Integration

A function F is called an antiderivative of a function f if $F'(x) = f(x)$ for every x in the domain of f.

If $F(x)$ is an antiderivative of $f(x)$, then $F(x)+c$ is also an antiderivative of $f(x)$. For if $\frac{d}{dx}F(x) = f(x)$ then it is also true that $\frac{d}{dx}(F(x)+c) = \frac{d}{dx}F(x) + \frac{d}{dx}c$

$$= f(x) + 0$$

$$= f(x)$$

Eg: $F(x) = x^3$, $G(x) = x^3 - 5$, $H(x) = x^3 + 0.3$ are all antiderivatives of $3x^2$

because

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^3-5) = \frac{d}{dx}(x^3+0.3) = 3x^2$$

The process of antidifferentiation determines a family of functions each differing from the other by a constant.

The antidifferentiation process is referred to as <u>integration</u> and is denoted by the symbol $\int$ called an integral sign. The symbol $\int f(x)\,dx$ is the indefinite integral of $f(x)$.

If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$ where $f(x)$ is called the integrand and c the constant of integration.

Basic Integration Rules

1. $\int dx = x + c$

2. $\int kf(x)\,dx = k\int f(x)\,dx$

3. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

4. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$, if $n \neq -1$

Eg 1.
$$\begin{aligned}\int (3x-7)\,dx &= \int 3x\,dx - \int 7\,dx \\ &= 3\int x\,dx - 7\int dx \\ &= 3\left(\frac{x^2}{2} + c_1\right) - 7(x + c_2)\end{aligned}$$

$\frac{3}{2}x^2 - 7x + 3c_1 - 7c_2 + \cancel{3c_4}$

$\frac{3}{2}x^2 - 7x + c$

② $\int \sqrt[3]{y}\, dy = \int y^{1/3} dy = \frac{y^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c$

$= \frac{y^{4/3}}{4/3} + \cancel{c} = \frac{3}{4} y^{4/3} + c$

③ $\int \left(\frac{3}{x^2} - \frac{1}{\sqrt{x^3}}\right) dx = \int (3x^{-2} - x^{-3/2})\, dx$

$= 3\int x^{-2} dx - \int x^{-3/2} dx$

$= 3\left(\frac{x^{-1}}{-1}\right) - \frac{x^{-1/2}}{-1/2} + c$

$= \frac{-3}{x} + \frac{2}{\sqrt{x}} + c$

④ Given $f'(x) = 6 - x^{1/2}$ and $f(1) = 4/3$ find $f(x)$

<u>Soln</u>

$f(x) = \int (6 - x^{1/2}) dx = 6x - \frac{x^{3/2}}{3/2} + c$

$= 6x - \frac{2}{3}x^{3/2} + c$

but $f(1) = 4/3 = 6(1) - \frac{2}{3}(1)^{3/2} + c$

$\Rightarrow c = -4$

$\therefore f(x) = 6x - \frac{2}{3}x^{3/2} - 4$

General Power Rule for Integration

$$\int (u_n(x))^n u'(x)\,dx = \frac{[u_n(x)]^{n+1}}{n+1} + C$$

where $n \neq -1$ and C is a constant

Examples

$$\int 2(1+2x)^3 dx = \int (1+2x)^3 \cdot 2\,dx$$
$$= \frac{(1+2x)^4}{4} + c$$

② $$\int x(3-4x^2)^2 dx = \int -\frac{1}{8}(3-4x^2)(-8x)dx$$
$$= -1/8\,\frac{(3-4x^2)^3}{3} + c$$
$$= -\frac{(3-4x^2)^3}{24} + c$$

Exx Find (i) $\int -8(3-4x^2)^2 dx$

(ii) $\int \frac{7x^2}{\sqrt{4x^3-5}} dx$

Using the power rule.

