UNIT 3

LAMINAR FLOW

Introduction

In this unit the effects of viscosity on an incompressible, internal flow will be studied. Such flows are of particular importance to engineers. Flow in a circular pipe is undoubtedly the most common internal fluid flow. It is encountered in the veins and arteries in the human body, in a city's water system, in a farmer's irrigation system, in piping systems transporting fluids in a factory, in the hydraulic lines of an aircraft, in the inkjet of a computer's printer, and at the bearings of machine parts. Viscous effects in a flow result in the introduction of the Reynold's number,

$$Re = \frac{V\rho l}{\mu} \qquad (3.1.1)$$

The Reynold's number represents the ratio of the inertial force to the viscous force in a flow field. Hence, when this ratio is large, it is expected that the inertial forces may dominate the viscous forces. This is usually true when short, sudden geometric changes occur, for long reaches of pipe or open channels, this is not the situation. When the surface areas, such as the wall area of pipe, are relatively large, viscous effects become quite important and must be included in the study.

Learning Objectives

After reading this unit you should be able to:

1. Explain the effects of viscosity on incompressible internal flows
2. State the characteristics of a developed flow
3. Explain the concept of laminar flow in a conduit
4. Explain the velocity distribution in steady incompressible developed laminar flow in a pipe with equations
5. Estimate loss of head due to friction in viscous flow

6. Estimate losses due to fittings, bends and junctions in pipe flow
7. Calculate frictional losses in bearings
8. Explain the method of viscosity determination.

Unit content

Session 1-3: Laminar flow and pipe flow quantities

1-3.1 Entrance flow and developed flow

When considering internal flows we are interested primarily in developed flows. A developed flow results when the velocity profile ceases to change in the flow direction, as shown in fig .35.

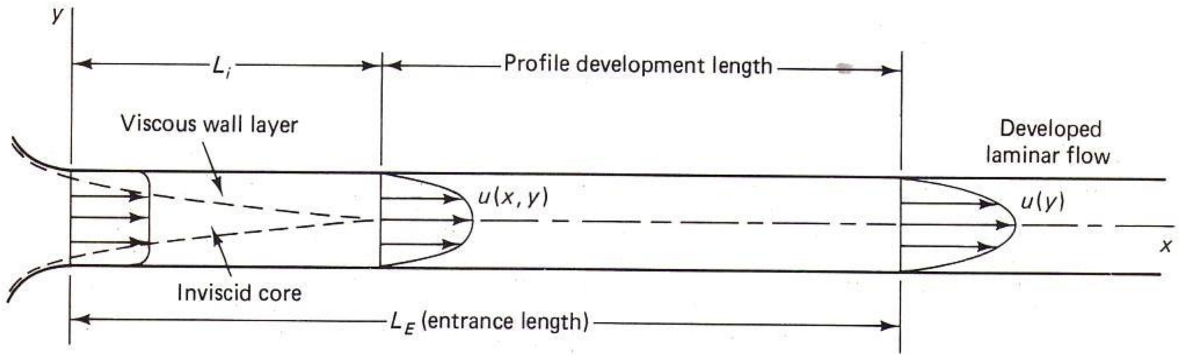


Fig 35 Laminar flow in conduit

The idealized flow from a reservoir begins at the inlet as a uniform flow (there is a thin boundary layer on the wall, as shown); a wall viscous layer then grows over the **inviscid core length** L_i until the viscous stresses dominate the entire cross section; the profile then continues to change in the profile development region due to viscous effects until a developed flow is achieved, the inviscid core length is approximately one-fourth of the entrance length L_E, depending on the conduit geometry, shape of the inlet, and Reynold's number.

For laminar flow in a circular pipe with a uniform profile at the inlet, the entrance length is given by

$$\frac{L_E}{D} = 0.065Re \qquad Re = \frac{VD}{\upsilon} \qquad (3.2.1)$$

For engineering applications of a value of 2000 is the highest Reynold's number for which laminar flow is assured; this is due to vibrations of the pipe, fluctuations in the flow, or roughness elements on the pipe wall.

For laminar flow between two parallel plates and a uniform profile at the inlet the entrance length is

$$\frac{L_E}{D} = 0.04Re \qquad Re = \frac{Vh}{\upsilon} \qquad (3.2.2)$$

Where the Reynold's number is based on the average velocity and the distance h between the plates. The inviscid core length is approximately one-third of the entance length. In a high-aspect-ratio channel (the aspect ratio is the width divided by the distance between the plates) a laminar flow cannot exist over Re=2200; for engineering situations a value of 1500 is often used as upper limit for laminar flow.

For a turbulent flow the situation is slightly different, as shown in fig 36. The inviscid core exists followed by the velocity profile development, region, which terminates at $x = L_d$.

