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Linear Algebra I 2005 Exam Solutions

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This is a core second-year course. All questions are mainly bookwork; only the details are unseen.

← Marks

SECTION A

A1. (a) $5A = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$ [1]

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$ [1]

$$AX = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$$ [1]

(b) $$AB = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 11$$ [1]

$$BA = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$ [1]

(c) $$AA^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}$$ [1]

$$A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 9 \end{pmatrix}$$ [1]

A2. $\det A = (1+12+12) - (9+4+4)$
[3] $= 25 - 17 = 8$

Matrix of minors of A is $\begin{pmatrix} -3 & -4 & 1 \\ -4 & -8 & -4 \\ 1 & -4 & -3 \end{pmatrix}$
(Use symmetry to avoid some calculation.)

Hence $A^{-1} = \frac{1}{8}\begin{pmatrix} -3 & 4 & 1 \\ 4 & -8 & 4 \\ 1 & 4 & -3 \end{pmatrix}$.

[4]

Check: (optional). $AA^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \frac{1}{8}\begin{pmatrix} -3 & 4 & 1 \\ 4 & -8 & 4 \\ 1 & 4 & -3 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

A3. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix} \xrightarrow[\text{swap}]{} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & -1 \end{pmatrix} \rightarrow$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

[5]

[1] Hence rank $= 4$

[1] One row swap $\Rightarrow \det = -(1)(1)(-2)(-2)$
$= -4$

A4. A linear combination of vectors in S is a vector of the form $\sum_{i=1}^{n} k_i v_i$ where the k_i are scalars. [2]

The vector space spanned by S is the set of all linear combinations of vectors in S. [1]

$(1,2,3,4)$ is a linear combination of the vectors in S if
$(1,2,3,4) = a(1,0,1,0) + b(0,1,0,1) + c(0,1,1,0) + d(1,0,0,1)$
can be solved for the scalars a, b, c, d. Then

$1 = a + d$ $\quad$ $1 = a - b$ $\quad$ These equations
$2 = b + c$ $\quad$ $3 = b - a$ $\quad$ are inconsistent.
$3 = a + c$
$4 = b + d$ $\quad$ Therefore $(1,2,3,4) \notin \langle S \rangle$. [3]

Moreover, $\langle S \rangle \neq \mathbb{R}^4$, since $(1,2,3,4) \in \mathbb{R}^4$ but $(1,2,3,4) \notin \langle S \rangle$. [1]

A5. A basis for V is a linearly independent subset of V that spans V. [2]
The dimension of V is the number of vectors in any basis for V. [1]

Suppose $v = \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i$.

Then $\sum_{i=1}^{n} (c_i - d_i) v_i = 0$. But $\{v_i\}$ is a linearly independent set since it is a basis, so $c_i - d_i = 0 \ \forall i$, i.e. $c_i = d_i \ \forall i$ and the representation is unique. [4]

A6. A map $\alpha : U \to V$ between two vector spaces U, V with the same field of scalars $\mathbb{K}$ is a map such that $\alpha(k_1 u_1 + k_2 u_2) = k_1 \alpha(u_1) + k_2 \alpha(u_2)$

[2] for all $u_1, u_2 \in U$ and all $k_1, k_2 \in \mathbb{K}$.

Let $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2)$

Then $\alpha(k_1 u_1 + k_2 u_2) = \alpha(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2, k_1 z_1 + k_2 z_2)$

$= ((k_1 x_1 + k_2 x_2) + (k_1 y_1 + k_2 y_2),\ (k_1 y_1 + k_2 y_2) + (k_1 z_1 + k_2 z_2),$
$(k_1 z_1 + k_2 z_2) + (k_1 x_1 + k_2 x_2))$

$= k_1 (x_1 + y_1, y_1 + z_1, z_1 + x_1) + k_2 (x_2 + y_2, y_2 + z_2, z_2 + x_2)$

$= k_1 \alpha(u_1) + k_2 \alpha(u_2)$.

This holds $\forall\, u_1, u_2 \in U$, $k_1, k_2 \in \mathbb{K}$.

[3] Hence α is linear.

$$\left.\begin{aligned} \alpha(1,0,0) &= (1,0,1) \\ \alpha(0,1,0) &= (1,1,0) \\ \alpha(0,0,1) &= (0,1,1) \end{aligned}\right\} \Rightarrow A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$ [2]

A7. If x is a nonzero vector in V such that $\alpha(x) = \lambda x$ for some $\lambda \in \mathbb{K}$ then λ is an

[2] eigenvalue of α with eigenvector x.