The Definite Integral

If a function f is continous on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where F is any function such that $F'(x) = f(x)$ for all x in $[a,b]$

Properties of Definite Integrals

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, k is a constant
2. $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $a<c<b$
3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

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Examples: $\int_1^2 (x^2-3)\,dx = \left[\frac{x^3}{3} - 3x\right]_1^2$

$= (8/3 - 6) - (1/3 - 3) = -2/3$

② $\int_{-8}^{-1} \frac{x+2x^2}{\sqrt[3]{x}}\,dx = \int_{-8}^{-1} \left(\frac{x}{x^{1/3}} + \frac{2x^2}{x^{1/3}}\right)dx$

$= \int_{-8}^{-1} (x^{2/3} + 2x^{5/3})\,dx$

$= \left[\frac{x^{5/3}}{5/3} + \frac{2x^{8/3}}{8/3}\right]_{-8}^{-1}$

$= (-3/5 + 3/4) - (\frac{-96}{5} + 192)$

$= -172.65$

③ $\int_0^2 |2x-1|\,dx = \int_0^{1/2} -(2x-1)\,dx + \int_{1/2}^{2} (2x-1)\,dx$

$= \left[-x^2+x\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^{2}$

$= 5/2$

NB: $|2x-1| = \begin{cases} -(2x-1), & x = 1/2 \\ (2x-1), & x \geq 1/2 \end{cases}$

Exx: Evaluate the ff definite integrals:

① $\int_{-1}^{1} (\sqrt[3]{t} - 2)\,dt$ ③ $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$

② $\int_1^4 \frac{u-2}{\sqrt{u}}\,du$ ④ $\int_0^2 (2-t)\sqrt{t}\,dt$

5. $\int_0^1 \frac{x-\sqrt{x}}{3}\,dx$ ⑦. $\int_{-2}^{-1}\left(\frac{-1}{u^2}+u\right)du$

6. $\int_{-1}^{0}(t^{1/3}-t^{2/3})\,dt$ ⑧ $\int_{-8}^{-1}\frac{x-x^2}{2\sqrt[3]{x}}\,dx$

9. $\int_0^1 x\sqrt{1-x^2}\,dx$ 10 $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$

Integration of Logarithmic Functions

If u is a differentiable function of x, then $\int \frac{u'}{u}\,dx = \ln|u| + c$.

In particular $\int \frac{1}{x}\,dx = \ln|x| + c$

Eg. $\int \frac{dx}{2x-1} = \frac{1}{2}\int \frac{2}{2x-1}\,dx = \frac{1}{2}\int \frac{u'}{u}\,dx$

$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x-1| + c$

② $\int_0^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^3 \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^3$

$= \frac{1}{2}(\ln 10 - \ln 1)$

$= \frac{1}{2}\ln 10 \simeq 1.15$

$$3.\ \int \frac{2x}{(x+1)^2}\,dx = \int \frac{2x+2-2}{(x^2+1)^2}\,dx$$

$$= \int \frac{2x+2}{(x+1)^2}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \int \frac{2(x+1)}{(x+1)^2}\,dx - 2\int (x+1)^{-2}\,dx$$

$$= \ln(x+1)^2 - \frac{2(x+1)^{-1}}{-1} + c$$

$$4)\ \int \frac{1}{x\ln x}\,dx = \int \frac{1/x}{\ln x}\,dx$$

$$= \ln|\ln x| + c$$

Exx

$1.\ \int \frac{x^2}{3-x^3}\,dx$ $2)\ \int \frac{2x}{(x-1)^2}\,dx$

$3.\ \int \frac{x+3}{x^2+6x+7}\,dx$ $3)\ \int \frac{1}{(x+1)^2}\,dx$

$5.\ \int \frac{\sqrt{x}}{1-x\sqrt{x}}\,dx$ $6)\ \int_{-2}^{-1} \frac{x+5}{x}\,dx$

Integrals of Exponential Functions

① $\int e^x dx = e^x + c$

② $\int a^x dx = \frac{1}{\ln a} a^x + c$

③ $\int e^u u' dx = e^u + c$

④ $\int a^u u' dx = \frac{1}{\ln a} a^u + c$

Eg $\int e^{3x+1} dx = \frac{1}{3}\int e^{3x+1}(3)\,dx$

$$= \frac{1}{3}\int e^u u' dx$$

$$= \frac{1}{3} e^u + c$$

$$= \frac{e^{3x+1}}{3} + c$$

NB $u = 3x+1$ and $u' = 3$

② $\int 5x e^{-x^2} = 5\int e^{-x^2} x \, dx$

$u = -x^2, \quad u' = -2x$

$5\int e^{-x^2} x\, dx = 5\left(-\frac{1}{2}\right)\int e^{-x^2}(-2x)\,dx$

$$= \frac{-5}{2}\int e^u u' dx$$
$$= -\frac{5}{2}e^{-x^2} + c$$

③ Evaluate $\int_0^2 \frac{e^x}{1+e^x}dx$

Soln let $u = 1+e^x$, $u' = e^x$

$$\int_0^2 \frac{e^x}{1+e^x}dx = \left[\ln|1+e^x|\right]_0^2$$
$$= \ln(1+e^2) - \ln 2$$
$$\simeq 1.434$$