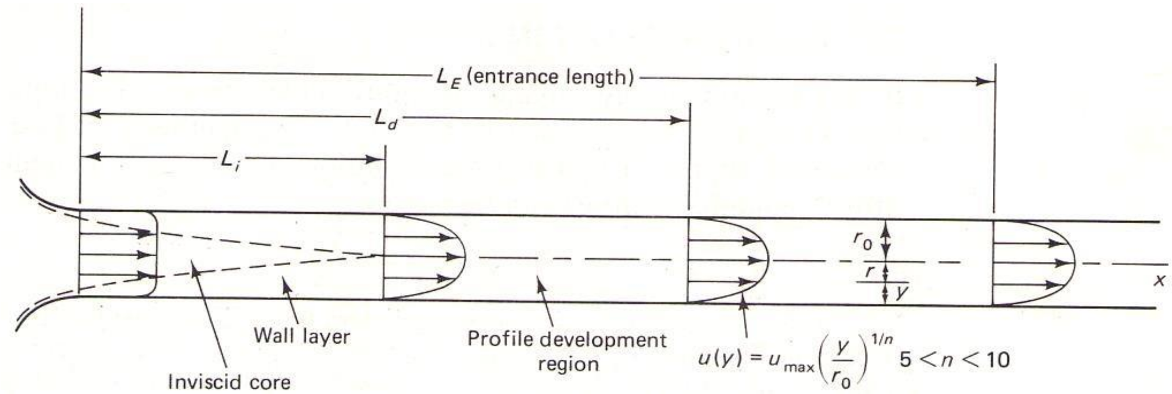


Figure 36 Velocity profile development in a turbulent pipe flow

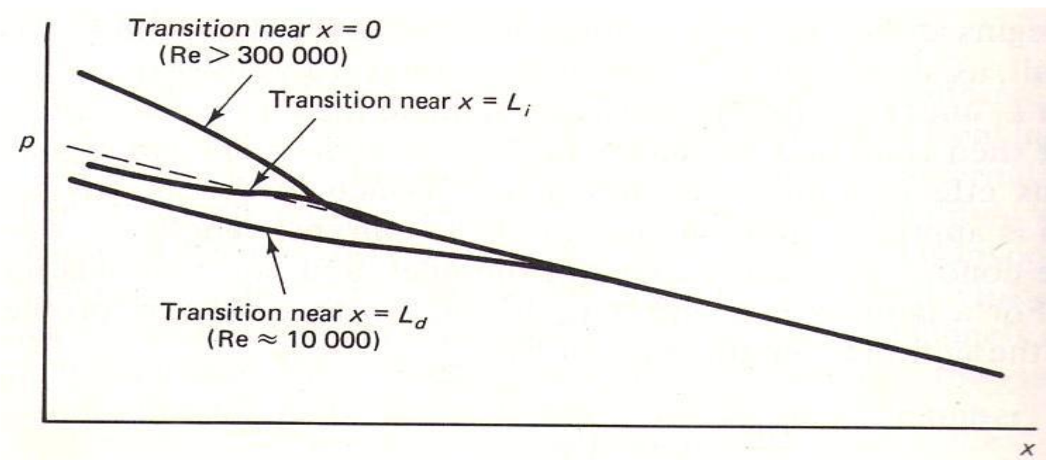


Figure 37 Pressure variation in a turbulent flow in a horizontal pipe

An additional length is needed, however, for the turbulent structure (e.g., the eddy size and frequency) to develop. For large Reynold's number flow ($Re > 10^5$) in a pipe, where the turbulence fluctuations initiate near $x = 0$, tests have yielded

$$\frac{L_i}{D} \backsimeq 10 \qquad \frac{L_d}{D} \backsimeq 40 \qquad \frac{L_E}{D} \backsimeq 120 \qquad (3.2.3)$$

For turbulent flow with $Re = 4000$ the forgoing developmental lengths would be substantially higher, perhaps five times the values listed; experimental data are not available for low Reynold's number turbulent flow.

In fig 37 the pressure variation is sketched. In the flow beyond a sufficiently large x it is noted that the pressure variation decreases linearly with x. If transition to turbulent flow occurs near the origin, the linear pressure variation begins near L_i and the pressure gradient in the inlet region is higher than in the developed region; if transition occurs nearL_d, as it does for low Re, the linear variation begins at the end of the transition process and the pressure gradient in the inlet region is less than that of developed flow. For a laminar flow, the pressure variation qualitatively resembles that associated with a large Reynold's number.

1-3.2.1 Laminar flow in a pipe

In this section we investigate incompressible, steady, developed laminar flow in a pipe, as sketched in fig 38. The elemental approach adopted for this analysis.

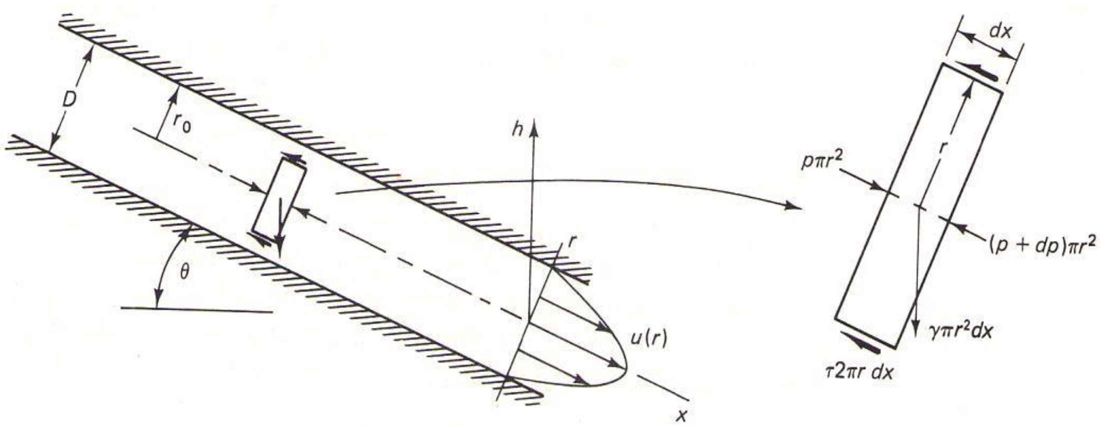


Figure 38 Developed flow in a circular pipe

1-3.2.2 Elemental Approach

An elemental volume of the fluid is shown in fig 38. This volume can be considered an infinitesimal control volume into which and from which fluid flows, or can be considered an infinitesimal fluid mass upon which forces are acting. If it is considered a control volume, we would apply the momentum equation; if it is a fluid mass, we would apply Newton's second law.

