SESSION 3-2: Iterative Method

3-1.1 Jacobi Method

Given a general set of n equations and n unknowns, we have

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side and the second equation is rewritten with x_2 on the left hand side and so on as follows:

$$
\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right) \\
x_2 &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right) \\
&\vdots \\
x_{n-1} &= \frac{1}{a_{n-1,n-1}}\left(b_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 - \cdots - a_{n-1,n}x_n\right) \\
x_n &= \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}\right)
\end{aligned}
$$

These equations can be rewritten in the summation form as

$$x_1 = \frac{1}{a_{11}}\left(b_1 - \sum_{j=1,\, j\neq 1}^{n} a_{1j}x_j\right)$$
$$x_2 = \frac{1}{a_{22}}\left(b_2 - \sum_{j=1,\, j\neq 2}^{n} a_{2j}x_j\right)$$
$$\vdots$$
$$x_{n-1} = \frac{1}{a_{n-1,n-1}}\left(b_{n-1} - \sum_{j=1,\, j\neq n-1}^{n} a_{n-1j}x_j\right)$$
$$x_n = \frac{1}{a_{nn}}\left(b_n - \sum_{j=1,\, j\neq n}^{n} a_{nj}x_j\right)$$

Hence for any row i, $x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}x_j\right), \quad i = 1, 2, \ldots, n.$

By assuming an initial guess for the x_i 's, one uses $x_i^{(k)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}x_j^{(k-1)}\right), \quad i = 1, 2, \ldots, n.$ to calculate the new values for the x_i 's. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as $\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$, where x_i^{new} is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.

When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

If the coefficient matrix A of the system $Ax = b$, is written as $A = L + D + U$ where D has the diagonal elements of A and L & U respectively have the lower diagonal and upper diagonal elements of A then the Jacobi scheme can be written in matrix form as

$$Dx = -(L+U)x + b$$
$$Dx^{(k)} = -(L+U)x^{(k-1)} + b$$

giving $x^{(k)} = -D^{-1}(L+U)x^{(k-1)} + D^{-1}b$, i.e., the Jacobi iterative matrix and the corresponding vector are given as $T_J = -D^{-1}(L+U)$ and $c_J = D^{-1}b$ respectively.

Example 1

Use Jacobi Method to solve $Ax = b$, where

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Assume an initial guess of the solution as $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}^T$. Perform two iterations only.

Solution

Rewriting the equations gives

$$x_1^{(k)} = \frac{106.8 - 5x_2^{(k-1)} - x_3^{(k-1)}}{25}$$

$$x_2^{(k)} = \frac{177.2 - 64x_1^{(k-1)} - x_3^{(k-1)}}{8}$$

$$x_3^{(k)} = \frac{279.2 - 144x_1^{(k-1)} - 12x_2^{(k-1)}}{1}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

we get

$$x_1^{(1)} = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$x_2^{(1)} = \frac{177.2 - 64(1) - (5)}{8} = 13.525$$

$$x_3^{(1)} = \frac{279.2 - 144(2) - 12(5)}{1} = - - 68.8$$

The absolute relative approximate error for each x_i then is

$$\left|\varepsilon_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| \times 100 = 72.76\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{13.525 - 2.0000}{13.525}\right| \times 100 = 85.21\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{-68.8 - 5.0000}{-68.8}\right| \times 100 = 108.53\%$$

At the end of the first iteration, the guess of the solution vector is

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 3.6720 \\ 13.525 \\ -68.8 \end{bmatrix}$$

and the maximum absolute relative approximate error is 108.53%.

