MATH 353

PROBABILITY AND STATISTICS

UNIT 3

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UNIT 3: OUTLINE

1. Introduction to Random Variables
2. Discrete Random Variables
3. Mean and Variance of Discrete R.V.
4. Continuous Random Variables
5. Mean and Variance of Continuous R.V.
6. Cumulative Distribution Functions

UNIT OBJECTIVES:

After completing this unit, you should be able to:

- Identify **random variables** (Discrete and Continuous)
- Explain **probability distributions** (Discrete and Continuous)
- Estimate **probabilities** using probability distribution (Discrete and Continuous)
- Explain **cumulative distribution functions** (Discrete and Continuous)
- Calculate **the mean, variance and standard deviation** (Discrete and Continuous)

INTRODUCTION TO RANDOM VARIABLES

- A **random variable** is the description of the outcome of a statistical experiment.
- A **random variable** that may assume a finite number of values or an infinite sequence of values is said to be **discrete**, one that may assume any value in an interval on the real number line is said to be **continuous.**
- A **random variable** is a variable that takes quantitative values representing the results of a probability experiments and thus its values are determined by chance.
- We denote random variables using capital letters such as X, Y or Z.
- The probability of observing one tail; $P(X=1)=0.25$ and obtaining two tails;$P(X=2)=0.25$

NOTATION FOR RANDOM VARIABLES

- **EXAMPLE** is considering an experiment of tossing a single fair die once.
- We define our random variable X to be the account of a single die roll.

a) Why is the variable X a random variable?

b) What are the possible values the random variable X can take?

c) What is the notation used for rolling 4?

d) Use a random variable notation to express the probability of rolling a 5.

NOTATION FOR RANDOM VARIABLES

- **EXAMPLE:** Give the following random variables:

a) The number of cars owned by the family

b) The length of your desk in the classroom

c) The number of games played by Chelsea football Club in 2020/2021 premier league fixtures

RULES FOR DISCRETE PROBABILITY DISTRIBUTION

- The sum of the probabilities of all the possible values of a discrete random variable must equals to 1.

  i.e. $\sum P(X) = 1$

- The probability of each variable X lies between 0 and 1.

  i.e. $0 \leq P(x) \leq 1$

- Calculating the probabilities of a discrete random variables

  $\sum P(X) = 1$ Or

  $\sum P_i = 1$, for $i = 1, 2, \dots n$

Example 3.1

- The discrete random variable X has the given probability distribution:

�	1	2	3	4	5
��= �	0.2	0.25	0.4	�	0.05

a) Find $P(1 \leq X \leq 3)$

b) Find $P(X > 2)$

c) Find $P(2 < X < 5)$

Solution 3.1

- $\sum_i P(X = x) = 1$

$$=> 0.2 + 0.25 + 0.4 + a + 0.05 = 1$$

$$a + 0.9 = 1$$

$$a = 0.1$$

Now:

�	1	2	3	4	5
���= �	0.2	0.25	0.4	0.1	0.05

Solution 3.1 cont'd.

- a) $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$$= 0.2 + 0.25 + 0.4 = 0.85$$

b) $P(X > 2) \quad = P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0.4 + 0.1 + 0.05 = 0.55$$

c) $P(2 < X < 5) = P(X = 3) + P(X = 4)$

$$= 0.4 + 0.1 = 0.5$$

Example 3.2

- The probability density function of a discrete random variable X is given by $P(X = x) = kx$ for $x = 12, 13, 14$.

  Write out the probability distribution and find the value of k.

Solution 3.2

To find k,

$$\sum P(X = x) = 1$$

$$12k + 13k + 14k = 1$$

$$k = 1/39$$

x	12	13	14
$P(X = x)$	$12k$	$13k$	$14k$

MEAN AND VARIABILITY OF DISCRETE R.V.

- Mean $E(X) = \mu = \sum_{all\ x} xP(X = x)$

  OR

$$\mu = \sum x_i p_i \ , i = 0,1,2, \dots n$$

- **Steps:**
  - Multiply each possible value of x by its probability
  - Add the resulting product

Example 3.3

- The probability distribution of the random variable X is shown in the table:

◆	0	1	2	3	4
◆◆= ◆	1/6	1/12	1/4	1/3	1/6

- Find $E(X)$

Solution 3.3

- $$E(X) = \sum_{all\ x} xP(X = x)$$

$$=> 0 \times \frac{1}{6} + 1 \times \frac{1}{12} + 2 \times \frac{1}{4} + 3 \times \frac{1}{3} + 4 \times \frac{1}{6} = 2.25$$

Example 3.4

- The random variable X has the probability density function (p.d.f): P(X=x) for x=5, 6,7,8,9 as defined in the table.

