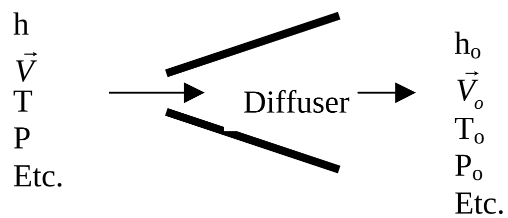
Chapter 16: Thermodynamics of High-Speed Gas Flow

Stagnation Properties

Consider a fluid flowing into a diffuser at a velocity $\vec{V}$, temperature *T*, pressure *P*, and enthalpy *h*, etc. Here the ordinary properties *T*, *P*, *h*, etc. are called the static properties; that is, they are measured relative to the flow at the flow velocity. The diffuser is sufficiently long and the exit area is sufficiently large that the fluid is brought to rest (zero velocity) at the diffuser exit while no work or heat transfer is done. The resulting state is called the stagnation state.



We apply the first law per unit mass for one entrance, one exit, and neglect the potential energies. Let the inlet state be unsubscripted and the exit or stagnation state have the subscript o.

$$q_{net} + h + \frac{\vec{V}^2}{2} = w_{net} + h_o + \frac{\vec{V}_o^2}{2}$$

Since the exit velocity, work, and heat transfer are zero,

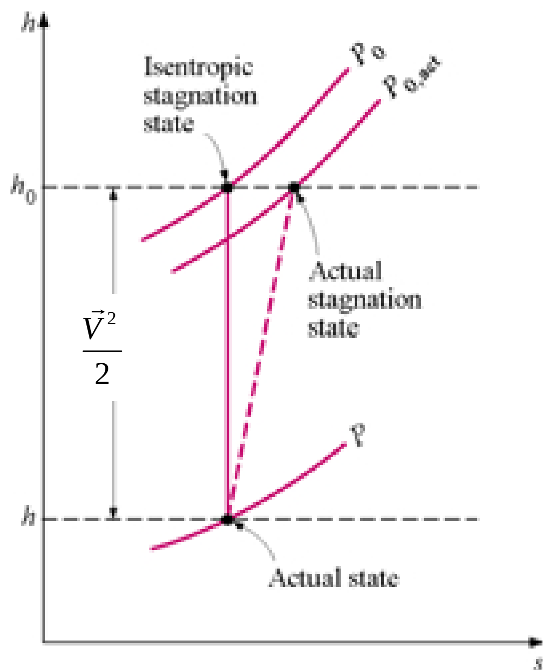
$$h_o = h + \frac{\vec{V}^2}{2}$$

The term h_o is called the stagnation enthalpy (some authors call this the total enthalpy). It is the enthalpy the fluid attains when brought to rest adiabatically while no work is done.

If, in addition, the process is also reversible, the process is isentropic, and the inlet and exit entropies are equal.

$$s_o = s$$

The stagnation enthalpy and entropy define the stagnation state and the isentropic stagnation pressure, P_o. The actual stagnation pressure for irreversible flows will be somewhat less than the isentropic stagnation pressure as shown below.



Example 16-1

Steam at 400°C, 1.0 MPa, and 300 m/s flows through a pipe. Find the properties of the steam at the stagnation state.

At T = 400°C and P = 1.0 Mpa,

h = 3263.9 kJ/kg $\qquad$ s = 7.4561 kJ/kg·K

Then

$$h_o = h + \frac{\vec{V}^2}{2}$$

$$= 3263.9\frac{kJ}{kg} + \frac{\left(300\frac{m}{s}\right)^2}{2}\frac{\frac{kJ}{kg}}{1000\frac{m^2}{s^2}}$$

$$= 3308.9\frac{kJ}{kg}$$

and

$$s_o \;=\; s \;=\; 7.4561\frac{kJ}{kg\cdot K}$$

$$h_o = h(P_o, s_o)$$

We can find P_o by trial and error (or try the EES solution for problem 2-26). The resulting stagnation properties are

$$P_o = 1.18\ MPa$$

$$T_o = 422.6^o C$$

$$\rho_o = \frac{1}{v_o} = 3.719\frac{kg}{m^3}$$

Ideal Gas Result

Rewrite the equation defining the stagnation enthalpy as

$$h_o - h = \frac{\vec{V}^2}{2}$$

For ideal gases with constant specific heats, the enthalpy difference becomes

$$C_P(T_o - T) = \frac{\vec{V}^2}{2}$$

where T_o is defined as the stagnation temperature.

$$T_o - T = \frac{\vec{V}^2}{2C_P}$$

For the isentropic process, the stagnation pressure can be determined from

$$\frac{T_o}{T} = \left(\frac{P_o}{P}\right)^{(k-1)/k}$$

or

$$\frac{P_o}{P} = \left(\frac{T_o}{T}\right)^{k/(k-1)}$$

Using variable specific heat data

$$\frac{P_o}{P} = \frac{P_o \,/\, P_{ref}}{P \,/\, P_{ref}} = \frac{P_{R@T_o}}{P_{R@T}}$$

Example 16-2

An aircraft flies in air at 5000 m with a velocity of 250 m/s. At 5000 m, air has a temperature of 255.7 K and a pressure of 54.05 kPa. Find T_o and P_o.

$$T_o = T + \frac{\vec{V}^2}{2C_P}$$

$$= 255.7\,K + \frac{\left(250\,\frac{m}{s}\right)^2}{2(1.005\,\frac{kJ}{kg\cdot K})}\,\frac{\frac{kJ}{kg}}{1000\,\frac{m^2}{s^2}}$$

$$= (255.7 + 31.1)\,K$$

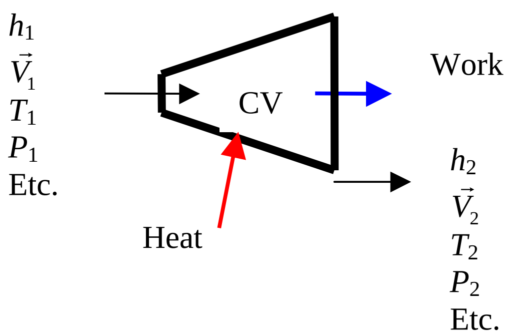
$$= 286.8\,K$$

$$P_o = P\left(\frac{T_o}{T}\right)^{k/(k-1)}$$

$$= 54.05\left(\frac{286.8\,K}{255.7\,K}\right)^{1.4/(1.4-1)}$$

$$= 80.77\,kPa$$

Conservation of Energy for Control Volumes Using Stagnation Properties



The steady-flow conservation of energy for the above figure is

$$\dot{Q}_{net} + \sum_{inlets} \dot{m}_i \left(h + \frac{\vec{V}^2}{2} + gz\right)_i = \dot{W}_{net} + \sum_{outlets} \dot{m}_e \left(h + \frac{\vec{V}^2}{2} + gz\right)_e$$

Since

$$h_o = h + \frac{\vec{V}^2}{2}$$

$$\dot{Q}_{net} + \sum_{inlets} \dot{m}_i \left(h_o + gz\right)_i = \dot{W}_{net} + \sum_{outlets} \dot{m}_e \left(h_o + gz\right)_e$$

For no heat transfer, one entrance, one exit, this reduces to

$$\dot{W}_{net} = \dot{m}\big((h_{o1} - h_{o2}) + g(z_1 - z_2)\big)$$

If we neglect the change in potential energy, this becomes

$$\dot{W}_{net} = \dot{m}\left(h_{o1} - h_{o2}\right)$$

For ideal gases we write this as

$$\dot{W}_{net} = \dot{m}C_P\left(T_{o1} - T_{o2}\right)$$

Conservation of Energy for a Nozzle

We assume steady-flow, no heat transfer, no work, one entrance, and one exit and neglect elevation changes; then the conservation of energy becomes

$$\dot{m}_1 h_{o1} = \dot{m}_2 h_{o2}$$

But

$$\dot{m}_1 = \dot{m}_2$$

Then

$$h_{o1} = h_{o2}$$

Thus the stagnation enthalpy remains constant throughout the nozzle. At any cross section in the nozzle, the stagnation enthalpy is the same as that at the entrance.

