14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

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14.3
Partial Derivatives

In this section, we will learn about:

Various aspects of partial derivatives.

INTRODUCTION

On a hot day, extreme humidity makes us think the temperature is higher than it really is.

In very dry air, though, we perceive the temperature to be lower than the thermometer indicates.

HEAT INDEX

The National Weather Service (NWS) has devised the heat index to describe the combined effects of temperature and humidity.

- This is also called the temperature-humidity index, or humidex, in some countries.

HEAT INDEX

The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H.

- So, I is a function of T and H.
- We can write $I = f(T, H)$.

HEAT INDEX

This table of values of I is an excerpt from a table compiled by the NWS.

		Relative humidity (%)								
	T \ H	50	55	60	65	70	75	80	85	90
Actual temperature (°F)	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

HEAT INDEX

Let's concentrate on the highlighted column.

- It corresponds to a relative humidity of *H* = 70%.
- Then, we are considering the heat index as a function of the single variable *T* for a fixed value of *H*.

Relative humidity (%)

	$T \backslash H$	50	55	60	65	70	75	80	85	90
Actual temperature (°F)	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

Let's write $g(T) = f(T, 70)$.

Then, $g(T)$ describes:

- How the heat index I increases as the actual temperature T increases when the relative humidity is 70%.

HEAT INDEX

The derivative of *g* when *T* = 96°F is the rate of change of *I* with respect to *T* when *T* = 96°F:

$$g'(96) = \lim_{h \to 0} \frac{g(96+h) - g(96)}{h}$$

$$= \lim_{h \to 0} \frac{f(96+h, 70) - f(96, 70)}{h}$$

HEAT INDEX

We can approximate $g'(96)$ using the values in the table by taking h = 2 and –2.

$$g'(96) \approx \frac{g(98)-g(96)}{2} = \frac{f(98,70)-f(96,70)}{2}$$

$$= \frac{133-125}{2} = 4$$

$$g'(96) \approx \frac{g(94)-g(96)}{-2} = \frac{f(94,70)-f(96,70)}{-2}$$

$$= \frac{118-125}{-2} = 3.5$$

HEAT INDEX

Averaging those values, we can say that the derivative $g'(96)$ is approximately 3.75

This means that:

- When the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises!

Now, let's look at the highlighted row.

- It corresponds to a fixed temperature of T = 96°F.

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

The numbers in the row are values of the function $G(H) = f(96, H)$.

- This describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96°F$.

HEAT INDEX

The derivative of this function when *H* = 70% is the rate of change of *I* with respect to *H* when *H* = 70%:

$$G'(70) = \lim_{h \to 0} \frac{G(70+h) - G(70)}{h}$$

$$= \lim_{h \to 0} \frac{f(96, 70+h) - f(96, 70)}{h}$$

HEAT INDEX

By taking *h* = 5 and –5, we approximate G'(70) using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96,75) - f(96,70)}{5}$$

$$= \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96,65) - f(96,70)}{-5}$$

$$= \frac{121 - 125}{-5} = 0.8$$

HEAT INDEX

By averaging those values, we get the estimate $G'(70) \approx 0.9$

This says that:

- When the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

- Then, we are really considering a function of a single variable x:

$$g(x) = f(x, b)$$

PARTIAL DERIVATIVE

If g has a derivative at a, we call it the partial derivative of f with respect to x at (a, b).

We denote it by:

$$f_x(a, b)$$

Equation 1

Thus,

$$f_x(a, b) = g'(a)$$

where $g(x) = f(a, b)$

By the definition of a derivative, we have:

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

Equation 2

So, Equation 1 becomes:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

PARTIAL DERIVATIVE

Similarly, the partial derivative of f with respect to y at (a, b), denoted by $f_y(a, b)$, is obtained by:

- Keeping x fixed $(x = a)$
- Finding the ordinary derivative at b of the function $G(y) = f(a, y)$

Equation 3

Thus,

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

With that notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when T = 96°F and H = 70% as:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 0.9$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

Equations 4

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

NOTATIONS

There are many alternative notations for partial derivatives.

- For instance, instead of f_x, we can write f_1 or D_1f (to indicate differentiation with respect to the first variable) or $\partial f/\partial x$.

- However, here, $\partial f/\partial x$ can't be interpreted as a ratio of differentials.

NOTATIONS FOR PARTIAL DERIVATIVES

If $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}$$

$$= f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}$$

$$= f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is:

- Remember from Equation 1 that the partial derivative with respect to x is just the ordinary derivative of the function g of a single variable that we get by keeping y fixed.

RULE TO FIND PARTIAL DERIVATIVES OF $z = f(x, y)$

Thus, we have this rule.

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1

If

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

find

$$f_x(2, 1) \text{ and } f_y(2, 1)$$

Example 1

Holding y constant and differentiating with respect to x, we get:

$$f_x(x, y) = 3x^2 + 2xy^3$$

- Thus,

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get:

$$f_y(x, y) = 3x^2y^2 - 4y$$

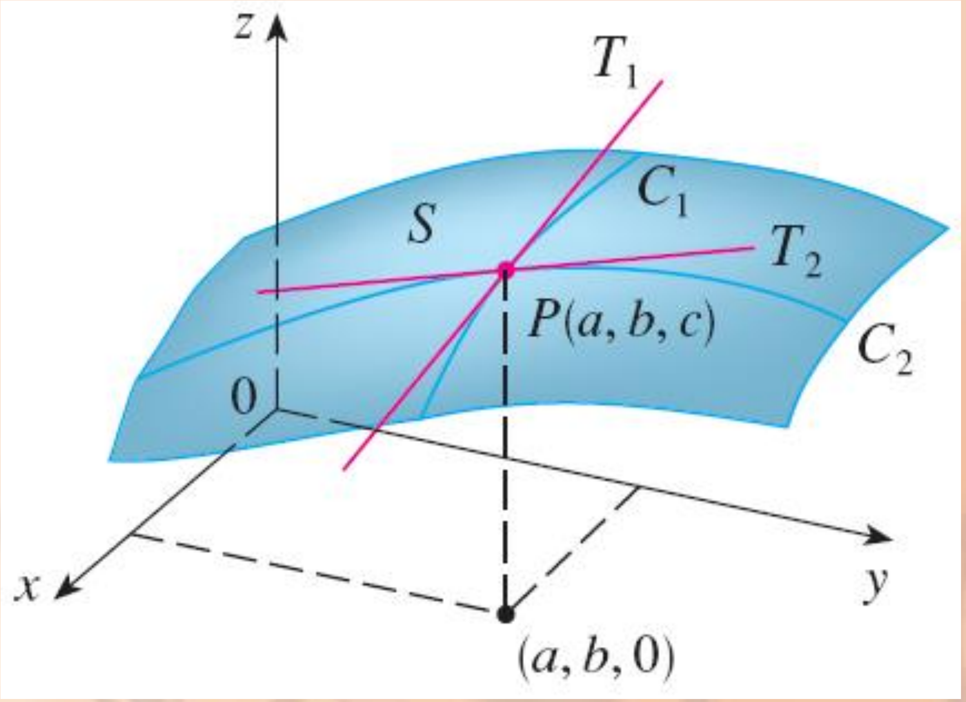
- Thus,

$$\begin{aligned} f_y(2, 1) &= 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 \\ &= 8 \end{aligned}$$

GEOMETRIC INTERPRETATION

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f).