Since the velocity profile does not change in the x-direction, the momentum flux in equals the momentum flux out and the resultant force is zero, since there is no acceleration of the mass element, the resultant force must also be zero. Consequently,

$$p\pi r^2 = (p + dp)\pi r^2 - \tau 2\pi r dx + \gamma \pi r^2 dx sin\theta = 0 \qquad (3.3.1)$$

Which can also be simplified to

$$\tau = -\frac{r}{2}\frac{d}{dx}(p + \gamma h) \qquad (3.3.2)$$

Where we have $sin\theta = -dh/dx$, the vertical direction being denoted by h. the shear stress in this flow is related to the velocity gradient and viscosity, giving

$$-\frac{du}{dr} = -\frac{r}{2}\frac{d}{dx}(p + \gamma h) \qquad (3.3.3)$$

Which can be integrated to give the velocity distribution,

$$u(r) = \frac{r^2}{4\mu}\frac{d}{dx}(p+\gamma h) + A \qquad (3.3.4)$$

Where A is a constant of integration. Using $u = 0$ at $r = r_o$, we can evaluate A and find the velocity distribution to be

$$u(r) = \frac{1}{4\mu}\frac{d(p+\gamma h)}{dx}(r^2 - r^2{}_0) \qquad (3.3.5)$$

a parabolic profile. It is often referred to a **Poiseulle flow.**

1-3.2.4 **Pipe flow quantities**

For steady, laminar, developed flow in a circular pipe, the velocity distribution has been shown to be

$$u(r) = \frac{1}{4\mu}\frac{d(p+\gamma h)}{dx}(r^2 - r^2{}_0) \qquad (3.3.6)$$

The average velocity V is found to be

$$V = \frac{Q}{A}\frac{\int_0^{r_o} u(r)2\pi r dr}{\pi {r_o}^2}$$

$$\frac{2}{{r_o}^2}\int_0^{r_o}\frac{1}{4\mu}\frac{d}{dx}(p+\gamma h)(r^2 - {r_o}^2)rdr = -\frac{{r_o}^2}{8\mu}\frac{d}{dx}(p+\gamma h) \qquad (3.3.7)$$

Or expressing the pressure drop Δp in terms of the average velocity, we have, for a horizontal pipe,

$$\Delta p = \frac{8\mu VL}{{r_o}^2} \qquad (3.3.8)$$

Where we have used $\Delta p/L = -dp/dx$ since dp/dx is a constant for developed flow. Note that the pressure drop is a positive quantity, whereas the pressure gradient is negative.

The maximum velocity $r = 0$ is

$$u_{max} = -\frac{{r_o}^2}{4\mu}\frac{d}{dx}(p+\gamma h) \qquad (3.3.9)$$

So that (see equ. 3.3.7)

$$V = \frac{1}{2}u_{max} \qquad (3.3.10)$$

The shearing stress is determined to be

$$\tau = -\mu\frac{du}{dr}$$

$$= -\frac{r}{2}\frac{d}{dx}(p+\gamma h) \qquad (3.3.11)$$

Letting $\tau = \tau_o \; at \; r = r_o$, we see that the pressure drop Δp over a length L of a horizontal section of pipe is

$$\Delta p = \frac{2\tau_o L}{r_o} \qquad (3.3.12)$$

Where we have again used $\Delta p/L = -dp/dx$. If we introduced the **friction factor** f, dimensionless wall shear, defined by

$$f = \frac{\tau_o}{\frac{1}{8} r_o V^2} \qquad (3.3.13)$$

We see that

$$\frac{\Delta p}{\gamma} = h_L = f \frac{L}{D} \frac{V^2}{2g} \qquad (3.3.14)$$

Where h_L is the head loss with dimension of length. This equation is often referred to as the **Darcy-Weisbach equation.** Combining eqs. 3.3.8, 3.3.12, 3.3.13, we find that

$$f = \frac{64}{Re} \qquad (3.3.15)$$

EXAMPLE 3.1

A small-diameter horizontal tube is connected to a supply reservoir as shown. If $6600\ mm^3$ is captured at the outlet in 10 s, calculate the viscosity of the water.

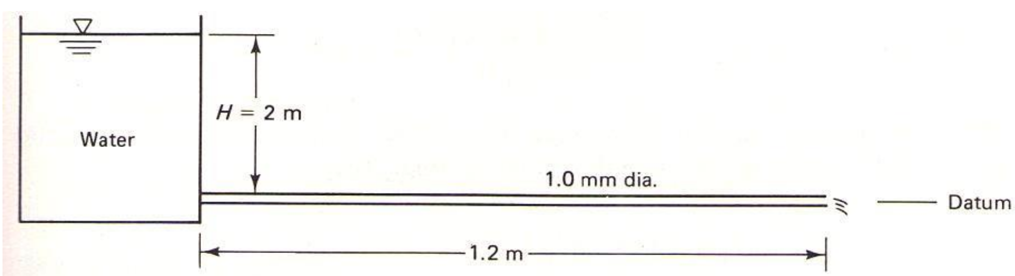


Figure 39 Solution 3.1

SOLUTION

The tube is very small, so we expect viscous effects to limit the velocity to a small value. Using Bernoulli's equations from the surface to the entrance to the tube, and neglecting the velocity head, we have, letting 0 be a point on the surface,

$$\frac{p_o}{\gamma} + H = \frac{V^2}{2g} + \frac{p}{\gamma}$$

where we have used gage pressure with $p_o = 0$. This becomes, assuming $V = 0$
$p = \gamma H$

