SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Learning Objectives

After reading this unit you should be able to:

1. State when LU Decomposition is numerically more efficient than Gaussian Elimination,
2. Decompose a non-singular matrix into LU,
3. Show how LU Decomposition is used to find matrix inverse

Unit content

SESSION 1-2: Matrices

1-2.1 Matrix Inversion and Cramer's Rule

1-2.2 Triangular System of Equations

1-2.3 Back Substitution

1-2.4 Division by zero:

1-2.5 Finding the determinant of a square matrix using Naïve Gauss Elimination methods

SESSION 2-2: Techniques for improving Naïve Gauss Elimination Method

2-2.1 Gaussian Elimination with partial pivoting

2-2.2 LU Decomposition

2-2.3 Decomposing a non-singular matrix A into the form A=LU.

2-2.4 Finding the inverse of a square matrix using LU Decomposition

SESSION 3-1: Iterative Method

3-1.1 Jacobi Method

3-1.2 Gauss-Seidel Method

3-1.3 Successive over–relaxation method

3-1.4 A necessary and sufficient condition for the convergence of iterative schemes

SESSION 1-2: MATRICES

1-2.1 Matrix Inversion and Cramer's Rule

A system of linear Algebraic equations is nothing but a system of n algebraic linear equations satisfied by a set of n unknown quantities. The aim is to find these *n* unknown quantities satisfying the *n* equations.

It is a very common practice to write the system of *n* equations in matrix form as $Ax = b$, where *A* is an $n \times n$, non-singular matrix and *x* and *b* are $n \times 1$ matrices out of which *b* is known. i.e.,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

For small *n* the elementary methods like matrix inversion or Cramers rule given below are very convenient to get the unknown vector *x* from the system $Ax = b$.

Let A be a nonsingular matrix of order *n* and *b* be an *n*-vector. The solution x of the system $Ax = b$ is given by:

a) Matrix Inversion as $x = A^{-1}.b$
b) Cramer's rule as $x_i = \dfrac{\det(A_i)}{\det(A)}, \qquad i = 1, \dots, n$

where A_i is a matrix obtained by replacing the i^{th} column of A by the vector b and $x = (x_1, x_2, \dots, x_n)^T$, i.e., Cramer's Rule or by matrix inversion formula given as $x = A^{-1}b$

However, for large ' n ' these methods will become computationally very expensive because of the evaluation of matrix determinants involved in these methods. Hence to make the solution methods computationally less expensive one has to find alternate means which doesn't require the evaluation of any determinants or inverses to find *x* form $Ax = b$.

1-2.2 Triangular System of Equations

The system of equations below is an upper triangular system of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \quad \cdots \quad + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \quad \cdots \quad + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \quad \cdots \quad + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n
\end{aligned}
$$

Now the equations are solved starting from the last equation as it has only one unknown.

$$x_n^* = \frac{b_n}{a_{nn}}$$

Then the second last equation, that is the $(n\text{-}1)^{th}$ equation, has two unknowns (x_n and x_{n-1}), but x_n^* is already known. This reduces the $(n\text{-}1)^{th}$ equation also to one unknown and we have:

$$x_{n-1} = \frac{1}{a_{n-1,n-1}}\left(b_{n-1} - a_{n-1,n}x_n^*\right)$$

Back substitution hence can be represented for all equations by the formula

$$x_n^* = \frac{b_n}{a_{nn}} \qquad \text{and} \quad x_i = \frac{1}{a_{i,i}}\left(b_i - \sum_{j=i+1}^{n} a_{i,j}x_j^*\right) \quad \text{for } i = n-1, n-2, \ldots, 1$$

Note $\sum_{j=n+1}^{n} = 0$.

1-2.3 Back Substitution

Now the equations are solved starting from the last equation as it has only one unknown.

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Then the second last equation, that is the $(n\text{-}1)^{th}$ equation, has two unknowns - x_n and x_{n-1}, but x_n is already known. This reduces the $(n\text{-}1)^{th}$ equation also to one unknown. Back substitution hence can be represented for all equations by the formula $x_i = \dfrac{1}{a_{ii}^{(i-1)}}\left(b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j \right)$ for $i = n, n-1, n-2, \ldots, 1$

Note: $\sum\limits_{j=n+1}^{n} \quad = 0$.

The procedure just described above is called the Naïve Gaussian Elimination method or the Gaussian Elimination without pivoting.

