MATH 252 (CALCULUS WITH SEVERAL VERIABLES)

LAST LECTURE NOTE

<u>B1</u>

If $f(x,y) = ax^2 + bxy + cy^2$

$$f(x,y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

Let $D = 4ac - b^2$

$$f(x,y) = a\left(x^2 + \frac{b}{a}xy + \frac{c}{a}y^2\right)$$

$$f(x,y) = a\left[x^2 + \frac{bxy}{a} + \left(\frac{1}{2}\frac{b\cancel{x}y}{a}\right)^2 - \left(\frac{1}{2}\frac{b\cancel{x}y}{a}\right)^2 + \frac{c}{a}y^2\right]$$

$$= a\left[\left(x + \frac{by}{2a}\right)^2 - \frac{b^2y^2}{4a^2} + \frac{cy^2}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}y\right)^2 + \frac{4acy^2 - b^2y^2}{4a^2}\right]$$

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<u>Definition</u>: Let D be a set in R^2 (a plane region). A vector field on R^2 is a function F that assigns to each point (x,y) in D a two-dimensional vector $F(x,y)$.

$$\cancel{F(x,y) = P(x,y)j}$$

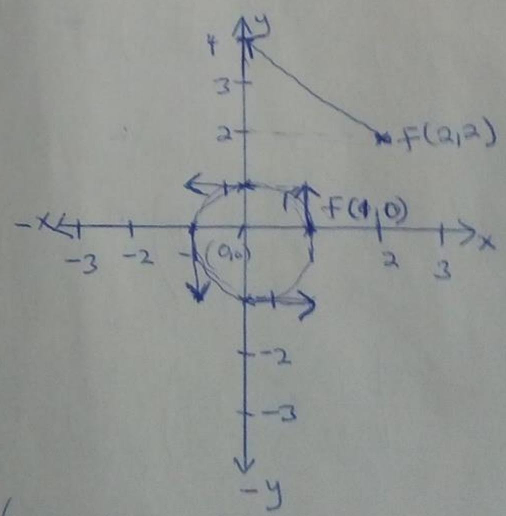
$$F(x,y) = P(x,y)\underline{i} + Q(x,y)\underline{j}$$

Example 1: A vector field on R^2 is defined by

$$F(x,y) = -y\underline{i} + x\underline{j}.$$ Plot the ~~phase~~ vector field $F(x,y)$.

Sol$^{\underline{n}}$

(x,y)	$F(x,y)$
$(1,0)$	$(0,1)$
$(2,2)$	$(-2,2)$
$(3,0)$	$(0,3)$
$(0,1)$	$(-1,0)$
$(-2,2)$	$(-2,-2)$
$(0,3)$	$(-3,0)$



$$(x,y) = (1,0)$$

$$F(1,0) = 0\underline{i} + j = \binom{0}{1}$$

Terminal pt = point + vector

$$= \binom{1}{0} + \binom{0}{1} = \binom{1}{1}$$

* Ex $F(2,2) = -2\underline{i} + 2\underline{j}$

Terminal point = Initial point + $F(2,2)$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

Terminal point = $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$F(0,2) = -2\underline{i} + 0\underline{j}$

$$TP = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$TP = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$F(0,1) = -\underline{i} + 0\underline{j}$

$$TP = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$TP = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$F(-1,0) = 0\underline{i} - \underline{j}$

$$TP = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$TP = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$F(0,-1) = \underline{i} + 0\underline{j}$

$$TP = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$TP = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$f(x,y) = -y\underline{i} + x\underline{j}$ is a per…

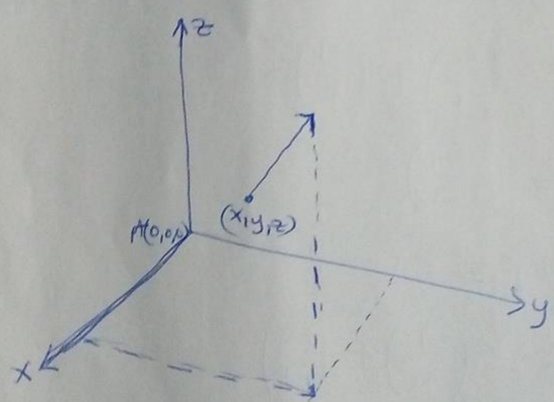
TRY
sketch the ff. vector fields.

