MA261-A Calculus III

2006 Fall

Homework 9 Solutions

Due 11/6/2006 8:00AM

12.1 #2 If $R = [-1,3] \times [0,2]$, use a Riemann sum with $m = 4$, $n = 2$ to estimate the value of $\iint\limits_R (y^2 - 2x^2)\, dA$. Take the sample point to be the upper right corner of each subrectangle.

[Solution]

By the definition of Riemann sum, since $m = 4$, we partition $[-1,3]$ into 4 pieces with $\Delta x = \frac{3-(-1)}{4} = 1$ and points

$$x_0 = a = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = b = 3.$$

Also, since $n = 2$, we partition $[0,2]$ into 2 pieces with $\Delta y = \frac{2-0}{2} = 1$ and points

$$y_0 = c = 0,\ y_1 = 1,\ x_2 = b = 2.$$

So, we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$.

In the region R_{ij}, the upper right corner has the coordinate $(x_i, y_j) = (i-1, j)$. (Check the Figure 3 in the textbook page 830.)

Let $f(x,y) = y^2 - 2x^2$. The Riemann sum becomes

$$\begin{aligned}
& \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A \\
= & \sum_{i=1}^{4}\sum_{j=1}^{2} f(i-1, j)(1) \\
= & \sum_{i=1}^{4}\sum_{j=1}^{2} \left((j)^2 - 2(i-1)^2\right) \\
= & \sum_{i=1}^{4} \left[\left((1)^2 - 2(i-1)^2\right) + \left((2)^2 - 2(i-1)^2\right)\right] \\
= & \sum_{i=1}^{4} \left(5 - 4(i-1)^2\right) \\
= & \left(5 - 4(1-1)^2\right) + \left(5 - 4(2-1)^2\right) + \left(5 - 4(3-1)^2\right) + \left(5 - 4(4-1)^2\right) \\
= & -36.
\end{aligned}$$

[Note that, by using the idea of the iterated integral, we have $\int_{-1}^{3}\int_{0}^{2} (y^2 - 2x^2)\, dy dx = -\frac{80}{3}$.]

■

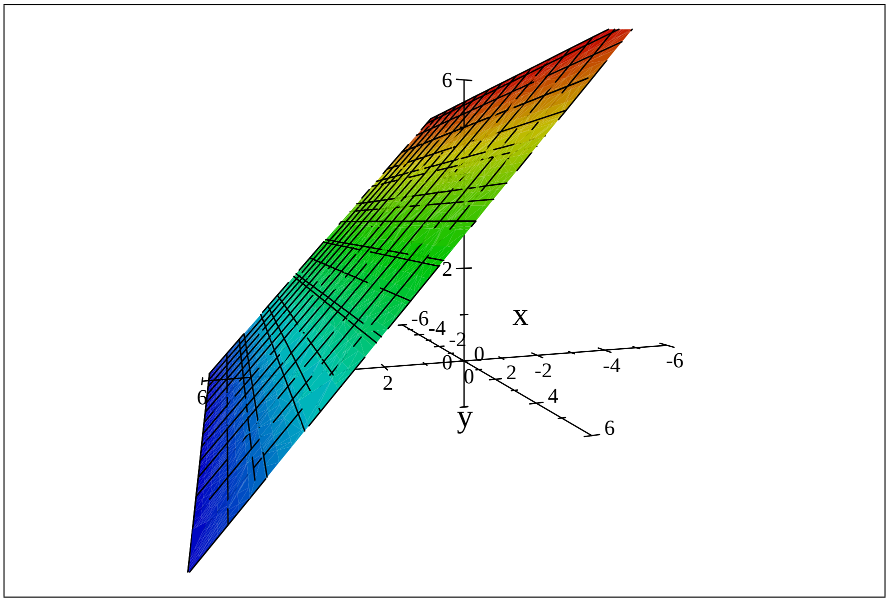
12.1 #12 Evaluate the double integral

$$\iint_R (5-x)\,dA,$$

where $R = \{(x,y) \mid 0 \le x \le 5, 0 \le y \le 3\}$ by first identifying it as the volume of a solid.

[Solution]

The solid we are looking for have the base R and the height $5-x$. Let $z = 5-x$. The graph of $z = 5-x$ looks like



So, the solid is a triangular prism.

Look at $y = 0$. We have a triangle formed by $x = 0$, $z = 0$ and $z = 5-x$. The three cornors of this triangle is $(5,0)$, $(0,0)$, and $(0,5)$ in the x-z plane. Thus, the area of this triangle is $\frac{1}{2}(5-0)(5-0) = \frac{25}{2}$.