Example 1

Use Naïve Gaussian Elimination to solve $Ax = b$, where

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution

Forward Elimination of Unknowns: Since there are three equations, there will be two steps of forward elimination of unknowns.

First step: Divide *Row 1* by 25 and then multiply it by 64, i.e.,

$\left[\dfrac{Row\ 1}{25}\right] \times (64) = Row\ 1 \times 2.56\,6$ gives *Row 1* as $[64 \quad 12.8 \quad 2.56] \qquad [273.408]$

Subtract the result from Row 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.81 \\ -96.21 \\ 279.2 \end{bmatrix}$$

Divide Row 1 by 25 and then multiply it by 144

$\left[\dfrac{Row\ 1}{25}\right] \times (144) = Row\ 1 \times 5.76$ gives *Row 1* as $[144 \quad 28.8 \quad 5.76] \qquad [615.2256]$

Subtract the result from Row 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ -336.0 \end{bmatrix}$$

<u>Second step</u>: We now divide Row 2 by –4.8 and then multiply by –16.8

$$\left[\frac{Row\ 2}{-4.8}\right]\times(-16.8) = Row\ 2\times 3.5 \text{ gives } \textit{Row 2} \text{ as } [0 \quad -16.8 \quad -5.46] \qquad [-336.735]$$

Subtract the result from Row 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

<u>Back substitution</u>: From the third equation

$$0.7x_3 = 0.735 \quad \Rightarrow \quad x_3 = \frac{0.735}{0.7} = 1.050$$

Substituting the value of x_3 in the second equation,

$$-4.8x_2 - 1.56x_3 = -96.21 \quad \Rightarrow \quad x_2 = \frac{-96.21 + 1.56x_3}{-4.8} = \frac{-96.21 + 1.56(1.050)}{-4.8} = 19.70$$

Substituting the value of x_2 and x_3 in the first equation,

$$25x_1 + 5x_2 + x_3 = 106.8 \quad \Rightarrow$$

$$x_1 = \frac{106.8 - 5x_2 - x_3}{25} = \frac{106.8 - 5(19.70) - 1.050}{25} = 0.2900$$

Hence the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Example 2
Use Naïve Gauss Elimination to solve

$$\begin{aligned} 10x_1 - 7x_2 \quad &= 7 \\ -3x_1 + 2.099x_2 + 6x_3 &= 3.901 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

Use <u>six</u> significant digits with chopping in your calculations.

Solution

Working in the matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

<u>Forward Elimination of Unknowns</u>

Dividing Row 1 by 10 and multiplying by –3, that is, multiplying Row 1 by -0.3, and subtract it from Row 2 would eliminate a_{21},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Again dividing Row 1 by 10 and multiplying by 5, that is, multiplying Row 1 by 0.5, and subtract it from Row 3 would eliminate a_{31},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, we would use Row 2 as the pivot equation and eliminate Row 3 – Column 2. Dividing Row 2 by –0.001 and multiplying by 2.5, that is multiplying Row 2 by –2500, and subtracting from Row 3 gives

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15005 \end{bmatrix}$$

This is the end of the forward elimination steps.

Back substitution

We can now solve the above equations by back substitution. From the third equation,

$$15005x_3 = 15005 \quad \Rightarrow \quad x_3 = \frac{15005}{15005} = 1$$

Substituting the value of x_3 in the second equation

$$-0.001x_2 + 6x_3 = 6.001 \quad \Rightarrow$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = \frac{6.001\text{-}6(1)}{-0.001} = \frac{0.001}{-0.001} = -1$$

Substituting the value of x_3 and x_2 in the first equation,

$$10x_1 - 7x_2 + 0x_3 = 7 \quad \Rightarrow \quad x_1 = \frac{7 + 7\,x_2 - 0x_3}{10} = \frac{7 + 7(-1) - 0(1)}{10} = 0$$

Hence the solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Note:

There are two pitfalls of Naïve Gauss Elimination method:

1. Division by zero,
2. Round-off error.

1-2.4 Division by zero

It is possible that division by zero may occur during forward elimination steps. For example for the set of equations

$$\begin{aligned} 10x_2 - 7x_3 &= 7 \\ 6x_1 + 2.099x_2 - 3x_3 &= 3.901 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

during the first forward elimination step, the coefficient of x_1 is zero and hence normalization would require division by zero.

Round-off error:

Naïve Gauss Elimination Method is prone to round-off errors. This is true when there are large numbers of equations as errors propagate. Also, if there is subtraction of numbers from each other, it may create large errors. See the example below.

