Math 351

Numerical Analysis

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Introduction

Physical phenomenon in Science and Engineering converted into mathematical models usually lead to problems with no analytical solutions.

Numerical Methods is the study, development and analysis of procedures/algorithms for solving problems with no analytical solutions using computers.

The main objective of this course is to equip students with the tools and skills to solve mathematical problems in Engineering/Science with no analytical solutions.

Introduction

There are three central concepts in the analysis of Numerical Techniques:

- *convergence*: whether the method approximates the solution,
- *order*: how well it approximates the solution, and
- *stability*: whether errors are damped out.

Objectives

The course is presented into four units:

- **Unit 1**: Numerical Solutions of Linear Systems using Direct and Iterative Techniques. Solution of Specil Systems:Positive Definite and Tridiagonal systems.
- **Unit 2**: Vector and Matrix Norms; Determinants and Inverses. Determination of Eigenvalues and Eigenvectors; Characteristic Equation Approach and Iterative Techniques.
- **Unit 3**: Interpolation Techniques. Numerical Techniques for Solving Non-Linear Systems.
- **Unit 4**: Numerical Differentiation amd Integration. Numerical Solutions Ordinary Differentiation.

Unit 1

System of Linear Equations

Consider an *nxn* Matrix *A*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The Matrix *A* is said to be a lower or upper triangular matrix if $a_{ij} = 0,\ i < j$ or $i > j$ respectively.

Solutions of Triangular Systems

Consider the Lower Triangular Systems: $Ly = b$, where

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Solutions of Triangular Systems

Writing the above in augmented form, we have

$$\left(\begin{array}{cccc|c} l_{11} & 0 & \cdots & 0 & b_1 \\ l_{21} & l_{22} & \ddots & \vdots & b_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} & b_n \end{array}\right)$$

and can be solved as follows:

$$y_1 = b_1 / l_{11}, \qquad y_2 = (b_2 - l_{21} y_1) / l_{22},$$

$$y_3 = (b_3 - l_{31} y_1 - l_{32} y_2) / l_{33}, \ldots$$

Solutions of Triangular Systems

The above shows that an algorithm can be developed to solve systems of such forms, hence the *forward substitution* algorithm. The algorithm is so named since y_1 is solved first followed by y_2 etc. Below is the *Forward Substitution* Algorithm:

For $i = 1, 2, ..., n$ do

$$y_i = \frac{1}{l_{ii}} \left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right)$$

Example 1 (Lower triangular system)

Solve the following lower triangular system, $Ly = b$ where

$$L = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 7 & 10 & 0 & 0 \\ 6 & 8 & 10 & 0 \\ 5 & 7 & 9 & 10 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ -1 \\ 2 \\ 4 \end{pmatrix}$$

Solution 1: (Lower triangular system)

$$i = 1$$

$$y_1 = \frac{1}{l_{11}}\left(b_1 - \sum_{j=1}^{0} l_{1j} y_j \right) = \frac{1}{5}(5 - 0) = \frac{5}{5} = 1$$

$$i = 2$$

$$y_2 = \frac{1}{l_{22}}\left(b_2 - \sum_{j=1}^{1} l_{2j} y_j \right) = \frac{1}{10}(-1 - 7 \times 1) = -\frac{8}{10} = -0.8$$

Solution 1 (Lower triangular system)

$$i = 3$$

$$y_3 = \frac{1}{l_{33}}\left(b_3 - \sum_{j=1}^{2} l_{3j} y_j\right) = \frac{1}{10}\left(2 - (6\times 1 + 8\times(-0.8))\right) = \frac{2.4}{10} = 0.24$$

$$i = 4$$

$$y_4 = \frac{1}{l_{44}}\left(b_4 - \sum_{j=1}^{3} l_{4j} y_j\right) = \frac{1}{10}\left(4 - (5\times 1 + 7\times(-0.8)) + 9\times 2.4\right) = \frac{2.44}{10} = 0.244$$