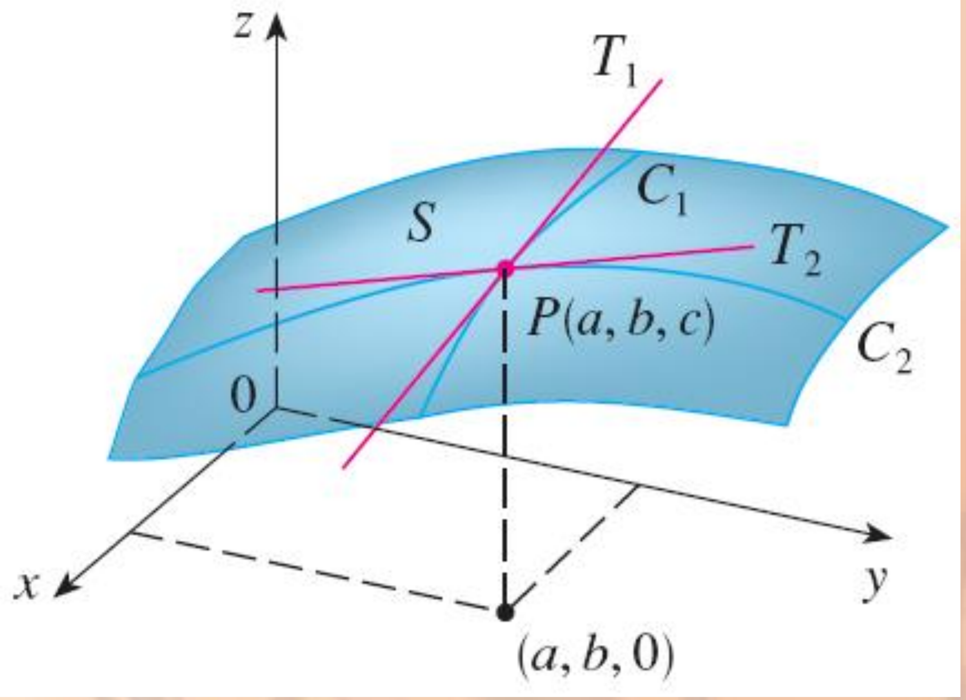
- If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.



GEOMETRIC INTERPRETATION

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S.

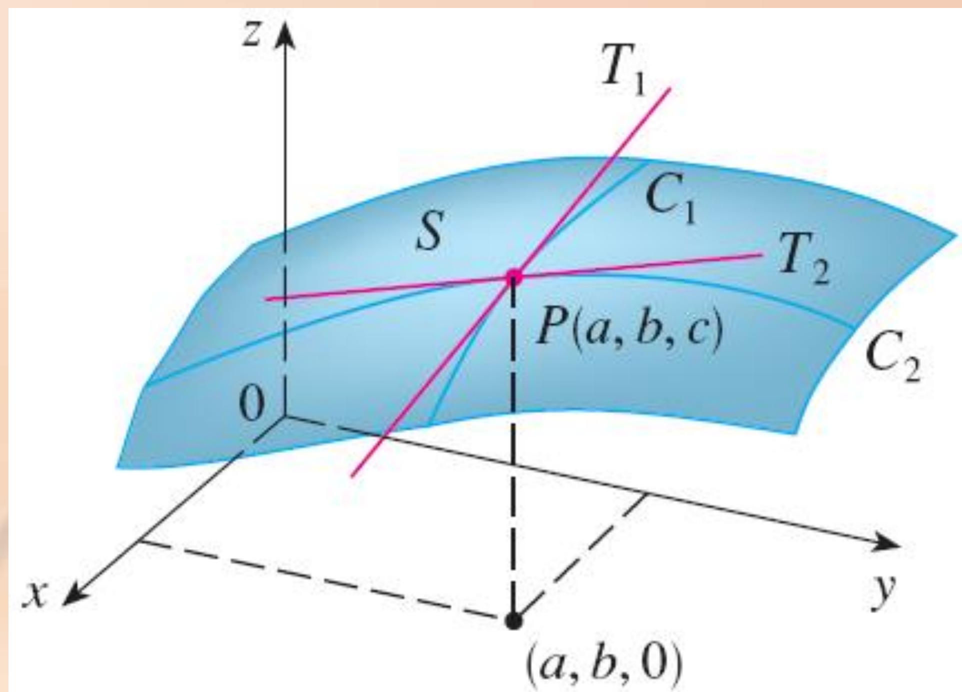
- That is, C_1 is the trace of S in the plane $y = b$.



GEOMETRIC INTERPRETATION

Likewise, the vertical plane $x = a$ intersects S in a curve C_2.

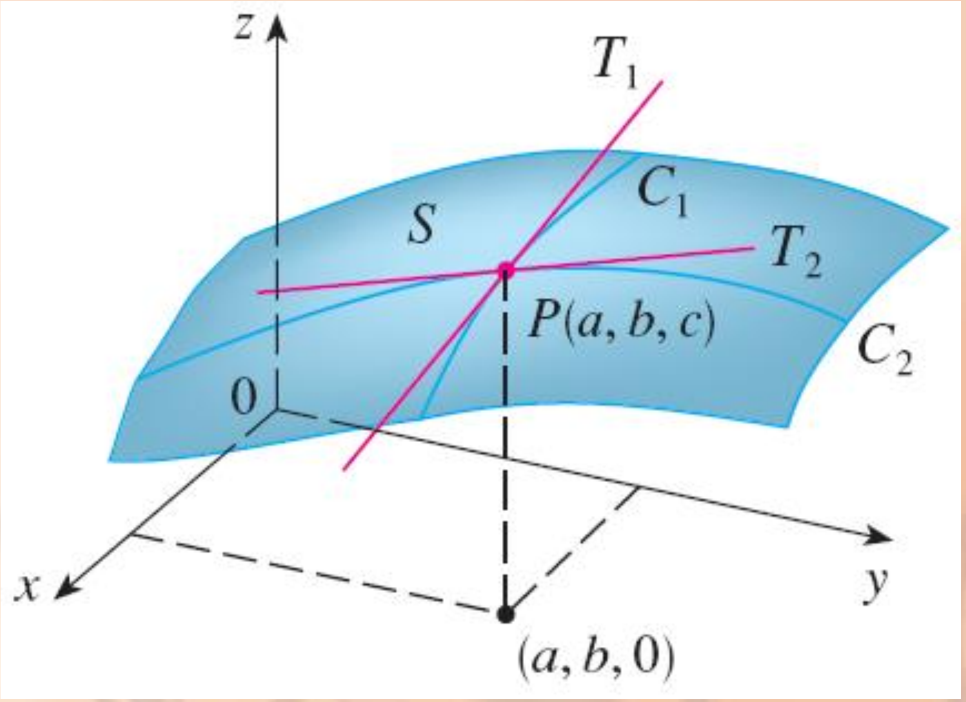
Both the curves C_1 and C_2 pass through P.



GEOMETRIC INTERPRETATION

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$.