[illegible]	5	6	7	8	9
[illegible]	3/11	2/11	1/11	2/11	3/11

- Find μ

Solution 3.4

- $$E(X) = \sum_{all\ x} xP(X = x)$$

$$=> \mu = 5 \times \frac{3}{11} + 6 \times \frac{2}{11} + 7 \times \frac{1}{11} + 8 \times \frac{2}{11} + 9 \times \frac{3}{11} = 7$$

VARIABILITY OF A DISCRETE RANDOM VARIABLE

- Variance or standard deviation of a random variable X measures whether a particular value of that random variable is unusual.
- As mean (μ) of random variable X is a measures center or central tendency, the variance (σ^2) and standard deviation (σ) measure the spread or the dispersion.
- The variance of a discrete random variable X is given by
- $\sigma^2 = E(X^2) - \mu^2$
- $\sigma^2 = E(X^2) - (E(X))^2$

VARIABILITY OF A DISCRETE RANDOM VARIABLE

- $\sigma^2 = E(X^2) - \left(E(X)\right)^2$
- $E\left(X^2\right) = \sum_{all\ x} x^2 P(X = x)$
- $E(X) = \sum_{all\ x} x P(X = x)$
- Standard deviation $(\sigma) = \sqrt{variance} = \sqrt{\sigma^2}$

Definition formulas	Computation formulas
[illegible]	[illegible]
[illegible]	[illegible]

Example 3.5

- The random variable X has a probability distribution as shown in the table:

[illegible]	1	2	3	4	5
[illegible]	0.1	0.3	0.2	0.3	0.1

a) Find $\mu = E(X)$

b) Find $E(X^2)$

c) Find the variance

d) Find the standard deviation

Solution 3.5

- (a) $\mu = E(X) = \sum_{all\ x} xP(X = x)$

$$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.3 + 5 \times 0.1 = 3$$

(b) $E(X^2) = \sum x^2 P(X = x)$

$$= 1^2 \times 0.1 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.3 + 5^2 \times 0.1$$

$$= 10.4$$

(c) Var(X)$= E(X^2) - \mu^2$

$$= 10.4 - 9 = 1.4$$

(d) $\sigma = \sqrt{1.4} = 1.18$

Example 3.6

- The discrete random variable X has a probability density function (pdf):

$P(X = 0) = 0.05, P(X = 1) = 0.45, P(X = 2) = 0.5$

(a) Find $\mu = E(X)$

(b) Find $E(X^2)$

(c) Var (X)

(d) $\sigma(X)$

Solution 3.6

[illegible]	0	1	2
[illegible]	0.05	0.45	0.5

(a) $\mu = \sum xP(X = x)$

$$= 0 \times 0.05 + 1 \times 0.45 + 2 \times 0.5 = 1.45$$

(b) $E(X^2) = 0^2 \times 0.05 + 1^2 \times 0.45 + 2^2 \times 0.5 = 2.45$

(c) Var $(X^2) = E(X^2) - \mu^2$

$$= 2.45 - (1.45)^2 = 0.35$$

(d) $\sigma = \sqrt{0.35} = 0.59$

Results Relating to Expectation and Variance

- Given that a and b are constant

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX + b) = aE(X) + b$
4. $Var(a) = 0$
5. $Var(aX) = a^2 Var(X)$
6. $Var(aX + b) = a^2 Var(X)$

Example 3.7

- The random variable X taking integer values only has probability density function

$P(X = x) = kx, \quad x = 1, 2, 3, 4, 5$

$P(X = x) = k(10 - x), \quad x = 6, 7, 8, 9$

(a) Find the value of the constant k

(b) Find $E(X)$

(c) Find Var(X)

(d) Find $E(2X - 3)$

(e) Find Var$(2X - 3)$

Solution 3.7

- To find k,

(a) $\sum P(X = x) = 1$

�	1	2	3	4	5	6	7	8	9
��= �	�	2�	3�	4�	5�	4�	3�	2�	�

$$k + 2k + 3k + 4k + 5k + 4k + 3k + 2k + k = 1$$

$$25k = 1$$

$$=> k = 1/25$$

�	1	2	3	4	5	6	7	8	9
��= �	1/25	2/25	3/25	4/25	5/25	4/25	3/25	2/25	1/25

Solution 3.7 cont'd.