Solutions of Triangular Systems

Consider the Upper Triangular Systems: $Ux = y$ where

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Solutions of Triangular Systems

Writing the above in augmented form, we have

$$\left(\begin{array}{cccc|c} u_{11} & u_{12} & \cdots & u_{1n} & y_1 \\ 0 & u_{22} & \ddots & \vdots & y_2 \\ \vdots & \ddots & \ddots & u_{n-1,n} & \vdots \\ 0 & \cdots & 0 & u_{nn} & y_n \end{array}\right)$$

and can be solved as follows:

$$x_n = y_n / u_{nn}, \qquad x_{n-1} = \left(y_{n-1} - u_{n-1,n}x_n\right) / u_{n-1,n-1},$$

$$x_{n-2} = \left(y_{n-2} - u_{n-2,n-1}x_{n-1} - u_{n-2,n}x_n\right) / u_{n-2,n-2}, \ldots$$

Solutions of Triangular Systems

The above shows that an algorithm can be developed to solve systems of such forms, hence the *back substitution* algorithm. The algorithm is so named since x_n is solved first followed by x_{n-1} etc. Below is the <u>*Back Substitution*</u> <u>Algorithm</u>:

For $i = n, n-1, ..., 1$ do

$$x_i = \frac{1}{u_{ii}} \left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j \right)$$

Example 2 (Upper triangular system)

Solve the following upper triangular system, $Ux = y$ where

$$U = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 0 & 10 & 8 & 7 \\ 0 & 0 & 10 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 5 \\ -1 \\ 2 \\ 4 \end{pmatrix}$$

Solution 2 (Upper triangular system)

$$i = 4$$

$$x_4 = \frac{1}{u_{44}}\left(y_4 - \sum_{j=5}^{4} u_{4j} x_j \right) = \frac{1}{10}(4 - 0) = \frac{4}{10} = 0.4$$

$$i = 3$$

$$x_3 = \frac{1}{u_{33}}\left(y_3 - \sum_{j=4}^{4} u_{3j} x_j \right) = \frac{1}{10}(2 - 9 \times 0.4) = -\frac{1.6}{10} = -0.16$$

Solution 2 (Upper triangular system)

$$i = 2$$

$$x_2 = \frac{1}{u_{22}}\left(y_2 - \sum_{j=3}^{4} u_{2j} x_j\right) = \frac{1}{10}\left(-1-(7\times0.4+8\times(-0.16))\right) = -\frac{2.52}{10} = -0.252$$

$$i = 1$$

$$x_1 = \frac{1}{u_{11}}\left(y_1 - \sum_{j=3}^{4} u_{1j} x_j\right) = \frac{1}{5}\left(5-(5\times0.4+6\times(-0.16)+7\times(-0.252))\right)$$

$$= \frac{5.724}{5} = 1.1448$$

Definitions: (Permutation/Elementary) Matrices

A nonzero matrix P is a Permutation matrix if there is exactly one nonzero entry in each row and column that is 1 and the rest are all zeros.

An elementary lower triangular matrix M of order n is a matrix of the form:

$$M = \overbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & & 0 & 0 \\ 0 & \cdots & 0 & m_{k+1,k} & 1 & \ddots & 0 & 0 \\ 0 & & 0 & m_{k+2,k} & 0 & \ddots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_{nk} & 0 & \cdots & 0 & 1 \end{pmatrix}}^{n \text{ elts}}$$

LU factorization

The Gaussian elimination methods for solving the linear system $Ax=b$ is based on the *LU* factorization of the matrix *A*. Three *LU* factorization methods are considered namely:

- *LU* factorization without pivoting,
- *LU* factorization with partial pivoting and
- *LU* factorization with complete pivoting.