For ideal gases this last result becomes

$$T_{o1} = T_{o2}$$

Thus the stagnation temperature remains constant through out the nozzle. At any cross section in the nozzle, the stagnation temperature is the same as that at the entrance.

Assuming an isentropic process for flow through the nozzle, we can write for the entrance and exit states

$$\frac{P_{o2}}{P_{o1}} = \left(\frac{T_{o2}}{T_{o1}}\right)^{k/(k-1)}$$

So we see that the stagnation pressure is also constant through out the nozzle for isentropic flow.

Velocity of Sound and Mach Number

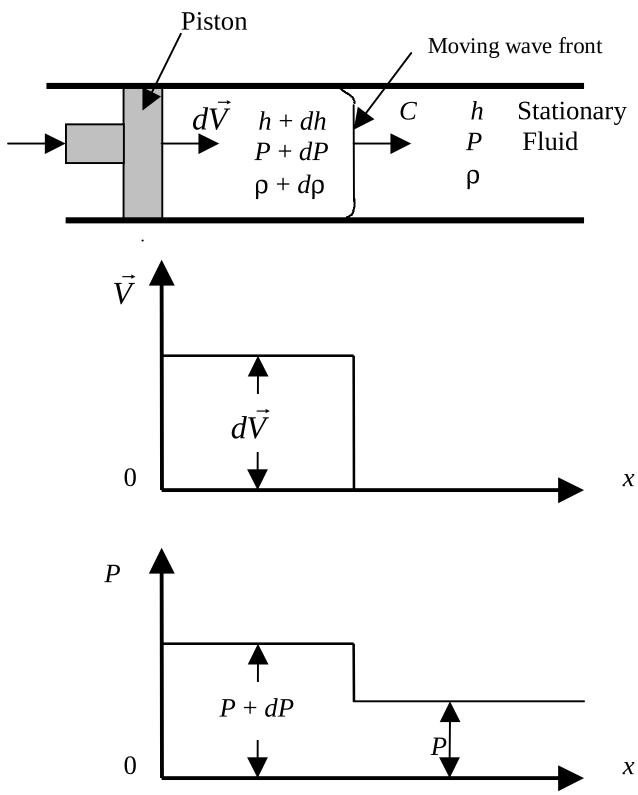
We want to show that the stagnation properties are related to the Mach number *M* of the flow where

$$M = \frac{\vec{V}}{C}$$

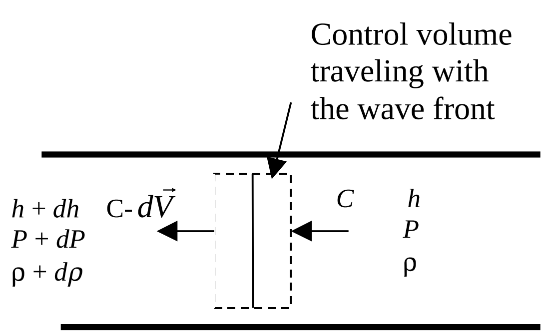
and *C* is the speed of sound in the fluid. But first we need to define the speed of sound in the fluid.

A pressure disturbance propagates through a compressible fluid with a velocity dependent upon the state of the fluid. The velocity with which this pressure wave moves through the fluid is called the velocity of sound, or the sonic velocity.

Consider a small pressure wave caused by a small piston displacement in a tube filled with an ideal gas as shown below.



It is easier to work with a control volume moving with the wave front as shown below.



Apply the conservation of energy for steady-flow with no heat transfer, no work, and neglect the potential energies.

$$h+\frac{C^2}{2}=(h+dh)+\frac{(C-d\vec{V})^2}{2}$$

$$h+\frac{C^2}{2}=(h+dh)+\frac{(C^2-2Cd\vec{V}+d\vec{V}^2)}{2}$$

Cancel terms and neglect $d\vec{V}^2$; we have

$$dh-Cd\vec{V}=0$$

Now, apply the conservation of mass or continuity equation $\dot{m}=\rho A\vec{V}$ to the control volume.

$$\rho AC=(\rho+d\rho)A(C-d\vec{V})$$

$$\rho AC=A(\rho C-\rho d\vec{V}+Cd\rho-d\rho\, d\vec{V})$$

Cancel terms and neglect the higher-order terms like $d\rho\, d\vec{V}$. We have

$$C\,d\rho-\rho\, d\vec{V}=0$$

Also, we consider the property relation

$$dh=T\,ds+v\,dP$$

$$dh=T\,ds+\frac{1}{\rho}dP$$

Let's assume the process to be isentropic; then *ds* = 0 and

$$dh = \frac{1}{\rho} dP$$

Using the results of the first law

$$dh = \frac{1}{\rho} dP = C\, d\vec{V}$$

From the continuity equation

$$d\vec{V} = \frac{C\, d\rho}{\rho}$$

Now

$$\frac{1}{\rho} dP = C\left(\frac{C\, d\rho}{\rho}\right)$$

Thus

$$\frac{dP}{d\rho} = C^2$$

Since the process is assumed to be isentropic, the above becomes

$$\left(\frac{\partial P}{\partial \rho}\right)_s = C^2$$

For a general thermodynamic substance, the results of Chapter 11 may be used to show that the speed of sound is determined from

$$C^2 = k\left(\frac{\partial P}{\partial \rho}\right)_T$$

where k is the ratio of specific heats, $k = C_P/C_V$.

Ideal Gas Result

For ideal gases

$$P = \rho RT$$

$$\left(\frac{\partial P}{\partial \rho}\right)_T = RT$$

$$C^2 = kRT$$

$$C = \sqrt{kRT}$$

Example 16-3

Find the speed of sound in air at an altitude of 5000 m.

At 5000 m, T = 255.7 K.

$$C = \sqrt{1.4(0.287\frac{kJ}{kg\cdot K})(255.7K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 320.5\frac{m}{s}$$

Notice that the temperature used for the speed of sound is the static (normal) temperature.

Example 16-4

Find the speed of sound in steam where the pressure is 1 Mpa and the temperature is 350°C.