Iteration #2

The estimate of the solution vector at the end of iteration #1 is

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 3.6720 \\ 13.525 \\ -68.8 \end{bmatrix}$$

Now we get

$$x_1^{(2)} = \frac{106.8 - 5(13.525) - 68.8}{25} = -1.185$$

$$x_2^{(2)} = \frac{177.2 - 64(3.6720) - 68.8}{8} = -15.862$$

$$x_3^{(2)} = \frac{279.2 - 144(3.6720) - 12(-68.8)}{1} = 576.032$$

The absolute relative approximate error for each x_i then is

$$\left|\in_a\right|_1 = \left|\frac{-1.185-3.6720}{-1.185}\right| \times 100 = 428.37\%$$

$$\left|\in_a\right|_2 = \left|\frac{-15.862-(13.525)}{-15.862}\right| \times 100 = 185.27\%$$

$$\left|\in_a\right|_3 = \left|\frac{576.032-(-68.8)}{576.032}\right| \times 100 = 111.94\%$$

At the end of second iteration the estimate of the solution is

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} -1.185 \\ -15.862 \\ 576.032 \end{bmatrix}$$

and the maximum absolute relative approximate error is 428.37%.

Example 2

Given the system of equations;

$$\begin{aligned} 12x_1 + 3x_2 - \ 5x_3 &= 1 \\ x_1 + 5x_2 + \ 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76 \end{aligned}$$

find the solution, given $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ as the initial guess.

Solution

Rewriting the equations, we get

$$x_1^{(k)} = \frac{1-3x_2^{(k-1)}+5x_3^{(k-1)}}{12}$$

$$x_2^{(k)} = \frac{28-x_1^{(k-1)}-3x_3^{(k-1)}}{5}$$

$$x_3^{(k)} = \frac{76-3x_1^{(k-1)}-7x_2^{(k-1)}}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

<u>Iteration #1:</u>

$$x_1^{(1)} = \frac{1-3(0)+5(1)}{12} = 0.50000$$
$$x_2^{(1)} = \frac{28-(1)-3(1)}{5} = 4.80000$$
$$x_3^{(1)} = \frac{76-3(1)-7(0)}{13} = 5.6154$$

The absolute relative approximate error at the end of first iteration is

$$\left|\varepsilon_a\right|_1 = \left|\frac{0.50000-1.0000}{0.50000}\right| \times 100 = 67.662\%$$
$$\left|\varepsilon_a\right|_2 = \left|\frac{4.8000-0}{4.8000}\right| \times 100 = 100\%$$
$$\left|\varepsilon_a\right|_3 = \left|\frac{5.6154-1.0000}{5.6154}\right| \times 100 = 82.19\%$$

The maximum absolute relative approximate error is 100.000%

<u>Iteration #2:</u>

$$x_1^{(2)} = \frac{1-3(4.8000)+5(5.6154)}{12} = 14.677$$
$$x_2^{(2)} = \frac{28-(0.5000)-3(5.6154)}{5} = 2.13076$$
$$x_3^{(2)} = \frac{76-3(0.5000)-7(4.800)}{13} = 3.14615$$

At the end of second iteration, the absolute relative approximate error is

$$\left|\varepsilon_a\right|_1 = \left|\frac{14.677-0.50000}{14.677}\right|\times 100 = 96.59\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{2.13076-4.8000}{2.13076}\right|\times 100 = 125.27\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{3.14615-5.6154}{3.14615}\right|\times 100 = 78.484\%$$

The maximum absolute relative approximate error is 125.27%.

3-1.2 Gauss-Seidel Method

If the set of n equations and n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

is rewritten as

$$\begin{aligned}
&a_{11}x_1 = b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n \\
&a_{21}x_1 + a_{22}x_2 = b_2 - a_{23}x_3 - \cdots - a_{2n}x_n \\
&\qquad\vdots \\
&a_{n-1,1}x_1 + a_{n-1,2}x_2 + \cdots + a_{n-1,n-1}x_{n-1} = b_{n-1} - a_{n-1,n}x_n \\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1} + a_{nn}x_n = b_n
\end{aligned}$$

From the above it is seen that the most recent updates of the x_i 's are used immediately they are available then the above reduces to

$$x_i^{(k)} = \frac{1}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}x_i^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_i^{(k-1)}\right\}, \quad i = 1,\ldots,n \qquad (*)$$

Again in the matrix notation, the coefficient matrix A of the system $Ax = b$, is split into $A = L + D + U$ where D has the diagonal elements of A and L & U respectively have the lower diagonal and upper diagonal elements of A then the Gauss-Seidel scheme can be written as

$(L+D)x = -Ux + b$

$(L+D)x^{(k)} = -Ux^{(k-1)} + b$

giving $x^{(k)} = -(L+D)^{-1}Ux^{(k-1)} + (L+D)^{-1}b$.

i.e., the Gauss-Seidel iterative matrix and the corresponding vector are given as $T_{GS} = -(L+D)^{-1}U$ and $c_{GS} = (L+D)^{-1}b$ respectively.