Example 3

Remember the previous example where we used Naïve Gauss Elimination to solve

$$\begin{aligned} 10x_1 - 7x_2 &= 7 \\ -3x_1 + 2.099x_2 + 6x_3 &= 3.901 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

using <u>six</u> significant digits with chopping in your calculations. Repeat the problem, but now use <u>five</u> significant digits with chopping in your calculations.

Solution

Writing in the matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

<u>Forward Elimination of Unknowns</u>

Dividing Row 1 by 10 and multiplying by –3, that is, multiplying Row 1 by -0.3, and subtract it from Row 2 would eliminate a_{21},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Again dividing Row 1 by 10 and multiplying by 5, that is, multiplying the Row 1 by 0.5, and subtract it from Row 3 would eliminate a_{31},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, we would use Row 2 as the pivoting equation and eliminate Row 3 – Column 2. Dividing Row 2 by –0.001 and multiplying by 2.5, that is, multiplying Row 2 by –2500, and subtract from Row 3 gives

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

This is the end of the forward elimination steps.

<u>Back substitution</u>

We can now solve the above equations by back substitution. From the third equation,

$$15005x_3 = 15004 \quad \Rightarrow \quad x_3 = \frac{15004}{15005} = 0.99993$$

Substituting the value of x_3 in the second equation

$$-0.001x_2 + 6x_3 = 6.001 \quad \Rightarrow$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = \frac{6.001\text{-}6(0.99993)}{\text{-}0.001} = \frac{6.001\text{-}5.9995}{\text{-}0.001} = \frac{0.0015}{-0.001} = -1.5$$

Substituting the value of x_3 and x_2 in the first equation, $10x_1 - 7x_2 + 0x_3 = 7 \quad \Rightarrow$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = \frac{7 + 7(-1.5) - 0(1)}{10} = \frac{7 + 7(-1.5) - 0(1)}{10}$$
$$= \frac{7 - 10.5 - 0}{10} = \frac{-3.5}{10} = -0.3500$$

Hence the solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Compare this with the exact solution of

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

1-2.5 Finding the determinant of a square matrix using Naïve Gaussian Elimination methods

One of the more efficient ways to find the determinant of a square matrix is by taking advantage of the following two theorems on a determinant of matrices coupled with Naïve Gauss Elimination.

Theorem 1:

Let *A* be a *n*x*n* matrix. Then, if *B* is a matrix that results from adding or subtracting a multiple of one row to another row, then det(*B*) = det(*A*). (The same is true for column operations also).

Theorem 2:

Let *A* be a *n*x*n* matrix that is upper triangular, lower triangular or diagonal, then $\det(A) = a_{11} * a_{22} * \cdots * a_{nn} = \prod_{i=1}^{n} a_{ii}$

This implies that if we apply the forward elimination steps of Naive Gauss Elimination method, the determinant of the matrix stays the same according the Theorem 1. Then since at the end of the forward elimination steps, the resulting matrix is upper triangular, the determinant will be given by Theorem 2.

Example 5

Find the determinant of

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Remember earlier in this chapter, we conducted the steps of forward elimination of unknowns using Naïve Gauss Elimination method on A to give

$$B = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

According to Theorem 2

$$\det(A) = \det(B) = (25)(-4.8)(0.7) = -84.00$$

Note:

If you cannot find the determinant of the matrix using Naive Gauss Elimination method due to a division by zero problems during Naïve Gauss Elimination method, you can apply Gaussian Elimination with Partial Pivoting. However, the determinant of the resulting upper triangular matrix may differ by a sign. The following theorem applies in addition to the previous two to find determinant of a square matrix.

Theorem 3:

Let A be a *nxn* matrix. Then, if B is a matrix that results from switching one row with another row, then $\det(A) = -\det(B)$.

Example 6

Find the determinant of

$$A = \begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}$$

Solution

Remember from that at the end of the forward elimination steps of Gaussian elimination with partial pivoting, we obtained

$$B = \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix}$$

$\det(B) = (10)(2.5)(6.002) = 150.05$

Since rows were switched once during the forward elimination steps of Gaussian elimination with partial pivoting, $\det(A) = -\det(B) = -150.05$

Prove that $\det(A) = \dfrac{1}{\det(A^{-1})}$

Proof:

$$AA^{-1} = I \quad \det(A\,A^{-1}) = \det(I) \quad \Rightarrow \quad \det(A)\det(A^{-1}) = 1 \quad \Rightarrow \quad \det(A) = \frac{1}{\det(A^{-1})}.$$

If A is a nxn matrix and det (A) $\neq 0$, what other statements are equivalent to it?