LU factorization (without Pivoting)

Algorithm

For $k = 1 \text{ to } n-1$ *do*

Find an elementary matrix M_k *such that* $A^{(k)} = M_k A^{(k-1)}$ *has zeros below (k,k) entry of* $\textit{the } k^{th} \textit{ column.}$

End For

Write $\begin{cases} L = \left(M_{n-1}M_{n-2} \cdots M_2 M_1\right)^{-1} \\ U = M_{n-1}M_{n-2} \cdots M_2 M_1 A \end{cases}$ *and* $A = LU$.

Example

Find the *LU* factorization of

$$A = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$$

using Gaussian Elimination without Pivoting.

Solution to Example

Step 1

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -7/5 & 1 & 0 & 0 \\ -6/5 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad A^{(1)} = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 0 & 1/5 & -2/5 & 0 \\ 0 & -2/5 & 14/5 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

Step 2

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A^{(2)} = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 0 & 1/5 & -2/5 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

Solution to Example

Step 3

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3/2 & 1 \end{pmatrix}, \qquad A^{(3)} = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 0 & 1/5 & -2/5 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7/5 & 1 & 0 & 0 \\ 6/5 & -2 & 1 & 0 \\ 1 & 0 & 3/2 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 0 & 1/5 & -2/5 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

LU factorization (with Partial Pivoting)

Algorithm

For $k = 1 \text{ to } n-1$ *do*

Scan the entries of the k^{th} column of the matrix $A^{(k-1)}$ *below the row* $(k-1)$ *identify the pivot* $a_{r_k,k}$, *such that*

$$\left|a_{r_k,k}\right| = \max_{k \le i \le n} \left|a_{ik}\right|.$$

Form the permutation matrix P_k and the elementary matrix M_k *such that* $A^{(k)} = M_k P_k A^{(k-1)}$ *has zeros* below the (k,k) *entry on the kth column.*

End For

Solution of the System *Ax*=*b*

Write $\begin{cases} L = P\left(M_{n-1}P_{n-1}M_{n-2}P_{n-2} \cdots M_2P_2M_1P_1\right)^{-1} \\ U = M_{n-1}P_{n-1}M_{n-2}P_{n-2} \cdots M_2P_2M_1P_1A \end{cases}$

where $P = \left(P_1P_2 \cdots P_{n-2}P_{n-1}\right)^{-1}$ *and* $PA = LU$.

Example

Find the *LU* factorization of

$$A = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$$

using Gaussian Elimination with partial Pivoting.

Solution

Step 1

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5/7 & 1 & 0 & 0 \\ -6/7 & 0 & 1 & 0 \\ -5/7 & 0 & 0 & 1 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} 7 & 10 & 8 & 7 \\ 0 & -1/7 & 2/7 & 0 \\ 0 & -4/7 & 22/7 & 3 \\ 0 & -1/7 & 23/7 & 5 \end{pmatrix}$$

Step 2

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/4 & 1 & 0 \\ 0 & -1/4 & 0 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 7 & 10 & 8 & 7 \\ 0 & -4/7 & 22/7 & 3 \\ 0 & 0 & -1/2 & -3/4 \\ 0 & 0 & 5/2 & 17/4 \end{pmatrix}$$

Solution

Step 3

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/5 & 1 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 7 & 10 & 8 & 7 \\ 0 & -4/7 & 22/7 & 3 \\ 0 & 0 & 5/2 & 17/4 \\ 0 & 0 & 0 & 1/10 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6/7 & 1 & 0 & 0 \\ 5/7 & 1/4 & 1 & 0 \\ 5/75 & 1/4 & -1/5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 7 & 10 & 8 & 7 \\ 0 & -4/7 & 22/7 & 3 \\ 0 & 0 & 5/2 & 17/4 \\ 0 & 0 & 0 & 1/10 \end{pmatrix}$$

LU factorization (with Complete Pivoting)

