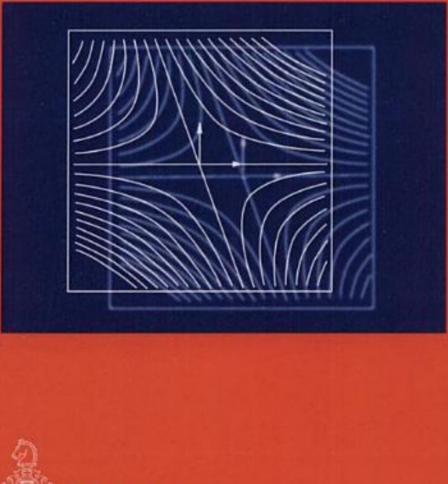
Joseph H. Spurk

Fluid Mechanics

Problems and Solutions



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1.2 Kinematics

Problem 1.2-1 Calculation of material coordinates for given pathlines

The material description of a flow is given by the motion

$$\begin{aligned} x_1 &= \xi_1\,, \\ x_2 &= k\,\xi_1^2\,t^2 + \xi_2\,, \\ x_3 &= \xi_3 \end{aligned}$$

with k as a constant having a dimension, such that the dimensional integrity of both sides of the above system of equations is preserved.

Show that the Jacobian determinant $J = \det(\partial x_i/\partial \xi_j)$ does not vanish and obtain the inverse $\vec{\xi} = \vec{\xi}(\vec{x}, t)$.

Solution
We obtain the necessary derivatives and insert them into the Jacobian determinant:

$$J = \det\begin{pmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_1} \\[2ex] \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_2} \\[2ex] \dfrac{\partial x_1}{\partial \xi_3} & \dfrac{\partial x_2}{\partial \xi_3} & \dfrac{\partial x_3}{\partial \xi_3} \end{pmatrix} = \det\begin{pmatrix} 1 & 2k\,\xi_1\,t^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1\,.$$

Since the Jacobian determinant does not vanish, the mappings $\vec{x} = \vec{x}(\vec{\xi}, t)$ and $\vec{\xi} = \vec{\xi}(\vec{x}, t)$ are unique inverses of each other. We obtain:

$$\begin{aligned} \xi_1 &= x_1 , \\ \xi_2 &= x_2 - k\, x_1^2 t^2 , \\ \xi_3 &= x_3 . \end{aligned}$$

At the time $t = 0$, $\xi_i = x_i$.

Problem 1.2-2 Velocity and acceleration in material and spatial coordinates with given pathlines

The fluid motion is described by:

$$x_1 = \xi_1 , \tag{1}$$

$$x_2 = \frac{1}{2}(\xi_2 + \xi_3)\,\mathrm{e}^{at} + \frac{1}{2}(\xi_2 - \xi_3)\,\mathrm{e}^{-at} , \tag{2}$$

$$x_3 = \frac{1}{2}(\xi_2 + \xi_3)\,\mathrm{e}^{at} - \frac{1}{2}(\xi_2 - \xi_3)\,\mathrm{e}^{-at} . \tag{3}$$

a) Show that the Jacobian determinant does not vanish.
b) Determine the velocity and acceleration components
   1) in material coordinates $u_i(\xi_j, t)$, $b_i(\xi_j, t)$,
   2) in spatial coordinates $u_i(x_j, t)$, $b_i(x_j, t)$.

Solution

a) The Jacobian determinant is:

$$\det \begin{pmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_1} \\ \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_2} \\ \dfrac{\partial x_1}{\partial \xi_3} & \dfrac{\partial x_2}{\partial \xi_3} & \dfrac{\partial x_3}{\partial \xi_3} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh at & \sinh at \\ 0 & \sinh at & \cosh at \end{pmatrix} = 1 ,$$

thus different from zero.

b) Velocity and acceleration components:

1) The velocity components in material coordinates are calculated from:

$$u_i(\xi_j\,,t) = \left(\frac{\partial x_i}{\partial t}\right)_{\xi_j} \,,$$

and thus

$$u_1 \;=\; 0\,, \tag{4}$$

$$u_2 \;=\; \frac{a}{2}(\xi_2+\xi_3)\,\mathrm{e}^{at} - \frac{a}{2}(\xi_2-\xi_3)\,\mathrm{e}^{-at}\,, \tag{5}$$

$$u_3 \;=\; \frac{a}{2}(\xi_2+\xi_3)\,\mathrm{e}^{at} + \frac{a}{2}(\xi_2-\xi_3)\,\mathrm{e}^{-at}\,. \tag{6}$$

Correspondingly, the acceleration components are

$$b_i(\xi_j\,,t) = \left(\frac{\partial u_i}{\partial t}\right)_{\xi_j} = \left(\frac{\partial^2 x_i}{\partial t^2}\right)_{\xi_j} \,,$$

written out

$$b_1 \;=\; 0\,, \tag{7}$$

$$b_2 \;=\; \frac{a^2}{2}(\xi_2+\xi_3)\,\mathrm{e}^{at} + \frac{a^2}{2}(\xi_2-\xi_3)\,\mathrm{e}^{-at}\,, \tag{8}$$

$$b_3 \;=\; \frac{a^2}{2}(\xi_2+\xi_3)\,\mathrm{e}^{at} - \frac{a^2}{2}(\xi_2-\xi_3)\,\mathrm{e}^{-at}\,. \tag{9}$$

2) We obtain the spatial description by extracting the material coordinates $\xi_j = \xi_j(x_k, t)$ from equations (1) to (3) and insert them into $u_i = u_i(\xi_j\,, t)$:

$$u_i = u_i(\xi_j(x_k,\,t),\,t) = u_i(x_k, t)\,.$$

$$\text{from (1)} \;\Rightarrow\; \xi_1 = x_1\,, \tag{10}$$

$$\text{from (2)+(3)} \;\Rightarrow\; (\xi_2+\xi_3)\,\mathrm{e}^{at} = x_2 + x_3\,, \tag{11}$$

$$\text{from (2)−(3)} \;\Rightarrow\; (\xi_2-\xi_3)\,\mathrm{e}^{-at} = x_2 - x_3\,. \tag{12}$$

It is not necessary to solve for ξ_2 and ξ_3 because $u_i(\xi_j, t)$ in equations (4), (5), and (6) contain ξ_2 and ξ_3 only in combined form as in (11) and (12). As a result, the velocity field is described by:

$$u_1 = 0 \,, \tag{13}$$

$$u_2 = \frac{a}{2}(x_2 + x_3) - \frac{a}{2}(x_2 - x_3) = a\,x_3 \,, \tag{14}$$

$$u_3 = \frac{a}{2}(x_2 + x_3) + \frac{a}{2}(x_2 - x_3) = a\,x_2 \,. \tag{15}$$

Similarly, the acceleration field $b_i(x_k\,,t)$ can be calculated from (7) – (9) and (10) – (12) and is:

$$\begin{aligned} b_1 &= 0 \,, \\ b_2 &= \frac{a^2}{2}(x_2 + x_3) + \frac{a^2}{2}(x_2 - x_3) = a^2\,x_2 \,, \\ b_3 &= \frac{a^2}{2}(x_2 + x_3) - \frac{a^2}{2}(x_2 - x_3) = a^2\,x_3 \,. \end{aligned}$$

As an alternative, the substantial derivatives $b_i(x_k\,,t) = \mathrm{D}u_i/\mathrm{D}t$ can be applied, where the acceleration components are calculated from

$$b_i = \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j}$$

as follows:

$$\begin{aligned} b_1 &= \frac{\partial u_1}{\partial t} + u_1\,\frac{\partial u_1}{\partial x_1} + u_2\,\frac{\partial u_1}{\partial x_2} + u_3\,\frac{\partial u_1}{\partial x_3} = 0 \,, \\ b_2 &= \frac{\partial u_2}{\partial t} + u_1\,\frac{\partial u_2}{\partial x_1} + u_2\,\frac{\partial u_2}{\partial x_2} + u_3\,\frac{\partial u_2}{\partial x_3} = a^2\,x_2 \,, \\ b_3 &= \frac{\partial u_3}{\partial t} + u_1\,\frac{\partial u_3}{\partial x_1} + u_2\,\frac{\partial u_3}{\partial x_2} + u_3\,\frac{\partial u_3}{\partial x_3} = a^2\,x_3 \,. \end{aligned}$$

Problem 1.2-3 Material description of a potential vortex flow

The motion of a fluid is given by the material description

$$
\begin{aligned}
x_1 &= (\xi_1^2+\xi_2^2)^{1/2}\cos\left[\frac{\Omega t}{\xi_1^2+\xi_2^2}+\arctan\left(\frac{\xi_2}{\xi_1}\right)\right]\,,\\
x_2 &= (\xi_1^2+\xi_2^2)^{1/2}\sin\left[\frac{\Omega t}{\xi_1^2+\xi_2^2}+\arctan\left(\frac{\xi_2}{\xi_1}\right)\right]\,,\\
x_3 &= \xi_3\,.
\end{aligned}
$$

a) Find the equation of the pathline in an implicit form and show that for $\vec{x}$ at time $t=0$ $x_1=\pm\xi_1$ and $x_2=\pm\xi_2$ holds.
b) Calculate the components of the velocity $u_i(\xi_j,t)$ and the acceleration $b_i(\xi_j,t)$.
c) Determine the velocity field $u_i(x_k,t)$ and the acceleration field $b_i(x_k,t)$.
d) Explain the equation of the streamline through the point (x_{10}, x_{20}).

Solution

a) The pathlines are in the plane $x_3=\xi_3$. We obtain their implicit form by squaring and adding the equations for x_1 and x_2

$$x_1^2+x_2^2=\xi_1^2+\xi_2^2\,. \tag{1}$$

The fluid particles $\vec{\xi}=$ const. describe circles around the x_3–axis in x_1, x_2– plane. Dividing the equations for x_2 and x_1 at time $t=0$ yields

$$\frac{x_2}{x_1}=\frac{\xi_2}{\xi_1}\,. \tag{2}$$

We present (1) in the form

$$x_1^2\left(1+\frac{x_2^2}{x_1^2}\right)=\xi_1^2\left(1+\frac{\xi_2^2}{\xi_1^2}\right)$$

and

$$x_2^2\left(1+\frac{x_1^2}{x_2^2}\right)=\xi_2^2\left(1+\frac{\xi_1^2}{\xi_2^2}\right)\,,$$

and find, using (2), $x_1=\pm\xi_1$ and $x_2=\pm\xi_2$.

b) The material description of the velocity and the acceleration:
The desired partial derivatives are

$$u_1 = \left(\frac{\partial x_1}{\partial t}\right)_{\xi_j} = -\frac{\Omega}{(\xi_1^2+\xi_2^2)^{1/2}} \sin\left[\frac{\Omega t}{\xi_1^2+\xi_2^2} + \arctan\left(\frac{\xi_2}{\xi_1}\right)\right] ,$$

$$u_2 = \left(\frac{\partial x_2}{\partial t}\right)_{\xi_j} = \frac{\Omega}{(\xi_1^2+\xi_2^2)^{1/2}} \cos\left[\frac{\Omega t}{\xi_1^2+\xi_2^2} + \arctan\left(\frac{\xi_2}{\xi_1}\right)\right] ,$$

$$u_3 = \left(\frac{\partial x_3}{\partial t}\right)_{\xi_j} = 0 ,$$

and

$$b_1 = \left(\frac{\partial u_1}{\partial t}\right)_{\xi_j} = -\frac{\Omega^2}{(\xi_1^2+\xi_2^2)^{3/2}} \cos\left[\frac{\Omega t}{\xi_1^2+\xi_2^2} + \arctan\left(\frac{\xi_2}{\xi_1}\right)\right] ,$$

$$b_2 = \left(\frac{\partial u_2}{\partial t}\right)_{\xi_j} = -\frac{\Omega^2}{(\xi_1^2+\xi_2^2)^{3/2}} \sin\left[\frac{\Omega t}{\xi_1^2+\xi_2^2} + \arctan\left(\frac{\xi_2}{\xi_1}\right)\right] ,$$

$$b_3 = \left(\frac{\partial u_3}{\partial t}\right)_{\xi_j} = 0 .$$

c) Velocity and acceleration in spatial coordinates:
To obtain the velocity components in spatial form $u_i(x_k, t)$, we replace the material coordinates in $u_i(\xi_j, t)$ by $\xi_j = \xi_j(x_k, t)$. For the sake of simplicity, we use (1) and the relations following from the material description

$$\sin\left[\frac{\Omega t}{\xi_1^2+\xi_2^2} + \arctan\left(\frac{\xi_2}{\xi_1}\right)\right] = \frac{x_2}{(\xi_1^2+\xi_2^2)^{1/2}} ,$$

$$\cos\left[\frac{\Omega t}{\xi_1^2+\xi_2^2} + \arctan\left(\frac{\xi_2}{\xi_1}\right)\right] = \frac{x_1}{(\xi_1^2+\xi_2^2)^{1/2}} .$$

The insertion leads to

$$u_1 = -\frac{\Omega x_2}{x_1^2+x_2^2} , \qquad u_2 = \frac{\Omega x_1}{x_1^2+x_2^2} , \qquad u_3 = 0 .$$

Similar procedure is applied for calculating the acceleration components

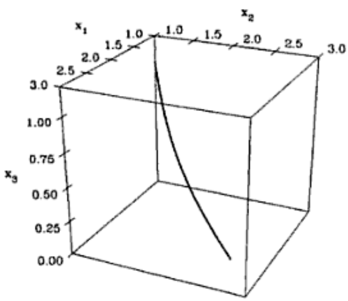
$$b_1 = -\frac{\Omega^2 x_1}{(x_1^2+x_2^2)^2} , \qquad b_2 = -\frac{\Omega^2 x_2}{(x_1^2+x_2^2)^2} , \qquad b_3 = 0 .$$

Using the substantial derivative $b_i = \mathrm{D}u_i/\mathrm{D}t$, the same acceleration components can be obtained.

d) Streamline equation:
The velocity field is steady, that means the streamline and pathline fall on the same curves. The streamline through the point x_{10}, x_{20} is

$$x_1^2 + x_2^2 = x_{10}^2 + x_{20}^2 \ .$$

Problem 1.2-4 Material description of an axisymmetric stagnation point flow



The motion of a fluid is described in material coordinates by:

$$x_1 = \xi_1 \, \mathrm{e}^{at} \ ,$$

$$x_2 = \xi_2 \, \mathrm{e}^{at} \ ,$$

$$x_3 = \xi_3 \, \mathrm{e}^{-2at}$$

with given $a = \text{const}$ and $\vec{\xi} = \vec{x}(t = 0)$.

a) Calculate the velocity and acceleration components $u_i(\xi_j, t)$ and $b_i(\xi_j, t)$ in material coordinates.
b) Determine the spatial description of the velocity and acceleration components $u_i(x_k, t)$ and $b_i(x_k, t)$ by eliminating the material coordinates $\xi_j = \xi_j(x_k, t)$ in the results obtained in a).
c) Find the acceleration components using the substantial derivatives of $u_i(x_k, t)$.
d) Is this a potential flow? If yes, find the potential function.

Solution

a) The material description of velocity and acceleration is determined using:

$$u_i(\xi_j \, , t) = \left(\frac{\partial x_i}{\partial t}\right)_{\xi_j} \ , \qquad b_i(\xi_j \, , t) = \left(\frac{\partial^2 x_i}{\partial t^2}\right)_{\xi_j} \ .$$

As a result, the velocity components are:

$$u_1 = a \, \xi_1 \, \mathrm{e}^{at} \ , \qquad u_2 = a \, \xi_2 \, \mathrm{e}^{at} \ , \qquad u_3 = -2a \, \xi_3 \, \mathrm{e}^{-2at} \ ,$$

while the acceleration components take the form:

$$b_1 = a^2\,\xi_1\,\mathrm{e}^{at}\,, \qquad b_2 = a^2\,\xi_2\,\mathrm{e}^{at}\,, \qquad b_3 = 4a^2\,\xi_3\,\mathrm{e}^{-2at}\,.$$

b) The spatial description of velocity and acceleration:
From

$$\xi_1 = x_1\,\mathrm{e}^{-at}\,, \qquad \xi_2 = x_2\,\mathrm{e}^{-at}\,, \qquad \xi_3 = x_3\,\mathrm{e}^{+2at}\,,$$

we find from a)

$$u_1 = a\,x_1\,, \qquad u_2 = a\,x_2\,, \qquad u_3 = -2a\,x_3\,,$$

and

$$b_1 = a^2\,x_1\,, \qquad b_2 = a^2\,x_2\,, \qquad b_3 = 4a^2\,x_3\,.$$

c) The acceleration components are the substantial derivatives of the velocity components $u_i(x_j, t)$:
Using

$$b_i = \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j}$$

and $u_1 = a\,x_1$, $u_2 = a\,x_2$, $u_3 = -2a\,x_3$ we obtain the acceleration components as

$$b_1 = a^2\,x_1\,, \qquad b_2 = a^2\,x_2\,, \qquad b_3 = 4\,a^2\,x_3\,.$$

d) Potential flow, Potential function:
The necessary and sufficient condition for the existence of a potential flow is the vanishing of the vorticity vector $\mathrm{curl}\vec{u}$ in the entire flow field:

$$\mathrm{curl}\vec{u} = \nabla \times \vec{u} = 0 \qquad \Leftrightarrow \qquad \epsilon_{ijk}\,\frac{\partial u_j}{\partial x_i} = 0\,.$$

The three components resulting from this condition are:

$$\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = 0\,, \qquad \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = 0\,, \qquad \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0\,.$$

The above results show that the flow under investigation is a potential flow. In fact, all six terms of the above equations here identically vanish.
From

$$u_i = \frac{\partial \Phi}{\partial x_i}$$

we find for the potential function Φ the partial differential equations:

$$\frac{\partial \Phi}{\partial x_1} = u_1 = a\,x_1\,, \qquad \frac{\partial \Phi}{\partial x_2} = u_2 = a\,x_2\,, \qquad \frac{\partial \Phi}{\partial x_3} = u_3 = -2a\,x_3\,.$$

Integrating the first differential equation yields

$$\Phi = \frac{a}{2}\, x_1^2 + h(x_2,\, x_3)\ ,$$

the second equation then gives

$$\frac{\partial \Phi}{\partial x_2} = \frac{\partial h}{\partial x_2} = a\, x_2 \qquad \Rightarrow \qquad h(x_2, x_3) = \frac{a}{2}\, x_2^2 + g(x_3)\ .$$

The arbitrary function $g(x_3)$ is determined by the last differential equation:

$$\frac{\partial \Phi}{\partial x_3} = \frac{\partial g}{\partial x_3} = -2a\, x_3 \qquad \Rightarrow \qquad g(x_3) = -a\, x_3^2 + \text{const}\ .$$

The potential function can be expressed as

$$\Phi = \frac{a}{2}\,(x_1^2 + x_2^2 - 2\, x_3^2) + \text{const}$$

where the constant may be omitted.

Problem 1.2-5 Pathlines, streamlines, and streaklines of an unsteady flow field

Given is the following unsteady velocity field:

$$\begin{aligned} u_1 &= \frac{1}{t_0 + t}\, x_1\ , \\ u_2 &= v_0\ , \\ u_3 &= 0 \qquad (t_0 = \text{const},\ v_0 = \text{const}). \end{aligned}$$

a) Find the equation of the streamline through the point $(x_{10},\, x_{20},\, x_{30})$ at time t.
b) Find the pathline equation of a fluid particle with the material coordinate $\vec{x}(t = 0) = \vec{\xi}$.
c) Determine the particle velocity along its pathline.
d) What happens to the fluid particles with the material coordinates $\xi_1 = 0\,,\ \xi_3 = 0$?
e) Find the equation for the streaklines.

Solution

a) The streamline through the point (x_{10}, x_{20}, x_{30}) at an arbitrarily fixed time t:

The streamlines are the solutions of the differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = \frac{u_i}{\sqrt{u_k u_k}} .$$

Instead of using the arc length s, it is appropriate to introduce the parameter η

$$\mathrm{d}s = \sqrt{u_k u_k}\,\mathrm{d}\eta , \qquad \eta(s=0) = 0 ,$$

and to rearrange the differential equation to

$$\frac{\mathrm{d}x_i}{\mathrm{d}\eta} = u_i\left(x_j, t\right) \quad , \quad t = \text{const} .$$

The components are:

$$\frac{\mathrm{d}x_1}{\mathrm{d}\eta} = \frac{x_1}{t_0 + t} , \quad \frac{\mathrm{d}x_2}{\mathrm{d}\eta} = v_0 , \quad \frac{\mathrm{d}x_3}{\mathrm{d}\eta} = 0 .$$

Integrating the above differential equations results in:

$$\begin{aligned} \ln x_1 &= \frac{\eta}{t_0 + t} + \ln C_1 \quad \Rightarrow \quad x_1 = C_1\,\mathrm{e}^{\eta/(t_0+t)} , \\ x_2 &= v_0\,\eta + C_2 , \\ x_3 &= C_3 . \end{aligned}$$

The streamline is in the plane $x_3 = C_3$, thus the field is plane. The above three integration constants are determined using the condition that the streamline goes through the point (x_{10}, x_{20}, x_{30}). Measuring η from the point (x_{10}, x_{20}, x_{30}) we have

$$\eta = 0 : \quad x_1 = x_{10} , \quad x_2 = x_{20} , \quad x_3 = x_{30} .$$

Thus, the constants read:

$$C_1 = x_{10} , \quad C_2 = x_{20} , \quad C_3 = x_{30} .$$

The parametric solution is obtained as

$$x_1 = x_{10}\,\mathrm{e}^{\eta/(t_0+t)} , \tag{1}$$

$$x_2 = v_0\,\eta + x_{20} , \tag{2}$$

$$x_3 = x_{30} . \tag{3}$$

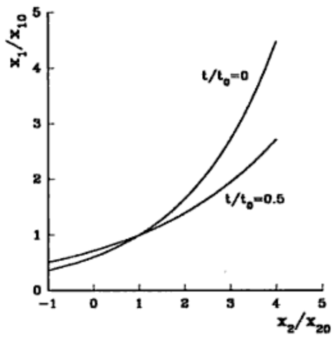
In (1) to (3), η is the curve parameter and x_{10}, x_{20}, and x_{30} are the family parameters. The explicit representation of the streamline in the plane $x_3 = x_{30}$ can be found by eliminating the curve parameter. From (2), we obtain

$$\eta = \frac{x_2 - x_{20}}{v_0}$$

and then

$$x_1 = x_{10} \exp\left(\frac{x_2 - x_{20}}{v_0\,(t_0 + t)}\right) \quad \text{or} \quad \frac{x_1}{x_{10}} = \exp\left(\frac{x_2/x_{20} - 1}{v_0\,t_0(1 + t/t_0)/x_{20}}\right) .$$

For the value $v_0\,t_0/x_{20} = 2$, the figure on the right shows the streamline through the point (x_{10}, x_{20}, x_{30}) at time $t = 0$ and $t = 0.5\,t_0$. Thus, we are dealing with an unsteady flow field with time dependent streamlines.



b) The pathline of a fluid particle:
The pathline differential equations are

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i\,(x_j, t) ,$$

in an extended form, they read

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{x_1}{t_0 + t} , \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = v_0 , \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = 0 .$$

The integration furnishes

$$x_1 = C_1\,(t_0 + t) , \quad x_2 = v_0\,t + C_2 , \quad x_3 = C_3 .$$

The above integration constants are determined from the initial conditions. At time $t = 0$, the fluid particle has the material coordinates

$$t = 0 : \quad x_1 = \xi_1 , \quad x_2 = \xi_2 , \quad x_3 = \xi_3 ,$$

resulting in

$$C_1 = \xi_1/t_0 , \quad C_2 = \xi_2 , \quad C_3 = \xi_3 .$$

The parametric solution is

$$x_1 = \xi_1 \left(1 + \frac{t}{t_0}\right) , \qquad (4)$$

$$x_2 = v_0 \, t + \xi_2 , \qquad (5)$$

$$x_3 = \xi_3 . \qquad (6)$$

In the above equations t is the curve parameter of the pathline; ξ_1, ξ_2, and ξ_3 are the family parameters. As shown previously, the explicit form of the pathline equation is obtained by eliminating the curve parameter t in the plane $x_3 = \xi_3$. We obtain from (5):

$$t = \frac{x_2 - \xi_2}{v_0}$$

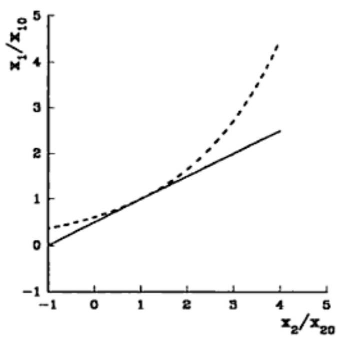
and with (4)

$$x_1 = \xi_1 \left(1 + \frac{x_2 - \xi_2}{v_0 \, t_0}\right) .$$

The above explicit form of the pathlines is a straight line for each fluid particle. An arbitrary fluid particle with the material coordinates $\xi_1 = x_{10}$, $\xi_2 = x_{20}$, $\xi_3 = x_{30}$ has the pathline

$$\frac{x_1}{x_{10}} = 1 + \frac{x_2/x_{20} - 1}{v_0 \, t_0/x_{20}} . \qquad (7)$$

This pathline has to be tangent to the streamline at time $t = 0$ and the space point (x_{10}, x_{20}, x_{30}). The pathline of the particle is plotted in the figure as a solid line. The streamline at time $t = 0$ is shown as a dashed line.



c) The velocity of a fluid particle is defined as the temporal change of the path coordinates at fixed $\vec{\xi}$

$$u_i(\xi_j, t) = \left(\frac{\partial x_i}{\partial t}\right)_{\xi_j} ,$$

and from the parametric representation of the pathline follow the velocity components

$$u_1(\xi_j, t) = \frac{\xi_1}{t_0}, \quad u_2(\xi_j, t) = v_0, \quad u_3(\xi_j, t) = 0.$$

Since the velocity components are constant, the pathline is again seen to be a straight line.

d) For a fluid particle with the material coordinates $\xi_1 = 0$ and $\xi_3 = 0$ we obtain the velocity components

$$u_1(\xi_j, t) = 0, \quad u_2(\xi_j, t) = v_0, \quad u_3(\xi_j, t) = 0,$$

and the pathline

$$x_1 = \xi_1 \left(1 + \frac{t}{t_0}\right) = 0, \quad x_3 = 0.$$

The above equations indicate that the fluid particle moves along the x_2-axis at constant speed.

e) Streaklines:
A streakline at a fixed time t is the connecting line or the locus of different fluid particles, which have or will pass through a fixed location $\vec{y}$ at time t'. The pathlines of the particles are given by $\vec{x} = \vec{x}(\vec{\xi}, t)$. Solving these equations for $\vec{\xi} = \vec{\xi}(\vec{x}, t)$ and replacing $\vec{x}$ by the coordinates of the fixed location $\vec{y}$, and setting $t = t'$, we locate the fluid particles $\vec{\xi}$, that were passing through the fixed location $\vec{y}$ at time t'. The equation assumes the form $\vec{\xi} = \vec{\xi}(\vec{y}, t')$. The pathlines of these particles are $\vec{x} = \vec{x}(\vec{\xi}(\vec{y}, t'), t)$. Thus, at the fixed time t und variable t' we obtain the streakline as a curve, which connects the fluid particles having passed through the fixed spatial location $\vec{y}$ at time t'. For this problem, all fluid particles remain in the plane $x_3 = \xi_3$. Inserting the coordinates $x_1 = y_1$ and $x_2 = y_2$ in the pathline equations (4), (5) expressed in parametric form, we obtain the material coordinates of the fluid particles which passed through the above location at time $t = t'$:

$$\xi_1 = \frac{y_1}{1 + \frac{t'}{t_0}},$$

$$\xi_2 = y_2 - v_0 t'.$$

Now, we insert these material coordinates into the pathline equations and obtain thus the parametric representation of the streaklines:

$$x_1 = y_1 \frac{1 + t/t_0}{1 + t'/t_0}, \tag{8}$$

$$x_2 = y_2 + v_0 t_0 \left(\frac{t}{t_0} - \frac{t'}{t_0}\right) . \tag{9}$$

In the above equations t represents the fixed actual time and t' the curve parameter. Eliminating the curve parameter t' from (9)

$$t' = t - \frac{x_2 - y_2}{v_0}$$

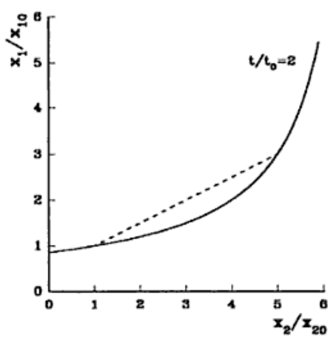
and inserting the above equation into (8), yields the explicit form of the streakline

$$x_1 = \frac{y_1}{1 - \dfrac{x_2 - y_2}{v_0 t_0 (1 + t/t_0)}} . \tag{10}$$

The streakline through the point $y_1 = x_{10}$, $y_2 = x_{20}$ is given by

$$\frac{x_1}{x_{10}} = \frac{1}{1 - \dfrac{x_2/x_{20} - 1}{v_0 t_0/x_{20}(1 + t/t_0)}} . \tag{11}$$

The figure shows the streakline for $v_0 t_0/x_{20} = 2$ at the fixed time $t/t_0 = 2$. The dashed line represents the pathline of the particle, which, at time $t'/t_0 = 0$ was located at point $x_1 = x_{10}$, $x_2 = x_{20}$. At time $t/t_0 = 2$ the fluid particle is at the point $x_1 = 3\,x_{10}$, $x_2 = 5\,x_{20}$. Using this, the streakline in parametric form (8), (9), gives for the parameter value $t'/t_0 = 0$ the point $x_1 = 3\,x_{10}$, $x_2 = 5\,x_{20}$.



Problem 1.2-6 Kinematics of an irrotational and divergence free flow field

The velocity field $u_i(\xi_j)$ is given by

$$u_1 = a\,(x_1 + x_2) ,$$

$$u_2 = a\,(x_1 - x_2) ,$$

$$u_3 = W$$

with the constants a and W.

Determine

a) the divergence $\nabla \cdot \vec{u}$ of the flow field,
b) the vorticity $\nabla \times \vec{u}$,
c) the parametric representation of the pathlines $x_i = x_i(\xi_j, t)$ with $\xi_j = x_j(t=0)$,
d) nonparametric representation of the projection of the pathlines in x_1, x_2-plane by eliminating the curve parameter t,
e) the projection of the streamlines in x_1, x_2-plane by integrating the differential equations for the streamlines.

Solution

a) The divergence of a vector field is defined as

$$\operatorname{div} \vec{u} = \nabla \cdot \vec{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \,.$$

For the present case, it follows

$$\frac{\partial u_i}{\partial x_i} = a - a + 0 = 0 \,,$$

thus the velocity field is divergence free.

b) Similarly, the three components of the vorticity vector disappear:

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2}\epsilon_{231} + \frac{\partial u_2}{\partial x_3}\epsilon_{321}\right) = 0 \,,$$

$$\omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\epsilon_{312} + \frac{\partial u_3}{\partial x_1}\epsilon_{132}\right) = 0 \,,$$

$$\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\epsilon_{123} + \frac{\partial u_1}{\partial x_2}\epsilon_{213}\right) = 0 \,.$$

As a result, the velocity field can be expressed in terms of a potential, which is obtained as (see also Problem 1.2.4):

$$\Phi = \frac{a}{2}\left(x_1^2 + 2\,x_1 x_2 - x_2^2\right) + W\,x_3 \,.$$

c) Parametric representation of pathlines:
The pathline differential equations are

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i \,,$$

or

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = u_1 = a\,(x_1 + x_2)\,, \tag{1}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = u_2 = a\,(x_1 - x_2)\,, \tag{2}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = u_3 = W\,. \tag{3}$$

Integrating the last differential equation ($x_3(t=0) = \xi_3$) leads us to

$$x_3(t) = W\,t + \xi_3\,. \tag{4}$$

The coupled equations (1) and (2) are first order differential equations. They may be reduced to one equation by a further differentiation. For this purpose, we differentiate (2)

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = a\,\frac{\mathrm{d}x_1}{\mathrm{d}t} - a\,\frac{\mathrm{d}x_2}{\mathrm{d}t}$$

and replace the derivatives on the right hand side by (1) and (2). As a result, we obtain

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = 2\,a^2\,x_2\,.$$

This is a second order linear differential equation with constant coefficients. It is solved by

$$x_2 = C\,\mathrm{e}^{\lambda\,t}\,.$$

The eigenvalues are found as

$$\lambda = \pm\sqrt{2}\,a$$

and the complete solution is:

$$x_2(t) = C_1\,\mathrm{e}^{\sqrt{2}a\,t} + C_2\,\mathrm{e}^{-\sqrt{2}a\,t}\,. \tag{5}$$

From equation (2) we extract

$$\begin{aligned} x_1(t) &= \frac{1}{a}\left(\sqrt{2}aC_1\,\mathrm{e}^{\sqrt{2}a\,t} - \sqrt{2}aC_2\,\mathrm{e}^{-\sqrt{2}a\,t}\right) + \\ &\quad + C_1\,\mathrm{e}^{\sqrt{2}a\,t} + C_2\,\mathrm{e}^{-\sqrt{2}a\,t}\,, \\ \Rightarrow \qquad x_1(t) &= (\sqrt{2}+1)C_1\,\mathrm{e}^{\sqrt{2}a\,t} - (\sqrt{2}-1)C_2\,\mathrm{e}^{-\sqrt{2}a\,t}\,. \end{aligned} \tag{6}$$

The two integration constants C_1 and C_2 are obtained from the initial conditions

$$\begin{aligned} x_1(t=0)=\xi_1 &= (\sqrt{2}+1)\,C_1-(\sqrt{2}-1)\,C_2\;, \\ x_2(t=0)=\xi_2 &= C_1+C_2\;, \end{aligned}$$

this yields

$$C_1 = \frac{1}{4}(2-\sqrt{2})\,\xi_2+\frac{\sqrt{2}}{4}\,\xi_1\;, \tag{7}$$

$$C_2 = \frac{1}{4}(2+\sqrt{2})\,\xi_2-\frac{\sqrt{2}}{4}\,\xi_1\;. \tag{8}$$

Equations (4), (5), (6), together with (7) and (8) describe the pathlines.

d) Nonparametric representation of the pathlines in x_1, x_2-plane:
To arrive at the nonparametric representation of the plane curve, we eliminate from equations (5) and (6) the pathline parameter t. For this purpose, we multiply (5) with $(\sqrt{2}-1)$ and add to (6):

$$\begin{aligned} (\sqrt{2}-1)\,x_2+x_1 &= ((\sqrt{2}+1)+(\sqrt{2}-1))\,C_1\,\mathrm{e}^{\sqrt{2}a\,t}=2\sqrt{2}C_1\,\mathrm{e}^{\sqrt{2}a\,t} \\ \Rightarrow \qquad \mathrm{e}^{\sqrt{2}a\,t} &= \frac{1}{2\sqrt{2}C_1}\,((\sqrt{2}-1)\,x_2+x_1) \\ \Rightarrow \qquad \mathrm{e}^{\sqrt{2}a\,t} &= \frac{1}{4C_1}\,((2-\sqrt{2})\,x_2+\sqrt{2}\,x_1)\;, \end{aligned}$$

and insert the final result into (5)

$$x_2=\frac{1}{4}\,((2-\sqrt{2})\,x_2+\sqrt{2}\,x_1)+4\,C_1C_2\,((2-\sqrt{2})\,x_2+\sqrt{2}\,x_1)^{-1}\;.$$

Thus, we obtain

$$\frac{1}{2}\,x_2^2+x_1x_2-\frac{1}{2}\,x_1^2=4\,C_1C_2\;. \tag{9}$$

The above equation together with (7) and (8) is the implicit equation of pathlines. The explicit equation is

$$x_2=-x_1\pm\sqrt{2\,x_1^2+8\,C_1C_2}=x_1\left(-1\pm\sqrt{2+\frac{8\,C_1C_2}{x_1^2}}\right)\;.$$

Setting $C_1C_2=0$ reduces the pathline equation to a linear one.

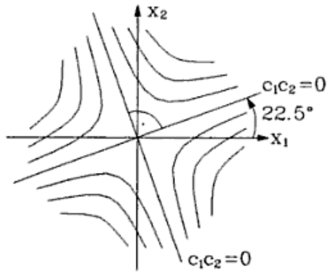
$$x_2=x_1\,(-1\pm\sqrt{2})\;,$$

with $\tan\alpha=x_2/x_1=-1\pm\sqrt{2}$ and therefore

$$\alpha_1 = -67.5° \quad \text{and}$$

$$\alpha_2 = 22.5° \ .$$

The figure shows the pathlines of a stagnation point flow in x_1, x_2-plane tilted at an angle of $\alpha_2 = 22.5°$.



e) Streamlines in x_1, x_2-plane:
The differential equations of streamlines are

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = \frac{u_i}{|\vec{u}|} \ ,$$

with the components in x_1 and x_2 direction

$$\frac{\mathrm{d}x_1}{\mathrm{d}s} = \frac{u_1}{|\vec{u}|} \ , \qquad \frac{\mathrm{d}x_2}{\mathrm{d}s} = \frac{u_2}{|\vec{u}|} \ .$$

Dividing the above equations by each other eliminates the streamline curve parameter s:

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{u_2}{u_1} \ .$$

Inserting the previous values for u_1 and u_2 gives us

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{a\,(x_1 - x_2)}{a\,(x_1 + x_2)} \quad \text{or}$$

$$(x_2 - x_1)\,\mathrm{d}x_1 + (x_1 + x_2)\,\mathrm{d}x_2 = 0 \ .$$

The last equation is an exact differential equation of the form

$$\mathrm{d}\Psi = \frac{\partial \Psi}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \Psi}{\partial x_2}\mathrm{d}x_2 = 0$$

with the solution $\Psi = \text{const.}$ To prove this statement we form the derivatives

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \Psi}{\partial x_1}\right) = \frac{\partial}{\partial x_2}(x_2 - x_1) = 1$$

and

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \Psi}{\partial x_2}\right) = \frac{\partial}{\partial x_1}(x_1 + x_2) = 1$$

As seen, the mixed derivatives $\partial^2\Psi/\partial x_1\partial x_2$ and $\partial^2\Psi/\partial x_2\partial x_1$ are equal. This is the necessary and sufficient condition for the differential equation to be exact. As a result, $\partial\Psi/\partial x_1$ and $\partial\Psi/\partial x_2$ are known. For the calculation of Ψ we first integrate

$$\frac{\partial\Psi}{\partial x_1} = x_2 - x_1$$

and obtain

$$\Psi = x_1 x_2 - \frac{1}{2}x_1^2 + h(x_2)$$

and the derivative

$$\frac{\partial\Psi}{\partial x_2} = x_1 + x_2 = x_1 + h'(x_2)\,.$$

Consequently

$$h'(x_2) = x_2$$

and thus

$$h(x_2) = \frac{x_2^2}{2} + C\,,$$

and we arrive at the final solution

$$\Psi = \frac{1}{2}x_2^2 + x_1 x_2 - \frac{1}{2}x_1^2 + C\,.$$

The lines $\Psi = \text{const}$ are the projection of the streamlines in the plane $x_3 = \text{const}$. Compared with (9) from Problem d) we recognize that for the present steady flow case, pathlines and streamlines coincide.

Problem 1.2-7 Kinematics of an unsteady, plane stagnation point flow

The velocity components of an unsteady, plane flow field are given by

$$u_1 = (a + b\sin\omega t)\,x_1\,,$$

$$u_2 = -(a + b\sin\omega t)\,x_2\,,$$

with the constants $a > b > 0$.

a) Find the equation of the streamline through the point $(x_{10},\, x_{20})$.
b) Find the equation of the pathline for a fluid particle which at the time $t = 0$ was at the place $\vec{x}(t=0) = \vec{\xi}$.
c) Find the equation of the streaklines through the origin ($\vec{y} = 0$).
d) What is the velocity change that a probe would measure if it moved along $x_{1P} = x_{2P} = c_0\,t$?

Solution

a) Streamlines (plane flow):
The differential equations of streamlines are

$$\frac{\mathrm{d}x_1}{\mathrm{d}s} = \frac{u_1}{|\vec{u}|}, \quad \frac{\mathrm{d}x_2}{\mathrm{d}s} = \frac{u_2}{|\vec{u}|},$$

and for this problem

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_2} = \frac{u_1}{u_2} = \frac{(a + b\sin\omega t)\,x_1}{-(a + b\sin\omega t)\,x_2} = -\frac{x_1}{x_2}.$$

The direction of the velocity field is time independent and therefore steady directional. In this case, the streamlines are also time independent. The differential equation can be solved by separating the variables

$$\int \frac{\mathrm{d}x_1}{x_1} = -\int \frac{\mathrm{d}x_2}{x_2}$$

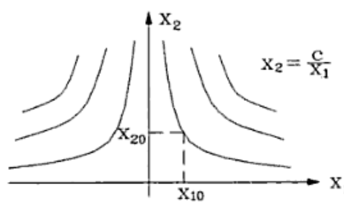
and after adding the integration constant we are led to

$$\ln x_1 = -\ln x_2 + \ln C$$

or

$$x_2 = \frac{C}{x_1}.$$

Thus, the integral curves are hyperbolae and represent the stagnation point flow in the upper half plane $x_2 \geq 0$. For the streamline equation through the fixed point (x_{10}, x_{20}), the integration constant is determined as $C = x_{10}\,x_{20}$ which leads to



$$x_2 = \frac{x_{10}\,x_{20}}{x_1}.$$

b) Pathlines:
Since the flow is steady directional, the streamlines coincide with the pathlines. The pathline equation is

$$x_2 = \frac{C}{x_1}.$$

We calculate the pathline again by integrating the differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i\,,$$

here

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = u_1 = (a + b\sin\omega t)\,x_1\ ,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = u_2 = -(a + b\sin\omega t)\,x_2\ .$$

Separating the variables yields

$$\int \frac{\mathrm{d}x_1}{x_1} = \int (a + b\sin\omega t)\,\mathrm{d}t\ ,$$

$$\int \frac{\mathrm{d}x_2}{x_2} = -\int (a + b\sin\omega t)\,\mathrm{d}t\ .$$

The integration leads to

$$\ln x_1 = \ln C_1 + \left(a\,t - \frac{b}{\omega}\cos\omega t\right) \quad\Rightarrow\quad x_1 = C_1\,\mathrm{e}^{\left(a\,t - \frac{b}{\omega}\cos\omega t\right)}\ ,$$

$$\ln x_2 = \ln C_2 - \left(a\,t - \frac{b}{\omega}\cos\omega t\right) \quad\Rightarrow\quad x_2 = C_2\,\mathrm{e}^{-\left(a\,t - \frac{b}{\omega}\cos\omega t\right)}\ .$$

The integration constants are determined from the initial conditions $\vec{x}(t=0) = \vec{\xi}$ as

$$C_1 = \xi_1\,\mathrm{e}^{\frac{b}{\omega}} \quad \text{and} \quad C_2 = \xi_2\,\mathrm{e}^{-\frac{b}{\omega}}\ .$$

Therefore the parametric representation of the pathlines is

$$x_1 = \xi_1\,\mathrm{e}^{\left(a\,t + \frac{b}{\omega}(1-\cos\omega t)\right)}\ , \tag{1}$$

$$x_2 = \xi_2\,\mathrm{e}^{-\left(a\,t + \frac{b}{\omega}(1-\cos\omega t)\right)}\ . \tag{2}$$

If we eliminate the pathline parameter t from the first equation

$$\mathrm{e}^{\left(a\,t + \frac{b}{\omega}(1-\cos\omega t)\right)} = \frac{x_1}{\xi_1}$$

and insert the result into the second equation, we arrive again at the equation

$$x_2 = \frac{\xi_1\,\xi_2}{x_1}\ .$$

c) Streaklines:
As mentioned previously, for a steady directional flow, the streamlines coincide with the pathlines. This is also true for the streaklines. Consequently, the streaklines are also described by the same equation, which is in this particular case a hyperbola

$$x_1\,x_2 = \text{const} = y_1\,y_2\ .$$

For $y_1 = y_2 = 0$ we get

$$x_1\,x_2 = 0\ .$$

The above equation implies that the coordinate axes ($x_1 = 0$ and $x_2 = 0$) are the wanted streaklines. From the pathline equations (1) and (2) we conclude that x_1–axis is the streakline of the fluid particles, which were located at the origin at the time $t' \to -\infty$. For the fluid particles that would arrive at the origin at the time $t' \to \infty$, the x_2–axis is the streakline.

d) Velocity change measured by a probe, which moves along the path $x_{1P} = x_{2P} = c_0\,t$:
The change of an arbitrary field quantity measured by the probe is given by the operator

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + c_j\,\frac{\partial}{\partial x_j}$$

with c_j as the absolute velocity of the probe. We apply the differential operator to the velocity vector u_i and arrive at the velocity changes measured by the probe on its pathline:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial u_i}{\partial t} + c_j\,\frac{\partial u_i}{\partial x_j}\ .$$

With $c_1 = c_2 = c_0$, the changes of the velocity components as seen by the probe are

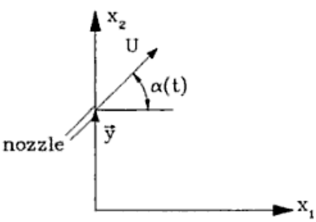
$$\begin{aligned}
\frac{\mathrm{d}u_1}{\mathrm{d}t} &= \frac{\partial u_1}{\partial t} + c_0\,\frac{\partial u_1}{\partial x_1} + c_0\,\frac{\partial u_1}{\partial x_2}\\
&= (b\omega\,\cos\omega t)\,x_1 + c_0\,(a + b\,\sin\omega t)\ ,\\
\frac{\mathrm{d}u_2}{\mathrm{d}t} &= \frac{\partial u_2}{\partial t} + c_0\,\frac{\partial u_2}{\partial x_1} + c_0\,\frac{\partial u_2}{\partial x_2}\\
&= (-b\omega\,\cos\omega t)\,x_2 + c_0\,[-(a + b\,\sin\omega t)]\ .
\end{aligned}$$

Along the path of the probe, we have $x_1 = x_{1_P} = c_0\, t$, $x_2 = x_{2_P} = c_0\, t$. This means that velocity changes expressed as functions of t are

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = c_0\,(b\,\omega\, t\,\cos\omega t + a + b\,\sin\omega t)\,,$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = -c_0\,(b\,\omega\, t\,\cos\omega t + a + b\,\sin\omega t)\,.$$

Problem 1.2-8 Streakline of a water jet



The nozzle of a water hose is located at $\vec{y} = h\,\vec{e}_2$ and oscillates with the angle $\alpha = \alpha(t)$. Water leaves the nozzle with a constant exit velocity U.

Neglecting the air forces exerted on the water jet, determine:

a) the velocity components $u_i(t)$ of a fluid particle which was at the nozzle exit at the time t',
b) its pathline for $\vec{x}(0) = \vec{\xi}$,
c) the equation of streaklines.
d) Has this type of flow streamlines?

Solution

a) The velocity of the fluid particle $u_i(t)$:
Neglecting the air forces exerted on the water jet, the fluid particles describe trajectories in form of parabolae, where the velocity components are given by

$$u_1 = C_1\,,$$

$$u_2 = C_2 - g\,t\,.$$

The constants C_1 and C_2 are determined from the condition that the fluid particle under consideration was located at the nozzle exit at time t' and had the velocity components

$$u_1(t') = U\,\cos\alpha(t')\,,$$

$$u_2(t') = U\,\sin\alpha(t')\,.$$

The constants are

$$C_1 = U \cos\alpha(t') ,$$

$$C_2 = U \sin\alpha(t') + g\,t' .$$

With the above constants, we recover the velocity components

$$u_1 = U \cos\alpha(t') ,$$

$$u_2 = U \sin\alpha(t') - g\,(t - t') .$$

b) Pathline of the particle with $\vec{x}(t=0) = \vec{\xi}$:
From the pathline differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = U \cos\alpha(t') \quad (= \text{const}) ,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = U \sin\alpha(t') - g\,(t - t') ,$$

follows by direct integration

$$x_1(t) = U \cos\alpha(t')\,t + C_3 ,$$

$$x_2(t) = U \sin\alpha(t')\,t - \frac{1}{2}\,g\,(t^2 - 2t'\,t) + C_4 .$$

For the fluid particle under consideration the integration constants are

$$x_1(0) = \xi_1 = C_3 ,$$

$$x_2(0) = \xi_2 = C_4 ,$$

therefore its pathline is

$$x_1(t) = U \cos\alpha(t')\,t + \xi_1 , \tag{1}$$

$$x_2(t) = U \sin\alpha(t')\,t - \frac{1}{2}\,g\,((t - t')^2 - t'^2) + \xi_2 . \tag{2}$$

c) Streakline equation:
Starting from a known pathline $x_i = x_i(\xi_j, t)$, we solve this equation for $\xi_j = \xi_j(x_i, t)$ and identify the fluid particle at time t' and position $\vec{y} = h\,\vec{e}_2$ through the equation $\xi_j = \xi_j(y_i, t')$. The resulting equation

$$x_i(t') = x_i(\xi_j(y_k, t'), t) = x_i(y_k, t', t)$$

gives for a fixed y_k and t the equation of a streakline. As we saw, the streakline is the connecting line of all particles having passed through the fixed location $\vec{y}$ at a time t'.
This is the step-by-step procedure:
We solve $x_i = x_i(\xi_j, t)$ for $\xi_j = \xi_j(x_i, t)$

$$\text{from (1):} \quad \xi_1 = x_1 - U \cos\alpha(t')\, t \,,$$

$$\text{from (2):} \quad \xi_2 = x_2 - U \sin\alpha(t')\, t + \frac{1}{2} g\,((t-t')^2 - t'^2) \,.$$

Particle identification ($t = t', x_1 = y_1 = 0, x_2 = y_2 = h$):

$$\xi_1 = -U \cos\alpha(t')\, t' \,,$$

$$\xi_2 = h - U \sin\alpha(t')\, t' - \frac{1}{2} g\, t'^2 \,.$$

Inserting the material coordinates into the pathline equation (1) and (2) leads to

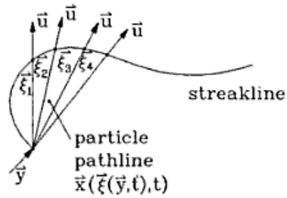
$$x_1 = U \cos\alpha(t')\,(t-t') \,,$$

$$x_2 = h + U \sin\alpha(t')\,(t-t') + -\frac{1}{2} g\,(t-t')^2 \,.$$

This is the parametric representation of the streakline at time t with t' as the curve parameter.

d) Are there streamlines?
The instantaneous picture of the water jet at time t is exactly the calculated streakline. The velocity vectors are not tangential to the streaklines.

To construct the curves tangential to the velocity vectors of different fluid particles at the same time t (streamlines), adjacent particles in the direction of the velocity vectors forming the streaklines must exist. This is not the case here, the streamlines degenerate to points. As a result, no streamline can be constructed for this particular problem.



Problem 1.2-9 Streamlines and Streaklines in cylindrical coordinates

The velocity vector of a plane, unsteady flow field is given in cylindrical coordinates (r, φ) by

$$\vec{u} = \frac{1}{r}\left(A_0\,\vec{e}_r + B_0\,(1+at)\,\vec{e}_\varphi\right)$$

with the dimensional constants (A_0, B_0, a).
Using cylindrical coordinates, calculate

a) the equation of streamline through the place $P(r = r_0, \varphi = 0)$ and
b) the pathline equation of a fluid particle, which was at time $t = 0$ at place P.

Solution

a) The equation of streamline:
In cylindrical coordinates a differential line element is given by (F. M. (B.2,b))

$$\mathrm{d}\vec{x} = \mathrm{d}r\,\vec{e}_r + r\,\mathrm{d}\varphi\,\vec{e}_\varphi + \mathrm{d}z\,\vec{e}_z\ , \qquad (1)$$

and the velocity vector (F. M. (B.2,c))

$$\vec{u} = u_r\,\vec{e}_r + u_\varphi\,\vec{e}_\varphi + u_z\,\vec{e}_z\ . \qquad (2)$$

Thus, we arrive at:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}s} = \frac{\vec{u}}{|\vec{u}|}\,,$$

the three components of the equation for the streamline are

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{u_r}{|\vec{u}|}\,, \quad r\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{u_\varphi}{|\vec{u}|}\,, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{u_z}{|\vec{u}|}\,.$$

The first two equations are sufficient for describing the streamlines in the plane flow in the r,φ-plane. To eliminate the curve parameter s, we divide and get

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{u_r}{u_\varphi} .$$

Introducing $B(t) = B_0(1 + at)$, we arrive at

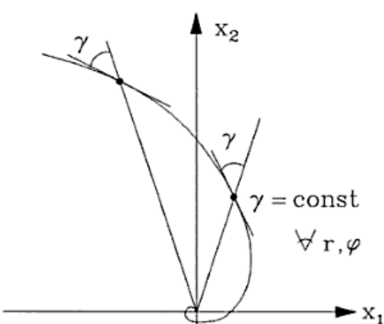
$$\frac{\mathrm{d}r}{r} = \frac{A_0}{B(t)}\,\mathrm{d}\varphi .$$

Since the streamlines are lines at fixed time, the time t and also $B(t)$ are considered as constants. The integration results for a streamline going through the point $(r_0, 0)$ in

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{r} = \frac{A_0}{B(t)} \int_0^{\varphi} \mathrm{d}\varphi$$

$$\Rightarrow \quad \ln\frac{r}{r_0} = \frac{A_0}{B(t)}\varphi \qquad \text{or} \tag{3}$$

$$r(\varphi) = r_0\,\mathrm{e}^{\frac{A_0}{B(t)}\varphi} . \tag{4}$$



As indicated previously, the auxiliary function $B(t)$ is a function of t and has only a parametric influence on the streamline. Equations (3) and (4) represent logarithmic spirals, i. e. all straight lines drawn from the origin, intersect the curve under the same angle γ.

b) Pathlines:
With (1) and (2) the differential equations are:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u_r , \quad r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = u_\varphi , \quad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z .$$

For the present plane flow, only the first two differential equations are used. For the velocity field in this problem, they assume the following form

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{A_0}{r}, \tag{5}$$

$$r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{B_0(1+at)}{r}. \tag{6}$$

Differential equation (6) is coupled with (5) via r. Since (5) is not coupled, we integrate it first:

$$\int r\,\mathrm{d}r = \int A_0\,\mathrm{d}t + C .$$

Introducing the initial conditions $r(t=0) = r_0$, the solution found is

$$r^2 = r_0^2 + 2\,A_0\,t . \tag{7}$$

Using (7), (6) can be integrated

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = B_0\,\frac{1+a\,t}{r_0^2+2A_0\,t} \quad \Rightarrow \quad \int\limits_0^{\varphi}\mathrm{d}\varphi = B_0\int\limits_0^{t}\frac{1+a\,t}{r_0^2+2A_0\,t}\,\mathrm{d}t \quad \Rightarrow$$

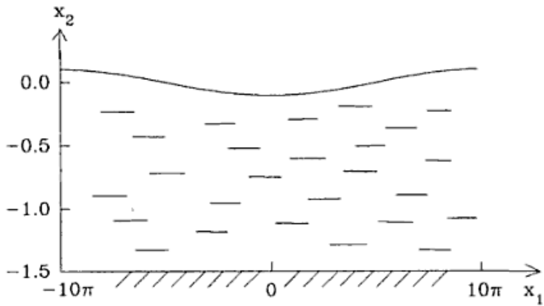
$$\begin{aligned}
\varphi(t) &= B_0\left[\frac{1}{2A_0}\,\ln(r_0^2+2A_0\,t) + a\left(\frac{t}{2A_0} - \frac{r_0^2}{4A_0^2}\,\ln(r_0^2+2A_0\,t)\right)\right]_0^t \\
&= \left[\left(\frac{B_0}{A_0} - \frac{B_0\,a\,r_0^2}{2\,A_0^2}\right)\frac{1}{2}\,\ln(r_0^2+2\,A_0\,t) + \frac{B_0\,a\,t}{2\,A_0}\right]_0^t \\
&= \left(\frac{B_0}{A_0} - \frac{B_0\,a\,r_0^2}{2A_0^2}\right)\ln\left(1+\frac{2A_0}{r_0^2}\,t\right)^{\frac{1}{2}} + \frac{B_0\,a\,t}{2A_0} .
\end{aligned}$$

Eliminating the pathline parameter t from (7) results in the following explicit representation

$$\varphi(r) = \left(\frac{B_0}{A_0} - \frac{B_0\,a\,r_0^2}{2A_0^2}\right)\ln\left(\frac{r}{r_0}\right) + \frac{B_0\,a}{4A_0^2}(r^2-r_0^2) . \tag{8}$$

It should be mentioned that by setting $a = 0$ ($\hat{=}B(t) = B_0$) a steady flow case is generated, for which the pathline equation (8) coincides with the streamline equation (4).

Problem 1.2-10 Streamlines and pathlines of standing gravity waves



The velocity field $u_i(x_j)$ with the components

$$
\begin{aligned}
u_1 &= -U \sin \Omega t \sin k x_1 \cosh k (x_2 + h) \,, \\
u_2 &= +U \sin \Omega t \cos k x_1 \sinh k (x_2 + h) \,, \\
u_3 &= 0
\end{aligned}
$$

describes a standing gravity wave in an horizontal liquid layer of depth h. The velocity U, the frequency Ω, the wave number k, and the depth h are constants.

a) Show that the velocity field represents a potential flow and determine the velocity potential $\Phi(x_1, x_2, t)$.

b) For sufficiently small amplitude the shape of free surface can be approximated by

$$\zeta = -\frac{1}{g} \left. \frac{\partial \Phi(x_1, x_2, t)}{\partial t} \right|_{x_2=0}$$

with g as the gravitational field strenght. Sketch the surface at the fixed time $t = 0$. Compute and sketch the streamlines and pathlines.

Given: U, frequency Ω, wave number k, depth h

Solution

a) The existence of a potential flow is proved since $\operatorname{curl} \vec{u} = 0$, i. e. :

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \,, \quad \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2} \,, \quad \frac{\partial u_3}{\partial x_1} = \frac{\partial u_1}{\partial x_3} \,.$$

The velocity potential Φ is obtained by integrating the differential equation $\partial \Phi / \partial x_i = u_i$

$$\Phi(x_1, \, x_2, \, t) = \frac{U}{k} \sin \Omega t \cos k x_1 \cosh k (x_2 + h) \,.$$

In the above equation the integration constant is omitted without loss of generality.

b) The surface at time $t = 0$:

$$\zeta = -\frac{U\,\Omega}{g\,k}\cosh kh\,\cos kx_1\ .$$

The unsteady flow field can be written in the form:

$$\vec{u}(\vec{x},t) = f(t)\,\vec{u}(\vec{x})\ .$$

The direction of the velocity vectors is thus time independent, i. e. the streamline curves coincide with the pathline's. These curves are calculated using the streamline differential equations in the following form

$$\frac{\mathrm{d}x_1}{u_1} = \frac{\mathrm{d}x_2}{u_2}\ .$$

Separation of the variables

$$-\frac{\mathrm{d}x_1}{\tan kx_1} = \frac{\mathrm{d}x_2}{\tanh k\,(x_2+h)}$$

and subsequent integration leads to

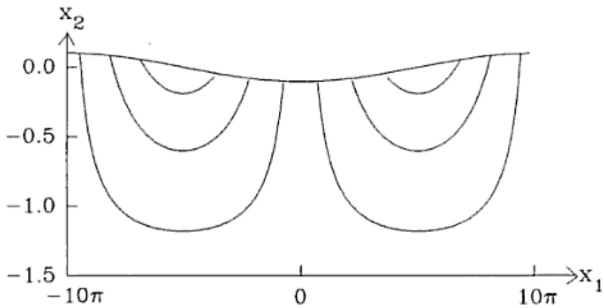
$$\begin{aligned} -\frac{1}{k}\ln(\sin kx_1) &= \frac{1}{k}\ln(\sinh k\,(x_2+h)) + \frac{1}{k}\ln C \\ \Rightarrow \qquad \frac{1}{C} &= \sinh k\,(x_2+h)\sin kx_1\ . \end{aligned}$$

We obtain the integration constant C from the requirement that the streamline is to go through the point $x_1 = x_{10}$, $x_2 = x_{20}$. As a result, we have

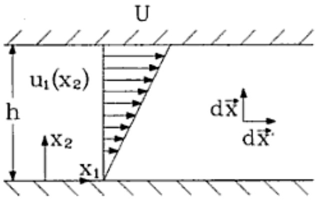
$$C = \frac{1}{\sinh k\,(x_{20}+h)\sin kx_{10}}\ .$$

Thus, the equation of streamlines in an explicit form is

$$x_2 = \frac{1}{k}\,\mathrm{arsinh}\left[\sinh k\,(x_{20}+h)\frac{\sin kx_{10}}{\sin kx_1}\right] - h\ .$$



Problem 1.2-11 Change of material line elements in a Couette-flow



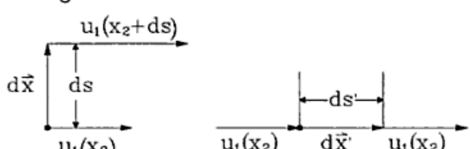
The velocity components of a Couette-flow is given by

$$u_1 = \frac{U}{h} x_2 \,, \quad u_2 = u_3 = 0 \,.$$

a) Determine the time rate of change of strain of material line elements $d\vec{x}$ and $d\vec{x}'$.
b) Obtain the angular velocities $D\varphi/Dt$ and $D\varphi'/Dt$ of the material line elements.
c) Determine the material change of the right angle between $d\vec{x}$ and $d\vec{x}'$.
d) Determine
   1) the velocity gradient $\partial u_i/\partial x_j$,
   2) the rate of deformation (rate of strain) tensor e_{ij} and
   3) the spin tensor Ω_{ij}.
e) Using the tensors from d), calculate the strain rate of the material elements $d\vec{x}$, $d\vec{x}'$, and the material change of the right angle between them.

Solution

a) The rate of change of material elements $d\vec{x}$ and $d\vec{x}'$:



The line elements have the directions of the coordinate axes

$$d\vec{x} = ds\,\vec{e}_2 \,,$$

$$d\vec{x}' = ds'\,\vec{e}_1 \,.$$

The velocity field is only x_2-dependent and only the x_1-component of $\vec{u}$ is different from zero.

The strain rate $1/\mathrm{d}s\,\mathrm{D}(\mathrm{d}s)/\mathrm{D}t$ corresponds to the component of the velocity difference $\mathrm{d}\vec{u}$ between both ends of the line element in direction of the element itself, divided by the length of the element $\mathrm{d}s$. Since $u_2 = u_3 = 0$, the velocity difference $\mathrm{d}\vec{u}$ has also only a x_1-component. It is

$$\mathrm{d}u_1 = u_1(x_2 + \mathrm{d}s) - u_1(x_2) = \frac{\mathrm{d}u_1}{\mathrm{d}x_2}\,\mathrm{d}s = \frac{U}{h}\,\mathrm{d}s\,. \tag{1}$$

The component in direction of the element $\mathrm{d}\vec{x}$ and thus the strain rate of $\mathrm{d}\vec{x}$ is zero.
For the element $\mathrm{d}\vec{x}'$, because of $u_1 = u_1(x_2)$ we may write

$$\mathrm{d}u_1' = u_1(x_2) - u_1(x_2) = 0\,, \tag{2}$$

i.e. the strain rate is zero.

b) Angular velocities of the elements:
The angular velocity of a line element $\mathrm{D}\varphi/\mathrm{D}t$ is calculated using the component of the velocity difference normal to the line element divided by the length of the element.
For $\mathrm{d}\vec{x}$ considering (1) we obtain

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = -\frac{\mathrm{d}u_1}{\mathrm{d}s} = -\frac{U}{h} = -\dot{\gamma}\,,$$

the negative sign refers to the convention that for positive $\mathrm{d}u$ the rotation occurs in a mathematically negative sense.
For $\mathrm{d}\vec{x}'$ follows from (2)

$$\frac{\mathrm{D}\varphi'}{\mathrm{D}t} = 0\,,$$

i.e. the element is neither stretched nor rotated.

c) The material change of the right angle between the line elements $\mathrm{d}\vec{x}$ and $\mathrm{d}\vec{x}'$:
The material change of the right angle is the difference of both angular velocities

$$\frac{\mathrm{D}\alpha_{12}}{\mathrm{D}t} = \frac{\mathrm{D}\varphi}{\mathrm{D}t} - \frac{\mathrm{D}\varphi'}{\mathrm{D}t} = -\frac{U}{h} - 0 = -\frac{U}{h} = -\dot{\gamma}\,.$$

d) Velocity gradient, rate of strain tensor, and the spin tensor:
   1) The only non-zero term of the velocity gradient $\partial u_i/\partial x_j$ is $\partial u_1/\partial x_2 = U/h = \dot{\gamma}$.
   2) The rate of deformation tensor e_{ij}:
   The rate of deformation tensor (rate of strain tensor) e_{ij} is the

symmetric part of the velocity gradient tensor $\partial u_i/\partial x_j$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) .$$

Since $e_{12} = e_{21}$, we have:

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\,\dot{\gamma} .$$

Again, the other components are zero.

3) Spin tensor Ω_{ij}:
The spin tensor is the antisymmetric part of $\partial u_i/\partial x_j$,

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) .$$

Thus

$$\Omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\frac{\partial u_1}{\partial x_2} = \frac{1}{2}\,\dot{\gamma} = -\Omega_{21} .$$

The other components disappear.

e) The strain rate:
The strain rate of a material element in direction of $\vec{l} = \mathrm{d}\vec{x}/\mathrm{d}s$ is

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = e_{ij}\, l_i\, l_j .$$

The direction vectors of both elements are

$$\mathrm{d}\vec{x}\ : \qquad \vec{l} = (0,1,0)\ ; \quad \mathrm{d}\vec{x}'\ : \qquad \vec{l}\,' = (1,0,0) .$$

Inserting e_{ij}, l_i, and l_j in the above relation, for the element $\mathrm{d}\vec{x}$ we obtain

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = e_{ij}\, l_i\, l_j = 0$$

and for $\mathrm{d}\vec{x}'$

$$\frac{1}{\mathrm{d}s'}\frac{\mathrm{D}(\mathrm{d}s')}{\mathrm{D}t} = e_{ij}\, l_i'\, l_j' = 0 .$$

In agreement with the results in a) we obtain both strain rates as zero. The material change of the right angle is calculated from:

$$\frac{\mathrm{D}\alpha_{12}}{\mathrm{D}t} = -2\, e_{12}$$

and $e_{12} = \frac{1}{2}\dot{\gamma}$:

$$\frac{\mathrm{D}\alpha_{12}}{\mathrm{D}t} = -\dot{\gamma}$$

which we already obtained under c).
Note:
For a plane flow, the only non-zero component of the angular velocity $\vec{\omega}$ is obtained from $\omega_n = \frac{1}{2}\,\epsilon_{ijn}\,\Omega_{ji}$ thus

$$\omega_3 = \frac{1}{2}\left(\epsilon_{123}\,\Omega_{21} + \epsilon_{213}\,\Omega_{12}\right) = \frac{1}{2}\left(-\frac{1}{2}\,\dot{\gamma} - \frac{1}{2}\,\dot{\gamma}\right) = -\frac{1}{2}\,\dot{\gamma}\,.$$

Problem 1.2-12 Change of material line elements in a three-dimensional flow

For a steady state flow field, the nondimensional velocity field is given by

$$\vec{u} = 3\,x_1^2\,x_2\,\vec{e}_1 + 2\,x_2^2\,x_3\,\vec{e}_2 + x_1\,x_2\,x_3^2\,\vec{e}_3\,.$$

Calculate at point $P = (1,1,1)$

a) the components of the velocity gradient $\partial u_i/\partial x_j$,
b) the components of the angular velocity of a fluid particle at P,
c) the components of the rate of deformation tensor e_{ij},
d) the strain rate in the x_1, x_2 and x_3-direction,
e) the material change of the right angle between $\mathrm{d}x_1$ and $\mathrm{d}x_2$ of a material volume element $\mathrm{d}V = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$,
f) the strain rate of a fluid element in its pathline direction,
g) the principal strain rates and their directions.

Solution

a) Velocity gradient:
The components of the velocity gradients are calculated from

$$\frac{\partial u_1}{\partial x_1} = 6x_1x_2 = 6 \qquad \frac{\partial u_2}{\partial x_1} = 0 \qquad \frac{\partial u_3}{\partial x_1} = x_2x_3^2 = 1$$
$$\frac{\partial u_1}{\partial x_2} = 3x_1^2 = 3 \qquad \frac{\partial u_2}{\partial x_2} = 4x_2x_3 = 4 \qquad \frac{\partial u_3}{\partial x_2} = x_1x_3^2 = 1$$
$$\frac{\partial u_1}{\partial x_3} = 0 \qquad \frac{\partial u_2}{\partial x_3} = 2x_2^2 = 2 \qquad \frac{\partial u_3}{\partial x_3} = 2x_1x_2x_3 = 2\,.$$

b) Angular velocity of a fluid particle:
From

$$\vec{\omega} = \frac{1}{2}\,\mathrm{curl}\,\vec{u} \Leftrightarrow \omega_i = \frac{1}{2}\,\epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

we obtain

$$\omega_1 = -\frac{1}{2}\,, \qquad \omega_2 = -\frac{1}{2}\,, \qquad \omega_3 = -\frac{3}{2}\,.$$

c) Rate of deformation tensor:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \Rightarrow \begin{cases} e_{11} = 6 & e_{12} = 3/2 & e_{13} = 1/2 \\ e_{21} = 3/2 & e_{22} = 4 & e_{23} = 3/2 \\ e_{31} = 1/2 & e_{32} = 3/2 & e_{33} = 2\,. \end{cases}$$

d) Strain rate in the coordinate directions:
For the material line element $\mathrm{d}\vec{x}$ with the direction vector $\vec{l} = \mathrm{d}\vec{x}/\mathrm{d}s$ we have

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = e_{ij}\, l_i\, l_j\,.$$

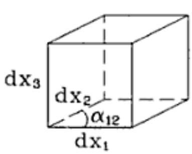
We successively insert for $\vec{l} = (1,0,0)$, $(0,1,0)$, $(0,0,1)$, and obtain the strain rates in the coordinate directions

$$\frac{1}{\mathrm{d}x_1}\frac{\mathrm{D}(\mathrm{d}x_1)}{\mathrm{D}t} = e_{11} = 6\,,$$

$$\frac{1}{\mathrm{d}x_2}\frac{\mathrm{D}(\mathrm{d}x_2)}{\mathrm{D}t} = e_{22} = 4\,,$$

$$\frac{1}{\mathrm{d}x_3}\frac{\mathrm{D}(\mathrm{d}x_3)}{\mathrm{D}t} = e_{33} = 2\,.$$

e) The material change of the right angle between $\mathrm{d}x_1$ and $\mathrm{d}x_2$ is



$$\frac{\mathrm{D}(\alpha_{12})}{\mathrm{D}t} = -2\,e_{12} = -3$$

similarly, the material changes of the right angles between $\mathrm{d}x_1$ and $\mathrm{d}x_3$, $\mathrm{d}x_2$ and $\mathrm{d}x_3$ respectively are

$$\frac{\mathrm{D}(\alpha_{13})}{\mathrm{D}t} = -2\,e_{13} = -1 \quad \text{and} \quad \frac{\mathrm{D}(\alpha_{23})}{\mathrm{D}t} = -2\,e_{23} = -3\,.$$

f) Strain rate in path direction:
The strain rate of a line element in the direction of the particle path is obtained by substituting the direction vector $\vec{l}$ in equation

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = e_{ij}\, l_i l_j$$

by the particle path direction, which is given by $\vec{t}$:

$$\vec{t} = \frac{\vec{u}}{|\vec{u}|}\,.$$

Since $\vec{u}(P = (1,1,1)) = (3,2,1)$, i.e. $|\vec{u}| = \sqrt{14}$, one can obtain the components of the normalized direction vector as:

$$t_1 = \frac{1}{\sqrt{14}}3\,, \qquad t_2 = \frac{1}{\sqrt{14}}2\,, \qquad t_3 = \frac{1}{\sqrt{14}}1\,.$$

Thus, we arrive at the strain rate

$$\begin{aligned}\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} &= e_{11}\,t_1\,t_1 + e_{12}\,t_1\,t_2 + e_{13}\,t_1\,t_3 + \\ &\quad + e_{21}\,t_2\,t_1 + e_{22}\,t_2\,t_2 + e_{23}\,t_2\,t_3 + \\ &\quad + e_{31}\,t_3\,t_1 + e_{32}\,t_3\,t_2 + e_{33}\,t_3\,t_3\,.\end{aligned}$$

Because of $e_{ij} = e_{ji}$, the above equation can be rearranged as

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = e_{11}\,t_1^2 + e_{22}\,t_2^2 + e_{33}\,t_3^2 + 2\,(e_{12}\,t_1\,t_2 + e_{13}\,t_1\,t_3 + e_{23}\,t_2\,t_3)\,.$$

At point $P = (1,1,1)$ the numerical value calculated is

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = 6\frac{9}{14} + 4\frac{4}{14} + 2\frac{1}{14} + \frac{2}{14}\left(\frac{3}{2}6 + \frac{1}{2}3 + \frac{3}{2}2\right) = \frac{99}{14}\,.$$

g) Principal strain rates and directions:
The calculation leads to the eigenvalue problem

$$(e_{ij} - e\delta_{ij})\,l_j = 0$$

with e as the principal strain (= eigenvalue) and $\vec{l}$ as unit vector in the principal direction (= eigenvector). Non-trivial solutions exist only if the determinant of the coefficient matrix $(e_{ij} - e\delta_{ij})$ identically vanishes, i.e. if

$$\det\begin{pmatrix} e_{11}-e & e_{12} & e_{13} \\ e_{21} & e_{22}-e & e_{23} \\ e_{31} & e_{32} & e_{33}-e \end{pmatrix} = 0\,.$$

This condition leads to the characteristic equation:

$$-e^3 + I_{1e}e^2 - I_{2e}e + I_{3e} = 0\,,$$

which allows the calculation of the three eigenvalues. The invariants of the strain tensor are

$$\begin{aligned} I_{1e} &= e_{ii} = 6 + 4 + 2 = 12\,, \\ I_{2e} &= \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ij}) \end{aligned}$$

with

$$\begin{aligned}
e_{ij}e_{ij} &= e_{11}e_{11} + e_{12}e_{12} + e_{13}e_{13} + \\
&+ e_{21}e_{21} + e_{22}e_{22} + e_{23}e_{23} + \\
&+ e_{31}e_{31} + e_{32}e_{32} + e_{33}e_{33} \\
&= e_{11}^2 + e_{22}^2 + e_{33}^2 + 2\left(e_{12}^2 + e_{13}^2 + e_{23}^2\right) \\
&= 6^2 + 4^2 + 2^2 + 2\left(\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2\right) = 65.5
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad I_{2e} &= \frac{1}{2}(12^2 - 65.5) = 39.25\ , \\
I_{3e} &= \det(e_{ij}) \\
\Rightarrow \quad I_{3e} &= 6\left(8 - \frac{9}{4}\right) + \frac{3}{2}\left(\frac{3}{4} - 3\right) + \frac{1}{2}\left(\frac{9}{4} - 2\right) = 31.25\ .
\end{aligned}$$

The three roots of the third order polynomial can be calculated using appropriate solution methods such as Newton's.

$$e^{(1)} = 1.180\ ,\ e^{(2)} = 3.741\ ,\ e^{(3)} = 7.079\ .$$

These solutions are the principal strain rates. The components of the strain rate tensor in direction of the principal axes are:

$$e'_{ij} = \begin{cases} e^{(i)} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j\ . \end{cases}$$

Using the above eigenvalues, the eigenvectors are calculated from the system of linear equations $(e_{ij} - e\delta_{ij})\, l_j = 0$. Since the determinant of this system of equations disappears, only two equations are linearly independent. A unique solution results from normalizing:

$$l_1^2 + l_2^2 + l_3^2 = 1\ .$$

A simpler alternative is to calculate first the vector $\vec{l}'$, which is not normalized and perform the normalization later. We delete the third equation and set instead $l'_3 = 1$. We obtain the first eigenvector from

$$i = 1 \ :\ (6 - 1.18)\, l'_1 + \frac{3}{2} l'_2 + \frac{1}{2} = 0\ ,$$

$$i = 2 \ :\ \frac{3}{2} l'_1 + (4 - 1.18)\, l'_2 + \frac{3}{2} = 0\ ,$$

using the Cramer's rule:

$$l'_1 = 0.07406\ , \quad l'_2 = -0.5713\ .$$

Thus the vector components of the non normalized eigenvector are

$$l'_1 = 0.07406\ , \qquad l'_2 = -0.5713\ , \qquad l'_3 = 1\ .$$

Normalizing the above components leads to

$$l_1^{(1)} = 0.06417\ , \qquad l_2^{(1)} = -0.4950\ , \qquad l_3^{(1)} = 0.8665\ .$$

The components of the second and third eigenvectors can be calculated in a similar way. For the second eigenvector we insert $e = e^{(2)} = 3.741$, which results in

$$l_1^{(2)} = -0.558\ , \qquad l_2^{(2)} = 0.702\ , \qquad l_3^{(2)} = 0.442\ .$$

The third eigenvector may be calculated the same way. It is worth noting that the calculation can be performed in a much simpler way by taking advantage of the fact that

$$\vec{l}^{(3)} = \vec{l}^{(1)} \times \vec{l}^{(2)} = -\begin{pmatrix} 0.825 \\ 0.515 \\ 0.233 \end{pmatrix}\ .$$

Calculating $\vec{l}^{(3)}$ from the above equation also determines the sign of $\vec{l}^{(3)}$. The vector $\vec{l}^{(1)}$ is in direction of $e^{(1)}$, $\vec{l}^{(2)}$ in direction of $e^{(2)}$, and $\vec{l}^{(3)}$ in direction of $e^{(3)}$.

Problem 1.2-13 Angular velocity vector and the change of material line elements in a two-dimensional flow field

Given is the velocity field:

$$\begin{aligned} u_1 &= -\frac{\omega}{h}\, x_2\, x_3\ , \\ u_2 &= +\frac{\omega}{h}\, x_1\, x_3\ , \\ u_3 &= 0\ . \end{aligned}$$

Determine the components of
a) the velocity gradient tensor,
b) rate of deformation tensor e_{ij} and the spin tensor Ω_{ij},

c) the angular velocity vector $\vec{\omega}$.

Calculate

d) the principal strain rates and the principal strain rate directions at point $P = (2,2,2)$ and

e) the pathline of the particle which at time $t = 0$ was at place $P = (2,2,2)$.

Solution

a) Velocity gradient tensor:

$$\frac{\partial u_1}{\partial x_1} = 0 \qquad \frac{\partial u_2}{\partial x_1} = x_3\,\frac{\omega}{h} \qquad \frac{\partial u_3}{\partial x_1} = 0$$

$$\frac{\partial u_1}{\partial x_2} = -x_3\,\frac{\omega}{h} \qquad \frac{\partial u_2}{\partial x_2} = 0 \qquad \frac{\partial u_3}{\partial x_2} = 0$$

$$\frac{\partial u_1}{\partial x_3} = -x_2\,\frac{\omega}{h} \qquad \frac{\partial u_2}{\partial x_3} = x_1\,\frac{\omega}{h} \qquad \frac{\partial u_3}{\partial x_3} = 0\,.$$

b) The components of the tensors e_{ij} and Ω_{ij}:

$$e_{11} = 0 \qquad e_{12} = 0 \qquad e_{13} = -\frac{x_2}{2}\,\frac{\omega}{h}$$

$$e_{21} = 0 \qquad e_{22} = 0 \qquad e_{23} = \frac{x_1}{2}\,\frac{\omega}{h}$$

$$e_{31} = -\frac{x_2}{2}\,\frac{\omega}{h} \qquad e_{32} = \frac{x_1}{2}\,\frac{\omega}{h} \qquad e_{33} = 0\,;$$

$$\Omega_{11} = 0 \qquad \Omega_{12} = -x_3\,\frac{\omega}{h} \qquad \Omega_{13} = -\frac{x_2}{2}\,\frac{\omega}{h}$$

$$\Omega_{21} = x_3\,\frac{\omega}{h} \qquad \Omega_{22} = 0 \qquad \Omega_{23} = \frac{x_1}{2}\,\frac{\omega}{h}$$

$$\Omega_{31} = \frac{x_2}{2}\,\frac{\omega}{h} \qquad \Omega_{32} = -\frac{x_1}{2}\,\frac{\omega}{h} \qquad \Omega_{33} = 0\,.$$

c) The components of the angular velocity vector $\vec{\omega}$:

$$\omega_i = \frac{1}{2}\,\epsilon_{ijk}\,\Omega_{kj}$$

with the individual components

$$\begin{aligned}
\omega_1 &= \frac{1}{2}\left(\epsilon_{123}\Omega_{32} + \epsilon_{132}\Omega_{23}\right) = \Omega_{32} = -\frac{x_1}{2}\,\frac{\omega}{h}\,,\\
\omega_2 &= \frac{1}{2}\left(\epsilon_{231}\Omega_{13} + \epsilon_{213}\Omega_{31}\right) = \Omega_{13} = -\frac{x_2}{2}\,\frac{\omega}{h}\,,\\
\omega_3 &= \frac{1}{2}\left(\epsilon_{312}\Omega_{21} + \epsilon_{321}\Omega_{12}\right) = \Omega_{21} = x_3\,\frac{\omega}{h}\,.
\end{aligned}$$

Note: Identifying the components Ω_{ij} of the spin tensor with the angular velocity components, we obtain

$$\begin{array}{lll} \Omega_{11} = 0 & \Omega_{12} = -\omega_3 & \Omega_{13} = \omega_2 \\ \Omega_{21} = \omega_3 & \Omega_{22} = 0 & \Omega_{23} = -\omega_1 \\ \Omega_{31} = -\omega_2 & \Omega_{32} = \omega_1 & \Omega_{33} = 0 \,. \end{array}$$

The three independent components of the antisymmetric spin tensor Ω_{ij} correspond to the components of $\vec{\omega}$.

d) Principal strains and principal strain rate directions:
The following eigenvalue problem is to be solved

$$(e_{ij} - e\,\delta_{ij})\,l_j = 0 \,.$$

The rate of deformation tensor at point $P = (2,2,2)$ is given by

$$\begin{array}{lll} e_{11} = 0 & e_{12} = 0 & e_{13} = -\dfrac{\omega}{h} \\ e_{21} = 0 & e_{22} = 0 & e_{23} = \dfrac{\omega}{h} \\ e_{31} = -\dfrac{\omega}{h} & e_{32} = \dfrac{\omega}{h} & e_{33} = 0 \,. \end{array}$$

The eigenvalues are calculated (with $\tilde{e} = e\,h/\omega$) from

$$\det(e_{ij} - \tilde{e}\delta_{ij}) = \det \begin{pmatrix} -\tilde{e} & 0 & -1 \\ 0 & -\tilde{e} & 1 \\ -1 & 1 & -\tilde{e} \end{pmatrix} \stackrel{!}{=} 0 \,,$$

that is, from the characteristic polynomial:

$$\begin{aligned} -\tilde{e}\,(\tilde{e}^2 - 1) - 1\,(-\tilde{e}) &= 0 \\ \Rightarrow \tilde{e}\,(\tilde{e}^2 - 2) &= 0 \,. \end{aligned}$$

$$\Rightarrow \quad \tilde{e} = 0\,, -\sqrt{2}\,, +\sqrt{2}\,.$$

Therefore the three eigenvalues are given by

$$e^{(1)} = -\sqrt{2}\,\frac{\omega}{h}\,, \quad e^{(2)} = 0\,, \quad e^{(3)} = +\sqrt{2}\,\frac{\omega}{h}$$

with $e = \tilde{e}\,\omega/h$.

With the above known eigenvalues, the eigenvectors (= principal strain rate direction) are determined from the following system of homogeneous equations

$$\begin{pmatrix} -\tilde{e} & 0 & -1 \\ 0 & -\tilde{e} & 1 \\ -1 & 1 & -\tilde{e} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

Since in the above system only two linearly independent equations exist, we may first delete one equation and set $l_1' = 1$. As a result, we obtain the inhomogeneous linear system

$$i = 2 \quad : \quad -\tilde{e}\, l_2' + l_3' = 0$$

$$i = 3 \quad : \quad l_2' - \tilde{e}\, l_3' - 1 = 0 \, ,$$

with the solution

$$l_2' = -\frac{1}{\tilde{e}^2 - 1} \, , \quad l_3' = -\frac{\tilde{e}}{\tilde{e}^2 - 1} \, .$$

The corresponding eigenvectors are calculated subsequently as

$$\begin{array}{lll} \tilde{e}^{(1)} = -\sqrt{2} & \tilde{e}^{(2)} = 0 & \tilde{e}^{(3)} = \sqrt{2} \\ l_2' = -1 \, , \; l_3' = \sqrt{2} & l_2' = 1 \, , \; l_3' = 0 & l_2' = -1 \, , \; l_3' = -\sqrt{2} \\ |\vec{l}'| = 2 & |\vec{l}'| = \sqrt{2} & |\vec{l}'| = 2 \, . \end{array}$$

The results in normalized form are:

$$\vec{l}^{(1)} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} , \qquad \vec{l}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} , \qquad \vec{l}^{(3)} = \pm\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix} .$$

To determine the sign of the vector $\vec{l}^{(3)}$ we require that $\vec{l}^{(1)}$, $\vec{l}^{(2)}$, and $\vec{l}^{(3)}$ should form a right-handed system, i.e. $\vec{l}^{(3)} = \vec{l}^{(1)} \times \vec{l}^{(2)}$:

$$\begin{aligned} \vec{l}^{(1)} \times \vec{l}^{(2)} &= \det \begin{pmatrix} \vec{e}^{(1)} & \vec{e}^{(2)} & \vec{e}^{(3)} \\ 1/2 & -1/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{pmatrix} \\ &= \vec{e}^{(1)} \left(-\frac{1}{2}\right) + \vec{e}^{(2)} \left(\frac{1}{2}\right) + \vec{e}^{(3)} \left(\frac{\sqrt{2}}{2}\right) \, . \end{aligned}$$

We obtain

$$l_1^{(3)} = -\frac{1}{2} \, , \quad l_2^{(3)} = \frac{1}{2} \, , \quad l_3^{(3)} = \frac{\sqrt{2}}{2} \, .$$

The rate of deformation tensor in principal axes system is

$$(e_{ij}) = \begin{pmatrix} e^{(1)} & 0 & 0 \\ 0 & e^{(2)} & 0 \\ 0 & 0 & e^{(3)} \end{pmatrix} .$$

The rotation matrix, which transfers the old coordinate system into the principal axes system is

$$a_{ij} = \vec{e}_i \cdot \vec{e}_j{}' = \vec{e}_i \cdot \vec{l}^{(j)} .$$

Thus, a_{ij} is the i-th component of the j-th eigenvector

$$(a_{ij}) = \begin{pmatrix} 1/2 & \sqrt{2}/2 & -1/2 \\ -1/2 & \sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} .$$

The eigenvectors form the columns of the rotation matrix.

e) Pathline through the point $P = (2, 2, 2)$:
The differential equations are

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\frac{\omega}{h} x_2 x_3 ,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\omega}{h} x_1 x_3 ,$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = 0 \quad \text{or} \quad x_3 = \text{const} = \xi_3 .$$

The first two equations along with the results of the third one are

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\frac{\omega}{h} \xi_3 x_2 ,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\omega}{h} \xi_3 x_1 .$$

The time t does not appear explicitly, i.e. pathlines and streamlines coincide. Thus, we divide the two equations by each other:

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_1}{x_2} .$$

Separation of variables and the integration yield

$$\frac{x_1^2}{2} = -\frac{x_2^2}{2} + \frac{C}{2} \quad \Rightarrow \quad x_1^2 + x_2^2 = C .$$

The pathlines are circles in the plane $x_3 = \xi_3$.
From the condition $\vec{x}(t=0) = \vec{\xi} = (2,2,2)$ then follows $C = 8$ and we get

$$x_1^2 + x_2^2 = 8\ , \quad x_3 = \xi_3 = 2\ .$$

Problem 1.2-14 Rate of deformation and spin tensors of an unsteady two-dimensional flow

The velocity components of an unsteady flow field are given as

$$\begin{aligned} u_1 &= 0\ , \\ u_2 &= A(x_1 x_2 - x_3^2)e^{-B(t-t_0)}\ , \\ u_3 &= A(x_2^2 - x_1 x_3)e^{-B(t-t_0)}\ . \end{aligned}$$

Determine the components of

a) the velocity gradient $\partial u_i/\partial x_j$,

b) the rate of deformation tensor e_{ij} and the spin tensor Ω_{ij}

as well as

c) $\operatorname{curl}\vec{u}$ at point $P = (1,0,3)$ and time $t = t_0$.

Solution

a) The components of the velocity gradient tensor are:

$$\begin{array}{lll} \dfrac{\partial u_1}{\partial x_1} = 0 & \dfrac{\partial u_2}{\partial x_1} = x_2\, A\, \mathrm{e}^{-B(t-t_0)} & \dfrac{\partial u_3}{\partial x_1} = -x_3\, A\, \mathrm{e}^{-B(t-t_0)} \\ \dfrac{\partial u_1}{\partial x_2} = 0 & \dfrac{\partial u_2}{\partial x_2} = x_1\, A\, \mathrm{e}^{-B(t-t_0)} & \dfrac{\partial u_3}{\partial x_2} = 2x_2\, A\, \mathrm{e}^{-B(t-t_0)} \\ \dfrac{\partial u_1}{\partial x_3} = 0 & \dfrac{\partial u_2}{\partial x_3} = -2x_3\, A\, \mathrm{e}^{-B(t-t_0)} & \dfrac{\partial u_3}{\partial x_3} = -x_1\, A\, \mathrm{e}^{-B(t-t_0)}\ . \end{array}$$

b) The rate of deformation tensor e_{ij} and the spin tensor Ω_{ij} are calculated as:

$$
\begin{aligned}
e_{11} &= 0\,, & e_{12} &= \frac{x_2}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
e_{13} &= -\frac{x_3}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,, & e_{21} &= \frac{x_2}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
e_{22} &= x_1\,A\,\mathrm{e}^{-B(t-t_0)}\,, & e_{23} &= (x_2-x_3)\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
e_{31} &= -\frac{x_3}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,, & e_{32} &= (x_2-x_3)\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
e_{33} &= -x_1\,A\,\mathrm{e}^{-B(t-t_0)}\,;
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{11} &= 0\,, & \Omega_{12} &= -\tfrac{x_2}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
\Omega_{13} &= \tfrac{x_3}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,, & \Omega_{21} &= \tfrac{x_2}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
\Omega_{22} &= 0\,, & \Omega_{23} &= -(x_2+x_3)\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
\Omega_{31} &= -\tfrac{x_3}{2}\,A\,\mathrm{e}^{-B(t-t_0)}\,, & \Omega_{32} &= (x_2+x_3)\,A\,\mathrm{e}^{-B(t-t_0)}\,,\\
\Omega_{33} &= 0\,.
\end{aligned}
$$

c) From

$$
\operatorname{curl}\vec{u} \;=\; \begin{pmatrix} 2x_2+2x_3 \\ 0+x_3 \\ x_2-0 \end{pmatrix} A\,\mathrm{e}^{-B(t-t_0)}\,,
$$

follows the $\operatorname{curl}\vec{u}$ at point $P=(1,0,3)$ and time $t=t_0$:

$$
\operatorname{curl}\vec{u} \;=\; \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} A\,.
$$

Problem 1.2-15 Time change of the kinetic energy of a fluid body

The velocity components of a two-dimensional flow with constant density ϱ are given in cylindrical coordinates:

$$u_r = \frac{A}{r}\,, \quad u_\varphi = u_z = 0 \qquad (A\text{=const})\,.$$

Consider a portion of the fluid at time $t = 0$ located between the surfaces of two concentric cylinders with radius $r = a$ and $r = b$ ($b > a$) where $0 \leq z \leq L$.

Calculate

a) the pathline of the particles located on the inner and outer cylinder surfaces at $t = 0$.
How does the material volume look like at time t?

b) the kinetic energy
$$K(t) = \iiint\limits_{(V(t))} \frac{\varrho}{2} u^2 \,\mathrm{d}V$$
and the momentum
$$\vec{P}(t) = \iiint\limits_{(V(t))} \varrho \vec{u} \,\mathrm{d}V$$
of the fluid under consideration at any arbitrarily chosen time t and the substantial changes $\mathrm{D}K/\mathrm{D}t$ and $\mathrm{D}\vec{P}/\mathrm{D}t$.

c) $\mathrm{D}K/\mathrm{D}t$, using Reynolds' transport theorem.

d) Describe the motion in material coordinates and calculate $\mathrm{D}K/\mathrm{D}t$ by transforming back to the volume V_0, which was occupied by the fluid at time $t = 0$.

Solution

a) Pathlines:
The differential equations of the pathlines in cylindrical coordinates are in general (see Problem 1.2-9)
$$\frac{\mathrm{d}r}{\mathrm{d}t} = u_r\,, \quad r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = u_\varphi\,, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z\,,$$
and here
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{A}{r}\,, \quad r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0\,, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0\,.$$
These three differential equations are decoupled and can be solved successively as follows:
$$\int r\,\mathrm{d}r = \int A\mathrm{d}t \quad \Leftrightarrow \quad \frac{r^2}{2} = At + C_1\,,$$

$$\int \mathrm{d}\varphi = 0 \qquad \Leftrightarrow \qquad \varphi = C_2 \,,$$

$$\int \mathrm{d}z = 0 \qquad \Leftrightarrow \qquad z = C_3 \,.$$

For a fluid particle located at time $t = 0$ on the inner cylinder ($r = a$), the integration constants are calculated as

$$C_1 = \frac{a^2}{2} \,, \quad C_2 = \varphi(t = 0) \,, \quad C_3 = z(t = 0) \,,$$

and the pathline is

$$r^2(t) = a^2 + 2\,A\,t \quad , \quad \varphi = \varphi(t = 0) \,, \quad z = z(t = 0) \,.$$

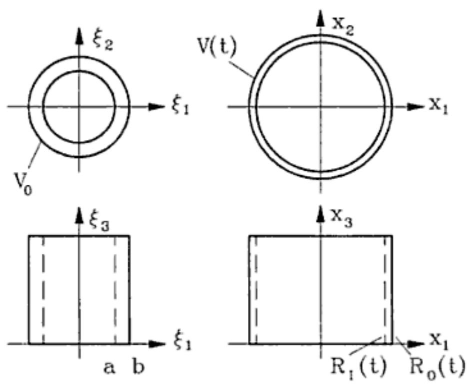
Correspondingly, we obtain for the particle on the outer cylinder ($r = b$, at time $t = 0$)

$$r^2(t) = b^2 + 2\,A\,t \,, \quad \varphi, z = \text{const} \,.$$

The boundaries of the material volume experience a displacement and the pathline equations are given as

$$R_I^2(t) = a^2 + 2\,A\,t \qquad \text{for the inner surface,}$$

$$R_O^2(t) = b^2 + 2\,A\,t \qquad \text{for the outer surface.}$$



The sketch shows the material volume V_0 and $V(t)$ occupied by the fluid body at time $t = 0$ and an arbitrary time t respectively.

b) Calculation of the kinetic energy $K(t)$, the momentum $\vec{P}(t)$, and their substantial changes $\mathrm{D}K/\mathrm{D}t$ and $\mathrm{D}\vec{P}/\mathrm{D}t$:

$$K(t) = \iiint\limits_{(V(t))} \frac{\varrho}{2} u^2 \,\mathrm{d}V = \iiint\limits_{(V(t))} \frac{\varrho}{2} (u_r^2 + u_\varphi^2 + u_z^2) \,\mathrm{d}V = \iiint\limits_{(V(t))} \frac{\varrho}{2} u_r^2 \,\mathrm{d}V$$

$$\Rightarrow \quad K(t) = \int_0^{2\pi} \int_0^L \int_{R_I(t)}^{R_O(t)} \frac{\varrho}{2} A^2 \frac{1}{r^2} r \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}\varphi$$

$$\Rightarrow \quad K(t) = \pi \varrho A^2 L \ln \left(\frac{b^2 + 2At}{a^2 + 2At} \right)^{\frac{1}{2}} .$$

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\pi}{2} \varrho A^2 L \ln \left(\frac{b^2 + 2At}{a^2 + 2At} \right) \right]$$

$$\Rightarrow \quad \frac{\mathrm{D}K}{\mathrm{D}t} = -\pi \varrho A^3 L \left(\frac{1}{R_I^2(t)} - \frac{1}{R_O^2(t)} \right) .$$

For the momentum we obtain

$$\vec{P}(t) = \iiint\limits_{(V(t))} \varrho \vec{u} \,\mathrm{d}V = \int_0^{2\pi} \int_0^L \int_{R_I(t)}^{R_O(t)} \varrho \frac{A}{r} \vec{e}_r \, r \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}\varphi$$

$$= \varrho A L (R_O(t) - R_I(t)) \int_0^{2\pi} \vec{e}_r \,\mathrm{d}\varphi$$

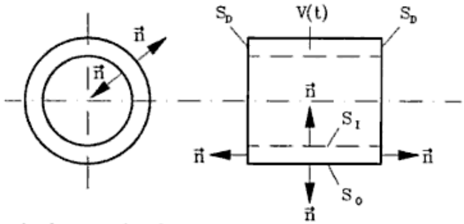
$$\Rightarrow \quad \vec{P}(t) = 0$$

and thus, $\mathrm{D}\vec{P}/\mathrm{D}t = 0$.

c) Calculation of $\mathrm{D}K/\mathrm{D}t$ using Reynolds' transport theorem:
The Reynolds transport theorem (see F.M. 1.96) applied to K is:

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \frac{\partial}{\partial t} \iiint\limits_{(V)} \frac{\varrho}{2} u^2 \,\mathrm{d}V + \iint\limits_{(S)} \frac{\varrho}{2} u^2 \, \vec{u} \cdot \vec{n} \,\mathrm{d}S .$$

Since the flow is steady, the first volume integral on the right hand side vanishes.



For the sketched control volume we get

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \iint\limits_{(S)} \frac{\varrho}{2} u^2 \, \vec{u} \cdot \vec{n} \, \mathrm{d}S$$

$$\begin{aligned}
\Rightarrow \quad \frac{\mathrm{D}K}{\mathrm{D}t} &= \iint\limits_{S_I} \frac{\varrho}{2} u_r^2 \, \vec{u} \cdot \vec{n} \, \mathrm{d}S + \iint\limits_{S_O} \frac{\varrho}{2} u_r^2 \, \vec{u} \cdot \vec{n} \, \mathrm{d}S + \iint\limits_{S_D} \frac{\varrho}{2} u_r^2 \, \vec{u} \cdot \vec{n} \, \mathrm{d}S \\
&= -\iint\limits_{S_I} \frac{\varrho}{2} u_r^3 \, \mathrm{d}S + \iint\limits_{S_O} \frac{\varrho}{2} u_r^3 \, \mathrm{d}S \\
&= -\iint\limits_{S_I} \frac{\varrho}{2} \frac{A^3}{r^3} \, \mathrm{d}S + \iint\limits_{S_O} \frac{\varrho}{2} \frac{A^3}{r^3} \, \mathrm{d}S \\
&= -\frac{\varrho}{2} \frac{A^3}{R_I^3} 2\pi \, R_I L + \frac{\varrho}{2} \frac{A^3}{R_O^3} 2\pi \, R_O L \\
\Rightarrow \quad \frac{\mathrm{D}K}{\mathrm{D}t} &= -\pi \, \varrho \, A^3 \, L \left(\frac{1}{R_I^2} - \frac{1}{R_O^2} \right) .
\end{aligned}$$

With $R_I = R_I(t)$ and $R_O = R_O(t)$, the above result is identical to the one found under b).

d) Using (ρ, ϕ, ζ) for the material (r, φ, z)-coordinates, the material description of motion is

$$r(t) = \sqrt{\rho^2 + 2\,A\,t} \,, \quad \varphi = \phi \,, \quad z = \zeta \,.$$

With the Jacobian determinant of this motion

$$\frac{\partial(r, \varphi, z)}{\partial(\rho, \phi, \zeta)} = \frac{\rho}{\sqrt{\rho^2 + 2\,A\,t}}$$

the integration over the time dependent volume $V(t)$ can be transformed back to the integral over a fixed control volume V_0:

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\int_0^{2\pi}\int_0^{L}\int_{\sqrt{a^2+2At}}^{\sqrt{b^2+2At}} \frac{\varrho}{2}\frac{A^2}{r^2}\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\varphi$$

$$= \frac{\mathrm{D}}{\mathrm{D}t}\int_0^{2\pi}\int_0^{L}\int_a^{b} \frac{\varrho}{2}\frac{A^2}{\rho^2+2At}\sqrt{\rho^2+2At}\,\frac{\rho}{\sqrt{\rho^2+2At}}\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,.$$

Because V_0 is time independent, the operator D/Dt can be taken under the integral sign. Thus, we have

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \pi L A^2 \varrho \int_a^b \frac{\mathrm{D}}{\mathrm{D}t}\frac{\rho}{\rho^2+2At}\,\mathrm{d}\rho$$

$$= -\pi L A^2 \varrho \int_a^b \left[\frac{\rho\, 2A}{(\rho^2+2At)^2}\right]\mathrm{d}\rho$$

$$= \pi L A^3 \varrho \left[\frac{1}{\rho^2+2At}\right]_a^b$$

$$\frac{\mathrm{D}K}{\mathrm{D}t} = -\pi\varrho A^3 L\left(\frac{1}{R_I^2(t)}-\frac{1}{R_O^2(t)}\right)$$

and obtain again the results from part b).

2 Fundamental Laws of Continuum Mechanics

2.1 Conservation of Mass, Equation of Continuity

Problem 2.1-1 One-dimensional unsteady flow with given density field

A one-dimensional unsteady flow is given by the following velocity

$$u = \frac{2}{\gamma+1}\left(\frac{x}{t} - a_0\right)$$

and density field

$$\frac{\varrho}{\varrho_0} = \left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{t}\,\frac{1}{a_0} + \frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} .$$

a) Calculate the substantial change of the density.
b) Check the validity of the continuity equation

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\frac{\partial u}{\partial x} = 0$$

for this flow field!

c) What is the change of density that a swimmer senses if he/she swims with the velocity $c = u + a$ or $c = u - a$ through the flow field? Use the relation

$$\frac{a}{a_0} = \left(\frac{\varrho}{\varrho_0}\right)^{\frac{\gamma-1}{2}} .$$

Solution

a) The substantial change of the density is

$$\begin{aligned}
\frac{\mathrm{D}\varrho}{\mathrm{D}t} &= \frac{\partial \varrho}{\partial t} + u\,\frac{\partial \varrho}{\partial x} \\
&= \frac{2\varrho_0}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}-1}\left(-\frac{\gamma-1}{\gamma+1}\,\frac{x}{a_0}\,\frac{1}{t^2}\right)+ \\
&+ \frac{2}{\gamma+1}\left(\frac{x}{t}-a_0\right)\frac{2\varrho_0}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{ta_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}-1}\left(\frac{\gamma-1}{\gamma+1}\,\frac{1}{t\,a_0}\right) \\
&= -\frac{2\varrho_0}{t\,(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}\ .
\end{aligned}$$

b) Check the validity of the continuity equation:
The second term of the continuity equation remains to be calculated. Performing the differentiation, we get

$$\varrho\,\frac{\partial u}{\partial x} = \varrho_0\left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{t}\,\frac{1}{a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}\ \frac{2}{\gamma+1}\,\frac{1}{t}\,.$$

We introduce the above results into the continuity equation and obtain

$$\begin{aligned}
\frac{\mathrm{D}\varrho}{\mathrm{D}t}+\varrho\,\frac{\partial u}{\partial x} &= -\frac{2\varrho_0}{t\,(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} \\
&\quad +\frac{2\varrho_0}{t\,(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\,\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}\ , \\
\Rightarrow \qquad \frac{\mathrm{D}\varrho}{\mathrm{D}t}+\varrho\,\frac{\partial u}{\partial x} &= 0\,.
\end{aligned}$$

c) Change of the density in the moving system:
The swimmer feels a density change described by the equation:

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \frac{\partial \varrho}{\partial t} + c\,\frac{\partial \varrho}{\partial x}\,.$$

With $c = u \pm a$ we have

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \frac{\partial \varrho}{\partial t} + u\,\frac{\partial \varrho}{\partial x} \pm a\,\frac{\partial \varrho}{\partial x} = \frac{\mathrm{D}\varrho}{\mathrm{D}t} \pm a\,\frac{\partial \varrho}{\partial x}\,.$$

Replacing the material derivative of density by the result from a), we find the expression

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\frac{2\varrho_0}{t(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} \pm a_0\left(\frac{\gamma-1}{\gamma+1}\frac{x}{t}\frac{1}{a_0}+\frac{2}{\gamma+1}\right)$$

$$\frac{2\varrho_0}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\frac{x}{t}\frac{1}{a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}-1}\left(\frac{\gamma-1}{\gamma+1}\frac{1}{t\,a_0}\right)$$

$$= -\frac{2\varrho_0}{t(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} +$$

$$\pm\frac{2\varrho_0}{t(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\frac{x}{t}\frac{1}{a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} .$$

Thus, the swimmer senses the following density changes:

for $c = u + a$: $$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = 0 ,$$

for $c = u - a$: $$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\frac{4\varrho_0}{t(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\frac{x}{t\,a_0}+\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} .$$

Problem 2.1-2 Plane, steady flow with a given density field

The density field of a plane, steady flow is given by

$$\varrho(x_i) = k\,x_1\,x_2 , \qquad k = \text{const} .$$

a) Determine the velocity field, for which the flow is incompressible.
b) Find the pathline equation.

Solution

a) For an incompressible flow the material derivative of the density must disappear. Using the equation (see F. M. (2.4))

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} = \frac{\partial\varrho}{\partial t} + u_i\,\frac{\partial\varrho}{\partial x_i} = 0$$

we first obtain the following relationship for the velocity components:

$$u_2 = -\frac{x_2}{x_1} u_1 , \tag{1}$$

since the density ϱ is not a function of time t. The continuity equation for an incompressible flow can be written as (see F. M. (2.5))

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 .$$

Introducing (1) into the above equation results in a first order partial differential equation in terms of the component u_1.

$$\frac{\partial u_1}{\partial x_1} - \frac{x_2}{x_1}\frac{\partial u_1}{\partial x_2} = \frac{u_1}{x_1} \tag{2}$$

We introduce now a parameter s and write the required solution in the form of $u_1(s) = u_1(x_1(s), x_2(s))$. The derivative

$$\frac{\mathrm{d}u_1}{\mathrm{d}s} = \frac{\partial u_1}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}s} + \frac{\partial u_1}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}s}$$

when compared with (2) results in a system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}s} = 1 , \tag{3}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}s} = -\frac{x_2}{x_1} ,$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}s} = \frac{u_1}{x_1} . \tag{4}$$

With equations (3) and (4) we arrive at

$$\frac{\mathrm{d}u_1}{\mathrm{d}x_1} = \frac{u_1}{x_1}$$

with the solution

$$u_1 = C\,x_1 ,$$

where C is the integration constant. Using (1), u_2 is now determined:

$$u_2 = -C\,x_2 .$$

b) From the differential equations of pathlines

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = C\,x_1 , \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -C\,x_2$$

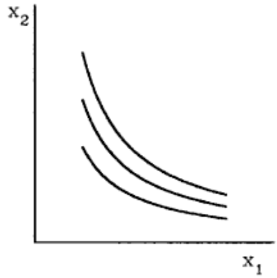
we calculate

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_2}{x_1} .$$

Separating the variables and integrating the above equation furnishes the pathline equation with $\tilde{C}$ as an integration constant:

$$x_1 \, x_2 = \tilde{C} = \text{const} .$$

This is the equation for a family of hyperbolae. Thus the flow is a stagnation point flow. The density does not change along the pathline:

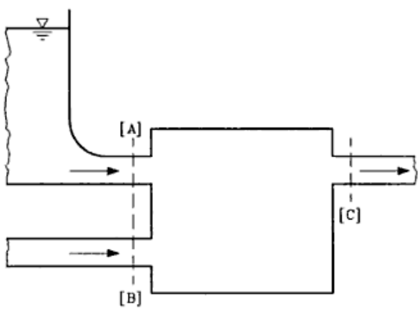


$$\varrho = k \, x_1 \, x_2 = k \, \tilde{C} .$$

However, it changes from one pathline to another, because the constant $\tilde{C}$ is different for each pathline.

Problem 2.1-3 Velocity at the exit of a container

The container (see figure) has two inlets and one outlet with circular cross sections. The flow is steady and the density is constant. At stations $[A]$ and $[C]$ the velocities are assumed to be constant over the cross-section, whereas the velocity at station $[B]$ has a parabolic distribution. The radii of the inlets and outlet are given as R_A, R_B, R_C, furthermore, the velocities u_A, $u_B = U_{B\,max}(1 - (r/R_B)^2)$.



Find the velocity u_C at station $[C]$.

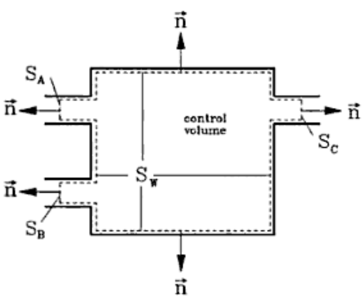
Solution

The velocity u_C can be obtained from continuity equation in integral form (see F. M. (2.7))

$$\frac{\partial}{\partial t}\iiint\limits_{(V)} \varrho \, \mathrm{d}V = -\iint\limits_{(S)} \varrho \vec{u}\cdot\vec{n}\, \mathrm{d}S \,.$$

The integrals have to be carried out over the fixed control volume. For the present steady state problem, the local derivative $\partial/\partial t = 0$. Furthermore, the constant density can be moved outside the integral resulting in

$$\iint\limits_{(S)} \vec{u}\cdot\vec{n}\, \mathrm{d}S = 0 \,.$$



We first place the control volume inside the container with the inlet control surfaces S_A, S_B, and the outlet control surface S_C and write the conservation equation in integral form:

$$\iint\limits_{S_A} \vec{u}\cdot\vec{n}\, \mathrm{d}S + \iint\limits_{S_B} \vec{u}\cdot\vec{n}\, \mathrm{d}S + + \iint\limits_{S_C} \vec{u}\cdot\vec{n}\, \mathrm{d}S + \iint\limits_{S_W} \vec{u}\cdot\vec{n}\, \mathrm{d}S = 0 \,.$$

$$\Rightarrow \quad -u_A \iint\limits_{S_A} \mathrm{d}S - \iint\limits_{S_B} u_B(r)\, \mathrm{d}S + u_C \iint\limits_{S_C} \mathrm{d}S = 0 \,.$$

At the wall S_W, the surface integral will disappear, because the scalar product $\vec{u}\cdot\vec{n} = 0$ there. We evaluate the second integral as follows

$$\begin{aligned}
\iint\limits_{S_B} u_B(r)\, \mathrm{d}S &= \int\limits_0^{2\pi}\int\limits_0^{R_B} U_{B_{max}} \left[1-\left(\frac{r}{R_B}\right)^2\right] r\, \mathrm{d}r\, \mathrm{d}\varphi \\
&= 2\pi\, U_{B_{max}} R_B^2 \int\limits_0^1 \left[1-\left(\frac{r}{R_B}\right)^2\right] \frac{r}{R_B}\, \mathrm{d}\left(\frac{r}{R_B}\right)
\end{aligned}$$

$$= 2\pi\, U_{B_{max}}\, R_B^2 \left[\frac{1}{2}\left(\frac{r}{R_B}\right)^2 - \frac{1}{4}\left(\frac{r}{R_B}\right)^4\right]_0^1$$

$$= \frac{U_{B_{max}}}{2}\,\pi\, R_B^2 \,.$$

Thus, continuity equation reduces to

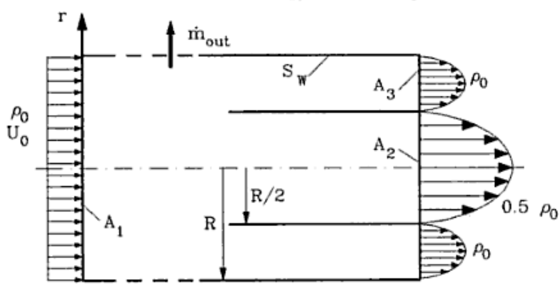
$$-u_A\,\pi\, R_A^2 - \frac{U_{B_{max}}}{2}\,\pi\, R_B^2 + u_C\,\pi\, R_C^2 = 0 \,.$$

A simple rearrangement results in

$$u_C = u_A \left(\frac{R_A}{R_C}\right)^2 + \frac{U_{B_{max}}}{2}\left(\frac{R_B}{R_C}\right)^2 \,.$$

Problem 2.1-4 Steady flow through a circular channel

Steady incompressible fluid flows through a circular channel with the inlet radius R. At the inlet cross-section A_1, the velocity $u = U_0$ is constant.



The density ϱ remains constant over the cross-sections A_1, A_2, and A_3. Inside the channel, a concentric pipe with a negligible thickness divides the flow regime into an inner cylindrical core and an outer annular portion with the exit cross-sections A_2 and A_3. The velocity distribution at A_2 is given by

$$\frac{u}{U_{2_{max}}} = 1 - \left(\frac{2r}{R}\right)^2 \,.$$

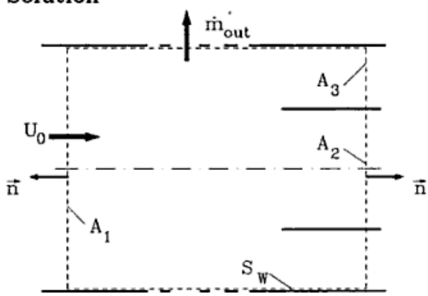
At the exit cross-section A_3, the velocity has the following distribution:

$$u(r) = \frac{U_0}{2}\left[1 - \left(\frac{2r}{R}\right)^2 + \frac{3}{\ln 2}\ln\left(\frac{2r}{R}\right)\right] .$$

Determine the mass flux $\dot{m}_{out}$ added or carried off within the channel.

Given: U_0, R, $\varrho|_{A_1} = \varrho|_{A_3} = \varrho_0$, $\varrho|_{A_2} = \varrho_0/2$, $U_{2_{max}} = 2.5\,U_0$

Solution



We apply the continuity equation (see F. M. (2.8)) for a steady flow to the control volume shown in the figure and obtain:

$$\iint\limits_{A_1} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{S_W} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{A_2} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{A_3} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S + \dot{m}_{out} = 0\ . \quad (1)$$

At the wall, because of $\vec{u}\cdot\vec{n} = 0$, the integration over S_W becomes zero. The evaluation of the integral at the inlet cross-section A_1 results in

$$\iint\limits_{A_1} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S = -\iint\limits_{A_1} \varrho_0\,U_0\,\mathrm{d}S = -\varrho_0\,U_0\,\pi\,R^2\ .$$

At the exit cross-section A_2 we find

$$\begin{aligned}
\iint\limits_{A_2} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S &= \iint\limits_{A_2} \frac{1}{2}\,\varrho_0\,U_{2_{max}}\left[1 - \left(\frac{2r}{R}\right)^2\right]\mathrm{d}S \\
&= \frac{1}{2}\,\varrho_0\,2.5\,U_0 2\pi \int\limits_0^{R/2} \left[1 - \left(\frac{2r}{R}\right)^2\right] r\,\mathrm{d}r \\
&= \frac{5}{32}\,\varrho_0\,U_0\,\pi\,R^2
\end{aligned}$$

and finally, at the exit cross-section A_3 we arrive at:

$$\iint\limits_{A_3} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S = \frac{\varrho_0}{2}\,U_0 \int\limits_0^{2\pi}\int\limits_{R/2}^{R} \left[1 - \left(\frac{2r}{R}\right)^2 + \frac{3}{\ln 2}\ln\left(\frac{2r}{R}\right)\right] r\,\mathrm{d}r\,\mathrm{d}\varphi\ .$$

To make the integration process easier, we substitute $\bar{r} = 2r/R$ and find

$$\iint_{A_3} \varrho \vec{u} \cdot \vec{n} \, \mathrm{d}S = \pi \varrho_0 U_0 \frac{R^2}{4} \int_1^2 \left[1 - \bar{r}^2 + \frac{3}{\ln 2} \ln \bar{r}\right] \bar{r} \, \mathrm{d}\bar{r}$$

$$= \varrho_0 U_0 \pi \frac{3R^2}{4} \frac{1}{4} \left(5 - \frac{3}{\ln 2}\right)$$

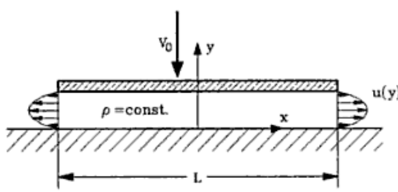
$$= 0.126 \, \varrho_0 U_0 \pi R^2 \, .$$

Introducing the above results in (1) we get

$$\dot{m}_{out} = 0.718 \, \varrho_0 U_0 \pi R^2 \, .$$

Problem 2.1-5 Squeeze film flow

The gap shown in the figure has the length L, the height $h(t)$, and is filled with a fluid of constant density. The top wall of the gap moves downward with the velocity V_0. The velocity distribution at the exit is



$$u(y) = 4U_0 \left\{ \frac{y}{h(t)} - \left(\frac{y}{h(t)}\right)^2 \right\} .$$

a) For $h(t=0) = h_0$, determine the function of the gap height.
b) Calculate the maximum velocity U_0 at the exit.

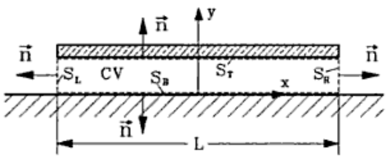
Given: V_0, h_0, L, ϱ

Solution

a) The gap height $h(t)$: To find the function of the gap height, we integrate the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -V_0 \quad \Rightarrow \quad h(t) = -V_0 \, t + h_0 \, .$$

b) The maximum velocity U_0:



The maximum velocity U_0 at the exits is calculated using the continuity equation in integral form (see F. M. (2.7))

$$\frac{\partial}{\partial t}\iiint\limits_{(V)} \varrho \,\mathrm{d}V = -\iint\limits_{(S)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S$$

The integration has to be carried out over the control volume (see figure), which coincides with the material volume at time t.
Since the integration domain is fixed, the first integral can be evaluated using $\varrho = \text{const}$:

$$\frac{\partial}{\partial t}\iiint\limits_{(V)} \varrho\,\mathrm{d}V = \iiint\limits_{(V)} \frac{\partial \varrho}{\partial t}\,\mathrm{d}V = 0\,.$$

Since the density ϱ is constant, it can be moved outside the second integral. We now split the integral into four parts

$$\iint\limits_{S_T} \vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{S_B} \vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{S_L} \vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{S_R} \vec{u}\cdot\vec{n}\,\mathrm{d}S = 0\,.$$

Since the normal velocity component must be equal to the velocity of the top surface S_T (otherwise the fluid would penetrate S_T)

$$\vec{u}\cdot\vec{n} = \vec{u}_{wall}\cdot\vec{n} = -V_0\,,$$

thus, the first integral is

$$\iint\limits_{S_T} \vec{u}\cdot\vec{n}\,\mathrm{d}S = -V_0 \iint\limits_{S_T} \mathrm{d}S = -V_0\,B\,L$$

with B as the plate depth. For the kinematic reasons explained above, at the bottom surface S_B: $\vec{u}\cdot\vec{n} = 0$. At the left and right control surfaces S_L and S_R, the normal velocity component is $\vec{u}\cdot\vec{n} = u(y)$. Introducing this result into the continuity equation, we find

$$-V_0\,B\,L \;+\; \iint\limits_{S_L} u(y)\,\mathrm{d}S + \iint\limits_{S_R} u(y)\,\mathrm{d}S = 0$$

$$\Rightarrow \quad V_0\,L \;=\; 2\int\limits_0^{h(t)} 4U_0\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]\mathrm{d}y$$

$$= 8U_0 \left[\frac{1}{2}\frac{y^2}{h} - \frac{1}{3}\frac{y^3}{h^2}\right]_0^{h(t)}$$

$$= \frac{4}{3} U_0\, h(t) \,.$$

We rearrange the above equation and obtain the velocity

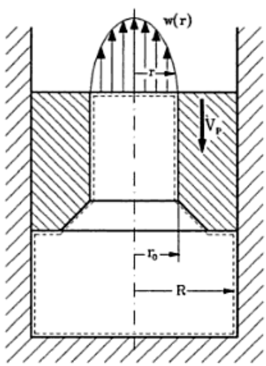
$$U_0 = \frac{3}{4}\frac{L}{h(t)} V_0 = \frac{3}{4}\frac{L\, V_0}{h_0 - V_0\, t} \,.$$

This equation indicates that the velocity U_0 approaches infinity for a gap height of $h(t) = 0$. However, in the reality, the top wall can not be made to move toward the bottom wall with a constant velocity V_0.

Problem 2.1-6 Moving Piston

A piston moves with the velocity V_P (see figure) inside a cylinder, which is filled with oil. The velocity distribution $w(r)$ of the exiting oil at the top surface is measured relative to the piston and is given by:

$$w(r) = W_0 \left\{1 - \left(\frac{r}{r_0}\right)^2\right\} \,.$$



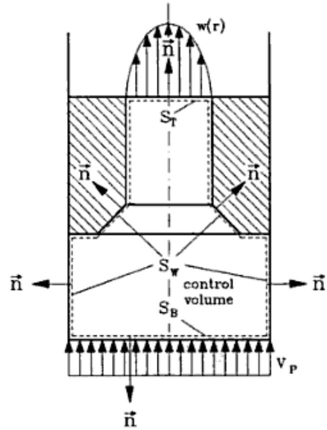
Determine the maximum velocity W_0 using
a) a piston-fixed coordinate system (a coordinate moving with the piston),
b) a space-fixed coordinate system.

Given: r_0, R, V_P, $\varrho = \text{const}$

Solution

a) Piston-fixed coordinate system: In a piston-fixed coordinate system the bottom cylinder wall moves with V_P upward. With $\varrho = \text{const}$ the continuity equation for a fixed control volume embedded in this relative coordinate system is written as:

$$\varrho \iint\limits_{(S)} \vec{w} \cdot \vec{n} \, \mathrm{d}S = 0 \, .$$



We subdivide the entire control surface into three surfaces:

$$\Rightarrow \quad \iint\limits_{S_W} \vec{w} \cdot \vec{n} \, \mathrm{d}S + \iint\limits_{S_B} \vec{w} \cdot \vec{n} \, \mathrm{d}S + $$
$$+ \iint\limits_{S_T} \vec{w} \cdot \vec{n} \, \mathrm{d}S = 0 \, .$$

At the bottom wall S_B whose velocity is $\vec{w}_W = -V_P\vec{n}$, the kinematic boundary conditions requires

$$\vec{w} \cdot \vec{n} = \vec{w}_W \cdot \vec{n} = -V_P$$

(otherwise the fluid would penetrate these walls), while $\vec{w} \cdot \vec{n}$ disappears at all solid walls. This results in the equation:

$$-V_P \iint\limits_{S_B} \mathrm{d}S + \iint\limits_{S_T} w(r) \, \mathrm{d}S = 0$$

or

$$V_P \, \pi \, R^2 = 2\,\pi \int\limits_0^{r_0} W_0 \left(1 - \left[\frac{r}{r_0}\right]^2\right) r \, \mathrm{d}r \, .$$

We substitute

$$\xi = \left(\frac{r}{r_0}\right)^2 \quad \Rightarrow \quad \mathrm{d}\xi = 2\frac{r}{r_0^2}\,\mathrm{d}r \quad \Rightarrow \quad \mathrm{d}r = \frac{r_0^2\,\mathrm{d}\xi}{2r}$$

and obtain

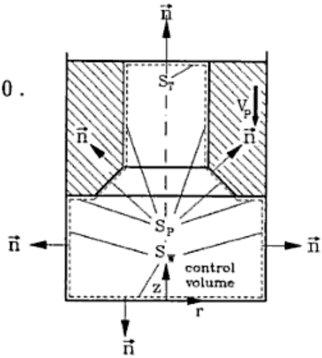
$$\int\limits_0^{r_0} w(r)\, r \, \mathrm{d}r = \int\limits_0^1 W_0\,(1-\xi)\, r \,\frac{r_0^2\,\mathrm{d}\xi}{2r} = \frac{W_0}{2} r_0^2 \int\limits_0^1 (1-\xi)\,\mathrm{d}\xi = \frac{W_0}{4} r_0^2$$

thus,

$$\begin{aligned} V_P \, \pi \, R^2 &= \pi \, r_0^2 \, \frac{W_0}{2} \\ \Rightarrow \quad W_0 &= 2V_P \left(\frac{R}{r_0}\right)^2 . \end{aligned}$$

b) Space-fixed coordinate system:
The continuity equation in this coordinate system is

$$\iint\limits_{S_W} \vec{c}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{S_P} \vec{c}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{S_T} \vec{c}\cdot\vec{n}\,\mathrm{d}S = 0\,.$$



On the fixed walls S_W, $\vec{c}\cdot\vec{n}$ disappears. The same condition applied to S_P results in

$$\vec{c}\cdot\vec{n} = \vec{c}_W\cdot\vec{n}\,,$$

with $\vec{c}_W = -V_P\,\vec{e}_z$

$$\vec{c}\cdot\vec{n} = -V_P\,\vec{e}_z\cdot\vec{n}\,.$$

As a consequence, the integral over S_P is

$$\iint\limits_{S_P} \vec{c}\cdot\vec{n}\,\mathrm{d}S = \iint\limits_{S_P} -V_P\,\vec{e}_z\cdot\vec{n}\,\mathrm{d}S\,.$$

$\vec{e}_z\cdot\vec{n}\,\mathrm{d}S$ is the projection of the surface element $\mathrm{d}S$ in z–direction, $\vec{e}_z\cdot\vec{n}\,\mathrm{d}S = \pm\mathrm{d}A_z$. The sign of the scalar product is determined from the angle between $\vec{n}$ and $\vec{e}_z$. If this angle is less than 90 degrees, the sign of the product will be positive (+), otherwise negative (−). At S_P $\operatorname{sgn}(\vec{e}_z\cdot\vec{n}) \geq 0$ (enclosed angle less than 90°), i. e. the positive sign must be applied. We evaluate the integral:

$$\iint\limits_{S_P} \vec{c}\cdot\vec{n}\,\mathrm{d}S = -V_P \iint\limits_{S_P} r\,\mathrm{d}r\,\mathrm{d}\varphi = -V_P\,\pi\,(R^2 - r_0^2)\,.$$

In a space-fixed coordinate system the absolute velocity at the oil exit is $c = w - V_P$, which is

$$c(r) = W_0\left[1-\left(\frac{r}{r_0}\right)^2\right] - V_P\,.$$

The integral over S_T is then

$$\iint\limits_{S_T} \vec{c}\cdot\vec{n}\,\mathrm{d}S = 2\pi\int\limits_0^{r_0}\left\{W_0\left[1-\left(\frac{r}{r_0}\right)^2\right] - V_P\right\} r\,\mathrm{d}r = \pi\,r_0^2\left(\frac{W_0}{2} - V_P\right)\,.$$

The continuity equation gives

$$-V_P\,\pi\,(R^2 - r_0^2) + \pi\,r_0^2\,(\frac{W_0}{2} - V_P) = 0\,,$$

or

$$V_P\,\pi\,R^2 \;=\; \pi\,r_0^2\,\frac{W_0}{2}$$

$$\Rightarrow \quad W_0 \;=\; 2\,V_P\left(\frac{R}{r_0}\right)^2\,,$$

which is identical with the result from a).

