}\right] \tag{2C.6-4}$$

Here $\mathcal{P}_i = p_i + \rho g L_i \cos\beta$.

2C.7 A simple rate-of-climb indicator (see Fig. 2C.7). Under the proper circumstances the simple apparatus shown in the figure can be used to measure the rate of climb of an airplane. The gauge pressure inside the Bourdon element is taken as proportional to the rate of climb. For the purposes of this problem the apparatus may be assumed to have the following properties: (i) the capillary tube (of radius R and length L, with $R << L$) is of negligible volume but appreciable flow resistance; (ii) the Bourdon element has a constant volume V and offers negligible resistance to flow; and (iii) flow in the capillary is laminar and incompressible, and the volumetric flow rate depends only on the conditions at the ends of the capillary.

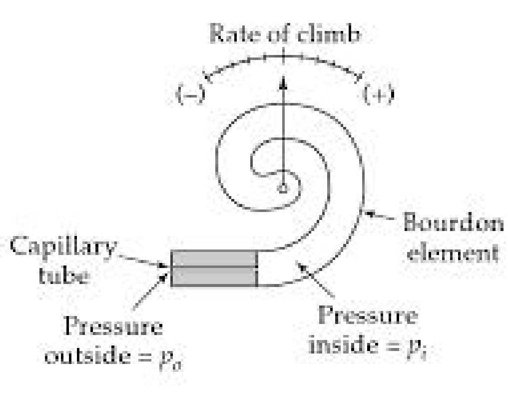

Fig. 2C.7 A rate-of-climb indicator.

(a) Develop an expression for the change of air pressure with altitude, neglecting temperature changes, and considering air to be an ideal gas of constant composition. (*Hint:* Write a shell balance in which the weight of gas is balanced against the static pressure.)

(b) By making a mass balance over the gauge, develop an approximate relation between gauge pressure $p_i - p_o$ and rate of climb v_z for a long continued constant-rate climb. Neglect change of air viscosity, and assume changes in air density to be small.

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In §3.2 the equation of motion is developed by making a momentum balance over a small element of volume and letting the volume element become infinitesimally small. Here again a partial differential equation is generated. This equation of motion can be used, along with some help from the equation of continuity, to set up and solve all the problems given in Chapter 2 and many more complicated ones. It is thus a key equation in transport phenomena.

In §3.3 and §3.4 we digress briefly to introduce the equations of change for mechanical energy and angular momentum. These equations are obtained from the equation of motion and hence contain no new physical information. However, they provide a convenient starting point for several applications in this book—particularly the macroscopic balances in Chapter 7.

In §3.5 we introduce the "substantial derivative." This is the time derivative following the motion of the substance (i.e., the fluid). Because it is widely used in books on fluid dynamics and transport phenomena, we then show how the various equations of change can be rewritten in terms of the substantial derivatives.

In §3.6 we discuss the solution of flow problems by use of the equations of continuity and motion. Although these are partial differential equations, we can solve many problems by postulating the form of the solution and then discarding many terms in these equations. In this way one ends up with a simpler set of equations to solve. In this chapter we solve only problems in which the general equations reduce to one or more ordinary differential equations. In Chapter 4 we examine problems of greater complexity that require some ability to solve partial differential equations. Then in Chapter 5 the equations of continuity and motion are used as the starting point for discussing turbulent flow. Later, in Chapter 8, these same equations are applied to flows of polymeric liquids, which are non-Newtonian fluids.

Finally, §3.7 is devoted to writing the equations of continuity and motion in dimensionless form. This makes clear the origin of the Reynolds number, Re, often mentioned in Chapter 2, and why it plays a key role in fluid dynamics. This discussion lays the groundwork for scale-up and model studies. In Chapter 6 dimensionless numbers arise again in connection with experimental correlations of the drag force in complex systems.

At the end of §2.2, we emphasized the importance of experiments in fluid dynamics. We repeat those words of caution here and point out that photographs and other types of flow visualization have provided us with a much deeper understanding of flow problems than would be possible by theory alone.[1] Keep in mind that when one derives a flow field from the equations of change, it is not necessarily the only physically admissible solution.

Vector and tensor notations are occasionally used in this chapter, primarily for the purpose of abbreviating otherwise lengthy expressions. The beginning student will find that only an elementary knowledge of vector and tensor notation is needed for reading this chapter and for solving flow problems. The advanced student will find Appendix A helpful in getting a better understanding of vector and tensor manipulations. With regard to the notation, it should be kept in mind that we use *lightface italic* symbols for scalars, **boldface Roman** symbols for vectors, and **boldface Greek** symbols for tensors. Also dot-product operations enclosed in ( ) are scalars, and those enclosed in [ ] are vectors.

[1] We recommend particularly M. Van Dyke, *An Album of Fluid Motion*, Parabolic Press, Stanford (1982); H. Werlé, *Ann. Rev. Fluid Mech.*, **5**, 361–382 (1973); D. V. Boger and K. Walters, *Rheological Phenomena in Focus*, Elsevier, Amsterdam (1993).

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Here we have made use of the definitions of the partial derivatives. Similar equations can be developed for the y- and z-components of the momentum balance:

$$\frac{\partial}{\partial t}\rho v_y = -\left(\frac{\partial}{\partial x}\phi_{xy} + \frac{\partial}{\partial y}\phi_{yy} + \frac{\partial}{\partial z}\phi_{zy}\right) + \rho g_y \tag{3.2-5}$$

$$\frac{\partial}{\partial t}\rho v_z = -\left(\frac{\partial}{\partial x}\phi_{xz} + \frac{\partial}{\partial y}\phi_{yz} + \frac{\partial}{\partial z}\phi_{zz}\right) + \rho g_z \tag{3.2-6}$$

By using vector-tensor notation, these three equations can be written as follows:

$$\frac{\partial}{\partial t}\rho v_i = -[\nabla \cdot \boldsymbol{\phi}]_i + \rho g_i \qquad i = x, y, z \tag{3.2-7}$$

That is, by letting i be successively x, y, and z, Eqs. 3.2-4, 5, and 6 can be reproduced. The quantities ρv_i are the Cartesian components of the vector $\rho\mathbf{v}$, which is the momentum per unit volume at a point in the fluid. Similarly, the quantities ρg_i are the components of the vector $\rho\mathbf{g}$, which is the external force per unit volume. The term $-[\nabla \cdot \boldsymbol{\phi}]_i$ is the ith component of the vector $-[\nabla \cdot \boldsymbol{\phi}]$.

When the ith component of Eq. 3.2-7 is multiplied by the unit vector in the ith direction and the three components are added together vectorially, we get

$$\frac{\partial}{\partial t}\rho\mathbf{v} = -[\nabla \cdot \boldsymbol{\phi}] + \rho\mathbf{g} \tag{3.2-8}$$

which is the differential statement of the law of conservation of momentum. It is the translation of Eq. 3.2-1 into mathematical symbols.

In Eq. 1.7-1 it was shown that the combined momentum flux tensor $\boldsymbol{\phi}$ is the sum of the convective momentum flux tensor $\rho\mathbf{vv}$ and the molecular momentum flux tensor $\boldsymbol{\pi}$, and that the latter can be written as the sum of $p\boldsymbol{\delta}$ and $\boldsymbol{\tau}$. When we insert $\boldsymbol{\phi} = \rho\mathbf{vv} + p\boldsymbol{\delta} + \boldsymbol{\tau}$ into Eq. 3.2-8, we get the following *equation of motion*:[2]

$$\boxed{\frac{\partial}{\partial t}\rho\mathbf{v} = -[\nabla \cdot \rho\mathbf{vv}] - \nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho\mathbf{g}} \tag{3.2-9}$$

$\frac{\partial}{\partial t}\rho\mathbf{v}$	$-[\nabla \cdot \rho\mathbf{vv}]$	$-\nabla p - [\nabla \cdot \boldsymbol{\tau}]$	$+\rho\mathbf{g}$
rate of increase of momentum per unit volume	rate of momentum addition by convection per unit volume	rate of momentum addition by molecular transport per unit volume	external force on fluid per unit volume

In this equation ∇p is a vector called the "gradient of (the scalar) p" sometimes written as "grad p." The symbol $[\nabla \cdot \boldsymbol{\tau}]$ is a vector called the "divergence of (the tensor) $\boldsymbol{\tau}$" and $[\nabla \cdot \rho\mathbf{vv}]$ is a vector called the "divergence of (the dyadic product) $\rho\mathbf{vv}$."

In the next two sections we give some formal results that are based on the equation of motion. The equations of change for mechanical energy and angular momentum are not used for problem solving in this chapter, but will be referred to in Chapter 7. Beginners are advised to skim these sections on first reading and to refer to them later as the need arises.

[2] This equation is attributed to A.-L. Cauchy, *Ex. de math.*, **2**, 108–111 (1827). **(Baron) Augustin-Louis Cauchy** (1789–1857) (pronounced "Koh-shee" with the accent on the second syllable), originally trained as an engineer, made great contributions to theoretical physics and mathematics, including the calculus of complex variables.

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Table 3.5-1 The Equations of Change for Isothermal Systems in the D/Dt-Form[a]
Note: At the left are given the equation numbers for the $\partial/\partial t$ forms.

(3.1-4)	$\dfrac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v})$	(A)
(3.2-9)	$\rho \dfrac{D\mathbf{v}}{Dt} = -\nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}$	(B)
(3.3-1)	$\rho \dfrac{D}{Dt}\left(\frac{1}{2}v^2\right) = -(\mathbf{v} \cdot \nabla p) - (\mathbf{v} \cdot [\nabla \cdot \boldsymbol{\tau}]) + \rho(\mathbf{v} \cdot \mathbf{g})$	(C)
(3.4-1)	$\rho \dfrac{D}{Dt}[\mathbf{r} \times \mathbf{v}] = -[\nabla \cdot \{\mathbf{r} \times p\boldsymbol{\delta}\}^{\dagger}] - [\nabla \cdot \{\mathbf{r} \times \boldsymbol{\tau}\}^{\dagger}] + [\mathbf{r} \times \rho \mathbf{g}]$	(D)[a]

[a] Equations (A) through (C) are obtained from Eqs. 3.1-4, 3.2-9, and 3.3-1 with *no assumptions*. Equation (*D*) is written for symmetrical $\boldsymbol{\tau}$ only.

The quantity in the second parentheses in the second line is zero according to the equation of continuity. Consequently Eq. 3.5-3 can be written in vector form as

$$\frac{\partial}{\partial t}(\rho f) + (\nabla \cdot \rho \mathbf{v} f) = \rho \frac{Df}{Dt} \tag{3.5-4}$$

Similarly, for any vector function $\mathbf{f}(x, y, z, t)$,

$$\frac{\partial}{\partial t}(\rho \mathbf{f}) + [\nabla \cdot \rho \mathbf{v} \mathbf{f}] = \rho \frac{D\mathbf{f}}{Dt} \tag{3.5-5}$$

These equations can be used to rewrite the equations of change given in §§3.1 to 3.4 in terms of the substantial derivative as shown in Table 3.5-1.

Equation A in Table 3.5-1 tells how the density is decreasing or increasing as one moves along with the fluid, because of the compression $[(\nabla \cdot \mathbf{v}) < 0]$ or expansion of the fluid $[(\nabla \cdot \mathbf{v}) > 0]$. Equation B can be interpreted as (mass) × (acceleration) = the sum of the pressure forces, viscous forces, and the external force. In other words, Eq. 3.2-9 is equivalent to Newton's second law of motion applied to a small blob of fluid whose envelope moves locally with the fluid velocity $\mathbf{v}$ (see Problem 3D.1).

We now discuss briefly the three most common simplifications of the equation of motion.[1]

(i) For *constant* ρ *and* μ, insertion of the Newtonian expression for $\boldsymbol{\tau}$ from Eq. 1.2-7 into the equation of motion leads to the very famous *Navier–Stokes equation*, first developed from molecular arguments by Navier and from continuum arguments by Stokes:[2]

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \quad \text{or} \quad \rho \frac{D}{Dt}\mathbf{v} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v} \tag{3.5-6, 7}$$

In the second form we have used the "modified pressure" $\mathcal{P} = p + \rho g h$ introduced in Chapter 2, where h is the elevation in the gravitational field and gh is the gravitational

[1] For discussions of the history of these and other famous fluid dynamics relations, see H. Rouse and S. Ince, *History of Hydraulics*, Iowa Institute of Hydraulics, Iowa City (1959).

[2] C.-L.-M.-H. Navier, *Mémoires de l'Académie Royale des Sciences*, **6**, 389–440 (1827); G. G. Stokes, *Proc. Cambridge Phil. Soc*, **8**, 287–319 (1845). The name Navier is pronounced "Nah-vyay."

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We now turn to the illustrative examples. The first two are problems that were discussed in the preceding chapter; we rework these just to illustrate the use of the equations of change. Then we consider some other problems that would be difficult to set up by the shell balance method of Chapter 2.

EXAMPLE 3.6-1

Steady Flow in a Long Circular Tube

Rework the tube-flow problem of Example 2.3-1 using the equations of continuity and motion. This illustrates the use of the tabulated equations for constant viscosity and density in cylindrical coordinates, given in Appendix B.

SOLUTION

We postulate that $\mathbf{v} = \boldsymbol{\delta}_z v_z(r, z)$. This postulate implies that there is no radial flow ($v_r = 0$) and no tangential flow ($v_\theta = 0$), and that v_z does not depend on θ. Consequently, we can discard many terms from the tabulated equations of change, leaving

equation of continuity $$\frac{\partial v_z}{\partial z} = 0 \tag{3.6-1}$$

r-equation of motion $$0 = -\frac{\partial \mathcal{P}}{\partial r} \tag{3.6-2}$$

θ-equation of motion $$0 = -\frac{\partial \mathcal{P}}{\partial \theta} \tag{3.6-3}$$

z-equation of motion $$0 = -\frac{\partial \mathcal{P}}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{3.6-4}$$

The first equation indicates that v_z depends only on r; hence the partial derivatives in the second term on the right side of Eq. 3.6-4 can be replaced by ordinary derivatives. By using the modified pressure $\mathcal{P} = p + \rho g h$ (where h is the height above some arbitrary datum plane), we avoid the necessity of calculating the components of $\mathbf{g}$ in cylindrical coordinates, and we obtain a solution valid for any orientation of the axis of the tube.

Equations 3.6-2 and 3.6-3 show that $\mathcal{P}$ is a function of z alone, and the partial derivative in the first term of Eq. 3.6-4 may be replaced by an ordinary derivative. The only way that we can have a function of r plus a function of z equal to zero is for each term individually to be a constant—say, C_0—so that Eq. 3.6-4 reduces to

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_0 = \frac{d\mathcal{P}}{dz} \tag{3.6-5}$$

The $\mathcal{P}$ equation can be integrated at once. The v_z-equation can be integrated by merely "peeling off" one operation after another on the left side (do not "work out" the compound derivative there). This gives

$$\mathcal{P} = C_0 z + C_1 \tag{3.6-6}$$

$$v_z = \frac{C_0}{4\mu} r^2 + C_2 \ln r + C_3 \tag{3.6-7}$$

The four constants of integration can be found from the boundary conditions:

B.C. 1 $$\text{at } z = 0, \quad \mathcal{P} = \mathcal{P}_0 \tag{3.6-8}$$

B.C. 2 $$\text{at } z = L, \quad \mathcal{P} = \mathcal{P}_L \tag{3.6-9}$$

B.C. 3 $$\text{at } r = R, \quad v_z = 0 \tag{3.6-10}$$

B.C. 4 $$\text{at } r = 0, \quad v_z = \text{finite} \tag{3.6-11}$$

The resulting solutions are:

$$\mathcal{P} = \mathcal{P}_0 - (\mathcal{P}_0 - \mathcal{P}_L)(z/L) \tag{3.6-12}$$

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{3.6-13}$$

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A novice might have a compelling urge to perform the differentiations in Eq. 3.6-21 before solving the differential equation, but this should not be done. All one has to do is "peel off" one operation at a time—just the way you undress—as follows:

$$\frac{1}{r}\frac{d}{dr}(rv_\theta) = C_1 \tag{3.6-23}$$

$$\frac{d}{dr}(rv_\theta) = C_1 r \tag{3.6-24}$$

$$rv_\theta = \frac{1}{2}C_1 r^2 + C_2 \tag{3.6-25}$$

$$v_\theta = \frac{1}{2}C_1 r + \frac{C_2}{r} \tag{3.6-26}$$

The boundary conditions are that the fluid does not slip at the two cylindrical surfaces:

B.C. 1 $\quad$ at $r = \kappa R$, $\quad v_\theta = 0$ $\quad$ (3.6-27)

B.C. 2 $\quad$ at $r = R$, $\quad v_\theta = \Omega_o R$ $\quad$ (3.6-28)

These boundary conditions can be used to get the constants of integration, which are then inserted in Eq. 3.6-26. This gives

$$v_\theta = \Omega_o R\,\frac{\left(\dfrac{r}{\kappa R} - \dfrac{\kappa R}{r}\right)}{\left(\dfrac{1}{\kappa} - \kappa\right)} \tag{3.6-29}$$

By writing the result in this form, with similar terms in the numerator and denominator, it is clear that both boundary conditions are satisfied and that the equation is dimensionally consistent.

From the velocity distribution we can find the momentum flux by using Table B.1:

$$\tau_{r\theta} = -\mu r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = -2\mu\Omega_o\left(\frac{R}{r}\right)^2\left(\frac{\kappa^2}{1-\kappa^2}\right) \tag{3.6-30}$$

The torque acting on the inner cylinder is then given by the product of the inward momentum flux ($-\tau_{r\theta}$), the surface of the cylinder, and the lever arm, as follows:

$$T_z = (-\tau_{r\theta})|_{r=\kappa R}\cdot 2\pi\kappa RL\cdot\kappa R = 4\pi\mu\Omega_o R^2 L\left(\frac{\kappa^2}{1-\kappa^2}\right) \tag{3.6-31}$$

The torque is also given by $T_z = k_t\theta_b$. Therefore, measurement of the angular velocity of the cup and the angular deflection of the bob makes it possible to determine the viscosity. The same kind of analysis is available for other rotational viscometers.[3]

For any viscometer it is essential to know when turbulence will occur. The critical Reynolds number $(\Omega_o R^2\rho/\mu)_{\text{crit}}$, above which the system becomes turbulent, is shown in Fig. 3.6-2 as a function of the radius ratio κ.

One might ask what happens if we hold the outer cylinder fixed and cause the inner cylinder to rotate with an angular velocity Ω_i (the subscript "i" stands for inner). Then the velocity distribution is

$$v_\theta = \Omega_i\kappa R\,\frac{\left(\dfrac{R}{r} - \dfrac{r}{R}\right)}{\left(\dfrac{1}{\kappa} - \kappa\right)} \tag{3.6-32}$$

This is obtained by making the same postulates (see before Eq. 3.6-20) and solving the same differential equation (Eq. 3.6-21), but with a different set of boundary conditions.

---

[3] J. R. Van Wazer, J. W. Lyons, K. Y. Kim, and R. E. Colwell, *Viscosity and Flow Measurement*, Wiley, New York (1963); K. Walters, *Rheometry*, Chapman and Hall, London (1975).

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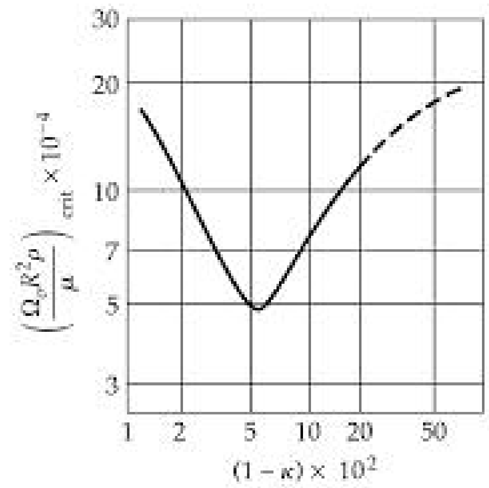

Fig. 3.6-2. Critical Reynolds number for the tangential flow in an annulus, with the outer cylinder rotating and the inner cylinder stationary [H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York (1955), p. 357].

Equation 3.6-32 describes the flow accurately for small values of Ω_i. However, when Ω_i reaches a critical value ($\Omega_{i,\text{crit}} \approx 41.3(\mu/R^2(1-\kappa)^{3/2}\rho)$ for $\kappa \approx 1$) the fluid develops a secondary flow, which is superimposed on the primary (tangential) flow and which is periodic in the axial direction. A very neat system of toroidal vortices, called *Taylor vortices*, is formed, as depicted in Figs. 3.6-3 and 3.6-4(b). The loci of the centers of these vortices are circles, whose centers are located on the common axis of the cylinders. This is still laminar motion—but certainly inconsistent with the postulates made at the beginning of the problem. When the angular velocity Ω_i is increased further, the loci of the centers of the vortices become traveling waves; that is, the flow becomes, in addition, periodic in the tangential direction [see Fig. 3.6-4(*c*)]. Furthermore, the angular velocity of the traveling waves is approximately $\frac{1}{3}\Omega_i$. When the angular velocity Ω_i is further increased, the flow becomes turbulent. Figure 3.6-5 shows the various flow regimes, with the inner and outer cylinders both rotating, determined for a specific apparatus and a

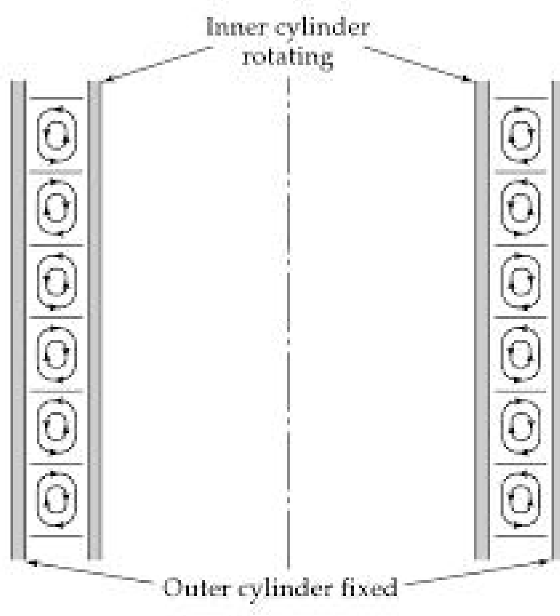

Fig. 3.6-3. Counter-rotating toroidal vortices, called *Taylor vortices*, observed in the annular space between two cylinders. The streamlines have the form of helices, with the axes wrapped around the common axis of the cylinders. This corresponds to Fig. 3.6-4(*b*).

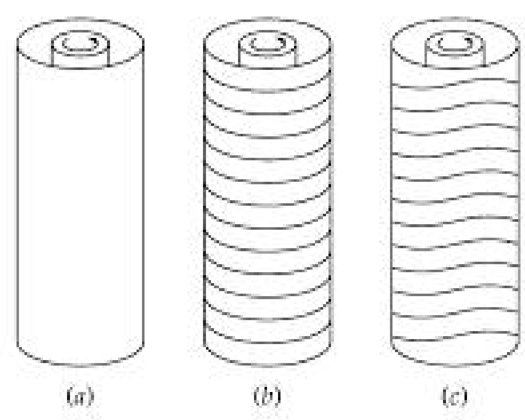

Fig. 3.6-4. Sketches showing the phenomena observed in the annular space between two cylinders: (*a*) purely tangential flow; (*b*) singly periodic flow (Taylor vortices); and (*c*) doubly periodic flow in which an undulatory motion is superposed on the Taylor vortices.

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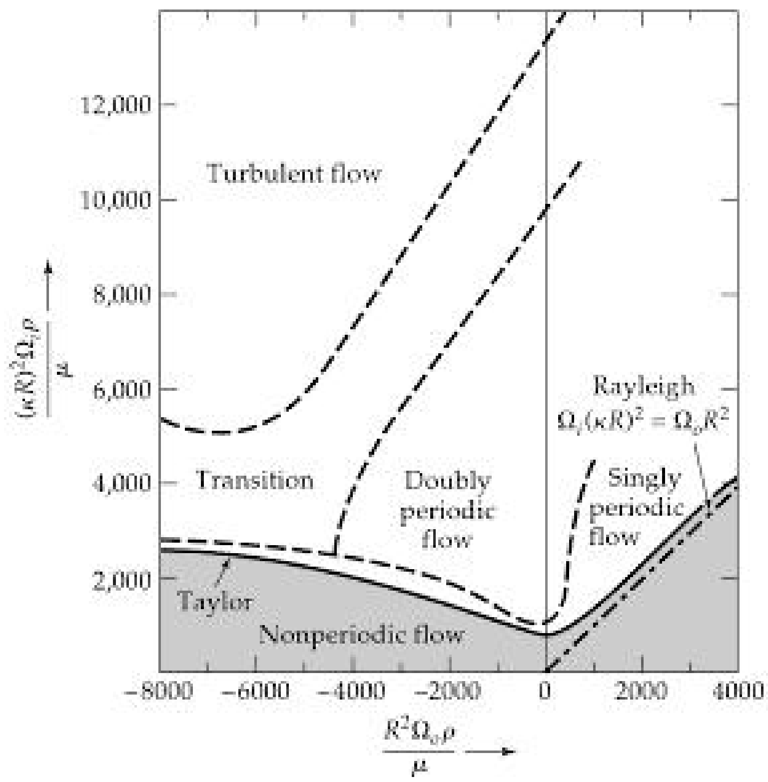

Fig. 3.6-5. Flow-regime diagram for the flow between two coaxial cylinders. The straight line labeled "Rayleigh" is Lord Rayleigh's analytic solution for an inviscid fluid. [See D. Coles, *J. Fluid. Mech.*, **21**, 385–425 (1965).]

specific fluid. This diagram demonstrates how complicated this apparently simple system is. Further details may be found elsewhere.[4,5]

The preceding discussion should serve as a stern warning that intuitive postulates may be misleading. Most of us would not think about postulating the singly and doubly periodic solutions just described. Nonetheless, this information *is* contained in the Navier–Stokes equations. However, since problems involving instability and transitions between several flow regimes are extremely complex, we are forced to use a combination of theory and experiment to describe them. Theory alone cannot yet give us all the answers, and carefully controlled experiments will be needed for years to come.

EXAMPLE 3.6-4

Shape of the Surface of a Rotating Liquid

A liquid of constant density and viscosity is in a cylindrical container of radius R as shown in Fig. 3.6-6. The container is caused to rotate about its own axis at an angular velocity Ω. The cylinder axis is vertical, so that $g_r = 0$, $g_\theta = 0$, and $g_z = -g$, in which g is the magnitude of the gravitational acceleration. Find the shape of the free surface of the liquid when steady state has been established.

---

[4] The initial work on this subject was done by **John William Strutt (Lord Rayleigh)** (1842–1919), who established the field of acoustics with his *Theory of Sound*, written on a houseboat on the Nile River. Some original references on Taylor instability are: J. W. Strutt (Lord Rayleigh), *Proc. Roy. Soc.*, **A93**, 148–154 (1916); G. I. Taylor, *Phil. Trans.*, **A223**, 289–343 (1923) and *Proc. Roy. Soc.* **A157**, 546–564 (1936); P. Schultz-Grunow and H. Hein, *Zeits. Flugwiss.*, **4**, 28–30 (1956); D. Coles, *J. Fluid. Mech.* **21**, 385–425 (1965). See also R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures in Physics*, Addison-Wesley, Reading, MA (1964), §41–6.

[5] Other references on Taylor instability, as well as instability in other flow systems, are: L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), pp. 99–106; S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press (1961), pp. 272–342; H. Schlichting and K. Gersten, *Boundary-Layer Theory*, 8th edition Springer-Verlag, Berlin (2000), Chapter 15; P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*, Cambridge University Press (1981); M. Van Dyke, *An Album of Fluid Motion*, Parabolic Press, Stanford (1982).

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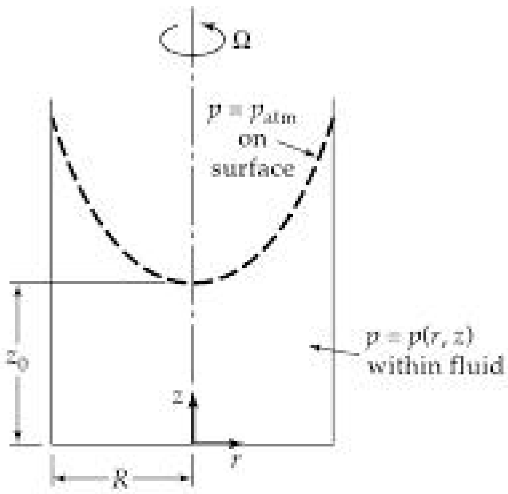

Fig. 3.6-6. Rotating liquid with a free surface, the shape of which is a paraboloid of revolution.

SOLUTION Cylindrical coordinates are appropriate for this problem, and the equations of change are given in Tables B.4 and B.6. At steady state we postulate that v_r and v_z are both zero and that v_θ depends only on r. We also postulate that p depends on z because of the gravitational force and on r because of the centrifugal force but not on θ.

These postulates give $0 = 0$ for the equation of continuity, and the equation of motion gives:

r-component
$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \tag{3.6-33}$$

θ-component
$$0 = \mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) \tag{3.6-34}$$

z-component
$$0 = -\frac{\partial p}{\partial z} - \rho g \tag{3.6-35}$$

The θ-component of the equation of motion can be integrated to give

$$v_\theta = \frac{1}{2}C_1 r + \frac{C_2}{r} \tag{3.6-36}$$

in which C_1 and C_2 are constants of integration. Because v_θ cannot be infinite at $r = 0$, the constant C_2 must be zero. At $r = R$ the velocity v_θ is $R\Omega$. Hence $C_1 = 2\Omega$ and

$$v_\theta = \Omega r \tag{3.6-37}$$

This states that each element of the rotating liquid moves as an element of a rigid body (we could have actually postulated that the liquid would rotate as a rigid body and written down Eq. 3.6-37 directly). When the result in Eq. 3.6-37 is substituted into Eq. 3.6-33, we then have these two equations for the pressure gradients:

$$\frac{\partial p}{\partial r} = \rho\Omega^2 r \quad \text{and} \quad \frac{\partial p}{\partial z} = -\rho g \tag{3.6-38, 39}$$

Each of these equations can be integrated, as follows:

$$p = \tfrac{1}{2}\rho\Omega^2 r^2 + f_1(\theta, z) \quad \text{and} \quad p = -\rho g z + f_2(r, \theta) \tag{3.6-40, 41}$$

where f_1 and f_2 are arbitrary functions of integration. Since we have postulated that p does not depend on θ, we can choose $f_1 = -\rho g z + C$ and $f_2 = \frac{1}{2}\rho\Omega^2 r^2 + C$, where C is a constant, and satisfy Eqs. 3.6-38 and 39. Thus the solution to those equations has the form

$$p = -\rho g z + \tfrac{1}{2}\rho\Omega^2 r^2 + C \tag{3.6-42}$$

The constant C may be determined by requiring that $p = p_{atm}$ at $r = 0$ and $z = z_0$, the latter being the elevation of the liquid surface at $r = 0$. When C is obtained in this way, we get

$$p - p_{atm} = -\rho g(z - z_0) + \tfrac{1}{2}\rho\Omega^2 r^2 \tag{3.6-43}$$

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This equation gives the pressure at all points within the liquid. Right at the liquid–air interface, $p = p_{atm}$, and with this substitution Eq. 3.6-43 gives the shape of the liquid–air interface:

$$z - z_0 = \left(\frac{\Omega^2}{2g}\right) r^2 \tag{3.6-44}$$

This is the equation for a parabola. The reader can verify that the free surface of a liquid in a rotating annular container obeys a similar relation.

EXAMPLE 3.6-5

Flow near a Slowly Rotating Sphere

A solid sphere of radius R is rotating slowly at a constant angular velocity Ω in a large body of quiescent fluid (see Fig. 3.6-7). Develop expressions for the pressure and velocity distributions in the fluid and for the torque T_z required to maintain the motion. It is assumed that the sphere rotates sufficiently slowly that it is appropriate to use the *creeping flow* version of the equation of motion in Eq. 3.5-8. This problem illustrates setting up and solving a problem in spherical coordinates.

SOLUTION

The equations of continuity and motion in spherical coordinates are given in Tables B.4 and B.6, respectively. We postulate that, for steady creeping flow, the velocity distribution will have the general form $\mathbf{v} = \boldsymbol{\delta}_\phi v_\phi(r, \theta)$, and that the modified pressure will be of the form $\mathcal{P} = \mathcal{P}(r, \theta)$. Since the solution is expected to be symmetric about the z-axis, there is no dependence on the angle ϕ.

With these postulates, the equation of continuity is exactly satisfied, and the components of the creeping flow equation of motion become

r-component
$$0 = -\frac{\partial \mathcal{P}}{\partial r} \tag{3.6-45}$$

θ-component
$$0 = -\frac{1}{r}\frac{\partial \mathcal{P}}{\partial \theta} \tag{3.6-46}$$

ϕ-component
$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right) \tag{3.6-47}$$

The boundary conditions may be summarized as

B.C. 1: at $r = R$, $v_r = 0,\ v_\theta = 0,\ v_\phi = R\Omega \sin\theta$ (3.6-48)

B.C. 2: as $r \to \infty$, $v_r \to 0,\ v_\theta \to 0,\ v_\phi \to 0$ (3.6-49)

B.C. 3: as $r \to \infty$, $\mathcal{P} \to p_0$ (3.6-50)

where $\mathcal{P} = p + \rho g z$, and p_0 is the fluid pressure far from the sphere at $z = 0$.

Equation 3.6-47 is a partial differential equation for $v_\phi(r, \theta)$. To solve this, we try a solution of the form $v_\phi = f(r) \sin\theta$. This is just a guess, but it is consistent with the boundary condition in Eq. 3.6-48. When this trial form for the velocity distribution is inserted into Eq. 3.6-47 we get the following ordinary differential equation for $f(r)$:

$$\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) - 2f = 0 \tag{3.6-51}$$

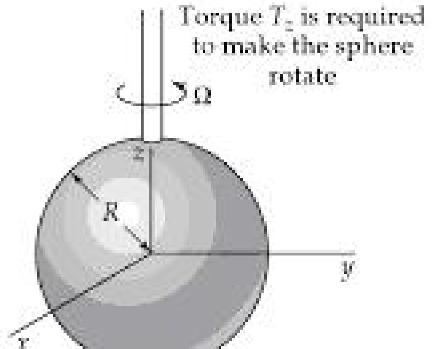

Fig. 3.6-7. A slowly rotating sphere in an infinite expanse of fluid. The primary flow is $v_\phi = \Omega R(R/r)^2 \sin\theta$.

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This is an "equidimensional equation," which may be solved by assuming a trial solution $f = r^n$ (see Eq. C.1-14). Substitution of this trial solution into Eq. 3.6-51 gives $n = 1, -2$. The solution of Eq. 3.6-51 is then

$$f = C_1 r + \frac{C_2}{r^2} \tag{3.6-52}$$

so that

$$v_\phi(r, \theta) = \left(C_1 r + \frac{C_2}{r^2}\right) \sin\theta \tag{3.6-53}$$

Application of the boundary conditions shows that $C_1 = 0$ and $C_2 = \Omega R^3$. Therefore the final expression for the velocity distribution is

$$v_\phi = \Omega R \left(\frac{R}{r}\right)^2 \sin\theta \tag{3.6-54}$$

Next we evaluate the torque needed to maintain the rotation of the sphere. This will be the integral, over the sphere surface, of the tangential force $(\tau_{r\phi}|_{r=R})R^2 \sin\theta d\theta d\phi$ exerted on the fluid by a solid surface element, multiplied by the lever arm $R \sin\theta$ for that element:

$$\begin{aligned} T_z &= \int_0^{2\pi}\int_0^{\pi} (\tau_{r\phi})|_{r=R}\,(R \sin\theta) R^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} (3\mu\Omega \sin\theta)(R\sin\theta)R^2 \sin\theta d\theta d\phi \\ &= 6\pi\mu\Omega R^3 \int_0^{\pi} \sin^3\theta d\theta \\ &= 8\pi\mu\Omega R^3 \end{aligned} \tag{3.6-55}$$

In going from the first to the second line, we have used Table B.1, and in going from the second to the third line we have done the integration over the range of the ϕ variable. The integral in the third line is $\frac{4}{3}$.