The triangular prism is also bounded by $y = 0$ and $y = 3$. Therefore, the volume of the triangular prism is $\frac{25}{2} \times (3-0) = \frac{75}{2}$. Thus, $\iint_R (5-x)\,dA = \frac{75}{2}$.

■

12.1 #18 If $R = [0,1] \times [0,1]$, show that $0 \le \iint_R \sin(x+y)\,dA \le 1$.

[Solution]

In the region $R = [0,1] \times [0,1]$, we have $0 \le x \le 1$ and $0 \le y \le 1$. Thus, we have $0 \le x+y \le 2$.

Since $2 > \frac{\pi}{2}$, we know that $x+y$ can form any number between 0 and $\frac{\pi}{2}$. Therefore, by the graph of sin, we know that $0 \le \sin(x+y) \le 1$ when $0 \le x+y \le \frac{\pi}{2}$. When $\frac{\pi}{2} \le x+y \le 2$, by the graph of sin, we also have $0 < \sin 2 \le \sin(x+y) \le 1$. Thus, we can conclude that $0 \le \sin(x+y) \le 1$.

The area of R is $A(R) = (1-0) \times (1-0) = 1$. So,

$$0 = 0 \cdot A(R) \le \iint_R \sin(x+y)\,dA \le 1 \cdot A(R) = 1.$$

■

12.2 #6 Calculate the iterated integral

$$\int_1^4 \int_0^2 (x + \sqrt{y})\, dxdy.$$

[Solution]

We have

$$\begin{aligned}
\int_1^4 \int_0^2 (x + \sqrt{y})\, dxdy &= \int_1^4 \left[\int_0^2 (x + \sqrt{y})\, dx\right] dy \\
&= \int_1^4 \left[\left(\frac{x^2}{2} + x\sqrt{y}\right)\Bigg|_{x=0}^{x=2}\right] dy \\
&= \int_1^4 \left[\left(\frac{2^2}{2} + 2\sqrt{y}\right) - \left(\frac{0^2}{2} + 0\sqrt{y}\right)\right] dy \\
&= \int_1^4 (2 + 2\sqrt{y})\, dy \\
&= \left[2y + 2\left(\frac{2}{3}y^{\frac{3}{2}}\right)\right]\Bigg|_{y=1}^{y=4} \\
&= \frac{46}{3}.
\end{aligned}$$

■

12.2 #12 Calculate the iterated integral

$$\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2} dydx.$$

[Solution]

When we treat x as a constant, we can calculate

$$\int_0^1 xy\sqrt{x^2 + y^2} dy$$

by using the substitution. Let $u = x^2 + y^2$. $du = 2ydy$. Thus, the intergal becomes

$$\begin{aligned}
\int_0^1 xy\sqrt{x^2 + y^2} dy &= \int_0^1 x\sqrt{u}\frac{du}{2} = \frac{x}{2}\int_0^1 \sqrt{u} du = \frac{x}{2}\left[\left(\frac{2}{3}u^{\frac{3}{2}}\right)\right]\Bigg|_{u=0}^{u=1} \\
&= \frac{x}{2}\left[\left(\frac{2}{3}(1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}}\right)\right] = \frac{x}{3}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2} dydx &= \int_0^1 \left[\int_0^1 xy\sqrt{x^2 + y^2} dy\right] dx = \int_0^1 \frac{x}{3} dx \\
&= \left[\left(\frac{x^2}{6}\right)\right]\Bigg|_{x=0}^{x=1} = \left(\frac{(1)^2}{6}\right) - \left(\frac{(0)^2}{6}\right) \\
&= \frac{1}{6}.
\end{aligned}$$

■

12.2 #14 Calculate the double integral

$$\iint_R \cos(x+2y)\, dA$$

where $R = \left\{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \frac{\pi}{2}\right\}$.

[Solution]

Since the region R is a rectangle in x-y plane, the double integral becomes an iterated integral

$$\int_0^{\pi} \int_0^{\frac{\pi}{2}} \cos(x+2y)\, dydx.$$

Let $u = x + 2y$. When we treat x as a constant, we have $du = 2dy$. Thus,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos(x+2y)\, dy &= \int_x^{\pi+x} \cos(u)\, \frac{du}{2} = \left(\frac{\sin u}{2}\right)\Bigg|_{u=x}^{u=\pi+x} \\
&= \left(\frac{\sin(\pi+x)}{2}\right) - \left(\frac{\sin(x)}{2}\right) \\
&= \left(\frac{-\sin(x)}{2}\right) - \left(\frac{\sin(x)}{2}\right) = -\sin x.
\end{aligned}$$

Therefore, the iterated integral becomes

$$\int_0^{\pi} \int_0^{\frac{\pi}{2}} \cos(x+2y)\, dydx = \int_0^{\pi} -\sin x dx = (\cos x)|_{x=0}^{x=\pi} = -1 - (1) = -2.$$