Example 1

Use Gauss-Seidel Method to solve $Ax = b$, where

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Assume an initial guess of the solution as $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}^T$.

Solution

Rewriting the equations gives

$$x_1^{(k)} = \frac{106.8 - 5x_2^{(k-1)} - x_3^{(k-1)}}{25}$$
$$x_2^{(k)} = \frac{177.2 - 64x_1^{(k)} - x_3^{(k-1)}}{8}$$
$$x_3^{(k)} = \frac{279.2 - 144x_1^{(k)} - 12x_2^{(k)}}{1}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

we get

$$x_1^{(1)} = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$
$$x_2^{(1)} = \frac{177.2 - 64\left(3.6720\right) - \left(5\right)}{8} = -7.8510$$
$$x_3^{(1)} = \frac{279.2 - 144\left(3.6720\right) - 12\left(-7.8510\right)}{1} = -155.36$$

The absolute relative approximate error for each x_i then is

$$\left|\varepsilon_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| \times 100 = 72.76\%$$
$$\left|\varepsilon_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| \times 100 = 125.47\%$$
$$\left|\varepsilon_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| \times 100 = 103.22\%$$

At the end of the first iteration, the guess of the solution vector is

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

and the maximum absolute relative approximate error is 125.47%.

Iteration #2

The estimate of the solution vector at the end of iteration #1 is

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

Now we get

$$x_1^{(2)} = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$
$$x_2^{(2)} = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$
$$x_3^{(2)} = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

The absolute relative approximate error for each x_i then is

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| \times 100 = 69.542\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| \times 100 = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| \times 100 = 80.54\%$$

At the end of second iteration the estimate of the solution is

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.34 \end{bmatrix}$$

and the maximum absolute relative approximate error is 85.695%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration k	$x_1^{(k)}$	$\left\|\varepsilon_a\right\|_1$ %	$x_2^{(k)}$	$\left\|\varepsilon_a\right\|_2$ %	$x_3^{(k)}$	$\left\|\varepsilon_a\right\|_3$ %
1	3.672	72.767	-7.8510	125.47	-155.36	103.22
2	12.056	67.542	-54.882	85.695	-798.34	80.540
3	47.182	74.448	-255.51	78.521	-3448.9	76.852
4	193.33	75.595	-1093.4	76.632	-14440	76.116
5	800.53	75.850	-4577.2	76.112	-60072	75.962
6	3322.6	75.907	-19049	75.971	-249580	75.931

As seen in the above table, the solution is not converging to the true solution of
$x_1 = 0.29048, \; x_2 = 19.690, \; x_3 = 1.0858$

The above system of equations does not seem to converge? Why?
A pitfall of most iterative methods is that they may or may not converge. However, certain classes of systems of simultaneous equations do always converge to a solution using Gauss-Seidel method. This class of system of equations is where the coefficient matrix A in $Ax = b$ is diagonally dominant, that is $\left|a_{ii}\right| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$ for all i and $\left|a_{ii}\right| > \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$ for at least one i.

If a system of equations has a coefficient matrix that is not diagonally dominant, it may or may not converge. Fortunately, many physical systems that result in simultaneous linear equations have diagonally dominant coefficient matrix, which then assures convergence for iterative methods such as Gauss-Seidel method of solving simultaneous linear equations.

Example 2

Given the system of equations;

$$\begin{aligned} 12x_1 + 3x_2 - \ 5x_3 &= 1 \\ x_1 + 5x_2 + \ 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76 \end{aligned}$$

find the solution. Given $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ as the initial guess.

Solution

The coefficient matrix

$$A = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence the solution should converge using Gauss-Seidel method.