④ Evaluate $\int \frac{2^{1/x}}{x^2}dx$

Soln let $u = 1/x$, $u' = -1/x^2$

$$\int \frac{2^{1/x}}{x^2}dx = -\int 2^{1/x}\left(-1/x^2\right)dx$$
$$= \left(-\frac{1}{\ln 2}\right)2^{1/x} + c$$

Exx

1. $\int_0^1 e^{-2x}dx$

2. $\int_1^2 e^{1-x}dx$

Fundamentals Integration Formulas

A major part of our integration problem is the recognition of which basic integration formula to use to solve the problem. Skills in recognizing what formula to use requires memorisation of the basic formulas and lots of practice in using them. The formulas are as follows:

1. $\int u^n u' dx = \frac{u^{n+1}}{n+1} + C, \; n \neq -1$

$\left(\int x^n dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1\right)$

2. $\int e^u u' dx = e^u + C, \left(\int a^u u' dx = \frac{1}{\ln a} a^u + C\right)$

3. $\int \frac{u'}{u} dx = \ln|u| + C, \; \left(\int \frac{1}{x} dx = \ln|x| + C\right)$

4. $\int (\sin u) u' dx = -\cos u + C$

5. $\int (\cos u) u' dx = \sin u + C$

6. $\int (\sec^2 u) u' dx = \tan u \, \cancel{\cot u} + C$

7. $\int (\csc^2 u)u'dx = -\cot u + c$

8. $\int (\sec u \tan u)u'dx = \sec u + c$

9. $\int (\csc u \cot u)u'dx = -\csc u + c$

10 $\int (\tan u)u'dx = -\ln|\cos u| + c$

11. $\int (\cot u)u'dx = \ln|\sin u| + c$

12. $\int (\sec u)u'dx = \ln|\sec u + \tan u| + c$

13. $\int (\csc u)u'dx = \ln|\csc - \cot u| + c$

14 $\int \frac{u'}{\sqrt{a^2-u^2}}dx = \text{Arc}\sin\frac{u}{a} + c$

15. $\int \frac{u'}{\sqrt{u^2 \pm a^2}}dx = \ln|u + \sqrt{u^2 \pm a^2}| + c$

16 $\int \frac{u'}{a^2+u^2}dx = \frac{1}{a}\text{Arc}\tan\frac{u}{a} + c$

17. $\int \frac{u'}{u^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$

18 $\int \frac{u'}{u\sqrt{u^2-a^2}}dx = \frac{1}{a}\text{Arcsec}\frac{|u|}{a} + c$

19 $\int \frac{u'}{u\sqrt{a^2 \pm u^2}}dx = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2 \pm u^2}}{|u|}\right) + c$

Eg: Evaluate $\int (2x+1)^3 dx$

Soln Let $u = 2x+1$ and use formula 1

Then $\int (2x+1)^3 dx = \frac{1}{2}\int (2x+1)^3 (2)\, dx$

$$= \frac{1}{2}\frac{(2x+1)^4}{4} + c$$

$$= \frac{(2x+1)^4}{8} + c$$

(2) Evaluate $\int 4\sec^2 3x\, dx$

Soln let $u = 3x$ and use formula 6

Then $\int 4\sec^2 3x\, dx = \frac{4}{3}\int (\sec^2 3x)(3)\, dx$

$$= \frac{4}{3}\tan 3x + c$$

(3) Evaluate the ff indefinite integrals

(a) $\int \frac{4}{x^2+2} dx$ (b) $\int \frac{4x}{x^2+2}$

(c) $\int \frac{4x^2}{x^2+2} dx$

Soln Considering formular 16,
Let $u = x$ and $a = \sqrt{2}$, then

$$\int \frac{4}{x^2+2}\,dx = 4\int \frac{1}{x^2+(\sqrt{2})^2}\,dx$$

$$= 4\left[\frac{1}{\sqrt{2}} \text{Arttan} \frac{x}{\sqrt{2}}\right] + c$$

$$= 2\sqrt{2}\, \text{Arctan} \frac{x}{\sqrt{2}} + c$$

(b) Here formular 16 does not apply because of the x in the numerator, considering formular 3, we let $u = x^2+2$