■

12.2 #16 Calculate the double integral

$$\iint_R \frac{1+x^2}{1+y^2} dA$$

where $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

[Solution]

Since the region R is a rectangle in x-y plane, the double integral becomes an iterated integral

$$\int_0^1 \int_0^1 \frac{1+x^2}{1+y^2} dydx = \int_0^1 \left[\int_0^1 \frac{1+x^2}{1+y^2} dy\right] dx = \int_0^1 \left[\left(1+x^2\right) \int_0^1 \frac{1}{1+y^2} dy\right] dx.$$

Note that $\int_0^1 \frac{1}{1+y^2}dy = (\arctan y)|_{y=0}^{y=1} = \frac{\pi}{4}$ is a real number (a constant with respect to x). Thus,

$$
\begin{aligned}
& \int_0^1 \int_0^1 \frac{1+x^2}{1+y^2} dydx \\
= & \int_0^1 \left[\left(1+x^2\right) \int_0^1 \frac{1}{1+y^2} dy \right] dx = \int_0^1 \left[\left(1+x^2\right) \frac{\pi}{4} \right] dx \\
= & \frac{\pi}{4} \int_0^1 \left(1+x^2\right) dx = \frac{\pi}{4} \left(x + \frac{x^3}{3} \right) \Bigg|_{x=0}^{x=1} \\
= & \frac{\pi}{4} \left[\left((1) + \frac{(1)^3}{3} \right) - \left((0) + \frac{(0)^3}{3} \right) \right] \\
= & \frac{\pi}{3}.
\end{aligned}
$$

■

12.2 #22 Find the volume of the solid that lies under the hyperbolic paraboloid $z = 4 + x^2 - y^2$ and above the square $R = [-1, 1] \times [0, 2]$.

[Solution]

The volume is

$$
\begin{aligned}
& \iint\limits_R \left(4 + x^2 - y^2\right) dA \\
= & \int_{-1}^1 \int_0^2 \left(4 + x^2 - y^2\right) dydx = \int_{-1}^1 \left[\int_0^2 \left(4 + x^2 - y^2\right) dy \right] dx \\
= & \int_{-1}^1 \left[\left(4y + x^2 y - \frac{y^3}{3} \right) \Bigg|_{y=0}^{y=2} \right] dx \\
= & \int_{-1}^1 \left[\left(4(2) + x^2(2) - \frac{(2)^3}{3} \right) - \left(4(0) + x^2(0) - \frac{(0)^3}{3} \right) \right] dx \\
= & \int_{-1}^1 \left(\frac{16}{3} + 2x^2 \right) dx \\
= & \left(\frac{16}{3}x + \frac{2x^3}{3} \right) \Bigg|_{x=-1}^{x=1} \\
= & \left(\frac{16}{3}(1) + \frac{2(1)^3}{3} \right) - \left(\frac{16}{3}(-1) + \frac{2(-1)^3}{3} \right) \\
= & 12.
\end{aligned}
$$

■

12.2 #26 Find the volume of the solid bounded by the elliptic paraboloid $z = 1 + (x-1)^2 + 4y^2$, the planes $x = 3$ and $y = 2$, and the coordinate planes.

[Solution]

Consider $z = 0$. The base of the solid is $[0, 3] \times [0, 2]$ since it is bounded by the lines $x = 3$ and $y = 2$, and the coordinate axes. The top of the solid is bounded by

$z = 1 + (x-1)^2 + 4y^2$. So, we can think z as the height at the point (x, y). Therefore, the volume is

$$
\begin{aligned}
& \iint\limits_R \left(1 + (x-1)^2 + 4y^2\right) dA \\
= & \int_0^3 \int_0^2 \left(1 + (x-1)^2 + 4y^2\right) dy dx \\
= & \int_0^3 \left[\int_0^2 \left(1 + (x-1)^2 + 4y^2\right) dy\right] dx \\
= & \int_0^3 \left[\left(y + (x-1)^2 y + \frac{4y^3}{3}\right)\Bigg|_{y=0}^{y=2}\right] dx \\
= & \int_0^3 \left[\left((2) + (x-1)^2 (2) + \frac{4(2)^3}{3}\right) - \left((0) + (x-1)^2 (0) + \frac{4(0)^3}{3}\right)\right] dx \\
= & \int_0^3 \left(2x^2 - 4x + \frac{44}{3}\right) dx \\
= & \left(2\frac{x^3}{3} - 2x^2 + \frac{44}{3}x\right)\Bigg|_{x=0}^{x=3} \\
= & \left(2\frac{(3)^3}{3} - 2(3)^2 + \frac{44}{3}(3)\right) - \left(2\frac{(0)^3}{3} - 2(0)^2 + \frac{44}{3}(0)\right) \\
= & \, 44.
\end{aligned}
$$

■

12.2 #32 Find the average value of $f(x, y) = e^y \sqrt{x + e^y}$ over the given rectangle $R = [0, 4] \times [0, 1]$.