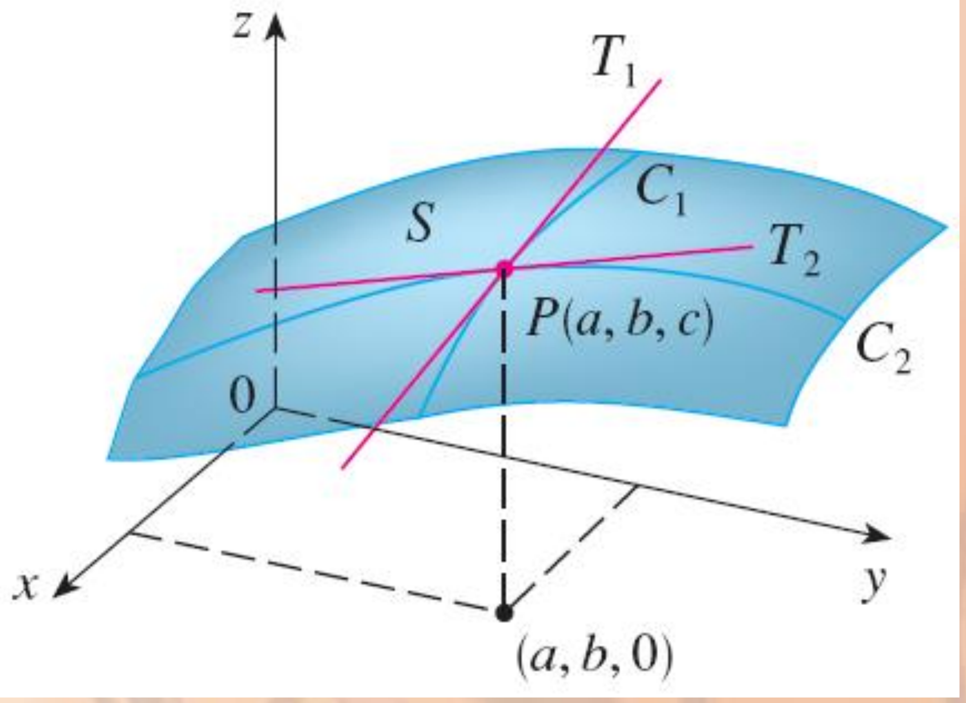
- So, the slope of its tangent T_1 at P is: $g'(a) = f_x(a, b)$



GEOMETRIC INTERPRETATION

The curve C_2 is the graph of the function $G(y) = f(a, y)$.

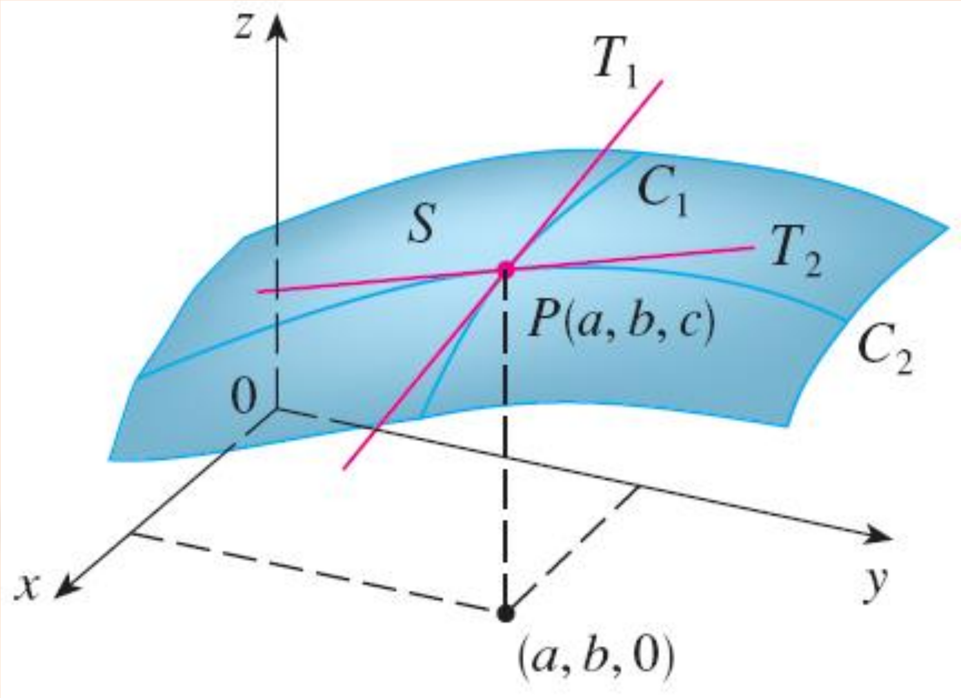
- So, the slope of its tangent T_2 at P is: $G'(b) = f_y(a, b)$



GEOMETRIC INTERPRETATION

Thus, the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as:

- The slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.



INTERPRETATION AS RATE OF CHANGE

As seen in the case of the heat index function, partial derivatives can also be interpreted as rates of change.

- If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed.
- Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

GEOMETRIC INTERPRETATION

Example 2

If

$$f(x, y) = 4 - x^2 - 2y^2$$

find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

We have:

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

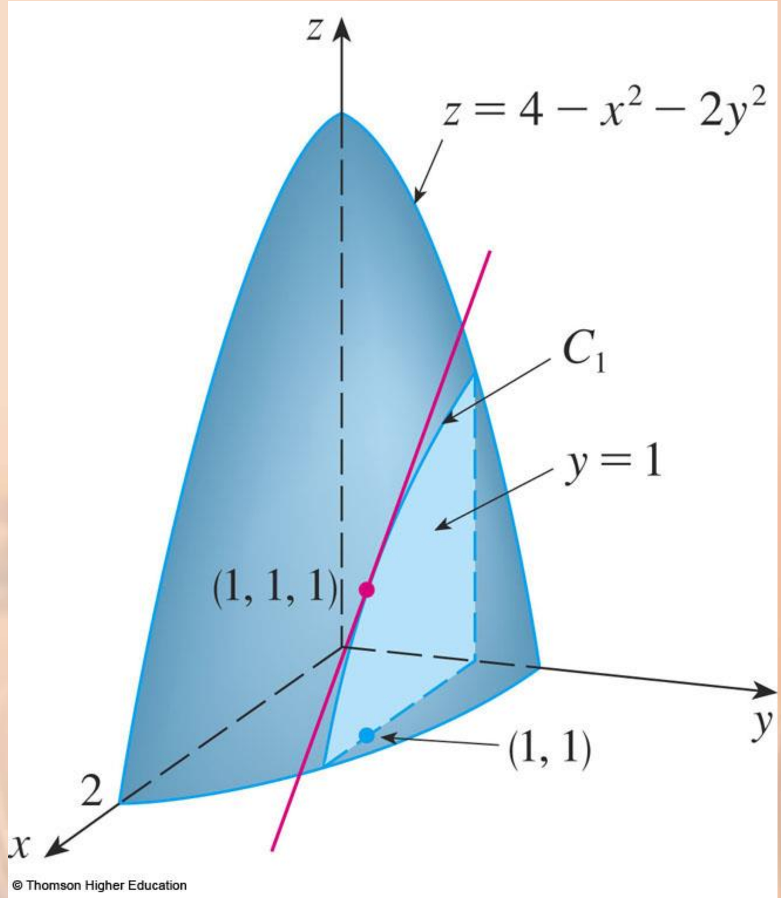
GEOMETRIC INTERPRETATION **Example 2**

The graph of f is the paraboloid

$$z = 4 - x^2 - 2y^2$$

The vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$.