① $F(x,y) = y\underline{i} + \sin(x)\underline{j}$

⑤ $f(x,y) = \ln(1+y^2)\underline{i} + \ln(1+x^2)\underline{j}$

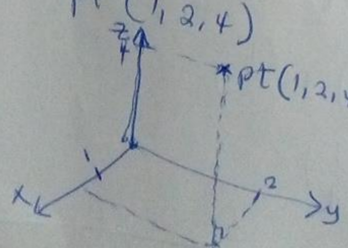
Definition: Let E be a subset in R^3. A vector field on R^3 is a function F that assigns to each point (x,y,z) in E a three-dimensional vector $F(x,y,z)$.

$$F(x,y,z) = P(x,y,z)\underline{i} + Q(x,y,z)\underline{j} + R(x,y,z)\underline{k}$$



TP = initial pt + vector.

pt (1, 2, 4)



B3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

parametric equations $x = a\cos(t)$, $y = b\sin(t)$, $0 \le t \le 2\pi$.

Type 4: Line Integral

$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz,$$

where $F = P\underline{i} + Q\underline{j} + R\underline{k}$.

TRY Questions

Q1. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line integral C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

Sol$^{\underline{n}}$

Along C_1

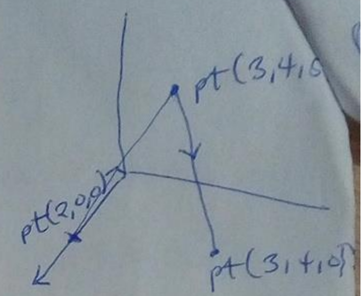
$$I_1 = \int_{C_1} y\,dx + z\,dy + x\,dz$$

$ST = (2,0,0)$, $EP = (3,4,5)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\left[\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$\cancel{= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+t \\ 4t \\ 5t \end{pmatrix}$$

$x = 2+t, \quad y = 4t, \quad z = 5t$

$0 \le t \le 1$ moving in x-direction

$$I_1 = \int_0^1 4t\,dt + 4(5t)\,dt + 5(2+t)\,dt$$

$$= \int_0^1 (10 + 29t)\,dt$$

$I_1 = 24.5\text{ J}$

Along C_2

$x = 3, \quad y = 4, \quad z = 5 - 5t, \quad 0 \le t \le 1.$

$$I_2 = \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$I_2 = \int_0^1 3(-5)\,dt = -15.$$

$$I = I_1 + I_2$$
$$= 24.5 - 15$$
$$= 9.5\text{ J}.$$

TRY

Q1. Find the workdone by the force $F(x,y) = x^2\underline{i} + xy\underline{j}$ on a particle that moves once around the circle $x^2+y^2=4$ oriented in the counter-clockwise direction.

Q2. The base of a circular fence with radius 10m is given by $x = 10\cos(t)$, $y = 10\sin(t)$. The height of the fence at position (x,y) is given by the function $h(x,y) = 4 + 0.01(x^2 - y^2)$,

so the height varries from 3m to 5m. Suppose that 1L of paint covers $100m^2$. Determine how much paint you will need if you paint both sides of the fence.

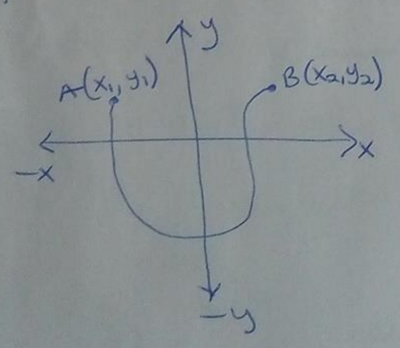
Fundamental theorem of Calculus

$$\int_a^b F'(x)dx = F(b) - F(a)$$

<u>Theorem</u>: Let C be a smooth curve given by the vector function $r(t)$, $a \le t \le b$. Let f be a differentiable function of two or more variables whose ~~gradient~~ gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Thus, the theorem says that we can evaluate the line integral of a ~~const~~ conservative vector field simply by knowing the value of f at the endpoints of C.



$$\int_C \nabla f \cdot dr = f(x_2, y_2) - f(x_1, y_1)$$

$$\int_C \nabla f \cdot dr = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

<u>Proof</u>

$$\int_C \nabla f \cdot dr = \int_a^b \nabla f(r(t)) \cdot r'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(r(t))\,dt$$

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

<u>Independence of Path</u>

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called paths) that have the same initial ~~poin/points and~~ point A and terminal point B, by the above theorem

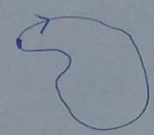
$$\int_{C_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot dr,$$

wherever ∇f is continuous.