[Solution]

The area of the rectangle R is $(4-0) \times (1-0) = 4$. The double integral is

$$\iint\limits_R \left(e^y \sqrt{x + e^y}\right) dA = \int_0^4 \int_0^1 \left(e^y \sqrt{x + e^y}\right) dy dx.$$

If we set $u = x + e^y$ and treat x as a constant, then we have $du = e^y dy$ and

$$
\begin{aligned}
\int_0^1 \left(e^y \sqrt{x + e^y}\right) dy &= \int_{x+1}^{x+e} \left(\sqrt{u}\right) du = \left(\frac{2}{3} u^{\frac{3}{2}}\right)\Bigg|_{u=x+1}^{u=x+e} \\
&= \left(\frac{2}{3}(x+e)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(x+1)^{\frac{3}{2}}\right).
\end{aligned}
$$

Thus, we have

$$\begin{aligned}
&\iint\limits_R \left(e^y\sqrt{x+e^y}\right) dA \\
&= \int_0^4 \left[\int_0^1 \left(e^y\sqrt{x+e^y}\right) dy\right] dx \\
&= \int_0^4 \left[\left(\frac{2}{3}(x+e)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(x+1)^{\frac{3}{2}}\right)\right] dx \\
&= \frac{2}{3}\int_0^4 \left[(x+e)^{\frac{3}{2}} - (x+1)^{\frac{3}{2}}\right] dx \\
&= \frac{2}{3}\left[\left(\frac{2}{5}(x+e)^{\frac{5}{2}} - \frac{2}{5}(x+1)^{\frac{5}{2}}\right)\Bigg|_{x=0}^{x=4}\right] \\
&= \frac{2}{3}\left[\left(\frac{2}{5}((4)+e)^{\frac{5}{2}} - \frac{2}{5}((4)+1)^{\frac{5}{2}}\right) - \left(\frac{2}{5}((0)+e)^{\frac{5}{2}} - \frac{2}{5}((0)+1)^{\frac{5}{2}}\right)\right] \\
&= \frac{4}{15}\left((4+e)^{\frac{5}{2}} - 5^{\frac{5}{2}} - e^{\frac{5}{2}} + 1\right) \\
&\approx 13.308.
\end{aligned}$$

So, the average value is

$$\begin{aligned}
\frac{\iint\limits_R \left(e^y\sqrt{x+e^y}\right) dA}{A(R)} &= \frac{\frac{4}{15}\left((4+e)^{\frac{5}{2}} - 5^{\frac{5}{2}} - e^{\frac{5}{2}} + 1\right)}{4} \\
&= \frac{1}{15}\left((4+e)^{\frac{5}{2}} - 5^{\frac{5}{2}} - e^{\frac{5}{2}} + 1\right) \\
&\approx 3.327.
\end{aligned}$$

■

12.3 #6 Evaluate the iterated integral

$$\int_0^1 \int_0^v \sqrt{1-v^2}\,du\,dv.$$

[Solution]
We have

$$\begin{aligned}
\int_0^1 \int_0^v \sqrt{1-v^2}\,du\,dv &= \int_0^1 \left[\int_0^v \sqrt{1-v^2}\,du\right] dv \\
&= \int_0^1 \left(u\sqrt{1-v^2}\right)\Big|_{u=0}^{u=v} dv \\
&= \int_0^1 \left[\left((v)\sqrt{1-v^2}\right) - \left((0)\sqrt{1-v^2}\right)\right] dv \\
&= \int_0^1 \left(v\sqrt{1-v^2}\right) dv.
\end{aligned}$$

Let $t = 1 - v^2$. Then, $dt = -2vdv$. So,

$$\begin{aligned}\int_0^1 \left(v\sqrt{1-v^2}\right) dv &= \int_1^0 \sqrt{t}\frac{dt}{-2} = -\frac{1}{2}\int_1^0 \sqrt{t}dt = \frac{1}{2}\int_0^1 \sqrt{t}dt \\ &= \frac{1}{2}\left[\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Big|_{t=0}^{t=1}\right] = \frac{1}{2}\left[\left(\frac{2}{3}(1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}}\right)\right] \\ &= \frac{1}{3}.\end{aligned}$$

Therefore,

$$\int_0^1 \int_0^v \sqrt{1-v^2}dudv = \int_0^1 \left(v\sqrt{1-v^2}\right) dv = \frac{1}{3}.$$