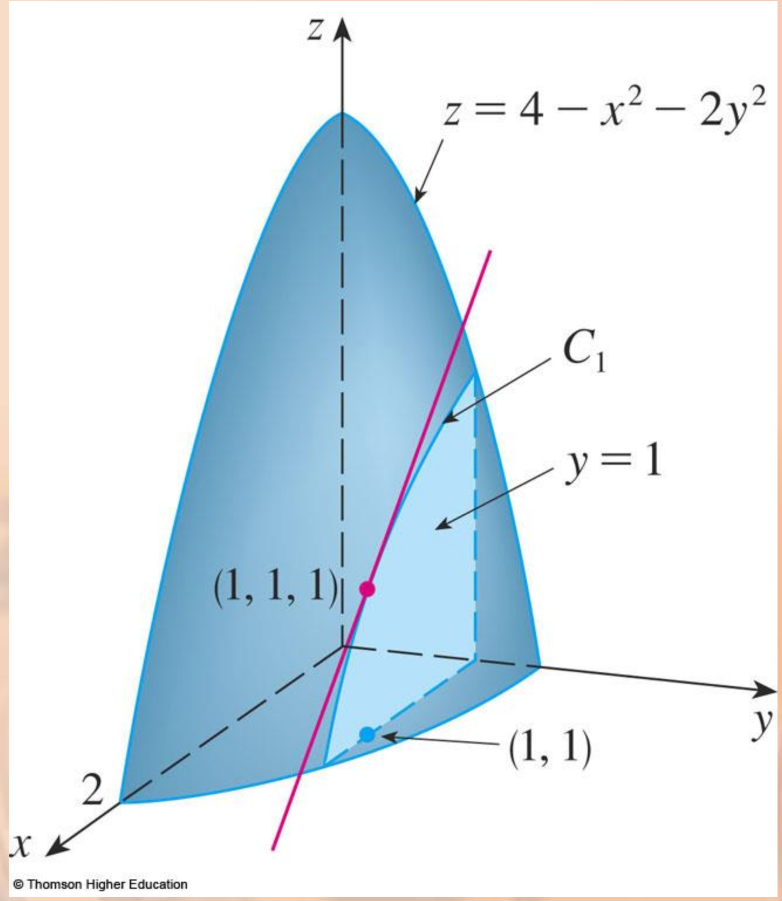
- As discussed, we label it C_1.



GEOMETRIC INTERPRETATION **Example 2**

The slope of the tangent line to this parabola at the point (1, 1, 1) is:

$$f_x(1, 1) = -2$$



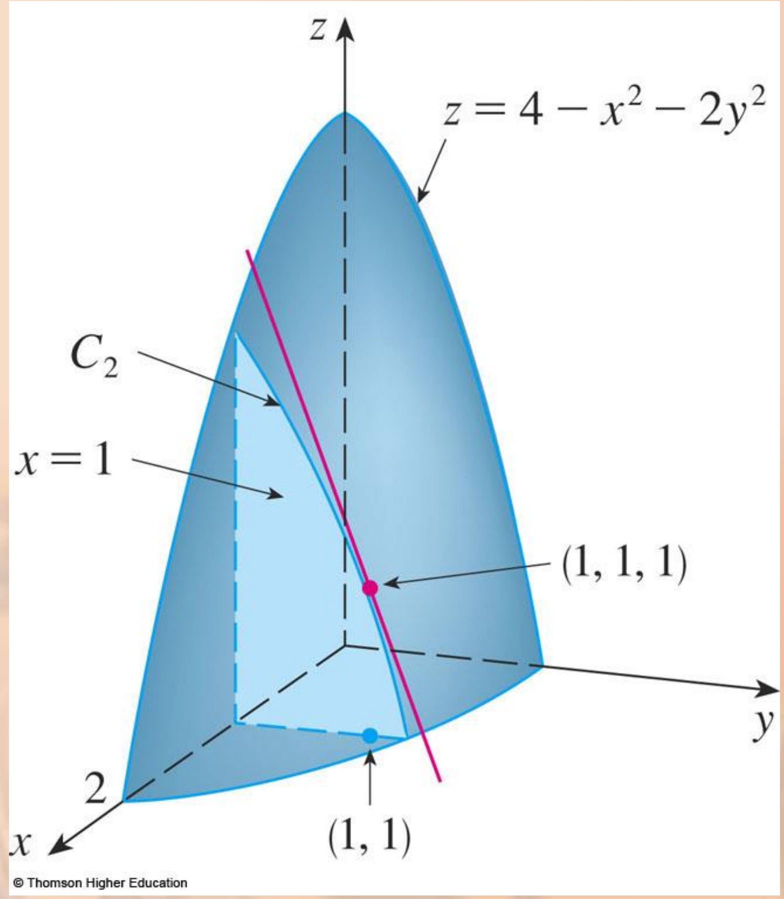
GEOMETRIC INTERPRETATION

Example 2

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$.

- The slope of the tangent line at (1, 1, 1) is:

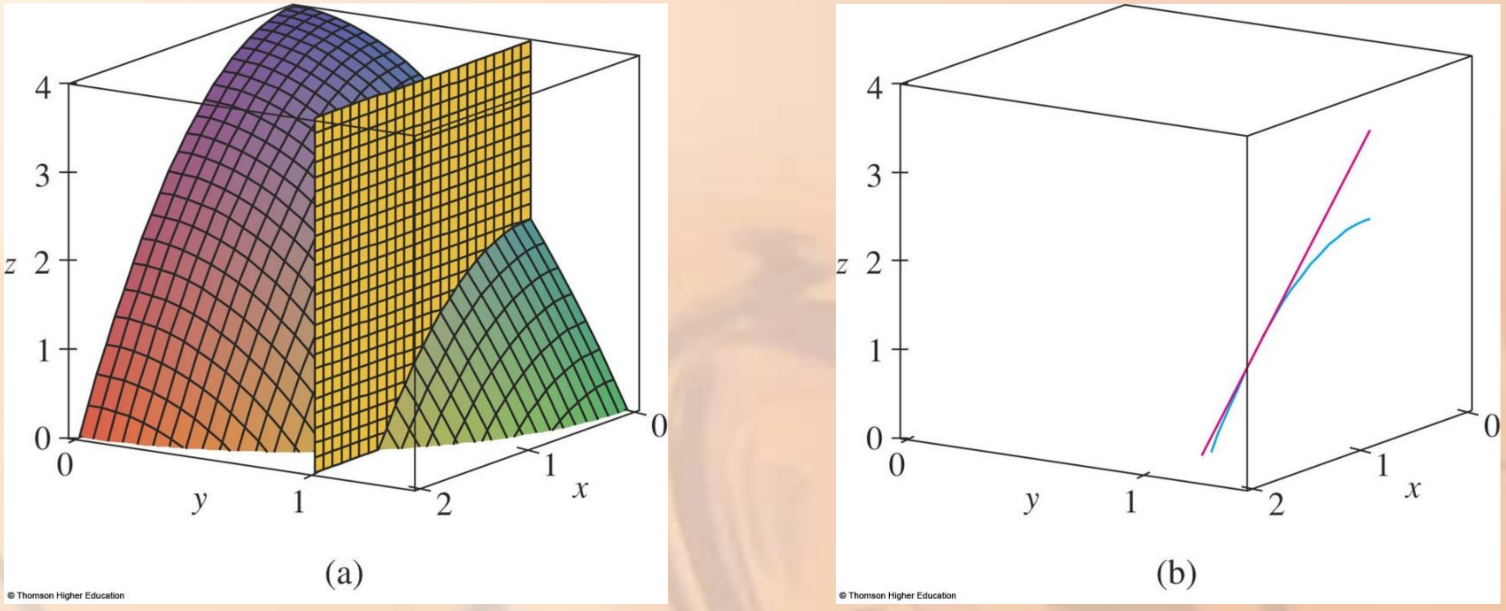
$$f_y(1, 1) = -4$$



GEOMETRIC INTERPRETATION

This is a computer-drawn counterpart to the first figure in Example 2.