As the angular velocity increases, deviations from the "primary flow" of Eq. 3.6-54 occur. Because of the centrifugal force effects, the fluid is pulled in toward the poles of the sphere and shoved outward from the equator as shown in Fig. 3.6-8. To describe this "secondary flow," one has to include the $[\mathbf{v} \cdot \nabla\mathbf{v}]$ term in the equation of motion. This can be done by the use of a stream-function method.[6]

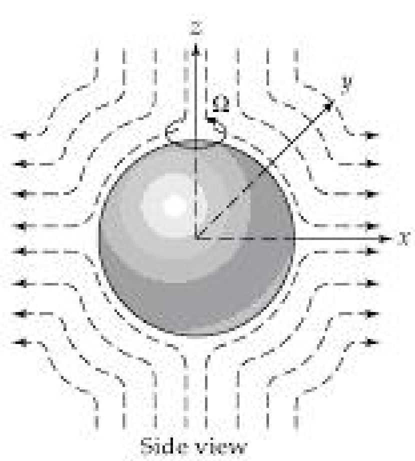

Fig. 3.6-8. Rough sketch showing the secondary flow which appears around a rotating sphere as the Reynolds number is increased.

[6] See, for example, the development by O. Hassager in R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1., Wiley-Interscience, New York, 2nd edition (1987), pp. 31–33. See also L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), p. 65; and L. G. Leal, *Laminar Flow and Convective Transport Processes*, Butterworth-Heinemann, Boston (1992), pp. 180–181.

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§3.7 DIMENSIONAL ANALYSIS OF THE EQUATIONS OF CHANGE

Suppose that we have taken experimental data on, or made photographs of, the flow through some system that cannot be analyzed by solving the equations of change analytically. An example of such a system is the flow of a fluid through an orifice meter in a pipe (this consists of a disk with a centered hole in it, placed in the tube, with a pressure-sensing device upstream and downstream of the disk). Suppose now that we want to scale up (or down) the experimental system, in order to build a new one in which exactly the same flow patterns occur [but appropriately scaled up (or down)]. First of all, we need to have *geometric similarity*: that is, the ratios of all dimensions of the pipe and orifice plate in the original system and in the scaled-up (or scaled-down) system must be the same. In addition, we must have *dynamic similarity*: that is, the dimensionless groups (such as the Reynolds number) in the differential equations and boundary conditions must be the same. The study of dynamic similarity is best understood by writing the equations of change, along with boundary and initial conditions, in dimensionless form.[1,2]

For simplicity we restrict the discussion here to fluids of constant density and viscosity, for which the equations of change are Eqs. 3.1-5 and 3.5-7

$$(\nabla \cdot \mathbf{v}) = 0 \tag{3.7-1}$$

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v} \tag{3.7-2}$$

In most flow systems one can identify the following "scale factors": a characteristic length l_0, a characteristic velocity v_0, and a characteristic modified pressure $\mathcal{P}_0 = p_0 + \rho g h_0$ (for example, these might be a tube diameter, the average flow velocity, and the modified pressure at the tube exit). Then we can define dimensionless variables and differential operators as follows:

$$\breve{x} = \frac{x}{l_0} \qquad \breve{y} = \frac{y}{l_0} \qquad \breve{z} = \frac{z}{l_0} \qquad \breve{t} = \frac{v_0 t}{l_0} \tag{3.7-3}$$

$$\breve{\mathbf{v}} = \frac{\mathbf{v}}{v_0} \qquad \breve{\mathcal{P}} = \frac{\mathcal{P} - \mathcal{P}_0}{\rho v_0^2} \quad \text{or} \quad \breve{\breve{\mathcal{P}}} = \frac{\mathcal{P} - \mathcal{P}_0}{\mu v_0 / l_0} \tag{3.7-4}$$

$$\breve{\nabla} = l_0 \nabla = \boldsymbol{\delta}_x (\partial / \partial \breve{x}) + \boldsymbol{\delta}_y (\partial / \partial \breve{y}) + \boldsymbol{\delta}_z (\partial / \partial \breve{z}) \tag{3.7-5}$$

$$\breve{\nabla}^2 = l_0^2 \nabla^2 = (\partial^2 / \partial \breve{x}^2) + (\partial^2 / \partial \breve{y}^2) + (\partial^2 / \partial \breve{z}^2) \tag{3.7-6}$$

$$D/D\breve{t} = (l_0 / v_0)(D/Dt) \tag{3.7-7}$$

We have suggested two choices for the dimensionless pressure, the first one being convenient for high Reynolds numbers and the second for low Reynolds numbers. When the equations of change in Eqs. 3.7-1 and 3.7-2 are rewritten in terms of the dimensionless quantities, they become

$$(\breve{\nabla} \cdot \breve{\mathbf{v}}) = 0 \tag{3.7-8}$$

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\breve{\nabla}\breve{\mathcal{P}} + \left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}^2 \breve{\mathbf{v}} \tag{3.7-9a}$$

or

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}\breve{\breve{\mathcal{P}}} + \left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}^2 \breve{\mathbf{v}} \tag{3.7-9b}$$

---

[1] G. Birkhoff, *Hydrodynamics*, Dover, New York (1955), Chapter IV. Our dimensional analysis procedure corresponds to Birkhoff's "complete inspectional analysis."

[2] R. W. Powell, *An Elementary Text in Hydraulics and Fluid Mechanics*, Macmillan, New York (1951), Chapter VIII; and H. Rouse and S. Ince, *History of Hydraulics*, Dover, New York (1963) have interesting historical material regarding the dimensionless groups and the persons for whom they were named.

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In these dimensionless equations, the four scale factors l_0, v_0, ρ, and μ appear in one dimensionless group. The reciprocal of this group is named after a famous fluid dynamicist[3]

$$\mathrm{Re} = \left[\frac{l_0 v_0 \rho}{\mu}\right] = \textit{Reynolds number} \tag{3.7-10}$$

The magnitude of this dimensionless group gives an indication of the relative importance of inertial and viscous forces in the fluid system.

From the two forms of the equation of motion given in Eq. 3.7-9, we can gain some perspective on the special forms of the Navier–Stokes equation given in §3.5. Equation 3.7-9a gives the Euler equation of Eq. 3.5-9 when $\mathrm{Re} \to \infty$ and Eq. 3.7-9b gives the creeping flow equation of Eq. 3.5-8 when $\mathrm{Re} << 1$. The regions of applicability of these and other asymptotic forms of the equation of motion are considered further in §§4.3 and 4.4.

Additional dimensionless groups may arise in the initial and boundary conditions; two that appear in problems with fluid–fluid interfaces are

$$\mathrm{Fr} = \left[\frac{v_0^2}{l_0 g}\right] = \textit{Froude number} \qquad (3.7\text{-}11)^4$$

$$\mathrm{We} = \left[\frac{l_0 v_0^2 \rho}{\sigma}\right] = \textit{Weber number} \qquad (3.7\text{-}12)^5$$

The first of these contains the gravitational acceleration g, and the second contains the interfacial tension σ, which may enter into the boundary conditions, as described in Problem 3C.5. Still other groups may appear, such as ratios of lengths in the flow system (for example, the ratio of tube diameter to the diameter of the hole in an orifice meter).

EXAMPLE 3.7-1

Transverse Flow around a Circular Cylinder[6]

The flow of an incompressible Newtonian fluid past a circular cylinder is to be studied experimentally. We want to know how the flow patterns and pressure distribution depend on the cylinder diameter, length, the approach velocity, and the fluid density and viscosity. Show how to organize the work so that the number of experiments needed will be minimized.

SOLUTION

For the analysis we consider an idealized flow system: a cylinder of diameter D and length L, submerged in an otherwise unbounded fluid of constant density and viscosity. Initially the fluid and the cylinder are both at rest. At time $t = 0$, the cylinder is abruptly made to move with velocity v_∞ in the negative x direction. The subsequent fluid motion is analyzed by using coordinates fixed in the cylinder axis as shown in Fig. 3.7-1.

The differential equations describing the flow are the equation of continuity (Eq. 3.7-1) and the equation of motion (Eq. 3.7-2). The initial condition for $t = 0$ is:

$$\text{I.C.} \qquad \text{if } x^2 + y^2 > \tfrac{1}{4}D^2 \text{ or if } |z| > \tfrac{1}{2}L, \qquad \mathbf{v} = \boldsymbol{\delta}_x v_\infty \tag{3.7-13}$$

The boundary conditions for $t \geq 0$ and all z are:

$$\text{B.C. 1} \qquad \text{as } x^2 + y^2 + z^2 \to \infty, \qquad \mathbf{v} \to \boldsymbol{\delta}_x v_\infty \tag{3.7-14}$$

$$\text{B.C. 2} \qquad \text{if } x^2 + y^2 \leq \tfrac{1}{4}D^2 \text{ and } |z| \leq \tfrac{1}{2}L, \qquad \mathbf{v} = 0 \tag{3.7-15}$$

$$\text{B.C. 3} \qquad \text{as } x \to -\infty \text{ at } y = 0, \qquad \mathcal{P} \to \mathcal{P}_\infty \tag{3.7-16}$$

[3] See fn. 1 in §2.2.

[4] **William Froude** (1810–1879) (rhymes with "food") studied at Oxford and worked as a civil engineer concerned with railways and steamships. The Froude number is sometimes defined as the square root of the group given in Eq. 3.7-11.

[5] **Moritz Weber** (1871–1951) (pronounced "Vayber") was a professor of naval architecture in Berlin; another dimensionless group involving the surface tension is the *capillary number*, defined as $\mathrm{Ca} = [\mu v_0/\sigma]$.

[6] This example is adapted from R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. II, Addison-Wesley, Reading, Mass. (1964), §41-4.

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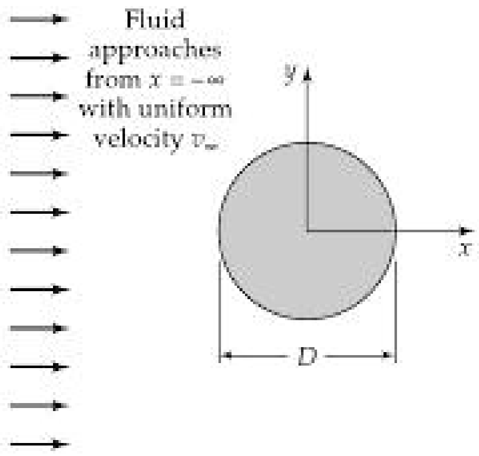

Fig. 3.7-1. Transverse flow around a cylinder.

Now we rewrite the problem in terms of variables made dimensionless with the characteristic length D, velocity v_∞, and modified pressure $\mathcal{P}_\infty$. The resulting dimensionless equations of change are

$$(\breve{\nabla}\cdot\breve{\mathbf{v}}) = 0, \quad \text{and} \quad \frac{\partial\breve{\mathbf{v}}}{\partial\breve{t}} + [\breve{\mathbf{v}}\cdot\breve{\nabla}\breve{\mathbf{v}}] = -\breve{\nabla}\breve{\mathcal{P}} + \frac{1}{\mathrm{Re}}\breve{\nabla}^2\breve{\mathbf{v}} \qquad (3.7\text{-}17, 18)$$

in which $\mathrm{Re} = Dv_\infty\rho/\mu$. The corresponding initial and boundary conditions are:

I.C.	if $\breve{x}^2 + \breve{y}^2 > \frac{1}{4}$ or if $\lvert\breve{z}\rvert > \frac{1}{2}(L/D)$,	$\breve{\mathbf{v}} = \boldsymbol{\delta}_x$	(3.7-19)
B.C. 1	as $\breve{x}^2 + \breve{y}^2 + \breve{z}^2 \to \infty$,	$\breve{\mathbf{v}} \to \boldsymbol{\delta}_x$	(3.7-20)
B.C. 2	if $\breve{x}^2 + \breve{y}^2 \le \frac{1}{4}$ and $\lvert\breve{z}\rvert \le \frac{1}{2}(L/D)$,	$\breve{\mathbf{v}} = 0$	(3.7-21)
B.C. 3	as $\breve{x} \to -\infty$ at $\breve{y} = 0$,	$\breve{\mathcal{P}} \to 0$	(3.7-22)

If we were bright enough to be able to solve the dimensionless equations of change along with the dimensionless boundary conditions, the solutions would *have* to be of the following form:

$$\breve{\mathbf{v}} = \breve{\mathbf{v}}(\breve{x}, \breve{y}, \breve{z}, \breve{t}, \mathrm{Re}, L/D) \quad \text{and} \quad \breve{\mathcal{P}} = \breve{\mathcal{P}}(\breve{x}, \breve{y}, \breve{z}, \breve{t}, \mathrm{Re}, L/D) \qquad (3.7\text{-}23, 24)$$

That is, the dimensionless velocity and dimensionless modified pressure can depend only on the dimensionless parameters Re and L/D and the dimensionless independent variables $\breve{x}$, $\breve{y}$, $\breve{z}$, and $\breve{t}$.

This completes the dimensional analysis of the problem. We have not solved the flow problem, but have decided on a convenient set of dimensionless variables to restate the problem and suggest the form of the solution. The analysis shows that if we wish to catalog the flow patterns for flow past a cylinder, it will suffice to record them (e.g., photographically) for a series of Reynolds numbers $\mathrm{Re} = Dv_\infty\rho/\mu$ and L/D values; thus, separate investigations into the roles of L, D, v_∞, ρ, and μ are unnecessary. Such a simplification saves a lot of time and expense. Similar comments apply to the tabulation of numerical results, in the event that one decides to make a numerical assault on the problem.[7,8]

[7] Analytical solutions of this problem at very small Re and infinite L/D are reviewed in L. Rosenhead (ed.), *Laminar Boundary Layers*, Oxford University Press (1963), Chapter IV. An important feature of this two-dimensional problem is the absence of a "creeping flow" solution. Thus the $[\mathbf{v}\cdot\nabla\mathbf{v}]$-term in the equation of motion has to be included, even in the limit as $\mathrm{Re} \to 0$ (see Problem 3B.9). This is in sharp contrast to the situation for slow flow around a sphere (see §2.6 and §4.2) and around other finite, three-dimensional objects.

[8] For computer studies of the flow around a long cylinder, see F. H. Harlow and J. E. From, *Scientific American*, **212**, 104–110 (1965), and S. J. Sherwin and G. E. Karniadakis, *Comput. Math.*, **123**, 189–229 (1995).

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periment on a scale model of diameter $D_{II} = 1$ ft. To have dynamic similarity, we must choose conditions such that $\text{Re}_{II} = \text{Re}_{I}$. Then if we use the same fluid in the small-scale experiment as in the large system, so that $\mu_{II}/\rho_{II} = \mu_I/\rho_I$, we find $(v_\infty)_{II} = 150$ ft/s as the required air velocity in the small-scale model. With the Reynolds numbers thus equalized, the flow patterns in the model and the full-scale system will look alike: that is, they are geometrically and dynamically similar.

Furthermore, if Re is in the range of periodic vortex formation, the dimensionless time interval $t_0 v_\infty / D$ between vortices will be the same in the two systems. Thus, the vortices will shed 25 times as fast in the model as in the full-scale system. The regularity of the vortex shedding at Reynolds numbers from about 10^2 to 10^4 is utilized commercially for precise flow metering in large pipelines.

EXAMPLE 3.7-2

Steady Flow in an Agitated Tank

It is desired to predict the flow behavior in a large, unbaffled tank of oil, shown in Fig. 3.7-3, as a function of the impeller rotation speed. We propose to do this by means of model experiments in a smaller, geometrically similar system. Determine the conditions necessary for the model studies to provide a direct means of prediction.

SOLUTION

We consider a tank of radius R, with a centered impeller of overall diameter D. At time $t = 0$, the system is stationary and contains liquid to a height H above the tank bottom. Immediately after time $t = 0$, the impeller begins rotating at a constant speed of N revolutions per minute. The drag of the atmosphere on the liquid surface is neglected. The impeller shape and initial position are described by the function $S_{\text{imp}}(r, \theta, z) = 0$.

The flow is governed by Eqs. 3.7-1 and 2, along with the initial condition

$$\text{at } t = 0, \text{ for } 0 \le r < R \text{ and } 0 < z < H, \qquad \mathbf{v} = 0 \tag{3.7-25}$$

and the following boundary conditions for the liquid region:

tank bottom	at $z = 0$ and $0 \le r < R$, $\quad \mathbf{v} = 0$	(3.7-26)
tank wall	at $r = R$, $\quad \mathbf{v} = 0$	(3.7-27)
impeller surface	at $S_{\text{imp}}(r, \theta - 2\pi Nt, z) = 0$, $\quad \mathbf{v} = 2\pi Nr\boldsymbol{\delta}_\theta$	(3.7-28)
gas–liquid interface	at $S_{\text{int}}(r, \theta, z, t) = 0$, $\quad (\mathbf{n} \cdot \mathbf{v}) = 0$	(3.7-29)
	and $\quad \mathbf{n}p + [\mathbf{n} \cdot \boldsymbol{\tau}] = \mathbf{n}p_{\text{atm}}$	(3.7-30)

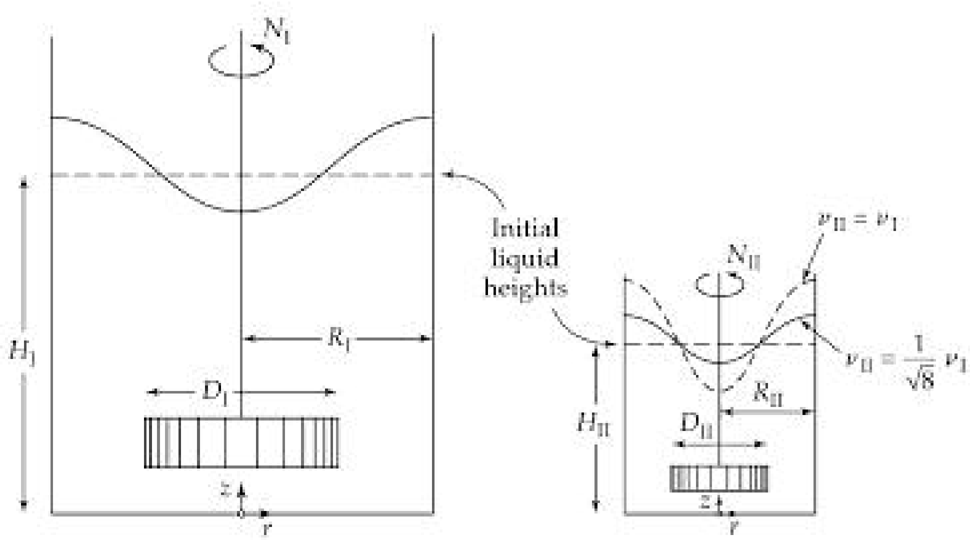

Fig. 3.7-3. Long-time average free-surface shapes, with $\text{Re}_I = \text{Re}_{II}$.

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Equations 3.7-26 to 28 are the no-slip and impermeability conditions; the surface $S_{\text{imp}}(r, \theta - 2\pi Nt, z) = 0$ describes the location of the impeller after Nt rotations. Equation 3.7-29 is the condition of no mass flow through the gas–liquid interface, described by $S_{\text{int}}(r, \theta, z, t) = 0$, which has a local unit normal vector $\mathbf{n}$. Equation 3.7-30 is a force balance on an element of this interface (or a statement of the continuity of the normal component of the momentum flux tensor $\boldsymbol{\pi}$) in which the viscous contributions from the gas side are neglected. This interface is initially stationary in the plane $z = H$, and its motion thereafter is best obtained by measurement, though it is also predictable in principle by numerical solution of this equation system, which describes the initial conditions and subsequent acceleration $D\mathbf{v}/Dt$ of every fluid element.

Next we nondimensionalize the equations using the characteristic quantities $v_0 = ND$, $l_0 = D$, and $\mathcal{P}_0 = p_{\text{atm}}$ along with dimensionless polar coordinates $\breve{r} = r/D$, θ, and $\breve{z} = z/D$. Then the equations of continuity and motion appear as in Eqs. 3.7-8 and 9, with $\text{Re} = D^2N\rho/\mu$. The initial condition takes the form

$$\text{at } \breve{t} = 0, \text{ for } 0 \leq \breve{r} < \left[\!\left[\frac{R}{D}\right]\!\right] \text{ and } 0 < \breve{z} < \left[\!\left[\frac{H}{D}\right]\!\right], \qquad \breve{\mathbf{v}} = 0 \tag{3.7-31}$$

and the boundary conditions become:

tank bottom
$$\text{at } \breve{z} = 0 \text{ and } 0 \leq \breve{r} < \left[\!\left[\frac{R}{D}\right]\!\right], \qquad \breve{\mathbf{v}} = 0 \tag{3.7-32}$$

tank wall
$$\text{at } \breve{r} = \left[\!\left[\frac{R}{D}\right]\!\right], \qquad \breve{\mathbf{v}} = 0 \tag{3.7-33}$$

impeller surface
$$\text{at } \breve{S}_{\text{imp}}(\breve{r}, \theta - 2\pi\breve{t}, \breve{z}) = 0, \qquad \breve{\mathbf{v}} = 2\pi\breve{r}\boldsymbol{\delta}_\theta \tag{3.7-34}$$

gas–liquid interface
$$\text{at } \breve{S}_{\text{int}}(\breve{r}, \theta, \breve{z}, \breve{t}) = 0, \qquad (\mathbf{n}\cdot\breve{\mathbf{v}}) = 0 \tag{3.7-35}$$

$$\text{and} \qquad \mathbf{n}\breve{\mathcal{P}} - \mathbf{n}\left[\!\left[\frac{g}{DN^2}\right]\!\right]\breve{z} - \left[\!\left[\frac{\mu}{D^2N\rho}\right]\!\right][\mathbf{n}\cdot\breve{\dot{\boldsymbol{\gamma}}}] = 0 \tag{3.7-36}$$

In going from Eq. 3.7-30 to 3.7-36 we have used Newton's law of viscosity in the form of Eq. 1.2-7 (but with the last term omitted, as is appropriate for incompressible liquids). We have also used the abbreviation $\dot{\boldsymbol{\gamma}} = \nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger$ for the rate-of-deformation tensor, whose dimensionless Cartesian components are $\breve{\dot{\gamma}} = (\partial\breve{v}_j/\partial\breve{x}_i) + (\partial\breve{v}_i/\partial\breve{x}_j)$.

The quantities in double brackets are known dimensionless quantities. The function $\breve{S}_{\text{imp}}(\breve{r}, \theta - 2\pi\breve{t}, \breve{z})$ is known for a given impeller design. The unknown function $\breve{S}_{\text{int}}(\breve{r}, \theta, \breve{z}, \breve{t})$ is measurable photographically, or in principle is computable from the problem statement.

By inspection of the dimensionless equations, we find that the velocity and pressure profiles must have the form

$$\breve{\mathbf{v}} = \breve{\mathbf{v}}\left(\breve{r}, \theta, \breve{z}, \breve{t}; \frac{R}{D}, \frac{H}{D}, \text{Re}, \text{Fr}\right) \tag{3.7-37}$$

$$\breve{\mathcal{P}} = \breve{\mathcal{P}}\left(\breve{r}, \theta, \breve{z}, \breve{t}; \frac{R}{D}, \frac{H}{D}, \text{Re}, \text{Fr}\right) \tag{3.7-38}$$

for a given impeller shape and location. The corresponding locus of the free surface is given by

$$\breve{S}_{\text{int}} = \breve{S}_{\text{int}}\left(\breve{r}, \theta, \breve{z}, \breve{t}; \frac{R}{D}, \frac{H}{D}, \text{Re}, \text{Fr}\right) = 0 \tag{3.7-39}$$

in which $\text{Re} = D^2N\rho/\mu$ and $\text{Fr} = DN^2/g$. For time-smoothed observations at large $\breve{t}$, the dependence on $\breve{t}$ will disappear, as will the dependence on θ for this axisymmetric tank geometry.

These results provide the necessary conditions for the proposed model experiment: the two systems must be (i) geometrically similar (same values of R/D and H/D, same impeller geometry and location), and (ii) operated at the same values of the Reynolds and Froude numbers. Condition (ii) requires that

$$\frac{D_{\text{I}}^2N_{\text{I}}}{\nu_{\text{I}}} = \frac{D_{\text{II}}^2N_{\text{II}}}{\nu_{\text{II}}} \tag{3.7-40}$$

$$\frac{D_{\text{I}}N_{\text{I}}^2}{g_{\text{I}}} = \frac{D_{\text{II}}N_{\text{II}}^2}{g_{\text{II}}} \tag{3.7-41}$$

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in which the kinematic viscosity $\nu = \mu/\rho$ is used. Normally both tanks will operate in the same gravitational field $g_{\text{I}} = g_{\text{II}}$, so that Eq. 3.7-41 requires

$$\frac{N_{\text{II}}}{N_{\text{I}}} = \left(\frac{D_{\text{I}}}{D_{\text{II}}}\right)^{1/2} \tag{3.7-42}$$

Substitution of this into Eq. 3.7-40 gives the requirement

$$\frac{\nu_{\text{II}}}{\nu_{\text{I}}} = \left(\frac{D_{\text{II}}}{D_{\text{I}}}\right)^{3/2} \tag{3.7-43}$$

This is an important result—namely, that the smaller tank (II) requires a fluid of smaller kinematic viscosity to maintain dynamic similarity. For example, if we use a scale model with $D_{\text{II}} = \frac{1}{2}D_{\text{I}}$, then we need to use a fluid with kinematic viscosity $\nu_{\text{II}} = \nu_{\text{I}}/\sqrt{8}$ in the scaled-down experiment. Evidently the requirements for dynamic similarity are more stringent here than in the previous example, because of the additional dimensionless group Fr.

In many practical cases, Eq. 3.7-43 calls for unattainably low values of ν_{II}. Exact scale-up from a single model experiment is then not possible. In some circumstances, however, the effect of one or more dimensionless groups may be known to be small, or may be predictable from experience with similar systems; in such situations, approximate scale-up from a single experiment is still feasible.[9]

This example shows the importance of including the boundary conditions in a dimensional analysis. Here the Froude number appeared only in the free-surface boundary condition Eq. 3.7-36. Failure to use this condition would result in the omission of the restriction in Eq. 3.7-42, and one might improperly choose $\nu_{\text{II}} = \nu_{\text{I}}$. If one did this, with $\text{Re}_{\text{II}} = \text{Re}_{\text{I}}$, the Froude number in the smaller tank would be too large, and the vortex would be too deep, as shown by the dotted line in Fig. 3.7-3.

EXAMPLE 3.7-3

Pressure Drop for Creeping Flow in a Packed Tube

Show that the mean axial gradient of the modified pressure $\mathscr{P}$ for creeping flow of a fluid of constant ρ and μ through a tube of radius R, uniformly packed for a length $L >> D_p$ with solid particles of characteristic size $D_p << R$, is

$$\frac{\Delta\langle\mathscr{P}\rangle}{L} = \frac{\mu\langle v_z\rangle}{D_p^2}K(\text{geom}) \tag{3.7-44}$$

Here $\langle\cdots\rangle$ denotes an average over a tube cross section within the packed length L, and the function K(geom) is a constant for a given bed geometry (i.e., a given shape and arrangement of the particles).

SOLUTION

We choose D_p as the characteristic length and $\langle v_z\rangle$ as the characteristic velocity. Then the interstitial fluid motion is determined by Eqs. 3.7-8 and 3.7-9b, with $\breve{\mathbf{v}} = \mathbf{v}/\langle v_z\rangle$ and $\breve{\mathscr{P}} = (\mathscr{P} - \mathscr{P}_0)D_p/\mu\langle v_z\rangle$, along with no-slip conditions on the solid surfaces and the modified pressure difference $\Delta\langle\mathscr{P}\rangle = \langle\mathscr{P}_0\rangle - \langle\mathscr{P}_L\rangle$. The solutions for $\breve{\mathbf{v}}$ and $\breve{\mathscr{P}}$ in creeping flow $(D_p\langle v_z\rangle\rho/\mu \to 0)$ accordingly depend only on $\breve{r}$, θ, and $\breve{z}$ for a given particle arrangement and shape. Then the mean axial gradient

$$\frac{D_p}{L}\int_0^{L/D_p}\left(-\frac{d\langle\breve{\mathscr{P}}\rangle}{d\breve{z}}\right)d\breve{z} = \frac{D_p}{L}\left(\langle\breve{\mathscr{P}}_0\rangle - \langle\breve{\mathscr{P}}_L\rangle\right) \tag{3.7-45}$$

depends only on the bed geometry as long as R and L are large relative to D_p. Inserting the foregoing expression for $\breve{\mathscr{P}}$, we immediately obtain Eq. 3.7-44.

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[9] For an introduction to methods for scale-up with incomplete dynamic similarity, see R. W. Powell, *An Elementary Text in Hydraulics and Fluid Mechanics*, Macmillan, New York (1951).

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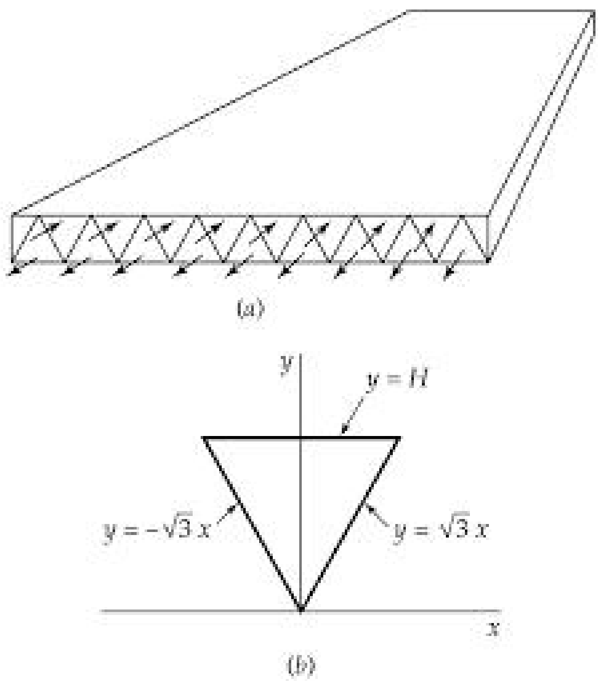

Fig. 3B.2. (*a*) Compact heat-exchanger element, showing channels of a triangular cross section; (*b*) coordinate system for an equilateral-triangular duct.

(b) From Eq. 3B.2-1 find the average velocity, maximum velocity, and mass flow rate.

Answers: **(b)** $\langle v_z \rangle = \dfrac{(\mathcal{P}_0 - \mathcal{P}_L)H^2}{60\mu L} = \dfrac{9}{20}v_{z,\max}$;

$$w = \frac{\sqrt{3}(\mathcal{P}_0 - \mathcal{P}_L)H^4\rho}{180\mu L}$$

3B.3 Laminar flow in a square duct.

(a) A straight duct extends in the z direction for a length L and has a square cross section, bordered by the lines $x = \pm B$ and $y = \pm B$. A colleague has told you that the velocity distribution is given by

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{4\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]\left[1 - \left(\frac{y}{B}\right)^2\right] \quad \text{(3B.3-1)}$$

Since this colleague has occasionally given you wrong advice in the past, you feel obliged to check the result. Does it satisfy the relevant boundary conditions and the relevant differential equation?

(b) According to the review article by Berker,[3] the mass rate of flow in a square duct is given by

$$w = \frac{0.563(\mathcal{P}_0 - \mathcal{P}_L)B^4\rho}{\mu L} \quad \text{(3B.3-2)}$$

Compare the coefficient in this expression with the coefficient that one obtains from Eq. 3B.3-1.

[3] R. Berker, *Handbuch der Physik*, Vol. VIII/2, Springer, Berlin (1963); see pp. 67–77 for laminar flow in conduits of noncircular cross sections. See also W. E. Stewart, *AIChE Journal*, **8**, 425–428 (1962).

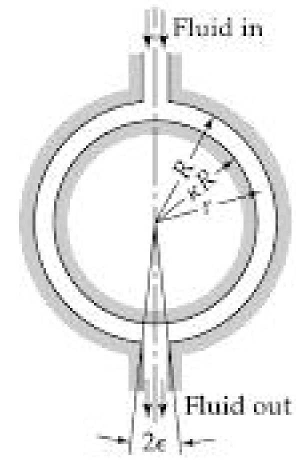

Fig. 3B.4. Creeping flow in the region between two stationary concentric spheres.

3B.4 Creeping flow between two concentric spheres (Fig. 3B.4). A very viscous Newtonian fluid flows in the space between two concentric spheres, as shown in the figure. It is desired to find the rate of flow in the system as a function of the imposed pressure difference. Neglect end effects and postulate that v_θ depends only on r and θ with the other velocity components zero.

(a) Using the equation of continuity, show that $v_\theta \sin\theta = u(r)$, where $u(r)$ is a function of r to be determined.

(b) Write the θ-component of the equation of motion for this system, assuming the flow to be slow enough that the $[\mathbf{v}\cdot\nabla\mathbf{v}]$ term is negligible. Show that this gives

$$0 = -\frac{1}{r}\frac{\partial\mathcal{P}}{\partial\theta} + \mu\left[\frac{1}{\sin\theta}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)\right] \quad \text{(3B.4-1)}$$

(c) Separate this into two equations

$$\sin\theta\frac{d\mathcal{P}}{d\theta} = B; \qquad \frac{\mu}{r}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = B \quad \text{(3B.4-2, 3)}$$

where B is the separation constant, and solve the two equations to get

$$B = \frac{\mathcal{P}_2 - \mathcal{P}_1}{2\ln\cot\frac{1}{2}\varepsilon} \quad \text{(3B.4-4)}$$

$$u(r) = \frac{(\mathcal{P}_1 - \mathcal{P}_2)R}{4\mu\ln\cot(\varepsilon/2)}\left[\left(1 - \frac{r}{R}\right) + \kappa\left(1 - \frac{R}{r}\right)\right] \quad \text{(3B.4-5)}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the values of the modified pressure at $\theta = \varepsilon$ and $\theta = \pi - \varepsilon$, respectively.

(d) Use the results above to get the mass rate of flow

$$w = \frac{\pi(\mathcal{P}_1 - \mathcal{P}_2)R^3(1-\kappa)^3\rho}{12\mu\ln\cot(\varepsilon/2)} \quad \text{(3B.4-6)}$$

3B.5 Parallel-disk viscometer (Fig. 3B.5). A fluid, whose viscosity is to be measured, is placed in the gap of thickness B between the two disks of radius R. One measures the torque T_z required to turn the upper disk at an angular velocity Ω. Develop the formula for deducing the viscosity from these measurements. Assume creeping flow.

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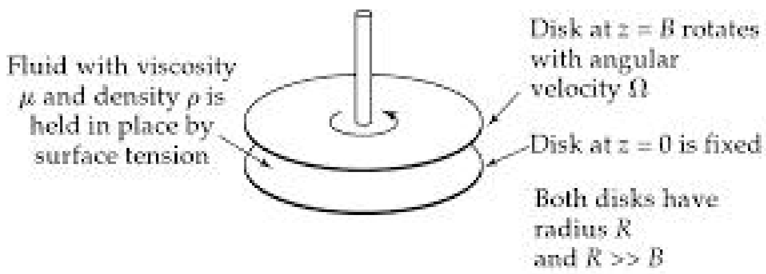

Fig. 3B.5. Parallel-disk viscometer.

(a) Postulate that for small values of Ω the velocity profiles have the form $v_r = 0$, $v_z = 0$, and $v_\theta = rf(z)$; why does this form for the tangential velocity seem reasonable? Postulate further that $\mathcal{P} = \mathcal{P}(r, z)$. Write down the resulting simplified equations of continuity and motion.

(b) From the θ-component of the equation of motion, obtain a differential equation for $f(z)$. Solve the equation for $f(z)$ and evaluate the constants of integration. This leads ultimately to the result $v_\theta = \Omega r(z/B)$. Could you have guessed this result?

(c) Show that the desired working equation for deducing the viscosity is $\mu = 2BT_z/\pi\Omega R^4$.

(d) Discuss the advantages and disadvantages of this instrument.

3B.6 Circulating axial flow in an annulus (Fig. 3B.6). A rod of radius κR moves upward with a constant velocity v_0 through a cylindrical container of inner radius R containing a Newtonian liquid. The liquid circulates in the cylinder, moving upward along the moving central rod and moving downward along the fixed container wall. Find the velocity distribution in the annular region, far from the end disturbances. Flows similar to this occur in the seals of some reciprocating machinery—for example, in the annular space between piston rings.