Problem 2.1-7 Flow between two inclined flat plates

Fluid with constant density ϱ is located between two flat plates of length L. Both plates turn symmetric to x-axis towards each other with a constant angular velocity Ω. The turning of the plates cause the fluid to move out of the plates. We assume that the plates in z-direction have an infinite width, justifying the assumption of plane flow. The velocity field in cylindrical coordinates is

$$\vec{u}(r,\varphi) = u_r(r,\varphi)\,\vec{e}_r + u_\varphi(r,\varphi)\,\vec{e}_\varphi$$

whose radial component is given by

$$u_r(r,\varphi) = f(r)\cos\left(\frac{\pi}{2\alpha}\varphi\right)$$

with the unknown function $f(r)$.

a) Determine the wall velocity $\vec{u}(r) = u_W(r)\,\vec{e}_\varphi$ for both plates.

b) Using the continuity equation in integral form, find the function $f(r)$ for $0 \le r \le L$.

c) Using the differential form, calculate $u_\varphi(r,\varphi)$ with the boundary equations $\varphi = \pm\alpha$.

Given: L, α, Ω

Solution

a) Wall velocity

The plates turn with an angular velocity Ω towards the x-axis. The wall velocities in cylindrical coordinates are

$$\text{for the top wall} \qquad \vec{u}(r) \;=\; -\Omega\,r\,\vec{e}_\varphi\,,$$

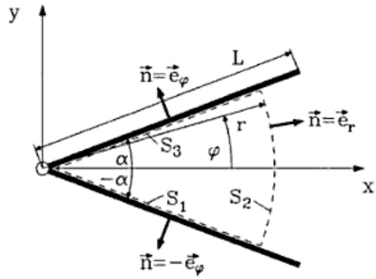
$$\text{for the bottom wall} \qquad \vec{u}(r) \;=\; \Omega\,r\,\vec{e}_\varphi\,.$$

b) Find the function $f(r)$

The continuity equation in integral form is

$$\iiint\limits_{(V)} \frac{\partial \varrho}{\partial t}\,\mathrm{d}V = -\iint\limits_{(S)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S\,. \tag{1}$$

The integrals must be carried out over the sketched control volume, which coincides with the material volume. The plates push at the instant being considered the fluid into the control volume via the surfaces S_1 and S_3. The left hand side of (1) can be written as



$$\iiint\limits_{(V)} \frac{\partial \varrho}{\partial t}\,\mathrm{d}V = 0\,, \quad \text{since} \quad \varrho = \text{const}\,.$$

The surface integral is carried out over the entire surfaces of the control volume

$$\iint\limits_{(S_1)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{(S_2)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S + \iint\limits_{(S_3)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S = 0\,.$$

The kinematic boundary condition requires that the normal velocity component $\vec{u}\cdot\vec{n}$ of the flow at the surfaces S_1 and S_3 must be equal to the normal component of the plate velocity. This requirement leads to

$$\int\limits_0^b\int\limits_0^r -\Omega\,r\,\mathrm{d}r\,\mathrm{d}z + \int\limits_0^b\int\limits_{-\alpha}^{\alpha} f(r)\cos\left(\frac{\pi}{2\alpha}\varphi\right)\,r\,\mathrm{d}\varphi\,\mathrm{d}z + \int\limits_0^b\int\limits_0^r -\Omega\,r\,\mathrm{d}r\,\mathrm{d}z = 0\,.$$

The integration $\int_0^b$ represents the extension in z-direction which is assumed to be infinity and can be canceled out. As a result, we get

$$-\Omega\,r^2 + r f(r)\frac{2\alpha}{\pi}\left[\sin\left(\frac{\pi}{2\alpha}\varphi\right)\right]_{-\alpha}^{\alpha} = 0\,,$$

and thus

$$f(r) = \frac{\pi}{4\alpha}\,\Omega\,r\,.$$

The radial velocity component is now

$$u_r(r,\varphi) = \frac{\pi}{4\alpha}\,\Omega\,r\cos\left(\frac{\pi}{2\alpha}\varphi\right)\,.$$

c) Calculation of $u_\varphi(r,\varphi)$
For a plane flow, the continuity equation in differential form is written in cylindrical coordinates (see F. M. , Appendix B.2)

$$\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\varphi}{\partial \varphi} = 0 \; .$$

Inserting the results from a), the following differential equation is obtained:

$$\frac{\partial u_\varphi}{\partial \varphi} = -\frac{\pi}{2\alpha}\,\Omega\, r \cos\left(\frac{\pi}{2\alpha}\varphi\right) \; .$$

The solution of the above differential equation is

$$u_\varphi(r,\varphi) = -\Omega\, r \sin\left(\frac{\pi}{2\alpha}\varphi\right) + C(r) \; .$$

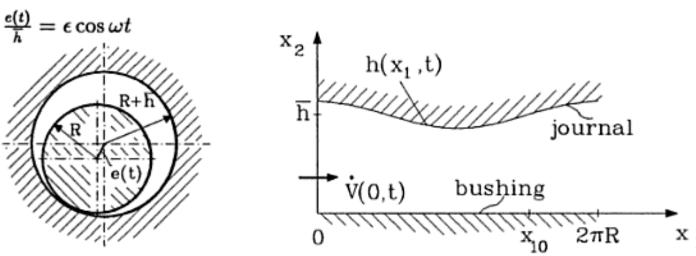
Considering the flow symmetry with respect to the x-axis, the integration constant is determined. Since u_φ is an odd function

$$u_\varphi(r,\varphi) = -u_\varphi(r,-\varphi) \; ,$$

and therefore

$$C(r) \equiv 0 \quad \Rightarrow \quad u_\varphi(r,\varphi) = -\Omega\, r \sin\left(\frac{\pi}{2\alpha}\varphi\right) \; .$$

Problem 2.1-8 Oscillating journal bearing



The figure shows an oscillating journal bearing with the eccentricity $e = e(t)$ and the shaft radius R. The shaft rotates with a constant rotational speed ω. We assume that the bearing has an infinite width in axial direction. For $\overline{h}/R \ll 1$, the clearance distribution $h(x_1,t)$ in x_1-direction can be unwrapped and the following assumption can be made:

$$\frac{h(x_1,t)}{\overline{h}} = 1 + \epsilon \cos \omega t \cos \frac{x_1}{R} \; .$$

The density of the fluid ϱ is constant and the volume flux per unit width $\dot{V}(0,t)$ at location $x_1 = 0$ is known.
Calculate the volume flux per unit width $\dot{V}(x_{10},t)$ as a function of time at x_{10}.
Given: $\dot{V}(0,t)$, ϵ, ω, R, $\overline{h}$

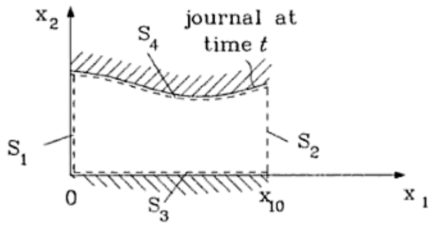
Solution
To calculate $\dot{V}(x_{10},t)$, we apply the continuity equation in integral form

$$\iiint\limits_{(V)} \frac{\partial \varrho}{\partial t}\,\mathrm{d}V = -\iint\limits_{(S)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S \qquad (1)$$

to the control volume sketched in the following figure.

The left hand side of (1) disappears, because of $\varrho =$ const. Therefore, the continuity equation reduces to

$$\iint\limits_{(S)} \vec{u}\cdot\vec{n}\,\mathrm{d}S = 0\,. \qquad (2)$$



The volume fluxes at S_1 and S_2 are

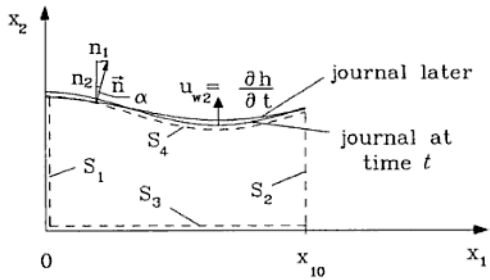
$$\iint\limits_{S_1} \vec{u}\cdot\vec{n}\,\mathrm{d}S = -b\,\dot{V}(0,t)$$

and

$$\iint\limits_{S_2} \vec{u}\cdot\vec{n}\,\mathrm{d}S = b\,\dot{V}(x_{10},t)\,.$$

Since the bushing is fixed and there is no mass flux through its wall, the integral over S_3 disappears.

However, the integration over S_4 has a non-zero value, because of the oscillating motion of the shaft. The flow velocity at S_4 is determined using the kinematic boundary condition. This condition requires that the normal component of the flow velocity must be the same as the normal component of the shaft-surface velocity.



Because of $\overline{h}/R \ll 1$

$$\tan\alpha = \frac{\partial h}{\partial x_1} = -\epsilon\,\frac{\overline{h}}{R}\cos\omega t\sin\frac{x_1}{R} \ll 1\ .$$

For the same reason, we have

$$n_1 = \vec{n}\cdot\vec{e}_1 = \sin\alpha \ll 1$$

and n_1 can be neglected compared with n_2. Thus, the normal component of the wall velocity can approximately be written as

$$\vec{u}_w\cdot\vec{n} = u_{w1}\,\vec{e}_1\cdot\vec{n} + u_{w2}\,\vec{e}_2\cdot\vec{n} \approx u_{w2}\,\vec{e}_2\cdot\vec{n}$$

and with $u_{w2} = \partial h/\partial t$ it follows that

$$\iint\limits_{S_4} \vec{u}\cdot\vec{n}\,\mathrm{d}S = \iint\limits_{S_4} \vec{u}_w\cdot\vec{n}\,\mathrm{d}S \approx \iint\limits_{S_4} \frac{\partial h}{\partial t}\,\vec{e}_2\cdot\vec{n}\,\mathrm{d}S\ .$$

The expression $\vec{e}_2\cdot\vec{n}\,\mathrm{d}S$ is the projection of $\mathrm{d}S$ in $\vec{e}_2$-direction, which is $\mathrm{d}x_1\,\mathrm{d}x_3$. Thus,

$$\iint\limits_{(S_4)} \vec{u}\cdot\vec{n}\,\mathrm{d}S \ \approx\ b\int\limits_0^{x_{10}} \frac{\partial h}{\partial t}\,\mathrm{d}x_1 = -b\,\overline{h}\,\omega\,\epsilon\sin\omega t\int\limits_0^{x_{10}}\cos\frac{x_1}{R}\,\mathrm{d}x_1$$

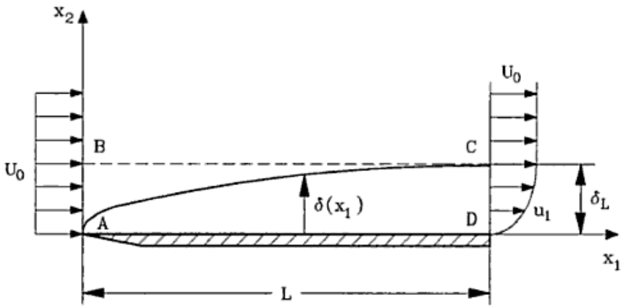
and

$$\iint\limits_{(S_4)} \vec{u}\cdot\vec{n}\,\mathrm{d}S \approx -b\,R\,\overline{h}\,\omega\,\epsilon\sin\omega t\sin\frac{x_{10}}{R}\ .$$

Inserting the above results into (2) gives

$$\dot{V}(x_{10},t) = R\,\overline{h}\,\omega\,\epsilon\sin\omega t\sin\frac{x_{10}}{R} + \dot{V}(0,t)\ .$$

Problem 2.1-9 Effect of boundary layer displacement thickness



Incompressible fluid flows over a flat plate (width b, length L) with constant velocity U_0. The viscosity effect causes a boundary layer with the thickness $\delta(x_1)$. Outside the boundary layer, the velocity is $u_1 = U_0 =$ const. We assume that the velocity distribution within the boundary layer follows a sine function with no-slip condition at the wall.

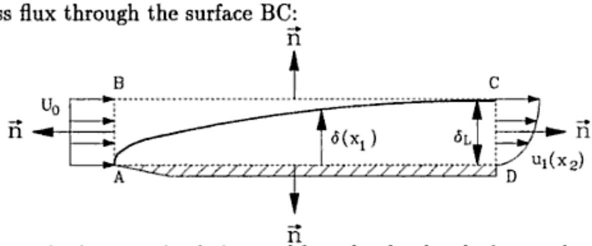
a) Determine the mass flux through the surface BC of the sketched control volume.
b) Calculate the velocity field within the boundary layer $u_i(x_j)$.
c) Calculate the mass flux through BC using $u_2(x_1, x_2 = \delta)$.

Given: $\delta = \delta(x_1)$, $\delta_L = \delta(x_1 = L)$,

$$u_1/U_0 = \begin{cases} \sin\left(\frac{1}{2}\pi x_2/\delta\right) & \text{for} \quad 0 \le x_2/\delta(x_1) \le 1 \\ 1 & \text{for} \quad x_2/\delta(x_1) > 1 \end{cases}$$

Solution

a) Mass flux through the surface BC:



The continuity equation in integral form for the sketched control volume gives:

$$\iint_{AB} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S + \iint_{BC} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S + \iint_{CD} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S + \iint_{AD} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S = 0\ . \quad (1)$$

The mass flux through the surface BC is $\dot{m}_{BC}$, thus

$$\iint_{BC} \varrho \vec{u} \cdot \vec{n} \, \mathrm{d}S = \dot{m}_{BC} \, .$$

We calculate the surface integrals

$$\begin{aligned}
\iint_{AB} \varrho \vec{u} \cdot \vec{n} \, \mathrm{d}S &= -\varrho \, b \, \delta_L \, U_0 \, , \\
\iint_{CD} \varrho \vec{u} \cdot \vec{n} \, \mathrm{d}S &= \varrho \, b \int_0^{\delta_L} U_0 \sin\left(\frac{\pi}{2} \frac{x_2}{\delta_L}\right) \, \mathrm{d}x_2 \\
&= \varrho \, b \, U_0 \, \frac{2\delta_L}{\pi} \left[-\cos\left(\frac{\pi}{2} \frac{x_2}{\delta_L}\right) \right]_0^{\delta_L} \\
&= \frac{2}{\pi} \, \varrho \, b \, \delta_L \, U_0 \\
\iint_{AD} \varrho \vec{u} \cdot \vec{n} \, \mathrm{d}S &= 0 \, ,
\end{aligned}$$

and obtain from (1)

$$\begin{aligned}
-\varrho \, b \, \delta_L \, U_0 + \dot{m}_{BC} + \frac{2}{\pi} \, \varrho \, b \, \delta_L \, U_0 &= 0 \\
\Rightarrow \quad \dot{m}_{BC} &= \varrho \, b \, \delta_L \, U_0 \left(1 - \frac{2}{\pi}\right) \, .
\end{aligned}$$

b) Velocity field:
To calculate $u_2(\vec{x})$ with a given u_1, we use the continuity equation in differential form (see F. M. (2.3a)):

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \frac{\partial u_i}{\partial x_i} = 0 \, .$$

Since the flow is incompressible, for the present two dimensional case $(\partial/\partial x_3 = 0)$ we have:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \quad \Rightarrow \quad u_2 = \int_0^{x_2} -\frac{\partial u_1}{\partial x_1} \, \mathrm{d}x_2 + f(x_1) \, . \qquad (2)$$

The above integration constant may be a function of x_1, but here we find $f(x_1) = 0$ from the boundary condition

$$u_2(x_2 = 0, \, x_1) = 0 \, .$$

Further we have

$$\frac{\partial u_1}{\partial x_1} = \begin{cases} -\frac{\pi}{2} U_0 \frac{x_2}{\delta^2} \delta' \cos\left(\frac{\pi}{2}\frac{x_2}{\delta}\right) & \text{for} \quad x_2 \le \delta(x_1)\,, \\ 0 & \text{for} \quad x_2 \ge \delta(x_1)\,, \end{cases}$$

with $\delta' = \frac{\mathrm{d}\delta}{\mathrm{d}x}$ and obtain from (2) for $x_2 \le \delta(x_1)$:

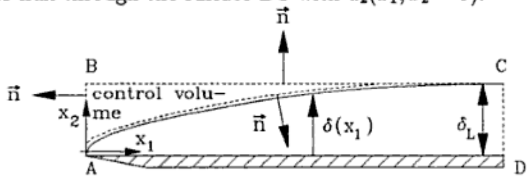
$$u_2(x_1, x_2) = \int_0^{x_2} \frac{\pi}{2} U_0 \frac{x_2}{\delta^2} \delta' \cos\left(\frac{\pi}{2}\frac{x_2}{\delta}\right) \mathrm{d}x_2$$

$$\Rightarrow \quad u_2(x_1, x_2) = U_0 \delta' \left\{ -\frac{2}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{x_2}{\delta}\right)\right] + \frac{x_2}{\delta}\sin\left(\frac{\pi}{2}\frac{x_2}{\delta}\right)\right\}$$

and for the edge of the boundary layer $x_2 = \delta(x_1)$

$$u_2(x_1, x_2) = U_0 \delta'(x_1)\left(1 - \frac{2}{\pi}\right) . \tag{3}$$

c) Mass flux through the surface BC with $u_2(x_1, x_2 = \delta)$:



We apply the continuity equation ($\partial/\partial t = 0$) to the sketched control volume:

$$\iint_{(S)} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S = 0$$

$$\Rightarrow \quad \iint_{AB} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S + \iint_{BC} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S + \iint_{AC} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S = 0 .$$

The first two integrals are already known from part a):

$$\Rightarrow \quad \dot{m}_{BC} = \varrho U_0 b \delta_L - \iint_{AC} \varrho \vec{u} \cdot \vec{n}\, \mathrm{d}S . \tag{4}$$

On the surface AC is $\vec{u} = U_0 \vec{e}_1 + u_2(x_1, \delta(x_1))\, \vec{e}_2$, thus:

$$\begin{aligned} \vec{u} \cdot \vec{n}\, \mathrm{d}S &= U_0 \vec{e}_1 \cdot \vec{n}\, \mathrm{d}S + u_2(x_1, \delta(x_1))\, \vec{e}_2 \cdot \vec{n}\, \mathrm{d}S \\ &= U_0\, \mathrm{d}x_2\, \mathrm{d}x_3 - u_2(x_1, \delta(x_1))\, \mathrm{d}x_1\, \mathrm{d}x_3 . \end{aligned}$$

With the above equations, the integral in (4) is calculated using (3)

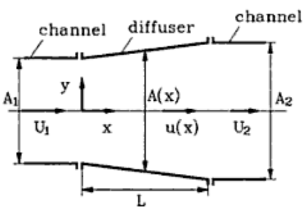
$$\iint_{AC} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S = \varrho\,U_0\,b\int_0^{\delta_L}\mathrm{d}x_2 - \varrho\,U_0\left(1-\frac{2}{\pi}\right)b\int_0^{L}\delta'(x_1)\,\mathrm{d}x_1$$

$$= \varrho\,U_0\,b\,\delta_L - \varrho\,b\,U_0\left(1-\frac{2}{\pi}\right)\delta_L\,,$$

we see that (4) gives the mass flux calculated previously:

$$\dot{m}_{BC} = \varrho\,b\,\delta_L\,U_0\left(1-\frac{2}{\pi}\right)\,.$$

Problem 2.1-10 Flow through a diffuser with a linear velocity change in flow direction



Two channels with the area A_1 and A_2 are connected with each other by a diffuser of the length L. The shape of the diffuser is designed in such a way that it allows a linear change of the velocity component u in flow direction x from U_1 to U_2, while it remains constant over the cross-section $A(x)$. The density ϱ should also remain constant.

a) Determine the distribution of the velocity component $u(x)$ in the channel. Find the change of the cross-section $A(x)$.

b) Calculate the local and convective acceleration in the diffuser for a constant inlet flow velocity U_1.

c) Answer part b), for the case that U_1 is time dependent with $\partial U_1/\partial t = a_1 = \text{const}$ as given.

Given: A_1, A_2, U_1, L, $\varrho = \text{const}$, a_1

Solution

a) The velocity distribution is linear $u(x) = m\,x + c$. We find the constants c and m using the boundary conditions

$$u(x=0) = U_1 \qquad \text{and} \qquad u(x=L) = U_2$$

$$c = U_1 \qquad \text{and} \qquad m = \frac{U_2 - U_1}{L}\,.$$

With the continuity equation in its integral form, we find U_2 from $U_2 = U_1\, A_1/A_2$ and the velocity distribution

$$u(x) = U_1 \left[\left(\frac{A_1}{A_2} - 1 \right) \frac{x}{L} + 1 \right] . \tag{1}$$

We consider a control volume between $x = 0$ and an arbitrary fixed location $0 < x < L$. The evaluation of the continuity equation

$$\iint\limits_{A(x)} \varrho\, \vec{u} \cdot \vec{n}\, \mathrm{d}A = - \iint\limits_{A_1} \varrho\, \vec{u} \cdot \vec{n}\, \mathrm{d}A$$

results in $u(x)\, A(x) = U_1\, A_1$, and determines the diffuser cross-sectional area as a function of x:

$$A(x) = \frac{A_1}{(A_1/A_2 - 1)\, x/L + 1} .$$

b) For the case that U_1 is not a function of t, we find from (1) $\partial u/\partial t = 0$, i. e. the local acceleration disappears. The convective acceleration is

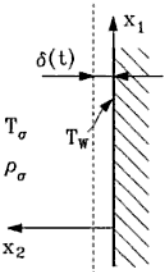
$$u\, \frac{\partial u}{\partial x} = \frac{U_1^2}{L} \left[\left(\frac{A_1}{A_2} - 1 \right) \frac{x}{L} + 1 \right] \left(\frac{A_1}{A_2} - 1 \right) . \tag{2}$$

c) With $U_1 = U_1(t)$, $\partial U_1/\partial t = a_1$ and from (1) $u = u(x,t)$. The local acceleration

$$\frac{\partial u}{\partial t} = a_1 \left[\left(\frac{A_1}{A_2} - 1 \right) \frac{x}{L} + 1 \right]$$

is not equal to zero.
The velocity is a function of x only through the diffuser geometry. Since the diffuser geometry does not change, we obtain the convective acceleration again in the form of equation (2).

Problem 2.1-11 Temperature boundary layer along a cold wall



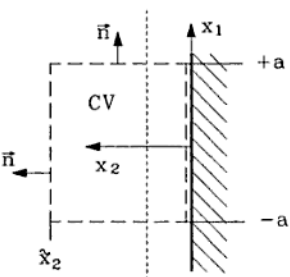
A gas at temperature T_σ and density ϱ_σ is suddenly brought in contact with a cold wall at temperature T_W. From the wall, a boundary layer develops into the gas with $\delta(t) = \sqrt{\nu\, t}$ (ν = const). The pressure is within the entire field constant. Inside the boundary layer, the gas temperature decreases linearly from T_σ to T_W. The gas density distribution is given by

$$\varrho = \begin{cases} \varrho_W - \dfrac{\varrho_W - \varrho_\sigma}{\delta(t)}\, x_2 & \text{for} \quad 0 \le x_2 \le \delta(t) \\ \varrho_\sigma & \text{for} \quad x_2 > \delta(t) \end{cases}$$

Relative to x_1- and x_3-direction, the wall extension can be considered as infinite. Determine the velocity outside the boundary layer. The velocity in x_1-direction is equal to zero within the entire field.

Given: ϱ_σ, ϱ_W

Solution



We apply the continuity equation in integral form to the sketched control volume (see F. M. (2.7))

$$\iiint\limits_{(V(t))} \frac{\partial \varrho}{\partial t}\, \mathrm{d}V = - \iint\limits_{(S)} \varrho\, u_i n_i \, \mathrm{d}S \,. \quad (1)$$

The left control surface is located outside the boundary layer at an arbitrary but fixed position $\tilde{x}_2$ from the wall. The surfaces with a distance a from the x_2-axis can be displaced to infinity. Equation (1) per unit depth in x_3-direction can be written as

$$\lim_{a\to\infty} \int\limits_{-a}^{a} \int\limits_{0}^{\tilde{x}_2} \frac{\partial \varrho}{\partial t}\, \mathrm{d}x_2 \mathrm{d}x_1 = - \lim_{a\to\infty} \int\limits_{-a}^{a} \varrho\, u_i(\tilde{x}_2) n_i \mathrm{d}x_1 \,. \quad (2)$$

The right hand side of the above equation already takes into account that no flow occurs through the cold wall and no velocity component in x_1-direction exists. Outside the boundary layer all flow quantities are uniform.

We form the local derivative of the density

$$\frac{\partial \varrho}{\partial t} = \begin{cases} (\varrho_W - \varrho_\sigma)\dfrac{x_2 \nu}{2(\nu\, t)^{3/2}} & \text{for} \quad 0 \le x_2 \le \delta(t) \\ 0 & \text{for} \quad x_2 > \delta \end{cases}$$

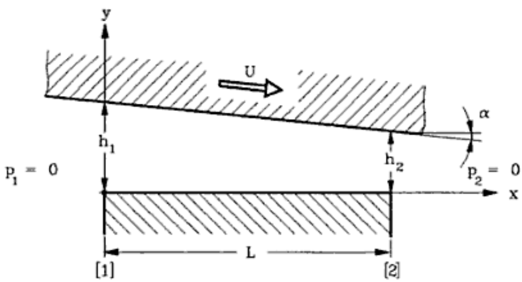
and obtain from (2)

$$\int\limits_0^{\sqrt{\nu t}} (\varrho_W - \varrho_\sigma)\frac{\nu}{2(\nu\, t)^{3/2}}\, x_2 \,\mathrm{d}x_2 = -\varrho_\sigma u_2(\tilde{x}_2)\,.$$

This equation is valid for arbitrary $\tilde{x}_2$ outside the boundary layer, where the velocity is

$$u_2 = -\left(\frac{\varrho_W}{\varrho_\sigma} - 1\right)\frac{\nu}{4\sqrt{\nu\, t}}\,.$$

Problem 2.1-12 Flow in a lubrication gap



The sketched "slide pad" with an infinite extension in z-direction has a gap height $h(x) = h_1 - \alpha\, x$, with

$$\alpha = (h_1 - h_2)/L \ll 1.$$

The slide pad, which is inclined at an angle α moves with a constant velocity U and drags the fluid with the density $\varrho = \text{const}$ into the gap. One would incorrectly expect that a linear velocity distribution $u(x,\, y)$ would develop inside the gap. The non-slip condition at the wall is accounted for by $u(x,\, 0) = 0$ and $u(x,\, h(x)) = U\, \cos\alpha \approx U$.
Hint: The velocity component in y-direction at the top wall is of the order of magnitude $\alpha\, U$ and can be neglected. Furthermore, the gap pressure is only a function of x.

a) Show that the volume flux in x-direction per unit depth

$$\dot{V} = \int_0^{h(x)} u(x,y)\,\mathrm{d}y \text{ , does not depend on } x.$$

b) The velocity distribution $u(x, y) = U\,y/h$ fulfills the required non-slip condition. Why is this velocity distribution not established?

c) We introduce in the velocity distribution from part b) a correction term, which depends on the pressure gradient $\mathrm{d}p(x)/\mathrm{d}x = -K(x)$ and has a quadratic term in y/h:

$$u(x, y) = U\,\frac{y}{h(x)} + \frac{K(x)\,h^2}{2\,\eta}\left(1 - \frac{y}{h(x)}\right)\frac{y}{h(x)} \ .$$

Determine the negative pressure gradient $K(x)$ with $K(0) = K_1$, such that the continuity equation is fulfilled.

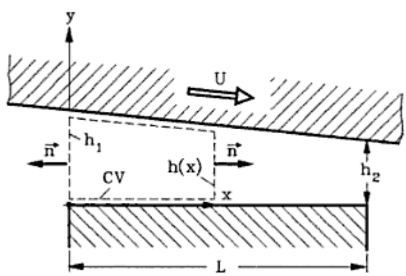
d) Obtain the pressure distribution in the gap by integrating $K(x)$. The integration constants and K_1 are calculated using the pressure boundary condition $p(0) = p_1 = p(L) = p_2 = 0$.

e) Determine the volume flux through the gap.

Given: η, h_1, h_2, L, U, $p_1 = p_2 = 0$

Solution

a) $\dot{V}$ =const :



For the sketched control volume, the continuity equation is

$$\int_0^{h_1} u(0, y)\,\mathrm{d}y = \int_0^{h(x)} u(x, y)\,\mathrm{d}y \ . \qquad (1)$$

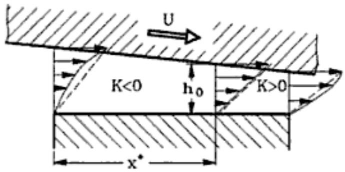
The right hand side of (1) is equal to the volume flux $\dot{V}$. Since the left hand side is constant, $\dot{V}$ is independent from x.

b) The velocity $u(x, y) = U\,y/h(x)$:
We calculate the volume flux at x:

$$\dot{V} = \int\limits_0^{h(x)} u(x, y)\,\mathrm{d}y = \int\limits_0^{h(x)} U\,\frac{y}{h(x)}\,\mathrm{d}y = \frac{1}{2}\,U\,h(x) = \frac{1}{2}\,U\,(h_1 - \alpha\,x)\,. \tag{2}$$

Thus, the volume flux is independent of x only for $\alpha = 0$. For $\alpha \neq 0$, the velocity distribution $u = U\,y/h$ does not satisfy the continuity requirement.

c) Determine $K(x)$:



Since, for a pure Couette flow the continuity requirement is not satisfied, a pressure distribution $p(x)$ must exist within the gap that causes a velocity profile at the gap inlet that is "thinner" and at the gap exit "thicker" with the result that $\dot{V}$ is now constant in the gap. The pressure gradient is determined by the continuity equation. We evaluate (1) for the given velocity distribution with $K(0) = K_1$:

$$\frac{1}{2}\,U\,h_1 + K_1\,\frac{h_1^3}{12\,\eta} = \frac{1}{2}\,U\,h(x) + K(x)\,\frac{h(x)^3}{12\,\eta} = \dot{V} \tag{3}$$

or

$$K(x) = -\frac{\mathrm{d}p}{\mathrm{d}x} = 6U\,\eta\left[\left(1 + \frac{K_1\,h_1^2}{6U\,\eta}\right)\frac{h_1}{h(x)^3} - \frac{1}{h(x)^2}\right]. \tag{4}$$

At the position $x = x^*$, with $K(x^*) = 0$, the pressure distribution has an extremum and the velocity profile is that of a pure Couette flow. We obtain from (4) thus an equation for $h(x^*)$:

$$h(x^*) = \left(1 + \frac{K_1\,h_1^2}{6U\,\eta}\right)h_1 = h_0\,,$$

with the special gap height h_0. We introduce this new constant into (4) and get

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -K(x) = 6U\,\eta\left[\frac{1}{h(x)^2} - \frac{h_0}{h(x)^3}\right]. \tag{5}$$

d) Pressure distribution $p(x)$:
The integration of the pressure gradient (5) with respect to x results in

$$p(x) = 6U\,\eta \left[\int\limits_0^x \frac{1}{h(\overline{x})^2}\,\mathrm{d}\overline{x} - h_0 \int\limits_0^x \frac{1}{h(\overline{x})^3}\,\mathrm{d}\overline{x} \right] ,$$

with

$$\int\limits_0^x \frac{1}{h(\overline{x})^2}\,\mathrm{d}\overline{x} = \frac{1}{\alpha}\left(\frac{1}{h(x)} - \frac{1}{h_1}\right)$$

and

$$\int\limits_0^x \frac{1}{h(\overline{x})^3}\,\mathrm{d}\overline{x} = \frac{1}{2\alpha}\left(\frac{1}{h(x)^2} - \frac{1}{h_1^2}\right) .$$

Thus

$$p(x) = 6\,\frac{U\,\eta}{\alpha}\left\{\frac{1}{h(x)} - \frac{1}{h_1} - \frac{h_0}{2}\left[\frac{1}{h(x)^2} - \frac{1}{h_1^2}\right]\right\} \tag{6}$$

and after a rearrangement (twice quadratic complement)

$$p(x) = 3\,\frac{U\,\eta}{\alpha\,h_0}\left\{\left[\frac{h_0}{h_1} - 1\right]^2 - \left[\frac{h_0}{h(x)} - 1\right]^2\right\} .$$

The pressure is zero at the right gap boundary. From this condition and (6) we obtain the unknown special height h_0:

$$h_0 = 2\,\frac{h_1\,h_2}{h_1 + h_2} . \tag{7}$$

e) Volume flux through the gap:
At $x = x^*$ the velocity profile is a pure Couette flow, thus

$$\dot{V} = \frac{1}{2}\,U\,h_0 = U\,\frac{h_1\,h_2}{h_1 + h_2} .$$

2.2 Balance of Momentum

Problem 2.2-1 Principal axes of a stress tensor

Given is the stress tensor in a non-dimensional form

$$\tau_{ij} = \begin{pmatrix} 5 & \sqrt{3} & 0 \\ \sqrt{3} & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Calculate:

a) the invariants $I_{1\tau}$, $I_{2\tau}$, and $I_{3\tau}$ of the tensor,
b) its principal stresses $\sigma^{(1)}$, $\sigma^{(2)}$, and $\sigma^{(3)}$
c) and its principal directions.
d) Determine the rotation matrix that transforms τ_{ij} to a diagonal form. Perform the transformation.

Solution

a) The invariants:

$$\begin{aligned} I_{1\tau} &= \tau_{ii} = \tau_{11} + \tau_{22} + \tau_{33} &&= 9 , \\ I_{2\tau} &= \tfrac{1}{2}(\tau_{ii}\tau_{jj} - \tau_{ij}\tau_{ij}) &&= 20 , \\ I_{3\tau} &= \det(\tau_{ij}) &&= 12 . \end{aligned}$$

b) Principal stresses:
The solutions of the characteristic equation

$$-\sigma^3 + I_{1\tau}\,\sigma^2 - I_{2\tau}\,\sigma + I_{3\tau} = 0 ,$$

which is here

$$-\sigma^3 + 9\,\sigma^2 - 20\,\sigma + 12 = 0 ,$$

are the required principal stresses

$$\sigma^{(1)} = 1 , \quad \sigma^{(2)} = 2 , \quad \sigma^{(3)} = 6 .$$

c) Principal directions:
The homogeneous system of equations

$$(\tau_{ij} - \sigma^{(k)}\,\delta_{ij})\,n_j^{(k)} = 0$$

has the solutions

$$\begin{array}{llll} \text{for } k = 1 : & n_1^{(1)} = 0 , & n_2^{(1)} = 0 , & n_3^{(1)} = \pm 1 , \\ \text{for } k = 2 : & n_1^{(2)} = \pm\frac{1}{2} , & n_2^{(2)} = \mp\sqrt{\frac{3}{4}} , & n_3^{(2)} = 0 , \\ \text{for } k = 3 : & n_1^{(3)} = \pm\sqrt{\frac{3}{4}} , & n_2^{(3)} = \pm\frac{1}{2} , & n_3^{(3)} = 0 . \end{array}$$

The solution vectors $\vec{n}^{(k)}$ are already normalized and are thus unit vectors. Their direction is fixed except the sign. The sign of two vectors can be chosen arbitrarily. The direction of the third vector is determined in such a way, that $\vec{n}^{(k)}$ constitutes a right handed coordinate system. Thus, the following condition must be satisfied:

$$\vec{n}^{(1)} \times \vec{n}^{(2)} \stackrel{!}{=} \vec{n}^{(3)} .$$

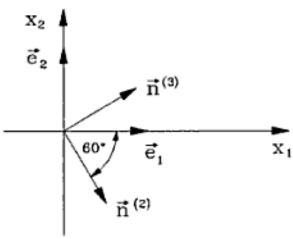
Choosing

$$\vec{n}^{(1)} = \vec{e}_3 , \quad \vec{n}^{(2)} = \frac{1}{2}\,\vec{e}_1 - \sqrt{\frac{3}{4}}\,\vec{e}_2$$

we find

$$\vec{n}^{(3)} = \sqrt{\frac{3}{4}}\,\vec{e}_1 + \frac{1}{2}\,\vec{e}_2 .$$

d) Principal axis transformation:



$\vec{n}^{(1)}$ coincide with $\vec{e}_3$, $\vec{n}^{(2)}$ and $\vec{n}^{(3)}$ are in the x_1x_2-plane.
The principal axis system is rotated relative to the original coordinate system. The rotation matrix is calculated from $a_{ij} = \cos(\angle\, x_i, x'_j)$ and is presented in matrix form as:

$$a_{ij} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} .$$

The columns of the transformation matrix are the components of the eigenvectors $\vec{n}^{(k)}$ (called modal matrix). The transformation yields by

$$\tau'_{kl} = a_{ik}\, a_{jl}\, \tau_{ij}$$

the values

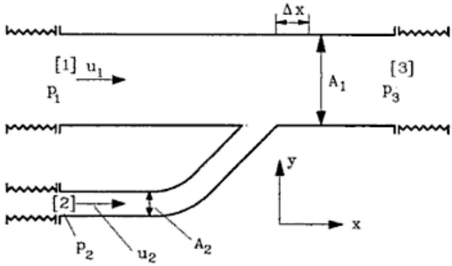
$$\tau'_{11} = 1 , \quad \tau'_{22} = 2 , \quad \tau'_{33} = 6 , \qquad \text{and} \qquad \tau'_{ij} = 0 \quad \text{for } i \neq j ,$$

such that the tensor takes on the following matrix form:

$$\tau'_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{pmatrix} .$$

The stress tensor in principal axes is a diagonal matrix with the principal stresses on the diagonal.

Problem 2.2-2 Fluid forces on a manifold



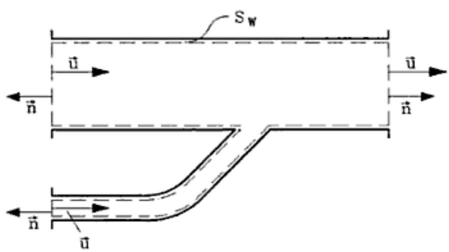
The sketched pipe branching is connected with the pipe line system by three flexible flanges (total spring stiffness c_{tot}) at locations [1], [2], and [3]. The pipe branching can move only in x-direction and the movement is considered frictionless.

a) Calculate the velocity u_3 for the case that the flow at [1], [2], and [3] is fully uniform.
b) Calculate the displacement Δx of the pipe branching with respect to the position of rest ($u_1 = u_2 = u_3 = 0$), when the flexible flanges in the position of rest are not preloaded.
c) Calculate the force acting on the pipe in y-direction.

Given: p_1, p_2, p_3, u_1, u_2, A_1, A_2, $\varrho = \text{const}$, c_{tot}

Solution

a) The velocity u_3:
To calculate u_3, we apply the continuity equation in integral form to the sketched control volume with S_W as the pipe walls. The flow is steady and incompressible, i. e.



$$\iint\limits_{(S)} \vec{u} \cdot \vec{n} \, \mathrm{d}S = 0 \, .$$

At the pipe walls S_W we have $\vec{u} \cdot \vec{n} = 0$. Since the velocities are uniform at the cross sections, we can write

$$u_1 A_1 + u_2 A_2 = u_3 A_1 \, ,$$

or

$$u_3 = u_1 + \frac{A_2}{A_1} u_2 \, .$$

b) Calculation of displacement Δx:
Using the integral form of the balance of momentum, we calculate the forces by the flow acting on the pipe branching by neglecting the body forces (see F. M. (2.43))

$$\iint_{(S)} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = \iint_{(S)} \vec{t}\,\mathrm{d}S\,.$$

For the control volume we find

$$\iint_{A_1} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint_{A_2} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint_{A_3} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint_{S_W} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S =$$

$$= \iint_{A_1} \vec{t}\,\mathrm{d}S + \iint_{A_2} \vec{t}\,\mathrm{d}S + \iint_{A_3} \vec{t}\,\mathrm{d}S + \iint_{S_W} \vec{t}\,\mathrm{d}S\,. \qquad (1)$$

The surface integrals are calculated term by term

$$\iint_{A_1} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = -\varrho u_1^2 A_1 \vec{e}_x\,, \qquad \iint_{A_1} \vec{t}\,\mathrm{d}S = -\iint_{A_1} p\vec{n}\,\mathrm{d}S = p_1 A_1 \vec{e}_x\,,$$

$$\iint_{A_2} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = -\varrho u_2^2 A_2 \vec{e}_x\,, \qquad \iint_{A_2} \vec{t}\,\mathrm{d}S = -\iint_{A_2} p\vec{n}\,\mathrm{d}S = p_2 A_2 \vec{e}_x\,,$$

$$\iint_{A_3} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = \varrho u_3^2 A_1 \vec{e}_x\,, \qquad \iint_{A_3} \vec{t}\,\mathrm{d}S = -\iint_{A_3} p\vec{n}\,\mathrm{d}S = -p_3 A_1 \vec{e}_x\,,$$

$$\iint_{S_W} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = 0\,,$$

with uniform flow at [1], [2], and [3] resulting in $\vec{t} = -p\vec{n}$. The integration of the stress vector $\vec{t}$ over the pipe wall S_W yields the force by the wall acting on the fluid. The reaction force is the required force $\vec{F}_{Fl.\to P.}$ by the flow exerting on the pipe branching.

$$\iint_{S_W} \vec{t}\,\mathrm{d}S = \vec{F}_{P.\to Fl.} = -\vec{F}_{Fl.\to P.}\,.$$

Thus, we obtain from (1)

$$\begin{aligned} -\varrho u_1^2 A_1 \vec{e}_x - \varrho u_2^2 A_2 \vec{e}_x + \varrho u_3^2 A_1 \vec{e}_x &= \\ = p_1 A_1 \vec{e}_x + p_2 A_2 \vec{e}_x - p_3 A_1 \vec{e}_x - \vec{F}_{Fl.\to P.}\,. \end{aligned} \qquad (2)$$

The force has only a component in x-direction:

$$F_x = (p_1 - p_3)A_1 + p_2 A_2 + \varrho(u_1^2 - u_3^2)A_1 + \varrho u_2^2 A_2\,.$$

The force equilibrium gives

$$F_x = c_{tot}\,\Delta x\ ,$$

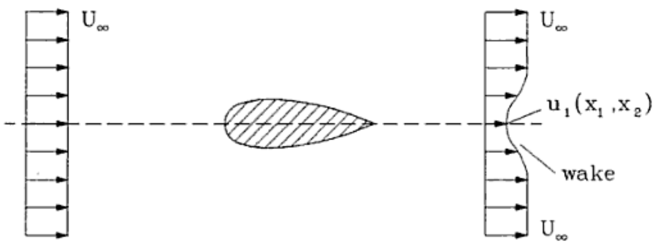
and thus, the displacement is calculated as

$$\Delta x = \frac{1}{c_{tot}}\left[(p_1 - p_3)A_1 + p_2 A_2 + \varrho(u_1^2 - u_3^2)A_1 + \varrho u_2^2 A_2\right]\ .$$

c) From (2) we obtain by a scalar multiplication with $\vec{e}_y$

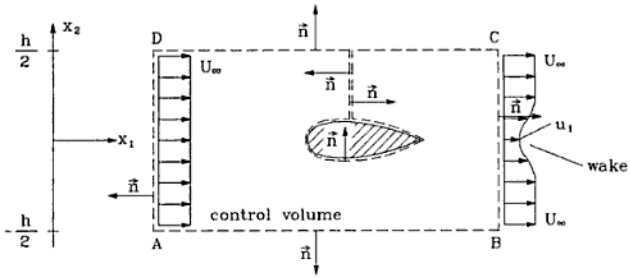
$$F_y = 0\ .$$

Problem 2.2-3 Calculation of drag force



Fluid with constant velocity U_∞ and density ϱ flows past an infinitely long symmetric cylindrical body. The flow direction coincides with the symmetry axis and the only force on the body is then the drag force F_D. Downstream of the body a wake flow is generated where the velocity u_1 is less than U_∞.
With a given u_1/U_∞ calculate the drag force F_D per unit depth acting on the body.

Solution
We choose a control volume with a control surface that encloses the body and then extends far enough from the body that all disturbances have died out sufficiently so that pressure differences to the undisturbed pressure vanish. Outside the wake flow we assume that the viscous stresses disappear.



To determine the drag force F_D we use the integral form of the balance of momentum (steady flow, no body forces)

$$\iint_{(S)} \varrho \vec{u}(\vec{u} \cdot \vec{n})\, \mathrm{d}S = \iint_{(S)} \vec{t}\, \mathrm{d}S\,. \tag{1}$$

The integral on the right hand side is decomposed into a surface integral along A, B, C, D, both sides of the slot, and one along the body surface S_B. The integrals over both sides of the slot will cancel each other, because of the opposite direction of the normal unit vectors.
According to the previous assumption, on A, B, C, D is $\vec{t} = -p_0 \vec{n}$ and we can write

$$\iint_{(S)} \vec{t}\, \mathrm{d}S = \iint_{\overline{ABCD}} -p_0\, \vec{n}\, \mathrm{d}S + \iint_{S_B} \vec{t}\, \mathrm{d}S\,.$$

The integral over A, B, C, D disappears, because no resultant force is acting on the closed surface, if $\vec{t}$ is the result of a constant pressure. The second integral is the force by the body on the fluid inside the control volume and is equal to the opposite force by the fluid on the body. We get from (1)

$$-\vec{F}_{\rightarrow body} = \iint_{(S)} \varrho\, \vec{u}\,(\vec{u} \cdot \vec{n})\, \mathrm{d}S\,,$$

from which we only need the x_1-component:

$$-F_D = -\vec{F}_{\rightarrow body} \cdot \vec{e}_1 = \iint_{(S)} \varrho\, u_1(\vec{u} \cdot \vec{n})\, \mathrm{d}S\,.$$

The surface integral of the momentum flux in x_1-direction can be decomposed in part integrals, thus we write

$$-F_D \;=\; \iint_{\overline{AB}} \varrho\, u_1(\vec{u} \cdot \vec{n})\, \mathrm{d}S + \iint_{\overline{BC}} \varrho\, u_1(\vec{u} \cdot \vec{n})\, \mathrm{d}S +$$

$$
\begin{aligned}
&+ \iint\limits_{\overline{CD}} \varrho u_1(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{\overline{DA}} \varrho u_1(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \\
&+ \iint\limits_{S_B} \varrho u_1(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{S_{slot}} \varrho u_1(\vec{u}\cdot\vec{n})\,\mathrm{d}S\,.
\end{aligned}
$$

On $\overline{AB}$, $\overline{CD}$, $\overline{DA}$, $u_1 = U_\infty$, on $\overline{BC}$, $(\vec{u}\cdot\vec{n}) = u_1(x_1,\,x_2)$, on $\overline{DA}$, $(\vec{u}\cdot\vec{n}) = -U_\infty$ and on S_B the product $(\vec{u}\cdot\vec{n})$ disappears leading to

$$
\begin{aligned}
-F_D &= U_\infty \iint\limits_{\overline{AB}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{\overline{BC}} \varrho u_1^2\,\mathrm{d}S + \\
&+ U_\infty \iint\limits_{\overline{CD}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S - \iint\limits_{\overline{DA}} \varrho U_\infty^2\,\mathrm{d}S \\
\Rightarrow F_D &= \iint\limits_{\overline{DA}} \varrho U_\infty^2\,\mathrm{d}S - \iint\limits_{\overline{BC}} \varrho u_1^2\,\mathrm{d}S + \\
&- U_\infty \left(\iint\limits_{\overline{AB}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{\overline{CD}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S \right)\,. \qquad (2)
\end{aligned}
$$

The integrals in parentheses are calculated using the continuity equation for steady flow $\iint\limits_{(S)} \varrho\,\vec{u}\cdot\vec{n}\,\mathrm{d}S = 0$:

$$
\iint\limits_{\overline{AB}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{\overline{BC}} \varrho\,\underbrace{(\vec{u}\cdot\vec{n})}_{u_1}\,\mathrm{d}S + \iint\limits_{\overline{CD}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{\overline{DA}} \varrho(\vec{u}\cdot\vec{n})\,\mathrm{d}S = 0
$$

$$
\Rightarrow \iint\limits_{\overline{AB}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{\overline{CD}} \varrho\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{\overline{DA}} \varrho U_\infty\,\mathrm{d}S - \iint\limits_{\overline{BC}} \varrho u_1\,\mathrm{d}S\,.
$$

The integrands are independent of x_3. The force per unit depth is therefore

$$
\begin{aligned}
F_D &= \varrho U_\infty^2 h - \int\limits_B^C \varrho u_1^2\,\mathrm{d}x_2 - \varrho U_\infty^2 h + \varrho U_\infty \int\limits_B^C u_1\,\mathrm{d}x_2 \\
\Rightarrow F_D &= \varrho U_\infty^2 \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{u_1}{U_\infty}\left(1 - \frac{u_1}{U_\infty}\right)\mathrm{d}x_2\,.
\end{aligned}
$$

Since the integrand disappears outside the wake flow, its value does not depend upon h provided that h is larger than the wake width. Therefore h can approach infinity $h \to \infty$ and the drag force per unit depth is calculated as

$$F_D = \varrho U_\infty^2 \int\limits_{-\infty}^{+\infty} \frac{u_1}{U_\infty}\left(1 - \frac{u_1}{U_\infty}\right) \mathrm{d}x_2 \,.$$

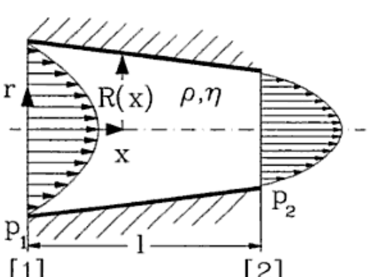
Since $F_D/\varrho U_\infty^2$ is a constant, the value of the integral does not depend upon x_1, although $u_1 = u_1(x_1, x_2)$ does.
Thus, the integral is a measure for the momentum deficiency caused by the viscous flow. In boundary layer theory, this integral, which has the dimension of a length, is called the momentum thickness.
$U_\infty - u_1 = u_d$ is called the velocity deficit u_d. Thus, we may write

$$\frac{F_D}{\varrho U_\infty^2} = \int\limits_{-\infty}^{+\infty} \left(1 - \frac{u_d}{U_\infty}\right) \frac{u_d}{U_\infty} \,\mathrm{d}x_2 \,.$$

Downstream of the body is $u_d/U_\infty \ll 1$ and the equation simplifies to

$$\frac{F_D}{\varrho U_\infty^2} = \int\limits_{-\infty}^{+\infty} \frac{u_d}{U_\infty} \,\mathrm{d}x_2 \,.$$

Problem 2.2-4 Force on a slender nozzle



A laminar flow (density ϱ, viscosity η) flows through the sketched slender nozzle:

$$R(x) = R_1 + (R_2 - R_1)\,\frac{x}{l} \,.$$

At stations [1] and [2] the stress vector is given by $\vec{t} = -p\,\vec{n}$. The pressures on both sides of the nozzle p_1 and p_2 were measured.

a) Calculate the velocity distribution inside the nozzle using the volume flux $\dot{V}$ and assuming a parabolic velocity profile with a mean velocity $\overline{U}$, which is half the maximum velocity.
b) Determine the force acting on the nozzle.

Given: p_1, p_2, $\dot{V}$, $R(x)$, l, ϱ

Solution

a) Velocity distribution:

The velocity distribution of a laminar flow through a slender nozzle is

$$u(r,x) = U_{max}\left[1-\left(\frac{r}{R(x)}\right)^2\right] .$$

The continuity equation is

$$\dot V = \overline{U}(x)\,A(x) \quad \Rightarrow \quad \overline{U}(x) = \frac{\dot V}{\pi\,R^2(x)} .$$

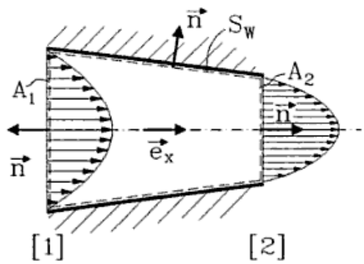
Assuming $\overline{U}(x) = U_{max}/2$, we obtain the velocity distribution

$$\begin{aligned} u(r,x) &= 2\,\frac{\dot V}{\pi\,R^2(x)}\left[1-\left(\frac{r}{R(x)}\right)^2\right] \\ &= 2\,\frac{\dot V}{\pi\,(R_1+(R_2-R_1)\,x/l)^2}\left[1-\left(\frac{r}{R_1+(R_2-R_1)\,x/l}\right)^2\right] . \end{aligned}$$

b) Force on the nozzle:

The momentum equation for the present case (constant density and no body forces) can be written as

$$\iint\limits_{(S)} \varrho\,\vec u(\vec u\cdot\vec n)\,\mathrm{d}S = \iint\limits_{(S)} \vec t\,\mathrm{d}S\,. \quad (1)$$



The flow force $\vec F_N$ on the nozzle has, because of flow symmetry, only a component in x-direction. To obtain this component, we multiply scalarly equation (1) with $\vec e_x$:

$$\iint\limits_{A_1} \varrho\,(-u_1^2(r))\,\mathrm{d}A + \iint\limits_{A_2} \varrho\,u_2^2(r)\,\mathrm{d}A = \iint\limits_{A_1} p_1\,\mathrm{d}A - \vec F_N\cdot\vec e_x + \iint\limits_{A_2} -p_2\,\mathrm{d}A\,. \quad (2)$$

The integration is to be carried out over $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\varphi$. The first integral on the left hand side is

$$-\int\limits_0^{2\pi}\int\limits_0^{R_1} \varrho\left\{\frac{2\,\dot V}{\pi\,R_1^2}\left[1-\left(\frac{r}{R_1}\right)^2\right]\right\}^2 r\,\mathrm{d}r\,\mathrm{d}\varphi =$$

$$= -2\pi\varrho\int_0^{R_1}\left(\frac{2\dot V}{\pi R_1^2}\right)^2\left\{1-\left(\frac{r}{R_1}\right)^2\right\}^2 r\,\mathrm{d}r = -\frac{4}{3}\varrho\frac{\dot V^2}{\pi R_1^2}$$

(Substitution: $t = 1-(r/R_1)^2$, $r\,\mathrm{d}r = -R_1^2/2\,\mathrm{d}t$)

and the second integral on the left hand side

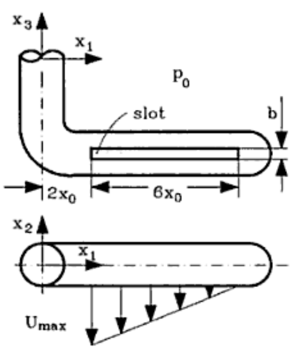
$$\int_0^{2\pi}\int_0^{R_2}\varrho\left\{\frac{2\dot V}{\pi R_2^2}\left[1-\left(\frac{r}{R_2}\right)^2\right]\right\}^2 r\,\mathrm{d}r\,\mathrm{d}\varphi = \frac{4}{3}\varrho\frac{\dot V^2}{\pi R_2^2}\,.$$

We solve (2) for the force vector:

$$\vec{F_N}\cdot\vec{e}_x = F_{N_x} = p_1\pi R_1^2 - p_2\pi R_2^2 + \frac{4}{3}\varrho\frac{\dot V^2}{\pi}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)\,.$$

2.3 Balance of Angular Momentum

Problem 2.3-1 Torque on pipe with slot



The sketched fixed pipe has in the horizontal part a narrow slot with the width b and the depth $6\,x_0$, where water (density ϱ) exits horizontally. The water velocity is a linear function of x_1. Viscous stresses may be neglected at all cross sections.

a) Determine the torque in x_3-direction as a function of $U_{\max}$ which is exerted by the flow on the pipe.

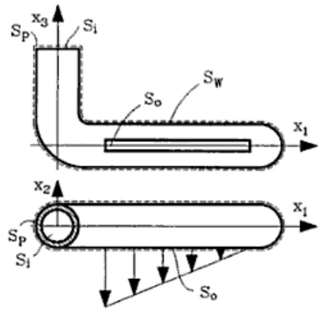
b) For the given volume flux of $\dot V$ find the maximum velocity of the water jet.

Given: b, x_0, $\dot V$, ϱ, p_0

Solution

a) The torque on the pipe:

The flow is steady, the body forces do not contribute to the moment of momentum in x_3–direction. Therefore, we use the balance of angular momentum in the form described in (F. M. (2.54b)) and form the third component by a scalar multiplication with $\vec{e}_3$



$$\iint\limits_{(S)} \varrho\, \vec{e}_3 \cdot (\vec{x} \times \vec{u})\,(\vec{u} \cdot \vec{n})\, \mathrm{d}S = \iint\limits_{(S)} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\, \mathrm{d}S\,. \tag{1}$$

The control volume we choose, encloses the pipe and intersects it at the surfaces S_i and S_P. We split the integration into the sketched parts and consider that $\vec{u} \cdot \vec{n} = 0$ at solid walls S_W and S_P:

$$\iint\limits_{S_i+S_o} \varrho\, \vec{e}_3 \cdot (\vec{x} \times \vec{u})\,(\vec{u} \cdot \vec{n})\, \mathrm{d}S = \iint\limits_{S_i+S_o+S_W} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\, \mathrm{d}S + \iint\limits_{S_P} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\, \mathrm{d}S\,. \tag{2}$$

To evaluate the integrals we consider individually the integrands.
Left hand side of (2):
At the inlet S_i with $\vec{u} = u_3\, \vec{e}_3$

$$\vec{e}_3 \cdot (\vec{x} \times \vec{u}) = -\vec{e}_3 \cdot (\vec{u} \times \vec{x}) = -(\vec{e}_3 \times \vec{u}) \cdot \vec{x} = 0\,,$$

at the outlet S_o with $\vec{u} = u_2\, \vec{e}_2$, $\vec{x} = x_1\, \vec{e}_1 + x_2\, \vec{e}_2 + x_3\, \vec{e}_3$, $\vec{n} = -\vec{e}_2$

$$\begin{aligned} \vec{e}_3 \cdot (\vec{x} \times \vec{u}) &= \vec{e}_3 \cdot ((x_1\, \vec{e}_1 + x_2\, \vec{e}_2 + x_3\, \vec{e}_3) \times u_2\, \vec{e}_2) \\ &= \vec{e}_3 \cdot (x_1\, u_2\, \vec{e}_3 - x_3\, u_2\, \vec{e}_1) = x_1\, u_2 \end{aligned}$$

and $\vec{u} \cdot \vec{n} = -u_2$.
The velocity distribution at the outlet is linear:

$$u_2 = u_o(x_1) = -U_{\max}\, \frac{8\; x_0 - x_1}{6\, x_0}\,, \quad \text{for } 2\, x_0 \le x_1 \le 8\, x_0\,.$$

For the left hand side of equation (1) we obtain

$$\begin{aligned} \iint\limits_{(S)} \varrho\, \vec{e}_3 \cdot (\vec{x} \times \vec{u})\,(\vec{u} \cdot \vec{n})\, \mathrm{d}S &= -\varrho\, U_{\max}^2 \int\limits_{2\,x_0}^{8\,x_0} \int\limits_{-b/2}^{+b/2} x_1 \left(\frac{8\; x_0 - x_1}{6\, x_0}\right)^2 \mathrm{d}x_3\, \mathrm{d}x_1 \\ &= -7\, \varrho\; b\, U_{\max}^2\, x_0^2\,. \end{aligned} \tag{3}$$

Now we consider the right hand side of equation (2):
Since the viscous stresses at S_i and S_o can be neglected ($P_{ij} = 0$), the stress vector at the inlet S_i assumes the form $\vec{t} = -p\,\vec{n}$ and at the outlet S_o we have $\vec{t} = -p_0\,\vec{n}$. At the wall S_W we also have $\vec{t} = -p_0\,\vec{n}$. Since the ambient pressure p_0 does not contribute to the torque, we may set $p_0 = 0$ and obtain

$$\iint\limits_{S_i+S_o+S_W} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\,\mathrm{d}S + \iint\limits_{S_P} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\,\mathrm{d}S =$$

$$\iint\limits_{S_i} \vec{e}_3 \cdot (\vec{x} \times (-p\,\vec{n}))\,\mathrm{d}S + \iint\limits_{S_P} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\,\mathrm{d}S\,. \tag{4}$$

The first integral on the left hand side over the inlet surface S_i disappears, because the vector product is normal to $\vec{e}_3$. The second integral is the moment in the intersection surface S_P, which is the reaction moment that correspond to the moment $M_{Fl\to P}$, which is exerted by the fluid on the pipe

$$\iint\limits_{S_P} \vec{e}_3 \cdot (\vec{x} \times \vec{t})\,\mathrm{d}S = M_{3_P} = -M_{Fl\to P}\,. \tag{5}$$

Thus, with (3), (4), and (5) we get the requested torque

$$M_{Fl\to P} = 7\,\varrho\,b\,x_0^2\,U_{\max}^2\,.$$

b) Maximum velocity $U_{\max}$:
With the given $\dot{V}$, we calculate $U_{\max}$ using the equation

$$\dot{V} = \iint\limits_{S_o} \vec{u} \cdot \vec{n}\,\mathrm{d}S$$

and obtain

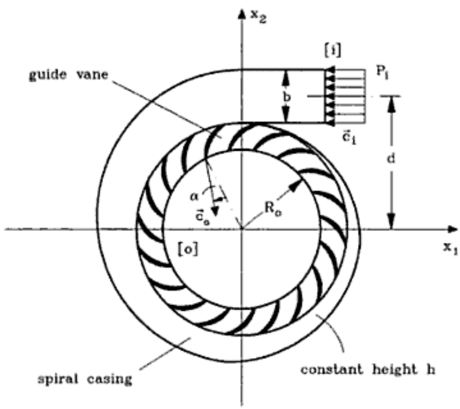
$$\dot{V} = b\,U_{\max} \int\limits_{2x_0}^{8x_0} \frac{8\,x_0 - x_1}{6\,x_0}\,\mathrm{d}x_1 = 3\,b\,U_{\max}\,x_0$$

$$\Rightarrow \qquad U_{\max} = \frac{\dot{V}}{3\,b\,x_0}\,.$$

Note: Frequently, it is appropriate to choose the control volume in such a way that it includes the fluid and the solid body. The concerns in connection with the Gauss theorem (used in Reynolds transport theorem) which requires differentiability of the quantities in control volume, can be circumvented by assuming that the transition between the fluid and the solid body is smooth but occurs with large gradients. As an alternative, one can place the control volume inside the pipe; the angular momentum on the wetted surface is the requested $M_{Fl \to P}$.

Problem 2.3-2 Moment exerted on the inlet guide vanes of a water turbine

The sketched inlet of a water turbine consists of a fixed spiral casing and the inlet guide vanes. The spiral casing is designed in such a way that the fluid ($\varrho = \text{const}$) can exit the vanes at a constant velocity and a constant exit flow angle α. We assume a steady flow with fully uniform velocity profile at the inlet and exit of the vanes and we also neglect the body forces.



a) Determine for a given volume flux $\dot{V}$, the magnitudes of the velocities $\vec{c}_i$ and $\vec{c}_o$.
b) Find the component of $\vec{c}_o$ in circumferential direction c_{uo} at the vane exit.
c) Calculate the torque in x_3–direction exerted by the flow on the entire inlet guides. (Hint: In evaluating the surface integrals over the inlet surface S_i, the terms linear in $\vec{x}$, i. e. $\vec{x} \times \vec{c}$ and $\vec{x} \times \vec{n}$ can be set equal to the corresponding mean values and be taken out of the integrals. It can be proved that the value of the surface integral is not affected by this simplifying assumption.)

Given: ϱ, $\dot{V}$, α, h, b, d, R_o, p_i

Solution

a) Velocity magnitudes c_i, c_o:
From the definition of volume flux we have

$$\dot{V} = -\iint_{S_i} \vec{c}\cdot\vec{n}\,\mathrm{d}S = \iint_{S_o} \vec{c}\cdot\vec{n}\,\mathrm{d}S\,.$$

The following conditions are obviously valid

$$\vec{c}\cdot\vec{n} = \begin{cases} -c_i & \text{on} \quad S_i \\ c_o\cos\alpha & \text{on} \quad S_o \end{cases}\,,$$

thus,

$$c_i = \frac{\dot{V}}{b\,h} \tag{1}$$

and

$$c_o = \frac{\dot{V}}{2\pi\,R_o\,h\,\cos\alpha}\,.$$

b) Components of $\vec{c}_o$ in circumferential direction c_{uo}:

$$c_{uo} = c_o\sin\alpha \quad \Rightarrow \quad c_{uo} = \frac{\dot{V}}{2\pi\,R_o\,h}\tan\alpha\,. \tag{2}$$

c) Moment exerted on the inlet guide vanes and spiral casing:
In the equation of angular momentum (steady flow, negligible body forces)

$$\iint_{(S)} \varrho\,\vec{x}\times\vec{c}(\vec{c}\cdot\vec{n})\,\mathrm{d}S = \iint_{(S)} \vec{x}\times\vec{t}\,\mathrm{d}S \tag{3}$$

we split the entire control surface (not shown in) in

$$S = S_i + S_o + S_W \quad (S_W = \text{wall surface of guide vanes})$$

and (3) yields, since, as a consequence of the kinematic boundary conditions,

$$\iint_{S_W} \varrho\,(\vec{x}\times\vec{c})\,(\vec{c}\cdot\vec{n})\,\mathrm{d}S = 0\,,$$

the following results

$$\iint_{S_i} \varrho\,\vec{x}\times\vec{c}(\vec{c}\cdot\vec{n})\,\mathrm{d}S + \iint_{S_o} \varrho\,\vec{x}\times\vec{c}(\vec{c}\cdot\vec{n})\,\mathrm{d}S$$

$$= \iint_{S_i} \vec{x}\times\vec{t}\,\mathrm{d}S + \iint_{S_o} \vec{x}\times\vec{t}\,\mathrm{d}S + \iint_{S_W} \vec{x}\times\vec{t}\,\mathrm{d}S\,. \tag{4}$$

The last integral on the right hand side is the moment, which the vanes and spiral casing exerts on the fluid. A sign change yields the reaction moment exerted by the fluid on the inlet guide vanes. The evaluation of the integrals gives the individual contributions:

1) On S_i, $\vec{x} \times \vec{c} = (x_{1i}\,\vec{e}_1 + d\,\vec{e}_2) \times (-c_i\,\vec{e}_1) = c_i d\,\vec{e}_3$

$$\Rightarrow \qquad \iint\limits_{S_i} \varrho\,\vec{x} \times \vec{c}(\vec{c}\cdot\vec{n})\,\mathrm{d}S = -\varrho\dot{V} c_i d\,\vec{e}_3\ .$$

2) On S_o, $\vec{x} \times \vec{c} = (R_o\,\vec{e}_r) \times (c_{ro}\,\vec{e}_r + c_{uo}\,\vec{e}_\varphi) = R_o c_{uo}\,\vec{e}_3$

$$\Rightarrow \qquad \iint\limits_{S_o} \varrho\,\vec{x} \times \vec{c}(\vec{c}\cdot\vec{n})\,\mathrm{d}S = \varrho\dot{V} R_o c_{uo}\,\vec{e}_3\ .$$

3) On S_i, the flow is uniform, i. e. $\vec{t} = -p\vec{n}$. Because $\vec{n} = \vec{e}_1$ we have therefore

$$\vec{x} \times \vec{t} = (x_{1i}\,\vec{e}_1 + d\,\vec{e}_2) \times (-p_i\,\vec{e}_1) = p_i d\,\vec{e}_3$$

$$\Rightarrow \qquad \iint\limits_{S_i} \vec{x} \times \vec{t}\,\mathrm{d}S = p_i b\,h\,d\,\vec{e}_3\ .$$

4) On S_o, the flow is also uniform, i. e. $\vec{t} = -p\,\vec{n} = p\,\vec{e}_r$, therefore

$$\vec{x} \times \vec{t} = (R_o\,\vec{e}_r) \times (p\,\vec{e}_r) = 0$$

$$\Rightarrow \qquad \iint\limits_{S_o} \vec{x} \times \vec{t}\,\mathrm{d}S = 0\ .$$

Thus, we obtain from (4) the moment exerted on the inlet guide vanes and spiral casing as

$$\vec{M}_{Fl.\to W} = \varrho\dot{V}(c_i d - c_{uo}R_o)\,\vec{e}_3 + p_i b\,h\,d\,\vec{e}_3\ ,$$

and the magnitude of the non-zero component is

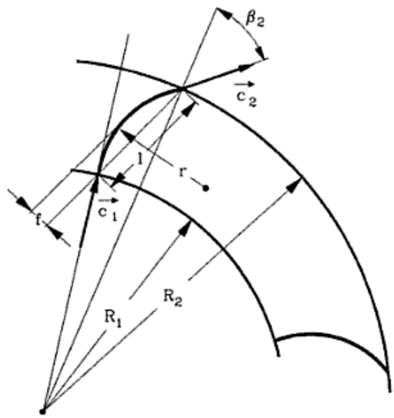
$$M_3 = \varrho\dot{V}(c_i d - c_{uo}R_o) + p_i b\,h\,d\ . \qquad (5)$$

The first term in (5) represents the change of angular momentum of the fluid and corresponds to the Euler turbine equation, the second term originates from the fact that the inlet surface is not a surface of revolution, where the stress vector (here $\vec{t} = -p\vec{n}$) generates a moment. Introducing in (5) c_i and c_{uo} from (1) respectively (2), we obtain the result

$$M_3 = \frac{\varrho\dot{V}^2}{h}\left(\frac{d}{b} - \frac{\tan\alpha}{2\pi}\right) + p_i b\,h\,d\ .$$

Problem 2.3-3 Curvature radius of circular arc profiles of a circular cascade

A swirl free fluid with density ϱ and the velocity c_1 enters the sketched fixed cascade that consists of 6 stationary blades. At the exit, the flow angle is β_2.



a) Assume a channel height H and calculate the mass flux $\dot{m}$ through the cascade.
b) Determine the torque by the flow on the cascade.
c) Find the force component in circumferential direction that is exerted on one blade, if the force acts at the radius r_k on the blade.
d) As we know, the lift force F_L in cascade flow is perpendicular to the mean velocity

$$\vec{U}_\infty = \frac{\vec{c}_1 + \vec{c}_2}{2} .$$

Calculate the angle γ between $\vec{c}_1$ and $\vec{U}_\infty$.
e) Determine the lift force on one of the blades.
f) For a small angle of attack, the lift force of a single circular arc profile can be calculated from the following relation:

$$c_L = \frac{F_L}{\varrho/2\, U_\infty^2\, l} = 2\pi \left(\alpha + 2\,\frac{f}{l}\right) , \qquad \alpha = \alpha_c - \gamma .$$

This relation can approximately be used for one blade within the cascade, provided that the blade spacing is much greater than the blade chord.

How should we choose the radius of curvature in order to determine the lift force calculated in e)? Use a circular arc profile of chord length l and radial angle α_c. α_c is the given angle between the chord and the radial direction. (Hint: For the circular arc profile the relation: $l^2 = 4\,(2\,f\,r - f^2)$ is valid)

Given: ϱ, c_1, α_c, β_2, R_1, R_2, r_k, H, l

Solution

a) Mass flux through the cascade:

$$\dot{m} = \varrho\, c_1\, 2\,\pi\, R_1\, H\ .$$

b) Torque T_{casc} on the cascade:
The Euler turbine equation (here for swirl free inlet flow condition) for the torque on the fluid gives

$$T = \dot{m}\,(R_2\, c_{u2})\ ,$$

where the unknown circumferential component of the exit velocity is calculated from (positive circumferential direction is anticlockwise):

$$c_{u2} = -c_{r2}\tan\beta_2 = \frac{-\dot{m}}{\varrho\, 2\,\pi\, R_2\, H}\tan\beta_2 = -c_1\,\frac{R_1}{R_2}\tan\beta_2\ .$$

For the torque on the cascade ($T_{casc} = -T$) we find

$$T_{casc} = c_1^2\, R_1^2\, \varrho\, 2\,\pi\, H\tan\beta_2\ .$$

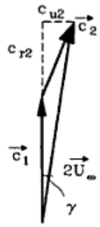
c) Blade force in circumferential direction:
The relationship between the torque and the force per blade in circumferential direction is given

$$T_{casc} = z\, F_u\, r_k\ , \qquad (z = \text{number of blades})$$

the circumferential component of the force on a blade is

$$F_u = \frac{T_{casc}}{z\, r_k} = \frac{c_1^2\, R_1^2\, H}{3\, r_k}\,\varrho\,\pi\tan\beta_2\ .$$

d) Angle between $\vec{c}_1$ and $\vec{U}_\infty$:

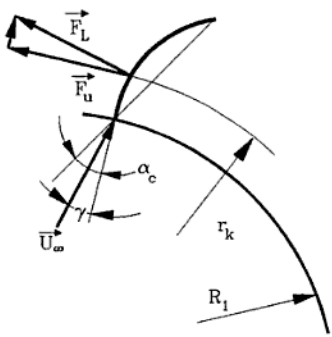


The result can be read from the sketch:

$$\tan\gamma = \frac{c_{u2}}{c_1 + c_{r2}} = \frac{c_1\, R_1/R_2 \tan\beta_2}{c_1 + c_1\, R_1/R_2}$$

$$\Rightarrow \qquad \gamma = \arctan\frac{\tan\beta_2}{R_2/R_1 + 1} .$$

e) Blade lift force:
The lift force is perpendicular to $\vec{U}_\infty$. Thus, we have for the relationship between the lift force and the circumferential component of the blade force



$$F_L = \frac{F_u}{\cos\gamma} .$$

Introducing the circumferential component, we obtain

$$F_L = \frac{T}{z\, r_k \cos\gamma} = \frac{c_1^2\, R_1^2\, H\, \varrho\, \pi \tan\beta_2}{3\, r_k \cos\left(\arctan\frac{\tan\beta_2}{R_2/R_1+1}\right)} .$$

f) Radius of curvature:
Using the relationship for the lift coefficient given in the problem definition we obtain the following expression for f

$$f = \frac{F_L}{2\pi\,\varrho\, U_\infty^2} - \frac{\alpha}{2}\, l .$$

The equation for the curvature radius can also be given as (see hint):

$$r = \frac{l^2}{8f} + \frac{f}{2}$$

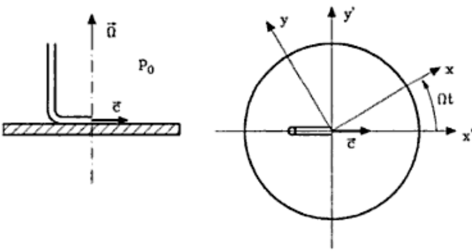
or else

$$r = \frac{l^2}{4}\left[\frac{R_1^2\,H\tan\beta_2}{3\,r_k\,(U_\infty^2/c_1^2)\cos\left(\arctan\frac{\tan\beta_2}{R_2/R_1+1}\right)} - \alpha\,l\right]^{-1} +$$

$$+\frac{1}{4}\left[\frac{R_1^2\,H\tan\beta_2}{3\,r_k\,(U_\infty^2/c_1^2)\cos\left(\arctan\frac{\tan\beta_2}{R_2/R_1+1}\right)} - \alpha\,l\right]\,.$$

2.4 Momentum and Angular Momentum in an Accelerating Frame

Problem 2.4-1 Fluid sprayed on a rotating disk



Inviscid fluid is sprayed on a disk rotating with angular velocity $\vec{\Omega}$. The spray nozzle with the tip located at the center, sprays a jet with the velocity $\vec{c} = c_0\,\vec{e}_{x'}$. The body forces $\varrho\,\vec{k}$ are neglected and the jet is subjected to ambient pressure.

a) Determine the path of the fluid particles in the inertial system (x', y').
b) Determine the same path in the rotating system (x, y) using the coordinate transformation.
c) Calculate the path directly by integrating the equation of motion in the rotating system.

Solution

a) The path in the inertial system:
The fluid is inviscid, the pressure p in the spray jet is constant, and the body forces are neglected, as a result, we obtain from Cauchy's law of

motion

$$\varrho \frac{\mathrm{D}\vec{c}}{\mathrm{D}t} = \varrho \vec{k} + \nabla \cdot \mathbf{T}$$

the differential equation for the pathline $\mathrm{D}\vec{c}/\mathrm{D}t = 0$, which, with the initial condition, yields the constant vector $\vec{c}$

$$\vec{c} = \frac{\mathrm{d}\vec{x}'}{\mathrm{d}t} = c_0 \, \vec{e}_{x'} \; .$$

The components in the inertial system ($\vec{x}' = x'\vec{e}_{x'} + y'\vec{e}_{y'}$) are written as

$$\frac{\mathrm{d}x'}{\mathrm{d}t} = c_0 \qquad \text{or} \qquad x' = c_0\, t + \text{const} \; ,$$

$$\frac{\mathrm{d}y'}{\mathrm{d}t} = 0 \qquad \text{or} \qquad y' = \text{const} \; .$$

With the initial conditions $x'(0) = 0$ and $y'(0) = 0$, the path of fluid particles in the inertial system is calculated as

$$\begin{aligned} x' &= c_0\, t \; , \\ y' &= 0 \; . \end{aligned}$$

b) The path in the rotating system:
For the coordinate transformation we have

$$x_j = a_{ji}\, x_i' \; .$$

The rotation matrix a_{ji} is

$$a_{ji} = \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix} \; .$$

This leads to

$$\begin{aligned} x &= \cos\Omega t\, x' + \sin\Omega t\, y' = c_0 t\, \cos\Omega t \; , \\ y &= -\sin\Omega t\, x' + \cos\Omega t y' = -c_0 t\, \sin\Omega t \; , \end{aligned}$$

and in cylindrical coordinate system

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = c_0\, t \; , \\ \varphi &= \arctan\left(\frac{y}{x}\right) = -\Omega t \; , \end{aligned}$$

and after eliminating t, we obtain

$$\varphi = -\frac{\Omega}{c_0} r \; .$$

c) Calculate the path directly integrating the equation of motion in the rotating system:
For the present case, we take from (F. M. (2.68)) the acceleration in the rotating system

$$\frac{\mathrm{D}\vec{w}}{\mathrm{D}t} = -2\vec{\Omega}\times\vec{w} - \vec{\Omega}\times(\vec{\Omega}\times\vec{x})\,.$$

With $\vec{w} = u\,\vec{e}_x + v\,\vec{e}_y$, $\vec{\Omega} = \Omega\,\vec{e}_z$ and $u = \dot{x}$, $v = \dot{y}$, the components in x- and y-direction are calculated:

$$\begin{aligned}\ddot{x} &= 2\,\Omega\,\dot{y} + \Omega^2\,x\,,\\ \ddot{y} &= -2\,\Omega\,\dot{x} + \Omega^2\,y\,.\end{aligned}$$

The two coupled, ordinary linear differential equations with constant coefficients can be rearranged using the following definitions

$$\mathbf{M} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,,\quad \mathbf{D} = \begin{pmatrix}0 & -2\Omega\\ 2\Omega & 0\end{pmatrix}\,,\quad \mathbf{K} = \begin{pmatrix}-\Omega^2 & 0\\ 0 & -\Omega^2\end{pmatrix}$$

$$\mathbf{M}\,\ddot{\vec{x}} + \mathbf{D}\,\dot{\vec{x}} + \mathbf{K}\,\vec{x} = \vec{0}\,. \tag{1}$$

The solution $\vec{x} = \vec{C}\,\mathrm{e}^{\lambda t}$ leads to the eigenvalue problem

$$(\lambda^2\,\mathbf{M} + \lambda\,\mathbf{D} + \mathbf{K})\,\vec{C} = \vec{0}\,. \tag{2}$$

Non-trivial solutions of the homogeneous systems (2) exist only if the determinant of the coefficient matrix disappears, i. e.

$$\begin{aligned} & & \det(\lambda^2\,\mathbf{M} + \lambda\,\mathbf{D} + \mathbf{K}) &= 0\\ \Rightarrow & & \det\begin{pmatrix}\lambda^2-\Omega^2 & -2\lambda\Omega\\ 2\lambda\Omega & \lambda^2-\Omega^2\end{pmatrix} &= 0\\ \Rightarrow & & (\lambda^2-\Omega^2)^2 + 4\lambda^2\Omega^2 = (\lambda^2+\Omega^2)^2 &= 0\,.\end{aligned}$$

$\lambda = \pm\mathrm{i}\Omega$ are each double eigenvalues, i. e. besides $\vec{C}\mathrm{e}^{\lambda t}$ also $\vec{C}t\mathrm{e}^{\lambda t}$ is a solution of (1). The eigenvectors $\vec{C}$ are calculated using the already known eigenvalues from (2). However, because the determinant is zero, only one equation can be used. We choose the first one

$$\begin{aligned}(\lambda^2-\Omega^2)\,C_1 - 2\,\lambda\,\Omega\,C_2 &= 0\\ \Rightarrow\qquad C_2 &= C_1\,\frac{\lambda^2-\Omega^2}{2\,\lambda\,\Omega}\end{aligned}$$

and obtain for $\lambda = \lambda_1 = +\mathrm{i}\Omega$ the first eigenvector:

$$C_2^{(1)} = C_1^{(1)} \frac{-2\,\Omega^2}{2\,\mathrm{i}\,\Omega^2} = C_1^{(1)}\mathrm{i} \qquad \Rightarrow \qquad \vec{C}^{(1)} = \alpha_1 \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \quad \text{with } C_1^{(1)} = \alpha_1$$

and for $\lambda = \lambda_2 = -\mathrm{i}\Omega$ the second one:

$$C_2^{(2)} = C_1^{(2)} \frac{-2\,\Omega^2}{-2\,\mathrm{i}\,\Omega^2} = -C_1^{(2)}\mathrm{i} \;\Rightarrow\; \vec{C}^{(2)} = \alpha_2 \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \quad \text{with } C_2^{(2)} = \alpha_2 \,.$$

The general solution of (1) is

$$\vec{x} = \alpha_1 \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}\,\Omega t} + \alpha_2 \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\,\Omega t} + t \left[\beta_1 \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}\,\Omega t} + \beta_2 \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\,\Omega t} \right] .$$

The constants $\alpha_1, \beta_1, \alpha_2, \beta_2$ which may be complex, can be determined from the initial conditions $\vec{x}(0) = \vec{0}$, $\dot{\vec{x}}(0) = c_0\,\vec{e}_x$ (if complex, then four equations for four complex constants). We get

$$\begin{aligned} \vec{x}(0) = \vec{0} &= \alpha_1 \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \\ \Rightarrow \qquad 0 &= \alpha_1 + \alpha_2 \,, \\ 0 &= \mathrm{i}\,(\alpha_1 - \alpha_2) \,. \end{aligned}$$

These two equations are satisfied only if $\alpha_1 = \alpha_2 = 0$, therefore we find

$$\dot{\vec{x}} = \beta_1 \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}\,\Omega t} + \beta_2 \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\,\Omega t} + t \left[\mathrm{i}\,\beta_1 \Omega \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}\,\Omega t} - \mathrm{i}\,\beta_2 \Omega \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\,\Omega t} \right] .$$

With $\dot{\vec{x}}(0) = (c_0, 0)^T$, we finally get

$$\begin{aligned} \dot{\vec{x}} = \begin{pmatrix} c_0 \\ 0 \end{pmatrix} &= \beta_1 \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \\ \Rightarrow \qquad c_0 &= \beta_1 + \beta_2 \,, \\ 0 &= \mathrm{i}\,(\beta_1 - \beta_2) \\ \Rightarrow \qquad \beta_1 = \beta_2 &= \frac{c_0}{2} \,, \end{aligned}$$

and considering the initial conditions, we arrive at

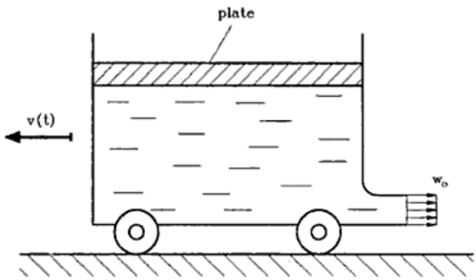
$$\vec{x} = t \left[\frac{c_0}{2} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}\,\Omega t} + \frac{c_0}{2} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\,\Omega t} \right] .$$

We decompose the above vector equation into its components and obtain the pathline in the form already know from part b):

$$x(t) = c_0\, t\left(\frac{1}{2}\cos\Omega t + \frac{\mathrm{i}}{2}\sin\Omega t + \frac{1}{2}\cos\Omega t - \frac{\mathrm{i}}{2}\sin\Omega t\right) = c_0\, t\, \cos\Omega t\ ,$$

$$y(t) = c_0\, t\left(\frac{\mathrm{i}}{2}\cos\Omega t - \frac{1}{2}\sin\Omega t - \frac{\mathrm{i}}{2}\cos\Omega t - \frac{1}{2}\sin\Omega t\right) = -c_0\, t\, \sin\Omega t\ .$$

Problem 2.4-2 Velocity of a moving container with a nozzle



A cart with a nozzle contains fluid of constant density. Its total mass at $t = 0$ is m_0. A slow motion of a heavy plate generates a constant mass flux $\dot{m}$ through the nozzle with an exit velocity w_o relative to the container. The flow in the relative system is steady. We assume that air drag and contact friction are negligible. Calculate the velocity $v(t)$ of the cart.

Given: m_0, $\dot{m}$, w_o, $v(t=0) = 0$

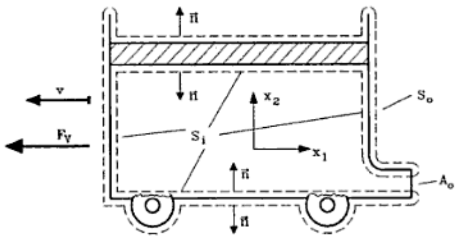
First Solution

The equation of motion for the container is

$$m_C\,\frac{\mathrm{d}v}{\mathrm{d}t} = F_v\ ,$$

with m_C as the mass of the container without fluid and F_v the force component in direction of container motion.

The force $\vec{F}$ acting only on the container is calculated by integrating the stress vector over the entire control surface



$$\vec{F} = \iint_{S_i} \vec{t}\,\mathrm{d}S + \iint_{S_o} \vec{t}\,\mathrm{d}S\,. \tag{1}$$

Neglecting the air drag,
we set the stress vector $\vec{t} = -p_0\,\vec{n}$ on S_o and obtain

$$\iint_{S_o} \vec{t}\,\mathrm{d}S = \iint_{S_o} -p_0\,\vec{n}\,\mathrm{d}S = \iint_{S_o+A_o} -p_0\,\vec{n}\,\mathrm{d}S - \iint_{A_o} -p_0\,\vec{n}\,\mathrm{d}S\,.$$

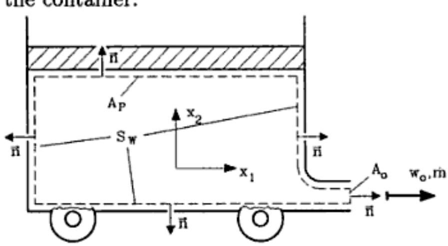
Since $S_o + A_o$ is a closed surface, the first integral is equal to zero, therefore

$$\iint_{S_o} -p_0\,\vec{n}\,\mathrm{d}S = p_0 \iint_{A_o} \vec{n}\,\mathrm{d}S = p_0\,A_o\,\vec{e}_1\,.$$

The force component in direction of the container motion (negative in x_1–direction) is calculated from (1)

$$F_v = \vec{F}\cdot(-\vec{e}_1) = -p_0\,A_o + \iint_{S_i} -t_1\,\mathrm{d}S\,. \tag{2}$$

The integral on the right hand side represents the fluid force that acts on the container.



The integral in (2) is carried out by applying the balance of momentum in integral form on the fluid in the container. We use the sketched accelerated frame.
The momentum equation for the control volume is (see F. M. (2.73))

$$\left(\frac{\partial}{\partial t}\iiint_{(V)} \varrho\,\vec{c}\,\mathrm{d}V\right)_A + \iint_{(S)} \varrho\,\vec{c}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \vec{\Omega}\times\iiint_{(V)} \varrho\,\vec{c}\,\mathrm{d}V = \iiint_{(V)} \varrho\,\vec{k}\,\mathrm{d}V + \iint_{(S)} \vec{t}\,\mathrm{d}S\,.$$

With

$$\vec{c} = \vec{w} + \vec{v} + \vec{\Omega} \times \vec{x} = \vec{w} + \vec{v}\,, \quad \vec{\Omega} = 0$$

and

$$S = S_W + A_P + A_o$$

we obtain the component in x_1–direction

$$\left(\frac{\partial}{\partial t} \iiint\limits_{(V)} \varrho\, c_1 \,\mathrm{d}V\right)_A + \iint\limits_{S_W} \varrho\, c_1(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{A_P} \varrho\, c_1(\vec{w}\cdot\vec{n})\,\mathrm{d}S +$$

$$+ \iint\limits_{A_o} \varrho\, c_1(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{S_W+A_P} t_1\,\mathrm{d}S + \iint\limits_{A_o} t_1\,\mathrm{d}S\,. \qquad (3)$$

At the wall S_W the product $\vec{w}\cdot\vec{n}$ and thus the integral will disappear. The first integral on the right hand side represents the force by the container exerted on the fluid. To obtain the reaction force, the sign needs to be changed. On the surfaces $S_W + A_P$, the normal unit vectors show an outward orientation, whereas on the surface S_i they are oriented inwardly. Therefore, we have

$$\iint\limits_{S_W+A_P} t_1\,\mathrm{d}S = \iint\limits_{S_i} -t_1\,\mathrm{d}S\,.$$

This expression corresponds to the required integral in (2). On A_o the flow is uniform, i. e. $\vec{t} = -p\,\vec{n}$. Furthermore, the jet pressure is equal to the ambient pressure p_0, consequently:

$$\iint\limits_{A_o} t_1\,\mathrm{d}S = \iint\limits_{A_o} -p_0 n_1\,\mathrm{d}S = -p_0 A_o\,.$$

We solve (3) for the required integral and obtain

$$\iint\limits_{S_i} -t_1\,\mathrm{d}S = p_0 A_o + \left(\frac{\partial}{\partial t} \iiint\limits_{(V)} \varrho\, c_1\,\mathrm{d}V\right)_A +$$

$$+ \iint\limits_{A_P} \varrho\, c_1\, \vec{w}\cdot\vec{n}\,\mathrm{d}S + + \iint\limits_{A_o} \varrho\, c_1\, \vec{w}\cdot\vec{n}\,\mathrm{d}S\,. \qquad (4)$$

With $c_1 = w_1 + v_1 = w_1 - v(t)$, we write the first integral on the right hand side

$$\left(\frac{\partial}{\partial t} \iiint\limits_{(V)} \varrho\, c_1\,\mathrm{d}V\right)_A = \iiint\limits_{(V)} \varrho\left(\frac{\partial w_1}{\partial t} - \frac{\mathrm{d}v}{\mathrm{d}t}\right)\,\mathrm{d}V\,.$$

The velocity w_1 inside the container can always be neglected $w_1 \approx 0$ and thus, also $\partial w_1/\partial t \approx 0$. At the exit nozzle the velocity w_1 is large but there it is steady $w_1 = w_o = \text{const}$. This leads to

$$\left(\frac{\partial}{\partial t}\iiint\limits_{(V)} \varrho\, c_1 \,\mathrm{d}V\right)_A = -\frac{\mathrm{d}v}{\mathrm{d}t}\iiint\limits_{(V)} \varrho \,\mathrm{d}V = -m_F\,\frac{\mathrm{d}v}{\mathrm{d}t}$$

with m_F as the fluid mass within the fixed control volume. For the second integral in (4) we obtain

$$\iint\limits_{A_P} \varrho\, c_1\, \vec{w}\cdot\vec{n}\,\mathrm{d}S = \iint\limits_{A_P} \varrho\,(w_1 - v(t))\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = -v(t)\iint\limits_{A_P} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S\,.$$

The last integral can be calculated using the continuity equation

$$\iiint\limits_{(V)} \frac{\partial\varrho}{\partial t}\,\mathrm{d}V = -\iint\limits_{(S)} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S\,,$$

with

$$\iiint\limits_{(V)} \frac{\partial\varrho}{\partial t}\,\mathrm{d}V = 0\,,$$

(the density ϱ is constant) and

$$\iint\limits_{S_W} \varrho\,\underbrace{\vec{w}\cdot\vec{n}}_{=0}\,\mathrm{d}S + \iint\limits_{A_P} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S + \underbrace{\iint\limits_{A_o} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S}_{=\dot{m}} = 0$$

it follows

$$\iint\limits_{A_P} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = -\dot{m}$$

and

$$\iint\limits_{A_P} \varrho\, c_1\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = \dot{m}\,v(t)\,.$$

Finally, the third integral on the right hand side in (4) is

$$\iint\limits_{A_o} \varrho\, c_1\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = \iint\limits_{A_o} \varrho\,(w_1 - v(t))\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = (w_o - v(t))\,\dot{m}\,,$$

such that from (4) the following equation can be generated

$$\iint\limits_{S_i} -t_1\,\mathrm{d}S = p_0\,A_o - m_F\,\frac{\mathrm{d}v}{\mathrm{d}t} + \dot{m}\,v(t) + \dot{m}(w_o - v(t))$$

which can be simplified as

$$\iint_{S_i} -t_1 \, \mathrm{d}S = p_0 \, A_o - m_F \frac{\mathrm{d}v}{\mathrm{d}t} + \dot{m} \, w_o \, .$$

From (2), we obtain the force acting on the moving container

$$F_v = -p_0 \, A_o + p_0 \, A_o - m_F \frac{\mathrm{d}v}{\mathrm{d}t} + \dot{m} \, w_o$$

thus, for the equation of motion

$$m_C \frac{\mathrm{d}v}{\mathrm{d}t} = -m_F \frac{\mathrm{d}v}{\mathrm{d}t} + \dot{m} \, w_o \, ,$$

or

$$(m_C + m_F) \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{m} \, w_o \, . \tag{5}$$

This equation shows that both the container mass as well as the fluid mass must be accelerated by the thrust of the jet ($\dot{m} \, w_o$) as we expected. The total mass $m(t) = m_C + m_F(t)$ decreases with time and since $\dot{m} =$ const, we have $m(t) = m_0 - \dot{m} \, t$, where m_0 is the total initial mass, i. e. $m_0 = m_C + m_F(t = 0)$. The differential equation for v is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\dot{m}}{m_0 - \dot{m} \, t} \, w_o \, .$$

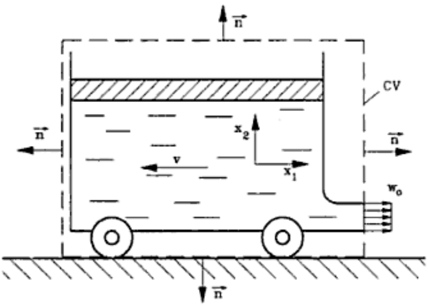
By integrating

$$\int_0^{v(t)} \mathrm{d}v = w_o \int_0^t \frac{\dot{m}}{m_0 - \dot{m} \, t} \, \mathrm{d}t$$

we obtain the requested velocity of the moving container

$$v(t) = w_o \ln \left(\frac{m_0}{m_0 - \dot{m} \, t} \right) \, .$$

Second Solution



We solve the problem using the container-fixed control volume (see figure): The momentum equation in the container-fixed (accelerated) reference system is

$$\frac{\partial}{\partial t}\left(\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V\right)_A + \iint\limits_{(S)} \varrho\,\vec{c}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \vec{\Omega}\times\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V = \iiint\limits_{(V)} \varrho\,\vec{k}\,\mathrm{d}V + \iint\limits_{(S)} \vec{t}\,\mathrm{d}S\;,$$

where the absolute velocity $\vec{c}$ can be calculated from

$$\vec{c} = \vec{w} + \vec{v} + \vec{\Omega}\times\vec{x}\;,$$

with $\vec{\Omega} = 0$ and $\vec{v} = -v(t)\,\vec{e}_1$. Furthermore, in computing the container velocity $v(t)$ the body forces $\varrho\,\vec{k}$ play no role. Therefore, we set $\vec{k} = 0$. We need only the x_1–momentum component, which we obtain by multiplying the momentum equation with $\vec{e}_1$:

$$\frac{\partial}{\partial t}\left(\iiint\limits_{(V)} \varrho\,c_1\,\mathrm{d}V\right)_A + \iint\limits_{(S)} \varrho\,c_1\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)} t_1\,\mathrm{d}S\;, \qquad (6)$$

where

$$c_1 = w_1 - v(t)\;.$$

The integral on the right hand side would contain the air drag forces, which we neglect here, and thus, the right hand side identically vanishes. For the first integral on the left hand side, we can write

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\iiint\limits_{(V)} \varrho\,c_1\,\mathrm{d}V\right)_A &= \iiint\limits_{(V)} \frac{\partial}{\partial t}\left(\varrho\,(w_1 - v(t))\right)\,\mathrm{d}V \\
&= \iiint\limits_{(V)} \frac{\partial\varrho}{\partial t}(w_1 - v(t))\,\mathrm{d}V + \iiint\limits_{(V)} \varrho\left(\frac{\partial w_1}{\partial t} - \frac{\partial v}{\partial t}\right)\,\mathrm{d}V\;.
\end{aligned}$$

Since at locations, where w_1 is different from zero (in the nozzle), the flow is steady, the following relation is valid

$$\frac{\partial \varrho}{\partial t} w_1 = \varrho \frac{\partial w_1}{\partial t} = 0 \, .$$

With the above equations we get

$$\frac{\partial}{\partial t}\left(\iiint\limits_{(V)} \varrho\, c_1 \,\mathrm{d}V\right)_A = -v(t) \iiint\limits_{(V)} \frac{\partial \varrho}{\partial t} \,\mathrm{d}V - \frac{\mathrm{d}v}{\mathrm{d}t} \iiint\limits_{(V)} \varrho \,\mathrm{d}V \, ,$$

where the last integral is the total mass m in the control volume: For the second integral in (6), we write (A_o is again the exit surface area)

$$\begin{aligned}
\iint\limits_{(S)} \varrho\, c_1 (\vec{w}\cdot\vec{n}) \,\mathrm{d}S &= \iint\limits_{A_o} \varrho (w_1 - v(t)) (\vec{w}\cdot\vec{n}) \,\mathrm{d}S \\
&= (w_o - v(t)) \iint\limits_{A_o} \varrho\, (\vec{w}\cdot\vec{n}) \,\mathrm{d}S \\
&= \dot{m}\, (w_o - v(t)) \, .
\end{aligned}$$

Thus, the momentum equation is:

$$-v(t) \iiint\limits_{(V)} \frac{\partial \varrho}{\partial t} \,\mathrm{d}V - m \frac{\mathrm{d}v}{\mathrm{d}t} + \dot{m}\, (w_o - v(t)) = 0 \, .$$

We apply the continuity equation to the control volume and get

$$\frac{\partial}{\partial t} \iiint\limits_{(V)} \varrho \,\mathrm{d}V = \iiint\limits_{(V)} \frac{\partial \varrho}{\partial t} \,\mathrm{d}V = -\iint\limits_{(S)} \varrho\, \vec{w}\cdot\vec{n} \,\mathrm{d}S = -\dot{m} \, .$$

This equation shows that the mass in the fixed control volume changes and the local change of density does not vanish. This local change occurs at the boundary surface between the plate and the fluid and is generated by the motion of the plate. Thus, we obtain again equation (5):

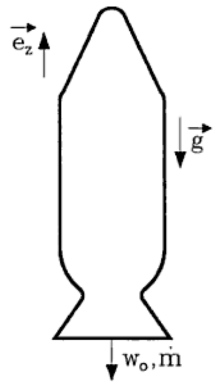
$$v(t)\, \dot{m} - m \frac{\mathrm{d}v}{\mathrm{d}t} + \dot{m}\, (w_o - v(t)) = 0$$

$$\Rightarrow \qquad m \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{m}\, w_o \, .$$

As expected, the problem is reduced to Newton's equation of motion for the acceleration of the total mass.

Problem 2.4-3 Acceleration and velocity of a rocket

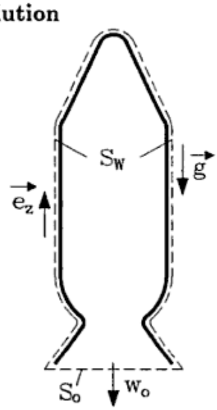
A rocket initially at rest is fired at time $t = 0$. The velocity of the exiting jet relative to the rocket is w_o, the mass flux is $\dot{m}$, and the starting mass is m_0. The rocket moves in vertical direction $\vec{e}_z$ and the air drag force is negligible. The supersonic nozzle is designed in such a way that at the exit, the jet pressure is equal to the ambient pressure.



a) Determine the initial acceleration of the rocket.
b) Calculate the velocity after t_0.

Given: w_o, $\dot{m}$, m_0, $\vec{g}$

Solution



a) Initial acceleration:
We apply the balance of momentum for an accelerated reference frame to the sketched control volume (see F. M. (2.73)). We do not take into account the ambient pressure, because it does not contribute to the net force on the control volume:

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho\,\vec{c}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \vec{\Omega}\times\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V = \iiint\limits_{(V)} \varrho\,\vec{k}\,\mathrm{d}V + \iint\limits_{(S)} \vec{t}\,\mathrm{d}S\,,$$

where

$$\vec{c} = \vec{w} + \vec{\Omega}\times\vec{x} + \vec{v} = \vec{w} + \vec{v}\,,$$

because no rotation occurs. The $\vec{e}_z$–component is the only interesting one, and including the body force ($\varrho\,\vec{k} = \varrho\,\vec{g}$) it is:

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_z\,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_z\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S$$

$$= \iiint\limits_{(V)} \varrho\,\vec{g}\cdot\vec{e}_z\,\mathrm{d}V + \iint\limits_{(S)} \vec{t}\cdot\vec{e}_z\,\mathrm{d}S\,. \qquad (1)$$

Evaluating the first integral gives:

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_z\,\mathrm{d}V\right]_A = \iiint\limits_{(V)} \left(\frac{\partial\varrho}{\partial t}(w_z+v_z) + \varrho\frac{\partial(w_z+v_z)}{\partial t}\right)\,\mathrm{d}V\,.$$

The relative velocity inside the rocket is approximately equal to zero and with

$$v_z = v_{\text{rocket}} = v_R\,,$$

and

$$\iiint\limits_{(V)} \frac{\partial\varrho}{\partial t}\,\mathrm{d}V = -\iint\limits_{S_o} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = -\dot{m}\,, \qquad \text{or} \qquad \iiint\limits_{(V)} \varrho\,\mathrm{d}V = m(t)$$

we get

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_z\,\mathrm{d}V\right]_A = -v_R\,\dot{m} + \frac{\mathrm{d}v_R}{\mathrm{d}t}\,m(t)\,.$$

The flux term

$$\iint\limits_{(S)} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_z\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S$$

is evaluated at the exit surface, since at the wall S_W the contribution vanishes because $\vec{w}\cdot\vec{n} = 0$. We arrive with $w_z + v_z = -w_o + v_R$ at

$$\iint\limits_{S_o} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_z\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = (v_R - w_o)\dot{m}\,.$$

The integrals on the right hand side of (1) are:

$$\iiint\limits_{(V)} \varrho\,\vec{g}\cdot\vec{e}_z\,\mathrm{d}V = -g\,m(t)\,,$$

$$\iint\limits_{(S)} \vec{t}\cdot\vec{e}_z\,\mathrm{d}S = F_{\text{drag force}} \approx 0\,,$$

so that one can recast (1) in the form

$$-v_R\,\dot{m} + \frac{\mathrm{d}v_R}{\mathrm{d}t}\,m(t) + \dot{m}\,(v_R - w_o) = -g\,m(t)\ .$$

Using the equation $m(t) = m_0 - \dot{m}\,t$, the acceleration follows to

$$\frac{\mathrm{d}v_R}{\mathrm{d}t} = \frac{1}{m_0/\dot{m} - t}\,w_o - g\ . \tag{2}$$

This equation furnishes the initial acceleration

$$\frac{\mathrm{d}v_R}{\mathrm{d}t} = \frac{\dot{m}}{m_0}\,w_o - g\ ,$$

which can be interpreted as

$$\text{Acceleration} = (\text{thrust} - \text{weight})/m_0.$$

b) Rocket velocity as a function of time:
We integrate the equation (2)

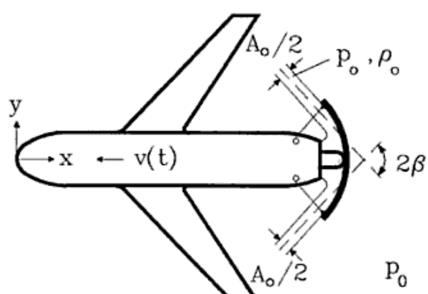
$$\int\limits_0^{v_R} \mathrm{d}v_R \;=\; \int\limits_0^{t_0} \frac{w_o}{m_0/\dot{m} - t}\,\mathrm{d}t - \int\limits_0^{t_0} g\,\mathrm{d}t$$

$$\Rightarrow \quad v_R \;=\; w_o \ln\left(\frac{m_0}{m_0 - \dot{m}\,t_0}\right) - g\,t_0$$

(compare Problem 2.4–2).

Problem 2.4-4 Thrust reversal

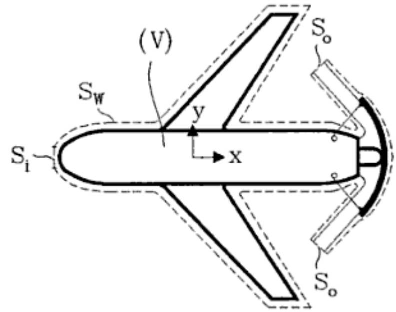
After touchdown of aircrafts (mass m_{tot}), the thrust reverser blocks the jet to the rear and directs it forward to produce reverse thrust. The exiting jet (subsonic jet $p_o = p_0$, ϱ_o, w_o, A_o) is subdivided into two symmetric jets with a deflection angle of $\pi - \beta$ (see figure). Thus, the aircraft experiences a deceleration $\vec{a} = a\,\vec{e}_x$. Body forces and the skin friction are neglected.



The engine inlet momentum flux, but not the mass flux can be neglected. Calculate the deceleration $\vec{a}$.

Solution

We choose an aircraft-fixed control volume, that includes the aircraft and the thrust reverser. We take the momentum equation in an accelerated reference frame (see F. M. (2.73)) with $\Omega = 0$ as



$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho \vec{c} \,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho \vec{c}(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)} \vec{t}\,\mathrm{d}S\,.$$

We consider only the x–component

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho \vec{c}\cdot\vec{e}_x \,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho (\vec{c}\cdot\vec{e}_x)(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S\,.$$

The flow in the aircraft-fixed frame is steady, i. e. $\partial\vec{w}/\partial t = 0$ and with $\vec{c} = \vec{w} + \vec{v}$ and $\vec{t} = -p_0\,\vec{n}$ on (S) we first obtain

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho \vec{v}\cdot\vec{e}_x \,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho (\vec{w}\cdot\vec{e}_x)(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{(S)} \varrho (\vec{v}\cdot\vec{e}_x)(\vec{w}\cdot\vec{n})\,\mathrm{d}S$$

$$= -\iint\limits_{(S)} p_0\,\vec{n}\cdot\vec{e}_x\,\mathrm{d}S \qquad .$$

Since a closed integral over a constant (here p_0) disappears, and the velocity v in each point of control volume is the same, we can simplify the above equation

$$\left[\frac{\partial\vec{v}}{\partial t}\cdot\vec{e}_x \iiint\limits_{(V)} \varrho \,\mathrm{d}V\right]_A + \vec{v}\cdot\vec{e}_x\left[\iiint\limits_{(V)} \frac{\partial\varrho}{\partial t}\,\mathrm{d}V\right]_A +$$

$$+\vec{v}\cdot\vec{e}_x \iint\limits_{(S)} \varrho(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{(S)} \varrho(\vec{w}\cdot\vec{e}_x)(\vec{w}\cdot\vec{n})\,\mathrm{d}S \;=\; 0\,.$$

Using the continuity equation and the following definitions

$$\frac{\partial \vec{v}}{\partial t} \cdot \vec{e}_x = a \,, \qquad \iiint\limits_{(V)} \varrho \, \mathrm{d}V = m_{\mathrm{tot}}$$

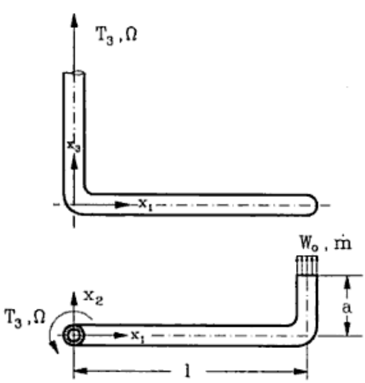
we can write

$$-a\, m_{\mathrm{tot}} = \iint\limits_{S_i} \varrho\, (\vec{w}\cdot\vec{e}_x)\, (\vec{w}\cdot\vec{n})\, \mathrm{d}S + \iint\limits_{S_W} \varrho\, (\vec{w}\cdot\vec{e}_x)\, (\vec{w}\cdot\vec{n})\, \mathrm{d}S + \iint\limits_{S_o} \varrho\, (\vec{w}\cdot\vec{e}_x)\, (\vec{w}\cdot\vec{n})\, \mathrm{d}S \,.$$

Neglecting the momentum through the inlet surface S_i, we arrive at the result

$$\begin{aligned} a\, m_{\mathrm{tot}} &= \varrho_o\, (w_o \cos\beta)\, w_o\, A_o \\ \Rightarrow \qquad a &= \frac{\varrho_o\, w_o^2\, A_o \cos\beta}{m_{\mathrm{tot}}} \,. \end{aligned}$$

Problem 2.4-5 Torque on a rotating bent pipe

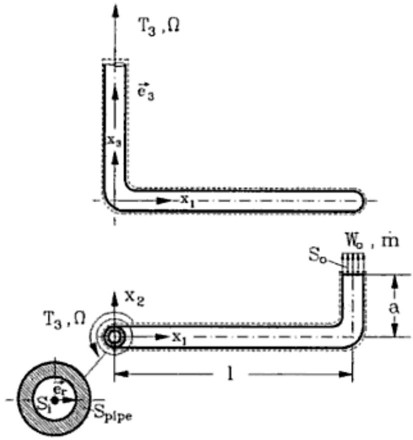


Fluid exits a thin bent pipe, which rotates with a constant angular velocity Ω about the x_3-axis. The exit velocity of the mass flux $\dot{m}$ is W_o. Calculate the torque T_3 that must be exerted on the pipe to maintain the rotational motion in the given direction.
Hint: The skin friction outside the pipe can be neglected.

Given: W_o, l, a, $\dot{m}$, Ω, on inlet $t_\varphi = \eta \dfrac{1}{r} \left(\dfrac{\partial w_z}{\partial \varphi} + r \dfrac{\partial w_\varphi}{\partial z} \right)$

Solution

We place the control surface S along the pipe and intersect the pipe at the inlet surface S_i, and close the control volume with a cross section at the exit jet. In this rotating system the flow is steady and the components of the angular momentum in $\vec{e}_3$–direction are



$$\vec{e}_3 \cdot \iint\limits_{(S)} \varrho\,(\vec{x}\times\vec{c})\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \vec{e}_3 \cdot \iint\limits_{(S)} \vec{x}\times\vec{t}\,\mathrm{d}S\,. \qquad (1)$$

The stress vector on the entire outer pipe surface and within the exiting jet is equal to zero, if we set the ambient pressure equal to zero, which is permissible in this particular problem. Only the integrals of the stress vectors over S_i and over the pipe cross section S_{pipe} must be evaluated. The later is the requested torque

$$T_3 = \vec{e}_3 \cdot \iint\limits_{S_{pipe}} \vec{x}\times\vec{t}\,\mathrm{d}S\,. \qquad (2)$$

On the inlet surface S_i, with $\vec{e}_z = \vec{e}_3$ the following relation is valid:

$$\vec{e}_z \cdot (\vec{x}\times\vec{t}) = r\,t_\varphi = r\,\tau_{\varphi z} = \eta\,\left(\frac{\partial w_z}{\partial \varphi} + r\,\frac{\partial w_\varphi}{\partial z}\right) = 0\,,$$

since the pipe flow is axisymmetric and w_φ is zero. From (1) with (2) follows:

$$T_3 = \iint\limits_{(S)} \varrho\,\vec{e}_3\cdot(\vec{x}\times\vec{c})\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S\,.$$

The normal component of the relative velocity $\vec{w}\cdot\vec{n}$ disappears everywhere, except on inlet and exit surfaces S_i and S_o. On S_i, $\vec{e}_3\cdot(\vec{x}\times\vec{c}) = 0$. Thus, we can write

$$T_3 = \iint\limits_{S_o} \varrho\,\vec{e}_3\cdot(\vec{x}\times\vec{c})\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S\,. \qquad (3)$$

On the exit surface S_o, we have

$$\vec{x} = l\,\vec{e}_1 + a\,\vec{e}_2\,,$$

$$\vec{w} = W_o\,\vec{e}_2\,,$$

thus, the absolute velocity, because of $\vec{v} = 0$ and $\vec{\Omega} = \Omega\,\vec{e}_3$ is:

$$\begin{aligned}\vec{c} &= W_o\,\vec{e}_2 + \Omega\,\vec{e}_3 \times (l\,\vec{e}_1 + a\,\vec{e}_2)\\ &= W_o\,\vec{e}_2 + \Omega l\,\vec{e}_3 \times \vec{e}_1 + \Omega a\,\vec{e}_3 \times \vec{e}_2\\ &= (W_o + \Omega l)\,\vec{e}_2 - \Omega a\,\vec{e}_1\end{aligned}$$

Calculating the moment of momentum (per unit mass) on the exit surface S_o (see Problem 2.3-2)

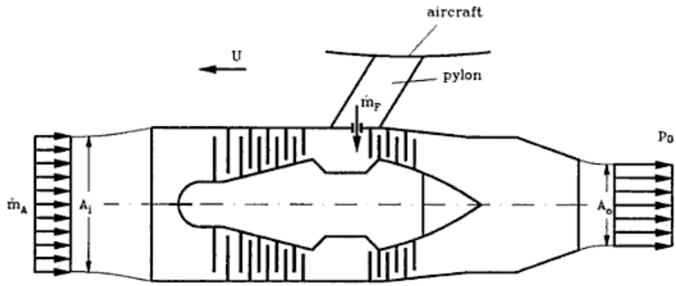
$$\begin{aligned}\vec{x} \times \vec{c} &= (l\,\vec{e}_1 + a\,\vec{e}_2) \times ((W_o + \Omega l)\,\vec{e}_2 - \Omega a\,\vec{e}_1)\\ &= \left[W_o l + \Omega(l^2 + a^2)\right]\,\vec{e}_3\,,\end{aligned}$$

we obtain from (3) the torque T_3 as

$$\begin{aligned}T_3 &= \iint\limits_{S_o} \varrho \left[W_o l + \Omega(l^2 + a^2)\right] W_o\,\mathrm{d}S\\ &= \left[W_o l + \Omega(l^2 + a^2)\right]\,\varrho\,W_o\,S_o\\ T_3 &= \dot{m}\,\left[W_o l + \Omega(l^2 + a^2)\right]\,.\end{aligned}$$

The same results can be obtained using Euler's turbine equation.

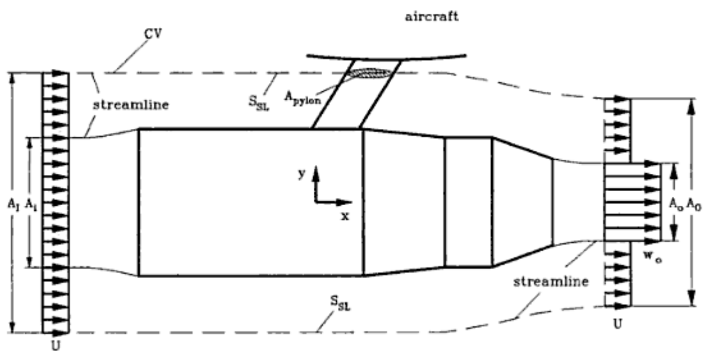
Problem 2.4-6 Thrust of a jet engine



An aircraft with the sketched jet engine flies with a constant velocity U through the air at rest. Air mass flux $\dot{m}_A$ enters the inlet diffuser of the engine with a fuel mass flux consumption of $\dot{m}_F$. The combustion gases exit the engine at the relative velocity w_o.
Calculate the thrust of the engine.

Given: U, w_o, $\dot{m}_A$, $\dot{m}_F$

Solution
We calculate the thrust using the balance of momentum. In the engine-fixed frame, we assume steady flow and place the control volume around the engine far enough that the disturbances have died out. As the control volume boundary, we choose a stream tube, then no mass flux crosses the boundary. Thus the momentum flux through the stream tube wall vanishes. We also neglect the momentum contribution of the fuel mass flux.



The momentum balance for an accelerated reference frame is (see F. M.

(2.73))

$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho\,\vec{c}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \vec{\Omega}\times\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V =$$

$$\iiint\limits_{(V)} \varrho\,\vec{k}\,\mathrm{d}V + \iint\limits_{(S)} \vec{t}\,\mathrm{d}S\,. \tag{1}$$

With $\vec{\Omega} = \vec{0}$, thus $\vec{c} = \vec{w} + \vec{v}$ and because $\vec{v} \neq \vec{v}(t)$, the above equation reduces, neglecting body forces, to

$$\iint\limits_{(S)} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{(S)} \varrho\,\vec{v}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)} \vec{t}\,\mathrm{d}S\,. \tag{2}$$

This is expected because the reference system is in fact not accelerated. The velocity $\vec{v}$ of the frame is for each point of the control volume the same. Considering the continuity equation, the second integral on the left hand side of (2) becomes zero:

$$\iint\limits_{(S)} \varrho\,\vec{v}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \vec{v}\iint\limits_{(S)} \varrho\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = 0\,,$$

the momentum equation (1) assumes the form

$$\iint\limits_{(S)} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)} \vec{t}\,\mathrm{d}S \tag{3}$$

and also indicates that the relative system in this case is an inertial frame. To calculate the momentum fluxes over the entire control surface S in (3), we subdivide the surface S in inlet surface A_i, exit (outlet) surface A_o, the ring surfaces $A_I - A_i$, $A_O - A_o$, and the surface of the external stream tube with S_{SL}. The left hand side of (3) becomes

$$\begin{aligned}\iint\limits_{(S)} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S \;=\; & \iint\limits_{A_i} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{A_o} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S \,+ \\ & + \iint\limits_{A_I-A_i} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{A_O-A_o} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S \,+ \\ & + \iint\limits_{S_{SL}} \varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S\,.\end{aligned}$$

Since the flow at the ring surfaces is uniform, i. e. flow velocity $w = U =$ const and the same fluid particles flow through inlet and exit ring surfaces, the contribution of momentum flux cancel each other. The momentum flux over the surface S_{SL} disappears, if we neglect the momentum contribution of the fuel mass flux and therefore the product $\vec{w} \cdot \vec{n}$ on the surface S_{SL} is zero.
The linear momentum equation (3) is then simplified as:

$$\iint_{A_i} \varrho \vec{w} (\vec{w} \cdot \vec{n}) \, dS + \iint_{A_o} \varrho \vec{w} (\vec{w} \cdot \vec{n}) \, dS = \iint_{(S)} \vec{t} \, dS \, . \tag{4}$$

Now we calculate the engine thrust. For this purpose, we subdivide the integral of the stress vector over the entire surface S in (4) into

$$\begin{aligned} \iint_{(S)} \vec{t} \, dS &= \iint_{S-A_{pylon}} \vec{t} \, dS + \iint_{A_{pylon}} \vec{t} \, dS + \\ &+ \iint_{A_{pylon}} -p_0 \vec{n} \, dS - \iint_{A_{pylon}} -p_0 \vec{n} \, dS \, . \end{aligned} \tag{5}$$

On the surface A_I and A_O the pressure is equal to the ambient pressure p_0. According to our previous assumption, the disturbances caused by the engine at surface S_{SL} are so small that the friction effects are neglected, and the stress vector is there $\vec{t} = -p_0 \vec{n}$. The first and third integral on the right hand side of (5) can be combined to a single integral over the closed total surface S with the integrand $p_0 \vec{n}$. Since p_0 is constant, the integral disappears. The sum of remaining integrals constitutes the force by the pylon on the engine which is equal and opposite of the requested force

$$-\vec{F}_{thrust} = \iint_{A_{pylon}} \vec{t} \, dS - \iint_{A_{pylon}} -p_0 \vec{n} \, dS \, .$$

Thus, we obtain from (4)

$$\iint_{A_i} \varrho \vec{w} (\vec{w} \cdot \vec{n}) \, dS + \iint_{A_o} \varrho \vec{w} (\vec{w} \cdot \vec{n}) \, dS = -\vec{F}_{thrust} \, . \tag{6}$$

The flow velocities are constant over the surfaces:

$$A_i \; : \quad \vec{w} = U \, \vec{e}_x \, , \qquad A_o \; : \quad \vec{w} = w_o \, \vec{e}_x \, .$$

We place the velocity out of the integral sign and obtain from (6)

$$-\vec{F}_{thrust} = U\,\vec{e}_x \iint\limits_{A_i} \varrho\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + w_o\,\vec{e}_x \iint\limits_{A_o} \varrho\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S\;,$$

and with

$$\iint\limits_{A_i} \varrho\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = -\dot{m}_A\;, \qquad \iint\limits_{A_o} \varrho\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \dot{m}_A + \dot{m}_F$$

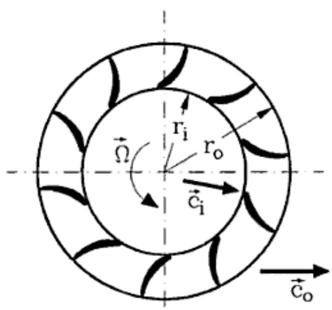
we finally get

$$\vec{F}_{thrust} = -\left[w_o\,\dot{m}_F + (w_o - U)\,\dot{m}_A\right]\vec{e}_x\;.$$

2.5 Applications to Turbomachines

Problem 2.5-1 Circulation around a blade profile in a circular cascade

For the circular cascade shown in the figure, the inlet and exit flow quantities as well as the torque exerted by the flow on the cascade are known.



Calculate the circulation around a blade profile of totally n profiles. Establish a relationship between the torque and the circulation around a single profile.

Solution

Using Euler's turbine equation, the torque on the fluid in the direction of positive c_u is

$$T = \dot{m}\,(r_o\,c_{uo} - r_i\,c_{ui}) \ .$$

With the circulations

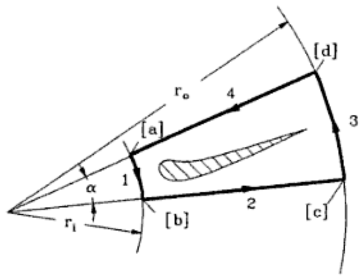
$$\begin{aligned}\Gamma_o &= \oint\limits_{r_o} \vec{c}\cdot\mathrm{d}\vec{x} = c_{uo}\,2\,\pi\,r_o\ ,\\ \Gamma_i &= \oint\limits_{r_i} \vec{c}\cdot\mathrm{d}\vec{x} = c_{ui}\,2\,\pi\,r_i\end{aligned}$$

we obtain

$$T = \frac{\dot{m}}{2\,\pi}(\Gamma_o - \Gamma_i)\ . \qquad (1)$$

The circulation is calculated as follows: Similar to a linear cascade with the spacing s as a measure of periodicity, the circular cascade is periodic with a spacing angle $\alpha = 2\pi/n$. The circulation around a blade is

$$\Gamma_{\text{blade}} = \oint\limits_{\text{blade}} \vec{c}\cdot\mathrm{d}\vec{x}$$



we first calculate the integrals

$$\int\limits_a^b \vec{c}\cdot\mathrm{d}\vec{x} = -c_{ui}\frac{2\,\pi\,r_i}{n}\ ,\quad \int\limits_c^d \vec{c}\cdot\mathrm{d}\vec{x} = c_{uo}\frac{2\,\pi\,r_o}{n}\ ,\quad \int\limits_b^c \vec{c}\cdot\mathrm{d}\vec{x} + \int\limits_d^a \vec{c}\cdot\mathrm{d}\vec{x} = 0\ .$$

The integrals over sides 2 and 4 cancel each other because of the cascade's periodicity. As a result, we have

$$\begin{aligned}\Gamma_{\text{blade}} &= \frac{1}{n}\,(c_{uo}\,2\,\pi\,r_o - c_{ui}\,2\,\pi\,r_i)\\ \Rightarrow\qquad n\,\Gamma_{\text{blade}} &= \Gamma_o - \Gamma_i\ .\end{aligned}$$

Thus, (1) becomes

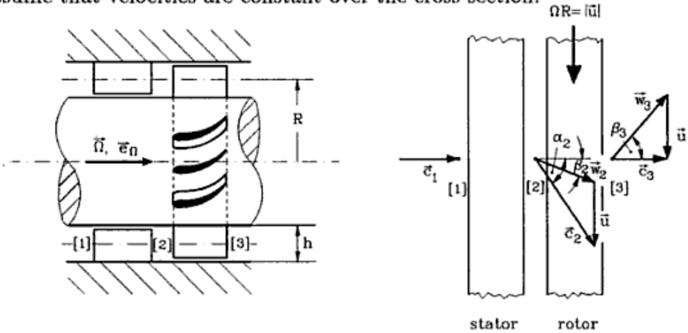
$$T = \frac{\dot{m}}{2\,\pi}\,n\,\Gamma_{\text{blade}}\ ,$$

and the circulation around the profile can be given as

$$\Gamma_{\text{blade}} = \frac{2\,\pi\,T}{n\,\dot{m}}\ .$$

Problem 2.5-2 Axial turbine stage

An axial turbine stage consists of a stator and a rotor. The turbine mass flux is $\dot{m}$, the rotational speed is n, the stage power is P, and the density ϱ is constant. Since the blade height h is much smaller than the mean blade radius R, the cascades can be represented by linear ones. Furthermore, we assume that velocities are constant over the cross-section.