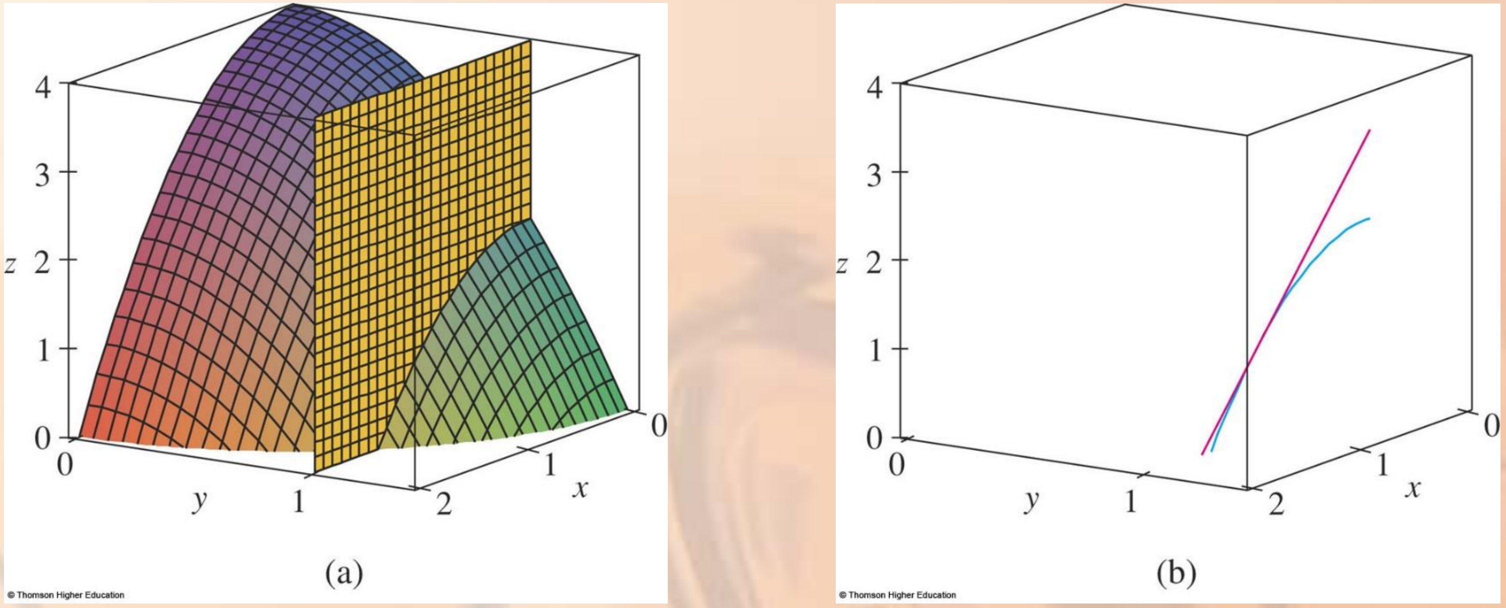
- The first part shows the plane $y = 1$ intersecting the surface to form the curve $C_{1.}$
- The second part shows C_1 and T_1.



GEOMETRIC INTERPRETATION

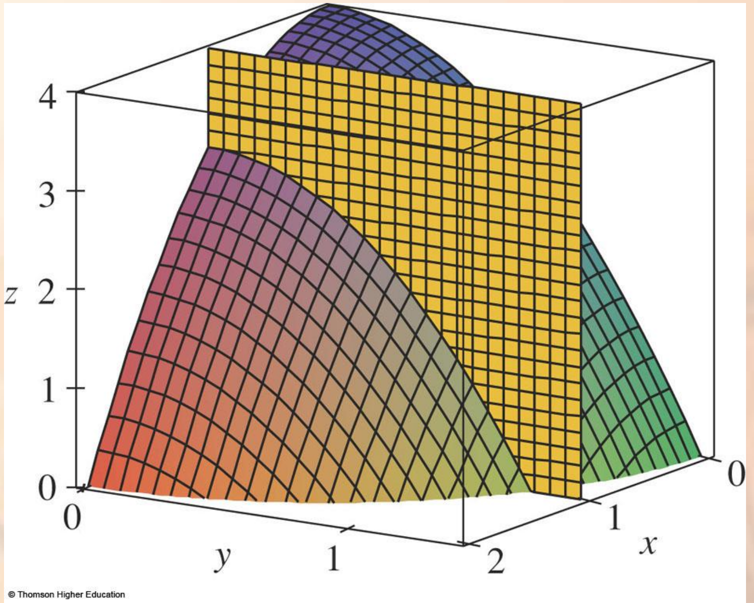
We have used the vector equations:

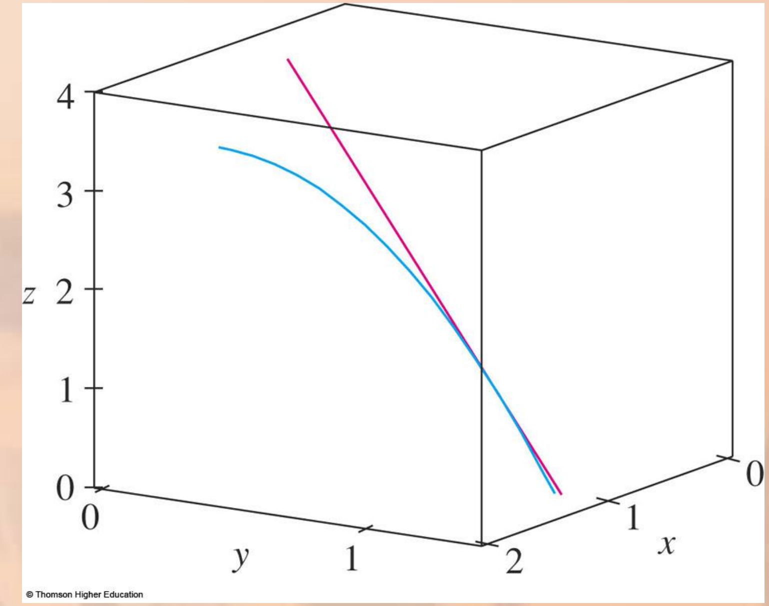
- $\mathbf{r}(t) = \langle t, 1, 2 - t^2 \rangle$ for C_1
- $\mathbf{r}(t) = \langle 1 + t, 1, 1 - 2t \rangle$ for T_1



GEOMETRIC INTERPRETATION

Similarly, this figure corresponds to the second figure in Example 2.