At P = 1 MPa, T = 350°C,

$$C = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$$
$$= \sqrt{\left(\frac{\partial P}{\partial\left(\frac{1}{v}\right)}\right)_s}$$

Here, we approximate the partial derivative by perturbating the pressure about 1 MPa. Consider using $P \pm 0.025$ MPa at the entropy value s = 7.3011 kJ/kg·K, to find the corresponding specific volumes.

$$C = \sqrt{\frac{(1025 - 975)kPa}{\left(\frac{1}{10.2773} - \frac{1}{10.2882}\right)\frac{kg}{m^3}} \frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}} \frac{kJ}{m^3 kPa}}$$
$$= 605.5\frac{m}{s}$$

What is the speed of sound for steam at 350°C assuming ideal-gas behavior?

Assume k = 1.3, then

$$C = \sqrt{1.3(0.4615\frac{kJ}{kg \cdot K})(350+273)K\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 611.4\frac{m}{s}$$

Mach Number

The Mach number *M* is defined as

$$M = \frac{\vec{V}}{C}$$

M <1 flow is subsonic
M =1 flow is sonic
M >1 flow is supersonic

Example 16-5

In the air and steam examples above, find the Mach number if the air velocity is 250 m/s and the steam velocity is 300 m/s.

$$M_{air} = \frac{250\frac{m}{s}}{320.5\frac{m}{s}} = 0.780$$

$$M_{steam} = \frac{300\frac{m}{s}}{605.5\frac{m}{s}} = 0.495$$

The flow parameters T_o/T, P_o/P, ρ_o/ρ, etc. are related to the flow Mach number. Let's consider ideal gases, then

$$T_o = T + \frac{\vec{V}^2}{2C_P}$$

$$\frac{T_o}{T} = 1 + \frac{\vec{V}^2}{2C_P T}$$

but

$$C_P = \frac{k}{k-1} R \quad or \quad \frac{1}{C_P} = \frac{k-1}{kR}$$

$$\frac{T_o}{T} = 1 + \frac{\vec{V}^2}{2T}\frac{(k-1)}{kR}$$

and

$$C^2 = kRT$$

so

$$\frac{T_o}{T} = 1 + \frac{(k-1)}{2}\frac{\vec{V}^2}{C^2}$$

$$= 1 + \frac{(k-1)}{2} M^2$$

The pressure ratio is given by

$$\frac{P_o}{P} = \left(\frac{T_o}{T}\right)^{k/(k-1)}$$

$$= \left(1 + \frac{(k-1)}{2} M^2\right)^{k/(k-1)}$$

We can show the density ratio to be

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{1/(k-1)}$$

$$= \left(1 + \frac{(k-1)}{2} M^2\right)^{1/(k-1)}$$

See Table A-15 for the inverse of these values (P/P_o, T/T_o, and ρ/ρ_o) when k = 1.4.

For the Mach number equal to 1, the sonic location, the static properties are denoted with a superscript "*". This condition, when M = 1, is called the sonic condition. When M = 1 and k = 1.4, the static-to-stagnation ratios are

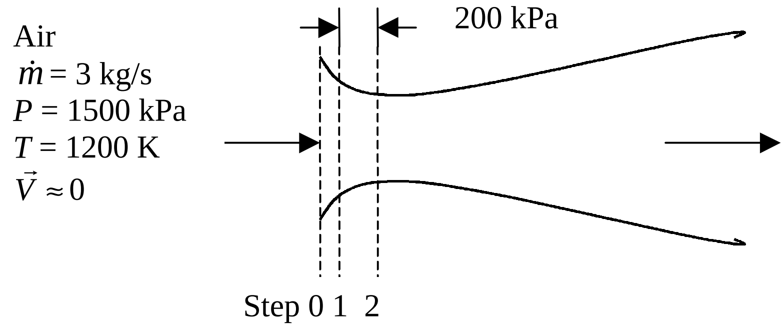
$$\frac{T^*}{T_o} = \frac{2}{k+1} = 0.83333$$

$$\frac{P^*}{P_o} = \left(\frac{2}{k+1}\right)^{k/(k-1)} = 0.52828$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{k+1}\right)^{1/(k-1)} = 0.63394$$

Effect of Area Changes on Flow Parameters

Consider the isentropic steady flow of an ideal gas through the nozzle shown below.



Air flows steadily through a varying-cross-sectional-area duct such as a nozzle at a flow rate of 3 kg/s. The air enters the duct at a low velocity at a pressure of 1500 kPa and a temperature of 1200 K and it expands in the duct to a pressure of 100 kPa. The duct is designed so that the flow process is isentropic. Determine the pressure, temperature, velocity, flow area, speed of sound, and Mach number at each point along the duct axis that corresponds to a pressure drop of 200 kPa.

Since the inlet velocity is low, the stagnation properties equal the static properties.

$$T_o = T_1 = 1200\,K, \quad P_o = P_1 = 1500\,kPa$$

After the first 200 kPa pressure drop, we have

$$T = T_o\left(\frac{P}{P_o}\right)^{(k-1)/k} = 1200K\left(\frac{1300\,kPa}{1500\,kPa}\right)^{(1.4-1)/1.4}$$
$$= 1151.9\,K$$

$$\vec{V} = \sqrt{2C_P(T_0 - T)}$$

$$= \sqrt{2(1.005\frac{kJ}{kg \cdot K})(1200 - 1151.9)K\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 310.77\frac{m}{s}$$

$$\rho = \frac{P}{RT} = \frac{(1300kPa)}{(0.287\frac{kJ}{kg \cdot K})(1151.9K)}\frac{kJ}{m^3 kPa}$$

$$= 3.932\frac{kg}{m^3}$$

$$A = \frac{\dot{m}}{\rho\vec{V}} = \frac{3\frac{kg}{s}}{(3.9322\frac{kg}{m^3})(310.77\frac{m}{s})}\frac{10^4 cm^2}{m^2}$$

$$= 24.55cm^2$$

$$C = \sqrt{kRT} = \sqrt{1.4(0.287\frac{kJ}{kg \cdot K})(1151.9K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 680.33\frac{m}{s}$$

$$M = \frac{\vec{V}}{C} = \frac{310.77\,\frac{m}{s}}{680.33\,\frac{m}{s}} = 0.457$$

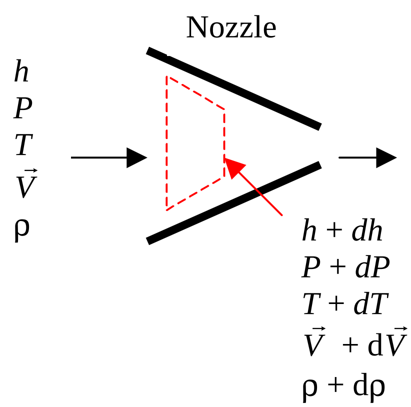
Now we tabulate the results for the other 200 kPa increments in the pressure.