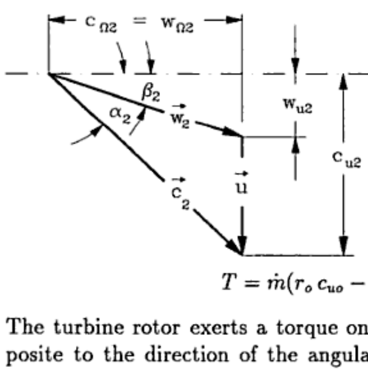


a) Calculate the rotor inlet flow angle α_2.
b) Determine the angle β_2 between the relative velocity $\vec{w}_2$ and the axial direction.
c) Calculate the relative exit flow angle β_3.
d) Sketch the stator and rotor blades for the case that the inlet flow is tangential to the blade leading edge (shock free incidence flow).

Given: $\dot{m}$, P, R, h, n

Solution

a) Rotor absolute flow angle α_2:



From the velocity triangle at the rotor inlet, we have

$$\tan\alpha_2 = \frac{c_{u_2}}{c_{\Omega_2}} .$$

The torque of the rotor can be found from Euler's turbine equation:

$$T = \dot{m}(r_o\, c_{uo} - r_i\, c_{ui}) .$$

The turbine rotor exerts a torque on the fluid whose direction is opposite to the direction of the angular velocity. Therefore $T = -T_T$.

Furthermore, the exit flow should be swirl free, i. e. $c_{uo} = c_{u_3} = 0$. Thus

$$T_T = \dot{m}\, R\, c_{u_2} \ .$$

Using the equation for the stage power $P = T_T\,\Omega$, the circumferential component of the inlet velocity vector is calculated as

$$c_{u_2} = \frac{P}{\dot{m}\,\Omega\, R} \ . \tag{1}$$

For a given mass flux

$$\dot{m} = \iint\limits_{S_2} \varrho\, \vec{c}\cdot\vec{n}\, \mathrm{d}S = \varrho\, c_{\Omega_2}\, 2\,\pi\, R\, h$$

and with $\vec{c}\cdot\vec{n} = c_{\Omega_2}$ we obtain the axial velocity component at the rotor inlet c_{Ω_2}

$$c_{\Omega_2} = \frac{\dot{m}}{\varrho\, 2\,\pi\, R\, h} \ . \tag{2}$$

This equation together with (1) provides a relation for the angle α_2

$$\tan\alpha_2 = \frac{P}{\dot{m}\,\Omega\, R}\,\frac{\varrho\, 2\,\pi\, R\, h}{\dot{m}} \ .$$

Since $\Omega/2\pi = n$, the above equation is further simplified to

$$\tan\alpha_2 = \frac{\varrho\, P\, h}{\dot{m}^2\, n} \ .$$

b) The angle β_2:
The relative exit flow angle β_2 is calculated from the velocity triangle:

$$\tan\beta_2 = \frac{w_{u_2}}{w_{\Omega_2}} = \frac{w_{u_2}}{c_{\Omega_2}} \ . \tag{3}$$

Since from the velocity triangle $\vec{c} = \vec{w} + \vec{u}$, the circumferential component is $c_u = w_u + \Omega\, R$. Thus, the circumferential component w_{u_2} of the relative velocity is

$$w_{u_2} = c_{u_2} - \Omega\, R = \frac{P}{\dot{m}\,\Omega\, R} - \Omega\, R \ ,$$

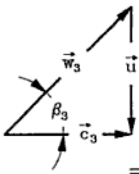
and after a simple rearrangement

$$w_{u_2} = \frac{P}{2\,\pi\,\dot{m}\, n\, R} - 2\,\pi\, n\, R \ . \tag{4}$$

We introduce equation (4) and (2) into equation (3) and obtain

$$\begin{aligned} \tan\beta_2 &= \frac{P}{2\,\pi\,\dot{m}\, n\, R}\,\frac{\varrho\, 2\,\pi\, R\, h}{\dot{m}} - 2\,\pi\, n\, R\,\frac{\varrho\, 2\,\pi\, R\, h}{\dot{m}} \\ \Rightarrow \qquad \tan\beta_2 &= \frac{\varrho\, P\, h}{\dot{m}^2\, n} - \frac{\varrho\,(2\,\pi\, R)^2\, n\, h}{\dot{m}} = \tan\alpha_2 - \frac{\varrho\,(2\,\pi\, R)^2\, n\, h}{\dot{m}} \ . \end{aligned}$$

c) The angle β_3:

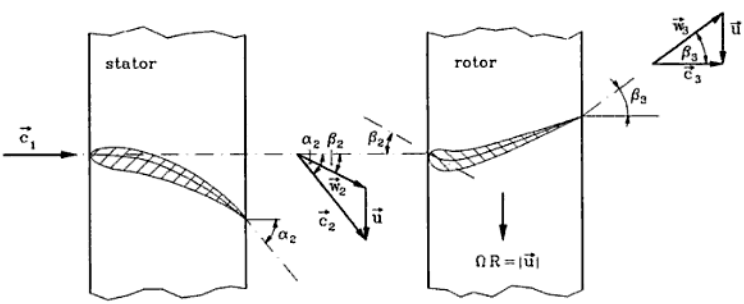


From the velocity triangle at the exit and the condition $c_{u_3} = 0$, it follows that

$$\tan\beta_3 = \frac{-w_{u_3}}{w_{\Omega_3}} = \frac{\Omega R}{c_{\Omega_3}} = 2\pi n R \frac{\varrho 2\pi R h}{\dot{m}} , \quad \text{or}$$

$$\Rightarrow \quad \tan\beta_3 = \frac{\varrho (2\pi R)^2 n h}{\dot{m}} .$$

d) Stator and rotor blades:
To design the blades, we assume for the design point a shock free incidence flow condition. We further assume that the exit flow angle is the blade trailing edge angle (no deviation). This flow situation corresponds to a hypothetical blade configuration with zero blade spacing.

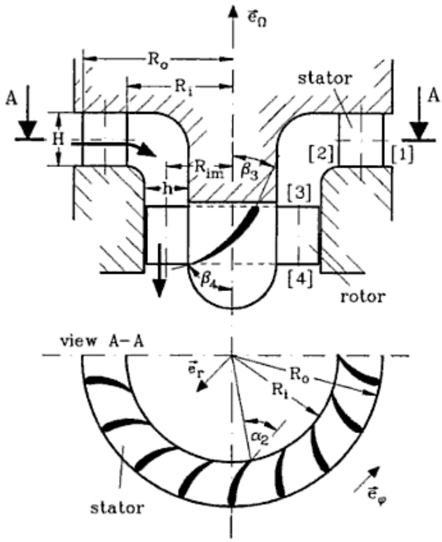


Problem 2.5-3 Kaplan turbine

The sketched turbine consists of a housing that includes the inlet guide vanes (IGV) and an impeller. The working fluid with constant volume flux $\dot{V}$ has a constant density ϱ. The impeller rotates with an angular velocity Ω and produces the power P_T. The fluid enters radially the IGV at station [1] at the angle $\alpha_1 = 0$. It then leaves IGV at station [2] with the velocity $\vec{c}_2$ and the angle α_2 with respect to the radial direction. The viscous stress between station [2] and [3] can be neglected.

In an impeller fixed system, the flow enters the blades at station [3] tangential to the blade leading edge. The flow at the exit station [4] has no swirl and is in the rotating frame tangential to the trailing edge. The flow at the inlet and exit is assumed to be uniform, thus the viscous stress is neglected.

Since the blade height h is much smaller than the mean blade radius R_{im}, the flow quantities at stations [3] and [4] may be calculated at the mean radius Radius R_{im}.



a) 1) Calculate the velocity $\vec{c}_1$ at station [1].
   2) Determine the components c_{r_2} of the velocity vector $\vec{c}_2$ at station [2].

b) Calculate the component c_{u_2} of the velocity vector $\vec{c}_2$ at station [2].

c) 1) Determine the circumferential velocity c_{u_3} at station [3].
   2) Calculate the axial component c_{Ω_3}.

d) Determine the turbine power P_T.

e) Determine the impeller blade angles β_3 and β_4.

Given: $\dot{V}$, ϱ, Ω, H, h, R_i, R_o, R_{im}, α_2

Solution

a) 1) Velocity $\vec{c}_1$ at the inlet station [1]:
The velocity vector is

$$\vec{c} = c_r\,\vec{e}_r + c_u\,\vec{e}_\varphi + c_\Omega\,\vec{e}_\Omega\ .$$

Since the flow at the inlet is purely radial, the absolute velocity at [1] is given by $\vec{c}_1 = c_{r_1}\,\vec{e}_r$. With the volume flux $\dot{V}$

$$\dot{V} = -\iint\limits_{S_1} \vec{c}\cdot\vec{n}\,\mathrm{d}S = -\iint\limits_{S_1} c_{r_1}\,\mathrm{d}S = -2\,\pi\,R_o\,H\,c_{r_1}$$

we obtain

$$\vec{c}_1 = -\frac{\dot{V}}{2\pi R_o H}\,\vec{e}_r\ .$$

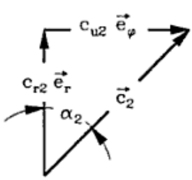
2) The radial component c_{r_2}:
The component c_{r_2} of $\vec{c}_2$ at the exit of the stator is expressed with the volume flux

$$\dot{V} = \iint\limits_{S_2} \vec{c}\cdot\vec{n}\,\mathrm{d}S = \iint\limits_{S_2} -c_{r_2}\,\mathrm{d}S = -2\,\pi\,R_i\,H\,c_{r_2}$$

as

$$c_{r_2} = -\frac{\dot{V}}{2\pi\,R_i H}\ .$$

b) The circumferential component c_{u_2} of $\vec{c}_2$ at station [2]:



From the velocity triangle we find

$$\tan\alpha_2 = \frac{c_{u_2}}{|c_{r_2}|}\ .$$

It determines for a given α_2 the circumferential component

$$c_{u_2} = \frac{\dot{V}}{2\pi\,R_i H}\tan\alpha_2$$

of the absolute velocity vector $\vec{c}_2$.

c) 1) The circumferential component c_{u_3} at the inlet of the impeller:
Since there are no blades in the flow channel between station [2] and [3], no torque is exerted on the flow, as the viscous stresses between [2] and [3] may also be neglected. Thus, using Euler's turbine equation

$$0 = \dot{m}\,(r_o c_{uo} - r_i c_{ui})$$

and the nomenclature in the figure

$$0 = \dot{m}\,(R_{im} c_{u_3} - R_i c_{u_2})\ ,$$

we obtain the circumferential component c_{u_3} as

$$c_{u_3} = \frac{R_i}{R_{im}}\,c_{u_2} = \frac{R_i}{R_{im}}\,\frac{\dot{V}}{2\,\pi\,R_i\,H}\tan\alpha_2 = \frac{\dot{V}}{2\,\pi\,R_{im}\,H}\tan\alpha_2\ .$$

2) Axial component c_{Ω_3} at the inlet of impeller:
The axial component follows from the velocity triangle and the definition of the volume flux to

$$c_{\Omega_3} = -\frac{\dot{V}}{2\,\pi\,R_{im}\,h}\,.$$

d) Turbine power P_T:
The power is

$$P = \vec{\Omega}\cdot\vec{T} = \Omega\dot{m}\,(r_o c_{uo} - r_i c_{ui})$$

and since it is generated by the fluid (power rejection), it is negative

$$P_T = -P = \Omega\dot{m}\,(r_i c_{ui} - r_o c_{uo})\,.$$

The flow at the exit is swirl free, thus using the nomenclature in the figure:

$$c_{uo} = c_{u_4} = 0\,,\quad r_i = R_{im}\,,\quad c_{ui} = c_{u_3}\,.$$

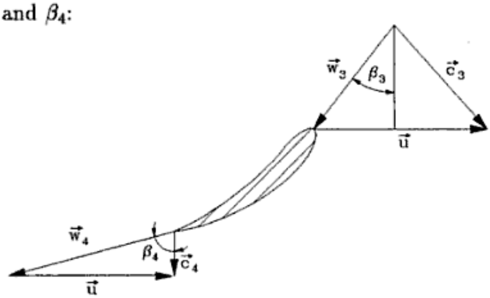
The power is therefore

$$\begin{aligned} P_T &= \Omega\,\dot{m}\,R_{im}\,c_{u_3} = \Omega\,\varrho\,\dot{V}\,R_{im}\,\frac{\dot{V}}{2\,\pi\,R_{im}\,H}\tan\alpha_2 \\ &= \varrho\,\Omega\,\frac{\dot{V}^2}{2\,\pi\,H}\tan\alpha_2\,. \end{aligned}$$

e) The blade angle β_3 and β_4:
From velocity triangle we read

$$\tan\beta_3 = \frac{|w_{u_3}|}{|w_{\Omega_3}|}\,,$$

$$w_{\Omega_3} = c_{\Omega_3}$$



and with $\vec{c}_3 = \vec{w}_3 + \vec{u}$ and $w_{u_3} = c_{u_3} - \Omega\,R_{im}$ then

$$\begin{aligned} \tan\beta_3 = \frac{|c_{u_3} - \Omega\,R_{im}|}{|c_{\Omega_3}|} &= \left|\frac{\dot{V}\tan\alpha_2}{2\,\pi\,R_{im}\,H} - \Omega\,R_{im}\right|\frac{2\,\pi\,R_{im}\,h}{\dot{V}} \\ &= \left|\frac{h}{H}\tan\alpha_2 - \frac{2\pi\,\Omega R_{im}^2 h}{\dot{V}}\right|\,. \end{aligned}$$

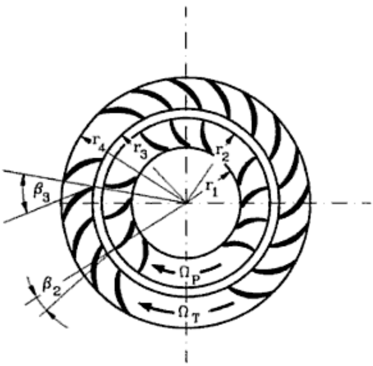
As shown in the figure

$$\tan\beta_4 = \frac{|\vec{u}|}{|c_{\Omega_4}|} \quad \text{with} \quad |\vec{u}| = \Omega\, R_{im} ,$$

while the continuity equation gives $c_{\Omega_4} = c_{\Omega_3}$, and thus for the angle β_4, we are led to

$$\tan\beta_4 = \frac{2\,\pi\,\Omega\,R_{im}^2\,h}{\dot{V}} .$$

Problem 2.5-4 Torque converter



The figure shows the working principle of a torque converter. The inner rotor cascade acts as a pump and the outer as a turbine. The working fluid for both cascades (height b) is oil (ϱ =const) with the mass flux $\dot{m}$. The converter is assumed to perform without losses. After leaving the turbine, the mass flux is guided back to the pump by a guide vane (not shown in figure).

a) Calculate the mass flux $\dot{m}$, if the acting torque on the pump is T_P, the inlet flow is swirl free, and the angular velocity is Ω_P.

b) Determine the swirl at the turbine exit, if the turbine torque is T_T.

c) Find the blade angle β_3 in such a way, that the mass flux can enter the turbine without incident.

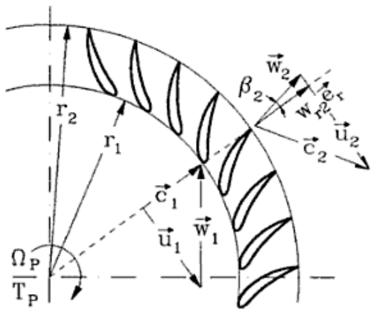
Given: r_1, r_2, r_3, r_4, Ω_P, T_P, T_T, β_2, ϱ, b

Solution

a) Calculation of mass flux:
Euler's turbine equation is written as

$$T_P = \dot{m}\,(r_o\,c_{uo} - r_i\,c_{ui}) ,$$

using the nomenclature in the figure and $c_{u_1} = 0$, Euler's equation is simplified as



$$T_P = \dot{m}\,(r_2\, c_{u_2} - 0)\,. \qquad (1)$$

From the velocity triangle at the pump outlet, we read

$$c_{u_2} = w_{u_2} + u_2 \qquad (2)$$

with

$$w_{u_2} = -w_{r_2}\,\tan\beta_2$$

and

$$u_2 = \Omega_P\, r_2\,.$$

We express the radial velocity components w_{r_2} by the mass flux

$$\dot{m} = \iint\limits_{S_2} \varrho\,\vec{w}\cdot\vec{n}\,\mathrm{d}S = \varrho\, w_{r_2}\, 2\,\pi\, r_2\, b$$

and get with (2)

$$c_{u_2} = -\frac{\dot{m}}{\varrho\, 2\,\pi\, r_2\, b}\,\tan\beta_2 + \Omega_P\, r_2\,,$$

so that (1) appears in the form

$$T_P = \dot{m}\,\Omega_P\, r_2^2 - \dot{m}^2\,\frac{\tan\beta_2}{\varrho\, 2\,\pi\, b}\,. \qquad (3)$$

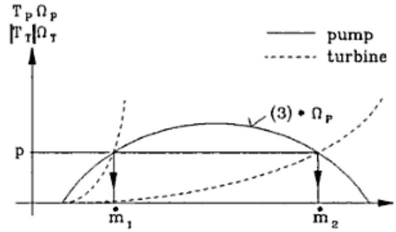
This quadratic equation for $\dot{m}$ has generally two solutions

$$\dot{m}_{1,2} = \frac{\varrho\,\pi\, b\,\Omega_P\, r_2^2}{\tan\beta_2}\left(1 \pm \sqrt{1 - \frac{2\,T_P\,\tan\beta_2}{\varrho\,\pi\, b\,\Omega_P^2\, r_2^4}}\right)\,.$$

The turbine performance map and the requirement that the pump and turbine power are the same (for loss free flow) decide, which mass flux is the proper one. We have:

$$T_P\,\Omega_P = |T_T|\,\Omega_T\,.$$

The situation is sketched in the diagram. Depending on the turbine performance map either $\dot{m}_1$ or $\dot{m}_2$ is possible.



b) Swirl at the turbine exit $r_4 c_{u_4}$
For the radial gap between the two rotors, the torque $T = 0$, thus $r\,c_u = \text{const}$, i. e.

$$r_3\,c_{u_3} = r_2\,c_{u_2} \ ,$$

so that (1) can be written as

$$r_3\,c_{u3} = \frac{T_P}{\dot{m}} \ . \tag{4}$$

From Euler's turbine equation for the turbine rotor

$$-T_T = \dot{m}\,(\,r_4\,c_{u_4} - r_3\,c_{u_3})$$

we conclude

$$-T_T = \dot{m}\,r_4\,c_{u_4} - T_P \ ,$$

and find a relationship for the swirl:

$$r_4\,c_{u_4} = \frac{T_P - T_T}{\dot{m}} \ .$$

This result can directly be obtained by applying the conservation of angular momentum on a control volume that reaches from pump inlet to the turbine exit. Since the fluid enters the pump without swirl, the guide vane must generate the torque $T_G = \dot{m}\,(-c_{u_4}\,r_4)$, such that

$$T_T = T_P + T_G$$

The difference between the converter and the hydraulic clutch should be clear. The hydraulic clutch does not have the guide vane. Therefore, $T_T = T_P$ but $T_P\,\Omega_P \neq T_T\,\Omega_T$.

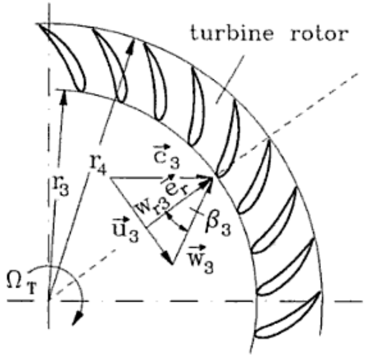
c) Blade angle β_3

From the figure we read

$$\tan\beta_3 = \frac{-w_{u_3}}{w_{r_3}}$$

and the expression for the mass flux becomes

$$w_{r_3} = \frac{\dot m}{2\,\pi\, r_3\, b\, \varrho}\ .$$



From $\vec{c}_3 = \vec{w}_3 + \vec{u}_3$ and $c_{u3} = w_{u3} + u_3$, we obtain the circumferential component of the relative velocity

$$w_{u_3} = c_{u_3} - \Omega_T r_3\ ,$$

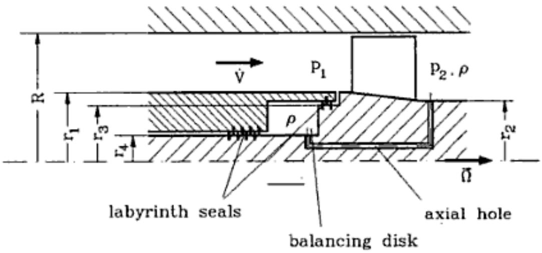
by replacing in c_{u_3} with (4) :

$$w_{u_3} = \frac{T_P}{r_3\,\dot m} - \Omega_T\, r_3\ .$$

This results in the blade angle:

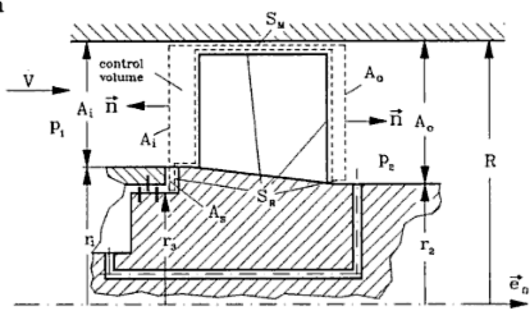
$$\tan\beta_3 = \frac{\Omega_T\, r_3 - \frac{T_P}{r_3\,\dot m}}{\dot m/(2\,\pi\, r_3\, b\, \varrho)} = \frac{\Omega_T\, r_3^2\, \varrho\, 2\,\pi\, b}{\dot m} - \frac{T_P\, \varrho\, 2\,\pi\, b}{\dot m^2}\ .$$

Problem 2.5-5 Balancing of axial thrust



To balance the axial thrust of the sketched axial pump, a balance disk is installed on the suction side of the pump, which is connected with the pressure side of the pump via an axial hole. The flow quantities are constant over the annular cross section of the channel, where the labyrinth seals prevent the flow from exiting. For the known pressures p_1 and p_2 and the volume flux $\dot{V}$, find r_3 such that the axial thrust is balanced.

Solution



We calculate first the axial thrust, which is exerted by the fluid on the runner without the balancing disk (see F. M. (2.108)):

$$\iint\limits_{A_i,A_o} \varrho\, \vec{e}_\Omega \cdot \vec{c}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{A_i,A_o,A_S} -p\,\vec{e}_\Omega \cdot \vec{n}\,\mathrm{d}S + F_a\,, \tag{1}$$

with F_a as the force exerted on the fluid by the wall. In contrast to the rotor treated in F. M. (2.108), the external surface S_M is a part of casing, i. e.

$$F_a = \iint\limits_{S_R,S_M} \vec{e}_\Omega \cdot \vec{t}\,\mathrm{d}S$$

is the total force over the surfaces S_R and S_M exerted on the fluid. The force F_a and the axial thrust differs by the term

$$\iint_{S_M} \vec{e}_\Omega \cdot \vec{t} \, \mathrm{d}S \, .$$

The component of the shear stress in axial direction $\vec{e}_\Omega$ is

$$\vec{e}_\Omega \cdot \vec{t} = t_\Omega = \tau_{r\Omega} \, n_r + \tau_{\varphi\Omega} \, n_\varphi + \tau_{\Omega\Omega} \, n_\Omega$$

and since on the surface S_M the normal vector is $\vec{n} = \vec{e}_r$, it then follows

$$\vec{e}_\Omega \cdot \vec{t} = \tau_{r\Omega} \, ,$$

and thus, the integral of the wall shear stress over the surfaces is found as

$$\iint_{S_M} \vec{e}_\Omega \cdot \vec{t} \, \mathrm{d}S = \iint_{S_M} \tau_{r\Omega} \, \mathrm{d}S \, .$$

In most cases, this integral is neglected compared to the pressure integrals in (1). Thus, in this case, where the casing is a cylindrical surface, F_a is the required axial thrust. (1) gives therefore

$$\iint_{A_i} \varrho \, \vec{e}_\Omega \cdot \vec{c} \, (\vec{w} \cdot \vec{n}) \, \mathrm{d}S + \iint_{A_o} \varrho \, \vec{e}_\Omega \cdot \vec{c} \, (\vec{w} \cdot \vec{n}) \, \mathrm{d}S \quad =$$

$$\iint_{A_i, A_S} -p \, \vec{e}_\Omega \cdot \vec{n} \, \mathrm{d}S + \iint_{A_o} -p \, \vec{e}_\Omega \cdot \vec{n} \, \mathrm{d}S + F_a \, ,$$

and furthermore

$$\varrho \, c_i \iint_{A_i} \vec{w} \cdot \vec{n} \, \mathrm{d}S + \varrho \, c_o \iint_{A_o} \vec{w} \cdot \vec{n} \, \mathrm{d}S = p_1 \, (A_i + A_S) - p_2 \, A_o + F_a \, .$$

We express the inlet and exit velocity components in axial direction using the volume flux

$$c_i = \frac{\dot{V}}{\pi \, (R^2 - r_1^2)} \, , \qquad c_o = \frac{\dot{V}}{\pi \, (R^2 - r_2^2)}$$

and obtain, after changing the sign of F_a, the force on the rotor as

$$F_{a \to \mathrm{rotor}} = \frac{\varrho \, \dot{V}^2}{\pi} \left(\frac{1}{R^2 - r_1^2} - \frac{1}{R^2 - r_2^2} \right) + p_1 \, \pi \, (R^2 - r_3^2) - p_2 \, \pi \, (R^2 - r_2^2) \, .$$

This force should now be compensated by the force exerted by the balancing disc on the rotor. Since the flow velocity is equal to zero inside the balancing tube as well as in the disc region, the pressure p_2 must be constant. The only force component on the disc is

$$F_D = \vec{F}_D \cdot \vec{e}_\Omega = -\iint_{A_D} p\,\vec{e}_\Omega \cdot \vec{n}\,\mathrm{d}S\ ,$$

or

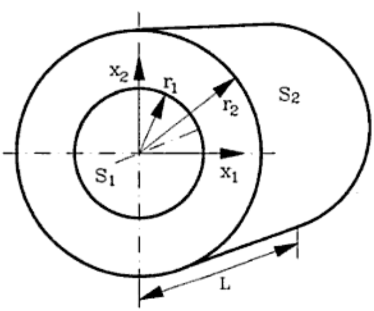
$$F_D = \pi\,(r_3^2 - r_4^2)\,p_2\ .$$

The requirement $F_D = F_{a\to\text{rotor}}$ gives the determining equation for the radius r_3:

$$r_3^2 = \frac{\varrho\,\dot{V}^2}{\pi^2(p_1+p_2)}\left(\frac{1}{R^2-r_1^2} - \frac{1}{R^2-r_2^2}\right) + \frac{p_1}{p_1+p_2}\,R^2 - \frac{p_2}{p_1+p_2}(R^2-r_2^2-r_4^2)\ .$$

2.6 Conservation of Energy

Problem 2.6-1 Cylinder with heat flux



Given is a two-dimensional velocity field of a steady, viscous flow

$$u_1 = \frac{-a\,x_2}{r^2}\ ,$$

$$u_2 = \frac{a\,x_1}{r^2}$$

with $r^2 = x_1^2 + x_2^2$. The change of internal energy $\mathrm{D}E/\mathrm{D}t$ is zero and body forces can be neglected.

a) Determine the change of kinetic energy of the fluid within an annular channel shown as the control volume with the length L (see figure). Hint:

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\iiint_{(V(t))} \varrho\,\frac{\vec{u}^2}{2}\,\mathrm{d}V = \iiint_{(V)} \frac{\varrho}{2}\,\frac{\mathrm{D}\vec{u}^2}{\mathrm{D}t}\,\mathrm{d}V\ .$$

b) Calculate the thermal energy added per unit time to the control volume through the cylinder surfaces $r = r_1$ and $r = r_2$, if the heat flux vector is given by

$$q_1 = -2\,\eta\,a^2\,\frac{x_1}{r^4}\,, \quad q_2 = -2\,\eta\,a^2\,\frac{x_2}{r^4}\,.$$

c) Calculate the work done by the surface forces per unit time on the fluid without integrating the expression

$$\iint\limits_{(S)} \vec{u}\cdot\vec{t}\,\mathrm{d}S\,.$$

Hint: Use the energy equation (see F. M. (2.113)).

Given: r_1, r_2, L, a, η, ϱ

Solution

a) For the change of the kinetic energy

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \iiint\limits_{(V)} \frac{\varrho}{2}\,\frac{\mathrm{D}\vec{u}^2}{\mathrm{D}t}\,\mathrm{d}V$$

we need to square the velocity:

$$\vec{u}^2 = \frac{a^2 x_2^2}{r^4} + \frac{a^2 x_1^2}{r^4} = \frac{a^2}{r^2}\,.$$

The material derivative of the kinetic energy disappears, i. e.

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\vec{u}^2}{2}\right) &= \frac{1}{2}\left(u_1\,\frac{\partial \vec{u}^2}{\partial x_1} + u_2\,\frac{\partial \vec{u}^2}{\partial x_2}\right)\\ &= \frac{1}{2}\left(-\frac{a\,x_2}{r^2}\left(-\frac{2\,a^2}{r^3}\right)\frac{x_1}{r} + \frac{a\,x_1}{r^2}\left(-\frac{2\,a^2}{r^3}\right)\frac{x_2}{r}\right) = 0\end{aligned}$$

thus leading to

$$\frac{\mathrm{D}K}{\mathrm{D}t} = 0\,.$$

b) The added heat flux $\dot{Q}$

$$\dot{Q} = -\iint\limits_{(S)} \vec{q}\cdot\vec{n}\,\mathrm{d}S$$

is calculated using the given heat flux vector

$$\vec{q} = -2\,\eta\,a^2\left(\frac{x_1}{r^4}\,\vec{e}_1 + \frac{x_2}{r^4}\,\vec{e}_2\right)$$

and the normal unit vector on the surfaces

$$\vec{n} = \pm\left(\frac{x_1}{r}\,\vec{e}_1 + \frac{x_2}{r}\,\vec{e}_2\right) ,$$

first for the inner surface ($r = r_1$) as

$$\begin{aligned}\dot{Q}_1 &= -\iint\limits_{(S_1)} \frac{2\,\eta\,a^2}{r^3}\,r\,\mathrm{d}\varphi\,\mathrm{d}x_3 = -\frac{2\,\eta\,a^2}{r_1^3}\,2\,\pi\,r_1\,L\,,\\ \dot{Q}_1 &= -\frac{4\,\eta\,a^2\,\pi}{r_1^2}\,L\,.\end{aligned}$$

At the outer surface, the normal unit vector is opposite to the normal unit vector at the inner surface. This results in a heat flux at the surface S_2

$$\dot{Q}_2 = \frac{4\,\eta\,a^2\,\pi}{r_2^2}\,L\,,$$

the total heat flux is then

$$\dot{Q} = 4\,\eta\,a^2\,\pi\,L\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) .$$

c) Conservation of energy (see F. M. (2.113))

$$\frac{\mathrm{D}}{\mathrm{D}t}(K + E) = P + \dot{Q}$$

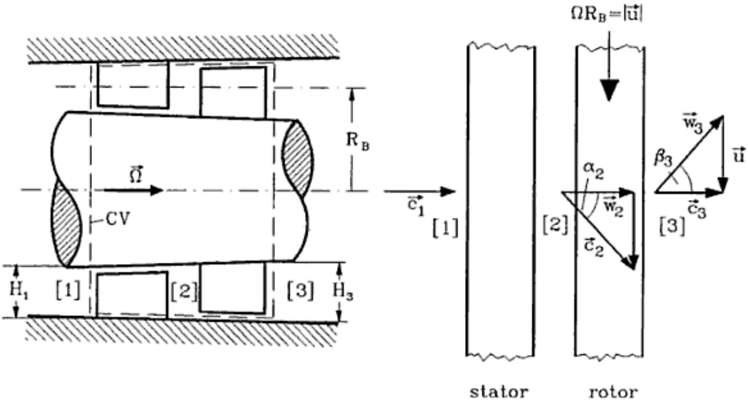
is reduced to

$$P = -\dot{Q}\,,$$

since by assumption $\mathrm{D}E/\mathrm{D}t = 0$ and as shown here $\mathrm{D}K/\mathrm{D}t = 0$, thus the power of the surface forces is

$$\iint\limits_{(S)} \vec{u}\cdot\vec{t}\,\mathrm{d}S = \iint\limits_{(S)} \vec{q}\cdot\vec{n}\,\mathrm{d}S = -4\,\eta\,a^2\,\pi\,L\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) .$$

Problem 2.6-2 Energy balance in an axial turbine stage

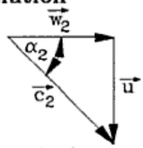


An axial turbine stage consists of a stator row, and a rotor row which rotates with an angular velocity of $\vec{\Omega} = \Omega\,\vec{e}_\Omega$. The working medium is air that can be considered as an ideal gas with given gas constant R and the specific heat ratio γ. The inlet velocity $\vec{c}_1$ at station [1] is purely axial $\vec{c}_1 = c_{ax}\vec{e}_\Omega$. At station [3] air exits also axially $\vec{c}_3 = c_{ax}\vec{e}_\Omega$. For small blade heights H_1, H_3 compared with R_B, the cascades may approximated as linear cascades. The flow quantities can be considered constant over the inlet and exit cross-sections. Body forces can be neglected.

a) The stator exit flow angle α_2 is given. Determine the angular velocity Ω of the rotor row, if the rotor has an axial velocity component only ($\vec{w}_2 = c_{ax}\vec{e}_\Omega$). Calculate the circumferential component c_{u_3} at the rotor exit.
b) Calculate the rotor exit flow angle β_3. Sketch qualitatively the stator and rotor blade profiles for the case that the inlet flow is tangential to the leading edge ("shock free incidence flow").
c) Calculate the mass flux $\dot{m}$.
d) Determine the turbine power using Euler's turbine equation.
e) For the sketched control volume, calculate the temperature T_3 using the energy equation in integral form, where no heat is added.
f) Calculate the density ϱ_3 and the height H_3 at the rotor exit.

Given: R, γ, R_B, H_1, c_{ax}, p_1, T_1, p_3, α_2

Solution

a) 

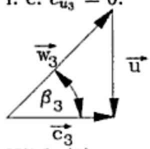
The angular velocity of the rotor row can be determined using the velocity triangle at the rotor exit:

$$\tan\alpha_2 = \frac{|\vec{u}|}{|\vec{w}_2|}\,, \tag{1}$$

with $\vec{w}_2 = c_{ax_2}\,\vec{e}_\Omega = c_{ax}\,\vec{e}_\Omega$ and $|\vec{u}| = \Omega\,R_B$ we get from (1)

$$\Omega = \frac{c_{ax}}{R_B}\tan\alpha_2\,. \tag{2}$$

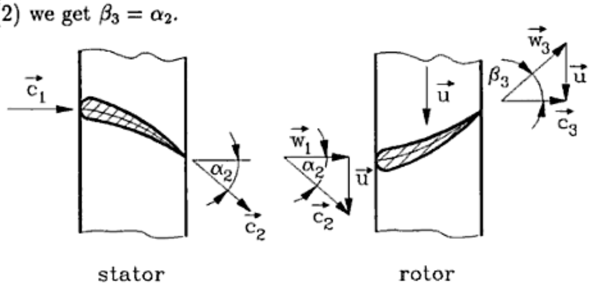
At station [3] the fluid exits the stage with an axial velocity $\vec{c}_3 = c_{ax}\,\vec{e}_\Omega$, i. e. $c_{u_3} = 0$.

b) 

We obtain the exit flow angle β_3 from the velocity triangle at the rotor exit:

$$\tan\beta_3 = \frac{|\vec{u}|}{|\vec{c}_3|} = \frac{\Omega\,R_B}{c_{ax}}\,.$$

With (2) we get $\beta_3 = \alpha_2$.



c) Mass flux $\dot{m}$ through the stage:
We call the inlet cross-section at station [1] S_1 (the normal unit vector is $\vec{n} = -\vec{e}_\Omega$, the flow quantities over S_1 are constant):

$$\dot{m} = -\iint_{(S_1)} \varrho\,\vec{c}\cdot\vec{n}\,\mathrm{d}S = \varrho_1\,c_{ax}\,2\pi\,R_B\,H_1\,.$$

Using the equation of state for a thermally ideal gas yields

$$\dot{m} = \frac{p_1}{R\,T_1}\,c_{ax}\,2\pi\,R_B\,H_1\,. \tag{3}$$

d) Stage power:
Euler's turbine equation for a rotor between station [2] and [3] is

$$T = \dot{m}\,(r_3\,c_{u3} - r_2\,c_{u2})\,. \tag{4}$$

The exit flow is purely axial $c_{u3} = 0$, with $r_2 = R_B$ we have

$$T = -\dot{m}\, R_B\, c_{u2} \; .$$

This torque is exerted by the rotor on the fluid:

$$\vec{T} = -\dot{m}\, R_B\, c_{u2}\, \vec{e}_\Omega \; .$$

The shaft power by the above torque is

$$P = \vec{T} \cdot \vec{\Omega} = -\dot{m}\, R_B\, c_{u2}\, \Omega\, \vec{e}_\Omega \cdot \vec{e}_\Omega \; ,$$

$$\text{i. e.} \quad P = -\dot{m}\, R_B\, c_{u2}\, \Omega \; . \tag{5}$$

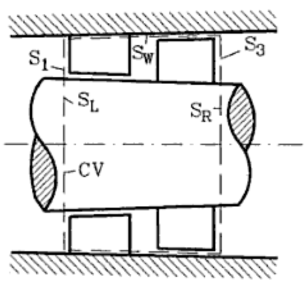
The vectors $\vec{T}$ and $\vec{\Omega}$ form an angle greater than $\pi/2$, thus the shaft power P is negative , i. e. the energy is transferred from the fluid to the rotor. With the equation (2) and (3) we extract from (5)

$$P = -\frac{p_1}{R T_1}\, c_{ax}^3\, 2\pi\, R_B\, H_1\, \tan^2 \alpha_2 \; . \tag{6}$$

e) The temperature T_3 at the rotor exit:
Under the assumption that no heat is added to or rejected from the fluid and the body forces are neglected, we apply the energy equation in integral form (see F. M. (2.114))

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint\limits_{(V(t))} \left[\frac{c_i\, c_i}{2} + e\right] \varrho\, \mathrm{d}V = \iint\limits_{(S)} c_i\, t_i\, \mathrm{d}S \; . \tag{7}$$

We use the sketched fixed control volume, apply Reynolds' transport theorem (see F. M. (1.96)), and obtain



$$\frac{\partial}{\partial t} \iiint\limits_{(V)} \left[\frac{c_i\, c_i}{2} + e\right] \varrho\, \mathrm{d}V \quad + \iint\limits_{(S)} \left[\frac{c_i\, c_i}{2} + e\right] \varrho\, (c_j\, n_j)\, \mathrm{d}S = \iint\limits_{(S)} c_i\, t_i\, \mathrm{d}S \; . \tag{8}$$

Within the integration domain (V) and at constant angular velocity $\vec{\Omega}$ the energy remains time independent, and we can set

$$\frac{\partial}{\partial t}\iiint\limits_{(V)}\left[\frac{c_i\,c_i}{2}+e\right]\varrho\,\mathrm{d}V=0\,.$$

At surfaces S_L, S_R, $c_j\,n_j=0$, on S_W, $c_j=0$ and the surface integral gives

$$\iint\limits_{S_1}\left[\frac{c_i\,c_i}{2}+e\right]\varrho\,(c_j\,n_j)\,\mathrm{d}S+\iint\limits_{S_3}\left[\frac{c_i\,c_i}{2}+e\right]\varrho\,(c_j\,n_j)\,\mathrm{d}S$$

$$=\iint\limits_{S_1}c_i\,t_i\,\mathrm{d}S+\iint\limits_{S_3}c_i\,t_i\,\mathrm{d}S+\iint\limits_{S_L}c_i\,t_i\,\mathrm{d}S+\iint\limits_{S_R}c_i\,t_i\,\mathrm{d}S\,. \qquad (9)$$

The sum of both last integrals on the right hand side is the shaft power P from d). The flow quantities at the surfaces S_1, S_3 are constant and we obtain with $\vec{t}=-p\,\vec{n}$, $e=c_v\,T$ by evaluating the integrals in (9):

$$\dot{m}\left(c_v\,T_3+\frac{p_3}{\varrho_3}-\left(c_v\,T_1+\frac{p_1}{\varrho_1}\right)\right)=P$$

or with $h=e+p/\varrho$

$$\dot{m}(h_3-h_1)=P\,. \qquad (10)$$

Using the caloric equation of state $h=c_p\,T$, we arrive at the temperature

$$T_3=\frac{P}{\dot{m}\,c_p}+T_1\,. \qquad (11)$$

f) With equation (11), the density ϱ_3 at the rotor exit becomes $\varrho_3=p_3/(R\,T_3)$. We use the continuity equation

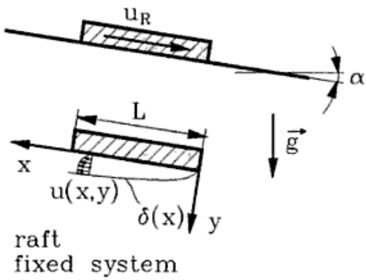
$$\varrho_1\,c_{ax}\,2\pi\,R_B\,H_1=\varrho_3\,c_{ax}\,2\pi\,R_B\,H_3$$

and obtain $H_3=(\varrho_1/\varrho_3)\,H_1$.

3 Constitutive equations

Problem 3-1 Velocity of a raft

A raft (length L, width b, mass m) floats with a constant velocity u_R on a river. The component of the gravitational force in direction of motion may be set equal to the friction force F_x on the raft. Far from the raft, the flow velocity U_∞, viscosity η, and the density ϱ are assumed to be constant. The free surface is inclined by the angle α (see figure).



The boundary layer profile at the raft bottom is:

$$u(x,y) = \Delta u \left[2\,\frac{y}{\delta(x)} - \left(\frac{y}{\delta(x)}\right)^2 \right] \qquad \text{for} \quad 0 \le y \le \delta(x)$$

$$\text{with} \qquad \delta(x) = \sqrt{30\,\frac{x\,\eta}{\varrho\,\Delta u}}\,, \qquad \Delta u = u_R - U_\infty\,.$$

a) Calculate the friction force F_x at the bottom of the raft (a direct integration of the shear stress would be an appropriate way)
b) Determine the velocity u_R of the raft.

Note: The raft attains a higher velocity than the displaced water even though the driving force $mg\sin x$ is the same. In the displaced water turbulent dissipation would take place, which is available here to compensate the power of the drag on the raft.
Given.: L, b, m, η, ϱ, g, α, U_∞

Solution

a) Friction force F_x:

The integration over the shear stress on the bottom of the raft gives:

$$F_{x\to\text{fluid}} = \iint\limits_{A_R} \tau_{yx}\, n_y \,\mathrm{d}S \,,$$

where the stress component of a Newtonian fluid is calculated from

$$\tau_{yx} = \tau_{xy} = 2\,\eta\, e_{xy} = 2\,\eta\,\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) .$$

Applied to the present problem

$$\frac{\partial v}{\partial x} = 0\,, \quad \frac{\partial u}{\partial y} = \Delta u\left(\frac{2}{\delta(x)} - \frac{2\,y}{\delta(x)^2}\right)$$

we obtain for $y = 0$

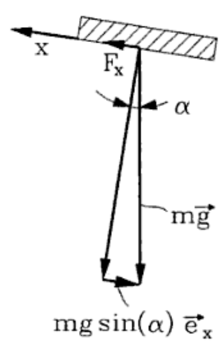
$$\tau_{yx} = 2\,\eta\,\Delta u\,\frac{1}{\delta(x)} .$$

With $n_y = -1$ the requested friction force is calculated as

$$\begin{aligned} F_x &= -F_{x\to\text{fluid}} = 2\,\eta\,\Delta u\, b\int\limits_0^L \frac{\mathrm{d}x}{\delta(x)} \\ &= 2\,\eta\,\Delta u\, b\int\limits_0^L \left(30\,\frac{x\,\eta}{\varrho\,\Delta u}\right)^{-1/2} \mathrm{d}x \\ &= \frac{4}{\sqrt{30}}\,\eta\,\Delta u\, b\sqrt{\mathrm{Re}_L} . \qquad (\text{with } \mathrm{Re}_L = \frac{\Delta u\, L\,\varrho}{\eta}) \end{aligned}$$

b) Raft velocity u_R:

Since the raft velocity is constant, the forces exerted on it are in equilibrium:

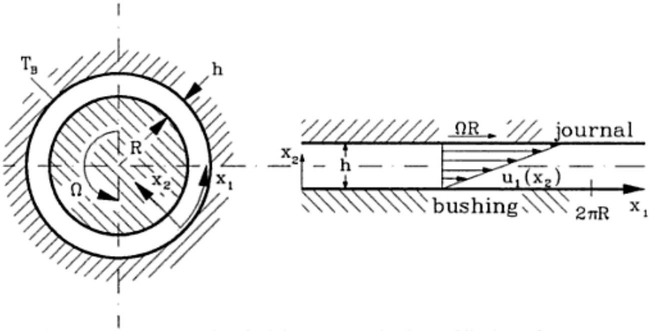


$$F_x\,\vec{e}_x - m\,g\sin\alpha\,\vec{e}_x = 0 .$$

With the results from a) we then have:

$$\begin{aligned} m\,g\sin\alpha &= 4\,\Delta u^{3/2}\, b\sqrt{\frac{L\,\eta\,\varrho}{30}} \\ \Rightarrow \quad u_R &= U_\infty + \left(\frac{m\,g\sin\alpha}{4\,b}\sqrt{\frac{30}{L\,\eta\,\varrho}}\right)^{2/3} . \end{aligned}$$

Problem 3-2 Energy balance in a journal bearing



The radial gap of an unloaded bearing which is filled with a Newtonian fluid can, for $h/R \ll 1$, be modeled by a two-dimensional gap with the coordinates x_1, x_2. The velocity distribution is approximated by

$$u_1(x_2) = \Omega\, R\, \frac{x_2}{h}\,, \qquad u_2 = u_3 = 0\,.$$

We assume a steady plane flow independent upon x_1. The material properties ϱ, η, and λ are constant. Body forces are neglected. All quantities are per unit depth.

a) Calculate the torque M_A exerted on the journal and the necessary power P_A.
b) Determine the dissipation function Φ for the given velocity distribution.
c) Calculate the energy P_L per unit time dissipated in the bearing gap by integrating the dissipation function over the gap volume. Compare the result with the driving power P_A.
d) Determine the heat flux $\dot{Q}_{rej}$ that must be rejected from the fluid in steady operation.
e) Calculate the temperature gradient at the bushing ($x_2 = 0$), if the total heat flux $\dot{Q}_{rej}$ flows through the bushing alone.
f) Determine the temperature distribution $T(x_2)$ in the gap, when the bushing is kept at constant temperature T_B.

Given: ϱ, η, λ, R, h, Ω, T_B

Solution

a) Torque and power:
The torque is

$$M_A = \iint\limits_{S_z} \tau_w\, R\, \mathrm{d}S\,.$$

With the wall shear stress

$$\tau_w = \eta \left. \frac{\partial u_1}{\partial x_2} \right|_{x_2=h} = \eta \frac{\Omega R}{h}$$

we obtain

$$M_A = \int_{\varphi=0}^{2\pi} \eta \frac{\Omega R}{h} R R \, \mathrm{d}\varphi = 2\pi \eta \frac{\Omega R^3}{h}$$

for the torque per unit depth, and thus for the power

$$P_A = M_A \Omega = 2\pi \eta \frac{\Omega^2 R^3}{h} .$$

b) Dissipation function Φ:
The dissipation function for Newtonian fluid is defined as (see F. M. (3.6a))

$$\Phi = \lambda^* e_{kk} e_{ii} + 2\eta e_{ij} e_{ij}$$

and represents the dissipated energy per unit volume and time. From

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} = 0 \qquad \text{follows} \qquad e_{kk} = e_{ii} = 0 .$$

The only non-zero components of the deformation tensor

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

are

$$e_{21} = e_{12} = \frac{1}{2} \frac{\Omega R}{h} .$$

Thus, follows

$$\Phi = 2\eta (e_{12}e_{12} + e_{21}e_{21}) = \eta \left(\frac{\Omega R}{h} \right)^2 .$$

c) Dissipated energy P_L:
The dissipated energy per unit time is obtained from the integration of Φ over the volume per unit depth $V_S = 2\pi R h$:

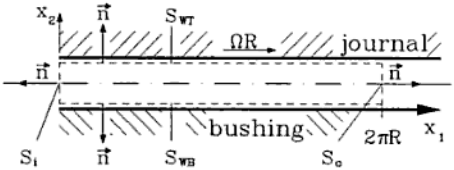
$$P_L = \iint_{V_S} \Phi \, \mathrm{d}V = \iint_{V_S} \eta \frac{\Omega^2 R^2}{h^2} \mathrm{d}V = \eta \frac{\Omega^2 R^2}{h^2} \iint_{V_S} \mathrm{d}V$$

or

$$P_L = 2\pi \eta \frac{\Omega^2 R^3}{h}$$

$$\Rightarrow \qquad P_L = P_A \, !$$

d) The heat flux $\dot{Q}_{rej}$:



The steady flow is described by $u_1 = u_1(x_2)$, $u_2 = u_3 = 0$, and $\partial/\partial x_1 = 0$. From the energy equation (see F. M. (2.113))

$$\iiint\limits_{(V(t))} \varrho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{u_i\,u_i}{2} + e\right)\,\mathrm{d}V = -\dot{Q}_{rej} + P_A$$

we infer $\dot{Q}_{rej} = P_A$, because local and convective changes of $(u_i\,u_i/2+e)$ disappear. With the result of part a), we obtain the heat flux as

$$\dot{Q}_{rej} = 2\,\pi\,\eta\,\frac{\Omega^2\,R^3}{h}\,.$$

e) The temperature gradient at the bushing ($x_2 = 0$):
We have for the rejected heat flux assuming Fourier's law of heat conduction (see F. M. (3.8))

$$\dot{Q}_{rej} = \iint\limits_{(S)} q_i\,n_i\,\mathrm{d}S = \iint\limits_{(S)} -\lambda\,\frac{\partial T}{\partial x_i}\,n_i\,\mathrm{d}S\,.$$

For the control volume from part d) we thus get:

$$\dot{Q}_{rej} = \iint\limits_{S_i+S_o} -\lambda\,\frac{\partial T}{\partial x_i}\,n_i\,\mathrm{d}S + \iint\limits_{S_{WT}} -\lambda\,\frac{\partial T}{\partial x_i}\,n_i\,\mathrm{d}S + \iint\limits_{S_{WB}} -\lambda\,\frac{\partial T}{\partial x_i}\,n_i\,\mathrm{d}S\,.$$

At the inlet and exit, we have $\vec{n} = \pm\vec{e}_1$ and $\partial T/\partial x_1 = 0$; furthermore, there is no heat exchange at the top wall and therefore

$$\dot{Q}_{rej} = \iint\limits_{S_{WB}} -\lambda\left(\frac{\partial T}{\partial x_2}\,n_2\right)\,\mathrm{d}S = \iint\limits_{S_{WB}} \lambda\,\frac{\partial T}{\partial x_2}\,\mathrm{d}S\,.$$

The temperature is a function of x_2 only. Therefore, $\partial T/\partial x_2$ is x_1 independent and it follows

$$\dot{Q}_{rej} = \lambda\,\left.\frac{\partial T}{\partial x_2}\right|_{x_2=0} 2\,\pi\,R$$

$$\Rightarrow \quad \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0} = \frac{\dot{Q}_{rej}}{\lambda\,2\,\pi\,R} = \frac{\eta}{\lambda}\,\frac{\Omega^2\,R^2}{h}\,.$$

f) The temperature distribution $T(x_2)$:
With $\mathrm{D}\varrho/\mathrm{D}t = 0$ and $\mathrm{D}e/\mathrm{D}t = 0$, the energy equation in differential form simplifies to (see F. M. (4.2)):

$$0 = \Phi + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i}\right) .$$

With $\lambda = \text{const}$ and $T = T(x_2)$ it then follows

$$\frac{\mathrm{d}^2 T}{\mathrm{d}{x_2}^2} = -\frac{\Phi}{\lambda} = -\frac{\eta}{\lambda}\frac{\Omega^2 R^2}{h^2} .$$

Integrating the above equation yields

$$T(x_2) = -\frac{\eta}{\lambda}\frac{\Omega^2 R^2}{h^2}\frac{1}{2}x_2^2 + C_1 x_2 + C_2 .$$

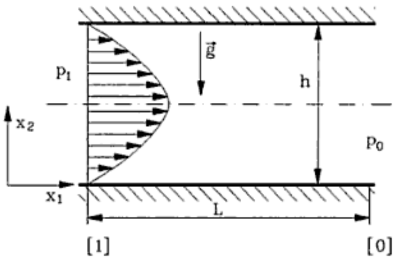
The integration constants are determined from the boundary conditions:

$$\begin{aligned} T(x_2 = 0) &= T_B = C_2 , \\ \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0} &= \frac{\eta}{\lambda}\frac{\Omega^2 R^2}{h} = C_1 , \end{aligned}$$

and we obtain for the temperature distribution

$$T(x_2) = \frac{\eta}{\lambda}\Omega^2 R^2 \left[\frac{x_2}{h} - \frac{1}{2}\left(\frac{x_2}{h}\right)^2\right] + T_B .$$

Problem 3-3 Pressure driven flow of paper pulp



Paper pulp can be considered as a generalized Newtonian fluid (see F. M. (3.16)). It is pumped through a duct of height h and length L. The flow within the duct is maintained by the first component of the pressure gradient $\partial p/\partial x_1$=const. The velocity profile does not depend upon x_1 and x_3 and is known. Furthermore, the stress tensor τ_{ij} is given. The velocity profile is given by:

$$u_1(x_2) = \frac{n}{n+1}\left(\frac{p_1-p_0}{m\,L}\right)^{1/n}\left(\left(\frac{h}{2}\right)^{(n+1)/n} - \left|\frac{h}{2}-x_2\right|^{(n+1)/n}\right),$$

$$u_2 = 0,$$

and the components of the stress tensor by:

$$\tau_{11} = \tau_{22} = -p\,; \quad \tau_{12} \neq f(x_1)\,.$$

a) Determine the pressure p_1 necessary to transport a given volume flux per unit width ($\tilde{\dot{V}} = \dot{V}/b$), if p_0 is known.
b) For constant kinetic energy K and internal energy E within the duct, show that the work done by the surface forces must be rejected as heat. (Hint: Use energy equation (see F. M. (2.114)))

Given: Material constants of the fluid: m, $n > 0$.

Solution

a) The volume flux is:

$$\dot{V} = \iint\limits_{(S)} u_1(x_2)\,\mathrm{d}A\,.$$

considering the symmetry of the velocity profile u_1 with respect to $x_2 = h/2$, we have

$$\dot{V} = 2\int_0^b\int_0^{h/2} \frac{n}{n+1}\left(\frac{p_1-p_0}{m\,L}\right)^{1/n}\left(\left(\frac{h}{2}\right)^{(n+1)/n} - \left(\frac{h}{2}-x_2\right)^{(n+1)/n}\right)\mathrm{d}x_2\,\mathrm{d}x_3\,.$$

After integration we obtain the following algebraic equation for the volume flux

$$\dot{V} = 2b\,\frac{n}{n+1}\left(\frac{p_1 - p_0}{m\,L}\right)^{1/n}\frac{n+1}{2n+1}\left(\frac{h}{2}\right)^{(2n+1)/n} .$$

A rearrangement

$$p_1 = p_0 + m\,L\left(\frac{2n+1}{h\,n}\,\dot{\tilde{V}}\right)^{n}\left(\frac{h}{2}\right)^{-(1+n)}$$

leads to the required pressure for the given volume flux $\dot{\tilde{V}}$ per unit width.

b) By the problem definition, the left hand side of the equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{(V(t))}\left(\frac{u_i u_i}{2} + e\right)\varrho\,\mathrm{d}V = \iiint\limits_{(V)} u_i k_i \varrho\,\mathrm{d}V + \iint\limits_{(S)} u_i t_i\,\mathrm{d}S - \iint\limits_{(S)} q_i n_i\,\mathrm{d}S$$

identically vanishes. Furthermore, the integral over body forces k_i, in this case, the gravitational force g_i, does not contribute, since g_i is orthogonal to u_i. Thus, the heat transfer per unit time over the control surface is calculated from:

$$\dot{Q} = -\iint\limits_{(S)} q_i n_i\,\mathrm{d}S = -\iint\limits_{(S)} \tau_{ij} u_j n_i\,\mathrm{d}S .$$

If we place the control volume between the stations [1] and [0], then the wall surface integrals disappear because of the no-slip condition $\vec{u} = 0$. The remaining expression is

$$\dot{Q} = -\iint\limits_{S_1} \tau_{ij} u_j n_i\,\mathrm{d}S - \iint\limits_{S_0} \tau_{ij} u_j n_i\,\mathrm{d}S$$

that gives

$$\dot{Q} = \iint\limits_{S_1} \tau_{11} u_1\,\mathrm{d}S - \iint\limits_{S_0} \tau_{11} u_1\,\mathrm{d}S .$$

The surfaces S_1 and S_0 are equal, the component τ_{11} is at location [1] $-p_1$ and at location [0] $-p_0$, thus

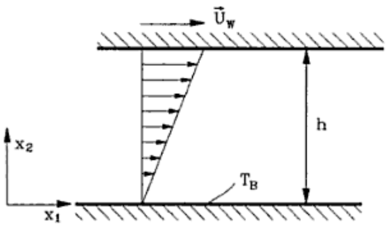
$$\dot{Q} = (-p_1 + p_0)\iint\limits_{(S)} u_1\,\mathrm{d}S = (p_0 - p_1)\dot{V} .$$

As we can see from a) the pressure $p_1 > p_0$ and thus the heat flux $\dot{Q}$ is negative, therefore heat must be rejected.

Problem 3-4 Flow of a non-Newtonian fluid

A non-Newtonian fluid flows through a channel of width b, whose top wall moves with a velocity U_W. The constitutive equation of the fluid is

$$\tau_{ij} = -p\delta_{ij} + 2\eta\, e_{ij} + 4\beta\, e_{ik}e_{kj} \ .$$



The flow is steady and incompressible. Its velocity field is given by

$$u_1 = \frac{U_W}{h} x_2 \ , \quad u_2 = 0 \ ,$$

and the temperature T_B of the bottom wall is known. Body forces are neglected.

a) Calculate the work per unit of time and width, which is necessary to move the top wall. (Hint: Calculate first the force acting on the wall).

b) Using the first law of thermodynamics (see F. M. (2.119))

$$\frac{\mathrm{D}e}{\mathrm{D}t} = \frac{1}{\varrho}\tau_{ij}e_{ij} - \frac{1}{\varrho}\frac{\partial q_i}{\partial x_i} \ , \quad e = cT \ , \quad c = \text{const}$$

calculate the temperature distribution $T(x_2)$ within the channel for the case that heat is only transferred to/from the top wall. Use the linear law for the heat flux vector (Fourier's law).

Given: b, h, U_W, η, β, λ, c, T_B, $T \neq T(x_1)$

Solution

a) The force of the moving wall on the fluid is

$$F_i = \iint\limits_{(S)} t_i \,\mathrm{d}S = \iint\limits_{(S)} \tau_{ij} n_j \,\mathrm{d}S$$

with $n_1 = 0$ and $n_2 = 1$. Only the force in x_1-direction

$$F_1 = \iint\limits_{(S)} t_1 \,\mathrm{d}S$$

is responsible for the power consumption. With the stress component $t_1 = \tau_{11}n_1 + \tau_{21}n_2$ constant over S, the necessary force per unit length is

$$F_1 = \tau_{12} b \ ,$$

i. e. only the component

$$\tau_{12} = 2\eta\, e_{12} + 4\beta(e_{12}e_{11} + e_{22}e_{12})$$

of the stress tensor is used. With

$$e_{11} = 0\ , \quad e_{22} = 0\ , \quad e_{12} = \frac{1}{2}\frac{\partial u_1}{\partial x_2} = \frac{1}{2}\frac{U_W}{h}$$

we obtain for the work per unit time and length:

$$P = F_1\, U_W = \eta\,\frac{b}{h}\,U_W^2\ .$$

b) In the first law of thermodynamics with $q_i = -\lambda\,\partial T/\partial x_i$, i. e.

$$c\left(\frac{\partial T}{\partial t} + u_1\frac{\partial T}{\partial x_1} + u_2\frac{\partial T}{\partial x_2}\right) = \frac{1}{\varrho}(\tau_{11}e_{11} + 2\tau_{12}e_{12} + \tau_{22}e_{22}) + \frac{\lambda}{\varrho}\left(\frac{\partial^2 T}{\partial {x_1}^2} + \frac{\partial^2 T}{\partial {x_2}^2}\right)$$

all terms except

$$0 = \eta\,\frac{U_W^2}{h^2} + \lambda\,\frac{\partial^2 T}{\partial {x_2}^2}$$

are cancelled. Since T is only a function of x_2, the temperature distribution follows from

$$\frac{\mathrm{d}^2 T}{\mathrm{d}{x_2}^2} = -\frac{\eta}{\lambda}\,\frac{U_W^2}{h^2}$$

by integration:

$$T(x_2) = -\frac{\eta}{2\lambda}\,\frac{U_W^2}{h^2}\,x_2^2 + C_1 x_2 + C_2\ ,$$

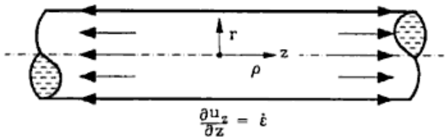
where the constants are determined from the boundary conditions. At the bottom wall the temperature T_B is known, consequently we have $C_2 = T_B$. Heat is transferred at the top wall. At the bottom wall the condition

$$\left.\frac{\partial T}{\partial x_2}\right|_{x_2=0} = 0$$

must be satisfied, since there $q_i n_i = (-\partial T/\partial x_i)n_i = 0$. From this requirement the final temperature distribution is calculated as

$$T(x_2) = T_B - \frac{\eta}{2\lambda}\,\frac{U_W^2}{h^2}\,x_2^2\ .$$

Problem 3-5 Extensional flow



A fluid cylinder with constant density in an extensional flow, experiences in axial direction an extension at a constant extension rate $\dot\varepsilon = \partial u_z/\partial z = \text{const.}$ The problem is axisymmetric, i. e. $\partial/\partial\varphi = 0$. The velocity field has the form $u_r = u_r(r)$, $u_\varphi = 0$, $u_z = u_z(z)$.

a) Calculate the radial velocity component u_r as a function of $\dot\varepsilon$.
b) Determine the components of the rate of deformation and the spin tensors $\mathbf{E}$ and $\mathbf{\Omega}$.
c) Give the components of the friction stress tensor $\mathbf{P}$ for a Newtonian fluid. The Trouton viscosity is defined by

$$\eta_T = \frac{P_{zz} - P_{rr}}{\dot\varepsilon} \,. \tag{1}$$

What is its value for a Newtonian fluid?

d) Determine the components of the friction stress tensor $\mathbf{P}$ for a viscoelastic fluid, which is described by the following constitutive equation

$$\mathbf{P} + \lambda_0 \frac{\mathcal{D}\mathbf{P}}{\mathcal{D}t} = 2\eta\,\mathbf{E} \,, \tag{2}$$

with the time derivative

$$\frac{\mathcal{D}\mathbf{P}}{\mathcal{D}t} = \frac{\mathrm{D}\mathbf{P}}{\mathrm{D}t} + \mathbf{\Omega}\cdot\mathbf{P} - \mathbf{P}\cdot\mathbf{\Omega} - \xi\,(\mathbf{P}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{P}) \,. \tag{3}$$

(For $\xi = 0$ we get from (3) Jaumann's (see F. M. 3.29), and for $\xi = 1$ Oldroyd's (see F. M. 3.30) derivative.)
Give the Trouton viscosity η_T for a viscoelastic fluid.

Solution

a) Velocity component u_r:
The continuity equation in cylindrical coordinates is (see F. M. , Appendix B.2)

$$\frac{\partial}{\partial r}(r\,u_r) = -r\,\frac{\partial u_z}{\partial z} = -r\,\dot\varepsilon \,.$$

Integration yields to

$$u_r\,r = -\frac{r^2}{2}\,\dot\varepsilon + C(z)$$

$$\text{or} \quad u_r = -\frac{r}{2}\,\dot\varepsilon + \frac{C(z)}{r} \,.$$

For $r \to 0$ the solution must be bounded ($\Rightarrow C(z) = 0$), i. e.

$$u_r(r) = -\frac{r}{2}\,\dot{\varepsilon}\,. \tag{4}$$

b) Rate of deformation and spin tensor:
The non-zero components of the rate of deformation tensor **E** (see F. M. , Appendix B.2) are

$$e_{rr} = \frac{\partial u_r}{\partial r} = -\frac{\dot{\varepsilon}}{2}\,, \quad e_{\varphi\varphi} = \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) = -\frac{\dot{\varepsilon}}{2}\,, \quad e_{zz} = \frac{\partial u_z}{\partial z} = \dot{\varepsilon}\,. \tag{5}$$

The spin tensor $\mathbf{\Omega}$:
We first obtain the velocity gradient (which is a second order tensor) in cylindrical coordinates by applying the nabla operator

$$\nabla = \vec{e}_r\,\frac{\partial}{\partial r} + \vec{e}_\varphi\,\frac{1}{r}\,\frac{\partial}{\partial \varphi} + \vec{e}_z\,\frac{\partial}{\partial z}$$

to the velocity vector $\vec{u} = u_r\,\vec{e}_r + u_\varphi\,\vec{e}_\varphi + u_z\,\vec{e}_z$ (tensor product). With the derivatives of the basis vectors

$$\begin{aligned}
&\frac{\partial \vec{e}_r}{\partial r} = 0\,, && \frac{\partial \vec{e}_\varphi}{\partial r} = 0\,, && \frac{\partial \vec{e}_z}{\partial r} = 0\,,\\
&\frac{\partial \vec{e}_r}{\partial \varphi} = \vec{e}_\varphi\,, && \frac{\partial \vec{e}_\varphi}{\partial \varphi} = -\vec{e}_r\,, && \frac{\partial \vec{e}_z}{\partial \varphi} = 0\,,\\
&\frac{\partial \vec{e}_r}{\partial z} = 0\,, && \frac{\partial \vec{e}_\varphi}{\partial z} = 0\,, && \frac{\partial \vec{e}_z}{\partial z} = 0\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
\nabla\vec{u} = \quad & \frac{\partial u_r}{\partial r}\,\vec{e}_r\vec{e}_r \;+\; \frac{\partial u_\varphi}{\partial r}\,\vec{e}_r\vec{e}_\varphi \;+\; \frac{\partial u_z}{\partial r}\,\vec{e}_r\vec{e}_z \;+\\
& \frac{1}{r}\left(\frac{\partial u_r}{\partial \varphi} - u_\varphi\right)\vec{e}_\varphi\vec{e}_r \;+\; \frac{1}{r}\left(u_r + \frac{\partial u_\varphi}{\partial \varphi}\right)\vec{e}_\varphi\vec{e}_\varphi \;+\; \frac{1}{r}\frac{\partial u_z}{\partial \varphi}\,\vec{e}_\varphi\vec{e}_z \;+\\
& \frac{\partial u_r}{\partial z}\,\vec{e}_z\vec{e}_r \;+\; \frac{\partial u_\varphi}{\partial z}\,\vec{e}_z\vec{e}_\varphi \;+\; \frac{\partial u_z}{\partial z}\,\vec{e}_z\vec{e}_z \quad .
\end{aligned}$$

The transposed tensor $(\nabla\vec{u})^T$ is

$$
\begin{aligned}
(\nabla\vec{u})^T = \; & \frac{\partial u_r}{\partial r}\,\vec{e}_r\vec{e}_r \;+\; \frac{1}{r}\left(\frac{\partial u_r}{\partial\varphi} - u_\varphi\right)\vec{e}_r\vec{e}_\varphi \;+\; \frac{\partial u_r}{\partial z}\,\vec{e}_r\vec{e}_z \;+ \\
& \frac{\partial u_\varphi}{\partial r}\,\vec{e}_\varphi\vec{e}_r \;+\; \frac{1}{r}\left(u_r + \frac{\partial u_\varphi}{\partial\varphi}\right)\vec{e}_\varphi\vec{e}_\varphi \;+\; \frac{\partial u_\varphi}{\partial z}\,\vec{e}_\varphi\vec{e}_z \;+ \\
& \frac{\partial u_z}{\partial r}\,\vec{e}_z\vec{e}_r \;+\; \frac{1}{r}\frac{\partial u_z}{\partial\varphi}\,\vec{e}_z\vec{e}_\varphi \;+\; \frac{\partial u_z}{\partial z}\,\vec{e}_z\vec{e}_z \,.
\end{aligned}
$$

The spin tensor $\mathbf{\Omega} = 1/2\left(\nabla\vec{u} - (\nabla\vec{u})^T\right)$ is then calculated as

$$
\begin{aligned}
\mathbf{\Omega} = \frac{1}{2}\Bigg\{ \; & \left(\frac{\partial u_\varphi}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\varphi} + \frac{1}{r}\,u_\varphi\right)\vec{e}_r\vec{e}_\varphi + \left(\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}\right)\vec{e}_r\vec{e}_z \\
+ \; & \left(\frac{1}{r}\frac{\partial u_r}{\partial\varphi} - \frac{\partial u_\varphi}{\partial r} - \frac{1}{r}\,u_\varphi\right)\vec{e}_\varphi\vec{e}_r + \left(\frac{1}{r}\frac{\partial u_z}{\partial\varphi} - \frac{\partial u_\varphi}{\partial z}\right)\vec{e}_\varphi\vec{e}_z \\
+ \; & \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_z\vec{e}_r + \left(\frac{\partial u_\varphi}{\partial z} - \frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right)\vec{e}_z\vec{e}_\varphi \Bigg\} \,.
\end{aligned}
$$

Since $u_\varphi = 0$, $u_r = u_r(r)$ and $u_z = u_z(z)$ the spin tensor disappears identically

$$\mathbf{\Omega} = 0 \,. \tag{6}$$

Alternative solution: Since the cylindrical coordinates are orthogonal, all expressions that do not contain derivatives of the basis vectors have the same form as in a cartesian coordinate system. In this case, the relation between the spin tensor and angular velocity (see F. M. (1.46)) is

$$\omega_k \epsilon_{ijk} = \Omega_{ji} \,.$$

The index 1 corresponds to r-, 2 to φ-, and 3 to z-direction. From Appendix B (see F. M.) and with $\omega_r = \omega_\varphi = \omega_z = 0$ then follows $\mathbf{\Omega} = 0$ immediately.

c) Behavior of Newtonian fluid:
For an incompressible Newtonian fluid, the friction stress tensor is directly proportional to the rate of deformation tensor $\mathbf{P} = 2\eta\,\mathbf{E}$ (see F. M. 3.2b). Since in case of an extensional flow, the fluid elements are extended parallel to the coordinate axes, only the normal components of the friction stress tensor

$$P_{rr} = -\eta\,\dot{\varepsilon}\,, \quad P_{\varphi\varphi} = -\eta\,\dot{\varepsilon}\,, \quad P_{zz} = 2\eta\,\dot{\varepsilon} \tag{7}$$

are different from zero. This statement is also valid for the viscoelastic material behavior treated in the next part. The Trouton viscosity η_T of a Newtonian fluid with (7) is

$$\eta_T = \frac{P_{zz} - P_{rr}}{\dot{\varepsilon}} = \frac{2\eta\,\dot{\varepsilon} + \eta\,\dot{\varepsilon}}{\dot{\varepsilon}} = 3\eta \ .$$

d) Viscoelastic material behavior:

Since the components of $\mathbf{E}$ are constant, the components of $\mathbf{P}$ are so too, i. e. $\mathrm{D}\mathbf{P}/\mathrm{D}t = 0$. With $\mathbf{\Omega} = 0$, it follows from (3)

$$\frac{\mathcal{D}\mathbf{P}}{\mathcal{D}t} = -\xi\,(\mathbf{P}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{P}) \ .$$

For $\xi = 0$, Jaumann's derivative disappears identically and the behavior of the fluid is described by the Newtonian constitutive equation. We perform the scalar multiplication and obtain

$$\begin{aligned}\frac{\mathcal{D}\mathbf{P}}{\mathcal{D}t} &= \xi\,\frac{\dot{\varepsilon}}{2}\left[\begin{pmatrix} P_{rr} & P_{r\varphi} & P_{rz} \\ P_{\varphi r} & P_{\varphi\varphi} & P_{\varphi z} \\ P_{zr} & P_{z\varphi} & P_{zz}\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2\end{pmatrix} + \right. \\ &\qquad \left. + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix} P_{rr} & P_{r\varphi} & P_{rz} \\ P_{\varphi r} & P_{\varphi\varphi} & P_{\varphi z} \\ P_{zr} & P_{z\varphi} & P_{zz}\end{pmatrix}\right] \\ &= \xi\,\frac{\dot{\varepsilon}}{2}\begin{pmatrix} 2P_{rr} & 2P_{r\varphi} & -P_{rz} \\ 2P_{\varphi r} & 2P_{\varphi\varphi} & -P_{\varphi z} \\ -P_{zr} & -P_{z\varphi} & -4\,P_{zz}\end{pmatrix} \ .\end{aligned}$$

From this form, we conclude that $\mathbf{P}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{P}$ is a symmetric tensor. With the rate of deformation tensor (5), the non-zero components of the friction stress tensor follow from (2):

$$P_{rr} + \lambda_0\,\xi\,\dot{\varepsilon}\,P_{rr} = 2\eta\,e_{rr} = -\eta\,\dot{\varepsilon} \quad \Rightarrow \quad P_{rr} = -\frac{\eta\,\dot{\varepsilon}}{1 + \lambda_0\,\xi\,\dot{\varepsilon}} \ ,$$

$$P_{\varphi\varphi} + \lambda_0\,\xi\,\dot{\varepsilon}\,P_{\varphi\varphi} = 2\eta\,e_{\varphi\varphi} = -\eta\,\dot{\varepsilon} \quad \Rightarrow \quad P_{\varphi\varphi} = -\frac{\eta\,\dot{\varepsilon}}{1 + \lambda_0\,\xi\,\dot{\varepsilon}} \ ,$$

$$P_{zz} + \lambda_0\,(-2\xi\,\dot{\varepsilon}\,P_{zz}) = 2\eta\,e_{zz} = 2\eta\,\dot{\varepsilon} \quad \Rightarrow \quad P_{zz} = \frac{2\eta\,\dot{\varepsilon}}{1 - 2\lambda_0\,\xi\,\dot{\varepsilon}} \ .$$

Thus, we obtain generally for a viscoelastic fluid the Trouton viscosity as

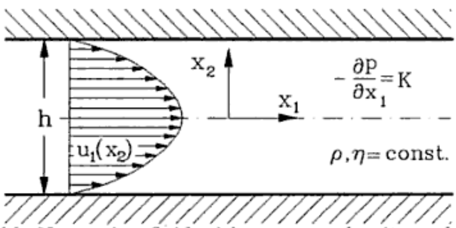
$$\eta_T = \frac{P_{zz} - P_{rr}}{\dot{\varepsilon}} = \frac{3\eta}{(1 + \xi\,\lambda_0\,\dot{\varepsilon})(1 - 2\xi\,\lambda_0\,\dot{\varepsilon})} \ .$$

For $\xi = 0$, we find the same Trouton viscosity as for a Newtonian fluid $\eta_T = 3\eta$.

4 Equation of Motion for Particular Fluids

4.1 Newtonian Fluids

Problem 4.1-1 Poiseuille flow



Incompressible Newtonian fluid with constant density and viscosity flows between two parallel plates with infinite width. Body forces are neglected. Given are the plate height h, the components of the pressure gradient,

$$\frac{\partial p}{\partial x_1} = -K \,, \quad \frac{\partial p}{\partial x_2} \equiv 0 \,, \quad \frac{\partial p}{\partial x_3} \equiv 0 \,,$$

and the velocity field between the plates (see F. M. (6.19) for $U \to 0$)

$$u_1(x_2) = \frac{K}{2\,\eta}\left(\frac{h^2}{4} - x_2^2\right) \,, \quad u_2 \equiv 0 \,, \quad u_3 \equiv 0 \,.$$

a) Show that the given velocity field satisfies the continuity and the Navier-Stokes equation.
b) Determine the components of the stress tensor.
c) Calculate the dissipation function Φ.
d) Find the energy per unit depth, length, and time dissipated in heat within the gap.

e) Calculate the principal stresses and their directions.

Given: $\partial p/\partial x_i$, u_i, h, ϱ, η

Solution

a) Inserting the velocity field and pressure gradient into the equations of continuity and Navier-Stokes, we will find that these equations are satisfied.

b) Stress tensor:
The constitutive equation for the Newtonian fluid is the Cauchy-Poisson law

$$\tau_{ij} = (-p + \lambda^* e_{kk})\,\delta_{ij} + 2\,\eta\,e_{ij}\ .$$

Because of $\mathrm{D}\varrho/\mathrm{D}t = 0$, the term $e_{kk} = \operatorname{div}\vec{u} = 0$, thus, with $e_{ij} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, the stress tensor becomes

$$\tau_{ij} = -p\,\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\ .$$

The components of the stress tensor are

$$\begin{aligned}
\tau_{11} &= -p + \eta\left(2\frac{\partial u_1}{\partial x_1}\right) = -p\,, &\quad \tau_{22} &= -p + \eta\left(2\frac{\partial u_2}{\partial x_2}\right) = -p\,,\\
\tau_{12} &= \eta\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = -K\,x_2\,, &\quad \tau_{23} &= \eta\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = 0\,,\\
\tau_{13} &= \eta\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0\,, &\quad \tau_{33} &= -p + \eta\left(2\frac{\partial u_3}{\partial x_3}\right) = -p\,,
\end{aligned}$$

or in matrix form

$$(\tau_{ij}) = \begin{pmatrix} -p & -K\,x_2 & 0\\ -K\,x_2 & -p & 0\\ 0 & 0 & -p \end{pmatrix}\ .$$

c) The dissipation function:
The dissipated energy per volume and time is in general $\Phi = P_{ij}\,e_{ij}$, for the special case of the Cauchy-Poisson law

$$\Phi = (\lambda^*\,e_{kk}\,\delta_{ij} + 2\,\eta\,e_{ij})\,e_{ij} = 2\,\eta\,e_{ij}\,e_{ij}\ .$$

With

$$e_{12} = e_{21} = \frac{1}{2}\frac{\partial u_1}{\partial x_2}\,,$$

$$e_{11} = e_{22} = e_{33} = 0\,,$$

the dissipation function becomes

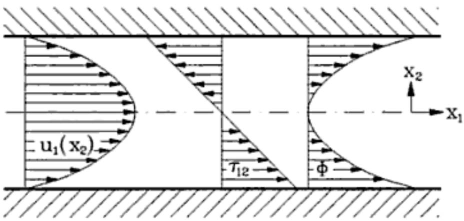
$$\begin{aligned}\Phi &= 2\eta\,(e_{i1}\,e_{i1}+e_{i2}\,e_{i2}+e_{i3}\,e_{i3})\\ &= \eta\left(\frac{\partial u_1}{\partial x_2}\right)^2=\frac{K^2\,x_2^2}{\eta}\,.\end{aligned}$$

d) We obtain the dissipated power as an integral over the volume occupied by the fluid:

$$P_D=\iiint\limits_{(V)}\Phi\,\mathrm{d}V$$

and find P_D per unit length and depth as:

$$P_D=\int\limits_{-\frac{h}{2}}^{\frac{h}{2}}\Phi\,\mathrm{d}x_2=\left.\frac{K^2\,x_2^3}{3\eta}\right|_{-\frac{h}{2}}^{\frac{h}{2}}=\frac{K^2\,h^3}{12\,\eta}\,.$$



The components of velocity, stress tensor, and dissipation function are plotted in the figure.

e) Principal stresses and their directions:
The eigenvalue problem is

$$(\tau_{ij}-\sigma\,\delta_{ij})\,l_j=0\,. \tag{1}$$

By looking at the stress tensor, we realize that $\vec{l}=(0,0,1)$ with $\sigma=-p$ is an eigenvector, respectively an eigenvalue (plane flow!). Therefore, we search only in the x_1,x_2-plane for the other two eigenvalues and eigenvectors. The characteristic equation is now

$$\det\begin{pmatrix}-(p+\sigma) & -K\,x_2\\ -K\,x_2 & -(p+\sigma)\end{pmatrix}=(p+\sigma)^2-K^2\,x_2^2=0\,.$$

This results in the following eigenvalues

$$(p+\sigma)=\pm K\,x_2$$

$$\Rightarrow \qquad \sigma^{(1)} = -p + K\,x_2\,, \qquad \sigma^{(2)} = -p - K\,x_2\,.$$

We calculate the eigenvectors using the first equation of the homogeneous system (1)

$$\begin{aligned} -(p+\sigma^{(k)})\,l_1^{(k)} - K\,x_2\,l_2^{(k)} &= 0 \\ \Rightarrow \qquad l_1^{(k)} &= \frac{-K\,x_2}{p+\sigma^{(k)}}\,l_2^{(k)} \end{aligned}$$

and the normalization condition $l_1^{(k)2} + l_2^{(k)2} = 1$, which we satisfy later by scaling, therefore we set first $l_2^{(k)} = 1$.
For $\sigma = \sigma^{(1)} = -p + K\,x_2$ we obtain

$$l_1^{(1)} = -1\,, \qquad |\vec{l}^{(1)}| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}\,,$$

and consequently, the normalized vector

$$\vec{l}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} .$$

For $\sigma = \sigma^{(2)} = -p - K\,x_2$ we have

$$l_1^{(2)} = +1\,, \qquad |\vec{l}^{(2)}| = \sqrt{(1)^2 + 1^2 + 0^2} = \sqrt{2}\,,$$

and therefore

$$\vec{l}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} .$$

To check the results, we calculate from the vector product $\vec{l}^{(1)} \times \vec{l}^{(2)}$ the eigenvector $\vec{l}^{(3)}$, which is already known.

$$\begin{aligned} \vec{l}^{(1)} \times \vec{l}^{(2)} &= \frac{1}{2} \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} \\ &= \frac{1}{2} \left[\vec{e}_1 \cdot 0 + \vec{e}_2 \cdot 0 + \vec{e}_3 \cdot (-2) \right] \\ &= -\vec{e}_3\,, \end{aligned}$$

which obviously corresponds to the negative eigenvector $\vec{l}^{(3)}$. In order to obtain $\vec{l}^{(3)} = (0,0,1)$, we have to change the sign of one of the two eigenvectors. We multiply $\vec{l}^{(1)}$ with -1 and find:

$$\sigma^{(1)} = -p + K\,x_2\,, \qquad \vec{l}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\,,$$

$$\sigma^{(2)} = -p - K\,x_2\,, \qquad \vec{l}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\,,$$

$$\sigma^{(3)} = -p\,, \qquad \vec{l}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,,$$

where now $\vec{l}^{(1)}$, $\vec{l}^{(2)}$, and $\vec{l}^{(3)}$ describe a right handed coordinate system. Even though the stress tensor depends on x_2, the principle stress directions are the same throughout the field. This is so because we are dealing locally with a simple shearing flow.

Problem 4.1-2 Temperature distribution in a Poiseuille flow

We reconsider the flow calculated in Problem 4.1-1. We assume a calorically perfect fluid with a constant heat conductivity λ. We further assume a constant temperature at the top wall T_0 and a full heat insulation at the bottom wall.

a) Calculate the temperature distribution $T(x_2)$ in the gap.
b) Find the temperature at the bottom wall.
c) Determine the heat flux per unit area through the top wall.
d) Calculate the entropy increase $\mathrm{D}s/\mathrm{D}t$ of the fluid inside the gap.

Solution

a) The temperature distribution in the channel:
The energy equation for Newtonian fluids is (see F. M. (4.2))

$$\varrho\,\frac{\mathrm{D}e}{\mathrm{D}t} - \frac{p}{\varrho}\frac{\mathrm{D}\varrho}{\mathrm{D}t} = \Phi + \frac{\partial}{\partial x_i}\left(\lambda\,\frac{\partial T}{\partial x_i}\right)\,, \tag{1}$$

with $T = T(x_2)$, $\mathrm{D}\varrho/\mathrm{D}t = 0$ and $e = c\,T$. The material change of the internal energy disappears here:

$$\frac{\mathrm{D}e}{\mathrm{D}t} = c\,\frac{\mathrm{D}T}{\mathrm{D}t} = c\left(\frac{\partial T}{\partial t} + u_1\,\frac{\partial T}{\partial x_1} + u_2\,\frac{\partial T}{\partial x_2} + u_3\,\frac{\partial T}{\partial x_3}\right) = 0\,. \tag{2}$$

Since λ =const and $T = T(x_2)$, equation (1) now assumes the form

$$\frac{\mathrm{d}^2 T}{\mathrm{d}{x_2}^2} = -\frac{\Phi}{\lambda}$$

and with $\Phi = K^2 x_2^2/\eta$ from Problem 4.1-1, we obtain the equation

$$\begin{aligned} \frac{\mathrm{d}^2 T}{\mathrm{d}{x_2}^2} &= -\frac{K^2}{\eta\,\lambda}\,x_2^2 \\ \Rightarrow \qquad \frac{\mathrm{d}T}{\mathrm{d}x_2} &= -\frac{K^2}{3\,\eta\,\lambda}\,x_2^3 + C_1 \qquad (3) \\ \Rightarrow \quad T(x_2) &= -\frac{K^2}{12\,\eta\,\lambda}\,x_2^4 + C_1\,x_2 + C_2\,. \qquad (4) \end{aligned}$$

On the bottom wall S_B is $x_2 = -h/2$ and $\vec{n} = (0,-1,0)$. Because it is insulated, the heat flux is zero:

$$\dot{Q}_{W_B} = \iint\limits_{S_B} -q_i n_i \,\mathrm{d}S = \iint\limits_{S_B} q_2 \,\mathrm{d}S = 0\,.$$

Since the integration domain is arbitrary, we must have $q_2(x_2 = -h/2) = 0$. From Fourier's heat conduction law

$$q_2 = -\lambda\,\frac{\partial T}{\partial x_2}$$

we infer the boundary condition

$$\left.\frac{\mathrm{d}T}{\mathrm{d}x_2}\right|_{x_2=-\frac{h}{2}} = 0\,,$$

which means that the temperature gradient disappears at the bottom wall. From (3), the first constant is

$$0 = \frac{K^2\,h^3}{24\,\lambda\,\eta} + C_1 \qquad \Rightarrow \qquad C_1 = -\frac{K^2\,h^3}{24\,\lambda\,\eta}\,.$$

The second integration constant is calculated from $T(x_2 = +h/2) = T_0$ and (4) as

$$C_2 = T_0 + \frac{5}{8}\,\frac{K^2\,h^4}{24\,\lambda\,\eta}\,,$$

Thus, the final solution is

$$T(x_2) = T_0 + \frac{K^2\,h^4}{24\,\lambda\,\eta}\left[-2\left(\frac{x_2}{h}\right)^4 - \frac{x_2}{h} + \frac{5}{8}\right]\,.$$

b) The temperature at the bottom wall:

$$T_B = T\left(x_2 = -\frac{h}{2}\right) = T_0 + \frac{K^2\,h^4}{24\,\lambda\,\eta}\left[-\frac{2}{16} + \frac{1}{2} + \frac{5}{8}\right]$$

$$\Rightarrow \qquad T_B = T_0 + \frac{K^2\,h^4}{24\,\lambda\,\eta}\,.$$

c) The heat flux through the top wall:
At the top wall S_T, $x_2 = h/2$ and $\vec{n} = (0,1,0)$, leading to the heat flux

$$\dot{Q}_{W_T} = \iint\limits_{S_T} -q_i n_i \,\mathrm{d}S = \iint\limits_{S_T} -q_2 \,\mathrm{d}S = \iint\limits_{S_T} \lambda\,\frac{\mathrm{d}T}{\mathrm{d}x_2}\,\mathrm{d}S\,.$$

With

$$\lambda\,\frac{\mathrm{d}T}{\mathrm{d}x_2}\bigg|_{x_2=\frac{h}{2}} = \frac{K^2\,h^3}{24\,\eta}\left[-8\left(\frac{x_2}{h}\right)^3 - 1\right]_{x_2=\frac{h}{2}} = -\frac{K^2\,h^3}{12\,\eta} = \text{const}\,,$$

the heat flux per unit area is

$$\frac{\dot{Q}}{A} = -q_2 = -\frac{K^2\,h^3}{12\,\eta} \qquad \text{(is rejected!)}\,,$$

which corresponds to the dissipated energy in Problem 4.1-1.

$$\int\limits_{-h/2}^{h/2} \Phi\,\mathrm{d}x_2 = \frac{K^2\,h^3}{12\,\eta}\,.$$

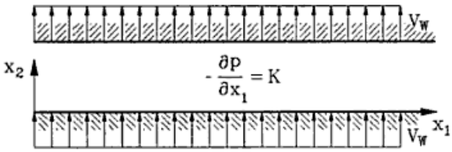
d) Entropy increase:
From Gibbs' relation we find for $\mathrm{D}s/\mathrm{D}t$:

$$T\,\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\mathrm{D}e}{\mathrm{D}t} + p\,\frac{\mathrm{D}v}{\mathrm{D}t} = 0 - \frac{p}{\varrho^2}\,\frac{\mathrm{D}\varrho}{\mathrm{D}t} = 0\,.$$

The substantial derivative $\mathrm{D}e/\mathrm{D}t$ disappears because of (2), $\mathrm{D}\varrho/\mathrm{D}t$ is also equal to zero. Thus, this frictional flow is isentropic. This is possible, because the heat generated by dissipation is immediately transferred through the top wall. This also can be seen from

$$\varrho\,T\,\frac{\mathrm{D}s}{\mathrm{D}t} = \Phi + \lambda\,\frac{\partial^2 T}{\partial x_2^2} = 0\,.$$

Problem 4.1-3 Pressure driven flow in a channel with porous walls



Newtonian fluid flows through the sketched channel with infinite extensions in x_1- and x_3-direction and the height h. The plane flow is steady, the density ϱ and the viscosity η are assumed to be constant, and body forces are neglected. The top and bottom wall are porous such that a constant normal velocity component V_W can be established at the walls. The pressure gradient in x_1-direction is constant ($\partial p/\partial x_1 = -K$). Because of the infinite extension of the channel, the velocity distribution does not depend upon x_1.

a) Using the continuity equation calculate the distribution of the velocity component in x_2-direction $u_2(x_2)$.
b) Simplify the x_1-component of the Navier-Stokes equation for this problem.
c) Give the boundary condition for the velocity component u_1.
d) Calculate the velocity distribution $u_1(x_2)$.
(Hint: After solving the homogeneous differential equation, the particular solution of the inhomogeneous differential equation can be found setting $u_{1_p} = \text{const} * x_2$.)

Given: ϱ, η, K, h, V_W

Solution

a) The velocity $u_2(x_2)$:
Since ϱ is an absolute constant, we also have $\mathrm{D}\varrho/\mathrm{D}t \equiv 0$. Consequently, we obtain from the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0 \, .$$

The term $\partial u_1/\partial x_1$ and all derivatives in x_3-direction disappear, and because of

$$\frac{\partial u_2}{\partial x_2} = 0$$

we conclude

$$u_2 = \text{const} \, .$$

With the boundary conditions

$$u_2(0) = u_2(h) = V_W$$

we obtain the velocity component:

$$u_2 = V_W \ .$$

b) The x_1-component of the Navier-Stokes equation:
With $\vec{k} = 0$, ϱ, η = const, steady flow, $\partial u_1/\partial x_1 = \partial u_1/\partial x_3 = 0$, $u_2 = V_W$, $u_3 = 0$ we have

$$\varrho V_W \frac{\partial u_1}{\partial x_2} = -\frac{\partial p}{\partial x_1} + \eta \frac{\partial^2 u_1}{\partial x_2{}^2} = K + \eta \frac{\partial^2 u_1}{\partial x_2{}^2} \ ,$$

and with $\nu = \eta/\varrho$ and $\partial u_1/\partial x_2 = \mathrm{d}u_1/\mathrm{d}x_2$ obtain

$$V_W \frac{\mathrm{d}u_1}{\mathrm{d}x_2} = \frac{1}{\varrho} K + \nu \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2{}^2} \ .$$

c) Boundary conditions:
The boundary conditions are the no-slip condition at the wall

$$u_1(x_2 = 0) = 0 \ , \qquad u_1(x_2 = h) = 0 \ .$$

d) The velocity distribution $u_1(x_2)$:
This is the solution of the ordinary, linear, inhomogeneous differential equation with constant coefficients from part b):

$$\nu \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2{}^2} - V_W \frac{\mathrm{d}u_1}{\mathrm{d}x_2} = -\frac{K}{\varrho} \ .$$

1) Solution of the homogeneous part:

$$\nu \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2{}^2} - V_W \frac{\mathrm{d}u_1}{\mathrm{d}x_2} = 0 \ .$$

Using $u_1(x_2) = C\,\mathrm{e}^{\lambda x_2}$ we obtain the characteristic polynomial

$$\nu \lambda^2 - V_W \lambda = 0 \ ,$$

with the roots

$$\lambda_1 = 0 \ , \qquad \lambda_2 = V_W/\nu \ ,$$

and thus,

$$u_{1_h} = C_1 + C_2\,\mathrm{e}^{\frac{V_W}{\nu} x_2} \ .$$

2) The particular solution

$$u_{1_p} = C_3\, x_2 + C_4$$

is inserted into the differential equation, and the constants are found as $C_3 = K/(\varrho V_W)$ and $C_4 = 0$.

The general solution

$$u_1(x_2) = u_{1_h} + u_{1_p} = C_1 + C_2\,\mathrm{e}^{\frac{V_W x_2}{\nu}} + \frac{K}{\varrho V_W}\,x_2$$

is subjected to the boundary conditions from part c)

$$\begin{aligned} u_1(0) &= C_1 + C_2 &= 0 \\ u_1(h) &= C_1 + C_2\,\mathrm{e}^{\frac{V_W h}{\nu}} + \frac{K h}{\varrho V_W} &= 0 \end{aligned}$$

$$\Rightarrow \qquad -C_1 = C_2 = \frac{K h}{\varrho V_W}\,\frac{1}{1-\mathrm{e}^{\frac{V_W h}{\nu}}}\,.$$

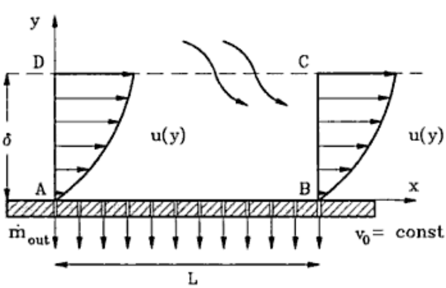
This results in the velocity distribution

$$u_1(x_2) = \frac{K h}{\varrho V_W}\left(\frac{x_2}{h} - \frac{1-\mathrm{e}^{\frac{V_W x_2}{\nu}}}{1-\mathrm{e}^{\frac{V_W h}{\nu}}}\right)\,.$$

Note:
In the limit $V_W \to 0$, we get the well known velocity profile of a plane channel flow

$$\lim_{V_W \to 0} u_1(x_2) = \frac{K h^2}{2\eta}\left(\frac{x_2}{h} - \left(\frac{x_2}{h}\right)^2\right)\,.$$

Problem 4.1-4 Boundary layer suction



Incompressible Newtonian fluid flows steadily over a flat plate with infinite extensions in x- and z-direction. A boundary layer develops which normally would grow with increasing x. But here suction is applied over the length L such that the boundary layer thickness remains constant. The pressure p is assumed to be constant. Far from the plate, the velocity component $u(y)$ has the value U_∞.

a) Give the boundary conditions for the velocity field.

b) Using the continuity equation, obtain the velocity component $v(y)$.

c) Simplify the x-component of the Navier-Stokes equation and calculate the velocity component $u(y)$.
d) Show that the mass flux entering the area $D-C$ is equal to the suction mass flux $\dot{m}_{out}$.
e) Calculate the drag per unit depth and plate length L by directly integrating the wall shear stress.
f) Calculate the drag force using the x-component of the balance of momentum applied to the control volume ABCD.

Given: U_∞, η, ϱ, v_0

Solution

a) The dynamic boundary condition at the wall requires

$$u(y=0)=0\ ,$$

and at infinity we have

$$\lim_{y\to\infty} u(y) = U_\infty\ .$$

At the wall, the velocity component v must be equal to the suction velocity:

$$v(y=0) = -v_0\ .$$

b) The density ϱ is constant, thus $\mathrm{D}\varrho/\mathrm{D}t=0$ and the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\ .$$

The flow field is planar ($\partial/\partial z=0$) and further $\partial u/\partial x=0$. Thus, we obtain

$$\frac{\partial v}{\partial y}=0 \qquad \text{or} \quad v=v(x)=\text{const}\ ,$$

thus, v does not depend on x. With the boundary condition $v(y=0)=-v_0$ the velocity component

$$v=-v_0\ ,$$

is therefore constant throughout the flow field.

c) With $v=-v_0$, $\partial u/\partial x=0$, $\partial u/\partial z=0$ as well as the constant pressure p, we obtain in case of a steady flow neglecting the body forces

$$-\varrho\, v_0\, \frac{\partial u}{\partial y} = \eta\, \frac{\partial^2 u}{\partial y^2}$$

or, since with $\partial u/\partial y = \mathrm{d}u/\mathrm{d}y$ and $\nu = \eta/\varrho$

$$-v_0 \frac{\mathrm{d}u}{\mathrm{d}y} = \nu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} .$$

This an ordinary second order, linear, homogeneous differential equation, whose solution can be taken from Problem 4.1-3:

$$u(y) = C_1 + C_2 \,\mathrm{e}^{\frac{-v_0}{\nu} y} .$$

The constants C_1 and C_2 follow from the boundary conditions as

$$C_1 = -C_2 = U_\infty .$$

Thus, for the velocity component we arrive at

$$u(y) = U_\infty (1 - \mathrm{e}^{\frac{-v_0}{\nu} y}) .$$

d) Mass flux:
We apply the equation of continuity in integral form to the sketched control volume V_{ABCD}

$$\iint_{AD} \varrho \vec{u} \cdot \vec{n} \,\mathrm{d}S + \iint_{BC} \varrho \vec{u} \cdot \vec{n} \,\mathrm{d}S + \iint_{DC} \varrho \vec{u} \cdot \vec{n} \,\mathrm{d}S + \iint_{AB} \varrho \vec{u} \cdot \vec{n} \,\mathrm{d}S = 0 .$$

Because of identical velocity profiles, the first two terms cancel each other; as a result we have

$$\dot{m}_{out} = \iint_{AB} \varrho \vec{u} \cdot \vec{n} \,\mathrm{d}S = - \iint_{DC} \varrho \vec{u} \cdot \vec{n} \,\mathrm{d}S = \dot{m}_{in} .$$

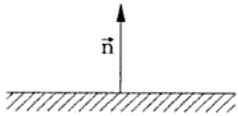
e) Drag force:
The drag force is

$$F_D = \vec{F}_{\text{fluid}\to\text{plate}} \cdot \vec{e}_x = \iint_{S_p} \vec{t} \cdot \vec{e}_x \,\mathrm{d}S ,$$

or using index notation with $t_1 = \tau_{j1}\, n_j$

$$F_D = F_1 = \iint_{S_p} t_1 \,\mathrm{d}S = \iint_{S_p} \tau_{j1} n_j \,\mathrm{d}S , \qquad j = 1, 2 .$$

With the components of the normal vector $n_1 = 0$, $n_2 = 1$ and $n_3 = 0$ we get



$$F_D = \iint\limits_{S_p} \tau_{21} \, \mathrm{d}S \, .$$

The constitutive equation of the Newtonian fluid is (Cauchy–Poisson law)

$$\tau_{ij} = -p\,\delta_{ij} + \lambda^*\, e_{kk}\,\delta_{ij} + 2\,\eta\, e_{ij} \, ,$$

therefore, the required component of the stress tensor is:

$$\tau_{21}\big|_{x_2=0} = 2\,\eta\, e_{21}\big|_{x_2=0} = \eta\, \frac{\partial u_1}{\partial x_2}\bigg|_{x_2=0} = \eta\, \frac{\partial u}{\partial y}\bigg|_{y=0} \, .$$

With

$$\frac{\partial u}{\partial y}\bigg|_{y=0} = U_\infty \frac{v_0}{\nu} \mathrm{e}^{\frac{-v_0}{\nu} y}\bigg|_{y=0} = U_\infty\, \frac{v_0}{\nu}$$

the shear stress becomes

$$\tau_{21}\big|_{y=0} = \varrho\, U_\infty\, v_0 \, ,$$

and the drag force

$$F_D = \iint\limits_{S_p} \varrho\, U_\infty\, v_0 \, \mathrm{d}S = \varrho\, U_\infty\, v_0\, L \, .$$

The drag force does not depend on the viscosity of the fluid!

f) Drag force using the momentum balance:
The x-component of the linear momentum applied to the control volume introduced in part b) leads to

$$\iint\limits_{AD} \varrho\,(\vec{u}\cdot\vec{e}_x)\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{BC} \varrho\,(\vec{u}\cdot\vec{e}_x)\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S \; +$$

$$+ \iint\limits_{DC} \varrho\,(\vec{u}\cdot\vec{e}_x)\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{AB} \varrho\,(\vec{u}\cdot\vec{e}_x)\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S \; =$$

$$\iint\limits_{AD} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S + \iint\limits_{BC} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S + \iint\limits_{DC} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S + \iint\limits_{AB} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S \, ,$$

where the integrals over the surfaces AD and BC cancel each other and the contribution at the wall AB vanishes because $\vec{u}\cdot\vec{e}_x = 0$. The last term on the right hand side represents the force exerted by the plate on the fluid:

$$\iint\limits_{AB} \vec{t} \cdot \vec{e}_x \, \mathrm{d}S = F_{x_{\text{plate}\to\text{fluid}}} = -F_D \, .$$

The remaining terms are

$$\iint\limits_{DC} \varrho(\vec{u} \cdot \vec{e}_x)(\vec{u} \cdot \vec{n}) \, \mathrm{d}S = \iint\limits_{DC} \vec{t} \cdot \vec{e}_x \, \mathrm{d}S - F_D \, .$$

With

$$\begin{aligned} \iint\limits_{DC} \varrho\,(\vec{u} \cdot \vec{e}_x)(\vec{u} \cdot \vec{n}) \, \mathrm{d}S &= -\iint\limits_{DC} \varrho\, U_\infty \,(1 - \mathrm{e}^{\frac{-v_0}{\nu} y})\, v_0 \, \mathrm{d}S \\ &= -\varrho\, U_\infty \, v_0 \,(1 - \mathrm{e}^{\frac{-v_0}{\nu} \delta})\, L \, , \end{aligned}$$

and

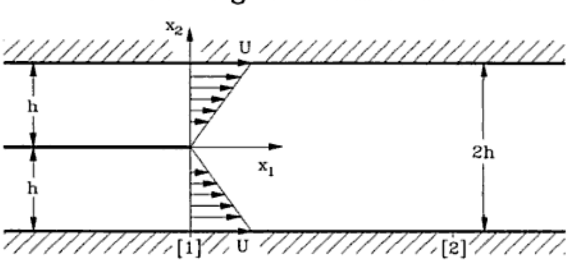
$$\iint\limits_{DC} \vec{t} \cdot \vec{e}_x \, \mathrm{d}S = \eta \iint\limits_{DC} \left.\frac{\partial u}{\partial y}\right|_{y=\delta} \mathrm{d}S = U_\infty \, \varrho\, v_0 \, L\, \mathrm{e}^{\frac{-v_0}{\nu}\delta}$$

we obtain the drag force

$$F_D = U_\infty \, \varrho\, v_0 \, L\, \mathrm{e}^{\frac{-v_0}{\nu}\delta} + \varrho\, U_\infty \, v_0 \,(1 - \mathrm{e}^{\frac{-v_0}{\nu} \delta})\, L = \varrho\, U_\infty \, v_0 \, L \, ,$$

which agrees with the result from part d). If the surface DC is located at $y \to \infty$, the integral of the stress vector over this surface vanishes and the drag force corresponds to the x-component of the momentum flux through DC.

Problem 4.1-5 Mixing of streams of fluids



Newtonian fluid ($\varrho, \eta = \text{const}$) flows steadily through a channel (height $2h$). In the middle of the channel, an infinitely thin splitter plate is mounted. The channel walls move with a constant velocity U in positive x_1-direction. The two fluid streams separated by the plate are mixed at the end of the plate. At station [2], a new velocity profile $u_1 = u_1(x_2)$ is developed that does not change anymore with x_1. The body forces can be neglected.

a) Using the equation of motion, show that the pressure gradient $\partial p/\partial x_1$ downstream of [2] does not change.
b) Calculate the volume flux per unit depth $\dot{V}$ at station [1].
c) Obtain the velocity profile $u_1 = u_1(x_2)$ at station [2] using the no-slip condition at $x_2 = \pm h$ and the requirement that the volume flux at stations [2] must be the same as at [1]. Show that the pressure gradient must be different from zero, resulting in a pressure driven Couette flow.
d) Calculate the pressure gradient.

Given: h, U, ϱ, η

Solution

a) Pressure gradient $\partial p/\partial x_1$:
For calculating the pressure gradient at [2], we start from Navier-Stokes equation (see F. M. (4.1)). The continuity equation (see F. M. (2.5))

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$$

yields at station [2], where u_1 by assumption is only a function of x_2:

$$\frac{\partial u_2}{\partial x_2} = 0 \ ,$$

and therefore $u_2 = u_2(x_1)$. The channel walls are impervious, i. e. $u_2 = 0$ at $x_2 = \pm h$. As a result, at [2] the velocity u_2 must vanish identically. The x_1-component of the Navier-Stokes equation is then simplified as

$$\frac{\partial^2 u_1}{\partial {x_2}^2} = \frac{1}{\eta}\frac{\partial p}{\partial x_1} \ . \tag{1}$$

We partially differentiate (1) with respect to x_1 and obtain:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial^2 u_1}{\partial {x_2}^2}\right) = \frac{1}{\eta}\frac{\partial^2 p}{\partial {x_1}^2} \ .$$

Since u_1 is a function of x_2 only, the left hand side of this equation is zero. Thus, $\partial p/\partial x_1$ at station [2] assumes a constant value which we call $-K$.

b) The volume flux per unit depth $\dot{V}$:
We obtain the volume flux per unit depth at [1] by integrating the velocity $u_1(x_2)$ over the channel height. Because of the no-slip condition at the wall and at station [1], we have the velocity profile of a Couette flow (simple shear flow):

$$-h < x_2 < 0 \ : \qquad u_1 = -U\,\frac{x_2}{h}\,,$$

$$0 < x_2 < h \ : \qquad u_1 = U\,\frac{x_2}{h}\,.$$

The integration yields:

$$\dot{V} = \int\limits_{-h}^{0} u_1\,\mathrm{d}x_2 + \int\limits_{0}^{h} u_1\,\mathrm{d}x_2 = U\,h\,.$$

c) Velocity profile $u_1 = u_1(x_2)$ at [2]:
To calculate the velocity profile at station [2] we solve the Navier-Stokes equation (1) by integrating twice and obtain the general solution

$$u_1(x_2) = -\frac{K}{2\,\eta}\,x_2^2 + C_1\,x_2 + C_2\,.$$

For vanishing pressure gradient ($K = 0$)

$$u_1(x_2) = C_1\,x_2 + C_2$$

and from the no-slip condition $u_1 = U$ for $x_2 = \pm h$

$$u_1(x_2) = C_2 = U\,, \quad C_1 = 0\,.$$

Thus the volume flux is $\dot{V} = 2\,U\,h$, which violates the continuity requirement. The continuity equation in integral form, however, shows that the volume flux $\dot{V}$ is independent of x_1. Thus the pressure gradient cannot be zero. For $K \neq 0$ we obtain the constants as

$$C_1 = 0\,, \quad C_2 = U + \frac{K}{2\,\eta}\,h^2$$

and thus,

$$\frac{u_1(x_2)}{U} = 1 + \frac{K}{2\,\eta}\,\frac{h^2}{U}\left[1 - \left(\frac{x_2}{h}\right)^2\right]\,.$$

d) Pressure gradient:
The constant $K = -\partial p/\partial x_1$ is determined from the requirement that the volume flux at [1] is equal to the one at [2]:

$$U\,h = \int_{-h}^{h} \left\{U + \frac{K}{2\eta}h^2\left[1-\left(\frac{x_2}{h}\right)^2\right]\right\}\mathrm{d}x_2$$

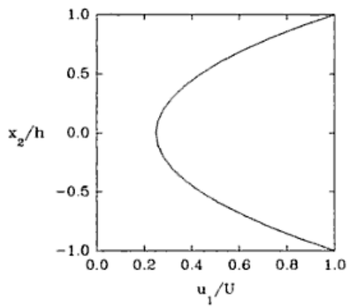
$$= U\,2h + \frac{K}{2\eta}h^2\left[2h - \frac{2}{3}h\right].$$

The requested constant is therefore

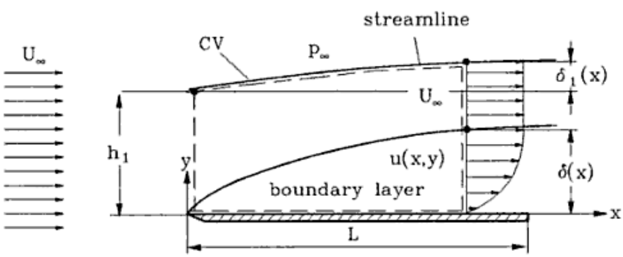
$$K = -\frac{3}{2}\eta\frac{U}{h^2}.$$

and the velocity profile at [2] is

$$\frac{u_1(x_2)}{U} = 1 - \frac{3}{4}\left[1-\left(\frac{x_2}{h}\right)^2\right].$$



Problem 4.1-6 Drag on a flat plate



Fluid with the velocity U_∞ flows steadily along a thin flat plate (width in z-direction b, length L ($b \gg L$)). A laminar boundary layer is developed which has the thickness $\delta(x) = \sqrt{30\,\nu x/U_\infty}$. We assume an incompressible flow of a Newtonian fluid at a high Reynolds number. The velocity profile on the upper half of the plate is described by

$$\frac{u(x,y)}{U_\infty} = \begin{cases} 2\,\dfrac{y}{\delta(x)} - \left(\dfrac{y}{\delta(x)}\right)^2 & \text{for} \quad 0 \le y \le \delta(x) \\ 1 & \text{for} \quad y > \delta(x)\,. \end{cases}$$

a) Determine the boundary layer displacement thickness $\delta_1(x)$.
b) Calculate the drag force F_D, on the upper half of the plate (length $x = L$),
   1) using the integral form of the momentum equation.
   2) by directly integrating $\iint\limits_{(S)} \vec{t}\,\mathrm{d}S$ over the plate surface.
c) Calculate Re, $\delta(L)$ and $\delta_1(L)$ for $U_\infty = 10\,\mathrm{m/s}$, $U_\infty = 50\,\mathrm{m/s}$ and $U_\infty = 100\,\mathrm{m/s}$.

Given: U_∞, $\nu = 15.6 * 10^{-6}\ \mathrm{m^2/s}$, $L = 1$ m, b, p_∞, $\varrho = \text{const}$

Solution

a) To calculate the displacement thickness $\delta_1(x)$, we apply the continuity equation to the sketched control volume and assume $\varrho = \text{const}$,

$$\iint\limits_{(S)} \vec{u}\cdot\vec{n}\,\mathrm{d}S = 0\,, \tag{1}$$

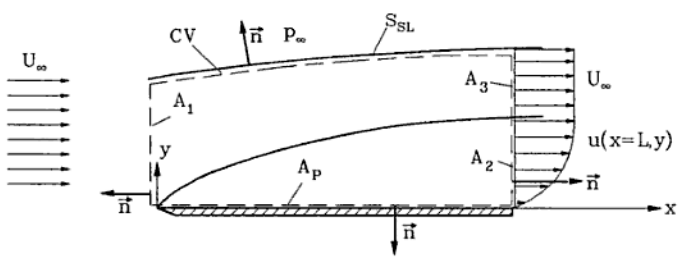
with the auxiliary quantity h_1 as the distance from the plate surface to the upper streamline at the inlet:

$$U_\infty\, b\, h_1 = U_\infty\, b\, \delta_1(x) + U_\infty\, b\,[h_1 - \delta(x)] + U_\infty\, b\, \delta(x) \int\limits_0^1 \frac{u}{U_\infty}\,\mathrm{d}(y/\delta)\,,$$

which yields

$$\delta_1(x) = \delta(x)\left\{1 - \int\limits_0^1 \left[2\,\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]\mathrm{d}(y/\delta)\right\} = \frac{1}{3}\,\delta(x)\,. \tag{2}$$

b) 1) Calculation of drag force F_D using the momentum equation:



For the calculation of the drag force F_D we apply the momentum equation in integral form (see F. M. (2.43)) in x-direction to the sketched control volume.

$$\iint_{(S)} \vec{t}\cdot\vec{e}_x \,\mathrm{d}S = \iint_{(S)} \varrho\,\vec{u}\cdot\vec{e}_x\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S\;. \tag{3}$$

We decompose the total surface S into the following parts:
$S = S_{SL} + A_1 + A_2 + A_3 + A_P$, with $A_1 = h_1\,b$, $A_2 = \delta(L)\,b$ and $A_3 = h_3\,b$.
The height h_3 is

$$h_3 = \delta_1(L) + h_1 - \delta(L) = h_1 - \frac{2}{3}\,\delta(L)\;.$$

The momentum flux through the control surfaces:
The control volume was placed between the upper stream surface and the surface of the plate. As a consequence, the momentum flux through the surface S_{SL} and A_P is zero and the right hand side of (3) becomes

$$\iint_{(S)} \varrho\,\vec{u}\cdot\vec{e}_x\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S =$$

$$= \varrho\,b\left\{-U_\infty^2\,h_1 + U_\infty^2\,h_3 + U_\infty^2\,\delta(L)\int_0^1\left(\frac{u}{U_\infty}\right)^2\mathrm{d}(y/\delta)\right\}$$

$$= \varrho\,b\,\delta(L)\,U_\infty^2\left\{\int_0^1\left(\frac{u}{U_\infty}\right)^2\mathrm{d}(y/\delta) - \frac{2}{3}\right\}$$

$$= \varrho\,b\,\delta(L)\,U_\infty^2\left\{\int_0^1\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]^2\mathrm{d}(y/\delta) - \frac{2}{3}\right\}$$

$$= -\frac{2}{15}\,\varrho\,b\,\delta(L)\,U_\infty^2\;. \tag{4}$$

Integral of t_x over the control surfaces:
The integral of the x-component of the stress vector over S is

$$\iint_{(S)} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S = \iint_{A_P} t_x\,\mathrm{d}S + \iint_{A_1} t_x\,\mathrm{d}S + \iint_{A_2} t_x\,\mathrm{d}S + \iint_{A_3} t_x\,\mathrm{d}S + \iint_{S_{SL}} t_x\,\mathrm{d}S\;. \tag{5}$$

The x-component of the stress vector at a control surface with the normal unit vectors n_x and n_y reads

$$\vec{t} \cdot \vec{e}_x = t_x = \tau_{xx}\, n_x + \tau_{xy}\, n_y \, . \tag{6}$$

The stresses τ_{xx} and τ_{xy} are given by the Cauchy-Poisson law (see F. M. (3.1a)) with ($e_{kk} = 0$):

$$\tau_{xx} = -p + 2\,\eta\,\frac{\partial u}{\partial x} \quad \text{and} \quad \tau_{xy} = \eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \, . \tag{7}$$

We neglect by previous agreement the velocity change in x-direction within the entire flow field, consider $p = p_\infty$, and are led to

$$t_x = -p_\infty\, n_x + \eta\,\frac{\partial u}{\partial y}\, n_y \, . \tag{8}$$

The integral of t_x over the plate surface A_P is equal to the negative drag force F_D:

$$F_D = -\iint_{A_P} t_x \,\mathrm{d}S \, . \tag{9}$$

In accordance with the boundary layer theory, outside the boundary layer the flow can be considered as uniform, since $\partial u/\partial y$ approaches zero. Thus, at the control surfaces A_1, A_3 and S_{SL} the x-component of the stress vector is $t_x = -p_\infty\, n_x$. For the integral of the stress vector we obtain

$$\begin{aligned}\iint_{(A_1+A_3+S_{SL})} t_x S &= p_\infty\, A_1 - p_\infty\, A_3 + p_\infty(A_2 + A_3 - A_1) \\ &= p_\infty\, A_2 = p_\infty\, \delta(L)\, b \, .\end{aligned}$$

At the control surface A_2 the components of the normal vector are $n_x = 1$ and $n_y = 0$. From (8), then

$$\iint_{A_2} t_x \,\mathrm{d}S = b \int_0^{\delta(\mathrm{L})} (-p_\infty)\,\mathrm{d}y = -p_\infty\, \delta(L)\, b \, .$$

The integral of t_x over the total control surface S is finally

$$\iint_{(S)} \vec{t} \cdot \vec{e}_x \,\mathrm{d}S = -F_D \, .$$

From (3) with (4) we obtain the drag force as

$$F_D = \frac{2}{15}\,\varrho\, b\, U_\infty^2\, \delta(L) = \frac{2\sqrt{30}}{15}\,\varrho\, b\, U_\infty^2\, L\,\frac{1}{\sqrt{Re}}\,. \tag{10}$$

The ratio of the drag from the solution above to the drag from the exact solution (see F. M. (12.50)) is

$$\frac{F_D}{F_{D_{exact}}} = 1.0998\,.$$

2) Calculation of F_D by direct integration of $\iint\limits_{(S)} \vec{t}\,\mathrm{d}S$:
From (9) and (8) and $n_x = 0$ and $n_y = -1$, we have

$$F_D = -\iint\limits_{A_P} t_x\,\mathrm{d}S = b\int\limits_0^L \eta\,\left.\frac{\partial u}{\partial y}\right|_{y=0}\mathrm{d}x\,. \tag{11}$$

The wall shear stress $\tau_w = \eta\ \partial u/\partial y|_{y=0}$ is

$$\tau_w = \eta\sqrt{\frac{U_\infty^3}{\nu x}}\,\frac{2}{\sqrt{30}}\,,$$

thus

$$F_D = \frac{4}{\sqrt{30}}\,b\,\eta\sqrt{\frac{U_\infty^3}{\nu}}\,\sqrt{L} = \frac{2\sqrt{30}}{15}\,\varrho\, U_\infty^2\, L\, b\,\frac{1}{\sqrt{Re}}\,.$$

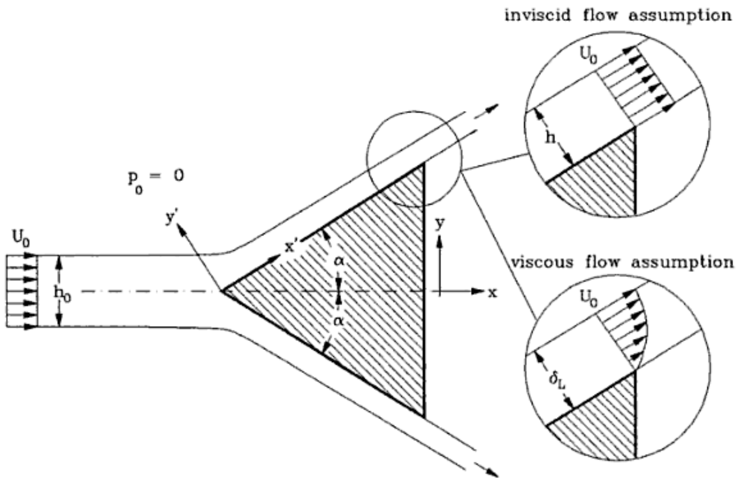
c) The following table summarizes the boundary layer thicknesses on a flat plate of 1 m length and the kinematic viscosity $\nu = 15.6 * 10^{-6}$ m^2/s (for dry air at 1 bar and 25°C (see F. M. Table D.2)) and different inlet velocities.

$U_\infty/\,(\mathrm{m/s})$	$Re = U_\infty\, L/\nu$	$\delta(L)/\,\mathrm{mm}$	$\delta_1(L)/\,\mathrm{mm}$
10	$0.64 * 10^6$	6.84	2.28
50	$3.70 * 10^6$	3.06	1.02
100	$6.41 * 10^6$	2.16	0.72

Problem 4.1-7 Two-dimensional water jet impinging on a wedge

A two-dimensional water jet symmetrically impinges on a wedge with included angle 2α. Far upstream, the jet has the velocity U_0 and a thickness of h_0. As a result of wall friction, at the end of the wedge, boundary layers with the following velocity distributions are developed

$$u(y') = U_0 \sin\left(\frac{\pi\, y'}{2\,\delta_L}\right) .$$

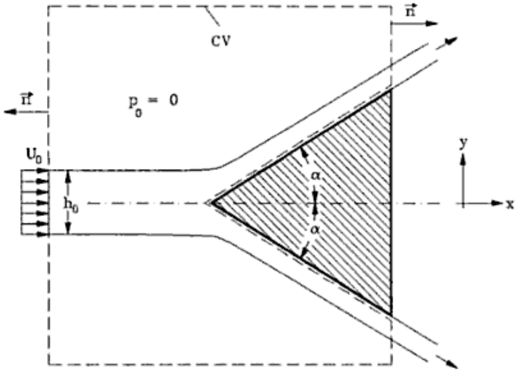


a) Calculate the thicknesses
   1) h for the inviscid flow and
   2) δ_L for the viscous flow.
b) Determine the force per unit depth exerted on the wedge for the case that
   1) the flow is inviscid,
   2) the flow is viscous.
c) Calculate the difference of the calculated forces for $\alpha = \pi/2$.

Given: U_0, h_0, α, ϱ

Solution

a)



To calculate the thickness of the fluid film at the end of the wedge, we apply the continuity equation to the sketched control volume with $\varrho =$ const,

$$\iint\limits_{(S)} \vec{u} \cdot \vec{n} \, \mathrm{d}S = 0 \,. \tag{1}$$

1) For an inviscid flow we obtain from (1)

$$U_0 \, h_0 = 2 \, U_0 \, h \quad \text{or} \quad h = \frac{1}{2} \, h_0 \,. \tag{2}$$

2) For the viscous flow, the continuity equation (1) with the given velocity profile becomes:

$$U_0 \, h_0 = 2 \int\limits_0^{\delta_L} U_0 \sin\left(\frac{\pi \, y'}{2 \, \delta_L}\right) \mathrm{d}y' = 2U_0 \, \frac{2 \, \delta_L}{\pi} \left[- \cos\left(\frac{\pi \, y'}{2 \, \delta_L}\right)\right]_0^{\delta_L} \,.$$

As a result:

$$\delta_L = \frac{\pi}{4} \, h_0 \,. \tag{3}$$

b) To determine the force per unit depth exerted on the wedge, we apply the momentum balance in integral form (see F. M. (2.43)):

$$\iint\limits_{(S)} \vec{t} \, \mathrm{d}S = \iint\limits_{S_W} \vec{t} \, \mathrm{d}S + \iint\limits_{S-S_W} -p_0 \, \vec{n} \, \mathrm{d}S = \iint\limits_{(S)} \varrho \, \vec{u} \, (\vec{u} \cdot \vec{n}) \, \mathrm{d}S \,.$$

The integral over the wedge surface S_W is equal to the negative of the force we are looking for, thus

$$-\vec{F} = \iint\limits_{(S)} \varrho\,\vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S\ . \tag{4}$$

1) For the case of inviscid flow we find from (4)

$$-\vec{F} = \left[-\varrho\,U_0^2\,h_0 + 2\,\varrho\,U_0^2\,h\,\cos\alpha\right]\,\vec{e}_x\ ,$$

and with (2) finally:

$$\vec{F} = \varrho\,U_0^2\,h_0\,(1-\cos\alpha)\,\vec{e}_x\ .$$

2) For the viscous flow we insert into (4) the given velocity profile and obtain:

$$-\vec{F} = \left[-\varrho\,U_0^2\,h_0 + 2\,\varrho\,U_0^2\,\cos\alpha\int\limits_0^{\delta_L}\sin^2\left(\frac{\pi\,y'}{2\,\delta_L}\right)\mathrm{d}y'\right]\vec{e}_x$$

$$= -\varrho\,U_0^2\,h_0\left[1-2\cos\alpha\,\frac{\delta_L}{h_0}\,\frac{1}{\pi}\left[\frac{\pi\,y'}{2\,\delta_L}-\sin\left(\frac{\pi\,y'}{2\,\delta_L}\right)\cos\left(\frac{\pi\,y'}{2\,\delta_L}\right)\right]_0^{\delta_L}\right]\vec{e}_x\ ,$$

and with (3)

$$\vec{F} = \varrho\,U_0^2\,h_0\left(1-\frac{\pi}{4}\,\cos\alpha\right)\,\vec{e}_x\ .$$

It should be pointed out that the force even in case of viscous flow does not depend on shear viscosity if the velocity profile is given.

c) The difference of both forces is:

$$\begin{aligned}\Delta\vec{F} &= \vec{F}_{\text{viscous}} - \vec{F}_{\text{inviscid}}\\ &= \varrho\,U_0^2\,h_0\left(1-\frac{\pi}{4}\right)\cos\alpha\,\vec{e}_x\ .\end{aligned}$$

For $\alpha = \pi/2$ the difference disappears.