If $\alpha(x) = \lambda x$ then by linearity $\alpha(kx) = k\alpha(x) = \lambda(kx)$. Therefore kx is

[2] also an eigenvector (with the same eigenvalue).

$\alpha(x) = \lambda x \Rightarrow \alpha^{(2)}(x) = \alpha(\lambda x) = \lambda \alpha(x) = \lambda^2 x$

Suppose $\alpha^{(k)}(x) = \lambda^k x$, which is true for $k = 1$.

Then $\alpha^{(k+1)}(x) = \alpha(\lambda^k x) = \lambda^k \alpha(x) = \lambda^{k+1} x$.

By induction, $\alpha^{(n)}(x) = \lambda^n x$, so x is an eigenvector of $\alpha^{(n)}$ with eigenvalue λ^n, where

[3] λ is the corresponding eigenvalue of α.

A8. The matrix A is symmetric if $A^T = A$
[2] and antisymmetric if $A^T = -A$.

[1] $(AB)^T = B^T A^T$

Let $A = xx^T$. Then $A^T = (x^T)^T x = xx^T$
Hence $A^T = A$ and A is symmetric.
The dimension of A the row dimension of
x, namely n, by the column dimension of
[3] x^T, also n. Hence A is an $n \times n$ matrix.

As a simple example, let $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

[1] Then $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

SECTION B

B1. (a) U is a vector subspace of V if U is a subset of V that is a vector space
[2] under the operations inherited from V.

It must satisfy the conditions that $0_V \in U$ and $au + bv \in U \; \forall \; u, v \in U$
[2] and all scalars a, b.

(b) The constraints on $U \Rightarrow x = -2y, \; z = \frac{2}{3}y$
hence $U = \{(-2, 1, \frac{2}{3})y \mid y \in \mathbb{R}\} = \langle(-6, 3, 2)\rangle$

$(0,0,0) = 0(-6, 3, 2) \in U$.
Let $u = \alpha(-6, 3, 2)$, $v = \beta(-6, 3, 2)$, $\alpha, \beta \in \mathbb{R}$.
Then $au + bv = (a\alpha + b\beta)(-6, 3, 2) \in U$
since $a\alpha + b\beta \in \mathbb{R}$.
[6] Hence U is a vector subspace of $\mathbb{R}^3$

$\{(-6, 3, 2)\}$ spans U and is linearly independent,
[3] so it is therefore a basis for U and $\dim U = 1$.

[1] (c) $S + T = \{s + t \mid s \in S, \; t \in T\}$
[1] $S \oplus T = S + T$ such that $S \cap T = \{\phi\}$.
[1] Choose* $\{(1,0,0), (0,1,0)\}$ as a basis for $W \subseteq \mathbb{R}^3$.
Then $U \cap W = \{(-6, 3, 2)y \mid y = 0\} = \{\phi\}$
[3] since $(-6, 3, 2)y \in \langle(1,0,0), (0,1,0)\rangle \Rightarrow y = 0$.
Moreover, $U + W = \{u + w \mid u = (-6, 3, 2)y,$
$w = a(1,0,0) + b(0,1,0), \; y, a, b \in \mathbb{R}\}$
[3] $= \langle(-6, 3, 2), (1,0,0), (0,1,0)\rangle = \mathbb{R}^3$.

* Any 2D vector subspace of $\mathbb{R}^3$ not containing $(-6, 3, 2)$ will suffice.

B2. (a) $\ker(\alpha) = \{u \in U \mid \alpha(u) = 0\}$
[2] $\quad \operatorname{im}(\alpha) = \{\alpha(u) \mid u \in U\}$

$\alpha(0) = 0 \Rightarrow 0 \in \ker(\alpha)$
$u, v \in \ker(\alpha) \Rightarrow \alpha(u) = \alpha(v) = 0$
Then $\alpha(au + bv) = a\alpha(u) + b\alpha(v) = 0$
$\Rightarrow au + bv \in \ker(\alpha)$ $\forall u, v \in \ker \alpha$, and
[3] all scalar a, b. So $\ker \alpha$ is a vector subspace of U.