- (b) $E(X) = 1 \times \frac{1}{25} + 2 \times \frac{2}{25} + 3 \times \frac{3}{25} + 4 \times \frac{4}{25} + 5 \times \frac{5}{25} + 6 \times \frac{4}{25} + 7 \times \frac{3}{25}$

  $+8 \times \frac{2}{25} + 9 \times \frac{1}{25}$

  $= 5$

  (c) Var (X) $= E(X^2) - (E(X)^2)$

  $E(X^2) = 1^2 \times \frac{1}{25} + 2^2 \times \frac{2}{25} + 3^2 \times \frac{3}{25} + 4^2 \times \frac{4}{25} + 5^2 \times \frac{5}{25} + 6^2 \times \frac{4}{25}$

  $+7^2 \times \frac{3}{25} + 8^2 \times \frac{2}{25} + 9^2 \times \frac{1}{25}$

  $= 29$

  $\therefore$ Var (X)$= 29 - 5^2$

  $= 4$

Solution 3.7 cont'd.

- (d) $E(2X-3) = 2E(X) - 3$

$$= 2(5) - 3 = 7$$

(e) Var $(2X-3) = 2^2 Var(X)$

$$= 4(4) = 16$$

Example 3.8

- The discrete random variable X has probability density function $P(X=x)=k$ for $x=1, 2, 3, 4, 5, 6$

  (a) Find $E(X)$

  (b) Find $E(X^2)$

  (c) Find $E(3X+4)$

  (d) Find Var(X)

  (e) Find $Var(2X+3)$

Solution 3.8

- To find k,

$$\sum_{all\ x} P(X = x) = 1$$

$$k + k + k + k + k + k = 1$$

$$k = \frac{1}{6}$$

[illegible]	1	2	3	4	5	6
[illegible]	k	k	k	k	k	k

Now:

[illegible]	1	2	3	4	5	6
[illegible]	1/6	1/6	1/6	1/6	1/6	1/6

Solution 3.8 cont'd

- (a) $E(X) = \sum_{all\ x} xP(X = x)$

$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$

(b) $E\left(X^2\right) = \sum_{all\ x} x^2\, P(X = x)$

$$= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} = \frac{91}{6}$$

(c) $E(3X + 4) = 3E(X) + 4$

$$= 3(3.5) + 4 = 14.5$$

(d) $Var\ (X) = E\left(X^2\right) - \left(E(X)\right)^2$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2$$

$$= 2.92$$

(e) $Var(2X + 3) = 4var(X)$

$$= 4(2.92) = 11.68$$

CUMULATIVE DISTRIBUTION FUNCTION FOR DISCRETE R.V.

- The cumulative frequencies are obtained by **summing all frequencies up to a particular value** say a.
- In probability distribution, the **probabilities up to value** (say a) are summed to give a cumulative probability.
- The cumulative probability function is written as $F(X)$.

$$P(X \leq a) = F(a) = \sum_{x=lower\ limit}^{a} P(X = x)$$

Example 3.9

- The probability distribution of the random variable Y is shown in the table. Find the CDF.

[illegible]	0.1	0.2	0.3	0.4	0.5
[illegible]	0.05	0.25	0.3	0.15	0.25

Solution 3.9

[illegible]	0.1	0.2	0.3	0.4	0.5
[illegible]	0.05	0.3	0.6	0.75	1

Example 3.10

- Given a discrete random variable R, the cumulative distribution function $F(r)$ is as shown in the table below.

◆	1	2	3	4
◆◆	0.13	0.54	0.75	1

Find

(a) $P(R = 2)$

(b) $P(R > 1)$

(c) $P(R \geq 3)$

(d) $P(R < 2)$

(e) $E(R)$

Solution 3.10

�	1	2	3	4
��= �	0.13	0.41	0.21	0.25

- (a) $P(R = 2) = 0.41$

(b) $P(R > 1) = P(R = 2) + P(R = 3) + P(R = 4)$

$$= 0.13 + 0.41 + 0.21 + 0.25 = 0.87$$

(c) $P(R \geq 3) = P(R = 3) + P(R = 4)$

$$= 0.21 + 0.25 = 0.46$$

(d) $P(R < 2) = P(R = 1)$

$$= 0.13$$

(e) $E(R) = \sum_{all\, r} rP(R = r)$

$$= 1 \times 0.13 + 2 \times 0.41 + 3 \times 0.21 + 4 \times 0.25$$
$$= 2.58$$

INDEPENDENT RANDOM VARIABLES

- Given two random variables X and Y and the constants a and b;