Then $\int \frac{4x}{x^2+2}\,dx = 2\int \frac{2x}{x^2+2}\,dx = 2\ln(x^2+2) + c$

(c) Since the degree of numerator equals degree of denominator, we first divide to obtain

$$\frac{4x^2}{x^2+2} = 4 - \frac{8}{x^2+2}$$

Thus $\int \frac{4x^2}{x^2+2}\,dx = \int \left(4 - \frac{8}{x^2+2}\right)dx$

$$= \int 4\,dx - 8\int \frac{1}{x^2+2}\,dx$$

$= 4x - 8\left[\frac{1}{\sqrt{2}} \text{Arctan} \frac{x}{\sqrt{2}}\right] + C$

$= 4x - 4\sqrt{2} \text{ Arctan}(\frac{x}{\sqrt{2}}) + C$ by formular 1 and 16.

4. find $\int \frac{x-3}{\sqrt{9x^2+4}} dx$

Soln

$$\int \frac{x-3}{\sqrt{9x^2+4}} dx = \int \frac{x\,dx}{\sqrt{9x^2+4}} - \int \frac{3dx}{\sqrt{9x^2+4}}$$

we can apply formulas 1 and 15 to obtain

$$\int \frac{x-3}{\sqrt{9x^2+4}} dx = \frac{1}{18}\int (9x^2+4)^{-1/2}(18x)\,dx - \int \frac{3dx}{\sqrt{9x^2+4}}$$

$$= \frac{1}{18}\left[\frac{(9x^2+4)^{1/2}}{1/2}\right] - \ln\left|3x + \sqrt{9x^2+4}\right| + c$$

$$= \frac{1}{9}\sqrt{9x^2+4} - \ln\left(3x + \sqrt{9x^2+4}\right) + c$$

5. Find $\int_0^{\pi/4} (\sin 2x) e^{-\cos 2x} dx$

Soln Considering formula 2, we let $u = -\cos 2x$, then $u' = 2\sin 2x$ and it follows that:

$$\int_0^{\pi/4} \sin 2x\, e^{-\cos 2x} dx = \frac{1}{2}\int_0^{\pi/4} (2\sin 2x) e^{-\cos 2x} dx$$

$$= \frac{1}{2} e^{-\cos 2x}\Big]_0^{\pi/4} = \frac{1}{2}\left(e^0 - e^{-1}\right) \approx \underline{0.1316}$$

6. Evaluate $\int \frac{1+\cos(e^{-2x})}{e^{2x}}\,dx$

Soln Quite often with a sum or difference in the numerator, we can separate the integrand into 2 or more parts. In this case we have

$$\int \frac{1+\cos(e^{-2x})}{e^{2x}}\,dx = \int \frac{dx}{e^{2x}} + \int \frac{\cos(e^{-2x})}{e^{2x}}\,dx$$

$$= \int e^{-2x}\,dx + \int [\cos(e^{-2x})](e^{-2x})\,dx$$

$$= -\tfrac{1}{2}\int e^{-2x}(-2)\,dx - \tfrac{1}{2}\int \cos(e^{-2x})(-2e^{-2x})\,dx$$

By formulas 1 and 5 we have

$$\int \frac{1+\cos(e^{-2x})}{e^{2x}}\,dx = -\tfrac{1}{2}e^{-2x} - \tfrac{1}{2}\sin(e^{-2x}) + c$$

(7) Evaluate $\int \frac{x^2}{\sqrt{16-x^6}}\,dx$

Soln $\int \frac{x^2}{\sqrt{16-x^6}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{16-(x^3)^2}}\,dx$

$$= ~~\tfrac{1}{3}\text{Arctan}~~$$

$$= \tfrac{1}{3}\text{Arcsin}\,\frac{x^3}{4} + c$$

8. Evaluate $\int \cot x \ln(\sin x)\,dx$

Soln Since this integral does not appear to fit any of our fundamental formulas, let $u = \ln(\sin x)$, then $u' = \frac{\cos x}{\sin x} = \cot x$