■

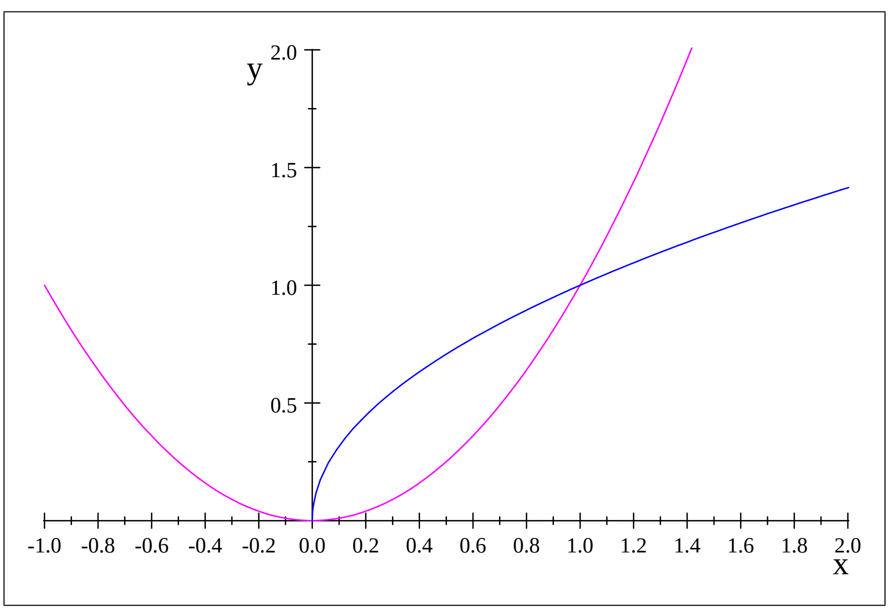
12.3 #12 Evaluate the double integral

$$\iint\limits_D (x+y)\, dA,$$

where D is bounded by $y = \sqrt{x}$ and $y = x^2$.

[Solution]

Set $\sqrt{x} = x^3$. We can solve for $x = 1$ or 0. The graph of $y = \sqrt{x}$(blue) and $y = x^2$(margenta) looks like



Thus,

$$
\begin{aligned}
&\iint\limits_D (x+y)\,dA \\
=\;& \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dydx = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (x+y)\,dy\right] dx \\
=\;& \int_0^1 \left[\left(xy + \frac{y^2}{2}\right)\Bigg|_{y=x^2}^{y=\sqrt{x}}\right] dx \\
=\;& \int_0^1 \left[\left(x\left(\sqrt{x}\right) + \frac{\left(\sqrt{x}\right)^2}{2}\right) - \left(x\left(x^2\right) + \frac{\left(x^2\right)^2}{2}\right)\right] dx \\
=\;& \int_0^1 \left[x^{\frac{3}{2}} + \frac{x}{2} - x^3 - \frac{x^4}{2}\right] dx \\
=\;& \left(\frac{2}{5}x^{\frac{5}{2}} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10}\right)\Bigg|_{x=0}^{x=1} \\
=\;& \left(\frac{2}{5}(1)^{\frac{5}{2}} + \frac{(1)^2}{4} - \frac{(1)^4}{4} - \frac{(1)^5}{10}\right) - \left(\frac{2}{5}(0)^{\frac{5}{2}} + \frac{(0)^2}{4} - \frac{(0)^4}{4} - \frac{(0)^5}{10}\right) \\
=\;& \frac{3}{10}.
\end{aligned}
$$

■

12.3 #16 Evaluate the double integral

$$\iint\limits_D 2xydA,$$

where D is the triangle region with vertices $(0,0)$, $(1,2)$, and $(0,3)$.

[Solution]