(1) $E(aX + bY) = aE(X) + bE(Y)$

(2) $E(aX - bY) = aE(X) - bE(Y)$

(3) $E(aX + b) = aE(X) + b$

(4) $E(aX - b) = aE(X) + b$

(5) $E(b) = b$

(6) $E(X_1 + X_2 + \cdots + X_n) = nE(X)$

INDEPENDENT RANDOM VARIABLES cont'd.

- If X and Y are independent, then

(7) $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$

(8) $Var(aX - bY) = a^2 Var(X) + b^2 Var(Y)$

(9) $Var(aX + b) = a^2 Var(X)$

(10) $Var(aX - b) = a^2 Var(X)$

(11) $Var(a) = 0$

(12) $Var(X_1 + X_2 + \cdots + X_n) = nVar(X)$

Example 3.11

- Independent random variables X and Y are such that

$E(X) = 4, \qquad E(Y) = 5, \qquad Var(X) = 1 \ and \qquad Var(Y) = 2$

(a) Find $E(4X + 2Y)$

(b) Find $E(5X - Y)$

(c) $Var(3X + 2Y)$

(d) $Var(5X - 3Y)$

Solution 3.11

- $E(X) = 4, E(Y) = 5, Var(X) = 1 \; and \; Var(Y) = 2$

(a) $E(4X + 2Y) = 4E(X) + 2E(Y) = 4 \times 4 + 2 \times 5 = 16 + 10 = 26$

(b) $E(5X - Y) = 5E(X) - E(Y) = 5 \times 4 - 5 = 20 - 5 = 15$

(c) $Var(3X + 2Y) = 3^2 Var(X) + 2^2 Var(Y) = 9 \times 1 + 4 \times 2 = 17$

(d) $Var(5X - 3Y) = 5^2 Var(X) + (-3)^2 Var(Y) = 25 \times 1 + 9 \times 2 = 43$

Example 3.12

- Independent random variables X and Y are such that

$E(X^2) = 14, E(Y^2) = 20, Var(X) = 10 \text{ and } Var(Y) = 11$

(a) Find $E(3X - 2Y)$

(b) Find $Var(5X - 2Y)$

Solution 3.12

- **<u>First solve for $E(X)$ and $E(Y)$.</u>**

$Var(X) = E(X^2) - E^2(X)$ and $Var(Y) = E(Y^2) - E^2(Y)$

$10 = 14 - E^2(X)$ $11 = 20 - E^2(Y)$

$E^2(X) = 14 - 10 = 4$ $E^2(Y) = 20 - 11 = 9$

$=> E(X) = \pm 2$ $=> E(Y) = \pm 3$

- Here both answers are **valid** and as such, a solution must be chosen based on the question as to ***whether $E(X) > 0$ and $E(Y) > 0$ or otherwise***.
- We choose $E(X) > 0$ and $E(Y) > 0$. **(NB: This is by choice!)**

(a) $E(3X - 2Y) = 3E(X) - 2E(Y) = 3 \times 2 - 2 \times 3 = 0$

(b) $Var(5X - 2Y) = 25Var(X) + 4Var(Y) = 25 \times 2 + 4 \times 3 = 62$

CONTINUOUS RANDOM VARIABLES

CONTINUOUS RANDOM VARIABLES

- **Examples of continuous random variables:**
- The mass, grams, of bag of sugar packaged by a particular machine.
- The time taken, in minutes, to perform a task.
- The height, in centimetres, of a ten-year old boy.
- The lifetime, in hours, of a 100-watt light bulb.

PROBABILITY DENSITY FUNCTION OF CONTINUOUS R.V.

- Continuous random variable X, with p.d.f, $f(x)$ is valid over the range, $a \leq x \leq b$ if
- $\int_a^b f(x)\,dx = 1$
- $for\ a \leq x_1 < x_2 \leq b$
- $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx$

Example 3.13

- A continuous random variable had p.d.f $f(x) = kx^2 \;\; for\; 0 \leq x \leq 4$.

a) Find the value of the constant k.

b) Find $P(1 \leq X \leq 3)$.