Problem 4.1-8 Rigid body rotation and potential vortex

Given is the velocity field of an incompressible, plane flow with negligible body forces:

$$\begin{aligned}u_1 &= -k\,x_2\,(x_1^2+x_2^2)^n\ ,\\ u_2 &= k\,x_1\,(x_1^2+x_2^2)^n\ .\end{aligned}$$

a) Calculate the streamlines and pathlines.
b) Calculate
   the velocity gradient,
   the rate of deformation tensor $e_{ij} = 1/2\,\{\partial u_i/\partial x_j + \partial u_j/\partial x_i\}$
   and the spin tensor $\Omega_{ij} = 1/2\,\{\partial u_i/\partial x_j - \partial u_j/\partial x_i\}$.
c) Investigate the case $n = 0$ and $n = -1$:
   For which value of n do we have a rigid body rotation, for which value a potential flow? Calculate the value of the constant in Bernoulli's equation for the potential vortex flow for different streamlines.
d) Sketch the velocity field for the rigid body rotation and the potential vortex.
e) Using Cauchy-Poisson law, calculate the stress tensor for $n = 0$ and $n = -1$.
f) Using Cauchy's law of motion, calculate for potential flow the acceleration of a fluid particle located on the x_1-axes.

Solution

a) The differential equations of the streamline are:

$$\frac{\mathrm{d}x_1}{\mathrm{d}s} = \frac{u_1}{\sqrt{u_k u_k}} \quad \text{and} \quad \frac{\mathrm{d}x_2}{\mathrm{d}s} = \frac{u_2}{\sqrt{u_k u_k}} \,.$$

We eliminate parameter s by performing the division

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_2} = \frac{u_1}{u_2} \,.$$

Inserting the given velocity field, we obtain:

$$x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2 = \frac{1}{2}\,\mathrm{d}\left(x_1^2\right) + \frac{1}{2}\,\mathrm{d}\left(x_2^2\right) = 0 \,.$$

Integration leads to

$$x_1^2 + x_2^2 = \text{const} \,.$$

Thus, the streamlines are concentric circles. Since the velocity field is steady, streamlines and pathlines coincide.

b) Velocity gradient $\partial u_i/\partial x_j$:

$$\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= -k\,x_2 n\,(x_1^2 + x_2^2)^{n-1}\,2x_1 \,, \\
\frac{\partial u_1}{\partial x_2} &= -k\,x_2 n\,(x_1^2 + x_2^2)^{n-1}\,2x_2 - k\,(x_1^2 + x_2^2)^n \,, \\
\frac{\partial u_2}{\partial x_1} &= k\,x_1 n\,(x_1^2 + x_2^2)^{n-1}\,2x_1 + k\,(x_1^2 + x_2^2)^n \,,
\end{aligned}$$

$$\frac{\partial u_2}{\partial x_2} = k\, x_1 n\, (x_1^2 + x_2^2)^{n-1}\, 2x_2 \,.$$

We decompose $\partial u_i/\partial x_j$

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \Omega_{ij} \,.$$

The components of the rate of deformation tensor are:

$$e_{11} = \frac{\partial u_1}{\partial x_1} = -2k\, n\, x_1\, x_2\, (x_1^2 + x_2^2)^{n-1} \,,$$

$$e_{12} = e_{21} = \frac{1}{2}\left\{\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right\} = k\, n\, (x_1^2 - x_2^2)\, (x_1^2 + x_2^2)^{n-1} \,,$$

$$\text{and} \quad e_{22} = \frac{\partial u_2}{\partial x_2} = 2k\, n\, x_1\, x_2\, (x_1^2 + x_2^2)^{n-1} \,.$$

The components of the spin tensor are:

$$\Omega_{11} = \Omega_{22} = 0 \,,$$

$$\Omega_{12} = -\Omega_{21} = \frac{1}{2}\left\{\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right\} = -k\, (1+n)\, (x_1^2 + x_2^2)^n \,.$$

c) Special cases $n = 0$ and $n = -1$:
$n = 0$: For $n = 0$ is $e_{ij} = 0$, i. e. the motion of the fluid has to be locally a rigid body motion. For $n = 0$, the spin tensor becomes

$$(\Omega_{ij}) = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} .$$

This tensor is independent of x_i. The flow is not just locally a rigid body rotation but everywhere. The angular velocity follows from

$$\omega_n = -\frac{1}{2}\, \Omega_{ij}\, \epsilon_{ijn} \tag{1}$$

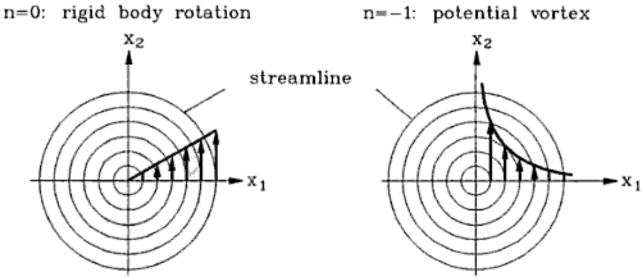
and since we are dealing with a plane flow, the only non-zero component is ω_3 :

$$\omega_3 = -\frac{1}{2}\, (\Omega_{12}\epsilon_{123} + \Omega_{21}\epsilon_{213}) = k \,.$$

The streamlines are concentric circles. No additional translational motion exists.

$n=-1$: For $n=-1$ is $\Omega_{ij}=0$ and therefore, with (1) $\omega_n=0$. The flow is irrotational, i. e. a potential flow.
For a potential flow the Bernoulli constant has the same value for all streamlines.

d) Sketch the streamlines and the velocity field:



e) Stress tensor:
The Cauchy-Poisson law has the form

$$\tau_{ij} = -p\,\delta_{ij} + \lambda^*\,e_{kk}\,\delta_{ij} + 2\eta\,e_{ij}\ .$$

For the incompressible flow we have $e_{kk}=0$. The components of the stress tensor are:

$$\begin{aligned}
\tau_{11} &= -p(x_i) + 2\eta\,e_{11} = -p(x_i) - 4\eta\,k\,n\,x_1\,x_2\,(x_1^2+x_2^2)^{(n-1)}\ ,\\
\tau_{12}=\tau_{21} &= 2\eta\,e_{12} = 2\eta\,k\,n\,(x_1^2-x_2^2)\,(x_1^2+x_2^2)^{(n-1)}\ ,\\
\tau_{22} &= -p(x_i) + 2\eta\,e_{22} = -p(x_i) + 4\eta\,k\,n\,x_1\,x_2\,(x_1^2+x_2^2)^{(n-1)}\ .
\end{aligned}$$

For the special cases $n=0$ and $n=-1$, we find:

$n=0$:

$$(\tau_{ij}) = \begin{pmatrix} -p(x_i) & 0 \\ 0 & -p(x_i) \end{pmatrix}\ .$$

This is identical to the stress state of a fluid at rest. Since there is no relative motion between the fluid particles, no viscous stresses are encountered even though the fluid is viscous.

$n=-1$:

$$\tau_{11} = -p(x_i) + 4\eta\,k\,\frac{x_1\,x_2}{(x_1^2+x_2^2)^2}\ ,$$

$$\tau_{12} = \tau_{21} \quad = \quad = -2\eta\, k \, \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} \,,$$

$$\tau_{22} \quad = \quad -p(x_i) - 4\eta\, k \, \frac{x_1\, x_2}{(x_1^2 + x_2^2)^2} \,.$$

f) Acceleration of a fluid particle located on the x_1-axis for $n = -1$ (potential flow):
We start from the first two components of Cauchy's law of motion neglecting body forces,

$$\varrho \, \frac{\mathrm{D}u_1}{\mathrm{D}t} = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} \quad \text{and} \quad \varrho \, \frac{\mathrm{D}u_2}{\mathrm{D}t} = \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} \,. \tag{2}$$

The derivatives of the components of the stress tensor in (2) with respect to the coordinates x_i at $x_2 = 0$ are:

$$\left.\frac{\partial \tau_{11}}{\partial x_1}\right|_{x_2=0} = \left[-\frac{\partial p}{\partial x_1} + 4\eta\, k \left(\frac{x_2}{(x_1^2 + x_2^2)^2} - \frac{2x_1\, x_2\, 2x_1}{(x_1^2 + x_2^2)^3} \right) \right]_{x_2=0}$$

$$= -\frac{\partial p}{\partial x_1} \,,$$

$$\left.\frac{\partial \tau_{12}}{\partial x_2}\right|_{x_2=0} = \left[-2\eta\, k \left(\frac{-2x_2}{(x_1^2 + x_2^2)^2} - \frac{2\,(x_1^2 - x_2^2)\, 2x_2}{(x_1^2 + x_2^2)^3} \right) \right]_{x_2=0}$$

$$= 0 \,,$$

$$\left.\frac{\partial \tau_{21}}{\partial x_1}\right|_{x_2=0} = \left[-2\eta\, k \left(\frac{2x_1}{(x_1^2 + x_2^2)^2} - \frac{2\,(x_1^2 - x_2^2)\, 2x_1}{(x_1^2 + x_2^2)^3} \right) \right]_{x_2=0}$$

$$= \frac{4\eta\, k}{x_1^3} \,,$$

$$\left.\frac{\partial \tau_{22}}{\partial x_2}\right|_{x_2=0} = \left[-\frac{\partial p}{\partial x_2} - 4\eta\, k \left(\frac{x_1}{(x_1^2 + x_2^2)^2} - \frac{2x_1\, x_2\, 2x_2}{(x_1^2 + x_2^2)^3} \right) \right]_{x_2=0}$$

$$= -\frac{\partial p}{\partial x_2} - \frac{4\eta\, k}{x_1^3} \,.$$

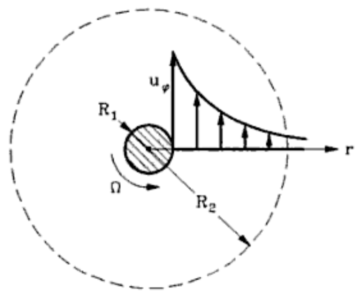
Inserted in (2) and solved for the acceleration components results in

$$\frac{\mathrm{D}u_1}{\mathrm{D}t} = -\frac{1}{\varrho} \frac{\partial p}{\partial x_1} \quad \text{and} \quad \frac{\mathrm{D}u_2}{\mathrm{D}t} = -\frac{1}{\varrho} \frac{\partial p}{\partial x_2} \,,$$

i. e. the acceleration of a fluid particle depends only on the pressure gradient and not on the viscous stresses. Thus, the first Cauchy's law of motion is reduced to Euler's equation. This is always the case in incompressible potential flow where the viscous stresses do not vanish, but the divergence of the viscous stress tensor disappears (see F. M. , page 103).

Problem 4.1-9 Energy balance in a potential vortex flow

An infinitely long circular cylinder with the radius R_1 rotates with a constant angular velocity Ω in a Newtonian fluid (ϱ, $\eta = \text{const}$). Because of the no-slip condition at the cylinder surface, for $r > R_1$ a flow field is generated that corresponds to a potential vortex flow. In cylindrical coordinates, the flow has the velocity components



$$u_r = 0\,, \quad u_\varphi = \frac{\Omega\, R_1^2}{r}\,, \quad \text{and} \quad u_z = 0\,.$$

a) Calculate the circulation Γ by evaluating the line integral at the cylinder circumference.
b) Give the dissipation function Φ in cylindrical coordinates.
(Hint: For Φ the following relation is valid $\Phi = 2\eta\,(e_{rr}^2 + 2e_{r\varphi}^2 + e_{\varphi\varphi}^2)$.)
c) Calculate the dissipated energy per unit time and depth, which in incompressible potential flow without heat conduction equals the change of internal energy E in (V)

$$\frac{\mathrm{D}}{\mathrm{D}t}E = \iiint\limits_{(V)} \Phi \;\mathrm{d}V$$

for (V) per unit depth, sketched as the annular area $S_R = \pi\,(R_2^2 - R_1^2)$ in the figure.
d) Show that the energy per unit time and depth which is dissipated as heat is equal to the power P per unit depth done on the surface of the fluid volume.

Given: ϱ, η, R_1, R_2, Ω

Solution

a) The circulation Γ along the cylinder surface $r = R_1$:
The circulation is defined as

$$\Gamma = \oint_{(C)} \vec{u} \cdot \mathrm{d}\vec{x} \, .$$

On the cylinder surface we have

$$\vec{u} = u_\varphi\big|_{r=R_1} \vec{e}_\varphi = \Omega \, R_1 \, \vec{e}_\varphi \quad \text{and} \quad \mathrm{d}\vec{x} = R_1 \, \mathrm{d}\varphi \, \vec{e}_\varphi \, .$$

We evaluate the line integral and obtain:

$$\Gamma = \int_0^{2\pi} \Omega \, R_1^2 \, \mathrm{d}\varphi = 2\pi \, \Omega \, R_1^2 \, .$$

Inserting this result into the given velocity field leads to the well known velocity field of a potential vortex

$$\vec{u} = \frac{\Gamma}{2\pi \, r} \vec{e}_\varphi \, . \tag{1}$$

b) Dissipation function Φ:
The dissipation function for a plane flow of an incompressible Newtonian fluid in cylindrical coordinates is expressed by

$$\Phi = 2\eta(e_{rr}^2 + 2e_{r\varphi}^2 + e_{\varphi\varphi}^2) \, . \tag{2}$$

We need the components of the rate of deformation tensor appearing in (2) in cylindrical coordinates (see F. M. Appendix B):

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} = 0 \, , \\ e_{\varphi\varphi} &= \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{1}{r} u_r = 0 \, , \\ e_{r\varphi} &= \frac{r}{2} \frac{\partial}{\partial r} \left(\frac{u_\varphi}{r} \right) + \frac{1}{2r} \frac{\partial u_r}{\partial \varphi} = -\frac{\Gamma}{2\pi \, r^2} \, . \end{aligned}$$

Thus, (2) becomes

$$\Phi = \eta \left(\frac{\Gamma}{\pi} \right)^2 \frac{1}{r^4} \, .$$

c) Dissipated energy:
The dissipated energy per unit time and depth is

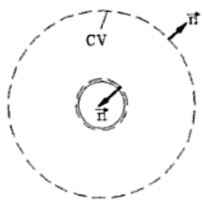
$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}E &= \iiint\limits_{(V)} \Phi\,\mathrm{d}V = \int\limits_{0}^{2\pi}\int\limits_{\mathrm{R}_1}^{\mathrm{R}_2} \eta\left(\frac{\Gamma}{\pi}\right)^2 \frac{1}{r^4}\,r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= \frac{\eta}{\pi}\left(\frac{\Gamma}{R_1}\right)^2\left(1-\left(\frac{R_1}{R_2}\right)^2\right) . \qquad (3)
\end{aligned}$$

The total dissipated energy per unit time and depth is obtained by taking the limit $R_2 \to \infty$:

$$\frac{\mathrm{D}}{\mathrm{D}t}E = \frac{\eta}{\pi}\left(\frac{\Gamma}{R_1}\right)^2 . \qquad (4)$$

d) The power performed on the surface of the fluid volume:
The power of the surface forces per unit depth on the control volume is:

$$P = \iint\limits_{(S)} \vec{u}\cdot\vec{t}\,\mathrm{d}S . \qquad (5)$$



We express the stress vector $\vec{t}$ by the stress tensor $\mathbf{T}$ from $\vec{t} = \vec{n}\cdot\mathbf{T}$. The stress tensor is calculated using the Cauchy-Poisson law for incompressible flow as

$$\mathbf{T} = -p\,\vec{e}_r\vec{e}_r - \frac{\eta\,\Gamma}{\pi\,r^2}\,\vec{e}_r\vec{e}_\varphi - p\,\vec{e}_\varphi\vec{e}_\varphi - \frac{\eta\,\Gamma}{\pi\,r^2}\,\vec{e}_\varphi\vec{e}_r .$$

We evaluate the integral (5) at $r = R_1$ and R_2:
$r = R_1$: At $r = R_1$, $\vec{n} = -\vec{e}_r$. The stress vector at the outer cylinder surface is:

$$\begin{aligned}
\vec{t} &= p\,\vec{e}_r\cdot\vec{e}_r\vec{e}_r + \frac{\eta\,\Gamma}{\pi\,R_1^2}\,\vec{e}_r\cdot\vec{e}_r\vec{e}_\varphi + \\
&\quad + p\,\vec{e}_r\cdot\vec{e}_\varphi\vec{e}_\varphi + \frac{\eta\,\Gamma}{\pi\,R_1^2}\,\vec{e}_r\cdot\vec{e}_\varphi\vec{e}_r \\
&= p\,\vec{e}_r + \frac{\eta\Gamma}{\pi\,R_1^2}\,\vec{e}_\varphi .
\end{aligned}$$

The velocity is $\vec{u} = \Gamma/(2\pi\,R_1)\,\vec{e}_\varphi$. The work per unit time and depth done on the cylinder surface of the control volume is

$$P_{R_1} = \int_0^{2\pi} \frac{\Gamma}{2\pi\, R_1} \frac{\eta\, \Gamma}{\pi\, R_1^2} R_1 \,\mathrm{d}\varphi = \frac{\eta}{\pi} \left(\frac{\Gamma}{R_1}\right)^2 .$$

It is equal to the total dissipated energy per unit time and depth (4).

$r = R_2$: At the outer boundary of the the control volume the unit vector is $\vec{n} = \vec{e}_r$. The stress vector at $r = R_2$ is therefore

$$\vec{t} = -p\,\vec{e}_r - \frac{\eta\, \Gamma}{\pi\, R_2^2}\,\vec{e}_\varphi$$

and the work per unit time and depth done by $\vec{t}$ at $r = R_2$ is

$$P_{R_2} = -\frac{\eta}{\pi} \left(\frac{\Gamma}{R_2}\right)^2 .$$

The power P is then

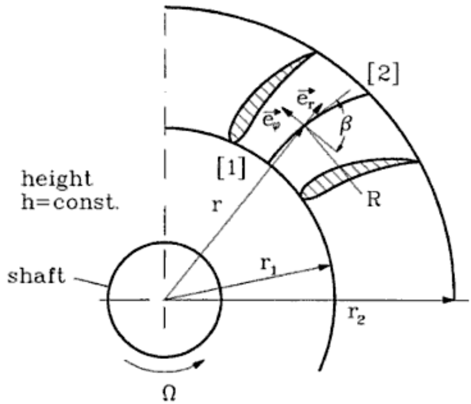
$$P = P_{R_1} + P_{R_2} = \frac{\eta}{\pi} \left(\frac{\Gamma}{R_1}\right)^2 \left(1 - \left(\frac{R_1}{R_2}\right)^2\right)$$

which is equal to the change of internal energy (3).

4.2 Inviscid flow

Problem 4.2-1 Pressure and energy increase of fluid in a centrifugal pump

The sketched radial pump rotates with a constant angular velocity $\vec{\Omega} = \Omega\,\vec{e}_z$.
The pump operates at a steady state with (ϱ = const). The gravitational (body) force $\varrho\,\vec{g} = -\varrho\,g\,\vec{e}_z$ is perpendicular to the drawing plane. The number of the blades mounted on the rotor row is chosen so large that the streamlines have the same curvature as the blade camberline.



a) For an inviscid flow determine the components of the pressure gradient
   – in flow direction $\partial p/\partial\sigma$ and
   – perpendicular to the flow direction $\partial p/\partial n$.
b) Specify the result from a) for the simple case, where the blade angle is $\beta(r) = \text{const} = 90°$.
c) With a given volume flux $\dot{V}$, determine for the above case the energy increase of the fluid.

Given: r_1, r_2, blade angle $\beta(r)$, curvature radius R, Ω, ϱ, $\dot{V}$, h

Solution

a) Components of the pressure gradient:
   The problem is treated in the rotating frame of reference, where the flow is steady. Cauchy's law of motion in the moving reference frame (see F. M. (2.68)) is:

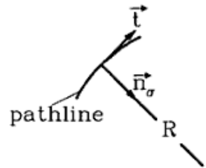
$$\varrho\left(\frac{\mathrm{D}\vec{w}}{\mathrm{D}t}\right)_A = \nabla\cdot\mathbf{T} + \vec{f}\,, \tag{1}$$

   with

$$\vec{f} = -\varrho\,g\,\vec{e}_z - [2\,\varrho\vec{\Omega}\times\vec{w} + \varrho\vec{\Omega}\times(\vec{\Omega}\times\vec{x})]\,.$$

We obtain the acceleration vector in the natural coordinate system (see F. M. (1.24)):

$$\frac{\mathrm{D}(w\vec{t})}{\mathrm{D}t} = \left(\frac{\partial w}{\partial t} + w\,\frac{\partial w}{\partial \sigma}\right)\vec{t} + \frac{w^2}{R}\,\vec{n}_\sigma$$



and in an inviscid flow ($\mathbf{T} = -p\mathbf{I}$) from equation (1) in the form

$$\varrho\left(w\,\frac{\partial w}{\partial \sigma}\,\vec{t} + \frac{w^2}{R}\,\vec{n}_\sigma\right) = -\nabla p + \vec{f}\,.$$

Using the basis vectors of the natural coordinate system, we can write

$$\vec{f} = f_\sigma\,\vec{t} + f_n\,\vec{n}_\sigma + f_b\,\vec{b}_\sigma$$

and obtain (see F. M. (4.43) - (4.45)) the components of Euler's equation in natural coordinates

$$\varrho\,w\,\frac{\partial w}{\partial \sigma} = f_\sigma - \frac{\partial p}{\partial \sigma}\,, \qquad (2)$$

$$\varrho\,\frac{w^2}{R} = f_n - \frac{\partial p}{\partial n}\,, \qquad (3)$$

$$0 = f_b - \frac{\partial p}{\partial b}\,. \qquad (4)$$

The entire body force is, with $\vec{w} = w\,\vec{t}$, $\vec{\Omega} = \Omega\,\vec{e}_z$, $\vec{x} = r\,\vec{e}_r$,

$$\vec{f} = -\varrho\,g\,\vec{e}_z - 2\,\varrho\,\Omega\,w\,(\vec{e}_z \times \vec{t}) - \varrho\,\Omega^2\,r\,\vec{e}_z \times (\vec{e}_z \times \vec{e}_r)\,,$$

and with

$$(\vec{e}_z \times \vec{t}) = -\vec{n}_\sigma, \quad \vec{e}_z \times (\vec{e}_z \times \vec{e}_r) = \vec{e}_z \times \vec{e}_\varphi = -\vec{e}_r, \quad \vec{e}_z = -\vec{b}_\sigma$$

then

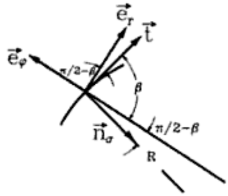
$$\vec{f} = \varrho\,g\,\vec{b}_\sigma + 2\,\varrho\,\Omega\,w\,\vec{n}_\sigma + \varrho\,\Omega^2\,r\,\vec{e}_r\,.$$

Decomposed in path direction therefore

$$f_\sigma = \vec{f} \cdot \vec{t} = \varrho\,\Omega^2\, r \sin\beta \qquad (5)$$

and normal to the path direction

$$f_n = \vec{f} \cdot \vec{n}_\sigma = 2\,\varrho\,\Omega\, w - \varrho\,\Omega^2\, r \cos\beta\,. \qquad (6)$$



Note that the Coriolis force has only a contribution normal to the pathline. From (2) and (3) we obtain with (5) and (6) the components of the pressure gradient as

$$\frac{\partial p}{\partial \sigma} = \varrho\,\Omega^2\, r \sin\beta - \varrho\, w\, \frac{\partial w}{\partial \sigma}\,,$$

$$\frac{\partial p}{\partial n} = 2\,\varrho\,\Omega\, w - \varrho\,\Omega^2\, r \cos\beta - \varrho\, \frac{w^2}{R}\,.$$

b) Special case $\beta(r) = \pi/2$:
The blades are not cambered, i. e. $R \to \infty$ and the pressure gradient is simplified as

$$\frac{\partial p}{\partial \sigma} = \varrho\,\Omega^2\, r - \varrho\, w\, \frac{\partial w}{\partial \sigma}\,, \qquad (7b)$$

$$\frac{\partial p}{\partial n} = 2\,\varrho\,\Omega\, w\,.$$

The flow through the blade channels is now purely radial, therefore, the change in the direction of the pathline corresponds to the change in radial direction. From (7b), we obtain the differential equation:

$$\frac{\partial p}{\partial r} = \varrho\,\Omega^2\, r - \varrho\, w\, \frac{\partial w}{\partial r}\,.$$

It can be integrated along the path (φ =const!) between the station [1] and [2]

$$\int_{r_1}^{r_2} \frac{\partial p}{\partial r} \mathrm{d}r = \int_{r_1}^{r_2} \varrho\,\Omega^2\, r\, \mathrm{d}r - \int_{r_1}^{r_2} \varrho\, w\, \frac{\partial w}{\partial r} \mathrm{d}r$$

and delivers the pressure difference

$$p_2 - p_1 = \varrho\,\Omega^2 \left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right) - \varrho \left(\frac{w_2^2}{2} - \frac{w_1^2}{2}\right)$$

between the station [1] and [2]. This equation is valid along the pathline and the streamline (steady flow). In fact, we are dealing with a special derivation of Bernoulli's equation in a rotating frame. Convince yourself that the Bernoulli's equation is derived from (1) for the general case.

c) Energy increase in the pump fluid:
From the energy equation (see F. M. (2.113))

$$\frac{\mathrm{D}}{\mathrm{D}t}(K + E) = P + \dot{Q}$$

we find for $\dot{Q} = 0$ in a control volume, which is placed in an frame at rest (work is done by the blades only in the frame at rest)

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{(V(t))}\left(\frac{\vec{c}\cdot\vec{c}}{2} + e\right)\varrho\,\mathrm{d}V = \iiint\limits_{(V)}\varrho\,\vec{c}\cdot\vec{k}\,\mathrm{d}V + \iint\limits_{(S)}\vec{c}\cdot\vec{t}\,\mathrm{d}S\,.$$

In an incompressible, frictionless, nonheatconducting flow, the change of internal energy disappears (see F. M. (9.61)) and we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{(V(t))}\varrho\,e\,\mathrm{d}V = \iiint\limits_{(V(t))}\varrho\,\frac{\mathrm{D}e}{\mathrm{D}t}\,\mathrm{d}V = 0\,.$$

The power of body forces disappears also. This, because the inner product of the absolute velocity

$$\vec{c} = \vec{w} + \vec{\Omega}\times\vec{x} = w\,\vec{e}_r + \Omega\,r\,\vec{e}_\varphi$$

and the body force vector

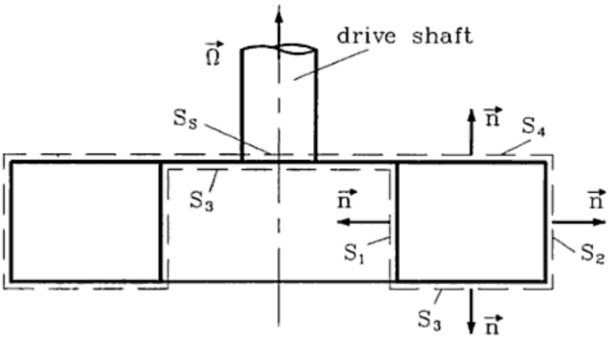
$$\vec{k} = -g\,\vec{e}_z$$

is zero. From Reynolds' transport theorem we have

$$\frac{\partial}{\partial t}\iiint\limits_{(V)}\frac{\vec{c}\cdot\vec{c}}{2}\varrho\,\mathrm{d}V + \iint\limits_{(S)}\varrho\,\frac{\vec{c}\cdot\vec{c}}{2}(\vec{c}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)}\vec{c}\cdot\vec{t}\,\mathrm{d}S\,.$$

Since the kinetic energy within the chosen control volume is time independent, if the angular velocity $\vec{\Omega}$ is constant, we have

$$\frac{\partial}{\partial t}\iiint\limits_{(V)}\frac{\vec{c}\cdot\vec{c}}{2}\varrho\,\mathrm{d}V = 0\,.$$



For a sufficiently large number of blades (small spacing), the flow quantities can be considered as constant over the inlet surface S_1 and the exit surface S_2. This allows the evaluation of

$$\iint\limits_{S_1} \varrho\,\frac{\vec{c}\cdot\vec{c}}{2}(\vec{c}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{S_2} \varrho\,\frac{\vec{c}\cdot\vec{c}}{2}(\vec{c}\cdot\vec{n})\,\mathrm{d}S + \iint\limits_{S_3+S_4+S_S} \varrho\,\frac{\vec{c}\cdot\vec{c}}{2}(\vec{c}\cdot\vec{n})\,\mathrm{d}S$$

$$= \iint\limits_{S_1} \vec{c}\cdot\vec{t}\,\mathrm{d}S + \iint\limits_{S_2} \vec{c}\cdot\vec{t}\,\mathrm{d}S + \iint\limits_{S_3+S_4} \vec{c}\cdot\vec{t}\,\mathrm{d}S + \iint\limits_{S_S} \vec{c}\cdot\vec{t}\,\mathrm{d}S$$

in a closed form. We realize that the third term on the left hand side is zero, because $\vec{c}\cdot\vec{n} = 0$ and the third term on the right hand side because of $\vec{t} = -p\,\vec{n}$. With the known volume flux $\dot{V}$, it then follows:

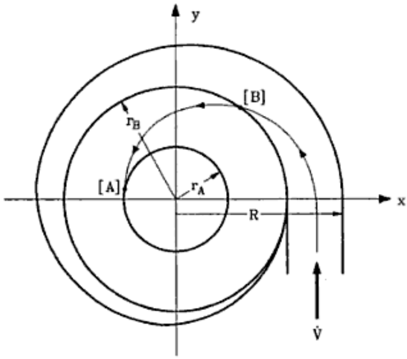
$$-\frac{\varrho}{2}\left(w_1^2 + \Omega^2 r_1^2\right)\dot{V} + \frac{\varrho}{2}\left(w_2^2 + \Omega^2 r_2^2\right)\dot{V} = p_1\,\dot{V} - p_2\,\dot{V} + \iint\limits_{S_S} \vec{c}\cdot\vec{t}\,\mathrm{d}S\,.$$

The remaining surface integral is the power which is transferred to the fluid by the shaft. A simple rearrangement gives

$$P_{B\rightarrow Fl.} = \frac{\varrho}{2}\,\dot{V}\left[\left(w_2^2 + \Omega^2 r_2^2\right) - \left(w_1^2 + \Omega^2 r_1^2\right)\right] + \dot{V}(p_2 - p_1)\,.$$

The first term on the right hand side represents the increase of the kinetic energy. The second one is the increase of pressure energy.

Problem 4.2-2 Pressure distribution within a spiral casing



The sketched spiral casing has a volume flux of $\dot{V}$. The fluid has a constant density and is inviscid. The spiral casing (constant height h) is designed such that the moment of momentum per unit mass $\vec{x} \times \vec{c}$ is constant over the circumference. Body forces are neglected. For the atmospheric pressure p_0 at the exit $[A]$, calculate the pressure p_B at the radial location r_B.

Given: R, r_A, r_B, $\dot{V}$, p_0, ϱ

Solution

Bernoulli's equation applied to a streamline from $[B]$ to $[A]$ and with $p_A = p_0$ (no body forces) gives

$$p_B + \frac{\varrho}{2}\, c_B^2 = p_0 + \frac{\varrho}{2}\, c_A^2 \,, \tag{1}$$

where $c^2 = c_r^2 + c_\varphi^2$ at station $[A]$ and $[B]$ is still unknown. From the volume flux through the annular cross section at r, we have

$$c_r = -\frac{\dot{V}}{2\,\pi\, r\, h} = -\frac{A}{r} \qquad \text{with} \qquad A = \frac{\dot{V}}{2\,\pi\, h} \,. \tag{2}$$

Since the fluid is inviscid and since there are no blades inside the casing, no torque is exerted on the fluid. So Euler's turbine equation gives

$$r\, c_u = K \qquad \text{or} \qquad c_u = c_\varphi = \frac{K}{r} \,.$$

The constant K is calculated from the condition that the outer contour of the casing is a streamline. The differential equation of this streamline

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{c_r}{c_\varphi} = -\frac{A}{K}$$

can be integrated immediately:

$$\ln r = -\frac{A}{K}\varphi + \ln C \,.$$

The initial condition $r(\varphi = 0) = R$ results in

$$\ln\frac{R}{r} = \frac{A}{K}\varphi \,,$$

and the unknown constant K is calculated from the condition $r(\varphi = 2\pi) = r_B$ as

$$K = \frac{2\pi A}{\ln\frac{R}{r_B}} \,. \tag{3}$$

Thus, we finally obtain

$$c_A^2 = (A^2 + K^2)\frac{1}{r_A^2} \,, \qquad c_B^2 = (A^2 + K^2)\frac{1}{r_B^2}$$

and therefore from Bernoulli's equation (1)

$$p_B = p_0 + \frac{\varrho}{2}(A^2 + K^2)\frac{1}{r_A^2}\left(1 - \left(\frac{r_A}{r_B}\right)^2\right) \,.$$

With (2) and (3) also

$$p_B = p_0 + \frac{\varrho}{2}\left(\frac{\dot{V}}{2\pi r_A h}\right)^2\left[1 + \left(\frac{2\pi}{\ln\frac{R}{r_B}}\right)^2\right]\left(1 - \left(\frac{r_A}{r_B}\right)^2\right) \,,$$

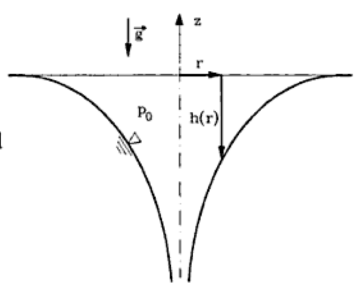
which shows that for $r_A \to 0$ a considerable pressure is produced.

Problem 4.2-3 Free surface in a potential vortex

The velocity potential of a potential vortex is:

$$\Phi = U_0 r_0 \arctan\left(\frac{y}{x}\right) = U_0 r_0 \varphi \,.$$

Calculate the velocity distribution and the lowering of the free surface of the potential vortex.



Given: U_0, r_0, ϱ, $\vec{g}$

Solution

From the given velocity potential $\Phi = U_0 \, r_0 \, \varphi$, we calculate the velocity field in cylindrical coordinates:

$$\begin{aligned} u_r &= \frac{\partial \Phi}{\partial r} = 0 \, , \\ u_\varphi &= \frac{1}{r}\frac{\partial \Phi}{\partial \varphi} = U_0 \, \frac{r_0}{r} \, , \\ u_z &= \frac{\partial \Phi}{\partial z} = 0 \, . \end{aligned}$$

The divergence of the velocity field disappears

$$\nabla \cdot \vec{u} = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \, ,$$

thus the flow is incompressible ($\mathrm{D}\varrho/\mathrm{D}t = 0$). If we take ϱ as an absolute constant, Bernoulli's equation in the form

$$p + \frac{\varrho}{2}\,u^2 + \varrho \, g \, z = \text{const}$$

is applicable, with

$$u^2 = u_\varphi^2 = U_0^2 \left(\frac{r_0}{r}\right)^2 \qquad \text{and} \qquad \vec{g} = -g \, \vec{e}_z \, .$$

Since we are dealing with a potential flow, Bernoulli's equation can be applied between arbitrary locations within the flow field. We apply it between a point on the free surface at ($r \to \infty$, $z = 0$, $p = p_0$) and a point on the free surface at ($z = -h(r)$, $p = p_0$):

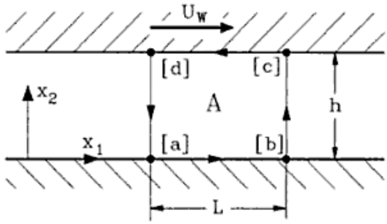
$$p_0 + \frac{\varrho}{2}\,u^2(r \to \infty) = p_0 + \frac{\varrho}{2}\,u^2(r) - \varrho \, g \, h(r)$$

and arrive with $u^2(r \to \infty) = 0$ at

$$h(r) = \frac{u^2(r)}{2\,g} = \frac{U_0^2}{2\,g}\,\left(\frac{r_0}{r}\right)^2 \, .$$

Problem 4.2-4 Circulation in a Couette flow

Newtonian fluid with constant density and viscosity occupies the space between two infinitely extended flat plates. The bottom plate is stationary, while the top one is moving with $U_W\,\vec{e}_1$.



a) Obtain the velocity distribution in the channel.

b) Calculate the circulation of the closed curve that includes the area A
   1) by evaluation of the line integral,
   2) by Stokes' integral theorem.

Given: U_W, L, h

Solution

a) Velocity distribution in the channel:
We are dealing with a simple shear flow or Couette flow, thus

$$\Rightarrow \qquad u_1 = \frac{U_W}{h}\,x_2\,, \quad u_2 = u_3 = 0\,.$$

b) Circulation along the line $\overline{abcda}$:
   1) Line integral:

$$\begin{aligned}\Gamma &= \oint\limits_{(C)} \vec{u}\cdot \mathrm{d}\vec{x} = \int\limits_a^b u_1\,\mathrm{d}x_1 + \int\limits_b^c u_2\,\mathrm{d}x_2 + \int\limits_c^d u_1\,\mathrm{d}x_1 + \int\limits_d^a u_2\,\mathrm{d}x_2 \\ &= \int\limits_c^d U_W\,\mathrm{d}x_1 \\ &= -U_W\,L\,.\end{aligned}$$

   2) Stokes' integral theorem:

$$\Gamma = \oint\limits_{(C)} \vec{u}\cdot\mathrm{d}\vec{x} = \iint\limits_{(S)} (\operatorname{curl}\vec{u})\cdot\vec{n}\,\mathrm{d}S\,.$$

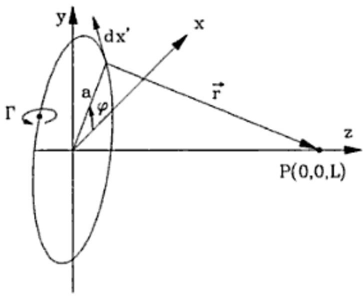
C is the line $\overline{abcda}$, S is the enclosed area A with the normal vector $\vec{e}_3$. With

$$\operatorname{curl}\vec{u} = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\vec{e}_3 = -\frac{U_W}{h}\,\vec{e}_3$$

we obtain

$$\Gamma = \iint_{(A)} -\frac{U_W}{h}\,\vec{e}_3 \cdot \vec{n}\,\mathrm{d}S = -U_W\,L\,.$$

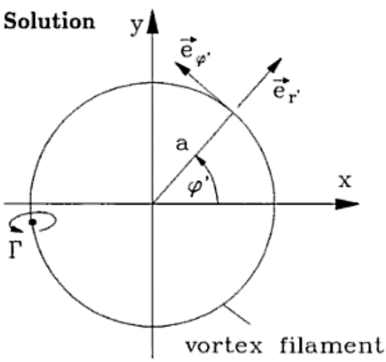
Problem 4.2-5 Velocity induced by a vortex ring



A circular vortex filament with the radius a and constant circulation Γ induces at point $P(0,0,L)$ the velocity $\vec{u}(0,0,L)$. Calculate this velocity by using the Biot-Savart law.

Given: a, Γ, L

Solution



In Biot-Savart's law

$$\vec{u} = \frac{\Gamma}{4\,\pi}\int_{\text{(filament)}} \frac{\mathrm{d}\vec{x}' \times \vec{r}}{r^3}$$

we have for the present case

$$\mathrm{d}\vec{x}' = a\,\mathrm{d}\varphi'\,\vec{e}_\varphi{}'\,,$$

and for $\vec{r} = \vec{x} - \vec{x}'$ we get

$$\vec{r} = L\,\vec{e}_z - a\,\vec{e}_r{}'\,,$$

$$\text{where} \qquad r^2 = L^2 + a^2\,.$$

From

$$\mathrm{d}\vec{x}' \times \vec{r} = a\,\mathrm{d}\varphi'\,\vec{e}_\varphi{}' \times (L\,\vec{e}_z - a\,\vec{e}_r{}')$$

we therefore obtain

$$\mathrm{d}\vec{x}' \times \vec{r} = (a\,L\,\vec{e}_r{}' + a^2\,\vec{e}_z)\,\mathrm{d}\varphi'\,.$$

Thus, the induced velocity can be written as

$$\vec{u} = \frac{\Gamma}{4\pi} \int\limits_{\varphi'=0}^{2\pi} \frac{a\, L\, \vec{e}_r{}'(\varphi') + a^2\, \vec{e}_z}{(L^2 + a^2)^{\frac{3}{2}}}\, \mathrm{d}\varphi'$$

or

$$\vec{u} = \frac{\Gamma}{4\pi} \frac{1}{(L^2 + a^2)^{\frac{3}{2}}} \left[a\, L \int\limits_{\varphi'=0}^{2\pi} \vec{e}_r{}'(\varphi')\, \mathrm{d}\varphi' + a^2\, \vec{e}_z \int\limits_{\varphi'=0}^{2\pi} \mathrm{d}\varphi' \right] .$$

The first integral disappears, since

$$\vec{e}_r{}' = \cos\varphi'\, \vec{e}_x + \sin\varphi'\, \vec{e}_y$$

and $\cos\varphi'$ as well as $\sin\varphi'$ integrated over a full period becomes zero. The second delivers 2π and we get

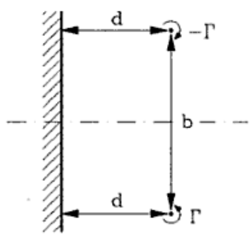
$$\vec{u} = \frac{\Gamma\, a^2}{2\,(L^2 + a^2)^{\frac{3}{2}}}\, \vec{e}_z\ .$$

The maximum velocity is obtained for $L = 0$, i. e. in the center of the circle:

$$\vec{u}_{max} = \frac{\Gamma}{2\,a}\, \vec{e}_z\ .$$

Problem 4.2-6 Two infinitely long vortex filaments near a wall

At the time $t = t_0$ two infinitely long vortex filaments, a distance b apart are located at a distance d from the wall.

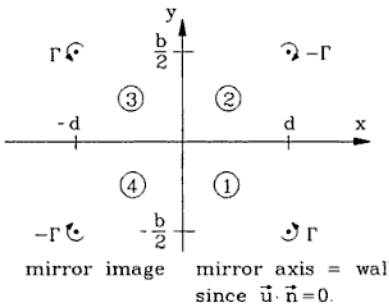


a) Satisfy the boundary condition ($\vec{u} \cdot \vec{n} = 0$) by using the method of images.
b) Calculate the velocities of the vortex filaments at $t = t_0$.
c) Sketch the path of both vortices qualitatively.

Given: d, b, Γ

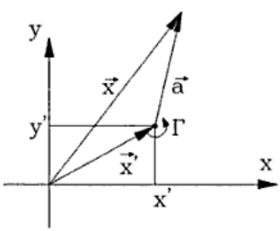
Solution

a) We replace the wall in the flow field (kinematic boundary condition $\vec{u} \cdot \vec{n} = 0$) using the method of images (see figure).



Thus, the kinematic condition $\vec{u} \cdot \vec{n} = 0$ on the y–axis (wall) is satisfied, because the normal component of the velocity induced by one vortex filament is compensated by the normal component of its image.

b)



A vortex filament with the circulation $+\Gamma$ located at point x', y' induces at point x, y the velocity

$$\vec{u} = \frac{\Gamma}{2\,\pi\, a}\, \vec{e}_z \times \frac{\vec{a}}{a}\,,$$

with $\vec{a} = \vec{x} - \vec{x}'$. The induced velocity is therefore

$$\vec{u} = \frac{\Gamma}{2\,\pi[(x - x')^2 + (y - y')^2]}\, [-(y - y')\, \vec{e}_x + (x - x')\, \vec{e}_y]\,.$$

At an arbitrary point $P(x, y)$, the vortex 1 induces the velocity

$$\vec{u}_1 = \frac{\Gamma}{2\,\pi \left[(x-d)^2 + \left(y + \frac{b}{2}\right)^2\right]} \left[-\left(y + \frac{b}{2}\right) \vec{e}_x + (x - d)\, \vec{e}_y\right]\,,$$

the contribution of vortex 2 at the same point P is

$$\vec{u}_2 = \frac{-\Gamma}{2\,\pi \left[(x-d)^2 + \left(y - \frac{b}{2}\right)^2\right]} \left[-\left(y - \frac{b}{2}\right) \vec{e}_x + (x - d)\, \vec{e}_y\right]\,.$$

Similarly for vortex 3 we have

$$\vec{u}_3 = \frac{\Gamma}{2\pi\left[(x+d)^2 + \left(y - \frac{b}{2}\right)^2\right]}\left[-\left(y - \frac{b}{2}\right)\vec{e}_x + (x+d)\,\vec{e}_y\right]$$

and finally for vortex 4 the velocity at P is

$$\vec{u}_4 = \frac{-\Gamma}{2\pi\left[(x+d)^2 + \left(y + \frac{b}{2}\right)^2\right]}\left[-\left(y + \frac{b}{2}\right)\vec{e}_x + (x+d)\,\vec{e}_y\right] .$$

The total induced velocity at point P is the superposition of the individual velocities:

$$\vec{u}_P = \vec{u}_1 + \vec{u}_2 + \vec{u}_3 + \vec{u}_4 .$$

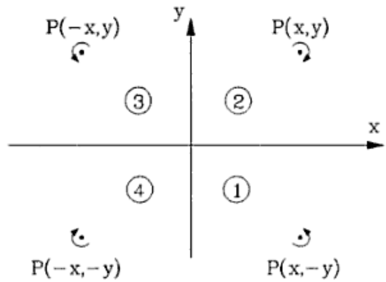
Since a straight vortex filament does not induce a translational velocity onto itself, the induced velocity at point $P(x = d, y = -b/2)$ is

$$\begin{aligned}\vec{u}_{P_1} &= 0 - \frac{\Gamma}{2\pi b}\vec{e}_x + \frac{\Gamma}{2\pi\,(4\,d^2+b^2)}(b\,\vec{e}_x + 2d\,\vec{e}_y) - \frac{\Gamma}{2\pi\,2\,d}\,\vec{e}_y \\ &= \frac{\Gamma}{2\pi}\left[\left(-\frac{1}{b} + \frac{b}{4\,d^2+b^2}\right)\vec{e}_x + \left(-\frac{1}{2\,d} + \frac{2\,d}{4\,d^2+b^2}\right)\vec{e}_y\right] .\end{aligned}$$

For $P(x = d, y = +b/2)$, we obtain the induced velocity on vortex 2

$$\begin{aligned}\vec{u}_{P_2} &= -\frac{\Gamma}{2\pi b}\vec{e}_x + 0 + \frac{\Gamma}{2\pi\,2\,d}\vec{e}_y - \frac{\Gamma}{2\pi\,(4\,d^2+b^2)}(-b\,\vec{e}_x + 2d\,\vec{e}_y) \\ &= \frac{\Gamma}{2\pi}\left[\left(-\frac{1}{b} + \frac{b}{4\,d^2+b^2}\right)\vec{e}_x + \left(\frac{1}{2\,d} - \frac{2\,d}{4\,d^2+b^2}\right)\vec{e}_y\right] .\end{aligned}$$

c) The equations for $\vec{u}_{P_1}$ and $\vec{u}_{P_2}$ are not only valid for the configuration $x = d$ and $y = b/2$ but also they are valid for any arbitrary x and y, which satisfy the same symmetry conditions.
If the induced velocities at x, y are known, we can immediately establish the equation of motion (see F. M. (4.145)). This is for vortex 2



$$
\begin{aligned}
\vec{u}_{P_2} &= \frac{\Gamma}{2\pi}\left[\left(-\frac{1}{2y}+\frac{2y}{4(x^2+y^2)}\right)\vec{e}_x+\left(\frac{1}{2x}-\frac{2x}{4(x^2+y^2)}\right)\vec{e}_y\right] \\
&= \frac{\mathrm{d}x}{\mathrm{d}t}\,\vec{e}_x+\frac{\mathrm{d}y}{\mathrm{d}t}\,\vec{e}_y
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\Gamma}{4\pi}\left(\frac{y^2-x^2-y^2}{y(x^2+y^2)}\right), \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\Gamma}{4\pi}\left(\frac{x^2+y^2-x^2}{x(x^2+y^2)}\right).
\end{aligned}
$$

Thus, we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y^3}{x^3},$$

with the general solution

$$\frac{1}{y^2}+\frac{1}{x^2}=\frac{1}{C^2}.$$

To determine the constants we prescribe the initial condition $y(x=d)=b/2$, and get the integration constant as

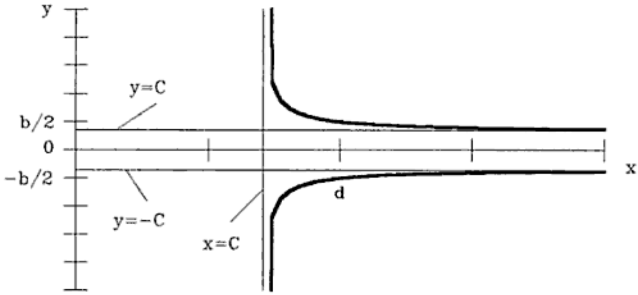
$$C=\sqrt{\frac{d^2\,b^2}{4\,d^2+b^2}}.$$

The explicit solution

$$y=\frac{C\,x}{\sqrt{x^2-C^2}}$$

shows immediately that the vortices can approach each other up to a finite distance different from zero:

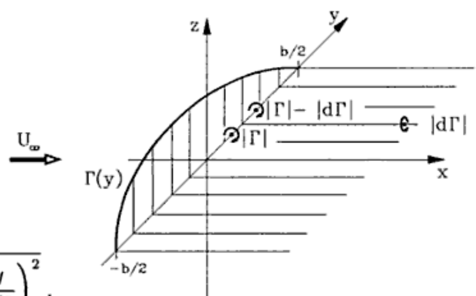
$$y\Big|_{x\to\infty} = C = \frac{d\,b}{\sqrt{4\,d^2 + b^2}}\,.$$



Problem 4.2-7 Wing with an elliptic spanwise distribution of circulation

To calculate the distribution of lift and induced velocity of a wing, we employ the model of an elliptic circulation distribution in spanwise direction:

$$\Gamma(y) = -\Gamma_0\sqrt{1-\left(\frac{y}{b/2}\right)^2}\,.$$



According to the first Helmholtz vortex theorem (F. M. , Chapter 4.2) at any point y' an infinitesimal free vortex with the strength $\mathrm{d}\Gamma = (\mathrm{d}\Gamma/\mathrm{d}y')\,\mathrm{d}y'$ is generated.

a) Calculate the induced velocity $w(y)$ (downwash) induced by the free vortices at the location of the bound vortex.

b) Calculate the induced angle of attack $\alpha_{ind} = w/U_\infty$ at the location of the bound vortex.

c) Using the Kutta-Joukowski theorem

$$L = \varrho U_\infty \int\limits_{-b/2}^{b/2} -\Gamma(y)\,\mathrm{d}y$$

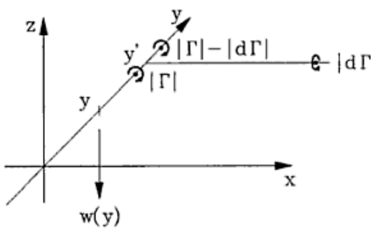
determine the lift of the wing.

d) Calculate the induced drag force D_{ind}.

Hint: $$\int\limits_{-1}^{1} \frac{t\,\mathrm{d}t}{(t-a)(1-t^2)^{\frac{1}{2}}} = \pi \qquad \text{for} \qquad -1 < a < +1.$$

Solution

a) Distribution of induced velocity $w(y)$:



At y' the value of the circulation of the wing is reduced by

$$\mathrm{d}\Gamma = \frac{\mathrm{d}\Gamma}{\mathrm{d}y'}\mathrm{d}y' \;=\; -\Gamma_0 \frac{-2\left(\frac{y'}{b/2}\right)}{2\sqrt{1-\left(\frac{y'}{b/2}\right)^2}}\,\frac{\mathrm{d}y'}{b/2}$$

$$\text{with} \quad \eta' \;=\; \frac{y'}{b/2}$$

$$\mathrm{d}\Gamma \;=\; \Gamma_0 \frac{\eta'\,\mathrm{d}\eta'}{\sqrt{1-\eta'^2}} \,.$$

The change of circulation requires, by the first Helmholz vortex theorem, that an infinitesimal vortex of the strength $\mathrm{d}\Gamma$ leaves the wing. This semi-infinite vortex induces at $(0, y, 0)$ a velocity of the magnitude

$$\mathrm{d}w = \frac{\mathrm{d}\Gamma}{4\,\pi\,(y'-y)} = \frac{\Gamma_0}{4\,\pi\,b/2}\,\frac{\eta'\,\mathrm{d}\eta'}{(\eta'-\eta)\sqrt{1-\eta'^2}} \,.$$

Integrating over all y', i. e. from $\eta' = -1$ to $+1$, we find the induced downwash at y as

$$w(y) \;=\; \frac{\Gamma_0}{2\,\pi\,b}\int\limits_{-1}^{1} \frac{\eta'\,\mathrm{d}\eta'}{(\eta'-\eta)\sqrt{1-\eta'^2}} \tag{1}$$

$$= \frac{\Gamma_0}{2b} = \text{const} .$$

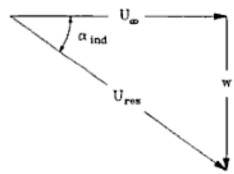
Note: To calculate Cauchy's principle value of the integral (1), we substitute $\eta' = \cos\varphi'$ with $0 \le \varphi' \le \pi$ and $\eta = \cos\varphi$ with $0 \le \varphi \le \pi$. The result is a Glauert integral (see F. M. (10.382) for $n = 1$):

$$\int_{-1}^{1} \frac{\eta' \, \mathrm{d}\eta'}{(\eta' - \eta)\sqrt{1 - \eta'^2}} = \int_{\pi}^{0} \frac{\cos\varphi'(-\sin\varphi')}{(\cos\varphi' - \cos\varphi)\sin\varphi'} \, \mathrm{d}\varphi'$$

$$= \int_{0}^{\pi} \frac{\cos\varphi'}{(\cos\varphi' - \cos\varphi)} \, \mathrm{d}\varphi' = \pi .$$

For an elliptic distribution of the circulation the downwash is thus constant along the span. It may be shown that the induced drag is then a minimum.

b) Induced angle of attack α_{ind}:



$$\tan\alpha_{ind} = \frac{w}{U_\infty} \quad \text{or}$$

$$\alpha_{ind} = \frac{w}{U_\infty} = \frac{\Gamma_0}{2\,b\,U_\infty} ,$$

since in general $\frac{w}{U_\infty} \ll 1$.

c) Lift of the wing:
Kutta-Joukowski theorem furnishes

$$L = \varrho U_\infty \int_{-b/2}^{b/2} -\Gamma(y) \, \mathrm{d}y = \varrho \Gamma_0 U_\infty \frac{b}{2} \int_{-1}^{1} \sqrt{1 - \eta^2} \, \mathrm{d}\eta$$

$$= \varrho \Gamma_0 U_\infty \frac{b}{2} \frac{1}{2} \left(\eta\sqrt{1 - \eta^2} + \arcsin\eta\right)\Big|_{-1}^{1} ,$$

i. e.

$$L = \varrho \Gamma_0 U_\infty b \frac{\pi}{4}$$

and the lift coefficient becomes

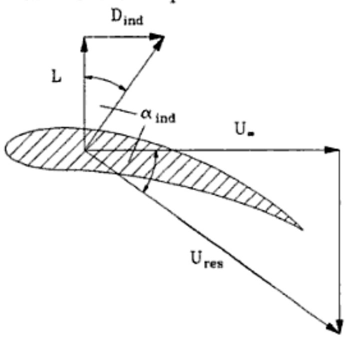
$$c_L = \frac{L}{\frac{\varrho}{2} U_\infty^2 S} = \frac{\pi}{2} \frac{\Gamma_0 b}{U_\infty S} ,$$

with S as the wing surface. We write with $\Lambda = b^2/S$

$$c_L = \frac{\pi}{2}\frac{\Lambda\,\Gamma_0}{U_\infty\, b} = \pi\,\Lambda\,\alpha_{ind}$$

and call Λ the aspect ratio.

d)



The induced drag is the component of the resultant force in the negative flight direction. Thus

$$\begin{aligned} D_{ind} &= L\tan\alpha_{ind} \approx L\,\alpha_{ind} \\ &= \varrho\,\Gamma_0\,U_\infty\,b\,\frac{\pi}{4}\,\frac{\Gamma_0}{2\,b\,U_\infty}\,, \end{aligned}$$

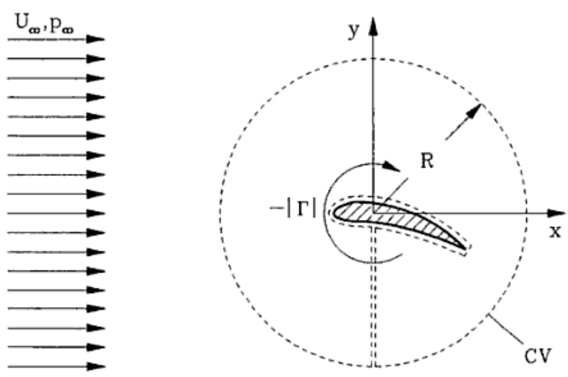
or

$$D_{ind} = \varrho\,\Gamma_0^2\,\frac{\pi}{8}\,.$$

The drag coefficient is

$$c_D = \frac{D_{ind}}{\frac{\varrho}{2}U_\infty^2\,S} = \frac{\pi}{4}\,\frac{\Gamma_0^2}{U_\infty^2\,S} = \frac{\pi}{4}\frac{\Gamma_0^2\,\Lambda}{U_\infty^2\,b^2} = \frac{c_L^2}{\pi\,\Lambda}$$

Problem 4.2-8 Airfoil in parallel flow



The velocity field sufficiently far from the (in z-direction infinitely long) wing may be described by the potential

$$\Phi = U_\infty\,r\,\cos\varphi + \frac{\Gamma}{2\,\pi}\,\varphi\,, \qquad (\Gamma < 0)\,.$$

This potential is the result of the superposition of a parallel flow and a potential vortex of the strength Γ (clockwise). The two-dimensional incompressible flow is inviscid and body forces are negligible. The pressure in the undisturbed flow region is p_∞.

a) Calculate the velocity and pressure field in cylindrical coordinates.
b) Determine the force per unit of depth exerted on the wing by using the balance of momentum.

Given: U_∞, Γ, ϱ, p_∞

Solution

a) Velocity and pressure field in cylindrical coordinates:

$$\vec{u} = \nabla\Phi = \frac{\partial \Phi}{\partial r}\,\vec{e}_r + \frac{1}{r}\,\frac{\partial \Phi}{\partial \varphi}\,\vec{e}_\varphi$$

$$\Rightarrow \qquad u_r = U_\infty \cos\varphi \;, \qquad u_\varphi = -U_\infty \sin\varphi + \frac{\Gamma}{2\pi r}\;. \tag{1}$$

The magnitude of the velocity is calculated as

$$u^2 = u_r^2 + u_\varphi^2 = U_\infty^2 - \frac{\Gamma\,U_\infty}{\pi r}\sin\varphi + \frac{\Gamma^2}{4\pi^2 r^2}\;. \tag{2}$$

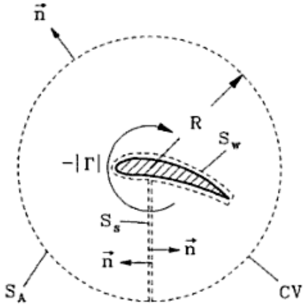
From Bernoulli's equation ($\psi = 0$, $\varrho = \text{const}$, $\partial/\partial t = 0$) we have

$$\begin{aligned} p(r\to\infty) + \frac{\varrho}{2}u^2(r\to\infty) &= p + \frac{\varrho}{2}u^2 \\ \Rightarrow \qquad p &= p_\infty - \frac{\varrho}{2}(u^2 - U_\infty^2) \end{aligned}$$

and with (2)

$$p = p_\infty - \frac{\varrho}{2}\left(-\frac{\Gamma\,U_\infty}{\pi r}\sin\varphi + \frac{\Gamma^2}{4\pi^2 r^2}\right)\;. \tag{3}$$

b) Force on the wing:



Balance of momentum:

$$\iint_{(S)} \varrho \vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = \iint_{(S)} \vec{t}\,\mathrm{d}S\,.$$

Decomposition of the control surface: $S = S_A + S_W + S_S$.
The integrals over both sides of the slot cancel each other, the momentum flux through the wing surface S_W is zero. The normal vector at S_A is $\vec{n} = \vec{e}_r$. Since $\vec{t} = -p\,\vec{n}$ (inviscid), we obtain

$$\vec{F} = \iint_{S_A} -p\,\vec{e}_r\,\mathrm{d}S - \iint_{S_A} \varrho(u_r^2\,\vec{e}_r + u_r u_\varphi\,\vec{e}_\varphi)\,\mathrm{d}S\,. \tag{4}$$

With $\mathrm{d}S = R\mathrm{d}\varphi$ and $p = p_\infty - (\varrho/2)(u_r^2 + u_\varphi^2 - U_\infty^2)$ we get

$$\vec{F} = \int_0^{2\pi} \left\{\left(-p_\infty + \frac{\varrho}{2}(u_r^2 + u_\varphi^2 - U_\infty^2) - \varrho\,u_r^2\right)\,\vec{e}_r - \varrho\,u_r\,u_\varphi\,\vec{e}_\varphi\right\} R\,\mathrm{d}\varphi$$

$$\Rightarrow\ \vec{F} = -\int_0^{2\pi}\left(p_\infty + \frac{\varrho}{2}U_\infty^2\right)\vec{e}_r\,R\,\mathrm{d}\varphi +$$

$$+\ \varrho\int_0^{2\pi}\left\{\frac{1}{2}\,(u_\varphi^2 - u_r^2)\,\vec{e}_r - u_r\,u_\varphi\,\vec{e}_\varphi\right\} R\,\mathrm{d}\varphi\,.$$

The first integral disappears ($\vec{e}_r = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y$), so that with u_r and u_φ from (1), it then follows

$$\vec{F} = \varrho\int_0^{2\pi}\left\{\frac{1}{2}\left(U_\infty^2(\sin^2\varphi - \cos^2\varphi) - \frac{\Gamma\,U_\infty}{\pi\,R}\,\sin\varphi + \frac{\Gamma^2}{4\,\pi^2\,R^2}\right)\vec{e}_r\right.$$

$$\left. +\ \left(U_\infty^2\,\sin\varphi\,\cos\varphi - \frac{\Gamma\,U_\infty}{2\,\pi\,R}\,\cos\varphi\right)\vec{e}_\varphi\right\} R\,\mathrm{d}\varphi\,.$$

Since $\vec{e}_r = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y$ and $\vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$, the integral reduces to

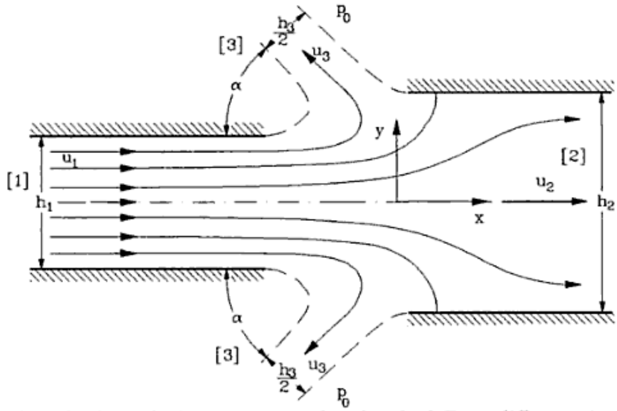
$$\vec{F} = \varrho \int_0^{2\pi} -\frac{\Gamma\,U_\infty}{2\,\pi\,R}(\sin\varphi\,\vec{e}_r + \cos\varphi\,\vec{e}_\varphi)\,R\,\mathrm{d}\varphi$$

and in accordance with the Kutta-Joukowski theorem, we obtain

$$\vec{F} = -\varrho\,\Gamma\,U_\infty\,\vec{e}_y\,, \quad (\Gamma < 0)\,.$$

For the flow and wing geometry sketched in the figure the circulation Γ is negative. This means that the force acts on the wing in positive y-direction.

Problem 4.2-9 Jet angle in a Betz diffuser



Fluid with the velocity u_1 enters the sketched Betz diffuser. At station [3] one portion of the entering mass flow is ejected into the environment (pressure p_0) at an angle α. We assume a two-dimensional inviscid potential flow, where Bernoulli's constant is the same on every streamline. Due to symmetry, the force on the diffuser in y-direction, F_y, is zero. The force on the diffuser in x-direction, F_x, is zero, because the flow is inviscid.

a) Calculate the thickness $h_3/2$ of the ejected jet at [3].

b) Calculate the pressure differences $p_1 - p_0$ and $p_2 - p_0$.

c) Considering $F_x = 0$, calculate the angle α.

Given: u_1, u_2, u_3, h_1, h_2, ϱ, p_0

Solution

a) Jet thickness $h_3/2$:

From the continuity equation in integral form we find

$$u_1\,h_1 = u_2\,h_2 + u_3\,h_3$$

$$\Rightarrow \qquad \frac{h_3}{2} = \frac{1}{2}\left(\frac{u_1}{u_3}\,h_1 - \frac{u_2}{u_3}\,h_2\right)\ . \tag{1}$$

b) Pressure differences $p_1 - p_0$ and $p_2 - p_0$:

The jet pressure at station [3] p_3 is equal to the ambient pressure p_0, since the streamlines are straight there. Bernoulli's equation for station [1] and [3] is

$$p_1 + \frac{\varrho}{2}\,u_1^2 = p_0 + \frac{\varrho}{2}\,u_3^2$$

$$\Rightarrow \qquad p_1 - p_0 = \frac{\varrho}{2}\left(u_3^2 - u_1^2\right)\ . \tag{2}$$

For station [1] and [2] we get:

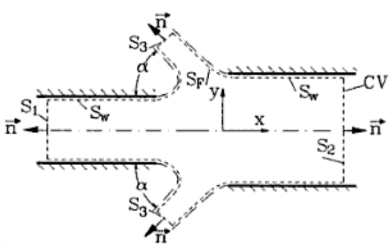
$$p_2 + \frac{\varrho}{2}\,u_2^2 \ = \ p_1 + \frac{\varrho}{2}\,u_1^2$$

$$\Rightarrow\ p_2 - p_0 \ = \ \frac{\varrho}{2}\left(u_1^2 - u_2^2\right) + (p_1 - p_0)$$

and with (2)

$$p_2 - p_0 = \frac{\varrho}{2}\left(u_3^2 - u_2^2\right)\ . \tag{3}$$

c) The jet angle α:



The balance of momentum in x-direction furnishes

$$\iint\limits_{(S)} \varrho\,\vec{u}\cdot\vec{e}_x\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S = \iint\limits_{(S)} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S\ . \tag{4}$$

Decomposition of the control surfaces: $S = S_F + S_W + S_1 + S_2 + S_3$.
The momentum flux integral over the wall surface S_W and the jet

boundaries disappears, since in both cases the normal vector is perpendicular to the velocity vector. We evaluate equation (4) per unit of depth as

$$-\varrho u_1^2 h_1 + \varrho u_2^2 h_2 - \varrho u_3^2 h_3 \cos\alpha = p_1 h_1 - p_2 h_2 + p_0 (h_2 - h_1)\,. \quad (5)$$

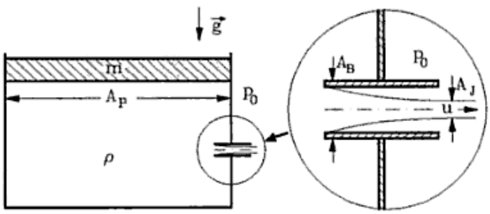
With (2), (3), and

$$h_3 = \frac{u_1}{u_3} h_1 - \frac{u_2}{u_3} h_2$$

the cosine of the desired jet angle α follows from (5) to

$$\cos\alpha = \left[1 + \left(\frac{u_2}{u_3}\right)^2\right] \frac{h_2}{2h_3} - \left[1 + \left(\frac{u_1}{u_3}\right)^2\right] \frac{h_1}{2h_3}\,.$$

Problem 4.2-10 Contraction coefficient of a Borda mouthpiece



A tank has a "Borda mouthpiece" (see figure) as an exit. The fluid (density ϱ) is under a piston of mass m and flows steadily through the Borda mouthpiece (cross-section A_B) with an exit velocity u (jet cross-section A_J). The ratio A_B/A_P is much smaller than 1, so the flow may be considered as quasisteady. Moreover body forces are neglected.

a) Find the exit velocity u.

b) Calculate the contraction coefficient $\alpha = A_J/A_B$.

Given: A_P, m, g, ϱ, p_0

Solution

a) Exit velocity u:

The pressure at the lower surface of the piston is obtained from force balance on the piston in the vertical direction

$$p_P = p_0 + \frac{m\,g}{A_P}\,. \quad (1)$$

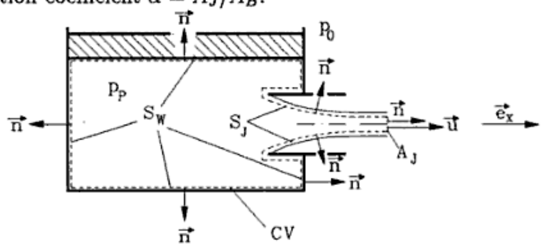
Since we neglected body forces in the liquid, the pressure p_P is constant inside the tank. The pressure in the jet is equal to the ambient pressure p_0 if the streamline curvature is neglected. Since $A_B/A_P \ll 1$, the velocity inside the tank can be taken as zero. Thus Bernoulli's equation is written as

$$p_P = p_0 + \frac{\varrho}{2} u^2 \, .$$

The exit velocity is obtained from (1):

$$u = \sqrt{\frac{2\,m\,g}{\varrho\,A_P}} \, . \qquad (2)$$

b) Contraction coefficient $\alpha = A_J/A_B$:



Momentum equation in x-direction:

$$\iint\limits_{(S)} \varrho \vec{u} \cdot \vec{e}_x \, (\vec{u} \cdot \vec{n}) \, \mathrm{d}S = \iint\limits_{(S)} \vec{t} \cdot \vec{e}_x \, \mathrm{d}S \, .$$

Decomposition of control surfaces: $S = S_W + S_J + A_J$

$$\Rightarrow \iint\limits_{S_W} \varrho \, \vec{u} \cdot \vec{e}_x \, (\vec{u} \cdot \vec{n}) \, \mathrm{d}S + \iint\limits_{S_J} \varrho \, \vec{u} \cdot \vec{e}_x \, (\vec{u} \cdot \vec{n}) \, \mathrm{d}S + \iint\limits_{A_J} \varrho \, \vec{u} \cdot \vec{e}_x \, (\vec{u} \cdot \vec{n}) \, \mathrm{d}S =$$

$$\iint\limits_{S_W} \vec{t} \cdot \vec{e}_x \, \mathrm{d}S - \iint\limits_{S_J} p_0 \, \vec{n} \cdot \vec{e}_x \, \mathrm{d}S - \iint\limits_{A_J} p_0 \, \vec{n} \cdot \vec{e}_x \, \mathrm{d}S \, . \qquad (3)$$

Since the fluid within the tank is at rest, the integral over the wall surface S_W disappears. Furthermore the momentum flux over the jet boundaries also disappears, since $\vec{u}$ is normal to $\vec{n}$. The remaining surface integral on the left hand side is $\varrho\, u^2 A_J$. We now evaluate the right hand side:
First integral:

$$\iint\limits_{S_W} \vec{t} \cdot \vec{e}_x \, \mathrm{d}S \; = \; - \iint\limits_{S_W} \underbrace{p_P}_{p_P = const} \vec{n} \cdot \vec{e}_x \, \mathrm{d}S$$

$$= -p_P \iint_{A_B} \underbrace{\vec{n} \cdot \vec{e}_x}_{-1} \, \mathrm{d}S$$

$$= p_P \, A_B \, .$$

$p_P \, A_B$ is the resultant force due to p_P on the wall opposite to A_B.
Second integral:

$$-\iint_{S_J} p_0 \, \vec{n} \cdot \vec{e}_x \, \mathrm{d}S = -\iint_{A_B - A_J} p_0 \, \mathrm{d}S = -p_0 \, (A_B - A_J) \, .$$

Third integral:

$$-\iint_{A_J} p_0 \, \vec{n} \cdot \vec{e}_x \, \mathrm{d}S = -p_0 \, A_J \, .$$

Inserting into equation (3) results in

$$\varrho \, u^2 \, A_J = p_P \, A_B - p_0 \, (A_B - A_J) - p_0 \, A_J = (p_P - p_0) \, A_B \, .$$

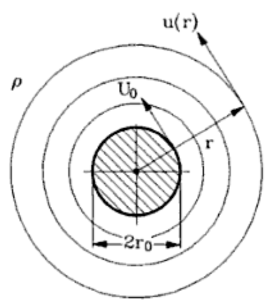
With the pressure difference $p_P - p_0 = \varrho \, u^2/2$ from part a) (equation (1)) we compute the contraction coefficient to

$$\alpha = \frac{1}{2} \, .$$

Problem 4.2-11 Pressure distribution in an inviscid and axisymmetric flow

Given is the velocity distribution in an inviscid, axisymmetric plane flow:

$$u(r) = U_0 \left(\frac{r}{r_0} \right)^n \, .$$



a) Determine the pressure distribution $p(r)$ with $p(r_0) = p_0$.

b) Find the exponent n, such that the Bernoulli constant assumes the same value throughout the flow field.

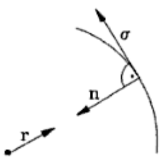
Given: r_0, U_0, n, ϱ, p_0

Solution

a) Pressure distribution $p(r)$:
The steady Euler equation without body forces is written in natural coordinates (see F. M. (4.43), (4.44)):

$$\varrho\, u\, \frac{\partial u}{\partial \sigma} = -\frac{\partial p}{\partial \sigma} \qquad \Rightarrow \qquad \text{in direction of pathline,} \tag{1}$$

$$\varrho\, \frac{u^2}{R} = -\frac{\partial p}{\partial n} \qquad \Rightarrow \qquad \text{normal to pathline.} \tag{2}$$



n should not be confused with the exponent n, its sign is positive when it is oriented toward the center of curvature. Since for symmetry reasons $\partial/\partial\sigma \equiv 0$, equation (1) is satisfied. In equation (2) we use the correspondences

$$n \hat{=} -r \qquad \text{and} \qquad R \hat{=} r\,,$$

and write (2) as an ordinary differential equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\varrho\, u^2}{r} \qquad \text{with} \qquad p(r_0) = p_0 \tag{3}$$

which now determines the pressure distribution. We insert the given velocity distribution and integrate over r to obtain

$$p(r) - p_0 = \frac{\varrho\, U_0^2}{r_0^{2n}} \int\limits_{r_0}^{r} \bar{r}^{2n-1}\, d\bar{r}\,. \tag{4}$$

Case study:

$n \neq 0$:

$$p(r) - p_0 = \frac{\varrho\, U_0^2}{2n} \left(\left(\frac{r}{r_0} \right)^{2n} - 1 \right)\,. \tag{5}$$

$n = 0$:

$$p(r) - p_0 = \varrho\, U_0^2 \ln \frac{r}{r_0}\,.$$

b) Determination of n for potential flow:
The streamlines are concentric circles with $r = 0$ as their center. Bernoulli's equation is valid on each streamline

$$p + \frac{\varrho}{2} u^2 = \text{const} = C\,.$$

We consider two streamlines:
For $r = r_0$

$$p_0 + \frac{\varrho}{2} U_0^2 = C_0 \tag{6}$$

and in general for $r > r_0$ using the given velocity distribution:

$$p(r) + \frac{\varrho}{2} U_0^2 \left(\frac{r}{r_0}\right)^{2n} = C(r) \, . \tag{7}$$

If the constant is to be the same on every streamline, the left hand side of equation (6) and (7) must be equal:

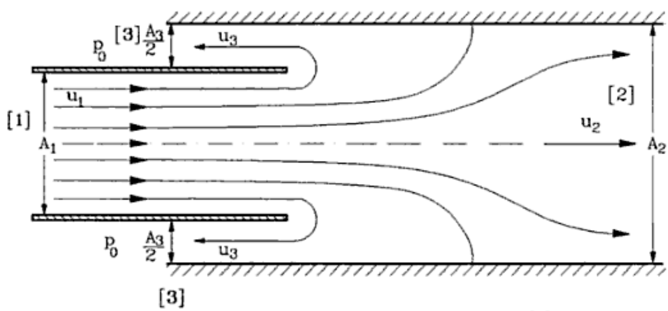
$$\begin{aligned} \Rightarrow \qquad p(r) - p_0 &= \frac{\varrho}{2} U_0^2 - \frac{\varrho}{2} U_0^2 \left(\frac{r}{r_0}\right)^{2n} \\ &= -\frac{\varrho U_0^2}{2} \left(\left(\frac{r}{r_0}\right)^{2n} - 1\right) \, . \end{aligned}$$

We compare with (5) and find

$$n = -1 \, .$$

$$\Rightarrow \qquad u(r) = U_0 \, \frac{r_0}{r} \, .$$

Problem 4.2-12 Increase of static pressure in a Betz diffuser



Air with constant density enters a Betz diffuser at station [1] with area A_1. 15% of the entering air mass flux is symmetrically ejected at [3] into the atmosphere with the ambient pressure p_0. At station [2] the streamlines are parallel. Since the pressure decreases along the wall in flow direction we do not anticipate flow separation at the diffuser walls. We may thus assume a steady flow without losses.

a) 1) Calculate the pressure difference $p_2 - p_1$.
 2) Calculate the total pressure change p_{tot} from [1] to [2].
b) Find the exit area A_3.

Given: $A_1 = 0.04\,\mathrm{m}^2$, $A_2 = 0.07\,\mathrm{m}^2$, $u_1 = 20\,\mathrm{m/s}$, $p_1 - p_0 = 100\,\mathrm{Pa}$, $\varrho = 1.15\,\mathrm{kg/m}^3$

Solution

a) Pressure difference $p_2 - p_1$ and $p_{tot_2} - p_{tot_1}$:
 1) Only 85% of the inlet mass flux $\dot{m}_1$ exits at [2]. The continuity equation furnishes

$$\dot{m}_2 = n\,\dot{m}_1 \qquad \text{with} \qquad n = 0.85\,.$$

$$\Rightarrow \qquad \varrho\, u_2\, A_2 = n\,\varrho\, u_1\, A_1$$

$$\Rightarrow \qquad \frac{u_2}{u_1} = n\,\frac{A_1}{A_2}\,. \tag{1}$$

Bernoulli's equation applied to a streamline between [1] and [2] leads to

$$p_1 + \frac{\varrho}{2}\,u_1^2 = p_2 + \frac{\varrho}{2}\,u_2^2$$

$$\Rightarrow \qquad p_2 - p_1 = \frac{\varrho}{2}\,u_1^2\left(1 - \left(\frac{u_2}{u_1}\right)^2\right)\,,$$

$$\text{with (1) follows:} \qquad p_2 - p_1 = \frac{\varrho}{2}\,u_1^2\left(1 - \left(n\,\frac{A_1}{A_2}\right)^2\right)$$

$$= 175.7\,\mathrm{N/m}^2\,.$$

 2) The total pressure is defined as

$$p_{tot} = p + \frac{\varrho}{2}\,u^2\,.$$

By assumption no losses between station [1] and [2] will occur:

$$p_2 + \frac{\varrho}{2}\,u_2^2 = p_1 + \frac{\varrho}{2}\,u_1^2$$

$$\Rightarrow \qquad p_{tot_2} = p_{tot_1}\,.$$

b) Exit area A_3:
First we calculate the exit velocity at station [3] using the continuity equation:

$$(1-n)\,\varrho\, u_1\, A_1 = \varrho\, u_3\, A_3$$

$$\Rightarrow \qquad \frac{u_3}{u_1} = (1-n)\,\frac{A_1}{A_3}\,. \tag{2}$$

The pressure at station [3] is equal to the ambient pressure ($p_3 = p_0$). Applying Bernoulli's equation to a streamline between station [1] and [3], it then follows:

$$p_1 + \frac{\varrho}{2}\,u_1^2 = p_0 + \frac{\varrho}{2}\,u_3^2$$

$$\Rightarrow \qquad \left(\frac{u_3}{u_1}\right)^2 = 1 - \frac{p_0 - p_1}{\frac{\varrho}{2}\,u_1^2}\,,$$

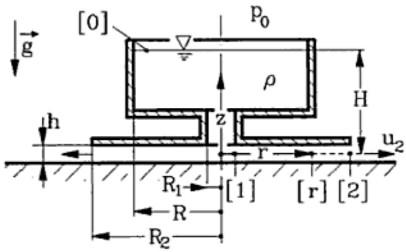
$$\text{with (2)}: \qquad \left((1-n)\,\frac{A_1}{A_3}\right)^2 = 1 - \frac{p_0 - p_1}{\frac{\varrho}{2}\,u_1^2}$$

$$\Rightarrow \qquad A_3 = A_1\,\frac{1-n}{\sqrt{1 - \frac{p_0 - p_1}{\frac{\varrho}{2}\,u_1^2}}}$$

$$= 50\,\mathrm{cm}^2\,.$$

Problem 4.2-13 Fluid flowing out of a tank

Fluid with constant density ϱ leaves steadily a tank of radius R. The exit consists of two circular disks with the radius R_2.



a) Find the volume flux $\dot V$.
b) Give the pressure distribution between the disks as a function of r.
c) Find the ratio R_2/R_1 such that the fluid at station [1] does not vaporize. The vapor pressure is p_V.

Given: H, h, R, R_2, p_0, ϱ, g

Solution

a) Calculation of exit volume flux $\dot V$:
The continuity equation for $\varrho = \mathrm{const}$ is

$$\dot V = u\,A = \mathrm{const}\,.$$

Thus :

$$\dot{V} = u_2 \, 2\pi \, R_2 \, h \qquad (1)$$

$$= u_0 \, \pi \, R^2 \, .$$

For the velocity ratio we get

$$\frac{u_0}{u_2} = \frac{2\,h}{R} \frac{R_2}{R} \, . \qquad (2)$$

The unknown exit velocity u_2 is calculated using Bernoulli's equation for the streamline from [0] to [2]:

$$p_0 + \varrho \, g \, z_0 + \frac{\varrho}{2} u_0^2 = p_2 + \varrho \, g \, z_2 + \frac{\varrho}{2} u_2^2 \, .$$

With $p_2 = p_0$, $z_0 = H$ and $z_2 = 0$, it follows

$$u_2^2 \left(1 - \left(\frac{u_0}{u_2}\right)^2\right) = 2 \, g \, H \, .$$

We solve for u_2, replace u_0/u_2 by (2) and find

$$u_2 = \sqrt{\frac{2\,g\,H}{1 - \left(\frac{R_2}{R} \frac{2\,h}{R}\right)^2}} \, . \qquad (3)$$

Introducing the above equation in (1) we obtain the volume flux

$$\dot{V} = \sqrt{\frac{2\,g\,H}{1 - \left(\frac{R_2}{R} \frac{2\,h}{R}\right)^2}} \, 2\pi \, R_2 \, h \, .$$

For $2\,h/R \ll 1$, therefore

$$\dot{V} = \sqrt{2\,g\,H} \, 2\pi \, R_2 \, h \, .$$

b) Pressure distribution $p(r)$:
To calculate the pressure distribution in the plane $z = 0$, we consider a streamline from station [r] extending between the disks to station [2]. Bernoulli's equation for this streamline is

$$p(r) + \frac{\varrho}{2} u(r)^2 = p_0 + \frac{\varrho}{2} u_2^2$$

$$\Rightarrow \quad p_0 - p(r) = \frac{\varrho}{2} \left(u(r)^2 - u_2^2\right) \, . \qquad (4)$$

The velocity $u(r)$ follows from the continuity equation

$$u(r)\,2\pi\,r\,h = u_2\,2\pi\,R_2\,h \qquad \Rightarrow \qquad u(r) = u_2\,\frac{R_2}{r} \tag{5}$$

where the exit velocity u_2 is already known from part a) (equation (3)). Introducing both velocities into (4), we get the pressure distribution:

$$p_0 - p(r) = \varrho\,g\,H\,\frac{\left(\frac{R_2}{r}\right)^2 - 1}{1 - \left(\frac{2\,h\,R_2}{R^2}\right)^2}\,. \tag{6}$$

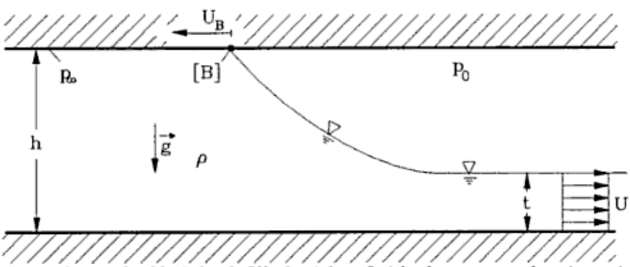
c) The radius ratio to avoid vaporization R_2/R_1:
At station [1] the radius is $r = R_1$ and the pressure should just reach the vapor pressure $p(R_1) = p_V$. Introducing this into (6) and solve for R_2/R_1, we find the maximum radius ratio as

$$\frac{R_2}{R_1} = \sqrt{1 + \frac{p_0 - p_V}{\varrho\,g\,H}\left[1 - \left(\frac{2h}{R}\,\frac{R_2}{R}\right)^2\right]}\,.$$

If the exit area $2\,\pi\,h\,R_2$ is much smaller than the tank surface $\pi\,R^2$, the ratio becomes

$$\frac{R_2}{R_1} = \sqrt{1 + \frac{p_0 - p_V}{\varrho\,g\,H}}\,.$$

Problem 4.2-14 Air bubble moving in a channel

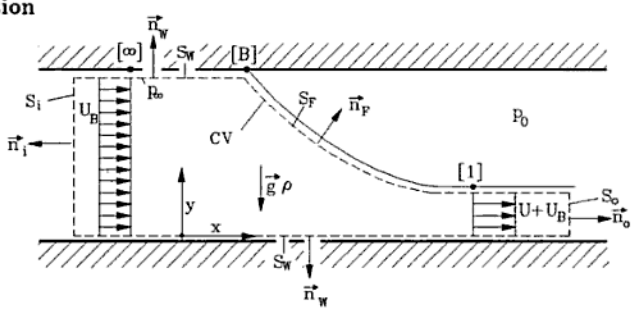


A long, plane channel of height h filled with a fluid of constant density ϱ is completely closed. Suddenly, the right side wall is removed. Under certain circumstances an air bubble enters the channel with velocity U_B which is therefore the velocity of point [B]. The liquid flows out of the channel with the velocity U. Choose a frame moving with velocity U_B. In this frame the flow is steady. Under the assumption of inviscid flow, calculate:

a) the height t of the exiting fluid layer,
b) the velocity U_B of point [B],
c) the exit velocity U, and
d) the pressure p_∞ at the upper channel boundary at [∞] far from [B].

Given: h, p_0, ϱ, g

Solution



a) The height t of the exiting fluid layer:
We choose a frame that is attached to point [B] and moves with U_B to the left. Point [B] becomes then a stagnation point. This is so because otherwise the velocity at [B] would be discontinuous. We place the control volume as shown in the figure. The continuity equation requires:

$$U_B\, h = (U + U_B)\, t \; . \tag{1}$$

The balance of momentum in x-direction is

$$\iint\limits_{S_i+S_o+S_F} \varrho\, \vec{u}\cdot\vec{e}_x\, (\vec{u}\cdot\vec{n})\, \mathrm{d}S = \iint\limits_{S_i+S_o+S_F} \vec{t}\cdot\vec{e}_x\, \mathrm{d}S \; . \tag{2}$$

For the inviscid flow the stress vector is $\vec{t} = -p\,\vec{n}$. Because of the body force $\vec{g}$, the pressure p depends on the coordinate y, and the hydrostatic pressure distribution at the inlet and exit cross-section is:

$$p(y) = p_\infty + \varrho\, g\, (h - y) \qquad \text{at} \quad S_i \; ,$$

$$p(y) = p_0 + \varrho\, g\, (t - y) \qquad \text{at} \quad S_o \; .$$

The momentum flux over the free surface S_F is zero, since $\vec{n}_F$ is normal to $\vec{u}$. The pressure at this surface is equal to the ambient pressure p_0. Furthermore, we note that $\vec{n}_F\cdot\vec{e}_x\, dS$ is the projection of the surface element $\mathrm{d}S$ of the free surface in the y-direction and therefore

$$-\iint\limits_{S_F} p\,\vec{n}_F\cdot\vec{e}_x\, \mathrm{d}S = -p_0\,(h - t) \; .$$

We are now in the position to evaluate the momentum equation (2):

$$-\varrho U_B^2 h + \varrho (U + U_B)^2 t = (p_\infty + \frac{\varrho}{2} g h) h - p_0 h - \frac{\varrho}{2} g t^2 . \quad (3)$$

The Bernoulli equation between [1] and [2] along a streamline is

$$p_1 + \frac{\varrho}{2} u_1^2 + \varrho g y_1 = p_2 + \frac{\varrho}{2} u_2^2 + \varrho g y_2 .$$

For the point [∞] and [B] which are located on a streamline we can write

$$p_\infty + \frac{\varrho}{2} U_B^2 = p_B = p_0 . \quad (4)$$

The same way we find for [B] and [1]

$$p_B + \varrho g (h - t) = p_0 + \frac{\varrho}{2} (U + U_B)^2 ,$$

$$\Rightarrow \qquad \varrho g (h - t) = \frac{\varrho}{2} (U + U_B)^2 . \quad (5)$$

The equations (1), (3), (4), and (5) form a system of equations with t, U_B, U, and p_∞ as unknowns. We now eliminate all unknowns except the fluid height t:
From (1)

$$U_B^2 = (U + U_B)^2 \left(\frac{t}{h}\right)^2$$

it follows then with (5)

$$U_B^2 = 2g (h - t) \left(\frac{t}{h}\right)^2 . \quad (6)$$

Inserting (6) in (4) gives

$$p_\infty - p_0 = -\varrho g (h - t) \left(\frac{t}{h}\right)^2 . \quad (7)$$

We introduce (5), (6), and (7) in the momentum equation (3) and get a quadratic equation

$$t^2 - \frac{3}{2} h t + \frac{1}{2} h^2 = 0$$

with two solutions

$$t_1 = h \qquad \text{and} \qquad t_2 = \frac{h}{2} .$$

$t_1 = h$ represents the trivial solution for the state at rest. The non-trivial solution is

$$t = \frac{h}{2} .$$

b) The velocity U_B of point [B] follows from (6) to

$$U_B = \frac{1}{2}\sqrt{g\,h}\ .$$

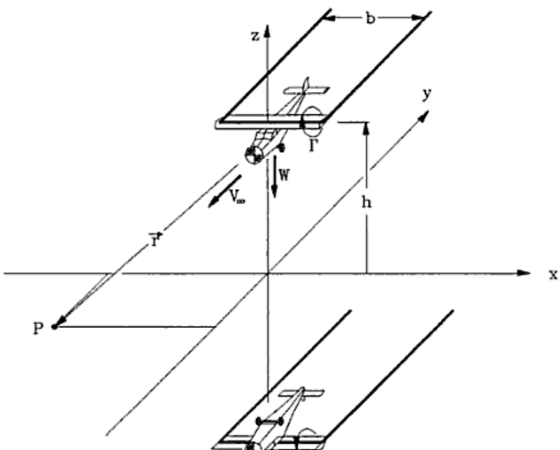
c) The exit velocity U can be calculated from the continuity equation (1). It is equal to the velocity U_B of the bubble moving into the channel:

$$U = U_B = \frac{1}{2}\sqrt{g\,h}\ .$$

d) From Bernoulli's equation (4), we finally obtain the pressure at the upper channel boundary as

$$p_\infty = p_0 - \frac{1}{8}\,\varrho\,g\,h\ .$$

Problem 4.2-15 Aircraft above the ground



An aircraft (weight W, span b) flies with a constant velocity V_∞ in an altitude of h above the ground (x-y-plane). If $b/h \ll 1$, the flow in the farfield and also in the ground region can be described by a horseshoe vortex. This vortex consists of a bound vortex located on the wing at ($-b/2 \le x \le b/2$; $y = 0$; $z = h$), two free vortices that are perpendicular to the wing and reach downstream in the positive y-direction. A second horseshoe vortex is the mirror image of the first one on the x-y-plane. The boundary condition requires that the component of the velocity normal to the ground vanishes. This boundary condition is fulfilled by the mirror imaging.

a) Calculate the pressure which is generated during fly-by at a point P of the ground surface.
b) Show that the pressure force exerted on the ground is equal to the weight of the aircraft.

1. Hint: Consider the problem in an aircraft-fixed cartesian coordinate system.
2. Hint:

$$\int [(a-x)^2+b]^{-\frac{3}{2}}\,\mathrm{d}x = -\frac{a-x}{b\sqrt{(a-x)^2+b}} + \text{const.}$$

Given: h, b, V_∞, p_∞, ϱ, W

Solution

a) The pressure at a point P located on the ground (coordinates x and y) is

$$p(x,y) = p_\infty + \Delta p(x,y)\ . \tag{1}$$

The horseshoe vortices induce the velocities u and v on the ground with

$$u \ll V_\infty \qquad \text{and} \qquad v \ll V_\infty\ . \tag{2}$$

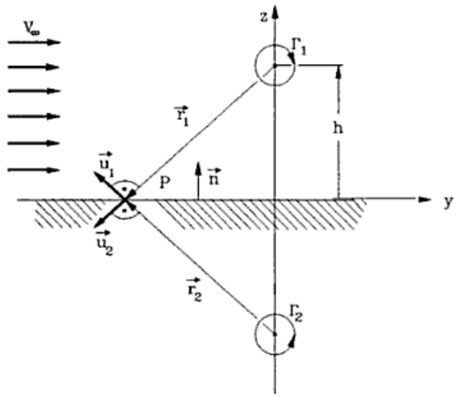
Bernoulli's equation between the point P with the coordinates x, y and a point in the undisturbed flow field is

$$p_\infty + \Delta p(x,\,y) + \frac{\varrho}{2}\left[u^2 + (v+V_\infty)^2\right] = p_\infty + \frac{\varrho}{2}\,V_\infty^2$$

$$\Rightarrow \qquad \Delta p(x,\,y) = -\varrho\,v\,V_\infty - \frac{\varrho}{2}\,(u^2+v^2)\ .$$

Because of (2), we can neglect terms quadratic in the induced velocities:

$$\Delta p(x,\,y) = -\varrho\,v\,V_\infty\ . \tag{3}$$



The induced velocity on the ground $\vec{u}$ has only a component in y-direction. The z-component disappears because of the boundary condition stated above. Furthermore, the velocities induced by the free vortices at the wing ends are neglected. This is so, because on the ground and for $b/h \ll 1$, the effect of both free vortices nearly cancel. Moreover the velocity induced by the free vortices has no component in y-direction. We obtain the velocities induced by the two horseshoe vortices at point P using the Biot-Savart law. The total induced velocity $\vec{u} = v\,\vec{e}_y$ is the sum of the induced velocities $\vec{u}_1$ and $\vec{u}_2$ of the two horseshoe vortices:

$$\begin{aligned} \vec{u} &= \vec{u}_1 + \vec{u}_2 \\ &= \frac{\Gamma_1}{4\pi} \int\limits_{(\text{filament}_1)} \frac{\mathrm{d}\vec{x}'_1 \times \vec{r}_1}{r_1^3} + \frac{\Gamma_2}{4\pi} \int\limits_{(\text{filament}_2)} \frac{\mathrm{d}\vec{x}'_2 \times \vec{r}_2}{r_2^3} \end{aligned} \tag{4}$$

with

$$\vec{x} = x\,\vec{e}_x + y\,\vec{e}_y$$

on the ground.

Vortex 1:

$$\begin{aligned} \Gamma_1 &= -\Gamma\,, \\ \vec{x}'_1 &= x'\,\vec{e}_x + h\,\vec{e}_z \qquad \Rightarrow \qquad \mathrm{d}\vec{x}'_1 = \mathrm{d}x'\,\vec{e}_x\,, \\ \vec{r}_1 &= \vec{x} - \vec{x}'_1 \end{aligned}$$

$$= (x - x')\,\vec{e}_x + y\,\vec{e}_y - h\,\vec{e}_z\ ,$$

$$\mathrm{d}\vec{x'}_1 \times \vec{r}_1 = \det\begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \mathrm{d}x' & 0 & 0 \\ (x-x') & y & -h \end{bmatrix}$$

$$= \mathrm{d}x'\,(h\,\vec{e}_y + y\,\vec{e}_z)\ .$$

Vortex 2:

$$\Gamma_2 = \Gamma\ ,$$

$$\vec{x'}_2 = x'\,\vec{e}_x - h\,\vec{e}_z \qquad \Rightarrow \qquad \mathrm{d}\vec{x'}_2 = \mathrm{d}x'\,\vec{e}_x\ ,$$

$$\vec{r}_2 = \vec{x} - \vec{x'}_2$$

$$= (x - x')\,\vec{e}_x + y\,\vec{e}_y + h\,\vec{e}_z\ ,$$

$$\mathrm{d}\vec{x'}_2 \times \vec{r}_2 = \det\begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \mathrm{d}x' & 0 & 0 \\ (x-x') & y & h \end{bmatrix}$$

$$= \mathrm{d}x'\,(-h\,\vec{e}_y + y\,\vec{e}_z)\ .$$

The magnitude of $\vec{r}_1$ and $\vec{r}_2$ is equal for both vortices:

$$r = \sqrt{(x-x')^2 + y^2 + h^2}\ .$$

The y-component is calculated from (4) as

$$v = \frac{\Gamma_1}{4\pi} \int\limits_{(\text{filament}_1)} \frac{h\,\mathrm{d}x'}{r_1^3} + \frac{\Gamma_2}{4\pi} \int\limits_{(\text{filament}_2)} \frac{-h\,\mathrm{d}x'}{r_2^3}$$

$$\text{or} \qquad v = -\frac{h\,\Gamma}{2\pi} \int\limits_{-b/2}^{b/2} r^{-3}\,\mathrm{d}x'\ ,$$

$$\text{therefore} \qquad v = -\frac{h\,\Gamma}{2\pi} \int\limits_{-b/2}^{b/2} \frac{1}{\left(\sqrt{(x-x')^2 + y^2 + h^2}\right)^3}\,\mathrm{d}x'\ . \qquad (5)$$

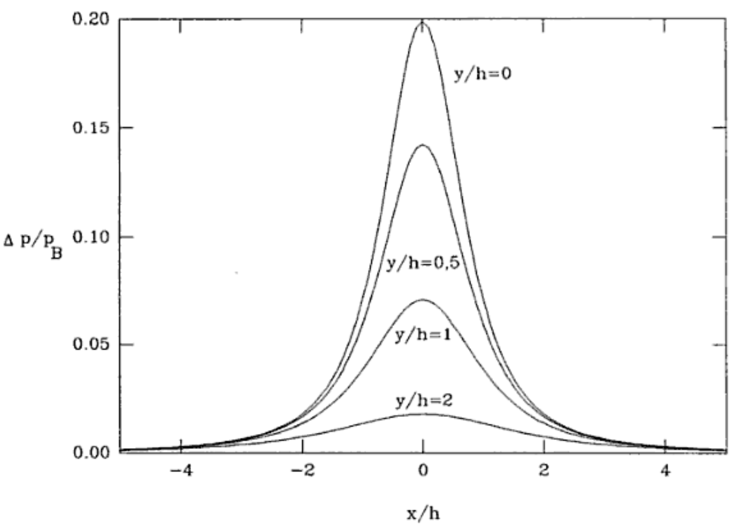
To integrate the above indefinite integral see above hints with $x \hat{=} x'$, $a \hat{=} x$ and $b \hat{=} h^2 + y^2$. The integration leads with (3) to the pressure

$$\Delta p = \frac{\varrho \Gamma V_\infty}{2\pi} \frac{h}{h^2+y^2} \left[\frac{x+b/2}{\sqrt{(x+b/2)^2+(h^2+y^2)}} + \right.$$
$$\left. - \frac{x-b/2}{\sqrt{(x-b/2)^2+(h^2+y^2)}} \right] \tag{6}$$

or

$$\frac{\Delta p}{p_B} = \frac{1}{1+(y/h)^2} \left[\frac{x/h+b/2h}{\sqrt{(x/h+b/2h)^2+(1+(y/h)^2)}} + \right.$$
$$\left. - \frac{x/h-b/2h}{\sqrt{(x/h-b/2h)^2+(1+(y/h)^2)}} \right]$$

with $p_B = \varrho \Gamma V_\infty/(2\pi h)$. In the figure below, $\Delta p/p_B$ is plotted as a function of x/h for $h/b = 5$ and y/h as a parameter.



The pressure distribution on the ground ($z = 0$) is

$$p(x,\, y) = p_\infty + \frac{\varrho\,\Gamma\, V_\infty}{2\pi}\,\frac{h}{h^2 + y^2}\left[\frac{x + b/2}{\sqrt{(x + b/2)^2 + (h^2 + y^2)}} + \right.$$

$$\left. - \frac{x - b/2}{\sqrt{(x - b/2)^2 + (h^2 + y^2)}}\right] .$$

b) The pressure distribution on the ground is symmetric with respect to the x- and y-axis. We obtain the force on the ground by integration

$$\vec{F}_{\text{aircraft}\to\text{ground}} = \iint\limits_{(S)} -\Delta p\, \vec{n}\, \mathrm{d}S .$$

Taking into account $\vec{n} = \vec{e}_z$ (see figure) and the fact that Δp is an even function, the integration can be carried out over the positive quadrant

$$\vec{F}_{\text{aircraft}\to\text{ground}} = -4 \int\limits_0^\infty\int\limits_0^\infty \Delta p(x,\, y)\, \mathrm{d}x\, \mathrm{d}y\, \vec{e}_z .$$

First we perform the integration over x and obtain

$$\int\limits_0^\infty \Delta p(x,\, y)\, \mathrm{d}x = \frac{b\, h\, \varrho\, \Gamma\, V_\infty}{2\pi\, (h^2 + y^2)} .$$

The integration over y gives

$$\int\limits_0^\infty \frac{b\, h\, \varrho\, \Gamma\, V_\infty}{2\pi\, (h^2 + y^2)}\, \mathrm{d}y = \frac{1}{4}\, b\, \varrho\, \Gamma\, V_\infty ,$$

thus

$$\vec{F}_{\text{aircraft}\to\text{ground}} = -\varrho\, \Gamma\, V_\infty\, b\, \vec{e}_z . \tag{7}$$

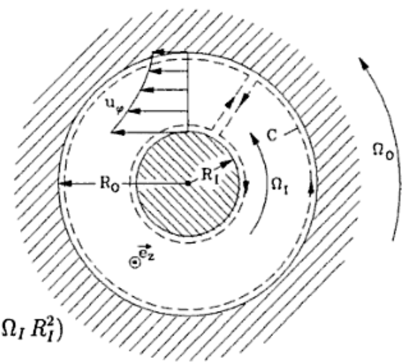
According to the Kutta-Joukowski theorem the lift is

$$\vec{L} = -\varrho\, \Gamma_1\, V_\infty\, b\, \vec{e}_z = \varrho\, \Gamma\, V_\infty\, b\, \vec{e}_z$$

and equal to the negative weight force $\vec{W}$. Thus the pressure distribution carries the weight of the aircraft.

Problem 4.2-16 Flow between two rotating cylinders, circulation and vorticity

Incompressible Newtonian fluid fills an annular gap between two infinitely long cylinders (R_I, R_O) that rotate with angular velocities Ω_I and Ω_O. The velocity field is given in cylindrical coordinates as



$$u_r = 0\,,$$

$$u_\varphi = \frac{1}{R_O^2 - R_I^2}\left[r\left(\Omega_O R_O^2 - \Omega_I R_I^2\right) - \frac{R_O^2 R_I^2}{r}\left(\Omega_O - \Omega_I\right)\right] \quad \text{and}$$

$$u_z = 0\,.$$

a) Show that this flow satisfies the continuity equation.

b) 1) Calculate the circulation along the curve C using

$$\Gamma = \oint_{(C)} \vec{u}\cdot \mathrm{d}\vec{x}$$

2) and using

$$\Gamma = \iint_{(S)} (\operatorname{curl}\vec{u})\cdot\vec{n}\,\mathrm{d}S\,.$$

c) For which ratio Ω_O/Ω_I is the flow irrotational?

Given: R_O, R_I, Ω_O, Ω_I

Solution

a) Continuity equation:

The continuity equation for an incompressible flow is:

$$\begin{aligned} 0 &= \operatorname{div}\vec{u} \\ &= \frac{1}{r}\left[\frac{\partial(u_r\, r)}{\partial r} + \frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial(u_z\, r)}{\partial z}\right]\,. \end{aligned} \qquad (1)$$

u_φ is only a function of r and the velocity components u_z and u_r are zero. The continuity equation (1) is thus satisfied.

b) Circulation Γ along the curve C:
The velocity field is given by

$$\begin{aligned}\vec{u} &= u_\varphi\,\vec{e}_\varphi \\ &= \left[\underbrace{\frac{\Omega_O\,R_O^2-\Omega_I\,R_I^2}{R_O^2-R_I^2}}_{A}\,r-\underbrace{\frac{R_O^2\,R_I^2\,(\Omega_O-\Omega_I)}{R_O^2-R_I^2}}_{B}\,\frac{1}{r}\right]\vec{e}_\varphi \\ &= \left(A\,r-\frac{B}{r}\right)\vec{e}_\varphi\,. \qquad (2)\end{aligned}$$

We are dealing with a plane flow in the r-φ-plane.

1) To calculate

$$\Gamma=\oint\limits_{(C)}\vec{u}\cdot\mathrm{d}\vec{x}\,, \qquad (3)$$

we express the line element $\mathrm{d}\vec{x}$ in cylindrical coordinates

$$\mathrm{d}\vec{x}=\mathrm{d}r\,\vec{e}_r+r\,\mathrm{d}\varphi\,\vec{e}_\varphi$$

and form the scalar product

$$\vec{u}\cdot\mathrm{d}\vec{x}=\left(A\,r-\frac{B}{r}\right)r\,\mathrm{d}\varphi\,.$$

The integrations over the radial curve pieces of C would cancel each other in any case, here they do not contribute to Γ, since the velocity vector $\vec{u}$ and the line element $\mathrm{d}\vec{x}$ are perpendicular to each other. With (3) we have

$$\Gamma=\oint\limits_{\varphi=0}^{2\pi}\left(A\,R_O-\frac{B}{R_O}\right)R_O\,\mathrm{d}\varphi+\oint\limits_{\varphi=2\pi}^{0}\left(A\,R_I-\frac{B}{R_I}\right)R_I\,\mathrm{d}\varphi$$

thus,

$$\Gamma=2\pi\left[\Omega_O\,R_O^2-\Omega_I\,R_I^2\right]\,.$$

2) Circulation using Stokes' integral theorem

$$\Gamma=\oint\limits_{(C)}\vec{u}\cdot\mathrm{d}\vec{x}=\iint\limits_{(S)}(\operatorname{curl}\vec{u})\cdot\vec{n}\,\mathrm{d}S\,. \qquad (4)$$

The curve C represents the boundary of the surface S. The surface normal vector $\vec{n}$ in (4) has to be chosen such that the direction of the

integration as seen from the positive side of the surface is positive counterclockwise. In this case, the normal vector is identical with the unit vector in z-direction, $\vec{e}_z$ (see figure).
Since we are dealing with a plane flow, the rotation of $\vec{u}$ has only one component in z-direction

$$\operatorname{curl}\vec{u} = \frac{1}{r}\left(\frac{\partial(u_\varphi r)}{\partial r} - \underbrace{\frac{\partial u_r}{\partial\varphi}}_{=0}\right)\vec{e}_z$$

and

$$\operatorname{curl}\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(A\,r^2 - B\right)\vec{e}_z$$

or

$$\operatorname{curl}\vec{u} = 2A\,\vec{e}_z \tag{5}$$

The rotation is constant within the entire flow field. The surface integral (4) gives, as expected,

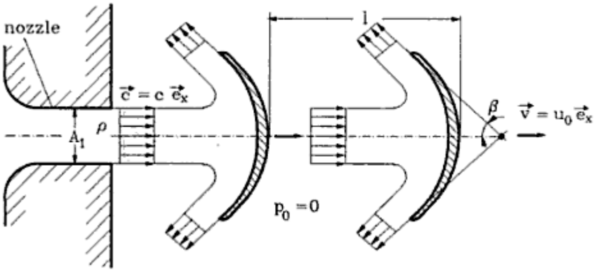
$$\begin{aligned}\Gamma &= 2A\,\vec{e}_z\cdot\vec{e}_z \iint_{(S)} \mathrm{d}S\\ \Gamma &= 2\pi\left[\Omega_O\,R_O^2 - \Omega_I\,R_I^2\right]\ .\end{aligned}$$

c) The flow is irrotational if $\operatorname{curl}\vec{u} = 0$ everywhere. Thus, we obtain the condition for the flow to be irrotational by setting (5) equal to zero:

$$\frac{\Omega_O}{\Omega_I} = \left(\frac{R_I}{R_O}\right)^2\ .$$

Problem 4.2-17 Power of a Pelton turbine

A water jet (density ϱ, cross-section A_1, velocity c) impinges symmetrically on blades of a Pelton turbine with a large diameter/blade height ratio. Thus, the motion of the blades can be considered as purely translational. We assume an inviscid, incompressible flow, where body forces are neglected. After each time $\Delta t_0 = l/u_0$ a new blade is fully exposed to the water jet. The portion of the jet, which is cut off by the new blade, continues to do work on the previously exposed blade as long as its surface remains wetted.



Hint: The force acting on a blade does not depend on the ambient pressure p_0. Without loss of generality, we set this pressure equal to zero.

a) Determine the force acting on a bade.
b) Find the turbine power.
c) Find the ratio c/u_0, for which the power has its maximum value.
d) Determine the efficiency of the plant.
e) Determine the turbine power using the Euler turbine equation.

Given: ϱ, u_0, c, l, A_1, $p_0 = 0$

Solution

a) Blade force:
We apply the balance of momentum in integral form to a blade-fixed control volume and neglect body forces (see F. M. (2.73)),

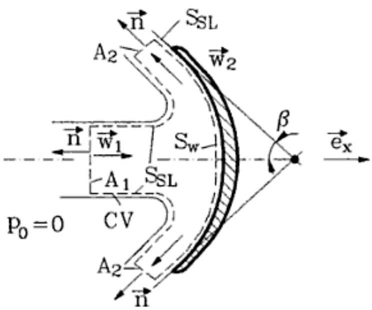
$$\frac{\partial}{\partial t}\left[\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V\right]_A + \iint\limits_{(S)} \varrho\,\vec{c}\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S + \vec{\Omega}\times\iiint\limits_{(V)} \varrho\,\vec{c}\,\mathrm{d}V = \iint\limits_{(S)} \vec{t}\,\mathrm{d}S\;, \tag{1}$$

with the absolute velocity $\vec{c} = \vec{w} + \vec{\Omega}\times\vec{x} + \vec{v}$. For an observer sitting on the blade, i. e. in the relative frame, the flow appears to be steady ($\partial\vec{w}/\partial t = 0$). With $\vec{\Omega} = 0$ (we consider the translational motion) and a constant translational velocity $\vec{v} = u_0\,\vec{e}_x$ of the frame, the first and third term on the left hand side of (1) disappear. Thus, the moving frame is an inertial one. The blade force has only a component F_x in direction of $\vec{e}_x$. Therefore, it is sufficient to apply the balance of momentum (1) in direction of $\vec{e}_x$:

$$\iint\limits_{(S)} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S \;=\; \iint\limits_{(S)} \varrho\,(\vec{w}+\vec{v})\cdot\vec{e}_x\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S\;,$$

Since the frame velocity $\vec{v} = u_0\,\vec{e}_x$ is constant and $\iint\limits_{(S)} \varrho\,(\vec{w}\cdot\vec{n})\;\mathrm{d}S$ vanishes because of the continuity equation, we find

$$\iint\limits_{(S)} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S = \iint\limits_{(S)} \varrho\,\vec{w}\cdot\vec{e}_x\,(\vec{w}\cdot\vec{n})\,\mathrm{d}S\;. \qquad (2)$$



Hence, one could have used the balance of momentum for the moving inertial reference frame. The total surface S is decomposed into the surfaces

$$S = A_1 + A_2 + S_{SL} + S_w\;.$$

$\varrho\,\vec{w}\,(\vec{w}\cdot\vec{n})$ and t_x are equal to zero at the boundaries of the jet S_{SL}. The momentum flux over the blade surface S_w is zero and the integral of t_x over S_w is equal to the negative of the desired blade force, i. e.

$$F_x = -\iint\limits_{S_w} \vec{t}\cdot\vec{e}_x\,\mathrm{d}S = \varrho\,w_1^2\,A_1 + 2\,\varrho\,w_2^2\,\cos\beta\,A_2\;. \qquad (3)$$

At the jet cross sections A_1, A_2, the flow is assumed to be uniform and the pressure equals the ambient pressure p_0. Thus, for an inviscid flow, the magnitudes of the velocity vectors $\vec{w}_1$, $\vec{w}_2$ are equal and the continuity equation gives $A_2 = A_1/2$. The blade force is calculated using the relative jet velocity $w_1 = c - u_0$ from (3) as

$$F_x = \varrho\,(c-u_0)^2\,(1+\cos\beta)\,A_1\;. \qquad (4)$$

b) To calculate the effective power, we take into account that several blades are acted upon simultaneously. Therefore, we decompose the time during which a blade is acted upon by the jet into a period of $\Delta t_0 = l/u_0$ which represents the time intervall between the immersion of two neighboring blades and the time $l/w_1 = l/(c-u_0)$ during which the cut off jet continues to act on the blade. Thus, the total time during which the fluid acts upon one blade is

$$\Delta t = \Delta t_0 + \Delta t_1 = \frac{l}{u_0} + \frac{l}{w_1}\;.$$

During this time, after each intervall Δt_0 another blade will be exposed to the jet. Thus, at any instant of time

$$n = \frac{\Delta t}{\Delta t_0} = \frac{c}{c - u_0} \tag{5}$$

blades are impinged upon. The power of a blade with the surface S_W can be calculated (see F. M. (2.111)) by

$$P_B = \iint_{S_W} u_0 \, \vec{e}_x \cdot \vec{t} \, \mathrm{d}S \, .$$

The velocity u_0 is uniform over S_W and we find using (3) for the power of one blade:

$$P_B = -u_0 \, F_x \, . \tag{6}$$

The power of n blades is then $P = n \, P_B$ or with (4), (5), (6)

$$P = -\varrho \, c^3 \, A_1 \, \frac{u_0}{c} \left(1 - \frac{u_0}{c}\right) (1 + \cos\beta) \, .$$

Since mechanical energy is rejected from the fluid, the power is negative.

c) The turbine power $P_T = -P$ done on the blades has for

$$\frac{\mathrm{d}P_T}{\mathrm{d}(u_0/c)} = \varrho \, c^3 \, A_1 \left(1 - 2 \, \frac{u_0}{c}\right) (1 + \cos\beta) = 0$$

$$\text{and therefore} \qquad \frac{c}{u_0} = 2$$

its maximum value $(\mathrm{d}^2 P_T / \mathrm{d}(u_0/c)^2 < 0)$:

$$P_{T_{max}} = \varrho \, c^3 \, A_1 \, \frac{1 + \cos\beta}{4} \, .$$

d) The maximum efficiency (at $c/u_0 = 2$) of the turbine amounts to

$$\eta_{max} = \frac{P_{Tmax}}{P_{in}} \, ,$$

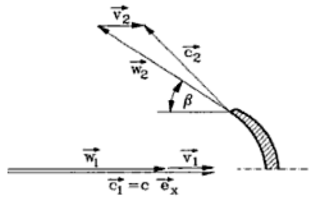
where the energy flux through the nozzle is

$$P_{in} = \iint_{A_1} \varrho \, \frac{\vec{c} \cdot \vec{c}}{2} (\vec{c} \cdot \vec{n}) \, \mathrm{d}S = \frac{\varrho}{2} \, c^3 \, A_1$$

and thus

$$\eta_{max} = \frac{1 + \cos\beta}{2} \, .$$

e) Euler's turbine equation:
We consider the blade system as a Pelton turbine with the blade radius r . The torque on the fluid is



$$T = \dot{m}\, r\,(c_{u2} - c_{u1})\ .$$

The circumferential component of the jet is $c_{u1} = c$. We obtain the circumferential component of the absolute velocity $\vec{c}_2$ from the velocity triangle of the leaving jet:

$$\begin{aligned} c_{u2} &= \vec{c}_2 \cdot \vec{e}_x = (\vec{w}_2 + \vec{v}_2) \cdot \vec{e}_x \\ &= -w_2 \cos\beta + u_0 = -(c - u_0)\cos\beta + u_0\ . \end{aligned}$$

The mass flux that impinges on n blades is

$$\dot{m} = -n \iint\limits_{A_1} \varrho\, \vec{w} \cdot \vec{n}\, \mathrm{d}S\ .$$

This mass flux is identical with the one through the nozzle:

$$\dot{m} = -\iint\limits_{A_1} \varrho\, \vec{c} \cdot \vec{n}\, \mathrm{d}S = \varrho\, c\, A_1\ .$$

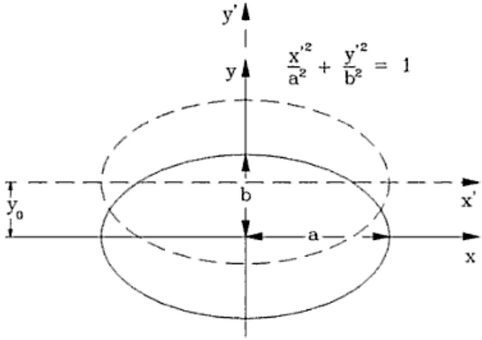
The torque is $T = -\varrho\, A_1\, c\, r\,(c - u_0)\,(1 + \cos\beta)$. Thus, the power $P = T\,\Omega$ with $u_0 = \Omega\, r$ is calculated as

$$P = -\varrho\, c^3\, A_1\, \frac{u_0}{c} \left(1 - \frac{u_0}{c}\right) (1 + \cos\beta)\ .$$

The translational motion we prescribed previously represents here the limit $r \to \infty$, $\Omega \to 0$, but $r\,\Omega = \mathrm{const} = u_0$.

4.3 Initial and Boundary Conditions

Problem 4.3-1 Oscillation of an elliptic cylinder in fluid



An elliptic cylinder oscillates in a fluid. The motion is given by

$$y(x,t) = y'(x') + y_0 \cos\omega t \ .$$

Find the kinematic boundary condition using the following relations

$$\left.\begin{array}{ll} \text{a)} & (u_i - u_{i(w)})\, n_i = 0 \\ \text{b)} & \mathrm{D}F/\mathrm{D}t = 0 \end{array}\right\} \qquad \text{at } F(\vec{x},t) = 0.$$

Given: a, b, y_0, ω

Solution

The surface of the cylinder in a body-fixed (moving with the cylinder) coordinate system x', y' is described by the implicit equation

$$F'(x',y') = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = 0 \ .$$

With the motion

$$y = y' + y_0 \cos\omega t \ , \qquad x = x'$$

we obtain the equation of the surface in a fixed coordinate system x, y as

$$F(x,y,t) = \frac{x^2}{a^2} + \frac{(y - y_0 \cos\omega t)^2}{b^2} - 1 = 0 \ . \tag{1}$$

a) $(u_i - u_{i(w)})\, n_i = 0$:
The application of the above equation to this problem results in

$$(u - 0)\, n_x + (v - v_w)\, n_y = 0$$

and with

$$v_w = \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_w = -y_0\, \omega\, \sin \omega t$$

and

$$\vec{n} = \frac{\nabla F}{|\nabla F|}\,, \qquad \nabla F = \frac{2\,x}{a^2}\,\vec{e}_x + \frac{2(y - y_0\,\cos\omega t)}{b^2}\,\vec{e}_y$$

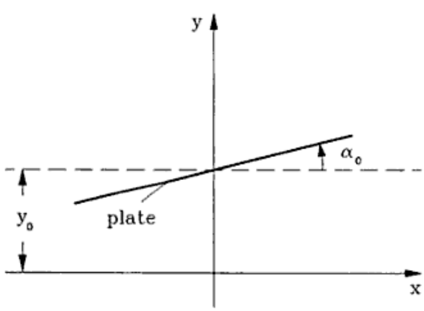
the boundary condition becomes

$$u\,\frac{2\,x}{a^2} + (v + y_0\,\omega\,\sin\omega t)\frac{2(y - y_0\,\cos\omega t)}{b^2} = 0 \qquad \text{(at the wall)}\,.$$

b) $\mathrm{D}F/\mathrm{D}t = 0$ at $F(\vec{x}, t) = 0$
The same result follows immediately

$$\frac{2(y - y_0\,\cos\omega t)(y_0\,\omega\,\sin\omega t)}{b^2} + u\,\frac{2\,x}{a^2} + v\,\frac{2(y - y_0\,\cos\omega t)}{b^2} = 0\,.$$

Problem 4.3-2 Flat plate with a pitching and oscillating motion



The motion of the sketched flat plate is described by a vertical oscillation ($y(t) = y_0 \cos\omega t$) and a small angular oscillation ($\alpha(t) = \alpha_0 \cos\omega t$). The small amplitude allows the approximation

$$\tan\alpha_0 \approx \alpha_0\,.$$

a) Find the implicit equation $F(x, y, t) = 0$ of the plate surface.
b) Give the kinematic boundary condition on plate.

Given: y_0, α_0, ω

Solution

a) Implicit equation of the surface:
The explicit form

$$y(x,\,t) = y_0\,\cos\omega t + x\,\tan(\alpha_0\,\cos\omega t) \approx (y_0 + \alpha_0 x)\,\cos\omega t\,,$$

leads immediately to the implicit form

$$F(x,y,t) = y - (y_0 + \alpha_0\,x)\,\cos\omega t = 0\,.$$

b) Boundary condition on the plate:
From

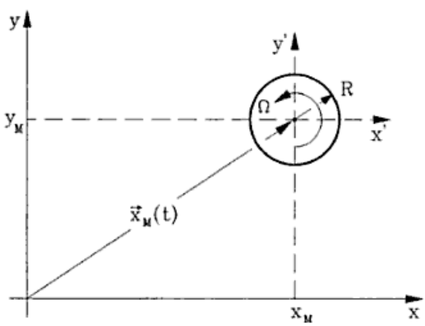
$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0 \qquad \text{at} \qquad F(x,y,t) = 0$$

we find

$$(y_0 + \alpha_0\,x)\omega\,\sin\omega t + u\,(-\alpha_0\,\cos\omega t) + v = 0 \quad \text{at} \quad F(x,y,t) = 0\,.$$

Problem 4.3-3 Rotating cylinder moving through fluid

A cylinder of Radius R rotates with the angular velocity Ω around its axis as it moves along the path $\vec{x}_M(t)$.



a) Find the kinematic
b) and dynamic boundary condition on the cylinder wall.