In general, if F is a continuous vector field in domain D, we say that the line integral $\int_C F \cdot dr$ is <u>independent of path</u> if

$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for any two paths C_1 and C_2 in D that have the same initial and terminal points.

Def: A curve is <u>closed</u> if its terminal point coincides with its initial point. That is $r(b) = r(a)$



<u>Theorem 2</u>: $\int_C F \cdot dr$ is <u>independent</u> of <u>path</u> in D if and only if $\int_C F \cdot dr = 0$, for every closed path in D.

The <u>line integral</u> of any <u>conservative vector field F</u> is <u>independent</u> of path. Thus, $\int_C F \cdot dr = 0$.

<u>Theorem 3</u>: Suppose F is a vector field that is continuous on an open connected region D. If $\int_C F \cdot dr$ is independent of path in D, then F is a <u>conservative vector field</u> on D; that is, there exists a function f such that $F = \nabla f$.

If $F = \nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} = P\underline{i} + Q\underline{j}$, then ①

F is conservative.

From equation ①,

$$P_{(x,y)} = \frac{\partial f}{\partial x}, \text{ and } Q_{(x,y)} = \frac{\partial f}{\partial y} \qquad (2)$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad (3)$$

Also, $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \qquad (4)$

By chairnut's theorem,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Theorem 4: Let $F(x,y) = P(x,y)\underline{i} + Q(x,y)\underline{j}$ be a vector field on an open simply-connected region D. Suppose that $P(x,y)$ and $Q(x,y)$ have continuous first-order partial derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. Then F is conservative.

Eg1. Determine whether or not the vector field

$$F(x,y) = (3+2xy)\underline{i} + (x^2-3y^2)\underline{j}$$

is conservative.

Sol^n

$P(x,y) = 3+2xy, \quad Q(x,y) = x^2-3y^2$

$$F(x,y) = P(x,y)\underline{i} + Q(x,y)\underline{j}$$

$$F(x,y) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}$$

$$\frac{\partial P}{\partial y} = 2x$$

$$\frac{\partial Q}{\partial x} = 2x$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$, it implies that

$F(x,y) = (3+2xy)\underline{i} + (x^2-3y^2)\underline{j}$ is conservative.

Eg2. Determine whether or not the vector field

$F = (x-y)\underline{i} + (x-2)\underline{j}$ is ~~conservative~~. conservative.

zvbZbn

Sol^{n}

$$F(x,y) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} = P(x,y)\underline{i} + Q(x,y)\underline{j}$$

$$P(x,y) = x - y, \quad Q(x,y) = x - 2$$

$$\frac{\partial P}{\partial y} = -1$$

$$\frac{\partial Q}{\partial x} = 1$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, it implies that $F(x,y) = (x-y)\underline{i} + (x-2)\underline{j}$ is not a conservative vector field.

Theorem 5: Let $F(x,y,z) = P(x,y,z)\underline{i} + Q(x,y,z)\underline{j} + R(x,y,z)\underline{k}$ be a vector field on an open simply-connected space D. Suppose that $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ have continuous first-order partial derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ throughout D. Then $F(x,y,z)$ is conservative.

Eg1. Determine whether or not the ff vector fields are conservative:

(a) $F(x, y, z) = (y^2 z + 2xz^2)\underline{i} + 2xyz\underline{j} + (xy^2 + 2x^2 z)\underline{k}$

(b) $F(x, y, z) = \sin(y)\underline{i} + (x\cos(y) + \cos(z))\underline{j} - y\sin(z)\underline{k}$

(c) $F(x, y, z) = yze^{xz}\underline{i} + e^{xz}\underline{j} + xye^{xz}\underline{k}$