Solution 3.13

- a) $\int_0^4 f(x)\,dx = 1$

$$\int_0^4 kx^2\,dx = 1$$

$$\left[\frac{kx^3}{3}\right]_0^4 = 1$$

$$\frac{64}{3}k = 1$$

$$k = \frac{3}{64}$$

$$\therefore f(x) = \frac{3x^2}{64}, for\ 0 \leq x \leq 4.$$

Solution 3.13 cont'd.

- b) $P(1 \leq X \leq 3) = \int_0^3 \frac{3}{64} x^2\, dx$

$$= \frac{3}{64}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{3}{64}\left[\frac{3^3}{3} - \frac{1^3}{3}\right]$$

$$= \frac{3}{64}\left[9 - \frac{1}{3}\right]$$

$$= 0.41$$

Example 3.14

- The continuous random variable X has pdf $f(x)$ where

$$f(x) = \begin{cases} k(x+2)^2 & -2 \le x < 0 \\ 4k & 0 \le x \le 2 \\ 0 & otherwise \end{cases}$$

a) Find the value of the constant k.

b) Find $P(-1 \le X \le 1)$.

c) Find $P(X > 1)$.

Solution 3.14

- a) To find k, $\int_{all\ x} f(x)\, dx = 1$
  - $\int_{-2}^{0} k(x+2)^2\, dx + \int_{0}^{2} 4k\, dx = 1$
  - $\frac{k}{3}[(x+2)^3]_{-2}^{0} + k[4x]_0^2 = 1$
  - $\frac{k}{3}[(0+2)^3 - (-2+2)^3] + 4k[2-0] = 1$
  - $\frac{k}{3}[8] + [8k] = 1$
  - $k\left[\frac{8}{3} + 8\right] = 1$
  - $\frac{32}{3}k = 1$
  - $k = \frac{3}{32}$

Solution 3.14 cont'd.

- b) $P(-1 \leq X \leq 1) = \int_{-1}^{0} \frac{3}{32}(x+2)^2\,dx + \int_{0}^{1} \frac{3}{32} \times 4\,dx$
- $= \int_{-1}^{0} \frac{3}{32}(x+2)^2\,dx + \int_{0}^{1} \frac{3}{8}\,dx$
- $= \left[\frac{3}{32} \times \frac{(x+2)^3}{3}\right]_{-1}^{0} + \left[\frac{3}{8}x\right]_{0}^{1}$
- $= \frac{1}{32}\left[(0+2)^3 - (-1+2)^3\right] + \frac{3}{8}[1-0]$
- $= \frac{1}{32}[8-1] + \frac{3}{8}$
- $= \frac{7}{32} + \frac{3}{8}$
- $= 0.59$

Solution 3.14 cont'd.

- c) $P(X > 1) = \int_1^2 \frac{3}{32} \times 4\,dx$
- $= \frac{3}{8}[x]_1^2$
- $= \frac{3}{8}[2 - 1]$
- $= \frac{3}{8}$
- $= 0.38$

EXPECTATION OF X, $E(X)$

- For a continuous random variable with pdf $f(x)$.

$$E(X) = \int_{all\ x} f(x)\, dx$$

- **Example 3.15**

X is a continuous random variable with pdf $f(x) = kx^2\ for$

$0 \leq x \leq 5.$

Find $E(X)$.

Solution 3.15

- To find k,
- $\int_0^5 kx^2\,dx = 1$
- $\left[\frac{kx^3}{3}\right]_0^5 = 1$
- $\left[\frac{125}{3}k - 0\right] = 1$
- $k = \frac{3}{125}$
- $\therefore f(x) = \frac{3}{125}x^2$
- $E(X) = \int_{all\,x} xf(x)\,dx$

- $=> xf(x) = x \cdot \frac{3}{125}x^2 = \frac{3}{125}x^3$
- $E(X) = \int_0^5 \frac{3}{125}x^3\,dx$

$$= \frac{3}{125}\left[\frac{x^4}{4}\right]_0^5$$
$$= \frac{3}{125}\left[\frac{5^4}{4} - 0\right]$$
$$= \frac{3}{125}\left[\frac{625}{4}\right]$$
$$= \frac{15}{4}$$
$$= 3.75$$

Example 3.16

- A random variable X has a pdf $f(x)$ given by
  $f(x) = cx(5 - x), 0 \le x \le 5.$
  Find the mean of X.