$= 98OO \times 2 = 19600\ Pa$

At the exit of the tube the pressure is zero; hence

$\frac{\Delta p}{L} = \frac{19600}{1.2} = 16300\ Pa/m$

The average velocity is found to be

$V = \frac{Q}{A}$

$= \frac{6600\times 10^2/10}{\pi \times 0.001^2/4} = 0.840\ m/s$

This is quite small ($V^2/2g = 0.036\ compared\ with\ p/\gamma = 2$), so the assumption of negligible velocity head is valid. Our pressure calculation is acceptable. Using eq. 3.3.8, we can find the viscosity to be

$$\mu = -\frac{{r_o}^2}{8V}\frac{dp}{dx}$$

$$= \frac{0.0005^2}{8 \times 0.84}(16300) = 6.06 \times 10^{-4}\ N.s/m^2$$

using $\Delta p/L = -dp/dx$. We should check the Reynolds number to determine if our assumption of a laminar flow was acceptable.

$$Re = \frac{\rho VD}{\mu}$$

$$\frac{1000\text{x}0.84\text{x}0.001}{6.06\text{x}10^{-4}} = 1390$$

This is obviously a laminar flow since Re < 2000, so the calculations are valid providing the inviscid core length is not too long. It is

$$L_i = 0.065 Re \frac{D}{2}$$

$$= 0.0065 \times 1390 \times \frac{0.001}{2} = 0.045\ m$$

This is approximately 4% of the total length, a rather small quantity; hence the calculations are assumed reliable.

EXAMPLE 3.2

Derive an expression for the velocity distribution between horizontal, concentric pipes for steady, incompressible developed flow.

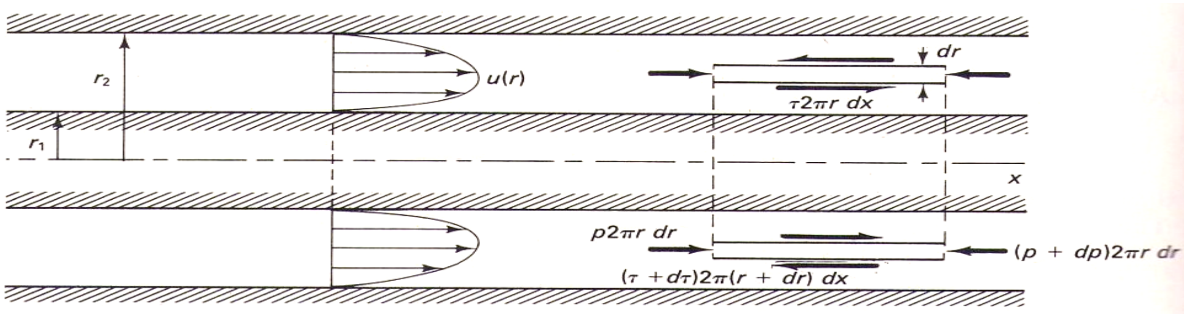


Figure 40 Solution 3.2

SOLUTION

$$p2\pi dr - (p + dp)2\pi rdr + \tau 2\pi rdx - (\tau + d\tau)2\pi(r + dr)dx = 0$$

Simplifying, there results

$$\frac{dp}{dx} = -\frac{\tau}{r} - \frac{d\tau}{dr}$$

Substituting $\tau = -\mu du/dr$ we have

$$\frac{dp}{dx} = \mu(\frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dr^2})$$

$$= \frac{\mu}{r}\frac{d}{dr}(r\frac{du}{dr})$$

This is integrated to yield

$$r\frac{du}{dr} = \frac{1}{2\mu}\frac{dp}{dx}r^2 + A$$

Integration again gives

$$u(r) = \frac{1}{4\mu}\frac{dp}{dx}r^2 + A\ In\ r + B$$

Where A and B are arbitrary constants. They are found by setting $u = o$ at $r = r_1$ and at $r = r_2$;

That is,

$$0 = \frac{1}{4\mu}\frac{dp}{dx}{r_1}^2 + A\ In\ r_1 + B$$

$$0 = \frac{1}{4\mu}\frac{dp}{dx}{r_2}^2 + A\,In\,r_2 + B$$

The solution is

$$A = \frac{1}{4\mu}\frac{dp}{dx}\frac{{r_1}^2 - {r_2}^2}{\ln(r_2/r_1)}$$

$$B = -A\,In\,r_2 - \frac{{r_2}^2}{4\mu}\frac{dp}{dx}$$

thus

$$u(r) = \frac{1}{4\mu}\frac{dp}{dx}[r^2 - {r_2}^2 + \frac{{r_2}^2 - {r_1}^2}{\ln(r_2/r_1)}\ln(r/r_2)]$$

This is integrated to give the flow rate

$$Q = \int_{r_1}^{r_2} u(r)2\pi r dr$$

$$= \frac{\pi}{8\mu}\frac{dp}{dx}[{r_2}^4 - {r_1}^4 + \frac{({r_2}^2 - {r_1}^2)^2}{\ln(r_2/r_1)}$$

As $r_1 \to 0$ the velocity distribution approaches the parabolic distribution of pipe flow. As $r_1 \to r_2$ this distribution approaches that of a parallel-plate flow.