$$\int \cot x \cdot \ln(\sin x)\,dx = \int \ln(\sin x)\cot x\,dx$$

$$= \int u u'\,dx = \frac{u^2}{2} + c$$

$$= \frac{[\ln(\sin x)]^2}{2} + c$$

9. Evaluate $\int_{\pi/4}^{\pi/2} \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx$

Soln Again this integral does not appear to fit any of our fundamental formulars. Let $u = 1-\cos 2x$, then $u' = 2\sin 2x$ and by the power Rule, we have

$$\int_{\pi/4}^{\pi/2} \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx = \frac{1}{2}\int_{\pi/4}^{\pi/2} (1-\cos 2x)^{-1/2}(2\sin 2x)\,dx$$

$$= \frac{1}{2}\left[\frac{(1-\cos 2x)^{1/2}}{1/2}\right]_{\pi/4}^{\pi/2}$$

$$= \left[\sqrt{1-\cos 2x}\right]_{\pi/4}^{\pi/2} = \sqrt{2} - 1$$

Completing the square

Eg: Evaluate $\int \frac{dx}{x^2-4x+7}$

<u>Soln</u> By completing the square, we obtain

$$x^2-4x+7 = (x^2-4x+4)-4+7 = (x-2)^2+3$$

$$\therefore \int \frac{dx}{x^2-4x+7} = \int \frac{dx}{(x-2)^2+3}$$

let $u = x-2$ and $a = \sqrt{3}$, we have

$$\int \frac{dx}{x^2-4x+7} = \frac{1}{\sqrt{3}} \text{Arctan} \frac{x-2}{\sqrt{3}} + c$$

(2) Evaluate $\int \frac{dx}{2x^2-x-3}$

<u>Soln</u>

$$2x^2-x-3 = 2\left(x^2-\frac{x}{2}-\frac{3}{2}\right)$$
$$= 2\left[x^2-\frac{x}{2}+(1/4)^2-(1/4)^2-3/2\right]$$
$$= 2\left[(x-1/4)^2-25/16\right]$$

$$\therefore \int \frac{dx}{2x^2-x-3} = \frac{1}{2}\int \frac{dx}{(x-1/4)^2-25/16}$$

let $u = x-1/4$ and $a = 5/4$, then by formular 17, we have

$$\int \frac{dx}{2x^2-x-3} = (1/2)\left[\frac{1}{2(5/4)}\right] \ln\left[\frac{x-1/4-5/4}{x-1/4+5/4}\right]+c$$

$$= \frac{1}{5}\ln\left|\frac{x-3/2}{x+1}\right| + c = \frac{1}{5}\ln\left|\frac{2x-3}{2x+2}\right| + c$$

Substitution

The technique of substitution involves a change of variable, which permits us to rewrite an integrand in a form to which we can apply a basic integration rule.