Solution 3.16

- 1. $E(X) = \int_{all\,x} xf(x)\,dx$ 2. $\int_{all\,x} f(x)\,dx = 1$
  - $\int_{all\,x} f(x)\,dx = 1$
  - $\int_0^5 cx(5-x)\,dx = 1$
  - $\int_0^5 c(5x - x^2)\,dx = 1$
  - $c\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = 1$
  - $c\left[\frac{125}{2} - \frac{125}{3}\right] = 1$
  - $c\left[\frac{125}{6}\right] = 1$
  - $c = \frac{6}{125}$

Solution 3.16 cont'd.

- $mean = E(X) = \int_{all\,x} xf(x)\,dx$
- $=> xf(x) = x \cdot \frac{6}{125} x(5-x) = \frac{6}{125} x^2(5-x)$
- $E(X) = \frac{6}{126} \int_0^5 x^2(5-x)\,dx$

$$= \frac{6}{126} \int_0^5 5x^2 - x^3\,dx$$

$$= \frac{6}{125} \left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5$$

$$= \frac{6}{125} \left[\frac{625}{3} - \frac{625}{4}\right] - 0$$

$$= \frac{6}{125} \cdot \frac{625}{12}$$

$$= 6 \cdot \frac{5}{12}$$

$$= 2.5$$

VARIANCE OF X, $Var(X)$

- If X is a continuous random variable with pdf $f(x)$ then
- $Var(X) = \int_{all x} x^2 f(x)\,dx - \mu^2$

where

- $\mu = E(X) = \int_{all\ x} xf(x)\,dx$
- The standard deviation of X is often written as σ.
- $\sigma = \sqrt{Var(X)}$

Example 3.17

-
  - The continuous random variable X has pdf $f(x) = \frac{1}{8}x, 0 \leq x \leq 4.$

a) Find $E(X)$.

b) Find $E(X^2)$.

c) Find $Var(X)$.

d) Find σ_X, the standard deviation of X.

e) Find $Var(3X + 2)$.

Solution 3.17

- a) $E(X) = \int_{allx} xf(x)\,dx$
$$= \int_0^4 \frac{1}{8}x^2\,dx$$
$$= \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4$$
$$= \frac{1}{8}\left[\frac{4^3}{3} - 0\right]$$
$$= 2.7$$

- b) $E(X^2) = \int_{allx} x^2 f(x)\,dx$
$$= \int_0^4 \frac{1}{8}x^3\,dx$$
$$= \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4$$
$$= \frac{1}{8}\left[\frac{4^4}{4} - 0\right]$$
$$= 8$$

Solution 3.17 cont'd.

- c) $Var(X) = E(X^2) - [E(X)]^2 = 8 - 2.7^2 = 0.89$

d) $\sigma_X = \sqrt{Var(X)} = \sqrt{0.89} = 0.94$

e) $Var(3X + 2) = Var(3X) + Var(2) = 3^2 Var(X) + 0 = 9 \times Var(X)$

$$= 9 \times 0.89$$

$$= 8.0$$

Example 3.18

- An experiment of temporary roundabout is installed at the crossroads, the time, X minutes, which vehicles have to wait before entering the roundabout has probability density function

$$f(x) = \begin{cases} 0.8 - 0.32x & 0 \leq x \leq 2.5 \\ 0 & otherwise \end{cases}$$

Find the mean and the standard deviation of X.

Solution 3.18

- $E(X) = \int_{all x} xf(x)\,dx$

$$= \int_0^{2.5} (0.8x - 0.32x^2)\,dx$$

$$= \left[\frac{0.8x^2}{2} - \frac{0.32x^3}{3}\right]_0^{2.5}$$

$$= \left[\frac{(0.8)2.5^2}{2} - \frac{(0.32)2.5^3}{3}\right] - 0$$

$$= 0.833\ minutes$$

$$= 50\ seconds$$

$$\therefore The\ mean\ time\ is\ 50\ seconds$$

Solution 3.18 cont'd.

- $E(X^2) = \int_{all x} x^2 f(x)\, dx$

$$= \int_0^{2.5} 0.8x^2 - 0.32x^3\, dx$$
$$= \left[\frac{0.8x^3}{3} - \frac{0.32x^4}{4}\right]_0^{2.5}$$
$$= \left[\frac{(0.8)2.5^3}{4} - \frac{(0.32)2.5^4}{4}\right] - 0$$
$$= 1.041$$

- $Var(X) = E(X^2) - E^2(X)$

$$= 1.041 - (0.833)^2$$
$$= 0.347$$

- $SD(X) = \sqrt{Var(X)} = \sqrt{0.347} = 0.589$
- The standard deviation is 35 seconds.