1-3-3 Loss of head due to friction in viscous flow

Perhaps the most calculated quantity in pipe flow is the head loss. If the head loss is known in a developed flow, the pressure change can be calculated; for a pipe the energy equation yields

$h_L = \frac{\Delta(p+\gamma h)}{\gamma}$ (3.4.1)

The head loss that results from the wall shear in a developed flow is related to the friction factor (see eq.3.4.14) by the Darcy- Weisbach equation, namely

$h_L = f\frac{L}{D}\frac{V^2}{2g}$ (3.4.2)

Consequently, if the friction factor were known, we could find the head loss and then the pressure drop

The friction factor f depends on the various quantities that affect flow, written as

$f = f(\rho, \mu, V, D, e)$ (3.4.3)

Where the average wall roughness height e accounts for the influence of the wall roughness elements. A dimensional analysis following the steps of unit 2 provides us with

$$f = f\left(\frac{\rho V D}{\mu}, \frac{e}{D}\right) \qquad (3.4.4)$$

Where e/D is the relative roughness.

Experimental data that relate the friction factor to the Reynold's number have been obtained for fully developed pipe flow over a wide range of wall roughness. The results of these data are presented in fig 41, which is commonly referred to as the **Moody diagram**.

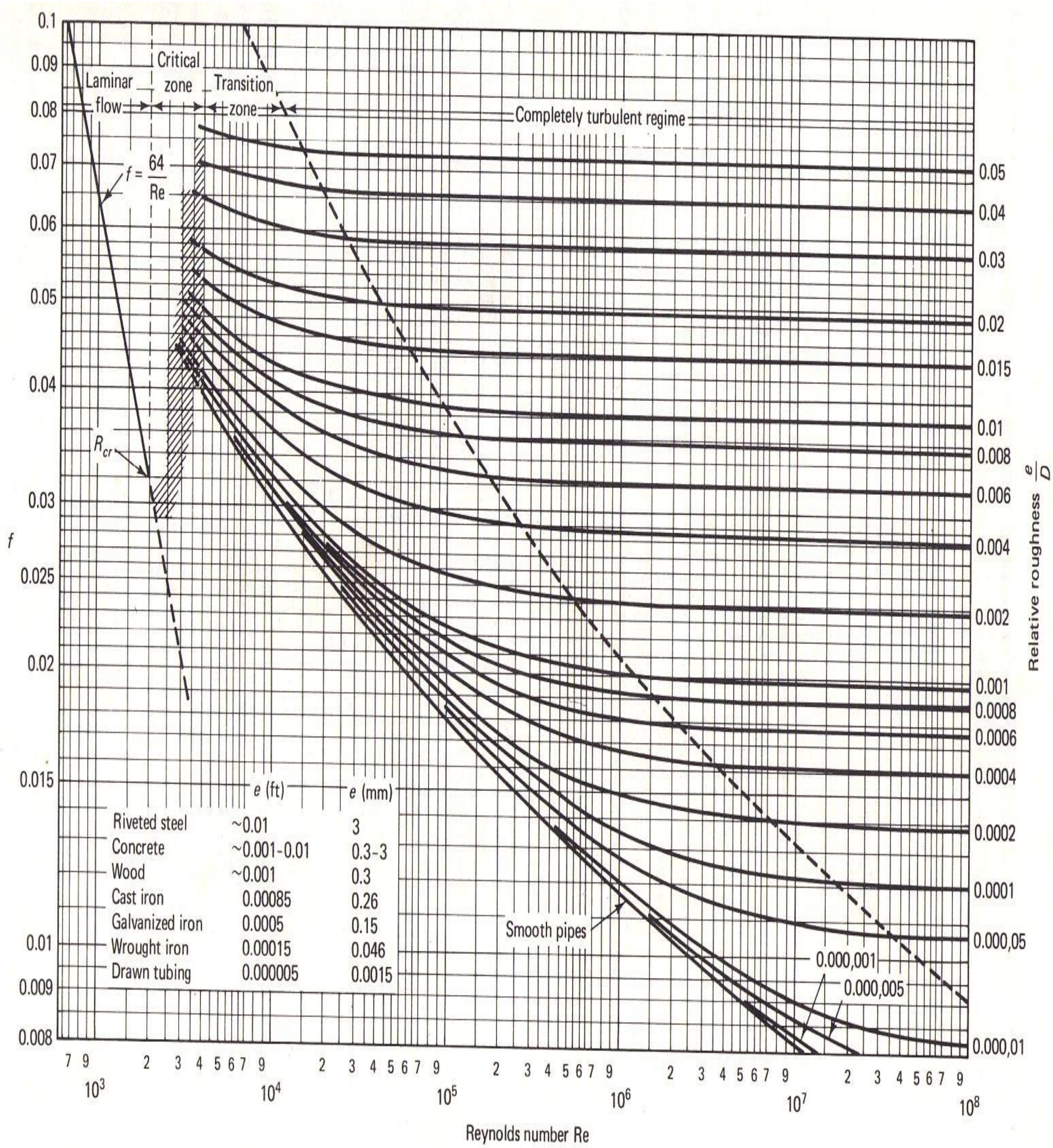


Figure 41 Moody diagram (From L.F. Moody, Trans. ASME, Vol. 66, 1944)

The Moody diagram is a graphical representation of the following empirical equations;

Smooth pipe $$\frac{1}{\sqrt{f}} = 0.86 ln\sqrt{f} - 0.8 \quad (3.4.5)$$

Completely turbulent zone $$\frac{1}{\sqrt{f}} = -0.86 ln\frac{e}{3.7D} \quad (3.4.6)$$

Transition zone $$\frac{1}{\sqrt{f}} = -0.86\ln\left(\frac{e}{3.7D} + \frac{2.51}{Re\sqrt{f}}\right) \quad (3.4.7)$$