Summary of Results for Nozzle Problem

Step	P kPa	T K	$\vec{V}$ m/s	ρ kg/m^3	C m/s	A cm^2	M
0	1500	1200	0	4.3554	694.38	∞	0
1	1300	1151.9	310.77	3.9322	680.33	24.55	0.457
2	1100	1098.2	452.15	3.4899	664.28	19.01	0.681
3	900	1037.0	572.18	3.0239	645.51	17.34	0.886
4	792.4	1000.0	633.88	2.7611	633.88	17.14	1.000
5	700	965.2	786.83	2.5270	622.75	17.28	1.103
6	500	876.7	805.90	1.9871	593.52	18.73	1.358
7	300	757.7	942.69	1.3796	551.75	23.07	1.709
8	100	553.6	1139.62	0.6294	471.61	41.82	2.416

Note that at P = 797.42 kPa, M = 1.000, and this state is the critical state.

Now let's see why these relations work this way. Consider the nozzle and control volume shown below.



The first law for the control volume is

$$dh + \vec{V}d\vec{V} = 0$$

The continuity equation for the control volume $\dot{m} = \rho A\vec{V}$ yields

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{d\vec{V}}{\vec{V}} = 0$$

Also, we consider the property relation for an isentropic process

$$Tds = dh - \frac{dP}{\rho} = 0$$

and the Mach Number relation

$$\frac{dP}{d\rho} = C^2 = \frac{\vec{V}^2}{M^2}$$

Putting these four relations together yields

$$\frac{dA}{A} = \frac{dP}{\rho\vec{V}^2}(1 - M^2)$$

A **nozzle** is a device that increases fluid velocity while causing its pressure to drop; thus, $d\vec{V} > 0$, $dP < 0$.

Nozzle Results

$$\frac{dA}{A} = \frac{dP}{\rho \vec{V}^2}(1 - M^2)$$

$$\text{Subsonic: } M < 1 \quad dP(1 - M^2) < 0 \quad dA < 0$$

$$\text{Sonic: } M = 1 \quad dP(1 - M^2) = 0 \quad dA = 0$$

$$\text{Supersonic: } M > 1 \quad dP(1 - M^2) > 0 \quad dA > 0$$

To accelerate subsonic flow, the nozzle flow area must first decrease in the flow direction. The flow area reaches a minimum at the point where the Mach number is unity. To continue to accelerate the flow to supersonic conditions, the flow area must increase.

Note that the throat of a nozzle is the minimum flow area.

A **diffuser** is a device that decreases fluid velocity while causing its pressure to rise; thus, $d\vec{V} < 0$, $dP > 0$.

Diffuser Results

$$\frac{dA}{A} = \frac{dP}{\rho \vec{V}^2}(1 - M^2)$$

$$\text{Subsonic: } M < 1 \quad dP(1 - M^2) > 0 \quad dA > 0$$

$$\text{Sonic: } M = 1 \quad dP(1 - M^2) = 0 \quad dA = 0$$

$$\text{Supersonic: } M > 1 \quad dP(1 - M^2) < 0 \quad dA < 0$$

To diffuse supersonic flow, the diffuser flow area must first decrease in the flow direction. The flow area reaches a minimum at the point where the Mach number is unity. To continue to diffuse the flow to subsonic conditions, the flow area must increase.

Equation of Mass Flow Rate through a Nozzle

Let's obtain an expression for the flow rate through a converging nozzle at any location as a function of the pressure at that location. The mass flow rate is given by

$$\dot{m} = \rho A \vec{V}$$

The velocity of the flow is related to the static and stagnation enthalpies.

$$\vec{V} = \sqrt{2(h_o - h)} = \sqrt{2C_P(T_0 - T)} = \sqrt{2C_P T_0 (1 - \frac{T}{T_o})}$$

and

$$\frac{T}{T_o} = \left(\frac{P}{P_o}\right)^{(k-1)/k}$$

$$\vec{V} = \sqrt{2C_P T_0 \left(1 - \left(\frac{P}{P_o}\right)^{(k-1)/k}\right)}$$

Write the mass flow rate as

$$\dot{m} = A\vec{V}\rho_o \frac{\rho}{\rho_o}$$

$$\frac{\rho}{\rho_o} = \left(\frac{P}{P_o}\right)^{1/k}$$

We note from the ideal-gas relations that

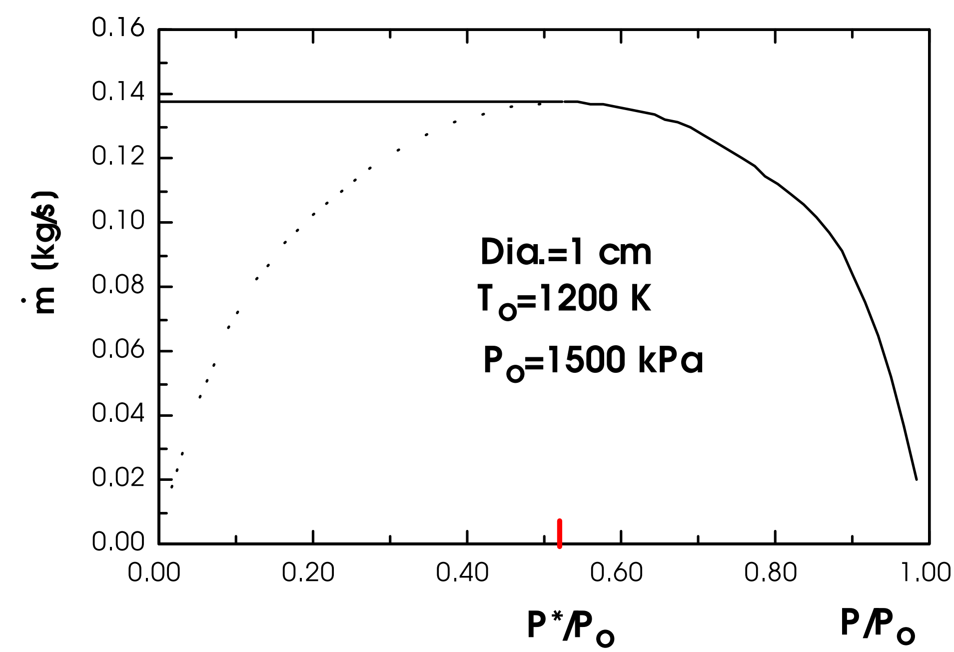
$$\rho_o = \frac{P_o}{RT_o}$$

$$\dot{m} = AP_o\sqrt{\frac{2k}{(k-1)RT_o}}\sqrt{\left(\frac{P}{P_o}\right)^{2/k} - \left(\frac{P}{P_o}\right)^{(k+1)/k}}$$

What pressure ratios make the mass flow rate zero?

Do these values make sense?

Now let's make a plot of mass flow rate versus the static-to-stagnation pressure ratio.



This plot shows there is a value of P/P_o that makes the mass flow rate a maximum. To find that mass flow rate, we note

$$\frac{d\dot{m}}{d\left(\dfrac{P}{P_o}\right)} = 0$$

The result is

$$\frac{P}{P_o} = \left(\frac{2}{k+1}\right)^{k/(k-1)} = \frac{P^*}{P_o}$$

So the pressure ratio that makes the mass flow rate a maximum is the same pressure ratio at which the Mach number is unity at the flow cross-sectional area. This value of the pressure ratio is called the critical pressure ratio for nozzle flow. For pressure ratios less than the critical value, the nozzle is said to be choked. When the nozzle is choked, the mass flow rate is the maximum possible for the flow area, stagnation pressure, and stagnation temperature. Reducing the pressure ratio below the critical value will not increase the mass flow rate.