Algorithm

For $k=1\ \text{to}\ n-1$ *do*

Scan the entries of the $A^{(k-1)}$ *below the row* $(k-1)$ and *to the right of the column* $(k-1)$ *to identify the pivot* a_{r_k,s_k}, *such that* $\left|a_{r_k,s_k}\right| = \max\limits_{\substack{k\le i\le n \\ k\le j\le n}} \left|a_{ij}\right|$. *Form the permutation* matrices P_k and Q_k, and the elementary matrix M_k *such that* $A^{(k)} = M_k P_k A^{(k-1)} Q_k$ *has zeros below the* (k,k) *entry on the* k^{th} *column.*

End For

LU factorization (with Complete Pivoting)

Write

$$\begin{cases} L = P\left(M_{n-1}P_{n-1}\cdots M_1P_1\right)^{-1} \\ U = M_{n-1}P_{n-1}\cdots M_1P_1AQ_1\cdots Q_{n-1} \end{cases}$$

where $P = \left(P_1P_2\cdots P_{n-2}P_{n-1}\right)^{-1}$, $Q = Q_1Q_2\cdots Q_{n-2}Q_{n-1}$

and $PAQ = LU$.

Example

Find the *LU* factorization of

$$A = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$$

using Gaussian Elimination with complete Pivoting.

Solution

Step 1

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -7/10 & 1 & 0 & 0 \\ -4/5 & 0 & 1 & 0 \\ -7/10 & 0 & 0 & 1 \end{pmatrix}, \quad A^{(1)} = M_1 P_1 A Q_1 = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 0 & 1/10 & 2/5 & 1/10 \\ 0 & 2/5 & 18/5 & 17/5 \\ 0 & 1/10 & 17/5 & 51/10 \end{pmatrix}$$

Solution

Step 2

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & -1/51 & 0 & 1 \end{pmatrix}, \quad A^{(2)} = M_2 P_2 A^{(1)} Q_2 = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 0 & 51/10 & 17/5 & 1/10 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/51 \end{pmatrix}$$

Solution

Step 3

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/4 & 1 \end{pmatrix}, \quad A^{(3)} = M_3 P_3 A^{(2)} Q_3 = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 0 & 51/10 & 17/5 & 1/10 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 1/68 \end{pmatrix}$$

Solution

and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7/10 & 1 & 0 & 0 \\ 4/5 & 2/3 & 1 & 0 \\ 7/10 & 1/51 & 1/4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 0 & 51/10 & 17/5 & 1/10 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 1/68 \end{pmatrix}$$

Solution of the System $Ax=b$

Gaussian Elimination without Pivoting

The solution of the system $Ax=b$ using LU factorization (without pivoting) can be achieved in two stages:

Step 1: Find an LU factorization of A.

Step 2: Solve two triangular systems: the lower triangular system $Ly=b$ first, followed by the upper triangular system $Ux=y$.

Solution of the System *Ax=b*

Once we have the factorization of *A,* the system becomes equivalent to two triangular systems:

$$Ax = b \quad \Rightarrow \quad LUx = b$$

$$\Rightarrow \quad Ly = b, \quad Ux = y$$

Example

Solve the system $Ax = b$ using Gaussian elimination without pivoting, where

$$A = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

Example

We compute *LU* as follows:

Step 1:

$$A=\begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}; \quad M_1=\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}; \quad A^{(1)}=M_1A=\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix}$$

Step 2:

$$A^{(1)}=\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix}; \quad M_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{6}{5} & 1 \end{pmatrix}; \quad A^{(2)}=M_2A^{(1)}=\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix}$$

i.e., $U=A^{(2)}=M_2M_1A$

Example

$$U = \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix}; \text{ and } L = (M_2 M_1)^{-1} = M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix}$$

$$Ly = b \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{15}{2} \\ -7 \end{pmatrix}$$

$$Ux = y \quad \Rightarrow \quad \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{15}{2} \\ -7 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{51}{2} \\ -\frac{115}{8} \\ \frac{7}{4} \end{pmatrix}$$

Solution of the System *Ax=b*

Gaussian Elimination with Partial Pivoting

The solution of the system $Ax=b$ using LU factorization (with partial pivoting) can be achieved in two stages:

Step 1: Find the LU factorization $PA=LU$.