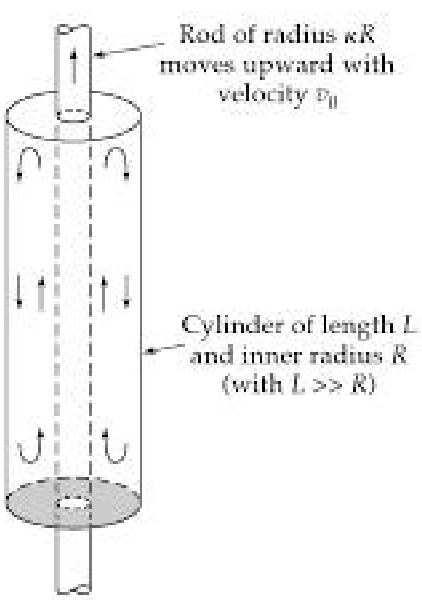

Fig. 3B.6. Circulating flow produced by an axially moving rod in a closed annular region.

(a) First consider the problem where the annular region is quite narrow—that is, where κ is just slightly less than unity. In that case the annulus may be approximated by a thin plane slit and the curvature can be neglected. Show that in this limit, the velocity distribution is given by

$$\frac{v_z}{v_0} = 3\left(\frac{\xi - \kappa}{1 - \kappa}\right)^2 - 4\left(\frac{\xi - \kappa}{1 - \kappa}\right) + 1 \tag{3B.6-1}$$

where $\xi = r/R$.

(b) Next work the problem without the thin-slit assumption. Show that the velocity distribution is given by

$$\frac{v_z}{v_0} = \frac{(1 - \xi^2)\left(1 - \dfrac{2\kappa^2}{1 - \kappa^2}\ln\dfrac{1}{\kappa}\right) - (1 - \kappa^2)\ln\dfrac{1}{\xi}}{(1 - \kappa^2) - (1 + \kappa^2)\ln\dfrac{1}{\kappa}} \tag{3B.6-2}$$

3B.7 Momentum fluxes for creeping flow into a slot (Fig. 3B.7). An incompressible Newtonian liquid is flowing very slowly into a thin slot of thickness $2B$ (in the y direction) and width W (in the z direction). The mass rate of flow in the slot is w. From the results of Problem 2B.3 it can be shown that the velocity distribution within the slot is

$$v_x = \frac{3w}{4BW\rho}\left[1 - \left(\frac{y}{B}\right)^2\right] \quad v_y = 0 \quad v_z = 0 \tag{3B.7-1}$$

at locations not too near the inlet. In the region outside the slot the components of the velocity for *creeping flow* are

$$v_x = -\frac{2w}{\pi W\rho}\frac{x^3}{(x^2 + y^2)^2} \tag{3B.7-2}$$

$$v_y = -\frac{2w}{\pi W\rho}\frac{x^2 y}{(x^2 + y^2)^2} \tag{3B.7-3}$$

$$v_z = 0 \tag{3B.7-4}$$

Equations 3B.7-1 to 4 are only approximate in the region near the slot entry for both $x \geq 0$ and $x \leq 0$.

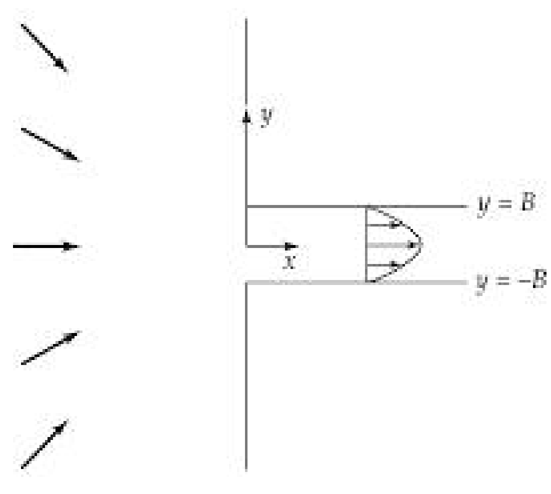

Fig. 3B.7. Flow of a liquid into a slot from a semi-infinite region $x < 0$.

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(a) Find the components of the convective momentum flux $\rho\mathbf{vv}$ inside and outside the slot.

(b) Evaluate the xx-component of $\rho\mathbf{vv}$ at $x = -a$, $y = 0$.

(c) Evaluate the xy-component of $\rho\mathbf{vv}$ at $x = -a$, $y = +a$.

(d) Does the total flow of kinetic energy through the plane $x = -a$ equal the total flow of kinetic energy through the slot?

(e) Verify that the velocity distributions given in Eqs. 3B.7-1 to 4 satisfy the relation $(\nabla \cdot \mathbf{v}) = 0$.

(f) Find the normal stress τ_{xx} at the plane $y = 0$ and also on the solid surface at $x = 0$.

(g) Find the shear stress τ_{yx} on the solid surface at $x = 0$. Is this result surprising? Does sketching the velocity profile v_y vs. x at some plane $y = a$ assist in understanding the result?

3B.8 Velocity distribution for creeping flow toward a slot (Fig. 3B.7).[4] It is desired to get the velocity distribution given for the upstream region in the previous problem. We postulate that $v_\theta = 0$, $v_z = 0$, $v_r = v_r(r, \theta)$, and $\mathcal{P} = \mathcal{P}(r, \theta)$.

(a) Show that the equation of continuity in cylindrical coordinates gives $v_r = f(\theta)/r$, where $f(\theta)$ is a function of θ for which $df/d\theta = 0$ at $\theta = 0$, and $f = 0$ at $\theta = \pi/2$.

(b) Write the r- and θ-components of the creeping flow equation of motion, and insert the expression for $f(\theta)$ from *(a)*.

(c) Differentiate the r-component of the equation of motion with respect to θ and the θ-component with respect to r. Show that this leads to

$$\frac{d^3 f}{d\theta^3} + 4\frac{df}{d\theta} = 0 \tag{3B.8-1}$$

(d) Solve this differential equation and obtain an expression for $f(\theta)$ containing three integration constants.

(e) Evaluate the integration constants by using the two boundary conditions in (a) and the fact that the total mass-flow rate through any cylindrical surface must equal w. This gives

$$v_r = -\frac{2w}{\pi W \rho r}\cos^2\theta \tag{3B.8-2}$$

(f) Next from the equations of motion in (b) obtain $\mathcal{P}(r, \theta)$ as

$$\mathcal{P}(r, \theta) = \mathcal{P}_\infty - \frac{2\mu w}{\pi W \rho r^2}\cos 2\theta \tag{3B.8-3}$$

What is the physical meaning of $\mathcal{P}_\infty$?

(g) Show that the total normal stress exerted on the solid surface at $\theta = \pi/2$ is

$$(p + \tau_{\theta\theta})|_{\theta=\pi/2} = p_\infty + \frac{2\mu w}{\pi W \rho r^2} \tag{3B.8-4}$$

(h) Next evaluate $\tau_{\theta r}$ on the same solid surface.

(i) Show that the velocity profile obtained in Eq. 3B.8-2 is the equivalent to Eqs. 3B.7-2 and 3.

3B.9 Slow transverse flow around a cylinder (see Fig. 3.7-1). An incompressible Newtonian fluid approaches a stationary cylinder with a uniform, steady velocity v_∞ in the positive x direction. When the equations of change are solved for creeping flow, the following expressions[5] are found for the pressure and velocity in the immediate vicinity of the cylinder (they are *not* valid at large distances):

$$p(r, \theta) = p_\infty - C\mu\frac{v_\infty \cos\theta}{r} - \rho g r \sin\theta \tag{3B.9-1}$$

$$v_r = Cv_\infty\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\cos\theta \tag{3B.9-2}$$

$$v_\theta = -Cv_\infty\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) + \frac{1}{4} - \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\sin\theta \tag{3B.9-3}$$

Here p_∞ is the pressure far from the cylinder at $y = 0$ and

$$C = \frac{2}{\ln(7.4/\mathrm{Re})} \tag{3B.9-4}$$

with the Reynolds number defined as $\mathrm{Re} = 2Rv_\infty\rho/\mu$.

(a) Use these results to get the pressure p, the shear stress $\tau_{r\theta}$, and the normal stress τ_{rr} at the surface of the cylinder.

(b) Show that the x-component of the force per unit area exerted by the liquid on the cylinder is

$$-p|_{r=R}\cos\theta + \tau_{r\theta}|_{r=R}\sin\theta \tag{3B.9-5}$$

(c) Obtain the force $F_x = 2C\pi L\mu v_\infty$ exerted in the x direction on a length L of the cylinder.

3B.10 Radial flow between parallel disks (Fig. 3B.10). A part of a lubrication system consists of two circular disks between which a lubricant flows radially. The flow takes place because of a modified pressure difference $\mathcal{P}_1 - \mathcal{P}_2$ between the inner and outer radii r_1 and r_2, respectively.

(a) Write the equations of continuity and motion for this flow system, assuming steady-state, laminar, incompressible Newtonian flow. Consider only the region $r_1 \le r \le r_2$ and a flow that is radially directed.

[4] Adapted from R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, Wiley-Interscience, New York, 2nd edition (1987), pp. 42–43.

[5] See G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press (1967), pp. 244–246, 261.

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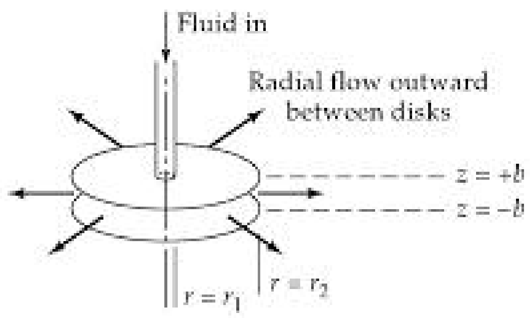

Fig. 3B.10. Outward radial flow in the space between two parallel, circular disks.

(b) Show how the equation of continuity enables one to simplify the equation of motion to give

$$-\rho \frac{\phi^2}{r^3} = -\frac{d\mathcal{P}}{dr} + \mu \frac{1}{r}\frac{d^2\phi}{dz^2} \tag{3B.10-1}$$

in which $\phi = rv_r$ is a function of z only. Why is ϕ independent of r?

(c) It can be shown that no solution exists for Eq. 3B.10-1 unless the nonlinear term containing ϕ is omitted. Omission of this term corresponds to the "creeping flow assumption." Show that for creeping flow, Eq. 3B.10-1 can be integrated with respect to r to give

$$0 = (\mathcal{P}_1 - \mathcal{P}_2) + \left(\mu \ln \frac{r_2}{r_1}\right)\frac{d^2\phi}{dz^2} \tag{3B.10-2}$$

(d) Show that further integration with respect to z gives

$$v_r(r, z) = \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu r \ln(r_2/r_1)}\left[1 - \left(\frac{z}{b}\right)^2\right] \tag{3B.10-3}$$

(e) Show that the mass flow rate is

$$w = \frac{4\pi(\mathcal{P}_1 - \mathcal{P}_2)b^3\rho}{3\mu \ln(r_2/r_1)} \tag{3B.10-4}$$

(f) Sketch the curves $\mathcal{P}(r)$ and $v_r(r, z)$.

3B.11 Radial flow between two coaxial cylinders. Consider an incompressible fluid, at constant temperature, flowing radially between two porous cylindrical shells with inner and outer radii κR and R.

(a) Show that the equation of continuity leads to $v_r = C/r$, where C is a constant.

(b) Simplify the components of the equation of motion to obtain the following expressions for the modified-pressure distribution:

$$\frac{d\mathcal{P}}{dr} = -\rho v_r \frac{dv_r}{dr} \qquad \frac{d\mathcal{P}}{d\theta} = 0 \qquad \frac{d\mathcal{P}}{dz} = 0 \tag{3B.11-1}$$

(c) Integrate the expression for $d\mathcal{P}/dr$ above to get

$$\mathcal{P}(r) - \mathcal{P}(R) = \tfrac{1}{2}\rho[v_r(R)]^2\left[1 - \left(\frac{R}{r}\right)^2\right] \tag{3B.11-2}$$

(d) Write out all the nonzero components of $\boldsymbol{\tau}$ for this flow.

(e) Repeat the problem for concentric spheres.

3B.12 Pressure distribution in incompressible fluids. Penelope is staring at a beaker filled with a liquid, which for all practical purposes can be considered as incompressible; let its density be ρ_0. She tells you she is trying to understand how the pressure in the liquid varies with depth. She has taken the origin of coordinates at the liquid–air interface, with the positive z-axis pointing away from the liquid. She says to you:

"If I simplify the equation of motion for an incompressible liquid at rest, I get $0 = -dp/dz - \rho_0 g$. I can solve this and get $p = p_{\text{atm}} - \rho_0 g z$. That seems reasonable—the pressure increases with increasing depth.

"But, on the other hand, the equation of state for any fluid is $p = p(\rho, T)$, and if the system is at constant temperature, this just simplifies to $p = p(\rho)$. And, since the fluid is incompressible, $p = p(\rho_0)$, and p must be a constant throughout the fluid! How can that be?"

Clearly Penelope needs help. Provide a useful explanation.

3B.13 Flow of a fluid through a sudden contraction.

(a) An incompressible liquid flows through a sudden contraction from a pipe of diameter D_1 into a pipe of smaller diameter D_2. What does the Bernoulli equation predict for $\mathcal{P}_1 - \mathcal{P}_2$, the difference between the modified pressures upstream and downstream of the contraction? Does this result agree with experimental observations?

(b) Repeat the derivation for the isothermal horizontal flow of an ideal gas through a sudden contraction.

3B.14 Torricelli's equation for efflux from a tank (Fig. 3B.14). A large uncovered tank is filled with a liquid to a height h. Near the bottom of the tank, there is a hole that allows the fluid to exit to the atmosphere. Apply Bernoulli's equation to a streamline that extends from the surface of the liquid at the top to a point in the exit

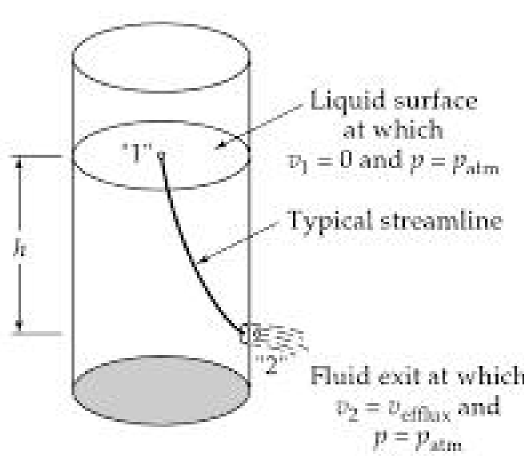

Fig. 3B.14. Fluid draining from a tank. Points "1" and "2" are on the same streamline.

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stream just outside the vessel. Show that this leads to an efflux velocity $v_{\text{efflux}} = \sqrt{2gh}$. This is known as *Torricelli's equation*.

To get this result, one has to assume incompressibility (which is usually reasonable for most liquids), and that the height of the fluid surface is changing so slowly with time that the Bernoulli equation can be applied at any instant of time (the quasi-steady-state assumption).

3B.15 Shape of free surface in tangential annular flow.

(a) A liquid is in the annular space between two vertical cylinders of radii κR and R, and the liquid is open to the atmosphere at the top. Show that when the inner cylinder rotates with an angular velocity Ω_i, and the outer cylinder is fixed, the free liquid surface has the shape

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2 R\Omega_i}{1-\kappa^2}\right)^2 (\xi^{-2} + 4\ln\xi - \xi^2) \tag{3B.15-1}$$

in which z_R is the height of the liquid at the outer-cylinder wall, and $\xi = r/R$.

(b) Repeat (a) but with the inner cylinder fixed and the outer cylinder rotating with an angular velocity Ω_o. Show that the shape of the liquid surface is

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2 R\Omega_o}{1-\kappa^2}\right)^2 [(\xi^{-2} - 1) + 4\kappa^{-2}\ln\xi - \kappa^{-4}(\xi^2 - 1)] \tag{3B.15-2}$$

(c) Draw a sketch comparing these two liquid-surface shapes.

3B.16 Flow in a slit with uniform cross flow (Fig. 3B.16). A fluid flows in the positive x-direction through a long flat duct of length L, width W, and thickness B, where $L >> W >> B$. The duct has porous walls at $y = 0$ and $y = B$, so that a constant cross flow can be maintained, with $v_y = v_0$, a constant, everywhere. Flows of this type are important in connection with separation processes using the sweep-diffusion effect. By carefully controlling the cross flow, one can concentrate the larger constituents (molecules, dust particles, etc.) near the upper wall.

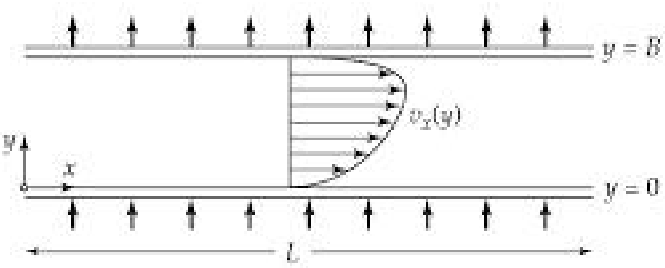

Fig. 3B.16. Flow in a slit of length L, width W, and thickness B. The walls at $y = 0$ and $y = B$ are porous, and there is a flow of the fluid in the y direction, with a uniform velocity $v_y = v_0$.

(a) Show that the velocity profile for the system is given by

$$v_x = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right) \tag{3B.16-1}$$

in which $A = Bv_0\rho/\mu$.

(b) Show that the mass flow rate in the x direction is

$$w = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W\rho}{\mu L}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right) \tag{3B.16-2}$$

(c) Verify that the above results simplify to those of Problem 2B.3 in the limit that there is no cross flow at all (that is, $A \to 0$).

(d) A colleague has also solved this problem, but taking a coordinate system with $y = 0$ at the midplane of the slit, with the porous walls located at $y = \pm b$. His answer to part (a) above is

$$\frac{v_x}{\langle v_x \rangle} = \frac{e^{\alpha\eta} - \eta\sinh\alpha - \cosh\alpha}{(1/\alpha)\sinh\alpha - \cosh\alpha} \tag{3B.16-3}$$

in which $\alpha = bv_0\rho/\mu$ and $\eta = y/b$. Is this result equivalent to Eq. 3B.16-1?

3C.1 Parallel-disk compression viscometer[6] (Fig. 3C.1). A fluid fills completely the region between two circular disks of radius R. The bottom disk is fixed, and the upper disk is made to approach the lower one very slowly with a constant speed v_0, starting from a height H_0 (and $H_0 << R$). The instantaneous height of the upper disk is $H(t)$. It is desired to find the force needed to maintain the speed v_0.

This problem is inherently a rather complicated unsteady-state flow problem. However, a useful approximate solution can be obtained by making two simplifications in

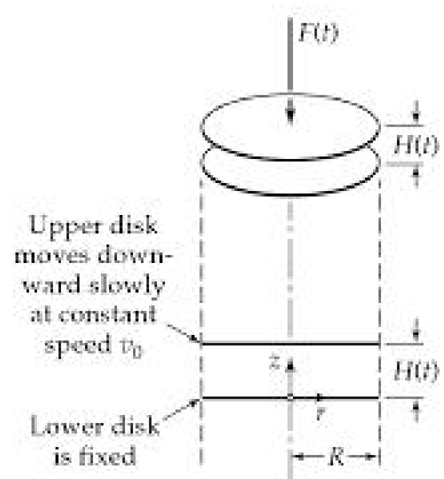

Fig. 3C.1. Squeezing flow in a parallel-disk compression viscometer.

[6] J. R. Van Wazer, J. W. Lyons, K. Y. Kim, and R. E. Colwell, *Viscosity and Flow Measurement*, Wiley-Interscience, New York (1963), pp. 292–295.

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rectangular fluid element. The Gauss theorem for a tensor is needed to complete the derivation.

This problem shows that applying Newton's second law of motion to an arbitrary moving "blob" of fluid is equivalent to setting up a momentum balance over an arbitrary fixed region of space through which the fluid is moving. Both (a) and (b) give the same result as that obtained in §3.2.

(c) Derive the equation of continuity using a volume element of arbitrary shape, both moving and fixed, by the methods outlined in (a) and (b).

3D.2 The equation of change for vorticity.

(a) By taking the curl of the Navier–Stokes equation of motion (in either the D/Dt form or the $\partial/\partial t$ form), obtain an equation for the *vorticity*, $\mathbf{w} = [\nabla \times \mathbf{v}]$ of the fluid; this equation may be written in two ways:

$$\frac{D}{Dt}\mathbf{w} = \nu\nabla^2\mathbf{w} + [\mathbf{w}\cdot\nabla\mathbf{v}] \tag{3D.2-1}$$

$$\frac{D}{Dt}\mathbf{w} = \nu\nabla^2\mathbf{w} + [\boldsymbol{\varepsilon}:\{(\nabla\mathbf{v})\cdot(\nabla\mathbf{v})\}] \tag{3D.2-2}$$

in which $\boldsymbol{\varepsilon}$ is a third-order tensor whose components are the permutation symbol ε_{ijk} (see §A.2) and $\nu = \mu/\rho$ is the kinematic viscosity.

(b) How do the equations in (a) simplify for two-dimensional flows?

3D.3 Alternate form of the equation of motion.[8] Show that, for an incompressible Newtonian fluid with constant viscosity, the equation of motion may be put into the form

$$4\nabla^2\mathcal{P} = \rho(\boldsymbol{\omega}:\boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}}) \tag{3D.3-1}$$

where

$$\dot{\boldsymbol{\gamma}} = \nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger \text{ and } \boldsymbol{\omega} = \nabla\mathbf{v} - (\nabla\mathbf{v})^\dagger \tag{3D.3-2}$$

Do any additional restrictions have to be placed on this result?

[8] P. G. Saffman, *Vortex Dynamics*, Cambridge University Press, corrected edition (1995).

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Chapter 4

Velocity Distributions with More Than One Independent Variable

In Chapter 2 we saw that viscous flow problems with straight streamlines can be solved by shell momentum balances. In Chapter 3 we introduced the equations of continuity and motion, which provide a better way to set up problems. The method was illustrated in §3.6, but there we restricted ourselves to flow problems in which only ordinary differential equations had to be solved.

In this chapter we discuss several classes of problems that involve the solutions of partial differential equations: unsteady-state flow (§4.1), viscous flow in more than one direction (§4.2), the flow of inviscid fluids (§4.3), and viscous flow in boundary layers (§4.4). Since all these topics are treated extensively in fluid dynamics treatises, we provide here only an introduction to them and illustrate some widely used methods for problem solving.

In addition to the analytical methods given in this chapter, there is also a rapidly expanding literature on numerical methods.[1] The field of computational fluid dynamics is already playing an important role in the field of transport phenomena. The numerical and analytical methods play roles complementary to one another, with the numerical methods being indispensable for complicated practical problems.

§4.1 TIME-DEPENDENT FLOW OF NEWTONIAN FLUIDS

In §3.6 only steady-state problems were solved. However, in many situations the velocity depends on both position and time, and the flow is described by partial differential equations. In this section we illustrate three techniques that are much used in fluid dynamics, heat conduction, and diffusion (as well as in many other branches of physics and engineering). In each of these techniques the problem of solving a partial differential equation is converted into a problem of solving one or more ordinary differential equations.

[1] R. W. Johnson (ed.), *The Handbook of Fluid Dynamics*, CRC Press, Boca Raton, Fla. (1998); C. Pozrikidis, *Introduction to Theoretical and Computational Fluid Dynamics*, Oxford University Press (1997).

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The first example illustrates the *method of combination of variables* (or the *method of similarity solutions*). This method is useful only for semi-infinite regions, such that the initial condition and the boundary condition at infinity may be combined into a single new boundary condition.

The second example illustrates the *method of separation of variables*, in which the partial differential equation is split up into two or more ordinary differential equations. The solution is then an infinite sum of products of the solutions of the ordinary differential equations. These ordinary differential equations are usually discussed under the heading of "Sturm-Liouville" problems in intermediate-level mathematics textbooks.[1]

The third example demonstrates the *method of sinusoidal response*, which is useful in describing the way a system responds to external periodic disturbances.

The illustrative examples are chosen for their physical simplicity, so that the major focus can be on the mathematical methods. Since all the problems discussed here are linear in the velocity, Laplace transforms can also be used, and readers familiar with this subject are invited to solve the three examples in this section by that technique.

EXAMPLE 4.1-1

Flow near a Wall Suddenly Set in Motion

A semi-infinite body of liquid with constant density and viscosity is bounded below by a horizontal surface (the xz-plane). Initially the fluid and the solid are at rest. Then at time $t = 0$, the solid surface is set in motion in the positive x direction with velocity v_0 as shown in Fig. 4.1-1. Find the velocity v_x as a function of y and t. There is no pressure gradient or gravity force in the x direction, and the flow is presumed to be laminar.

SOLUTION

For this system $v_x = v_x(y, t)$, $v_y = 0$, and $v_z = 0$. Then from Table B.4 we find that the equation of continuity is satisfied directly, and from Table B.6 we get

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{4.1-1}$$

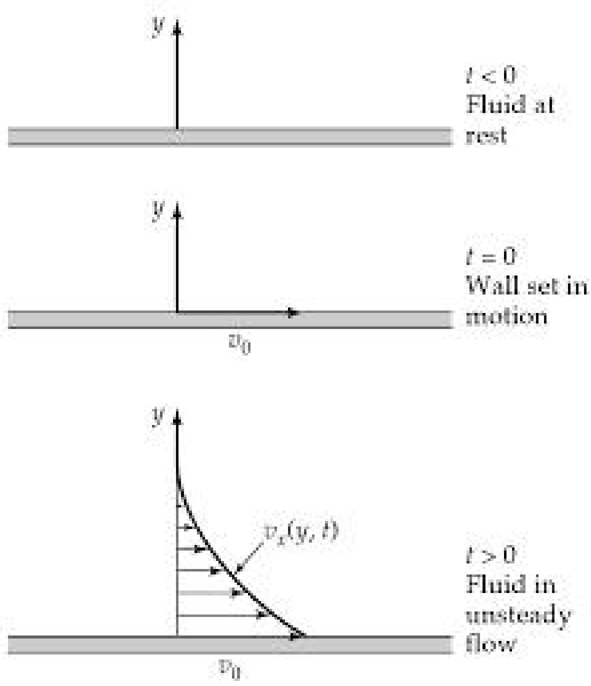

Fig. 4.1-1. Viscous flow of a fluid near a wall suddenly set in motion.

[1] See, for example, M. D. Greenberg, *Foundations of Applied Mathematics*, Prentice-Hall, Englewood Cliffs, N.J. (1978), §20.3.

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in which $\nu = \mu/\rho$. The initial and boundary conditions are

I.C.:	at $t \leq 0$,	$v_x = 0$	for all y	(4.1-2)
B.C. 1:	at $y = 0$,	$v_x = v_0$	for all $t > 0$	(4.1-3)
B.C. 2:	at $y = \infty$,	$v_x = 0$	for all $t > 0$	(4.1-4)

Next we introduce a dimensionless velocity $\phi = v_x/v_0$, so that Eq. 4.1-1 becomes

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2} \tag{4.1-5}$$

with $\phi(y, 0) = 0$, $\phi(0, t) = 1$, and $\phi(\infty, t) = 0$. Since the initial and boundary conditions contain only pure numbers, the solution to Eq. 4.1-5 has to be of the form $\phi = \phi(y, t; \nu)$. However, since ϕ is a dimensionless function, the quantities y, t, and ν must always appear in a dimensionless combination. The only dimensionless combinations of these three quantities are $y/\sqrt{\nu t}$ or powers or multiples thereof. We therefore conclude that

$$\phi = \phi(\eta), \qquad \text{where } \eta = \frac{y}{\sqrt{4\nu t}} \tag{4.1-6}$$

This is the "method of combination of (independent) variables." The "4" is included so that the final result in Eq. 4.1-14 will look neater; we know to do this only after solving the problem without it. The form of the solution in Eq. 4.1-6 is possible essentially because there is no characteristic length or time in the physical system.

We now convert the derivatives in Eq. 4.1-5 into derivatives with respect to the "combined variable" η as follows:

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}\frac{d\phi}{d\eta} \tag{4.1-7}$$

$$\frac{\partial \phi}{\partial y} = \frac{d\phi}{d\eta}\frac{\partial \eta}{\partial y} = \frac{d\phi}{d\eta}\frac{1}{\sqrt{4\nu t}} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = \frac{d^2\phi}{d\eta^2}\frac{1}{4\nu t} \tag{4.1-8}$$

Substitution of these expressions into Eq. 4.1-5 then gives

$$\frac{d^2\phi}{d\eta^2} + 2\eta\frac{d\phi}{d\eta} = 0 \tag{4.1-9}$$

This is an ordinary differential equation of the type given in Eq. C.1-8, and the accompanying boundary conditions are

B.C. 1:	at $\eta = 0$,	$\phi = 1$	(4.1-10)
B.C. 2:	at $\eta = \infty$,	$\phi = 0$	(4.1-11)

The first of these boundary conditions is the same as Eq. 4.1-3, and the second includes Eqs. 4.1-2 and 4. If now we let $d\phi/d\eta = \psi$, we get a first-order separable equation for ψ, and it may be solved to give

$$\psi = \frac{d\phi}{d\eta} = C_1 \exp(-\eta^2) \tag{4.1-12}$$

A second integration then gives

$$\phi = C_1 \int_0^\eta \exp(-\bar{\eta}^2)\, d\bar{\eta} + C_2 \tag{4.1-13}$$

The choice of 0 for the lower limit of the integral is arbitrary; another choice would lead to a different value of C_2, which is still undetermined. Note that we have been careful to use an overbar for the variable of integration ($\bar{\eta}$) to distinguish it from the η in the upper limit.

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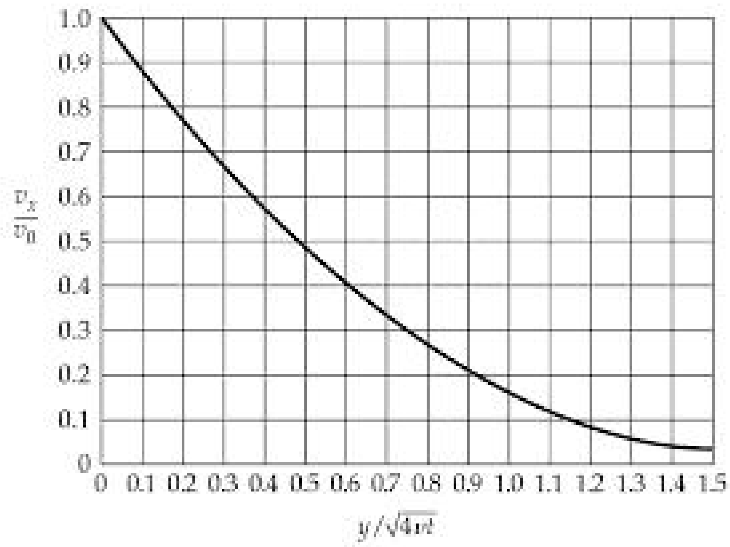

Fig. 4.1-2. Velocity distribution, in dimensionless form, for flow in the neighborhood of a wall suddenly set in motion.

Application of the two boundary conditions makes it possible to evaluate the two integration constants, and we get finally

$$\phi(\eta) = 1 - \frac{\int_0^\eta \exp(-\bar{\eta}^2)\, d\bar{\eta}}{\int_0^\infty \exp(-\bar{\eta}^2)\, d\bar{\eta}} = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\bar{\eta}^2)\, d\bar{\eta} = 1 - \text{erf}\ \eta \tag{4.1-14}$$

The ratio of integrals appearing here is called the *error function*, abbreviated erf η (see §C.6). It is a well-known function, available in mathematics handbooks and computer software programs. When Eq. 4.1-14 is rewritten in the original variables, it becomes

$$\frac{v_x(y, t)}{v_0} = 1 - \text{erf}\, \frac{y}{\sqrt{4\nu t}} = \text{erfc}\, \frac{y}{\sqrt{4\nu t}} \tag{4.1-15}$$

in which erfc η is called the *complementary error function*. A plot of Eq. 4.1-15 is given in Fig. 4.1-2. Note that, by plotting the result in terms of dimensionless quantities, only one curve is needed.

The complementary error function erfc η is a monotone decreasing function that goes from 1 to 0 and drops to 0.01 when η is about 2.0. We can use this fact to define a "boundary-layer thickness" δ as that distance y for which v_x has dropped to a value of $0.01v_0$. This gives $\delta = 4\sqrt{\nu t}$ as a natural length scale for the diffusion of momentum. This distance is a measure of the extent to which momentum has "penetrated" into the body of the fluid. Note that this boundary-layer thickness is proportional to the square root of the elapsed time.

EXAMPLE 4.1-2

Unsteady Laminar Flow Between Two Parallel Plates

It is desired to re-solve the preceding illustrative example, but with a fixed wall at a distance b from the moving wall at $y = 0$. This flow system has a steady-state limit as $t \to \infty$, whereas the problem in Example 4.1-1 did not.

SOLUTION

As in Example 4.1-1, the equation for the x-component of the velocity is

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{4.1-16}$$

The boundary conditions are now

I.C.:	at $t \leq 0$,	$v_x = 0$	for $0 \leq y \leq b$	(4.1-17)
B.C. 1:	at $y = 0$,	$v_x = v_0$	for all $t > 0$	(4.1-18)
B.C. 2:	at $y = b$,	$v_x = 0$	for all $t > 0$	(4.1-19)

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It is convenient to introduce the following dimensionless variables:

$$\phi = \frac{v_x}{v_0}; \qquad \eta = \frac{y}{b}; \qquad \tau = \frac{\nu t}{b^2} \tag{4.1-20}$$

The choices for dimensionless velocity and position ensure that these variables will go from 0 to 1. The choice of the dimensionless time is made so that there will be no parameters occurring in the transformed partial differential equation:

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2} \tag{4.1-21}$$

The initial condition is $\phi = 0$ at $\tau = 0$, and the boundary conditions are $\phi = 1$ at $\eta = 0$ and $\phi = 0$ at $\eta = 1$.

We know that at infinite time the system attains a steady-state velocity profile $\phi_\infty(\eta)$ so that at $\tau = \infty$ Eq. 4.1-21 becomes

$$0 = \frac{d^2\phi_\infty}{d\eta^2} \tag{4.1-22}$$

with $\phi_\infty = 1$ at $\eta = 0$, and $\phi_\infty = 0$ at $\eta = 1$. We then get

$$\phi_\infty = 1 - \eta \tag{4.1-23}$$

for the steady-state limiting profile.

We then can write

$$\phi(\eta, \tau) = \phi_\infty(\eta) - \phi_t(\eta, \tau) \tag{4.1-24}$$

where ϕ_t is the transient part of the solution, which fades out as time goes to infinity. Substitution of this expression into the original differential equation and boundary conditions then gives for ϕ_t

$$\frac{\partial \phi_t}{\partial \tau} = \frac{\partial^2 \phi_t}{\partial \eta^2} \tag{4.1-25}$$

with $\phi_t = \phi_\infty$ at $\tau = 0$, and $\phi_t = 0$ at $\eta = 0$ and 1.

To solve Eq. 4.1-25 we use the "method of separation of (dependent) variables," in which we assume a solution of the form

$$\phi_t = f(\eta)g(\tau) \tag{4.1-26}$$

Substitution of this trial solution into Eq. 4.1-25 and then division by the product fg gives

$$\frac{1}{g}\frac{dg}{d\tau} = \frac{1}{f}\frac{d^2 f}{d\eta^2} \tag{4.1-27}$$

The left side is a function of τ alone, and the right side is a function of η alone. This means that both sides must equal a constant. We choose to designate the constant as $-c^2$ (we could equally well use c or $+c^2$, but experience tells us that these choices make the subsequent mathematics somewhat more complicated). Equation 4.1-27 can then be separated into two equations

$$\frac{dg}{d\tau} = -c^2 g \tag{4.1-28}$$

$$\frac{d^2 f}{d\eta^2} + c^2 f = 0 \tag{4.1-29}$$

These equations have the following solutions (see Eqs. C.1-1 and 3):

$$g = Ae^{-c^2\tau} \tag{4.1-30}$$

$$f = B \sin c\eta + C \cos c\eta \tag{4.1-31}$$

in which A, B, and C are constants of integration.