Given: R, Ω, $\vec{x}_M(t)$

Solution

a) Kinematic boundary condition:

$$\vec{u}\cdot\vec{n}=\vec{u}_w\cdot\vec{n} \qquad \text{or} \qquad \frac{\mathrm{D}F}{\mathrm{D}t}=\frac{\partial F}{\partial t}+u_i\,\frac{\partial F}{\partial x_i}=0 \quad \text{(at the wall)}\,.$$

The implicit equation of the surface is

$$F=x'^2+y'^2-R^2=0\,,$$

with

$$x'=x-x_M \qquad \text{and} \qquad y'=y-y_M$$

it then follows

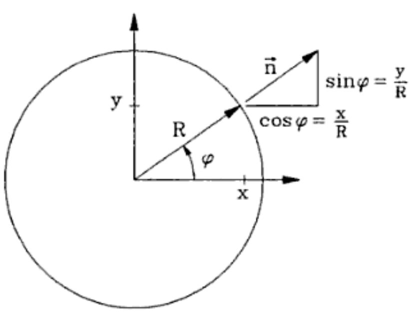
$$F(x,y,t)=(x-x_M)^2+(y-y_M)^2-R^2=0\,.$$

We form the material (substantial) derivative of this equation

$$\frac{\mathrm{D}F}{\mathrm{D}t}=2\,(x-x_M)(-\dot{x}_M)+2\,(y-y_M)(-\dot{y}_M)+u\,2\,(x-x_M)+v\,2\,(y-y_M)\,,$$

where the dot refers to the time derivative. The kinematic boundary condition is

$$(x-x_M)(u-\dot{x}_M)+(y-y_M)(v-\dot{y}_M)=0 \quad \text{at} \quad F(x,y,t)=0\,.$$



For the special case

$$\vec{x}_M(t)\equiv 0\,,$$

and after a division by R, the boundary condition is reduced to

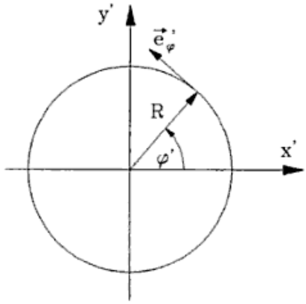
$$\frac{x}{R}\,u+\frac{y}{R}\,v=0\,,$$

or

$$u\,n_x+v\,n_y=\vec{u}\cdot\vec{n}=0 \quad \text{at the cylinder}\,.$$

b) Dynamic boundary condition in general:
We have

$$\vec{u} = \vec{u}_w \qquad \text{at the wall}\,.$$



At the cylinder surface

$$\vec{u}_w = \dot{\vec{x}}_M + \Omega\, R\, \vec{e}_\varphi{}'\,.$$

From $\vec{e}_\varphi{}' = \vec{e}_z \times \vec{n}$ we have

$$\vec{e}_\varphi{}' = -\sin\varphi'\,\vec{e}_x + \cos\varphi'\,\vec{e}_y$$

with

$$\sin\varphi' = \frac{y'}{R} = \frac{y - y_M}{R}$$

$$\cos\varphi' = \frac{x'}{R} = \frac{x - x_M}{R}\,,$$

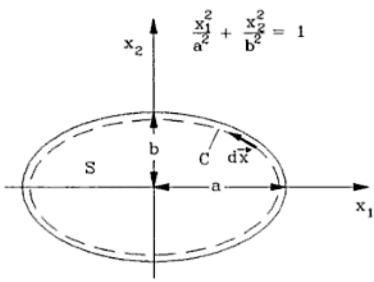
as a result, the boundary condition at the cylinder surface, i. e. at $F(x,y,t) = 0$, assumes the form

$$\vec{u} = \vec{u}_w = (\dot{x}_M - (y - y_M)\,\Omega)\,\vec{e}_x + (\dot{y}_M + (x - x_M)\,\Omega)\,\vec{e}_y\,.$$

Note: The fact that the cylinder rotates is only relevant to the dynamic boundary condition.

Problem 4.3-4 Vortical flow inside an elliptic cylinder

The velocity field of a vortical flow inside an elliptic cylinder is given by:



$$u_1(x_1, x_2) = -\frac{2K\,x_2}{b^2}\,,$$

$$u_2(x_1, x_2) = \frac{2K\,x_1}{a^2}\,.$$

The equation of the cylinder is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\,.$$

a) Show that no mass flux crosses the cylinder wall.
b) Calculate the circulation along the curve C.
Hint: The easiest way is to use Stokes' theorem

$$\oint_{(C)} \vec{u}\cdot \mathrm{d}\vec{x} = \iint_{(S)} (\mathrm{curl}\,\vec{u})\cdot \vec{n}\,\mathrm{d}S\,.$$

Given: a, b, K

Solution

a) No mass flux through the wall:
We show that the condition

$$(u_i - u_{i(w)})\,n_i = 0 \qquad \text{(at the wall)} \tag{1}$$

is fulfilled. Since $u_{i(w)} = 0$ it follows that $u_i\,n_i = 0$.
The cylinder surface equation in implicit form is

$$F(x_1,x_2) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0\,. \tag{2}$$

Because

$$u_i n_i = u_i\,\frac{\partial F}{\partial x_i}\ \Big/ \sqrt{\frac{\partial F}{\partial x_j}\,\frac{\partial F}{\partial x_j}} = 0$$

we find

$$u_1\,\frac{\partial F}{\partial x_1} + u_2\,\frac{\partial F}{\partial x_2} = -\frac{2K\,x_2}{b^2}\,\frac{2\,x_1}{a^2} + \frac{2K\,x_1}{a^2}\,\frac{2\,x_2}{b^2} = 0\,.$$

b) Circulation Γ:
The circulation is given by

$$\Gamma = \oint_{(C)} \vec{u}\cdot \mathrm{d}\vec{x} = \iint_{(S)} (\mathrm{curl}\,\vec{u})\cdot \vec{n}\,\mathrm{d}S\,. \tag{3}$$

As seen from the positive side of the surface S the integration along the curve C is in the mathematically positive sense. In this case, the normal vector points away from the paper plane and coincides with $\vec{e}_3$. Thus the integrand of the surface integral in (3)

$$(\mathrm{curl}\,\vec{u})\cdot\vec{n} = (\mathrm{curl}\,\vec{u})_3 = \varepsilon_{3jk}\,\frac{\partial u_k}{\partial x_j} = \frac{2K}{a^2} + \frac{2K}{b^2}\,.$$

With the known expression for the surface of an ellipse $\pi\,a\,b$, we obtain the circulation as

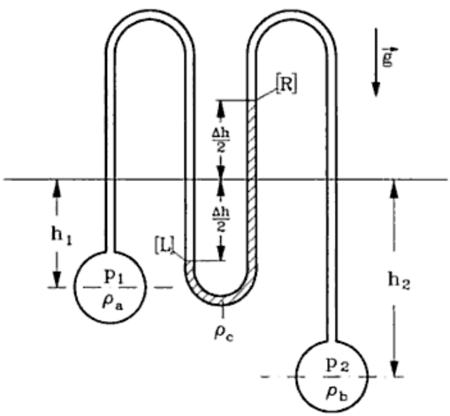
$$\Gamma = 2\pi\,K\frac{a^2+b^2}{a\,b}\,.$$

5 Hydrostatics

5.1 Hydrostatic Pressure Distribution

Problem 5.1-1 U-tube manometer

Two containers filled with different fluids of densities ϱ_a and ϱ_b are connected with each other via a manometer as shown in the figure. The density of the manometer fluid is ϱ_c. Determine the pressure difference $p_1 - p_2$ as a function of column height Δh.



Given: h_1, h_2, Δh, ϱ_a, ϱ_b, ϱ_c, g

Solution

Pressure distribution in a fluid at rest in an inertial system (see F. M. (5.15))

$$p + \varrho\, g\, z = \text{const.}$$

The pressure p_L at station L, coming from a is

$$p_L = p_1 + \varrho_a\, g \left(\frac{\Delta h}{2} - h_1\right) \ ,$$

the pressure at station p_R at station R, coming from b is

$$p_R = p_2 - \varrho_b\, g \left(\frac{\Delta h}{2} + h_2\right) .$$

Thus, the difference is

$$p_L - p_R = p_1 - p_2 + \frac{\Delta h}{2}\, g\,(\varrho_a + \varrho_b) - g\,(\varrho_a\, h_1 - \varrho_b\, h_2) ,$$

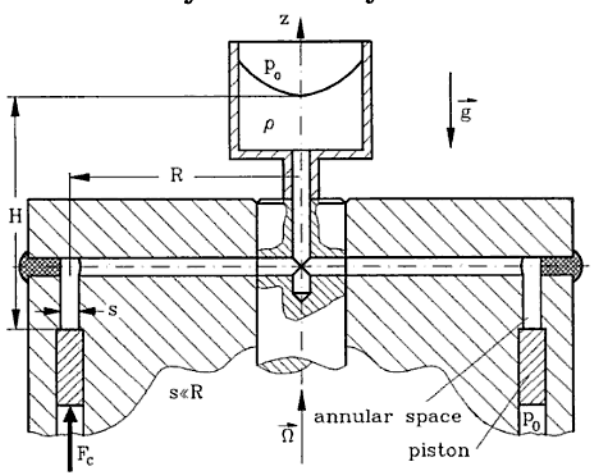
while on the other hand, the pressure difference at these locations in the liquid c is

$$p_L - p_R = \varrho_c\, g\, \Delta h .$$

Equating the differences, we get

$$p_1 - p_2 = \varrho_c\, g\, \Delta h \left(1 - \frac{\varrho_a + \varrho_b}{2\, \varrho_c}\right) + g\,(\varrho_a\, h_1 - \varrho_b\, h_2) .$$

Problem 5.1-2 Hydraulic safety clutch



A hydraulic safety clutch should be decoupled at a certain angular velocity. The frequency adjustments occur by way of the filling height H. The required coupling force F_C is produced by the fluid pressure acting on a annular piston (radius R and width s). Assuming $s \ll R$, the pressure can be considered constant over the piston surface.

a) Find the speed, at which the coupling force F_C is just overcome.
b) How is the filling height to be changed in order to double the coupling speed?

Given: H, F_C, R, s, ϱ, p_0, g

Solution

a) Limiting angular velocity:
The pressure distribution in a rotating frame (see F. M. (5.15)) is

$$p + \varrho g z - \frac{\varrho}{2} \Omega^2 r^2 = \text{const} .$$

The piston pressure is therefore

$$p_P = p_0 + \varrho g H + \frac{\varrho}{2} \Omega^2 R^2 .$$

Since p_P is constant over the piston surface, we have

$$p_P = \frac{F_P}{2 \pi R s} ,$$

with F_P as the fluid force acting on the piston. With the force balance $F_P = F_C + p_0 2 \pi R s$, we obtain the limiting angular velocity Ω^*:

$$\frac{F_C}{2 \pi R s} + p_0 = p_P^* = p_0 + \varrho g H + \frac{\varrho}{2} \Omega^{*2} R^2$$

$$\Rightarrow \quad \Omega^* = \left(\frac{F_C}{\varrho \pi R^3 s} - \frac{2 g H}{R^2} \right)^{\frac{1}{2}} .$$

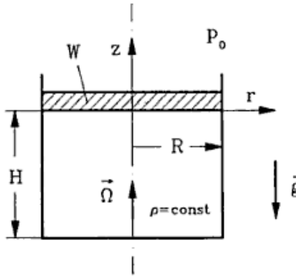
b) Changing the filling height:
Using H' for the filling height at double the coupling angular velocity $\Omega' = 2 \Omega^*$, we have:

$$\left(\frac{F_C}{\varrho \pi R^3 s} - \frac{2 g H'}{R^2} \right)^{\frac{1}{2}} = 2 \left(\frac{F_C}{\varrho \pi R^3 s} - \frac{2 g H}{R^2} \right)^{\frac{1}{2}}$$

$$\Rightarrow \quad H' = 4 H - \frac{3}{2} \frac{F_C}{\varrho g \pi R s} .$$

Problem 5.1-3 Rotating container filled with fluid

An axisymmetric container (radius R, fluid height H) is filled with a fluid of density ϱ and is closed with a piston of weight W. The container rotates around its vertical axis with an angular velocity of Ω. The ambient pressure is p_0.



a) Find for $\vec{\Omega} = 0$ the pressure p_2 at the container base.
b) Calculate the pressure distribution inside the container for $\vec{\Omega} \neq 0$ (the constant in Bernoulli's equation is determined by the force balance on the piston).
c) At which location on the base is the pressure p_2 from part a) encountered when the container is rotating?
d) At which location and angular velocity is the vapor pressure $p_V = 0.2\,p_0$ reached first?

Given: R, H, p_0, W, ϱ, Ω, g

Solution

a) Pressure p_2 at the container base for $\vec{\Omega} = 0$:
We call the pressure under the piston ($z = 0$) p_1 and find the pressure distribution in the fluid as

$$p(z) + \varrho\, g\, z = p_1 \ .$$

With the force balance on the piston

$$p_1\, \pi\, R^2 = W + p_0\, \pi\, R^2$$

we obtain

$$p(z) + \varrho\, g\, z = \frac{W}{\pi\, R^2} + p_0$$

and particularly for the pressure p_2 at the container base ($z = -H$)

$$p_2 = \frac{W}{\pi\, R^2} + p_0 + \varrho\, g\, H \ .$$

b) Pressure distribution for $\vec{\Omega} \neq 0$:
The pressure distribution on the lower piston side ($z = 0$) is obtained from Bernoulli's equation in the coordinate system, in which the fluid is at rest

$$p(r, z) + \varrho\, g\, z - \frac{\varrho}{2}\, \Omega^2\, r^2 = C$$

$$p_1(r) = C + \frac{\varrho}{2}\,\Omega^2\,r^2\,.$$

The force balance on the piston is

$$\begin{aligned} \iint\limits_{S_P} p_1(r)\,\mathrm{d}A &= W + p_0\,A \\ \Rightarrow \qquad C &= \frac{W}{\pi\,R^2} + p_0 - \frac{\varrho}{4}\,\Omega^2\,R^2\,. \end{aligned}$$

Thus, the pressure distribution inside the container is calculated as

$$p(r,z) = \frac{W}{\pi\,R^2} + p_0 - \varrho\,g\,z + \frac{\varrho}{2}\,\Omega^2\left(r^2 - \frac{R^2}{2}\right)$$

and particularly at the container base ($z = -H$)

$$p(r,-H) = \frac{W}{\pi\,R^2} + p_0 + \varrho\,g\,H + \frac{\varrho}{2}\,\Omega^2\left(r^2 - \frac{R^2}{2}\right)\,.$$

c) Where is $p(r,-H) = p_2$?

$$\begin{aligned} \frac{W}{\pi\,R^2} + p_0 + \varrho\,g\,H &= \frac{W}{\pi\,R^2} + p_0 + \varrho\,g\,H + \frac{\varrho}{2}\,\Omega^2\left(r^2 - \frac{R^2}{2}\right) \\ \Rightarrow r &= \frac{R}{\sqrt{2}}\,. \end{aligned}$$

Note that the pressure is independent of Ω on the circles $r = R/\sqrt{2}$.

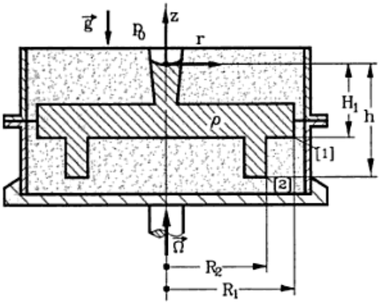
d) Where is the vapor pressure reached first?
The pressure decreases with z and increases with r, i. e. the lowest pressure in the container is found at $r = z = 0$. Thus, Ω is determined by the equation

$$p_V = p(0,0) = \frac{W}{\pi\,R^2} + p_0 - \frac{\varrho}{4}\,\Omega^2\,R^2\,.$$

With $p_V = 0.2\,p_0$ it then follows

$$\Omega = \left(\frac{4\,W}{\varrho\,\pi\,R^4} + \frac{3.2\,p_0}{\varrho\,R^2}\right)^{\frac{1}{2}}\,.$$

Problem 5.1-4 Centrifugal casting process



An axisymmetric part to be manufactured by centrifugal casting process utilizes the sketched mold. During the casting process the mold rotates with a constant angular velocity Ω about its vertical axis.

a) Calculate the pressure p_1 at station [1] as a function of Ω.
b) Determine the maximum angular velocity Ω if the maximum sand pressure p_{max} at station [1] is not to be exceeded.
c) Calculate for this Ω_{max} the maximum height h such that at station [2] the pressure p_{max} is not exceeded.

Given: R_1, R_2, H_1, ϱ, p_0, p_{max}, g

Solution

The pressure distribution with regard to a rotating frame of reference about the vertical axis is (see F. M. (5.15))

$$p = p_0 - \varrho\, g\, z + \frac{1}{2}\,\varrho\,\Omega^2 r^2\,. \tag{1}$$

a) The pressure at [1] is calculated by introducing in the above equation $r = R_1$ and $z = -H_1$:

$$p_1 = p_0 + \varrho\, g\, H_1 + \frac{1}{2}\,\varrho\,\Omega^2 R_1^2\,. \tag{2}$$

b) The maximum admissible angular velocity Ω_{max} is found by setting in equation (2) $p_1 = p_{max}$ and solve the equation for Ω_{max}:

$$\Omega_{max} = \frac{1}{R_1}\sqrt{\frac{2}{\varrho}(p_{max} - p_0) - 2\, g\, H_1}\,.$$

c) The corresponding admissible height h (at [2]) is calculated by replacing the quantities in equation (1) as follows:

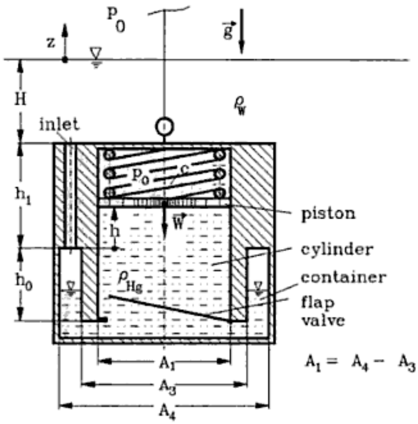
$$p = p_{max}\,, \quad z = -h\,, \quad \Omega = \Omega_{max} \quad \text{and} \quad r = R_2\,.$$

The unknown height h is then calculated from this equation as

$$h = \frac{p_{max} - p_0}{\varrho\, g}\left(1 - \left(\frac{R_2}{R_1}\right)^2\right) + \left(\frac{R_2}{R_1}\right)^2 H_1\,.$$

Problem 5.1-5 Depth gauge

A depth gauge consists of a container filled with mercury (density ϱ_{Hg}) which is connected with a cylinder by way of a flap valve. Water can enter the gauge through the inlet (see figure). Submerging the gauge causes mercury to flow resulting in an upward movement of the piston. The water depth can be read after the device has been pulled out.



a) Determine the weight of the piston such that for a water depth of $H = 0$ and an unloaded spring the level of mercury in the container is as high as in the cylinder. (The gauge is filled with mercury such that at $H = 0$ the surface of discontinuity water-mercury is at $z = -h_1$.)

b) Find the spring stiffness c such that the gauge is operational for a depth up to H_{max} without water flowing into the cylinder.
Hint: The effect of the air compression can be neglected compared with the spring force.

c) Find the relation between the water depth H and the mercury level in the cylinder.

Given: A_1, $A_2 = A_4 - A_3 = A_1$, h_0, h_1, p_0, ϱ_W, ϱ_{Hg}, g

Solution

We assume that the ambient pressure p_0 is acting on the upper side of the piston and thus has no influence and may be set $p_0 = 0$.

a) Piston weight W:
Since the areas A_1 and A_2 are equal, the mercury level will rise with increasing depth H by the same amount h with which the level in the container decreases. The distance of the surface of discontinuity water-

mercury from the water surface is therefore

$$z_D = -(H + h_1 + h) .$$

The pressure at this location is

$$\begin{aligned} p_D &= -\varrho_W \, g \, z_D \, , \\ p_D &= \varrho_W \, g \, (H + h_1 + h) \, . \end{aligned} \tag{1}$$

This pressure is equal to the pressure inside of the cylinder at $z = -z_D$:

$$p_{Cy} = p_D \, . \tag{2}$$

The pressure p_{Cy} is caused by spring force $c\,h$ (unknown stiffness c), the still unknown weight W, and the mercury column height $2\,h$. The force balance for the piston with mercury column is:

$$\begin{aligned} p_{Cy} \, A_1 &= W + c\,h + 2\,\varrho_{Hg} \, g \, h \, A_1 \\ \Rightarrow p_{Cy} &= \frac{W + c\,h}{A_1} + 2\,\varrho_{Hg} \, g \, h \, . \end{aligned} \tag{3}$$

For the problem part a) we have in particular $h = 0$ and $H = 0$. We insert this into equation (1) and (3):

$$\begin{aligned} p_D &= \varrho_W \, g \, h_1 \, , \\ p_{Cy} &= \frac{W}{A_1} \, . \end{aligned}$$

If the above results are inserted into equation (2) and this equation is solved for weight W, we get:

$$W = \varrho_W \, g \, h_1 \, A_1 \, . \tag{4}$$

b) Stiffness c:
The mercury level within the container may sink by h_0 (at a measurable depth of H_{max}) in order for water not to penetrate into the cylinder. In the same time, the piston rises at $H = H_{max}$ to $h = h_0$ (see part a). To calculate the stiffness, we set $H = H_{max}$ and $h = h_0$ in equation (1) and (3) and get

$$p_D = \varrho_W \, g \, (H_{max} + h_1 + h_0) \, ,$$

$$p_{Cy} = \frac{W + c\,h_0}{A_1} + 2\,\varrho_{Hg} \, g \, h_0$$

or

$$p_{C_y} = \varrho_W \, g \, h_1 + \frac{c \, h_0}{A_1} + 2 \, \varrho_{Hg} \, g \, h_0 \, .$$

With equation (2) we find the stiffness c as

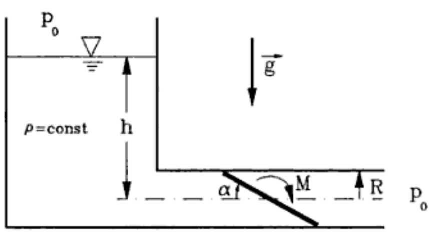
$$c = A_1 \, g \, \varrho_W \left(\frac{H_{max}}{h_0} + 1 - 2 \, \frac{\varrho_{Hg}}{\varrho_W} \right) . \tag{5}$$

c) Relation between the water depth H and the mercury level h:
With (4) and (5) the piston height h can be eliminated from (3). Using (1) h can be expressed by known quantities:

$$\frac{H}{h} = \frac{H_{max}}{h_0} \, .$$

5.2 Hydrostatic Lift, Force on Walls

Problem 5.2-1 Force and moment on a throttle valve



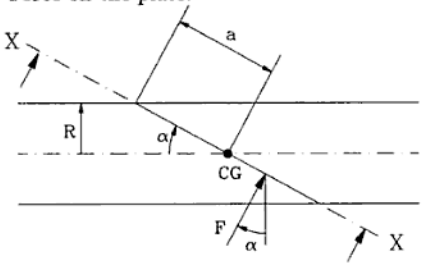
The sketched container is connected with a pipe that includes a throttle valve with an elliptic plate. In closed position, the plate angle with respect to the pipe axis is α. The axis of the plate is perpendicular to the drawing plane. In closed position, the plate is under a moment M to overcome the water pressure.

a) Determine the force and its axial component acting on the plate.
b) Find the necessary moment M to keep the throttle closed.

Given: h, R, α, ϱ, g

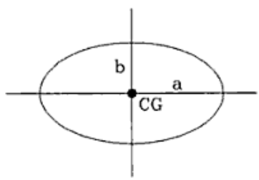
Solution

a) Force on the plate:



The ambient pressure p_0 has no influence on the force acting on the plate. Thus, in the following calculation, we set $p_0 = 0$. The plate has an elliptic form with the axes

$$a = \frac{R}{\sin\alpha} \quad \text{and} \quad b = R\,.$$



The centroid of area CG is on the pipe axis due to the symmetry, i. e. the pressure at the centroid of area p_{CG} is ($p_0 = 0$):

$$p_{CG} = \varrho\, g\, h\,.$$

The area of the ellipse is

$$A = \pi\, a\, b = \pi\, \frac{R^2}{\sin\alpha}\,.$$

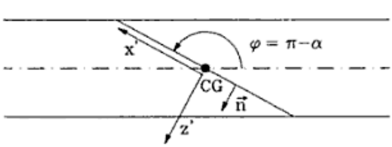
Therefore, the force on the plate is

$$F = p_{CG}\, A = \varrho\, g\, h\, \pi\, \frac{R^2}{\sin\alpha}\,.$$

The force is normal to the plate surface and its component in direction of the pipe axis is

$$F_x = F\, \sin\alpha = \varrho\, g\, h\, \pi\, R^2\,.$$

b) Moment on the plate:

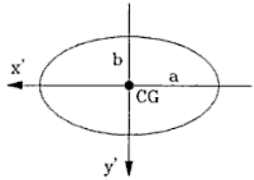


We utilize the sketched coordinate system and calculate the moment with respect to the coordinate origin ($\vec{x}_p = 0$) using the formula (see F. M. (5.41))

$$\vec{M}_p = (\varrho\, g\, \sin\varphi\; \mathrm{I}_{x'y'} + y'_p\, p_{CG}\, A)\, \vec{e}_x{}' - (\varrho\, g\, \sin\varphi\; \mathrm{I}_{y'} + x'_p\, p_{CG}\, A)\, \vec{e}_y{}'$$

as

$$\vec{M}_0 = \varrho g \sin\varphi \, (\mathrm{I}_{x'y'} \, \vec{e}_x{}' - \mathrm{I}_{y'} \, \vec{e}_y{}') \, . \qquad (1)$$



The area moments of inertia of an ellipse are (see engineering handbooks)

$$\mathrm{I}_{x'y'} = 0 \qquad \text{(for symmetry reasons)} \, ,$$

$$\mathrm{I}_{y'} = \frac{\pi}{4} a^3 b = \frac{\pi}{4} \frac{R^4}{\sin^3\alpha} \, .$$

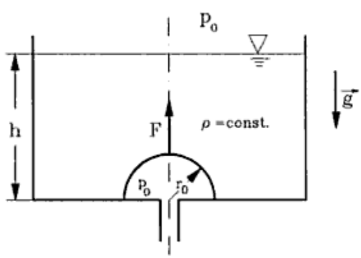
We insert this in (1) and obtain, because of $\sin\varphi = \sin(\pi - \alpha) = \sin\alpha$,

$$\vec{M}_0 = -\varrho g \sin\alpha \frac{\pi}{4} \frac{R^4}{\sin^3\alpha} \vec{e}_y{}' = -\varrho g \frac{\pi}{4} \frac{R^4}{\sin^2\alpha} \vec{e}_y{}' \, .$$

Since the direction of $\vec{e}_y{}'$ is into the plane of drawing, the moment due to the fluid pressure acts in an opening mode.

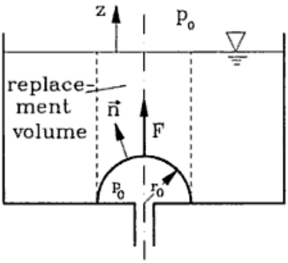
Problem 5.2-2 Half sphere closing an orifice

The exit pipe of a container filled with water (height h) is closed by a half sphere (weight W, radius r_0). Calculate the force F necessary to open the exit pipe.



Given: h, ϱ, r_0, g

Solution



For the calculation we use the sketched replacement volume. The ambient pressure has no influence and can be set $p_0 = 0$. The force on the inner surface is then equal to zero. The force on the outer surface is calculated using the formula (see F. M. (5.45))

$$F_z = p_0 A_z + \varrho g V \, . \qquad (1)$$

By deriving the above equation we assumed that $\vec{n} \cdot \vec{e}_z < 0$, i. e. the lower surface to be S. Here we consider the upper surface such that in (1) the sign must be reversed. Thus, with $p_0 = 0$,

$$F_z = -\varrho g V$$

is the fluid force that acts on the half sphere. The force balance on the half sphere is

$$\begin{aligned} F_z - W + F &= 0 \\ \Rightarrow \qquad F &= W - F_z = W + \varrho g V \, . \end{aligned}$$

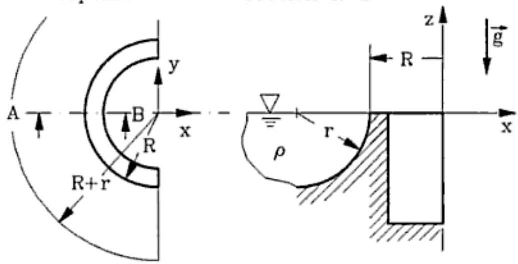
The volume content of the replacement volume is

$$V = V_{\text{cylinder}} - V_{\text{half sphere}} = \pi r_0^2 h - \frac{2}{3} \pi r_0^3 \, .$$

Thus, we finally obtain the requested force

$$F = W + \varrho g h \pi r_0^2 \left(1 - \frac{2}{3} \frac{r_0}{h}\right) \, . \qquad (h \geq r_0!)$$

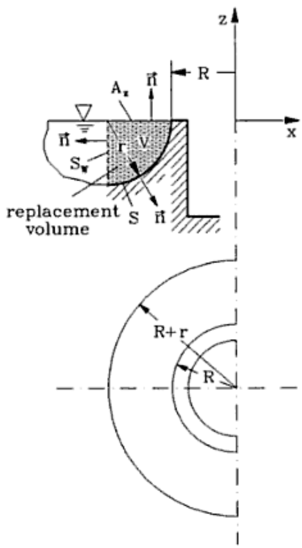
Problem 5.2-3 Force on a dam



Calculate the water force (F_x, F_z) on the sketched circular dam.
Given: ϱ, r, R, g

Solution

Since the ambient pressure does not have any influence on the force acting on the dam, we set it equal to zero. We calculate first the z-component of the force and consider the fluid volume V with the surface S_{tot}. This surface consists of S, S_W, and A_z, where S is the wetted surface under consideration.



From Archimedes' principle it follows that the force on the replacement volume is

$$\vec{F}_{\rightarrow\text{repl. vol.}} = \iint\limits_{(S+S_W+A_z)} -p\vec{n}\,\mathrm{d}S = -\varrho\vec{g}V$$

and for the z-component ($\vec{g}\cdot\vec{e}_z = -g$)

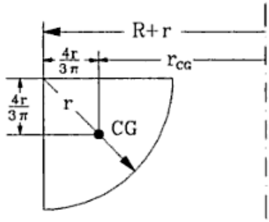
$$F_{z_{\rightarrow\text{repl.vol.}}} = \iint\limits_{(S+S_W+A_z)} -p\vec{n}\cdot\vec{e}_z\,\mathrm{d}S = \varrho\,g\,V\ .$$

Since on S_W $\vec{n}\cdot\vec{e}_z = 0$ and on A_z $p = p_0 = 0$ we have $F_{z_{\rightarrow\text{repl. vol.}}} = -F_{z_{\rightarrow\text{ wall}}}$, and obtain

$$F_z = F_{z_{\rightarrow\text{wall}}} = -\varrho\,g\,V\ .$$

V is the volume of a segment of the sketched torus. We calculate this volume by Guldin's rule:
Volume = cross-section area multiplied with the path, the centroid makes generating the axisymmetric body, i. e.

$$V = A_{\text{quarter circle}}\, \pi\, r_{CG} = \frac{1}{4}\pi^2 r^2 r_{CG}\,.$$



With

$$r_{CG} = (R+r) - \frac{4}{3\pi} r$$

the volume is

$$V = \frac{1}{4}\pi^2 r^2 \left[R + r\left(1 - \frac{4}{3\pi}\right)\right]$$

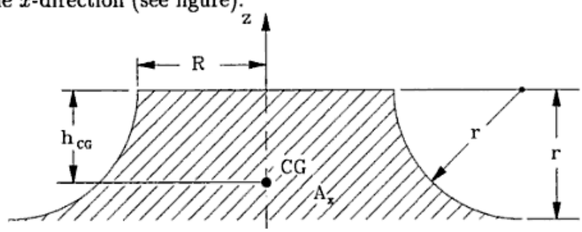
thus, the z-component of the force:

$$F_z = -\varrho g \frac{\pi^2}{4} r^2 \left[R + r\left(1 - \frac{4}{3\pi}\right)\right]\,.$$

The x-component (the y-component disappears due to symmetry) is

$$F_x = \iint\limits_{(S)} p\,\vec{n}\cdot\vec{e}_x\,\mathrm{d}S = \iint\limits_{(A_x)} p\,\mathrm{d}A\,.$$

F_x is also the force acting on the plane surface A_x, generated by projecting S in the x-direction (see figure):



$$F_x = p_{CG} A_x = \varrho g\, h_{CG} A_x\,.$$

Considering that the centroid of area is

$$-z_{CG} = h_{CG} = \frac{\sum A_i\,(-z_{CG_i})}{A_x}\,,$$

we find

$$F_x = \varrho g \sum A_i\,(-z_{CG_i})\,.$$

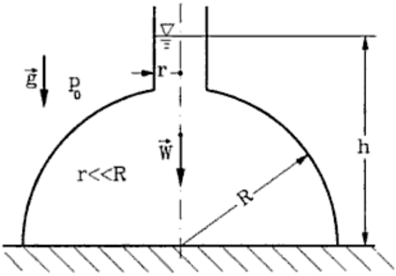
The partial surfaces A_i consist of a rectangle $2(R+r)\,r$ and the two quarter circles that must be subtracted:

$$F_x = \varrho g \left[2(R+r)\,r\left(\frac{r}{2}\right) - 2\frac{\pi}{4}r^2\left(\frac{4r}{3\pi}\right)\right]$$

$$\Rightarrow \quad F_x = \varrho g\left(\frac{1}{3}r^3 + r^2 R\right)\,.$$

Problem 5.2-4 Half sphere cup sealing by its own weight

A half sphere with radius R filled with fluid (density ϱ) has an inlet (radius r with $r \ll R$). It is layed on a flat plate and seals with its own weight W. Determine the fluid height h, at which there is no leakage.



Given: r, R, W, ϱ, g

Solution

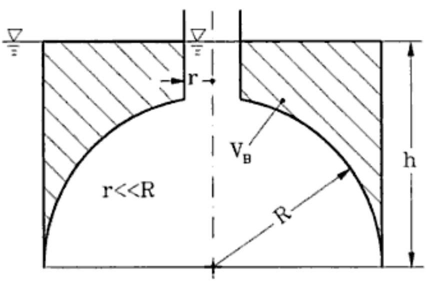
The half sphere is able to seal if the resultant force F_L from the pressure integral over the inner wall of the half sphere is just equal to the weight

$$F_L = W \; . \tag{1}$$

The ambient pressure p_0 acts on both sides of the sphere and can be set $p_0 = 0$. The pressure integration is easily carried out by calculating the lift for an appropriate replacement volume V_B using

$$F_L = \varrho \, g \, V_B \; . \tag{2}$$

Case a: The filling height is $h > R$:



The volume of the replacement body is

$$V_B = \pi R^2 h - \frac{2}{3}\pi R^3 - \pi r^2 (h - R).$$

We insert this into equation (1) and (2) and obtain the permissible filling height with respect to the radius R:

$$\frac{h}{R} = \frac{W}{\pi \varrho g R^3} \frac{1}{1 - \left(\frac{r}{R}\right)^2} + \frac{\frac{2}{3} - \left(\frac{r}{R}\right)^2}{1 - \left(\frac{r}{R}\right)^2}.$$

We assume that $r \ll R$ and simplify the result as

$$\frac{h}{R} = \frac{W}{\pi \varrho g R^3} + \frac{2}{3}.$$

In this equation the limit $W \to 0$ is not permissible because of $h/R > 1$.

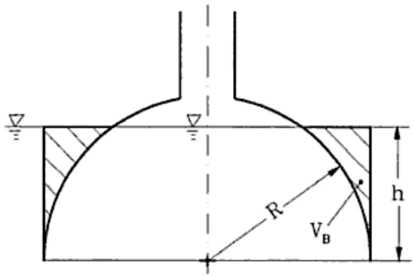
Case b: The filling height is $h \leq R$:

The replacement body has the volume

$$V_B = \pi R^2 h - V_{\text{spherical layer}} ,$$

with the volume of the spherical layer (see engineering handbooks)

$$V_{\text{spherical layer}} = \pi R^2 h - \frac{1}{3}\pi h^3 .$$



Similar to case a), we obtain

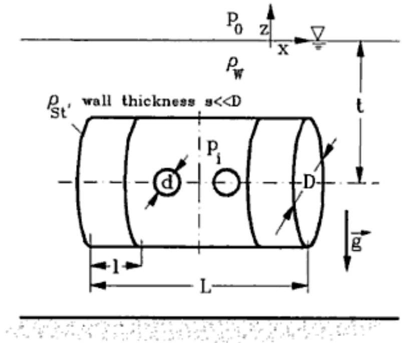
$$W = \pi \varrho g \left(R^2 h - R^2 h + \frac{1}{3} h^3\right) .$$

We solve the equation for the height h and find

$$h = \left(\frac{3W}{\pi \varrho g}\right)^{\frac{1}{3}} .$$

Problem 5.2-5 Cylindrical submarine

The sketched submarine consists of a container manufactured from steel sheet of thickness s, where $s \ll D$. It has two side tanks. In the horizontal plane, the sub has circular windows (diameter d). The pressure inside the sub is p_i.



a) The tanks are filled with air and the sub floats without load immersed to a height $D/2$. Determine the density ratio of steel and water (ϱ_{St}/ϱ_W).
b) The sub is now loaded with weight W and is now completely submerged.
   1) Determine the water volume ΔV flooded into the tanks.
   2) Find l for the case that the tanks are filled.
c) Determine the force exerted on a window at a depth of t .

Given: L, D, d, s, ϱ_W, p_0, g

Solution

a) Density ratio ϱ_{St}/ϱ_W:
The sum of vertical forces acting on the sub is zero:

$$F_{L_1} - W_{St} = 0 \,. \tag{1}$$

The lift force is calculated from

$$F_{L_1} = V_{fl_1}\, \varrho_W\, g \,.$$

Since the sub is half submerged, the volume V_{fl_1} is equal to the half of total volume:

$$V_{fl_1} = \frac{1}{2} V \quad , \text{with } V = \pi \frac{D^2}{4} L \,.$$

The weight of the empty sub, which is equal to the weight of the steel sheet is calculated as follows:

$$W_{St} = V_{St}\, \varrho_{St}\, g \,,$$

with the steel volume for $s \ll D$

$$V_{St} \;=\; \pi\, D\, L\, s + \pi\, D^2\, s$$

$$= \pi D^2 s \left(1 + \frac{L}{D}\right)$$

$$= 4 V \frac{s}{L} \left(1 + \frac{L}{D}\right) .$$

We insert this into equation (1) and solve for the requested density ratio

$$\frac{\varrho_{St}}{\varrho_W} = \frac{1}{8} \frac{L}{s} \left(1 + \frac{L}{D}\right)^{-1} .$$

b) 1) Flooded water volume ΔV:
The sum of all vertical forces acting on the submerged submarine loaded with W is

$$F_{L_2} - W_{St} - W = 0 . \qquad (2)$$

The lift force is

$$F_{L_2} = V_{fl_2} \varrho_W g ,$$

with the displaced water volume

$$V_{fl_2} = V - \Delta V .$$

W_{St} in equation (2) can be expressed as $1/2 V \varrho_W g$ using (1). Inserting into equation (2) delivers the requested volume ΔV relative to the total volume

$$\frac{\Delta V}{V} = \frac{1}{2} - \frac{W}{V \varrho_W g} .$$

(Note: As we can see the inequality $W < V \varrho_w g/2$ must be kept, in order for the sub not to sink to the bottom)

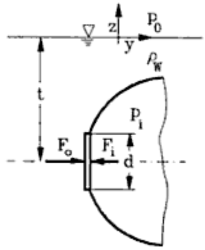
2) Calculate the length l at filled tanks:
For completely filled tanks the tank length l is calculated from

$$\Delta V = \frac{\pi}{2} D^2 l .$$

With the above result, the tank length is calculated as

$$\frac{l}{L} = \frac{1}{2} \left(\frac{1}{2} - \frac{W}{V \varrho_W g}\right) .$$

c) Force on a window:
The center of the windows is at depth t under the water surface. Inside the sub a constant pressure p_i acts on the window. The force



$$F_i = p_i \frac{\pi}{4} d^2$$

acts on the window centroid. At this point (depth t) the fluid pressure is

$$p_{CG} = p_0 + \varrho_W \, g \, t \, .$$

The fluid force on a window therefore

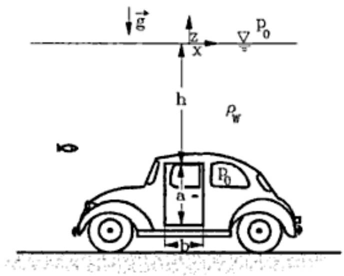
$$F_o = p_{CG} \frac{\pi}{4} d^2$$

and thus the resultant force is found as

$$F_y = (p_0 - p_i + \varrho_W \, g \, t) \frac{\pi}{4} d^2 \, .$$

Problem 5.2-6 Car under water

The sketched car has just fallen into a river such that its inner pressure is equal to the atmospheric pressure p_0. The shape of the car door can be approximated by a rectangle with the sides a and b. The height of the water level above the top edge of the door is h.



a) Calculate the force F necessary to open the door. Assume that the force is normal to the door surface and acts at a distance $3/4\, b$ from the door axis.
b) Up to which height x must the water rise inside the car to allow a passenger with the muscle force F_M to open the door?

Given: $h = 5\,\mathrm{cm}$, $a = 95\,\mathrm{cm}$, $b = 60\,\mathrm{cm}$, $\varrho_W = 10^3\,\mathrm{kg/m^3}$, $g = 9.81\,\mathrm{m/s^2}$, $F_M = 500\,\mathrm{N}$

Solution

Since the atmospheric pressure acts on both sides of the door, its effect is cancelled.

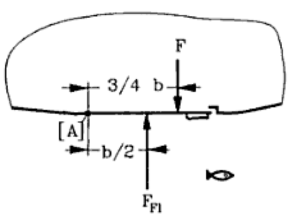
a) Force F:

The pressure at the area centroid z_{CG} of the door is

$$\begin{aligned} p_{CG} &= -\varrho_W \, g \, z_{CG} \\ &= \varrho_W \, g \left(h + \frac{1}{2} a \right) . \end{aligned}$$

The magnitude of the force acting on the door is the product of this pressure and the door surface A:

$$\begin{aligned} F_{Fl} &= p_{CG} \, A \\ &= \varrho_W \, g \, a \, b \, h \left(1 + \frac{a}{2h} \right) . \end{aligned} \tag{1}$$

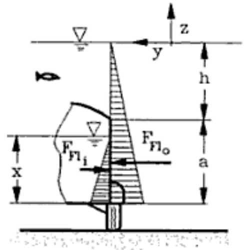


The necessary force is obtained using the balance of angular momentum about the door axis [A]:

$$\begin{aligned} F \, \frac{3}{4} \, b &= F_{Fl} \, \frac{1}{2} \, b \\ \Rightarrow F &= \frac{2}{3} \, F_{Fl} \qquad (2) \\ &= \frac{2}{3} \, \varrho_W \, g \, h \, a \, b \left(1 + \frac{a}{2h} \right) \\ &= 1.957 \, \mathrm{kN} . \end{aligned}$$

b) Water height x in car:
The resultant force is

$$F_y = F_{Fl_o} - F_{Fl_i} \, .$$



The force F_{Fl_o} is equal to F_{Fl} from part a) (equation (1)). The force produced by the fluid inside the car is

$$F_{Fl_i} = \frac{1}{2} \varrho_W \, g \, b \, x^2 \, .$$

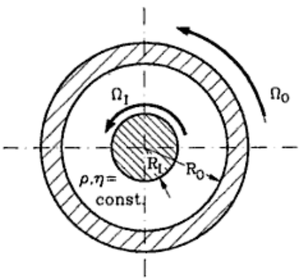
We introduce the resultant force F_y into the right hand side of equation (2) and the muscle force F_M into the left hand side and solve for the the water height x. As a result, we obtain

$$\begin{aligned} x &= \sqrt{2 \, a \, h \left(1 + \frac{a}{2 \, h}\right) - \frac{3 \, F_M}{\varrho_W \, g \, b}} \\ &= 86 \, \mathrm{cm} \, . \end{aligned}$$

For $x = 0$ we find again the result from a).

6 Laminar Unidirectional Flow

Problem 6-1 Flow in an annular gap



Incompressible Newtonian fluid (ϱ, η) flows steadily within the annular gap of two infinitely long cylinders (R_O, R_I). The outer cylinder rotates with Ω_O, the inner one with Ω_I. For the case that the axial component of the velocity is equal to zero, calculate

a) the velocity and pressure field,
b) the moments acting on both cylinders,
c) the dissipated power within the gap.
d) For what ratio Ω_O/Ω_I is the flow a potential flow?

Given: R_O, R_I, Ω_O, Ω_I, ϱ, η

Solution

a) Velocity and pressure field in cylindrical coordinates:
The only non-zero component is known (see F. M. (6.42))

$$\begin{aligned} u_\varphi(r) &= \frac{\Omega_O R_O^2 - \Omega_I R_I^2}{R_O^2 - R_I^2}\, r + \frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2}\, \frac{1}{r} \\ &= C_1\, r + C_2\, \frac{1}{r} \end{aligned}$$

with the constants

$$C_1 = \frac{\Omega_O R_O^2 - \Omega_I R_I^2}{R_O^2 - R_I^2}\,, \qquad C_2 = \frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2}\,. \tag{1}$$

From the r-component of Navier-Stokes' equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\varrho}{r}\left(C_1 r + \frac{C_2}{r}\right)^2$$

the pressure distribution can be calculated:

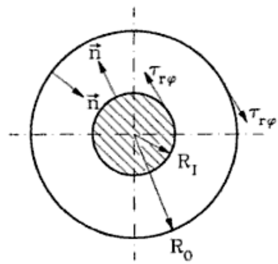
$$\begin{aligned} p(r) &= \frac{\varrho}{2}\left(\frac{\Omega_O R_O^2 - \Omega_I R_I^2}{R_O^2 - R_I^2}\right)^2 r^2 \\ &+ 2\varrho \frac{\Omega_O R_O^2 - \Omega_I R_I^2}{R_O^2 - R_I^2} \frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2} \ln r \\ &- \frac{\varrho}{2}\left(\frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2}\right)^2 \frac{1}{r^2} + \text{const} \end{aligned}$$

(In incompressible flow without pressure boundary conditions, the pressure can be determined only up to a constant).

b) Moment on cylinders:
For symmetry reasons the moments on both cylinders have only z-components:

$$M_I = \tau_{r\varphi}|_{r=R_I} 2\pi R_I^2 \qquad (2)$$

$$M_O = -\tau_{r\varphi}|_{r=R_O} 2\pi R_O^2 . \qquad (3)$$



The component $\tau_{r\varphi}$ of the stress tensor is

$$\tau_{r\varphi} = 2\eta\, e_{r\varphi} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \varphi}\right)$$

$$\Rightarrow \qquad \tau_{r\varphi} = -\eta \frac{2}{r^2} \frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2} .$$

From (2) and (3) it then follows

$$M_I = -4\pi\eta \frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2} ,$$

$$M_O = 4\pi\eta \frac{(\Omega_I - \Omega_O) R_I^2 R_O^2}{R_O^2 - R_I^2} .$$

The moments are equal and opposite as is also a consequence of the momentum balance.

c) Dissipated power:
For the flow to be steady, the dissipated power within the gap must be transferred as heat to the environment:

$$P_D = \iiint\limits_V \Phi \, \mathrm{d}V = -\dot{Q} \, .$$

We apply the energy equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(K+E) = P + \dot{Q}$$

to a control volume that contains the fluid within the gap. Since the flow is steady and there is no fluid flow through the cylinder walls, the left hand side of this equation is zero. This is clearly explained by applying Reynolds' transport theorem to the left hand side

$$\frac{\mathrm{D}}{\mathrm{D}t}(K+E) = \frac{\partial}{\partial t}\iiint\limits_{(V)} \varrho \left(\frac{u^2}{2}+e\right)\,\mathrm{d}V + \iint\limits_{(S)} \varrho \left(\frac{u^2}{2}+e\right)\vec{u}\cdot\vec{n}\,\mathrm{d}S \, .$$

The power is dissipated as heat (the power of the body forces is zero):

$$-\dot{Q} = P_D = P = \iint\limits_{(S)} t_i \, u_i \, \mathrm{d}S$$

$$\Rightarrow P_D = \iint\limits_{S_I} t_\varphi \, u_\varphi \, \mathrm{d}S + \iint\limits_{S_O} t_\varphi \, u_\varphi \, \mathrm{d}S \, .$$

Because of the no-slip condition $u_\varphi = \Omega_I R_I$ on S_I and $u_\varphi = \Omega_O R_O$ on S_O:

$$\Rightarrow \qquad P_D = \Omega_I \iint\limits_{S_I} t_\varphi \, R_I \, \mathrm{d}S + \Omega_O \iint\limits_{S_O} t_\varphi \, R_O \, \mathrm{d}S \, .$$

The integrals represent the moments by the cylinders on the fluid. Thus, we find

$$P_D = -\Omega_I \, M_I - \Omega_O \, M_O$$

and with moments already calculated

$$P_D = 4\,\pi\,\eta \, \frac{R_I^2 \, R_O^2}{R_O^2 - R_I^2} \, (\Omega_I - \Omega_O)^2 \, .$$

The dissipated power is independent of the rigid body contribution $C_1 r$ in the velocity field as indeed must be the case.

d) Ω_O/Ω_I for irrotational flow (potential flow):
The plane flow can have a non-zero curl component in z-direction only:

$$\begin{aligned}\mathrm{curl}\vec{u} &= \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\varphi) - \frac{\partial u_r}{\partial \varphi}\right)\vec{e_z} \\ &= \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(C_1 r^2 + C_2\right)\vec{e_z} \\ &= 2\,C_1\,\vec{e}_z\,.\end{aligned}$$

For irrotational flow we require

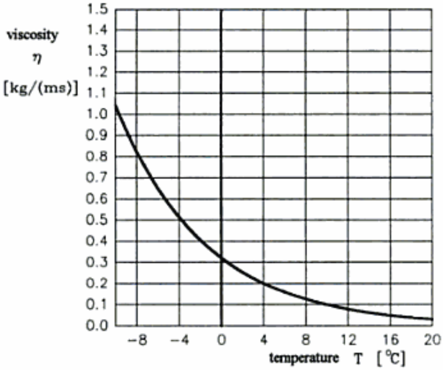
$$C_1 = 0\,.$$

From (1) it follows

$$\Omega_O\,R_O^2 = \Omega_I\,R_I^2 \qquad \Rightarrow \qquad \frac{\Omega_O}{\Omega_I} = \left(\frac{R_I}{R_O}\right)^2\,. \tag{4}$$

An important special case of (4) is $R_O \to \infty$, i. e. $\Omega_O = 0$. The calculated flow is the exact solution of the Navier-Stokes equations for a long cylinder (R_I, Ω_I) rotating in infinite space. For the case that (4) is satisfied, the flow is a viscous potential flow.

Problem 6-2 Crude oil transport through pipeline



At very low atmospheric temperature the crude oil can be transported through pipelines at an acceptable pressure drop only because the heat generated by the dissipation causes the viscosity to drop. The relationship between viscosity and temperature is plotted in the figure. The flow is laminar, incompressible, and the temperature can be assumed constant across the cross-sectional area. Furthermore, we assume that flow quantities are constant in axial direction. The mean velocity of oil $\overline{U}$, the pipe diameter R and the atmospheric temperature T_A are known. The heat loss to the surroundings per unit of length can be approximated by the formula

$$\dot{Q} = k\,(T - T_A)\,2\,\pi\,R \tag{1}$$

where T is the mean oil temperature.

a) Find the velocity profile $u_z(r)$.

b) Determine the dissipation function Φ. (Hint: write the equation of Φ in symbolic form and use cylindrical coordinates.)

c) Find the dissipated energy P_D per unit of length and time as a function of the viscosity η.

d) In order for the flow to be independent from the coordinate in axial direction, the dissipated energy must be transferred to the surroundings. The condition for the heat flux to satisfy equation (1) provides a relationship between the viscosity and the oil temperature. Plot this relationship into the figure and find T and η.

e) Calculate the pressure gradient $\partial p/\partial z$.

Given: $\overline{U} = 3\,\mathrm{m/s}$, $R = 0.5\,\mathrm{m}$, $T_A = -40°\,\mathrm{C}$ (Alaska), $k = 0.8\,\mathrm{W/(m^2K)}$ (insulated!)

Solution

a) Velocity profile $u_z(r)$:
The flow is laminar and the viscosity is constant over the cross-section. Consequently, the velocity profile is a Hagen-Poiseuille profile:

$$u_z(r) = 2\overline{U}\left(1 - \left(\frac{r}{R}\right)^2\right) . \tag{2}$$

b) Dissipation function Φ:
Φ is given by

$$\Phi = \lambda^* (\operatorname{sp}\mathbf{E})^2 + 2\,\eta\,\operatorname{sp}(\mathbf{E}^2) . \tag{3}$$

For incompressible flow $\operatorname{sp}\mathbf{E} = e_{ii} = 0$. The rate of deformation tensor $\mathbf{E}$ has in cylindrical coordinates the components

$$\mathbf{E} = \begin{pmatrix} e_{rr} & e_{r\varphi} & e_{rz} \\ e_{\varphi r} & e_{\varphi\varphi} & e_{\varphi z} \\ e_{zr} & e_{z\varphi} & e_{zz} \end{pmatrix} .$$

With $e_{rz} = (1/2)\,\partial u_z/\partial r$ (see F. M. (B.2)), all the other components are zero, their matrix is

$$\mathbf{E} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\partial u_z}{\partial r} & 0 & 0 \end{pmatrix} \qquad \text{with } \frac{\partial u_z}{\partial r} = -4\,\overline{U}\,\frac{r}{R^2} . \tag{4}$$

We find for $\mathbf{E}^2$

$$\mathbf{E}^2 = \begin{pmatrix} \frac{1}{4}\left(\frac{\partial u_z}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\left(\frac{\partial u_z}{\partial r}\right)^2 \end{pmatrix}$$

and from that

$$\operatorname{sp}(\mathbf{E}^2) = \frac{1}{4}\left(\frac{\partial u_z}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{\partial u_z}{\partial r}\right)^2 = \frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2 .$$

From (3) and (4) we get

$$\Phi = 2\,\eta\,\frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2 = \eta\left(\frac{\partial u_z}{\partial r}\right)^2 = 16\,\eta\,\overline{U}^2\,\frac{r^2}{R^4} . \tag{5}$$

c) Dissipated energy per unit of length and time:

$$P_D = \iiint\limits_{(V)} \Phi\,\mathrm{d}V = \int\limits_{r=0}^{R} 16\,\eta\,\overline{U}^2\,\frac{r^2}{R^4}\,2\,\pi\,r\,\mathrm{d}r = 8\,\pi\,\eta\,\overline{U}^2 . \tag{6}$$

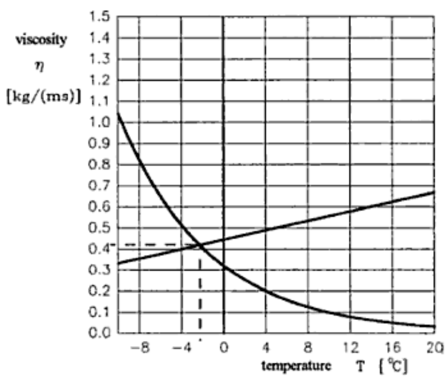
d) Determination of T and η:
The dissipated energy is equal to the heat transferred to the surroundings

$$P_D = \dot{Q}$$

$$\Rightarrow \quad 8\,\pi\,\eta\,\overline{U}^2 = k\,(T - T_A)\,2\,\pi\,R$$

$$\Rightarrow \quad \eta = \frac{k\,R}{4\,\overline{U}^2}\,(T - T_A)\,. \quad (T_A = -40^\circ\,\mathrm{C}) \tag{7}$$

We plot the line given by (7) with $(k\,R)/(4\,\overline{U}^2) = 1/90\,(\mathrm{kg/msK})$ into the figure and read at the intersection point of the two curves (see figure) the values $T \approx -2.3^\circ\,\mathrm{C}$, $\eta \approx 0.42\,\mathrm{kg/(ms)}$.

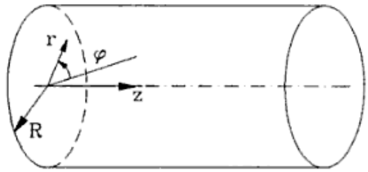


e) Pressure gradient:
For the Hagen-Poiseuille flow (see F. M. (6.63)) we have

$$\dot{V} = \frac{\pi}{8}\,\frac{R^4}{\eta}\,\frac{\Delta p}{l} = -\frac{\pi}{8}\,\frac{R^4}{\eta}\,\frac{\partial p}{\partial z}$$

$$\Rightarrow \qquad -\frac{\partial p}{\partial z} = \frac{8\,\dot{V}\,\eta}{\pi\,R^4} = 8\,\eta\,\frac{\overline{U}}{R^2} = 0.4032\,\frac{\mathrm{bar}}{\mathrm{km}}\,.$$

Problem 6-3 Oscillating pipe flow



Incompressible Newtonian flow (density ϱ, viscosity η) through an infinitely long straight pipe is subjected to a periodic pressure gradient. The radius of the pipe is R. The pressure gradient is given by $\partial p/\partial z = -\varrho \overline{K} \cos(\omega t)$, with $\overline{K} = \text{const.}$ To simplify the equation of motion we refer to the discussion in F. M. (Chapter 6.1.5). Consider the steady-state oscillating flow. Body forces are neglected. Calculate the velocity distribution.

Given: $\overline{K}$, R, ϱ, η

Solution

We use cylindrical coordinates. At the wall $r = R$ the velocity is $u_r = u_\varphi = 0$. Similar to the Hagen-Poiseuille flow (see F. M. Chap. 6.1.5) we assume u_r and u_φ are equal to zero within the entire flow field. Since the flow is axisymmetric the derivatives of all flow quantities in φ-direction disappear ($\partial/\partial\varphi = 0$). From the continuity equation (see F. M. , Appendix B.2) we then obtain

$$\frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad u_z = u_z(r,t) \, .$$

The r-component of the Navier-Stokes equation gives

$$0 = \frac{\partial p}{\partial r} \quad \Rightarrow \quad p = p(z,t) \, .$$

All terms of the φ-component disappear identically. With the given pressure gradient we find from the z-component the equation

$$\frac{\partial u_z}{\partial t} = \overline{K} \cos \omega t + \nu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) \tag{1}$$

which is subject to the boundary condition

$$u_z(r = R, t) = 0 \, .$$

To solve the differential equation (1) we use the complex notation

$$-\frac{1}{\varrho} \frac{\partial p}{\partial z} = \overline{K} e^{i \omega t} \, , \tag{2}$$

where only the real part has physical meaning. This suggests the following velocity distribution

$$u_z(r,t) = f(r)\mathrm{e}^{\mathrm{i}\omega t} \,. \tag{3}$$

Inserting (2) and (3) into (1), we obtain a nonhomogeneous zero-th order Bessel's differential equation for the function $f(r)$:

$$r^2 f''(r) + r f'(r) - \mathrm{i}\,\frac{\omega}{\nu}\, r^2 f(r) = -\frac{\overline{K}}{\nu}\, r^2 \,. \tag{4}$$

We find a particular solution $f_p(r)$ of this equation by using an expression that has the type of the right hand side, i. e.

$$f_p(r) = -\mathrm{i}\,\frac{\overline{K}}{\omega} \,. \tag{5}$$

The general solution of the homogeneous differential equation is

$$f_h(r) = C_1 J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\, r\right) + C_2 Y_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\, r\right) \,. \tag{6}$$

This solution can be found in textbooks on ordinary differential equations. The solution is a linear combination of Bessel functions of the first kind of order zero

$$J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\, r\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sqrt{\frac{-\mathrm{i}\,\omega}{\nu}}\,\frac{r}{2}\right)^{2n}$$

and the Bessel functions of the second kind of order zero

$$Y_0(z) = \frac{2}{\pi}\, J_0(z)(\ln\frac{z}{2} + \gamma) = -\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right) \,,$$

here $\gamma \approx 0.5772$ is the Euler-Máscheroni constant. The function $Y_0(\sqrt{-\mathrm{i}\omega/\nu}\, r)$ has a logarithmic singularity for $r = 0$. Since the velocity must remain finite within the entire pipe, the function $Y_0(z)$ cannot be a solution, thus $C_2 = 0$. As a result, we obtain from (3), (5), and (6)

$$u_z(r,t) = \left[C_1 J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\, r\right) - \mathrm{i}\,\frac{\overline{K}}{\omega}\right] \mathrm{e}^{\mathrm{i}\omega t} \tag{7}$$

and with the no-slip condition at the pipe wall

$$u_z(r=R,t) = 0 = \left[C_1 J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\, R\right) - \mathrm{i}\,\frac{\overline{K}}{\omega}\right] \mathrm{e}^{\mathrm{i}\omega t} \,,$$

we obtain the constant

$$C_1 = \frac{\mathrm{i}\,\overline{K}/\omega}{J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\, R\right)}$$

and thus the solution

$$u_z(r,t) = -\mathrm{i}\,\frac{\overline{K}}{\omega}\mathrm{e}^{\mathrm{i}\omega t}\left(1 - \frac{J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\,r\right)}{J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\,R\right)}\right) .$$

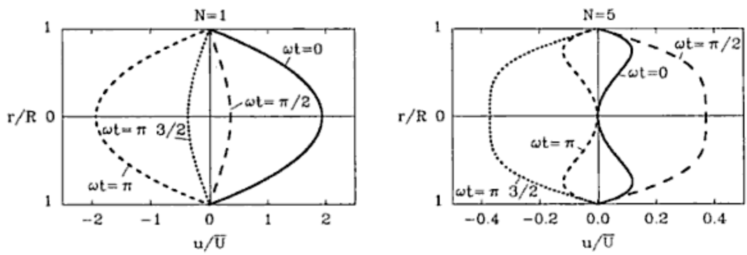
Since only the real part has physical meaning, we write

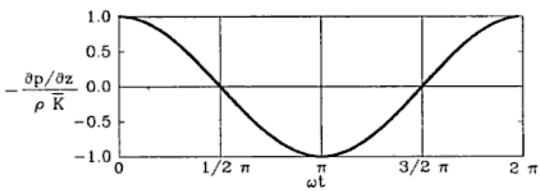
$$u_z(r,t) = \Re\left[-\mathrm{i}\,\frac{\overline{K}}{\omega}\mathrm{e}^{\mathrm{i}\omega t}\left(1 - \frac{J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\,r\right)}{J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}}\,R\right)}\right)\right] . \tag{8}$$

To plot this function, we nondimensionalize the velocity component $u_z(r,t)$ with the mean velocity $\overline{U} = \overline{K}\,R^2/(8\nu) = \overline{K}\,N^2/(8\omega)$ (see F. M. (6.58))

$$\frac{u_z(r,t)}{\overline{U}} = \Re\left[-\mathrm{i}\,\frac{8}{N^2}\mathrm{e}^{\mathrm{i}\omega t}\left(1 - \frac{J_0\left(\sqrt{-\mathrm{i}}\,N\,\frac{r}{R}\right)}{J_0\left(\sqrt{-\mathrm{i}}\,N\right)}\right)\right] , \tag{9}$$

where we introduced the parameter $N = \sqrt{\omega/\nu}\,R$. In the figures, the velocity distribution is plotted for the two parameter values of $N = 1$ and $N = 5$, corresponding to two different frequencies ω.

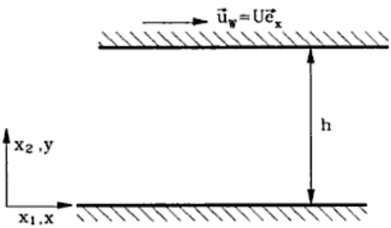




For small frequencies, the velocity distribution has the same phase as the time dependent pressure gradient. The velocity distribution is parabolic as in the case of the Hagen-Poiseuille flow. Increasing the frequency causes a phase delay of the flow in the middle of the pipe compared to the flow close to the wall. The velocity amplitude in the middle of the pipe decreases, the fluid velocity oscillates with a phase delay of a quarter of a period compared to the driving pressure gradient.

Problem 6-4 Comparison of a Couette-Poiseuille flow of a Newtonian fluid, a Stokes fluid, and a Bingham material

Incompressible fluid of constant density ϱ flows between two parallel plates infinitely extended in x- and z-directions.



The top plate moves with a constant velocity $\vec{u}_W = U\,\vec{e}_x$. The motion of the plate and the x-component of the pressure gradient $\partial p/\partial x = -K$ cause a steady Couette-Poiseuille flow. Body forces are neglected. Calculate

a) the dissipation function Φ and
b) the dissipated energy E_d per unit of length, depth, and time between the plates

for

1) a Newtonian fluid,
2) a fluid that follows the constitutive equation

$$\tau_{ij} = -p\,\delta_{ij} + 2\alpha\, e_{ij} + 4\beta\, e_{ik}e_{kj}$$

(incompressible Stokes fluid) and for

3) a Bingham material (ϑ, η_1, G).

Vary at fixed $K > 0$, $U > 0$ and h the material constants η, α, β, η_1, and ϑ and compare the results.

Given: h, U, K, ϱ, η, η_1, ϑ, α, $\beta = \text{const}$

Solution

a) The dissipation function Φ is in general

$$\Phi = P_{ij} e_{ij} \ .$$

The calculation requires the constitutive equation and the velocity field. In the present case, the flow is steady and unidirectional such that

$$u_1 = u_1(x_2) \ , \quad u_2 = u_3 = 0 \ .$$

Thus, the deformation tensor e_{ij} has only two non-zero components, namely

$$e_{12} = e_{21} = \frac{1}{2}\frac{\partial u_1}{\partial x_2} = \frac{1}{2}\frac{\mathrm{d}u_1}{\mathrm{d}x_2} \ ,$$

and the dissipation function is simplified to

$$\Phi = P_{12}e_{12} + P_{21}e_{21} = 2P_{12}e_{12} \ , \qquad (1)$$

where the symmetry of the friction stress tensor P_{ij} has been used.

1) The friction stress tensor P_{ij} for a Newtonian fluid is (see F. M. (3.2a))

$$P_{ij} = \lambda^* e_{kk}\delta_{ij} + 2\eta\, e_{ij} \ ,$$

and can be simplified for incompressible flow ($e_{kk} = 0$) as

$$P_{ij} = 2\eta\, e_{ij} \ .$$

Thus, we obtain from (1)

$$\Phi = 2P_{12}e_{12} = 4\eta\, e_{12}e_{12} = \eta \left(\frac{\mathrm{d}u_1}{\mathrm{d}x_2}\right)^2 \ .$$

The velocity component $u_1(x_2) = u(y)$ of the Couette-Poiseuille flow (see F. M. (6.19)) is

$$\frac{u(y)}{U} = \frac{y}{h} + \frac{K\,h^2}{2\eta\,U}\left(1 - \frac{y}{h}\right)\frac{y}{h} \ ,$$

from which we obtain

$$\frac{\mathrm{d}u_1}{\mathrm{d}x_2} = \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{U}{h} + \frac{K\,h}{2\eta}\left(1 - 2\frac{y}{h}\right) \ .$$

The dissipation function becomes

$$\Phi(y) = \eta\left[\frac{U}{h} + \frac{K\,h}{2\eta}\left(1 - 2\frac{y}{h}\right)\right]^2 \ ,$$

or in nondimensional form

$$\tilde{\Phi}_N = \frac{\Phi}{K\,U/2} = A_N\left(1 + \frac{1}{A_N}(1 - 2\tilde{y})\right)^2 \ , \qquad (2)$$

with $A_N = 2U\,\eta/(K\,h^2)$ and $\tilde{y} = y/h$.

2) From the stress tensor for a Stokes fluid

$$\tau_{ij} = -p\,\delta_{ij} + 2\alpha\, e_{ij} + 4\beta\, e_{ik}e_{jk}$$

and its decomposition (see F. M. (2.35))

$$\tau_{ij} = -p\,\delta_{ij} + P_{ij}$$

we obtain the friction stress tensor as

$$P_{ij} = 2\alpha\, e_{ij} + 4\beta\, e_{ik}e_{jk}\ . \tag{3}$$

For the dissipation function we have

$$\Phi = 2P_{12}e_{12}\ ,$$

where P_{12} will be calculated from equation (3)

$$P_{12} = 2\alpha\, e_{12} + 4\beta(\underbrace{e_{11}e_{21}}_{=0} + \underbrace{e_{12}e_{22}}_{=0} + \underbrace{e_{13}e_{23}}_{=0}) = 2\alpha\, e_{12}\ .$$

Thus, the dissipation function is obtained as

$$\Phi = 4\alpha\, e_{12}e_{12} = \alpha\left(\frac{\mathrm{d}u_1}{\mathrm{d}x_2}\right)^2\ . \tag{4}$$

To calculate the velocity field, we start from the Cauchy equation (see F. M. (2.38a))

$$\varrho\,\frac{\mathrm{D}u_i}{\mathrm{D}t} = \varrho\, k_i + \frac{\partial \tau_{ji}}{\partial x_j}\ ,$$

and simplify it for a steady unidirectional flow, where body forces are neglected

$$0 = \frac{\partial \tau_{ji}}{\partial x_j}\ . \tag{5}$$

The first component of equation (5)

$$0 = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}$$

together with the stress components τ_{ij}

$$\tau_{11} = -p\,\delta_{11} + 4\beta(\underbrace{e_{11}e_{11}}_{=0} + e_{12}e_{12} + \underbrace{e_{13}e_{13}}_{=0}) = -p + 4\beta\, e_{12}^2\ ,$$

$$\tau_{12} = 2\alpha\, e_{12} + 4\beta(\underbrace{e_{11}e_{21}}_{=0} + \underbrace{e_{12}e_{22}}_{=0} + \underbrace{e_{13}e_{23}}_{=0}) = 2\alpha\, e_{12} = \tau_{21}$$

delivers a differential equation for the unknown velocity $u_1(x_2)$:

$$0 = -\frac{\partial p}{\partial x_1} + 2\alpha\frac{\partial e_{12}}{\partial x_2},$$

$$0 = K + \alpha\frac{\mathrm{d}^2 u_1}{\mathrm{d}{x_2}^2}$$

or $\mathrm{d}^2u/(\mathrm{d}y)^2 = -K/\alpha$. These differential equations for $u(y)$ and the corresponding boundary conditions

$$u(y=0) = 0\ , \quad u(y=h) = U$$

are the same as for the Couette-Poiseuille flow. Hence, we get the same velocity field and from (4), the same expression for the dimensionless dissipation function Φ:

$$\tilde{\Phi}_S = \frac{\Phi}{K\,U/2} = A_S\left(1 + \frac{1}{A_S}(1-2\tilde{y})\right)^2\ , \tag{6}$$

where now

$$A_S = \frac{2U\,\alpha}{K\,h^2}\ ,$$

if we use α instead of η.

Note 1: Calculating the Couette-Poiseuille flow (see F. M. Chap. 6.1.2), we conclude from the y-component of the Navier-Stokes equation that the pressure p is only a function of x. This restriction follows from the material law for Newtonian fluids. But, for the Couette-Poiseuille flow of a Stokes fluid, we obtain from the second component of (4)

$$0 = -\frac{\partial p}{\partial x_2} + 4\beta\frac{\partial e_{12}^2}{\partial x_2}$$

and thus,

$$\frac{\partial p}{\partial y} = \beta\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\ ,$$

resulting in the pressure distribution

$$p(x,y) = \beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - K\,x + C\ .$$

The pressure distribution here is also a function of the coordinate y. The integration constant C cannot be obtained for an incompressible flow without a boundary condition on the pressure.

Note 2: The material constant β has no influence on the dissipation function, however, it does have an influence on the pressure distribution.

3) In a Couette-Poiseuille flow with a Bingham material as working medium (see F. M. , Chap. 6.4.1) energy is dissipated only in the flow zones. From the constitutive relation for a Bingham material we obtain for the flow case

$$P_{ij} = 2\eta\, e'_{ij} \quad \text{with} \quad \eta = \eta_1 + \frac{\vartheta}{\sqrt{2e'_{ij}e'_{ij}}} .$$

In calculating P_{12} we focus our attention to the summation procedure in the expression $e'_{ij}e'_{ij}$, where we sum over the indices i and j noting that e'_{ij} is the deviatoric part of the rate of deformation tensor e_{ij}. We get

$$P_{12} = P_{xy} = 2\left(\eta_1 + \frac{\vartheta}{\sqrt{4e'_{xy}e'_{xy}}}\right) e'_{xy} .$$

With $e'_{xy} = \frac{1}{2}\mathrm{d}u/\mathrm{d}y$ we obtain

$$P_{xy} = \eta_1 \frac{\mathrm{d}u}{\mathrm{d}y} + \vartheta\,\mathrm{sgn}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)$$

and thus the dissipation function

$$\Phi = 2P_{xy}e_{xy} = \eta_1\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + \vartheta\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right| . \qquad (7)$$

From the velocity distribution in the flow zones (see F. M. (6.197), (6.198)) it follows for the first flow zone

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{K\,h}{\eta_1}\left(\kappa_1 - \frac{y}{h}\right) \geq 0$$

and the second zone

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{K\,h}{\eta_1}\left(\kappa_2 - \frac{y}{h}\right) \leq 0 .$$

With the dimensionless parameters $A = 2U\eta_1/(K\,h^2)$ and $B = 2\vartheta/(K\,h)$, using $\tilde{y} = y/h$ we calculate the dimensionless dissipation function in the first flow zone $(0 < \tilde{y} < \kappa_1)$ as

$$\tilde{\Phi}_{B_1} = \frac{\Phi}{K\,U/2} = \frac{2}{A}\left(2(\kappa_1 - \tilde{y})^2 + B(\kappa_1 - \tilde{y})\right) \qquad (8)$$

and in the second flow zone ($\kappa_2 < \tilde{y} < 1$)

$$\tilde{\Phi}_{B_2} = \frac{\Phi}{K\,U/2} = \frac{2}{A}\left(2(\kappa_2 - \tilde{y})^2 - B(\kappa_2 - \tilde{y})\right) \;. \tag{9}$$

With known A and B the boundaries of the flow zones are given by

$$\kappa_1 = \frac{A + (1-B)^2}{2(1-B)} \qquad \text{(F. M. (6.203))}$$

and

$$\kappa_2 = \frac{A + (1-B^2)}{2(1-B)} \,. \qquad \text{(F. M. (6.204))}$$

b) The dissipated energy E_d per unit of length, depth, and time between the plates is found by integrating the dissipation function Φ:

$$E_d = \int_0^h \Phi(y)\mathrm{d}y = \frac{K\,U}{2}\int_0^h \tilde{\Phi}(y)\mathrm{d}y \,.$$

With $\tilde{y} = y/h$ and $\mathrm{d}y = h\mathrm{d}\tilde{y}$ the dimensionless dissipated energy

$$\tilde{E}_d = \frac{E_d}{K\,U\,h/2} = \int_0^1 \tilde{\Phi}(\tilde{y})\mathrm{d}\tilde{y}$$

takes on the following forms:

1) For Newtonian fluids in connection with equation (2) we have

$$\tilde{E}_{d_N} = \int_0^1 \tilde{\Phi}_N(\tilde{y})\mathrm{d}\tilde{y} = A_N + \frac{1}{3A_N} \,, \tag{10}$$

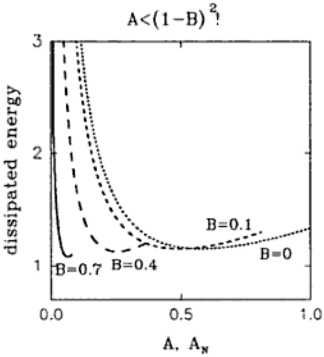
2) For the Stokes fluid from equation (6)

$$\tilde{E}_{d_S} = \int_0^1 \tilde{\Phi}_S(\tilde{y})\mathrm{d}\tilde{y} = A_S + \frac{1}{3A_S} \,, \tag{11}$$

3) For the Bingham material from equation (8) and (9)

$$\begin{aligned} \tilde{E}_{d_B} &= \int_0^{\kappa_1} \tilde{\Phi}_{B_1}(\tilde{y})\mathrm{d}\tilde{y} + \int_{\kappa_2}^1 \tilde{\Phi}_{B_2}(\tilde{y})\mathrm{d}\tilde{y} \\ &= \frac{2 + 6A^2 - 7B - 3A^2B + 8B^2 - 2B^3 - 2B^4 + B^5}{6A(B-1)^2} \,. \end{aligned} \tag{12}$$

Concluding the comparison, we keep U, K, h constant and vary the material constants η, α, η_1, and ϑ. The constant β, as shown, has no effect on the dissipation. The Stokes fluid behaves like a Newtonian one; α plays the role of η. To compare the Bingham material and Newtonian fluids, we plot equations (10) and (12) as functions of dimensionless viscosity A_N and A. The dimensionless yield stress B for the Bingham material is family parameter.



As seen, each curve has a minimum. In case of a Newtonian flow this minimum can be calculated from equation (2) as

$$A_{N_{\min}} = \sqrt{\tfrac{1}{3}}\,.$$

For $B = 0$, the Bingham material behaves like a Newtonian fluid. The intersection point $\hat{A}$, from which on more energy is dissipated in a Bingham material than in Couette-Poiseuille flow, is obtained as a function of B by equating equations (10) and (12):

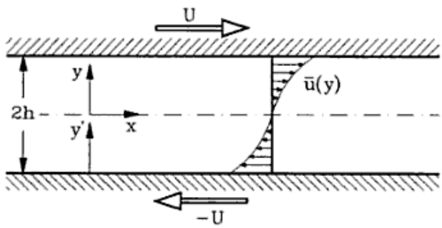
$$\hat{A} = \sqrt{\frac{3 - 6B + 2B^2 + 2B^3 - B^4}{9 - 6B}}\,.$$

The steep slope of the dissipated energy for $A_N \to 0$ and $A \to 0$ is caused by an increased volume flux and a strong velocity gradient as a result of a fixed pressure gradient in x-direction. The special case $\eta = 0 \to A = 0$ (frictionless flow) cannot be described, since then $K = 0$ and the wall motion does not generate any flow.

7 Fundamentals of Turbulent Flows

Problem 7-1 Turbulent Couette flow

The velocity field of turbulent Couette flow between two, infinitely long plates, moving in opposite directions, shall be calculated. The flow with a constant density ϱ has a time averaged steady velocity field that is a function of y only. Body forces are neglected. The turbulent apparent stresses (Reynolds' stresses) are calculated using the Prandtl mixing length theory, where for the distribution of the mixing length the following relationship is used:



$$l(y) = K\,(h^2 - y^2)\,.$$

a) Determine the constant K such that the condition

$$-\left.\frac{\mathrm{d}l}{\mathrm{d}y}\right|_{y=\pm h} = \pm\kappa$$

is satisfied.

b) Find the equation of the turbulent shear stress τ_t for the given mixing length distribution.

c) Since in Couette flow $\overline{p}$ is constant, the shear stress is also constant. As a result, we have

$$\eta\,\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \varrho\,\overline{u'\,v'} = \varrho\,u_*^2 = \text{const}\ .$$

Outside the viscous sublayer, the viscous shear stress $\eta\,\mathrm{d}\overline{u}/\mathrm{d}y$ can be neglected compared to the turbulent shear stress. Calculate the velocity profile $\overline{u}(y)$ (Hint: $\overline{u}(y=0)$ follows from the symmetry condition).

d) Give the velocity distribution as a function of the distance from the bottom wall $y' = y + h$.

e) Show that for small y' the logarithmic law in a dimensionally homogeneous form emerges (Hint: The velocity distribution from d) is at the edge of the viscous sublayer ($y'u_*/\nu = \beta$) equal to the velocity distribution of the viscous sublayer).

Given: h, κ, U, u_*, ϱ, ν

Solution

a) The constant K:

Close to the wall the mixing length has to be

$$l = \kappa\, y' \,, \qquad \left(\frac{y'}{h} \ll 1\right) \,.$$

Hence,

$$\left.\frac{\mathrm{d}l}{\mathrm{d}y}\right|_{y=-h} = \kappa \,, \qquad \left.\frac{\mathrm{d}l}{\mathrm{d}y}\right|_{y=+h} = -\kappa \,.$$

With the given mixing length it then follows

$$\left.\frac{\mathrm{d}l}{\mathrm{d}y}\right|_{y=\mp h} = K\,(-2\,y)\Big|_{y=\mp h} = \pm 2\,K\,h = \pm\kappa$$

$$\Rightarrow \qquad K = \frac{\kappa}{2\,h} \,. \tag{1}$$

b) The turbulent shear stress:

Prandtl's mixing length formula:

$$\tau_t = -\varrho\,\overline{u'\,v'} = \varrho\, l^2 \left|\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right| \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} \,.$$

In the present case $\mathrm{d}\overline{u}/\mathrm{d}y$ is always positive. We introduce the given distribution of $l(y)$ with K from (1) into the above equation and find

$$\tau_t = -\varrho\,\overline{u'\,v'} = \varrho\left[\frac{\kappa}{2\,h}\left(h^2 - y^2\right)\right]^2 \left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2 \,. \tag{2}$$

c) Velocity profile $\overline{u}(y)$:

Because the pressure gradient vanishes the entire shear stress consisting of viscous and turbulent shear stress is constant over the channel height:

$$\eta\,\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \varrho\,\overline{u'\,v'} = \tau_w = \varrho\, u_*^2 = \text{const} \,. \tag{3}$$

Outside the viscous sublayer and the buffer layer the viscous contribution in (3) can be neglected. Thus, for the turbulent portion we may write

$$-\varrho\,\overline{u'v'} = \varrho\,u_*^2\ ,$$

with (2)

$$\begin{aligned}
\varrho\left[\frac{\kappa}{2\,h}\left(h^2-y^2\right)\right]^2\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2 &= \varrho\,u_*^2\\
\Rightarrow\qquad \frac{\kappa}{2\,h}\left(h^2-y^2\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} &= u_*\\
\Rightarrow\qquad \frac{1}{u_*}\int \mathrm{d}\overline{u} &= \frac{2\,h}{\kappa}\int\frac{\mathrm{d}y}{h^2-y^2}+\mathrm{const}
\end{aligned}$$

and integrated:

$$\frac{\overline{u}}{u_*} = \frac{1}{\kappa}\ln\left(\frac{h+y}{h-y}\right)+\mathrm{const}\ ,$$

where the constant is zero, since for symmetry reasons we have $\overline{u}(y=0)=0$.

d) $\overline{u}=\overline{u}(y')$:
With $y'=y+h$ we obtain

$$\frac{\overline{u}(y')}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y'}{2h-y'}\right) = \frac{1}{\kappa}\ln\left(\frac{y'/h}{2-(y'/h)}\right)\ . \tag{4}$$

e) Rearranging equation (4) in the following form

$$\frac{\overline{u}(y')}{u_*} = \frac{1}{\kappa}\left[\ln\left(\frac{y'}{2h}\right)-\ln\left(1-\frac{y'}{2h}\right)\right]\ , \tag{5}$$

and considering $y'/h \ll 1$, we arrive at

$$\frac{\overline{u}(y')}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y'}{2h}\right)\ .$$

This equation is now written in the form

$$\frac{\overline{u}(y')}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y'u_*}{\nu}\right)+\frac{1}{\kappa}\ln\left(\frac{\nu}{2hu_*}\right)\ . \tag{6}$$

At the edge of the viscous sublayer this velocity distribution must coincide with the one of viscous sublayer. The velocity distribution within the viscous sublayer is linear (see F. M. (7.54)), here we have

$$\frac{\overline{u}+U}{u_*} = \frac{y'u_*}{\nu} = y_*\ ,$$

where y_* represents the dimensionless distance from the bottom wall and U the magnitude of the wall velocity

$$\Rightarrow \qquad \frac{\overline{u}}{u_*} = y_* - \frac{U}{u_*} \,. \tag{7}$$

With β as the dimensionless thickness of the viscous sublayer, we obtain from (6) and (7)

$$\beta - \frac{U}{u_*} = \frac{1}{\kappa} \ln \beta + \frac{1}{\kappa} \ln \left(\frac{\nu}{2hu_*} \right)$$

and identify the constant of the logarithmic law B as

$$\frac{1}{\kappa} \ln \left(\frac{\nu}{2hu_*} \right) + \frac{U}{u_*} = \beta - \frac{1}{\kappa} \ln \beta = B \,.$$

Thus, we get from (6)

$$\frac{\overline{u} + U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y' u_*}{\nu} \right) + B \,. \tag{8}$$

This is the logarithmic law of the wall, which differs from the usual form through the superimposed plate velocity U/u_*.

Problem 7-2 Velocity distribution in turbulent Couette flow with given Reynolds number

The velocity distribution $\overline{u}(y')/U$ in Problem 7-1 shall be calculated for a Reynolds number of $Re = 2h\,U/\nu = 34\,000$. From the inner law and the logarithmic law of the wall (equation (6) and (8) in Problem 7-1) we determine first an implicit equation for u_*. From this equation obtain the number $2h\,u_*/\nu$ and the velocity distribution $\overline{u}(y')/U$ for $0 < y' < 2h$.

Solution

From the equation

$$\frac{\overline{u}(y')}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y' u_*}{\nu} \right) + \frac{1}{\kappa} \ln \left(\frac{\nu}{2hu_*} \right)$$

and

$$\frac{\overline{u} + U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y' u_*}{\nu} \right) + B$$

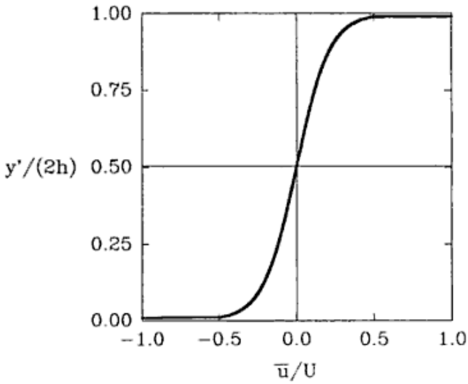
we obtain by subtracting the relation

$$\frac{2hU}{\nu}\frac{\nu}{u_* 2h} = B - \frac{1}{\kappa}\ln\left(\frac{\nu}{2hu_*}\right) .$$

With $B = 5.0$, $\kappa = 0.4$ and $2hU/\nu = 34\,000$ we get the numerical solution of this implicit equation as $u_* 2h/\nu = 1\,464.11$. The velocity distribution (equation (5), Problem 7-1) between the plates is written in the form

$$\frac{\overline{u}(y')}{U} = \frac{u_* 2h}{\nu}\frac{\nu}{2hU}\frac{1}{\kappa}\left[\ln\left(\frac{y'}{2h}\right) - \ln\left(1-\frac{y'}{2h}\right)\right]$$

and is plotted in the intervall $0 < y'/(2h) < 1$, with the values $\kappa = 0.4$, $u_* 2h/\nu = 1464$ and $2hU/\nu = 34\,000$ in the figure.



Problem 7-3 Turbulent pipe flow

Incompressible fluid flows through a pipe with the diameter d. The pipe is hydraulically smooth ($k/d = 0$) and the volume flux $\dot{V}$ is known.

a) Determine the velocity $\overline{U}$ averaged over the cross-section and the Reynolds number. Is the flow laminar or turbulent?

b) Calculate the friction factor λ. Determine the friction velocity u_* and the maximum velocity $U_{\max}$ in the pipe center. Estimate the thickness of the viscous sublayer δ_V.

c) Calculate the wall shear stress τ_w and the pressure gradient $\partial p/\partial x$.

d) Give the value of the turbulent shear stress at the wall and in the center of the pipe. Sketch qualitatively the distribution of the total and the turbulent shear stress.

Given: $\dot{V} = 0.07854\,\mathrm{m^3/s}$, $d = 2R = 0.1\,\mathrm{m}$, $\nu = 10^{-6}\,\mathrm{m^2/s}$, $\varrho = 10^3\,\mathrm{kg/m^3}$

Solution

a) The average velocity is

$$\overline{U} = \dot{V}/\pi R^2 = 10\,\mathrm{m/s}\ ,$$

and the Reynolds number

$$Re = \overline{U}\,d/\nu = 10^6\ ,$$

i. e. the flow is turbulent.

b) The friction factor is calculated numerically using the implicit equation:

$$\begin{aligned} \frac{1}{\sqrt{\lambda}} &= 2.03\,\mathrm{lg}\left(\mathrm{Re}\sqrt{\lambda}\right) - 0.8 \\ \Rightarrow \qquad \lambda &= 0.011308\ . \end{aligned}$$

The friction velocity follows from

$$\lambda = 8\,(u_*/\overline{U})^2 \qquad \text{to} \qquad u_* = 0.375\,\mathrm{m/s}\ ,$$

thus, we obtain

$$U_{\max} = \overline{U} + 3.75\,u_* = 11.41\,\mathrm{m/s}\ .$$

An estimate for the thickness δ_V of the viscous sublayer follows from the inequality

$$0 \le \frac{\delta_V\,u_*}{\nu} \le 5 \qquad \text{to} \qquad \delta_V = \frac{5\,\nu}{u_*} = 0.013\,\mathrm{mm}\ .$$

c) The wall shear stress and the pressure gradient:

$$\begin{aligned} \tau_w &= \varrho\,{u_*}^2 = 140.6\frac{\mathrm{N}}{\mathrm{m}^2}\ , \\ \frac{\partial\overline{p}}{\partial x} &= -\frac{2}{R}\,\tau_w = -0.0562\frac{\mathrm{bar}}{\mathrm{m}}\ . \end{aligned}$$

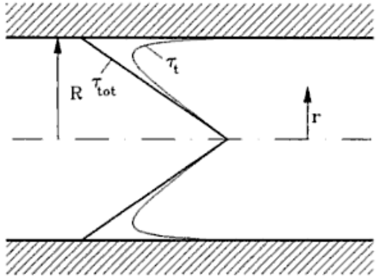
d) The shear stress distribution:
The total shear stress consists of a viscous and a turbulent part

$$\tau_{tot} = \tau_{vis} + \tau_t = \eta\,\frac{\mathrm{d}\overline{u}}{\mathrm{d}r} - \varrho\,\overline{u'\,v'}\ .$$

The turbulent shear stress

$$\tau_t = -\varrho\, \overline{u'v'}$$

disappears at the wall because of the no-slip condition and in the pipe center for symmetry reasons:



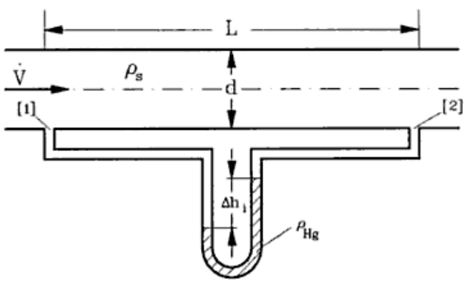
$$\tau_t|_{r=R} = 0 \,, \qquad \tau_t|_{r=0} = 0 \,.$$

The distribution of the total shear stress is linear

$$\tau_{tot} = -\tau_w \frac{r}{R} \,.$$

Problem 7-4 Crystal growth on pipe walls

A fluid containing salt (density ϱ_S) causes salt crystallization at the wall while flowing through a long pipe. To control the crystal growth a manometer is installed between stations [1] and [2]. The manometer fluid, mercury, has the density ϱ_{Hg}. The column heights Δh_1, Δh_2, and Δh_3 are measured in equal time intervals.



a) Find the pressure difference $\Delta p_i(\Delta h_i)$.
b) Determine the loss factor $\zeta_i(\Delta h_i)$.
c) Calculate the three Reynolds numbers.
d) Determine the averaged crystal heights that correspond to the pressure differences.

Given: $L = 10\,\mathrm{m}$, $d = 1\,\mathrm{m}$, $\dot V = 4.3\,\mathrm{m^3/s}$, $\varrho_S = 1184\,\mathrm{kg/m^3}$, $\eta_S = 0.01296\,\mathrm{kg/(ms)}$, $\varrho_{Hg} = 13550\,\mathrm{kg/m^3}$, $\Delta h_1 = 41.68\,\mathrm{mm}$, $\Delta h_2 = 64.00\,\mathrm{mm}$, $\Delta h_3 = 95.08\,\mathrm{mm}$, $g = 9.81\,\mathrm{m/s^2}$

Solution

a) Pressure differences $\Delta p_i(\Delta h_i)$:
Hydrostatics:
(p_L, p_R denote the pressures acting on the left and right surfaces of the manometer fluid)

$$\begin{aligned} p_L &= p_1 + \varrho_S\, g\,(z_1 - z_l)\,, \\ p_R &= p_2 + \varrho_S\, g\,(z_2 - z_r)\,. \end{aligned}$$

With $z_1 = z_2$ and $z_r - z_l = \Delta h$ we obtain the pressure difference

$$\begin{aligned} \Rightarrow \qquad p_L - p_R &= \varrho_{Hg}\, g\, \Delta h = p_1 - p_2 + \varrho_S\, g\, \Delta h \\ \Rightarrow \qquad p_1 - p_2 &= (\varrho_{Hg} - \varrho_S)\, g\, \Delta h\,. \end{aligned}$$

We set for different measurements $\Delta p_i = (p_1 - p_2)_i$, thus

$$\Delta p_i = (\varrho_{Hg} - \varrho_S)\, g\, \Delta h_i\,.$$

We get

$$\begin{aligned} \Delta h_1 &= 41.68 \cdot 10^{-3}\,\mathrm{m} \qquad \Rightarrow \qquad \Delta p_1 = 5056\,\mathrm{N/m^2}\,, \\ \Delta h_2 &= 64.00 \cdot 10^{-3}\,\mathrm{m} \qquad \Rightarrow \qquad \Delta p_2 = 7764\,\mathrm{N/m^2}\,, \\ \Delta h_3 &= 95.08 \cdot 10^{-3}\,\mathrm{m} \qquad \Rightarrow \qquad \Delta p_3 = 11534\,\mathrm{N/m^2}\,. \end{aligned}$$

b) Calculation of loss factors:

$$\zeta_i = \frac{\Delta p_i}{\varrho_S/2\,\overline{U}^2} \qquad \text{with} \qquad \overline{U} = \frac{\dot{V}}{A} = \frac{4\,\dot{V}}{\pi\, d^2} = 5.475\,\mathrm{m/s}$$

$$\Rightarrow \qquad \zeta_1 = 0.285\,, \qquad \zeta_2 = 0.438\,, \qquad \zeta_3 = 0.650\,.$$

c) Reynolds numbers:

$$Re = \frac{\overline{U}\, d\, \varrho_S}{\eta_S}$$

For all three measurements we find the same Reynolds number:

$$Re = 500185\,.$$

d) Crystallization height:

$$\lambda = \zeta\, \frac{d}{L} = \frac{\zeta}{10}$$

From the resistance law $\lambda = \lambda(Re, k/d)$, the known values of the friction factor λ_i and the Reynolds number Re we find the corresponding k_i/d. We read from the resistance diagram (see F. M. Fig. 7.4) ($Re \approx 5 \cdot 10^5$) the values in the following table:

i	λ_i	k_i/d	k_i[mm]
1	0.0285	0.004	4
2	0.0438	0.015	15
3	0.0650	0.040	40

Problem 7-5 Comparison of momentum and energy flux in laminar and turbulent flow in a pipe

For a hydraulically smooth circular pipe with radius R the following ratios for a) laminar and b) turbulent pipe flow are to be calculated:

$$1)\quad U_{\max}/\overline{U}\,,\quad 2)\quad \frac{\overline{U}^2 A}{\iint\limits_A \overline{u}^2\,\mathrm{d}A}\,,\quad 3)\quad \frac{\overline{U}^3 A}{\iint\limits_A \overline{u}^3\,\mathrm{d}A}\,.$$

In the above relations $\overline{U}$ denotes the area averaged and $\overline{u}$ the time averaged velocity. In the case of turbulent pipe flow we use the velocity defect law:

$$\frac{\overline{u}}{u_*} = \frac{U_{\max}}{u_*} + \frac{1}{\kappa}\ln\frac{y}{R}$$

The Reynolds number $Re = 2300$ is given.

Solution

a) Laminar pipe flow:

1) Using the ratio from (see F. M. (6.57))

$$\frac{U_{\max}}{\overline{U}} = 2\,.$$

2) With $\overline{u} = u(r) = K/(4\eta)\,(R^2 - r^2)$ we obtain first

$$\iint\limits_A \overline{u}^2\,\mathrm{d}A = \int\limits_0^{2\pi}\int\limits_0^{R} \overline{u}^2\,r\mathrm{d}r\mathrm{d}\varphi = \frac{\pi}{3}\left(\frac{K}{4\eta}\right)^2 R^6\,,$$

and with

$$\overline{U}^2 A = \left(\frac{K\,R^2}{8\eta}\right)^2 \pi R^2 = \frac{\pi}{4}\left(\frac{K}{4\eta}\right)^2 R^6$$

it then follows

$$\frac{\overline{U}^2 A}{\iint\limits_A \overline{u}^2 \,\mathrm{d}A} = \frac{3}{4}\ .$$

This means that the momentum flux that uses the averaged velocity is only 3/4 of the real momentum flux in a laminar flow.

3) With

$$\int\limits_0^{2\pi}\int\limits_0^{R} \left(\frac{K}{4\eta}(R^2-r^2)\right)^3 r\mathrm{d}r\mathrm{d}\varphi = 2\pi\left(\frac{K}{4\eta}\right)^3 \frac{R^8}{8}$$

and

$$\overline{U}^3 A = \pi\left(\frac{K}{4\eta}\right)^3 \frac{R^8}{8}$$

we obtain

$$\frac{\overline{U}^3 A}{\iint\limits_A \overline{u}^3 \,\mathrm{d}A} = \frac{1}{2}\ .$$

The energy flux that uses the averaged velocity is only half of the real energy flux through the pipe.

b) Turbulent pipe flow:

1) With the equations (see F. M. (7.83) and (7.87)) we have

$$\frac{U_{\max}}{\overline{U}} = 1 + 3.75\sqrt{\frac{\lambda}{8}}\ .$$

The friction factor λ is calculated for the Reynolds number $Re = 2300$ from the equation

$$\frac{1}{\sqrt{\lambda}} = 2.03\lg\left(Re\sqrt{\lambda}\right) - 0.8$$

(see F. M. (7.89)) as $\lambda = 0.0459257$ thus, we obtain

$$\frac{U_{\max}}{\overline{U}} = 1.28413\ .$$

For a turbulent pipe flow the difference between the maximum velocity and the mean velocity is smaller because the velocity profile in turbulent flow is fuller than that in the laminar one.

2) To compare the momentum fluxes we replace in the velocity defect law $U_{\max}$ by

$$U_{\max} = \overline{U} + 3.75u_*$$

and write

$$\frac{\overline{U}^2 A}{\iint\limits_A \overline{u}^2 \,\mathrm{d}A} = \frac{\overline{U}^2 \pi R^2}{u_*^2 2\pi \int\limits_0^R \left(\frac{\overline{U}}{u_*} + 3.75 + \frac{1}{\kappa}\ln\frac{y}{R}\right)^2 (R-y)\mathrm{d}y}$$

$$= \frac{8}{\lambda} \frac{R^2}{2\int\limits_0^R \left(\sqrt{\frac{8}{\lambda}} + 3.75 + \frac{1}{\kappa}\ln\frac{y}{R}\right)^2 (R-y)\mathrm{d}y} .$$

For $\kappa = 0.4$ the integration leads to the expression $(3.90625 + 4/\lambda)\,R^2$ and thus

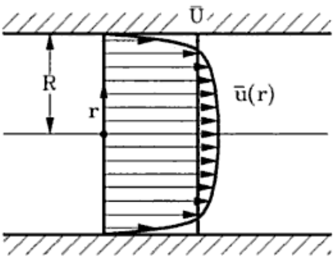
$$\frac{\overline{U}^2 A}{\iint\limits_A \overline{u}^2 \,\mathrm{d}A} = \frac{4}{3.90625\lambda + 4} = 0.957076 .$$

3) For the ratio of energy fluxes we get

$$\frac{\overline{U}^3 A}{\iint\limits_A \overline{u}^3 \,\mathrm{d}A} = \sqrt{\left(\frac{8}{\lambda}\right)^3} \frac{R^2}{2\int\limits_0^R \left(\sqrt{\frac{8}{\lambda}} + 3.75 + \frac{1}{\kappa}\ln\frac{y}{R}\right)^3 (R-y)\mathrm{d}y}$$

$$= \frac{8\sqrt{2}}{-17.5781\sqrt{\lambda^3} + 23.4375\sqrt{2}\lambda + 8\sqrt{2}} = 0.89345 .$$

Problem 7-6 Velocity distribution in a turbulent pipe flow resulting from the Blasius friction law



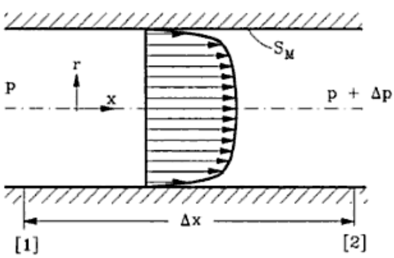
The friction factor λ of a turbulent pipe flow within a Reynolds number range $5000 < Re < 10^5$ can be calculated using the Blasius formula $\lambda = 0.316\, Re^{-1/4}$. The velocity distribution inside the pipe with the radius R has the form $\overline{u}(r) = C\,(R-r)^m$.

a) Calculate the wall shear stress.
b) Calculate the average velocity $\overline{U}$ using the above velocity distribution.
c) Show that the exponent m must have the value $m = 1/7$ to be consistent with the Blasius formula.
d) Calculate the constant C and plot the velocity distribution $\overline{u}(r)/\overline{U}$.

Solution

a) Wall shear stress:
We consider a pipe element of the length Δx. The Bernoulli equation containing the loss term Δp_l is written for a streamline between stations [1] and [2] as

$$p_1 + \frac{\varrho}{2}\overline{U}_1^2 = p_2 + \frac{\varrho}{2}\overline{U}_2^2 + \Delta p_l \,. \tag{1}$$



The continuity equation requires that $\overline{U}_1 = \overline{U}_2 = \overline{U}$. The pressure loss Δp_l is obtained using the friction factor λ:

$$\Delta p_l = \frac{\varrho}{2}\overline{U}^2\,\lambda\,\frac{\Delta x}{2R} \,.$$

The pressure drop between [1] and [2] follows from (1) as:

$$p_2 - p_1 = \Delta p = -\Delta p_l = -\frac{\varrho}{2}\overline{U}^2\,\lambda\,\frac{\Delta x}{2R} \,. \tag{2}$$

In a fully developed pipe flow the momentum fluxes cancel each other out, thus, the momentum balance is reduced to

$$\vec{e}_x \cdot \iint\limits_{(S)} \vec{t}\,\mathrm{d}S = 0$$

or

$$\iint\limits_{S_M} \tau_{rx}\, n_r\, \mathrm{d}S + \iint\limits_{S_1} \tau_{xx}\, n_x\, \mathrm{d}S + \iint\limits_{S_2} \tau_{xx}\, n_x\, \mathrm{d}S = 0\ ,$$

i. e.

$$2\pi\, R\, \Delta x\, \tau_{rx}(R) - \Delta p\, \pi\, R^2 = 0\ .$$

$$\tau_{rx}(R) = +\frac{\Delta p}{\Delta x}\,\frac{R}{2} = -\tau_w\ ,$$

with τ_w as the wall shear stress, which is by definition positive (see F. M. (7.85)). With (2) we have the expression

$$\tau_w = \frac{1}{8}\,\varrho\,\overline{U}^2\,\lambda\ ,$$

in which we can replace the friction factor λ by the Blasius formula ($Re = \overline{U}\,2R/\nu$):

$$\tau_w = \frac{0.316}{8 * 2^{1/4}}\,\varrho\,\nu^{1/4}\,R^{-1/4}\,\overline{U}^{7/4}\ .$$

The solution for the average velocity results in

$$\overline{U} = A^{4/7}\,R^{1/7} \quad \text{with} \quad A = \frac{8 * 2^{1/4}}{0.316}\,\frac{\tau_w}{\varrho\,\nu^{1/4}}\ . \tag{3}$$

b) Average velocity $\overline{U}$:
The average velocity $\overline{U}$ follows also from

$$\overline{U} = \frac{1}{A}\iint\limits_{(A)} u(r)\,\mathrm{d}S = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^{R} C\,(R-r)^m\,r\,\mathrm{d}r\,\mathrm{d}\varphi$$

as

$$\overline{U} = \frac{2C}{(m+1)(m+2)}\,R^m\ . \tag{4}$$

c) A comparison of the exponents of R in (3) and in (4) shows that the value of m must be $m = 1/7$ to be consistent with the Blasius formula.

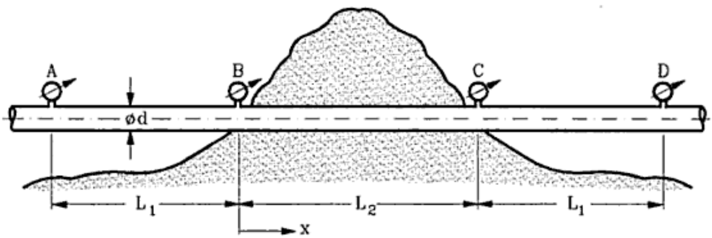
d) Velocity distribution:
With $m = 1/7$ it follows from (4)

$$\overline{U} = \frac{2\,C}{8/7 * 15/7}\,R^{1/7} \quad \text{or} \quad C \approx 1.2\,\frac{\overline{U}}{R^{1/7}}\ .$$

The required velocity distribution is therefore

$$\frac{\overline{u}}{\overline{U}} = 1.2\left(1 - \frac{r}{R}\right)^{1/7}\ .$$

Problem 7-7 Location of a pipe leakage



To test for leaks of a hydraulically smooth water pipe (diameter d) passing through a hill, static pressure measurements at stations A, B, C, and D are carried out. No leakage was found for the accessible portions AB and CD.

a) Under the assumption that the pipe flow is turbulent, calculate the volume fluxes between AB and CD from the given data.
b) In the case that you find a leakage, give the leakage volume flux.
c) From the given data determine the location of the leak x_L and the inner pressure p_L at the leak location (for example by extrapolating the pressure distribution).

Given: $d = 0.05\,\mathrm{m}$, $L_1 = 1000\,\mathrm{m}$, $L_2 = 1500\,\mathrm{m}$, $p_A = 6\,\mathrm{bar}$, $p_B = 4\,\mathrm{bar}$, $p_C = 1.5\,\mathrm{bar}$, $p_D = 1\,\mathrm{bar}$, $\varrho = 1000\,\mathrm{kg/m^3}$, $\nu = 10^{-6}\,\mathrm{m^2/s}$

Solution

a) Volume fluxes between AB und CD:

Pipe portion AB:

From the measured data the pressure loss between stations A and B is calculated as $\Delta p_l = p_A - p_B = 2$ bar. This pressure loss can be calculated using the friction factor λ, which is at this time unknown

$$\Delta p_l = \frac{\varrho}{2}\overline{U}^2\,\lambda\,\frac{L_1}{d}\,. \tag{1}$$

For a hydraulically smooth pipe the friction factor is determined by the implicit formula (see (7.89))

$$\frac{1}{\sqrt{\lambda}} = 2.03\ \lg(Re\sqrt{\lambda}) - 0.8 \tag{2}$$

as a function of Reynolds number $Re = \overline{U}\,d/\nu$. From equations (1) and (2) we eliminate the friction factor λ and obtain the average velocity

$$\overline{U} = \sqrt{\frac{2\,\Delta\,p\,d}{\varrho\,L_1}}\left[2.03\lg\left(\frac{d}{\nu}\sqrt{\frac{2\,\Delta\,p\,d}{\varrho\,L_1}}\right) - 0.8\right]\,.$$

As a result, we get $\overline{U} = 0.992\,\mathrm{m/s}$ and the Reynolds number as $Re = 0.992 * 0.05/10^{-6} = 49600 \gg Re_{\mathrm{crit}}$. We conclude that the flow is turbulent. The volume flux in the portion AB is

$$\dot{V}_{AB} = \overline{U}\,\frac{\pi}{4}\,d^2 = 1.95 * 10^{-3}\,\mathrm{m^3/s}\ .$$

In the portion CD:

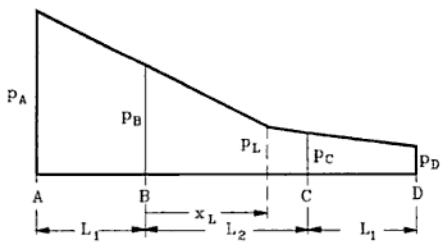
The measured pressure loss is $\Delta p_l = p_C - p_D = 0.5\,\mathrm{bar}$. We get $\overline{U} = 0.453\,\mathrm{m/s}$ and $Re = 22650 \gg Re_{\mathrm{crit}}$. Thus, it is justifiable to assume a turbulent flow in the pipe portion CD. The volume flux in CD is

$$\dot{V}_{CD} = \overline{U}\,\frac{\pi}{4}\,d^2 = 0.89 * 10^{-3}\,\mathrm{m^3/s}\ .$$

b) The leakage volume flux is

$$\dot{V}_L = \dot{V}_{AB} - \dot{V}_{CD} = 1.06 * 10^{-3}\,\mathrm{m^3/s}\ .$$

c) Location of the leak x_L and the pressure at the leakage location p_L: Equations (1) and (2) indicate that at a constant velocity $\overline{U}$, the pressure loss in the pipe is a linear function of the pipe length (see sketch).



To determine the location of the leak x_L and p_L we apply twice the linear interpolation formula

$$\frac{p_B - p_L}{p_A - p_L} = \frac{x_L}{L_1 + x_L} \tag{3}$$

and

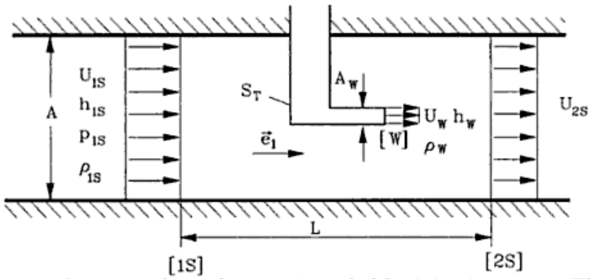
$$\frac{p_L - p_D}{p_C - p_D} = \frac{L_1 + L_2 - x_L}{L_1}\ . \tag{4}$$

The solution of the system (3), (4) is

$$p_L = p_B + \frac{(p_B - p_A)(L_1(p_B - p_C) + L_2(p_D - p_C))}{L_1(p_A + p_D - (p_B + p_C))} = \frac{5}{3}\,\mathrm{bar}\ ,$$

$$x_L = \frac{L_1(p_B - p_C) + L_2(p_D - p_C)}{p_A + p_D - (p_B + p_C)} = \frac{3500}{3}\,\mathrm{m}\ .$$

Problem 7-8 Cooling of superheated steam by water injection



In a power plant superheated steam is cooled by injecting water. The flow quantities of the steam at [1S] and of the water at [W] are known. The flow can be considered as uniform at stations [1S], [2S], and [W]. At [2S] the entire injected water is evaporated. The force (in $\vec{e}_1$-direction) from the steam on the injection tube was measured as

$$F_{S\to T} = \iint\limits_{(S_T)} \vec{t}\cdot\vec{e}_1 \,\mathrm{d}S = 500\,\mathrm{N}\ .$$

a) Estimate the total force on the wall $F_{S\to W}$ in $\vec{e}_1$-direction using Blasius' friction law
$$\frac{\tau_w}{\varrho \overline{U}^2} = 0.0395 Re^{-1/4}$$
Make a decission whether the pressure losses due to friction at the pipe wall should be considered or not.

b) Evaluate the linear momentum balance such that the pressure p_{2S} can be determined. Decide whether the force $F_{S\to T}$ in the momentum balance should be considered or not.

c) Simplify the energy equation in integral form so that the quantities u_i and h at the surface of a control volume can be evaluated assuming the flow to be uniform at the surfaces.

d) Can the kinetic energy be neglected compared to the enthalpy?

e) Establish the system of equations, that determines the unknowns ϱ_{2S}, u_{2S}, p_{2S}, and h_{2S}. For $h = h(\varrho, p)$ use the Mollier diagram from Problem 9.2-3.

Given.: $A = 2.4 * 10^5$ mm^2, $A_W = 5.3 * 10^2$ mm^2, $U_{1S} = 80$ m/s, $U_W = 20$ m/s, $\varrho_{1S} = 3.26$ kg/m^3, $\varrho_W = 916$ kg/m^3, $p_{1S} = 10$ bar, $h_{1S} = 3264$ kJ/kg, $h_W = 632$ kJ/kg, $\nu_{1S} = 7.5 * 10^{-6}$ m^2/s, L=3 m

Solution

a) The force on the wall as a result of wall shear stress τ_w is

$$F_{S\to W} = \tau_w \pi \, d \, L$$

and consequently with $\overline{U} = U_{1S}$, $d = 2\sqrt{A/\pi}$ and $Re = \overline{U} d/\nu$

$$F_{S\to W} = 87\,\mathrm{N} \,.$$

With (see F. M. (7.87))

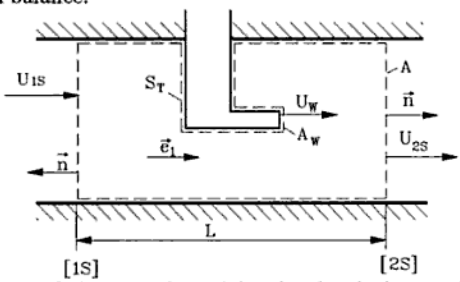
$$\lambda = 8\,\frac{\tau_w}{\varrho \overline{U}^2} = 0.3164 Re^{-1/4}$$

we obtain the pressure drop

$$\Delta p = \lambda \frac{l}{d} \frac{\varrho}{2} \overline{U}^2 = 0.0037\,\mathrm{bar} \,.$$

The pressure loss is negligible compared to the pressure at station [1S] $p_{1S} = 10\,\mathrm{bar}$.

b) Momentum balance:



The momentum balance evaluated for the sketched control volume in $\vec{e}_1$-direction is

$$\iint\limits_{(S)} \varrho\, \vec{u} \cdot \vec{e}_1 (\vec{u} \cdot \vec{n})\, \mathrm{d}S = \iint\limits_{(S)} \vec{t} \cdot \vec{e}_1 \, \mathrm{d}S \,.$$

Neglecting the force $F_{S\to W}$ on the wall we get

$$-\varrho_{1S} U_{1S}^2 A - \varrho_W U_W^2 A_W + \varrho_{2S} U_{2S}^2 A = p_{1S} A + p_W A_W - p_{2S} A - F_{S\to T}$$

or, since $F_{S\to T} \ll p_{1S}\, A = 2.5 * 10^5\,\mathrm{N}$, $F_{S\to T}$ is negligible small, and with $p_W = p_{1S}$:

$$p_{2S} = \varrho_{1S}\, U_{1S}^2 + \varrho_W\, U_W^2 \frac{A_W}{A} - \varrho_{2S}\, U_{2S}^2 + p_{1S} \left(1 + \frac{A_W}{A}\right) \,. \qquad (1)$$

c) The energy equation in integral form (see F. M. (2.114)) for time averaged steady flow, with $q_i = 0$, $k_i = 0$, $t_i = \tau_{ij}\, n_j$, reads

$$\iiint\limits_{(V)} \frac{\partial}{\partial x_j}\left[\varrho\, u_j \left(\frac{u_i u_i}{2} + e\right)\right]\, \mathrm{d}V = \iint\limits_{(S)} u_i\, \tau_{ij} n_j\, \mathrm{d}S$$

and becomes with the Gauss' theorem

$$\iint\limits_{(S)} \varrho\, u_j n_j \left(\frac{u_i u_i}{2} + e\right)\, \mathrm{d}S = \iint\limits_{(S)} u_i\, \tau_{ij} n_j\, \mathrm{d}S\,.$$

We use the same control volume, as in part b). At the solid walls the velocity is $u_i = 0$. As a result, only the integrals over the cross sections are different from zero. Since the flow is uniform, $\tau_{ij} n_j = -p\, n_i$ and we get

$$\iint\limits_{(S)} \varrho\, u_j n_j \left(\frac{u_i u_i}{2} + e + \frac{p}{\varrho}\right)\, \mathrm{d}S = 0\,,$$

or, with $h = e + p/\varrho$,

$$\iint\limits_{(S)} \varrho\, u_j n_j \left(\frac{u_i u_i}{2} + h\right)\, \mathrm{d}S = 0\,. \tag{2}$$

d) The kinetic energy of the steam per unit of mass $u_i\, u_i/2 = U_{1S}^2/2$ at station [1S] is

$$\frac{u_{1S}^2}{2} = 3.2\,\mathrm{kJ/kg}$$

and can be neglected compared to the enthalpy $h_{1S} = 3264$ kJ/kg. The kinetic energy of the water

$$\frac{u_W^2}{2} = 0.2\frac{\mathrm{kJ}}{\mathrm{kg}}$$

is also negligibly small compared with the enthalpy of the water $h_W =$ 632 kJ/kg. As a result, we conclude that at station [2S] the kinetic energy can be neglected compared with the enthalpy.
Thus, equation (2) is simplified as

$$\iint\limits_{(S)} \varrho\, u_j n_j\, h\, \mathrm{d}S = 0\,.$$

Evaluating the energy equation for the sketched control volume from part b), we get

$$\varrho_{2S}\, U_{2S}\, A\, h_{2S} = \varrho_{1S}\, U_{1S}\, A\, h_{1S} + \varrho_W\, U_W\, A_W\, h_W\,,$$

or with $\dot{m}_{1S} = \varrho_{1S} U_{1S} A$, $\dot{m}_{2S} = \varrho_{2S} U_{2S} A$ and $\dot{m}_W = \varrho_W U_W A_W$

$$h_{2S} = \frac{\dot{m}_{1S} h_{1S} + \dot{m}_W h_W}{\dot{m}_{2S}} \,. \tag{3}$$

e) To determine the four flow quantities at station [2S] (U_{2S}, ϱ_{2S}, p_{2S}, h_{2S}) we apply the three conservation equations, namely momentum (1), energy (3), and continuity equation

$$\iint\limits_{(S)} \varrho\, u_i n_i \,\mathrm{d}S = 0 \quad \Rightarrow \dot{m}_{2S} = \dot{m}_{1S} + \dot{m}_W \tag{4}$$

and the thermodynamic relation $h = h(\varrho, p)$, which is given by the Mollier diagram (Problem 9.2-3). $\dot{m}_{2S}$ follows directly from (4): $\dot{m}_{2S} =$ 72.3 kg/s. From the energy equation (3) we get $h_{2S} = 2910$ kJ/kg. U_{2S}, ϱ_{2S} and p_{2S} are found iteratively. We first assume an isobaric mixing (iteration step 0)

$$\Rightarrow \quad p_{2S}^{(0)} = p_{1S} = 10\,\mathrm{bar}\,.$$

From the Mollier diagram, it follows: $\varrho_{2S} \approx 4.5\,\mathrm{kg/m^3}$;

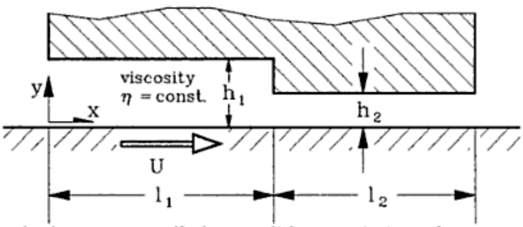
the continuity equation gives: $u_{2S} = 67\,\dfrac{\mathrm{m}}{\mathrm{s}}$;

from the momentum balance we have: $p_{2S} = 10.037$ bar .

Since the pressure has changed only marginally, we anticipate also very small changes in ϱ_{2S}, u_{2S} such that further iterations are not necessary.

8 Hydrodynamic Lubrication

Problem 8-1 Bearing with step slider



The sketch shows a so called step slider consisting of two sections of constant gap height $h(x)$. Considering the simplification in lubrication theory, calculate

a) the pressure distribution $p(x)$ in the gap (Hint: observe that the volume flux is continuous at the transition point $x = l_1$),
b) the load capacity,
c) the necessary force to move the lower wall by integrating the stress vector over
   1) the top wall,
   2) the bottom wall.

Given: h_1, h_2, l_1, l_2, η, U

Solution

a) Pressure distribution:
From the Reynolds equation it follows for each section

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\eta}\frac{\partial p}{\partial x}\right) = 0$$

($\partial h/\partial x = 0$) and thus, for the pressure gradient

$$\frac{\partial p}{\partial x} = \frac{A\,\eta}{h^3}\,, \tag{1}$$

where A is the integration constant. The pressure distribution is then

$$p(x) = \frac{A\,\eta}{h^3}\,x + B\,. \tag{2}$$

The volume flux is calculated from (see F. M. (6.22))

$$\dot{V} = \frac{h\,U}{2} - \frac{\partial p}{\partial x}\,\frac{h^3}{12\,\eta}$$

$$\text{to} \qquad \dot{V} = \frac{h\,U}{2} - \frac{A}{12}\,. \tag{3}$$

For bearing section 1 $(0 \le x \le l_1)$:
Using the pressure boundary condition $p(0) = 0$ we obtain from (2) $B_1 = 0$, hence

$$p(x) = \frac{A_1\,\eta}{h_1^3}\,x\,, \qquad 0 \le x \le l_1 \tag{4}$$

and from (3)

$$\dot{V}_1 = \frac{h_1\,U}{2} - \frac{A_1}{12}\,.$$

The subscript of the integration constant denotes the individual bearing section. For section 2 we obtain, because of $p(l_1 + l_2) = 0$,

$$B_2 = -\frac{A_2\,\eta}{h_2^3}\,(l_1 + l_2)$$

thus

$$p(x) = -\frac{A_2\,\eta}{h_2^3}\,(l_1 + l_2 - x)\,, \qquad l_1 \le x \le l_2 \tag{5}$$

and

$$\dot{V}_2 = \frac{h_2\,U}{2} - \frac{A_2}{12}\,.$$

Continuity of pressure at station $x = l_1$ gives

$$\frac{A_1\,\eta}{h_1^3}\,l_1 = -\frac{A_2\,\eta}{h_2^3}\,l_2$$

and of volume flux

$$\frac{h_1\,U}{2} - \frac{A_1}{12} = \frac{h_2\,U}{2} - \frac{A_2}{12}\,,$$

from which the constants are calculated as

$$A_1 = \frac{6\,U\,(h_1 - h_2)\,l_2\,h_1^3}{l_2\,h_1^3 + l_1\,h_2^3}$$

and

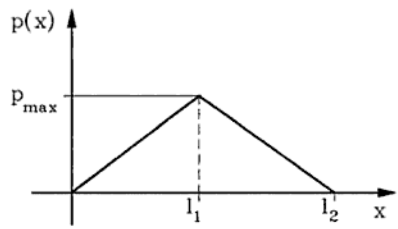
$$A_2 = -A_1 \frac{l_1}{l_2} \left(\frac{h_2}{h_1}\right)^3 = -\frac{6\,U\,(h_1 - h_2)\,l_1\,h_2^3}{l_2\,h_1^3 + l_1\,h_2^3} \,.$$

Thus, the pressure distribution is

$$p(x) = \begin{cases} \dfrac{6\,\eta\,U\,(h_1 - h_2)\,l_2}{l_2\,h_1^3 + l_1\,h_2^3}\,x & \text{for } 0 \le x \le l_1 \,, \\[2ex] \dfrac{6\,\eta\,U\,(h_1 - h_2)\,l_1}{l_2\,h_1^3 + l_1\,h_2^3}\,(l_1 + l_2 - x) & \text{for } l_1 \le x \le l_2 \,. \end{cases}$$

The pressure distribution has a triangular shape with the maximum value

$$p_{\max} = \frac{6\,\eta\,U\,(h_1 - h_2)\,l_1\,l_2}{l_2\,h_1^3 + l_1\,h_2^3} \,. \tag{6}$$



b) Load capacity per unit of depth:
From the sketched pressure distribution we read for the integral

$$F_y = \int\limits_{(S)} \tau_{yy}\,n_y\,\mathrm{d}S = \int\limits_{x=0}^{l_1+l_2} p(x)\,\mathrm{d}x \qquad \text{(area of the triangle)}$$

the value

$$F_y = \frac{l_1 + l_2}{2}\,p_{\max} = \frac{3\,\eta\,U\,(h_1 - h_2)\,l_1\,l_2\,(l_1 + l_2)}{l_2\,h_1^3 + l_1\,h_2^3} \,.$$

c) Drag force:
The velocity distribution in the gap is the same as the velocity distribution of the Couette-Poiseuille flow (see F. M. (6.16)):

$$u = \frac{\partial p}{\partial x}\,\frac{y^2}{2\,\eta} + C_1\,y + C_2 \,,$$

with the boundary conditions $u(y=0) = U$, $u(y=h) = 0$:

$$u = U\left(1 - \frac{y}{h}\right) + \frac{\mathrm{d}p}{\mathrm{d}x}\,\frac{1}{2\,\eta}\,(y^2 - y\,h) \,.$$

Thus, the shear stress $\tau_{xy}(y)$ is

$$\tau_{xy} = \eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\eta\,\frac{U}{h} + \frac{\mathrm{d}p}{\mathrm{d}x}\,\left(y - \frac{1}{2}\,h\right)$$

and with $\mathrm{d}p/\mathrm{d}x$ from (1)

$$\tau_{xy} = -\eta\,\frac{U}{h} + \frac{A\,\eta}{h^3}\,\left(y - \frac{h}{2}\right)\,.$$

At the bottom wall the shear stress is

$$\tau_{xy}(y=0) = -\eta\,\frac{U}{h} - \frac{A\,\eta}{2\,h^2} \tag{7}$$

similarly, at the top wall we have

$$\tau_{xy}(y=h) = -\eta\,\frac{U}{h} + \frac{A\,\eta}{2\,h^2}\,. \tag{8}$$

The integral

$$F_x = \int\limits_{(S)} t_x\,\mathrm{d}S = \int\limits_{(S)} \tau_{jx}\,n_j\,\mathrm{d}S$$

gives at the bottom wall ($n_j = (0,\,-1,\,0)$, $\mathrm{d}S = \mathrm{d}x$)

$$F_{xb} = -\int \tau_{xy}(0)\,\mathrm{d}x \tag{9}$$

and at the top wall ($n_j = (0,\,1,\,0)$, $\mathrm{d}S = \mathrm{d}x$ and at the location of the step $n_j = (1,\,0,\,0)$, $\mathrm{d}S = \mathrm{d}y$)

$$F_{xt} = \int \tau_{xy}(h)\,\mathrm{d}x + \tau_{xx}(l)\,(h_1 - h_2)\,, \tag{10}$$

where $\tau_{xx} = -p(l_1) = -p_{\max}$ using (6) and where the v-component of the velocity is zero by assumption. Since the shear stress is constant for each bearing section, equation (9), (10) furnishes

$$F_{xb} = -\tau_{xy1}(0)\,l_1 - \tau_{xy2}(0)\,l_2\,,$$

$$F_{xt} = \tau_{xy1}(h_1)\,l_1 + \tau_{xy2}(h_2)\,l_2 - p_{\max}\,(h_1 - h_2)\,.$$

With (7) and (8) then

$$F_{xb} = \eta\,\frac{U\,l_1}{h_1} + \frac{A_1\,\eta\,l_1}{2\,h_1^2} + \eta\,\frac{U\,l_2}{h_2} + \frac{A_2\,\eta\,l_2}{2\,h_2^2}\,,$$

$$F_{xt} = -\eta\,\frac{U\,l_1}{h_1} + \frac{A_1\,\eta\,l_1}{2\,h_1^2} - \eta\,\frac{U\,l_2}{h_2} + \frac{A_2\,\eta\,l_2}{2\,h_2^2} - p_{\max}\,(h_1 - h_2)\,.$$

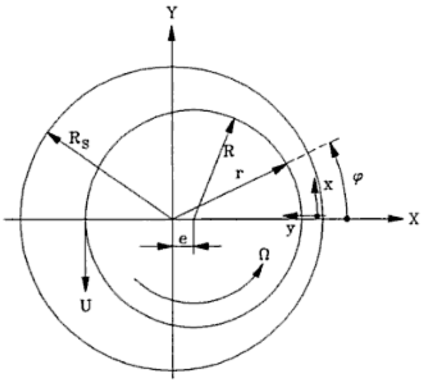
Introducing the values of A_1, A_2, and $p_{\max}$, we obtain

$$F_{xt} = -F_{xb} = -\eta\,U\,\left[\frac{l_1}{h_1} + \frac{l_2}{h_2} + \frac{3\,(h_1 - h_2)^2\,l_1\,l_2}{l_2\,h_1^3 + l_1\,h_2^3}\right]\,.$$

The forces are equal and opposite, as expected.