THE CUMULATIVE DISTRIBUTION FUNCTION, $F(x)$

- If $f(x)$ is valid in the range $a \leq x \leq b$

Then

$$F(t) = \int_a^t f(x)\,dx$$

where $a = lower\ limit$ and $t = arbitrary\ value > a$

NOTE:

- $F(b) = P(X \leq b) = \int_a^b f(x)\,dx = 1$
- Using $F(x)$ to find $P(x_1 \leq X \leq x_2)$.

$$P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1)$$

THE CUMULATIVE DISTRIBUTION FUNCTION, $F(x)$

- Median $= \int_a^m f(x)\,dx = 0.5$

  i.e. $F(m) = 0.5$
- Lower quartile $= \int_a^{q_1} f(x)\,dx = 0.25$

  i.e. $F(q_1) = 0.25$
- Upper quartile $= \int_a^{q_3} f(x)\,dx = 0.75$

  i.e. $F(q_3) = 0.75$
- Percentiles $F(nth\ percentile) = \frac{n}{100}$

Example 3.19

- X is a continuous random variable with pdf as

$$f(x) = \frac{1}{8}x, for\ 0 \leq x \leq 4.$$

a) Find the cumulative distribution function, $F(x)$.

b) Find $P(0.3 \leq X \leq 1.8)$.

c) Find the median, m.

d) Find the interquartile range, $(q_3 - q_1)$.

Solution 3.19

- (a) For values of t between 0 and 4.
  - $F(t) = \int_0^t \frac{1}{8} x \, dx$

    $= \left[\frac{x^2}{16}\right]_0^t$

    $= \frac{t^2}{16}$
  - NB: $F(4) = \frac{4^2}{16} = 1$, as expected.

Solution 3.19 cont'd.

- • The cumulative distribution function can now be written in terms of x as follows:

- $$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{16} & 0 \leq x \leq 4 \\ 1 & x \geq 4 \end{cases}$$

(b) $P(0.3 \leq X \leq 1.8) = F(1.8) - F(0.3) F(1.8) = \frac{1.8^2}{16} = 0.2025$

$$F(0.3) = \frac{0.3^2}{16} = 0.005625$$

$$=> P(0.3 \leq X \leq 1.8) = 0.2025 - 0.005625$$

$$= 0.197$$

Solution 3.19 cont'd.

- (c) For the median:

$$F(m) = 0.5$$

$$\frac{m^2}{16} = 0.5$$

$$m^2 = 0.5 \times 16 = 8$$

$m = \sqrt{8} = 2\sqrt{2} = 2.828$ ∴ The median is 2.83

- NB: Take the positive square root, since $0 \leq m \leq 4$.

Solution 3.19 cont'd.

- (d) For the lower quartile $F(q_1) = 0.25$

$$\frac{q_1^2}{16} = 0.25$$
$$q_1^2 = 0.25 \times 16 = 4$$
$$\therefore q_1 = 2$$

For the upper quartile

$$F(q_3) = 0.25$$

$$\frac{q_3^2}{16} = 0.25$$
$$q_3^2 = 0.75 \times 16 = 12$$
$$\therefore q_3 = 3.2164$$

- $Interquartile\ Range = q_3 - q_1 = 3.2164 - 2 = 1.5$
- NB: Semi-Interquartile Range = $\frac{q_3 - q_1}{2}$

OBTAINING THE PDF, $f(x)$, FROM THE CDF, $F(x)$.

- *Use the relationship below to obtain the pdf.*

$$f(x) = \frac{d}{dx}\left(F(x)\right)$$

$$= F'(x)$$

Example 3.20

- The continuous random variable X has cdf, $F(x)$ where

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^4}{64} & 0 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$$

- Find $f(x)$.

Solution 3.20

- $$f(x) = \frac{d}{dx}F(x)$$
$$= \frac{d}{dx}\left(\frac{x^4}{64}\right)$$
$$= \frac{4x^3}{64}$$
$$= \frac{x^3}{16}$$
$$\therefore f(x) = \frac{x^3}{16}, \qquad 0 \leq x \leq 4$$

End of Slides

Thank You