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Here v° is chosen to be a *complex* function of y, so that $v_x(y, t)$ will differ from $v_x(0, t)$ both in amplitude and phase. We substitute this trial solution into Eq. 4.1-44 and obtain

$$\Re\{v^\circ i\omega e^{i\omega t}\} = \nu\Re\left\{\frac{d^2v^\circ}{dy^2}e^{i\omega t}\right\} \tag{4.1-49}$$

Next we make use of the fact that, if $\Re\{z_1w\} = \Re\{z_2w\}$, where z_1 and z_2 are two complex quantities and w is an arbitrary complex quantity, then $z_1 = z_2$. Then Eq. 4.1-49 becomes

$$\frac{d^2v^\circ}{dy^2} - \left(\frac{i\omega}{\nu}\right)v^\circ = 0 \tag{4.1-50}$$

with the following boundary conditions:

B.C. 1: at $y = 0$, $v^\circ = v_0$ (4.1-51)

B.C. 2: at $y = \infty$, $v^\circ = 0$ (4.1-52)

Equation 4.1-50 is of the form of Eq. C.1-4 and has the solution

$$v^\circ = C_1e^{\sqrt{i\omega/\nu}y} + C_2e^{-\sqrt{i\omega/\nu}y} \tag{4.1-53}$$

Since $\sqrt{i} = \pm(1/\sqrt{2})(1 + i)$, this equation can be rewritten as

$$v^\circ = C_1e^{\sqrt{\omega/2\nu}(1+i)y} + C_2e^{-\sqrt{\omega/2\nu}(1+i)y} \tag{4.1-54}$$

The second boundary condition requires that $C_1 = 0$, and the first boundary condition gives $C_2 = v_0$. Therefore the solution to Eq. 4.1-50 is

$$v^\circ = v_0e^{-\sqrt{\omega/2\nu}(1+i)y} \tag{4.1-55}$$

From this result and Eq. 4.1-48, we get

$$\begin{aligned} v_x(y, t) &= \Re\{v_0e^{-\sqrt{\omega/2\nu}(1+i)y}e^{i\omega t}\} \\ &= v_0e^{-\sqrt{\omega/2\nu}y}\Re\{e^{-i(\sqrt{\omega/2\nu}y-\omega t)}\} \end{aligned} \tag{4.1-56}$$

or finally

$$v_x(y, t) = v_0e^{-\sqrt{\omega/2\nu}y}\cos(\omega t - \sqrt{\omega/2\nu}y) \tag{4.1-57}$$

In this expression, the exponential describes the *attenuation* of the oscillatory motion—that is, the decrease in the amplitude of the fluid oscillations with increasing distance from the plate. In the argument of the cosine, the quantity $-\sqrt{\omega/2\nu}y$ is called the *phase shift;* that is, it describes how much the fluid oscillations at a distance y from the wall are "out-of-step" with the oscillations of the wall itself.

Keep in mind that Eq. 4.1-57 is not the complete solution to the problem as stated in Eqs. 4.1-44 to 47, but only the "periodic-steady-state" solution. The complete solution is given in Problem 4D.1.

§4.2 SOLVING FLOW PROBLEMS USING A STREAM FUNCTION

Up to this point the examples and problems have been chosen so that there was only one nonvanishing component of the fluid velocity. Solutions of the complete Navier–Stokes equation for flow in two or three dimensions are more difficult to obtain. The basic procedure is, of course, similar: one solves simultaneously the equations of continuity and motion, along with the appropriate initial and boundary conditions, to obtain the pressure and velocity profiles.

However, having both velocity and pressure as dependent variables in the equation of motion presents more difficulty in multidimensional flow problems than in the simpler ones discussed previously. It is therefore frequently convenient to eliminate the

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pressure by taking the curl of the equation of motion, after making use of the vector identity $[\mathbf{v} \cdot \nabla \mathbf{v}] = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - [\mathbf{v} \times [\nabla \times \mathbf{v}]]$, which is given in Eq. A.4-23. For fluids of constant viscosity and density, this operation gives

$$\frac{\partial}{\partial t}[\nabla \times \mathbf{v}] - [\nabla \times [\mathbf{v} \times [\nabla \times \mathbf{v}]]] = \nu \nabla^2 [\nabla \times \mathbf{v}] \tag{4.2-1}$$

This is the *equation of change for the vorticity* $[\nabla \times \mathbf{v}]$; two other ways of writing it are given in Problem 3D.2.

For viscous flow problems one can then solve the vorticity equation (a third-order vector equation) together with the equation of continuity and the relevant initial and boundary conditions to get the velocity distribution. Once that is known, the pressure distribution can be obtained from the Navier–Stokes equation in Eq. 3.5-6. This method of solving flow problems is sometimes convenient even for the one-dimensional flows previously discussed (see, for example, Problem 4B.4).

For planar or axisymmetric flows the vorticity equation can be reformulated by introducing the *stream function* ψ. To do this, we express the two nonvanishing components of the velocity as derivatives of ψ in such a way that the equation of continuity is automatically satisfied (see Table 4.2-1). The component of the vorticity equation corresponding to the direction in which there is no flow then becomes a fourth-order scalar equation for ψ. The two nonvanishing velocity components can then be obtained after the equation for the scalar ψ has been found. The most important problems that can be treated in this way are given in Table 4.2-1.[1]

The stream function itself is not without interest. Surfaces of constant ψ contain the *streamlines*,[2] which in steady-state flow are the paths of fluid elements. The volumetric rate of flow between the surfaces $\psi = \psi_1$ and $\psi = \psi_2$ is proportional to $\psi_2 - \psi_1$.

In this section we consider, as an example, the steady, creeping flow past a stationary sphere, which is described by the Stokes equation of Eq. 3.5-8, valid for Re $<<$ 1 (see the discussion right after Eq. 3.7-9). For creeping flow the second term on the left side of Eq. 4.2-1 is set equal to zero. The equation is then linear, and therefore there are many methods available for solving the problem.[3] We use the stream function method based on Eq. 4.2-1.

EXAMPLE 4.2-1

Creeping Flow around a Sphere

Use Table 4.2-1 to set up the differential equation for the stream function for the flow of a Newtonian fluid around a stationary sphere of radius R at Re $<<$ 1. Obtain the velocity and pressure distributions when the fluid approaches the sphere in the positive z direction, as in Fig. 2.6-1.

---

[1] For a technique applicable to more general flows, see J. M. Robertson, *Hydrodynamics in Theory and Application*, Prentice-Hall, Englewood Cliffs, N.J. (1965), p. 77; for examples of three-dimensional flows using two stream functions, see Problem 4D.5 and also J. P. Sørensen and W. E. Stewart, *Chem. Eng. Sci.*, **29**, 819–825 (1974). A. Lahbabi and H.-C. Chang, *Chem. Eng. Sci.*, **40**, 434–447 (1985) dealt with high-Re flow through cubic arrays of spheres, including steady-state solutions and transition to turbulence. W. E. Stewart and M. A. McClelland, *AIChE Journal*, **29**, 947–956 (1983) gave matched asymptotic solutions for forced convection in three-dimensional flows with viscous heating.

[2] See, for example, G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press (1967), §2.2. Chapter 2 of this book is an extensive discussion of the kinematics of fluid motion.

[3] The solution given here follows that given by L. M. Milne-Thomson, *Theoretical Hydrodynamics*, Macmillan, New York, 3rd edition (1955), pp. 555–557. For other approaches, see H. Lamb, *Hydrodynamics*, Dover, New York (1945), §§337, 338. For a discussion of unsteady flow around a sphere, see R. Berker, in *Handbuch der Physik*, Volume VIII-2, Springer, Berlin (1963), §69; or H. Villat and J. Kravtchenko, *Leçons sur les Fluides Visqueux*, Gauthier-Villars, Paris (1943), Chapter VII. See also F. Sy, J. W. Taunton, and E. N. Lightfoot, *AIChE Journal*, **16**, 386–391 (1970). The problem of finding the forces and torques on objects of arbitrary shapes is discussed thoroughly by S. Kim and S. J. Karrila, *Microhydrodynamics: Principles and Selected Applications*, Butterworth-Heinemann, Boston (1991), Chapter II.

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Table 4.2-1 Equations for the Stream Function[a]

Type of motion	Coordinate system	Velocity components	Differential equations for ψ which are equivalent to the Navier–Stokes equation[b]	Expressions for operators
Two-dimensional (planar)	Rectangular with $v_z = 0$ and no z-dependence	$v_x = -\dfrac{\partial\psi}{\partial y}$ $v_y = +\dfrac{\partial\psi}{\partial x}$	$\dfrac{\partial}{\partial t}(\nabla^2\psi) + \dfrac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = \nu\nabla^4\psi$ (A)	$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ $\nabla^4\psi = \nabla^2(\nabla^2\psi)$ $= \left(\dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2\partial y^2} + \dfrac{\partial^4}{\partial y^4}\right)\psi$
	Cylindrical with $v_z = 0$ and no z-dependence	$v_r = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$ $v_\theta = +\dfrac{\partial\psi}{\partial r}$	$\dfrac{\partial}{\partial t}(\nabla^2\psi) + \dfrac{1}{r}\dfrac{\partial(\psi, \nabla^2\psi)}{\partial(r, \theta)} = \nu\nabla^4\psi$ (B)	$\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$
Axisymmetrical	Cylindrical with $v_\theta = 0$ and no θ-dependence	$v_z = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}$ $v_r = +\dfrac{1}{r}\dfrac{\partial\psi}{\partial z}$	$\dfrac{\partial}{\partial t}(E^2\psi) - \dfrac{1}{r}\dfrac{\partial(\psi, E^2\psi)}{\partial(r, z)} - \dfrac{2}{r^2}\dfrac{\partial\psi}{\partial z}E^2\psi = \nu E^4\psi$ (C)	$E^2 = \dfrac{\partial^2}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$ $E^4\psi = E^2(E^2\psi)$
	Spherical with $v_\phi = 0$ and no ϕ-dependence	$v_r = -\dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta}$ $v_\theta = +\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r}$	$\dfrac{\partial}{\partial t}(E^2\psi) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial(\psi, E^2\psi)}{\partial(r, \theta)}$ $- \dfrac{2E^2\psi}{r^2\sin^2\theta}\left(\dfrac{\partial\psi}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^4\psi$ (D)	$E^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right)$

[a] Similar relations in general orthogonal coordinates may be found in S. Goldstein, *Modern Developments in Fluid Dynamics*, Dover, N.Y. (1965), pp. 114–115; in this reference, formulas are also given for axisymmetrical flows with a nonzero component of the velocity around the axis.

[b] Here the Jacobians are designated by

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix}$$

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SOLUTION For steady, creeping flow, the entire left side of Eq. D of Table 4.2-1 may be set equal to zero, and the ψ equation for axisymmetric flow becomes

$$E^4\psi = 0 \tag{4.2-2}$$

or, in spherical coordinates

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = 0 \tag{4.2-3}$$

This is to be solved with the following boundary conditions:

B.C. 1: at $r = R$, $\quad v_r = -\dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta} = 0 \quad$ (4.2-4)

B.C. 2: at $r = R$, $\quad v_\theta = +\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r} = 0 \quad$ (4.2-5)

B.C. 3: as $r \to \infty$, $\quad \psi \to -\frac{1}{2}v_\infty r^2 \sin^2\theta \quad$ (4.2-6)

The first two boundary conditions describe the no-slip condition at the sphere surface. The third implies that $v_z \to v_\infty$ far from the sphere (this can be seen by noting that $v_r = v_\infty \cos\theta$ and $v_\theta = -v_\infty \sin\theta$ far from the sphere).

We now postulate a solution of the form

$$\psi(r, \theta) = f(r)\sin^2\theta \tag{4.2-7}$$

since it will at least satisfy the third boundary condition in Eq. 4.2-6. When it is substituted into Eq. 4.2-3, we get

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f = 0 \tag{4.2-8}$$

The fact that the variable θ does not appear in this equation suggests that the postulate in Eq. 4.2-7 is satisfactory. Equation 4.2-8 is an "equidimensional" fourth-order equation (see Eq. C.1-14). When a trial solution of the form $f(r) = Cr^n$ is substituted into this equation, we find that n may have the values -1, 1, 2, and 4. Therefore $f(r)$ has the form

$$f(r) = C_1 r^{-1} + C_2 r + C_3 r^2 + C_4 r^4 \tag{4.2-9}$$

To satisfy the third boundary condition, C_4 must be zero, and C_3 has to be $-\frac{1}{2}v_\infty$. Hence the stream function is

$$\psi(r, \theta) = (C_1 r^{-1} + C_2 r - \tfrac{1}{2}v_\infty r^2)\sin^2\theta \tag{4.2-10}$$

The velocity components are then obtained by using Table 4.2-1 as follows:

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \left(v_\infty - 2\frac{C_2}{r} - 2\frac{C_1}{r^3}\right)\cos\theta \tag{4.2-11}$$

$$v_\theta = +\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \left(-v_\infty + \frac{C_2}{r} - \frac{C_1}{r^3}\right)\sin\theta \tag{4.2-12}$$

The first two boundary conditions now give $C_1 = -\frac{1}{4}v_\infty R^3$ and $C_2 = \frac{3}{4}v_\infty R$, so that

$$v_r = v_\infty\left(1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\cos\theta \tag{4.2-13}$$

$$v_\theta = -v_\infty\left(1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right)\sin\theta \tag{4.2-14}$$

These are the velocity components given in Eqs. 2.6-1 and 2 without proof.

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To get the pressure distribution, we substitute these velocity components into the r- and θ-components of the Navier–Stokes equation (given in Table B.6). After some tedious manipulations we get

$$\frac{\partial \mathcal{P}}{\partial r} = 3\left(\frac{\mu v_\infty}{R^2}\right)\left(\frac{R}{r}\right)^3 \cos\theta \tag{4.2-15}$$

$$\frac{\partial \mathcal{P}}{\partial \theta} = \frac{3}{2}\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2 \sin\theta \tag{4.2-16}$$

These equations may be integrated (cf. Eqs. 3.6-38 to 41), and, when use is made of the boundary condition that as $r \to \infty$ the modified pressure $\mathcal{P}$ tends to p_0 (the pressure in the plane $z = 0$ far from the sphere), we get

$$p = p_0 - \rho g z - \frac{3}{2}\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2 \cos\theta \tag{4.2-17}$$

This is the same as the pressure distribution given in Eq. 2.6-4.

In §2.6 we showed how one can integrate the pressure and velocity distributions over the sphere surface to get the drag force. That method for getting the force of the fluid on the solid is general. Here we evaluate the "kinetic force" F_k by equating the rate of doing work on the sphere (force × velocity) to the rate of viscous dissipation within the fluid, thus

$$F_k v_\infty = -\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} (\boldsymbol{\tau}:\nabla\mathbf{v}) r^2 dr \sin\theta\, d\theta\, d\phi \tag{4.2-18}$$

Insertion of the function $(-\boldsymbol{\tau}:\nabla\mathbf{v})$ in spherical coordinates from Table B.7 gives

$$\begin{aligned} F_k v_\infty = \mu \int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} \Bigg[& 2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + 2\left(\frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)^2 \\ & + \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2\Bigg] r^2 dr \sin\theta\, d\theta\, d\phi \end{aligned} \tag{4.2-19}$$

Then the velocity profiles from Eqs. 4.2-13 and 14 are substituted into Eq. 4.2-19. When the indicated differentiations and integrations (lengthy!) are performed, one finally gets

$$F_k = 6\pi\mu v_\infty R \tag{4.2-20}$$

which is *Stokes' law*.

As pointed out in §2.6, Stokes' law is restricted to $\text{Re} < 0.1$. The expression for the drag force can be improved by going back and including the $[\mathbf{v}\cdot\nabla\mathbf{v}]$ term. Then use of the *method of matched asymptotic expansions* leads to the following result[4]

$$F_k = 6\pi\mu v_\infty R[1 + \tfrac{3}{16}\text{Re} + \tfrac{9}{160}\text{Re}^2(\ln\tfrac{1}{2}\text{Re} + \gamma + \tfrac{5}{3}\ln 2 - \tfrac{323}{360}) + \tfrac{27}{640}\text{Re}^3\ln\tfrac{1}{2}\text{Re} + O(\text{Re}^3)] \tag{4.2-21}$$

where $\gamma = 0.5772$ is Euler's constant. This expression is good up to Re of about 1.

[4] I. Proudman and J. R. A. Pearson, *J. Fluid Mech.*, **2**, 237–262 (1957); W. Chester and D. R. Breach, *J. Fluid. Mech.*, **37**, 751–760 (1969). A solution to the unsteady-state analog of Eq. 4.2-2 was obtained by F. Sy, J. W. Taunton, and E. N. Lightfoot, *AIChE Journal*, **16**, 386–391 (1970).

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Similar elimination of ϕ gives

$$G(x, y, \psi) = 0 \tag{4.3-14}$$

Setting ϕ = a constant in Eq. 4.3-13 gives the equations for the equipotential lines for *some* flow problem, and setting ψ = constant in Eq. 4.3-14 gives equations for the streamlines. The velocity components can be obtained from

$$-\frac{dz}{dw} = \frac{v_x + iv_y}{v_x^2 + v_y^2} \tag{4.3-15}$$

Thus from any analytic function $w(z)$, or its inverse $z(w)$, we can construct a flow net with streamlines ψ = constant and equipotential lines ϕ = constant. The task of finding $w(z)$ or $z(w)$ to satisfy a given flow problem is, however, considerably more difficult. Some special methods are available[4,5] but it is frequently more expedient to consult a table of conformal mappings.[6]

In the next two illustrative examples we show how to use the complex potential $w(z)$ to describe the potential flow around a cylinder, and the inverse function $z(w)$ to solve the problem of the potential flow into a channel. In the third example we solve the flow in the neighborhood of a corner, which is treated further in §4.4 by the boundary-layer method. A few general comments should be kept in mind:

(a) The streamlines are everywhere perpendicular to the equipotential lines. This property, evident from Eqs. 4.3-10, 11, is useful for the approximate construction of flow nets.

(b) Streamlines and equipotential lines can be interchanged to get the solution of another flow problem. This follows from (a) and the fact that both ϕ and ψ are solutions to the two-dimensional Laplace equation.

(c) Any streamline may be replaced by a solid surface. This follows from the boundary condition that the normal component of the velocity of the fluid is zero at a solid surface. The tangential component is not restricted, since in potential flow the fluid is presumed to be able to slide freely along the surface (the complete-slip assumption).

EXAMPLE 4.3-1

Potential Flow around a Cylinder

(a) Show that the complex potential

$$w(z) = -v_\infty R\left(\frac{z}{R} + \frac{R}{z}\right) \tag{4.3-16}$$

describes the potential flow around a circular cylinder of radius R, when the approach velocity is v_∞ in the positive x direction.

(b) Find the components of the velocity vector.

(c) Find the pressure distribution on the cylinder surface, when the modified pressure far from the cylinder is $\mathcal{P}_\infty$.

SOLUTION

(a) To find the stream function and velocity potential, we write the complex potential in the form $w(z) = \phi(x, y) + i\psi(x, y)$:

$$w(z) = -v_\infty x\left(1 + \frac{R^2}{x^2 + y^2}\right) - iv_\infty y\left(1 - \frac{R^2}{x^2 + y^2}\right) \tag{4.3-17}$$

---

[5] J. Fuka, Chapter 21 in K. Rektorys, *Survey of Applicable Mathematics*, MIT Press, Cambridge, Mass. (1969).

[6] H. Kober, *Dictionary of Conformal Representations*, Dover, New York, 2nd edition (1957).

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Hence the stream function is

$$\psi(x, y) = -v_\infty y\left(1 - \frac{R^2}{x^2 + y^2}\right) \tag{4.3-18}$$

To make a plot of the streamlines it is convenient to rewrite Eq. 4.3-18 in dimensionless form

$$\Psi(X, Y) = -Y\left(1 - \frac{1}{X^2 + Y^2}\right) \tag{4.3-19}$$

in which $\Psi = \psi/v_\infty R$, $X = x/R$, and $Y = y/R$.

In Fig. 4.3-1 the streamlines are plotted as the curves Ψ = constant. The streamline $\Psi = 0$ gives a unit circle, which represents the surface of the cylinder. The streamline $\Psi = -\frac{3}{2}$ goes through the point $X = 0$, $Y = 2$, and so on.

(b) The velocity components are obtainable from the stream function by using Eqs. 4.3-6 and 7. They may also be obtained from the complex velocity according to Eq. 4.3-12, as follows:

$$\begin{aligned}\frac{dw}{dz} &= -v_\infty\left(1 - \frac{R^2}{z^2}\right) = -v_\infty\left(1 - \frac{R^2}{r^2}e^{-2i\theta}\right) \\ &= -v_\infty\left(1 - \frac{R^2}{r^2}(\cos 2\theta - i \sin 2\theta)\right)\end{aligned} \tag{4.3-20}$$

Therefore the velocity components as functions of position are

$$v_x = v_\infty\left(1 - \frac{R^2}{r^2}\cos 2\theta\right) \tag{4.3-21}$$

$$v_y = -v_\infty\left(\frac{R^2}{r^2}\sin 2\theta\right) \tag{4.3-22}$$

(c) On the surface of the cylinder, $r = R$, and

$$\begin{aligned}v^2 &= v_x^2 + v_y^2 \\ &= v_\infty^2[(1 - \cos 2\theta)^2 + (\sin 2\theta)^2] \\ &= 4v_\infty^2 \sin^2\theta\end{aligned} \tag{4.3-23}$$

When θ is zero or π, the fluid velocity is zero; such points are known as *stagnation points*. From Eq. 4.3-5 we know that

$$\tfrac{1}{2}\rho v^2 + \mathcal{P} = \tfrac{1}{2}\rho v_\infty^2 + \mathcal{P}_\infty \tag{4.3-24}$$

Then from the last two equations we get the pressure distribution on the surface of the cylinder

$$(\mathcal{P} - \mathcal{P}_\infty) = \tfrac{1}{2}\rho v_\infty^2(1 - 4\sin^2\theta) \tag{4.3-25}$$

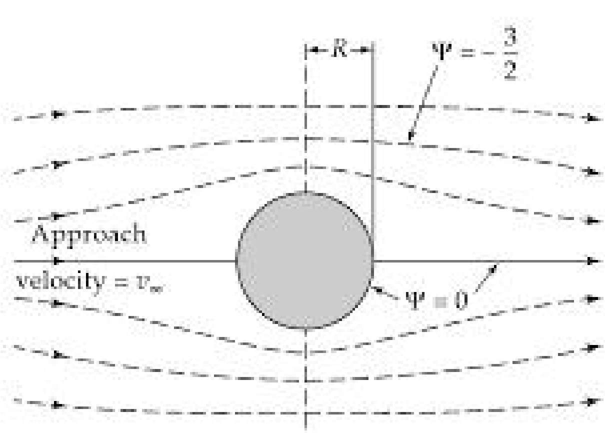

Fig. 4.3-1. The streamlines for the potential flow around a cylinder according to Eq. 4.3-19.

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Note that the modified pressure distribution is symmetric about the x-axis; that is, for potential flow there is no form drag on the cylinder (*d'Alembert's paradox*).[7] Of course, we know now that this is not really a paradox, but simply the result of the fact that the inviscid fluid does not permit applying the no-slip boundary condition at the interface.

EXAMPLE 4.3-2

Flow Into a Rectangular Channel

Show that the inverse function

$$z(w) = \frac{w}{v_\infty} + \frac{b}{\pi}\exp(\pi w/bv_\infty) \tag{4.3-26}$$

represents the potential flow into a rectangular channel of half-width b. Here v_∞ is the magnitude of the velocity far downstream from the entrance to the channel.

SOLUTION

First we introduce dimensionless distance variables

$$X = \frac{\pi x}{b} \qquad Y = \frac{\pi y}{b} \qquad Z = X + iY = \frac{\pi z}{b} \tag{4.3-27}$$

and the dimensionless quantities

$$\Phi = \frac{\pi\phi}{bv_\infty} \qquad \Psi = \frac{\pi\psi}{bv_\infty} \qquad W = \Phi + i\Psi = \frac{\pi w}{bv_\infty} \tag{4.3-28}$$

The inverse function of Eq. 4.3-26 may now be expressed in terms of dimensionless quantities and split up into real and imaginary parts

$$Z = W + e^W = (\Phi + e^\Phi \cos\Psi) + i(\Psi + e^\Phi \sin\Psi) \tag{4.3-29}$$

Therefore

$$X = \Phi + e^\Phi \cos\Psi \qquad Y = \Psi + e^\Phi \sin\Psi \tag{4.3-30, 31}$$

We can now set Ψ equal to a constant, and the streamline $Y = Y(X)$ is expressed parametrically in Φ. For example, the streamline $\Psi = 0$ is given by

$$X = \Phi + e^\Phi \qquad Y = 0 \tag{4.3-32, 33}$$

As Φ goes from $-\infty$ to $+\infty$, X also goes from $-\infty$ to $+\infty$; hence the X-axis is a streamline. Next, the streamline $\Psi = \pi$ is given by

$$X = \Phi - e^\Phi \qquad Y = \pi \tag{4.3-34, 35}$$

As Φ goes from $-\infty$ to $+\infty$, X goes from $-\infty$ to -1 and then back to $-\infty$; that is, the streamline doubles back on itself. We select this streamline to be one of the solid walls of the rectangular channel. Similarly, the streamline $\Psi = -\pi$ is the other wall. The streamlines $\Psi = C$, where $-\pi < C < \pi$, then give the flow pattern for the flow into the rectangular channel as shown in Fig. 4.3-2.

Next, from Eq. 4.3-29 the derivative $-dz/dw$ can be found:

$$-\frac{dz}{dw} = -\frac{1}{v_\infty}\frac{dZ}{dW} = -\frac{1}{v_\infty}(1 + e^W) = -\frac{1}{v_\infty}(1 + e^\Phi \cos\Psi + ie^\Phi \sin\Psi) \tag{4.3-36}$$

Comparison of this expression with Eq. 4.3-15 gives for the velocity components

$$\frac{v_x v_\infty}{v^2} = -(1 + e^\Phi \cos\Psi) \qquad \frac{v_y v_\infty}{v^2} = -(e^\Phi \sin\Psi) \tag{4.3-37}$$

These equations have to be used in conjunction with Eqs. 4.3-30 and 31 to eliminate Φ and Ψ in order to get the velocity components as functions of position.

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[7] Hydrodynamic paradoxes are discussed in G. Birkhoff, *Hydrodynamics*, Dover, New York (1955).

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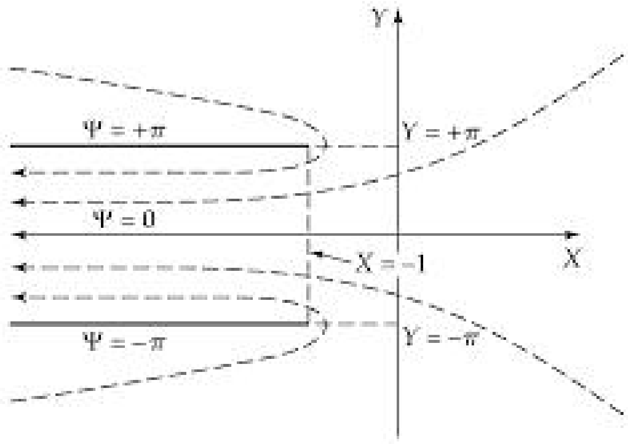

Fig. 4.3-2. The streamlines for the potential flow into a rectangular channel, as predicted from potential flow theory in Eqs. 4.3-30 and 31. A more realistic flow pattern is shown in Fig. 4.3-5.

EXAMPLE 4.3-3

Flow Near a Corner[8]

Figure 4.3-3 shows the potential flow in the neighborhood of two walls that meet at a corner at O. The flow in the neighborhood of this corner can be described by the complex potential

$$w(z) = -cz^{\alpha} \tag{4.3-38}$$

in which c is a constant. We can now consider two situations: (i) an "interior corner flow," with $\alpha > 1$; and (ii) an "exterior corner flow," with $\alpha < 1$.

(a) Find the velocity components.

(b) Obtain the tangential velocity at both parts of the wall.

(c) Describe how to get the streamlines.

(d) How can this result be applied to the flow around a wedge?

SOLUTION

(a) The velocity components are obtained from the complex velocity

$$\frac{dw}{dz} = -c\alpha z^{\alpha-1} = -c\alpha r^{\alpha-1}e^{i(\alpha-1)\theta} \tag{4.3-39}$$

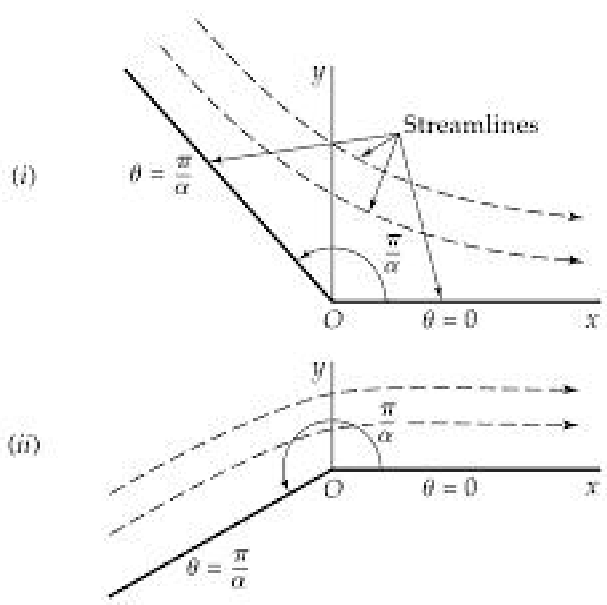

Fig. 4.3-3. Potential flow near a corner. On the left portion of the wall, $v_r = -c\alpha r^{\alpha-1}$, and on the right, $v_r = +c\alpha r^{\alpha-1}$. (i) Interior-corner flow, with $\alpha > 1$; and (ii) exterior-corner flow, with $\alpha < 1$.

[8] R. L. Panton, *Incompressible Flow*, Wiley, New York, 2nd edition (1996).

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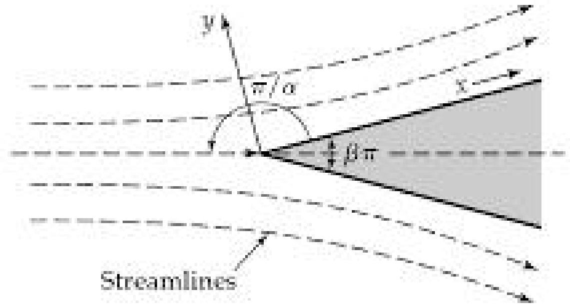

Fig. 4.3-4. Potential flow along a wedge. On the upper surface of the wedge, $v_x = c\alpha x^{\alpha-1} = c'x^{\beta/(2-\beta)}$, where $c' = c[2/(2-\beta)]$. The quantities α and β are related by $\beta = (2/\alpha)(\alpha - 1)$.

Hence from Eq. 4.3-12 we get

$$v_x = +c\alpha r^{\alpha-1}\cos(\alpha-1)\theta \tag{4.3-40}$$

$$v_y = -c\alpha r^{\alpha-1}\sin(\alpha-1)\theta \tag{4.3-41}$$

(b) The tangential velocity at the walls is

at $\theta = 0$: $\quad v_x = v_r = c\alpha r^{\alpha-1} = c\alpha x^{\alpha-1}$ (4.3-42)

at $\theta = \pi/\alpha$:

$$\begin{aligned} v_r &= v_x\cos\theta + v_y\sin\theta \\ &= +c\alpha r^{\alpha-1}\cos(\alpha-1)\theta\cos\theta - c\alpha r^{\alpha-1}\sin(\alpha-1)\theta\sin\theta \\ &= c\alpha r^{\alpha-1}\cos\alpha\theta \\ &= -c\alpha r^{\alpha-1} \end{aligned} \tag{4.3-43}$$

Hence, in Case (i), the incoming fluid at the wall decelerates as it approaches the junction, and the departing fluid accelerates as it moves away from the junction. In Case (ii) the velocity components become infinite at the corner as $\alpha - 1$ is then negative.

(c) The complex potential can be decomposed into its real and imaginary parts

$$w = \phi + i\psi = -cr^{\alpha}(\cos\alpha\theta + i\sin\alpha\theta) \tag{4.3-44}$$

Hence the stream function is

$$\psi = -cr^{\alpha}\sin\alpha\theta \tag{4.3-45}$$

To get the streamlines, one selects various values for the stream function—say, $\psi_1, \psi_2, \psi_3 \cdots$—and then for each value one plots r as a function of θ.

(d) Since for ideal flow any streamline may be replaced by a wall, and vice versa, the results found here for $\alpha > 1$ describe the inviscid flow over a wedge (see Fig. 4.3-4). We make use of this in Example 4.4-3.

A few words of warning are in order concerning the applicability of potential-flow theory to real systems:

a. For the flow around a cylinder, the streamlines shown in Fig. 4.3-1 do not conform to any of the flow regimes sketched in Fig. 3.7-2.

b. For the flow into a channel, the predicted flow pattern of Fig. 4.3-2 is unrealistic inside the channel and just upstream from the channel entrance. A much better approximation to the actual behavior is shown in Fig. 4.3-5.

Both of these failures of the elementary potential theory result from the phenomenon of *separation*: the departure of streamlines from a boundary surface.

Separation tends to occur at sharp corners of solid boundaries, as in channel flow, and on the downstream sides of bluff objects, as in the flow around a cylinder. Generally, separation is likely to occur in regions where the pressure increases in the direction

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boundary layer. The success of the method depends on the thinness of the boundary layer, a condition that is met at high Reynolds number.

We consider the steady, two-dimensional flow of a fluid with constant ρ and μ around a submerged object, such as that shown in Fig. 4.4-1. We assert that the main changes in the velocity take place in a very thin region, the boundary layer, in which the curvature effects are not important. We can then set up a Cartesian coordinate system with x pointing downstream, and y perpendicular to the solid surface. The continuity equation and the Navier–Stokes equations then become:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{4.4-1}$$

$$\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{4.4-2}$$

$$\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \tag{4.4-3}$$

Some of the terms in these equations can be discarded by order-of-magnitude arguments. We use three quantities as "yardsticks": the approach velocity v_∞, some linear dimension l_0 of the submerged body, and an average thickness δ_0 of the boundary layer. The presumption that $\delta_0 << l_0$ allows us to make a number of rough calculations of orders of magnitude.

Since v_x varies from zero at the solid surface to v_∞ at the outer edge of the boundary layer, we can say that

$$\frac{\partial v_x}{\partial y} = \mathrm{O}\left(\frac{v_\infty}{\delta_0}\right) \tag{4.4-4}$$

where O means "order of magnitude of." Similarly, the maximum variation in v_x over the length l_0 of the surface will be v_∞, so that

$$\frac{\partial v_x}{\partial x} = \mathrm{O}\left(\frac{v_\infty}{l_0}\right) \text{ and } \frac{\partial v_y}{\partial y} = \mathrm{O}\left(\frac{v_\infty}{l_0}\right) \tag{4.4-5}$$

Here we have made use of the equation of continuity to get one more derivative (we are concerned here only with orders of magnitude and not the signs of the quantities). Integration of the second relation suggests that $v_y = O((\delta_0/l_0)v_\infty) << v_x$. The various terms in Eq. 4.4-2 may now be estimated as

$$v_x \frac{\partial v_x}{\partial x} = \mathrm{O}\left(\frac{v_\infty^2}{l_0}\right);\ v_y \frac{\partial v_x}{\partial y} = \mathrm{O}\left(\frac{v_\infty^2}{l_0}\right) \qquad \frac{\partial^2 v_x}{\partial x^2} = \mathrm{O}\left(\frac{v_\infty}{l_0^2}\right) \qquad \frac{\partial^2 v_x}{\partial y^2} = \mathrm{O}\left(\frac{v_\infty}{\delta_0^2}\right) \tag{4.4-6}$$

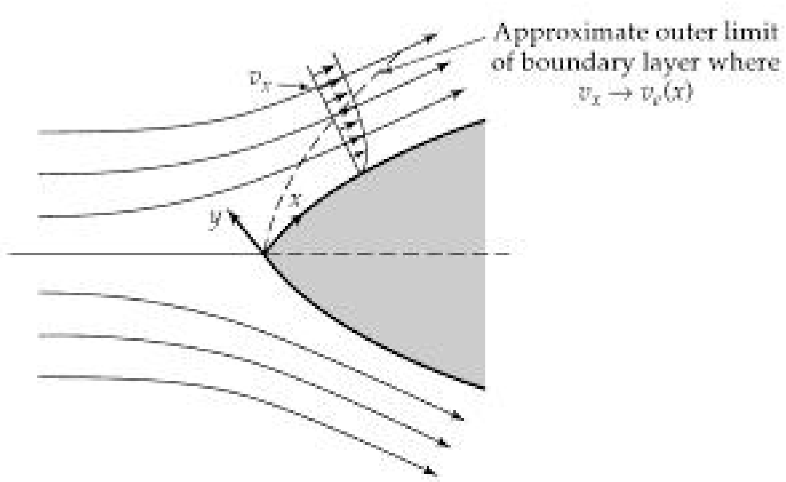

Fig. 4.4-1. Coordinate system for the two-dimensional flow around a submerged object. The boundary-layer thickness is greatly exaggerated for purposes of illustration. Because the boundary layer is in fact quite thin, it is permissible to use rectangular coordinates locally along the curved surface.