What is the expression for mass flow rate when the nozzle is choked?

Using

$$\frac{P_o}{P} = \left(1 + \frac{k-1}{2}M^2\right)^{k/(k-1)}$$

The mass flow rate becomes

$$\dot{m} = AP_o\sqrt{\frac{k}{RT_o}}\left[\frac{M}{\left(1+\frac{k-1}{2}M^2\right)^{(k+1)/[2(k-1)]}}\right]$$

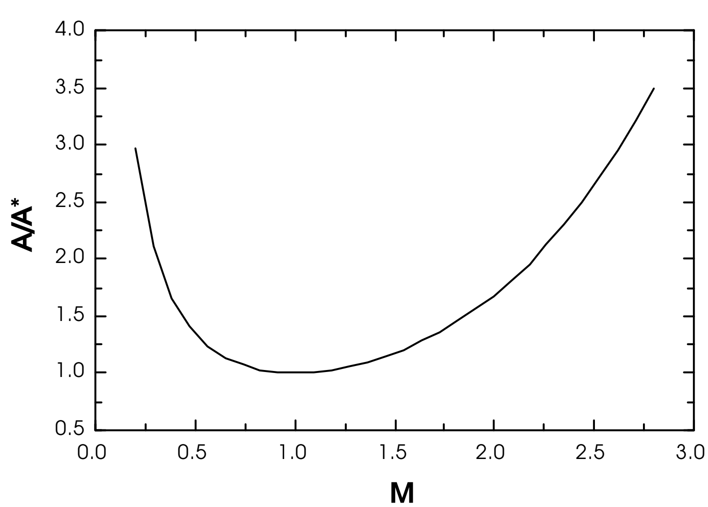
When the Mach number is unity, $M = 1$, $A = A^*$

$$\dot{m} = A^* P_o\sqrt{\frac{k}{RT_o}}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

Taking the ratio of the last two results gives the ratio of the area of the flow A at a given Mach number to the area where the Mach number is unity, A^*.

Then

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}M^2\right)\right]^{(k+1)/[2(k-1)]}$$

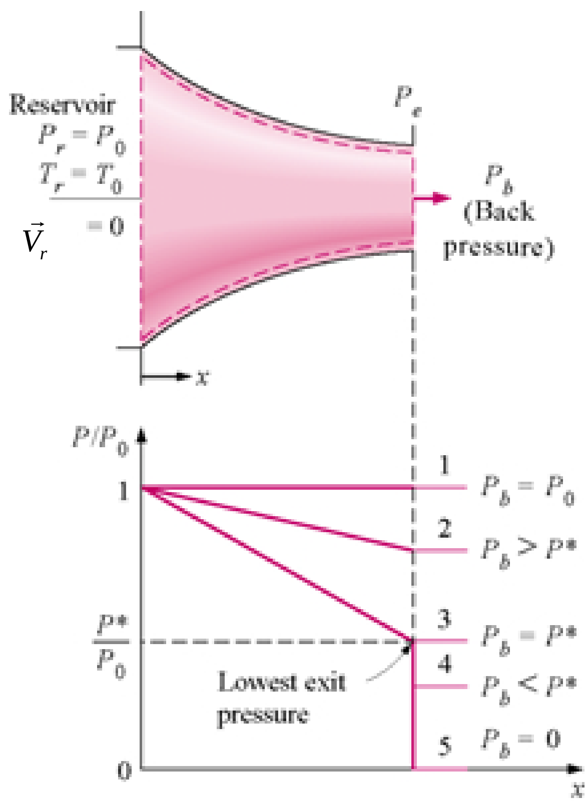


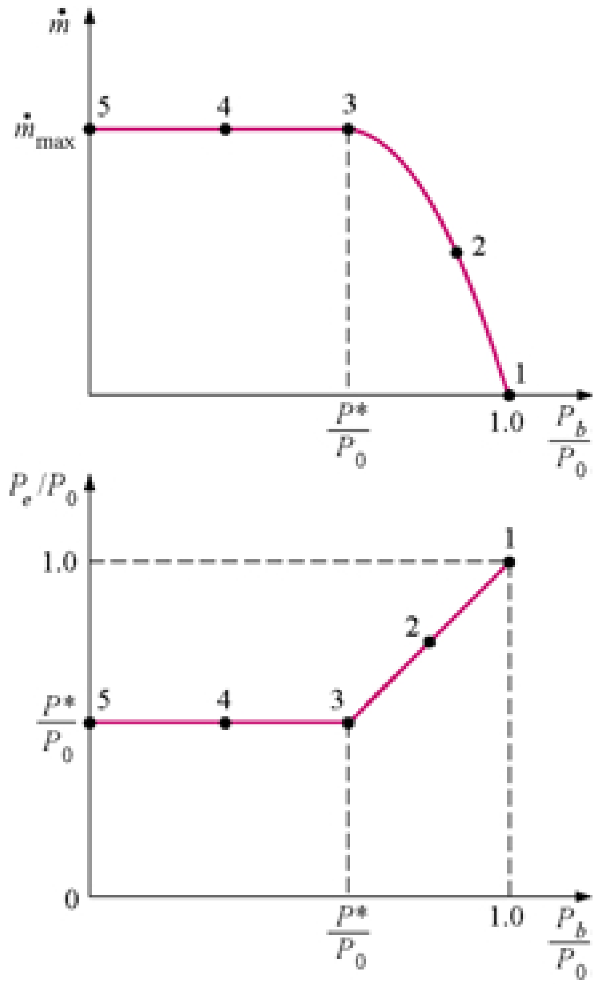
From the above plot we note that for each A/A^* there are two values of M: one for subsonic flow at that area ratio and one for supersonic flow at that area ratio. The area ratio is unity when the Mach number is equal to one.

Effect of Back Pressure on Flow through a Converging Nozzle

Consider the converging nozzle shown below. The flow is supplied by a reservoir at pressure P_r and temperature T_r. The reservoir is large enough that the velocity in the reservoir is zero.

Let's plot the ratio P/P_o along the length of the nozzle, the mass flow rate through the nozzle, and the exit plane pressure P_e as the back pressure P_b is varied. Let's consider isentropic flow so that P_o is constant throughout the nozzle.





1. $P_b = P_o$, $P_b/P_o = 1$. No flow occurs. $P_e = P_b$, $M_e=0$.