Example 3

If

$$f(x, y) = \sin\left(\frac{x}{1+y}\right)$$

calculate

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

Example 3

Using the Chain Rule for functions of one variable, we have:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example 4

Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Example 4

To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

- Solving for $\partial z/\partial x$, we obtain:

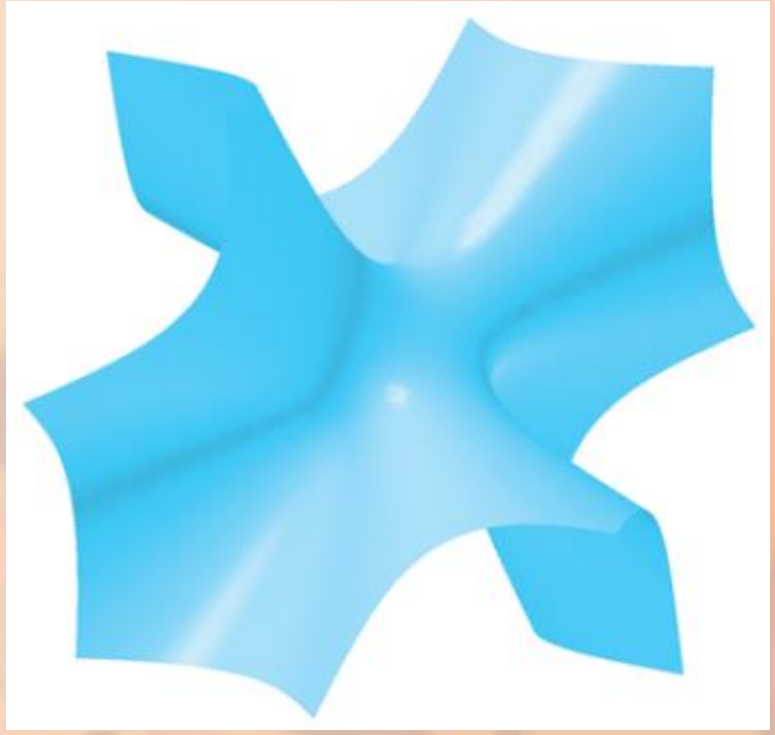
$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to *y* gives:

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Some computer algebra systems can plot surfaces defined by implicit equations in three variables.

- The figure shows such a plot of the surface defined by the equation in Example 4.



Partial derivatives can also be defined for functions of three or more variables.

- For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

It is found by:

- Regarding y and z as constants.
- Differentiating $f(x, y, z)$ with respect to x.

FUNCTIONS OF MORE THAN TWO VARIABLES

If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed.

- However, we can’t interpret it geometrically since the graph of f lies in four-dimensional space.

FUNCTIONS OF MORE THAN TWO VARIABLES

In general, if *u* is a function of *n* variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the *i* th variable x_i is:

$$\frac{\partial u}{\partial x_i} =$$

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

Then, we also write:

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

 Example 5

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$

- Holding y and z constant and differentiating with respect to x, we have:

$$f_x = ye^{xy} \ln z$$

- Similarly,

$$f_y = xe^{xy} \ln z \qquad f_z = e^{xy}/z$$

HIGHER DERIVATIVES

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.

SECOND PARTIAL DERIVATIVES

So, we can consider their partial derivatives

$$(f_x)_x\,,\ (f_x)_y\,,\ (f_y)_x\,,\ (f_y)_y$$

These are called the second partial derivatives of *f*.

NOTATION

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

SECOND PARTIAL DERIVATIVES

Thus, the notation f_{xy} (or $\partial^2 f/\partial y \partial x$) means that we first differentiate with respect to x and then with respect to y.

In computing f_{yx} , the order is reversed.

 Example 6

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

- In Example 1, we found that:

$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$

- Hence,

$$f_{xx} = \frac{\partial}{\partial x}\left(3x^2 + 2xy^3\right) = 6x + 2y^3$$

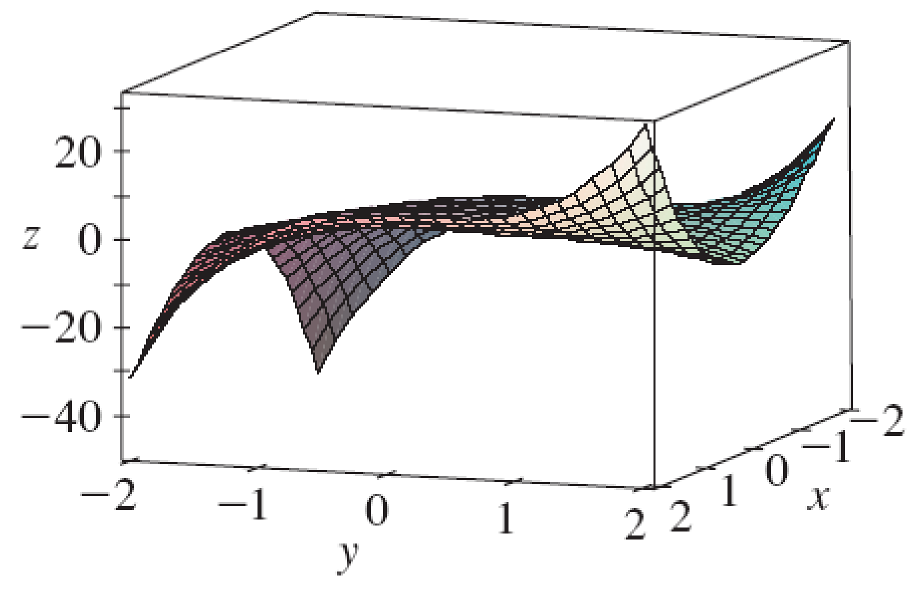
$$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}\left(3x^2y^2 - 4y\right) = 6x^2y - 4$$