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This suggests that $\partial^2 v_x/\partial x^2 << \partial^2 v_x/\partial y^2$, so that the former may be safely neglected. In the boundary layer it is expected that the terms on the left side of Eq. 4.4-2 should be of the same order of magnitude as those on the right side, and therefore

$$\frac{v_\infty^2}{l_0} = O\left(\nu \frac{v_\infty}{\delta_0^2}\right) \quad \text{or} \quad \frac{\delta_0}{l_0} = O\left(\sqrt{\frac{\nu}{v_\infty l_0}}\right) = O\left(\frac{1}{\sqrt{\text{Re}}}\right) \tag{4.4-7}$$

The second of these relations shows that the boundary-layer thickness is small compared to the dimensions of the submerged object in high-Reynolds-number flows.

Similarly it can be shown, with the help of Eq. 4.4-7, that three of the derivatives in Eq. 4.4-3 are of the same order of magnitude:

$$v_x \frac{\partial v_y}{\partial x}, v_y \frac{\partial v_y}{\partial y}, \nu \frac{\partial^2 v_y}{\partial y^2} = O\left(\frac{v_\infty^2 \delta_0}{l_0^2}\right) >> \nu \frac{\partial^2 v_y}{\partial x^2} \tag{4.4-8}$$

Comparison of this result with Eq. 4.4-6 shows that $\partial\mathcal{P}/\partial y << \partial\mathcal{P}/\partial x$. This means that the y-component of the equation of motion is not needed and that the modified pressure can be treated as a function of x alone.

As a result of these order-of-magnitude arguments, we are left with the *Prandtl boundary layer equations*:[1]

(continuity)
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{4.4-9}$$

(motion)
$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{d\mathcal{P}}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \tag{4.4-10}$$

The modified pressure $\mathcal{P}(x)$ is presumed known from the solution of the corresponding potential-flow problem or from experimental measurements.

The usual boundary conditions for these equations are the no-slip condition ($v_x = 0$ at $y = 0$), the condition of no mass transfer from the wall ($v_y = 0$ at $y = 0$), and the statement that the velocity merges into the external (potential-flow) velocity at the outer edge of the boundary layer ($v_x(x, y) \to v_e(x)$). The function $v_e(x)$ is related to $\mathcal{P}(x)$ according to the potential-flow equation of motion in Eq. 4.3-5. Consequently the term $-(1/\rho)(d\mathcal{P}/dx)$ in Eq. 4.4-10 can be replaced by $v_e(dv_e/dx)$ for steady flow. Thus Eq. 4.4-10 may also be written as

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v_e \frac{dv_e}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \tag{4.4-11}$$

The equation of continuity may be solved for v_y by using the boundary condition that $v_y = 0$ at $y = 0$ (i.e., no mass transfer), and then this expression for v_y may be substituted into Eq. 4.4-11 to give

$$v_x \frac{\partial v_x}{\partial x} - \left(\int_0^y \frac{\partial v_x}{\partial x} dy\right) \frac{\partial v_x}{\partial y} = v_e \frac{dv_e}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \tag{4.4-12}$$

This is a partial differential equation for the single dependent variable v_x.

[1] **Ludwig Prandtl** (1875–1953) (pronounced "Prahn-t'l), who taught in Hannover and Göttingen and later served as the Director of the Kaiser Wilhelm Institute for Fluid Dynamics, was one of the people who shaped the future of his field at the beginning of the twentieth century; he made contributions to turbulent flow and heat transfer, but his development of the boundary-layer equations was his crowning achievement. L. Prandtl, *Verhandlungen des III Internationalen Mathematiker-Kongresses* (Heidelberg, 1904), Leipzig, pp. 484–491; L. Prandtl, *Gesammelte Abhandlungen*, **2**, Springer-Verlag, Berlin (1961), pp. 575–584. For an introductory discussion of matched asymptotic expressions, see D. J. Acheson, *Elementary Fluid Mechanics*, Oxford University Press (1990), pp. 269–271. An exhaustive discussion of the subject may be found in M. Van Dyke, *Perturbation Methods in Fluid Dynamics*, The Parabolic Press, Stanford, Cal. (1975).

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This equation may now be multiplied by ρ and integrated from $y = 0$ to $y = \infty$ to give the *von Kármán momentum balance*[2]

$$\mu \frac{\partial v_x}{\partial y}\bigg|_{y=0} = \frac{d}{dx}\int_0^\infty \rho v_x(v_e - v_x)dy + \frac{dv_e}{dx}\int_0^\infty \rho(v_e - v_x)dy \tag{4.4-13}$$

Here use has been made of the condition that $v_x(x, y) \to v_e(x)$ as $y \to \infty$. The quantity on the left side of Eq. 4.4-13 is the shear stress exerted by the fluid on the wall: $-\tau_{yx}|_{y=0}$.

The original Prandtl boundary-layer equations, Eqs. 4.4-9 and 10, have thus been transformed into Eq. 4.4-11, Eq. 4.4-12, and Eq. 4.4-13, and any of these may be taken as the starting point for solving two-dimensional boundary-layer problems. Equation 4.4-13, with assumed expressions for the velocity profile, is the basis of many "approximate boundary-layer solutions" (see Example 4.4-1). On the other hand, the analytical or numerical solutions of Eqs. 4.4-11 or 12 are called "exact boundary-layer solutions" (see Example 4.4-2).

The discussion here is for steady, laminar, two-dimensional flows of fluids with constant density and viscosity. Corresponding equations are available for unsteady flow, turbulent flow, variable fluid properties, and three-dimensional boundary layers.[3-6]

Although many exact and approximate boundary-layer solutions have been obtained and applications of the theory to streamlined objects have been quite successful, considerable work remains to be done on flows with adverse pressure gradients (i.e., positive $\partial\mathcal{P}/\partial x$) in Eq. 4.4-10, such as the flow on the downstream side of a blunt object. In such flows the streamlines usually separate from the surface before reaching the rear of the object (see Fig. 3.7-2). The boundary-layer approach described here is suitable for such flows only in the region upstream from the separation point.

EXAMPLE 4.4-1

Laminar Flow along a Flat Plate (Approximate Solution)

Use the von Kármán momentum balance to estimate the steady-state velocity profiles near a semi-infinite flat plate in a tangential stream with approach velocity v_∞ (see Fig. 4.4-2). For this system the potential-flow solution is $v_e = v_\infty$.

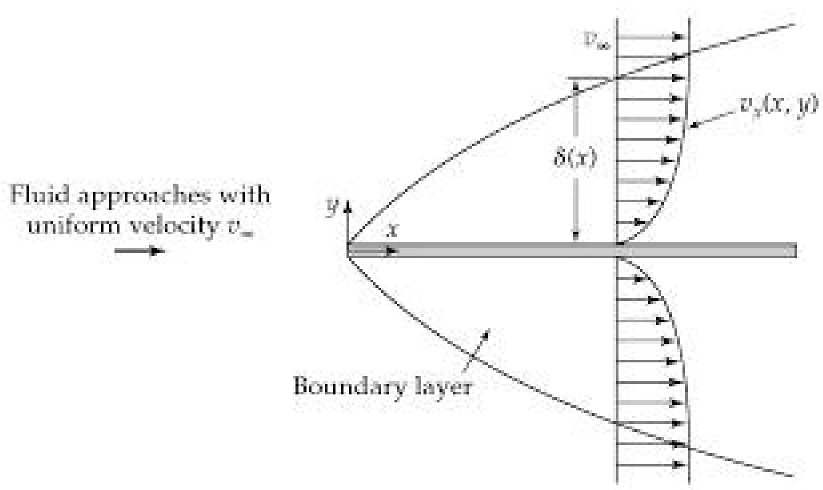

Fig. 4.4-2. Boundary-layer development near a flat plate of negligible thickness.

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[2] Th. von Kármán, *Zeits. für angew. Math. u. Mech.*, **1**, 233–252 (1921). Hungarian-born **Theodor von Kármán** taught in Göttingen, Aachen, and California Institute of Technology; he contributed much to the theory of turbulence and aerodynamics.

[3] H. Schlichting and K. Gersten, *Boundary-Layer Theory*, Springer Verlag, Berlin, 8th edition (2000).

[4] L. Rosenhead, *Laminar Boundary Layers*, Oxford University Press, London (1963).

[5] K. Stewartson, *The Theory of Laminar Boundary Layers in Compressible Fluids*, Oxford University Press (1964).

[6] W. H. Dorrance, *Viscous Hypersonic Flow*, McGraw-Hill, New York (1962).

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SOLUTION

We know intuitively what the velocity profile $v_x(y)$ looks like. Hence we can guess a form for $v_x(y)$ and substitute it directly into the von Kármán momentum balance. One reasonable choice is to let $v_x(y)$ be a function of y/δ, where $\delta(x)$ is the "thickness" of the boundary layer. The function is so chosen that $v_x = 0$ at $y = 0$ and $v_x = v_e$ at $y = \delta$. This is tantamount to assuming geometrical similarity of the velocity profiles for various values of x. When this assumed profile is substituted into the von Kármán momentum balance, an ordinary differential equation for the boundary-layer thickness $\delta(x)$ is obtained. When this equation has been solved, the $\delta(x)$ so obtained can then be used to get the velocity profile and other quantities of interest.

For the present problem a plausible guess for the velocity distribution, with a reasonable shape, is

$$\frac{v_x}{v_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad \text{for } 0 \le y \le \delta(x) \qquad \text{(boundary–layer region)} \tag{4.4-14}$$

$$\frac{v_x}{v_\infty} = 1 \qquad \text{for } y \ge \delta(x) \qquad \text{(potential flow region)} \tag{4.4-15}$$

This is "reasonable" because this velocity profile satisfies the no-slip condition at $y = 0$, and $\partial v_x/\partial y = 0$ at the outer edge of the boundary layer. Substitution of this profile into the von Kármán integral balance in Eq. 4.4-13 gives

$$\frac{3}{2}\frac{\mu v_\infty}{\delta} = \frac{d}{dx}\left(\frac{39}{280}\rho v_\infty^2 \delta\right) \tag{4.4-16}$$

This first-order, separable differential equation can now be integrated to give for the boundary-layer thickness

$$\delta(x) = \sqrt{\frac{280}{13}\frac{\nu x}{v_\infty}} = 4.64\sqrt{\frac{\nu x}{v_\infty}} \tag{4.4-17}$$

Therefore, the boundary-layer thickness increases as the square root of the distance from the upstream end of the plate. The resulting approximate solution for the velocity distribution is then

$$\frac{v_x}{v_\infty} = \frac{3}{2}\left(y\sqrt{\frac{13}{280}\frac{v_\infty}{\nu x}}\right) - \frac{1}{2}\left(y\sqrt{\frac{13}{280}\frac{v_\infty}{\nu x}}\right)^3 \tag{4.4-18}$$

From this result we can estimate the drag force on a plate of finite size wetted on both sides. For a plate of width W and length L, integration of the momentum flux over the two solid surfaces gives:

$$F_x = 2\int_0^W \int_0^L \left(+\mu \frac{\partial v_x}{\partial y}\right)\bigg|_{y=0} dx\, dz = 1.293\sqrt{\rho\mu L W^2 v_\infty^3} \tag{4.4-19}$$

The exact solution, given in the next example, gives the same result, but with a numerical coefficient of 1.328. Both solutions predict the drag force within the scatter of the experimental data. However, the exact solution gives somewhat better agreement with the measured velocity profiles.[3] This additional accuracy is essential for stability calculations.

EXAMPLE 4.4-2

Laminar Flow along a Flat Plate (Exact Solution)[7]

Obtain the exact solution for the problem given in the previous example.

SOLUTION

This problem may be solved by using the definition of the stream function in the first row of Table 4.2-1. Inserting the expressions for the velocity components in Eq. 4.4-11, we get

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -\nu\frac{\partial^3\psi}{\partial y^3} \tag{4.4-20}$$

[7] This problem was treated originally by H. Blasius, *Zeits. Math. Phys.*, **56**, 1–37 (1908).

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The boundary conditions for this equation for $\psi(x, y)$ are

$$\text{B.C. 1:} \qquad \text{at } y = 0, \quad \frac{\partial \psi}{\partial x} = v_y = 0 \quad \text{for } x \geq 0 \tag{4.4-21}$$

$$\text{B.C. 2:} \qquad \text{at } y = 0, \quad \frac{\partial \psi}{\partial y} = -v_x = 0 \quad \text{for } x \geq 0 \tag{4.4-22}$$

$$\text{B.C. 3:} \qquad \text{as } y \to \infty, \quad \frac{\partial \psi}{\partial y} = -v_x \to -v_\infty \quad \text{for } x \geq 0 \tag{4.4-23}$$

$$\text{B.C. 4:} \qquad \text{at } x = 0, \quad \frac{\partial \psi}{\partial y} = -v_x = -v_\infty \quad \text{for } y > 0 \tag{4.4-24}$$

Inasmuch as there is no characteristic length appearing in the above relations, the method of combination of independent variables seems appropriate. By dimensional arguments similar to those used in Example 4.1-1, we write

$$\frac{v_x}{v_\infty} = \Pi(\eta), \qquad \text{where } \eta = y\sqrt{\frac{1}{2}\frac{v_\infty}{\nu x}} \tag{4.4-25}$$

The factor of 2 is included to avoid having any numerical factors occur in the differential equation in Eq. 4.4-27. The stream function that gives the velocity distribution in Eq. 4.4-25 is

$$\psi(x, y) = -\sqrt{2v_\infty \nu x}f(\eta), \qquad \text{where } f(\eta) = \int_0^\eta \Pi(\bar{\eta})d\bar{\eta} \tag{4.4-26}$$

This expression for the stream function is consistent with Eq. 4.4-25 as may be seen by using the relation $v_x = -\partial\psi/\partial y$ (given in Table 4.2-1). Substitution of Eq. 4.4-26 into Eq. 4.4-20 gives

$$-ff'' = f''' \tag{4.4-27}$$

Substitution into the boundary conditions gives

$$\text{B.C. 1 and 2:} \qquad \text{at } \eta = 0, \quad f = 0 \quad \text{and} \quad f' = 0 \tag{4.4-28}$$

$$\text{B.C. 3 and 4:} \qquad \text{as } \eta \to \infty, \quad f' \to 1 \tag{4.4-29}$$

Thus the determination of the flow field is reduced to the solution of one third-order ordinary differential equation.

This equation, along with the boundary conditions given, can be solved by numerical integration, and accurate tables of the solution are available.[3,4] The problem was originally solved by Blasius[7] using analytic approximations that proved to be quite accurate. A plot of his solution is shown in Fig. 4.4-3 along with experimental data taken subsequently. The agreement between theory and experiment is remarkably good.

The drag force on a plate of width W and length L may be calculated from the dimensionless velocity gradient at the wall, $f''(0) = 0.4696\ldots$ as follows:

$$\begin{aligned}
F_x &= 2\int_0^W \int_0^L \left(+\mu\frac{\partial v_x}{\partial y}\right)\bigg|_{y=0} dx\,dz \\
&= 2\int_0^W \int_0^L \left(+\mu v_\infty \frac{df'}{d\eta}\frac{\partial \eta}{\partial y}\right)\bigg|_{y=0} dx\,dz \\
&= 2\int_0^W \int_0^L \mu v_\infty f''(0)\sqrt{\frac{1}{2}\frac{v_\infty}{\nu x}}\,dx\,dz \\
&= 1.328\sqrt{\rho\mu L W^2 v_\infty^3}
\end{aligned} \tag{4.4-30}$$

This result has also been confirmed experimentally.[3,4]

Because of the approximations made in Eq. 4.4-10, the solution is most accurate at large local Reynolds numbers; that is, $\text{Re}_x = xv_\infty/\nu >> 1$. The excluded region of lower Reynolds numbers is small enough to ignore in most drag calculations. More complete

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3. What happens in Example 4.1-2 if one tries to solve Eq. 4.1-21 by the method of separation of variables without first recognizing that the solution can be written as the sum of a steady-state solution and a transient solution?
4. What happens if the separation constant after Eq. 4.1-27 is taken to be c or c^2 instead of $-c^2$?
5. Try solving the problem in Example 4.1-3 using trigonometric quantities in lieu of complex quantities.
6. How is the vorticity equation obtained and how may it be used?
7. How is the stream function defined, and why is it useful?
8. In what sense are the potential flow solutions and the boundary-layer flow solutions complementary?
9. List all approximate forms of the equations of change encountered thus far, and indicate their range of applicability.

PROBLEMS

4A.1 Time for attainment of steady state in tube flow.

(a) A heavy oil, with a kinematic viscosity of $3.45 \times 10^{-4}\ \mathrm{m^2/s}$, is at rest in a long vertical tube with a radius of 0.7 cm. The fluid is suddenly allowed to flow from the bottom of the tube by virtue of gravity. After what time will the velocity at the tube center be within 10% of its final value?

(b) What is the result if water at 68°F is used?

Note: The result shown in Fig. 4D.2 should be used.

Answers: **(a)** 6.4×10^{-2} s; **(b)** 22 s

4A.2 Velocity near a moving sphere. A sphere of radius R is falling in creeping flow with a terminal velocity v_∞ through a quiescent fluid of viscosity μ. At what horizontal distance from the sphere does the velocity of the fluid fall to 1% of the terminal velocity of the sphere?

Answer: About 37 diameters

4A.3 Construction of streamlines for the potential flow around a cylinder. Plot the streamlines for the flow around a cylinder using the information in Example 4.3-1 by the following procedure:

(a) Select a value of $\Psi = C$ (that is, select a streamline).

(b) Plot $Y = C + K$ (straight lines parallel to the X-axis) and $Y = K(X^2 + Y^2)$ (circles with radius $1/2K$, tangent to the X-axis at the origin).

(c) Plot the intersections of the lines and circles that have the same value of K.

(d) Join these points to get the streamline for $\Psi = C$.

Then select other values of C and repeat the process until the pattern of streamlines is clear.

4A.4 Comparison of exact and approximate profiles for flow along a flat plate. Compare the values of v_x/v_∞ obtained from Eq. 4.4-18 with those from Fig. 4.4-3, at the following values of $y\sqrt{v_\infty/\nu x}$: (a) 1.5, (b) 3.0, (c) 4.0. Express the results as the ratio of the approximate to the exact values.

Answers: (a) 0.96; (b) 0.99; (c) 1.01

4A.5 Numerical demonstration of the von Kármán momentum balance.

(a) Evaluate the integrals in Eq. 4.4-13 numerically for the Blasius velocity profile given in Fig. 4.4-3.

(b) Use the results of (a) to determine the magnitude of the wall shear stress $\tau_{yx}|_{y=0}$.

(c) Calculate the total drag force, F_x, for a plate of width W and length L, wetted on both sides. Compare your result with that obtained in Eq. 4.4-30.

Answers: **(a)** $\displaystyle\int_0^\infty \rho v_x(v_e - v_x)dy = 0.664\sqrt{\rho\mu v_\infty^3 x}$

$$\int_0^\infty \rho(v_e - v_x)dy = 1.73\sqrt{\rho\mu v_\infty x}$$

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4A.6 Use of boundary-layer formulas. Air at 1 atm and 20°C flows tangentially on both sides of a thin, smooth flat plate of width $W = 10$ ft, and of length $L = 3$ ft in the direction of the flow. The velocity outside the boundary layer is constant at 20 ft/s.

(a) Compute the local Reynolds number $\text{Re}_x = xv_\infty/\nu$ at the trailing edge.

(b) Assuming laminar flow, compute the approximate boundary-layer thickness, in inches, at the trailing edge. Use the results of Example 4.4-1.

(c) Assuming laminar flow, compute the total drag on the plate in lb_f. Use the results of Examples 4.4-1 and 2.

4A.7 Entrance flow in conduits.

(a) Estimate the entrance length for laminar flow in a circular tube. Assume that the boundary-layer thickness δ is given adequately by Eq. 4.4-17, with v_∞ of the flat-plate problem corresponding to $v_{\max}$ in the tube-flow problem. Assume further that the entrance length L_e can be taken to be the value of x at which $\delta = R$. Compare your result with the expression for L_e cited in §2.3—namely, $L_e = 0.035D$ Re.

(b) Rewrite the transition Reynolds number $xv_\infty/\nu \approx 3.5 \times 10^5$ (for the flat plate) by inserting δ from Eq. 4.4-17 in place of x as the characteristic length. Compare the quantity $\delta v_\infty/\nu$ thus obtained with the corresponding minimum transition Reynolds number for the flow through long smooth tubes.

(c) Use the method of (a) to estimate the entrance length in the flat duct shown in Fig. 4C.1. Compare the result with that given in Problem 4C.1(d).

4B.1 Flow of a fluid with a suddenly applied constant wall stress. In the system studied in Example 4.1-1, let the fluid be at rest before $t = 0$. At time $t = 0$ a constant force is applied to the fluid at the wall in the positive x direction, so that the shear stress τ_{yx} takes on a new constant value τ_0 at $y = 0$ for $t > 0$.

(a) Differentiate Eq. 4.1-1 with respect to y and multiply by $-\mu$ to obtain a partial differential equation for $\tau_{yx}(y, t)$.

(b) Write the boundary and initial conditions for this equation.

(c) Solve using the method in Example 4.1-1 to obtain

$$\frac{\tau_{yx}}{\tau_0} = 1 - \text{erf}\,\frac{y}{\sqrt{4\nu t}} \tag{4B.1-1}$$

(d) Use the result in (c) to obtain the velocity profile. The following relation[1] will be helpful

$$\int_x^\infty (1 - \text{erf}\,u)du = \frac{1}{\sqrt{\pi}}e^{-x^2} - x(1 - \text{erf}\,x) \tag{4B.1-2}$$

4B.2 Flow near a wall suddenly set in motion (approximate solution) (Fig. 4B.2). Apply a procedure like that of Example 4.4-1 to get an approximate solution for Example 4.1-1.

(a) Integrate Eq. 4.1-1 over y to get

$$\int_0^\infty \frac{\partial v_x}{\partial t}dy = \nu \left.\frac{\partial v_x}{\partial y}\right|_0^\infty \tag{4B.2-1}$$

Make use of the boundary conditions and the Leibniz rule for differentiating an integral (Eq. C.3-2) to rewrite Eq. 4B.2-1 in the form

$$\frac{d}{dt}\int_0^\infty \rho v_x dy = \tau_{yx}|_{y=0} \tag{4B.2-2}$$

Interpret this result physically.

[1] A useful summary of error functions and their properties can be found in H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, 2nd edition (1959), Appendix II.

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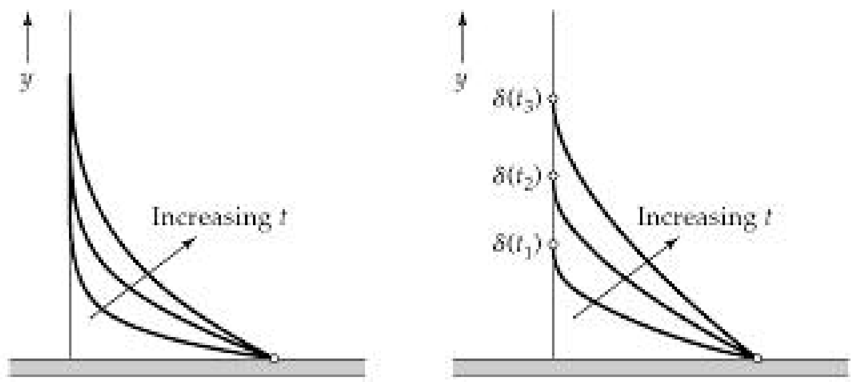

(a) True solution (b) Boundary-layer approximation

Fig. 4B.2. Comparison of true and approximate velocity profiles near a wall suddenly set in motion with velocity v_0.

(b) We know roughly what the velocity profiles look like. We can make the following reasonable postulate for the profiles:

$$\frac{v_x}{v_0} = 1 - \frac{3}{2}\frac{y}{\delta(t)} + \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3 \qquad \text{for } 0 \le y \le \delta(t) \tag{4B.2-3}$$

$$\frac{v_x}{v_0} = 1 \qquad \text{for } y \ge \delta(t) \tag{4B.2-4}$$

Here $\delta(t)$ is a time-dependent boundary-layer thickness. Insert this approximate expression into Eq. 4B.2-2 to obtain

$$\delta\frac{d\delta}{dt} = 4\nu \tag{4B.2-5}$$

(c) Integrate Eq. 4B.2-5 with a suitable initial value of $\delta(t)$, and insert the result into Eq. 4B.2-3 to get the approximate velocity profiles.

(d) Compare the values of v_x/v_0 obtained from (c) with those from Eq. 4.1-15 at $y/\sqrt{4\nu t}$ = 0.2, 0.5, and 1.0. Express the results as the ratio of the approximate value to the exact value.

Answer **(d)** 1.015, 1.026, 0.738

4B.3 Creeping flow around a spherical bubble. When a liquid flows around a gas bubble, circulation takes place within the bubble. This circulation lowers the interfacial shear stress, and, to a first approximation, we may assume that it is entirely eliminated. Repeat the development of Ex. 4.2-1 for such a gas bubble, assuming it is spherical.

(a) Show that B.C. 2 of Ex. 4.2-1 is replaced by

$$\text{B.C. 2:} \qquad \text{at } r = R, \qquad \frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}\right) + 2\frac{f}{r^4} = 0 \tag{4B.3-1}$$

and that the problem set-up is otherwise the same.

(b) Obtain the following velocity components:

$$v_r = v_\infty\left[1 - \left(\frac{R}{r}\right)\right]\cos\theta \tag{4B.3-2}$$

$$v_\theta = -v_\infty\left[1 - \frac{1}{2}\left(\frac{R}{r}\right)\right]\sin\theta \tag{4B.3-3}$$

(c) Next obtain the pressure distribution by using the equation of motion:

$$p = p_0 - \rho g h - \left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\cos\theta \tag{4B.3-4}$$

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(d) Evaluate the total force of the fluid on the sphere to obtain

$$F_z = \tfrac{4}{3}\pi R^3 \rho g + 4\pi\mu R v_\infty \tag{4B.3-5}$$

This result may be obtained by the method of §2.6 or by integrating the z-component of $-[\mathbf{n}\cdot\boldsymbol{\pi}]$ over the sphere surface ($\mathbf{n}$ being the outwardly directed unit vector normal to the surface of the sphere).

4B.4 Use of the vorticity equation.

(a) Work Problem 2B.3 using the y-component of the vorticity equation (Eq. 3D.2-1) and the following boundary conditions: at $x = \pm B$, $v_z = 0$ and at $x = 0$, $v_z = v_{z,\max}$. Show that this leads to

$$v_z = v_{z,\max}[1 - (x/B)^2] \tag{4B.4-1}$$

Then obtain the pressure distribution from the z-component of the equation of motion.

(b) Work Problem 3B.6(*b*) using the vorticity equation, with the following boundary conditions: at $r = R$, $v_z = 0$ and at $r = \kappa R$, $v_z = v_0$. In addition an integral condition is needed to state that there is no net flow in the z direction. Find the pressure distribution in the system.

(c) Work the following problems using the vorticity equation: 2B.6, 2B.7, 3B.1, 3B.10, 3B.16.

4B.5 Steady potential flow around a stationary sphere.[2] In Example 4.2-1 we worked through the creeping flow around a sphere. We now wish to consider the flow of an incompressible, inviscid fluid in irrotational flow around a sphere. For such a problem, we know that the velocity potential must satisfy Laplace's equation (see text after Eq. 4.3-11).

(a) State the boundary conditions for the problem.

(b) Give reasons why the velocity potential ϕ can be postulated to be of the form $\phi(r,\theta) = f(r)\cos\theta$.

(c) Substitute the trial expression for the velocity potential in (*b*) into Laplace's equation for the velocity potential.

(d) Integrate the equation obtained in (c) and obtain the function $f(r)$ containing two constants of integration; determine these constants from the boundary conditions and find

$$\phi = -v_\infty R\left[\left(\frac{r}{R}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^2\right]\cos\theta \tag{4B.5-1}$$

(e) Next show that

$$v_r = v_\infty\left[1 - \left(\frac{R}{r}\right)^3\right]\cos\theta \tag{4B.5-2}$$

$$v_\theta = -v_\infty\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\sin\theta \tag{4B.5-3}$$

(f) Find the pressure distribution, and then show that at the sphere surface

$$\mathcal{P} - \mathcal{P}_\infty = \tfrac{1}{2}\rho v_\infty^2 (1 - \tfrac{9}{4}\sin^2\theta) \tag{4B.5-4}$$

4B.6 Potential flow near a stagnation point (Fig. 4B.6).

(a) Show that the complex potential $w = -v_0 z^2$ describes the flow near a plane stagnation point.

(b) Find the velocity components $v_x(x, y)$ and $v_y(x, y)$.

(c) Explain the physical significance of v_0.

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[2] L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Boston, 2nd edition (1987), pp. 21–26, contains a good collection of potential-flow problems.

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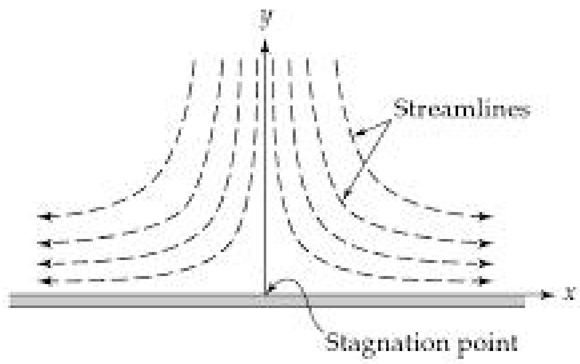

Fig. 4B.6. Two-dimensional potential flow near a stagnation point.

4B.7 Vortex flow.

(a) Show that the complex potential $w = (i\Gamma/2\pi)\ln z$ describes the flow in a vortex. Verify that the tangential velocity is given by $v_\theta = \Gamma/2\pi r$ and that $v_r = 0$. This type of flow is sometimes called a *free vortex*. Is this flow irrotational?

(b) Compare the functional dependence of v_θ on r in (a) with that which arose in Example 3.6-4. The latter kind of flow is sometimes called a *forced vortex*. Actual vortices, such as those that occur in a stirred tank, have a behavior intermediate between these two idealizations.

4B.8 The flow field about a line source. Consider the symmetric radial flow of an incompressible, inviscid fluid outward from an infinitely long uniform source, coincident with the z-axis of a cylindrical coordinate system. Fluid is being generated at a volumetric rate Γ per unit length of source.

(a) Show that the Laplace equation for the velocity potential for this system is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \tag{4B.8-1}$$

(b) From this equation find the velocity potential, velocity, and pressure as functions of position:

$$\phi = -\frac{\Gamma}{2\pi}\ln r \qquad v_r = \frac{\Gamma}{2\pi r} \qquad \mathscr{P}_\infty - \mathscr{P} = \frac{\rho\Gamma^2}{8\pi^2 r^2} \tag{4B.8-2}$$

where $\mathscr{P}_\infty$ is the value of the modified pressure far away from the source.

(c) Discuss the applicability of the results in (b) to the flow field about a well drilled into a large body of porous rock.

(d) Sketch the flow net of streamlines and equipotential lines.

4B.9 Checking solutions to unsteady flow problems.

(a) Verify the solutions to the problems in Examples 4.1-1, 2, and 3 by showing that they satisfy the partial differential equations, initial conditions, and boundary conditions. To show that Eq. 4.1-15 satisfies the differential equation, one has to know how to differentiate an integral using the *Leibniz formula* given in §C.3.

(b) In Example 4.1-3 the initial condition is not satisfied by Eq. 4.1-57. Why?

4C.1 Laminar entrance flow in a slit[3] (Fig. 4C.1). Estimate the velocity distribution in the entrance region of the slit shown in the figure. The fluid enters at $x = 0$ with $v_y = 0$ and $v_x = \langle v_x \rangle$, where $\langle v_x \rangle$ is the average velocity inside the slit. Assume that the velocity distribution in the entrance region $0 \le x \le L_e$ is

$$\frac{v_x}{v_e} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \text{(boundary layer region, } 0 \le y \le \delta\text{)} \tag{4C.1-1}$$

$$\frac{v_x}{v_e} = 1 \qquad \text{(potential flow region, } \delta \le y \le B\text{)} \tag{4C.1-2}$$

in which δ and v_e are functions of x, yet to be determined.

[3] A numerical solution to this problem using the Navier–Stokes equation has been given by Y. L. Wang and P. A. Longwell, *AIChE Journal*, **10**, 323–329 (1964).

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Show that the problem can now be restated as follows:

(bob) $$\frac{d^2\theta_R}{d\tau^2} = -\theta_R + M\left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0} \qquad \text{at } \tau = 0, \theta_R = 0; d\theta_R/d\tau = 0 \tag{4C.2-11}$$

(fluid) $$\frac{\partial\phi}{\partial\tau} = M\frac{\partial^2\phi}{\partial x^2} \qquad \begin{cases}\text{at } \tau = 0, & \phi = 0\\ \text{at } x = 0, & \phi = A(d\theta_R/d\tau)\\ \text{at } x = 1, & \phi = A(d\theta_{aR}/d\tau)\end{cases} \tag{4C.2-12}$$

From these two equations we want to get θ_R and ϕ as functions of x and τ, with M and A as parameters.

(e) Obtain the "sinusoidal steady-state" solution by taking the input function θ_{aR} (the displacement of the cup) to be of the form

$$\theta_{aR}(\tau) = \theta_{aR}^{\circ}\Re\{e^{i\bar{\omega}\tau}\} \qquad (\theta_{aR}^{\circ} \text{ is real}) \tag{4C.2-13}$$

in which $\bar{\omega} = \omega/\omega_0 = \omega\sqrt{I/k}$ is a dimensionless frequency. Then postulate that the bob and fluid motions will also be sinusoidal, but with different amplitudes and phases:

$$\theta_R(\tau) = \Re\{\theta_R^{\circ}e^{i\bar{\omega}\tau}\} \qquad (\theta_R^{\circ} \text{ is complex}) \tag{4C.2-14}$$

$$\phi(x, \tau) = \Re\{\phi^{\circ}(x)e^{i\bar{\omega}\tau}\} \qquad (\phi^{\circ}(x) \text{ is complex}) \tag{4C.2-15}$$

Verify that the amplitude ratio is given by $|\theta_R^{\circ}|/\theta_{aR}^{\circ}$, where $|\cdots|$ indicates the absolute magnitude of a complex quantity. Further show that the phase angle α is given by $\tan\alpha = \Im\{\theta_R^{\circ}\}/\Re\{\theta_R^{\circ}\}$, where $\Re$ and $\Im$ stand for the real and imaginary parts, respectively.

(f) Substitute the postulated solutions of (e) into the equations in (d) to obtain equations for the complex amplitudes θ_R° and $\phi^{\circ}(x)$.

