18.06 Professor Strang/Ingerman Final Exam December 17, 2002

Your name is: ______________________________

The ten questions are worth 10 points each.

Thank you for taking 18.06!

1 The 4 by 6 matrix A has all 2's below the diagonal and elsewhere all 1's:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

(a) By elimination factor A into L (4 by 4) times U (4 by 6).

(b) Find the rank of A and a basis for its nullspace (the special solutions would be good).

2 Suppose you know that the 3 by 4 matrix A has the vector $\boldsymbol{s} = (2, 3, 1, 0)$ as a basis for its nullspace.

(a) What is the *rank* of A and the complete solution to $A\boldsymbol{x} = \mathbf{0}$?

(b) What is the exact row reduced echelon form R of A?

3 The following matrix is a *projection matrix*:

$$P = \frac{1}{21}\begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}.$$

(a) What subspace does P project onto?

(b) What is the *distance* from that subspace to $\boldsymbol{b} = (1, 1, 1)$?

(c) What are the three eigenvalues of P? Is P diagonalizable?

4 **(a)** Suppose the product of A and B is the zero matrix: $AB = 0$. Then the $\underline{\quad\textbf{(1)}\quad}$ space of A contains the $\underline{\quad\textbf{(2)}\quad}$ space of B. Also the $\underline{\quad\textbf{(3)}\quad}$ space of B contains the $\underline{\quad\textbf{(4)}\quad}$ space of A. Those blank words are

(1) ______________ **(2)** ______________ **(3)** ______________ **(4)** ______________

(b) Suppose that matrix A is 5 by 7 with rank r, and B is 7 by 9 of rank s. What are the dimensions of spaces **(1)** and **(2)**? From the fact that space **(1)** contains space **(2)**, what do you learn about $r + s$?

5 Suppose the 4 by 2 matrix Q has orthonormal columns.

(a) Find the least squares solution $\widehat{\boldsymbol{x}}$ to $Q\boldsymbol{x} = \boldsymbol{b}$.

(b) Explain why QQ^{T} is not positive definite.

(c) What are the (nonzero) singular values of Q, and why?

6 Let S be the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$.

(a) Find an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for S by Gram-Schmidt.

(b) Write down the 3 by 3 matrix P which projects vectors perpendicularly onto S.

(c) Show how the properties of P (what are they?) lead to the conclusion that $P\boldsymbol{b}$ is orthogonal to $\boldsymbol{b} - P\boldsymbol{b}$.

7 **(a)** If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis for $\mathbf{R}^3$ then the matrix with those three columns is _____.

(b) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ span $\mathbf{R}^3$, give all possible ranks for the matrix with those four columns. ______________.

(c) If $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ form an orthonormal basis for $\mathbf{R}^3$, and T is the transformation that projects every vector $\boldsymbol{v}$ onto the plane of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, what is the matrix for T in this basis? Explain.

8 Suppose the n by n matrix A_n has 3's along its main diagonal and 2's along the diagonal below and the $(1, n)$ position:

$$A_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

Find by cofactors of row 1 or otherwise the determinant of A_4 and then the determinant of A_n for $n > 4$.

9 There are six 3 by 3 permutation matrices P.

(a) What numbers can be the *determinant* of P? What numbers can be *pivots*?

(b) What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

10 Suppose A is a 4 by 4 upper triangular matrix with $1, 2, 3, 4$ on its main diagonal. (You could put all 1's above the diagonal.)

(a) For $A - 3I$, which columns have pivots? Which components of the eigenvector $\boldsymbol{x}_3$ (the special solution in the nullspace) are definitely zero?

(b) Using part (a), show that the eigenvector matrix S is also upper triangular.