The equation of the line passes through $(0,0)$ and $(0,3)$ is

$$x = 0.$$

The equation of the line passes through $(0,0)$ and $(1,2)$ is

$$y - 0 = \frac{2-0}{1-0}(x-0),$$

or,

$$y = 2x.$$

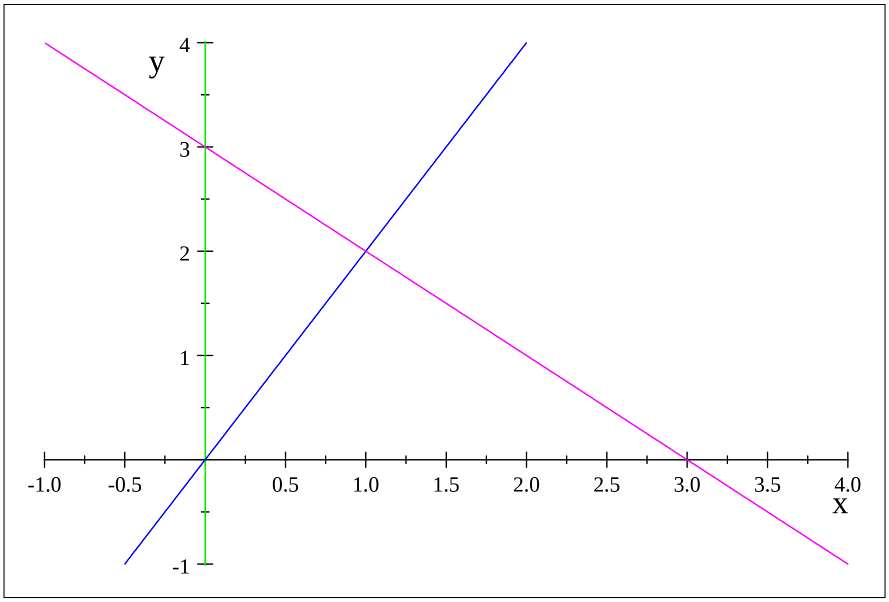
The equation of the line passes through $(1,2)$ and $(0,3)$ is

$$y - 3 = \frac{3-2}{0-1}(x-0),$$

or,

$$y = 3 - x.$$

Thus, D is bounded by $x = 0$(green), $y = 2x$(blue) and $y = 3 - x$(margenta). The graph looks like



Note that the intersection point of $y = 2x$ and $y = 3 - x$ is $(1, 2)$. So, we can describe D as

$$\{(x, y) \mid 0 \le x \le 1, 2x \le y \le 3 - x\}.$$

The double integral becomes

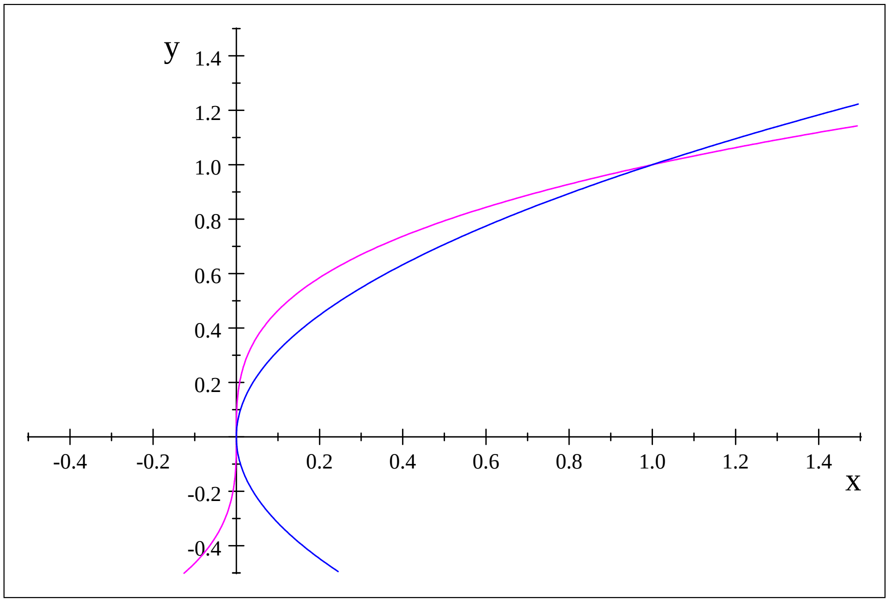
$$\begin{aligned}
\iint_D 2xy\,dA &= \int_0^1 \int_{2x}^{3-x} 2xy\,dy\,dx = \int_0^1 \left[\int_{2x}^{3-x} 2xy\,dy\right] dx \\
&= \int_0^1 \left[\left(xy^2\right)\Big|_{y=2x}^{y=3-x}\right] dx = \int_0^1 \left[\left(x(3-x)^2\right) - \left(x(2x)^2\right)\right] dx \\
&= \int_0^1 \left(9x - 6x^2 - 3x^3\right) dx = \left(9\frac{x^2}{2} - 6\frac{x^3}{3} - 3\frac{x^4}{4}\right)\Bigg|_{x=0}^{x=1} \\
&= \left(9\frac{(1)^2}{2} - 6\frac{(1)^3}{3} - 3\frac{(1)^4}{4}\right) - \left(9\frac{(0)^2}{2} - 6\frac{(0)^3}{3} - 3\frac{(0)^4}{4}\right) \\
&= \frac{7}{4}.
\end{aligned}$$

■

12.3 #18 Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$.