Problem 8-2 Friction torque transmitted by the shaft to the journal

For an infinitely long, plain journal bearing that is statically loaded calculate the friction torques transmitted to the journal and to the bearing shell by the friction stress.



Show that the difference of these torques is equal to the moment generated by the bearing force F_Y about the eccentricity e. Because of $R_S^2 = R^2(1+\overline{h}/R)^2$ and $\overline{h}/R \ll 1$, it is sufficient to calculate the torque exerted on the bearing shell using the shaft radius R.

Solution

The friction torque on the journal is (see F. M. (8.38))

$$M_{\text{journal}} = \frac{\eta\,\Omega\,R^2}{\Psi}\left(4\,I_1 - 3\frac{I_2^2}{I_3}\right) , \tag{1}$$

where I_1, I_2, I_3 are given by the equations (F. M. (8.40), (8.41), (8.42)). To obtain the torque on the bearing shell we calculate first the friction stress on the bearing. Using (F. M. (8.10)) we obtain

$$\tau_{xy}\big|_{y=0} = \eta\,\frac{\partial u}{\partial y}\bigg|_{y=0} = \eta\,U\left(\frac{1}{h(x)} - \frac{\partial p}{\partial x}\frac{h(x)}{2\eta\,U}\right) . \tag{2}$$

With equation (F. M. (8.26))

$$h(x) = h(\varphi) = \overline{h}\,(1 - \epsilon\,\cos\varphi)$$

and (F. M. (8.28))

$$\frac{\partial p}{\partial x} = 6\,\frac{\eta\,\Omega\,R}{h^2(\varphi)}\left(1 - \frac{\overline{h}}{h(\varphi)}\,\frac{I_2}{I_3}\right) ,$$

with $m = \pi/c(2n-1)$. For a square cross-section ($b = c$, area $A = c^2$), the volume flux is

$$\dot{V} = \overline{U}\,A = \overline{U}\,c^2 = \frac{K\,c^4}{4\,\eta}\left\{\frac{1}{3} - \frac{64}{\pi^5}\sum_{n=1}^{\infty}\frac{\tanh(m\,b/2)}{(2n-1)^5}\right\} .$$

If we replace

$$\frac{K}{\eta} \quad \text{by} \quad -\frac{12\,\eta\,\dot{h}}{h^3} ,$$

we obtain the load on a slider with square cross-section (side length c)

$$\begin{aligned} F_{y_S} &= \frac{-3\eta\,\dot{h}}{h^3}\,c^4\left\{\frac{1}{3} - \frac{64}{\pi^5}\sum_{n=1}^{\infty}\frac{\tanh(\pi/2\,(2n-1))}{(2n-1)^5}\right\} \\ &= -0.4217\,\frac{\eta\,\dot{h}}{h^3}\,c^4 , \end{aligned} \tag{2}$$

where the expression in the parentheses was evaluated numerically.

2) Triangular cross-section:
For an equilateral triangle (height d, surface $A = d^2/\sqrt{3}$), we obtain similar to part 1) from equation (F. M. (6.94)):

$$\dot{V} = \overline{U}\,A = \overline{U}\,\frac{d^2}{\sqrt{3}} = \frac{1}{60\sqrt{3}}\,\frac{K\,d^4}{\eta}$$

and

$$F_{y_T} = -\frac{1}{5\sqrt{3}}\,\frac{\eta\,\dot{h}}{h^3}\,d^4 . \tag{3}$$

3) Elliptic cross-section:
For an elliptic cross-section (half axes a, b; area $A = \pi\,a\,b$) we calculate using equation (F. M. (6.99))

$$\dot{V} = \overline{U}\,A = \overline{U}\,\pi\,a\,b = \frac{K}{4\,\eta}\,\pi\,\frac{a^3\,b^3}{a^2+b^2}$$

and

$$F_{y_E} = -3\pi\,\frac{\eta\,\dot{h}}{h^3}\,\frac{a^3\,b^3}{a^2+b^2} . \tag{4}$$

b) For the same cross-section area but different slider shape we have

$$\begin{aligned} &\text{square:} && c^2 = \pi\,R^2 && \Rightarrow && c = \sqrt{\pi}\,R , \\ &\text{triangle:} && d^2/\sqrt{3} = \pi\,R^2 && \Rightarrow && d = \sqrt[4]{3\,\pi^2}\,R , \\ &\text{ellipse:} && \pi\,a\,b = \pi\,R^2 && \Rightarrow && a = R/b\,R . \end{aligned}$$

We obtain from equation (2)–(4):

$$F_{y_S} = -0.4217\,\pi^2\,\frac{\eta}{h^3}\dot{h}\,R^4\,,\quad F_{y_T} = -\frac{\sqrt{3}}{5}\,\pi^2\,\frac{\eta}{h^3}\dot{h}\,R^4\,,$$

$$F_{y_E} = -3\,\pi\,\frac{\eta}{h^3}\dot{h}\,\frac{R^2\,b^2}{R^4+b^4}\,R^4\,.$$

With respect to the load of a slider with circular cross-section, we find

square: $F_{y_S}/F_{y_C} = 0.8832$,
triangle: $F_{y_T}/F_{y_C} = 0.726$,
ellipse: $F_{y_E}/F_{y_C} = 2\,R^2\,b^2/(R^4+b^4)$.

From

$$\frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{F_{y_E}}{F_{y_C}}\right) = 2R^2\,\frac{2b\,(R^4+b^4)-4b^5}{(R^4+b^4)^2} = 0$$

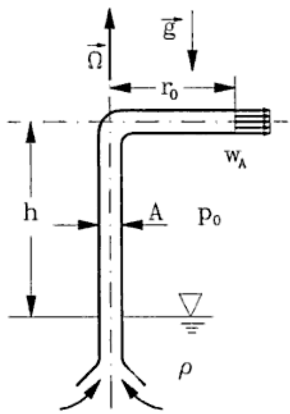
we obtain the values $b=0$ and $b=R$, for which the load of an elliptic cross-section slider assumes extremal values. In the case $b=0$, the ellipse degenerates in an infinitely long line, the load approaches zero. In the case $b=R$ the ellipse becomes a circle and $F_{y_E}/F_{y_C} = 1$. As a result of the comparison, we find that the slider with circular cross-section has the highest load capacity.

9 Stream filament theory

9.1 Incompressible Flow

Problem 9.1-1 Rotating tube acting as pump

The sketched pipe (area A, total length l), whose one end is immersed in a fluid (density $\varrho = \text{const}$) acts as a pump, while rotating with a constant angular velocity Ω about its vertical axis.

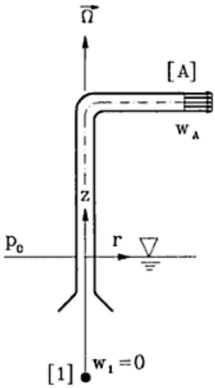


a) Determine the maximum Ω such that the pipe pressure remains always above the vapor pressure p_V.
b) At what acceleration $a(t)$ does the water start moving, if the pipe was initially closed by a valve suddenly opened at time $t = 0$?
c) Give the exit velocity w_A as a function of time for the valve opening.

Given.: Ω, h, l, A, r_0, ϱ, p_V, p_0, g

Solution

a) Maximum value for Ω, such that $p(r,z) > p_V$:



We consider the steady operation of the pump. Bernoulli's equation in the rotating reference frame is for this case

$$p_1 + \frac{\varrho}{2} w_1^2 - \frac{\varrho}{2} \Omega^2 r_1^2 + \varrho g z_1 = p + \frac{\varrho}{2} w^2 - \frac{\varrho}{2} \Omega^2 r^2 + \varrho g z \, .$$

The hydrostatic pressure distribution in the fluid at rest is

$$p_1 + \varrho g z_1 = p_0 \, ,$$

resulting in

$$p_0 = p + \frac{\varrho}{2} w^2 - \frac{\varrho}{2} \Omega^2 r^2 + \varrho g z \, .$$

For a pipe with constant cross-section area the continuity equation requires $w = w_A$ and the pressure distribution within the pipe is

$$p(r,z) = p_0 - \frac{\varrho}{2} w_A^2 + \frac{\varrho}{2} \Omega^2 r^2 - \varrho g z \, .$$

At the exit ($r = r_0$, $z = h$), the pressure is $p = p_0$ which leads to an equation for w_A:

$$\frac{\varrho}{2} w_A^2 = \frac{\varrho}{2} \Omega^2 r_0^2 - \varrho g h \, ,$$

such that the pressure distribution assumes the form

$$p(r,z) = p_0 + \varrho g (h - z) - \frac{\varrho}{2} \Omega^2 (r_0^2 - r^2) \, .$$

The pressure is minimal for $z = h$ and $r = 0$. For the case that it should be greater than the vapor pressure p_V, following relation must hold:

$$p(0,h) = p_0 - \frac{\varrho}{2} \Omega^2 r_0^2 > p_V$$

$$\Rightarrow \quad p_0 - p_V > \frac{\varrho}{2} \Omega^2 r_0^2$$

$$\Rightarrow \quad \Omega < \sqrt{\frac{2(p_0 - p_V)}{\varrho r_0^2}} \, .$$

yields with $u_1 = u_3$ and $z_1 = z_3$

$$p_3 = p_1 - \Delta p_l = p_1 - \frac{\varrho}{2} u_1^2 \left(\frac{A_1}{A_2} - 1\right)^2 .$$

c) Force on the orifice:
The balance of momentum in the form used in stream filament theory for steady flow ($\partial/\partial t = 0$) is

$$-\varrho_1 u_1^2 A_1 \vec{\tau}_1 + \varrho_3 u_3^2 A_3 \vec{\tau}_3 = p_1 A_1 \vec{\tau}_1 - p_3 A_3 \vec{\tau}_3 - \vec{F} ,$$

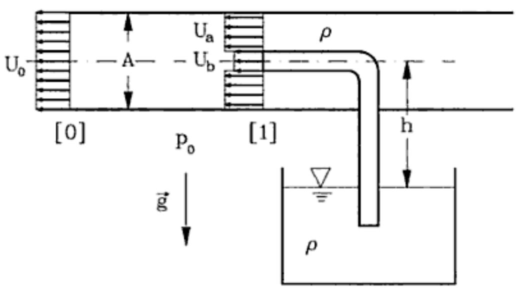
with $\vec{F}$ as the force exerted on the wall. In this case $\vec{\tau}_1 = \vec{\tau}_3 = \vec{\tau}$, $\varrho_1 = \varrho_3 = \varrho$ and we get ($A_1 = A_3$)

$$\vec{F} = \vec{\tau} \left[(p_1 - p_3) A_1 + \varrho u_1^2 A_1 \left(1 - \left(\frac{u_3}{u_1}\right)^2\right)\right] .$$

The last expression in the bracket disappears and we have

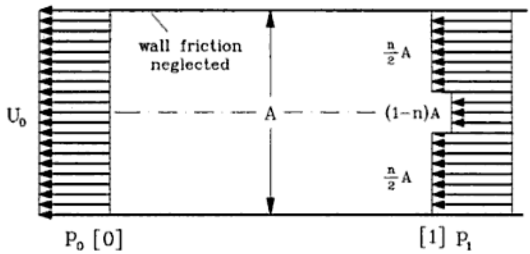
$$\vec{F} = \vec{\tau} \, \Delta p_l A_1 = \vec{\tau} \frac{\varrho}{2} u_1^2 \left(\frac{A_1}{A_2} - 1\right)^2 A_1 .$$

Problem 9.1-3 Injector pump



Inside a pipe with the cross-section A, a tube is installed that has an area of $(1 - n) A$ and is immersed in a large container as shown. The working fluid within the pipe has the density ϱ and the velocity U_a at station [1]. It sucks fluid of the same density and the velocity U_b from the tube. At station [0] a uniform velocity distribution is established. Wall shear stresses can be neglected.
Give the relationship between U_a and U_b.
Given.: A, n, h, ϱ

Solution



For the mixing problem the pressure increase is (see F. M. (9.59)):

$$\Delta p = n\,(1-n)\,\epsilon^2\,\varrho\,u_1^2\,,$$

where u_1 corresponds to U_b and

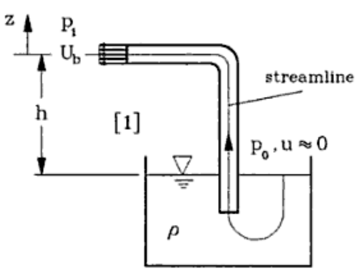
$$\Delta p = p_0 - p_1\,.$$

With

$$U_a = (1-\epsilon)\,U_b \qquad (\text{here}\,\epsilon < 0\,!)$$

we have:

$$p_0 - p_1 = n\,(1-n)\,\varrho\,(U_a - U_b)^2\,. \tag{1}$$



The pressure p_1 is obtained from Bernoulli's equation along the sketched streamline:

$$p_1 + \frac{\varrho}{2}\,U_b^2 = p_0 - \varrho\,g\,h\,,$$

$$\Rightarrow \quad p_1 = p_0 - \frac{\varrho}{2}\,U_b^2 - \varrho\,g\,h\,. \tag{2}$$

We introduce (2) in (1)

$$\frac{\varrho}{2}\,U_b^2 + \varrho\,g\,h = (n-n^2)\,\varrho\,(U_a - U_b)^2$$

and solve the equation for U_a:

$$U_a = U_b \pm \sqrt{\frac{\frac{1}{2}\,U_b^2 + g\,h}{n-n^2}}\,,$$

Solution

a) The pressures p_2, p_3, and p_4:

We find the pressure p_2 using Bernoulli's equation in an inertial system along the streamline from the pump inlet (p_1, c_1) to the impeller inlet (c_2):

$$p_1 + \frac{\varrho}{2} c_1^2 = p_2 + \frac{\varrho}{2} c_2^2 ,$$

$$\Rightarrow \qquad p_2 = p_1 + \frac{\varrho}{2} (c_1^2 - c_2^2) . \qquad (1)$$

To determine p_3, we use Bernoulli's equation in a rotating frame of reference from the impeller inlet (p_2, c_2, R_2) to the impeller exit (c_3, R_3):

$$p_2 + \frac{\varrho}{2} w_2^2 - \frac{\varrho}{2} \Omega^2 R_2^2 = p_3 + \frac{\varrho}{2} w_3^2 - \frac{\varrho}{2} \Omega^2 R_3^2 ,$$

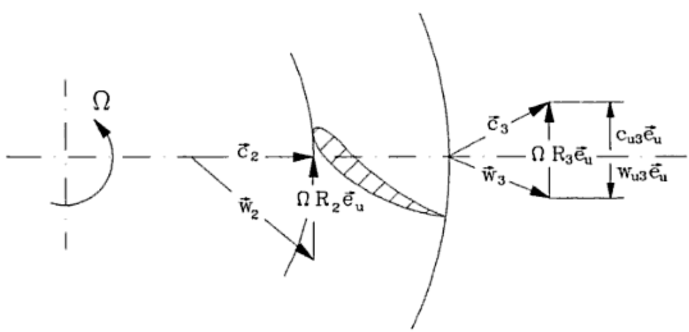
$$\Rightarrow \qquad p_3 = p_2 + \frac{\varrho}{2} (w_2^2 - w_3^2) + \frac{\varrho}{2} \Omega^2 (R_3^2 - R_2^2) . \quad (2)$$

The relative velocities w_2 and w_3 can be expressed in terms of absolute velocities and the circumferential velocity.
From $\vec{c} = \vec{w} + \vec{u}$, i. e. $\vec{w} = \vec{c} - \vec{u}$ we have

$$\vec{w} \cdot \vec{e}_r = w_r = c_r , \qquad \vec{w} \cdot \vec{e}_u = w_u = c_u - \Omega R ,$$

$$\vec{w} \cdot \vec{w} = \vec{c} \cdot \vec{c} + \vec{u} \cdot \vec{u} - 2 \vec{u} \cdot \vec{c} ,$$

$$\Rightarrow \qquad w^2 = c^2 - 2 \Omega R c_u + \Omega^2 R^2 .$$



At the impeller inlet ($c_{u_2} = 0$) we thus have

$$w_2^2 = c_2^2 + \Omega^2 R_2^2 \,, \tag{3}$$

and at the exit

$$w_3^2 = c_3^2 - 2\,\Omega\, R_3\, c_{u_3} + \Omega^2 R_3^2 \,, \tag{4}$$

which also follows directly from the above velocity triangles. We introduce (3) and (4) into (2), eliminate p_2 by (1), and find

$$\begin{aligned} p_3 &= p_1 + \frac{\varrho}{2}(c_1^2 - c_2^2) + \frac{\varrho}{2}\Omega^2 (R_3^2 - R_2^2) + \\ &\quad + \frac{\varrho}{2}(c_2^2 + \Omega^2 R_2^2 - c_3^2 + 2\,\Omega\, R_3\, c_{u_3} - \Omega^2 R_3^2) \,, \\ \Rightarrow \qquad p_3 &= p_1 + \frac{\varrho}{2}(c_1^2 - c_3^2 + 2\,\Omega\, R_3\, c_{u_3}) \,. \end{aligned} \tag{5}$$

In the inertial system from the impeller exit (p_3, c_3) to the stator exit (p_4, c_4) we apply Bernoulli's equation:

$$\begin{aligned} p_3 + \frac{\varrho}{2} c_3^2 &= p_4 + \frac{\varrho}{2} c_4^2 \,, \\ \Rightarrow \qquad p_4 &= p_3 + \frac{\varrho}{2}(c_3^2 - c_4^2) \,, \end{aligned}$$

and with p_3 from (5), finally arrive at

$$\begin{aligned} p_4 &= p_1 + \frac{\varrho}{2}(c_1^2 - c_3^2 + 2\,\Omega\, R_3\, c_{u_3}) + \frac{\varrho}{2}(c_3^2 - c_4^2) \,, \\ \Rightarrow \qquad p_4 &= p_1 + \frac{\varrho}{2}(c_1^2 - c_4^2 + 2\,\Omega\, R_3\, c_{u_3}) \,. \end{aligned} \tag{6}$$

The energy equation (7) combined with (8) and (9) yields

$$\varrho \dot{V} \left(\frac{c_4^2}{2} - \frac{c_1^2}{2} \right) = \dot{V} (p_1 - p_4) + P_D ,$$

$$\Rightarrow \qquad P_D = \dot{V} \left(\frac{\varrho}{2} (c_4^2 - c_1^2) + p_4 - p_1 \right) ,$$

with $p_4 - p_1$ from (6)

$$P_D = \dot{V} \left(\frac{\varrho}{2} (c_4^2 - c_1^2) + \frac{\varrho}{2} (c_1^2 - c_4^2 + 2 \Omega R_3 c_{u_3}) \right) ,$$

$$= \varrho \dot{V} \Omega R_3 c_{u_3} .$$

($\dot{V} = c_1 \pi R_1^2$, as R_1 as the radius of the inlet)

c) Drive power from Euler's turbine equation:
With Euler's turbine equation we find

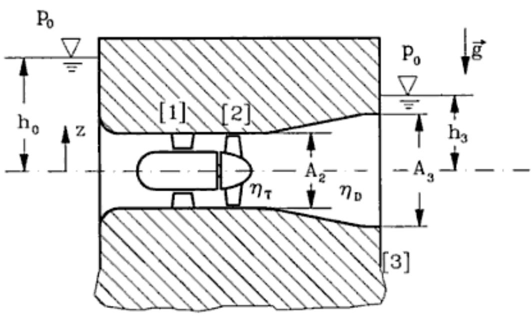
$$T = \dot{m} (R_o c_{u_o} - R_i c_{u_i}) ,$$

$$= \varrho \dot{V} (R_3 c_{u_3} - R_2 c_{u_2}) .$$

Since the inlet flow has no swirl, the velocity component $c_{u_2} = 0$, and the second term disappears,

$$\Rightarrow \qquad P_D = T \Omega = \varrho \dot{V} \Omega R_3 c_{u_3} .$$

Problem 9.1-5 Bulb turbine



The sketch shows a bulb turbine of a river power plant. The geometry data, volume flux $\dot{V}$, the mechanical efficiency η_T, and the diffuser efficiency η_D are known. The flow up- and downstream of the turbine is uniform and in axial direction.

or with $c_3 = c_2\, A_2/A_3$

$$p_2 = p_0 + \varrho\, g\, h_3 - \eta_D \frac{\varrho}{2}\, c_2^2 \left(1 - \left(\frac{A_2}{A_3}\right)^2\right) . \tag{6}$$

Introducing p_2 from (6) in the energy equation (2) yields

$$P_{in} = \dot{V}\left\{p_0 - p_0 - \varrho g h_3 + \eta_D \frac{\varrho}{2} c_2^2 \left(1 - \left(\frac{A_2}{A_3}\right)^2\right) + \varrho\, g\, h_0 - \frac{\varrho}{2}\, c_2^2\right\}$$

$$\Rightarrow\ P_{in} = \varrho\, \dot{V}\left\{g(h_0 - h_3) - \frac{1}{2}\left(\frac{\dot{V}}{A_2}\right)^2 \left[1 - \eta_D \left(1 - \left(\frac{A_2}{A_3}\right)^2\right)\right]\right\} , \tag{7}$$

where we replaced c_2 by $\dot{V}/A_2$ (axial, uniform flow). From the above expression we conclude that the maximum power input for the ideal case is

$$P_{\text{ideal}} = \varrho\, \dot{V}\, g\, \Delta h\ . \tag{8}$$

With the given values we calculate

$$P_{in} = 10^3 * 100\left\{9.81*(12-7) - \frac{1}{2}\left(\frac{100}{18}\right)^2 \left[1 - 0.85\left(1 - \left(\frac{18}{54}\right)^2\right)\right]\right\}\text{W}$$

$$= 4.9050\,\text{MW} - 0.3772\,\text{MW} = 4.5278\,\text{MW}\ .$$

b) P_{ideal} :

$$P_{\text{ideal}} = 4.9050\,\text{MW}$$

c) η_H :
With equation (7) and (8) we obtain the hydraulic efficiency to

$$\eta_H = \frac{P_{in}}{P_{\text{ideal}}} = \left\{1 - \frac{\frac{1}{2}\left(\frac{\dot{V}}{A_2}\right)^2}{g\,\Delta h}\left[1 - \eta_D \left(1 - \left(\frac{A_2}{A_3}\right)^2\right)\right]\right\} .$$

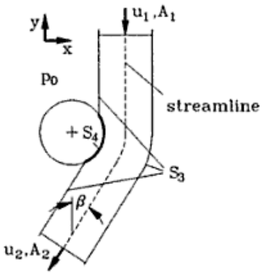
$$\eta_H = \frac{P_{in}}{P_{\text{ideal}}} = \frac{4.5278}{4.9050} = 0.92$$

d) P_{out} :

$$P_{out} = \eta_T\, P_{in} = 0.95 * 4.5278\,\text{MW} = 4.3014\,\text{MW}$$

Problem 9.1-6 Coanda effect

The effect that fluid attaches to a curved wall is termed Coanda effect. Body and viscous forces are neglected.



Calculate the deflection angle β of a jet (density ϱ, cross section A_1, velocity u_1). Determine the total force necessary to keep the cylinder in place provided that the x-component of the force F_x is known.

Given: F_x, ϱ, A_1, u_1

Solution

Bernoulli's equation along the streamline of the jet is ($p_1 = p_2 = p_0$)

$$p_1 + \frac{\varrho}{2} u_1^2 = p_2 + \frac{\varrho}{2} u_2^2$$

$$\Rightarrow \quad u_1 = u_2 \, .$$

The balance of momentum for steady flow is written as

$$\iint\limits_{(S)} \varrho \vec{u} \, (\vec{u} \cdot \vec{n}) \, \mathrm{d}S = \iint\limits_{(S)} \vec{t} \, \mathrm{d}S \, ,$$

here, with the control surface as $S = A_1 + A_2 + S_3 + S_4$ we have

$$\begin{aligned} \varrho \, u_1^2 \, A_1 \, \vec{e}_y - \varrho \, u_2^2 \, A_2 \, (\sin\beta \vec{e}_x + \cos\beta \, \vec{e}_y) &= \\ \varrho \, u^2 \, A \, (-\sin\beta \, \vec{e}_x + (1 - \cos\beta) \, \vec{e}_y) &= \iint\limits_{(S)} \vec{t} \, \mathrm{d}S \, , \end{aligned} \tag{1}$$

where the continuity equation with $A_1 = A_2 = A$ has been used. For the integral we write

$$\iint\limits_{(S)} \vec{t} \, \mathrm{d}S = - \iint\limits_{A_1, A_2, S_3} p_0 \, \vec{n} \, \mathrm{d}S - \iint\limits_{S_4} p \, \vec{n} \, \mathrm{d}S \, ,$$

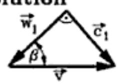
and note that $-\iint\limits_{S_4} p \, \vec{n} \, \mathrm{d}S$ represents the force acting on the fluid by the cylinder. Thus

$$\iint\limits_{(S)} \vec{t} \, \mathrm{d}S = - \iint\limits_{S_{tot}} p_0 \, \vec{n} \, \mathrm{d}S + \iint\limits_{S_4} p_0 \, \vec{n} \, \mathrm{d}S + \vec{F}_{C \to Fl} \, . \tag{2}$$

a) Calculate the velocity $\vec{v}$ of the coordinate system.
b) Find the thicknesses h_2 and h_3 of the exiting jets.
c) Find the absolute velocities $\vec{c}_2$ and $\vec{c}_3$.
d) Calculate the mass fluxes $\dot{m}_2$ and $\dot{m}_3$ per unit of width.

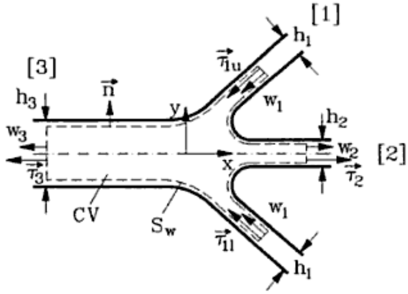
Given: $c_1 = |\vec{c}_1|$, h_1, β, ϱ

Solution

a) 

We add to the velocity $\vec{c}_1$ of the top fluid layer the velocity $-\vec{v} = -(c_1/\sin\beta)\,\vec{e}_x$. Thus, this fluid layer flows steadily under the angle β with the velocity $\vec{w}_1 = \vec{c}_1 - \vec{v}$ against the symmetry plane. The same is true for the bottom layer. The flow process is steady in a relative system that moves with the velocity $\vec{v} = (c_1/\sin\beta)\,\vec{e}_x$ in the positive x-direction.

b) The jet thicknesses h_2 and h_3:



We use a control volume which is attached to the relative system. w_1, w_2, and w_3 denote the magnitude of the relative velocities. With $\varrho =$ const we obtain from the continuity equation

$$2\,w_1\,h_1 = w_2\,h_2 + w_3\,h_3\ . \tag{1}$$

The relative system moves with a constant velocity, thus, it is an inertial system. The flow is steady; the stress vector has on the total surface of the control volume the form $\vec{t} = -p_0\,\vec{n}$, so that the application of the momentum balance (see F. M. (9.43)) leads to

$$-\,w_1^2\,h_1\,\vec{\tau}_{1l} - w_1^2\,h_1\,\vec{\tau}_{1u} + w_2^2\,h_2\,\vec{\tau}_2 + w_3^2\,h_3\,\vec{\tau}_3 = 0\ . \tag{2}$$

Multiplication with $\vec{e}_x$ results in

$$2\,w_1^2\,h_1\,\cos\beta + w_2^2\,h_2 - w_3^2\,h_3 = 0\ . \tag{3}$$

Bernoulli's equation applied to a streamline between stations [1] and [2]

$$p_0 + \frac{\varrho}{2}\, w_1^2 = p_0 + \frac{\varrho}{2}\, w_2^2$$

as well as between [1] and [3]

$$p_0 + \frac{\varrho}{2}\, w_1^2 = p_0 + \frac{\varrho}{2}\, w_3^2$$

gives

$$w_1 = w_2 = w_3 \,. \tag{4}$$

With equations (1), (3), and (4) the jet thicknesses are

$$h_2 = h_1\,(1 - \cos\beta) \quad \text{and} \quad h_3 = h_1\,(1 + \cos\beta)\,. \tag{5}$$

c) The absolute velocities $\vec{c}_2$, $\vec{c}_3$ are obtained from the velocity triangles $\vec{c} = \vec{w} + \vec{v}$. With $\vec{v} = (c_1/\sin\beta)\,\vec{e}_x$ and $\vec{w}_2 = w_2\,\vec{e}_x = w_1\,\vec{e}_x = c_1\,\cot\beta\,\vec{e}_x$, we get

$$\vec{c}_2 = \vec{w}_2 + \vec{v} = \frac{c_1}{\sin\beta}(1 + \cos\beta)\,\vec{e}_x \tag{6}$$

and with $\vec{w}_3 = -w_3\,\vec{e}_x = -w_1\,\vec{e}_x = -c_1\,\cot\beta\,\vec{e}_x$

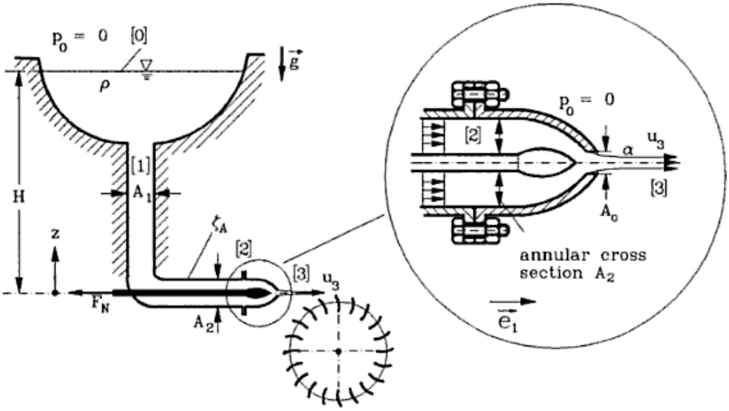
$$\vec{c}_3 = \vec{w}_3 + \vec{v} = \frac{c_1}{\sin\beta}(1 - \cos\beta)\,\vec{e}_x \,. \tag{7}$$

d) The mass fluxes $\dot{m}_2$, $\dot{m}_3$ are calculated using (5) as

$$\begin{aligned} \dot{m}_2 &= \varrho\, w_2\, h_2 = \varrho\, \frac{c_1}{\sin\beta}\, \cos\beta\,(1 - \cos\beta)\, h_1 \,, \\ \dot{m}_3 &= \varrho\, w_3\, h_3 = \varrho\, \frac{c_1}{\sin\beta}\, \cos\beta\,(1 + \cos\beta)\, h_1 \,. \end{aligned}$$

Equation (6) shows, that for $\beta \to 0$, very high velocities c_2 can be produced. One can show easily that the magnitude of the momentum of this jet is given by $c_2\, \dot{m}_2 = 2\,\varrho\, c_1^2\, h_1$.

Problem 9.1-8 Penstock and nozzle of a Pelton turbine



A nozzle is mounted at the end of a penstock. The nozzle controls the water mass flux to a Pelton turbine. The jet contraction coefficient α at the nozzle exit is known.

The pressure loss factor ζ_A includes all losses in the penstock up to station [2]. The losses from station [2] to the exit and the friction stresses at the nozzle needle are negligible. To keep the nozzle needle in the sketched position, a force F_N is necessary.

a) Determine the exit velocity u_3.
b) Find u_2 and p_2.
c) Calculate the force F_B, exerted on the bolts (because of symmetry, only a force component in direction $\vec{e}_1$ is present).

Given: $p_0 = 0$, ϱ, A_1, A_2, A_o, α, H, ζ_A, F_N

Solution

a) Exit velocity u_3:
We apply Bernoulli's equation to a streamline from point [0] to point [3]

$$p_0 + \varrho\, g\, H = p_0 + \frac{\varrho}{2}\, u_3^2 + \Delta p_l \;, \tag{1}$$

with the pressure loss $\Delta p_l = \zeta_A\, \varrho/2\, u_1^2$. From continuity

$$u_1\, A_1 = u_3\, A_3 \quad \text{or} \quad u_1 = \alpha\, \frac{A_o}{A_1}\, u_3$$

follows, with $A_3 = \alpha A_o$. From (1) we have therefore

$$u_3 = \sqrt{\frac{2g\,H}{1+\zeta_A\,(\alpha\,A_o/A_1)^2}}\,. \tag{2}$$

b) Velocity and pressure at station [2]:
From the continuity equation $u_2\,A_2 = u_3\,A_3$ and with (2) it follows

$$u_2 = \alpha\,\frac{A_o}{A_2}\sqrt{\frac{2g\,H}{1+\zeta_A\,(\alpha\,A_o/A_1)^2}}\,. \tag{3}$$

From Bernoulli's equation for a streamline from [2] to [3]

$$p_2 + \frac{\varrho}{2}\,u_2^2 = p_0 + \frac{\varrho}{2}\,u_3^2\,,$$

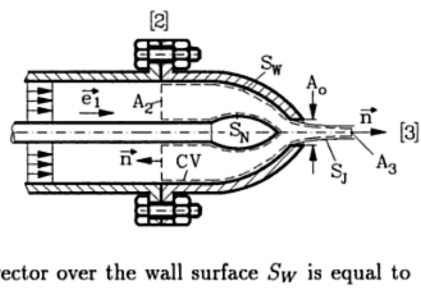
and (2), (3), the pressure at station [2] follows with $p_0 = 0$ as

$$\begin{aligned} p_2 &= \left[1-\left(\alpha\,\frac{A_o}{A_2}\right)^2\right]\frac{\varrho}{2}\,u_3^2 \\ &= \varrho\,g\,H\,\frac{1-(\alpha\,A_o/A_2)^2}{1+\zeta_A\,(\alpha\,A_o/A_1)^2}\,. \end{aligned} \tag{4}$$

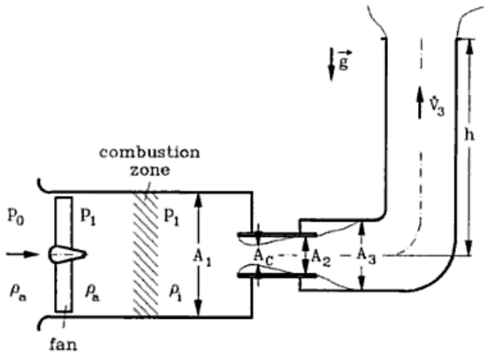
c) Bolt force F_B:
We evaluate the balance of momentum in integral form in the direction of $\vec{e}_1$ for the sketched control volume. For this purpose, we form the scalar multiplication of the linear momentum equation with $\vec{e}_1$:

$$\iint\limits_{(S)} \vec{t}\cdot\vec{e}_1\,\mathrm{d}S = \iint\limits_{(S)} \varrho\,\vec{u}\cdot\vec{e}_1\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S\,. \tag{5}$$

The total surface S of the control volume is decomposed in A_2, S_W, S_N, S_J, and A_3. The momentum flux over S_W, S_N, and S_J is equal to zero, since the normal vectors on these surfaces are perpendicular to the velocity vector.



The integral of the stress vector over the wall surface S_W is equal to the force, which is exerted by the wall on the fluid. Thus, the requested bolt force is equal and opposite to this force. The integral of $\vec{t}$ over S_N



a) Give the functional relationship for the volume flux $\dot{V}_3 = u_3 A_3$

$$\dot{V}_3 = \text{fn}(\varrho_a\,,\ \varrho_i\,,\ \alpha_1\,,\ \alpha_2\,,\ \alpha_3\,,\ A_3\,,\ h\,,\ \Delta p_F)\ ,$$

where $\alpha_1 = A_3/A_1$, $\alpha_2 = A_C/A_2$, $\alpha_3 = A_2/A_3$.

b) Determine the pressure difference Δp_F that must be generated by the fan in order to burn the air volume flux $\dot{V}_A$. (receiver characteristic $\Delta p_F(\dot{V}_A)$)

c) With the given fan rotational speed n and the volume flux $\dot{V}_A$ the pressure increase Δp_F can be determined. Furthermore, the fan pressure increase is influenced by the density ϱ_a and the fan diameter $d = \sqrt{A_1 4/\pi}$. Given is the idealized dimensionless performance map

$$\Psi = 1 - \varphi^2$$

in terms of the dimensionless products

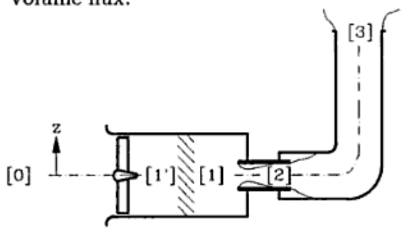
$$\text{pressure coefficient } \Psi = \frac{2\,\Delta p_F/\varrho_a}{n^2\pi^2 d^2} \text{ and flow coefficient } \varphi = \frac{4\,\dot{V}_A}{n\,\pi^2 d^3}\,.$$

Give the receiver characteristic from b) in terms of Ψ and φ. Now find the operating point Ψ_B and φ_B by graphical and numerical methods.

Given: ϱ_a, ϱ_i, α_1, α_2, α_3, A_3, Δp_F, h, g

Solution

a) Volume flux:



With Bernoulli's equation along a streamline between station [0] far upstream of the fan and station [1′] just in front of the combustion zone we obtain

$$p_0 + \Delta p_F = p'_1 + \frac{\varrho_a}{2} u'^2_1 \,. \tag{1}$$

Correspondingly, taking losses into account, from station [1] to [3]

$$\begin{aligned} p_1 + \frac{\varrho_i}{2} u_1^2 &= (p_0 - \varrho_a g h) + \frac{\varrho_i}{2} u_3^2 + \varrho_i g h + \\ &+ \frac{\varrho_i}{2} u_2^2 \left[\left(\frac{A_2}{A_C} - 1 \right)^2 + \left(1 - \frac{A_2}{A_3} \right)^2 \right] \,. \end{aligned} \tag{2}$$

In equation (1) we replace the velocity u'_1 by $u'_1 = u_1 \varrho_i/\varrho_a$ using the continuity equation, in (2) u_2 by $u_2 = u_3 A_3/A_2$. The combustion process is assumed isobaric, i. e. we have $p_1 = p'_1$. Thus, we get from (1) and (2)

$$\begin{aligned} p_0 + \Delta p_F - \frac{\varrho_a}{2} \left(\frac{\varrho_i}{\varrho_a} \right)^2 u_1^2 = p_0 + (\varrho_i - \varrho_a) g h - \frac{\varrho_i}{2} u_1^2 + \\ + \frac{\varrho_i}{2} u_3^2 \left[1 + \left(\frac{A_3}{A_2} \left(\frac{A_2}{A_C} - 1 \right) \right)^2 + \left(\frac{A_3}{A_2} - 1 \right)^2 \right] \,. \end{aligned} \tag{3}$$

We replace u_1 by $u_1 = u_3 A_3/A_1$ using the continuity equation, and write instead of the area ratios the corresponding α_i, then solve for u_3. Hence we get for the volume flux $\dot{V}_3 = A_3 u_3$

$$\dot{V}_3 = A_3 \sqrt{\frac{2 \dfrac{\Delta p_F}{\varrho_i} + 2 \left(\dfrac{\varrho_a}{\varrho_i} - 1 \right) g h}{\left(\dfrac{\varrho_i}{\varrho_a} - 1 \right) \alpha_1^2 + \left[1 + \left(\dfrac{1}{\alpha_3} \left(\dfrac{1}{\alpha_2} - 1 \right) \right)^2 + \left(\dfrac{1}{\alpha_3} - 1 \right)^2 \right]}} \,. \tag{4}$$

b) The receiver characteristic $\Delta p_F(\dot{V}_A)$:
In the combustion process, the volume flux $\dot{V}_A$ is consumed which is equal to the volume flux at station [1′]. We determine $\dot{V}_A$ by applying

Shear stress on the walls: To overcome the shear stress on the duct walls, the pressure drop

$$\Delta p_{l_D} = \lambda \frac{l_1 + l_2}{d} \frac{\varrho}{2} u_D^2$$

is necessary.

Exit loss: The sudden cross-section change entering the chamber causes a Carnot shock loss

$$\Delta p_{l_C} = \frac{\varrho}{2} u_D^2 \, ,$$

where we have assumed that the cross-section of the duct is much smaller than that of the chamber.

The total loss is therefore

$$\Delta p_{l_{tot}} = \Delta p_{l_B} + \Delta p_{l_D} + \Delta p_{l_C} = \frac{\varrho}{2} u_D^2 \left(4\,\zeta_B + \frac{l_1 + l_2}{d} \lambda + 1 \right) . \quad (3)$$

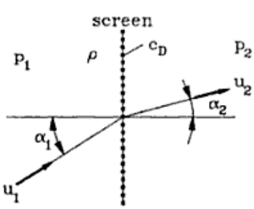
With (1) we find from (2) and (3) the velocity in the duct:

$$u_D = \frac{u_G}{\sqrt{2 + 4\,\zeta_B + (l_1 + l_2)/d\,\lambda}} \, . \quad (4)$$

b) Portion of exhaust gas through the analyser:
For the ratio of volume flux through the analyser to volume flux through the exhaust pipe we have

$$\frac{\dot{V}_D}{\dot{V}_G} = \frac{\pi/4\, d^2\, u_D}{A_G\, u_G} = \frac{\pi}{4} \frac{d^2}{A_G} \frac{1}{\sqrt{2 + 4\,\zeta_B + (l_1 + l_2)/d\,\lambda}} \, .$$

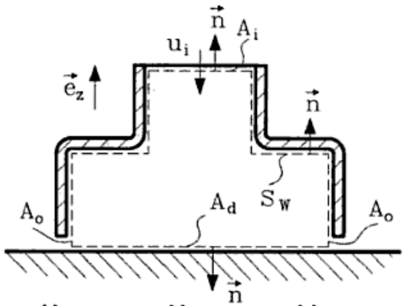
Problem 9.1-12 Flow deflection through a screen



Fluid with the density ϱ flows through a screen. The velocity u_1, the inlet flow angle α_1, and the pressure p_1 are given upstream of the screen. In flow direction, a resistance force per unit area $D = c_D\, \varrho\, u_1^2/2$ is acting on the screen. The resistance coefficient c_D is given.

Calculate the flow quantities u_2, α_2, and p_2 downstream of the screen.
Given: ϱ, u_1, p_1, α_1, c_D

To compute the force acting on the inside of the craft, we use the balance of momentum for the sketched control volume. Note that there is no component of momentum flux at A_o in $\vec{e}_z$-direction.



$$\iint\limits_{A_i} \varrho\,\vec{u}\cdot\vec{e}_z(\vec{u}\cdot\vec{n})\,\mathrm{d}A = \iint\limits_{A_i} \vec{t}\cdot\vec{e}_z\,\mathrm{d}A + \iint\limits_{A_o} \vec{t}\cdot\vec{e}_z\,\mathrm{d}A + \iint\limits_{A_d} \vec{t}\cdot\vec{e}_z\,\mathrm{d}S + \iint\limits_{S_W} \vec{t}\cdot\vec{e}_z\,\mathrm{d}S\,.$$

The last integral gives the force of the craft onto the fluid. Since at A_i, A_o, and A_d, the frictional stresses are neglected, thus $\vec{t} = p\,\vec{n}$, we arrive at

$$F_{\to\text{craft}} = -\iint\limits_{S_W} \vec{t}\cdot\vec{e}_z\,\mathrm{d}S = +\iint\limits_{A_d} p\,\mathrm{d}S - \iint\limits_{A_i} p\,\mathrm{d}A - \iint\limits_{A_i} \varrho\,u^2\,\mathrm{d}A\,,$$

which together with (1) furnishes

$$-W - p_0\,(A_d - A_i) - p_i\,A_i - \varrho\,u_i^2\,A_i + p_d\,A_d = 0\,. \tag{2}$$

The velocities u_i and u_o follow from the continuity equation:

$$u_i = \frac{\dot{V}}{A_i}\,, \qquad u_o = \frac{\dot{V}}{A_o}\,. \tag{3}$$

Bernoulli's equation between ambient and blower intake

$$p_0 + \frac{\varrho}{2}\,u_0^2 + \varrho\,g\,z_0 = p_i + \frac{\varrho}{2}\,u_i^2 + \varrho\,g\,z_i$$

leads us with (3) to

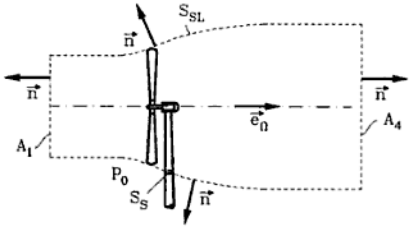
$$p_i = p_0 - \frac{\varrho}{2}\left(\frac{\dot{V}}{A_i}\right)^2\,.$$

The pressure inside the craft is also computed from Bernoulli's equation taken from the inside to the exit area A_o where $p_o = p_0$:

$$\begin{aligned} p_d + \frac{\varrho}{2}\,u_d^2 &= p_0 + \frac{\varrho}{2}\,u_o^2 \\ \Rightarrow\qquad p_d &= p_0 + \frac{\varrho}{2}\left(\frac{\dot{V}}{A_o}\right)^2\,, \end{aligned}$$

Solution

a) Axial force:



To determine the axial force we apply the balance of momentum to the sketched fixed control volume. On the streamsurface S_{SL} the mass flux is zero, since $\vec{u}\cdot\vec{n}=0$. We find, under the assumption of stream filament theory, from the momentum equation (see F. M. (9.41)) the component in $\vec{e}_\Omega$-direction

$$-\varrho\, u_1^2\, A_1 + \varrho\, u_4^2\, A_4 = p_1\, A_1 - p_4\, A_4 + \iint\limits_{S_{SL}+S_S} \vec{t}\cdot\vec{e}_\Omega\, \mathrm{d}S\,. \tag{1}$$

On the surface S_{SL} we neglect the friction stresses and thus $\vec{t}=-p_0\,\vec{n}$. Therefore, a constant pressure $p=p_0$ =const acts on the entire closed control surface $S=A_1+A_4+S_{SL}+S_S$. As a result, the pressure does not have any contribution in the momentum balance. The remaining surface integral

$$\iint\limits_{S_S} \vec{t}\cdot\vec{e}_\Omega\, \mathrm{d}S = -F_{\mathbf{axial}} \tag{2}$$

on the right hand side of (1) is the force in the intersection S_S of the support on the flow, $F_{\mathbf{axial}}$, therefore, the force of the flow on S_S in $\vec{e}_\Omega$-direction, i. e. the force on the support.

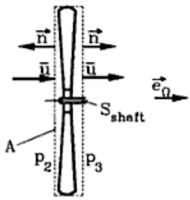
With (2) we obtain from (1)

$$F_{\mathbf{axial}} = \varrho\,(u_1^2\, A_1 - u_4^2\, A_4)$$

expressed in terms of mass flux $\dot{m}$

$$F_{\mathbf{axial}} = \dot{m}\,(u_1 - u_4)\,. \tag{3}$$

b) Mass flux $\dot{m}$ and averaged velocity $\overline{u}$:



Based on the above assumption, we consider the wind turbine to be a flat disk, through which air streams with the average velocity $\overline{u}$ (the shaft cross-section is small compared to $A = \pi/4\, D^2$). We apply the balance of momentum in $\vec{e}_\Omega$-direction to the sketched control volume and get

$$(p_2 - p_3)\, A = -\iint\limits_{S_{\text{shaft}}} \vec{t}\cdot\vec{e}_\Omega \,\mathrm{d}S = F_{\text{axial}}$$

and with (3)

$$(p_2 - p_3)\, A = \dot{m}\,(u_1 - u_4)\ . \tag{4}$$

Bernoulli's equation applied to a streamline between station [1] and [2] upstream and [3] and [4] downstream of the wheel gives

$$p_0 + \frac{\varrho}{2}\, u_1^2 = p_2 + \frac{\varrho}{2}\, u_2^2\ , \tag{5}$$

$$p_3 + \frac{\varrho}{2}\, u_3^2 = p_0 + \frac{\varrho}{2}\, u_4^2\ . \tag{6}$$

According to the assumption made previously, we neglect the circumferential component of the velocity at station [3]. This leads to $u_2 = u_3 = \overline{u}$ and from (5) and (6) it follows that

$$p_3 - p_2 = \frac{\varrho}{2}\,(u_4 + u_1)\,(u_4 - u_1)\ .$$

We insert this equation into (4) and obtain the mass flux as

$$\dot{m} = \frac{\varrho}{2}\, A\,(u_1 + u_4) \tag{7}$$

and with $\dot{m} = \varrho\, A\, \overline{u}$ the average velocity as

$$\overline{u} = \frac{u_1 + u_4}{2}\ . \tag{8}$$

c) Optimum efficiency η^*:
To judge the aerodynamic quality of the wind turbine to convert wind energy per unit time into shaft power, we first define an ideal turbine power

$$P_{\text{ideal}} = \iint\limits_{(A)} \frac{\varrho}{2}\,\vec{u}\cdot\vec{u}\,(\vec{u}\cdot\vec{n})\,\mathrm{d}A = \frac{\varrho}{2}\, u_1^3\, A\ , \tag{9}$$

which is the kinetic energy flowing through A per unit time in the absence of a turbine. On the other hand, the energy per unit time which is extracted from the flow is

$$P = \Delta p \, \dot{V} = \Delta p \, A \, \overline{u} = F_{\text{axial}} \, \overline{u}.$$

We obtain the wind turbine aerodynamic efficiency with (3) and (8)

$$\eta = \frac{P}{P_{\text{ideal}}} = \frac{1}{2}\left[1 - \left(\frac{u_4}{u_1}\right)^2\right]\left[1 + \frac{u_4}{u_1}\right] \tag{10}$$

as a function of the velocity ratio u_4/u_1. To find the maximum efficiency η^* we set the derivative $\mathrm{d}\eta/\mathrm{d}(u_4/u_1)$ of equation (10) equal to zero

$$\left(\frac{u_4}{u_1}\right)^2 + \frac{2}{3}\frac{u_4}{u_1} - \frac{1}{3} = 0 \, ,$$

and find the roots

$$\frac{u_4}{u_1} = -\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{3}{9}} \, .$$

Because of $u_1 > 0$, $u_4 > 0$, only the positive sign is possible. Thus the maximum efficiency η^* is obtained for the velocity ratio

$$\frac{u_4}{u_1} = \frac{1}{3} \tag{11}$$

Inserting this ration into (10), leads to $\eta^* = 0.59$. This implies that at most an efficiency of about 60 % is possible for a wind turbine without any losses.

d) Power and force at optimum efficiency:
Power:

$$P = \eta^* \, P_{\text{ideal}} = 0.59 \, \frac{\varrho}{2} \, A \, u_1^3 = 709.61 \, \text{kW} \, .$$

Support force:

$$F_{\text{axial}} = \dot{m} \, (u_1 - u_4) = \frac{\varrho}{2} \, A \, (u_1^2 - u_4^2) = \frac{4}{9} \, \varrho \, A \, u_1^2 = 106.9 \, \text{kN} \, .$$

2) Conduit with quadratic cross-section:
We obtain the side length a of the quadratic cross-section from the equal area requirement as $a = d\sqrt{\pi/4}$. The hydraulic diameter is equal to the side length

$$d_h = a = 0.125 * \sqrt{\pi/4} = 0.111\,\mathrm{m}$$

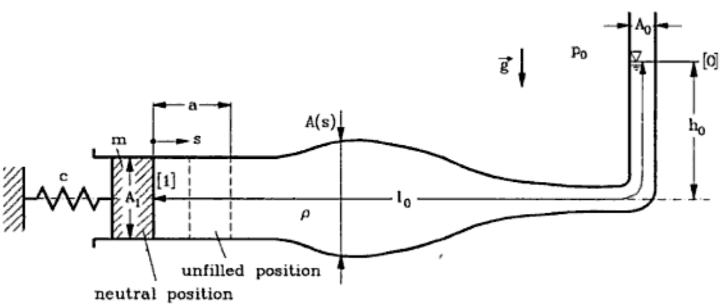
and k/d_h is

$$\frac{k}{d_h} = 0.0004 * \frac{0.125}{0.111} = 0.00045\ .$$

Using the above values, the solution is listed in the following table

	triangular	quadratic
$\bar{U}/\,(\mathrm{m/s})$	9.16	9.58
λ	0.0174	0.017
Re	$0.89 * 10^6$	$1.1 * 10^6$
$\dot{V}/\,(\mathrm{m^3/s})$	0.112	0.118
$\dot{V}_0/\dot{V}$	<u>1.089</u>	<u>1.033</u>

Problem 9.1-16 Vibrating system consisting of a fluid column and a spring suspended piston



A pipe with variable cross-section $A(s)$ is filled with fluid of constant density ϱ and is sealed with a piston at station [1]. The piston of mass m has a cross-section area A_1 and is connected with a spring which has stiffness c. In the sketched neutral position, the distance of the fluid level from the horizontal pipe axis is h_0 and the mean streamline has the length l_0.

its derivative $\partial u_1/\partial t = \ddot{x}$, we get from (5)

$$\int\limits_{[1]}^{[2]} \frac{\partial u}{\partial t}\,\mathrm{d}s = \ddot{x}\left\{\int\limits_0^{l_0} \frac{A_1}{A(s)}\,\mathrm{d}s + \int\limits_{l_0}^{l_0+h} \frac{A_1}{A_0}\,\mathrm{d}s - \int\limits_0^{x} \mathrm{d}s\right\}$$

$$= \ddot{x}\left\{L + x\left[\left(\frac{A_1}{A_0}\right)^2 - 1\right]\right\},$$

and then obtain from (4)

$$p_1 - p_0 = \varrho\,\ddot{x}\left\{L + x\left[\left(\frac{A_1}{A_0}\right)^2 - 1\right]\right\} + \frac{\varrho}{2}\,\dot{x}^2\left[\left(\frac{A_1}{A_0}\right)^2 - 1\right] + \\ + \varrho\,g\left[h_0 + \frac{A_1}{A_0}\,x\right]. \tag{6}$$

If we introduce (6) into equation (2) and consider (1), we find a second order nonlinear differential equation for the piston motion $x(t)$:

$$\left\{\frac{m}{A_1} + \varrho\,L + \varrho\left[\left(\frac{A_1}{A_0}\right)^2 - 1\right]x\right\}\ddot{x} + \\ + \frac{\varrho}{2}\left[\left(\frac{A_1}{A_0}\right)^2 - 1\right]\dot{x}^2 + \left(\varrho\,g\,\frac{A_1}{A_0} + \frac{c}{A_1}\right)x = 0\,. \tag{7}$$

c) Special case $A_1 = A_0$:
In this case equation (7) is converted into a second order linear differential equation

$$\left(\frac{m}{A_1} + \varrho\,L\right)\ddot{x} + \left(\frac{c}{A_1} + \varrho\,g\right)x = 0\,, \tag{8}$$

with the initial values

$$t = 0: \quad x = a \quad \text{and} \quad \dot{x} = 0\,. \tag{9}$$

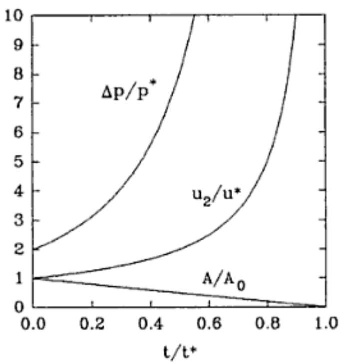
As a possible solution we set $x(t) = \mathrm{e}^{\mathrm{i}\lambda t}$ and find

$$\lambda_1 = \sqrt{\frac{c + \varrho\,A_1\,g}{m + \varrho\,A_1\,L}} =: \omega \quad \text{and} \quad \lambda_2 = -\omega\,.$$

The general solution of (8) is therefore

$$x(t) = C_1\,\mathrm{e}^{\mathrm{i}\omega t} + C_2\,\mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{10}$$

Hence
$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{-1}{1-\tau}\frac{1}{t^*}$$
and
$$\frac{1}{A}\frac{\mathrm{d}^2A}{\mathrm{d}t^2}=0\,.$$
The exit velocity (2) is
$$u_2=\frac{1}{1-\tau}\frac{l}{2t^*}$$
or
$$\frac{u_2}{u^*}=\frac{1}{1-\tau}\,,$$
with the reference velocity $u^*=l/(2\,t^*)$.



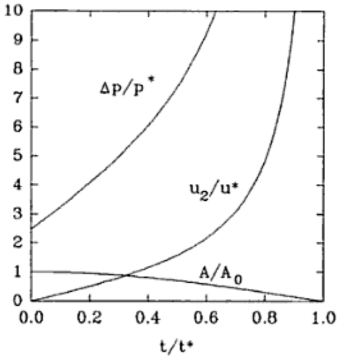
The pressure difference (4) is
$$\Delta p=\frac{\varrho}{2}\left(\frac{l}{2\,t^*}\right)^2\frac{2}{(1-\tau)^2}$$
or
$$\frac{\Delta p}{p^*}=\frac{2}{(1-\tau)^2}$$
with the reference pressure $p^*=\varrho/2\,[l/(2\,t^*)]^2$.
In the figure the temporal distributions of A/A_0, u_2/u^*, and $\Delta p/p^*$ are sketched.

2.) Harmonic decrease of the cross-section area:
Using the dimensionless cross-section area
$$\frac{A}{A_0}=\cos\left(\frac{\pi}{2}\tau\right)\,,$$
we obtain
$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t}=-\frac{\pi}{2}\tan\left(\frac{\pi}{2}\tau\right)\frac{1}{t^*}$$
and
$$\frac{1}{A}\frac{\mathrm{d}^2A}{\mathrm{d}t^2}=-\left(\frac{\pi}{2}\right)^2\left(\frac{1}{t^*}\right)^2\,.$$
With $u^*=l/(2\,t^*)$ as the reference velocity, the dimensionless exit velocity is
$$\frac{u_2}{u^*}=\frac{\pi}{2}\tan\left(\frac{\pi}{2}\tau\right)\,,$$

and with $p^* = \varrho/2\,[l/(2\,t^*)]^2$ as the reference pressure, the dimensionless pressure difference (4) is found as

$$\frac{\Delta p}{p^*} = \left(\frac{\pi}{2}\right)^2 \left[2\tan^2\left(\frac{\pi}{2}\,\tau\right) + 1\right] .$$

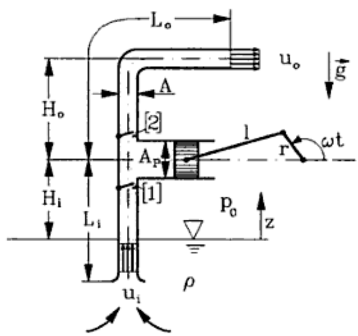
The temporal distribution of A/A_0, u_2/u^*, and $\Delta p/p^*$ are sketched in the figure.



Problem 9.1-18 Plunger pump

The sketched plunger pump, whose characteristic dimensions are small compared with the heights H_o and H_i, pumps incompressible fluid (density ϱ) through a pipe (cross-section area A, length L_i, and L_o). The piston velocity u_P is given by the geometry (crank radius r, piston rod length l) and the angular velocity ω:

$$u_P(t) = \omega\, r \left[\sin(\omega\, t) + \frac{1}{2}\,\frac{r}{l}\,\sin(2\,\omega\, t)\right].$$



a) Give the inlet and exit velocity u_i and u_o as a function of time during a cycle.
b) Determine the pressure $p_2(t)$ at station [2] immediately above the exit valve at the pressure side of the pump.
c) Determine the pressure $p_1(t)$ at station [1] immediately below the inlet valve at the suction side of the pump.

with

$$\varrho \int_{[2]}^{[o]} \frac{\partial u}{\partial t}\,\mathrm{d}s = \varrho\, L_o\, \frac{\mathrm{d}u_o}{\mathrm{d}t}\ .$$

Since the pipe cross-section area A is constant, the flow velocity in the region between [2] and the exit cross-section is $u_o(t)$. Therefore $u_2 = u_o$ and we obtain from (3)

$$p_2(t) = p_0 + \varrho\, g\, H_o + \varrho\, L_o\, \frac{\mathrm{d}u_o}{\mathrm{d}t}\ . \tag{4}$$

The case distinction leads to:

Exhaust stroke, i. e. $0 \le \omega\, t \le \pi$:

$$p_2(t) = p_0 + \varrho\, g\, H_o + \varrho\, L_o\, u_0\, \omega\, [\cos(\omega\, t) + 2\lambda\, \cos(2\, \omega\, t)]\ .$$

Intake stroke, i. e. $\pi \le \omega\, t \le 2\pi$:

$$p_2(t) = p_0 + \varrho\, g\, H_o\ .$$

c) Pressure at station [1]:
Bernoulli's equation from a point [0] on the free surface of the fluid with $u_0 = 0$, to the point [1], with $u_1 = u_i$ gives

$$p_0 = p_1(t) + \frac{\varrho}{2}\, u_i(t)^2 + \varrho\, g\, H_i + \varrho \int_{[0]}^{[1]} \frac{\partial u}{\partial t}\mathrm{d}s\ , \tag{5}$$

with

$$\varrho \int_{[0]}^{[1]} \frac{\partial u}{\partial t}\,\mathrm{d}s = \varrho\, L_i\, \frac{\mathrm{d}u_i}{\mathrm{d}t}\ .$$

For the pressure we get from (5)

$$p_1(t) = p_0 - \varrho\, g\, H_i - \frac{\varrho}{2}\, u_i(t)^2 - \varrho\, L_i\, \frac{\mathrm{d}u_i}{\mathrm{d}t}\ . \tag{6}$$

Using the inlet velocity calculated in part a) leads to:

Exhaust stroke, i. e. $0 \le \omega\, t \le \pi$:

$$p_1(t) = p_0 - \varrho\, g\, H_i\ .$$

Intake stroke, i. e. $\pi \le \omega t \le 2\pi$:

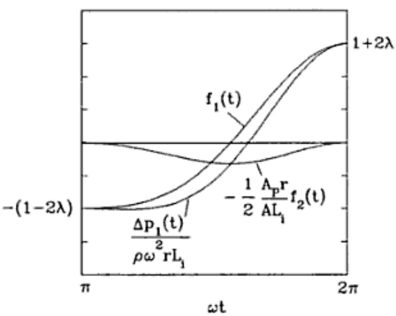
$$p_1(t) = p_0 - \varrho g H_i - \frac{\varrho}{2} u_0^2 [\sin(\omega t) + \lambda \sin(2\omega t)]^2 + + \varrho L_i u_0 \omega [\cos(\omega t) + 2\lambda \cos(2\omega t)] . \quad (7)$$

The difference between the total pressure and the hydrostatic pressure at station [1], $\Delta p_1 = p_1 - (p_0 - \varrho g H_i)$, follows from (7) for $\pi \le \omega t \le 2\pi$ to

$$\frac{\Delta p_1(t)}{u_0 \varrho L_i \omega} = f_1(t) - \frac{1}{2} \frac{A_P}{A} \frac{r}{L_i} f_2(t) \quad (8)$$

with the functions

$$f_1(t) = \cos(\omega t) + 2\lambda \cos(2\omega t) \quad \text{and} \quad f_2(t) = [\sin(\omega t) + \lambda \sin(2\omega t)]^2 .$$



The distribution of (8) and the functions f_1 and f_2 versus ωt are plotted for $A_P/A{*}r/L_i = 1/2$ in the figure. For $0 < \omega t < \pi$ (exhaust stroke) there is no dynamic pressure at station [1].

Plunger pumps are used for example in the oil industry for pumping oil from wells. Here usually $r \ll L_i$, so that the second term on the right hand side of (8) can be neglected.

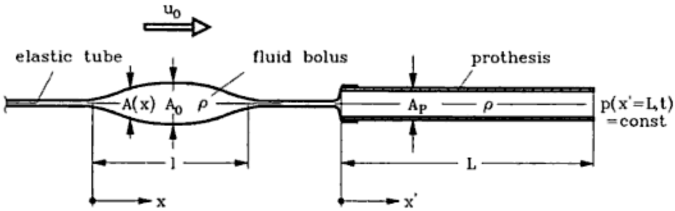
d) Calculation of $H_{i\max}$ for $A_P/A * r/L_i \ll 1$:
Looking at the pressure distribution shown in the figure from part c), we find that at time $t = \pi/\omega$ the pressure p_1 becomes a minimum:

$$p_{1_{\min}} = p_1(\omega t = \pi) = p_0 - \varrho g H_i - u_0 \varrho L_i \omega (1 - 2\lambda) .$$

This pressure should be greater than the vapor pressure p_V. This condition establishes an inequality for the height H_i and therefore for the maximum head:

$$H_i \le H_{i_{\max}} = \frac{p_0 - p_V}{\varrho g} - \frac{u_0 L_i \omega}{g} (1 - 2\lambda) .$$

Problem 9.1-19 Flow within an urethra prothesis

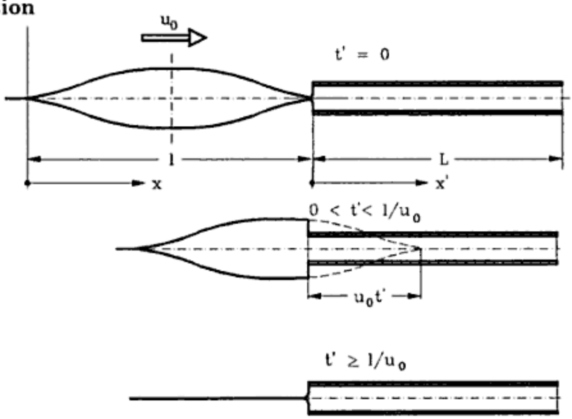


A fluid bolus (density ϱ) moves within an elastic tube (urethra) with the velocity u_0 toward a rigid tube (prothesis) filled with the same fluid. The bolus has the length l and the cross-section area $A(x) = A_0\,(x/l)\,(1 - x/l)$, where x is the bolus fixed coordinate. We assume that the shape of the bolus remains the same even if it reaches the pipe inlet (with sudden change in cross-section area at $x' = 0$). We further assume that the urethra cross-section area is zero, when the bolus is not present. The prothesis has the length L and the constant cross-section A_P.

Determine the pressure distribution $p(x' = 0, t)$ as a function of time, if the bolus reaches the prothesis at $t = 0$ and the pressure $p(x' = L, t)$ is constant at any time.

Given: ϱ, u_0, l, L, A_P, A_0, $p(x' = L)$

Solution

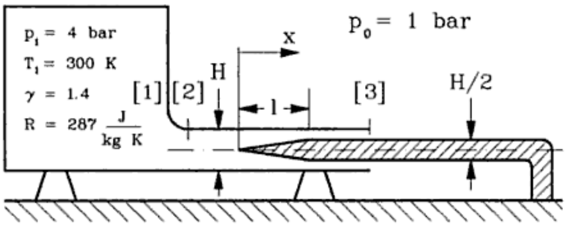


The fluid volume moves with the velocity $u_0 = \text{const}$ toward the rigid tube (prothesis) and reaches the position $x' = 0$ at time $t = 0$. Up to this point the velocity inside the rigid tube is zero and the pressure is constant and

9.2 Steady Compressible Flow

Problem 9.2-1 Force on a plate in subsonic flow

Air considered as an ideal gas flows from a large container [1] through a duct with rectangular cross-section $b*H$, in which a flat plate (cross-section $b*H/2$) with a sharp leading edge is installed as shown in figure.



The plate holder is installed far downstream from point [3]. The flow between [1] and [3] is isentropic and the pressure ratio is $p_1/p_0 = 4$.

a) Calculate the exit velocity u_3 and the pressure p_3 of the gas.
b) Determine p_2, u_2, and M_2.
c) Calculate the flow force in x–direction on the plate.
d) Plot the course of $M(x/l)$, $p(x/l)$, and $u(x/l)$. (Hint: Choose few points, for example, $x/l = 1/4,\ 1/2,\ 3/4$, and then work with the Table in F. M. Appendix C)

Given: $p_1 = 4$ bar, $T_1 = 300$ K, $\gamma = 1.4$, $R = 287$ J/(kg K)

Solution

a) u_3 and p_3:

The pressure ratio of container pressure/atmospheric pressure is supercritical

$$\frac{p_1}{p_0} = 4 \, .$$

Thus, the Mach number at the exit is

$$M_3 = 1 \, ,$$

the pressure ratio is therefore critical:

$$\frac{p_3}{p_1} = \frac{p^*}{p_t} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528$$

$$\Rightarrow \quad p_3 = 0.528 * 4 \text{ bar} = 2.112 \text{ bar} \, .$$

The speed of sound at the exit (exit velocity) is

$$\begin{aligned} \frac{a_3}{a_1} &= \frac{a^*}{a_t} = \sqrt{\frac{2}{\gamma+1}} = 0.913\,, \\ a_1 &= a_t = \sqrt{\gamma\,R\,T_1} = 347\ \mathrm{m/s} \\ \Rightarrow \qquad u_3 &= a_3 = 0.913 * a_t = 317\ \mathrm{m/s}\,. \end{aligned}$$

b) Flow quantities at station [2]:
The critical cross-section area at station [3] with the Mach number $M = 1$ is

$$A^* = A_3 = \frac{1}{2}\,b\,H\,,$$

thus, for station [2] we can write

$$\frac{A^*}{A_2} = \frac{A_3}{A_2} = \frac{1}{2}\,.$$

With this data, we read from the gas dynamics table F. M. Appendix C (subsonic flow) the quantities:

$$\begin{aligned} M_2 &= 0.306\,, \\ \frac{p_2}{p_t} &= 0.937\,, \end{aligned}$$

and thus,

$$p_2 = 0.937\,p_1 = 3.748\ \mathrm{bar}\,.$$

Furthermore, we have

$$\frac{a_2}{a_t} = 0.991$$

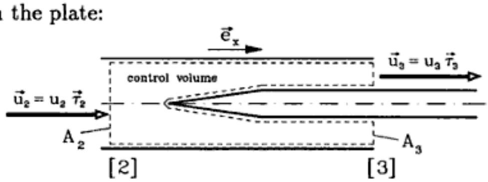
and thus,

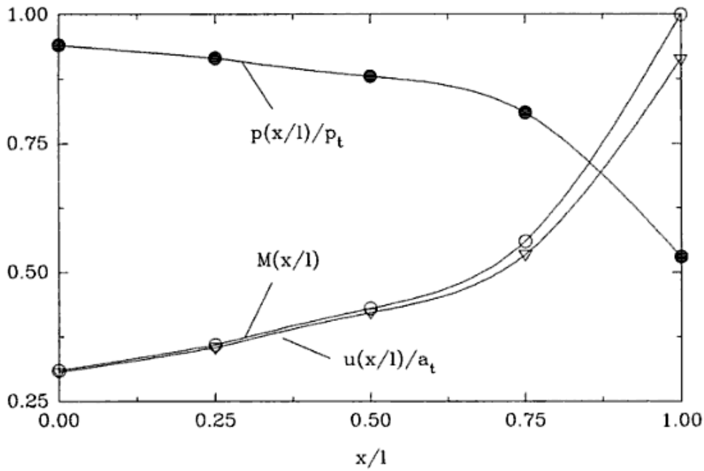
$$a_2 = 0.991\,a_1 = 344\ \mathrm{m/s}\,.$$

The flow velocity is

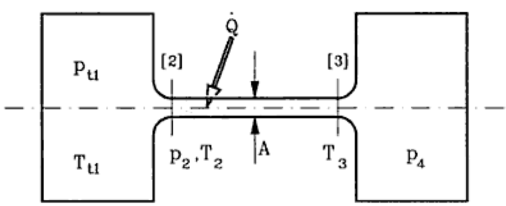
$$u_2 = M_2\,a_2 = 105\ \mathrm{m/s}\,.$$

c) Force on the plate:





Problem 9.2-2 Channel flow with heat addition



Air exits a large container (p_{t_1}, T_{t_1}) steadily and isentropically. At station [2] the pressure p_2 and temperature T_2 are known. Heat is added to the channel between stations [2] and [3] (cross-section A) such that the temperature at [3] is twice as high as at [2]. The heated air enters then a second large container (p_4). Air is considered a calorically perfect gas. The Mach number is everywhere smaller than one.

a) Determine the velocity, density, and the mass flux at [2].
b) Calculate the state properties (pressure and temperature) at [3] and the heat added per unit time.
c) Determine the total temperature T_{t_4} and the density ratio ϱ_{t_1}/ϱ_4.

Given: γ, R, c_p, p_{t_1}, T_{t_1}, p_2, T_2, T_3, A, p_4

For a calorically perfect gas ($h = c_p\,T$) we get

$$\dot{Q} = \dot{m}\left[c_p\,(T_3 - T_{t_1}) + \frac{1}{2}\,u_3^2\right] ,$$

or with u_3 from (3) and $T_3 = 2\,T_2$, we have

$$\dot{Q} = \dot{m}\left[c_p\,(2\,T_2 - T_{t_1}) + \frac{1}{2}\,u_2^2\left(\frac{2\,p_2}{p_4}\right)^2\right] ,$$

where $\dot{m}$ and u_2 follow from (2) and (1).

c) Total temperature T_{t4} and total density ϱ_{t4}:
With the energy equation from [1] to [4], we find

$$h_{t_1} + \frac{\dot{Q}}{\dot{m}} = h_4 \,, \qquad \text{with } h = c_p\,T \,,$$

and therefore

$$T_{t4} = T_{t_1} + \frac{\dot{Q}}{c_p\,\dot{m}} \,.$$

The density is calculated from the total temperature and pressure as

$$\varrho_4 = \frac{p_4}{R\,T_{t4}} = \frac{p_{t_1}}{R\,T_{t_1}}\,\frac{p_{t4}}{p_{t_1}}\,\frac{T_{t_1}}{T_{t4}}$$

$$\Rightarrow \qquad \frac{\varrho_{t_1}}{\varrho_4} = \frac{p_{t_1}}{p_4}\,\frac{T_{t4}}{T_{t_1}} = \frac{p_{t_1}}{p_4}\left(1 + \frac{\dot{Q}}{c_p\,T_{t_1}\,\dot{m}}\right) .$$

This leads to the flow velocity

$$u_1 = M_1 \, a_1 = 2 \, a_1 = 1054 \, \frac{\mathrm{m}}{\mathrm{s}} \, .$$

c) State properties downstream of the shock:
To determine the thermodynamic properties downstream of the shock we have the following equations

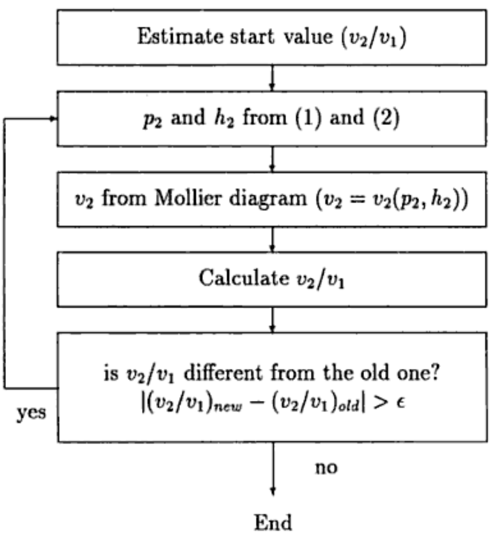
$$\text{(momentum)} \qquad p_2 \;=\; p_1 + \frac{u_1^2}{v_1}\left(1 - \frac{v_2}{v_1}\right) \, , \tag{1}$$

$$\text{(energy)} \qquad h_2 \;=\; h_1 + \frac{u_1^2}{2}\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) \, . \tag{2}$$

These are two equations for the three unknowns p_2, h_2, and v_2. The third relation is available in terms of the Mollier diagram

$$v_2 = v_2(p_2, h_2) \, .$$

The properties can be found iteratively. The iteration procedure has the following scheme:



Inserting the numerical values of u_1, v_1, p_1, h_1 in (1) and (2) gives

$$p_2 = \left\{1 + 5.5546\left(1 - \frac{v_2}{v_1}\right)\right\} \text{bar},$$

$$h_2 = \left\{2818 + 555.46\left(1 - \left(\frac{v_2}{v_1}\right)^2\right)\right\} \frac{\text{kJ}}{\text{kg}}.$$

The iteration step with the start value $v_2/v_1 = 1/6$ (this would be the maximum compression ratio for an ideal gas with $\gamma = 1.4$) gives the following numerical values:

Iteration step	v_2/v_1	p_2 from (1) [bar]	h_2 from (2) [kJ/kg]	$v_2 = v_2(p_2, h_2)$ [m^3/kg]
0	1/6	5.63	3358	0.68
1	0.34	4.67	3309	0.72
2	0.36	4.55	3301	0.75
3	0.375	4.47	3295	0.76

d) State [2] in the Mollier diagram:
We read off

$$s_2 \approx 7.9\,\text{kJ/(kg K)} \qquad \text{and} \qquad T_2 \approx 415°\,\text{C}.$$

e) Flow velocity downstream of the shock:
The continuity equation requires

$$\varrho_1 u_1 = \varrho_2 u_2 \qquad \Rightarrow \qquad u_2 = u_1 \frac{v_2}{v_1} = 1054 * \frac{0.76}{2.0} \frac{\text{m}}{\text{s}} = 400.5 \frac{\text{m}}{\text{s}}.$$

$$= \left[\frac{\gamma-1}{2}\,\frac{\gamma+1+(\gamma-1)\,(M_\infty^2-1)}{\gamma+1+2\,\gamma\,(M_\infty^2-1)}+1\right]^{\frac{\gamma}{\gamma-1}}\left(1+\frac{2\,\gamma}{\gamma+1}\,(M_\infty^2-1)\right)$$

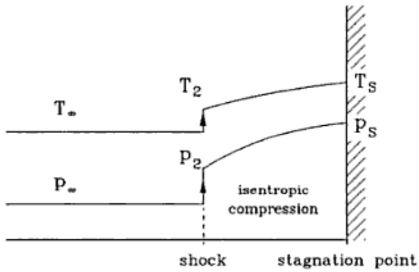
$$\Rightarrow \qquad \frac{p_s}{p_\infty}=\left[\frac{\gamma+1}{2}\,M_\infty^2\right]^{\frac{\gamma}{\gamma-1}}\left[1+\frac{2\,\gamma}{\gamma+1}(M_\infty^2-1)\right]^{-\frac{1}{\gamma-1}}\ .$$

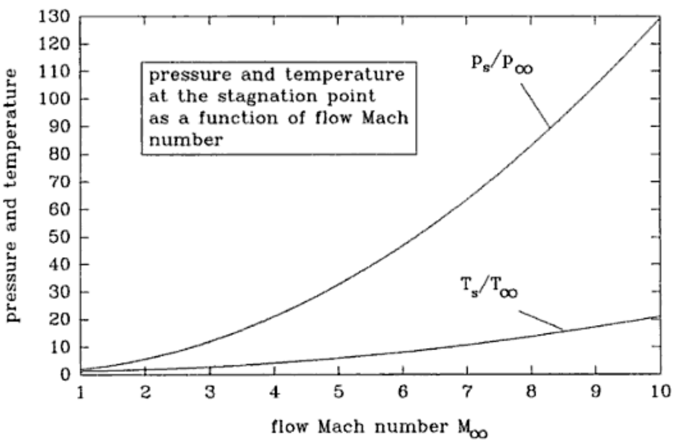
The stagnation point temperature is calculated by using the energy equation along the streamline. The stagnation enthalpy does not change through the shock:

$$h_s=h_t=h_\infty+\frac{1}{2}\,u_\infty^2 \quad\Rightarrow\quad c_p\,T_s=c_p\,T_\infty+\frac{1}{2}\,M_\infty^2\,\gamma R T_\infty\ ,$$

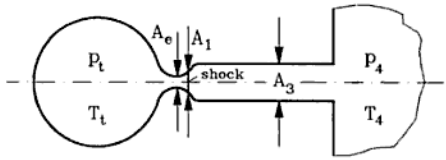
$$\text{with}\quad\left(\frac{\gamma\,R}{c_p}=\gamma-1\right)\quad\Rightarrow\quad\frac{T_s}{T_\infty}=1+\frac{\gamma-1}{2}\,M_\infty^2\ .$$

The qualitative distribution of T and p along the stagnation streamline are shown in the sketch. The distributions of p_s/p_∞ and T_s/T_∞ are plotted in the following figure as a function of the Mach number M_∞.





Problem 9.2-5 Shock waves in the divergent part of a Laval nozzle



Ideal gas ($\gamma = 1.4$) exits a large container at stagnation pressure $p_t = 2$ bar and stagnation temperature $T_t = 500\,\mathrm{K}$ through a Laval nozzle (throat A_e), which is connected via a pipe of constant area ($A_3 = 5\,A_e$) with another large container. In the divergent part of the nozzle at $A_1 = 2\,A_e$ a normal shock is developed. With the exception of the normal shock and mixing within the second container, the flow is considered isentropic.

a) Find the Mach numbers M_1, M_2, the pressures p_1, p_2, and the temperatures T_1 and T_2 up- and downstream of the normal shock.

b) Calculate the Mach number M_3, pressure p_3, and temperature T_3 inside the pipe.

c) Determine the stagnation temperature T_4 and pressure p_4 in the second container.

Given: p_t, T_t, γ, A_1/A_e, A_3/A_e

Solution

a) M_1, p_1, T_1, and M_2, p_2, T_2:

At station [1] there exists a supersonic flow regime, since a shock can be developed only for $M_1 > 1$. With the known area ratio

$$\frac{A^*}{A_1} = \frac{A_e}{A_1} = \frac{1}{2}$$

we read from the Table C.1 (see F. M.):

$$M_1 = 2.2\ , \quad \frac{p_1}{p_t} = 0.09352\ , \quad \frac{T_1}{T_t} = 0.50813\ .$$

With $p_t = 2\,\mathrm{bar}$, $T_t = 500\,\mathrm{K}$ therefore

$$p_1 = \frac{p_1}{p_t}\,p_t = 0.09352 * 2\,\mathrm{bar} = 0.1870\,\mathrm{bar}\ ,$$

$$T_1 = \frac{T_1}{T_t}\,T_t = 0.50813 * 500\,\mathrm{K} = 254\,\mathrm{K}\ .$$

The quantities at station [2] (immediately downstream of the shock) can be read from Table C.2 (see F. M.) knowing the Mach number upstream of the shock ($M_1 = 2.2$):

$$M_2 = 0.5471\ , \quad \frac{p_2}{p_1} = 5.4800\ , \quad \frac{T_2}{T_1} = 1.85686\ .$$

Solution

a) Mass flux $\dot{m}$:

$$\begin{aligned}\dot{m} &= \varrho_1 u_1 A_1 = \varrho_1 M_1 a_1 A_1 = \frac{p_1}{RT_1} M_1 \sqrt{\gamma R T_1}\, A_1 = M_1 p_1 \sqrt{\frac{\gamma}{RT_1}}\, \pi \frac{d^2}{4} \\ &= 1 * 1.5 * 10^5 \frac{\mathrm{N}}{\mathrm{m}^2} * \sqrt{\frac{1.4}{287 * 400} \frac{\mathrm{s}}{\mathrm{m}}} * \pi * \frac{3^2 * 10^{-6}}{4} \mathrm{m}^2 \\ &= 3.703 \frac{\mathrm{g}}{\mathrm{s}} .\end{aligned}$$

b) M_2, p_2 and M_3, p_3:

At the nozzle inlet, station [1], the Mach number is $M_1 = 1$.

$$A^* = A_1 = \frac{\pi}{4} d^2 .$$

To obtain the Mach number and the pressure at station [2], the area ratio A^*/A_2 must be known. We have

$$\frac{A^*}{A_2} = \left(\frac{d}{d_2}\right)^2 .$$

The diameter d_2 is linearly related to l, thus

$$\begin{aligned} d_2 &= d + \frac{l}{L}(D - d) \\ \Rightarrow \quad \frac{d_2}{d} &= 1 + \frac{l}{L}\left(\frac{D}{d} - 1\right) .\end{aligned}$$

Inserting the numerical values, we find

$$\begin{aligned} \frac{d_2}{d} &= 1 + \frac{11}{16}\left(\frac{7}{3} - 1\right) = \frac{23}{12} \\ \Rightarrow \quad \frac{A^*}{A_2} &= \left(\frac{12}{23}\right)^2 = 0.2722 . \qquad (1)\end{aligned}$$

From Table C.1 (see F. M.) we read

$$M_2 = 2.85 , \qquad \frac{p_2}{p_t} = 0.03415 .$$

The total pressure p_t is still unknown. It can be determined from the data at station [1] with the known Mach number $M_1 = 1$:

$$\begin{aligned} \frac{p_1}{p_t} &= \frac{p^*}{p_t} = 0.5283 \\ \Rightarrow \quad p_t &= \frac{p_1}{0.5283} = \frac{1.5}{0.5283} \mathrm{bar} = 2.8393\, \mathrm{bar} .\end{aligned}$$

Solution

a) The flow from [1] to [2] is isentropic. As a result, Bernoulli's equation can be applied for compressible flow in form of

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\,\frac{p_1}{\varrho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1}\,\frac{p_1}{\varrho_1}\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} .$$

Solving for the velocity

$$u_2 = \sqrt{u_1^2 + \frac{2\gamma}{\gamma-1}\,R\,T_1\left(1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right)} ,$$

we find $u_2 = 124.14$ m/s. With the isentropic relation

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} ,$$

we get $T_2 = 310.13\,\mathrm{K}$. From the equation of state for an ideal gas we obtain

$$\varrho_2 = \frac{p_2}{R\,T_2} = 1.4044\,\frac{\mathrm{kg}}{\mathrm{m}^3} .$$

Because the heat addition is isobaric and fluid is inviscid no pressure changes and no particle acceleration occur in the combustion chamber. As a consequence, we have $u_2 = u_3$.

b) Since we have the heat addition q_{23} the flow from [2] to [3] is not adiabatic and we obtain from the energy equation

$$\frac{u_2^2}{2} + h_2 + q = \frac{u_3^2}{2} + h_3$$

with $h = c_p T$ and $u_2 = u_3$:

$$q_{23} = c_p\,(T_3 - T_2) .$$

Using the specific heat at constant pressure $c_p = \gamma/(\gamma-1)\,R$, we arrive at the result

$$T_3 = \frac{q_{23}(\gamma-1)}{\gamma R} + T_2 = 608.79\mathrm{K} .$$

The density follows from the equation of state:

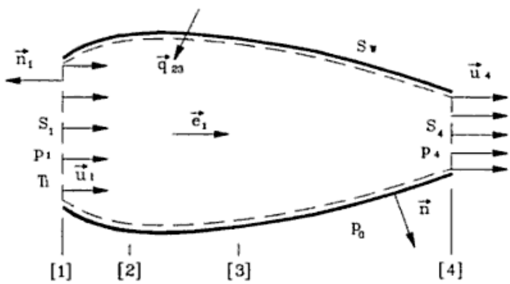
$$\varrho_3 = \frac{p_3}{R\,T_3} = 0.7154\,\frac{\mathrm{kg}}{\mathrm{m}^3} .$$

c) From station [3] to [4] the flow is isentropic. Thus, the velocity u_4 is found in a similar way described in part a) as $u_4 = 402.28$ m/s. Evaluating the isentropic relation

$$\frac{\varrho_4}{\varrho_3} = \left(\frac{p_4}{p_3}\right)^{1/\gamma}$$

gives $\varrho_4 = 0.5201$ kg/m³.

d)



The thrust is the force acting on the engine. Applying the balance of momentum

$$\iint_{(S)} \varrho\, \vec{u}(\vec{u}\cdot\vec{n})\, \mathrm{d}S = \iint_{(S)} \vec{t}\, \mathrm{d}S$$

to the sketched control volume ($S_{total} = S_1 + S_w + S_4$) leads to

$$\iint_{S_1} \varrho_1 \vec{u}_1(\vec{u}_1\cdot\vec{n})\, \mathrm{d}A + \iint_{S_w} \varrho\, \vec{u}(\vec{u}\cdot\vec{n})\, \mathrm{d}S + \iint_{S_4} \varrho_4 \vec{u}_4(\vec{u}_4\cdot\vec{n})\, \mathrm{d}S =$$

$$= -\iint_{S_1} p_1 \vec{n}\, \mathrm{d}A - \iint_{S_4} p_4 \vec{n}\, \mathrm{d}S + \iint_{S_w} \vec{t}\, \mathrm{d}S\,.$$

The second integral disappears (wall boundary condition). Because of the flow uniformity, the stress vector of the first two terms on the right hand side is $-p\vec{n}$. The last surface integral is the force exerted by the engine inner wall on the fluid. The force on the engine $\vec{F}_{\text{engine}_i}$ is equal and opposite to the flow force and is

$$-\varrho_1 u_1^2 A_1 \vec{e}_1 + \varrho_4 u_4^2 A_4 \vec{e}_1 = p_1 A_1 \vec{e}_1 - p_4 A_4 \vec{e}_1 - \vec{F}_{\text{engine}_i}$$

With $p_1 = p_4 = p_0$ the above equation is

$$\vec{F}_{\text{engine}_i} = \vec{e}_1 [p_0(A_1 - A_4) + \varrho_1 u_1^2 A_1 - \varrho_4 u_4^2 A_4]\,.$$

We are interested in the thrust component in $\vec{e}_1$-direction. The force in this direction on the outside wall of the engine is

$$F_{\text{engine}_o} = \vec{F}_{\text{engine}_o} \cdot \vec{e}_1 = -\iint\limits_{S_w} p_0\, \vec{n} \cdot \vec{e}_1 \, \mathrm{d}S = -p_0(A_1 - A_4)\ .$$

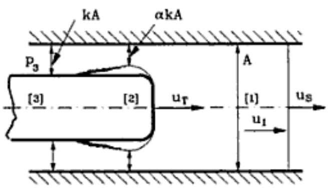
The total thrust of the engine

$$F_T = \vec{F}_{\text{engine}_i} \cdot \vec{e}_1 + \vec{F}_{\text{engine}_o} \cdot \vec{e}_1 = \varrho_1 u_1^2 A_1 - \varrho_4 u_4^2 A_4$$

can now be calculated by using the continuity equation $\varrho_1 u_1 A_1 = \varrho_4 u_4 A_4$:

$$F_T = \varrho_1 u_1 A_1 (u_1 - u_4) = -31.332\,\text{kN}\ .$$

Problem 9.2-8 High speed train in a tunnel



A high speed train moves with constant velocity u_T through a tunnel. At the front the flow experiences a separation as shown in the sketch (jet contraction coefficient α). The flow is compressible and the fluid is considered as a perfect gas.

Far upstream at station [1], the pressure p_1 and density ϱ_1 are known. At this station the velocity is $u_1 \neq u_T$.

a) Calculate the velocity u_2, pressure p_2, and density ϱ_2 at station [2] by considering an isentropic flow from [1] to [2].

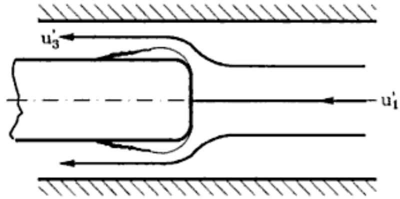
b) Calculate the velocity u_3 by applying the balance of momentum from [2] to [3] with p_3 as given.

Given: u_1, p_1, ϱ_1, A, k, α, u_T

Solution

a) When the train arrives at station [3] (tunnel inlet) then for a sufficiently small area ratio $k \ll 1$ a shock forms which propagates into the tunnel. Downstream of the shock the pressure p_1 and the density ϱ_1 are given. In addition, the unsteady shock causes air to move with the velocity u_1. To be investigated is the case, where the train, with the velocity u_T, moves through the tunnel, but its end has not yet passed station [3]. The flow inside the tunnel is unsteady. In a train fixed system, however, it is steady. From the tunnel fixed system one can arrive at the train fixed system by superimposing the train velocity u_T to all velocities. Thus, the train velocity becomes zero.

$$u_1' = u_T - u_1 \,,$$
$$u_2' = u_T - u_2 \,,$$
$$u_3' = u_T - u_3 \,.$$



The flow around the train from [1] to [2] generates a contraction in cross-section, which can be approximated as an isentropic nozzle flow. With the flow Mach number M_1' toward the train

$$M_1' = \frac{u_1'}{a_1} = \frac{u_1'}{\sqrt{\gamma \frac{p_1}{\varrho_1}}} \,,$$

we obtain the area ratio A^*/A from Table C.1 (see F. M.) or from the explicit equation

$$\frac{A^*}{A} = M_1' \left[\frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}\, M_1'^2 + 1\right)\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

for a given M_1'. The Mach number M_2' at [2] can be determined for a given area ratio $A^*/(\alpha\, k\, A)$ from the same equation

$$\frac{A^*}{\alpha\, k\, A} = M_2' \left[\frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}\, M_2'^2 + 1\right)\right]^{-\frac{\gamma+1}{2(\gamma-1)}} \,.$$

However, for the calculation of M_2' an iterative solution procedure is necessary. As an alternative solution, for a given area ratio, we can read the Mach number M_2' from the Table C.1. Applying twice the equation (F. M. (9.94))

$$\frac{p_t}{p} = \left(\frac{\gamma-1}{2}\, M^2 + 1\right)^{\frac{\gamma}{\gamma-1}}$$

gives the pressure ratio

$$\frac{p_2}{p_1} = \left(\frac{\frac{\gamma-1}{2}\, M_1'^2 + 1}{\frac{\gamma-1}{2}\, M_2'^2 + 1}\right)^{\frac{\gamma}{\gamma-1}} \,,$$

where it is self-evident that the thermodynamic quantities are frame indifferent. With the density ϱ_2 from

$$\frac{\varrho_2}{\varrho_1} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$

we obtain the velocity

$$u_2' = M_2' \sqrt{\frac{\gamma\, p_2}{\varrho_2}}$$

relative to the train and then the velocity

$$u_2 = -u_2' + u_T = -M_2' \sqrt{\gamma \frac{p_2}{\varrho_2}} + u_T$$

in the tunnel.

Numerical example: Given are the Mach number $M_1' = 0.3$ and $\alpha = 0.7$; $k = 0.8$; $\gamma = 1.4$. From the shock table we obtain $A^*/A = 0.4914$. The Mach number M_2' is read from the table with the area ratio

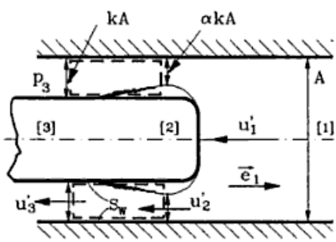
$$\frac{A^*}{\alpha\, k\, A} = 0.8775$$

as $M_2' \approx 0.65$. The pressure ratio p_2/p_1 is 0.8013 and the density ratio ϱ_2/ϱ_1 is calculated as 0.8536. For the velocity we expand

$$\frac{u_2'}{u_1'} = \frac{M_2'}{M_1'} \frac{a_2}{a_1} = \frac{M_2'}{M_1'} \frac{a_2}{a_t} \frac{a_t}{a_1}$$

and with the sound speeds from the Table C.1 we get $u_2' = 2.0993 u_1'$.

b)



The flow separation leads to a jet spreading, which is associated with mixing and losses. The flow from [2] to [3] is then no longer isentropic. Therefore, we apply the linear momentum balance in order to calculate the flow quantities (flow state) at station [3]. The velocity in the separated regime is very small and can be set equal to zero. Applying the balance of momentum

$$\iint\limits_{(S)} \varrho\, \vec{u}(\vec{u} \cdot \vec{n})\, \mathrm{d}S = \iint\limits_{(S)} \vec{t}\, \mathrm{d}S \tag{1}$$

for the sketched control volume, we first calculate the left hand side ($\vec{L}$) momentum fluxes

$$\vec{L} = \iint\limits_{k\,A} \varrho\, \vec{u}(\vec{u} \cdot \vec{n})\, \mathrm{d}A + \iint\limits_{\alpha\, k\, A} \varrho\, \vec{u}(\vec{u} \cdot \vec{n})\, \mathrm{d}A + \iint\limits_{S_W} \varrho\, \vec{u}(\vec{u} \cdot \vec{n})\, \mathrm{d}S$$

a) Calculate the leakage mass flux.
b) Determine the thermodynamic state (p_3, T_3) within the labyrinth chamber.
c) Determine the clearance height ratio H_2/H_4 to achieve a supercritical pressure ratio for p_1/p_3 .

Given: p_1, T_1, p_5, T_5, R, H_2, H_4, γ

Solution

The labyrinth clearances may be considered as convergent nozzles, within which the working medium flows in a steady, quasi one-dimensional, and isentropic fashion. The pressure ratios p_1/p_3 and p_3/p_5 are supercritical with the Mach number $M = 1$ at the throat. Thus, the thermodynamic properties are determined by the critical ones. From [2] to [3] the gas is expanded supercritically to the pressure p_3. The sudden expansion inside the labyrinth chamber produces vortices that totally dissipate the kinetic energy of the medium resulting in a considerable reduction of flow velocity ($D \gg H_2, H_4$ and $L \gg H_2, H_4$). Similar flow process takes place from [4] to [5].

a) In an steady flow case, the leakage mass flux does not change:

$$\dot{m} = \dot{m}_2 = \dot{m}_4 = A_2 u_2 \varrho_2 \ . \tag{1}$$

Because of the design geometry $H_2 \ll R$, we can approximate the area by

$$A_2 = 2\pi \, R \, H_2 \ .$$

In the throat the sonic speed is reached with the Mach number $M = 1$, therefore $u_2 = a_2$, and the flow quantities within the clearance are critical quantities. Since $p_1 = p_t$ and $\varrho_1 = \varrho_t$ are the stagnation quantities, we have for the velocity

$$a_2^{*\,2} = u_2^2 = \gamma \, \frac{p_1}{\varrho_1} \, \frac{2}{\gamma + 1}$$

and for the density ratio

$$\frac{\varrho_2}{\varrho_1} = \frac{\varrho_2^*}{\varrho_1} = \left(\frac{2}{\gamma + 1} \right)^{1/(\gamma - 1)} \ .$$

Thus, for the leakage mass flux we obtain

$$\dot{m} = 2\pi \sqrt{\gamma \, \varrho_1 p_1} \, R \, H_2 \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \ . \tag{2}$$

This is the maximum mass flux. In a critical or supercritical case, this mass flux depends on the area and the stagnation quantities only and not on the condition in the chamber.

b) The continuity equation requires that the same mass flux $\dot{m}_2$ must exit the clearance [4]. The thermodynamic state in [3] adjust itself correspondingly, so that the continuity requirement is fulfilled. The flow from [3] to [4] can be considered similar to the one from [1] to [2]. Replacing the indices in equation (2), we find

$$\dot{m}_4 = 2\pi\, R\, H_4 \varrho_3 a_3 \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} .$$

Because of (1) a comparison of (2) with the above equation gives

$$\frac{\varrho_1 a_1 H_2}{\varrho_3 a_3 H_4} = 1 , \tag{3}$$

and with $a^2 = \gamma\, p/\varrho$ and $p/\varrho = RT$, it follows from equation (3)

$$\frac{p_1}{p_3}\sqrt{\frac{T_3}{T_1}} = \frac{H_4}{H_2} .$$

For a given clearance height ratio and the known temperature T_3 the pressure p_3 can be calculated. T_3 follows from the energy equation for an adiabatic system (no heat transfer) with $u_1 = u_3 = 0$:

$$h_1 = h_3 \quad \Rightarrow \quad T_3 = T_1$$

$$\Rightarrow \quad p_3 = p_1 \frac{H_2}{H_4} .$$

c) We have the inequality

$$\frac{p_3}{p_1} \leq \frac{p_2^*}{p_1} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} ,$$

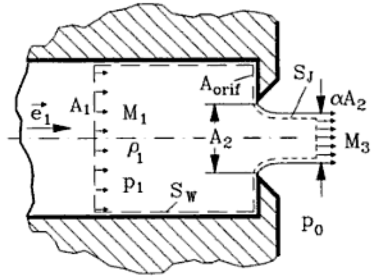
where the equal sign refers to the critical state. With this inequality we arrive at the result

$$\frac{H_2}{H_4} \leq \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0.5283 , \tag{4}$$

for $\gamma = 1.4$. The clearance at station [4] must therefore be larger than the one at [2].

Problem 9.2-10 Gas flow through an orifice

Ideal gas ($\gamma = 1.4$) exits a pipe (cross-section area A_1) through an orifice with an opening A_2, where a contraction of the exiting jet cross-section to $\alpha(M_3)\,A_2 = A_3$ takes place. Friction can be neglected.
The relation $\alpha(M_3)$ is given by the approximation



$$\alpha(M_3) = \frac{\pi}{\pi + \dfrac{2}{\left(1 + \frac{\gamma-1}{2}\, M_3^2\right)^{1/(\gamma-1)}}}\,,$$

which leads for $M_3 \to 0$ to the known values for incompressible flow (see F. M. (10.310)).

a) For the given thermodynamic state at station [1] and geometry, determine the quantities p_3, M_3, $\alpha(M_3)$.

b) Calculate the force exerted by the flow on the orifice.

Given: $p_0 = 1$ bar, $p_1 = 1.2$ bar, $\varrho_1 = 1.3\ \mathrm{kg/m^3}$, $M_1 = 0.2$, $A_1 = 10\ \mathrm{cm^2}$, $A_2 = 5\ \mathrm{cm^2}$, $\gamma = 1.4$

Solution

a) With the isentropic relations

$$\frac{p_t}{p} = \left(\frac{\gamma-1}{2}\, M^2 + 1\right)^{\gamma/(\gamma-1)}\,, \qquad \frac{\varrho_t}{\varrho} = \left(\frac{\gamma-1}{2}\, M^2 + 1\right)^{1/(\gamma-1)}\,, \quad (1)$$

and the given state properties at [1], we determine the stagnation quantities or read from Table C.1 (see F. M.):

$$\frac{p_1}{p_t} = 0.9725 \qquad \Rightarrow \quad p_t = 1.2339\,\mathrm{bar}\,,$$

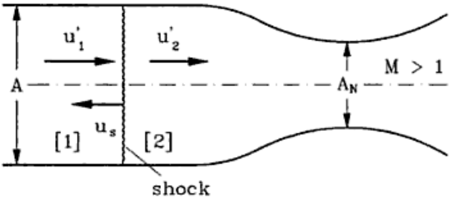
$$\frac{\varrho_1}{\varrho_t} = 0.9803 \qquad \Rightarrow \quad \varrho_t = 1.3261\ \frac{\mathrm{kg}}{\mathrm{m}^3}\,.$$

From the pressure ratio $p_0/p_t = 0.8104$ we find, after solving the isentropic relations (1) or from the above mentioned table, the Mach number $M_3 = 0.5564$ and further

$$\frac{\varrho_3}{\varrho_t} = 0.8634 \quad \Rightarrow \quad \varrho_3 = 1.145\ \frac{\mathrm{kg}}{\mathrm{m}^3}\,.$$

9.3 Unsteady Compressible Flow

Problem 9.3-1 Traveling normal shock in a pipe



Ideal gas exiting from a pipe enters a Laval nozzle with the Mach number $M_1' = 0.6$. The pressure and the speed of sound (p_1, a_1) as well as the area A at station [1] are known. A sudden decrease in nozzle throat A_N, causes an increase in pressure p_2 at the nozzle inlet resulting in a normal shock that propagates into the pipe (see sketch). Downstream of the shock front a steady flow is developed that has the Mach number $M_2' = u_2'/a_2$.

a) Give the cross section A_N, before the sudden change occurred.
b) Calculate the Mach number M_2', downstream of the shock, i. e. upstream of the nozzle if the throat is reduced to $0.6A_N$.
c) Determine numerically the shock Mach number M_S, the Mach number M_2 in the shock fixed system, and the ratio of speed of sound a_2/a_1.
d) Find p_2 and T_2.

Given: $\gamma = 1.4$, $R = 287\,\mathrm{J/(kg\,K)}$, $M_1' = 0.6$, $p_1 = 3\,\mathrm{bar}$, $T_1 = 300\,\mathrm{K}$, $A = 10\,\mathrm{cm}^2$

Solution

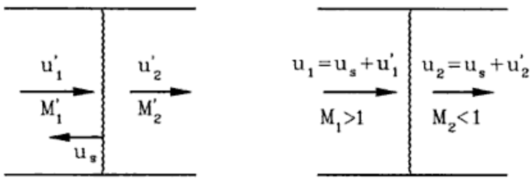
a) The throat A_N before the change:
With $M_1' = 0.6$ from Table C.1 (see F. M.) follows:

$$\frac{A^*}{A} = \frac{A_N}{A} = 0.8416 \qquad \Rightarrow \qquad A_N = 0.8416\,A = 8.416\,\mathrm{cm}^2 \,.$$

b) The Mach number M_2' after the change:
The new cross section is $A^* = 0.6\,A_N$, the new area ratio is then:

$$\frac{A^*}{A} = \frac{0.6\,A_N}{A} = 0.5050 \qquad \Rightarrow \qquad M_2' \approx 0.31 \,.$$

c) The shock Mach number M_S, the Mach number M_2, and the ratio a_2/a_1:
The Mach numbers measured in a laboratory fixed coordinate system up- and downstream of the shock, M_1' and M_2', are known. The corresponding velocities $u_1' = M_1'\,a_1$ and $u_2' = M_2'\,a_2$ must be transferred into the shock fixed system (shock at rest):



laboratory system shock fixed system

The corresponding Mach numbers in the shock fixed system are:

$$M_1 = \frac{u_1}{a_1} = \frac{u_S + u_1'}{a_1} = \frac{u_S}{a_1} + \frac{u_1'}{a_1} = M_S + M_1' \,, \tag{1}$$

$$M_2 = \frac{u_2}{a_2} = \frac{u_S + u_2'}{a_2} = \frac{u_S}{a_1}\frac{a_1}{a_2} + \frac{u_2'}{a_2} = \frac{a_1}{a_2} M_S + M_2' \,. \tag{2}$$

Equations (1) and (2) are two equations in four unknowns M_S, M_1, M_2, and a_1/a_2. The shock relations (see F. M. (9.139) and (9.141)) provide the missing equations:

$$\frac{a_1}{a_2} = \frac{(\gamma+1)\, M_1}{\sqrt{(2\gamma M_1^2 - (\gamma-1))(2 + (\gamma-1)\, M_1^2)}} \tag{3}$$

and

$$M_2 = \left(\frac{\gamma + 1 + (\gamma-1)(M_1^2 - 1)}{2\gamma M_1^2 - (\gamma - 1)}\right)^{1/2} \,. \tag{4}$$

Eliminating M_S in (2) using (1)

$$M_2 - M_2' = \frac{a_1}{a_2}(M_1 - M_1')$$

and inserting the shock relation (3) and (4), we obtain with given M_1', M_2' an equation for M_1. With the numerical values $\gamma = 1.4$, $M_1' = 0.6$, $M_2' = 0.31$ we have the numerical solution $M_1 = 1.17703$.
From (1) we find the shock Mach number

$$M_S = M_1 - M_1' = 0.58 \,,$$

from (2) the Mach number behind the shock

$$M_2 = 0.86 \ ,$$

and from (3), finally, we obtain the ratio of sound speed

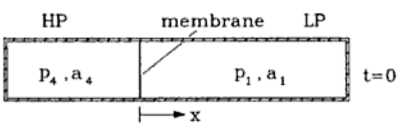
$$\frac{a_2}{a_1} = 1.055 \ .$$

d) Pressure and temperature downstream of the shock:
The thermodynamic properties p_2 and T_2 are found with M_1 from Table C.1 (see F. M.) as

$$\frac{p_2}{p_1} = 1.45 \ , \quad \frac{T_2}{T_1} = 1.1154$$

$$\Rightarrow \quad p_2 = 4.35\,\text{bar} \quad \text{and} \quad T_2 = 334.6\,\text{K} \ .$$

Problem 9.3-2 Shock tube



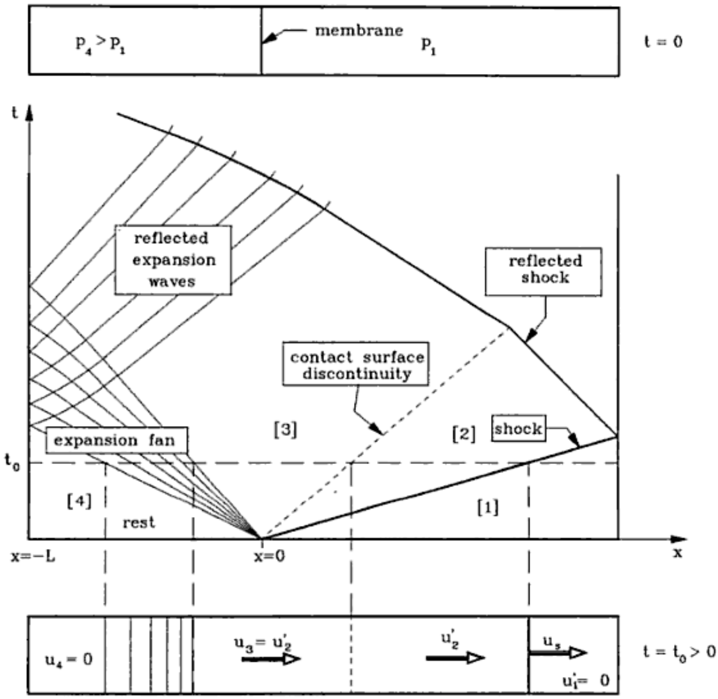
A shock tube consists of a long cylindrical tube, in which a thin membrane separates a high pressure (HP) part from a low pressure (LP) one. The two parts are filled with gas of different thermodynamic conditions p_1, a_1, and p_4, a_4, and $p_4 > p_1$. A sudden rupture of the membrane causes a shock wave to travel in the LP-part which is followed by a contact discontinuity. Into the HP-part a centered expansion wave travels. Shock and expansion waves are reflected at the tube ends.

a) Sketch the flow process in a x-t-diagram.
b) Under the ideal gas assumption ($\gamma_{\text{HP}} = \gamma_{\text{LP}} = \gamma$) give the equations necessary for determining the thermodynamic properties downstream of the shock and expansion fan.

Given: p_1, a_1, p_4, a_4, γ

Solution

a) x-t diagram:



b) State [2] behind the shock and [3] behind the expansion fan:
All primed quantities pertain to a laboratory fixed frame of reference. The equations essential for describing the process are:

1) Velocity behind a shock traveling into a gas at rest (see F. M. (9.154))

$$u_2' = \frac{2}{\gamma+1}\, a_1 \left(M_s - \frac{1}{M_s}\right)$$

or because of $M_s = M_1$ ($u_1' = 0 \quad \Rightarrow \quad u_1 = u_s$)

$$u_2' = \frac{2}{\gamma+1}\, a_1 \left(M_1 - \frac{1}{M_1}\right) . \tag{1}$$

2) Pressure ratio across the shock (see F. M. (9.137))

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} . \tag{2}$$

3) Pressure ratio across the expansion fan
(The contact discontinuity, which moves with $u_2' = u_3$, acts as a piston that moves to the right relative to the expansion fan (see F. M. (9.198)). This means that $|u_P|$ there must be replaced by u_3 here):

$$\frac{p_3}{p_4} = \left(1 - \frac{\gamma-1}{2}\frac{u_3}{a_4}\right)^{\frac{2\gamma}{\gamma-1}} . \tag{3}$$

Furthermore, the following boundary conditions are valid for the contact discontinuity:

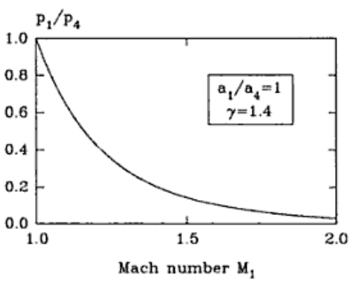
$$u_2' = u_3 , \tag{4}$$

$$p_2 = p_3 . \tag{5}$$

(1) to (5) are five equations for five unknowns u_2, u_3, p_2, p_3, M_1. The calculation procedure is:
Inserting (5) into (3), we obtain $p_2/p_4 = \mathrm{f}\,(u_3/a_4)$. We divide this equation by (2) and find $p_1/p_4 = \mathrm{f}\,(M_1, u_3/a_4)$. In this equation we replace $u_3 = u_2'$ by (1). So, we obtain an equation in terms of $p_1/p_4 = \mathrm{f}\,(M_1)$, from which M_1 can be determined. This equation is

$$\frac{p_1}{p_4} = \left[1 - \frac{\gamma-1}{\gamma+1}\frac{a_1}{a_4}\left(M_1 - \frac{1}{M_1}\right)\right]^{\frac{2\gamma}{\gamma-1}} \frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)} .$$

For given a_1/a_4 and p_1/p_4 the highest possible Mach number can be read from this equation. The following figure exhibits the distribution of $p_1/p_4 = \mathrm{f}\,(M_1)$ for $a_1/a_4 = 1$ and $\gamma = 1.4$.



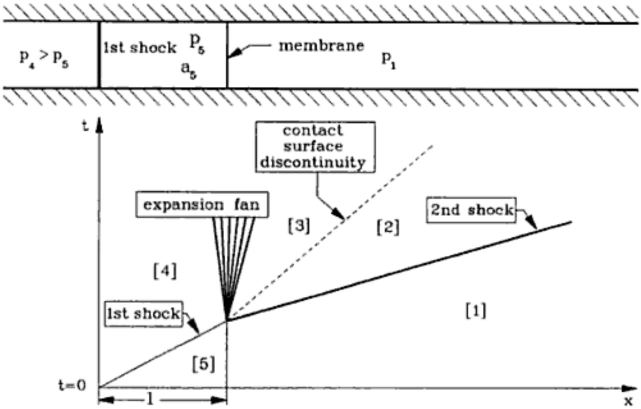
With M_1 known it follows from (1) $u_2' = u_3$, and then from (3) $p_3/p_4 = p_2/p_4$. The ratio a_3/a_4 can be calculated from the isentropic relation valid across the expansion fan:

$$\frac{a_3}{a_4} = \left(\frac{p_3}{p_4}\right)^{\frac{\gamma-1}{2\gamma}} .$$

The ratio a_2/a_1 is gotten with the known M_1 from the shock relation (F. M. (9.139))

$$\frac{a_2}{a_1} = \left(\frac{T_2}{T_1}\right)^{1/2} = \frac{\sqrt{(2\gamma M_1^2 - (\gamma-1))\,(2 + (\gamma-1)M_1^2)}}{(\gamma+1)M_1} .$$

Problem 9.3-5 Principle of an expansion tube



In an infinitely long tube filled with ideal gas ($\gamma = 1.4$), a very thin membrane with negligible mass is located at $x = l$. The gas pressure p_1 on the right hand side of the membrane and the properties p_5 and a_5 on the left hand side are given. From left, a shock wave propagates through the tube and increases the pressure from p_5 to p_4 behind the shock. The shock wave ruptures the membrane as soon as it reaches the location of membrane. As a result, a contact discontinuity is developed which moves with a known velocity of $u_C = u_2' = u_3'$.

a) Determine the Mach number M_{S_1} of the first shock.
b) Give the passing time t_0 necessary for the shock to reach the membrane.
c) Determine the speed of sound a_3.
d) Find the pressure at the contact discontinuity.
e) Calculate the Mach number of the second shock.
f) Determine the speed of sound a_1.

Given: $p_1 = 1$ bar, $p_4 = 18$ bar, $p_5 = 4$ bar, $u_C = 575$ m/s, $a_5 = 300$ m/s, $l = 1$ m, γ=1.4

Solution

a) With the pressure ratio p_4/p_5, we determine the shock Mach number $M_{S_1} = u_S/a_1$ using Table C.2 (see F. M.) or the shock relations (see F. M. (9.137)):

$$\frac{p_4}{p_5} = 1 + 2\frac{\gamma}{\gamma+1}\left(M_{S_1}^2 - 1\right) \quad \Rightarrow \quad M_{S_1} = 2\,.$$

b) The shock velocity is

$$u_S = M_{S_1} a_5 = 600\,\mathrm{m/s}\ ,$$

thus, the time t_0, needed for the shock to cover the distance l, is $t_0 = 10^{-2}/6$ s.

c) Along the C$^+$ charakteristic,

$$u_4' + \frac{2}{\gamma - 1} a_4 = u_3' + \frac{2}{\gamma - 1} a_3\ . \tag{1}$$

We determine the velocity in region [4] u_4 from (see F. M. (9.154))

$$u_4' = \frac{2a_5}{\gamma + 1}\left(M_{S_1} - \frac{1}{M_{S_1}}\right)$$

as $u_4 = 375$ m/s. The speed of sound in this region is calculated from the shock relation (see F. M. (9.139)) or Table C.2 (see F. M.) as $a_4 = 389.7$ m/s. Then, the speed of sound in region [3] is calculated from (1) as $a_3 = 349.7$ m/s.

d) The pressure in region [3] results from the isentropic expansion in region [4]

$$\frac{p_3}{p_4} = \left(\frac{a_3}{a_4}\right)^{\frac{2\gamma}{\gamma-1}}\ ,$$

and p_3 is calculated as 8.4341 bar.

e) For the Mach number M_{S_2} of the second shock, because of $p_2 = p_3$, one can use the shock relation (see F. M. (9.137)) or again Table C.2 (see F. M.). With $p_2/p_1 = 8.4341$ it follows

$$M_{S_2} = 2.7152\ .$$

f) The known flow velocity downstream of the second shock ($u_2' = u_C$) is connected by (see F. M. (9.154))

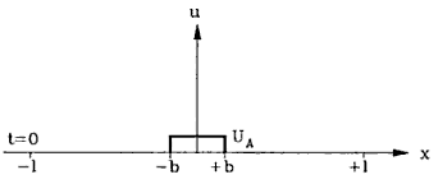
$$u_C = \frac{2a_1}{\gamma + 1}\left(M_{S_2} - \frac{1}{M_{S_2}}\right)$$

with the speed of sound upstream of the second shock. The solution gives $a_1 = 294.0$ m/s.

Problem 9.3-6 Propagation of acoustic waves in a closed tube

In a closed tube of length $2l$, which is filled with ideal gas, the initial distributions of flow velocity $u(x, t)$ and speed of sound $a(x, t)$ are given:

$$u(x, 0) = \begin{cases} 0 & \text{for} \quad x > b \\ U_A & \text{for} \quad |x| \le b \\ 0 & \text{for} \quad x < -b \end{cases}, \qquad a(x, 0) = a_4 \, .$$



Hints:
Flow in the tube is homentropic.
Flow velocity is small compared with the speed of sound.
Flow should be calculated using the method of characteristics.

a) Give the boundary conditions for the velocity u.
b) Calculate the flow velocity in the tube before the disturbance reaches the wall. Give the time t_0 at which the velocity $u(0, t_0) = 0$.
c) Calculate the flow in the tube after the reflection at the wall.

Given: U_A, a_4, l, b

Solution

a) Boundary conditions:
At both tube ends, there is no flow, i. e. :

$$u(l, t) = 0 \, , \quad u(-l, t) = 0 \, .$$

Thus, we are dealing with an initial and boundary value problem.

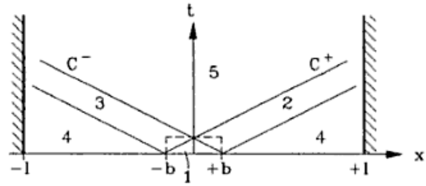
b) Velocity distribution in the tube without reflections:
Since $u \ll a$, the differential equation for the C^+ and C^- characteristics are $\mathrm{d}x/\mathrm{d}t = \pm a_4$. After integration, we arrive at the equation for the C^+ characteristic

$$x(t) = a_4 \, t + \text{const}$$

and the equation of the C^- characteristic

$$x(t) = -a_4 \, t + \text{const} \, .$$

Disturbances propagate along the characteristics as shown in the time–distance diagram. To calculate the velocity $u(x, t)$ in the individual regions, the compatibility condition for a homentropic flow (see F. M. (9.174), (9.175))



$$2r = u(x, t) + \frac{2}{\gamma - 1}\, a(x, t)\,, \tag{1}$$

$$\text{and} \quad -2s = u(x, t) - \frac{2}{\gamma - 1}\, a(x, t) \tag{2}$$

are used, where (1) is applied along the C^+ and (2) along the C^- characteristic. The Riemann invariants $2r$ and $-2s$ are constant along the C^+ and C^- characteristics and are determined using the initial distributions. Two characteristics go through each point of zone [1]; one C^+ characteristic, along which

$$2r = u(x, 0) + \frac{2}{\gamma - 1}\, a(x, t = 0) = U_A + \frac{2}{\gamma - 1}\, a_4$$

holds and one C^- characteristic, along which

$$-2s = u(x, 0) - \frac{2}{\gamma - 1}\, a(x, t = 0) = U_A - \frac{2}{\gamma - 1}\, a_4$$

holds. The velocity u_1 inside zone [1] is obtained by adding equations (1) and (2):

$$u_1 = r - s = U_A\,.$$

In an analogous way we now calculate in zone [2]

$$\begin{aligned} C^+ : \quad 2r &= u(x, 0) + \frac{2}{\gamma - 1}\, a(x, 0) \\ &= U_A + \frac{2}{\gamma - 1}\, a_4\,, \\ C^- : \quad -2s &= u(x, 0) - \frac{2}{\gamma - 1}\, a(x, 0) \\ &= 0 - \frac{2}{\gamma - 1}\, a_4 \end{aligned}$$

(each C^- characteristic in zone [2] starts at a point $x > b$ on the x-axis) and get

$$u_2 = r - s = \frac{1}{2} U_A .$$

In zone [3] all right-running characteristics start at $x < -b$ on the x-axis:

$$\begin{aligned} C^+ : \quad 2r &= 0 + \frac{2}{\gamma - 1} a_4 , \\ C^- : \quad -2s &= U_A - \frac{2}{\gamma - 1} a_4 \end{aligned}$$

$$\Rightarrow \quad u_3 = r - s = \frac{U_A}{2} .$$

The zones [1], [2], and [3] represent the ranges of influence. For the zone [4] and [5] we obtain

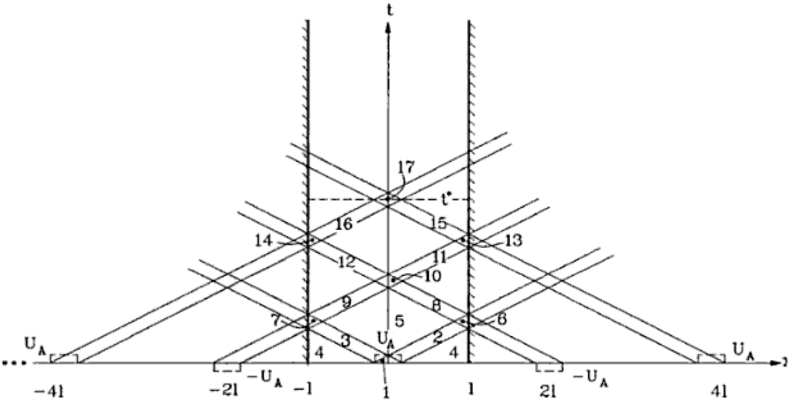
$$\begin{aligned} C^+ : \quad 2r &= 0 + \frac{2}{\gamma - 1} a_4 , \\ C^- : \quad -2s &= 0 - \frac{2}{\gamma - 1} a_4 , \end{aligned}$$

$$\text{and} \quad u_4 = r - s = 0 , \quad \text{as well as} \quad u_5 = r - s = 0 .$$

The time t_0 at which the flow at the location $x = 0$ is at rest, is the point of intersection of the C^+ characteristic, which starts at $(t = 0, x = -b)$ and the time axis:

$$C^+ : \quad x(t) = a_4 \, t - b , \quad x(t_0) = a_4 \, t_0 - b = 0 \quad \Rightarrow \quad t_0 = \frac{b}{a_4} .$$

c) So far, we treated the flow as a purely initial value problem with the given initial value distributions. However, it is apparent that the velocities u_2 and u_3, calculated in point b), do not satisfy the boundary conditions $u(\pm l, t)$. To solve the problem, we assume fictitious disturbances of the width $2b$ and the velocity u_B at $x = \pm 2l$:



Thus, the initial-boundary value problem is converted into a purely initial value problem, where only the interval $|x| \leq l$ has a physical meaning. We get

$$u_{11} = u_{12} = -\frac{1}{2} U_A , \quad u_{13} = u_{14} = 0 ,$$

$$u_{15} = u_{16} = -\frac{1}{2} U_A , \quad u_{17} = U_A .$$

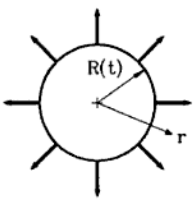
After the disturbance has reached zone [17] (after reflecting twice), the flow pattern repeats itself. The time t^* corresponds to the intersection of the right-running characteristic, which starts at point ($x = -4l$, $t = 0$) with the time axis:

$$C^+ : \quad x(t) = a_4 t - 4l , \quad x(t^*) = a_4 t^* - 4l = 0 \quad \Rightarrow \quad t^* = 4 \frac{l}{a_4} .$$

10 Potential Flow

10.3 Incompressible Potential Flow

Problem 10.3-1 Expanding sphere



An expanding sphere, whose surface is described by $F(r,\, t) = r - R(t) = 0$ is placed in an incompressible fluid.

a) Calculate the velocity potential $\Phi(r)$. (Hint: on the sphere surface the kinematic boundary condition must be satisfied and for $r \to \infty$ the disturbances must vanish.)
b) Calculate the pressure distribution $p(r, t)$.
c) Sketch for the function

$$R(t) = R_0 \left(1 + \frac{t}{t_0}\right)$$

the velocity $u_r(r, t)$ and the pressure distribution $p(r, t)$ at times $t = 0$, $1/2\, t_0$.