Eg: Evaluate $\int x\sqrt{x-1}\,dx$

Soln let $u = \sqrt{x-1} \Rightarrow u^2 = x-1 \Rightarrow x = u^2+1$

$$2u\frac{du}{dx} = 1 \Rightarrow 2u\,du = dx$$

$$\Rightarrow \int x\sqrt{x-1}\,dx = \int (u^2+1)\,u\,2u\,du$$

$$= \int (2u^4 + 2u^2)\,du$$

$$= \frac{2u^5}{5} + \frac{2u^3}{3} + c$$

$$= \frac{2}{5}(\sqrt{x-1})^5 + \cancel{2}\,\frac{2}{3}(\sqrt{x-1})^3 + c$$

$$= \frac{6}{15}(x-1)^{5/2} + \frac{10}{15}(x-1)^{3/2} + c$$

$$= \frac{2}{15}(x-1)^{3/2}[3(x-1)+5] + c$$

$\frac{2}{15}(x-1)^{3/2}(3x+2)+c$

(2) Evaluate $\int \frac{1}{3\sqrt{x}+1}\,dx$

Soln Let $u=\sqrt{x}$, then $u^2=x$ and

$$2u\,du = dx$$

$$\int \frac{1}{3\sqrt{x}+1}\,dx = \int \frac{1}{3u+1}(2u\,du)$$

$$= \int \frac{2u\,du}{3u+1}$$

$$= \int\left(\frac{2}{3} - \frac{2/3}{3u+1}\right)du$$

$$= \frac{2}{3}\int du - \frac{2}{3}\int \frac{du}{3u+1}$$

$$= \frac{2}{3}u - \frac{2}{9}\ln|3u+1| + c$$

$$= \frac{2}{3}\sqrt{x} - \frac{2}{9}\ln(3\sqrt{x}+1) + c$$

(3) Find $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

Soln Let $u=\sqrt{2x-1}$, then $u^2=2x-1$

$$x = \frac{u^2+1}{2} \Rightarrow u\,du = dx$$

when $x = 5$, $u = \sqrt{10-1} = 3$

when $x = 1$ $u = \sqrt{2-1} = 1$

$$\Rightarrow \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{1}{2}\left(\frac{u^2+1}{u}\right) u\,du =$$

$$= \frac{1}{2}\int_1^3 (u^2+1)\,du$$

$$= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 = 16/3$$

Exx

$\int \sqrt{2x+3}\,dx$ ② $\int \frac{dx}{\sqrt{x+2} - \sqrt{x}}$

Partial Fractions

This technique involves the decomposition of a rational function into the sum of two or more "~~simpleer~~" "simples" rational functions.

Examples: Evaluate $\int \frac{x+7}{x^2-x-6}\,dx$

Soln $\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}$

$$\int \frac{x+7}{x^2-x-6}\,dx = \int \left(\frac{2}{x-3} - \frac{1}{x+2}\right)dx$$

$$= 2\int \frac{dx}{x-3} - \int \frac{1}{x+2}\,dx$$

$$= 2\ln|x-3| - \ln|x+2| + c$$

(2) $$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\Rightarrow \int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int \frac{6}{x}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{9}{(x+1)^2}\,dx$$

$$= 6\ln|x| - \ln|x+1| + 9\,\frac{(x+1)^{-1}}{-1} + c$$

$$= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + c$$

(3) $$\int \frac{2x^3-4x-8}{(x^2+x)(x^2+4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right)dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\,\mathrm{Arctan}\,\frac{x}{2} + c$$

$$= \ln\left[\frac{x^2(x^2+4)}{(x-2)^2}\right] + 2\,\mathrm{Arctan}\,\frac{x}{2} + c$$

Exx Find

(1) $$\int \frac{2x^5-5x}{(x^2+2)^2}\,dx$$

(2) $$\int \frac{x^2}{(x^2+1)^2}\,dx$$

Trigonometric Integrals

Power Rules For Trig Functions

$$\int \sin^n u\,(\cos u)\,u'\,dx = \frac{\sin^{n+1} u}{n+1} + C,\ n \neq -1$$

$$\int \cos^n u\,(-\sin u)\,u'\,dx = \frac{\cos^{n+1} u}{n+1} + C,$$

$$\int \sec^n u\,(\sec u \tan u)\,u'\,dx = \frac{\sec^{n+1} u}{n+1} + C$$

$$\int \tan^n u\,(\sec^2 u)\,u'\,dx = \frac{\tan^{n+1} u}{n+1} + C$$

In general, to evaluate integrals of the form $\int (\sin^m u \cos^n u)\,u'\,dx$ we may proceed as follows;

① if $m = 2k+1$ is odd and positive, we have

$$\int (\sin^m u \cos^n u)\,u'\,dx = \int \sin^{2k} u \cos^n u\,(\sin u)\,u'\,dx$$

$$= \int (\sin^2 u)^k \cos^n u\,(\sin u)\,u'\,dx$$

$$= \int (1-\cos^2 u)^k \cos^n u\,(\sin u)\,u'\,dx$$

then expand and integrate.

2. If $n = 2k+1$ is odd and positive, write $\int (\sin^m u \cos^n u)\, u' dx$

$$= \int \sin^m u \cos^{2k} u (\cos u) u' dx$$

$$= \int \sin^m u (\cos^2 u)^k (\cos u) u' dx$$

$$= \int \sin^m u (1-\sin^2 u)^k (\cos u) u' dx$$

then expand and integrate.

③ If both m and n are even and +ve, make repeated use of the identities

$$\sin^2 u = \frac{1-\cos 2u}{2} \quad \text{and} \quad \cos^2 u = \frac{1+\cos 2u}{2}$$

to convert the integrand to odd powers on the cosine and then proceed as in step 2.