[Solution]

The graph of $x = y^2$(blue) and $x = y^3$(margenta) looks like



We know that when we set $y^2 = y^3$, we get the intersection point $(1,1)$. Thus, the region can be describled as

$$\left\{(x,y) \mid 0 \le x \le 1, \sqrt{x} \le y \le \sqrt[3]{x}\right\}$$

or

$$\left\{(x,y) \mid 0 \le y \le 1, y^3 \le x \le y^2\right\}.$$

Therefore, the volume is

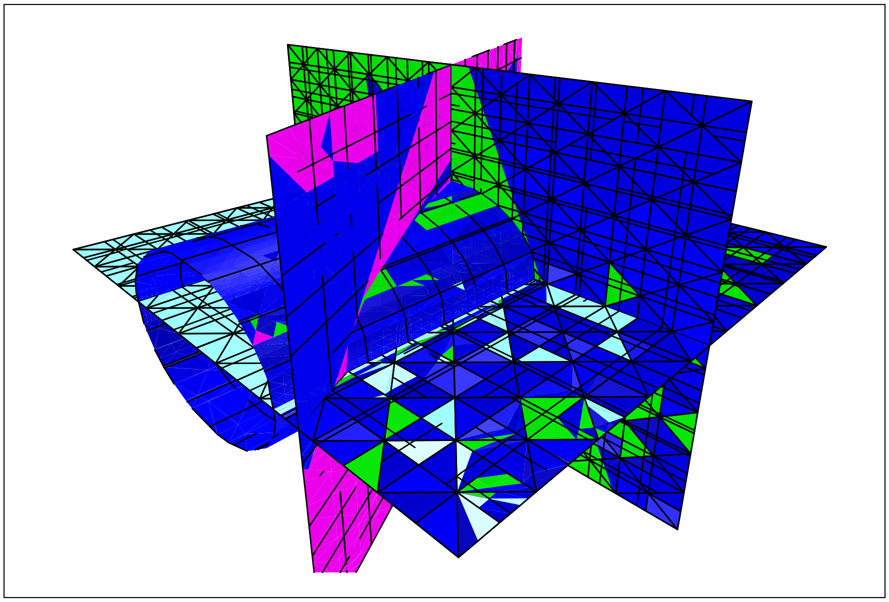
$$\begin{aligned}
& \int_0^1 \int_{y^3}^{y^2} \left(2x + y^2\right) dx dy \\
= & \int_0^1 \left[\int_{y^3}^{y^2} \left(2x + y^2\right) dx\right] dy \\
= & \int_0^1 \left[\left(x^2 + xy^2\right)\Big|_{x=y^3}^{x=y^2}\right] dx \\
= & \int_0^1 \left[\left(\left(y^2\right)^2 + \left(y^2\right) y^2\right) - \left(\left(y^3\right)^2 + \left(y^3\right) y^2\right)\right] dx \\
= & \int_0^1 \left(2y^4 - y^5 - y^6\right) dx = \left(2\frac{y^5}{5} - \frac{y^6}{6} - \frac{y^7}{7}\right)\Bigg|_{y=0}^{y=1} \\
= & \left(2\frac{(1)^5}{5} - \frac{(1)^6}{6} - \frac{(1)^7}{7}\right) - \left(2\frac{(0)^5}{5} - \frac{(0)^6}{6} - \frac{(0)^7}{7}\right) \\
= & \frac{19}{210}.
\end{aligned}$$

■

12.3 #24 Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

[Solution]

The solid is bounded by the cylinder $y^2 + z^2 = 4$(blue) and the planes $x = 2y$(magenta), $x = 0$(green), $z = 0$(cyan) in the first octant. The graph looks like



Since the solid is in the first octant, we have $x \geq 0$, $y \geq 0$, and $z \geq 0$. By $y^2 + z^2 = 4$, we know the maximum y is 2. So, we can say $0 \leq y \leq 2$. x is bounded by $x = 0$ and $x = 2y$. Thus, $0 \leq x \leq 2y$. Therefore, the base region D can be describe as

$$D = \{(x, y) \mid 0 \leq y \leq 2, 0 \leq x \leq 2y\}.$$

The solid is bounded below by $z = 0$ and above by $z = \sqrt{4 - y^2}$. Hence, the volume is

$$\begin{aligned}
\int_0^2 \int_0^{2y} \sqrt{4 - y^2} dx dy &= \int_0^2 \left[\int_0^{2y} \sqrt{4 - y^2} dx \right] dy \\
&= \int_0^2 \left[\left(x\sqrt{4 - y^2} \right) \Big|_{x=0}^{x=2y} \right] dy \\
&= \int_0^2 \left[\left((2y) \sqrt{4 - y^2} \right) - \left((0) \sqrt{4 - y^2} \right) \right] dy \\
&= \int_0^2 \left(2y\sqrt{4 - y^2} \right) dy.
\end{aligned}$$

Set $u = 4 - y^2$. We have $du = -2y dy$. So,

$$\begin{aligned}
\int_0^2 \left(2y\sqrt{4 - y^2} \right) dy &= \int_4^0 -\sqrt{u} du = \int_0^4 \sqrt{u} du \\
&= \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Big|_{u=0}^{u=4} = \left(\frac{2}{3} (4)^{\frac{3}{2}} \right) - \left(\frac{2}{3} (0)^{\frac{3}{2}} \right) \\
&= \frac{16}{3}.
\end{aligned}$$

Thus, the volume is $\frac{16}{3}$.