Rewriting the equations, we get

$$x_1^{(k)} = \frac{1 - 3x_2^{(k-1)} + 5x_3^{(k-1)}}{12}$$

$$x_2^{(k)} = \frac{28 - x_1^{(k)} - 3x_3^{(k-1)}}{5}$$

$$x_3^{(k)} = \frac{76 - 3x_1^{(k)} - 7x_2^{(k)}}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration #1:

$$x_1^{(1)} = \frac{1-3(0)+5(1)}{12} = 0.50000$$
$$x_2^{(1)} = \frac{28-(0.5)-3(1)}{5} = 4.90000$$
$$x_3^{(1)} = \frac{76-3(0.50000)-7(4.9000)}{13} = 3.0923$$

The absolute relative approximate error at the end of first iteration is

$$|\varepsilon_a|_1 = \left|\frac{0.50000-1.0000}{0.50000}\right| \times 100 = 67.662\%$$
$$|\varepsilon_a|_2 = \left|\frac{4.9000-0}{4.9000}\right| \times 100 = 100\%$$
$$|\varepsilon_a|_3 = \left|\frac{3.0923-1.0000}{3.0923}\right| \times 100 = 67.662\%$$

The maximum absolute relative approximate error is 100.000%

Iteration #2:

$$x_1^{(2)} = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$
$$x_2^{(2)} = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$
$$x_3^{(2)} = \frac{76-3(0.14679)-7(4.900)}{13} = 3.8118$$

At the end of second iteration, the absolute relative approximate error is

$$\left|\varepsilon_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.62\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.887\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.876\%$$

The maximum absolute relative approximate error is 240.62%. This is greater than the value of 67.612% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Iteration k	$x_1^{(k)}$	$\left\|\varepsilon_a\right\|_1$	$x_2^{(k)}$	$\left\|\varepsilon_a\right\|_2$	$x_3^{(k)}$	$\left\|\varepsilon_a\right\|_3$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 3

Given the system of equations

$$\begin{aligned} 3x_1 + 7x_2 + 13x_3 &= 76 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 12x_1 + 3x_2 - 5x_3 &= 1 \end{aligned}$$

find the solution using Gauss-Seidal method. Use $\left[x_1, x_2, x_3\right]^T = \left[1 \quad 0 \quad 1\right]^T$ as the initial guess.

Solution

Rewriting the equations, we get

$$x_1^{(k)} = \frac{76 - 7x_2^{(k-1)} - 13x_3^{(k-1)}}{3}$$

$$x_2^{(k)} = \frac{28 - x_1^{(k)} - 3x_3^{(k-1)}}{5}$$

$$x_3^{(k)} = \frac{1 - 12x_1^{(k)} - 3x_3^{(k)}}{-5}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below

Iteration k	$x_1^{(k)}$	$\lvert\varepsilon_a\rvert_1$	$x_2^{(k)}$	$\lvert\varepsilon_a\rvert_2$	$x_3^{(k)}$	$\lvert\varepsilon_a\rvert_3$
1	21.000	110.71	0.80000	100.00	5.0680	98.027
2	-196.15	109.83	14.421	94.453	-462.30	110.96
3	-1995.0	109.90	-116.02	112.43	4718.1	109.80
4	-20149	109.89	1204.6	109.63	-47636	109.90
5	$2.0364\text{x}10^5$	109.90	-12140	109.92	$4.8144\text{x}10^5$	109.89
6	$-2.0579\text{x}10^5$	1.0990	$1.2272\text{x}10^5$	109.89	$-4.8653\text{x}10^6$	109.89

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$A = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{11}| = |3| = 3 \le |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence Gauss-Seidal method may or may not converge.

However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.

So it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. But it is not possible for all cases. For example, the following set of equations.

$$\begin{aligned} x_1 + \ \ x_2 + \ \ x_3 &= 3 \\ 2x_1 + 3x_2 + 4x_3 &= 9 \\ x_1 + 7x_2 + \ \ x_3 &= 9 \end{aligned}$$

cannot be rewritten to make the coefficient matrix diagonally dominant.