1. A is invertible.
2. A^{-1} exists.
3. $Ax = b$ has a unique solution.
4. $Ax = 0$ solution is $x = 0$
5. $AA^{-1} = I = A^{-1}A$.

SESSION 2-2: Techniques for improving Naïve Gauss Elimination Method

As seen in the example, round off errors were large when five significant digits were used as opposed to six significant digits. So, one way of decreasing round off error would be to use more significant digits, that is, use double or quad precision. However, this would not avoid division by zero errors in Naïve Gauss Elimination. To avoid division by zero as well as reduce (not eliminate) round off error, Gaussian Elimination with partial pivoting is the method of choice.

2-2.1 Gaussian Elimination with partial pivoting

The Gaussian elimination with partial pivoting and the Naïve Gauss elimination methods are the same, except in the beginning of each step of forward elimination; a row switching is done based on the following criterion. If there are n equations, then there are $(n-1)$ forward elimination steps. At the beginning of the k^{th} step of forward elimination, one finds the maximum of

$|a_{kk}|, |a_{k+1,k}|, \ldots, |a_{nk}|$

Then if the maximum of these values is $|a_{pk}|$ in the p^{th} row, $k \le p \le n$, then switch rows p and k. The other steps of forward elimination are the same as Naïve Gauss elimination method. The back substitution steps stay exactly the same as Naïve Gauss Elimination method.

Example 4

In the previous two examples, we used Naïve Gauss Elimination to solve

$$\begin{aligned}
10x_1 - 7x_2 \quad &= 7 \\
-3x_1 + 2.099x_2 + 6x_3 &= 3.901 \\
5x_1 - x_2 + 5x_3 &= 6
\end{aligned}$$

using five and six significant digits with chopping in the calculations. Using <u>five</u> significant digits with chopping, the solution found was

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

This is different from the exact solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Find the solution using Gaussian elimination with partial pivoting using five significant digits with chopping in your calculations.

Solution

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.901 \\ 6 \end{bmatrix}$$

<u>Forward Elimination of Unknowns</u>

Now for the first step of forward elimination, the absolute values of first column elements are $|10|, |-3|, |5|$ or 10, 3, 5.

So the largest absolute value is in the Row 1. So as per Gaussian Elimination with partial pivoting, the switch is between Row 1 and Row 1 to give

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Dividing Row 1 by 10 and multiplying by –3, that is, multiplying the Row 1 by -0.3, and subtract it from Row 2 would eliminate a_{21},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Again dividing Row 1 by 10 and multiplying by 5, that is, multiplying the Row 1 by 0.5, and subtract it from Row 3 would eliminate a_{31},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, the absolute value of the second column elements below the Row 2 is $|-0.001|$, $|2.5|$ or 0.001, 2.5

So the largest absolute value is in Row 3. So the Row 2 is switched with the Row 3 to give

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Dividing row 2 by 2.5 and multiplying by –0.001, that is multiplying by $0.001/2.5 = –0.0004$, and then subtracting from Row 3 gives

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

<u>Back substitution</u>

$$6.002x_3 = 6.002 \quad \Rightarrow \quad x_3 = \frac{6.002}{6.002} = 1$$

Substituting the value of x_3 in Row 2

$$2.5x_2 + 5x_3 = 2.5 \quad \Rightarrow \quad x_2 = \frac{2.5 - 5x_2}{2.5} = \frac{2.5 - 5}{2.5} = -1$$

Substituting the value of x_3 and x_2 in Row 1

$$10x_1 - 7x_2 + 0x_2 = 7 \quad \Rightarrow$$
$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = \frac{7 + 7(-1) - 0(1)}{10} = \frac{7 - 7 - 0}{10} = \frac{0}{10} = 0$$

So the solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

This, in fact, is the exact solution. By coincidence only, in this case, the round off error is fully removed.

2-2.2 LU Decomposition

We already studied two numerical methods of finding the solution to simultaneous linear equations – Naïve Gauss Elimination and Gaussian Elimination with Partial Pivoting. To appreciate why *LU* Decomposition could be a better choice than the Gaussian Elimination techniques in some cases, let us discuss first what *LU* Decomposition is about.

For any non-singular matrix A on which one can conduct Naïve Gaussian Elimination or forward elimination steps, one can always write it as $A = LU$

where

L = Lower triangular matrix

U = Upper triangular matrix

Then if one is solving a set of equations $Ax = b$, it will imply that $LUx = b$ since $A = LU$.