■

12.3 #36 Sketch the region of integration

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x, y) \, dx dy,$$

and change the order of integration.

[Solution]
The region can be read from the integral by

$$\left\{(x,y) \mid 0 \le y \le 3, 0 \le x \le \sqrt{9-y}\right\}.$$

From $x \le \sqrt{9-y}$, we have $x^2 \le 9-y$, a parabolic curve. So, our region is bounded by $y = 0$(red), $y = 3$(blue), $x = 0$(green), and $x = \sqrt{9-y}$(margenta). The region looks like
So, from the diagram, we know that $0 \le x \le 3$. But, there are two different descriptions for the top of the y. Thus, we divide $[0,3]$ into two parts. First, we need the intersection point of $x = \sqrt{9-y}$(or, $y = 9 - x^2$) and $y = 3$. It is $\left(\sqrt{6}, 3\right)$. Therefore,

$$D = \left\{(x,y) \mid 0 \le x \le \sqrt{6}, 0 \le y \le 3\right\} \cup \left\{(x,y) \mid \sqrt{6} \le x \le 3, 0 \le y \le 9 - x^2\right\}.$$

The integral becomes that

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\,dxdy = \int_0^{\sqrt{6}} \int_0^3 f(x,y)\,dydx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x,y)\,dydx.$$

■

12.3 #40 Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy$$

by reversing the order of integration.
[Solution]
The region can be read from the integral by

$$\{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\}.$$

From $\sqrt{y} \le x$, we have $y \le x^2$, a parabolic curve. So, our region is bounded by $y = 0$(red), $y = 1$(blue), $x = \sqrt{y}$(margenta), and $x = 1$(green). The region looks likeThus, we can describe the region as

$$\left\{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\right\}.$$

The integral becomes

$$\begin{aligned}
\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy &= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}dydx = \int_0^1 \left[\int_0^{x^2} \sqrt{x^3+1}dy\right] dx \\
&= \int_0^1 \left[\left(y\sqrt{x^3+1}\right)\Big|_{y=0}^{y=x^2}\right] dx \\
&= \int_0^1 \left[\left(\left(x^2\right)\sqrt{x^3+1}\right) - \left((0)\sqrt{x^3+1}\right)\right] dx \\
&= \int_0^1 \left(x^2\sqrt{x^3+1}\right) dx.
\end{aligned}$$

Let $u = x^3 + 1$. Then, $du = 3x^2dx$. So, we have

$$\begin{aligned}\int_0^1 \left(x^2\sqrt{x^3+1}\right)dx &= \int_1^2 \left(\sqrt{u}\right)\frac{du}{3} = \frac{1}{3}\left[\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{u=1}^{u=2}\right] \\ &= \frac{1}{3}\left[\left(\frac{2}{3}(2)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right)\right] \\ &= \frac{4\sqrt{2}-2}{9}.\end{aligned}$$

Hence,

$$\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy = \frac{4\sqrt{2}-2}{9}.$$

■

12.3 #48 Use Property 11 to estimate the value of the integral

$$\iint\limits_D e^{x^2+y^2}dA$$

where D is the disk with center the origin and radius $\frac{1}{2}$.

[Solution]

The Property 11 says if $m \le e^{x^2+y^2} \le M$ for all (x, y) in D, then

$$mA(D) \le \iint\limits_D e^{x^2+y^2}dA \le MA(D).$$

Since D is the disk with center the origin and radius $\frac{1}{2}$, we know that $A(D) = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

For all (x, y) in D, since D is the disk with center the origin and radius $\frac{1}{2}$, we have $x^2+y^2 \le \left(\frac{1}{2}\right)^2$. Also, it is obviously that $0 \le x^2+y^2$ in D. Thus, we get $0 \le x^2+y^2 \le \frac{1}{4}$. By applying the exponential function on both sides, we have $e^0 \le e^{x^2+y^2} \le e^{\frac{1}{4}}$, or, $1 \le e^{x^2+y^2} \le e^{-4}$.

By Property 11, we can estimate

$$1 \times \frac{\pi}{4} \le \iint\limits_D e^{x^2+y^2}dA \le e^{-4} \times \frac{\pi}{4},$$

or,

$$\frac{\pi}{4} \le \iint\limits_D e^{x^2+y^2}dA \le \frac{\pi}{4e^4}.$$

■