Step 2: Solve two triangular systems: the lower triangular system $Ly = b'$ first, followed by the upper triangular system $Ux = y$ where $b' = Pb$.

Solution of the System A*x*=b

Once we have the factorization of *A*, the system becomes equivalent to two triangular systems:

$$Ax = b \quad \Rightarrow \quad PAx = LUx = Pb = b'$$

$$\Rightarrow \quad Ly = b', \quad Ux = y$$

Example

Solve the system $Ax=b$ using Gaussian elimination with partial pivoting, where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

Solution

Step 1:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ P_1A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix};\ M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix};\ A^{(1)} = M_1P_1A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

Step 2:

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad P_2A^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix};$$

$$U = A^{(2)} = M_2P_2A^{(1)} = M_2P_2M_1P_1A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Note: Defining $P = \left(P_1P_2\right)^{-1}$ and $L = P(M_2P_2M_1P_1)^{-1}$, we have $PA = LU$.

Solution

$$P = \left(P_1 P_2\right)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \quad L = P(M_2 P_2 M_1 P_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Verify: $PA = LU$.

$$Ly = Pb = b' \;\; \Rightarrow \;\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \;\; \Rightarrow \;\; \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}$$

$$Ux = y \;\; \Rightarrow \;\; \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix} \;\; \Rightarrow \;\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution of the System *Ax=b*

Gaussian Elimination with Complete Pivoting

The solution of the system $Ax=b$ using *LU* factorization (with complete pivoting) can be achieved in two stages:

Step 1: Find the *LU* factorization $PAQ=LU$.

Step 2: Solve the two triangular systems: the lower triangular system $Ly = b'$ where $b' = Pb$ first, followed by the upper triangular system $Ux' = y$, and write the solution as $x = Qx'$.

Solution of the System *Ax*=*b*

Once we have the factorization of *A*, the system becomes equivalent to two triangular systems:

$$Ax = b \;\Rightarrow\; PAQx' = LUx' = Pb = b'$$

$$\Rightarrow \quad Ly = b', \;\; Ux' = y, \quad x = Qx'$$

Example

Solve the system $Ax=b$ using Gaussian elimination with complete pivoting, where

$$A=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad b=\begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

Solution

Step 1:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ P_1A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix};\ Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$P_1AQ_1 = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix};\ M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix};\ A^{(1)} = M_1P_1AQ_1 = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Solution

$Step\ 2:$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad P_2 A^{(1)} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}; \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$P_2 A^{(1)} Q_2 = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix};$$

$$U = A^{(2)} = M_2 P_2 A^{(1)} Q_2 = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Solution

$$P=\left(P_1P_2\right)^{-1}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix};\ Q=Q_1Q_2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix};\ \text{and}$$

$$L=P(M_2P_2M_1P_1)^{-1}=\begin{pmatrix}1&0&0\\\frac{1}{3}&1&0\\\frac{1}{3}&-\frac{1}{2}&1\end{pmatrix}$$

Verify: $PAQ=LU$.

Solution

$$Ly = Pb = b' \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$$Ux' = y \Rightarrow \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ \frac{1}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x = Qx' = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution of $Ax = b$ without Explicit Factorization

In this case the solution can be obtained by solving an upper triangular system obtained by processing the matrix A and the vector b simultaneously.

An augmented matrix $(A|b)$ is formed, and triangularized and the solution is then obtained by back substitution.