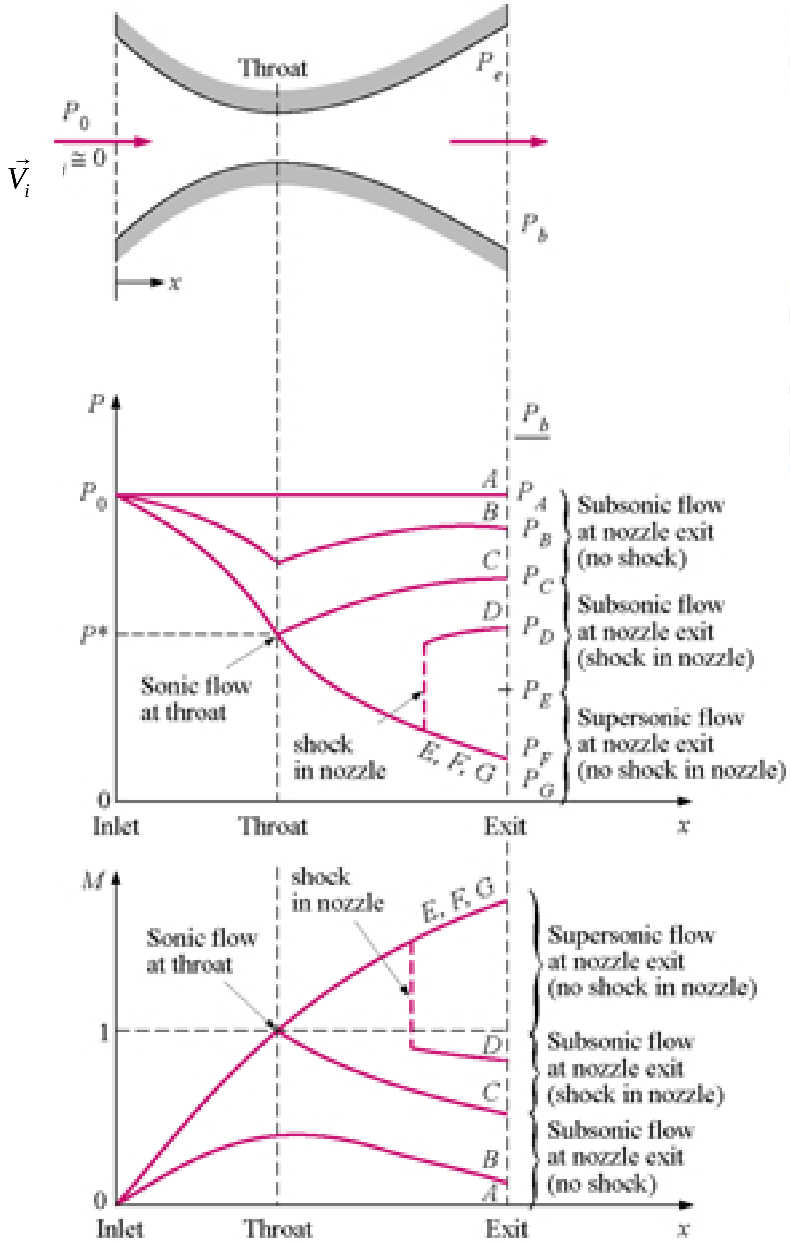
2. $P_b > P^*$ or $P^*/P_o < P_b/P_o < 1$. Flow begins to increase as the back pressure is lowered. $P_e = P_b$, $M_e < 1$.

3. $P_b = P^*$ or $P^*/P_o = P_b/P_o < 1$. Flow increases to the choked flow limit as the back pressure is lowered to the critical pressure. $P_e = P_b$, $M_e=1$.

4. $P_b < P^*$ or $P_b/P_o < P^*/P_o < 1$. Flow is still choked and does not increase as the back pressure is lowered below the critical pressure, pressure drop from P_e to P_b occurs outside the nozzle. $P_e = P^*$, $M_e=1$.

5. $P_b = 0$. Results are the same as for item 4.

Consider the converging-diverging nozzle shown below.



Let's plot the ratio P/P_o and the Mach number along the length of the nozzle as the back pressure P_b is varied. Let's consider isentropic flow so that P_o is constant throughout the nozzle.

- $P_A = P_o$, or $P_A/P_o = 1$. No flow occurs. $P_e = P_b$, $M_e = 0$.

- $P_o > P_B > P_C > P^*$ or $P^*/P_o < P_C/P_o < P_B/P_o < 1$. Flow begins to increase as the back pressure is lowered. The velocity increases in the converging section but $M < 1$ at the throat; thus, the diverging section acts as a diffuser with the velocity decreasing and pressure increasing. The flow remains subsonic through the nozzle. $P_e = P_b$ and $M_e < 1$.

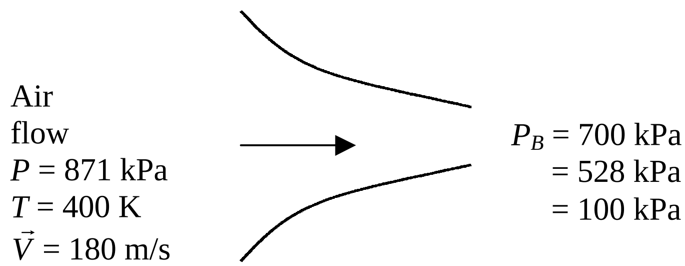
- $P_b = P_C = P^*$ or $P^*/P_o = P_b/P_o = P_C/P_o$ and P_b is adjusted so that M=1 at the throat. Flow increases to its maximum value at choked conditions; velocity increases to the speed of sound at the throat, but the converging section acts as a diffuser with velocity decreasing and pressure increasing. $P_e = P_b$, $M_e < 1$.

- $P_C > P_b > P_E$ or $P_E/P_o < P_b/P_o < P_C/P_o < 1$. The fluid that achieved sonic velocity at the throat continues to accelerate to supersonic velocities in the diverging section as the pressure drops. This acceleration comes to a sudden stop, however, as a normal shock develops at a section between the throat and the exit plane. The flow across the shock is highly irreversible. The normal shock moves downstream away from the throat as P_b is decreased and approaches the nozzle exit plane as P_b approaches P_E. When $P_b = P_E$, the normal shock forms at the exit plane of the nozzle. The flow is supersonic through the entire diverging section in this case, and it can be approximated as isentropic. However, the fluid velocity drops to subsonic levels just before leaving the nozzle as it crosses the normal shock.

- $P_E > P_b > 0$ or $0 < P_b/P_o < P_E/P_o < 1$. The flow in the diverging section is supersonic, and the fluids expand to P_F at the nozzle exit with no normal shock forming within the nozzle. Thus the flow through the nozzle can be approximated as isentropic. When $P_b = P_F$, no shocks occur within or outside the nozzle. When $P_b < P_F$, irreversible mixing and expansion waves occur downstream of the exit plane or the nozzle. When $P_b > P_F$, however, the pressure of the fluid increases from P_F to P_b irreversibly in the wake or the nozzle exit, creating what are called oblique shocks.

Example 16-6

Air leaves the turbine of a turbojet engine and enters a convergent nozzle at 400 K, 871 kPa, with a velocity of 180 m/s. The nozzle has an exit area of 730 cm^2. Determine the mass flow rate through the nozzle for back pressures of 700 kPa, 528 kPa, and 100 kPa, assuming isentropic flow.