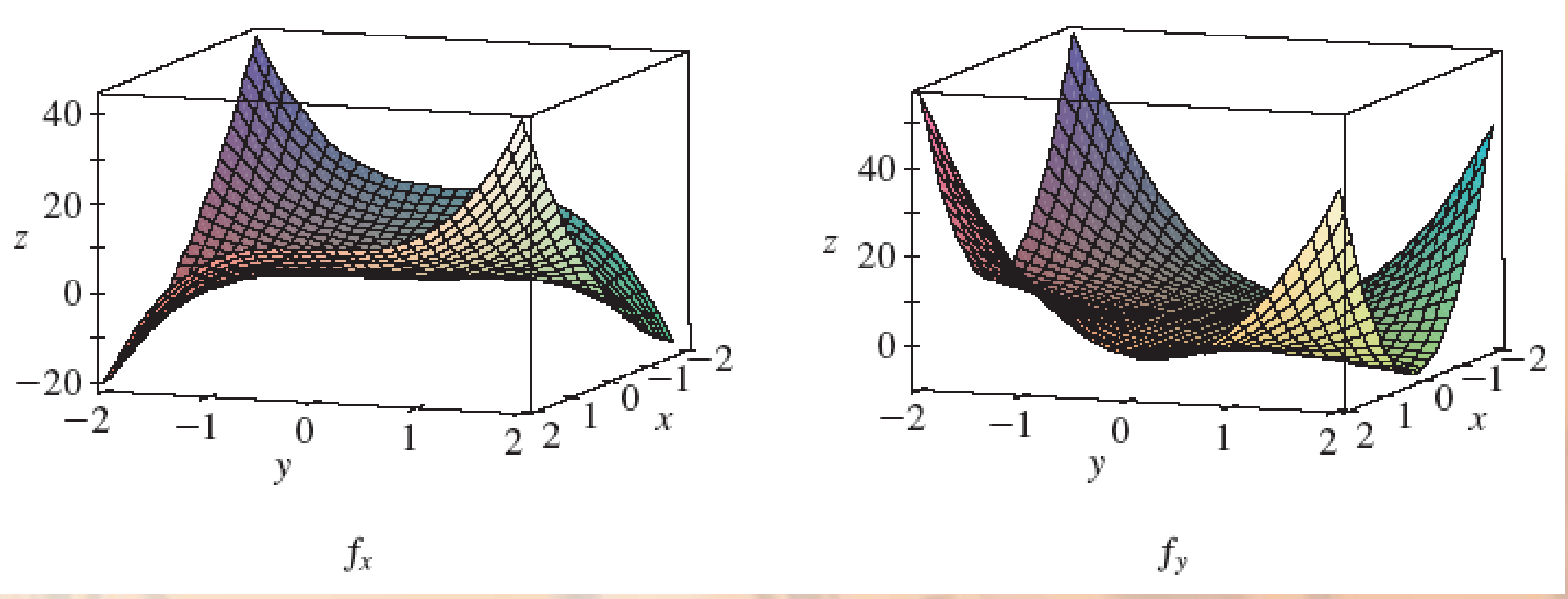
The figure shows the graph of the function *f* in Example 6 for:

$$-2 \leq x \leq 2, -2 \leq y \leq 2$$



f

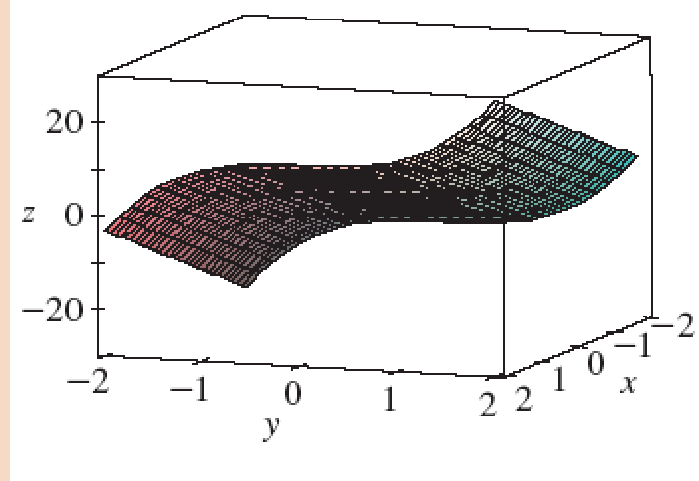
These show the graphs of its first-order partial derivatives.



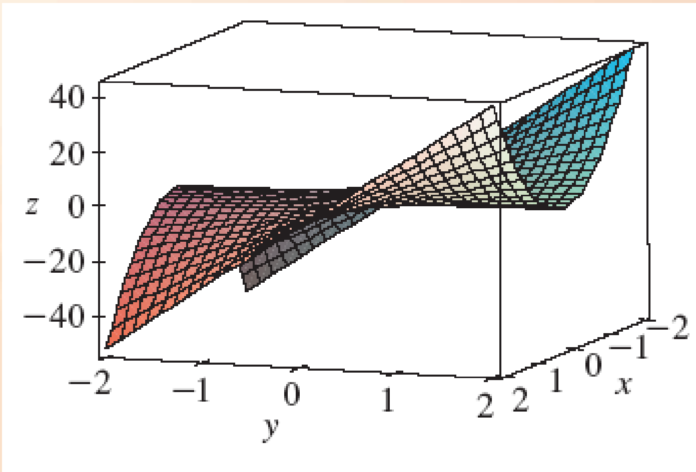
f_x f_y

SECOND PARTIAL DERIVATIVES

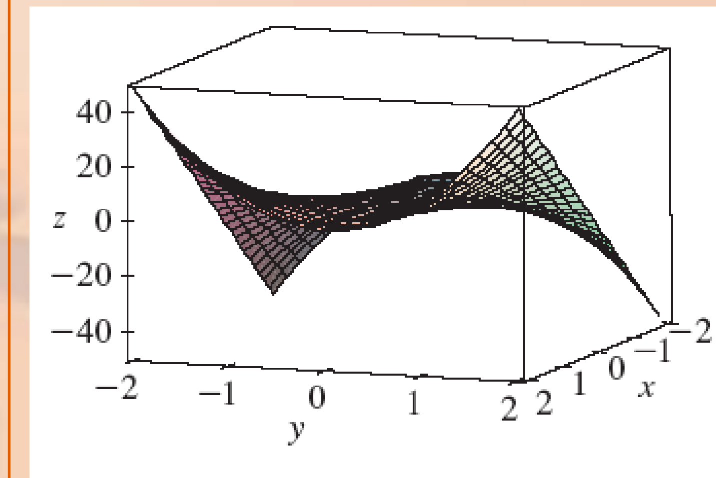
These show the graphs of its second-order partial derivatives.



f_{xx}



f_{yy}



$f_{xy} = f_{yx}$

Notice that $f_{xy} = f_{yx}$ in Example 6.

- This is not just a coincidence.
- It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

- The proof is given in Appendix F.

CLAIRAUT'S THEOREM

Suppose f is defined on a disk D that contains the point (a, b).

If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 or higher can also be defined.

- For instance, $f_{xyy} = \left(f_{xy} \right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 f}{\partial y\, \partial x} \right) = \dfrac{\partial^3 f}{\partial y^2\, \partial x}$
- Using Clairaut's Theorem, it can be shown that

$$f_{xyy} = f_{yxy} = f_{yyx}$$

if these functions are continuous.

Example 7

Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$

- $f_x = 3\cos(3x + yz)$
- $f_{xx} = -9\sin(3x + yz)$
- $f_{xxy} = -9z\cos(3x + yz)$
- $f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$

PARTIAL DIFFERENTIAL EQUATIONS

Partial derivatives occur in partial differential equations that express certain physical laws.

LAPLACE'S EQUATION

For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called Laplace's equation after

Pierre Laplace (1749–1827).

HARMONIC FUNCTIONS

Solutions of this equation are called harmonic functions.

- They play a role in problems of heat conduction, fluid flow, and electric potential.

LAPLACE'S EQUATION

Example 8

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

- $u_x = e^x \sin y$
- $u_y = e^x \cos y$
- $u_{xx} = e^x \sin y$
- $u_{yy} = -e^x \sin y$
- $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$

- Thus, u satisfies Laplace's equation.