Example

: Solve the system $Ax=b$ without explicit factorization where *A* and *b* are given as follows:

$$A=\begin{pmatrix}1&1&1\\1&5&5\\1&5&14\end{pmatrix},\quad b=\begin{pmatrix}2\\6\\3\end{pmatrix}$$

Solution:

$k=1$:

The pivot entry is $a_{11}=1,\ r_1=1.$

$$(A\,|\,b)=\left(\begin{array}{ccc|c}1&1&1&2\\1&5&5&6\\1&5&14&3\end{array}\right),\ m_{21}=-\frac{a_{21}}{a_{11}}=-1,\ \left(A^{(1)}\,|\,b^{(1)}\right)=\left(\begin{array}{ccc|c}1&1&1&2\\0&4&4&4\\0&4&13&1\end{array}\right)$$

Solution:

$k=2$:

The pivot entry is $a_{22}=4,\ r_2=2.$

$$m_{32}=-\frac{a_{32}}{a_{22}}=-1,\qquad \left(A^{(2)}\,|\,b^{(2)}\right)=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 9 & -3 \end{array}\right)$$

The reduced triangular system $A^{(2)}x=b^{(2)}$ is

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 4x_2 + 4x_3 &= 4 \\ 9x_3 &= -3 \end{aligned}$$

The solution is $x_3=-\frac{1}{3},\ x_2=\frac{4}{3},\ x_1=1.$

Example

Solve the system $Ax = b$ using *Partial Pivoting without Explicit Factorization* where *A* and *b* are given as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

Solution

$k = 1:$

The pivot entry is $a_{31} = 4,\ r_1 = 3.$ Interchanged rows 3 and 1 of *A* and *b*.

$$(A \mid b) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 2 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{array}\right), \quad m_{21} = -\frac{a_{21}}{a_{11}} = -\frac{1}{2}, \quad \left(A^{(1)} \mid b^{(1)}\right) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 0 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 0 & 1 & 1 & 2 \end{array}\right)$$

Solution Cont'd

$k = 2:$

The pivot entry is $a_{22} = \frac{3}{2},\ r_2 = 2.$

$$m_{32} = -\frac{a_{32}}{a_{22}} = -\frac{2}{3}, \qquad \left(A^{(2)} \mid b^{(2)}\right) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 0 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 0 & 0 & -\frac{2}{3} & -1 \end{array}\right)$$

The reduced triangular system $A^{(2)}x = b^{(2)}$ is

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 3 \\ \tfrac{3}{2}x_2 + \tfrac{5}{2}x_3 &= \tfrac{9}{2} \\ -\tfrac{2}{3}x_3 &= -1 \end{aligned}$$

The solution is $x_3 = \frac{3}{2},\ x_2 = \frac{1}{2},\ x_1 = \frac{1}{4}.$

Example

Solve the system $Ax = b$ using *Complete Pivoting without Explicit Factorization* where *A* and *b* are given as follows:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

Solution

$k = 1:$

The pivot entry is $a_{32} = 4,\ r_1 = 3,\ s_1 = 2.$

Interchanging rows 3 and 1 of *A* and *b*, and columns 2 and 1 of *A, we have*

$$(A \mid b) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 2 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{array}\right), \quad m_{21} = -\frac{a_{21}}{a_{11}} = -\frac{1}{2}, \quad \left(A^{(1)} \mid b^{(1)}\right) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 0 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 0 & 1 & 1 & 2 \end{array}\right)$$

Solution Cont'd

$k = 2:$

The pivot entry is $a_{23} = \frac{5}{2},\ r_2 = 2,\ s_2 = 3.$ No rows interchanges needed. Interchanging columns 3 and 2 of *A*, we have

$$\left(A^{(1)} \mid b^{(1)}\right) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 0 & \frac{5}{2} & \frac{3}{2} & \frac{9}{2} \\ 0 & 1 & 1 & 2 \end{array}\right),\ m_{32} = -\frac{1}{5/2} = -\frac{2}{5},\ \left(A^{(2)} \mid b^{(2)}\right) = \left(\begin{array}{ccc|c} 4 & 1 & 1 & 3 \\ 0 & \frac{5}{2} & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & \frac{5}{2} & \frac{1}{5} \end{array}\right)$$