The transition zone equation (3.4.7) that couples the smooth pipe equation to the completely turbulent regime equation is known as the **Colebrook equation.**

Good approximations can be made for the head loss in conduits with noncircular cross sections by using the **hydraulic radius R**, defined by

$$R = \frac{A}{P} \quad (3.4.8)$$

For a circular pipe the hydraulic radius is $R = r_0/2$. Hence we simply replace the radius r_0 with 2R and use the Moody diagram with

$$Re = \frac{4VR}{v} \qquad relative\ roughness = \frac{e}{4R}$$

$$h_L = f\frac{L}{4R}\frac{V^2}{2g} \quad (3.4.9)$$

To use this hydraulic radius technique the cross section should be fairly ”open”, such as a rectangle with aspect ratio less than 4:1, and equilateral triangle, or an oval. For other shapes, such as an annulus, the error would be significant.

Table 4 Three categories of problems can be identified for turbulent flow in a pipe of length L

Category	Known	Unknown
1	$Q,\ D,\ e,\ v$	h_L
2	$D,\ e,\ v,\ h_L$	Q
3	$Q,\ e,\ v,\ h_L$	D

A category 1 problem is straightforward and requires no iteration procedure when using the Moody diagram. Category 2 and 3 problems are more like problems encountered in engineering design situations and require an iterative trial-and-error process when using the Moody diagram. Each of these types will be illustrated with an example.

An alternative to using the Moody diagram is made possible by empirically derived formulas. The best of such formulas were presented by Swamee and Jain; an explicit expression that provides an approximate value for the unknown in each category above is as follows:

$$h_L = 1.07 \frac{Q^2 L}{g D^5}\{\ln[\frac{e}{3.7D} + 4.62 \left(\frac{vD}{Q}\right)^{0.9}]\}^{-2} \qquad 10^{-6} < e/D < 10^{-2}$$

$$3000 < Re < 3\times 10^{8} \qquad (3.4.10)$$

$$Q = -0.965 \left(\frac{gD^5 h_L}{L}\right)^{0.5} \ln\left[\frac{e}{3.7D} + \left(\frac{3.17v^2 L}{gD^3 h_L}\right)^{0.5}\right] \qquad Re > 2000 \qquad (3.4.11)$$

$$D = 0.6[e^{1.25}\left(\frac{LQ^2}{gh_L}\right)^{4.75} + vQ^{9.4}\left(\frac{L}{gh_L}\right)^{5.2}]^{0.04} \qquad 10^{-6} < e/D < 10^{-2}$$

$$5000 < Re < 3 \times 10^{8} \qquad (3.4.12)$$

Equation (3.4.11) is as accurate as the Moody diagram, and eqs. 3.4.10 and 3.4.12 accurate to within approximately 2% of the Moody diagram. These tolerances are acceptable for engineering calculations. It is important to realize that the Moody diagram is based on experimental data that likely is accurate to within no more than 5%. Hence the foregoing three formulas of Swamee and Jain, which can easily be input on programmable hand-held calculator, are often used by design engineers.

EXAMPLE 3.3

Water at 20^{o}C is transported for 500 m in a 4-cm-diameter wrought iron horizontal pipe with a flow rate of 0.003 m^3/s. Calculate the pressure drop over the 500-m length of pipe.

SOLUTION

The average velocity is

$$V = \frac{Q}{A}$$

$$V = \frac{0.003}{\pi\times 0.02^2} = 2.39\ m/s$$

The Reynolds number is

$$Re = \frac{VD}{\nu}$$

$$= \frac{2.39 \times 0.04}{10^{-6}} = 9.6 \times 10^4$$

Obtaining e from Fig. 44, we have, using $D = 40\ mm$,

$$\frac{e}{D} = \frac{0.046}{40} = 0.00115$$

The friction factor is read from the Moody diagram to be

$$f = 0.023$$

The head loss is calculated as

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

$$= 0.023\frac{500}{0.04}\frac{2.39^2}{2\ x\ 9.81} = 84m$$

This answer is given to two significant numbers since the friction factor is known to at most two significant numbers. The pressure drop is found to be

$$\Delta p = \gamma h_L$$
$$= 9800\ x\ 84 = 820000\ Pa$$

Using eqn. (3.4.10) we find

$$h_L = 1.07\frac{Q^2L}{gD^5}\{\ln[\frac{e}{3.7D} + 4.62(\frac{\nu D}{Q})^{0.9}\}^{-2}$$

$$= 1.07\ x\ 4480\ x\ 0.01731 = 83m$$

This value is within 1.2% of the value using the Moody diagram.

EXAMPLE 3.4

A pressure drop of 700 kPa is measured over a 300-m length of horizontal, 10-cm-diameter wrought iron pipe that transports oil

$S = 0.9, v = 10^{-5}\ m^2/s$

Calculate the flow rate.