Multiplying both side by L^{-1}, we have

$L^{-1}LUx = L^{-1}b$

$\Rightarrow\ IUx = L^{-1}x$ since $\left(L^{-1}L = I\right)$,

$\Rightarrow\ Ux = L^{-1}x$ since $\left(IU = U\right)$

Let $L^{-1}b = z$ then $Lz = b$ (1)

And $Ux = z$ (2)

So we can solve equation (1) first for z and then use equation (2) to calculate x.

The computational time required to decompose the A matrix to LU form is proportional to $\frac{n^3}{3}$, where n is the number of equations (size of A matrix). Then to solve the $Lz = b$, the computational time is proportional to $\frac{n^2}{2}$. Then to solve the $Ux = z$, the computational time is proportional to $\frac{n^2}{2}$. So the total computational time to solve a set of equations by *LU* decomposition is proportional to $\frac{n^3}{3} + n^2$.

In comparison, Gaussian elimination is computationally more efficient. It takes a computational time proportional to $\frac{n^3}{3} + \frac{n^2}{2}$, where the computational time for forward elimination is proportional to $\frac{n^3}{3}$ and for the back substitution the time is proportional to $\frac{n^2}{2}$.

Finding the inverse of the matrix A reduces to solving *n* sets of equations with the *n* columns of the identity matrix as the RHS vector. For calculations of each column of the inverse of the A

matrix, the coefficient matrix A matrix in the set of equation $Ax = b$ does not change. So if we use LU Decomposition method, the $A = LU$ decomposition needs to be done only once and the use of equations (1) and (2) still needs to be done 'n' times.

So the total computational time required to find the inverse of a matrix using LU decomposition is proportional to $\frac{n^3}{3} + n(n^2) = \frac{4n^3}{3}$.

In comparison, if Gaussian elimination method were applied to find the inverse of a matrix, the time would be proportional to $n\left(\frac{n^3}{3} + \frac{n^2}{2}\right) = \frac{n^4}{3} + \frac{n^3}{2}$.

For large values of n, $\frac{n^4}{3} + \frac{n^3}{2} \rangle\rangle \frac{4n^3}{3}$

2-2.3 Decomposing a non-singular matrix A into the form A=LU.

***L U* Decomposition Algorithm**:

In these methods the coefficient matrix A of the given system of equation $Ax = b$ is written as a product of a Lower triangular matrix L and an Upper triangular matrix U, such that $A = LU$ where the elements of $L = (l_{ij} = 0;\ \text{for } i < j)$ and the elements of $U = (u_{ij} = 0;\ \text{for } i > j)$ that is,

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}.$$

Now using the rules of matrix multiplication $a_{ij} = \sum_{k=1}^{\min(i,j)} l_{ik}u_{kj}, \quad i, j = 1, \dots, n$

This gives a system of n^2 equations for the $n^2 + n$ unknowns (the non-zero elements in L and U). To make the number of unknowns and the number of equations equal one can fix the diagonal element either in L or in U such as '1's then solve the n^2 equations for the remaining n^2 unknowns in L and U. This leads to the following algorithm:

Algorithm

The factorization $A = LU$, where $L = \left(l_{ij}\right)_{n\times n}$ is a lower triangular and $U = \left(u_{ij}\right)_{n\times n}$ an upper triangular, can be computed directly by the following algorithm (provided zero divisions are not encountered):

Algorithm

For $k = 1$ to n *do specify* $\left(l_{kk} \text{ or } u_{kk}\right)$ *and compute the other such that* $l_{kk}u_{kk} = a_{kk} - \sum_{m=1}^{k-1} l_{km}u_{mk}$.

Compute the k^{th} *column of* L *using* $l_{ik} = \dfrac{1}{u_{kk}}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im}u_{mk}\right) \quad \left(k < i \le n\right)$, *and compute the* k^{th} *row of* U *using* $u_{kj} = \dfrac{1}{l_{kk}}\left(a_{kj} - \sum_{m=1}^{k-1} l_{km}u_{mj}\right) \quad \left(k < j \le n\right)$

End

Note:

The procedure is called ***Doolittle*** or ***Crout*** Factorization when $l_{ii} = 1 \ \left(1 \le i \le n\right)$ or $u_{jj} = 1 \ \left(1 \le j \le n\right)$ respectively.