The stagnation temperature and stagnation pressure are

$$T_o = T + \frac{\vec{V}^2}{2C_P}$$

$$T_o = 400\,K + \frac{(180 m/s)^2}{2\left(1.005\dfrac{kJ}{kg\cdot K}\right)} \frac{\dfrac{kJ}{kg}}{1000\dfrac{m^2}{s^2}}$$

$$= (400 + 16.1)\,K = 416.1\,K$$

$$P_o = P\left(\frac{T_o}{T}\right)^{\frac{k}{k-1}} = 871\,kPa\left(\frac{416.1\,K}{400\,K}\right)^{\frac{1.4}{1.4-1}}$$

$$= 1000\,kPa$$

For air k = 1.4 and Table A-15 applies. The critical pressure ratio is $P^*/P_o = 0.528$.

The critical pressure for this nozzle is

$$
\begin{aligned}
P^* &= 0.528 P_o \\
&= 0.528(1000\, kPa) = 528\, kPa
\end{aligned}
$$

Therefore, for a back pressure of 528 kPa, M = 1 at the nozzle exit and the flow is choked. For a back pressure of 700 kPa, the nozzle is not choked. The flow rate will not increase for back pressures below 528 kPa.

For the back pressure of 700 kPa,

$$
\frac{P_B}{P_o} = \frac{700\, kPa}{1000\, kPa} = 0.700 > \frac{P^*}{P_o}
$$

Thus, $P_E = P_B$ = 700 kPa. For this pressure ratio Table A-15 gives

$$
\begin{aligned}
M_E &= 0.7324 \\
\frac{T_E}{T_o} &= 0.9031 \\
T_E &= 0.9031 T_o = 0.9031(416.1\, K) = 375.8\, K
\end{aligned}
$$

$$C_E = \sqrt{kRT_E}$$

$$= \sqrt{1.4(0.287\frac{kJ}{kg \cdot K})(375.8K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 388.6\frac{m}{s}$$

$$\vec{V}_E = M_E C_E = (0.7324)(388.6\frac{m}{s})$$

$$= 284.6\frac{m}{s}$$

$$\rho_E = \frac{P_E}{RT_E} = \frac{(700kPa)}{(0.287\frac{kJ}{kg \cdot K})(375.8K)}\frac{kJ}{m^3 kPa}$$

$$= 6.4902\frac{kg}{m^3}$$

Then

$$\dot{m} = \rho_E A_E \vec{V}_E$$

$$= 6.4902\frac{kg}{m^3}(730\,cm^2)(284.6\frac{m}{s})\frac{m^2}{(100\,cm)^2}$$

$$= 134.8\frac{kg}{s}$$

For the back pressure of 528 kPa,

$$\frac{P_E}{P_o} = \frac{528\,kPa}{1000\,kPa} = 0.528 = \frac{P^*}{P_o}$$

This is the critical pressure ratio and $M_E = 1$ and $P_E = P_B = P^* = 528$ kPa.

$$\frac{T_E}{T_o} = \frac{T^*}{T_o} = 0.8333$$

$$T_E = 0.8333\,T_o = 0.8333(416.1\,K) = 346.7\,K$$

And since $M_E = 1$,

$$\vec{V}_E = C_E = \sqrt{kRT_E}$$

$$= \sqrt{1.4(0.287\frac{kJ}{kg \cdot K})(346.7\,K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 373.2\frac{m}{s}$$

$$\rho_E = \rho^* = \frac{P^*}{RT^*} = \frac{(528kPa)}{(0.287\frac{kJ}{kg \cdot K})(346.7\,K)}\frac{kJ}{m^3 kPa}$$

$$= 5.3064\frac{kg}{m^3}$$

$$\dot{m} = \rho_E A_E \vec{V}_E$$

$$= 5.3064 \frac{kg}{m^3}(730\,cm^2)(373.2\frac{m}{s})\frac{m^2}{(100\,cm)^2}$$

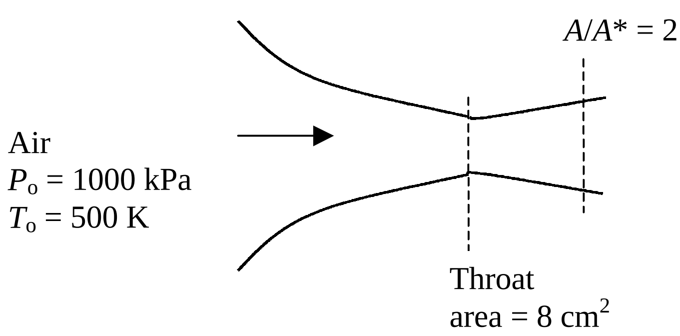
$$= 144.6 \frac{kg}{s}$$

For a back pressure less than the critical pressure, 528 kPa in this case, the nozzle is choked and the mass flow rate will be the same as that for the critical pressure. Therefore, at a back pressure of 100 kPa the mass flow rate will be 144.6 kg/s.

Example 16-7

A converging-diverging nozzle has an exit-area-to-throat area ratio of 2. Air enters this nozzle with a stagnation pressure of 1000 kPa and a stagnation temperature of 500 K. The throat area is 8 cm^2. Determine the mass flow rate, exit pressure, exit temperature, exit *M*ach number, and exit velocity for the following conditions:

- Sonic velocity at the throat, diverging section acting as a nozzle.
- Sonic velocity at the throat, diverging section acting as a diffuser.



For $A/A^* = 2$, Table A-15 yields two *M*ach numbers, one > 1 and one < 1.

When the diverging section acts as a supersonic nozzle, we use the value for $M > 1$. Then, for $A_E/A^* = 2.0$, $M_E = 2.197$, $P_E/P_o = 0.0939$, and $T_E/T_o = 0.5089$,

$$P_E = 0.0939 P_o = 0.0939(1000\, kPa) = 93.9\, kPa$$
$$T_E = 0.8333 T_o = 0.5089(500\, K) = 254.5\, K$$

$$C_E = \sqrt{kRT_E}$$
$$= \sqrt{1.4(0.287\frac{kJ}{kg \cdot K})(254.5K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$
$$= 319.7\frac{m}{s}$$

$$\vec{V}_E = M_E C_E = 2.197(319.7\frac{m}{s}) = 702.5\frac{m}{s}$$

The mass flow rate can be calculated at any known cross-sectional area where the properties are known. It normally is best to use the throat conditions. Since the flow has sonic conditions at the throat, $M_t = 1$, and

$$\frac{T_t}{T_o} = \frac{T^*}{T_o} = 0.8333$$
$$T_t = 0.8333 T_o = 0.8333(500\, K) = 416.6\, K$$

$$\vec{V}_t = C_t = \sqrt{kRT_t}$$

$$= \sqrt{1.4(0.287\frac{kJ}{kg \cdot K})(416.6K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$

$$= 409.2\frac{m}{s}$$

$$\frac{P_t}{P_o} = \frac{P^*}{P_o} = 0.528$$

$$P_t = 0.528\,P_o = 0.528(1000\,kPa) = 528\,kPa$$

$$\rho_t = \rho^* = \frac{P^*}{RT^*} = \frac{(528kPa)}{(0.287\frac{kJ}{kg \cdot K})(416.6K)}\frac{kJ}{m^3kPa}$$

$$= 4.416\frac{kg}{m^3}$$

$$\dot{m} = \rho_t A_t \vec{V}_t$$

$$= 4.416\frac{kg}{m^3}(8\,cm^2)(409.2\frac{m}{s})\frac{m^2}{(100\,cm)^2}$$

$$= 1.446\frac{kg}{s}$$

When the diverging section acts as a diffuser, we use $M < 1$. Then, for $A_E/A^* = 2.0$, $M_E = 0.308$, $P_E/P_o = 0.936$, and $T_E/T_o = 0.9812$,

$$P_E = 0.0939\,P_o = 0.936(1000\,kPa) = 936\,kPa$$
$$T_E = 0.8333\,T_o = 0.9812(500\,K) = 490.6\,K$$