$\alpha(0) = 0 \Rightarrow 0 \in \operatorname{im}(\alpha)$ since $0 \in U$.
$u, v \in \operatorname{im}(\alpha) \Rightarrow \exists\, u', v' \in U$ s.t. $u = \alpha(u')$,
$v = \alpha(v')$. Then $au' + bv' \in U$
$\Rightarrow \alpha(au' + bv') = au + bv \in \operatorname{im}(\alpha)$
$\forall u, v \in \operatorname{im}(\alpha)$ and all scalars a, b.
[3] So $\operatorname{im}(\alpha)$ is a vector subspace of V

(b) Let $\{u_1, \dots, u_n\}$ be a spanning set for U.
Then $u = \sum_{i=1}^{n} k_i u_i$ for any $u \in U$ and scalars k_i.
Any $v \in \operatorname{im}(\alpha)$ is the image of some $u \in U$, i.e.
$v = \alpha(u) = \sum_{i=1}^{n} k_i \alpha(u_i)$ by linearity.
[3] Therefore $\{\alpha(u_1), \dots \alpha(u_n)\}$ is a spanning set for $\operatorname{im}(\alpha)$.

[1] (c) $\dim U = \dim \ker(\alpha) + \dim \operatorname{im}(\alpha)$.

Proof. Let $\{v_1, \dots v_m\}$ be a basis for $\ker(\alpha)$ and extend this to a basis $\{v_1, \dots v_m, u_1, \dots u_n\}$ for U.
Then $\dim U = m + n = \dim \ker \alpha + n$.
By part (b), $\{\alpha(v_1), \dots \alpha(v_m), \alpha(u_1), \dots \alpha(u_n)\}$ is a spanning set for $\operatorname{im}(\alpha)$. But $\alpha(v_i) = 0$ since $v_i \in \ker(\alpha)$. Hence $\{\alpha(u_1), \dots \alpha(u_n)\}$ is
[2] a spanning set for $\operatorname{im}(\alpha)$.

Suppose $k_1\alpha(u_1)+\dots+k_n\alpha(u_n)=0$
Then $\alpha(k_1u_1+\dots+k_nu_n)=0$ by linearity
$\Rightarrow k_1u_1+\dots+k_nu_n \in \ker(\alpha)$.
But $u_i \notin \ker(\alpha)$, hence $k_1=\dots=k_n=0$ is the only solution. Therefore $\{\alpha(u_1),\dots,\alpha(u_n)\}$ is a linearly independent set and so is a basis for $\operatorname{im}(\alpha)$ and $\dim \operatorname{im}(\alpha)=n$.
[2] Therefore $\dim U = \dim \ker(\alpha) + \dim \operatorname{im}(\alpha)$.

(d) $\ker(\alpha)=\{(x,y,z)\mid x+y=0,\ y+z=0\}\subseteq \mathbb{R}^3$
The constraints imply $x=-y$, $z=-y$, hence
$\ker(\alpha)=\{(-1,1,-1)y \mid y\in\mathbb{R}\}=\langle(-1,1,-1)\rangle$
This set is clearly linearly independent, so
[2] a basis for $\ker(\alpha)$ is $\{(-1,1,1)\}$ and $\dim\ker(\alpha)=1$.

$\operatorname{im}(\alpha)=\{(x+y,\ y+z)\mid x,y,z\in\mathbb{R}\}$
$=\{x(1,0)+y(1,1)+z(0,1)\mid x,y,z\in\mathbb{R}\}$
$=\langle(1,0),(0,1)\rangle$. Clearly linearly independent
[2] so a basis for $\operatorname{im}(\alpha)$ is $\{(1,0),(0,1)\}$ and $\dim\operatorname{im}(\alpha)=2$.

$U=\mathbb{R}^3$ so $\dim U = 3 = \dim\ker(\alpha)$
[2] $+\dim\operatorname{im}(\alpha) = 1+2$.