If forward elimination steps of Naïve Gauss elimination methods can be applied on a non-singular matrix, then A can be decomposed into LU as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \ell_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \ell_{n1} & \cdots & \ell_{n-1,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & \mathrm{u}_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix} = LU$$

1. The elements of the U matrix are exactly the same as the coefficient matrix one obtains at the end of the forward elimination steps in Naïve Gauss Elimination.
2. The lower triangular matrix L has 1 in its diagonal entries. The non-zero elements below the diagonal in L are multipliers that made the corresponding entries zero in the upper triangular matrix U during forward elimination.

Solving Ax=b in pure matrix Notations

Solving systems of linear equations (AX=b) using LU factorization can be quite cumbersome, although it seem to be one of the simplest ways of finding the solution for the system, Ax=b. In pure matrix notation, the upper triangular matrix, U, can be calculated by constructing specific permutation matrices and elementary matrices to solve the Elimination process with both partial and complete pivoting.

The elimination process is equivalent to pre multiplying A by a sequence of lower-triangular matrices M_k as follows:

$$M_{k-1}M_{k-2}\dots M_1A = U$$

Where $M_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & 0 & 1 \end{bmatrix}$, $\mathrm{M}_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & m_{n2} & \cdots & 1 \end{bmatrix}$ with $m_{ij} = -\dfrac{a_{ij}^{(j-1)}}{a_{jj}}$ known as the multiplier

In solving Gaussian elimination without partial pivoting to triangularize *A,* the process yields the factorization, $MA = U$. In this case, the system *Ax* = *b* is equivalent to the triangular system $Ux = Mb = b'$ where $M = M_{k-1}M_{k-2}M_{k-3}\dots M_1$

The elementary matrices M_1 and M_2 are called first and second Gaussian transformation matrix respectively with M_k being the k^{th} Gaussian transformation matrix.
Generally, to solve $Ax = b$ using Naïve Gaussian elimination without partial pivoting by this approach, a permutation matrix is introduced to perform the pivoting strategies:

First We find the factorization $MA = U$ by the triangularization algorithm using partial pivoting. We then solve the triangular system by back substitution as follows $Ux = Mb = b'$.

Note that $M = M_{n-1}P_{n-1}M_{n-2}P_{n-2}\cdots M_2P_2M_1P_1$
The vector $b' = Mb = M_{n-1}P_{n-1}M_{n-2}P_{n-2}\cdots M_2P_2M_1P_1b$
Generally if we set $s_1 = b = (b_1, b_2, \dots, b_n)^T$
Then For $k = 1, 2, \dots, n-1$ do

$$s_{k+1} = M_kP_ks_k$$

Example

If $n = 3$, $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}$, then $s_2 = M_1P_1s_1 = \begin{pmatrix} s_1^{(2)} \\ s_2^{(2)} \\ s_3^{(2)} \end{pmatrix}$

If $P_1 s_1 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then the entries of $s^{(2)}$ are given by:

$$s_1^{(2)} = b_1$$
$$s_2^{(2)} = m_{21}b_1 + b_3$$
$$s_3^{(2)} = m_{31}b_1 + b_2$$

In the same way, to solve **Ax = b** Using Gaussian Elimination with Complete Pivoting, we modify the previous construction to include another permutation matrix, Q such that when post multiplied by A, we can perform column interchange. This results in $M(PAQ) = U$

Example 1

Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The U matrix is the same as found at the end of the forward elimination of Naïve Gauss elimination method, that is

Forward Elimination of Unknowns: Since there are three equations, there will be two steps of forward elimination of unknowns.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

First step: Divide Row 1 by 25 and then multiply it by 64 and subtract the results from Row 2

$$Row\ 2 - \left[\frac{64}{25}\right] \times (Row\ 1) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Here the multiplier , $m_{21} = -\frac{64}{25}$

Divide Row 1 by 25 and then multiply it by 144 and subtract the results from Row 3

$$Row\ 3 - \left[\frac{144}{25}\right] \times (Row\ 1) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Here the multiplier , $m_{31} = -\frac{144}{25}$, hence the first Gaussian transformation matrix is given by:

$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2.56 & 1 & 0 \\ -5.76 & 0 & 1 \end{bmatrix}$ And the corresponding product is given by:

$A^{(1)} = M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2.60 & 1 & 0 \\ -5.76 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$ (by a single multiplication)

<u>Second step</u>: We now divide Row 2 by -4.8 and then multiply by -16.8 and subtract the results from Row 3