SOLUTION

The relative roughness is

$\frac{e}{D} = \frac{0.046}{100} = 0.00046$

Assuming that the flow rate is completely turbulent (Re is not needed), the Moody diagram gives

$$f = 0.0165$$

The head loss is found to be

$h_L = \frac{\Delta p}{\gamma}$

$= \frac{700000}{9800\ x 0.9} = 79.4\ m$

The velocity is calculated from eqn. (3.4.2) to be

$$V = \left(\frac{2gDh_L}{fL}\right)^{1/2}$$

$$= \left(\frac{2 \times 9.8 \times 0.1 \times 79.4}{0.0165 \times 300}\right)^{1/2} = 5.61\ m/s$$

This provides us with a Reynold’s number of

$$Re = \frac{VD}{v}$$

$$= \frac{5.61\ x\ 0.1}{0.1\ x\ 10^{-5}} = 5.61\ x\ 10^4$$

Using this Reynold's number and e /D = 0.00046, the moody diagram gives the friction factor as $f = 0.023$

This corrects the original value for f the velocity is recalculated to be

$$V = \left(\frac{2 \times 9.8 \times 0.1 \times 79.4}{0.023 \times 300}\right)^{1/2} = 4.75\ x\ 10^4\ m/s$$

The Reynold's number is then

$$Re = \frac{4.75\ x\ 0.1}{10^{-5}} = 4.75\, m/s$$

From the Moody diagram $f = 0.023$ appears to be satisfactory. Thus the flow rate is

$$Q = VA$$

$$= 4.75\ x\ \pi\ x\ 0.05^2 = 0.37\, m^3/s$$

Only two significant numbers are given since f is known to at most two significant numbers.

Using the explicit relationship (3.4.11), we can directly calculate Q to be

$$Q = -0.965\left(\frac{9.8 \times 0.1^2 79.4}{300}\right)^{0.5} \ln\left[\frac{0.00046}{3.7\ x\ 0.1} + \left(\frac{3.17 \times 10^{-10}\ x\ 300}{9.8 \times 0.1^3 \times 79.4}\right)^{0.5}\right]$$

$= 0.965\ x\ 5.096\ x\ 10^{-3}\ x\ (-7.655) = 0.038\ \ m^3/s$

This value is within 2.5% of the value using the Moody diagram

Example 3.5

Drawn tubing of what diameter should be selected to transport $0.002\, m^3/s$ of 20^0C water over a 400-m length so that the head loss does not exceed 30 m?

SOLUTION

In this problem we do not know D. Thus, a trial-and –error solution is anticipate*d.* The average velocity is related to D by

$$V = \frac{Q}{A}$$

$$= \frac{0.002}{\pi D^2/4} = \frac{0.00255}{D^2}$$

This friction factor and D are related as follows:

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

$$30 = f\frac{400}{D}\frac{\left(0.00255/D^2\right)^2}{2\ x\ 9.8}$$

$$D^5 = 4.42\ x\ 10^{-6} f$$

The Reynold’s number is

$$Re = \frac{VD}{\upsilon}$$

$$= \frac{0.00255D}{D^2\ x\ 10^{-6}} = \frac{2550}{D}$$

Now, let us simply guess a value for f and check with the relations above and the Moody diagram. The first guess of $f = 0.03$ and the correction is listed in Table 5.

Table 5 Guessed values of f

f	$D(m)$	Re	e/D	f (*Fig*. 44)
0.03	0.0421	6.06 x10^4	0.000036	0.02
0.02	0.0388	6.57 x10^4	0.000039	0.02

The value of $f = 0.02$ is acceptable, yielding a diameter of 3.88 cm. Since this diameter would undoubtedly not be standard, a diameter of

$D = 4cm$

would be the tube size selected. This tube would have a head loss less than the limit $h_L = 30m$ imposed in the problem statement. Any larger-diameter tube would also satisfy this criterion but would be more costly, so should not be selected.

Using the explicit relationship (3.4.12), we can directly calculate D to be

$$= 0.6[(1.5 \times 10^{-6})^{1.25}\left(\frac{400 \times 0.002^2}{9.8 \times 30}\right)^{4.75} + \times\, 0.002^{9.4}\left(\frac{400}{9.8 \times 30}\right)^{5.2}]^{0.04} \qquad = 0.039\ m$$

Hence D = 4 cm would be the tube size selected. This is the same tube size as that selected using the Moody diagram.

EXAMPLE 3.6

Air at standard conditions is to be transported through 500 m of a smooth, horizontal, 30cm x 20cm rectangular duct at a flow rate of 0.24m^3/s. calculate the pressure drop.

SOLUTION

The hydraulic radius is

$$R = \frac{A}{P}$$

$$= \frac{0.3\ x\ 0.2}{(0.3+0.2)x\ 2} = 0.06m$$

The average velocity is

$$V = \frac{Q}{A}$$

$$= \frac{0.24}{0.3\ x\ 2} = 4.0\ m/s$$

This gives a Reynolds number of

$$Re = \frac{4VR}{v}$$

$$= \frac{4\ x\ 4\ x\ 0.06}{1.6\ x\ 10^4} = 6\ x\ 10^4$$

Using the smooth pipe curve of the Moody diagram, we have

$$f = 0.0198$$

Hence,

$$h_L = f\frac{L}{4R}\frac{V^2}{2g}$$

$$= 0.0198\frac{500}{4\ x\ 0.06}\frac{4^2}{2\ x\ 9.8} = 33.7\ m$$

The pressure drop is

$$\Delta p = \rho g h_L$$

$$= 1.23\ x\ 9.8\ x 33.7 = 406\ Pa$$

1-3.3.1 Minor losses in pipe flow

Pipe systems include valves, elbows, enlargements, contractions, inlets, outlets, bends, and fittings that cause additional losses, referred to as **minor losses**. Each of these devices cause a change in the magnitude and/or the direction of the velocity vectors and hence results in losses. In general, if the flow is gradually accelerated by a device, the losses are very small; relatively large losses are associated with sudden enlargements because of the separated regions that result (a separated flow occurs when the primary flow separates from wall).

