Line Integrals: Practice Problems

EXPECTED SKILLS:

- Understand how to evaluate a line integral to calculate the mass of a thin wire with density function $f(x, y, z)$ or the work done by a vector field $\mathbf{F}(x, y, z)$ in pushing an object along a curve.
- Be able to evaluate a given line integral over a curve C by first parameterizing C.
- Given a conservative vector field, $\mathbf{F}$, be able to find a potential function f such that $\mathbf{F} = \nabla f$.
- Be able to apply the Fundamental Theorem of Line Integrals, when appropriate, to evaluate a given line integral.
- Know how to evaluate Green's Theorem, when appropriate, to evaluate a given line integral.

PRACTICE PROBLEMS:

1. Evaluate the following line integrals.

   (a) $\displaystyle\int_C (xy + z^3)\, ds$, where C is the part of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ from $t = 0$ to $t = \pi$

   $\boxed{\dfrac{\pi^4\sqrt{2}}{16}}$

   (b) $\displaystyle\int_C \left(\frac{x}{1 + y^2}\right) ds$ where C is given parametrically by $x = 1+2t$, $y = t$, for $0 \le t \le 1$

   $\boxed{\sqrt{5}\left(\dfrac{\pi}{4} + \ln 2\right)}$

2. Find the mass of a thin wire in the form of $y = \sqrt{9 - x^2}$ $(0 \le x \le 3)$ if the density function is $f(x, y) = x\sqrt{y}$.

   $\boxed{6\sqrt{3}}$

3. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2, xy\rangle$ and $C : \mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$ for $0 \le t \le \pi$

   $\boxed{0}$

4. For each of the following, compute the work done by the vector field $\mathbf{F}$ on the particle that moves along C.

(a) $\mathbf{F}(x, y) = (xy)\mathbf{i} + x^2\mathbf{j}$ where C is the portion of $x = y^2$ from $(0, 0)$ to $(1, 1)$

$\boxed{\dfrac{3}{5}}$

(b) $\mathbf{F}(x, y, z) = (x+y)\mathbf{i} + xy\mathbf{j} - z^2\mathbf{k}$ where C consists of the line segment from $(0, 0, 0)$ to $(1, 3, 1)$ followed by the line segment from $(1, 3, 1)$ to $(2, -1, 4)$

$\boxed{-\dfrac{37}{2}}$

5. Evaluate the following line integrals.

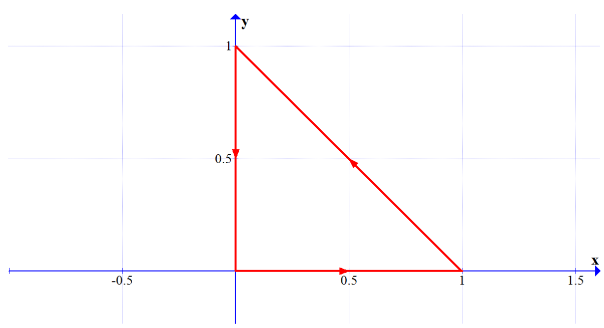
(a) $\displaystyle\int_C (x + 2y)\, dx + (x - y)\, dy$ where $C : x = 2\cos t,\ y = 4\sin t,\ 0 \leq t \leq \dfrac{\pi}{4}$

$\boxed{-\dfrac{9}{2} - \pi}$

(b) $\displaystyle\int_C (y - x)\, dx + (xy)\, dy$ where C is the line segment from $(3, 4)$ to $(2, 1)$

$\boxed{-\dfrac{39}{2}}$

(c) $\displaystyle\int_C y\, dx - x\, dy$ where C is as shown below



$\boxed{-1}$

6. For each of the following, determine whether the given vector field in the plane is a conservative vector field. If so, find a potential function.

(a) $\mathbf{F}(x, y) = \langle x, y \rangle$

$\boxed{\text{Yes; } f(x, y) = \dfrac{1}{2}(x^2 + y^2) + C}$

(b) $\mathbf{F}(x, y) = \langle 3y^2, 6xy \rangle$

$\boxed{\text{Yes; } f(x, y) = 3xy^2 + C}$

(c) $\mathbf{F}(x,y) = \langle x^2y, 5xy^2 \rangle$

$\boxed{\text{No}}$

(d) $\mathbf{F}(x,y) = \langle e^x \cos y, -e^x \sin y \rangle$

$\boxed{\text{Yes; } f(x,y) = e^x \cos y + C}$

7. Compute a potential function for $\mathbf{F}(x,y,z) = \langle e^z, 2y, xe^z \rangle$

$\boxed{f(x,y,z) = xe^z + y^2 + C}$

8. For each of the following, apply the fundamental theorem of line integrals to evaluate the given integral.

(a) $\displaystyle\int_C 3y\,dx + 3x\,dy$ where C is any curve from $(1,2)$ to $(4,0)$

$\boxed{-6}$

(b) $\displaystyle\int_C e^x \sin y\,dx + e^x \cos y\,dy$ where C is any curve from $(0,0)$ to $\left(1, \frac{\pi}{2}\right)$

$\boxed{e}$

(c) $\displaystyle\int_C \left(e^x \ln y - \frac{e^y}{x}\right) dx + \left(\frac{e^x}{y} - e^y \ln x\right) dy$ where $x > 0$, $y > 0$, and C is any curve from $(1,1)$ to $(3,3)$

$\boxed{0}$

9. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle e^y + ye^x, xe^y + e^x \rangle$ and $C : \mathbf{r}(t) = \left\langle \sin\left(\frac{\pi t}{2}\right), \ln t \right\rangle$ for $1 \le t \le 2$

$\boxed{-1 + \ln 2}$

10. Compute the area of the region which is bounded by $y = 4x$ and $y = x^2$ using the indicated method.

(a) By evaluating an appropriate double integral.

$\boxed{\dfrac{32}{3}}$

(b) By evaluating one or more appropriate line integrals.

$\boxed{\dfrac{32}{3}}$

11. Evaluate the following line integrals using Green's Theorem. Unless otherwise stated, assume that all curves are oriented counterclockwise.

(a) $\oint_C 2xy\,dx + y^2\,dy$ where C is the closed curve formed by $y = \dfrac{x}{2}$ and $y = \sqrt{x}$

$\boxed{-\dfrac{64}{15}}$

(b) $\oint_C xy\,dx + (x+y)\,dy$ where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,1)$

$\boxed{\dfrac{1}{3}}$

(c) $\oint_C (e^3 x + 2y)\,dx + (x^3 + \sin y)\,dy$ where C is the rectangle with vertices $(2,1)$, $(6,1)$, $(6,4)$ and $(2,4)$.

$\boxed{600}$

(d) $\oint_C \ln(1+y)\,dx - \dfrac{xy}{1+y}\,dy$ where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,4)$

$\boxed{-4}$