$Row\ 3 - \left[\frac{-16.8}{-4.8}\right] \times (Row\ 2) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$ which produces $U = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

Here the multiplier, $m_{32} = -\frac{-16.8}{-4.8}$, hence the 2nd Gaussian transformation matrix is given by:

$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3.5 & 1 \end{bmatrix}$ And the corresponding product is given by:

$A^{(2)} = M_2 A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = U$ (by a single multiplication)

To find ℓ_{21} and ℓ_{31}, what multiplier was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination of Naïve Gauss Elimination Method It was

$$\ell_{21} = -m_{21} = \frac{64}{25} = 2.56$$

$$\ell_{31} = -m_{31} = \frac{144}{25} = 5.76$$

To find ℓ_{32}, what multiplier was used to make a_{32} element zero. Remember a_{32} element was made zero in the second step of forward elimination. The A matrix at the beginning of the second step of forward elimination was

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

So

$$\ell_{32} = -m_{32} = \frac{-16.8}{-4.8} = 3.5$$

Hence

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Confirm $LU = A$.

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Example 2

Use *LU* decomposition method to solve the following linear system of equations.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution

Recall that $Ax = b$ and if $A = LU$ then first solving $Lz = b$ and then $Ux = z$ gives the solution vector x.

Now in the previous example, we showed

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

First solve $Lz = b$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

to give

$$\begin{aligned} z_1 \qquad\qquad &= 106.8 \\ 2.56z_1 + \quad z_2 \qquad &= 177.2 \\ 5.76z_1 + 3.5z_2 + z_3 &= 279.2 \end{aligned}$$

Forward substitution starting from the first equation gives

$$\begin{aligned} z_1 &= 106.8 \\ z_2 &= 177.2 - 2.56z_1 = 177.2 - 2.56(106.8) = -96.2 \\ z_3 &= 279.2 - 5.76z_1 - 3.5z_2 = 279.2 - 5.76(106.8) - 3.5(-96.21) = 0.735 \end{aligned}$$

Hence

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

This matrix is same as the right hand side obtained at the end of the forward elimination steps of Naïve Gauss elimination method. Is this a coincidence?

Now solve $Ux = z$, i.e.,

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

$$\begin{aligned} 25x_1 + \ 5x_2 + \quad x_3 &= 106.8 \\ -4.8x_2 - 1.56x_3 &= -96.21 \\ 0.7x_3 &= 0.735 \end{aligned}$$

From the third equation $0.7x_3 = 0.735 \ \Rightarrow \ x_3 = \dfrac{0.735}{0.7} = 1.050$

Substituting the value of a_3 in the second equation,

$$-4.8x_2 - 1.56x_3 = -96.21 \ \Rightarrow \ x_2 = \frac{-96.21 + 1.56x_3}{-4.8} = \frac{-96.21 + 1.56(1.050)}{-4.8} = 19.70$$

Substituting the value of x_2 and x_3 in the first equation,

$$25x_1 + 5x_2 + x_3 = 106.8 \ \Rightarrow \ x_1 = \frac{106.8 - 5x_2 - x_3}{25}$$

$$= \frac{106.8 - 5(19.70) - 1.050}{25} = 0.2900$$

The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Example 2.2
Solve

$$Ax = b \text{ with } A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

(a) using partial pivoting and (b) using complete pivoting.

Solution:
(a) Partial pivoting:
We compute U as follows:

$Step 1:$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_1A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix};$$

$$A^{(1)} = M_1P_1A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix};$$

$Step 2:$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_2A^{(1)} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}; \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix};$$

$$U = A^{(2)} = M_2P_2A^{(1)} = M_2P_2M_1P_1A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$Note$: Defining $P = P_2P_1$ and $L = P(M_2P_2M_1P_1)^{-1}$, we have $PA = LU$.

We compute b' as follows:

$Step 1:$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_1b = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}; \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \quad M_1P_1b = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix};$$

$Step 2:$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_2M_1P_1b = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}; \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \quad b' = M_2P_2M_1P_1b = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix};$$

The solution of the system

$$Ux = b' \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \text{ and } x_1 = x_2 = x_3 = 1$$

(b) Complete pivoting: We compute U as follows:

Step 1:

$$P_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix};\quad Q_1=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix};\quad P_1AQ_1=\begin{pmatrix}3&2&1\\1&1&0\\1&1&1\end{pmatrix};\quad M_1=\begin{pmatrix}1&0&0\\-\frac{1}{3}&1&0\\-\frac{1}{3}&0&1\end{pmatrix};$$

$$A^{(1)}=M_1P_1AQ_1=\begin{pmatrix}1&2&3\\0&\frac{1}{3}&-\frac{1}{3}\\0&\frac{1}{3}&\frac{2}{3}\end{pmatrix}$$