B3. (a) P is the matrix of the identity map on V with basis $\mathcal{B}$ in its domain and $\mathcal{C}$ in its

[1] codomain, i.e. $P = (Id_V, \mathcal{B}, \mathcal{C})$.

$$Id(1,0,0) = (1,0,0) = -\tfrac{1}{2}(0,1,1) + \tfrac{1}{2}(1,0,1) + \tfrac{1}{2}(1,1,0)$$
$$Id(0,1,0) = (0,1,0) = \tfrac{1}{2}(0,1,1) - \tfrac{1}{2}(1,0,1) + \tfrac{1}{2}(1,1,0)$$
$$Id(0,0,1) = (0,0,1) = \tfrac{1}{2}(0,1,1) + \tfrac{1}{2}(1,0,1) - \tfrac{1}{2}(1,1,0)$$

by inspection

Hence $P = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

[4]

$$P^{-1} = (Id_V, \mathcal{C}, \mathcal{B})$$
$$Id(0,1,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$
$$Id(\ ,0,1) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$
$$Id(1,1,0) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

Hence $P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ [or invert P by any convenient method]

[4]

Check: (optional). $PP^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ✓

(b) $$v' = Pv = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

[3]

This satisfies the definition of the coordinate vector wrt basis $\mathcal{C}$ because

[2] $(1,2,3) = 2(0,1,1) + 1(1,0,1) + 0(1,1,0)$

(c) $$A' = PAP^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

[4]

$$A'v' = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$Av = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

Hence $$P(Av) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

[4] Thus $A'v' = P(Av)$ (as it should!)

B4. (a) A similarity transformation of a square matrix A is a transformation of the form PAP^{-1} where P is an invertible matrix.

[1]

If a square matrix A is $n \times n$ and has n linearly independent eigenvectors x_i then $Ax_i = \lambda_i x_i$, $i = 1..n$. This set of n equations can be written as $AX = X\Lambda$ where the columns of the matrix X are the n eigenvectors x_i, $i = 1..n$, and Λ is a diagonal matrix with the eigenvalues λ_i, $i = 1..n$, on the diagonal. Since the columns of X are linearly independent, X has maximal rank and so is invertible. Therefore $X^{-1}AX = \Lambda$ is diagonal and $X^{-1}AX$ has the form PAP^{-1} where $P = X^{-1}$ and so is a similarity transformation.

[5]

(b) A set $\{v_i, i=1..n\}$ of vectors is orthogonal if $\langle v_i, v_j \rangle = 0$ for $i \neq j$, where $\langle , \rangle$ denotes the inner product. The set of vectors is orthonormal if $\langle v_i, v_j \rangle = 1$ if $i = j$ and 0 otherwise.

[2]

If $Ax = \lambda x$ then $x^{*T}Ax = \lambda x^{*T}x$, where * denotes complex conjugate. Transpose and conjugate to give $x^{*T}A^{*T}x = \lambda^* x^{*T}x$.
But A is real symmetric so $A^{*T} = A$.
Subtracting $\Rightarrow$ $(\lambda - \lambda^*)\, x^{*T}x = 0$.
$x^{*T}x = \sum_i |x_i|^2 \neq 0$ since x is an eigenvector

[3] Hence $\lambda = \lambda^*$ $\Rightarrow$ λ is real.

Suppose $Ax = \lambda x$ and $Ay = \mu y$
Then $y^T A x = \lambda y^T x$ and $x^T A y = \mu x^T y$
Transposing the second equation and subtracting gives $(\lambda - \mu) y^T x = 0$. If $\mu \neq \lambda$ then
[3] $y^T x = 0$, i.e. $\langle y, x \rangle = 0$, so x & y are orthogonal

(c) An orthogonal matrix Q is a square matrix such that $QQ^T = Q^T Q = I$ or
[1] equivalently $Q^{-1} = Q^T$.

Denote the ith column of Q by q_i.
Then $Q^T Q = \begin{pmatrix} \vdots \\ q_i^T \\ \vdots \end{pmatrix} \begin{pmatrix} \cdots & q_j & \cdots \end{pmatrix} = \begin{pmatrix} q_i^T q_j \end{pmatrix} = I$

$\Rightarrow q_i^T q_j = 1$ if $i = j$ and 0 otherwise,
[3] i.e. the columns of Q are orthonormal.

(d) If an $n \times n$ real symmetric matrix A has n distinct eigenvalues then it has n orthogonal eigenvectors. Normalize each eigenvector before constructing the matrix X as in part (a). Then $X^T A X = \Lambda$ is diagonal since $X^{-1} = X^T$ and $X^T A X$ has the form PAP^T where $P = X^T$ and so is a real
[4] orthogonal similarity transformation.