Example 4

Find an approximation to the solution of $Ax = b$ by performing two iterations of the Gauss-Seidel method where $A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution

Using $x_i^{(k)} = \dfrac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{2} a_{ij} x_j^{(k-1)}\right)$, $i = 1, 2$, $k = 1, 2, \ldots$, we have

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \ A_{\text{off}} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$k = 1$

$$x_1^{(1)} = \frac{1}{4}\left(1 - [0, 2] \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -\frac{1}{4}, \ \ x_2^{(1)} = 1\left(2 - [1, 0] \times \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}\right) = \frac{9}{4}$$

$$\vec{x_1} = \begin{bmatrix} -1/4 \\ 9/4 \end{bmatrix}$$

$k = 2$

$$x_1^{(2)} = \frac{1}{4}\left(1 - [0, 2] \times \begin{bmatrix} -1/4 \\ 9/4 \end{bmatrix}\right) = -7/8, \ \ x_2^{(2)} = 1\left(2 - [1, 0] \times \begin{bmatrix} -7/8 \\ 9/4 \end{bmatrix}\right) = 23/8$$

$$\vec{x_2} = \begin{bmatrix} -7/8 \\ 23/8 \end{bmatrix}$$

3-1.3 Successive Over-Relaxation Method

If the Gauss-Seidel iteration equations (*) is written as

$$x_i^{(k)} = x_i^{(k-1)} + \frac{1}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}x_i^{(k)} - \sum_{j=i}^{n} a_{ij}x_i^{(k-1)}\right\}, \quad i = 1,\dots,n$$

Then multiplying the second term of the right hand side of the above by ω, we have

$$\begin{aligned} x_i^{(k)} &= x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}x_i^{(k)} - \sum_{j=i}^{n} a_{ij}x_i^{(k-1)}\right\}, \quad i = 1,\dots,n \\ \Rightarrow \quad x_i^{(k)} &= \frac{\omega}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}x_i^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_i^{(k-1)}\right\} + (1-\omega)x_i^{(k-1)}, \quad i = 1,\dots,n \end{aligned} \tag{**}$$

which gives the SOR iteration equations where the factor ω is called the acceleration parameter or relaxation factor, which generally lies in the range $0 < \omega < 2$. The determination of the optimum value of ω for maximum rate of convergence is again a matter of discussion and is not considered. When $\omega = 1$ gives the Gauss-Seidel iteration, $0 < \omega < 1$ we have **under relaxation** and $1 < \omega < 2$ we have **over relaxation**.

From (**), we can write the matrix notation for the SOR method as follows:

$$\begin{aligned} & x^{(k)} - x^{(k-1)} = \omega D^{-1}\left(-Lx^{(k)} - Ux^{(k-1)} + b - Dx^{(k-1)}\right) \\ \Rightarrow \quad & \left(I + \omega D^{-1}L\right)x^{(k)} = \omega D^{-1}\left(-U - D\right)x^{(k-1)} + x^{(k-1)} + \omega D^{-1}b \\ \Rightarrow \quad & \left(D + \omega L\right)x^{(k)} = \omega\left(-U - D\right)x^{(k-1)} + Dx^{(k-1)} + \omega b \end{aligned}$$

Therefore,

$$x^{(k)} = \left(D + \omega L\right)^{-1}\left(\left(1-\omega\right)D - \omega U\right)x^{(k-1)} + \omega\left(D + \omega L\right)^{-1} b$$

i.e., the SOR iterative matrix and the corresponding vector are given as

$T_{SOR} = \left(D + \omega L\right)^{-1}\left(\left(1-\omega\right)D - \omega U\right)$ and $c_{SOR} = \omega\left(D + \omega L\right)^{-1} b$ respectively.

Example 1

Find an approximation to the solution of $Ax = b$ by performing two iterations of the SOR method where $A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $\omega = 1.2$.