(g) Solve the equation for $\phi^{\circ}(x)$ and verify that

$$\left.\frac{d\phi^{\circ}}{dx}\right|_{x=0} = -\frac{A(i\bar{\omega})^{3/2}}{\sqrt{M}}\left(\frac{\theta_R^{\circ}\cosh\sqrt{i\bar{\omega}/M} - \theta_{aR}^{\circ}}{\sinh\sqrt{i\bar{\omega}/M}}\right) \tag{4C.2-16}$$

(h) Next, solve the θ_R° equation to obtain

$$\frac{\theta_R^{\circ}}{\theta_{aR}^{\circ}} = \frac{AMi\bar{\omega}}{(1-\bar{\omega}^2)\dfrac{\sinh\sqrt{i\bar{\omega}/M}}{\sqrt{i\bar{\omega}/M}} + AMi\bar{\omega}\cosh\sqrt{i\bar{\omega}/M}} \tag{4C.2-17}$$

from which the amplitude ratio $|\theta_R^{\circ}|/\theta_{aR}^{\circ}$ and phase shift α can be found.

(i) For high-viscosity fluids, we can seek a power series by expanding the hyperbolic functions in Eq. 4C.2-17 to get a power series in $1/M$. Show that this leads to

$$\frac{\theta_{aR}^{\circ}}{\theta_R^{\circ}} = 1 + \frac{i}{M}\left(\frac{\bar{\omega}^2-1}{A\bar{\omega}} + \frac{\bar{\omega}}{2}\right) - \frac{1}{M^2}\left(\frac{\bar{\omega}^2-1}{6A} + \frac{\bar{\omega}^2}{24}\right) + O\left(\frac{1}{M^3}\right) \tag{4C.2-18}$$

From this, find the amplitude ratio and the phase angle.

(j) Plot $|\theta_R^{\circ}|/\theta_{aR}^{\circ}$ versus $\bar{\omega}$ for $\mu/\rho = 10^{-2}$ cm^2/s, $L = 25$ cm, $R = 5.5$ cm, $I = 2500$ g cm^2, $k = 4 \times 10^6$ dyn cm, $\rho = 1$ g/cm^3, $(a-1)R = 10^{-2}$ cm. Where is the maximum in the curve?

4C.3 **Darcy's equation for flow through porous media.** For the flow of a fluid through a porous medium, the equations of continuity and motion may be replaced by

smoothed continuity equation $$\varepsilon\frac{\partial\rho}{\partial t} = -(\nabla\cdot\rho\mathbf{v}_0) \tag{4C.3-1}$$

Darcy's equation[5] $$\mathbf{v}_0 = -\frac{\kappa}{\mu}(\nabla p - \rho\mathbf{g}) \tag{4C.3-2}$$

[5] **Henry Philibert Gaspard Darcy** (1803–1858) studied in Paris and became famous for designing the municipal water-supply system in Dijon, the city of his birth. H. Darcy, *Les Fontaines Publiques de la Ville de Dijon*, Victor Dalmont, Paris (1856). For further discussions of "Darcy's law," see J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Martinus Nihjoff, Dordrecht (1983); and H. Brenner and D. A. Edwards, *Macrotransport Processes*, Butterworth-Heinemann, Boston (1993).

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in which ε, the *porosity*, is the ratio of pore volume to total volume, and κ is the *permeability* of the porous medium. The velocity $\mathbf{v}_0$ in these equations is the *superficial velocity*, which is defined as the volume rate of flow through a unit cross-sectional area of the solid plus fluid, averaged over a small region of space—small with respect to the macroscopic dimensions in the flow system, but large with respect to the pore size. The density and pressure are averaged over a region available to flow that is large with respect to the pore size. Equation 4C.3-2 was proposed empirically to describe the slow seepage of fluids through granular media.

When Eqs. 4C.3-1 and 2 are combined we get

$$\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial\rho}{\partial t} = (\nabla \cdot \rho(\nabla p - \rho\mathbf{g})) \tag{4C.3-3}$$

for constant viscosity and permeability. This equation and the equation of state describe the motion of a fluid in a porous medium. For most purposes we may write the *equation of state* as

$$\rho = \rho_0 p^m e^{\beta p} \tag{4C.3-4}$$

in which ρ_0 is the fluid density at unit pressure, and the following parameters have been given:[6]

1. Incompressible liquids	$m = 0$	$\beta = 0$
2. Compressible liquids	$m = 0$	$\beta \neq 0$
3. Isothermal expansion of gases	$\beta = 0$	$m = 1$
4. Adiabatic expansion of gases	$\beta = 0$	$m = C_V/C_p = 1/\gamma$

Show that Eqs. 4C.3-3 and 4 can be combined and simplified for these four categories to give (for gases it is customary to neglect the gravity terms since they are small compared with the pressure terms):

Case 1. $$\nabla^2\mathcal{P} = 0 \tag{4C.3-5}$$

Case 2. $$\left(\frac{\varepsilon\mu\beta}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho - (\nabla \cdot \rho^2\beta\mathbf{g}) \tag{4C.3-6}$$

Case 3. $$\left(\frac{2\varepsilon\mu\rho_0}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho^2 \tag{4C.3-7}$$

Case 4. $$\left(\frac{(m+1)\varepsilon\mu\rho_0^{1/m}}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho^{(1+m)/m} \tag{4C.3-8}$$

Note that Case 1 leads to *Laplace's equation*, Case 2 without the gravity term leads to the *heat-conduction* or *diffusion equation*, and Cases 3 and 4 lead to nonlinear equations.[7]

4C.4 Radial flow through a porous medium (Fig. 4C.4). A fluid flows through a porous cylindrical shell with inner and outer radii R_1 and R_2, respectively. At these surfaces, the pressures are known to be p_1 and p_2, respectively. The length of the cylindrical shell is h.

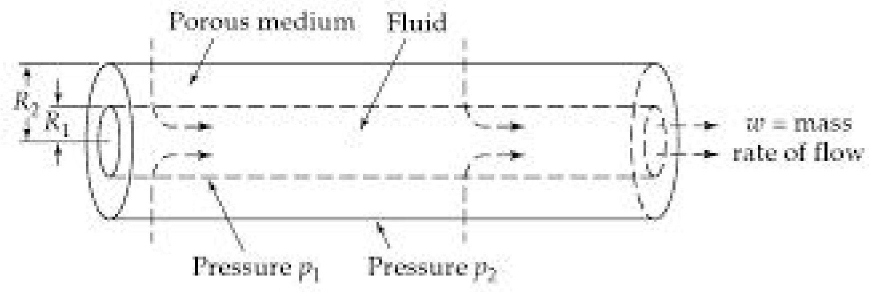

Fig. 4C.4. Radial flow through a porous medium.

[6] M. Muskat, *Flow of Homogeneous Fluids Through Porous Media*, McGraw-Hill (1937).

[7] For the boundary condition at a porous surface that bounds a moving fluid, see G. S. Beavers and D. D. Joseph, *J. Fluid Mech.*, **30**, 197–207 (1967) and G. S. Beavers, E. M. Sparrow, and B. A. Masha, *AIChE Journal*, **20**, 596–597 (1974).

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(a) Find the pressure distribution, radial flow velocity, and mass rate of flow for an incompressible fluid.

(b) Rework (a) for a compressible liquid and for an ideal gas.

Answers: **(a)** $\dfrac{\mathcal{P}-\mathcal{P}_1}{\mathcal{P}_2-\mathcal{P}_1}=\dfrac{\ln(r/R_1)}{\ln(R_2/R_1)}$ $\quad v_{0r}=-\dfrac{\kappa}{\mu r}\dfrac{\mathcal{P}_2-\mathcal{P}_1}{\ln(R_2/R_1)}$ $\quad w=\dfrac{2\pi\kappa h(\mathcal{P}_2-\mathcal{P}_1)\rho}{\mu\ln(R_2/R_1)}$

4D.1 Flow near an oscillating wall.[8] Show, by using Laplace transforms, that the complete solution to the problem stated in Eqs. 4.1-44 to 47 is

$$\frac{v_x}{v_0}=e^{-\sqrt{\omega/2\nu}y}\cos(\omega t-\sqrt{\omega/2\nu}y)-\frac{1}{\pi}\int_0^\infty e^{-\bar{\omega}t}(\sin\sqrt{\bar{\omega}/\nu}y)\frac{\omega}{\omega^2+\bar{\omega}^2}d\bar{\omega} \tag{4D.1-1}$$

4D.2 Start-up of laminar flow in a circular tube (Fig. 4D.2). A fluid of constant density and viscosity is contained in a very long pipe of length L and radius R. Initially the fluid is at rest. At time $t=0$, a pressure gradient $(\mathcal{P}_0-\mathcal{P}_L)/L$ is imposed on the system. Determine how the velocity profiles change with time.

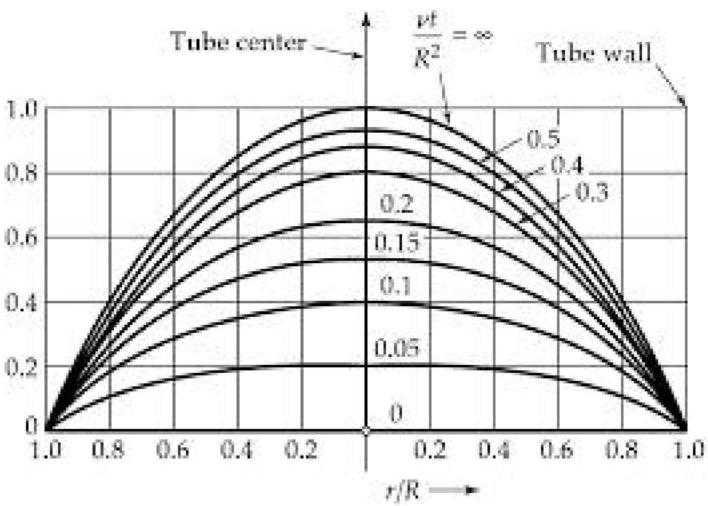

Fig. 4D.2. Velocity distribution for the unsteady flow resulting from a suddenly impressed pressure gradient in a circular tube [P. Szymanski, *J. Math. Pures Appl.*, Series 9, **11**, 67–107 (1932)].

(a) Show that the relevant equation of motion can be put into dimensionless form as follows:

$$\frac{\partial\phi}{\partial\tau}=4+\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right) \tag{4D.2-1}$$

in which $\xi=r/R$, $\tau=\mu t/\rho R^2$, and $\phi=[(\mathcal{P}_0-\mathcal{P}_L)R^2/4\mu L]^{-1}v_z$.

(b) Show that the asymptotic solution for large time is $\phi_\infty=1-\xi^2$. Then define ϕ_t by $\phi(\xi,\tau)=\phi_\infty(\xi)-\phi_t(\xi,\tau)$, and solve the partial differential equation for ϕ_t by the method of separation of variables.

(c) Show that the final solution is

$$\phi(\xi,\tau)=(1-\xi^2)-8\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\xi)}{\alpha_n^3J_1(\alpha_n)}\exp(-\alpha_n^2\tau) \tag{4D.2-2}$$

in which $J_n(\xi)$ is the nth order Bessel function of ξ, and the α_n are the roots of the equation $J_0(\alpha_n)=0$. The result is plotted in Fig. 4D.2.

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[8] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, 2nd edition (1959), p. 319, Eq. (8), with $\varepsilon=\frac{1}{2}\pi$ and $\bar{\omega}=\kappa u^2$.

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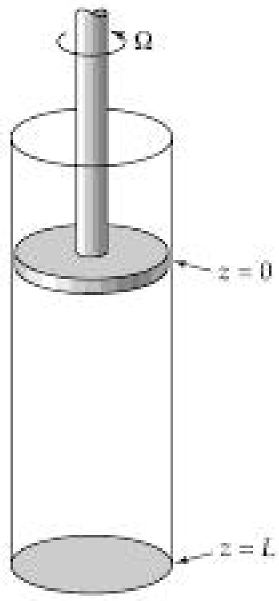

Fig. 4D.3. Rotating disk in a circular tube.

4D.3 Flows in the disk-and-tube system (Fig. 4D.3).[9]

(a) A fluid in a circular tube is caused to move tangentially by a tightly fitting rotating disk at the liquid surface at $z = 0$; the bottom of the tube is located at $z = L$. Find the steady-state velocity distribution $v_\theta(r, z)$, when the angular velocity of the disk is Ω. Assume that creeping flow prevails throughout, so that there is no secondary flow. Find the limit of the solution as $L \to \infty$.

(b) Repeat the problem for the unsteady flow. The fluid is at rest before $t = 0$, and the disk suddenly begins to rotate with an angular velocity Ω at $t = 0$. Find the velocity distribution $v_\theta(r, z, t)$ for a column of fluid of height L. Then find the solution for the limit as $L \to \infty$.

(c) If the disk is oscillating sinusoidally in the tangential direction with amplitude Ω_0, obtain the velocity distribution in the tube when the "oscillatory steady state" has been attained. Repeat the problem for a tube of infinite length.

4D.4 Unsteady annular flows.

(a) Obtain a solution to the Navier–Stokes equation for the start-up of *axial* annular flow by a sudden impressed pressure gradient. Check your result against the published solution.[10]

(b) Solve the Navier–Stokes equation for the unsteady *tangential* flow in an annulus. The fluid is at rest for $t < 0$. Starting at $t = 0$ the outer cylinder begins rotating with a constant angular velocity to cause laminar flow for $t > 0$. Compare your result with the published solution.[11]

4D.5 Stream functions for three-dimensional flow.

(a) Show that the velocity functions $\rho\mathbf{v} = [\nabla \times \mathbf{A}]$ and $\rho\mathbf{v} = [(\nabla\psi_1) \times (\nabla\psi_2)]$ both satisfy the equation of continuity identically for steady flow. The second function also describes unsteady incompressible flows. The functions ψ_1, ψ_2, and $\mathbf{A}$ are arbitrary, except that their derivatives appearing in $(\nabla \cdot \rho\mathbf{v})$ must exist.

(b) Show that the expression $\mathbf{A}/\rho = -\boldsymbol{\delta}_3\psi/h_3$ reproduces the velocity components for the four incompressible flows of Table 4.2-1. Here h_3 and $\boldsymbol{\delta}_3$ are the scale factor and unit vector for the velocity component not shown in the table. (Read the general vector $\mathbf{v}$ of Eq. A.7-18 here as $\mathbf{A}$.)

(c) Show that the streamlines of $[(\nabla\psi_1) \times (\nabla\psi_2)]$ are given by the intersections of the surfaces ψ_1 = constant and ψ_2 = constant. Sketch such a pair of surfaces for the flow in Fig. 4.3-1.

(d) Use Stokes' theorem (Eq. A.5-4), and the definition of $\mathbf{A}$ from (a), to obtain an expression in terms of $\mathbf{A}$ for the mass flow rate through a surface S bounded by a closed curve C. Show that the vanishing of $\mathbf{v}$ on C does not imply the vanishing of $\mathbf{A}$ on C.

---

[9] W. Hort, *Z. tech. Phys.*, **10**, 213 (1920); C. T. Hill, J. D. Huppler, and R. B. Bird, *Chem. Engr. Sci.*, **21**, 815–817 (1966).

[10] W. Müller, *Zeits. für angew. Math. u. Mech.*, **16**, 227–238 (1936).

[11] R. B. Bird and C. F. Curtiss, *Chem. Engr. Sci*, **11**, 108–113 (1959).

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Chapter 5

Velocity Distributions in Turbulent Flow

In the previous chapters we discussed laminar flow problems only. We have seen that the differential equations describing laminar flow are well understood and that, for a number of simple systems, the velocity distribution and various derived quantities can be obtained in a straightforward fashion. The limiting factor in applying the equations of change is the mathematical complexity that one encounters in problems for which there are several velocity components that are functions of several variables. Even there, with the rapid development of computational fluid dynamics, such problems are gradually yielding to numerical solution.

In this chapter we turn our attention to turbulent flow. Whereas laminar flow is orderly, turbulent flow is chaotic. It is this chaotic nature of turbulent flow that poses all sorts of difficulties. In fact, one might question whether or not the equations of change given in Chapter 3 are even capable of describing the violently fluctuating motions in turbulent flow. Since the sizes of the turbulent eddies are several orders of magnitude larger than the mean free path of the molecules of the fluid, the equations of change *are* applicable. Numerical solutions of these equations are obtainable and can be used for studying the details of the turbulence structure. For many purposes, however, we are not interested in having such detailed information, in view of the computational effort required. Therefore, in this chapter we shall concern ourselves primarily with methods that enable us to describe the time-smoothed velocity and pressure profiles.

In §5.1 we start by comparing the experimental results for laminar and turbulent flows in several flow systems. In this way we can get some qualitative ideas about the main differences between laminar and turbulent motions. These experiments help to define some of the challenges that face the fluid dynamicist.

In §5.2 we define several *time-smoothed* quantities, and show how these definitions can be used to time-average the equations of change over a short time interval. These equations describe the behavior of the time-smoothed velocity and pressure. The time-smoothed equation of motion, however, contains the *turbulent momentum flux*. This flux

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Over the same range of Reynolds numbers the mass rate of flow and the pressure drop are no longer proportional but are related approximately by

$$\mathscr{P}_0 - \mathscr{P}_L \approx 0.0198\left(\frac{2}{\pi}\right)^{7/4}\left(\frac{\mu^{1/4}L}{\rho R^{19/4}}\right)w^{7/4} \qquad (10^4 < \text{Re} < 10^5) \qquad (5.1\text{-}6)$$

The stronger dependence of pressure drop on mass flow rate for turbulent flow results from the fact that more energy has to be supplied to maintain the violent eddy motion in the fluid.

The laminar–turbulent transition in circular pipes normally occurs at a *critical Reynolds number* of roughly 2100, although this number may be higher if extreme care is taken to eliminate vibrations in the system.[2] The transition from laminar flow to turbulent flow can be demonstrated by the simple experiment originally performed by Reynolds. One sets up a long transparent tube equipped with a device for injecting a small amount of dye into the stream along the tube axis. When the flow is laminar, the dye moves downstream as a straight, coherent filament. For turbulent flow, on the other hand, the dye spreads quickly over the entire cross section, similarly to the motion of particles in Fig. 2.0-1, because of the eddying motion (turbulent diffusion).

Noncircular Tubes

For developed laminar flow in the triangular duct shown in Fig. 3B.2(b), the fluid particles move rectilinearly in the z direction, parallel to the walls of the duct. By contrast, in turbulent flow there is superposed on the time-smoothed flow in the z direction (the *primary flow*) a time-smoothed motion in the xy-plane (the *secondary flow*). The secondary flow is much weaker than the primary flow and manifests itself as a set of six vortices arranged in a symmetric pattern around the duct axis (see Fig. 5.1-2). Other noncircular tubes also exhibit secondary flows.

Flat Plate

In §4.4 we found that for the laminar flow around a flat plate, wetted on both sides, the solution of the boundary layer equations gave the drag force expression

$$F = 1.328\sqrt{\rho\mu L W^2 v_\infty^3} \qquad \text{(laminar) } 0 < \text{Re}_L < 5 \times 10^5 \qquad (5.1\text{-}7)$$

in which $\text{Re}_L = Lv_\infty\rho/\mu$ is the Reynolds number for a plate of length L; the plate width is W, and the approach velocity of the fluid is v_∞.

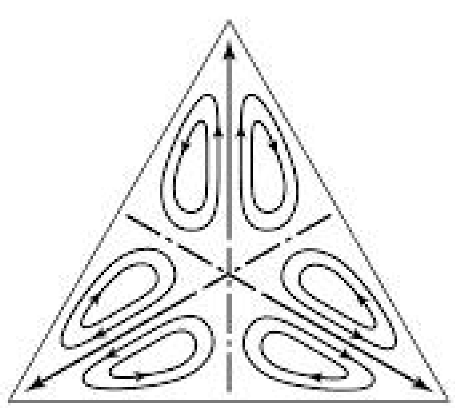

Fig. 5.1-2. Sketch showing the secondary flow patterns for turbulent flow in a tube of triangular cross section [H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), p. 613].

[2] O. Reynolds, *Phil. Trans. Roy. Soc.*, **174**, Part III, 935–982 (1883). See also A. A. Draad and F. M. T. Nieuwstadt, *J. Fluid Mech.*, **361**, 297–308 (1998).

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Table 5.1-1 Dependence of Jet Parameters on Distance z from Wall

	Laminar flow			Turbulent flow		
	Width of jet	Centerline velocity	Mass flow rate	Width of jet	Centerline velocity	Mass flow rate
Circular jet	z	z^{-1}	z	z	z^{-1}	z
Plane jet	$z^{2/3}$	$z^{-1/3}$	$z^{1/3}$	z	$z^{-1/2}$	$z^{1/2}$

For turbulent flow, on the other hand, the dependence on the geometrical and physical properties is quite different:[1]

$$F \approx 0.74\sqrt[5]{\rho^4\mu L^4 W^5 v_\infty^9} \qquad \text{(turbulent)}\ (5\times10^5 < \mathrm{Re}_L < 10^7) \tag{5.1-8}$$

Thus the force is proportional to the $\frac{3}{2}$-power of the approach velocity for laminar flow, but to the $\frac{9}{5}$-power for turbulent flow. The stronger dependence on the approach velocity reflects the extra energy needed to maintain the irregular eddy motions in the fluid.

Circular and Plane Jets

Next we examine the behavior of jets that emerge from a flat wall, which is taken to be the xy-plane (see Fig. 5.6-1). The fluid comes out from a circular tube or a long narrow slot, and flows into a large body of the same fluid. Various observations on the jets can be made: the width of the jet, the centerline velocity of the jet, and the mass flow rate through a cross section parallel to the xy-plane. All these properties can be measured as functions of the distance z from the wall. In Table 5.1-1 we summarize the properties of the circular and two-dimensional jets for laminar and turbulent flow.[1] It is curious that, for the circular jet, the jet width, centerline velocity, and mass flow rate have exactly the same dependence on z in both laminar and turbulent flow. We shall return to this point later in §5.6.

The above examples should make it clear that the gross features of laminar and turbulent flow are generally quite different. One of the many challenges in turbulence theory is to try to explain these differences.

§5.2 TIME-SMOOTHED EQUATIONS OF CHANGE FOR INCOMPRESSIBLE FLUIDS

We begin by considering a turbulent flow in a tube with a constant imposed pressure gradient. If at one point in the fluid we observe one component of the velocity as a function of time, we find that it is fluctuating in a chaotic fashion as shown in Fig. 5.2-1(a). The fluctuations are irregular deviations from a mean value. The actual velocity can be regarded as the sum of the mean value (designated by an overbar) and the fluctuation (designated by a prime). For example, for the z-component of the velocity we write

$$v_z = \bar{v}_z + v_z' \tag{5.2-1}$$

which is sometimes called the *Reynolds decomposition*. The mean value is obtained from $v_z(t)$ by making a time average over a large number of fluctuations

$$\bar{v}_z = \frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} v_z(s)\,ds \tag{5.2-2}$$

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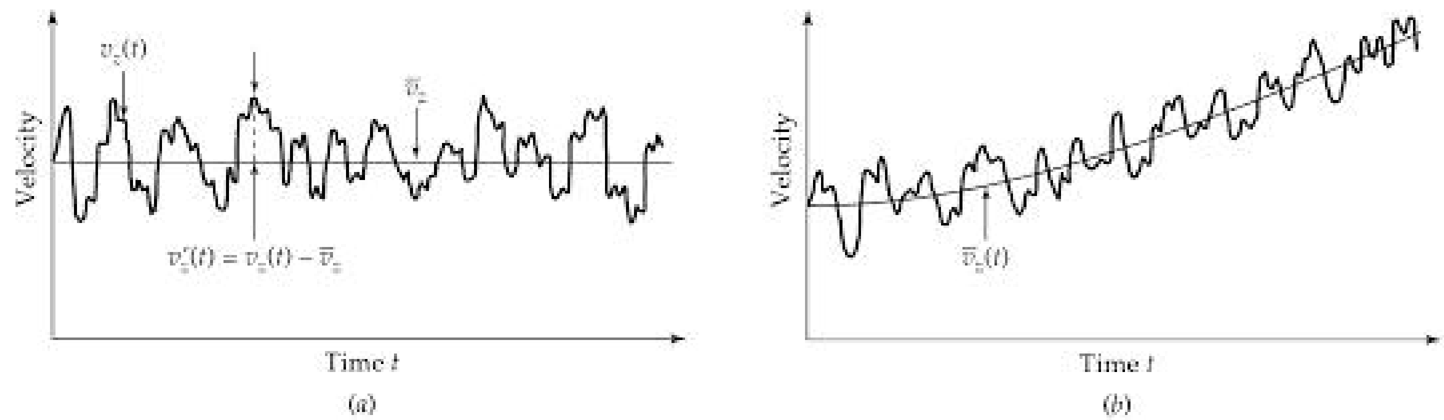

Fig. 5.2-1. Sketch showing the velocity component v_z as well as its time-smoothed value $\overline{v}_z$ and its fluctuation v_z' in turbulent flow (*a*) for "steadily driven turbulent flow" in which $\overline{v}_z$ does not depend on time, and (*b*) for a situation in which $\overline{v}_z$ does depend on time.

the period t_0 being long enough to give a smooth averaged function. For the system at hand, the quantity $\overline{v}_z$, which we call the *time-smoothed velocity*, is independent of time, but of course depends on position. When the time-smoothed velocity does not depend on time, we speak of *steadily driven turbulent flow*. The same comments we have made for velocity can also be made for pressure.

Next we consider turbulent flow in a tube with a time-dependent pressure gradient. For such a flow one can define time-smoothed quantities as above, but one has to understand that the period t_0 must be small with respect to the changes in the pressure gradient, but still large with respect to the periods of fluctuations. For such a situation the time-smoothed velocity and the actual velocity are illustrated in Fig. 5.2-1(*b*).[1]

According to the definition in Eq. 5.2-2, it is easy to verify that the following relations are true:

$$\overline{v_z'} = 0 \qquad \overline{\overline{v}_z} = \overline{v}_z \qquad \overline{\overline{v}_z v_z'} = 0 \qquad \overline{\frac{\partial}{\partial x} v_z} = \frac{\partial}{\partial x}\overline{v}_z \qquad \overline{\frac{\partial}{\partial t} v_z} = \frac{\partial}{\partial t}\overline{v}_z \tag{5.2-3}$$

The quantity $\overline{v_z'^2}$ will not, however, be zero, and in fact the ratio $\sqrt{\overline{v_z'^2}}/\langle \overline{v}_z \rangle$ can be taken to be a measure of the magnitude of the turbulent fluctuations. This quantity, known as the *intensity of turbulence*, may have values from 1 to 10% in the main part of a turbulent stream and values of 25% or higher in the neighborhood of a solid wall. Hence, it must be emphasized that we are not necessarily dealing with tiny disturbances; sometimes the fluctuations are actually quite violent and large.

Quantities such as $\overline{v_x' v_y'}$ are also nonzero. The reason for this is that the local motions in the x and y directions are *correlated*. In other words, the fluctuations in the x direction are not independent of the fluctuations in the y direction. We shall see presently that these time-smoothed values of the products of fluctuating properties have an important role in turbulent momentum transfer. Later we shall find similar correlations arising in turbulent heat and mass transport.

[1] One can also define the "overbar" quantities in terms of an "ensemble average." For most purposes the results are equivalent or are assumed to be so. See, for example, A. A. Townsend, *The Structure of Turbulent Shear Flow*, Cambridge University Press, 2nd edition (1976). See also P. K. Kundu, *Fluid Mechanics*, Academic Press, New York (1990), p. 421, regarding the last of the formulas given in Eq. 5.2-3.

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Equation 5.2-11 is an extra equation obtained by subtracting Eq. 5.2-10 from the original equation of continuity.

The principal result of this section is that the equation of motion in terms of the stress tensor, summarized in Appendix Table B.5, can be adapted for time-smoothed turbulent flow by changing all v_i to $\bar{v}_i$ and p to $\bar{p}$ as well as τ_{ij} to $\bar{\tau}_{ij} = \bar{\tau}_{ij}^{(v)} + \bar{\tau}_{ij}^{(t)}$ in any of the coordinate systems given.

We have now arrived at the main stumbling block in the theory of turbulence. The Reynolds stresses $\bar{\tau}_{ij}^{(t)}$ above are not related to the velocity gradients in a simple way as are the time-smoothed viscous stresses $\bar{\tau}_{ij}^{(v)}$ in Eq. 5.2-9. They are, instead, complicated functions of the position and the turbulence intensity. To solve flow problems we must have experimental information about the Reynolds stresses or else resort to some empirical expression. In §5.4 we discuss some of the empiricisms that are available.

Actually one can also obtain equations of change for the Reynolds stresses (see Problem 5D.1). However, these equations contain quantities like $\overline{v_i'v_j'v_k'}$. Similarly, the equations of change for the $\overline{v_i'v_j'v_k'}$ contain the next higher-order correlation $\overline{v_i'v_j'v_k'v_l'}$, and so on. That is, there is a never-ending hierarchy of equations that must be solved. To solve flow problems one has to "truncate" this hierarchy by introducing empiricisms. If we use empiricisms for the Reynolds stresses, we then have a "first-order" theory. If we introduce empiricisms for the $\overline{v_i'v_j'v_k'}$, we then have a "second-order theory," and so on. The problem of introducing empiricisms to get a closed set of equations that can be solved for the velocity and pressure distributions is referred to as the "closure problem." The discussion in §5.4 deals with closure at the first order. At the second order the "k-ε empiricism" has been extensively studied and widely used in computational fluid mechanics.[2]

§5.3 THE TIME-SMOOTHED VELOCITY PROFILE NEAR A WALL

Before we discuss the various empirical expressions used for the Reynolds stresses, we present here several developments that do not depend on any empiricisms. We are concerned here with the fully developed, time-smoothed velocity distribution in the neighborhood of a wall. We discuss several results: a Taylor expansion of the velocity near the wall, and the universal logarithmic and power law velocity distributions a little further out from the wall.

The flow near a flat surface is depicted in Fig. 5.3-1. It is convenient to distinguish four regions of flow:

- the *viscous sublayer* very near the wall, in which viscosity plays a key role
- the *buffer layer* in which the transition occurs between the viscous and inertial sublayers
- the *inertial sublayer* at the beginning of the main turbulent stream, in which viscosity plays at most a minor role
- the *main turbulent stream*, in which the time-smoothed velocity distribution is nearly flat and viscosity is unimportant

It must be emphasized that this classification into regions is somewhat arbitrary.

[2] J. L. Lumley, *Adv. Appl. Mech.*, **18**, 123–176 (1978); C. G. Speziale, *Ann. Revs. Fluid Mech.*, **23**, 107–157 (1991); H. Schlichting and K. Gersten, *Boundary-Layer Theory*, Springer, Berlin, 8th edition (2000), pp. 560–563.

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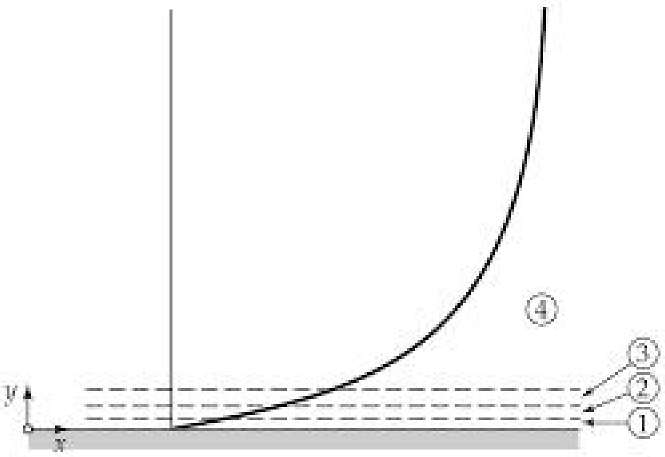

Fig. 5.3-1. Flow regions for describing turbulent flow near a wall: ① viscous sublayer, ② buffer layer, ③ inertial sublayer, ④ main turbulent stream.

The Logarithmic and Power Law Velocity Profiles in the Inertial Sublayer[1–4]

Let the time-smoothed shear stress acting on the wall $y = 0$ be called τ_0 (this is the same as $-\bar{\tau}_{yx}|_{y=0}$). Then the shear stress in the inertial sublayer will not be very different from the value τ_0. We now ask: On what quantities will the time-smoothed velocity gradient $d\bar{v}_x/dy$ depend? It should not depend on the viscosity, since, out beyond the buffer layer, momentum transport should depend primarily on the velocity fluctuations (loosely referred to as "eddy motion"). It may depend on the density ρ, the wall shear stress τ_0, and the distance y from the wall. The only combination of these three quantities that has the dimensions of a velocity gradient is $\sqrt{\tau_0/\rho}/y$. Hence we write

$$\frac{d\bar{v}_x}{dy} = \frac{1}{\kappa}\sqrt{\frac{\tau_0}{\rho}}\frac{1}{y} \tag{5.3-1}$$

in which κ is an arbitrary dimensionless constant, which must be determined experimentally. The quantity $\sqrt{\tau_0/\rho}$ has the dimensions of velocity; it is called the *friction velocity* and given the symbol v_*. When Eq. 5.3-1 is integrated we get

$$\bar{v}_x = \frac{v_*}{\kappa}\ln y + \lambda' \tag{5.3-2}$$

λ' being an integration constant. To use dimensionless groupings, we rewrite Eq. 5.3-2 as

$$\frac{\bar{v}_x}{v_*} = \frac{1}{\kappa}\ln\left(\frac{yv_*}{\nu}\right) + \lambda \tag{5.3-3}$$

in which λ is a constant simply related to λ'; the kinematic viscosity ν was included in order to construct the dimensionless argument of the logarithm. Experimentally it has been found that reasonable values of the constants[2] are $\kappa = 0.4$ and $\lambda = 5.5$, giving

$$\frac{\bar{v}_x}{v_*} = 2.5\ln\left(\frac{yv_*}{\nu}\right) + 5.5 \qquad \frac{yv_*}{\nu} > 30 \tag{5.3-4}$$

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[1] L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), pp. 172–178.

[2] H. Schlichting and K. Gersten, *Boundary-Layer Theory*, Springer-Verlag, Berlin, 8th edition (2000), §17.2.3.

[3] T. von Kármán, *Nachr. Ges. Wiss. Göttingen, Math-Phys. Klasse*, Series 5, 58–76 (1930); L. Prandtl, *Ergeb. Aerodyn. Versuch. Göttingen*, Series 4, 18–29 (1932).

[4] G. I. Barenblatt and A. J. Chorin, *Proc. Nat. Acad. Sci. USA*, **93**, 6749–6752 (1996) and *SIAM Rev.*, **40**, 265–291 (1981); G. I. Barenblatt, A. J. Chorin, and V. M. Prostokishin, *Proc. Nat. Acad. Sci. USA*, **94**, 773–776 (1997). See also G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press (1992), §10.2.

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This is called the *von Kármán-Prandtl universal logarithmic velocity distribution*;[3] it is intended to apply only in the inertial sublayer. Later we shall see (in Fig. 5.5-3) that this function describes moderately well the experimental data somewhat beyond the inertial sublayer.

If Eq. 5.3-1 were correct, then the constants κ and λ would indeed be "universal constants," applicable at any Reynolds number. However, values of κ in the range 0.40 to 0.44 and values of λ in the range 5.0 to 6.3 can be found in the literature, depending on the range of Reynolds numbers. This suggests that the right side of Eq. 5.3-1 should be multiplied by some function of Reynolds number and that y could be raised to some power involving the Reynolds number. Theoretical arguments have been advanced[4] that Eq. 5.3-1 should be replaced by

$$\frac{d\bar{v}_x}{dy} = \frac{v_*}{y}\left(B_0 + \frac{B_1}{\ln \mathrm{Re}}\right)\left(\frac{yv_*}{\nu}\right)^{\beta_1/\ln \mathrm{Re}} \tag{5.3-5}$$

in which $B_0 = \frac{1}{2}\sqrt{3}$, $B_1 = \frac{15}{4}$, and $\beta_1 = \frac{3}{2}$. When Eq. 5.3-5 is integrated with respect to y, the *Barenblatt–Chorin universal velocity distribution* is obtained:

$$\frac{\bar{v}_x}{v_*} = \left(\frac{1}{\sqrt{3}}\ln \mathrm{Re} + \frac{5}{2}\right)\left(\frac{yv_*}{\nu}\right)^{3/(2\ln \mathrm{Re})} \tag{5.3-6}$$

Equation 5.3-6 describes regions ③ and ④ of Fig. 5.3-1 better than does Eq. 5.3-4.[4] Region ① is better described by Eq. 5.3-13.