Step 2:

$$P_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix};\quad Q_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix};\quad P_2A^{(1)}Q_2=\begin{pmatrix}3&1&2\\0&\frac{2}{3}&\frac{1}{3}\\0&-\frac{1}{3}&\frac{1}{3}\end{pmatrix};\quad M_2=\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{1}{2}&1\end{pmatrix};$$

$$U=A^{(2)}=M_2P_2A^{(1)}Q_2=M_2P_2M_1P_1AQ_1Q_2=\begin{pmatrix}3&1&2\\0&\frac{2}{3}&\frac{1}{3}\\0&0&\frac{1}{2}\end{pmatrix}$$

Note: Defining $P=P_2P_1$, $Q=Q_1Q_2$ and $L=P(M_2P_2M_1P_1)^{-1}$, we have $PAQ=LU$.

We compute b' as follows:

Step 1:

$$P_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix};\quad P_1b=\begin{pmatrix}6\\2\\3\end{pmatrix};\quad M_1=\begin{pmatrix}1&0&0\\-\frac{1}{3}&1&0\\-\frac{1}{3}&0&1\end{pmatrix};\quad M_1P_1b=\begin{pmatrix}6\\0\\1\end{pmatrix}$$

Step 2:

$$P_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix};\quad P_2M_1P_1b=\begin{pmatrix}6\\1\\0\end{pmatrix};\quad M_2=\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{1}{2}&1\end{pmatrix};\quad M_1P_1b=\begin{pmatrix}6\\0\\1\end{pmatrix};$$

$$b'=M_2P_2M_1P_1b=\begin{pmatrix}6\\1\\\frac{1}{2}\end{pmatrix}$$

The solution of the system

$$Uy=b'\quad\Rightarrow\quad\begin{pmatrix}3&1&2\\0&\frac{2}{3}&\frac{1}{3}\\0&0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}6\\1\\\frac{1}{2}\end{pmatrix}$$

is $y_1=y_2=y_3=1$. Because $\{x_k\}$, $k=1,2,3$ is simply the rearrangement of $\{y_k\}$, we have $x_1=x_2=x_3=1$.

2-2.4 Finding the inverse of a square matrix using LU Decomposition

A matrix B is the inverse of A if $AB = I = BA$. First assume that the first column of B (the inverse of A is $[b_{11}\ b_{21}\ \cdots\ b_{n1}]^T$ then from the above definition of inverse and definition of matrix multiplication.

$$A\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly the second column of B is given by

$$A\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly, all columns of B can be found by solving n different sets of equations with the column of the right hand sides being the n columns of the identity matrix.

Example 3

Use LU decomposition to find the inverse of

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Knowing that

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

We can solve for the first column of $B = A^{-1}$ by solving for

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First solve $Lz = c$, that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

to give

$$\begin{aligned} z_1 \quad\quad\quad\quad\quad &= 1 \\ 2.56z_1 + \quad z_2 \quad\quad &= 0 \\ 5.76z_1 + 3.5z_2 + z_3 &= 0 \end{aligned}$$

Forward substitution starting from the first equation gives

$$\begin{aligned} &z_1 = 1 \\ &z_2 = 0\text{-}2.56z_1 = 0 - 2.56(1) = -256 \\ &z_3 = 0 - 5.76z_1 - 3.5z_2 = 0 - 5.76(1) - 3.5(-2.56) = 3.2 \end{aligned}$$

Hence

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Now solve $Ux = z$, that is

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ \text{-}\,2.56 \\ 3.2 \end{bmatrix} \Rightarrow \begin{aligned} 25b_{11} + 5b_{21} + b_{31} &= 1 \\ -4.8b_{21} - 1.56b_{31} &= -2.56 \\ 0.7b_{31} &= 3.2 \end{aligned}$$

Backward substitution starting from the third equation gives

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8} = \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25} = \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

Hence the first column of the inverse of A is

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Similarly by solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

and solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{b}_{13} \\ \mathrm{b}_{23} \\ \mathrm{b}_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} 0.4762 & 0.08333 & 0.0357 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.050 & 1.429 \end{bmatrix}$$

Exercise

Show that $AA^{-1} = I = A^{-1}A$ for the above example.