Solution

Using $x_i^{(k)} = \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{2} a_{ij}x_j^{(k-1)}\right) + (1-\omega)x_i^{(k-1)}$, $i = 1, 2,$ $k = 1, 2, \dots,$ we have

$$D=\begin{bmatrix}4 & 0\\ 0 & 1\end{bmatrix},\ A_{\text{off}}=\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}$$

$k=1$

$$x_1^{(1)}=\frac{1.2}{4}\left(1-[0,2]\times\begin{bmatrix}1\\1\end{bmatrix}\right)-0.2=-0,5,\ \ x_2^{(1)}=1.2\left(2-[1,0]\times\begin{bmatrix}-0.5\\1\end{bmatrix}\right)-0.2=2.8$$

$$\vec{x}_1=\begin{bmatrix}-0.5\\2.8\end{bmatrix}$$

$k=2$

$$x_1^{(2)}=\frac{1.2}{4}\left(1-[0,2]\times\begin{bmatrix}-0.5\\2.8\end{bmatrix}\right)+0.1=-1.28,\ \ x_2^{(2)}=1.2\left(2-[1,0]\times\begin{bmatrix}-1.28\\2.8\end{bmatrix}\right)-0.56=3.376$$

$$\vec{x}_2=\begin{bmatrix}-1.28\\3.376\end{bmatrix}$$

3-1.4 A necessary and sufficient condition for the convergence of iterative schemes

Consider any iterative scheme $x^{(k)}=Tx^{(k-1)}+c$, where T is the iterative matrix and c is the constant vector of known values. If $e^{(k)}$ is the error in the k^{th} approximation to the exact solution then it can be written as: $e^{(k)}=x-x^{(k)}$.

Similarly, $e^{(k+1)}=x-x^{(k+1)}$. Therefore $e^{(k+1)}=Te^{(k)}$ or $e^{(k)}=Te^{(k-1)}=T^2e^{(k-2)}=\cdots=T^ke^{(0)}$.

Since the sequence of approximations $\{x^{(0)},x^{(1)},x^{(2)},\ldots,x^{(k)},\ldots\}$ converges to x as n tends to infinity, we have $\lim\limits_{k\to\infty}e^{(k)}=0 \ \Rightarrow\ \lim\limits_{k\to\infty}T^ke^{(0)}=e^{(0)}\lim\limits_{k\to\infty}T^k=0$

If T has n linearly independent Eigenvectors $\nu_r,\ \ r=1,\ldots,n$ then these n vectors can be used as a basis for any n dimensional space and hence any vector in this n dimensional space can be represented in terms of these n vectors. In particular, $e^{(0)}=\sum_{r=1}^{n}c_r\nu_r$, where $c_r,\ \ r=1,\ldots,n$ are scalars. Hence $e^{(1)}=Te^{(0)}=\sum_{r=1}^{n}c_rT\nu_r$.

But $T\nu_r=\lambda_r\nu_r$ by the definition of an eigenvalue, where λ_r is the eigenvalue corresponding to the eigenvector ν_r. Hence $e^{(1)}=\sum_{r=1}^{n}c_r\lambda_r\nu_r$

Similarly $e^{(k)} = \sum_{r=1}^{n} c_r \lambda_r^k v_r$

Therefore $e^{(k)}$ will tend to the null vector as *k* tends to infinity, for arbitrary $e^{(0)}$, if and only if $|\lambda_r| < 1$ for all *r*. In other words the iteration will converge for arbitrary $x^{(0)}$ if and only if, the spectral radius $\rho(T)$ of T is less than unity.

As a corollary to this result a sufficient condition for convergence is that $\|T\| < 1$. To prove this we have that $Tv_r = \lambda_r v_r$. Hence

$$\|Tv_r\| = \|\lambda_r v_r\| = |\lambda_r| \|v_r\|$$
$$\|\lambda_r v_r\| \leq \|T\| \|v_r\|$$

But for any matrix norm that is compatible with a vector norm $\|v_r\|$, $\|Tv_r\| \leq \|T\| \|v_r\|$.

Therefore $\|\lambda_r v_r\| \leq \|T\| \|v_r\|$ so $|\lambda_r| \leq \|T\|, \quad r = 1, \ldots, n$.

It follows from this that a sufficient condition for convergence is that $\|T\| < 1$. It is not a necessary condition because the norm of *T* can exceed one even when $\rho(T) < 1$.