Taylor-Series Development in the Viscous Sublayer

We start by writing a Taylor series for $\bar{v}_x$ as a function of y, thus

$$\bar{v}_x(y) = \bar{v}_x(0) + \left.\frac{\partial \bar{v}_x}{\partial y}\right|_{y=0} y + \frac{1}{2!}\left.\frac{\partial^2 \bar{v}_x}{\partial y^2}\right|_{y=0} y^2 + \frac{1}{3!}\left.\frac{\partial^3 \bar{v}_x}{\partial y^3}\right|_{y=0} y^3 + \cdots \tag{5.3-7}$$

To evaluate the terms in this series, we need the expression for the time-smoothed shear stress in the vicinity of a wall. For the special case of the steadily driven flow in a slit of thickness $2B$, the shear stress will be of the form $\bar{\tau}_{yx} = \bar{\tau}_{yx}^{(v)} + \bar{\tau}_{yx}^{(t)} = -\tau_0[1 - (y/B)]$. Then from Eqs. 5.2-8 and 9, we have

$$+\mu\frac{\partial \bar{v}_x}{\partial y} - \rho\overline{v_x' v_y'} = \tau_0\left(1 - \frac{y}{B}\right) \tag{5.3-8}$$

Now we examine one by one the terms that appear in Eq. 5.3-7:[5]

(i) The first term is zero by the no-slip condition.

(ii) The coefficient of the second term can be obtained from Eq. 5.3-8, recognizing that both v_x' and v_y' are zero at the wall so that

$$\left.\frac{\partial \bar{v}_x}{\partial y}\right|_{y=0} = \frac{\tau_0}{\mu} \tag{5.3-9}$$

(iii) The coefficient of the third term involves the second derivative, which may be obtained by differentiating Eq. 5.3-8 with respect to y and then setting $y = 0$, as follows,

$$\left.\frac{\partial^2 \bar{v}_x}{\partial y^2}\right|_{y=0} = \frac{\rho}{\mu}\left.\left(\overline{v_x'\frac{\partial v_y'}{\partial y} + v_y'\frac{\partial v_x'}{\partial y}}\right)\right|_{y=0} - \frac{\tau_0}{\mu B} = -\frac{\tau_0}{\mu B} \tag{5.3-10}$$

since both v_x' and v_y' are zero at the wall.

[5] A. A. Townsend, *The Structure of Turbulent Shear Flow*, Cambridge University Press, 2nd edition (1976), p. 163.

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in which $\mu^{(t)}$ is the *turbulent viscosity* (often called the *eddy viscosity*, and given the symbol ε). As one can see from Table 5.1-1, for at least one of the flows given there, the circular jet, one might expect Eq. 5.4-1 to be useful. Usually, however, $\mu^{(t)}$ is a strong function of position and the intensity of turbulence. In fact, for some systems[2] $\mu^{(t)}$ may even be negative in some regions. It must be emphasized that the viscosity μ is a property of the *fluid*, whereas the eddy viscosity $\mu^{(t)}$ is primarily a property of the *flow*.

For two kinds of turbulent flows (i.e., flows along surfaces and flows in jets and wakes), special expressions for $\mu^{(t)}$ are available:

(i) Wall turbulence: $$\mu^{(t)} = \mu\left(\frac{yv_*}{14.5\nu}\right)^3 \qquad 0 < \frac{yv_*}{\nu} < 5 \tag{5.4-2}$$

This expression, derivable from Eq. 5.3-13, is valid only very near the wall. It is of considerable importance in the theory of turbulent heat and mass transfer at fluid–solid interfaces.[3]

(ii) Free turbulence: $$\mu^{(t)} = \rho\kappa_0 b(\bar{v}_{z,\max} - \bar{v}_{z,\min}) \tag{5.4-3}$$

in which κ_0 is a dimensionless coefficient to be determined experimentally, b is the width of the mixing zone at a downstream distance z, and the quantity in parentheses represents the maximum difference in the z-component of the time-smoothed velocities at that distance z. Prandtl[4] found Eq. 5.4-3 to be a useful empiricism for jets and wakes.

The Mixing Length of Prandtl

By assuming that eddies move around in a fluid very much as molecules move around in a low-density gas (not a very good analogy) Prandtl[5] developed an expression for momentum transfer in a turbulent fluid. The "mixing length" l plays roughly the same role as the mean free path in kinetic theory (see §1.4). This kind of reasoning led Prandtl to the following relation:

$$\bar{\tau}_{yx}^{(t)} = -\rho l^2 \left|\frac{d\bar{v}_x}{dy}\right| \frac{d\bar{v}_x}{dy} \tag{5.4-4}$$

If the mixing length were a universal constant, Eq. 5.4-4 would be very attractive, but in fact l has been found to be a function of position. Prandtl proposed the following expressions for l:

(i) Wall turbulence: $l = \kappa_1 y$ (y = distance from wall) (5.4-5)

(ii) Free turbulence: $l = \kappa_2 b$ (b = width of mixing zone) (5.4-6)

in which κ_1 and κ_2 are constants. A result similar to Eq. 5.4-4 was obtained by Taylor[6] by his "vorticity transport theory" some years prior to Prandtl's proposal.

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[2] J. O. Hinze, *Appl. Sci. Res.*, **22**, 163–175 (1970); V. Kruka and S. Eskinazi, *J. Fluid. Mech.*, **20**, 555–579 (1964).

[3] C. S. Lin, R. W. Moulton, and G. L. Putnam, *Ind. Eng. Chem.*, **45**, 636–640 (1953).

[4] L. Prandtl, *Zeits. f. angew. Math. u. Mech.*, **22**, 241–243 (1942).

[5] L. Prandtl, *Zeits. f. angew. Math. u. Mech.*, **5**, 136–139 (1925).

[6] G. I. Taylor, *Phil. Trans.* **A215**, 1–26 (1915), and *Proc. Roy. Soc.* (London), **A135**, 685–701 (1932).

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(b) For the flow between parallel plates, we can use the expression found in Eq. 5.3-12 for the time-smoothed velocity profile to get the turbulent momentum flux:

$$\begin{aligned}\bar{\tau}_{yx}^{(t)} = \rho\overline{v_x' v_y'} &= -\tau_0\left(1 - \frac{y}{B}\right) + \mu\frac{d\bar{v}_x}{dy} \\ &= -\tau_0\left(1 - \frac{y}{B}\right) + \left(\tau_0 - \tau_0\frac{y}{B} + Ay^3 + \cdots\right)\end{aligned} \tag{5.4-13}$$

where $A = 4C(v_*/\nu)^4$. This is in accord with Eq. 5.4-12.

§5.5 TURBULENT FLOW IN DUCTS

We start this section with a short discussion of experimental measurements for turbulent flow in rectangular ducts, in order to give some impressions about the Reynolds stresses. In Figs. 5.5-1 and 2 are shown some experimental measurements of the time-smoothed quantities $\overline{v_z'^2}$, $\overline{v_x'^2}$, and $\overline{v_x' v_z'}$ for the flow in the z direction in a rectangular duct.

In Fig. 5.5-1 note that quite close to the wall, $\sqrt{\overline{v_z'^2}}$ is about 13% of the time-smoothed centerline velocity $\bar{v}_{z,\max}$, whereas $\sqrt{\overline{v_x'^2}}$ is about 5%. This means that, near the wall, the velocity fluctuations in the flow direction are appreciably greater than those in the transverse direction. Near the center of the duct, the two fluctuation amplitudes are nearly equal and we say that the turbulence is nearly *isotropic* there.

In Fig. 5.5-2 the turbulent shear stress $\bar{\tau}_{xz}^{(t)} = \rho\overline{v_x' v_z'}$ is compared with the total shear stress $\bar{\tau}_{xz} = \bar{\tau}_{xz}^{(t)} + \bar{\tau}_{xz}^{(v)}$ across the duct. It is evident that the turbulent contribution is the

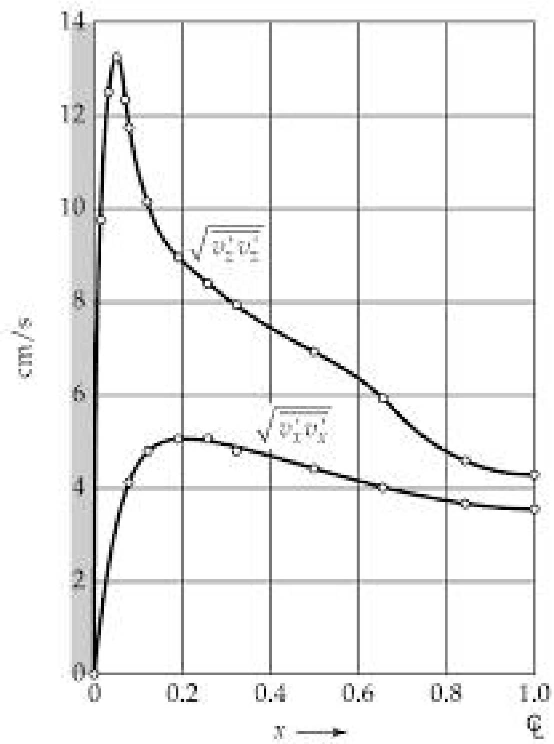

Fig. 5.5-1. Measurements of H. Reichardt [*Naturwissenschaften*, 404 (1938), *Zeits. f. angew. Math. u. Mech.*, **13**, 177–180 (1933), **18**, 358–361 (1938)] for the turbulent flow of air in a rectangular duct with $\bar{v}_{z,\max} = 100$ cm/s. Here the quantities $\sqrt{\overline{v_x'v_x'}}$ and $\sqrt{\overline{v_z'v_z'}}$ are shown.

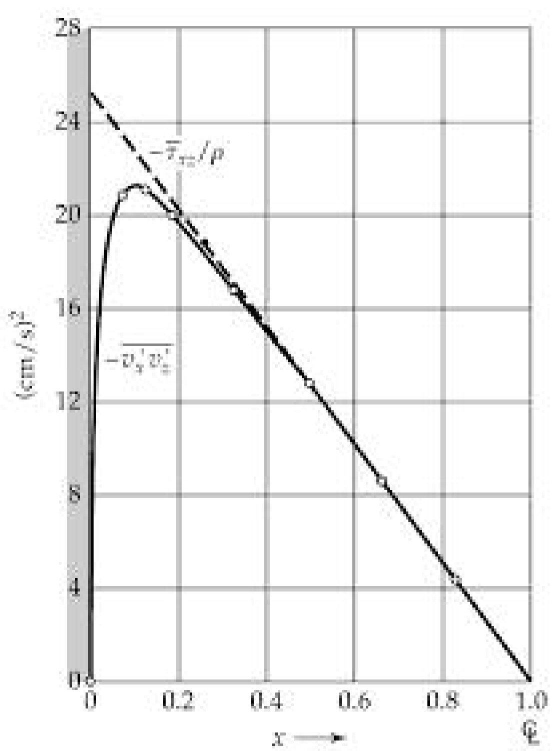

Fig. 5.5-2. Measurements of Reichardt (see Fig. 5.5-1) for the quantity $\overline{v_x'v_z'}$ in a rectangular duct. Note that this quantity differs from $\bar{\tau}_{xz}/\rho$ only near the duct wall.

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This result may be solved for $\mu^{(t)}/\mu$ and the result can be expressed in terms of dimensionless variables:

$$\begin{aligned}\frac{\mu^{(t)}}{\mu} &= \frac{1}{\mu}\frac{\bar{\tau}_{rz}}{d\bar{v}_z/dy} - 1 \\ &= \frac{1}{\mu}\frac{\tau_0[1-(y/R)]}{d\bar{v}_z/dy} - 1 \\ &= \frac{[1-(y/R)]}{dv^+/dy^+} - 1 \end{aligned} \tag{5.5-8}$$

where $y^+ = yv_*\rho/\mu$ and $v^+ = \bar{v}_z/v_*$. When $y = R/2$, the value of y^+ is

$$y^+ = \frac{yv_*\rho}{\mu} = \frac{(R/2)\sqrt{\tau_0/\rho}\,\rho}{\mu} = 477 \tag{5.5-9}$$

For this value of y^+, the logarithmic distribution in the caption of Fig. 5.5-3 gives

$$\frac{dv^+}{dy^+} = \frac{2.5}{477} = 0.00524 \tag{5.5-10}$$

Substituting this into Eq. 5.5-8 gives

$$\frac{\mu^{(t)}}{\mu} = \frac{1/2}{0.00524} - 1 = 94 \tag{5.5-11}$$

This result emphasizes that, far from the tube wall, molecular momentum transport is negligible in comparison with eddy transport.

§5.6 TURBULENT FLOW IN JETS

In the previous section we discussed the flow in ducts, such as circular tubes; such flows are examples of *wall turbulence*. Another main class of turbulent flows is *free turbulence*, and the main examples of these flows are jets and wakes. The time-smoothed velocity in these types of flows can be described adequately by using Prandtl's expression for the eddy viscosity in Eq. 5.4-3, or by using Prandtl's mixing length theory with the empiricism given in Eq. 5.4-6. The former method is simpler, and hence we use it in the following illustrative example.

EXAMPLE 5.6-1

***Time-Smoothed Velocity Distribution in a Circular Wall Jet*[1–4]**

A jet of fluid emerges from a circular hole into a semi-infinite reservoir of the same fluid as depicted in Fig. 5.6-1. In the same figure we show roughly what we expect the profiles of the z-component of the velocity to look like. We would expect that for various values of z the profiles will be similar in shape, differing only by a scale factor for distance and velocity. We also can imagine that as the jet moves outward, it will create a net radial inflow so that some of the surrounding fluid will be dragged along. We want to find the time-smoothed velocity distribution in the jet and also the amount of fluid crossing each plane of constant z. Before working through the solution, it may be useful to review the information on jets in Table 5.1-1.

[1] H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), pp. 747–750.

[2] A. A. Townsend, *The Structure of Turbulent Shear Flow*, Cambridge University Press, 2nd edition (1976), Chapter 6.

[3] J. O. Hinze, *Turbulence*, McGraw-Hill, New York, 2nd edition (1975), Chapter 6.

[4] S. Goldstein, *Modern Developments in Fluid Dynamics*, Oxford University Press (1938), and Dover reprint (1965), pp. 592–597.

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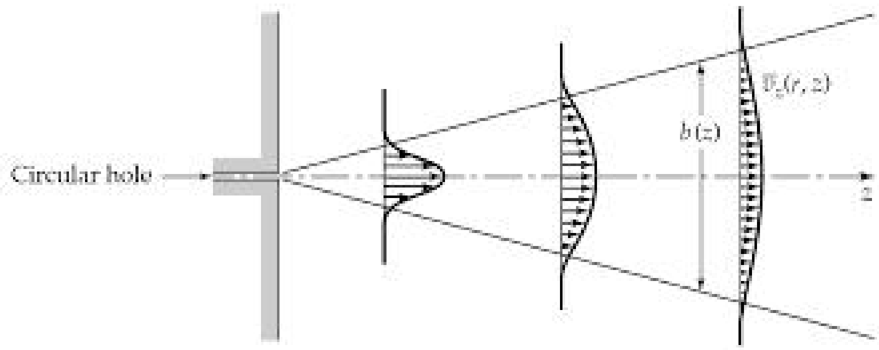

Fig. 5.6-1. Circular jet emerging from a plane wall.

SOLUTION

In order to use Eq. 5.4-3 it is necessary to know how b and $\bar{v}_{z,\max} - \bar{v}_{z,\min}$ vary with z for the circular jet. We know that the total rate of flow of z-momentum J will be the same for all values of z. We presume that the convective momentum flux is much greater than the viscous momentum flux. This permits us to postulate that the jet width b depends on J, on the density ρ and the kinematic viscosity ν of the fluid, and on the downstream distance z from the wall. The only combination of these variables that has the dimensions of length is $b \propto Jz/\rho\nu^2$, so that the jet width is proportional to z.

We next postulate that the velocity profiles are "similar," that is,

$$\frac{\bar{v}_z}{\bar{v}_{z,\max}} = f(\xi) \qquad \text{where } \xi = \frac{r}{b(z)} \tag{5.6-1}$$

which seems like a plausible proposal; here $\bar{v}_{z,\max}$ is the velocity along the centerline. When this is substituted into the expression for the rate of momentum flow in the jet (neglecting the contribution from $\bar{\tau}_{zz}$)

$$J = \int_0^{2\pi}\int_0^\infty \rho\bar{v}_z^2 r\,dr\,d\theta \tag{5.6-2}$$

we find that

$$J = 2\pi\rho b^2 \bar{v}_{z,\max}^2 \int_0^\infty f^2\xi d\xi = \text{constant} \times \rho b^2 \bar{v}_{z,\max}^2 \tag{5.6-3}$$

Since J does not depend on z and since b is proportional to z, then $\bar{v}_{z,\max}$ has to be inversely proportional to z.

The $\bar{v}_{z,\min}$ in Eq. 5.4-3 occurs at the outer edge of the jet and is zero. Therefore because $b \propto z$ and $\bar{v}_{z,\max} \propto z^{-1}$, we find from Eq. 5.4-3 that $\mu^{(t)}$ is a constant. Thus we can use the equations of motion for laminar flow and replace the viscosity μ by the eddy viscosity $\mu^{(t)}$, or ν by $\nu^{(t)}$.

In the jet the main motion is in the z direction; that is $|\bar{v}_r| << |\bar{v}_z|$. Hence we can use a boundary layer approximation (see §4.4) for the time-smoothed equations of change and write

continuity:
$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}_r) + \frac{\partial \bar{v}_z}{\partial z} = 0 \tag{5.6-4}$$

motion:
$$\bar{v}_r\frac{\partial \bar{v}_z}{\partial r} + \bar{v}_z\frac{\partial \bar{v}_z}{\partial z} = \nu^{(t)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{v}_z}{\partial r}\right) \tag{5.6-5}$$

These equations are to be solved with the following boundary conditions:

B.C. 1: at $r = 0$, $\bar{v}_r = 0$ (5.6-6)

B.C. 2: at $r = 0$, $\partial\bar{v}_z/\partial r = 0$ (5.6-7)

B.C. 3: as $z \to \infty$, $\bar{v}_z \to 0$ (5.6-8)

The last boundary condition is automatically satisfied, inasmuch as we have already found that $\bar{v}_{z,\max}$ is inversely proportional to z. We now seek a solution to Eq. 5.6-5 of the form of Eq. 5.6-1 with $b = z$.

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To avoid working with two dependent variables, we introduce the stream function as discussed in §4.2. For axially symmetric flow, the stream function is defined as follows:

$$\bar{v}_z = -\frac{1}{r}\frac{\partial\psi}{\partial r} \qquad \bar{v}_r = \frac{1}{r}\frac{\partial\psi}{\partial z} \tag{5.6-9, 10}$$

This definition ensures that the equation of continuity in Eq. 5.6-4 is satisfied. Since we know that $\bar{v}_z$ is $z^{-1} \times$ some function of ξ, we deduce from Eq. 5.6-9 that ψ must be proportional to z. Furthermore ψ must have dimensions of (velocity) $\times$ (length)^2, hence the stream function must have the form

$$\psi(r, z) = \nu^{(t)} z F(\xi) \tag{5.6-11}$$

in which F is a dimensionless function of $\xi = r/z$. From Eqs. 5.6-9 and 10 we then get

$$\bar{v}_z = -\frac{\nu^{(t)}}{z}\frac{F'}{\xi} \qquad \bar{v}_r = \frac{\nu^{(t)}}{z}\left(\frac{F}{\xi} - F'\right) \tag{5.6-12, 13}$$

The first two boundary conditions may now be rewritten as

B.C. 1: $$\text{at } \xi = 0, \qquad \frac{F}{\xi} - F' = 0 \tag{5.6-14}$$

B.C. 2: $$\text{at } \xi = 0, \qquad \frac{F''}{\xi} - \frac{F'}{\xi^2} = 0 \tag{5.6-15}$$

If we expand F in a Taylor series about $\xi = 0$,

$$F(\xi) = a + b\xi + c\xi^2 + d\xi^3 + e\xi^4 + \cdots \tag{5.6-16}$$

then the first boundary condition gives $a = 0$, and the second gives $b = d = 0$. We will use this result presently.

Substitution of the velocity expressions of Eqs. 5.6-12 and 13 into the equation of motion in Eq. 5.6-5 then gives a third-order differential equation for F,

$$\frac{d}{d\xi}\left(\frac{FF'}{\xi}\right) = \frac{d}{d\xi}\left(F'' - \frac{F'}{\xi}\right) \tag{5.6-17}$$

This may be integrated to give

$$\frac{FF'}{\xi} = F'' - \frac{F'}{\xi} + C_1 \tag{5.6-18}$$

in which the constant of integration must be zero; this can be seen by using the Taylor series in Eq. 5.6-16 along with the fact that a, b, and d are all zero.

Equation 5.6-18 was first solved by Schlichting.[5] After multiplying the equation by ξ, the left side may be written as $\frac{1}{2}(F^2)'$, and the first term on the right side as $(\xi F')' - F'$. Then the equation may be integrated term by term to give:

$$\xi F' = 2F + \tfrac{1}{2}F^2 + C_2 \tag{5.6-19}$$

Once again, knowing the behavior of F near $\xi = 0$, we conclude that the second constant of integration is zero. Equation 5.6-19 is then a first-order separable equation, and it may be solved to give

$$F(\xi) = -\frac{(C_3\xi)^2}{1 + \frac{1}{4}(C_3\xi)^2} \tag{5.6-20}$$

[5] H. Schlichting, *Zeits. f. angew. Math. u. Mech.*, **13**, 260–263 (1933).

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in which C_3 is the third constant of integration. Substitution of this into Eqs. 5.6-12 and 13 then gives

$$\bar{v}_z = \frac{\nu^{(t)}}{z}\frac{2C_3^2}{[1+\frac{1}{4}(C_3 r/z)^2]^2} \tag{5.6-21}$$

$$\bar{v}_r = \frac{C_3\nu^{(t)}}{z}\frac{(C_3 r/z)-\frac{1}{4}(C_3 r/z)^3}{[1+\frac{1}{4}(C_3 r/z)^2]^2} \tag{5.6-22}$$

When the above expression for $\bar{v}_z$ is substituted into Eq. 5.6-2 for J, we get an expression for the third integration constant in terms of J:

$$C_3 = \sqrt{\frac{3}{16\pi}}\sqrt{\frac{J}{\rho}}\frac{1}{\nu^{(t)}} \tag{5.6-23}$$

The last three equations then give the time-smoothed velocity profiles in terms of J, ρ, and $\nu^{(t)}$.

A measurable quantity in jet flow is the radial position corresponding to an axial velocity one-half the centerline value; we call this half-width $b_{1/2}$. From Eq. 5.6-21 we then obtain

$$\frac{\bar{v}_z(b_{1/2}, z)}{\bar{v}_{z,\max}(z)} = \frac{1}{2} = \frac{1}{[1+\frac{1}{4}(C_3 b_{1/2}/z)^2]^2} \tag{5.6-24}$$

Experiments indicate[6] that $b_{1/2} = 0.0848z$. When this is inserted into Eq. 5.6-24, it is found that $C_3 = 15.1$. Using this value, we can get the turbulent viscosity $\nu^{(t)}$ as a function of J and ρ from Eq. 5.6-23.

Figure 5.6-2 gives a comparison of the above axial velocity profile with experimental data. The calculated curve obtained from the Prandtl mixing length theory is also shown.[7] Both methods appear to give reasonably good curve fits of the experimental profiles. The

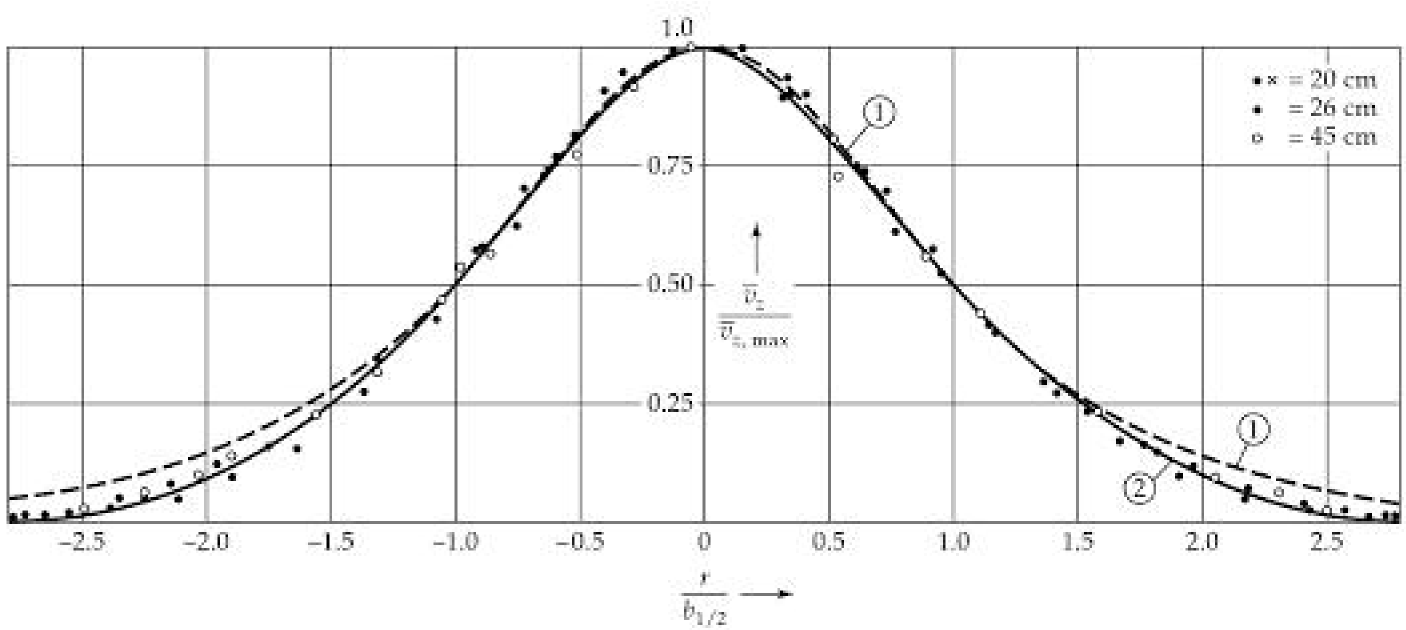

Fig. 5.6-2. Velocity distribution in a circular jet in turbulent flow [H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), Fig. 24.9]. The eddy viscosity calculation (curve 1) and the Prandtl mixing length calculation (curve 2) are compared with the measurements of H. Reichardt [*VDI Forschungsheft*, 414 (1942), 2nd edition (1951)]. Further measurements by others are cited by S. Corrsin ["Turbulence: Experimental Methods," in *Handbuch der Physik*, Vol. VIII/2, Springer, Berlin (1963)].

[6] H. Reichardt, *VDI Forschungsheft*, **414** (1942).

[7] W. Tollmien, *Zeits. f. angew. Math. u. Mech.*, **6**, 468–478 (1926).

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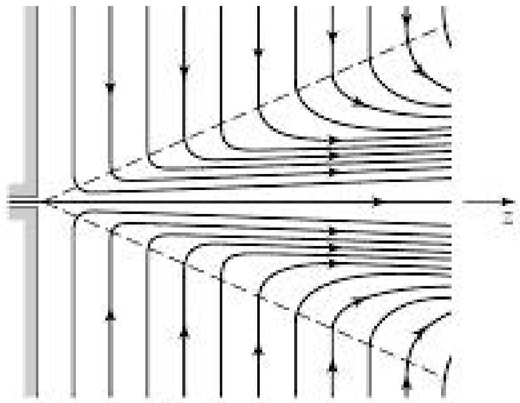

Fig. 5.6-3. Streamline pattern in a circular jet in turbulent flow [H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), Fig. 24.10].

eddy viscosity method seems to be somewhat better in the neighborhood of the maximum, whereas the mixing length results are better in the outer part of the jet.

Once the velocity profiles are known, the streamlines can be obtained. From the streamlines, shown in Fig. 5.6-3, it can be seen how the jet draws in fluid from the surrounding mass of fluid. Hence the mass of fluid carried by the jet increases with the distance from the source. This mass rate of flow is

$$w = \int_0^{2\pi}\int_0^{\infty} \rho\bar{v}_z r\, dr\, d\theta = 8\pi\rho\nu^{(t)}z \tag{5.6-25}$$

This result corresponds to an entry in Table 5.1-1.

The two-dimensional jet issuing from a thin slot may be analyzed similarily. In that problem, however, the turbulent viscosity is a function of position.

QUESTIONS FOR DISCUSSION

1. Compare and contrast the procedures for solving laminar flow problems and turbulent flow problems.
2. Why must Eq. 5.1-4 *not* be used for evaluating the velocity gradient at the solid boundary?
3. What does the logarithmic profile of Eq. 5.3-4 give for the fluid velocity at the wall? Why does this not create a problem in Example 5.5-1 when the logarithmic profile is integrated over the cross section of the tube?
4. Discuss the physical interpretation of each term in Eq. 5.2-12.
5. Why is the absolute value sign used in Eq. 5.4-4? How is it eliminated in Eq. 5.5-5?
6. In Example 5.6-1, how do we know that the momentum flow through any plane of constant z is a constant? Can you imagine a modification of the jet problem in which that would not be the case?
7. Go through some of the volumes of *Ann. Revs. Fluid Mech.* and summarize the topics in turbulent flow that are found there.
8. In Eq. 5.3-1 why do we investigate the functional dependence of the velocity gradient rather than the velocity itself?
9. Why is turbulence such a difficult topic?

PROBLEMS

5A.1 Pressure drop needed for laminar-turbulent transition. A fluid with viscosity 18.3 cp and density 1.32 g/cm^3 is flowing in a long horizontal tube of radius 1.05 in. (2.67 cm). For what pressure gradient will the flow become turbulent?

Answer: 26 psi/mi (1.1×10^5 Pa/km)

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5A.2 Velocity distribution in turbulent pipe flow. Water is flowing through a long, straight, level section of smooth 6.00 in. i.d. pipe, at a temperature of 68°F. The pressure gradient along the length of the pipe is 1.0 psi/mi.

(a) Determine the wall shear stress τ_0 in psi ($lb_f/in.^2$) and Pa.

(b) Assume the flow to be turbulent and determine the radial distances from the pipe wall at which $\bar{v}_z/\bar{v}_{z,max} = 0.0, 0.1, 0.2, 0.4, 0.7, 0.85, 1.0$.

(c) Plot the complete velocity profile, $\bar{v}_z/\bar{v}_{z,max}$ vs. $y = R - r$.

(d) Is the assumption of turbulent flow justified?

(e) What is the mass flow rate?

5B.1 Average flow velocity in turbulent tube flow.

(a) For the turbulent flow in smooth circular tubes, the function[1]

$$\frac{\bar{v}_z}{\bar{v}_{z,max}} = \left(1 - \frac{r}{R}\right)^{1/n} \tag{5B.1-1}$$

is sometimes useful for curve-fitting purposes: near $Re = 4 \times 10^3$, $n = 6$; near $Re = 1.1 \times 10^5$, $n = 7$; and near $Re = 3.2 \times 10^6$, $n = 10$. Show that the ratio of average to maximum velocity is

$$\frac{\langle \bar{v}_z \rangle}{\bar{v}_{z,max}} = \frac{2n^2}{(n+1)(2n+1)} \tag{5B.1-2}$$

and verify the result in Eq. 5.1-5.

(b) Sketch the logarithmic profile in Eq. 5.3-4 as a function of r when applied to a circular tube of radius R. Then show how this function may be integrated over the tube cross section to get Eq. 5.5-1. List all the assumptions that have been made to get this result.

5B.2 Mass flow rate in a turbulent circular jet.

(a) Verify that the velocity distributions in Eqs. 5.6-21 and 22 do indeed satisfy the differential equations and boundary conditions.

(b) Verify that Eq. 5.6-25 follows from Eq. 5.6-21.

5B.3 The eddy viscosity expression in the viscous sublayer. Verify that Eq. 5.4-2 for the eddy viscosity comes directly from the Taylor series expression in Eq. 5.3-13.

5C.1 Two-dimensional turbulent jet. A fluid jet issues forth from a slot perpendicular to the xy-plane and emerges in the z direction into a semi-infinite medium of the same fluid. The width of the slot in the y direction is W. Follow the pattern of Example 5.6-1 to find the time-smoothed velocity profiles in the system.

(a) Assume the similar profiles

$$\bar{v}_z/\bar{v}_{z,max} = f(\xi) \qquad \text{with } \xi = x/z \tag{5C.1-1}$$

Show that the momentum conservation statement leads to the fact that the centerline velocity must be proportional to $z^{-1/2}$.

(b) Introduce a stream function ψ such that $\bar{v}_z = -\partial\psi/\partial x$ and $\bar{v}_x = +\partial\psi/\partial z$. Show that the result in (a) along with dimensional considerations leads to the following form for ψ:

$$\psi = z^{1/2}\sqrt{J/\rho W}F(\xi) \tag{5C.1-2}$$

Here $F(\xi)$ is a dimensionless stream function, which will be determined from the equation of motion for the fluid and J is the total momentum flow defined analogously to Eq. 5.6-2.

[1] H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), pp. 596–600.

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Reynolds numbers.[3] Assume further that κ in Eq. 5.3-3 is the same for the inner and outer walls.

(a) Show that direct application of Eq. 5.3-3 leads immediately to the following velocity profiles[4] in the region $r \leq bR$ (designated by <) and $r \geq bR$ (designated by >):

$$\frac{\overline{v}_z^<}{v_*^<} = \frac{1}{\kappa}\ln\left(\frac{(r - aR)v_*^<}{\nu}\right) + \lambda^< \qquad \text{where } v_*^< = v_{**}\sqrt{\frac{b^2 - a^2}{a}} \tag{5C.2-2}$$

$$\frac{\overline{v}_z^>}{v_*^>} = \frac{1}{\kappa}\ln\left(\frac{(R - r)v_*^>}{\nu}\right) + \lambda^> \qquad \text{where } v_*^> = v_{**}\sqrt{1 - b^2} \tag{5C.2-3}$$

in which $v_{**} = \sqrt{(\mathcal{P}_0 - \mathcal{P}_L)R/2L\rho}$.

(b) Obtain a relation between the constants $\lambda^<$ and $\lambda^>$ by requiring that the velocity be continuous at $r = bR$.

(c) Use the results of (b) to show that the mass flow rate through the annulus is

$$w = \pi R^2 \rho v_{**}\left\{\left[\sqrt{1-b^2}\,(1 - a^2)\right]\left[\frac{1}{\kappa}\ln\frac{R(1 - b)\sqrt{1 - b^2}v_{**}}{\nu} + \lambda^>\right] - B\right\} \tag{5C.2-4}$$

in which B is

$$B = \frac{(b^2 - a^2)^{3/2}}{\kappa\sqrt{a}}\left(\frac{a}{a + b} + \frac{1}{2}\right) + \frac{(1 - b^2)^{3/2}}{\kappa}\left(\frac{1}{1 + b} + \frac{1}{2}\right) \tag{5C.2-5}$$

5C.3 Instability in a simple mechanical system (Fig. 5C.3).

(a) A disk is rotating with a constant angular velocity Ω. Above the center of the disk a sphere of mass m is suspended by a massless rod of length L. Because of the rotation of the disk, the sphere experiences a centrifugal force and the rod makes an angle θ with the vertical. By making a force balance on the sphere, show that

$$\cos\theta = \frac{g}{\Omega^2 L} \tag{5C.3-1}$$

What happens when Ω goes to zero?

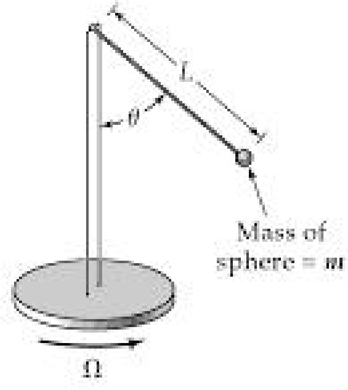

Fig. 5C.3. A simple mechanical system for illustrating concepts in stability.

[3] J. G. Knudsen and D. L. Katz, *Fluid Dynamics and Heat Transfer*, McGraw-Hill, New York (1958); R. R. Rothfus (1948), J. E. Walker (1957), and G. A. Whan (1956), Doctoral theses, Carnegie Institute of Technology (now Carnegie-Mellon University), Pittsburgh, Pa.