Examples: Evaluate $\int \sin^2 x \cos^5 x\, dx$

Since 'n' is odd, we write

$$\int \sin^2 x \cos^5 x\, dx = \int \sin^2 x \cos^4 x (\cos x)\, dx$$

$$= \int \sin^2 x (\cos^2 x)^2 \cos x\, dx$$

$$= \int \sin^2 x (1-\sin^2 x)^2 \cos x\, dx$$

$$= \int \sin^2 x (1-2\sin^2 x + \sin^4 x) \cos x\, dx$$

$$= \int [\sin^2 x - 2\sin^4 x + \sin^6 x] \cos x \, dx$$

Applying the power rule for $\sin x$,

$$\int \sin^2 x \cos^5 x \, dx = \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

② Evaluate $\int_0^{\pi/6} \sin^5 3x \, dx$

<u>Soln</u> It does not matter that $n = 0$; m is odd and we write

$$\int_0^{\pi/6} \sin^5 3x \, dx = \int_0^{\pi/6} \sin^4 3x (\sin 3x) dx$$

$$= \int_0^{\pi/6} (\sin^2 3x)^2 (\sin 3x) dx$$

$$= \int_0^{\pi/6} (1 - \cos^2 3x)^2 \sin 3x \, dx$$

$$= \int_0^{\pi/6} (1 - 2\cos^2 3x + \cos^4 3x) \sin 3x \, dx$$

Now if we consider $u = \cos 3x$, then $u' = -3\sin 3x$, and we write

$$\int_0^{\pi/6} \sin^5 3x \, dx = -\frac{1}{3} \int_0^{\pi/6} [1 - 2\cos^2 3x + \cos^4 3x] (-3 \sin 3x \, dx)$$

$$= -\frac{1}{3} \left[\cos 3x - \frac{2\cos^3 3x}{3} + \frac{\cos^5 3x}{5} \right]_0^{\pi/6} = 8/45$$

Exx Evaluate $\int \cos^4 x\, dx$

To evaluate integrals of the form $\int (\sec^m u \tan^n u) u' dx$, we proceed as follows:

① If $m = 2k$ is even and positive, write

$$\int (\sec^m u \tan^n u) u' dx = \int (\sec u)^{2k-2} \tan^n u (\sec^2 u) u' dx$$

$$= \int (\sec^2 u)^{k-1} \tan^n u (\sec^2 u) u' dx$$

$$= \int (1 + \tan^2 u)^{k-1} \tan^n u (\sec^2 u) u' dx$$

then expand and integrate.

② If $n = 2k+1$ is odd and positive, write $\int (\sec^m u \tan^n u) u' dx = \int \sec^{m-1} u \tan^{2k} u (\sec u \tan u) u' dx$

$$= \int \sec^{m-1} u (\tan^2 u)^k (\sec u \tan u) u' dx$$

$$= \int \sec^{m-1} u (\sec^2 u - 1)^k (\sec u \tan u) u' dx$$

then expand and integrate.

③ If $m = 0$, write

$$\int (\tan^n) u' dx = \int \tan^{n-2} u (\tan^2 u) u' dx$$

$$= \int \tan^{n-2} u (\sec^2 u - 1) u' dx$$

$$= \int \tan^{n-2} u (\sec^2 u) u' dx - \int \tan^{n-2} u) u' dx$$

Now repeate this ~~prodv~~ procedure for the integral $\int (\tan^{n-2} u) u' dx$.

(4) If none of the first 3 cases apply, try rewriting the integrand in terms of sines and cosines. For integrals involving powers of the cotangent and cosecant, we follow a similar strategy by making use of the identity $\csc^2 u = 1 + \cot^2 u$

<u>Examples</u>: Evaluate $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

<u>Soln</u> Since n is odd, we write

$$\int (\sec x)^{-1/2} \tan^3 x \, dx = \int (\sec x)^{-3/2} (\tan^2 x)(\sec x \tan x) dx$$

$$= \int (\sec x)^{-3/2} (\sec^2 x - 1)(\sec x \tan x) dx$$

$$= \int \left[(\sec x)^{1/2} - (\sec x)^{-3/2} \right] (\sec x \tan x) dx$$

$$= \frac{2}{3} (\sec x)^{3/2} + 2 (\sec x)^{-1/2} + c$$

(2) Evaluate $\int \sec^4 3x \tan^3 3x \, dx$

<u>Soln</u> Since m is even and n is odd, we may use procedure 1 or procedure 2.