The reduced triangular system $A^{(2)}x = b^{(2)}$ is

$$\begin{aligned} 4y_1 + y_2 + y_3 &= 3 \\ \tfrac{5}{2} y_2 + \tfrac{3}{2} y_3 &= \tfrac{9}{2} \\ \tfrac{2}{5} y_3 &= \tfrac{1}{5} \end{aligned}$$

The solution is $y_3 = x_1 = \frac{1}{2},\ y_2 = x_3 = \frac{3}{2},\ y_1 = x_2 = \frac{1}{4}.$

Vector and Matrix Norms

A vector norm on a vector space X is a real-valued function $x \to \|x\|_p$, on X having the following properties:

- $\|x\|_p > 0$, for all nonzero vectors x.
- $\|\alpha x\|_p = |\alpha| \|x\|_p$, for all vectors x and scalars α.
- $\|x + y\|_p \le \|x\|_p + \|x\|_p$, for all vectors x and y.

On R^n, the simplest vector norms are:

- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, $\quad (l_1 - \text{vector norm})$
- $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$, $\quad (\text{Euclidean}/l_2 - \text{vector norm})$
- $\|x\|_\infty = \max_{1 \le i \le n} \{|x_i|\}$, $\quad (l_\infty - \text{vector norm})$

where x_i denotes the i^{th} component of the vector x.

Vector and Matrix Norms

When a vector norm has been specified on R^n, the related matrix norm for an $n \times n$ matrix A is defined as follows:

$$\|A\|_p = \sup\left\{\|Ax\|_p : x \in R^n, 0 \neq \|x\|_p \leq 1\right\} = \max_{x \neq 0}\left\{\frac{\|Ax\|_p}{\|x\|_p}\right\}$$

This matrix norm is the subordinate norm to the given vector norm or the norm induced by the given vector norm.

On R^n, the simplest matrix norms are:

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|,$ $\quad (l_1 - matrix \text{ norm})$
- $\|A\|_2 = \max_{1 \leq k \leq n} \sigma_k,$ $\quad (\text{Spectral radius}/l_2 - matrix \text{ norm})$
- $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|,$ $\quad (l_\infty - matrix \text{ norm})$
- $\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2},$ $\quad (Frobenius - matrix \text{ norm})$

Vector and Matrix Norms

A vector norm on a vector space X is a real-valued function $x \to \|x\|_p$, on X having the following properties:

- $\|x\|_p > 0,$ for all nonzero vectors x.
- $\|\alpha x\|_p = |\alpha| \|x\|_p$, for all vectors x and scalars α.
- $\|x + y\|_p \leq \|x\|_p + \|x\|_p$, for all vectors x and y.

The Power Method

Let σ be an approximation to a real eigenvalue λ_1 such that ; $|\lambda_1 - \sigma| = |\lambda_i - \sigma| \quad (i \neq 1)$; that is, σ is much closer to λ_1 than to the other eigenvalues.

$$
\begin{aligned}
&\mathit{Step}\ 1: && \text{Choose } x_0 \\
&\mathit{Step}\ 2: && \text{For } k = 1, 2, 3, \ldots \text{ do} \\
& && \hat{x}_k = A x_{k-1} \\
& && x_k = \hat{x}_k / \max(\hat{x}_k)
\end{aligned}
$$

The sequence $\{x_k\}$ converges to the direction of the eigenvector corresponding to λ_1 .

The Inverse Power Method/ Inverse Iteration

Let σ be an approximation to a real eigenvalue λ_1 such that ; $|\lambda_1 - \sigma| = |\lambda_i - \sigma| \quad (i \neq 1)$; that is, σ is much closer to λ_1 than to the other eigenvalues.

Step 1: Choose x_0

Step 2: For $k = 1, 2, 3, \ldots$ do

Solve $\left(A - \sigma I\right)\hat{x}_k = x_{k-1}$ for $\hat{x}_k$,

(using Gaussian elimination with partial pivoting.)

$$x_k = \hat{x}_k / \max\left(\hat{x}_k\right)$$

The sequence $\{x_k\}$ converges to the direction of the eigenvector corresponding to λ_1 .