[4] W. Tiedt, *Berechnung des laminaren u. turbulenten Reibungswiderstandes konzentrischer u. exzentrischer Ringspalten*, Technischer Bericht Nr. 4, Inst. f. Hydraulik u. Hydraulogie, Technische Hochschule, Darmstadt (1968); D. M. Meter and R. B. Bird, *AIChE Journal*, 7, 41–45 (1961) did the same analysis using the Prandtl mixing length theory.

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(b) Show that, if Ω is below some threshold value Ω_{thr}, the angle θ is zero. Above the threshold value, show that there are two admissible values for θ. Explain by means of a carefully drawn sketch of θ vs. Ω. Above Ω_{thr} label the two curves *stable* and *unstable*.

(c) In (a) and (b) we considered only the steady-state operation of the system. Next show that the equation of motion for the sphere of mass m is

$$mL\frac{d^2\theta}{dt^2} = m\Omega^2 L \sin\theta\cos\theta - mg\sin\theta \tag{5C.3-2}$$

Show that for steady-state operation this leads to Eq. 5C.3-1. We now want to use this equation to make a small-amplitude stability analysis. Let $\theta = \theta_0 + \theta_1$, where θ_0 is a steady-state solution (independent of time) and θ_1 is a very small perturbation (dependent on time).

(d) Consider first the lower branch in (b), which is $\theta_0 = 0$. Then $\sin\theta = \sin\theta_1 \approx \theta_1$ and $\cos\theta = \cos\theta_1 \approx 1$, so that Eq. 5C.3-2 becomes

$$\frac{d^2\theta_1}{dt^2} = \left(\Omega^2 - \frac{g}{L}\right)\theta_1 \tag{5C.3-3}$$

We now try a small-amplitude oscillation of the form $\theta_1 = A\Re\{e^{-i\omega t}\}$ and find that

$$\omega_{\pm} = \pm i\sqrt{\Omega^2 - \frac{g}{L}} \tag{5C.3-4}$$

Now consider two cases: (i) If $\Omega^2 < g/L$, both ω_+ and ω_- are real, and hence θ_1 oscillates; this indicates that for $\Omega^2 < g/L$ the system is stable. (ii) If $\Omega^2 > g/L$, the root ω_+ is positive imaginary and $e^{-i\omega t}$ will increase indefinitely with time; this indicates that for $\Omega^2 > g/L$ the system is unstable with respect to infinitesimal perturbations.

(e) Next consider the upper branch in (b). Do an analysis similar to that in (d). Set up the equation for θ_1 and drop terms in the square of θ_1 (that is, linearize the equation). Once again try a solution of the form $\theta_1 = A\Re\{e^{-i\omega t}\}$. Show that for the upper branch the system is stable with respect to infinitesimal perturbations.

(f) Relate the above analysis, which is for a system with one degree of freedom, to the problem of laminar-turbulent transition for the flow of a Newtonian fluid in the flow between two counter-rotating cylinders. Read the discussion by Landau and Lifshitz[5] on this point.

5D.1 Derivation of the equation of change for the Reynolds stresses. At the end of §5.2 it was pointed out that there is an equation of change for the Reynolds stresses. This can be derived by (a) multiplying the ith component of the vector form of Eq. 5.2-5 by v_j' and time smoothing, (b) multiplying the jth component of the vector form of Eq. 5.2-5 by v_i' and time smoothing, and (c) adding the results of (a) and (b). Show that one finally gets

$$\begin{aligned}\rho\frac{D}{Dt}\overline{\mathbf{v}'\mathbf{v}'} = &-\rho\left[\overline{\mathbf{v}'\mathbf{v}'}\cdot\nabla\bar{\mathbf{v}}\right] - \rho\left[\overline{\mathbf{v}'\mathbf{v}'}\cdot\nabla\bar{\mathbf{v}}\right]^{\dagger} - \rho\left[\nabla\cdot\overline{\mathbf{v}'\mathbf{v}'\mathbf{v}'}\right] \\ &-\left[\overline{\mathbf{v}'\nabla p'}\right] - \left[\overline{\mathbf{v}'\nabla p'}\right]^{\dagger} + \mu\left\{\overline{\mathbf{v}'\nabla^2\mathbf{v}'} + \left[\overline{\mathbf{v}'\nabla^2\mathbf{v}'}\right]^{\dagger}\right\}\end{aligned} \tag{5D.1-1}$$

Equations 5.2-10 and 11 will be needed in this development.

5D.2 Kinetic energy of turbulence. By taking the trace of Eq. 5D.1-1 obtain the following:

$$\frac{D}{Dt}\left(\tfrac{1}{2}\rho\overline{v'^2}\right) = -\rho(\overline{\mathbf{v}'\mathbf{v}'}:\nabla\bar{\mathbf{v}}) - (\nabla\cdot\tfrac{1}{2}\rho\overline{v'^2\mathbf{v}'}) - (\nabla\cdot\overline{p'\mathbf{v}'}) + \mu(\overline{\mathbf{v}'\cdot\nabla^2\mathbf{v}'}) \tag{5D.2-1}$$

Interpret the equation.[6]

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[5] L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), §§26–27.

[6] H. Tennekes and J. L. Lumley, *A First Course in Turbulence*, MIT Press, Cambridge, Mass. (1972), §3.2.

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Chapter 6

Interphase Transport in Isothermal Systems

In Chapters 2–4 we showed how laminar flow problems may be formulated and solved. In Chapter 5 we presented some methods for solving turbulent flow problems by dimensional arguments or by semiempirical relations between the momentum flux and the gradient of the time-smoothed velocity. In this chapter we show how flow problems can be solved by a combination of dimensional analysis and experimental data. The technique presented here has been widely used in chemical, mechanical, aeronautical, and civil engineering, and it is useful for solving many practical problems. It is a topic worth learning well.

Many engineering flow problems fall into one of two broad categories: flow in channels and flow around submerged objects. Examples of channel flow are the pumping of oil through pipes, the flow of water in open channels, and extrusion of plastics through dies. Examples of flow around submerged objects are the motion of air around an airplane wing, motion of fluid around particles undergoing sedimentation, and flow across tube banks in heat exchangers.

In channel flow the main object is usually to get a relationship between the volume rate of flow and the pressure drop and/or elevation change. In problems involving flow around submerged objects the desired information is generally the relation between the velocity of the approaching fluid and the drag force on the object. We have seen in the preceding chapters that, if one knows the velocity and pressure distributions in the system, then the desired relationships for these two cases may be obtained. The derivation of the Hagen–Poiseuille equation in §2.3 and the derivation of the Stokes equation in §2.6 and §4.2 illustrate the two categories we are discussing here.

For many systems the velocity and pressure profiles cannot be easily calculated, particularly if the flow is turbulent or the geometry is complicated. One such system is the flow through a packed column; another is the flow in a tube in the shape of a helical coil. For such systems we can take carefully chosen experimental data and then construct "correlations" of dimensionless variables that can be used to estimate the flow behavior in geometrically similar systems. This method is based on §3.7.

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We start in §6.1 by defining the "friction factor," and then we show in §§6.2 and 6.3 how to construct friction factor charts for flow in circular tubes and flow around spheres. These are both systems we have already studied and, in fact, several results from earlier chapters are included in these charts. Finally in §6.4 we examine the flow in packed columns, to illustrate the treatment of a geometrically complicated system. The more complex problem of fluidized beds is not included in this chapter.[1]

§6.1 DEFINITION OF FRICTION FACTORS

We consider the steadily driven flow of a fluid of constant density in one of two systems: (*a*) the fluid flows in a straight conduit of uniform cross section; (*b*) the fluid flows around a submerged object that has an axis of symmetry (or two planes of symmetry) parallel to the direction of the approaching fluid. There will be a force $\mathbf{F}_{f\to s}$ exerted by the fluid on the solid surfaces. It is convenient to split this force into two parts: $\mathbf{F}_s$, the force that would be exerted by the fluid even if it were stationary; and $\mathbf{F}_k$, the additional force associated with the motion of the fluid (see §2.6 for the discussion of $\mathbf{F}_s$ and $\mathbf{F}_k$ for flow around spheres). In systems of type (*a*), $\mathbf{F}_k$ points in the same direction as the average velocity $\langle\mathbf{v}\rangle$ in the conduit, and in systems of type (*b*), $\mathbf{F}_k$ points in the same direction as the approach velocity $\mathbf{v}_\infty$.

For both types of systems we state that the magnitude of the force $\mathbf{F}_k$ is proportional to a characteristic area A and a characteristic kinetic energy K per unit volume; thus

$$F_k = AKf \tag{6.1-1}$$ [1]

in which the proportionality constant f is called the *friction factor*. Note that Eq. 6.1-1 is *not* a law of fluid dynamics, but only a definition for f. This is a useful definition, because the dimensionless quantity f can be given as a relatively simple function of the Reynolds number and the system shape.

Clearly, for any given flow system, f is not defined until A and K are specified. Let us now see what the customary definitions are:

(a) For *flow in conduits*, A is usually taken to be the wetted surface, and K is taken to be $\frac{1}{2}\rho\langle v\rangle^2$. Specifically, for circular tubes of radius R and length L we define f by

$$F_k = (2\pi RL)(\tfrac{1}{2}\rho\langle v\rangle^2)f \tag{6.1-2}$$

Generally, the quantity measured is not F_k, but rather the pressure difference $p_0 - p_L$ and the elevation difference $h_0 - h_L$. A force balance on the fluid between 0 and L in the direction of flow gives for fully developed flow

$$\begin{aligned} F_k &= [(p_0 - p_L) + \rho g(h_0 - h_L)]\pi R^2 \\ &= (\mathcal{P}_0 - \mathcal{P}_L)\pi R^2 \end{aligned} \tag{6.1-3}$$

Elimination of F_k between the last two equations then gives

$$f = \frac{1}{4}\left(\frac{D}{L}\right)\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2}\rho\langle v\rangle^2}\right) \tag{6.1-4}$$

[1] R. Jackson, *The Dynamics of Fluidized Beds*, Cambridge University Press (2000).

[1] For systems lacking symmetry, the fluid exerts both a force and a torque on the solid. For discussions of such systems see J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Martinus Nijhoff, The Hague (1983), Chapter 5; H. Brenner, in *Adv. Chem. Engr.*, **6**, 287–438 (1966); S. Kim and S. J. Karrila, *Microhydrodynamics: Principles and Selected Applications*, Butterworth-Heinemann, Boston (1991), Chapter 5.

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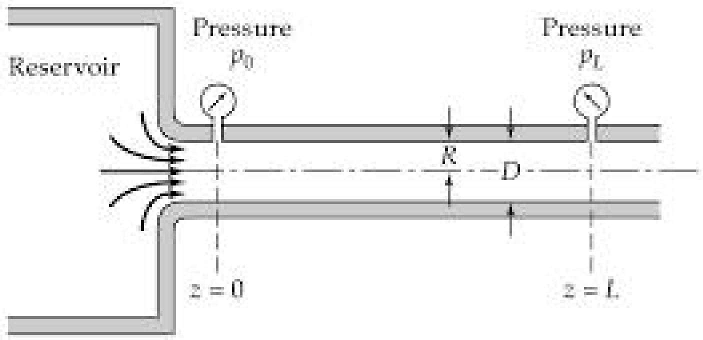

Fig. 6.2-1. Section of a circular pipe from $z = 0$ to $z = L$ for the discussion of dimensional analysis.

The system is either in steady laminar flow or steadily driven turbulent flow (i.e., turbulent flow with a steady total throughput). In either case the force in the z direction of the fluid on the inner wall of the test section is

$$F_k(t) = \int_0^L \int_0^{2\pi} \left(-\mu \frac{\partial v_z}{\partial r} \right)\bigg|_{r=R} R\, d\theta\, dz \tag{6.2-1}$$

In turbulent flow the force may be a function of time, not only because of the turbulent fluctuations, but also because of occasional ripping off of the boundary layer from the wall, which results in some disturbances with long time scales. In laminar flow it is understood that the force will be independent of time.

Equating Eqs. 6.2-1 and 6.1-2, we get the following expression for the friction factor:

$$f(t) = \frac{\displaystyle\int_0^L \int_0^{2\pi} \left(-\mu \frac{\partial v_z}{\partial r} \right)\bigg|_{r=R} R\, d\theta\, dz}{(2\pi RL)(\frac{1}{2}\rho\langle v_z\rangle^2)} \tag{6.2-2}$$

Next we introduce the dimensionless quantities from §3.7: $\breve{r} = r/D$, $\breve{z} = z/D$, $\breve{v}_z = v_z/\langle v_z\rangle$, $\breve{t} = \langle v_z\rangle t/D$, $\breve{\mathcal{P}} = (\mathcal{P} - \mathcal{P}_0)/\rho\langle v_z\rangle^2$, and $\text{Re} = D\langle v_z\rangle\rho/\mu$. Then Eq. 6.2-2 may be rewritten as

$$f(\breve{t}) = \frac{1}{\pi}\frac{D}{L}\frac{1}{\text{Re}} \int_0^{L/D} \int_0^{2\pi} \left(-\frac{\partial \breve{v}_z}{\partial \breve{r}} \right)\bigg|_{\breve{r}=1/2} d\theta\, d\breve{z} \tag{6.2-3}$$

This relation is valid for laminar or turbulent flow in smooth circular tubes. We see that for flow systems in which the drag depends on viscous forces alone (i.e., no "form drag") the product of $f\text{Re}$ is essentially a dimensionless velocity gradient averaged over the surface.

Recall now that, in principle, $\partial\breve{v}_z/\partial\breve{r}$ can be evaluated from Eqs. 3.7-8 and 9 along with the boundary conditions[1]

B.C. 1:	at $\breve{r} = \frac{1}{2}$,	$\breve{\mathbf{v}} = 0$	for $z > 0$	(6.2-4)
B.C. 2:	at $\breve{z} = 0$,	$\breve{\mathbf{v}} = \boldsymbol{\delta}_z$		(6.2-5)
B.C. 3:	at $\breve{r} = 0$ and $\breve{z} = 0$,	$\breve{\mathcal{P}} = 0$		(6.2-6)

[1] Here we follow the customary practice of neglecting the $(\partial^2/\partial\breve{z}^2)\breve{\mathbf{v}}$ terms of Eq. 3.7-9, on the basis of order-of-magnitude arguments such as those given in §4.4. With those terms suppressed, we do not need an outlet boundary condition on $\breve{\mathbf{v}}$.

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and appropriate initial conditions. The uniform inlet velocity profile in Eq. 6.2-5 is accurate except very near the wall, for a well-designed nozzle and upstream system. If Eqs. 3.7-8 and 9 could be solved with these boundary and initial conditions to get $\breve{\mathbf{v}}$ and $\breve{\mathcal{P}}$, the solutions would necessarily be of the form

$$\breve{\mathbf{v}} = \breve{\mathbf{v}}(\breve{r}, \theta, \breve{z}, \breve{t}; \text{Re}) \tag{6.2-7}$$

$$\breve{\mathcal{P}} = \breve{\mathcal{P}}(\breve{r}, \theta, \breve{z}, \breve{t}; \text{Re}) \tag{6.2-8}$$

That is, the functional dependence of $\breve{\mathbf{v}}$ and $\breve{\mathcal{P}}$ must, in general, include all the dimensionless variables and the one dimensionless group appearing in the differential equations. No additional dimensionless groups enter via the preceding boundary conditions. As a consequence, $\partial\breve{v}_z/\partial\breve{r}$ must likewise depend on $\breve{r}$, θ, $\breve{z}$, $\breve{t}$, and Re. When $\partial\breve{v}_z/\partial\breve{r}$ is evaluated at $\breve{r} = \frac{1}{2}$ and then integrated over $\breve{z}$ and θ in Eq. 6.2-3, the result depends only on $\breve{t}$, Re, and L/D (the latter appearing in the upper limit in the integration over $\breve{z}$). Therefore we are led to the conclusion that $f(\breve{t}) = f(\text{Re}, L/D, \breve{t})$, which, when time averaged, becomes

$$f = f(\text{Re}, L/D) \tag{6.2-9}$$

when the time average is performed over an interval long enough to include any long-time turbulent disturbances. The measured friction factor then depends only on the Reynolds number and the length-to-diameter ratio.

The dependence of f on L/D arises from the development of the time-average velocity distribution from its flat entry shape toward more rounded profiles at downstream z values. This development occurs within an entrance region, of length $L_e \approx 0.03D$ Re for laminar flow or $L_e \approx 60D$ for turbulent flow, beyond which the shape of the velocity distribution is "fully developed." In the transportation of fluids, the entrance length is usually a small fraction of the total; then Eq. 6.2-9 reduces to the long-tube form

$$f = f(\text{Re}) \tag{6.2-10}$$

and f can be evaluated experimentally from Eq. 6.1-4, which was written for fully developed flow at the inlet and outlet.

Equations 6.2-9 and 10 are useful results, since they provide a guide for the systematic presentation of data on flow rate versus pressure difference for laminar and turbulent flow in circular tubes. For *long* tubes we need only a single curve of f plotted versus the single combination $D\langle\bar{v}_z\rangle\rho/\mu$. Think how much simpler this is than plotting pressure drop versus the flow rate for separate values of D, L, ρ, and μ, which is what the uninitiated might do.

There is much experimental information for pressure drop versus flow rate in tubes, and hence f can be calculated from the experimental data by Eq. 6.1-4. Then f can be plotted versus Re for smooth tubes to obtain the *solid* curves shown in Fig. 6.2-2. These solid curves describe the laminar and turbulent behavior for fluids flowing in *long, smooth, circular* tubes.

Note that the *laminar* curve on the friction factor chart is merely a plot of the *Hagen–Poiseuille* equation in Eq. 2.3-21. This can be seen by substituting the expression for $(\mathcal{P}_0 - \mathcal{P}_L)$ from Eq. 2.3-21 into Eq. 6.1-4 and using the relation $w = \rho\langle\bar{v}_z\rangle\pi R^2$; this gives

$$f = \frac{16}{\text{Re}} \begin{Bmatrix} \text{Re} < 2100 & \text{stable} \\ \text{Re} > 2100 & \text{usually unstable} \end{Bmatrix} \tag{6.2-11}$$

in which $\text{Re} = D\langle\bar{v}_z\rangle\rho/\mu$; this is exactly the laminar line in Fig. 6.2-2.

Analogous *turbulent* curves have been constructed by using *experimental data*. Some analytical curve-fit expressions are also available. For example, Eq. 5.1-6 can be put into the form

$$f = \frac{0.0791}{\text{Re}^{1/4}} \qquad 2.1 \times 10^3 < \text{Re} < 10^5 \tag{6.2-12}$$

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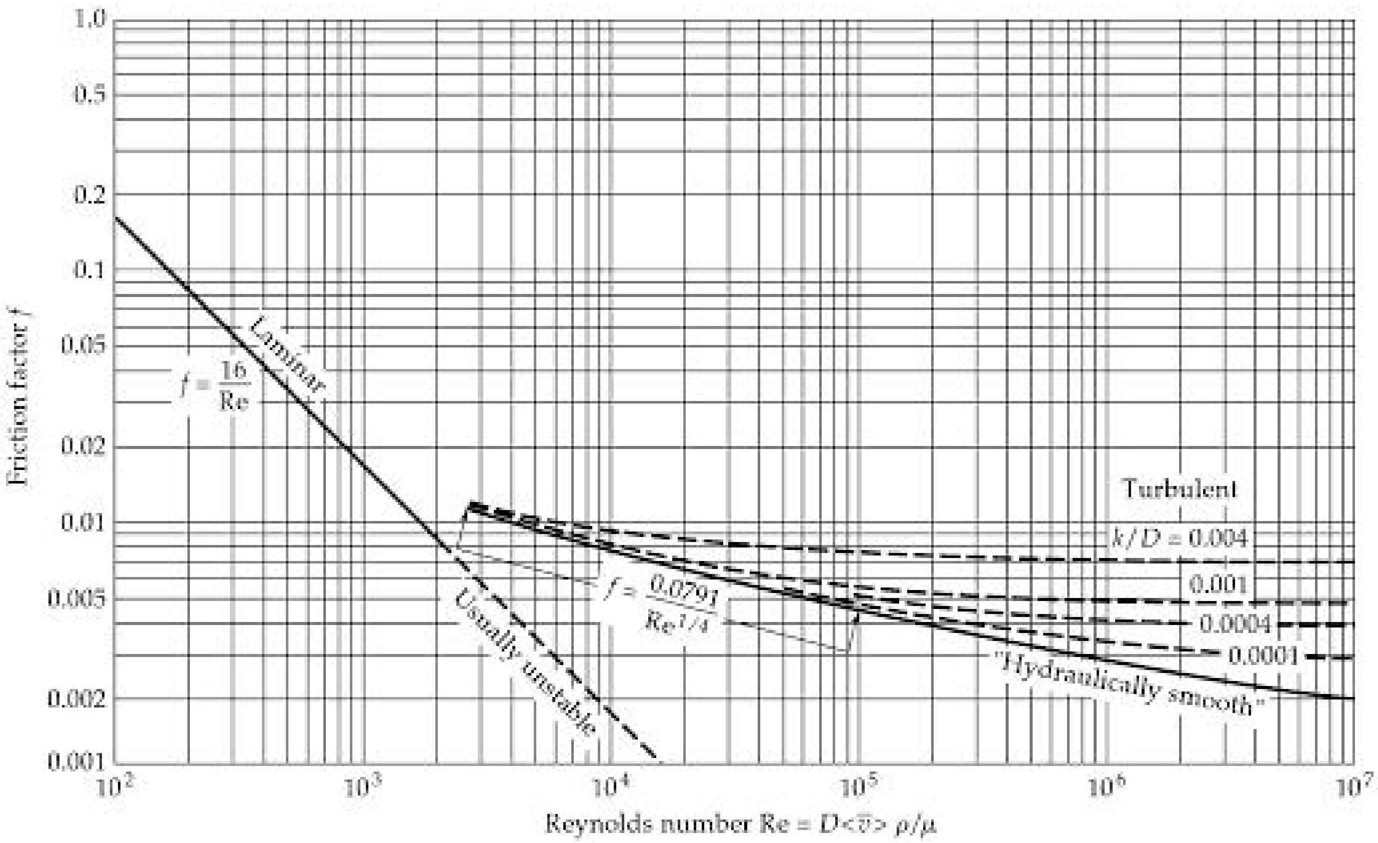

Fig. 6.2-2. Friction factor for tube flow (see definition of f in Eqs. 6.1-2 and 6.1-3). [Curves of L. F. Moody, *Trans. ASME*, **66**, 671–684 (1944) as presented in W. L. McCabe and J. C. Smith, *Unit Operations of Chemical Engineering*, McGraw-Hill, New York (1954).]

which is known as the *Blasius formula*.[2] Equation 5.5-1 (with 2.5 replaced by 2.45 and 1.75 by 2.00) is equivalent to

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10}(\text{Re}\sqrt{f}) - 0.4 \qquad 2.3 \times 10^3 < \text{Re} < 4 \times 10^6 \tag{6.2-13}$$

which is known as the *Prandtl formula*.[3] Finally, corresponding to Eq. 5.5-2, we have

$$f = \frac{2}{\Psi^{2/(\alpha+1)}} \qquad \text{where} \qquad \Psi = \frac{e^{3/2}(\sqrt{3} + 5\alpha)}{2^{\alpha}\alpha(\alpha + 1)(\alpha + 2)} \tag{6.2-14}$$

and $\alpha = 3/(2 \ln \text{Re})$. This has been found to represent the experimental data well for $3.07 \times 10^3 < \text{Re} < 3.23 \times 10^6$. Equation 6.2-14 is called the *Barenblatt formula*.[4]

A further relation, which includes the dashed curves for rough pipes in Fig. 6.2-2, is the empirical *Haaland equation*[5]

$$\frac{1}{\sqrt{f}} = -3.6 \log_{10}\left[\frac{6.9}{\text{Re}} + \left(\frac{k/D}{3.7}\right)^{10/9}\right] \qquad \begin{cases} 4 \times 10^4 < \text{Re} < 10^8 \\ 0 < k/D < 0.05 \end{cases} \tag{6.2-15}$$

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[2] H. Blasius, *Forschungsarbeiten des Ver. Deutsch. Ing.*, no. 131 (1913).

[3] L. Prandtl, *Essentials of Fluid Dynamics*, Hafner, New York (1952), p. 165.

[4] G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press (1996), §10.2.

[5] S. E. Haaland, *Trans. ASME, JFE*, **105**, 89–90 (1983). For other empiricisms see D. J. Zigrang and N. D. Sylvester, *AIChE Journal*, **28**, 514–515 (1982).

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This equation is stated[5] to be accurate within 1.5%. As can be seen in Fig. 6.2-2, the frictional resistance to flow increases with the height, k, of the protuberances. Of course, k has to enter into the correlation in a dimensionless fashion and hence appears via the ratio k/D.

For *turbulent flow* in *noncircular tubes* it is common to use the following empiricism: First we define a "mean hydraulic radius" R_h as follows:

$$R_h = S/Z \tag{6.2-16}$$

in which S is the cross-sectional area of the conduit and Z is the wetted perimeter. Then we can use Eq. 6.1-4 and Fig. 6.2-2, with the diameter D of the circular pipe replaced by $4R_h$. That is, we calculate pressure differences by replacing Eq. 6.1-4 by

$$f = \left(\frac{R_h}{L}\right)\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2}\rho\langle v_z\rangle^2}\right) \tag{6.2-17}$$

and getting f from Fig. 6.2-2 with a Reynolds number defined as

$$\text{Re}_h = \frac{4R_h\langle v_z\rangle\rho}{\mu} \tag{6.2-18}$$

For laminar flows in noncircular passages, this method is less satisfactory.

EXAMPLE 6.2-1

Pressure Drop Required for a Given Flow Rate

What pressure gradient is required to cause diethylaniline, $C_6H_5N(C_2H_5)_2$, to flow in a horizontal, smooth, circular tube of inside diameter $D = 3$ cm at a mass flow rate of 1028 g/s at 20°C? At this temperature the density of diethylaniline is $\rho = 0.935$ g/cm^3 and its viscosity is $\mu = 1.95$ cp.

SOLUTION

The Reynolds number for the flow is

$$\begin{aligned}\text{Re} &= \frac{D\langle \bar{v}_z\rangle\rho}{\mu} = \frac{Dw}{(\pi D^2/4)\mu} = \frac{4w}{\pi D\mu} \\ &= \frac{4(1028\text{ g/s})}{\pi(3\text{ cm})(1.95\times 10^{-2}\text{ g/cm}\cdot\text{s})} = 2.24\times 10^4\end{aligned} \tag{6.2-19}$$

From Fig. 6.2-2, we find that for this Reynolds number the friction factor f has a value of 0.0063 for smooth tubes. Hence the pressure gradient required to maintain the flow is (according to Eq. 6.1-4)

$$\begin{aligned}\frac{p_0 - p_L}{L} &= \left(\frac{4}{D}\right)\left(\frac{1}{2}\rho\langle\bar{v}_z\rangle^2\right)f = \frac{2}{D}\rho\left(\frac{4w}{\pi D^2\rho}\right)^2 f \\ &= \frac{32w^2 f}{\pi^2 D^5\rho} = \frac{(32)(1028)^2(0.0063))}{\pi^2(3.0)^5(0.935)} \\ &= 95(\text{dyne/cm}^2)/\text{cm} = 0.071(\text{mm Hg})/\text{cm}\end{aligned} \tag{6.2-20}$$

EXAMPLE 6.2-2

Flow Rate for a Given Pressure Drop

Determine the flow rate, in lb$_m$/hr, of water at 68°F through a 1000-ft length of horizontal 8-in. schedule 40 steel pipe (internal diameter 7.981 in.) under a pressure difference of 3.00 psi. For such a pipe use Fig. 6.2-2 and assume that $k/D = 2.3 \times 10^{-4}$.

SOLUTION

We want to use Eq. 6.1-4 and Fig. 6.2-2 to solve for $\langle v_z\rangle$ when $p_0 - p_L$ is known. However, the quantity $\langle v_z\rangle$ appears explicitly on the left side of the equation and implicitly on the right side in f, which depends on $\text{Re} = D\langle v_z\rangle\rho/\mu$. Clearly a trial-and-error solution can be found.

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When these expressions are substituted into Eqs. 6.3-5 and 6, it is then evident that the friction factor in Eq. 6.3-4 must have the form $f(\breve{t}) = f(\text{Re}, \breve{t})$, which, when time averaged over the turbulent fluctuations, simplifies to

$$f = f(\text{Re}) \tag{6.3-13}$$

by using arguments similar to those in §6.2. Hence from the definition of the friction factor and the dimensionless form of the equations of change and the boundary conditions, we find that f must be a function of Re alone.

Many experimental measurements of the drag force on spheres are available, and when these are plotted in dimensionless form, Fig. 6.3-1 results. For this system there is no sharp transition from an unstable laminar flow curve to a stable turbulent flow curve as for long tubes at a Reynolds number of about 2100 (see Fig. 6.2-2). Instead, as the approach velocity increases, f varies smoothly and moderately up to Reynolds numbers of the order of 10^5. The kink in the curve at about $\text{Re} = 2 \times 10^5$ is associated with the shift of the boundary layer separation zone from in front of the equator to in back of the equator of the sphere.[1]

We have juxtaposed the discussions of tube flow and flow around a sphere to emphasize the fact that various flow systems behave quite differently. Several points of difference between the two systems are:

Flow in Tubes	**Flow Around Spheres**
• Rather well defined laminar–turbulent transition at about Re = 2100	• No well defined laminar–turbulent transition
• The only contribution to f is the friction drag (if the tubes are smooth)	• Contributions to f from both friction and form drag
• No boundary layer separation	• There is a kink in the f vs. Re curve associated with a shift in the separation zone

The general shape of the curves in Figs. 6.2-2 and 6.3-1 should be carefully remembered.

For the *creeping flow region*, we already know that the drag force is given by *Stokes' law*, which is a consequence of solving the continuity equation and the Navier–Stokes equation of motion without the $\rho D\mathbf{v}/Dt$ term. Stokes' law can be rearranged into the form of Eq. 6.1-5 to get

$$F_k = (\pi R^2)(\tfrac{1}{2}\rho v_\infty^2)\left(\frac{24}{Dv_\infty\rho/\mu}\right) \tag{6.3-14}$$

Hence for *creeping flow* around a sphere

$$f = \frac{24}{\text{Re}} \qquad \text{for Re} < 0.1 \tag{6.3-15}$$

and this is the straight-line asymptote as $\text{Re} \rightarrow 0$ of the friction factor curve in Fig. 6.3-1.

For higher values of the Reynolds number, Eq. 4.2-21 can describe f accurately up to about Re = 1. However, the empirical expression[2]

$$f = \left(\sqrt{\frac{24}{\text{Re}}} + 0.5407\right)^2 \qquad \text{for Re} < 6000 \tag{6.3-16}$$

[1] R. K. Adair, *The Physics of Baseball*, Harper and Row, New York (1990).
[2] F. F. Abraham, *Physics of Fluids*, 13, 2194 (1970); M. Van Dyke, *Physics of Fluids*, 14, 1038–1039 (1971).

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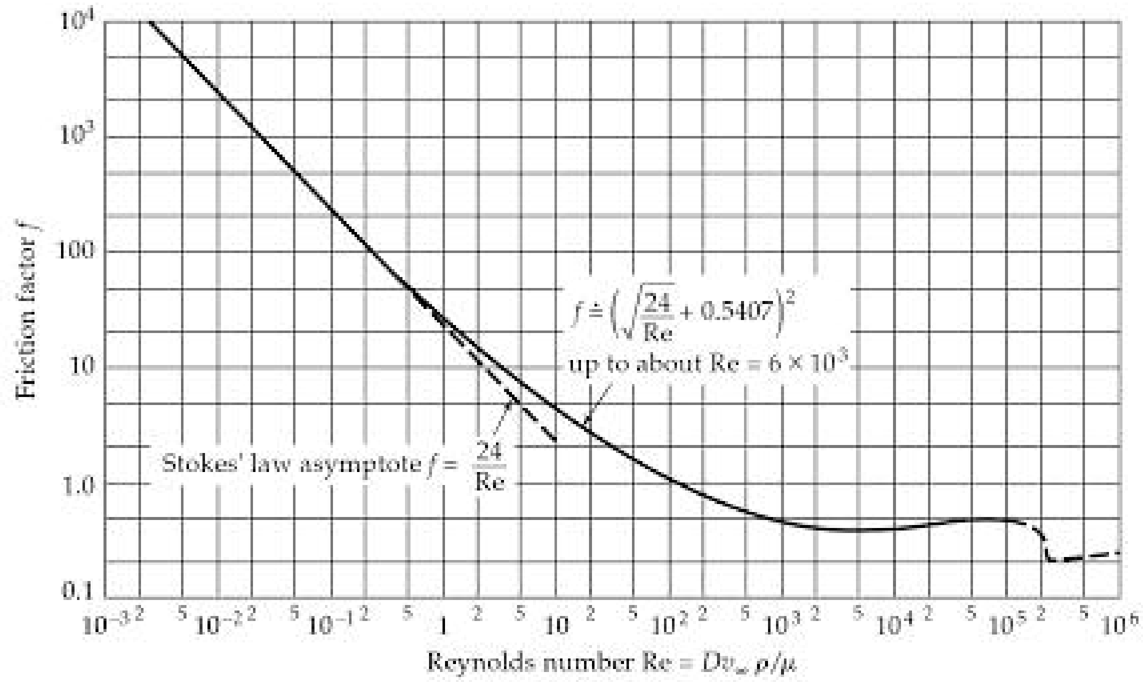

Fig. 6.3-1. Friction factor (or drag coefficient) for spheres moving relative to a fluid with a velocity v_∞. The definition of f is given in Eq. 6.1-5. [Curve taken from C. E. Lapple, "Dust and Mist Collection," in *Chemical Engineers' Handbook*, (J. H. Perry, ed.), McGraw-Hill, New York, 3rd edition (1950), p. 1018.]

is both simple and useful. It is important to remember that

$$f \approx 0.44 \qquad \text{for } 5 \times 10^2 < \mathrm{Re} < 1 \times 10^5 \tag{6.3-17}$$

which covers a remarkable range of Reynolds numbers. Eq. 6.3-17 is sometimes called *Newton's resistance law*; it is handy for a seat-of-the-pants calculation. According to this, the drag force is proportional to the square of the approach velocity of the fluid.

Many extensions of Fig. 6.3-1 have been made, but a systematic study is beyond the scope of this text. Among the effects that have been investigated are wall effects[3] (see Prob. 6C.2), fall of droplets with internal circulation,[4] hindered settling (i.e., fall of clusters of particles[5] that interfere with one another), unsteady flow,[6] and the fall of nonspherical particles.[7]

EXAMPLE 6.3-1

Determination of the Diameter of a Falling Sphere

Glass spheres of density $\rho_{sph} = 2.62\ \mathrm{g/cm^3}$ are to be allowed to fall through liquid CCl_4 at 20°C in an experiment for studying human reaction times in making time observations with stopwatches and more elaborate devices. At this temperature the relevant properties of CCl_4 are $\rho = 1.59\ \mathrm{g/cm^3}$ and $\mu = 9.58$ millipoises. What diameter should the spheres be to have a terminal velocity of about 65 cm/s?

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[3] J. R. Strom and R. C. Kintner, *AIChE Journal*, **4**, 153–156 (1958).

[4] L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), pp. 65–66; S. Hu and R. C. Kintner, *AIChE Journal*, **1**, 42–48 (1955).

[5] C. E. Lapple, *Fluid and Particle Mechanics*, University of Delaware Press, Newark, Del. (1951), Chapter 13; R. F. Probstein, *Physicochemical Hydrodynamics*, Wiley, New York, 2nd edition (1994), §5.4.

[6] R. R. Hughes and E. R. Gilliland, *Chem. Eng. Prog.*, **48**, 497–504 (1952); L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), pp. 90–91.

[7] E. S. Pettyjohn and E. B. Christiansen, *Chem. Eng. Prog.*, **44**, 147 (1948); H. A. Becker, *Can. J. Chem. Eng.*, **37**, 885–891 (1959); S. Kim and S. J. Karrila, *Microhydrodynamics: Principles and Selected Applications*, Butterworth-Heinemann, Boston (1991), Chapter 5.

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