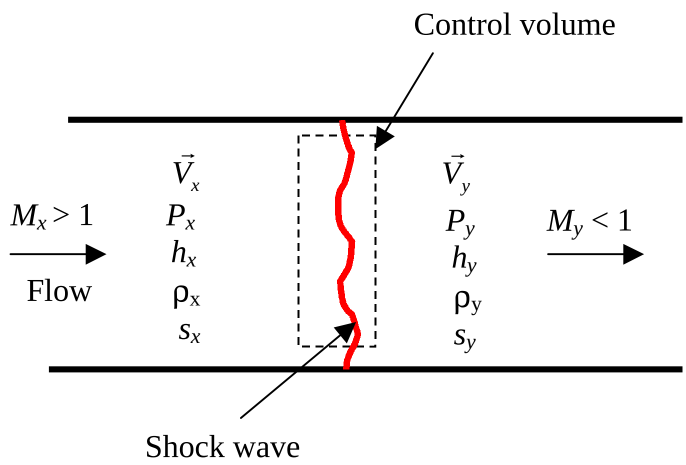
$$C_E = \sqrt{kRT_E}$$
$$= \sqrt{1.4(0.287\frac{kJ}{kg\cdot K})(490.6K)\frac{1000\frac{m^2}{s^2}}{\frac{kJ}{kg}}}$$
$$= 444.0\frac{m}{s}$$

$$\vec{V}_E = M_E C_E = 0.308(444.0\frac{m}{s}) = 136.7\frac{m}{s}$$

Since $M = 1$ at the throat, the mass flow rate is the same as that in the first part because the nozzle is choked.

Normal Shocks

In some range of back pressure, the fluid that achieved a sonic velocity at the throat of a converging-diverging nozzle and is accelerating to supersonic velocities in the diverging section experiences a *normal shock*. The normal shock causes a sudden rise in pressure and temperature and a sudden drop in velocity to subsonic levels. Flow through the shock is highly irreversible, and thus it cannot be approximated as isentropic. The properties of an ideal gas with constant specific heats before (subscript x) and after (subscript y) a shock are related by



We assume steady-flow with no heat and work interactions and no potential energy changes. We have the following

Conservation of mass

$$\rho_x A\vec{V}_x = \rho_y A\vec{V}_y$$
$$\rho_x \vec{V}_x = \rho_y \vec{V}_y$$

Conservation of energy

$$h_x + \frac{\vec{V}_x^{\,2}}{2} = h_y + \frac{\vec{V}_y^{\,2}}{2}$$
$$h_{ox} = h_{oy}$$
$$for\ ideal\ gases:\ T_{ox} = T_{oy}$$

Conservation of momentum

Rearranging Eq. 16-14 and integrating yield

$$A(P_x - P_y) = \dot{m}(\vec{V}_y - \vec{V}_x)$$

Increase of entropy

$$s_y - s_x \geq 0$$

Thus, we see that from the conservation of energy, the stagnation temperature is constant across the shock. However, the stagnation pressure decreases across the shock because of irreversibilities. The ordinary (static) temperature rises drastically because of the conversion of kinetic energy into enthalpy due to a large drop in fluid velocity.

We can show that the following relations apply across the shock.

$$\frac{T_y}{T_x} = \frac{1 + M_x^2 (k-1)/2}{1 + M_y^2 (k-1)/2}$$

$$\frac{P_y}{P_x} = \frac{M_x \sqrt{1 + M_x^2 (k-1)/2}}{M_y \sqrt{1 + M_y^2 (k-1)/2}}$$

$$M_y^2 = \frac{M_x^2 + 2/(k-1)}{2 M_x^2 k/(k-1) - 1}$$

The entropy change across the shock is obtained by applying the entropy-change equation for an ideal gas, constant properties, across the shock:

$$s_y - s_x = C_p \ln \frac{T_y}{T_x} - R \ln \frac{P_y}{P_x}$$

Example 16-8

Air flowing with a velocity of 600 m/s, a pressure of 60 kPa, and a temperature of 260 K undergoes a normal shock. Determine the velocity and static and stagnation conditions after the shock and the entropy change across the shock.

The Mach number before the shock is

$$M_x = \frac{\vec{V}_x}{C_x} = \frac{\vec{V}_x}{\sqrt{kRT_x}}$$

$$= \frac{600\dfrac{m}{s}}{\sqrt{1.4(0.287\dfrac{kJ}{kg\cdot K})(260K)\dfrac{1000\dfrac{m^2}{s^2}}{\dfrac{kJ}{kg}}}}$$

$$= 1.856$$

For M_x = 1.856, Table A-15 gives

$$\frac{P_x}{P_{ox}} = 0.1597, \quad \frac{T_x}{T_{ox}} = 0.5921$$

For M_x = 1.856, Table A-16 gives the following results.

$$M_y = 0.6045, \ \frac{P_y}{P_x} = 3.852, \ \frac{\rho_y}{\rho_x} = 2.4473$$

$$\frac{T_y}{T_x} = 1.574, \ \frac{P_{oy}}{P_{ox}} = 0.7875, \ \frac{P_{oy}}{P_x} = 4.931$$

From the conservation of mass with $A_y = A_x$

$$\vec{V}_y \rho_y = \vec{V}_x \rho_x$$

$$\vec{V}_y = \frac{\vec{V}_x}{\frac{\rho_y}{\rho_x}} = \frac{600\,\frac{m}{s}}{2.4473} = 245.2\,\frac{m}{s}$$

$$P_y = P_x \frac{P_y}{P_x} = 60\,kPa(3.852) = 231.1\,kPa$$

$$T_y = T_x \frac{T_y}{T_x} = 260\,K(1.574) = 409.2\,K$$

$$T_{ox} = \frac{T_x}{\left(\frac{T_x}{T_{ox}}\right)} = \frac{260\,K}{0.5921} = 439.1\,K = T_{oy}$$

$$P_{ox} = \frac{P_x}{\left(\frac{P_x}{P_{ox}}\right)} = \frac{60\,kPa}{0.1597} = 375.6\,kPa$$

$$P_{oy} = P_{ox} \frac{P_{oy}}{P_{ox}} = 375.6\,kPa(0.7875) = 295.8\,kPa$$

The entropy change across the shock is

$$s_y - s_x = C_P \ln\left(\frac{T_y}{T_x}\right) - R \ln\left(\frac{P_y}{P_x}\right)$$

$$s_y - s_x = 1.005 \frac{kJ}{kg \cdot K} \ln(1.574) - 0.287 \frac{kJ}{kg \cdot K} \ln(3.852)$$

$$= 0.0688 \frac{kJ}{kg \cdot K}$$