Math 263 Assignment 6 Solutions

Problem 1. Find the volume of the solid bounded by the surfaces $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Solution. The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$ or $x^2 + y^2 = 1$. Wrting down the given volume first in Cartesian coordinates and then converting into polar form we find that

$$\begin{aligned} V &= \iint_{x^2+y^2\le 1} \left[(4 - x^2 - y^2) - (3x^2 + 3y^2)\right] \, dA \\ &= \int_0^{2\pi} \int_0^1 4(1 - r^2) r dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3) dr = 2\pi. \end{aligned}$$

□

Problem 2. Sketch the region enclosed by the curve $r = b + a\cos\theta$ and compute its area. Here a and b are positive constants, $b > a$.

Solution. The curve is a cardioid symmetric about the x-axis. The area enclosed by it is

$$\begin{aligned} A &= 2 \int_{\theta=0}^{\pi} \int_{r=0}^{b+a\cos\theta} r \, dr \, d\theta \\ &= \int_0^{\pi} (b + a\cos\theta)^2 \, d\theta \\ &= \int_0^{\pi} \left[b^2 + \frac{a^2}{2}(1 + \cos(2\theta)) + 2ab\cos\theta \right] \, d\theta \\ &= \left(b^2 + \frac{a^2}{2} \right) \pi. \end{aligned}$$

□

Problem 3. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely poportional to its distance from the origin.

Solution. The circles $x^2 + y^2 = 2y$ and $x^2 + y^2 = 1$ may be written in polar coordinates as $r = 2\sin\theta$ and $r = 1$ respectively. They intersect at two points, where $\sin\theta = \frac{1}{2}$, so that $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ at these points. Further the density function is $\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r$, where k is the constant of proportionality. Therefore

$$\begin{aligned}
\text{mass} = m &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{1}^{2\sin\theta} \frac{k}{r} r dr\, d\theta \\
&= k \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin\theta - 1)\, d\theta \\
&= 2k(\sqrt{3} - \frac{\pi}{3}).
\end{aligned}$$

By symmetry of the domains and the function $f(x) = x$, we know that $M_y = 0$, and

$$\begin{aligned}
M_x &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{1}^{2\sin\theta} kr\sin\theta dr\, d\theta \\
&= \frac{k}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4\sin^3\theta - \sin\theta)\, d\theta \\
&= \sqrt{3}k.
\end{aligned}$$

Hence $(\overline{x}, \overline{y}) = (0, \frac{3\sqrt{3}}{2(3\sqrt{3}-\pi)})$. □

Problem 4. Evaluate the triple integral

$$\iiint_E zdV,$$

where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$ and $z = 0$ in the first octant.

Solution.

$$\begin{aligned}
\iiint_E zdV &= \int_0^1 \int_{3x}^{3} \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx \\
&= \int_0^1 \int_{3x}^{3} \frac{1}{2}(9 - y^2)\, dy\, dx \\
&= \int_0^1 \left[\frac{9y}{2} - \frac{y^3}{6}\right]_{y=3x}^{y=3} \\
&= \int_0^1 \left[9 - \frac{27}{2}x + \frac{9}{2}x^3\right] dx = \frac{27}{8}.
\end{aligned}$$

□

Problem 5. Find the volume of the solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$ and $y = 9$.

Solution.

$$
\begin{aligned}
V = \iiint_E dV &= \int_{-3}^{3} \int_{x^2}^{9} \int_0^4 dz\, dy\, dx \\
&= 4 \int_{-3}^{3} \int_{x^2}^{9} dy\, dx \\
&= 4 \int_{-3}^{3} (9 - x^2)\, dx \\
&= 144.
\end{aligned}
$$

$\square$

Problem 6. Sketch the solid whose volume is given by the iterated integral

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy.$$

Solution. The triple integral is the volume of $E = \{(x, y, z) : 0 \leq y \leq 2,\ 0 \leq z \leq 2-y,\ 0 \leq x \leq 4 - y^2\}$, the solid bounded by the three coordinate planes, the plane $z = 2 - y$, and the cylindrical surface $x = 4 - y^2$. $\square$

Problem 7. Rewrite the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

as an equivalent iterated integral in five other orders.

Solution. The projection of E onto the xy plane is the right triangle bounded by the coordinate axes and the straight line $x + y = 1$. On the other hand, the projection onto the xz plane is the region bounded by the coordinate axes and the parabola $z = 1 - x^2$. Therefore the given iterated integral may also be written as

$$
\begin{aligned}
\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx &= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z) dy\, dx\, dz \\
&= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z) dz\, dx\, dy \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z)\, dz\, dy\, dx.
\end{aligned}
$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1 - y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz plane (which is the unit square) into two regions, whose boundary is the curve above. The given integral is therefore also equal to

$$\left[\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y}\right] f(x,y,z)\,dx\,dy\,dz$$
$$= \left[\int_0^1 \int_0^{2y-y^2} \int_0^{1-y} + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}}\right] f(x,y,z)\,dx\,dz\,dy.$$

$\square$