Given: R_0, t_0, ϱ

Solution

a) Velocity potential $\Phi(r)$:
The surface of the sphere is described by

$$F(r, t) = r - R(t) = 0$$

which contains only the independent spatial variable r. As an appropriate coordinate system we choose spherical coordinates. Since through the boundary condition no dependency on the independent variables φ and ϑ appears, the problem is expected to be spherically symmetric and the Laplace equation for Φ is reduced to

$$\Delta\Phi = 0 = \frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) .$$

The first integration yields

$$\frac{\partial\Phi}{\partial r} = \frac{C_1(t)}{r^2} ,$$

the second integration leads to

$$\Phi = -\frac{C_1(t)}{r} + C_2(t) . \tag{1}$$

At infinity, a finite sphere cannot generate any flow and, without restriction of generality, we may set $\Phi = 0$ for $r \to \infty$ and conclude that $C_2(t) = 0$. The surface of the sphere is a material surface, since no fluid particle can flow through the surface, thus $\mathrm{D}F/\mathrm{D}t = 0$. The kinematic boundary condition leads therefore to

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + \vec{u}\cdot\nabla F = -\dot{R} + u_r|_R\,\frac{\partial F}{\partial r} = -\dot{R} + \left.\frac{\partial\Phi}{\partial r}\right|_R = 0 ,$$

where the Nabla operator in spherical coordinates (see F. M., Appendix B)

$$\nabla = \vec{e}_r\,\frac{\partial}{\partial r} + \vec{e}_\vartheta\,\frac{1}{r}\,\frac{\partial}{\partial\vartheta} + \vec{e}_\varphi\,\frac{1}{r\sin\vartheta}\,\frac{\partial}{\partial\varphi}$$

has been used to calculate the convective change of the surface:

$$\vec{u}\cdot\nabla F = (u_r\vec{e}_r)\cdot\left(\frac{\partial F}{\partial r}\vec{e}_r\right) = u_r\frac{\partial F}{\partial r} .$$

With the potential (1), the boundary condition yields for the constant $C_1(t)$

$$C_1(t) = \dot{R}\,R^2 ,$$

thus, the potential is

$$\Phi = -\frac{\dot{R}\,R^2}{r} , \tag{2}$$

from which the velocity vector is calculated as

$$\vec{u} = u_r\,\vec{e}_r = \frac{\partial\Phi}{\partial r}\,\vec{e}_r = \frac{\dot{R}R^2}{r^2}\,\vec{e}_r \tag{3}$$

b) Pressure distribution $p(r,t)$:
Bernoulli's equation for incompressible flow without volume forces is (see F. M. (10.59))

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + \frac{p}{\varrho} = \text{const} .$$

We calculate the terms of the equation in turn:

$$\frac{\partial \Phi}{\partial t} = -\frac{\ddot{R}\,R^2}{r} - \frac{\dot{R}\,2\,R\,\dot{R}}{r} = -\frac{R}{r}\,(\ddot{R}\,R + 2\,\dot{R}^2) ,$$

$$\frac{1}{2}\nabla\Phi \cdot \nabla\Phi = \frac{\dot{R}^2\,R^4}{2r^4}$$

and get from Bernoulli's equation the expression

$$-\frac{R}{r}\,(\ddot{R}\,R + 2\,\dot{R}^2) + \frac{\dot{R}^2\,R^4}{2\,r^4} + \frac{p}{\varrho} = \text{const} ,$$

where the Bernoulli constant is given by the condition at infinity ($u_r = 0$, $p = p_\infty$), i. e.

$$\text{const} = \frac{p_\infty}{\varrho} .$$

Thus, we obtain the equation

$$\frac{p_\infty - p}{\varrho} = -\frac{R}{r}\,(\ddot{R}\,R + 2\,\dot{R}^2) + \frac{\dot{R}^2}{2}\left(\frac{R}{r}\right)^4 , \qquad (4)$$

which represents for given pressure difference $p_\infty - p(r = R)$ a differential equation for the sphere radius $R(t)$ and for the given sphere radius $R(t)$ an equation for the pressure field. For the sphere radius in form of

$$R(t) = R_0\left(1 + \frac{t}{t_0}\right)$$

the time derivative becomes $\dot{R} = R_0/t_0$ and $\ddot{R} = 0$, from which we obtain the pressure field

$$\frac{p_\infty - p}{\varrho} = -2\,\frac{R_0}{r}\left(1 + \frac{t}{t_0}\right)\left(\frac{R_0}{t_0}\right)^2 + \frac{1}{2}\left(\frac{R_0}{t_0}\right)^2\frac{R_0^4}{r^4}\left(1 + \frac{t}{t_0}\right)^4 .$$

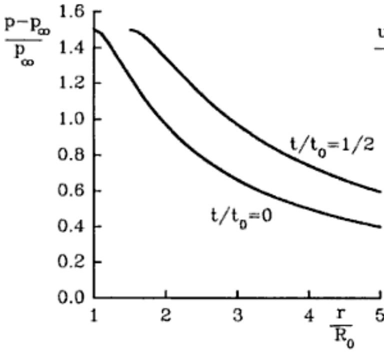
The above equation indicates that the pressure at the surface $r = R(t)$ remains constant in time. Choosing $(R_0/t_0)^2\,\varrho/p_\infty = 1$, the above equations become

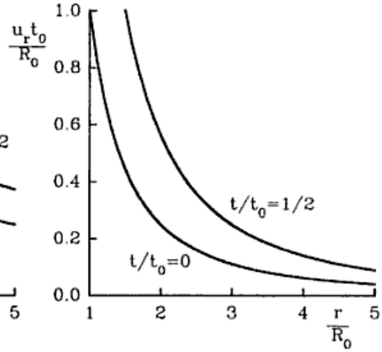
$$\frac{p - p_\infty}{p_\infty} = 2\,\frac{R_0}{r}\left(1 + \frac{t}{t_0}\right) - \frac{1}{2}\left(\frac{R_0}{r}\right)^4\left(1 + \frac{t}{t_0}\right)^4$$

and

$$u_r = \frac{R_0}{t_0}\left(\frac{R_0}{r}\right)^2\left(1 + \frac{t}{t_0}\right)^2 .$$

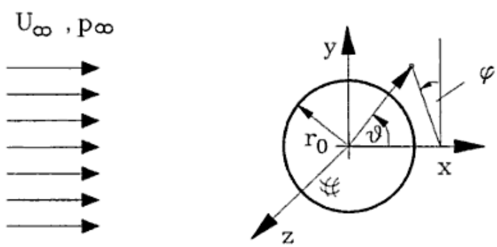
c) The dimensionless plots of u_r and p are:





Problem 10.3-2 Sphere in a translational flow

A sphere of radius r_0 is subjected to a steady, irrotational, inviscid flow.
At infinity, the flow is undisturbed and translational:
$\vec{u} = U_\infty \vec{e}_x$, $p = p_\infty$.



a) Find the velocity potential Φ.
Hints: Use spherical coordinates, where the surface r = const can be described very easily. Solve the Laplace equation subject to the appropriate boundary conditions.
Because of the axisymmetric nature of the problem (x as the symmetry axis) we have $\partial\Phi/\partial\varphi = 0$.

b) Find the resultant force on the sphere.

Solution

a) Potential of a steady irrotational flow around a sphere:
The problem is axisymmetric with x as the polar axis, such that the Laplace equation in spherical coordinates (see F. M. , Appendix B.3) is written as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial \Phi}{\partial \vartheta}\right)=0\ . \tag{1}$$

We solve this equation using the boundary condition

$$\left.\frac{\partial \Phi}{\partial n}\right|_{r=r_0}=\left.\frac{\partial \Phi}{\partial r}\right|_{r=r_0}=0 \tag{2}$$

and the condition at infinity

$$\Phi \sim U_\infty x = U_\infty r\cos\vartheta\ ,\ \text{for } r\to\infty\ . \tag{3}$$

Separating the variables in form of a product,

$$\Phi(r,\vartheta)=R(r)*F(\vartheta)$$

it follows from (1) after multiplication with $\dfrac{r^2}{RF}$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{1}{F}\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right)\ . \tag{4}$$

The left hand side is only a function of r, the right hand side is only a function of ϑ, both sides are thus equal to the separation constant k. The equation of the left hand side

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}-kR=0 \tag{5}$$

is an Euler differential equation and the equation on the right hand side is a Legendre equation.

$$\cos\vartheta\frac{\mathrm{d}F}{\mathrm{d}\vartheta}+\sin\vartheta\frac{\mathrm{d}^2F}{\mathrm{d}\vartheta^2}+kF\sin\vartheta=0\ . \tag{6}$$

As we know, $R(r)=r^\alpha$ is the solution of Euler's equation, if α satisfies

$$\alpha^2+\alpha=k\ .$$

We obtain the roots

$$\begin{aligned}\alpha_1 &= -\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+k}=:n\ ,\\ \alpha_2 &= -\tfrac{1}{2}-\sqrt{\tfrac{1}{4}+k}=-n-1\end{aligned}$$

and the solution of (5) as

$$R(r)=A'_n r^n+B'_n r^{-(n+1)}\ . \tag{7}$$

With $\alpha^2 + \alpha = k = n(n+1)$ equation (6) becomes

$$\cos\vartheta \frac{\mathrm{d}F}{\mathrm{d}\vartheta} + \sin\vartheta \frac{\mathrm{d}^2F}{\mathrm{d}\vartheta^2} + n(n+1)\sin\vartheta\, F = 0 \ . \tag{8}$$

Substituting $\mu := \cos\vartheta$ results in

$$\frac{\mathrm{d}F}{\mathrm{d}\vartheta} = \frac{\mathrm{d}F}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\vartheta} = -\sin\vartheta \frac{\mathrm{d}F}{\mathrm{d}\mu}$$

and

$$\frac{\mathrm{d}^2F}{\mathrm{d}\vartheta^2} = \sin^2\vartheta \frac{\mathrm{d}^2F}{\mathrm{d}\mu^2} - \cos\vartheta \frac{\mathrm{d}F}{\mathrm{d}\mu} \ .$$

With $\sin^2\vartheta = 1 - \cos^2\vartheta = 1 - \mu^2$ we obtain from (8) the Legendre differential equation:

$$(1-\mu^2)\frac{\mathrm{d}^2F}{\mathrm{d}\mu^2} - 2\mu\frac{\mathrm{d}F}{\mathrm{d}\mu} + n(n+1)F = 0 \ . \tag{9}$$

If n is a positive integer ($n \geq 0$), the general solution of (9) can be written in the form

$$F(\mu) = C'P_n(\mu) + D'Q_n(\mu) \ .$$

The functions $P_n(\mu)$ are polynomials of degree n, and are called Legendre polynomials. $Q_n(\mu)$ are called Legendre functions of the second kind, they are singular at $\mu = \pm 1$, i. e. $Q_n(\mu)$ becomes infinite on the x-axis:

$$\mu = \cos\vartheta \quad , \quad \vartheta = 0 \ \text{or} \ \pi \quad \Rightarrow \quad \mu = 1 \ \text{or} \ -1 \ .$$

Because the potential for finite r on the x-axis is finite (the stagnation points are located on the x-axis), we have

$$D' \equiv 0 \ .$$

If n is not a positive integer, the functions $P_n(\mu)$ also are singular at $\mu = \pm 1$ and for the solution to remain bounded on the x-axis, we must have

$$F(\mu) = C'P_n(\mu) \quad \text{with } n \ \text{ integer and} \quad n \geq 0 \ . \tag{10}$$

The Legendre polynomials can be generated by the formula

$$P_n(\mu) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}(\mu^2-1)^n$$

and are

$$\begin{aligned} P_0(\mu) &= 1\,, \\ P_1(\mu) &= \mu = \cos\vartheta\,, \\ P_2(\mu) &= \frac{1}{2}(3\mu^2-1) = \frac{1}{2}(3\cos^2\vartheta - 1) \\ &\vdots \qquad\quad \vdots \end{aligned}$$

From (7) and (10) it follows

$$\Phi_n = \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\mu)\,,$$

for every n integer ≥ 0 and thus we also have

$$\Phi = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\mu)$$

as a solution of the potential equation (with $A_n = A'_n C'_n$, $B_n = B'_n C'_n$).
The constants A_n and B_n are calculated from the boundary conditions and the condition at infinity.
On the sphere surface we find from

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=r_0} = \sum_{n=0}^{\infty} \left(nA_n r_0^{n-1} - (n+1)B_n r_0^{-(n+2)}\right) P_n(\mu) = 0$$

$$\Rightarrow \quad B_n = \frac{n}{n+1} r_0^{(2n+1)} A_n$$

$$\Rightarrow \quad \Phi = \sum_{n=0}^{\infty} A_n \left(r^n + \frac{n}{n+1} r_0^{(2n+1)} r^{-(n+1)}\right) P_n(\mu)\,.$$

For $r \to \infty$ we have

$$\Phi \sim \sum_{n=0}^{\infty} A_n r^n P_n(\mu)\,,$$

and since the asymptotic behavior of Φ for $r \to \infty$ is

$$\Phi \sim U_\infty r \cos\vartheta\,,$$

the constants take on the values

$$A_1 = U_\infty \ \text{ and } \ A_n = 0 \ \text{ for } \ n \neq 1\,,$$

and thus the solution is

$$\Phi = U_\infty \left(r + \frac{1}{2}\frac{r_0^3}{r^2}\right) \cos\vartheta\,.$$

This is the known potential of the flow around a sphere (see F. M. (10.139)).

b) Force on the sphere:
Bernoulli's equation is

$$p_\infty + \frac{\varrho}{2}U_\infty^2 = p + \frac{\varrho}{2}\vec{u}\cdot\vec{u}\,.$$

On the sphere we have $u_r = 0$ (boundary condition)

$$u_\vartheta = \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\bigg|_{r=r_0} = -\frac{3}{2}U_\infty \sin\vartheta$$

$$\Rightarrow p - p_\infty = \frac{\varrho}{2}U_\infty^2\left(1 - \frac{9}{4}\sin^2\vartheta\right)$$

and the force on the sphere is

$$\vec{F} = \iint\limits_{S_{sphere}} -(p - p_\infty)\vec{n}\,\mathrm{d}S\ ,\ \vec{n} = \vec{e}_r$$

$$\vec{F} = -\frac{\varrho}{2}U_\infty^2 \int_0^{2\pi}\int_0^{\pi}\left(1 - \frac{9}{4}\sin^2\vartheta\right)\vec{e}_r r_0^2 \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\ ,$$

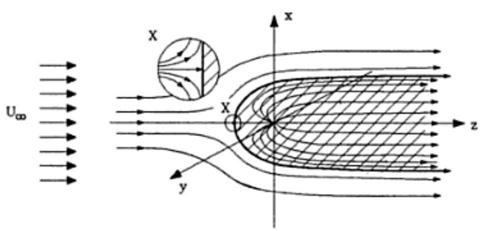
with $\vec{e}_r = \cos\vartheta\vec{e}_x + \sin\vartheta\cos\varphi\vec{e}_y + \sin\vartheta\sin\varphi\vec{e}_z$ we obtain

$$\vec{F} \equiv 0\ ,$$

in accordance with d'Alembert's paradoxon.

Problem 10.3-3 Flow near the stagnation point of a body in parallel flow

Superimposing a point source and a parallel flow results in a flow around an infinitely long body as shown in the sketch.



a) Find the velocity potential of the flow.
b) By expanding the velocity components, show that the flow in the neighbourhood of the stagnation point corresponds to a stagnation point flow with z as the symmetry axis.

Given: U_∞, m

Solution

a) The velocity potential is

$$\Phi = U_\infty z - \frac{m}{4\pi r} \, .$$

Different from F. M. (10.92), we use here z as the pole axis to establish a relationship with the rotationally symmetric stagnation point flow. The velocity components are:

$$\begin{aligned} u &= \frac{\partial \Phi}{\partial x} &&= \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} = \frac{m}{4\pi r^2}\frac{2x}{2r} = \frac{m}{4\pi}\frac{x}{r^3} \, , \\ v &= \frac{\partial \Phi}{\partial y} &&= \frac{m}{4\pi}\frac{y}{r^3} \, , \\ w &= \frac{\partial \Phi}{\partial z} &&= U_\infty + \frac{m}{4\pi}\frac{z}{r^3} \, . \end{aligned}$$

b) Velocity at the stagnation point:
The stagnation point is calculated from $\vec{u} = 0$ as:

$$\begin{aligned} u = 0 \quad &\Rightarrow \quad x_s = 0 \, , \\ v = 0 \quad &\Rightarrow \quad y_s = 0 \, , \\ w = 0 \quad &\Rightarrow \quad U_\infty + \frac{m}{4\pi}\frac{z_s}{z_s^2\sqrt{z_s^2}} = 0 \, . \end{aligned}$$

The last equation has a real solution only on the negative z-axis, with $z_s = -|z_s|$ we get

$$|z_s|^2 = \frac{m}{4\pi U_\infty} \quad z_s = -\sqrt{\frac{m}{4\pi U_\infty}} \, .$$

The Taylor series of the velocity about the stagnation point (x_s, y_s, z_s) neglecting the higher order terms, is in index notation

$$u_i = u_i\big|_s + \frac{\partial u_i}{\partial x_j}\bigg|_s (x_j - x_{js}) \, .$$

The calculation of the velocity gradient gives

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial x_i}\right) = -\frac{m}{4\pi}\frac{\partial}{\partial x_j}\left[\frac{\partial(1/r)}{\partial x_i}\right] ,$$

furthermore,

$$-\frac{m}{4\pi}\frac{\partial}{\partial x_j}\left[\frac{\partial(1/r)}{\partial x_i}\right] = \frac{m}{4\pi}\frac{\partial}{\partial x_j}\left(\frac{x_i}{r^3}\right) =$$

$$= \frac{m}{4\pi}\left\{\frac{1}{r^3}\frac{\partial x_i}{\partial x_j} - 3\frac{x_i x_j}{r^5}\right\} = \frac{m}{4\pi r^3}\left\{\delta_{ij} - 3\frac{x_i x_j}{r^2}\right\} .$$

We evaluate the velocity gradient at the stagnation point, i. e. at $x = x_s = 0$, $y = y_s = 0$ and $z = z_s$:

$$\left.\frac{\partial u_1}{\partial x_1}\right|_s = \underbrace{\frac{m}{4\pi\,|z_s|^3}}_{a} \quad ; \quad \left.\frac{\partial u_2}{\partial x_1}\right|_s = 0 \quad ; \quad \left.\frac{\partial u_3}{\partial x_1}\right|_s = 0$$

$$\left.\frac{\partial u_1}{\partial x_2}\right|_s = 0 \quad ; \quad \left.\frac{\partial u_2}{\partial x_2}\right|_s = a \quad ; \quad \left.\frac{\partial u_3}{\partial x_2}\right|_s = 0$$

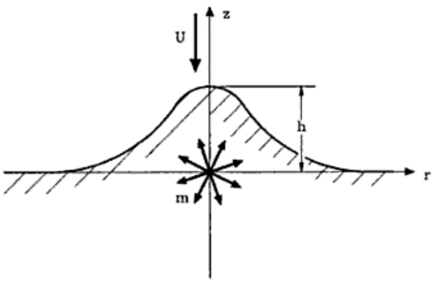
$$\left.\frac{\partial u_1}{\partial x_3}\right|_s = 0 \quad ; \quad \left.\frac{\partial u_1}{\partial x_2}\right|_s = 0 \quad ; \quad \left.\frac{\partial u_3}{\partial x_3}\right|_s = -2a$$

$$\Rightarrow \quad u = ax \; ; \; v = ay \; ; \; w = -2a(z - z_s) .$$

We realize that the velocity components in a coordinate system $(z' = z - z_s)$ which is located at the stagnation point coincides with velocity components of an rotationally symmetric stagnation point flow with:

$$u = ax' \; ; \; v = ay' \; ; \; w = -2az' .$$

Problem 10.3-4 Point source in a rotationally symmetric stagnation point flow



Steady incompressible flow over the sketched body is considered as frictionless potential flow. The flow is the result of a superposition of a rotationally symmetric stagnation point flow and the flow generated by a point source located in the origin as shown in the figure.

a) Find the velocity potential of the flow around the body.
b) Determine the velocity field $\vec{u}(r,z)$.
c) Find the source strength m.
d) Find the stream function.
e) Determine the body shape by calculating the stagnation streamline.
f) Determine the stagnation points of the flow, if there is a sink ($m < 0$) located in the origin. Sketch the streamlines for this case.

Solution

a) We find the total potential by superimposing the rotationally symmetric stagnation point flow

$$\Phi_S = \frac{a}{2}(x^2 + y^2 - 2z^2)$$

with source potential at the origin

$$\Phi_m = -\frac{m}{4\pi}\frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

as

$$\Phi_{tot} = \frac{a}{2}(x^2 + y^2 - 2z^2) - \frac{m}{4\pi}\frac{1}{\sqrt{x^2 + y^2 + z^2}} \,.$$

b) The velocity field $\vec{u}(r,z)$ in cylindrical coordinates is ($r^2 = x^2 + y^2$)

$$u_r = \frac{\partial \Phi_{tot}}{\partial r}\,, \quad u_\varphi = \frac{1}{r}\frac{\partial \Phi_{tot}}{\partial \varphi}\,, \quad u_z = \frac{\partial \Phi_{tot}}{\partial z}\,.$$

Hence,

$$u_r \;=\; ar + \frac{m}{4\,\pi}\frac{r}{(r^2 + z^2)^{3/2}}\,, \tag{1}$$

$$u_\varphi = 0 ,$$

$$u_z = -2\,a\,z + \frac{m}{4\,\pi}\,\frac{z}{(r^2+z^2)^{3/2}} . \qquad (2)$$

c) Strength m:
For symmetry reasons, the stagnation point must be located at $r = r_s = 0$ and $z = z_s = h$.

$$\Rightarrow \quad u_r(r=0, z=h) = 0 ,$$

$$u_z(r=0, z=h) = 0 = -2ah + \frac{m}{4\pi}\frac{1}{h^2}$$

$$\Rightarrow \quad m = 8\pi a h^3 .$$

d) Stream function $\Psi = \Psi(r,z)$ for the rotationally symmetric flow:
For the streamlines, we have $\Psi(r,z) = \text{const}$

$$\Rightarrow \quad \mathrm{d}\Psi = \frac{\partial\Psi}{\partial r}\mathrm{d}r + \frac{\partial\Psi}{\partial z}\mathrm{d}z = 0 ,$$

from which follows

$$\frac{\partial^2\Psi}{\partial z\partial r} = \frac{\partial^2\Psi}{\partial r\partial z} ,$$

provided the derivatives are continuous. The continuity equation $\operatorname{div}\vec{u} = 0$ in cylindrical coordinates is

$$\frac{\partial(u_r r)}{\partial r} + \frac{\partial(u_z r)}{\partial z} = 0 .$$

Comparing the last two equations we infer the equations

$$\frac{\partial\Psi}{\partial z} = u_r r \quad \text{and} \quad \frac{\partial\Psi}{\partial r} = -u_z r$$

(see F. M. (10.103), (10.104), (10.105) for the case of spherical coordinates). With the velocity component u_z already known the integration with respect to r is

$$\Psi = azr^2 + \frac{m}{4\pi}\frac{z}{\sqrt{r^2+z^2}} + C(z)$$

and from

$$\frac{\partial\Psi}{\partial z} = ar^2 + \frac{m}{4\pi}\frac{r^2}{(r^2+z^2)^{3/2}} + \frac{\mathrm{d}C}{\mathrm{d}z} = u_r r ,$$

with u_r known, it follows

$$u_r r = ar^2 + \frac{m}{4\pi}\frac{r^2}{(r^2+z^2)^{3/2}} ,$$

b) The velocity field is

$$\vec{u} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\vec{e}_\varphi + \frac{\partial\Phi}{\partial z}\vec{e}_z \ ;$$

$$\begin{aligned}
u_r &= \frac{\partial\Phi}{\partial r} = \frac{m}{4\pi}\left[\frac{r}{[r^2+(z-a)^2]^{3/2}} + \frac{r}{[r^2+(z+a)^2]^{3/2}}\right] \ , \\
u_\varphi &= \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = 0 \ , \\
u_z &= \frac{\partial\Phi}{\partial z} = \frac{m}{4\pi}\left[\frac{z-a}{[r^2+(z-a)^2]^{3/2}} + \frac{z+a}{[r^2+(z+a)^2]^{3/2}}\right] \ ,
\end{aligned}$$

and particularly at the wall ($z = 0$):

$$u_r = \frac{m}{4\pi}\frac{2r}{[r^2+a^2]^{3/2}}$$

and

$$u_z = \frac{m}{4\pi}\left[\frac{-a}{[r^2+a^2]^{3/2}} + \frac{a}{[r^2+a^2]^{3/2}}\right] = 0 \ .$$

The last equation shows that the kinematic boundary condition is satisfied. For $(x, y, z) = (0, 0, 0)$ the velocity $\vec{u} = 0$, i. e. the origin is a stagnation point.

c) Pressure distribution at the wall:
We call the pressure at the stagnation point p_0. Since we are dealing with a potential flow, the constant in Bernoulli's equation is throughout the entire flow field an absolute constant, thus the equation is also valid "across streamlines":

$$p + \frac{\varrho}{2}(u_r^2 + u_\varphi^2 + u_z^2) = p_0 \ .$$

At the wall is

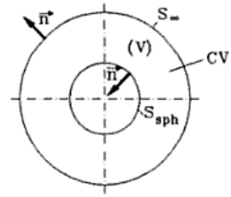
$$u_\varphi = u_z = 0 \ , \quad u_r^2 = \left(\frac{m}{4\pi}\right)^2 \frac{4r^2}{[r^2+a^2]^3} \ ,$$

and thus, the pressure distribution is

$$p - p_0 = -\frac{\varrho}{2}\left(\frac{m}{4\pi}\right)^2 \frac{4r^2}{[r^2+a^2]^3} \ .$$

Solution

a) Kinetic energy of the fluid:
The kinetic energy of the fluid inside the volume V is



$$K = \frac{\varrho}{2}\iiint\limits_{(V)} u_r^2\,\mathrm{d}V = \frac{\varrho}{2}\iiint\limits_{(V)} \nabla\Phi\cdot\nabla\Phi\,\mathrm{d}V$$

$$= \frac{\varrho}{2}\iiint\limits_{(V)} [\nabla\cdot(\Phi\,\nabla\Phi) - \Phi\,\Delta\Phi]\,\mathrm{d}V\,.$$

With $\Delta\Phi = 0$ and the Gauss' theorem we have

$$K = \frac{\varrho}{2}\iint\limits_{(S)} \Phi\,\frac{\partial\Phi}{\partial n}\,\mathrm{d}S\,. \tag{1}$$

The kinetic energy outside the sphere is obtained by evaluating (1) for the sketched control volume. The outer boundary surface S_∞ is everywhere at infinity ($r\to\infty$). The normal derivatives of Φ are

$$\frac{\partial\Phi}{\partial n} = \begin{cases} -\dfrac{\partial\Phi}{\partial r} & \text{at} \quad S_{sph} \\[2ex] \dfrac{\partial\Phi}{\partial r} & \text{at} \quad S_\infty \end{cases}$$

with $\partial\Phi/\partial r = \dot{R}\,(R/r)^2$. From (1) and with the surface element in spherical coordinates, $\mathrm{d}S_r = r^2\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$, we find

$$K = \frac{\varrho}{2}\left(\int_0^{2\pi}\!\!\int_0^{\pi} -\Phi\,\frac{\partial\Phi}{\partial r}\,R^2\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi + \lim_{r\to\infty}\int_0^{2\pi}\!\!\int_0^{\pi} \Phi\,\frac{\partial\Phi}{\partial r}\,r^2\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\right)$$

$$= \varrho\,\pi\left(\int_0^{\pi} \dot{R}^2\,R^3\,\sin\vartheta\,\mathrm{d}\vartheta + \lim_{r\to\infty}\int_0^{\pi} -\frac{(\dot{R}\,R^2)^2}{r}\,\sin\vartheta\,\mathrm{d}\vartheta\right)$$

$$= 2\pi\,\varrho\,\dot{R}^2\,R^3\,. \tag{2}$$

b) Since in inviscid, nonconducting flow $\mathrm{D}e/\mathrm{D}t = 0$ (see F. M. (2.119)), the energy equation (see F. M. (2.113)) is reduced to $\mathrm{D}K/\mathrm{D}t = P$. The power P of the stress vector $\vec{t} = -p(r,\,t)\,\vec{n}$ at the control surface is

$$P = \iint\limits_{S_{sph}} u_r\,p(R,t)\,\mathrm{d}S\,, \tag{3}$$

where we took into account the stress vector on the surface S_∞ being zero. The pressure distribution follows from the unsteady Bernoulli equation. For $r = R$, we have

$$p(R, t) = \varrho \left(\ddot{R}\, R + \frac{3}{2}\,\dot{R}^2\right) \quad \text{and} \quad u_r(R, t) = \dot{R}\,.$$

(3) becomes

$$P = 4\pi\, R^2\, p(R, t)\, u_r(R, t) = 4\pi\, \varrho\, R^2 \dot{R} \left(\ddot{R}\, R + \frac{3}{2}\,\dot{R}^2\right)\,.$$

Calculating $\mathrm{D}K/\mathrm{D}t$ using (2) leads to

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \frac{\partial K}{\partial t} + u_r\,\frac{\partial K}{\partial r} = \frac{\mathrm{d}K}{\mathrm{d}t} = 4\pi\, \varrho\, R^2 \dot{R} \left(\ddot{R}\, R + \frac{3}{2}\,\dot{R}^2\right)\,.$$

The energy equation $\mathrm{D}K/\mathrm{D}t = P$ is therefore satisfied.

c) Since we are dealing with an incompressible potential flow, the Navier-Stokes equation is satisfied. The velocity vector on the sphere is $\vec{u}_w = \dot{R}\,\vec{e}_r$ and is equal to the flow velocity vector $\vec{u} = \dot{R}\,R^2/r^2|_R\,\vec{e}_r = \dot{R}\,\vec{e}_r$. This means that the dynamic boundary condition (see F. M. (4.159)) is satisfied. Thus, we are dealing with an exact solution of the Navier-Stokes equation. Although the divergence of the viscous stress tensor disappears (see F. M. , Page 103), the viscous stresses do not vanish. The tensor of the viscous stresses for an incompressible flow assumes the form: (F. M. (3.2b))

$$\mathbf{P} = 2\,\eta\,\mathbf{E}$$

The rate of deformation tensor is (see F. M. , Appendix B.3)

$$\mathbf{E} = \frac{\partial u_r}{\partial r}\vec{e}_r\,\vec{e}_r + \frac{u_r}{r}\vec{e}_\varphi\,\vec{e}_\varphi + \frac{u_r}{r}\vec{e}_\vartheta\,\vec{e}_\vartheta$$

and thus, the viscous stress vector $\vec{t}_{vis}$ on the sphere surface S_{sph}

$$\vec{t}_{vis} = \vec{n}\cdot\mathbf{P} = -\vec{e}_r\cdot\mathbf{P} = -2\,\eta\,\frac{\partial u_r}{\partial r}\,\vec{e}_r\,,$$

and on S_∞ is

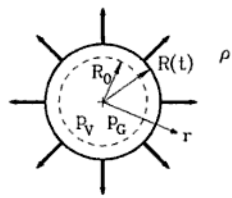
$$\vec{t}_{vis} = \vec{n}\cdot\mathbf{P} = \vec{e}_r\cdot\mathbf{P} = 2\,\eta\,\frac{\partial u_r}{\partial r}\,\vec{e}_r\,.$$

or

$$\iiint\limits_{(V)} \Phi \, \mathrm{d}V = 16\,\pi\,\eta\,\dot{R}^2\,R = \iiint\limits_{(V)} \varrho\,\frac{\mathrm{D}e}{\mathrm{D}t}\,\mathrm{d}V\,.$$

The dissipated energy, which is equal to the increase of internal energy, is caused by the viscous stress work. In the energy equation (see F. M. (2.114)), the contribution of the viscous stress work and the increase of internal energy cancel each other. Thus, as in the inviscid flow, the work by the pressure serves to increase (or to decrease) the kinetic energy of the flow.

Problem 10.3-8 Growth of a vapor filled cavity



In a fluid at rest, the pressure has been dropped so far below the vapor pressure that a spherosymmetric cavity of radius $R(t)$ is formed. Cavitation nuclei are often undissolved gases in form of bubbles, that are attached to solid particles. We assume a gas vapor mixture inside the cavity where gas and vapor are under their partial pressures p_G and p_V. The expansion of the cavity with the time dependent change of the cavity radius $\mathrm{d}R/\mathrm{d}t = \dot{R}$ causes a sphero-symmetrical flow of the fluid. The pressure distribution

$$\frac{p(r,\,t) - p_\infty(t)}{\varrho} = \frac{R}{r}\,(R\,\ddot{R} + 2\,\dot{R}^2) - \frac{1}{2}\left(\frac{R^2\,\dot{R}}{r^2}\right)^2$$

of this flow was calculated in Problem 10.3-1.

a) In inviscid flow, the pressure at the cavity boundary is given as (compare F. M. (5.53)):

$$p(R,\,t) = p_G + p_V - 2\,\frac{C}{R}\,.$$

Furthermore, the partial pressure of the gas p_G, the partial pressure of vapor p_V, and the capillary constant C (see F. M. (5.53)) are also given.

Find the equation of motion of the cavity surface.

b) For small disturbances of the cavity radius R about R_0, i. e. $R(t) = R_0 + \varepsilon\,R_1(t)$ with $\varepsilon \ll 1$, examine the stability of the cavity growth, if the partial pressure of the gas p_G changes isothermally.

Solution

a) Equation of motion of the cavity surface:
The equation of motion of the cavity boundary follows from the pressure distribution in the field for $r = R$:

$$\frac{p(R,\,t) - p_\infty(t)}{\varrho} = R\,\ddot{R} + 2\,\dot{R}^2 - \frac{1}{2}\,\dot{R}^2 = R\,\ddot{R} + \frac{3}{2}\,\dot{R}^2\,.$$

With the given pressure at the surface we find

$$R\,\ddot{R} + \frac{3}{2}\,\dot{R}^2 = \frac{1}{\varrho}\left(p_G + p_V - \frac{2C}{R} - p_\infty(t)\right)\,. \tag{1}$$

From the equation of state

$$p_G = \frac{m}{V}\,\mathcal{R}\,T\,,$$

with m as the constant mass of the gas and the cavity volume $V = (4/3)\,\pi\,R^3$, the gas partial pressure is

$$p_G = \frac{3}{4}\,\frac{m\,\mathcal{R}}{\pi}\,T\,\frac{1}{R^3} = A\,T\,\frac{1}{R^3}\,, \quad A = \text{const}\,. \tag{2}$$

Inserting (2) into (1) leads to a differential equation for the cavity radius $R(t)$:

$$R\,\ddot{R} + \frac{3}{2}\,\dot{R}^2 - \frac{A\,T}{\varrho}\,\frac{1}{R^3} + \frac{1}{\varrho}\,\frac{2C}{R} = \frac{p_V - p_\infty(t)}{\varrho}\,. \tag{3}$$

b) Stability of the cavity growth for an isothermal gas pressure change:
We consider the case of constant temperature in the cavity ($T = \text{const}$) and calculate from (2) the gas partial pressure as

$$p_G = p_{G_0}\left(\frac{R_0}{R}\right)^3\,,$$

which leads to the equation of motion

$$R\,\ddot{R} + \frac{3}{2}\,\dot{R}^2 - \frac{p_{G_0}}{\varrho}\left(\frac{R_0}{R}\right)^3 + \frac{1}{\varrho}\,\frac{2C}{R} = \frac{p_V - p_\infty(t)}{\varrho}\,. \tag{4}$$

To obtain $R(t)$, the above ordinary, nonlinear, second order differential equation is solved numerically by using the initial conditions

$$t = 0: \quad R = R_0 \quad \text{and} \quad \dot{R} = 0\,.$$

We focus here on the stability aspects of the cavity growth and expand the function $R(t)$ for small disturbances about R_0. From the problem definition we get

$$\dot{R} = \varepsilon\,\dot{R}_1 \quad \text{and} \quad \ddot{R} = \varepsilon\,\ddot{R}_1\,.$$

We insert the above equations into (4) and neglect higher order terms in ε. This leads to a linearized equation of motion

$$\varepsilon\, R_0 \ddot{R}_1 + \varepsilon\, \frac{R_1}{\varrho\, R_0}\left(3\, p_{G_0} - 2\frac{C}{R_0}\right) = \frac{1}{\varrho}\left(p_V + p_{G_0} - \frac{2C}{R_0} - p_\infty(t)\right)\,, \quad (5)$$

where R_0 and R_1 are now of the same order of magnitude. The order of magnitude of the terms in the differential equation is given by powers of ε . Comparing terms of equal magnitude in (5) is identical to comparing terms with equal exponents in ε.

Terms of order ε^0: From (5) follows the equation

$$0 = \frac{1}{\varrho}\left(p_V + p_{G_0} - \frac{2C}{R_0} - p_\infty(t)\right)$$

which describes the pressure equilibrium on the surface of the cavity at rest, thus, we have

$$R_0 = \frac{2\,C}{p_V - p_\infty + p_{G_0}}\,.$$

Comparing terms of order ε results in the linear differential equation

$$\ddot{R}_1 + \frac{R_1}{\varrho\, R_0^2}\left(3\, p_{G_0} - 2\,\frac{C}{R_0}\right) = 0\,, \quad (6)$$

that can be solved by the setting $R_1 = \mathrm{e}^{\lambda t}$. For a positive real value of λ, a disturbance grows exponentially, the cavity is unstable. From (6) we get

$$\lambda^2 = \frac{1}{\varrho\, R_0^2}\left(2\,\frac{C}{R_0} - 3\, p_{G_0}\right)$$

and thus, a positive real part for

$$2\,\frac{C}{R_0} > 3 p_{G_0}$$

and with (2) also

$$\frac{2\,C}{R_0} > \frac{3\,A\,T}{R_0^3}$$

or

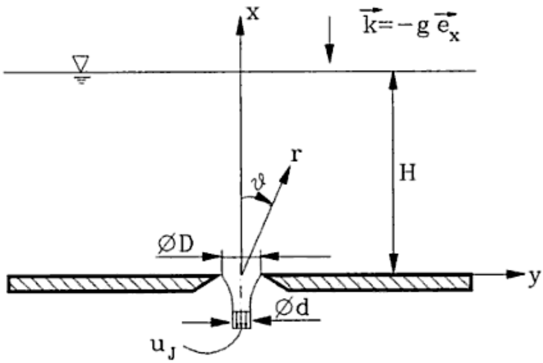
$$R_0 > \sqrt{\frac{3}{2}\,\frac{A\,T}{C}} = R_{0_{\text{crit}}}\,.$$

If the equilibrium radius is greater than the critical radius, the cavity will grow exponentially. For an initial radius of $R_0 = R_{0_{\text{crit}}}$ the cavity maintains the same radius.

For $R_0 < R_{0_{\text{crit}}}$ the eigenvalue λ is imaginary. The cavity will oscillate with constant amplitude. For the case that the gas pressure p_{G_0} is equal to zero or much smaller than the vapor pressure, the cavity will grow to establish thermodynamic equilibrium.

Problem 10.3-9 Contraction coefficient for a circular orifice

Incompressible fluid (density ϱ) flows through the sketched circular opening (diameter D), which is located in the bottom of a large container. The exiting jet experiences a radius contraction to d. The flow is steady and inviscid. Calculate the contraction coefficient $\alpha = d^2/D^2$.



Hint: If D is sufficiently small compared with the height H, the fluid can move purely radially in a sufficiently large distance from the opening and may be replaced by a point sink that has the velocity field

$$\vec{u} = \frac{m}{4\pi}\frac{1}{r^2}\vec{e}_r\,, \quad m < 0\,.$$

a) Calculate the velocity u_J of the exiting jet.
b) Find the contraction coefficient α and the diameter d of the jet.

Given: H, D, ϱ, p_0

Solution

a) The height H of the fluid does not change and Torricelli's discharge formula can be applied:

$$u_J = \sqrt{2\,g\,H}\,. \tag{1}$$

b) The control volume for the momentum balance contains a half sphere (radius R_{sph}) with the center in the coordinate origin and the exiting jet. The control surface intersects the jet at a point, where the jet diameter does not change anymore.

with $\vec{u} = (A/r^2)\,\vec{e}_r$, where $A = m/4\pi$, $m < 0$, $\vec{n} = \vec{e}_r$, and with the surface element (in spherical coordinates) $\mathrm{d}S = r^2 \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$. The second integral on the left hand side of (2) becomes with $\vec{u} = -u_J\,\vec{e}_x$, $\vec{n} = -\vec{e}_x$ and $\mathrm{d}S = r\,\mathrm{d}\varphi\,\mathrm{d}r$

$$\iint\limits_{A_J} \varrho\,\vec{u}\cdot\vec{e}_x\,(\vec{u}\cdot\vec{n})\,\mathrm{d}A = \int\limits_0^{2\pi}\int\limits_0^{\frac{d}{2}} -\varrho\,u_J^2\,r\,\mathrm{d}r\,\mathrm{d}\varphi = -\pi\,\varrho\,u_J^2\,\frac{d^2}{4}\,.$$

By evaluating the volume integral we note that ϱg is constant and arrive at

$$-\iiint\limits_V \varrho\,g\,\mathrm{d}V = -\varrho\,g\iiint\limits_V \mathrm{d}V\,,$$

with $\iiint\limits_V \mathrm{d}V$ as the volume capacity of the control volume. For sufficiently large radius R_{sph} the volume of the jet can be neglected compared with the volume of the half sphere, thus

$$-\iiint\limits_V \varrho\,g\,\mathrm{d}V = -\varrho\,g\,\frac{2}{3}\,\pi\,R_{sph}^3\,.$$

To calculate the integral of the stress vector $\vec{t} = -p\,\vec{n}$ over the surfaces S_{sph} and A_B we need the pressure distributions. These are determined by applying Bernoulli's equation.
Pressure distribution on S_{sph}:
Bernoulli's equation applied to the streamline from $x = H$ to $r = R_{sph}$:

$$\varrho\,\frac{u_{sph}^2}{2} + \varrho\,g\,R_{sph}\cos\vartheta + p_{sph} = \varrho\,g\,H + p_0\,.$$

From the velocity field of the sink

$$u_{sph} = \frac{A}{R_{sph}^2}$$

follows the pressure

$$p_{sph}(R_{sph},\,\vartheta) = \varrho\,g\,H + p_0 - \frac{\varrho}{2}\,\frac{A^2}{R_{sph}^4} - \varrho\,g\,R_{sph}\cos\vartheta\,.$$

The pressure distribution at the bottom is calculated from Bernoulli's equation

$$\varrho\,\frac{u_B^2}{2} + p_B = \varrho\,g\,H + p_0\,.$$

gives

$$A = u_J \frac{d^2}{8} . \tag{3}$$

With equations (1), (2), and (3) as well as the definition for the contraction coefficient $\alpha = d^2/D^2$, we obtain a quadratic equation for the contraction coefficient α:

$$1 - 2\alpha + \frac{\alpha^2}{4} = 0 .$$

The solution is

$$\alpha_{1/2} = \pm\sqrt{12} + 4 ,$$

where the value

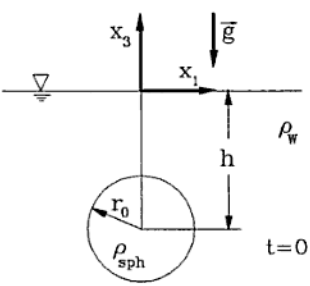
$$\alpha = -\sqrt{12} + 4 = 0.536 < 1$$

is the correct one. The diameter of the jet d is then

$$d = \sqrt{\alpha}\, D = 0.732\, D .$$

The result is exact only for $D \to 0$ and $H \to \infty$, since only in this case can the flow be modeled as a point sink flow.

Problem 10.3-10 Sphere rising in water



A sphere having the density ϱ_{sph} and the volume V_{sph} (sphere radius r_0) is suddenly released at time $t = 0$ from a depth of h under the surface of water (density ϱ_W).

a) Give the equation of motion of the sphere path $\vec{x}(t)$.
b) For the case $\varrho_{sph} < \varrho_W$, determine the time for the sphere to reach the water surface and find its kinetic energy.
c) Determine the change of results from b), if the acceleration resistance of the surrounding water is neglected.

Given: ϱ_{sph}, ϱ_W, h, V_{sph}, g

Solution

a) Path of the sphere:

The equation of motion for the sphere is

$$\vec{X} = (M + M')\frac{\mathrm{d}\vec{U}}{\mathrm{d}t},$$

where $\vec{X}$ is the external force, M the mass of the sphere, and M' the virtual mass (see F. M. (10.175)). For the mass of the sphere we write

$$M = \varrho_{sph} V_{sph} = \frac{4}{3}\pi r_0^3 \varrho_{sph}$$

and for the virtual mass $M' = \frac{2}{3}\pi r_0^3 \varrho_W$. The external force consists of the lift and the weight of the sphere.

$$\vec{X} = -V_{sph}(\varrho_{sph} - \varrho_W) g \vec{e}_3 .$$

Both forces are parallel to the gravitation vector $g\,\vec{e}_3$. The sphere acceleration itself is

$$\frac{\mathrm{d}\vec{U}}{\mathrm{d}t} = \frac{\mathrm{d}^2 x_3}{\mathrm{d}t^2}\vec{e}_3 ,$$

and the following equation is obtained:

$$-V_{sph}(\varrho_{sph} - \varrho_W) g = V_{sph}(\varrho_{sph} + \frac{1}{2}\varrho_W)\frac{\mathrm{d}^2 x_3}{\mathrm{d}t^2} .$$

It leads to

$$\frac{\mathrm{d}^2 x_3}{\mathrm{d}t^2} = \frac{\varrho_W - \varrho_{sph}}{\varrho_W/2 + \varrho_{sph}} g$$

and after integration results in

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{\varrho_W - \varrho_{sph}}{\varrho_W/2 + \varrho_{sph}} g t + C_1 .$$

The integration constant is determined from the initial condition

$$\left.\frac{\mathrm{d}x_3}{\mathrm{d}t}\right|_{t=0} = U(t=0) = 0 ,$$

and thus, is

$$C_1 = 0 .$$

Further integration and consideration of the initial condition $x_3(t=0) = -h$ results in

$$x_3(t) = \frac{\varrho_W - \varrho_{sph}}{\varrho_W/2 + \varrho_{sph}} \frac{1}{2} g t^2 - h .$$

follows, or

$$m_{11} = \varrho\pi r_0^2 L \, .$$

$$m_{22} = -\varrho L \int\limits_0^{2\pi} \left(-\frac{r_0^2}{r_0^2}x_2\right)\sin\beta \, r_0 \mathrm{d}\beta \, ,$$

with $x_2 = r_0 \sin\beta$:

$$m_{22} = \varrho L r_0^2 \int\limits_0^{2\pi} \sin^2\beta \, \mathrm{d}\beta = \varrho\pi r_0^2 L \, .$$

$$m_{21} = m_{12} = \varrho L r_0^2 \int\limits_0^{2\pi} \cos\beta \, \sin\beta \, \mathrm{d}\beta = 0 \, .$$

b) Required force:
The equation of motion is generally

$$X_i = (M\delta_{ij} + m_{ij})\frac{\mathrm{d}U_j}{\mathrm{d}t}$$

and here because of

$$m_{ij} = M'\delta_{ij} \, , \quad M' = \varrho\pi r_0^2 L$$

and

$$M\delta_{ij} = \varrho_C \pi r_0^2 L \delta_{ij} \qquad \text{(mass of the cylinder)}$$

$$X_i = \pi r_0^2 L(\varrho_C + \varrho)\delta_{ij}\frac{\mathrm{d}U_j}{\mathrm{d}t}$$

or

$$X_i = \pi r_0^2 L(\varrho_C + \varrho)\frac{\mathrm{d}U_i}{\mathrm{d}t} \, .$$

The direction of the force coincides with the direction of the acceleration. With

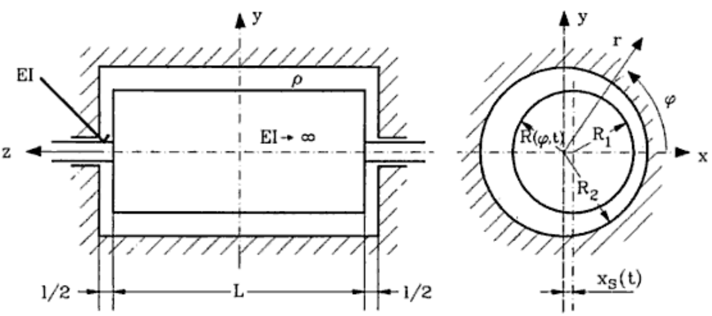
$$\frac{\mathrm{d}U_1}{\mathrm{d}t} = -\frac{U}{t_0} \, , \quad \frac{\mathrm{d}U_2}{\mathrm{d}t} = \frac{\mathrm{d}U_3}{\mathrm{d}t} = 0 \, ,$$

we obtain the required force as

$$\vec{X} = X_1\vec{e}_1 = -\pi r_0^2 L(\varrho_C + \varrho)\frac{U}{t_0}\vec{e}_1 \, ,$$

$$\vec{X} = -\pi r_0^2 L \varrho_C \left(1 + \frac{\varrho}{\varrho_C}\right)\frac{U}{t_0}\vec{e}_1 \, .$$

Problem 10.3-12 Rotor oscillating in an inviscid fluid



A cylindrical rotor (mass m, radius R_1, length L, resistance to bending $EI \to \infty$) is mounted symmetrically via two shafts with negligible mass (length $l/2$, resistance to bending EI) within a cylindrical casing (radius R_2). Inside the casing, there is incompressible inviscid fluid (density ϱ). Because of $L/R_2 \gg 1$ and neglecting corner flow, the problem can be considered in the plane $z = \text{const}$.

a) Find the potential flow in the gap, if the rotor executes small oscillations in x-direction.
b) Determine the kinetic energy of the fluid in the gap.
c) Determine the virtual mass M'.
d) Find the differential equation for the rotor motion.
e) Compare the eigenfrequency of the rotor with the frequency of the rotor without fluid in the casing.

Given: R_1, R_2, l, L, m, ϱ, EI, $\dot{x}_S$

Solution

a) Potential of the plane, incompressible flow:
The Laplace equation in cylindrical coordinates is

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0\ , \tag{1}$$

where $\partial/\partial z \equiv 0$ since we are dealing with plane flow. To solve the Laplace equation, we apply the method of separation of variables. This method can lead to a solution only if the boundary conditions are on coordinate surfaces. The kinematic boundary condition on the casing $(r = R_2)$

$$\vec{u}\cdot\vec{n} = -u_r = 0 \quad \text{at} \quad r = R_2\ , \tag{2}$$

Since the flow is periodic in φ-direction, i. e.

$$G(\varphi) = G(\varphi + 2\pi) ,$$

the constant k however can only be an integer. This is seen when the cos-term, for example, is written out:

$$\cos k\varphi = \cos k(\varphi + 2\pi) = \cos k\varphi \ \cos k2\pi - \sin k\varphi \ \sin k2\pi .$$

Looking at the sum of the terms on the right hand side, the left hand side can be verified only if the constant k assumes integer values $k = 1, 2, \ldots$. Because of this and the linearity of the Laplace equation (1), the sum of all individual solutions is also a solution of the Laplace equation:

$$\Phi = \sum_{k=0}^{\infty} \left(A_k r^k + B_k r^{-k}\right) (C_k \cos k\varphi + D_k \sin k\varphi) .$$

We impose the boundary condition (2) and (4); for u_r we get

$$u_r = \frac{\partial \Phi}{\partial r} = \sum_{k=0}^{\infty} k \left(A_k r^{k-1} - B_k r^{-k-1}\right) (C_k \ \cos k\varphi + D_k \ \sin k\varphi) , \quad (10)$$

and from (2) follows

$$u_r(R_2) = 0 = \sum_{k=0}^{\infty} k \left(A_k R_2^{k-1} - B_k R_2^{-(k+1)}\right) (C_k \cos k\varphi + D_k \sin k\varphi) .$$

From this equation we conclude that

$$B_k = A_k R_2^{2k} .$$

The boundary condition (4) now becomes

$$u_r(R_1) = \dot{x}_S \cos\varphi = \sum_{k=0}^{\infty} k \, A_k \left(R_1^{k-1} - R_2^{2k} R_1^{-k-1}\right) (C_k \ \cos k\varphi + D_k \ \sin k\varphi) .$$

From this equation we find $D_k \equiv 0$ and $k = 1$. Writing the product $A_1 \, C_1$ now $\overline{A_1}$, we have

$$\dot{x}_S \cos\varphi = \overline{A_1} \left(1 - R_2^2 R_1^{-2}\right) \cos\varphi ,$$

which leads to

$$\overline{A_1} = \frac{\dot{x}_S R_1^2}{R_1^2 - R_2^2} \quad \text{and} \quad B_1 = \frac{\dot{x}_S R_1^2 R_2^2}{R_1^2 - R_2^2} .$$

Finally, the solution satisfying the boundary conditions is:

$$\Phi = -\dot{x}_S(t) \frac{R_1^2}{R_2^2 - R_1^2} \left(r + \frac{R_2^2}{r}\right) \cos\varphi . \quad (11)$$

b) Kinetic energy of the fluid in the gap:
From the definition

$$K = \iiint\limits_{(V)} \frac{\varrho}{2} \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}V$$

we get with

$$\frac{\partial}{\partial x_i}\left(\Phi \frac{\partial \Phi}{\partial x_i}\right) = \frac{\partial \Phi}{\partial x_i}\frac{\partial \Phi}{\partial x_i} + \Phi \Delta \Phi \, ,$$

$$K = \iiint\limits_{(V)} \frac{\varrho}{2} \frac{\partial}{\partial x_i}\left(\Phi \frac{\partial \Phi}{\partial x_i}\right) \, \mathrm{d}V = \iint\limits_{(S)} \frac{\varrho}{2} \Phi \frac{\partial \Phi}{\partial x_i} n_i \, \mathrm{d}S \, .$$

The surface of the volume consists of the rotor surface where the condition $(\partial\Phi/\partial x_i)\, n_i = \partial\Phi/\partial n = -\partial\Phi/\partial r$, and the surface of the casing where the condition $\partial\Phi/\partial n = \partial\Phi/\partial r = 0$ is applied. Thus, we find for the kinetic energy

$$K = - \iint\limits_{\mathrm{S_{rotor}}} \frac{\varrho}{2} \Phi \frac{\partial \Phi}{\partial r} R_1 \, \mathrm{d}\varphi \mathrm{d}z \, .$$

We evaluate first $\Phi\, \partial\Phi/\partial r$, at $r = R_1$:

$$\begin{aligned} \left(\Phi \frac{\partial \Phi}{\partial r}\right)\Big|_{R_1} &= + \left(\dot{x}_S(t) \frac{R_1^2}{R_2^2 - R_1^2}\right)^2 \left(R_1 + \frac{R_2^2}{R_1}\right) \left(1 - \left(\frac{R_2}{R_1}\right)^2\right) \cos^2\varphi = \\ &= -\dot{x}_S^2(t) R_1 \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \cos^2 \varphi \, , \end{aligned}$$

and calculate the kinetic energy in the gap as

$$K = \frac{\varrho}{2}\, \dot{x}_S^2(t) R_1^2 \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} L \int\limits_{\varphi=0}^{2\pi} \cos^2\varphi \mathrm{d}\varphi = \frac{\varrho}{2}\, \dot{x}_S^2(t) \pi R_1^2 L \, \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \, .$$

c) The virtual mass M':
The kinetic energy of the tensor of the virtual mass is $K = \frac{1}{2} U_i U_j m_{ij}$ and since

$$U_l = \begin{cases} \dot{x}_S & \text{for } l = 1 \\ 0 & \text{else} \end{cases} \, ,$$

the comparison shows that the virtual mass M' is

$$M' = \frac{K}{\dot{x}_S^2/2} = \varrho \pi R_1^2 L \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}$$

and with the rotor volume $V_R = \pi R_1^2 L$, also

$$M' = \varrho V_R \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} .$$

For the cases $R_1 \to R_2$ and $R_2 \to \infty$ we have

$R_1 \to R_2 \quad \Rightarrow \quad M' \to \infty$ (however, the friction can then no longer be neglected!)

$R_2 \to \infty \quad \Rightarrow \quad M' \to \varrho V_R$ (virtual mass of a circular cylinder in infinite space) .

d) Differential equation for the rotor motion:
With two springs of length $a = l/2$ the stiffness c is given by

$$c = 2\frac{12EI}{(l/2)^3} = 192\frac{EI}{l^3}$$

so that the differential equation can be written as

$$(m + M')\ddot{x} + cx = F_x(t),$$

from which we read off

e) the eigenfrequency of the rotor as

$$\omega = \left(\frac{c}{m + M'}\right)^{1/2} = \left(\frac{c}{\varrho_R V_R \left(1 + (\varrho/\varrho_R)\,(R_2^2 + R_1^2)/(R_2^2 - R_1^2)\right)}\right)^{1/2} ,$$

where ($\varrho_R = m/V_R$). A rearrangement leads to

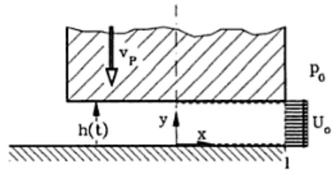
$$\omega = \omega_0 \left(\frac{R_2^2 - R_1^2}{R_2^2 - R_1^2 + (\varrho/\varrho_R)(R_2^2 + R_1^2)}\right)^{1/2} ,$$

where $\omega_0 = \sqrt{c/(\varrho_R V_R)}$ represents the eigenfrequency without considering the virtual mass.

10.4 Plane Potential Flow

Problem 10.4-1 Flow in the squeeze gap between a moving piston and a wall

A piston moves with the constant velocity v_P toward a wall as shown in the figure. The flow can be considered as a plane, incompressible flow. Calculate the velocity between the moving piston and the wall. For a high Reynolds number $Re = v_P\, h/\nu \to \infty$ we neglect friction effects and then assume that the problem can be treated as a potential flow problem.



Taking advantage of the symmetry with respect to the y-axis, we focus our attention to the right half of the flow domain. On the line $x = l$, the velocity component in x-direction is $u = U_o$. On the surfaces $x = 0$, $x = l$, $y = 0$, and $y = h$, constant velocities are prescribed as boundary condition. This suggests separation of variables in additive form, where the potential is assumed as $\Phi = f(x) + g(y)$.

a) Calculate the potential function Φ and the exit velocity U_o.
b) Find the pressure distribution $p(x, y, t)$, if the exit pressure is $p_o = p(x = l) = p_0$.
c) Determine the force on the piston.

Given: v_P, $h(t)$, l, ϱ

Solution

a) Potential function Φ, exit velocity U_o:
The Laplace equation is written as

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \qquad \text{(plane flow problem)}$$

and delivers with the above separation, i. e.

$$\begin{aligned} \Phi(x,y) &= f(x) + g(y) \\ \Delta\Phi &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0 \, . \end{aligned}$$

The first term depends on x only and the second one on y. The differential equation can be correct only if both terms are constant:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 g}{\partial y^2} = K \, .$$

These are two ordinary differential equations, which have the following solutions:

$$f(x) = \frac{K}{2}\,x^2 + C_1\,x + A \qquad \text{and} \qquad g(y) = -\frac{K}{2}\,y^2 + C_2\,y + B \, ,$$

thus

$$\Phi(x,y) = (f(x) + g(y)) = \frac{K}{2}\,(x^2 - y^2) + C_1\,x + C_2\,y + C_3 \, .$$

i. e. the pressure field does not depend on y . The Bernoulli constant follows from the known quantities at the exit

$$C = \left(\frac{v_P}{h}\, l\right)^2 + \frac{p_0}{\varrho}\,,$$

such that the above equation becomes

$$p(x,t) = p_0 + \varrho\left(\frac{v_P}{h}\right)^2 (l^2 - x^2)\,.$$

c) Force on the piston:

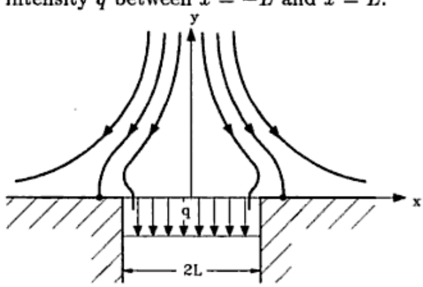
$$F_y = \iint_{(S)} -p\vec{n}\cdot\vec{e}_y\,\mathrm{d}S = 2\int_{x=0}^{l} p(x,t)\,\mathrm{d}x \qquad \text{(per unit of depth)}$$

$$= 2\int_{x=0}^{l}\left[p_0 + \varrho\left(\frac{v_P}{h}\right)^2 (l^2 - x^2)\right]\mathrm{d}x$$

$$\Rightarrow \qquad F_y = 2\,p_0\,l + \frac{4}{3}\,\varrho\left(\frac{v_P}{h}\right)^2 l^3\,.$$

Problem 10.4-2 Sink distribution in a stagnation point flow

The sketched flow is the result of the superposition of the potential of a plane stagnation point flow and the potential of a sink distribution of constant sink intensity q between $x = -L$ and $x = L$.



a) Find the potential of the sink distribution.
b) Give the total potential.

c) Determine the velocity components u and v. Hint: Differentiate first and then integrate using the substitution

$$t = (x - x')^2 + y^2 \quad \text{and} \quad t = \frac{x - x'}{y} .$$

d) Find the equation for the x-coordinate of the stagnation points at $(y = 0)$.
e) Describe the behavior of the velocity component $u(x, 0)$ as $x \to \pm L$.

Solution

a) Potential of the sink distribution:
The infinitesimal sink strength $\mathrm{d}m$ of an intensity distribution on the line element $\mathrm{d}x'$ is

$$\mathrm{d}m = q(x')\mathrm{d}x' .$$

Hence the contribution to the potential is

$$\mathrm{d}\Phi_{Si}(x, y) = \frac{q(x')\mathrm{d}x'}{2\pi} \ln \sqrt{(x - x')^2 + y^2}$$

and thus, the potential of the sink distribution:

$$\Phi_{Si}(x, y) = \frac{q}{2\pi} \int\limits_{-L}^{+L} \ln \sqrt{(x - x')^2 + y^2}\mathrm{d}x' , \quad q < 0 .$$

b) With the known potential of the stagnation point flow, the total potential becomes

$$\Phi(x, y) = \frac{a}{2}(x^2 - y^2) + \frac{q}{2\pi} \int\limits_{-L}^{+L} \ln \sqrt{(x - x')^2 + y^2}\mathrm{d}x' .$$

c) The velocity components

$$u = \frac{\partial \Phi}{\partial x} , \quad v = \frac{\partial \Phi}{\partial y} .$$

For $y \neq 0$, we may interchange the order of differentiation and integration and get

$$\frac{\partial \Phi}{\partial x} = ax + \frac{q}{2\pi} \int\limits_{-L}^{+L} \frac{x - x'}{(x - x')^2 + y^2}\mathrm{d}x' ,$$

with the substitution $t = (x - x')^2 + y^2$, $\mathrm{d}t = -2(x - x')\mathrm{d}x'$

$$\frac{\partial \Phi}{\partial x} = ax - \frac{q}{2\pi}\frac{1}{2}\int\limits_{(x+L)^2+y^2}^{(x-L)^2+y^2} \frac{\mathrm{d}t}{t}$$

and thus

$$\frac{\partial \Phi}{\partial x} = ax - \frac{q}{2\pi}\ln\sqrt{\frac{(x-L)^2+y^2}{(x+L)^2+y^2}} .$$

In the same way as above

$$\frac{\partial \Phi}{\partial y} = -ay + \frac{q}{2\pi}\int\limits_{-L}^{L} \frac{y}{(x-x')^2+y^2}\mathrm{d}x'$$

and with the substitution $t = (x - x')/y$, $\mathrm{d}t = -\mathrm{d}x'/y$,

$$\frac{\partial \Phi}{\partial y} = -ay - \frac{q}{2\pi}\int\limits_{(x+L)/y}^{(x-L)/y} \frac{\mathrm{d}t}{1+t^2}$$

and then

$$\frac{\partial \Phi}{\partial y} = -ay - \frac{q}{2\pi}\left[\arctan\frac{x-L}{y} - \arctan\frac{x+L}{y}\right] .$$

d) The stagnation points are located on the x-axis for $x^2 > L^2$, since there v disappears. We rewrite v in the form

$$v = -ay + \frac{q}{2\pi}\arctan\frac{2Ly}{x^2 - L^2 + y^2} , \quad \text{for} \quad \frac{x^2 - L^2}{y^2} > -1$$

and obtain for $y \to 0$ the limit

$$\lim_{y \to 0} v(x, y) = 0 , \quad \text{for} \quad x^2 - L^2 > 0 .$$

The x-coordinate is calculated from $u(x, 0) = 0$:

$$ax = \frac{q}{2\pi}\frac{1}{2}\ln\frac{(x-L)^2}{(x+L)^2} \quad \text{with } |x| > L .$$

One can show easily that with x also $-x$ is a solution.

e) $\lim\limits_{x \to \pm L} u(x, 0)$:

$$\lim_{x \to L} u(x,0) = \lim_{x \to L}\left\{ax - \frac{q}{2\pi}\frac{1}{2}\ln\frac{(x-L)^2}{(x+L)^2}\right\} \to -\infty \qquad q < 0!$$

$$\lim_{x \to -L} u(x,0) = \lim_{x \to -L}\left\{ax - \frac{q}{2\pi}\frac{1}{2}\ln\frac{(x-L)^2}{(x+L)^2}\right\} \to +\infty \qquad q < 0!$$

Problem 10.4-3 Circle theorem

In an incompressible, plane, frictionless, potential flow with the potential $F_1(z)$, a circular cylinder with the radius a is inserted at the origin. As a result, we find from the so called circle theorem the potential $F_2(z)$ of the new flow as

$$F_2(z) = F_1(z) + \overline{F_1}\left(\frac{a^2}{z}\right) ,$$

where $\overline{F_1}$ is the conjugate complex potential.

a) Calculate the complex potential of a circular cylinder (radius a) at $z = 0$ in a source flow (strength m, source at $z = b$).
b) Show here that the circle $z = ae^{i\varphi}$ is a streamline.
c) Sketch few streamlines.
d) Calculate the velocity potential. Where is the stagnation point located?
e) Calculate the force on the cylinder with Blasius' theorem.

Solution

Given is the complex potential $F_1(z)$ of an incompressible frictionless flow. Under the condition that all singularities of the function $F_1(z)$ are located in a distance larger than a from the origin, the circle theorem states that: If a circular cylinder with the radius a is introduced into the field of flow, then the potential of the modified flow is expressed by

$$F_2(z) = F_1(z) + \overline{F_1}\left(\frac{a^2}{z}\right) .$$

a) Calculation of the potential:
The potential of the source located at point $z = b$ is:

$$F_1(z) = \frac{m}{2\pi}\ln(z-b) \quad , \qquad (|b| > a) .$$

First we replace in F_1 z by a^2/z and i by $-$i and get $\overline{F_1}(a^2/z)$. The potential of the modified flow is

$$F_2(z) = \frac{m}{2\pi}\ln(z-b) + \frac{m}{2\pi}\ln\left(\frac{a^2}{z} - b\right) .$$

b) The circle $z = ae^{i\varphi}$ is streamline:
If the circle is a streamline, then for $z = ae^{i\varphi}$ and from

$$F_2(z) = \Phi + i\Psi$$

it must follow that $\Psi = \text{const}$:

$$F_2(ae^{i\varphi}) \quad = \quad \frac{m}{2\pi}\left[\ln(ae^{i\varphi} - b) + \ln(ae^{-i\varphi} - b)\right] ,$$

$$F_2(a\mathrm{e}^{\mathrm{i}\varphi}) = \frac{m}{2\pi}\ln\left(a^2+b^2-ab(\mathrm{e}^{\mathrm{i}\varphi}+\mathrm{e}^{-\mathrm{i}\varphi})\right) ,$$

$$F_2(a\mathrm{e}^{\mathrm{i}\varphi}) = \frac{m}{2\pi}\ln\left(a^2+b^2-2ab\cos\varphi\right) ,$$

which shows that

$$F_2(a\mathrm{e}^{\mathrm{i}\varphi}) \quad \text{is purely real}$$

and thus

$$\Psi = 0 ,$$

Q.E.D.

(This can be seen immediately: On the circle $z = a\mathrm{e}^{\mathrm{i}\varphi}$ and thus $a^2/z = \overline{z}$. Since the sum of two conjugate complex numbers, i. e. $F(z) + \overline{F}(\overline{z})$ is real, $\Psi = 0$ and the circle is streamline.)

c) Sketch a few streamlines:

It is appropriate to rearrange the potential:

$$F_2(z) = \frac{m}{2\pi}\left[\ln(z-b)+\ln\left(\frac{a^2}{z}-b\right)\right]$$

or

$$F_2(z) = \frac{m}{2\pi}\left[\ln(z-b)+\ln(a^2-bz)-\ln z\right]$$

and finally, we get

$$F_2(z) = \frac{m}{2\pi}\left[\ln(z-b)+\ln\left(z-\frac{a^2}{b}\right)+\ln(-b)-\ln z\right] .$$

Since $(m/2\pi)\ln(-b)$ is an additive constant, it can be set equal to zero, without loss of generality. It follows that

$$F_2(z) = \frac{m}{2\pi}\left[\ln(z-b)+\ln\left(z-\frac{a^2}{b}\right)-\ln z\right] ,$$

which shows that the potential consists of a

source at point $\quad z = b \quad$ with the strength $\quad m > 0$

and a

source at point $\quad z = \dfrac{a^2}{b} \quad$ with the strength $\quad m > 0$

as well as a

sink at point $\quad z = 0 \quad$ with the strength $\quad -m$.

The streamline graph is obtained by plotting the contour diagram of the imaginary part of $F_2(z)$. The following graph shows the streamlines for $b = -1$ and $a = 1/3$.

Location of the stagnation points :
$v(x,y)=0$ if $y=0$, i. e. the stagnation points are located on the x-axis. From $u(x,0)=0$ it follows:

$$0=\frac{1}{x-b}+\frac{1}{x-(a^2/b)}-\frac{1}{x}\ ,$$

which leads to a quadratic equation with the solutions

$$x=\pm a\ .$$

Thus, we have two stagnation points:

$$y_s=0 \quad ,x_s=\pm a\ .$$

e) Force on the cylinder using the Blasius theorem (see F. M. (10.260))

$$F_x-\mathrm{i}F_y=\mathrm{i}\frac{\varrho}{2}\oint_C\left[\frac{\mathrm{d}F_2}{\mathrm{d}z}\right]^2\mathrm{d}z\ ,$$

with C as the closed curve around the cylinder. We first evaluate the integrand

$$\begin{aligned}\left(\frac{\mathrm{d}F_2}{\mathrm{d}z}\right)^2 &= \left(\frac{m}{2\pi}\right)^2\left[\frac{2}{(z-b)\,(z-(a^2/b))}-\frac{2}{z(z-b)}-\frac{2}{z\,(z-(a^2/b))}+\right.\\ &\left.+\frac{1}{(z-b)^2}+\frac{1}{(z-(a^2/b))^2}+\frac{1}{z^2}\right]\end{aligned}$$

and then apply the residue theorem to solve the integral:

$$\int_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_k \operatorname*{Res}_{z_k} f(z)\ . \qquad (z_k \text{ inside of } C)$$

Rule: If the function $f(z)$ has at point z_0 a pole of order n, then its residue is given by

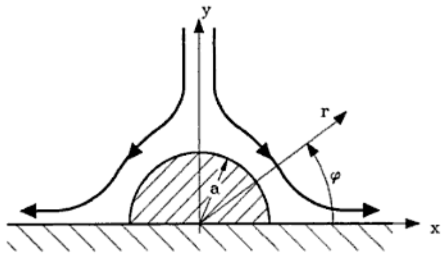
$$\operatorname*{Res}_{z_0} f(z)=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[(z-z_0)^n f(z)\right]$$

The first term of $(\mathrm{d}F_2/\mathrm{d}z)^2$

$$\frac{2}{(z-b)(z-(a^2/b))}$$

Problem 10.4-4 Half cylinder in stagnation point flow

Incompressible, inviscid fluid flows against a plane wall, where a half cylinder ($r = a$) is located.



a) Formulate the boundary conditions necessary to find the potential.
b) Solve the Laplace differential equation for the boundary conditions found above.
c) Find the stream function.

Solution

a) Boundary conditions:
   1) $\vec{u} \cdot \vec{n} = 0$ at the walls, i. e. $\nabla\Phi \cdot \vec{n} = \partial\Phi/\partial n = 0$.
   2) At infinity, the disturbance by the half cylinder must vanish, i. e. the prevailing potential there is a stagnation point flow

$$\Phi_S = \frac{b}{2}(x^2 - y^2) .$$

With

$$x = r\cos\varphi , \quad y = r\sin\varphi \quad \text{and} \quad x^2 - y^2 = r^2\cos 2\varphi$$

the potential has the asymptotic form

$$\Phi(r,\varphi) \sim \frac{b}{2}r^2\cos 2\varphi , \quad \text{for} \quad r \to \infty .$$

b) Solution of the Laplace equation:

$$\Phi = \Phi(r,\varphi) ,$$

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0 . \tag{1}$$

The method of separation $\Phi(r,\varphi) = R(r)H(\varphi)$ leads to

$$\Delta\Phi = R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0 ,$$

where the prime $'$ refers to the derivative with respect to the corresponding argument. Since r and φ are independent, it follows that

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{H''}{H} = k^2 = \text{const}\ ,$$

from which two ordinary differential equations result:

$$H''(\varphi) + k^2 H(\varphi) = 0 \quad \text{(differential equation of a linear oscillator)}\ ,$$

$$r^2 R''(r) + rR'(r) - k^2 R(r) = 0 \qquad \text{(Euler's differential equation)}\ .$$

The boundary condition on the cylinder ($r = a$, $0 < \varphi < \pi$) is

$$\left.\frac{\partial\Phi}{\partial n}\right|_{r=a} = \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = R'(a)H(\varphi) = 0\ ,$$

thus, for arbitrary φ

$$R'(a) = 0\ .$$

At the wall ($r > a$, $\varphi = 0$) the boundary condition is

$$\left.\frac{\partial\Phi}{\partial n}\right|_{\varphi=0} = \left.\frac{\partial\Phi}{\partial\varphi}\right|_{\varphi=0} = R(r)H'(0) = 0\ ,$$

and thus, for arbitrary r

$$H'(0) = 0\ ,$$

at the wall ($r > a$, $\varphi = \pi$) we have

$$\left.\frac{\partial\Phi}{\partial n}\right|_{\varphi=\pi} = -\left.\frac{\partial\Phi}{\partial\varphi}\right|_{\varphi=\pi} = -R(r)H'(\pi) = 0\ ,$$

thus for arbitrary r

$$H'(\pi) = 0$$

and at infinity, i. e. $r \to \infty$

$$\Phi(r,\varphi) = R(r)H(\varphi) \sim \frac{b}{2}r^2\cos 2\varphi\ .$$

From the general solution for the linear oscillator

$$H(\varphi) = A\sin k\varphi + B\cos k\varphi$$

we obtain with $H'(\varphi) = Ak\cos k\varphi - Bk\sin k\varphi$ and the boundary conditions $H'(0) = 0$, the special solution

$$H(\varphi) = B\,\cos(k\,\varphi)\ ,$$

since $A \equiv 0$. In order to avoid trivial solutions, the boundary condition $H'(\pi) = 0$, i. e.

$$B\,k \sin(k\,\pi) = 0\ ,$$

requires that the separation constant k assume the eigenvalues

$$k = 0,\ \pm 1, \pm 2, \pm 3 \ldots$$

Thus, the special solution is

$$H_k(\varphi) = B_k\ \cos k\varphi\ , \quad k = 1, 2, 3 \ldots$$

in which B_k is still unknown. The solution of the Euler differential equation reads

$$R_k(r) = C_k r^k + D_k r^{-k}$$

and the boundary condition

$$R_k'(a) = 0$$

gives, because of

$$R_k'(a) = C_k k a^{k-1} - D_k k a^{-k-1} = 0$$

$$D_k = C_k a^{2k}$$

the result

$$R_k(r) = C_k(r^k + a^{2k} r^{-k})\ .$$

Thus the general solution is written as

$$\Phi(r,\varphi) = \sum_{k=0}^{\infty} C_k(r^k + a^{2k} r^{-k})\, B_k\ \cos k\varphi\ .$$

The condition for $r \to \infty$

$$\Phi(r,\varphi) \sim \frac{b}{2} r^2 \cos 2\varphi$$

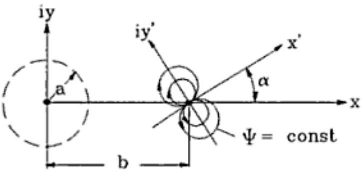
requires that

$$k = 2 \quad \text{and} \quad B_2\, C_2 = \frac{b}{2}\ ,$$

and therefore the solution reads

$$\Phi(r,\varphi) = a^2 \frac{b}{2} \cos 2\varphi \left[\left(\frac{r}{a}\right)^2 + \left(\frac{a}{r}\right)^2 \right]\ . \tag{2}$$

Problem 10.4-5 Dipol flow around a circular cylinder



Consider an incompressible, inviscid, plane potential flow. A dipole (dipole moment M) is located at $z = b$, which is oriented toward the x-axis at an angle α. If we insert in the origin of the coordinate system a circular cylinder with the radius a ($a < b$), then we find the potential of the new flow by using the circle theorem (compare Problem 10.4-3).

a) Find the complex potential $F_1(z)$ of the dipole at $z = b$. The angle between the dipole axis and the x-axis is α.
b) Calculate the complex potential of a circular cylinder ($r = a$) at $z = 0$ in a dipole flow (dipole moment M, dipole at $z = b$).
c) Find the stream function Ψ and its value on the circle ($r = a$).
d) Calculate the force on the cylinder.
 Explain the dependency of the force upon the orientation of the dipole.

Solution

a) Potential of the dipole:
 The complex potential of a dipole in the primed coordinate system (see figure and also F. M. (10.225)) is

$$F_1(z') = \frac{M}{2\pi}\frac{1}{z'}$$

with $z' = x' + \mathrm{i}\, y'$ and M as the magnitude of the dipole moment. Using the transformation

$$z' = (z - b)\,\mathrm{e}^{-\mathrm{i}\alpha}$$

we get the potential of the dipole in the unprimed system:

$$F_1(z) = \frac{M}{2\pi}\frac{1}{z-b}\mathrm{e}^{\mathrm{i}\alpha}\,. \tag{1}$$

b) Potential of a circular cylinder in a dipole flow:
 From equation (1) we obtain the complex conjugate potential $\overline{F_1}(z)$, by changing the sign of all "i " that explicitly occur in $F_1(z)$:

$$\overline{F_1}(z) = \frac{M}{2\pi}\frac{1}{z-b}\mathrm{e}^{-\mathrm{i}\alpha}\,.$$

The circle theorem yields

$$F_2(z) \;=\; F_1(z) + \overline{F_1}\left(\frac{a^2}{z}\right)$$

$$= \frac{M}{2\pi}\,\frac{1}{z-b}\,\mathrm{e}^{\mathrm{i}\,\alpha} + \frac{M}{2\pi}\,\frac{1}{a^2/z-b}\,\mathrm{e}^{-\mathrm{i}\,\alpha}$$

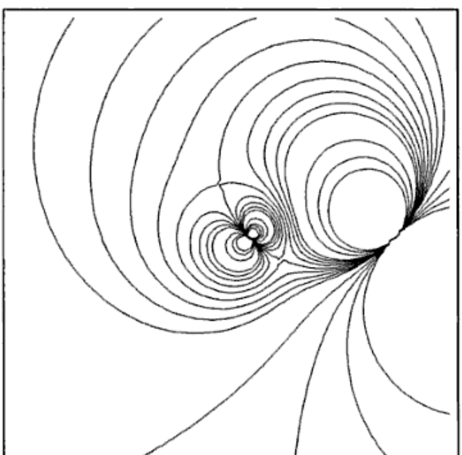
$$= \frac{M}{2\pi}\left(\frac{1}{z-b}\,\mathrm{e}^{\mathrm{i}\,\alpha} + \frac{z}{a^2-b\,z}\,\mathrm{e}^{-\mathrm{i}\,\alpha}\right) . \qquad (2)$$

c) Stream function Ψ:
We obtain the stream function Ψ as the imaginary part of the complex potential $F_2(z)$ by decomposing $F_2(z) = \Phi + \mathrm{i}\,\Psi$

$$\Psi(x,y) = \frac{M}{2\pi}\left[\frac{(x-b)\,\sin\alpha - y\,\cos\alpha}{(x-b)^2+y^2} + \right.$$

$$\left. + \frac{a^2\,y\,\cos\alpha - (a^2\,x - b\,x^2 - b\,y^2)\,\sin\alpha}{(a^2-b\,x)^2+(b\,y)^2}\right] . \qquad (3)$$

The streamlines are lines, along which the value of the stream function is constant. Introducing in (3) the contour of the circle $y^2 = a^2 - x^2$, we obtain the value of the stream function as $\Psi = \text{const} = 0$, which proves that the circle is a streamline. As an alternative, if we insert for the complex variable $z = a\,\mathrm{e}^{i\,\varphi}$ in (2), we realize that (2) is purely real. The circle separates two flow zones; for $|z| < a$ the stream function is always less than zero, whereas for $|z| > a$ it is always positive.



The streamlines are shown in the figure for $\alpha = \pi/4$ and $b/a = 3$.

d) Force on the cylinder:
The force on the cylinder is calculated using the first Blasius theorem (see F. M. (10.263))

$$F_x - \mathrm{i}\,F_y = \mathrm{i}\,\frac{\varrho}{2}\oint\limits_{(C)}\left(\frac{\mathrm{d}F_2}{\mathrm{d}z}\right)^2 \mathrm{d}z\ .$$

From (2) the integrand follows as

$$\left(\frac{\mathrm{d}F_2}{\mathrm{d}z}\right)^2 = \left(\frac{M}{2\pi}\right)^2\left[\left(\frac{a}{b}\right)^4 \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{(a^2/b-z)^4} + - 2\left(\frac{a}{b}\right)^2\frac{1}{(a^2/b-z)^2\,(z-b)^2} + \frac{\mathrm{e}^{2\mathrm{i}\alpha}}{(z-b)^4}\right]\ . \qquad (4)$$

The integration must be carried out along the contour of the circle. The integral is evaluated using the residue theorem

$$\oint\limits_{(C)} f(z)\,\mathrm{d}z = 2\pi\,\mathrm{i}\sum_k \operatorname*{Res}_{z_k} f(z) \quad (z_k \text{ inside of } (C))\ .$$

When evaluating the residues, the singular point inside the circle, here $z = a^2/b$, must be considered:

$$\oint\limits_{(C)}\left(\frac{\mathrm{d}F_2}{\mathrm{d}z}\right)^2 \mathrm{d}z = 2\pi\,\mathrm{i}\operatorname*{Res}_{z=a^2/b}\left(\frac{\mathrm{d}F_2}{\mathrm{d}z}\right)^2\ . \qquad (5)$$

Similar to the procedure in Problem 10.4-6 we obtain the individual terms of (4)

$$\operatorname*{Res}_{z=a^2/b}\frac{1}{(a^2/b-z)^4} = 0\ , \qquad \operatorname*{Res}_{z=a^2/b}\frac{1}{(a^2/b-z)^2\,(z-b)^2} = -\frac{2}{(a^2/b-b)^3}$$

$$\text{and}\quad \operatorname*{Res}_{z=a^2/b}\frac{1}{(z-b)^4} = 0\ ,$$

where the last term does not have any contribution to the integral, since the singularity $z = b$ is located outside the integration domain. Thus, we obtain the force

$$F_x - \mathrm{i}\,F_y = \varrho\,\frac{M^2}{\pi}\,\frac{a^2\,b}{(b^2-a^2)^3}$$

and

$$F_x = \varrho\,\frac{M^2}{\pi}\,\frac{a^2\,b}{(b^2-a^2)^3} \quad \text{and} \quad F_y = 0\ .$$

Independent from the orientation of the dipole, the force acts always in the positive x-direction.

The square of the magnitude of the velocity is

$$u^2 + v^2 = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\overline{F}}{\mathrm{d}\overline{z}} = \frac{a^2}{\sqrt{z\,\overline{z}}} = \frac{a^2}{r}\,,$$

using this and Bernoulli's equation, the pressure is calculated as

$$p(r) = p_\infty - \frac{\varrho}{2}\,(u^2 + v^2) = p_\infty - \frac{\varrho}{2}\,\frac{a^2}{r}\,. \tag{3}$$

c) Equation of the streamline $\Psi = \Psi_0$ and the range of φ values:
From (1), the equation of the streamline in polar coordinates is

$$r(\varphi) = \left(\frac{\Psi_0}{2a}\right)^2 \frac{1}{\sin^2(\varphi/2)}\,. \tag{4}$$

The limiting angle ϵ is the angle of the intersection between the straight line $x = L$ written in polar coordinates as

$$\frac{L}{r(\varphi)} = \cos\varphi$$

and the streamline (4). To determine ϵ, we solve the equation of the straight line for $r(\varphi)$ and insert the result into (4). We use the identity

$$\sin^2(\varphi/2) = \frac{1}{2}\,(1 - \cos\varphi)\,, \tag{5}$$

and obtain

$$\cos\epsilon = \left(1 + \frac{\Psi_0^2}{2a^2L}\right)^{-1}\,. \tag{6}$$

On $\Psi = \Psi_0$ the following relation holds

$$\epsilon \le \varphi \le 2\pi - \epsilon \qquad \text{for} \qquad x \le L\,.$$

d) Force on the contour:
Due to the symmetry, the force acts only in x-direction:

$$F_x = \iint\limits_{(S)} -p\,\vec{n}\cdot\vec{e}_x\,\mathrm{d}S$$

with $\vec{n}\cdot\vec{e}_x\,\mathrm{d}S = -\mathrm{d}y\,\mathrm{d}z$. For the force on the streamline $\Psi = \Psi_0$ per unit of depth, the above equation yields

$$F_x = \int p(r)\,\mathrm{d}y = \int p(r(\varphi))\,\mathrm{d}y\,,$$

which suggests the integration variable φ. From the equation of the streamline (4) it follows with (5)

$$r = \frac{y}{\sin\varphi} = \frac{\Psi_0^2}{2a^2}\,\frac{1}{1-\cos\varphi}$$

and with the identity

$$\frac{\sin\varphi}{1-\cos\varphi} = \cot(\varphi/2)$$

we obtain

$$y(\varphi) = \left(\frac{\Psi_0}{2a}\right)^2 2\,\cot(\varphi/2)\ ,$$

such that on the streamline under consideration

$$\mathrm{d}y = -\left(\frac{\Psi_0}{2a}\right)^2 \frac{1}{\sin^2(\varphi/2)}\,\mathrm{d}\varphi\ . \tag{7}$$

With (3), (4), and considering the negative sign in (7), the integration is

$$\begin{aligned}
F_x &= \int\limits_{\epsilon}^{2\pi-\epsilon} \left[p_\infty - \frac{\varrho}{2}\,\frac{a^2}{(\Psi_0/2a)^2}\,\sin^2(\varphi/2)\right]\left(\frac{\Psi_0}{2a}\right)^2 \frac{\mathrm{d}\varphi}{\sin^2(\varphi/2)} \\
&= \int\limits_{\epsilon}^{2\pi-\epsilon} \left[p_\infty \left(\frac{\Psi_0}{2a}\right)^2 \frac{1}{\sin^2(\varphi/2)} - \frac{\varrho}{2}\,a^2\right]\mathrm{d}\varphi \\
&= p_\infty \left(\frac{\Psi_0}{2a}\right)^2 \underbrace{\int\limits_{\epsilon}^{2\pi-\epsilon} \frac{1}{\sin^2(\varphi/2)}\,\mathrm{d}\varphi}_{-2\,\cot(\varphi/2)|_{\epsilon}^{2\pi-\epsilon} = 4\cot(\epsilon/2)} - \left[\frac{\varrho}{2}\,a^2\,\varphi\right]_{\epsilon}^{2\pi-\epsilon}
\end{aligned}$$

$$\Rightarrow\quad F_x = p_\infty\,\frac{\Psi_0^2}{a^2}\,\cot(\epsilon/2) - \varrho\,a^2\,(\pi-\epsilon)\ . \tag{8}$$

In this equation ϵ must be replaced by (6). With

$$\cot(\epsilon/2) = \frac{\sin\epsilon}{1-\cos\epsilon} = \frac{\sqrt{(1/\cos^2\epsilon)-1}}{(1/\cos\epsilon)-1}$$

it follows from equation (6)

$$\cot(\epsilon/2) = \sqrt{\frac{4a^2\,L}{\Psi_0^2} + 1}\ .$$

Thus, the equation describing the force on the streamline $\Psi = \Psi_0$ is:

$$F_x = p_\infty \frac{\Psi_0}{a}\sqrt{4L + \frac{\Psi_0^2}{a^2}} - \varrho\, a^2 \left[\pi - \overbrace{\arccos\left(1 + \frac{\Psi_0^2}{2a^2\,L}\right)^{-1}}^{=\epsilon}\right]\ . \qquad (9)$$

e) Suction force on the infinitely thin plate:
We obtain this force from (9) by taking the limit $\Psi_0 \to 0$:

$$F_x = -\pi\,\varrho\, a^2\ .$$

f) Suction force calculated by using the Blasius theorem:
The conjugate complex force on the body is

$$F_x - \mathrm{i}F_y = \mathrm{i}\frac{\varrho}{2}\oint\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2 \mathrm{d}z = \mathrm{i}\frac{\varrho}{2}\oint \frac{a^2}{z}\,\mathrm{d}z\ .$$

For any domain that contains $z = 0$, the residue theorem yields

$$\oint \frac{a^2}{z}\,\mathrm{d}z = 2\pi\mathrm{i}\ \underset{z=0}{\mathrm{Res}}\ f(z) = 2\pi\mathrm{i}a^2\ .$$

Thus, the total force is:

$$F_x - \mathrm{i}F_y = -\pi\,\varrho\, a^2\ .$$

The force acting on the body has the components

$$F_x = -\pi\,\varrho\, a^2 \qquad \text{and} \qquad F_y = 0\ ,$$

as expected.

Problem 10.4-7 Airfoil over a fixed wall

A slender airfoil with the circulation $-\Gamma_0$ is located at a distance a above a fixed wall. The airfoil is immersed in an incompressible, inviscid, potential flow with velocity U_∞ and pressure p_∞ at infinity.

a) Using the balance of momentum, show that the force acting on the wall is equal to the force on the airfoil.
b) Calculate the force on the wall by pressure integration along the x-axis.
c) Using the results from part b), show that the Kutta-Joukowski theorem is valid for $\Gamma_0/(U_\infty a) \ll 1$.

Given: ϱ, U_∞, p_∞, Γ_0, a

the vortex on the x-axis. The complex potential is then

$$F(z) = \underbrace{U_\infty z}_{\substack{\text{parallel}\\ \text{flow}}} + \underbrace{\frac{\mathrm{i}\Gamma_0}{2\pi}\ln(z - \mathrm{i}a)}_{\substack{\text{potential vortex at}\\ z=\mathrm{i}a \text{ with the cir-}\\ \text{culation } -\Gamma_0 \text{ clock-}\\ \text{wise}}} - \underbrace{\frac{\mathrm{i}\Gamma_0}{2\pi}\ln(z + \mathrm{i}a)}_{\substack{\text{potential vortex at}\\ z=-\mathrm{i}a \text{ with the}\\ \text{circulation } \Gamma_0 \text{ coun-}\\ \text{terclockwise}}}$$

From the conjugate complex velocity

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} = u - \mathrm{i}v &= U_\infty + \frac{\mathrm{i}\Gamma_0}{2\pi}\frac{1}{z - \mathrm{i}a} - \frac{\mathrm{i}\Gamma_0}{2\pi}\frac{1}{z + \mathrm{i}a}\,, \\ \overline{w} &= U_\infty + \frac{\mathrm{i}\Gamma_0}{2\pi}\frac{z + \mathrm{i}a - z + \mathrm{i}a}{z^2 + a^2}\,, \\ \overline{w} &= U_\infty - \frac{\Gamma_0}{\pi}\frac{a}{z^2 + a^2} \end{aligned}$$

we obtain with Bernoulli's equation the pressure distribution on the wall as

$$p(x, y = 0) + \frac{\varrho}{2} w\overline{w}(x, y = 0) = \frac{\varrho}{2}U_\infty^2 + p_\infty$$

or

$$p(x, y = 0) - p_\infty = \frac{\varrho}{2}U_\infty^2 - \frac{\varrho}{2}\left(U_\infty - \frac{\Gamma_0}{\pi}\frac{a}{x^2 + a^2}\right)^2$$

and

$$p(x, y = 0) - p_\infty = \varrho\, U_\infty \frac{\Gamma_0}{\pi}\frac{a}{x^2 + a^2} - \frac{\varrho}{2}\frac{\Gamma_0^2}{\pi^2}\frac{a^2}{(x^2 + a^2)^2}\,.$$

The force on the wall generated by the inviscid flow follows alone from the pressure integration

$$F_{y \to W} = \iint\limits_{S_W} -(p - p_\infty)(\vec{n} \cdot \vec{e}_y)\, \mathrm{d}S\,, \qquad \text{with} \quad \vec{n} = \vec{e}_y\,.$$

Per unit of depth, it becomes

$$F_{y \to W} = \int\limits_{-\infty}^{\infty} -\varrho U_\infty \frac{\Gamma_0}{\pi}\frac{a}{x^2 + a^2}\mathrm{d}x + \frac{\varrho}{2}\int\limits_{-\infty}^{\infty} \frac{\Gamma_0^2}{\pi^2}\frac{a^2}{(x^2 + a^2)^2}\mathrm{d}x\,.$$

The total potential of this flow is

$$\begin{aligned}F(z) &= U_\infty z + \frac{m}{2\pi}\,[\ln z + \ln(z+\mathrm{i}\,a) + \ln(z-\mathrm{i}\,a) + \\ &\qquad + \ln(z+\mathrm{i}\,2a) + \ln(z-\mathrm{i}\,2a) + \cdots] \\ &= U_\infty z + \frac{m}{2\pi}\left[\ln z + \ln(z^2+a^2) + \ln(z^2+4a^2) + \cdots\right] \\ &= U_\infty z + \frac{m}{2\pi}\left[\ln\left\{z\prod_{k=1}^{\infty}\left[1+\left(\frac{z}{k\,a}\right)^2\right]\right\} + \ln a^2 + \ln(4a^2) + \cdots\right] \qquad (1)\end{aligned}$$

The sum $\ln a^2 + \ln(4a^2) + \cdots$ is a constant and can be left out without loss of generality. Using the substitution $\zeta = (z/a)\,\pi$, the product is written as

$$z\prod_{k=1}^{\infty}\left[1+\left(\frac{z}{k\,a}\right)^2\right] = \frac{a}{\pi}\zeta\prod_{k=1}^{\infty}\left[1+\left(\frac{\zeta}{k\,\pi}\right)^2\right] = \frac{a}{\pi}\sinh\zeta = \frac{a}{\pi}\sinh\frac{\pi\,z}{a}\,.$$

Thus, we obtain form (1) the flow potential as

$$F(z) = U_\infty z + \frac{m}{2\pi}\ln\left[\sinh\frac{\pi\,z}{a}\right]\,, \qquad (2)$$

where we left out the additive constant $m/(2\pi)\,\ln(a/\pi)$.

b) Velocity at the walls:
The conjugate complex velocity is

$$\frac{\mathrm{d}F}{\mathrm{d}z} = u - \mathrm{i}\,v = U_\infty + \frac{m}{2\pi}\frac{\cosh(\pi\,z/a)}{\sinh(\pi\,z/a)}\frac{\pi}{a} = U_\infty + \frac{m}{2a}\coth\frac{\pi\,z}{a}\,. \qquad (3)$$

With the addition theorem

$$\coth(x \pm \mathrm{i}\,y) = \frac{\sinh 2x \mp \mathrm{i}\sin 2y}{\cosh 2x - \cos 2y}$$

we get from (3)

$$\frac{\mathrm{d}F}{\mathrm{d}z} = U_\infty + \frac{m}{2a}\,\frac{\sinh(2\pi\,x/a)}{\cosh(2\pi\,x/a)-\cos(2\pi\,y/a)} - \mathrm{i}\frac{m}{2a}\,\frac{\sinh(2\pi\,y/a)}{\cosh(2\pi\,x/a)-\cos(2\pi\,y/a)}$$

or

$$u = U_\infty + \frac{m}{2a}\,\frac{\sinh(2\pi\,x/a)}{\cosh(2\pi\,x/a)-\cos(2\pi\,y/a)}$$

and

$$v = \frac{m}{2a}\,\frac{\sinh(2\pi\,y/a)}{\cosh(2\pi\,x/a)-\cos(2\pi\,y/a)}\,. \qquad (4)$$

At the walls ($y = \pm a/2$) we find thereafter

$$u = U_\infty + \frac{m}{2a}\frac{\sinh(2\pi\, x/a)}{\cosh(2\pi\, x/a) + 1} \quad \text{and} \quad v = 0\,.$$

To calculate the oncoming flow velocity which adjusts itself according to (4), we take the limit

$$U_1 = \lim_{x \to -\infty}\left[U_\infty + \frac{m}{2a}\frac{\sinh(2\pi\, x/a)}{\cosh(2\pi\, x/a) - \cos(2\pi\, y/a)}\right] = U_\infty - \frac{m}{2a}\,.$$

c) Stagnation point:
For $v = 0$, we obtain from (4) $y_S = 0$.
From the condition $u(x_S,\, y_S) = 0$ we get

$$0 = U_\infty + \frac{m}{2a}\frac{\sinh(2\pi\, x_S/a)}{\cosh(2\pi\, x_S/a) - 1}$$

which leads with $\alpha = m/(2a\,U_\infty)$ and $\cosh^2 x = 1 + \sinh^2 x$ to

$$1 + \sinh^2(2\pi\, x_S/a) = (1 - \alpha\, \sinh(2\pi\, x_S/a))^2$$

or

$$\sinh(2\pi\, x_S/a)\, \Big[(1 - \alpha^2)\, \sinh(2\pi\, x_S/a) + 2\alpha\Big] = 0\,.$$

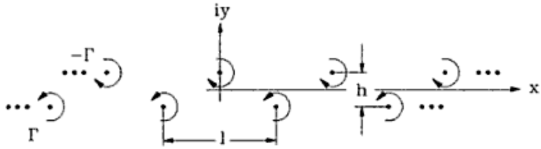
This results in

$$\sinh(2\pi\, x_S/a) = -\frac{2\alpha}{1 - \alpha^2}\,.$$

The oncoming flow velocity $U_1 = U_\infty\,(1 - \alpha)$ is positive only for $0 < \alpha < 1$. For x_S/a we find:

$$\frac{x_S}{a} = -\frac{1}{2\pi}\,\text{arsinh}\left(\frac{2\alpha}{1 - \alpha^2}\right)\,.$$

Problem 10.4-9 Kármán's vortex street



Flow at a sufficiently high Reynolds number around a plane, finite body causes alternatively at the upper and lower side of the body vortex generation and separation with the circulation strength of $-\Gamma$ and Γ, respectively. The vortices, which are idealized as infinite long straight vortex filaments, form two parallel vortex rows that are apart by the distance h. If the incoming flow is steady, then the distance between two vortices that pertain to each row is l. As a consequence of alternating separation, the two vortex rows are offset by the distance $l/2$. If we place the temporal and spatial location of the vortex generation at infinity, then the two infinitely long vortex rows will form an infinitely long "Kármán vortex street". The configuration is stable only for a certain ratio of h/l (= 0.281). In the following, we consider l and h as given.

a) Determine the complex potential of a vortex row, whose individual vortices have the circulation Γ and are located at $z_k = z_0$, $z_0 \pm l$, $z_0 \pm 2\,l \ldots$.
Hint: Use the following relation

$$\pi\,\zeta \prod_{k=1}^{\infty}\left[1-\left(\frac{\zeta}{k}\right)^2\right] = \sin(\pi\,\zeta)\,.$$

b) Find the velocity field induced by the vortex row.
Do the vortices of the row move?

c) Using the result from part a) determine the complex potential $F(z)$ of the Kármán vortex street. The upper vortices of circulation $-\Gamma$ are now located at $z_k = k\,l + \mathrm{i}h/2$ and the lower ones of circulation Γ are located at $z_k = 1/2\,l\,(2\,k+1) - \mathrm{i}\,h/2$ ($k = 0, \pm 1, \pm 2, \ldots$).

d) Show that both the vortex rows move in the negative x-direction with the velocity $\Gamma/(2l)\,\tanh(\pi\,h/l)$.
Hint:

$$\cot\pi(\xi + \mathrm{i}\,\eta) = \frac{\sin(2\pi\,\xi)}{\cosh(2\pi\,\eta) - \cos(2\pi\,\xi)} - \mathrm{i}\,\frac{\sinh(2\pi\,\eta)}{\cosh(2\pi\,\eta) - \cos(2\pi\,\xi)}$$

Given: Γ, h, l

Since the vortex row is infinitely long, each vortex must move with the same velocity. As a result, it is sufficient to investigate the vortex at $z = z_0$, which does not induce any velocity on itself. The first term in (4) does not contribute to the velocity at z_0 and all remaining terms cancel each other in pairs for $z = z_0$.

c) Potential of the Kármán vortex street:
Adding the potential of the upper vortex row, for which it holds

$$z_1 = \mathrm{i}\,h/2\,, \qquad \Gamma_1 = -\Gamma\,,$$

with the lower one

$$z_2 = l/2 - \mathrm{i}\,h/2\,, \qquad \Gamma_2 = \Gamma\,,$$

we find the total potential as

$$F(z) = \frac{\Gamma}{2\pi\mathrm{i}}\left\{\ln\left[\sin\frac{\pi}{l}\left(z - \frac{l}{2} + \mathrm{i}\,\frac{h}{2}\right)\right] - \ln\left[\sin\frac{\pi}{l}\left(z - \mathrm{i}\,\frac{h}{2}\right)\right]\right\}\,. \quad (5)$$

To investigate the motion of the vortex street, it is sufficient to investigate the motion of a single vortex. The vortices of the row, to which the vortex under investigation belongs, induce, as we saw above, on it no velocity. Consequently, the motion of the vortex is caused by the second vortex row. The induced velocity on the vortex located at $z = \mathrm{i}\,h/2$ of the upper row is

$$\begin{aligned}
(u_1 - \mathrm{i}\,v_1)|_{z=i\,h/2} &= \frac{\Gamma}{2\,l\,\mathrm{i}}\cot\frac{\pi}{l}\left(z - \frac{l}{2} + \mathrm{i}\,\frac{h}{2}\right)\bigg|_{z=i\,h/2} \\
&= \frac{\Gamma}{2\,l\,\mathrm{i}}\cot\frac{\pi}{l}\left(-\frac{l}{2} + \mathrm{i}\,h\right) \\
&= \frac{\Gamma}{2\,l\,\mathrm{i}}(-\mathrm{i})\tanh\frac{\pi\,h}{l} \\
&= -\frac{\Gamma}{2\,l}\tanh\frac{\pi\,h}{l}\,,
\end{aligned}$$

i. e.

$$u_1 = u_2 = -\frac{\Gamma}{2\,l}\tanh\frac{\pi\,h}{l} \qquad \text{and} \qquad v_1 = v_2 = 0\,,$$

using the circle theorem

$$F(z) = F_1(z) + \overline{F}_1\left(\frac{r_0^2}{z}\right)$$

and immediately obtain

$$F(z) = U_\infty e^{-i\alpha} z + U_\infty e^{+i\alpha}\frac{r_0^2}{z} \qquad \text{(see above)} .$$

b) Stagnation points and streamline plot:
From the stagnation point condition

$$\frac{dF}{dz} = u - iv = 0$$

it follows that

$$e^{2i\alpha} = \left(\frac{z}{r_0}\right)^2$$

or

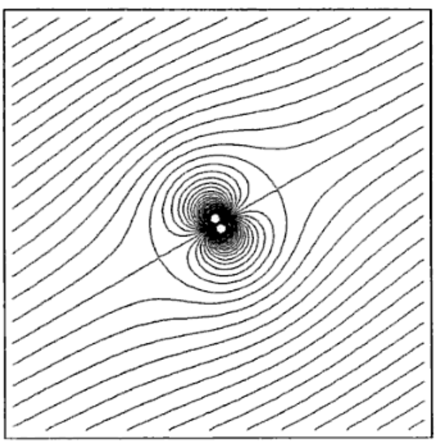
$$\frac{z}{r_0} = e^{(i/2)(2\alpha+k2\pi)} = e^{i(\alpha+k\pi)} ,$$

therefore

$$z_s = r_0 e^{i(\alpha+k\pi)} \quad ; \qquad k \in N .$$

Based on the following values for k, there are two separate stagnation points:

$$k = 0 \qquad \Rightarrow \qquad z_{s1} = r_0 e^{i\alpha} ,$$
$$k = 1 \qquad \Rightarrow \qquad z_{s2} = r_0 e^{i(\alpha+\pi)} .$$



c) Body obtained using the mapping $\zeta = (z + r_0^2/z)\,\mathrm{e}^{-\mathrm{i}\alpha}$:
The body contour in the z-plane is the circle

$$z_C = r_0 \mathrm{e}^{\mathrm{i}\varphi}$$

and in the ζ-plane

$$\begin{aligned}\zeta_C &= \left(r_0 \mathrm{e}^{\mathrm{i}\varphi} + \frac{r_0^2}{r_0}\mathrm{e}^{-\mathrm{i}\varphi}\right)\mathrm{e}^{-\mathrm{i}\alpha}\ ,\\ \zeta_C &= 2r_0\cos\varphi\,\mathrm{e}^{-\mathrm{i}\alpha}\ .\end{aligned}$$

In the ζ-plane, we thus get a plate of the length $4r_0$, through the origin under the angle $-\alpha$ with respect to the ξ-axis.

d) Stagnation points in the ζ-plane:
The stagnation points remain stagnation points, if $\mathrm{d}\zeta/\mathrm{d}z$ at z_s is non-singular. This is indeed the case:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \left(1 - \frac{r_0^2}{z^2}\right)\mathrm{e}^{-\mathrm{i}\alpha} \neq 0 \quad \text{for} \quad z_s = r_0 \mathrm{e}^{\mathrm{i}(\alpha + k\pi)}\ .$$

The stagnation points in the ζ-plane are found by determining the points ζ_s as the image of the points $z = z_s$

$$\begin{aligned}\zeta_s &= \left(r_0 \mathrm{e}^{\mathrm{i}(\alpha+k\pi)} + \frac{r_0^2}{r_0}\mathrm{e}^{-\mathrm{i}(\alpha+k\pi)}\right)\mathrm{e}^{-\mathrm{i}\alpha} =\\ &= 2r_o\cos(\alpha + k\pi)\mathrm{e}^{-\mathrm{i}\alpha}\ ,\end{aligned}$$

$$\begin{aligned}k = 0: &\qquad \zeta_{s1} = 2r_0\cos\alpha\,\mathrm{e}^{-\mathrm{i}\alpha}\ ,\\ k = 1: &\qquad \zeta_{s2} = 2r_0\cos(\alpha+\pi)\mathrm{e}^{-\mathrm{i}\alpha} = -2r_0\cos\alpha\,\mathrm{e}^{-\mathrm{i}\alpha}\ .\end{aligned}$$

e) Flow for $|\zeta| \to \infty$:
The conjugate complex velocity in the ζ-plane is

$$\begin{aligned}\overline{w}_\zeta &= \frac{\mathrm{d}F}{\mathrm{d}z}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^{-1}\\ &= U_\infty\left(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{r_0^2}{z^2}\mathrm{e}^{\mathrm{i}\alpha}\right)\left(1 - \frac{r_0^2}{z^2}\right)^{-1}\mathrm{e}^{\mathrm{i}\alpha}\\ &= U_\infty\frac{1 - (r_0/z)^2\mathrm{e}^{2\mathrm{i}\alpha}}{1 - (r_0/z)^2}\end{aligned}$$

and since for $\zeta \to \infty$, $z \to \infty$, it follows that

$$\overline{w}_\zeta(\zeta \to \infty) = U_\infty$$

$$\text{or} \qquad u_\zeta(\zeta \to \infty) = U_\infty \quad ; \qquad v_\zeta(\zeta \to \infty) = 0\ .$$

This means that at infinity, there is a parallel flow in the ζ-plane.

f) Pressure distribution along the body contour in the z-plane:
From Bernoulli's equation

$$\frac{w\overline{w}}{2} + \frac{p}{\varrho} = \frac{U_\infty^2}{2} + \frac{p_\infty}{\varrho}$$

and from the conjugate complex velocity on the circle contour $z = r_0 e^{i\varphi}$

$$\overline{w} = U_\infty e^{-i\alpha}\left(1 - e^{2i(\alpha-\varphi)}\right)$$

and the complex velocity

$$w = U_\infty e^{i\alpha}\left(1 - e^{-2i(\alpha-\varphi)}\right) ,$$

we first obtain

$$w\overline{w} = U_\infty^2\,(2 - 2\,\cos 2(\alpha - \varphi))$$

and then the pressure

$$\frac{p - p_\infty}{\varrho} = \frac{U_\infty^2}{2} - U_\infty^2\,(1 - \cos 2(\alpha - \varphi))$$

or the pressure coefficient

$$c_p = \frac{p - p_\infty}{\frac{\varrho}{2}U_\infty^2} = 2\,\cos 2(\alpha - \varphi) - 1 .$$

Along the body contour in the ζ-plane

$$\zeta_C = 2r_0\cos\varphi\, e^{-i\alpha}$$

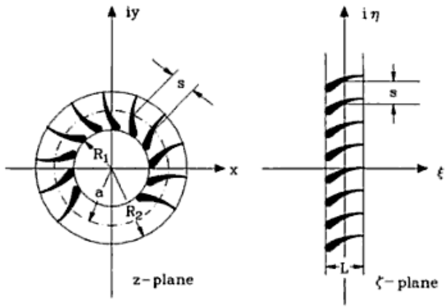
we obtain the pressure distribution by mapping a point of the circle $z_C = r_0 e^{i\varphi}$ onto ζ_C. At this point the velocity is $\overline{w}_\zeta = \overline{w}_z(d\zeta/dz)^{-1}$, from which the complex velocity w_ζ can be calculated. The pressure distribution is determined by using Bernoulli's equation.

Problem 10.4-11 Plane circular cascade

A plane circular cascade can be mapped onto a rectilinear cascade by the mapping

$$\zeta = K \ln\frac{z}{a} .$$

(K, a: positive and real, $R_1 < a < R_2$)



a) How do the images of circles ($r = \text{const}$) and straight lines through the origin ($\varphi = \text{const}$) of the z-plane look like in the ζ-plane?
b) Determine the constant a such that the cascade's leading edge and trailing edge in the ζ-plane have the same distance from the η-axis.
c) How should the constant K be chosen in order for the blade spacing s on the mean radius a of the z-plane to remain the same under mapping? Find the resulting mapping function.
d) Find the cascade length L in the ζ-plane.
e) The cascade flow in the z-plane is obtained by superimposing a source with a potential vortex ("vortex source"). Find the flow in the transformed plane.

Solution

a) Image of cylinders and straight lines through the origin:
The complex variable $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$ is transformed to the complex variable

$$\zeta = K \ln\left(\frac{z}{a}\right) = K \ln\left(\frac{r}{a}\mathrm{e}^{\mathrm{i}\varphi}\right) ,$$

$$\zeta = \xi + \mathrm{i}\eta = K \ln\frac{r}{a} + \mathrm{i}K\varphi$$

whose real part is

$$\xi = K \ln\frac{r}{a} \tag{1}$$

and the imaginary part

$$\eta = K\varphi . \tag{2}$$

Therefore, circles ($r = \text{const}$) are mapped onto straight lines $\xi = \text{const}$ and the lines through the origin ($\varphi = \text{const}$) are mapped onto the lines $\eta = \text{const}$.

b) Determine a:
From (1) we find:

$$\xi(r = R_1) = K \ln\frac{R_1}{a} = -K \ln\frac{a}{R_1} < 0 , \tag{3}$$

$$\xi(r = R_2) = K \ln \frac{R_2}{a} > 0\,. \tag{4}$$

Considering $-\xi(r = R_1) = \xi(r = R_2)$, we obtain:

$$K \ln \frac{a}{R_1} = K \ln \frac{R_2}{a}$$

$$\Rightarrow \quad \frac{a}{R_1} = \frac{R_2}{a} \quad \Rightarrow \quad a = \sqrt{R_1 R_2}\,. \tag{5}$$

The constant a corresponds to the geometric mean of both radii.

c) Determine K:
The spacing $s = (2\pi a)/N$ should remain the same under mapping. In the ζ-plane, the spacing is the difference of η-values between two adjacent blades. Using (2) for $\Delta\varphi = 2\pi/N$ (angle of spacing) therefore

$$s = \frac{2\pi a}{N} = K\frac{2\pi}{N} \quad \Rightarrow \quad K = a = \sqrt{R_1 R_2}\,. \tag{6}$$

With (5) and (6), the mapping is found as

$$\zeta = \sqrt{R_1 R_2} \ln\left(\frac{z}{\sqrt{R_1 R_2}}\right)\,. \tag{7}$$

d) Cascade length in the ζ-plane:
The cascade length L is calculated as

$$L = \xi(r = R_2) + |\xi(r = R_1)|\,,$$

thus with (3), (4)

$$L = K \ln \frac{R_2}{a} + K \ln \frac{a}{R_1} = K \ln\left(\frac{R_2}{R_1}\right)$$

and with K from (6) :

$$L = \sqrt{R_1 R_2} \ln\left(\frac{R_2}{R_1}\right)\,. \tag{8}$$

e) Approach flow in the ζ-plane:
The flow in the z-plane corresponds to the flow image of a vortex source, i. e. the superposition of the source flow and a potential vortex (see F. M. (10.217) and (10.221))

$$F(z) = \frac{m}{2\pi} \ln \frac{z}{a} - \mathrm{i}\frac{\Gamma}{2\pi} \ln \frac{z}{a}\,,$$

$$F(z) = \left(\frac{m}{2\pi} - \mathrm{i}\frac{\Gamma}{2\pi}\right) \ln \frac{z}{a} \,. \tag{9}$$

The inverse of the mapping (7) is

$$\frac{z}{a} = \mathrm{e}^{\zeta/a} \,,$$

such that the potential in the ζ-plane reads

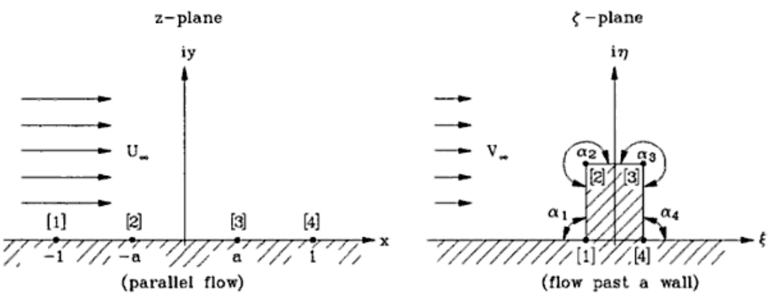
$$F(z(\zeta)) = F(\zeta) = \left(\frac{m}{2\pi} - \mathrm{i}\frac{\Gamma}{2\pi}\right) \frac{\zeta}{a} \,. \tag{10}$$

The vortex source in the z-plane corresponds a translational flow with

$$U_\infty = \frac{m}{2\pi a} \quad , \qquad V_\infty = \frac{\Gamma}{2\pi a}$$

in the transformed plane.

Problem 10.4-12 Schwarz-Christoffel transformation of a wall of infinite extent



The sketched wall of a finite width in the ζ-plane is mapped onto the z-plane using the Schwarz-Christoffel transformation.

a) Determine the Schwarz-Christoffel transformation for this problem.
b) Determine the image points, on which the conjugate complex velocity in the ζ-plane $\overline{w}_\zeta$ becomes infinite.
c) Give the relationship between U_∞ and V_∞.
d) Show that the velocity at the wall is tangential to the wall.
e) Find the pressure distribution on the wall surface.

Given: K, a, V_∞, ϱ, p_∞

d) Kinematic boundary condition:
To examine the kinematic boundary condition on the wall, we calculate $\overline{w}_\zeta$ for the straight lines [1] → [2], [2] → [3], and [3] → [4]. If we let ζ go from the corner [1] to the corner [2], then z goes along the x-axis from -1 to $-a$ and we obtain from (4)

$$\overline{w}_\zeta(\zeta) = -\mathrm{i}\, V_\infty \sqrt{\frac{1-x^2}{x^2-a^2}}\,,$$

i. e. only a component tangential to the wall. The complex velocity distributions at all lines representing the wall are summarized in the following table:

Line	Range of values for x	$\overline{w}_\zeta / V_\infty$
[1] → [2]	$-1 < x < -a$	$-\mathrm{i}\sqrt{(1-x^2)/(x^2-a^2)}$
[2] → [3]	$-a < x < a$	$\sqrt{(1-x^2)/(a^2-x^2)}$
[3] → [4]	$a < x < 1$	$\mathrm{i}\sqrt{(1-x^2)/(x^2-a^2)}$.

Thus, the kinematic boundary condition at the wall surface is fulfilled.

e) Pressure distribution:
We determine the pressure distribution on the wall surface by applying Bernoulli's equation:

$$p_\infty + \frac{\varrho}{2} V_\infty^2 = p + \frac{\varrho}{2} w_\zeta\, \overline{w}_\zeta\ .$$

The pressure coefficient is

$$c_p = \frac{p - p_\infty}{1/2\, \varrho\, V_\infty^2} = 1 - \frac{w_\zeta\, \overline{w}_\zeta}{V_\infty^2}\ .$$

Using the wall velocity calculated in part d), we obtain the pressure coefficients listed in the table.

Line	Range of values for x	c_p
[1] → [2]	$-1 < x < -a$	$1-(1-x^2)/(x^2-a^2)$
[2] → [3]	$-a < x < a$	$1-(1-x^2)/(a^2-x^2)$
[3] → [4]	$a < x < 1$	$1-(1-x^2)/(x^2-a^2)$.

the corresponding point, without loss of generality, in the origin of the z-plane, thus $x_1 = 0$. In the same way, one after the other, we have the angle at point [4]

$$\alpha_2 = \frac{3}{4}\pi$$

and place the corresponding point of the z-plane on

$$x_2 = a^4 \ .$$

We have the angle at point [5]

$$\alpha_3 = \frac{5}{4}\pi$$

and place the corresponding point of the z-plane on

$$x_3 = 1 \ .$$

We place point [6] in the z-plane at $+\infty$. The angle α_4 at point [6] does then not appear in the mapping. This can be realized, if, instead of the constant K, the constant $K(-x_4)^{-(\alpha_4/\pi)+1}$ is introduced. Then at point [6] with angle α_4 the factor $(-x_4)^{-(\alpha_4/\pi)+1}\,(z-x_4)^{(\alpha_4/\pi)-1}$ is generated, which for $x_4 \to \infty$ approaches unity. Thus, the Schwarz-Christoffel transformation assumes the form

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = K\,(z-0)^{0-1}\,(z-a^4)^{(3/4)-1}\,(z-1)^{(5/4)-1}$$

or

$$f' = \frac{K}{z}\left(\frac{z-1}{z-a^4}\right)^{1/4} \ .$$

b) Potential in the z-plane:
In the ζ-plane a channel flow should be generated which has the velocity U at the points [2] and [3], i. e. the volume flux is $\dot{V} = UH$. Now, we seek a potential in the z-plane such that between the points [2] and [3], which will be mapped onto $x_1 = 0$ (of the z-plane), a volume flux $\dot{V} = UH$ is generated. This can be achieved by placing a source in the origin of the z-plane:

$$F(z) = \frac{m}{2\pi}\ln z \ ,$$

with a source strength of

$$m = 2UH \ .$$

The factor 2 in the above equation indicates that in the z-plane half of the strength flows in the negative y-direction which is compensated by the factor 2. Thus, the potential is

$$F(z) = \frac{UH}{\pi}\ln z \ ,$$

from which

c) we find the velocity in the z-plane as

$$\overline{w}_z(z) = \frac{\mathrm{d}F}{\mathrm{d}z} = \frac{UH}{\pi z} = u - \mathrm{i}v \; .$$

The velocity at corresponding points of the ζ-plane is then (see F. M. (10.290))

$$\overline{w}_\zeta(\xi) = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} \; .$$

d) Determine K:
K is to be determined in such a way that the velocity at point [6] of the ζ-plane is equal to the velocity at point [6] of the z-plane:

$$\overline{w}_\zeta(\zeta) = \frac{UH}{h} \; ,$$

which follows immediately from the continuity equation. With

$$\overline{w}_\zeta(\zeta) = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{UH}{\pi z}\frac{z}{K}\left(\frac{z-a^4}{z-1}\right)^{1/4}$$

follows therefore

$$\lim_{z\to\infty} \overline{w}_\zeta(\zeta) = \frac{UH}{\pi K} = \frac{UH}{h}$$

and thus

$$K = \frac{h}{\pi} \; .$$

e) Determination of the coordinate a^4:
The condition that at point [2] of the ζ-plane we have the velocity U, furnishes

$$\overline{w}_\zeta(\zeta) = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{UH}{\pi z}\frac{z}{K}\left(\frac{z-a^4}{z-1}\right)^{1/4} = U$$

and since point [2] ($\zeta \to -\infty$) of the ζ-plane corresponds to point [2] ($z \to 0$) of the z-plane we find

$$\overline{w}_\zeta(\zeta \to -\infty) = \frac{UH}{\pi K}\frac{a}{1} = U \; ,$$

from which follows

$$a = \frac{K\pi}{H} = \frac{h}{H} \; .$$

f) Mapping function:
To obtain the mapping function $\zeta = \zeta(z)$ we integrate

$$\mathrm{d}\zeta = \frac{K}{z}\left(\frac{z-1}{z-a^4}\right)^{1/4}\mathrm{d}z$$

using the substitution $t^4 = (z - a^4)/(z - 1)$ or $z = (t^4 - a^4)/(t^4 - 1)$ and first obtain

$$\mathrm{d}\zeta = \frac{K}{z}\left(\frac{z-1}{z-a^4}\right)^{1/4}\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 4z\left(\frac{t^3}{t^4-a^4} - \frac{t^3}{t^4-1}\right)$$

$$\mathrm{d}\zeta = 2K\left(\frac{(1/2)a}{t-a} - \frac{(1/2)a}{t+a} - \frac{1/2}{t-1} + \frac{1/2}{t+1} + \frac{1}{t^2+a^2} - \frac{1}{t^2+1}\right)\mathrm{d}t$$

and therefore finally

$$\zeta = K\left[-\frac{1}{a}\ln\left(\frac{t+a}{t-a}\right) + \ln\left(\frac{t+1}{t-1}\right) + \frac{2}{a}\arctan\left(\frac{t}{a}\right) - 2\arctan(t)\right] + \text{const.}$$

The constant of integration is fixed by the requirement that point

$$z = a^4 \quad \text{or} \quad t = 0$$

is mapped onto the origin of the ζ-plane.

$$\Rightarrow \quad 0 = K\left(-\frac{1}{a}\ln(-1) + \ln(-1)\right) + \text{const}$$

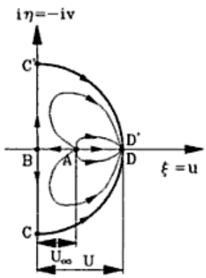
since $\ln(-1) = \mathrm{i}\pi$ also

$$\text{const} = \mathrm{i}\pi K\left(\frac{1}{a} - 1\right) = \mathrm{i}(H-h)$$

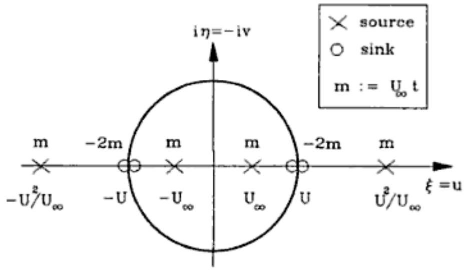
and therefore the mapping

$$\zeta = K\left[-\frac{1}{a}\ln\left(\frac{t+a}{t-a}\right) + \ln\left(\frac{t+1}{t-1}\right) + \frac{2}{a}\arctan\left(\frac{t}{a}\right) - 2\arctan(t)\right] + \mathrm{i}(H-h)\,.$$

b) Streamlines in the hodograph plane $\zeta = u - iv$:



streamlines in the hodograph plane



singularity distribution

c) Singularity distribution in the ζ-plane:
We place a source of strength $U_\infty t$ in point A ($\zeta = U_\infty$). The volume flux originating from this source is absorbed by a sink of strength $-U_\infty t$ located at point D ($\zeta = U$).
Since the line CC' is a streamline, the singularity distribution must be symmetric to the line $\xi = 0$; we therefore introduce the mirror images of the source and sink obtained so far, where $\xi = 0$ is the reflecting surface. The circle $|\zeta| = U$ is a streamline. To achieve this, we apply to the potential

$$F_1(\zeta) = \frac{U_\infty t}{2\pi}\ln(\zeta - U_\infty) + \frac{U_\infty t}{2\pi}\ln(\zeta + U_\infty) - \frac{U_\infty t}{2\pi}\ln(\zeta - U) - \frac{U_\infty t}{2\pi}\ln(\zeta + U)$$

the circle theorem

$$F(\zeta) = F_1(\zeta) + \overline{F}_1\left(\frac{U^2}{\zeta}\right) .$$

The resulting potential $F(\zeta)$ is

$$\begin{aligned} F(\zeta) \;=\; \frac{U_\infty t}{2\pi} \{ & \ln(\zeta + U^2/U_\infty) + \ln(\zeta - U^2/U_\infty) - 2\ln(\zeta + U) + \\ & -2\ln(\zeta - U) + \ln(\zeta + U_\infty) + \ln(\zeta - U_\infty)\} \, . \end{aligned}$$

d) Relationship $f(b/t, U_\infty/U) = 0$:
From $\mathrm{d}F/\mathrm{d}z = \zeta$ it follows

$$z = \int \frac{\mathrm{d}F}{\mathrm{d}\zeta}\frac{1}{\zeta}\mathrm{d}\zeta + C . \quad (1)$$

e) For $b/t = 0.68$ we find the velocity ratio as $U_\infty/U = 0.2$.
From the continuity equation $U_\infty t = U(t-h)$ it follows that
$h = t(1 - U_\infty/U) = 0.8t$.