WAVE EQUATION

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

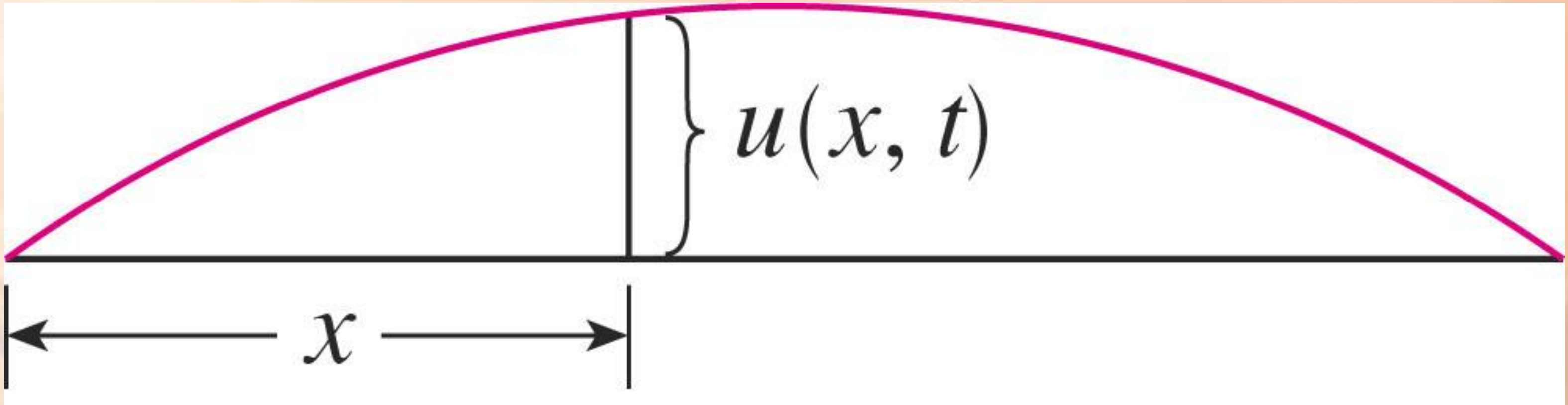
describes the motion of a waveform.

- This could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

WAVE EQUATION

For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string, then

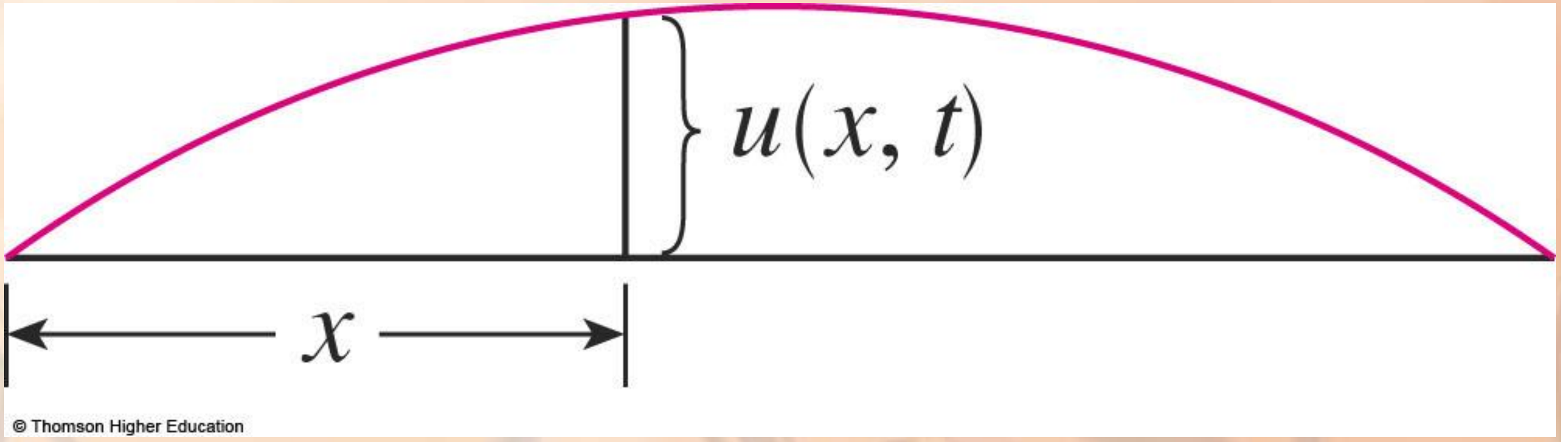
- $u(x, t)$ satisfies the wave equation.



WAVE EQUATION

Here, the constant *a* depends on:

- Density of the string
- Tension in the string



Example 9

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

- $u_x = \cos(x - at)$
- $u_{xx} = -\sin(x - at)$
- $u_t = -a\cos(x - at)$
- $u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$

- So, u satisfies the wave equation.

COBB-DOUGLAS PRODUCTION FUNCTION

In Example 3 in Section 14.1, we described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of:

- Amount of labor L
- Capital investment K

Here, we use partial derivatives to show how:

- The particular form of their model follows from certain assumptions they made about the economy.

MARGINAL PRODUCTIVITY OF LABOR

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\partial P / \partial L$ is the rate at which production changes with respect to the amount of labor.

- Economists call it the marginal production with respect to labor or the marginal productivity of labor.

MARGINAL PRODUCTIVITY OF CAPITAL

Likewise, the partial derivative $\partial P/\partial K$ is the rate of change of production with respect to capital.

- It is called the marginal productivity of capital.

COBB-DOUGLAS ASSUMPTIONS

In these terms, the assumptions made by Cobb and Douglas can be stated as follows.

i. If either labor or capital vanishes, so will production.

ii. The marginal productivity of labor is proportional to the amount of production per unit of labor.

iii. The marginal productivity of capital is proportional to the amount of production per unit of capital.

COBB-DOUGLAS ASSUMPTION ii

Since the production per unit of labor is *P*/*L*, assumption ii says that:

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant *α*.

COBB-DOUGLAS ASSUMPTION ii **Equation 5**

If we keep K constant ($K = K_0$), this partial differential equation becomes an ordinary differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

COBB-DOUGLAS ASSUMPTION ii **Equation 6**

If we solve this separable differential equation by the methods of Section 9.3, we get:

$$P(L, K_0) = C_1(K_0)L^{\alpha}$$

- Notice that we have written the constant C_1 as a function of K_0 since it could depend on the value of K_0.

COBB-DOUGLAS ASSUMPTION iii

Similarly, assumption iii says that:

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

COBB-DOUGLAS ASSUMPTION iii **Equation 7**

We can solve that differential equation to get:

$$P(L_0, K) = C_2(L_0)K^{\beta}$$

COBB-DOUGLAS ASSUMPTIONS

Equation 8

Comparing Equations 6 and 7,

we have:

$$P(L, K) = bL^{\alpha}K^{\beta}$$

where b is a constant that is independent of both L and K.

- Assumption i shows that $\alpha > 0$ and $\beta > 0$

COBB-DOUGLAS ASSUMPTIONS

Notice from Equation 8 that, if labor and capital are both increased by a factor *m*, then

$$\begin{aligned} P(mL, mK) &= b(mL)^{\alpha}(mK)^{\beta} \\ &= m^{\alpha+\beta} b L^{\alpha} K^{\beta} \\ &= m^{\alpha+\beta} P(L, K) \end{aligned}$$

COBB-DOUGLAS ASSUMPTIONS

If $\alpha + \beta = 1$, then

$$P(mL, mK) = mP(L, K)$$

- This means that production is also increased by a factor of m.

COBB-DOUGLAS ASSUMPTIONS

That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

- This is the Cobb-Douglas production function we discussed in Section 14.1