Problem 10.4-15 Representation of a slender body by a source distribution

A slender, symmetric, two-dimensional profile is described by the following equation:

$$y = \pm f(x) = \pm 2\epsilon \frac{x}{l}(l-x)$$

($0 \le x \le l$, $\epsilon \ll 1$). The profile is positioned in an incompressible, inviscid potential flow with approach velocity U_∞ parallel to the x-axis.

a) Determine the source-sink distribution using the slender body theory for incompressible flow.
b) Calculate the perturbation velocity on the profile.
c) Calculate the pressure coefficient c_p.
d) For the source distribution in part a) determine the exact contour equation of the wing section.

Solution

a) Source-sink distribution:
The source-sink distribution for the profile shape is given by (see F. M. (10.358))

$$q(x) = 2\frac{\mathrm{d}f}{\mathrm{d}x}U_\infty \ .$$

With $f(x) = 2\epsilon(l-x)\,x/l$

$$q(x) = 4U_\infty\epsilon\,\frac{1}{l}(l-2x) \ .$$

b) Perturbation velocities on the profile:
The y-component of the perturbation velocity on the upper and lower profile sides is

$$v(x,0^+) = \frac{\mathrm{d}f}{\mathrm{d}x}U_\infty \ , \quad v(x,0^-) = -\frac{\mathrm{d}f}{\mathrm{d}x}U_\infty \ ,$$

thus

$$v(x,0) = \pm 2U_\infty \epsilon \frac{1}{l}(l-2x) = \pm \frac{q(x)}{2}\,.$$

We calculate the x-component ($u = U - U_\infty$) of the perturbation velocity from the potential (see F. M. (10.342))

$$\Phi = U_\infty x + \frac{1}{2\pi} \int_0^l q(x') \ln \sqrt{(x-x')^2 + y^2}\, \mathrm{d}x'$$

as

$$\frac{u(x,y)}{U_\infty} = \frac{1}{2\pi} \int_0^l \frac{q(x')}{U_\infty} \frac{x-x'}{(x-x')^2 + y^2} \mathrm{d}x'\,.$$

To find $u(x, y = f(x))/U_\infty$ we use the Taylor expansion of u around $y = 0$, i. e.

$$\frac{u(x,y)}{U_\infty} = \frac{u(x,0)}{U_\infty} + \left.\frac{\partial u}{\partial y}\right|_{y=0} \frac{y}{U_\infty} + \cdots$$

and estimate the order of magnitude of the second term as

$$\frac{\partial u}{\partial y}\frac{y}{U_\infty} = \frac{\partial v}{\partial x}\frac{y}{U_\infty} \sim \frac{v}{U_\infty}\frac{d}{l} \sim \epsilon^2\,.$$

This means we can determine u on the x-axis instead of on the profile contour. The estimated error is of the order $O(\epsilon^2)$ and may be neglected. With $q(x')$ from part a) we find

$$u(x,0) = \frac{2U_\infty \epsilon}{l\,\pi} \int_0^l \frac{l-2x'}{x-x'} \mathrm{d}x'\,.$$

This integral becomes singular for $x = x'$ therefore, we take Cauchy's principal value:

$$u(x,0) = \frac{2U_\infty \epsilon}{l\,\pi} \left(\underbrace{\int_0^l \frac{l}{x-x'} \mathrm{d}x'}_{\mathrm{I}} - \underbrace{2 \int_0^l \frac{x'}{x-x'} \mathrm{d}x'}_{\mathrm{II}} \right),$$

and with $\kappa > 0$ we find for the first integral

$$\mathrm{I} \;=\; l \lim_{\kappa \to 0} \left\{ \int_0^{x-\kappa} \frac{\mathrm{d}x'}{x-x'} + \int_{x+\kappa}^{l} \frac{\mathrm{d}x'}{x-x'} \right\}$$

For $\epsilon \ll 1$ the terms $\epsilon^2 Y^2$, $\epsilon^2 Y$ may be neglected and we obtain

$$0 = \epsilon Y + \frac{\epsilon}{\pi} 2x(x-1)\pi ,$$

thus

$$y = 2\epsilon x(1-x) ,$$

which in dimensional form gives the original profile shape:

$$\frac{y}{l} = 2\,\epsilon\,\frac{x}{l}\left(1 - \frac{x}{l}\right) .$$

The velocity follows from (1) as

$$\frac{\mathrm{d}F}{\mathrm{d}z} = U_\infty + u - \mathrm{i}\,v ,$$

$$\frac{u}{U_\infty l} = \frac{4\epsilon}{\pi} + \frac{\epsilon}{\pi}(1-2x)\ln\!\left(\frac{x^2+y^2}{(x-1)^2+y^2}\right) + \frac{4\epsilon}{\pi} y \left(\arctan\!\left(\frac{-y}{x(x-1)+y^2}\right) - \pi\right),$$

$$\frac{v}{U_\infty l} = -\frac{2\epsilon}{\pi}(1-2x)\left(\arctan\!\left(\frac{-y}{x(x-1)+y^2}\right) - \pi\right) + \frac{2\epsilon}{\pi} y \ln\!\left(\frac{x^2+y^2}{(x-1)^2+y^2}\right),$$

It should be pointed out that x, y are still dimensionless quantities. On the axis ($y = 0$) we obtain the results from part b).

Problem 10.4-16 Distribution of vortex intensity and mean camber line of a slender airfoil

The camber line $y = f(x)$ of a slender, slightly cambered profile, can be found by prescribing the vortex intensity $\gamma(x)$. This is called the inverse method. If, however, the camber line geometry is given, the vortex distribution can be obtained. This method is called the direct method.

a) Determine the equation of camber line for the given vortex distribution

$$\gamma(x) = 2U_\infty C = \text{const} .$$

b) For the camber line $y = f(x) = \epsilon\, x(1 - x/l)$ calculate the vortex distribution.

c) Calculate for part b) the lift coefficient c_L and the moment coefficient c_M.

Solution

a) I. Inverse method:

Given is the vortex distribution $\gamma = 2U_\infty C = \text{const}$ and the desired camber line $f(x)$ satisfies the equation (see F. M. (10.372))

$$U_\infty \frac{\mathrm{d}f}{\mathrm{d}x} - \alpha U_\infty = -\frac{1}{2\pi}\int\limits_0^l \gamma(x')\frac{1}{x-x'}\mathrm{d}x'$$

or

$$U_\infty \frac{\mathrm{d}f}{\mathrm{d}x} - \alpha U_\infty = -\frac{CU_\infty}{\pi}\int\limits_0^l \frac{1}{x-x'}\mathrm{d}x' \,.$$

In dimensionless form $f = \overline{f}l$, $x = \overline{x}l$, where in the following the overbars are omitted, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \alpha - \frac{C}{\pi}\int\limits_0^1 \frac{\mathrm{d}x'}{x-x'} \,.$$

The resulting integral is an improper integral, whose Cauchy's principal value

$$\mathrm{I} = \int\limits_0^1 \frac{\mathrm{d}x'}{x-x'} = \lim_{\epsilon\to 0}\left[\int\limits_0^{x-\epsilon} \frac{\mathrm{d}x'}{x-x'} + \int\limits_{x+\epsilon}^1 \frac{\mathrm{d}x'}{x-x'}\right]$$

is determined as

$$\begin{aligned}
\mathrm{I} &= \lim_{\epsilon\to 0}\left[-\left(\ln(x-x')\right)\Big|_0^{x-\epsilon} - \left(\ln(x-x')\right)\Big|_{x+\epsilon}^1\right] \,, \\
\mathrm{I} &= \lim_{\epsilon\to 0}\left[-\ln(\epsilon) + \ln(x) - \ln(1-x) + \ln(\epsilon)\right] \,, \\
\mathrm{I} &= \ln\left(\frac{x}{1-x}\right) \,.
\end{aligned}$$

From

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \alpha + \frac{C}{\pi}\left(\ln\frac{1-x}{x}\right)$$

the camber line follows by integration:

$$f = \alpha x + \frac{C}{\pi}\left[(x-1)\ln(1-x) - x\ln x\right] + K \,.$$

Here we have the relationship $x/l = \overline{x} = (1/2)(1 + \cos\varphi)$, $f/l = \overline{f}$ and $\gamma/U_\infty = \overline{\gamma}$, the bars are again omitted. To calculate the coefficients for the given camber line we first compute

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \epsilon(1 - 2x) = \epsilon(1 - 1 - \cos\varphi) = -\epsilon\cos\varphi$$

and find for the coefficients

$$A_0 = \alpha + \frac{\epsilon}{\pi}\int\limits_0^\pi \cos\varphi\,\mathrm{d}\varphi = \alpha$$

and

$$A_1 = \frac{2\epsilon}{\pi}\int\limits_0^\pi \cos^2\varphi\,\mathrm{d}\varphi = \frac{2\epsilon}{\pi}\frac{\pi}{2} = \epsilon\,.$$

Since we have the derivative $\mathrm{d}f/\mathrm{d}x = -\epsilon\cos\varphi$, for higher coefficients we deal with the integrals of the form

$$\int\limits_0^\pi \cos\varphi\,\cos(n\varphi)\,\mathrm{d}\varphi\,.$$

These integrals disappear however for $n > 1$, and the vortex distribution is given by the first two terms of the sum:

$$\gamma(\varphi) = 2\alpha\tan\frac{\varphi}{2} + 2\epsilon\sin\varphi\,.$$

With

$$\tan\frac{\varphi}{2} = \frac{\sin\varphi}{1 + \cos\varphi} = \frac{\sin\varphi}{2x}$$

and

$$\sin\varphi = \sqrt{1 - \cos^2\varphi} = \sqrt{1 - (2x - 1)^2} = 2\sqrt{x(1 - x)}$$

we find the vortex distribution

$$\gamma(x) = 4\sqrt{x(1 - x)}\left(\frac{\alpha}{2x} + \epsilon\right)$$

as a function of x.

c) Lift and moment coefficient

$$c_L = \pi(2A_0 + A_1) = \pi(2\alpha + \epsilon)$$

$$c_M = -\frac{\pi}{4}(2A_0 + 2A_1 + \underbrace{A_2}_{=0}) = -\frac{\pi}{4}(2\alpha + 2\epsilon)$$

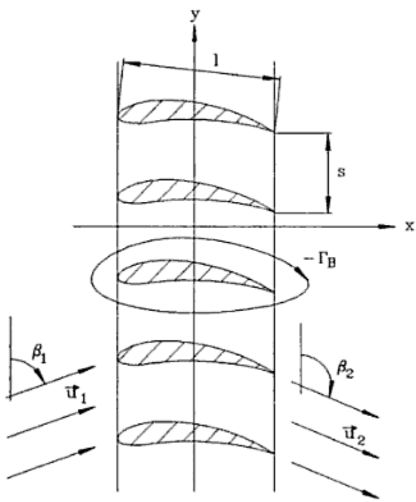
Note:
The perturbation velocities for part b) are:

$$u(x,0^{\pm}) = \pm\frac{1}{2}\gamma(x) = \pm 2\sqrt{x(1-x)}\left(\frac{\alpha}{2x}+\epsilon\right) ,$$

$$v(x,0) = \frac{df}{dx} - \alpha = \epsilon(1-2x) - \alpha ,$$

where $v(x,0)$ is determined by the boundary condition (see F. M. (10.371)) and γ, u, and v are nondimensionalized with U_∞.

Problem 10.4-17 Straight cascade



A plane flow through an infinite plane cascade ($-\infty \le y \le \infty$), is obtained for example by unwrapping a cylinder cut of an annular blade cascade.The flow in a sufficiently large distance from the cascade can be represented by a bound vortex sheet along the y-axis from $-\infty$ to ∞. This procedure is used to determine the inlet and exit velocity $\vec{u}_1$, $\vec{u}_2$ as well as the inlet and exit flow angle β_1, β_2 of a plane cascade with a small blade spacing s ($s/l \ll 1$; l is the blade chord). The undisturbed flow velocity at infinity ($x \to -\infty$), i. e. without the cascade is $\vec{W}_\infty = U_\infty\,\vec{e}_x + V_\infty\,\vec{e}_y$.
The velocities U_∞, V_∞, the circulation of a blade Γ_B, and the spacing s are known.

a) Calculate the velocities $\vec{u}_1 = u_1\,\vec{e}_x + v_1\,\vec{e}_y$, $\vec{u}_2 = u_2\,\vec{e}_x + v_2\,\vec{e}_y$ and the corresponding angles β_1, β_2.
b) For an incompressible, inviscid potential flow, determine the force $\vec{F}$ acting on the blade. Show that $\vec{F}$ is perpendicular to $\vec{W}_\infty$.

$$= \frac{\varrho}{2}\left(U_\infty^2 + \left(V_\infty + \frac{\gamma}{2}\right)^2 - U_\infty^2 - \left(V_\infty - \frac{\gamma}{2}\right)^2\right)$$

$$= \varrho\,\gamma\,V_\infty\,,$$

the mass flux in (7) by $\dot{m} = s\,\varrho\,U_\infty$, and write the force in vector form

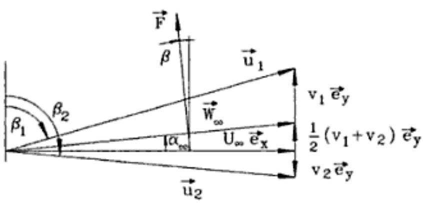
$$\vec{F} = s\,\varrho\,\gamma\,(-V_\infty\,\vec{e}_x + U_\infty\,\vec{e}_y)\,.$$

The scalar product between $\vec{F}$ and the undisturbed flow velocity

$$\vec{W}_\infty = U_\infty\,\vec{e}_x + V_\infty\,\vec{e}_y$$

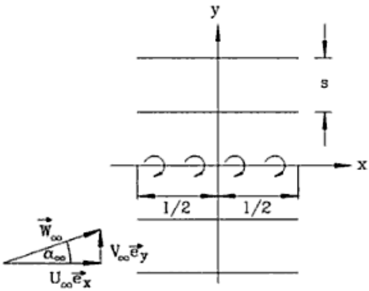
is zero, i. e. the force on the blade is perpendicular to $\vec{W}_\infty$. The angle β between the force and the cascade axis in (y-direction) is

$$\tan\beta = \frac{|F_y|}{|F_x|} = \frac{U_\infty}{V_\infty}\,. \tag{8}$$



From the sketch and equation (8), we recognize that β is equal to the angle α_∞ between the velocity $\vec{W}_\infty$ and the x-axis.

Problem 10.4-18 Vortex distribution of a flat-plate cascade



A flat-plate cascade (spacing s, plate length l) is located in a flow. The undisturbed velocity is $\vec{W}_\infty$, the undisturbed (small) angle α_∞. With (F. M. (10.221) and (10.361)) we find the complex disturbance potential for an individual plate, which is located on the x-axis

$$F(z) = \frac{\mathrm{i}}{2\pi} \int_{-l/2}^{l/2} \gamma(x') \ln(z - x') \, \mathrm{d}x'.$$

By summing over all plates we find the perturbation potential

$$F(z) = \frac{\mathrm{i}}{2\pi} \int_{-l/2}^{l/2} \gamma(x') \ln\left[\sinh\left(\frac{\pi}{s}(z - x')\right)\right] \mathrm{d}x'$$

and the perturbation velocities

$$u(x,y) = \frac{1}{2s} \int_{-l/2}^{l/2} \gamma(x') \frac{\sin \frac{2\pi}{s} y}{\cosh \frac{2\pi}{s}(x - x') - \cos \frac{2\pi}{s} y} \, \mathrm{d}x' ,$$

$$v(x,y) = -\frac{1}{2s} \int_{-l/2}^{l/2} \gamma(x') \frac{\sinh \frac{2\pi}{s}(x - x')}{\cosh \frac{2\pi}{s}(x - x') - \cos \frac{2\pi}{s} y} \, \mathrm{d}x' .$$

a) Calculate the vortex distribution $\gamma(x)$ for an undisturbed flow angle $\alpha_\infty = 3°$, by satisfying the boundary condition only at three points $y = 0$: $x/l = -3/12, 1/12, 5/12$.
b) Calculate the lift coefficient.
   Hint: Use the transformation $x = -(l/2)\cos(\varphi)$ and set

$$\gamma(\varphi) = 2\,U_\infty\,(A_0 \operatorname{cotan}(\varphi/2) + A_1 \sin\varphi + A_2 \sin 2\varphi) .$$

c) Calculate the components of the real inlet velocity u_1, v_1, and the inlet flow angle α_1.

where V_∞, U_∞ are the components of unperturbed velocity. For a small angle α_∞ and $\mathrm{d}f/\mathrm{d}x = 0$ we obtain for $y = 0$

$$v(x,0) = -\alpha_\infty U_\infty = -\frac{1}{2s} \int\limits_{-l/2}^{+l/2} \gamma(x') \frac{\sinh\left(\frac{2\pi}{s}(x-x')\right)}{\cosh\left(\frac{2\pi}{s}(x-x')\right)-1} \mathrm{d}x' . \tag{5}$$

The Kutta condition requires at station $x = l/2$ (trailing edge) equal velocities on the upper and lower surface

$$\Delta u(x = l/2, 0) = u^+(l/2, 0) - u^-(l/2, 0) = 0 . \tag{6}$$

We calculate first the boundary values $u^\pm(x,0)$. For $|x| > l/2$, the integral in (1) is regular and disappears for $y \to 0^\pm$. For $-l/2 \le x \le l/2$ we split off the singular part of the integral in (3) and write

$$\int\limits_{-l/2}^{+l/2} \gamma(x') \frac{\sin\left(\frac{2\pi}{s} y\right)}{\cosh\left(\frac{2\pi}{s}(x-x')\right) - \cos\left(\frac{2\pi}{s} y\right)} \mathrm{d}x' =$$

$$= \frac{s}{\pi} \int\limits_{-l/s}^{l/2} \gamma(x') \frac{y}{(x-x')^2+y^2} \mathrm{d}x' +$$

$$+ \int\limits_{-l/2}^{+l/2} \gamma(x') \left\{ \frac{\sin\frac{2\pi}{s} y}{\cosh\frac{2\pi}{s}(x-x') - \cos\left(\frac{2\pi}{s} y\right)} - \frac{s}{\pi} \frac{y}{(x-x')^2+y^2} \right\} \mathrm{d}x' .$$

The first integral on the right hand side gives, because of a formal equality, with (F. M. (10.352)) the value $\pm s\,\gamma(x)$, the second integral approaches zero for $y \to 0$. Thus, we obtain

$$u(x,0) = \pm\frac{\gamma(x)}{2} \quad \text{or} \quad \Delta u = \gamma(x) \tag{7}$$

and the Kutta condition

$$\Delta u|_{x=l/2} = \gamma(l/2) = 0 . \tag{8}$$

Similar to an individual profile (see F. M. Chapter 10.4.9), we also split off here the flow around the leading edge and obtain with (see F. M. (10.378))

$$\gamma_0(x) = 2a \sqrt{\frac{l/2 - x}{x + l/2}} , \quad a = \text{const}, \tag{9}$$

case is excluded because of the boundary condition (4), i. e. $X(\overline{x})$ must have the form

$$X(\overline{x}) = C_1 \cos\left(\frac{2\pi\overline{x}}{L}\right) + C_2 \sin\left(\frac{2\pi\overline{x}}{L}\right) .$$

We introduce the above formulation into the differential equation and consider the boundary conditions (3), (4) to obtain the solution as

$$\overline{\varphi}(\overline{x},\overline{y}) = -U_\infty \overline{A} \cos\left(\frac{2\pi\overline{x}}{L}\right) e^{-(2\pi\overline{y})/L} ,$$

this is the potential of an incompressible flow over a wavy wall with the amplitude $\overline{A}$.

We find the potential in the physical plane by introducing (1), (2), and $\overline{A} = A\sqrt{1-M_\infty^2}$ as

$$\varphi(x,y) = -\frac{U_\infty}{\sqrt{1-M_\infty^2}} A \cos\left(\frac{2\pi x}{L}\right) e^{-\sqrt{1-M_\infty^2}(2\pi y)/L} .$$

The velocity components (u and v are the perturbation velocities) are:

$$\begin{aligned} U_\infty + u = \frac{\partial\Phi}{\partial x} &= U_\infty + \frac{U_\infty}{\sqrt{1-M_\infty^2}} 2\pi \frac{A}{L} \sin\left(\frac{2\pi x}{L}\right) e^{-\sqrt{1-M_\infty^2}(2\pi y)/L} , \\ v = \frac{\partial\Phi}{\partial y} &= 2\pi U_\infty \frac{A}{L} \cos\left(\frac{2\pi x}{L}\right) e^{-\sqrt{1-M_\infty^2}(2\pi y)/L} . \end{aligned} \tag{5}$$

b) Supersonic solution $M_\infty > 1$:

For $M_\infty > 1$ we rewrite the differential equation in the form

$$(M_\infty^2 - 1)\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial y^2} .$$

The solution is (see F. M. (11.20))

$$\varphi(x,y) = -\frac{U_\infty}{\beta} f(x - \beta y) ,$$

where

$$\beta = \sqrt{M_\infty^2 - 1} .$$

With the known wall contour

$$f(x) = A \sin\left(\frac{2\pi}{L} x\right)$$

the perturbation potential assumes the form

$$\varphi(x,y) = -\frac{U_\infty}{\beta} A \sin\left(\frac{2\pi}{L}(x-\beta\, y)\right) .$$

The velocity components are calculated as

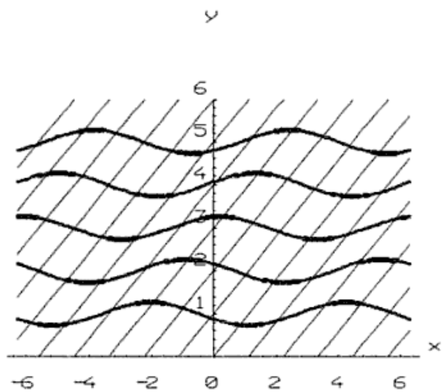
$$U_\infty + u = \frac{\partial\Phi}{\partial x} = U_\infty - \frac{U_\infty}{\beta} 2\pi \frac{A}{L} \cos\left(\frac{2\pi}{L}(x-\beta\, y)\right) , \qquad (6)$$

$$v = \frac{\partial\Phi}{\partial y} = U_\infty 2\pi \frac{A}{L} \cos\left(\frac{2\pi}{L}(x-\beta\, y)\right) .$$

The values of φ, u, and v do not change on the Mach lines $x - \beta\, y =$ const, which may be written as

$$y = \frac{x}{\beta} + \text{const}$$

The perturbation velocity u is plotted qualitatively in the figure for different wall distances. The perturbations in a supersonic flow field do not decrease for $y \to \infty$.



c) Pressure distribution and drag per wavelength:
The pressure coefficient is (see F. M. (10.404)) given by

$$c_P = -2\frac{u}{U_\infty} \qquad (7)$$

and is of the order of magnitude $\epsilon = A/L$. For the subsonic flow, the pressure distribution follows from (5) ($y \approx 0$ along the contour)

$$c_P = -\frac{4\pi}{\sqrt{1-M_\infty^2}} \frac{A}{L} \sin\left(\frac{2\pi x}{L}\right) .$$

The pressure distribution and the wall contour are in phase. Therefore, in accordance with the d'Alembert paradoxon, we do not anticipate a drag. For supersonic flow we obtain from (6)

$$c_P = \frac{4\pi}{\sqrt{M_\infty^2 - 1}} \frac{A}{L} \cos\left(\frac{2\pi x}{L}\right) .$$

For $M_\infty > 1$ the pressure distribution is offset by the phase angle $\pi/2$ compared with the wall contour. As a result, a force in x-direction (drag) acts on the wavy wall. The drag per wave length is calculated from

$$F_x = \iint_{(S)} -p \underbrace{\vec{n} \cdot \vec{e}_x \, \mathrm{d}S}_{\mp \mathrm{d}y} .$$

The sign of $\vec{n} \cdot \vec{e}_x$ is positive for $\dfrac{\mathrm{d}f}{\mathrm{d}x} < 0$ and negative for $\dfrac{\mathrm{d}f}{\mathrm{d}x} > 0$ and we write for the force per unit of depth

$$F_x = \int_{(S)} p \frac{\mathrm{d}y}{\mathrm{d}x} \, \mathrm{d}x = \int_{x=0}^{L} p f'(x) \, \mathrm{d}x$$

or

$$F_x = \int_0^L (p - p_\infty) f'(x) \, \mathrm{d}x + p_\infty \underbrace{\int_0^L f'(x) \, \mathrm{d}x}_{=0} .$$

Using the definition of c_P, therefore

$$F_x = \frac{\rho_\infty}{2} U_\infty^2 L \int_0^1 c_P(x) f'(x) \, \mathrm{d}\left(\frac{x}{L}\right) .$$

Hence, the drag coefficient is calculated from

$$c_D = \frac{F_x}{(\rho_\infty/2) U_\infty^2 L} = \int_0^1 c_P(x) f'(x) \, \mathrm{d}\left(\frac{x}{L}\right) .$$

With $f'(x) = 2\pi \frac{A}{L} \cos\left(\frac{2\pi x}{L}\right)$ we get for the subsonic flow case:

$$c_D = -\frac{8\pi^2}{\sqrt{1 - M_\infty^2}} \left(\frac{A}{L}\right)^2 \underbrace{\int\limits_0^1 \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) \mathrm{d}\left(\frac{x}{L}\right)}_{=0}$$

$$\Rightarrow \quad c_D = 0 \quad \text{(subsonic flow)}$$

and for supersonic flow:

$$c_D = \frac{8\pi^2}{\sqrt{M_\infty^2 - 1}} \left(\frac{A}{L}\right)^2 \underbrace{\int\limits_0^1 \cos^2\left(\frac{2\pi x}{L}\right) \mathrm{d}\left(\frac{x}{L}\right)}_{=1/2},$$

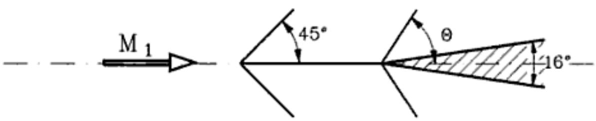
$$c_D = \frac{\left(2\pi(A/L)\right)^2}{\sqrt{M_\infty^2 - 1}} \quad \text{(supersonic)}\,.$$

11 Supersonic Flow

11.1 Oblique Shock Waves

Problem 11.1-1 Wedge with a thin plate in front of it

A wedge with a thin plate in front of it has an angle of 16° and is subjected to a plane supersonic flow of an ideal gas ($\gamma = 1.4$). The approach flow is parallel to the plate such that the plate's leading edge causes only a small perturbation (Mach wave).



a) Measurements indicate an angle of 45° between the plate upper surface and the Mach wave. Determine the approach flow Mach number M_1.
b) Find the shock angle Θ, the Mach number M_2 downstream of the first oblique shock, the pressure ratio p_2/p_1, and the temperature ratio T_2/T_1.
c) Sketch the streamlines.

Solution

a) Mach number M_1:
Using the relationship for the Mach angle

$$\sin\mu = \frac{1}{M}$$

the Mach number follows immediately:

$$M_1 = \frac{1}{\sin\mu_1} = \frac{1}{\sin 45°} = 1.4142\ .$$

b) Θ, M_2, p_2/p_1, T_2/T_1:
The deflection angle $\delta = 8°$ is equal to the wedge angle. From the graphic representation of the angle $\Theta = \Theta(M, \delta)$ (see F. M. , Diagram C.1), we find

$$\Theta = \Theta(M_1 = 1.41,\ \delta = 8°) \approx 58°$$

and Diagram C.2

$$M_2 = M_2(M_1 = 1.41,\ \delta = 8°) \approx 1.02\ .$$

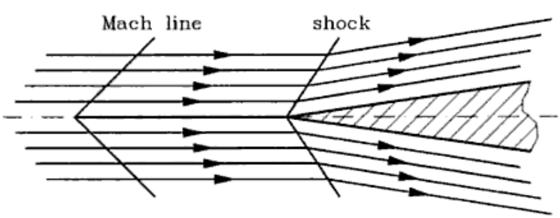
The Mach number normal to the shock follows from the shock angle Θ and the approach Mach number M_1 as

$$M_{1n} = \frac{w_1}{a_1} = \frac{u_1}{a_1} \sin\Theta = M_1 \sin\Theta = 1.2\ .$$

With the Mach number normal to the shock we find the variables of state p_2 and T_2 either by using the equations for a normal shock or by taking the values from the shock tables (see F. M. , Table C.2). Thus, we obtain

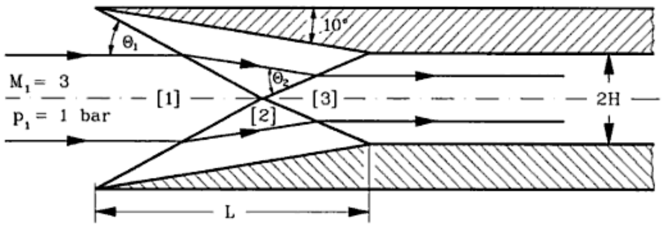
$$M_{1n} = 1.2\,, \quad \frac{p_2}{p_1} = 1.513 \quad \text{and} \quad \frac{T_2}{T_1} = 1.128\ .$$

c) Streamlines:



Problem 11.1-2 Inlet of a plane channel

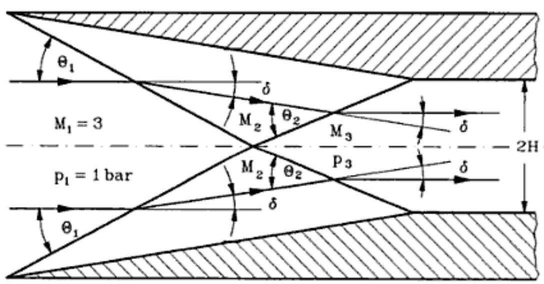
The supersonic flow at the inlet of a plane channel generates two crossing oblique shocks of equal strengths as shown below. The shocks are not reflected at the corners of the convergent part of the inlet (deflection angle $\delta = 10°$). The undisturbed Mach number is $M_1 = 3$, the undisturbed pressure $p_1 = 1\,\text{bar}$. The working medium is air considered as an ideal gas with $\gamma = 1.4$, $R = 287\ \text{J/(kgK)}$).



a) Determine the shock angle Θ_1 of the weak shocks before crossing and the Mach number M_2 in the region between the shocks.
b) Find the shock angle Θ_2 of the weak shocks after crossing and the Mach number M_3 downstream of the shocks.
c) Find the pressure at station [3] behind the shocks.
d) Calculate the entropy increase.
e) Find the ratio L/H such that the sketched flow pattern can be established.

Given: $M_1 = 3$, $p_1 = 1$ bar, $\delta = 10°$, $R = 287\ \text{J/(kgK)}$, $\gamma = 1.4$

Solution



a) Shock angle Θ_1 and Mach number M_2:
With the given Mach number M_1 and the deflection angle we read the shock angle from Diagram C.1 (see F. M.):

The integration leads to

$$s_3 - s_1 = R\left(\frac{1}{\gamma-1}\ln\frac{T_3}{T_1} + \ln\frac{v_3}{v_1}\right) ,$$

where $c_v = R/(\gamma - 1)$ has been used. With the normal shock Mach number $M_n = M\sin\Theta$, the tables for the normal shock may be used:

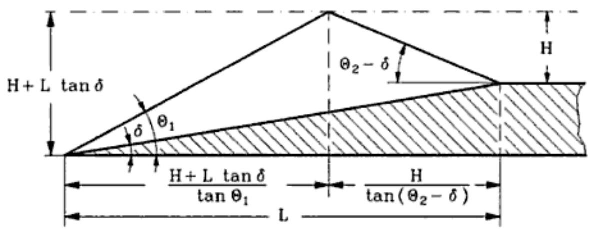
$$\frac{T_3}{T_1} = \frac{T_3}{T_2}\frac{T_2}{T_1} = 1.261 * 1.203 = 1.517 ,$$

$$\frac{v_3}{v_1} = \frac{\varrho_1}{\varrho_3} = (1.707 * 1.551)^{-1} = 0.378$$

and thus

$$\Delta s = 19.8\ \mathrm{J/(kgK)} .$$

e) L/H for the design point:



We find from the sketch

$$L = \frac{H + L\tan\delta}{\tan\Theta_1} + \frac{H}{\tan(\Theta_2 - \delta)}$$

and therefore

$$L\left(1 - \frac{\tan\delta}{\tan\Theta_1}\right) = H\left(\frac{1}{\tan\Theta_1} + \frac{1}{\tan(\Theta_2 - \delta)}\right)$$

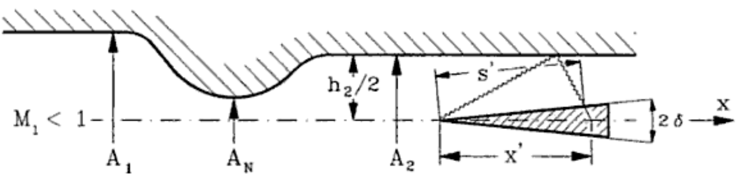
or

$$\frac{L}{H} = \frac{\tan(\Theta_2 - \delta) + \tan\Theta_1}{\tan(\Theta_2 - \delta)(\tan\Theta_1 - \tan\delta)} = 6.5 .$$

11.3 Reflection of Oblique Shock Waves

Problem 11.3-1 Flow over a wedge in a supersonic wind tunnel

In a supersonic wind tunnel (cross-section $A_2 = 0.5\ \mathrm{m}^2$, height $h_2 = 0.7$ m) the state in the test section is given as: pressure $p_2 = 0.1$ bar, temperature $T_2 = 300$ K, Mach number $M_2 = 2.5$.



a) Find the cross-section A_N the throat must have.
b) Find the state at the inlet ($A_1 = 1.0\ \mathrm{m}^2$) for the given state in the test section.
c) In the test section a wedge ($\delta = 5°$) is installed. Give the distance x', where the extension of the reflected shock intersects with the x-axis. To what extent does the pressure distribution on the wedge surface in the range of $0 \leq s \leq s'$ differ from the pressure on a wedge in free flight?

Solution

a) In the Laval nozzle of the wind tunnel the flow is accelerated from subsonic ($M_1 < 1$) to supersonic ($M_2 > 1$). The throat A_N is then the critical area $A_N = A^*$.
With $M_2 = 2.5$ we read from the Table C.1 (see F. M. (supersonic range)):

$$\frac{A^*}{A_2} = 0.379 \quad \text{or} \quad A^* = 0.1895\ \mathrm{m}^2 \ .$$

b) Inlet state: M_1, p_1, T_1
Following relations are used:

$$\frac{p_1}{p_2} = \frac{p_1}{p_t}\frac{p_t}{p_2} = \frac{f(M_1)}{f(M_2)} \ ,$$

$$\frac{T_1}{T_2} = \frac{T_1}{T_t}\frac{T_t}{T_2} = \frac{g(M_1)}{g(M_2)} \ .$$

or

$$M_{down} = 2.2 \ .$$

For the second shock we now have:

$$\delta = 5^\circ \quad \text{and} \quad M_{up} = 2.2 \ ,$$

from Diagram C.1:

$$\Theta_2 = 32^\circ \ ,$$

from Diagram C.2:

$$1 - \frac{1}{M_{down}} = 0.5 \ ,$$

or

$$M_{down} = 2.0$$

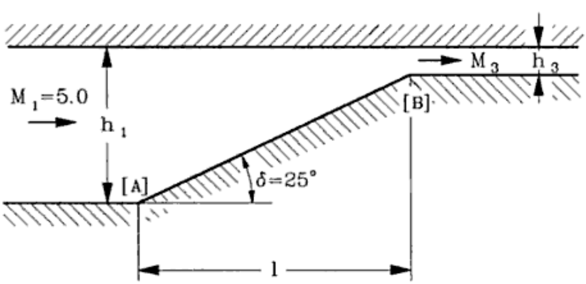
thus the distance is

$$x' = 1.345 \text{ m} \ .$$

The state downstream of the first reflected shock ($0 \leq s \leq s'$) is determined by the upstream Mach number and the deflection angle δ only and is the same as on a wedge in free flight. The influence of the wall is felt only downstream of the reflected shock.

Problem 11.3-2 Supersonic flow in a convergent channel

The lower wall of a plane channel turns at [A] and [B] reducing the channel height from h_1 to h_3. (see also Problem 10.4-13 for incompressible channel flow). The working medium is an ideal gas ($\gamma = 1.4$) with the Mach number $M_1 = 5.0$.

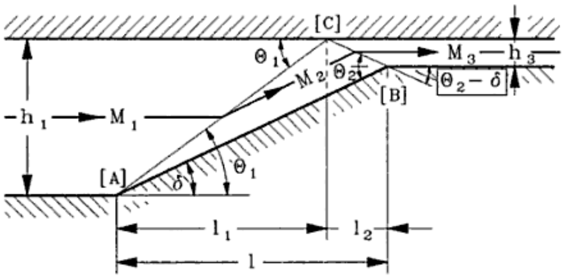


a) Determine the distance l between the points [A] and [B] such that the downstream flow at point [3] is parallel and uniform. Find the channel height h_3.
b) Find the value of the downstream Mach number M_3.

Given: h_1, $M_1 = 5.0$, $\delta = 25°$, $\gamma = 1.4$

Solution

a) Distance l:



From the sketch we read $l = l_1 + l_2$,

$$l_1 = h_1 \cot \Theta_1 \ , \quad l_2 = h_3 \cot(\Theta_2 - \delta)$$

and

$$l = (h_1 - h_3) \cot \delta \ , \tag{1}$$

which means

$$h_1 \cot \Theta_1 + h_3 \cot(\Theta_2 - \delta) = h_1 \cot \delta - h_3 \cot \delta$$

or

$$\frac{h_3}{h_1} = \frac{\cot \delta - \cot \Theta_1}{\cot(\Theta_2 - \delta) + \cot \delta} \tag{2}$$

and with (1)

$$\frac{l}{h_1} = \left(1 - \frac{h_3}{h_1}\right) \cot \delta \ . \tag{3}$$

The shock angles Θ_1 and Θ_2 are taken from the Diagram C.1 (see F. M.)

$$\Theta_1 = f(M_1, \delta)$$

$$\left.\begin{array}{ll} \delta & = 25° \\ M_1 & = 5.0 \end{array}\right\} \Rightarrow \quad \text{Diagram C.1} \quad \Rightarrow \quad \Theta_1 = 35.7° \ .$$

The downstream Mach number M_2 is found from Diagram C.2 (see F. M.)

Radius:

$$R_B = \frac{h_1}{\sin\mu_1} = h_1 M_1 = 0.3 \text{ m} * 1.6 = 0.48 \text{ m} . \tag{1}$$

c) M_2, p_2, T_2, ϱ_2, and u_2:
Prandtl-Meyer function (see F. M. , Table C.3): From the above table we take the auxiliary angle ν_1 which corresponds to M_1. This is the deflection angle which produces the Mach number M_1 from an approach Mach number $M = 1$.

$$M_1 = 1.6 , \quad \nu_1 = 14.861° ,$$

with the given deflection

$$\delta = 30°$$

the total deflection from $M = 1$ is

$$\nu = \nu_1 + \delta = 44.861° ,$$

which produces the Mach number

$$M_2 = 2.76 .$$

The corresponding characteristic angle is

$$\mu_2 = 21.24° .$$

The necessary temperature and pressure for $M_2 = 2.76$ in an isentropic expansion

$$\frac{p_2}{p_t} = 0.0392 , \quad \frac{T_2}{T_t} = 0.3963$$

are taken from Table C.1 (see F. M.). From the same table we obtain for $M_1 = 1.6$

$$\frac{p_1}{p_t} = 0.2353 , \quad \frac{T_1}{T_t} = 0.6614 ,$$

such that

$$\frac{p_2}{p_1} = \frac{0.0392}{0.2353} = 0.1665$$

and the pressure is found as

$$p_2 = 0.066 \text{ bar} .$$

Similarly

$$\frac{T_2}{T_1} = \frac{0.3963}{0.6614} = 0.5992 \quad \Rightarrow \quad T_2 = 149.8 \text{ K} ,$$

furthermore

$$\varrho_2 = \frac{p_2}{RT_2} = 0.1549 \frac{\text{kg}}{\text{m}^3}$$

and

$$u_2 = M_2\sqrt{\gamma RT_2} = 677.1 \text{ m/s} .$$

d) Equation of channel contour:
The upper channel contour is a streamline of the Prandtl-Meyer flow. In a coordinate system, in which at $\varphi^* = \pi/2$ the Mach number is one, the differential equation for the streamline using u_r, u_φ from F. M. (11.46), (11.47) is

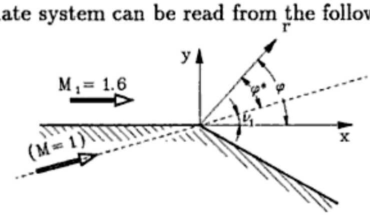
$$\frac{\mathrm{d}r}{r\mathrm{d}\varphi^*} = \frac{u_r}{u_\varphi} = \frac{\sqrt{2/(\gamma-1)}a_t \sin(\sqrt{(\gamma-1)/(\gamma+1)}((\pi/2)-\varphi^*))}{-\sqrt{2/(\gamma+1)}a_t \cos(\sqrt{(\gamma-1)/(\gamma+1)}((\pi/2)-\varphi^*))}$$

$$\Rightarrow \quad \frac{\mathrm{d}r}{r\mathrm{d}\varphi^*} = -\frac{1}{m}\tan\left(m\left(\frac{\pi}{2}-\varphi^*\right)\right) \qquad (2)$$

with

$$m = \sqrt{\frac{\gamma-1}{\gamma+1}} . \qquad (3)$$

The relationship between φ^* and the polar angle of the present coordinate system can be read from the following sketch as



$$\varphi^* = \varphi - \nu_1$$

and the differential equation (2) after separation of variables is

$$\frac{\mathrm{d}r}{r} = -\frac{1}{m}\tan\left(m\left(\frac{\pi}{2}+\nu_1-\varphi\right)\right)\mathrm{d}\varphi .$$

From integrating

$$\int\limits_{r=R_B}^{r} \frac{\mathrm{d}r}{r} = -\frac{1}{m}\int\limits_{\varphi=\mu_1}^{\varphi} \tan\left(m\left(\frac{\pi}{2}+\nu_1-\varphi\right)\right)\mathrm{d}\varphi$$

we obtain

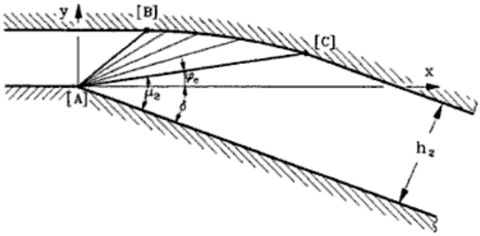
$$\ln\frac{r}{R_B} = -\frac{1}{m^2}\ln\left(\frac{\cos m((\pi/2)+\nu_1-\varphi)}{\cos m((\pi/2)+\nu_1-\mu_1)}\right) ,$$

$$\text{or} \qquad \frac{r}{R_B} = \left(\frac{\cos m((\pi/2)+\nu_1-\mu_1)}{\cos m((\pi/2)+\nu_1-\varphi)}\right)^{1/m^2} .$$

With (1) and (3) we find

$$r(\varphi) = M_1 h_1 \left(\frac{\cos(\sqrt{(\gamma-1)(\gamma+1)}((\pi/2)+\nu_1-\mu_1))}{\cos(\sqrt{(\gamma-1)(\gamma+1)}((\pi/2)+\nu_1-\varphi))} \right)^{(\gamma+1)/(\gamma-1)} . \quad (4)$$

This equation describes the streamline $r = r(\varphi)$ within the expansion fan in the domain $\varphi_B \geq \varphi \geq \varphi_C$. The angle φ_C is read from the sketch:



$$\varphi_C = \mu_2 - \delta = -8.76° .$$

At the end point [C] it follows from (4)

$$R_C = r(\varphi_C) = 2.2272 \text{ m}$$

and from it the channel height behind the expansion fan

$$h_2 = R_C \sin\mu_2 = \frac{R_C}{M_2} = 0.807 \text{ m} ,$$

which can easily be checked using the continuity equation

$$h_2 = \frac{\dot{m}}{\varrho_2 u_2} = 0.807 \text{ m} .$$

11.6 Shock Expansion Theory

Problem 11.6-1 Airfoil in supersonic flow

The following asymmetric profile is exposed to a supersonic flow with Mach number $M_\infty = 1.7$ ($\gamma = 1.4$).

$$\left.\begin{array}{lll} M_3 = 1.51 & \Rightarrow & \dfrac{p_3}{p_{t3,4}} = 0.268 \\ M_4 = 1.68 & \Rightarrow & \dfrac{p_4}{p_{t3,4}} = 0.209 \end{array}\right\} \Rightarrow \quad \frac{p_4}{p_3} = 0.78 \, ,$$

$$\frac{p_3}{p_\infty} = 1.32 \quad \Rightarrow \quad \frac{p_4}{p_\infty} = 0.78 * 1.32 = 1.03 \, .$$

c) Lift- and drag coefficient (per unit of depth):
The pressure on each of the surfaces A_i $(i = 1, 2, 3, 4)$ is constant such that

$$\vec{F}_{(i)} = \iint\limits_{(S_i)} -p\vec{n}\,\mathrm{d}S = -p_{(i)}A_{(i)}\vec{n}_{(i)} \quad (i = \text{number of the surface}) \, .$$

The force magnitude is $p_{(i)}A_{(i)}$. Because of the assumption of inviscid flow, the force vector is normal to the surfaces. If β_i is the angle between the individual surfaces and the profile chord, the surface area per unit of depth is

$$A_{(i)} = \frac{l/2}{\cos\beta_{(i)}} \, ,$$

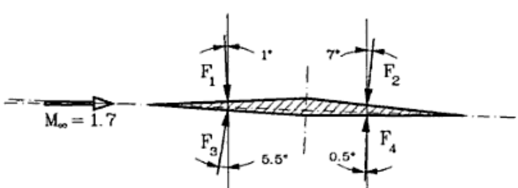
and

$$F_{(i)} = |\vec{F}_{(i)}| = p_\infty \frac{p_i}{p_\infty} \frac{l/2}{\cos\beta_{(i)}} \, .$$

The numerical values are

i	1	2	3	4
$F_{(i)}/(lp_\infty)$	0.526	0.341	0.661	0.515

The lift and drag force are calculated using the known forces from the following sketch:



$$L = F_3\cos 5.5° + F_4\cos 0.5° - F_1\cos 1° - F_2\cos 7° = 0.309\,p_\infty l \, ,$$

$$D = F_1\sin 1° + F_3\sin 5.5° + F_4\sin 0.5° - F_2\sin 7° = 0.0355 p_\infty l \, .$$

Lift and drag coefficients are the respective forces referred to the upstream dynamic pressure, i. e.

$$\frac{\varrho_\infty}{2}U_\infty^2 = \frac{1}{2}\varrho_\infty a_\infty^2 M_\infty^2 = \frac{1}{2}\varrho_\infty \gamma \frac{p_\infty}{\varrho_\infty} M_\infty^2 = \frac{1}{2}\gamma p_\infty M_\infty^2$$

times the profile chord l:

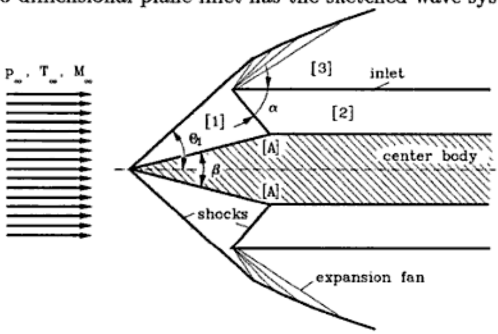
$$c_L = \frac{A}{(\gamma/2)p_\infty l M_\infty^2} = 0.153$$

and

$$c_D = \frac{W}{(\gamma/2)p_\infty l M_\infty^2} = 0.0175 .$$

Problem 11.6-2 Inlet of a supersonic jet engine

The two-dimensional plane inlet has the sketched wave system:



a) Determine the shock angle Θ_1 and M_1, p_1, T_1, and u_1.
b) Calculate the angle α (see figure), M_2, p_2, T_2, and u_2.
c) Determine M_3, p_3, T_3, and u_3.
d) Why is there no shock reflection?

Given: $M_\infty = 2.6$, $p_\infty = 1\,\text{bar}$, $T_\infty = 300\,\text{K}$, $\gamma = 1.4$, $R = 287\,\text{J/(kg K)}$, $\beta = 28°$

Solution

a) State at region [1]:
The flow is deflected by the angle $\delta = \beta/2 = 14°$. With the upstream Mach number $M_\infty = 2.6$ it follows from (see F. M. , Diagram C.1)

$$\Theta_1 = \Theta_1(M_\infty = 2.6,\ \delta = 14°) \approx 35°$$

and from (see F. M. , Diagram C.2)

$$M_1 = M_1(M_\infty = 2.6,\ \delta = 14°) \approx 2 .$$

The Mach number normal to the shock front is

$$M_{1n} = M_\infty \sin\Theta = 1.49 \; .$$

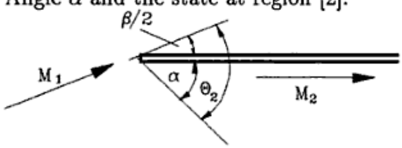
Thus, for the normal shock we obtain from the shock relation (see F. M. , Table C.2):

$$\frac{p_1}{p_\infty} = 2.42 \quad \Rightarrow \quad p_1 = 2.42 * 1\,\mathrm{bar} = 2.42\,\mathrm{bar} \; ,$$

$$\frac{T_1}{T_\infty} = 1.31 \quad \Rightarrow \quad T_1 = 1.31 * 300\,\mathrm{K} = 393\,\mathrm{K}$$

$$\Rightarrow \quad u_1 = M_1\, a_1 = M_1 \sqrt{\gamma\, R\, T_1} = 795\,\mathrm{m/s} \; .$$

b) Angle α and the state at region [2]:



We read the deflection angle from the sketch:

$$\delta = \frac{\beta}{2} = 14°$$

and the shock angle as:

$$\Theta_2 = \alpha + \frac{\beta}{2} \; .$$

Since

$$\Theta_2 = \Theta_2(M_1 = 2,\, \delta = 14°) = 44°$$

(see F. M. , Diagram C.1), one obtains α as

$$\alpha = \Theta_2 - \frac{\beta}{2} = 44° - 14° = 30° \; .$$

From (see F. M. , Diagram C.2) we read the Mach number

$$M_2 = M_2(M_1 = 2,\, \delta = 14°) = 1.5 \; .$$

With the normal Mach number

$$M_{1n} = M_1 \sin\Theta = 2\sin(44°) = 1.39$$

we get p_2, T_2, and u_2 from the shock relations (see F. M. , Table C.2)

$$\frac{p_2}{p_1} = 2.09 \quad \Rightarrow \quad p_2 = 2.09 * 2.42\mathrm{bar} = 5.06\,\mathrm{bar} \; ,$$

$$\frac{T_2}{T_1} = 1.25 \quad \Rightarrow \quad T_2 = 1.25 * 393\,\mathrm{K} = 491\,\mathrm{K}$$

$$\Rightarrow \quad u_2 = M_2\, a_2 = M_2 \sqrt{\gamma\, R\, T_2} = 666\,\mathrm{m/s} \; .$$

Solution

Derivation of the momentum equation

$$\frac{\mathrm{d}\delta_2}{\mathrm{d}x} + \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x}(2\delta_2 + \delta_1) = \frac{\tau_w}{\varrho U^2} \,.$$

Integrating the boundary layer equations over y from 0 to ∞.
(see F. M. (12.19–12.21))

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \,, \tag{1}$$

$$0 = \frac{\partial p}{\partial y} \,, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,. \tag{3}$$

Because of (2), the pressure is independent of y, i. e. it is the same as in the inviscid outer flow, where the component of the pressure gradient in (1), using Euler's equation evaluated at the wall, can be replaced by:

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = U\frac{\mathrm{d}U}{\mathrm{d}x} \,.$$

The friction term in (1) is in the context of the boundary layer theory the only non-zero component of the divergence of the stress tensor.

$$\underbrace{\frac{\eta}{\varrho}}_{=\nu}\frac{\partial^2 u}{\partial y^2} = \frac{1}{\varrho}\frac{\partial}{\partial y}\tau_{xy} \,.$$

Thus, equation (1) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{\partial U}{\partial x} - \frac{1}{\varrho}\frac{\partial \tau_{xy}}{\partial y} = 0 \,.$$

We integrate this equation from $y = 0$ to $y = h > \delta(x)$:

$$\int\limits_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{\partial U}{\partial x}\right) \mathrm{d}y = \frac{1}{\varrho}\int\limits_0^h \frac{\partial \tau_{xy}}{\partial y}\,\mathrm{d}y \,. \tag{4}$$

Using the continuity equation (3), we replace the component v of the boundary layer velocity by

$$v = -\int\limits_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y + f(x) \,,$$

where $f(x)$ disappears since $v(x,\, y=0)=0$. With $\tau_{xy}\,(y=0)=\tau_w$ and $\tau_{xy}\,(y=h)=0$ we find from (4)

$$\int\limits_0^h \Big[u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\int\limits_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y-U\frac{\partial U}{\partial x}\Big]\,\mathrm{d}y=-\frac{\tau_w}{\varrho}\,.$$

We rearrange the second term by partial integration:

$$\int\limits_0^h u\frac{\partial u}{\partial x}\,\mathrm{d}y-\Big[u\int\limits_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y\Big]_0^h+\int\limits_0^h u\frac{\partial u}{\partial x}\,\mathrm{d}y-\int\limits_0^h U\frac{\partial U}{\partial x}\,\mathrm{d}y=-\frac{\tau_w}{\varrho}$$

and obtain

$$-2\int\limits_0^h u\frac{\partial u}{\partial x}\,\mathrm{d}y+\int\limits_0^h U\frac{\partial u}{\partial x}\,\mathrm{d}y+\int\limits_0^h U\frac{\partial U}{\partial x}\,\mathrm{d}y=+\frac{\tau_w}{\varrho}\,. \tag{5}$$

Equation (5) can be simplified further by adding and subtracting $u\dfrac{\partial U}{\partial x}$

$$\int\limits_0^h \Big(-2u\frac{\partial u}{\partial x}+U\frac{\partial u}{\partial x}+u\frac{\partial U}{\partial x}-u\frac{\partial U}{\partial x}+U\frac{\partial U}{\partial x}\Big)\,\mathrm{d}y=\frac{\tau_w}{\varrho}\,.$$

As a result, we have

$$\int\limits_0^h \frac{\partial}{\partial x}[u(U-u)]\,\mathrm{d}y+\frac{\partial U}{\partial x}\int\limits_0^h (U-u)\,\mathrm{d}y=\frac{\tau_w}{\varrho}\,.$$

Outside the boundary layer the integrands are zero, since $u=U$; we can also integrate from $y=0$ to $y\to\infty$. In the first integral we interchange the order of the integration and differentiation (this operation is admissible since the integration boundaries are independent of x):

$$\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\underbrace{\int\limits_0^\infty u(U-u)\,\mathrm{d}y}_{\delta_2 U^2}+\frac{\mathrm{d}U}{\mathrm{d}x}\underbrace{\int\limits_0^\infty (U-u)\,\mathrm{d}y}_{\delta_1 U}=\frac{\tau_w}{\varrho}\,.$$

Solution

The momentum equation reads $\left(\dfrac{\mathrm{d}}{\mathrm{d}x} \mathrel{\hat{=}} {}'\right)$

$$\delta_2' + \frac{U'}{U}(2\delta_2 + \delta_1) = \frac{\tau_w}{\varrho U^2}$$

with

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right) \mathrm{d}y \qquad \text{“displacement thickness”}$$

and

$$\delta_2 = \int_0^\delta \left(1 - \frac{u}{U}\right) \frac{u}{U} \,\mathrm{d}y \qquad \text{“momentum thickness”} \ .$$

The velocity in the boundary layer has the assumed distribution

$$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \ .$$

Thus, for the displacement thickness we have

$$\frac{\delta_1}{\delta} = \int_0^1 \left[1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] \mathrm{d}\left(\frac{y}{\delta}\right) = \frac{\pi - 2}{\pi}$$

and for the momentum thickness

$$\frac{\delta_2}{\delta} = \int_0^1 \left[1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \mathrm{d}\left(\frac{y}{\delta}\right) = \frac{4 - \pi}{2\pi} \ ,$$

$$\tau_w = \eta \frac{\partial u}{\partial y}\bigg|_{y=0} = \eta U \frac{\partial}{\partial y}\left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]_{y=0} = \eta \frac{\pi}{2}\frac{U}{\delta}$$

expressed in terms of momentum thickness we get

$$\frac{\delta_2}{\delta} = \frac{4 - \pi}{2\pi} \quad \Rightarrow \quad \tau_w = \eta \frac{4 - \pi}{4}\frac{U}{\delta_2} \ .$$

Inserting τ_w and δ_1 in the momentum equation results in an ordinary differential equation for the momentum thickness δ_2. With

$$\delta_1 = \frac{\delta_1}{\delta}\frac{\delta}{\delta_2}\delta_2 = \frac{\pi - 2}{\pi}\frac{2\pi}{4 - \pi}\delta_2$$

follows

$$\delta_2' + \frac{U'}{U} 2\delta_2 \left(1 + \frac{\pi - 2}{4 - \pi}\right) = \frac{4 - \pi}{4} \frac{U}{\delta_2} \frac{\eta}{\varrho U^2}$$

or

$$\delta_2 \delta_2' + \frac{4}{4 - \pi} \frac{U'}{U} \delta_2^2 = \frac{4 - \pi}{4} \frac{\nu}{U} \,. \tag{1}$$

a) Integration of the momentum equation (1) for the wedge flow:

$$U = C x^m \,, \quad U' = C m x^{m-1} \quad \Rightarrow \quad \frac{U'}{U} = \frac{m}{x}$$

and because of $\delta_2 \delta_2' = \dfrac{1}{2} \dfrac{\mathrm{d}\delta_2^2}{\mathrm{d}x}$ a first order differential equation linear in δ_2^2 is obtained:

$$\frac{1}{2} \frac{\mathrm{d}\delta_2^2}{\mathrm{d}x} + \frac{4}{4 - \pi} \frac{m}{x} \delta_2^2 = \frac{4 - \pi}{4} \frac{\nu}{C x^m}$$

or

$$x^m f' + a \frac{x^m}{x} f = b$$

with $\quad f = \delta_2^2 \,; \quad a = \dfrac{8m}{4 - \pi} \,; \quad b = \dfrac{4 - \pi}{2} \dfrac{\nu}{C} \,.$

General solution of the homogeneous differential equation:

$$x^m \frac{\mathrm{d}f}{\mathrm{d}x} + a \frac{x^m}{x} f = 0$$

or

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -a \frac{f}{x}$$

and after separation of variables

$$\frac{\mathrm{d}f}{f} = -a \frac{\mathrm{d}x}{x} \,,$$

such that the homogeneous solution reads

$$f_h = K_1 x^{-a} = K_1 x^{-(8m)/(4-\pi)} \,.$$

We find the particular solution of the inhomogeneous differential equation by variation of the constant:

$$f_p = K_2(x) x^{-a} \,,$$

leading to

$$x^m (K_2' x^{-a} - a K_2 x^{-a-1} + a K_2 x^{-a-1}) = b \,.$$

The integral method used to calculate the boundary layer is an approximate solution. For the present problem, it is possible to solve the equation exactly. We compare the calculated wall shear stress (2) with the exact solution:
Flat plate ($m = 0$):

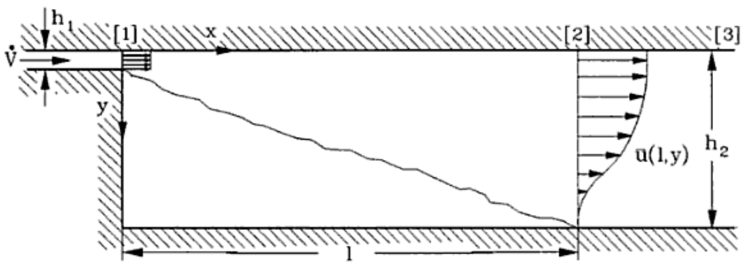
$$A = \frac{1}{2}\sqrt{\frac{4-\pi}{2}} = 0.3276 \; ; \qquad A_{\text{exact}} = 0.3321 \; .$$

Stagnation point flow ($m = 1$):

$$A = 1 \; ; \qquad A_{\text{exact}} = 1.2326 \; .$$

Problem 12-3 Diffuser with discontinuous change of the cross-section

Fluid with volume flux $\dot{V}$ and constant density ϱ flows through a plane channel, whose height at station [1] changes discontinuously from h_1 to h_2.



The velocity at station [1] is fully uniform. At the location of the sudden cross-section increase, flow separation occurs that at $x = l$ reattaches at station [2]. At station [3] which is located sufficiently far downstream from station [2], the flow can be considered as fully uniform. The velocity distribution between [1] and [2] in a sufficient distance from [1] can be approximated by half of the profile of a plane turbulent free jet:

$$\overline{u}(x,y) = \sqrt{\frac{K}{x}}\frac{1}{\cosh^2(\sigma\,\frac{y}{x})} \quad \text{with} \;\; \sigma = 7.67 \; ,$$

where the x-axis is the symmetry line of the jet. We assume that at the top wall no flow separation occurs. Furthermore, we consider the pressure at each location x within the channel to be constant over the cross-section.

a) Calculate the mean velocity at station [1] and give the velocity distribution at [2] $\overline{u}(l, y)$.
b) Determine the pressure difference $\Delta p = p_2 - p_1$ between station [1] and [2]. The velocity profile of the free jet is given at station [2]; at station [1] the flow is fully uniform and all wall shear stresses are neglected.
c) Consider now the shear stresses at the top wall ($y = 0$), where beginning from station [1] a turbulent boundary layer is developed. The velocity at the edge of the boundary layer can be approximated by the velocity, the free jet profile has at $y = 0$. Calculate now $p_2 - p_1$ and by comparing with the result from part b) show that the friction force at the top wall can be neglected.
d) Determine the pressure recovery $p_2 - p_1$ under the assumption that the flow at station [2] is uniform and the wall shear stresses are neglected.

Given: $\dot{V} = 0.3\,\mathrm{m^2/s}$, $h_1 = 0.02$ m, $h_2 = 10\,h_1 = 0.2$ m, $l = 23\,h_1 = 0.46$ m, $\varrho = 1000$ kg/m^3, $\nu = 10^{-6}$ m^2/s

Solution

a) The velocity at station [1] is calculated using the volume flux $\dot{V}$ as

$$\overline{u}_1 = \frac{\dot{V}}{h_1} = 15\,\mathrm{m/s}\,.$$

At station [2] we have the same volume flux:

$$\dot{V} = \iint\limits_{(S)} \vec{u}\cdot\vec{n}\,\mathrm{d}S = \int\limits_0^{h_2} \overline{u}(x = l, y)\mathrm{d}y = \int\limits_0^{h_2} \sqrt{\frac{K}{l}}\,\frac{\mathrm{d}y}{\cosh^2\left(\sigma\,\frac{y}{l}\right)}$$

$$= \left[\sqrt{\frac{K}{l}}\,\frac{l}{\sigma}\,\tanh\left(\sigma\,\frac{y}{l}\right)\right]_0^{h_2} = \frac{\sqrt{K\,l}}{\sigma}\,\tanh\left(\sigma\,\frac{h_2}{l}\right)\,.$$

Thus, the unknown constant K of the profile is determined as

$$K = \frac{1}{l}\left(\frac{\dot{V}\,\sigma}{\tanh\left(\sigma\,\frac{h_2}{l}\right)}\right)^2 = 11.57\ \mathrm{m^3/s^2}\,.$$

The velocity profile at station $x = l$ is

$$\overline{u}(l, y) = \frac{\dot{V}\,\sigma}{l\tanh\left(\sigma\,\frac{h_2}{l}\right)}\,\frac{1}{\cosh^2\left(\sigma\,\frac{y}{l}\right)} = \frac{A}{\cosh^2\left(\sigma\,\frac{y}{l}\right)} \qquad (1)$$

with

$$A = \frac{\dot{V}\,\sigma}{l\tanh\left(\sigma\,\frac{h_2}{l}\right)}\,.$$

The evaluation of the integral on the right hand side gives

$$\varrho A^2 \int\limits_0^{h_2} \frac{\mathrm{d}y}{\cosh^4\left(\sigma \frac{y}{l}\right)} = \varrho \frac{A^2 l}{3\sigma} \tanh\left(\sigma \frac{h_2}{l}\right)\left[3 - \tanh^2\left(\sigma \frac{h_2}{l}\right)\right]$$

$$= \varrho \frac{\sigma \dot{V}^2}{l}\left[\coth\left(\sigma \frac{h_2}{l}\right) - \frac{1}{3}\tanh\left(\sigma \frac{h_2}{l}\right)\right] .$$

Inserting the evaluated integral in equation (2), we find the pressure difference we are looking for:

$$(p_2 - p_1)\, h_2 = \varrho \frac{\dot{V}^2}{h_1} - \varrho A^2 \int\limits_0^{h_2} \frac{\mathrm{d}y}{\cosh^4\left(\sigma \frac{y}{l}\right)} \quad \text{or}$$

$$(p_2 - p_1)\, h_2 = \varrho \frac{\dot{V}^2}{h_1} - \varrho \frac{\dot{V}^2 \sigma}{l}\left[\coth\left(\sigma \frac{h_2}{l}\right) - \frac{1}{3}\tanh\left(\sigma \frac{h_2}{l}\right)\right]$$

or

$$p_2 - p_1 = \varrho \frac{\dot{V}^2}{h_1 h_2}\left[1 - \sigma \frac{h_1}{l}\left(\coth\left(\sigma \frac{h_2}{l}\right) - \frac{1}{3}\tanh\left(\sigma \frac{h_2}{l}\right)\right)\right] \quad (3)$$

$$p_2 - p_1 = 0.776\, \varrho \frac{\dot{V}^2}{h_1 h_2} = 0.175\,\text{bar} .$$

c) We include now the wall shear stresses at the upper wall in the momentum balance. This results in an additional contribution, since t_x on the partial surface S_U of the wall S_W is different from zero:

$$F_{xw \to Fl.} = \iint\limits_{S_W} t_x \,\mathrm{d}S = p_1 (h_2 - h_1) + \iint\limits_{S_U} t_x \,\mathrm{d}S .$$

The x-component of the stress vector at the upper wall S_U is

$$t_x = \tau_{xx}\, n_x + \tau_{xy}\, n_y$$

and because of $n_x = 0$ and $n_y = -1$

$$t_x = -\tau_{xy} = -\tau_W .$$

d) If we assume a fully uniform flow at station [2], we can calculate the pressure difference using the balance of momentum:

$$p_2 - p_1 = \varrho \frac{\dot{V}^2}{h_1 h_2}\left(1 - \frac{h_1}{h_2}\right)$$

$$= 0.9\,\varrho \frac{\dot{V}^2}{h_1 h_2} = 0.205\,\text{bar}\ ,$$

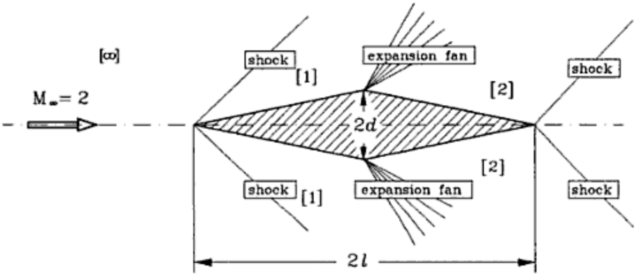
which corresponds to the Carnot shock loss (see F. M. (9.52)).
The pressure rise in this case is expectedly higher than the pressure rise obtained in part b), since the velocity profile is smoothed out between station [2] and [3], which causes a further pressure rise. Choosing another velocity profile at [2] will always result in a pressure difference, which is smaller than the Carnot shock loss, since the rectangular profile is the profile with smallest momentum for a given volume flux $\dot{V}$.

Problem 12-4 Drag coefficient of a diamond airfoil

The drag coefficient

$$c_D = \frac{F_D}{\varrho_\infty/2\ U_\infty^2\, l}$$

of the sketched diamond airfoil is to be calculated for a supersonic flow ($M_\infty = 2$).



According to the Froude's hypothesis, which is very well confirmed by experiments, the drag coefficient c_D consists of the friction drag coefficient c_f, which depends on the Mach and Reynolds number and the drag coefficient c_{wd}, which depends only on the Mach number and is called the coefficient of wave drag.

$$c_D = c_f(Re, M) + c_{wd}(M)\ .$$

Solution

a) State in the regions [1] and [2] :

1) Oblique shocks from [∞] to [1]:
From $d/l = 0.176 = \tan\delta$, the deflection angle follows: $\delta = 10°$, the upstream Mach number is $M_\infty = 2$, and we find from Diagram C.1 and C.2 (see F. M. , Appendix C):

$$M_1 = 1.64\ ,\quad \Theta_1 = 39.31° \quad\Rightarrow\quad M_{n\infty} = 1.267\ .$$

The quantities behind the shock are (see F. M. , Table C.2):

$$p_1 = 1.69\ p_\infty\ ,\quad \varrho_1 = 1.45\ \varrho_\infty\ ,\quad T_1 = 1.17\ T_\infty\ .$$

As a result, we have

$$a_1 = 1.08\ a_\infty\ ,\quad U_1 = 0.885\ U_\infty\ ,\quad \eta_1 = \eta_\infty \frac{T_1}{T_\infty} = 1.17\ \eta_\infty\ ,$$

$$\nu_1 = \frac{\eta_1}{\varrho_1} = 0.807\ \nu_\infty\ .$$

2) Prandtl-Meyer expansion from [1] to [2]:
With $M_1 = 1.64$, we read from (see F. M. , Table C.3) the deflection angle $\nu_1 = 16.04°$ that corresponds to the upstream Mach number $M = 1$. Thus, the total deflection is $\nu = \nu_1 + 20° = 36.04°$ and the Mach number in region [2] is $M_2 = 2.37$. The flow quantities behind the expansion wave are calculated using Table C.1 (see F. M.). We find

$$p_2 = 0.329\ p_1 = 0.56\ p_\infty\ ,\quad \varrho_2 = 0.445\ \varrho_1 = 0.65\ \varrho_\infty\ ,$$

$$T_2 = 0.725\ T_1 = 0.85\ T_\infty\ ,$$

from which we get

$$a_2 = 0.92\ a_\infty\ ,\quad U_2 = 1.09\ U_\infty\ ,\quad \eta_2 = \eta_\infty \frac{T_2}{T_\infty} = 0.85\ \eta_\infty\ ,$$

$$\nu_2 = \frac{\eta_2}{\varrho_2} = 1.31\ \nu_\infty\ .$$

The above calculated values are equally valid for model and prototype. In the following, the model values are denoted with a prime ($'$).

The state of gas upstream of the prototype is ($M_\infty = 2$)

$$\varrho_\infty = 1.16\,\mathrm{kg/m^3},\quad p_\infty = 1\,\mathrm{bar},\quad T_\infty = 300\,\mathrm{K}$$

c) Reynolds numbers:
The length of a profile section is $\sqrt{l^2 + d^2} \approx l$ and the Reynolds numbers in the individual regions can be calculated with

$$Re = \frac{U\,l}{\nu}\,.$$

1) For the prototype we find with:

$$l = 2\,\mathrm{m}, \quad U_\infty = 694.4\,\mathrm{m/s}, \quad \nu_\infty = 16 * 10^{-6}\,\mathrm{m^2/s}\,,$$

$$Re_1 = \frac{U_1\,l}{\nu_1} = \frac{0.885 U_\infty\,l}{0.807\nu_\infty} = 9.5 * 10^7\,,$$

and

$$Re_2 = \frac{U_2\,l}{\nu_2} = \frac{1.09 U_\infty\,l}{1.31\nu_\infty} = 7.2 * 10^7\,.$$

2) For the model (scale factor $l/l' = 100$) with:

$$l' = 0.02\,\mathrm{m}, \quad U'_\infty = 517\,\mathrm{m/s}, \quad \nu'_\infty = 38.3 * 10^{-6}\,\mathrm{m^2/s}$$

therefore

$$Re'_1 = \frac{U'_1\,l'}{\nu_1} = \frac{0.885 U'_\infty\,l'}{0.807\nu'_\infty} = 2.96 * 10^5$$

as well as

$$Re'_2 = \frac{U'_2\,l'}{\nu'_2} = \frac{1.09 U'_\infty\,l'}{1.31\nu'_\infty} = 2.24 * 10^5\,.$$

Because of different Reynolds numbers, the model may have a laminar boundary layer, whereas the boundary layer on the prototype may be turbulent.

d) Coefficient of friction:
The total coefficient of friction is the sum of the coefficient of friction of the four profile surfaces

$$c_f = 2c_{f1} \;+\; 2c_{f2} \;=\; 2\,(c_{f1} \;+\; c_{f2})\,,$$

where for the coefficient of friction of a profile surface, the coefficient of a flat plate is used.

1) Model:
The boundary layer is laminar and under the given conditions we have $c_{f\mathrm{compressible}} = c_{f\mathrm{incompressible}}$. In the present case, we use the Blasius friction law for laminar flow along a flat plate (see F. M. (12.51))

$$c_f = 1.33\;Re^{-1/2}\;\mathrm{mit}\;Re = \frac{Ul}{\nu}$$

and obtain in region [1]:

$$c'_{f1} = 1.33\ Re_1'^{-1/2} = 0.0024 = \frac{F_D}{\varrho_1'/2\ U_1'^2 l'}$$

and in region [2]:

$$c'_{f2} = 1.33\ Re_2'^{-1/2} = 0.0027 = \frac{F_D}{\varrho_2'/2\ U_2'^2 l'} .$$

Relative to the approach flow $\varrho'_\infty/2\ U'^2_\infty l'$, we have

$$c'_{f1} = \frac{F_D}{\varrho'_\infty/2\ U'^2_\infty l'} = 0.0027$$

and

$$c'_{f2} = \frac{F_D}{\varrho'_\infty/2\ U'^2_\infty l'} = 0.0022 .$$

Thus, the total friction coefficient is

$$c'_f = 0.0098 .$$

2) Prototype:
The boundary layer is turbulent. We use the resistance law for incompressible flow, however, we introduce ϱ and the viscosity η at the reference temperature, in this case, the wall temperature. We utilize the local coefficient of friction (see F. M. (12.186)):

$$\frac{\tau_W}{\varrho/2\ U_\infty^2} = 0.024\ Re_x^{-1/7}$$

with the local Reynolds number $Re = U_\infty x/\nu$, and find after the integration over the plate length l

$$\frac{F_D}{\varrho/2\ U_\infty^2} = 0.024 \int\limits_0^l \left(\frac{U_\infty x}{\nu}\right)^{-1/7} \mathrm{d}x = 0.028 \left(\frac{U_\infty l}{\nu}\right)^{-1/7} l$$

and for the coefficient of friction

$$c_f = \frac{F_D}{\varrho/2\ U_\infty^2 l} = 0.028\ Re_l^{-1/7} . \qquad (1)$$

In incompressible flow, the density and viscosity are introduced at the temperature of the undisturbed approach flow. By applying to compressible flow, however, the above quantities must be taken at reference temperature T^*: It is

$$\eta \sim T \Rightarrow \frac{\eta^*}{\eta_\infty} = \frac{T^*}{T_\infty} ,$$

$$\varrho \sim \frac{1}{T} \Rightarrow \quad \frac{\varrho^*}{\varrho_\infty} = \frac{T_\infty}{T^*} .$$

Thus, we get

$$\varrho^* = \frac{T_\infty}{T^*}\varrho_\infty \quad \text{and} \quad \nu^* = \frac{\eta^*}{\varrho^*} = \left(\frac{T^*}{T_\infty}\right)^2 \nu_\infty .$$

In equation (1), the Reynolds number is based on ν^*:

$$\frac{F_D}{\varrho^*/2\ U_\infty^2 l} = 0.028 \ \left(\frac{U_\infty l}{\nu^*}\right)^{-1/7} ,$$

which furnishes with the Reynolds number of the approach flow

$$\frac{F_D}{\varrho_\infty/2\ U_\infty^2 l} = 0.028 \ \left(\frac{U_\infty l}{\nu_\infty}\right)^{-1/7} \left(\frac{T_\infty}{T^*}\right)^{5/7} .$$

This is the friction law for the turbulent boundary layer in compressible flow. The reference temperature (in this case, the wall temperature) is calculated in region [1] with $M_1 = 1.64$ as

$$T_{W1} = T_1(1 + r\frac{\gamma - 1}{2}M_1^2) = 1.47\ T_1$$

and in region [2] with $M_2 = 2.37$ as

$$T_{W2} = T_2(1 + r\frac{\gamma - 1}{2}M_2^2) = 1.99\ T_2 .$$

The friction coefficient in [1] is then

$$c_{f1} = \frac{F_D}{\varrho_1/2\ U_1^2 l} = 0.028 \ \left(\frac{U_1 l}{\nu_1}\right)^{-1/7} \left(\frac{T_1}{T_{W1}}\right)^{5/7} = 0.00154$$

and in [2]

$$c_{f2} = \frac{F_D}{\varrho_2/2\ U_2^2 l} = 0.028 \ \left(\frac{U_2 l}{\nu_2}\right)^{-1/7} \left(\frac{T_2}{T_{W2}}\right)^{5/7} = 0.00129 .$$

Based on $\varrho_\infty/2\ U_\infty^2 l$ we get

$$c_{f1} = \frac{F_D}{\varrho_\infty/2\ U_\infty^2 l} = 0.00175$$

and

$$c_{f2} = \frac{F_D}{\varrho_\infty/2\ U_\infty^2 l} = 0.00099 ,$$

Solution

a) In general $x_i\,y_j \neq x_j\,y_i$, from which follows the assertion by summing.
b) Relabeling the dummy indices and using $x_i\,x_j = x_j\,x_i$ we obtain

$$\begin{aligned} b_{ij}\,x_i\,x_j &= b_{ji}\,x_j\,x_i \\ &= b_{ji}\,x_i\,x_j \;. \end{aligned}$$

c) Again relabeling the dummy indices $x_i\,x_j = x_j\,x_i$ yields

$$\begin{aligned} (b_{ij}+b_{ji})\,x_i\,y_j - 2\,b_{ji}\,x_i\,y_j &= \\ b_{ij}\,x_i\,y_j - b_{ji}\,x_i\,y_j &= \\ b_{ij}\,x_i\,y_j - b_{ij}\,x_j\,y_i &= \\ b_{ij}\,x_i\,y_j - b_{ij}\,y_i\,x_j &\neq 0 \qquad \text{by a)}\;. \end{aligned}$$

d) follows from b)
e)

$$\begin{aligned} k=1 &: \quad \epsilon_{231}\,\tau_{23} + \epsilon_{321}\,\tau_{32} = \tau_{23} - \tau_{32} = 0 \\ k=2 &: \quad \epsilon_{312}\,\tau_{31} + \epsilon_{132}\,\tau_{13} = \tau_{31} - \tau_{13} = 0 \\ k=3 &: \quad \epsilon_{123}\,\tau_{12} + \epsilon_{213}\,\tau_{21} = \tau_{12} - \tau_{21} = 0 \\ &\Rightarrow \tau_{ij} = \tau_{ji} \end{aligned}$$

Problem A-3

δ_{ij} is the Kronecker-symbol, ϵ_{ijk} the ϵ-tensor. Find the numerical values of the following expressions $(i,j,k = 1,2,3)$:
a) $\delta_{ii} = \ldots$, $\delta_{ij}\,\delta_{ji} = \ldots$, $\delta_{ij}\,\delta_{ik}\,\delta_{jk} = \ldots$, $\epsilon_{ijk}\,\epsilon_{ijk} = \ldots$
b) Simplify the expression

$$a_i = \delta_{jl}\,\delta_{km}\,\epsilon_{ilm}\,b_j\,c_k \;.$$

Give the expression in symbolic notation and write out the components.

Solution

a)

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3\,,$$

$$\delta_{ij}\delta_{ji} = \delta_{ii} = 3\,, \qquad \delta_{ij}\delta_{ik}\delta_{jk} = \delta_{ij}\delta_{ji} = \delta_{ii} = 3\,,$$

$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{ijk} &= \epsilon_{123}\,\epsilon_{123} + \epsilon_{231}\,\epsilon_{231} + \epsilon_{312}\,\epsilon_{312} + \\ &\quad + \epsilon_{132}\,\epsilon_{132} + \epsilon_{213}\,\epsilon_{213} + \epsilon_{321}\,\epsilon_{321} \\ &= 6\,. \end{aligned}$$

b)

$$\begin{aligned} a_i &= \delta_{jl}\delta_{km}\epsilon_{ilm} b_j c_k \\ &= \epsilon_{ijk} b_j c_k\,. \end{aligned}$$

We are dealing with the cross product: $\vec{a} = \vec{b} \times \vec{c}$.

$$\begin{aligned} i=1\,: \qquad a_1 &= \epsilon_{123}\, b_2c_3 + \epsilon_{132}\, b_3c_2 \\ a_1 &= b_2c_3 - b_3c_2 \\ i=2\,: \qquad a_2 &= \epsilon_{231}\, b_3c_1 + \epsilon_{213}\, b_1c_3 \\ a_2 &= b_3c_1 - b_1c_3 \\ i=3\,: \qquad a_3 &= \epsilon_{312}\, b_1c_2 + \epsilon_{321}\, b_2c_1 \\ a_3 &= b_1c_2 - b_2c_1 \end{aligned}$$

Problem A-4

Using

$$\epsilon_{pqs}\,\epsilon_{mnr} = \det \begin{bmatrix} \delta_{mp} & \delta_{mq} & \delta_{ms} \\ \delta_{np} & \delta_{nq} & \delta_{ns} \\ \delta_{rp} & \delta_{rq} & \delta_{rs} \end{bmatrix}.$$

Prove the following identities

a) $\epsilon_{pqs}\,\epsilon_{snr} = \delta_{pn}\,\delta_{qr} - \delta_{pr}\,\delta_{qn}$

b) $\epsilon_{pqs}\,\epsilon_{sqr} = -2\,\delta_{pr}$

Solution

a) We expand the determinant using the first row

$$\epsilon_{pqs}\,\epsilon_{mnr} = \delta_{mp}\,(\delta_{nq}\delta_{rs} - \delta_{ns}\delta_{rq}) - \delta_{mq}\,(\delta_{np}\delta_{rs} - \delta_{ns}\delta_{rp}) + \delta_{ms}\,(\delta_{np}\delta_{rq} - \delta_{nq}\delta_{rp})$$

and put $m = s$:

$$\epsilon_{pqs}\,\epsilon_{snr} = \delta_{sp}\,(\delta_{nq}\delta_{rs} - \delta_{ns}\delta_{rq}) - \delta_{sq}\,(\delta_{np}\delta_{rs} - \delta_{ns}\delta_{rp}) + \delta_{ss}\,(\delta_{np}\delta_{rq} - \delta_{nq}\delta_{rp})$$

$$= \delta_{nq}\,\delta_{rp} - \delta_{np}\,\delta_{rq} - \delta_{np}\,\delta_{rq} + \delta_{nq}\,\delta_{rp} + 3\,\delta_{np}\,\delta_{rq} - 3\,\delta_{nq}\,\delta_{rp}\,,$$

$$\epsilon_{pqs}\,\epsilon_{snr} = \delta_{np}\,\delta_{rq} - \delta_{nq}\,\delta_{rp} = \delta_{pn}\,\delta_{qr} - \delta_{pr}\,\delta_{qn}\,.$$

b) In the result from a) we put $n = q$:

$$\epsilon_{pqs}\,\epsilon_{sqr} = \delta_{pq}\,\delta_{qr} - \delta_{pr}\,\delta_{qq} = \delta_{pr} - 3\,\delta_{pr} = -2\,\delta_{pr}\,.$$

Problem A-5

Prove in index notation the following identities:

a) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})\,(\vec{d} \cdot \vec{b}) - (\vec{a} \cdot \vec{d})\,(\vec{b} \cdot \vec{c})$

b) $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

Solution

a)

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) \mathrel{\hat{=}} \epsilon_{ijk}\,a_i\,b_j\,\epsilon_{lmk}\,c_l\,d_m\,.$$

From $\epsilon_{ijk}\,\epsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$ follows

$$\begin{aligned}\epsilon_{ijk}\,\epsilon_{lmk}\,a_i\,b_j\,c_l\,d_m &= \delta_{il}\,\delta_{jm}\,a_i\,b_j\,c_l\,d_m - \delta_{im}\,\delta_{jl}\,a_i\,b_j\,c_l\,d_m \\ &= a_l\,c_l\,b_m\,d_m - a_m\,d_m\,b_l\,c_l \\ &\mathrel{\hat{=}} (\vec{a} \cdot \vec{c})\,(\vec{d} \cdot \vec{b}) - (\vec{a} \cdot \vec{d})\,(\vec{b} \cdot \vec{c})\,.\end{aligned}$$

b)

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) \mathrel{\hat{=}} \epsilon_{lkn}\,a_l\,\epsilon_{ijk}\,b_i\,c_j + \epsilon_{lkn}\,b_l\,\epsilon_{ijk}\,c_i\,a_j + \epsilon_{lkn}\,c_l\,\epsilon_{ijk}\,a_i\,b_j\,.$$

With $\epsilon_{lkn}\,\epsilon_{ijk} = -\epsilon_{lnk}\,\epsilon_{kij} = -(\delta_{li}\,\delta_{nj} - \delta_{lj}\,\delta_{ni})$, we obtain

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) \;\hat{=}\; -\delta_{li}\,\delta_{nj}\,a_l\,b_i\,c_j+\delta_{lj}\,\delta_{ni}\,a_l\,b_i\,c_j+$$
$$-\delta_{li}\,\delta_{nj}\,b_l\,c_i\,a_j+\delta_{lj}\,\delta_{ni}\,b_l\,c_i\,a_j+$$
$$-\delta_{li}\,\delta_{nj}\,c_l\,a_i\,b_j+\delta_{lj}\,\delta_{ni}\,c_l\,a_i\,b_j$$
$$= -a_i\,b_i\,c_n+a_l\,b_l\,c_n+$$
$$+a_l\,c_l\,b_n-a_i\,c_i\,b_n+$$
$$-b_i\,c_i\,a_n+b_l\,c_l\,a_n \;=\; 0\,.$$

Problem A-6

Show that $\omega_{ik}=\epsilon_{ikm}\,x_m$ is an antisymmetric tensor of second order.

Solution

For ω_{ik} to be a second order tensor the equation $\omega'_{rs}=a_{ir}\,a_{ks}\,\omega_{ik}$ must be satisfied where a_{ij} are the elements of the rotation matrix.
Since x_m is a first order tensor and ϵ_{ikm} a third order tensor we have

$$x'_t=a_{mt}\,x_m \quad , \quad \epsilon'_{rst}=a_{ir}\,a_{ks}\,a_{mt}\,\epsilon_{ikm}\,.$$

Thus $\epsilon'_{rst}\,x'_t=a_{ir}\,a_{ks}\,a_{mt}\,a_{nt}\,\epsilon_{ikm}\,x_n$ and since $a_{mt}\,a_{nt}=\delta_{mn}$ we find

$$a_{ir}\,a_{ks}\,\epsilon_{ikm}\,x_m \;=\; a_{ir}\,a_{ks}\,\omega_{ik} \;=\; \omega'_{rs}\,,$$

hence ω_{ik} obeys the transformation rule for second order tensors. If ω_{ik} is antisymmetric then $\omega_{ik}=-\omega_{ki}$. Since $\omega_{ik}=\epsilon_{ikm}\,x_m$, $\omega_{ki}=\epsilon_{kim}\,x_m$ and $\epsilon_{kim}=-\epsilon_{ikm}$ we get

$$\begin{aligned}\omega_{ik} &= \epsilon_{ikm}\,x_m\\ &= -\epsilon_{kim}\,x_m \;=\; -\omega_{ki}\,,\end{aligned}$$

i. e. ω_{ik} is antisymmetric.

Problem A-7

Show that the i-th component of

$$\nabla\times(\nabla\times\vec{a}) \quad \text{equals} \quad \frac{\partial^2 a_j}{\partial x_i\partial x_j}-\frac{\partial^2 a_i}{\partial x_j\partial x_j}$$

b) We wish to show that the gradient transforms as a first order tensor

$$\left(\frac{\partial \lambda}{\partial x_l}\right)' = a_{pl}\,\frac{\partial \lambda}{\partial x_p}\,.$$

Using

$$\begin{aligned} x'_m &= a_{tm}\,x_t\,, \\ A'_{ml} &= a_{rm}\,a_{sl}\,A_{rs}\,, \\ A'_{lm} &= a_{pl}\,a_{qm}\,A_{pq} \end{aligned}$$

we transform the right side of (1) into the new coordinate system:

$$\begin{aligned} (A'_{lm} + A'_{ml})\,x'_m &= (a_{pl}\,a_{qm}\,A_{pq} + a_{rm}\,a_{sl}\,A_{rs})\,a_{tm}\,x_t \\ &= (a_{pl}\,a_{qm}\,a_{tm}\,A_{pq} + a_{rm}\,a_{sl}\,a_{tm}\,A_{rs})\,x_t \\ &= (a_{pl}\,\delta_{qt}\,A_{pq} + a_{sl}\,\delta_{rt}\,A_{rs})\,x_t \\ &= (a_{pl}\,A_{pt} + a_{sl}\,A_{ts})\,x_t \\ &= a_{pl}\,(A_{pt} + A_{tp})\,x_t = a_{pl}\,\frac{\partial \lambda}{\partial x_p}\,, \end{aligned}$$

i. e.

$$\left(\frac{\partial \lambda}{\partial x_l}\right)' = a_{pl}\,\frac{\partial \lambda}{\partial x_p}\,.$$

Problem A-9

Show, using cartesian coordinates that the identity

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{x}) = -\frac{1}{2}\nabla(\vec{\Omega} \times \vec{x})^2$$

holds (see F. M. (4.77)). (To prove this it is not necessary to eliminate the ϵ-tensor. But note: $\vec{\Omega} \neq \vec{\Omega}(\vec{x})$.)

Solution

$$\vec{I} := \vec{\Omega} \times (\vec{\Omega} \times \vec{x}) \quad , \quad \vec{II} := -\frac{1}{2}\nabla(\vec{\Omega} \times \vec{x})^2$$

The k-th component of $(\vec{\Omega} \times \vec{x})$ is $\epsilon_{ijk}\,\Omega_i\,x_j$, and so the r-th component of $\vec{I}$ and $\vec{II}$:

$$I_r = \epsilon_{mkr}\,\Omega_m\,\epsilon_{ijk}\,\Omega_i\,x_j\,,$$

$$II_r = -\frac{1}{2}\frac{\partial}{\partial x_r}\epsilon_{ijk}\,\Omega_i\,x_j\,\epsilon_{mlk}\,\Omega_m\,x_l\ .$$

Since $\vec{\Omega} \neq \vec{\Omega}(\vec{x})$

$$\begin{aligned} II_r &= -\frac{1}{2}\epsilon_{ijk}\,\epsilon_{mlk}\,\Omega_i\Omega_m\,\frac{\partial}{\partial x_r}(x_j\,x_l) \\ &= -\frac{1}{2}\left\{\epsilon_{ijk}\,\epsilon_{mlk}\,\Omega_i\,\Omega_m\,x_j\delta_{rl} + \epsilon_{ijk}\,\epsilon_{mlk}\,\Omega_i\,\Omega_m\,x_l\delta_{rj}\right\} \\ &= -\frac{1}{2}\left\{\epsilon_{ijk}\,\epsilon_{mrk}\,\Omega_i\,\Omega_m\,x_j + \epsilon_{irk}\,\epsilon_{mlk}\,\Omega_i\,\Omega_m\,x_l\right\} \\ &= -\epsilon_{ijk}\,\epsilon_{mrk}\,\Omega_i\,\Omega_m\,x_j\ . \end{aligned}$$

By $\epsilon_{mrk} = -\epsilon_{mkr}$, we have $II_r = I_r = \epsilon_{ijk}\,\epsilon_{mkr}\,\Omega_i\,\Omega_m\,x_j$.

Problem A-10

For the scalar field $\Phi(r)$, with $r = \sqrt{x_i\,x_i}$ show that

a) $\dfrac{\partial^2\Phi}{\partial x_i \partial x_j} = \dfrac{\Phi'}{r}\left(\delta_{ij} - \dfrac{x_i\,x_j}{r^2}\right) + \dfrac{x_i\,x_j}{r^2}\,\Phi''$, where $\Phi'(r) = \dfrac{\mathrm{d}\Phi}{\mathrm{d}r}$, and further that

b) $\dfrac{\partial^2\Phi}{\partial x_i \partial x_j}$ is a tensor of second order whose contraction is given by

c) $\dfrac{\partial^2\Phi}{\partial x_i \partial x_i} = \dfrac{1}{r}\,\dfrac{\mathrm{d}^2(r\,\Phi)}{\mathrm{d}r^2}$.

Solution

a)
$$\frac{\partial\Phi}{\partial x_i} = \frac{\partial\Phi}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\mathrm{d}\Phi}{\mathrm{d}r}\frac{\partial r}{\partial x_i}\,,$$

where in the last term the partial derivative has been replaced by a total derivative since $\Phi = \Phi(r)$ only. Using

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}(\sqrt{x_j\,x_j}) = \frac{1}{2}\,(x_j\,x_j)^{-1/2}\,2\,x_i = \frac{x_i}{r}$$

we have
$$\frac{\partial\Phi}{\partial x_i} = \frac{\mathrm{d}\Phi}{\mathrm{d}r}\frac{x_i}{r} = \Phi'\frac{x_i}{r}\ .$$

Differentiation with respect to x_j yields

$$\frac{\partial^2\Phi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\Phi'\,\frac{x_i}{r}\right)$$

$$
\begin{aligned}
&= \Phi'' \frac{\partial r}{\partial x_j} \frac{x_i}{r} + \Phi' \frac{1}{r} \frac{\partial x_i}{\partial x_j} + \Phi' x_i \frac{\partial}{\partial r}\left(\frac{1}{r}\right) \frac{\partial r}{\partial x_j} \\
&= \Phi'' \frac{x_i x_j}{r^2} + \Phi' \frac{\delta_{ij}}{r} - \Phi' \frac{x_i x_j}{r^3} , \\
\frac{\partial^2 \Phi}{\partial x_i \partial x_j} &= \frac{\Phi'}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) + \Phi'' \frac{x_i x_j}{r^2} .
\end{aligned}
$$

b) We wish to show that $\dfrac{\partial^2 \Phi}{\partial x_k \partial x_l}$ transforms like a second order tensor

$$
\frac{\partial^2 \Phi}{\partial x_i' \partial x_j'} = a_{ki}\, a_{lj} \frac{\partial^2 \Phi}{\partial x_k \partial x_l} .
$$

With the orthogonal transformation $x_k = a_{ki}\, x_i'$ we find

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x_i'} &= \frac{\partial \Phi(x_k(x_i'))}{\partial x_i'} = \frac{\partial \Phi}{\partial x_k} \frac{\partial x_k}{\partial x_i'} = \frac{\partial \Phi}{\partial x_k} a_{ki} , \\
\frac{\partial^2 \Phi}{\partial x_i' \partial x_j'} &= \frac{\partial}{\partial x_j'} \left(\frac{\partial \Phi}{\partial x_k} a_{ki}\right) = \frac{\partial}{\partial x_l} \left(\frac{\partial \Phi(x_l(x_j'))}{\partial x_k}\right) \frac{\partial x_l}{\partial x_j'} a_{ki} \\
&= a_{ki}\, a_{lj} \frac{\partial^2 \Phi}{\partial x_k \partial x_l} .
\end{aligned}
$$

c) Laplace operator on $\Phi(r)$:

$$
\begin{aligned}
\frac{\partial^2 \Phi}{\partial x_i \partial x_i} &= \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \delta_{ij} \\
&= \frac{\Phi'}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \delta_{ij} + \Phi'' \frac{x_i x_j}{r^2} \delta_{ij} \\
&= \frac{\Phi'}{r}\left(\delta_{ii} - \frac{x_i x_i}{r^2}\right) + \Phi'' \frac{x_i x_i}{r^2} \\
&= \frac{\Phi'}{r}(3-1) + \Phi'' \\
&= \frac{1}{r}(2\,\Phi' + r\,\Phi'') \\
&= \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r}(\Phi + r\,\Phi') ,
\end{aligned}
$$

$$= \quad \Phi \operatorname{curl} \vec{u} \quad + \quad \operatorname{grad} \Phi \times \vec{u} \, .$$

b) Equation (1) using the Nabla operator

$$\nabla = \frac{\partial}{\partial x_1} \vec{e}_1 + \frac{\partial}{\partial x_2} \vec{e}_2 + \frac{\partial}{\partial x_3} \vec{e}_3$$

reads

$$\nabla \times (\Phi \vec{u}) = \Phi \nabla \times \vec{u} + \nabla \Phi \times \vec{u} \, .$$

c) The k-th component of (1) is

$$\epsilon_{ijk} \frac{\partial(\Phi u_j)}{\partial x_i} = \Phi \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} + \epsilon_{ijk} \frac{\partial \Phi}{\partial x_i} u_j$$

$$= \Phi \left\{ \epsilon_{12k} \frac{\partial u_2}{\partial x_1} + \epsilon_{13k} \frac{\partial u_3}{\partial x_1} + \epsilon_{21k} \frac{\partial u_1}{\partial x_2} + \epsilon_{23k} \frac{\partial u_3}{\partial x_2} + \epsilon_{31k} \frac{\partial u_1}{\partial x_3} + \epsilon_{32k} \frac{\partial u_2}{\partial x_3} \right\} +$$

$$+ \left\{ \epsilon_{12k} \frac{\partial \Phi}{\partial x_1} u_2 + \epsilon_{13k} \frac{\partial \Phi}{\partial x_1} u_3 + \epsilon_{21k} \frac{\partial \Phi}{\partial x_2} u_1 + \right.$$

$$\left. + \epsilon_{23k} \frac{\partial \Phi}{\partial x_2} u_3 + \epsilon_{31k} \frac{\partial \Phi}{\partial x_3} u_1 + \epsilon_{32k} \frac{\partial \Phi}{\partial x_3} u_2 \right\} \, .$$

For $k = 1, 2, 3$ we get:

$$k = 1: \quad \epsilon_{ij1} \frac{\partial(\Phi u_j)}{\partial x_i} = \Phi \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \left(\frac{\partial \Phi}{\partial x_2} u_3 - \frac{\partial \Phi}{\partial x_3} u_2 \right) \, .$$

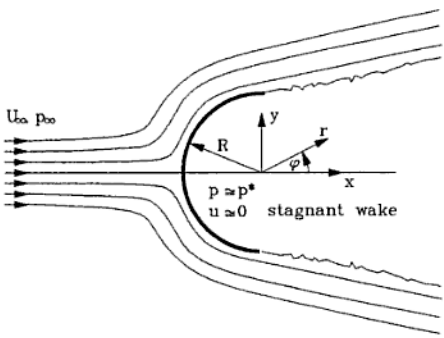
$$k = 2: \quad \epsilon_{ij2} \frac{\partial(\Phi u_j)}{\partial x_i} = \Phi \left(-\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) + \left(-\frac{\partial \Phi}{\partial x_1} u_3 + \frac{\partial \Phi}{\partial x_3} u_1 \right) \, .$$

$$k = 3: \quad \epsilon_{ij3} \frac{\partial(\Phi u_j)}{\partial x_i} = \Phi \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) + \left(\frac{\partial \Phi}{\partial x_1} u_2 - \frac{\partial \Phi}{\partial x_2} u_1 \right) \, .$$

Problem B-2 Drag of a half cylinder shell

A half cylinder shell is exposed to a uniform, plane, incompressible flow. Volume forces need not be considered. Ahead of the body the flow may be taken as a potential flow:

$$\Phi(r,\varphi) = U_\infty \, \cos\varphi \, \left(r + \frac{R^2}{r}\right).$$



The flow separates from the edges of the shell and a stagnant region is formed behind the shell where the velocity is nearly zero and the pressure $p = p^*$ nearly constant. Far in front of the body the flow is uniform, the velocity is U_∞ and the pressure p_∞. From experiments the value of the drag coefficient

$$c_D = \frac{F_x}{\varrho U_\infty^2 \, R}$$

was found as $c_D = 1.2$.

a) Compute the velocity $\vec{u}(r,\varphi)$ ahead of the body.
b) Convince yourself that flow is irrotational.
c) Find the pressure distribution $p(R,\varphi)$ from Bernoulli's equation.
d) Find the stagnation pressure p_S.
e) Compute the force F_x per unit depth exerted on the shell by direct integration of the stress vector. Assume that p^* in the stagnant flow region is equal to pressure at the edge of the shell $p(R, \frac{\pi}{2})$. Compute now the drag coefficient.
f) Compare with the measured coefficient and find the pressure p^* from the measured value.

Given: R, ϱ, U_∞, p_∞, $c_D = 1.2$

Solution

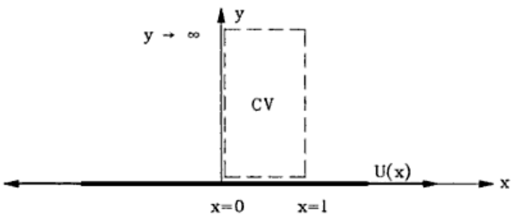
a) $\vec{u} = U_\infty \, \cos\varphi \left(1 - \dfrac{R^2}{r^2}\right) \vec{e}_r - U_\infty \, \sin\varphi \left(1 + \dfrac{R^2}{r^2}\right) \vec{e}_\varphi$

Solution

a) $\dot{m}_{CD} = \dfrac{1}{2}\,\varrho\, U_0\, \delta$

b) $(F_{\rightarrow\text{awning}})_{x_1} = \varrho\, U_0^2 \dfrac{\delta}{8}$

Problem B-4 Stretching of a foil



A foil of infinite extension in x- and z-direction is stretched in x-direction. The velocity is $U(x) = a\, x$. Because of the no-slip condition the Newtonian fluid (ν = const, ϱ = const) above the foil is set in motion and a plane, steady flow is established. The x-component of the velocity field $\vec{u} = u\,\vec{e}_x + v\,\vec{e}_y$ is given:

$$u(x,y) = a\; x\; \mathrm{e}^{-y\sqrt{a/\nu}}\;.$$

Volume forces are neglected and the pressure p depends only on y.

a) Find from the continuity equation in differential form the velocity component $v(x,y)$ in y-direction.
b) What is the limiting behavior of u and v as $y \to \infty$?
c) Find the x-component of the force per unit depth exerted by the foil on the fluid in the domain $0 \le x \le l$. Compute the force by direct integration of the stress vector.
d) Now find the same force using the balance of momentum for the shown control volume.

Given: ϱ, ν, l, $a > 0$

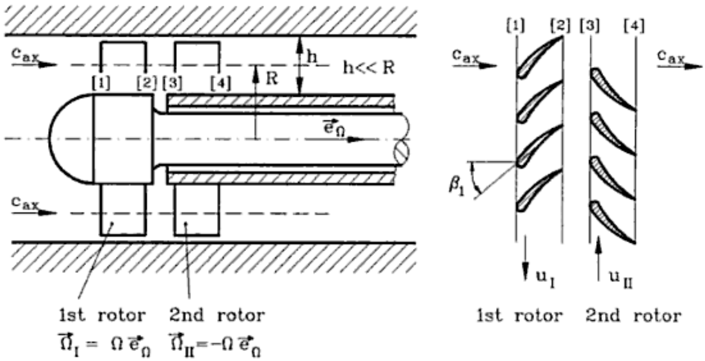
Solution

a) $v(x,y) = v(y) = -\sqrt{a\,\nu}\,\left(1 - \mathrm{e}^{-y\,\sqrt{a/\nu}}\right)$

b) $\lim_{y\to\infty} u(x,y) = 0\;,\quad \lim_{y\to\infty} v(x,y) = -\sqrt{a\,\nu} = \text{const}$

c) $F_x = \varrho\, a\,\sqrt{a\,\nu}\, l^2/2$

Problem B-5 Single stage, axial blower



In this blower the stator is replaced by a rotor. The rotors spin in opposite direction ($\vec{\Omega}_I = \Omega\,\vec{e}_\Omega$, $\vec{\Omega}_{II} = -\Omega\,\vec{e}_\Omega$) at constant speed. The approach flow to the first rotor is purely axial. The density ϱ is constant.

a) Find the angle β_1 of the approach flow $\vec{w}_1$ to the first rotor.
b) Determine the circumferential component of the absolute velocity c_{u2} at the exit of the rotor if the relative velocity $\vec{w}_1$ has been turned by 10° in the first rotor blading.
c) Compute the power P_I delivered to the first rotor.
d) Find the power P_{II} delivered to the second rotor if the absolute velocity leaving the second rotor is purely axial.
e) By how much has then the relative velocity $\vec{w}_1$ been turned in the second rotor?

Given: $\Omega = 324.6$ 1/s, $\dot{m} = 30$ kg/s, $c_{ax} = 100$ m/s, $R = 0.308$ m

Solution

a) $\beta_1 = 45°$

b) $c_{u2} = 30$ m/s

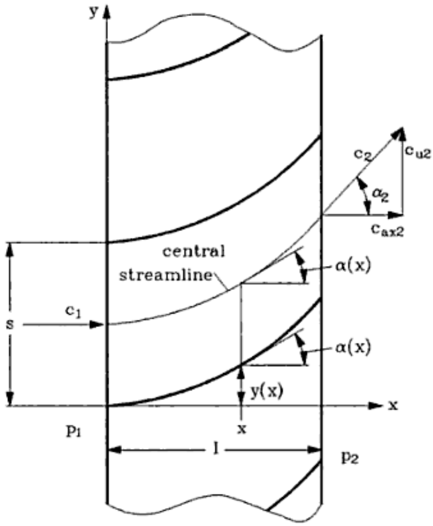
c) $P_I = 90$ kW

d) $P_{II} = P_I$

e) $\Delta\beta_{43} = \beta_3 - \beta_4 = 7.4°$

Problem B-6 Blade profile for given pressure distribution

The blade profile for a turbine stator is to be designed in such a way that the pressure along the central streamline decreases linearly with x from p_1 at the cascade entrance to p_2 at the exit. The blade spacing is very small so that the direction of the central streamline coincides with the blade direction. The plane, inviscid approach flow of constant density ϱ is in axial direction. Volume forces need not be considered.



a) Give the function $p(x)$.
b) Determine the axial component c_{ax} of the flow velocity as a function of x.
c) Give the distribution $c_u(x)$.
d) Find the distribution $\tan\alpha(x)$ of the blade profile.
e) Now compute the profile $y(x)$.
f) Give the force $\vec{F}$ per unit depth on one blade.

Given: c_1, p_1, p_2, s, l, ϱ

Solution

a) $p(x) = p_1 - \Delta p\, x/l\,,\qquad$ with $\Delta p = p_1 - p_2$

b) $c_{ax}(x) = c_1 = \text{const}$

c) $c_u(x) = \sqrt{2\,\Delta\,p/\varrho}\,(x/l)^{1/2}$

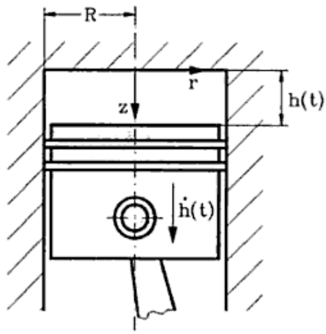
d) $\tan\alpha(x) = A\,(x/l)^{1/2}\,,\qquad$ with $A = \sqrt{2\Delta\,p/(\varrho\,c_1^2)}$

e) $y/l = 2/3\,A\,(x/l)^{3/2}$

f) $F_x = t\,(p_1 - p_2)\,, \quad F_y = -\varrho\, c_1\, t\sqrt{2\,\Delta p/\varrho}$

Problem B-7 Combustion chamber of a piston engine

We wish to determine approximately the flow field in the combustion chamber of a piston engine. To this end we assume that the flow is inviscid and the density homogeneous ($\varrho = \varrho(t)$). We take the velocity field in the form



$$\vec{u}(r,z,t) = \frac{A(t)}{r}\vec{e}_\varphi + u_z(z,t)\vec{e}_z$$

where the first term accounts for the swirl generated during intake.
The geometry and the height $h(t)$ of the combustion chamber as well as $\dot{h} = \mathrm{d}h/\mathrm{d}t$ are given. Volume forces are not present.

a) Give the density $\varrho(t)$, where m is the total mass in the combustion chamber.
b) Compute the density change $\mathrm{D}\varrho/\mathrm{D}t$ of a fluid particle.
c) Now find $u_z(z,t)$ from continuity equation and the fact that the walls of the chamber are impermeable.
d) Determine the material change of the z-component of the angular momentum.
e) Find the dependence of angular momentum on $A(t)$ by integrating over the domain occupied by the gas.
f) Determine from d) and e) $A(t)$ with the initial condition

$$A(t{=}0) = \frac{\Gamma}{2\pi}\,.$$

Given: m, R, $h(t)$, $\dot{h}(t)$, Γ

Solution

a) $\varrho(t) = m/(\pi R^2\, h(t))$

b) $\mathrm{D}\varrho/\mathrm{D}t = -\varrho\,\dot{h}/h$

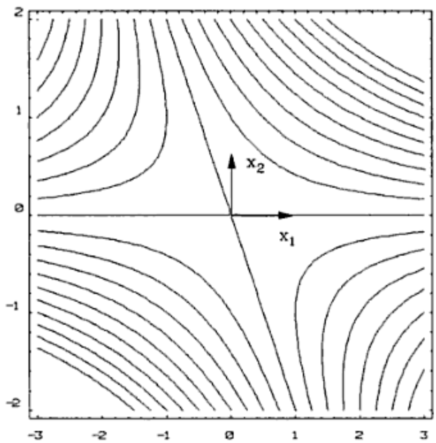
c) $u_z = \dot{h}/h\; z$

d) $\frac{\mathrm{D}}{\mathrm{D}t}(D_z) = 0$

e) $D_z = A(t)\, m$

f) $A = \Gamma/(2\pi)$

Problem B-8 Two-dimensional oblique stagnation point flow



The sketch shows the streamlines of two oblique jets in Newtonian flow. The velocity field is given by:

$$u_1(x_1, x_2) = a\, x_1 \; + 2\, b\, x_2\ , \quad u_2(x_1, x_2) = -a\, x_2$$

where a, b are dimensional constants.

a) Show that the flow is incompressible.
b) Find the curl of the velocity field.
c) Determine the pressure distribution $p(x_1, x_2)$ from the Navier-Stokes equation if the pressure at the origin $p(x_1 = 0,\, x_2 = 0)$ is p_t. Volume forces have no influence.
d) Compute the components t_1 and t_2 of the stress vector $\vec{t}$ at the place $(0,\, 0)$ on a plane surface with the unit normal $\vec{n} = (0,\, 1)$.

e) Find the pathline in parameter form and then explicit $x_2 = f(x_1)$. Give the equation for the streamlines.

Given: ϱ, η, a, b, p_t

Solution

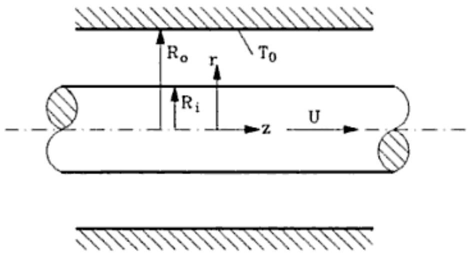
b) $\operatorname{curl}\vec{u} = -2\,b\,\vec{e}_3$

c) $p(x_1, x_2) = p_t - \varrho\,\dfrac{a^2}{2}\,(x_1^2 + x_2^2)$

d) $t_1(0,\,0) = 2\eta\,b\,,\quad t_2(0,\,0) = -p_t - 2\eta\,a$

e) $x_2 = -\dfrac{a}{b}\dfrac{x_1}{2} \pm \sqrt{\left(\dfrac{a}{b}\dfrac{x_1}{2}\right)^2 - D}\,,\quad$ with $D = -AC\dfrac{a}{b}$

Problem B-9 Generalized Hagen-Poiseuille flow



Newtonian fluid (ϱ, η, λ = const) in an infinitely long pipe (radius R_o) is dragged along by a cylinder (radius R_i) with velocity U. The flow caused by the moving cylinder is steady and rotationally symmetric. The pipe is at a constant temperature T_0, the cylinder is thermally isolated ($\vec{q}=0$), therefore $\partial T/\partial z = \partial T/\partial\varphi = 0$. The heat capacity of the incompressible fluid is c.

a) Find the velocity field of the unidirectional laminar flow ($u_z = u_z\,(r)$, $u_\varphi = u_r = 0$).

b) Compute the dissipation function Φ using the nonvanishing components of $\mathbf{E}$: $e_{rz} = e_{zr}$.

c) Using the energy equation

$$\varrho\frac{\mathrm{D}e}{\mathrm{D}t} - \frac{p}{\varrho}\frac{\mathrm{D}\varrho}{\mathrm{D}t} = \Phi + \lambda\Delta T\,,$$

find the differential equation for the temperature $T(r)$.

d) Determine the solution for the homogeneous equation.

e) Find the particular solution (Hint: Use $T_p \sim (\ln r)^2$) and then fit the solution to the boundary conditions.

f) What is the heat flux to the pipe wall?

g) What is the cylinder temperature?

Given: U, η, λ, R_i, R_o, T_0

Solution

a) $u_z(r) = U\,\dfrac{\ln(r/R_o)}{\ln(R_i/R_o)}$

b) $\Phi = \eta\,\dfrac{U^2}{r^2}\,\dfrac{1}{\ln^2(R_i/R_o)}$

c) $\dfrac{\mathrm{d}^2T}{\mathrm{d}r^2} + \dfrac{1}{r}\,\dfrac{\mathrm{d}T}{\mathrm{d}r} = -\,\dfrac{A}{r^2}\,, \quad$ with $A = \dfrac{\eta}{\lambda}\,\dfrac{U^2}{\ln^2(R_i/R_o)}$

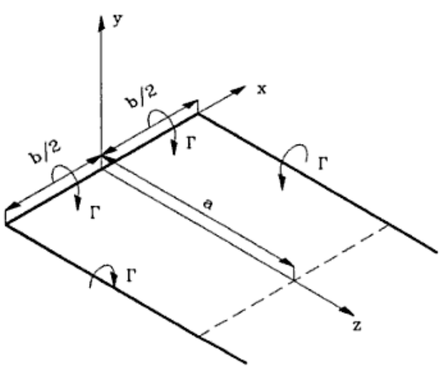
d) $T_h(r) = C_1\,\ln(r/R_o) + C_2$

e) $T_p(r) = -\dfrac{1}{2}\,A\,\ln^2\left(\dfrac{r}{R_o}\right)\,, \quad T(r) = T_0 + A\left[\ln\left(\dfrac{R_i}{R_o}\right)\,\ln\left(\dfrac{r}{R_o}\right) - \dfrac{1}{2}\,\ln^2\left(\dfrac{r}{R_o}\right)\right]$

f) $\vec{q}(r = R_o) = \dfrac{\eta\,U^2}{R_o\,\ln(R_o/R_i)}\vec{e}_r$

g) $T(r = R_i) = T_0 + \dfrac{A}{2}\,\ln^2\left(\dfrac{R_o}{R_i}\right)$

Problem B-10 Induced velocity of a horse-shoe vortex



The most simple representation of a finite wing is a horse-shoe vortex.

a) Find the downwash along the line $z = a\,, -b/a \le x \le b/a$ using Biot-Savart law.
Hint: Compute the contribution for each straight vortex filament to get the total downwash.

b) Compute the induced velocity along the z-axis and give the result for $a \to \infty$?

Given: b, Γ

Solution

a) $\vec{u} = -\left(|\vec{u}|_{(1)} + |\vec{u}|_{(2)} + |\vec{u}|_{(3)}\right)\,\vec{e}_y\,,$ with

$$|\vec{u}|_{(1)} = \frac{\Gamma}{4\pi}\,\frac{1}{b/2+x}\left(1+\frac{a}{\sqrt{a^2+(b/2+x)^2}}\right)\,,$$

$$|\vec{u}|_{(2)} = \frac{\Gamma}{4\pi}\,\frac{1}{a}\left(\frac{b/2-x}{\sqrt{a^2+(b/2-x)^2}}+\frac{b/2+x}{\sqrt{a^2+(b/2+x)^2}}\right)\,,$$

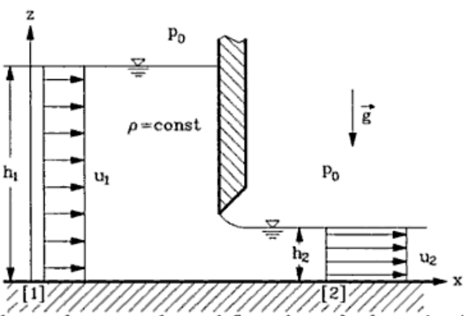
$$|\vec{u}|_{(3)} = \frac{\Gamma}{4\pi}\,\frac{1}{b/2-x}\left(1+\frac{a}{\sqrt{a^2+(b/2-x)^2}}\right)\,.$$

b)

$$\vec{u}(x=0,\,a) = -\frac{\Gamma}{4\pi}\left(\frac{4}{b}+\frac{4}{a\,b}\sqrt{a^2+(b/2)^2}\right)\vec{e}_y\,, \qquad (1)$$

$$\vec{u}(x=0, a\to\infty) = -\frac{2\Gamma}{\pi\,b}\,\vec{e}_y \qquad (2)$$

Problem B-11 Open channel flow through a weir



The sketch shows the open channel flow through the weir. Assume inviscid flow of constant density ϱ. The streamlines at stations [1] and [2] are parallel.

a) Find h_2.

b) Determine the pressure distribution $p(z)$ at [1] and [2] from the component of Euler's equation in z-direction and include the gravity volume force.

c) Does the ambient pressure p_0 make a contribution to the force in x-direction?
d) Compute the force F_x per unit depth on the weir.

Given: p_0, ϱ, h_1, u_1, u_2, g

Solution

a) $h_2 = \dfrac{u_1}{u_2}\, h_1$

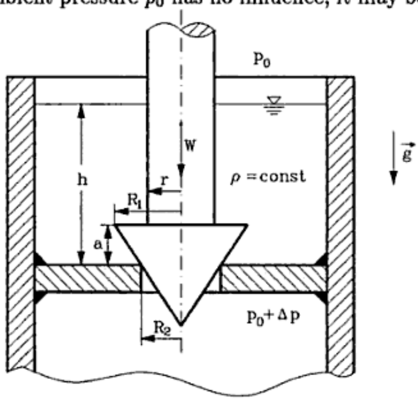
b) $p_1 = p_0 + \varrho g\,(h_1 - z)\,, \quad p_2 = p_0 + \varrho g\,(h_2 - z)$

c) No

d) $F_x = \varrho u_1^2\, h_1 - \varrho u_2^2\, h_2 + \varrho g\, h_1^2/2 - \varrho g\, h_2^2/2$

Problem B-12 Safety valve

The safety valve sketched below closes a container with interior pressure Δp above the ambient pressure p_0. To tune the opening pressure, the space above the valve has been filled with a liquid of constant density ϱ to height h. Since the ambient pressure p_0 has no influence, it may be taken as zero.



a) Find the force of the liquid onto the valve in vertical direction.
Hint: Make use of the hydrostatic lift of a suitable replacement body.
b) Give the total force resulting from the interior pressure Δp and the pressure above the valve.
c) Choose h such that the valve just opens at Δp.

Given: Δp, $p_0 = 0$, ϱ, g, r, R_1, R_2, a

Solution

a) No-slip condition: $\vec{u}(y=0) = 0$, $\vec{u}(y=h) = 0$,
i. e. $y=0 : u = v = 0$; $y=h : u = v = 0$

b) $v \equiv 0$

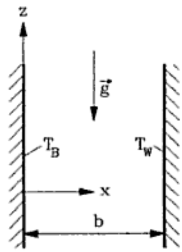
c) $\tau_{xx} = \tau_{yy} = -p$, $\tau_{xy} = \tau_{yx} = \eta \dfrac{\partial u}{\partial y}$

d) $\tau_{xy} = -K\,y + C_1$

e) $u(y) = \dfrac{1}{\eta_L} \dfrac{K\,h^2}{\alpha\,\Delta T} \left\{ \dfrac{e^{\alpha\,\Delta T\,y/h} - 1}{1 - e^{-\alpha\,\Delta T}} - \dfrac{y}{h}\,e^{\alpha\,\Delta T\,y/h} \right\}$

Problem B-17 Temperature induced flow

Water (η = const) is contained between two plane walls held at constant temperature T_B (at $x=0$) and T_W (at $x=b$). Since walls and liquid extend to infinity in y- and z-direction, the temperature distribution $T(x)$ is only dependent on x. The density of water is dependent on temperature:



$$\varrho = \overline{\varrho} + \alpha(\overline{T} - T(x)) ,$$

where $\overline{\varrho}$ is the density at $\overline{T} = 1/2(T_B + T_W)$.

a) Prove that as a consequence of the temperature gradient, hydrostatic equilibrium is not possible and that the water is set in motion.
Hint: The flow is steady and plane.

b) Show on the basis of the continuity equation that the x-component of the velocity vanishes identically.

c) Find the temperature $T(x)$ in the water. (The dissipation is much smaller than the heat flux due to the temperature gradient and may be neglected.)

d) Prove that the pressure is only a function of x.

e) Prove then that $\partial p/\partial z$ is a constant and find from this the velocity distribution $w(x)$.

Given: η, $\overline{\varrho}$, T_B, T_W, α, b, g, K

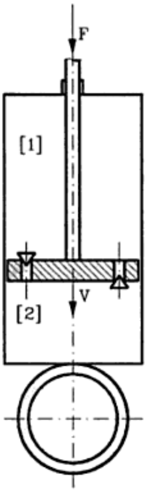
Solution

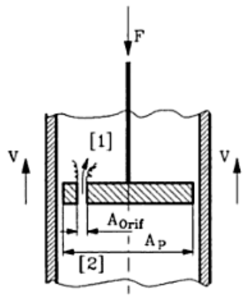
a) $\nabla\,\varrho \times \vec{k} = \partial\varrho/\partial x\,g\,\vec{e}_y \neq 0$

c) $T(x) = T_B + (T_W - T_B)\,x/b$

e) $w(x) = \frac{(\overline{\varrho} g + k)\, b^2}{2\eta}\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right] + \frac{\alpha\, g(T_W - T_B)\, b^2}{\eta}\left[\frac{1}{4}\left(\frac{x}{b}\right)^2 - \frac{1}{6}\left(\frac{x}{b}\right)^3 - \frac{1}{12}\frac{x}{b}\right]$

Problem B-18 Shock absorber





The force F causes the piston to move downwards. The liquid (ϱ = const) in the cylinder flows from the lower chamber to the upper chamber through the orifice on the left side of the piston. Underneath the piston at position [2], the pressure is p_2 and above the piston at position [1] it is p_1. The velocities in the two chambers, i. e. u_1 and u_2, are small compared to the velocities in the orifice and may be neglected. The cross-section of the piston rod may be neglected. The flow through area A_{Orif} of an orifice depends on the pressure difference $\Delta p = p_2 - p_1$, giving a progressive damping characteristic. The relationship is

$$A_{Orif} = A_0 \left(\frac{\Delta p}{\hat{p}}\right)^m, \qquad \text{where} \quad m > 0$$

is an adjustable parameter. Only exit losses need to be considered. Friction at the walls is neglected.

a) Give the force in terms of Δp. Use the balance of momentum in a frame fixed to the piston.
b) Give the velocity w_{Orif} in the piston fixed frame.
c) Determine the pressure difference Δp from Bernoulli's equation.
d) Find the piston velocity V in terms of the force F. For what value of the parameter m is a linear damping $V(F)$ realized?

Given: A_0, A_P, $\hat{p}$, ϱ, F

This collection of over 200 detailed worked exercises adds to and complements the textbook Fluid Mechanics by the same author, and illustrates the teaching material through examples. In the exercises the fundamental concepts of Fluid Mechanics are applied to obtaining the solution of diverse concrete problems, and in doing this the student's skill in the mathematical modeling of practical problems is developed.
In addition, 30 challenging questions without detailed solutions have been included, and while lecturers will find these questions suitable for examinations and tests, the student himself can use them to check his understanding of the subject.

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