Minor loss is expressed in terms of a loss coefficient K, defined by

$$h_L = K\frac{V^2}{2g} \qquad (3.4.13)$$

Values of K have been determined experimentally for the various fittings and geometry changes of interest in piping systems. One exception is the sudden expansion from area A_1 to area A_2 for which the loss can be calculated as

$$h_L = \left(1 - \frac{A_1}{A_2}\right)^2\frac{{V_1}^2}{2g} \qquad (3.4.14)$$

Thus, for the sudden expansion

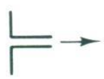
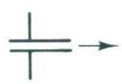
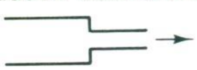
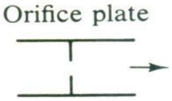
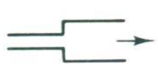
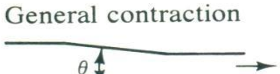
$$K = \left(1 - \frac{A_1}{A_2}\right)^2 \qquad (3.4.15)$$

If A_2 is extremely large (e.g., a pipe existing into a reservoir), K=1.0.

The loss coefficients for various geometries are presented in Table 6 and fig 42. A globe valve may be used to control the flow rate by introducing large losses by partially closing the valve. The other types of valves should not be used to control the flow; damage could result.

Loss coefficients for sudden contractions and orifice plates can be approximated by neglecting the losses in the converging flow up to vena contracta and calculating the losses in the diverging flow using the loss coefficient for a sudden expansion.

Table 6 Nominal loss coefficient (Turbulent flow)

Type of Fitting		Screwed			Flanged		
Diameter		1 in.	2 in.	4 in.	2 in.	4 in.	8 in.
Globe valve (fully open)		8.2	6.9	5.7	8.5	6.0	5.8
(half open)		20	17	14	21	15	14
(one-quarter open)		57	48	40	60	42	41
Angle valve (fully open)		4.7	2.0	1.0	2.4	2.0	2.0
Swing check valve (fully open)		2.9	2.1	2.0	2.0	2.0	2.0
Gate valve (fully open)		0.24	0.16	0.11	0.35	0.16	0.07
Return bend		1.5	.95	.64	0.35	0.30	0.25
Tee (branch)		1.8	1.4	1.1	0.80	0.64	0.58
Tee (line)		0.9	0.9	0.9	0.19	0.14	0.10
Standard elbow		1.5	0.95	0.64	0.39	0.30	0.26
Long sweep elbow		0.72	0.41	0.23	0.30	0.19	0.15
45° elbow		0.32	0.30	0.29			
Square-edged entrance				0.5			
Reentrant entrance				0.8			
Well-rounded entrance				0.03			
Pipe exit				1.0			
	Area ratio						
Sudden contraction[b] 	2 : 1			0.25			
	5 : 1			0.41			
	10 : 1			0.46			
	Area ratio A/A_0						
Orifice plate 	1.5 : 1			0.85			
	2 : 1			3.4			
	4 : 1			29			
	≥6 : 1			$2.78\left(\frac{A}{A_0} - 0.6\right)^2$			
Sudden enlargement[c]				$\left(1 - \frac{A_1}{A_2}\right)^2$			
90° miter bend (without vanes)				1.1			
(with vanes)				0.2			
General contraction 	(30° included angle)			0.02			
	(70° included angle)			0.07			

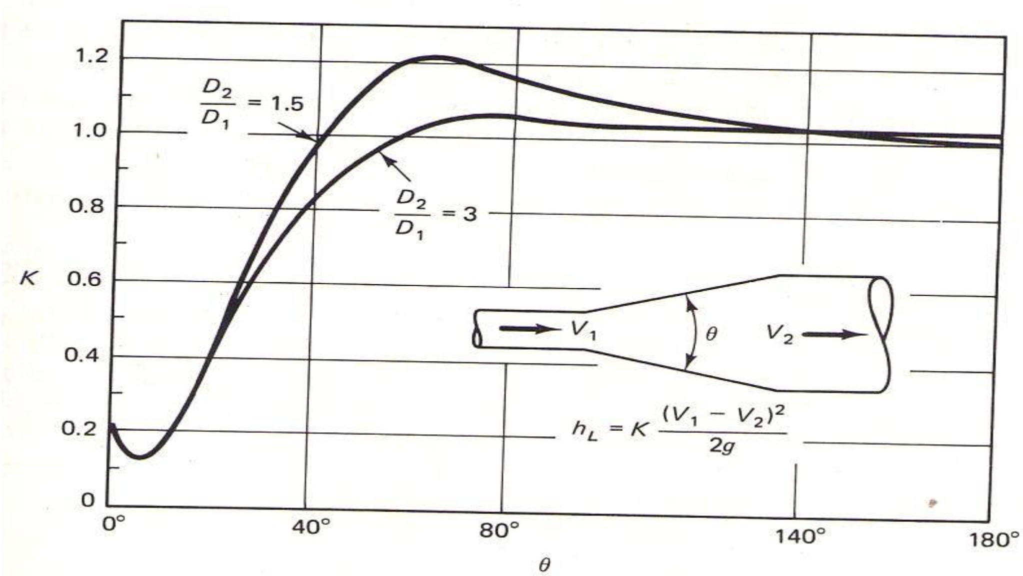


Figure 42 Loss coefficient in a conical expansion (From A.H. Gibson, Engineering, Vol. 93, 1912)

Fig. 43 provides the information necessary to establish the area of the **vena contracta**, the minimum area; this minimum area results when the converging streamlines begin to expand to fill the downstream area.

It is often the practice to express a loss coefficient as an equivalent length L_e of pipe. This is done by equating (3.4.13) to eqn. (3.4.2):

$$K\frac{V^2}{2g} = f\frac{L_e}{D}\frac{V^2}{2g}$$

2

Giving the relationship

$$L_e = K\frac{D}{f}$$

Hence the pipe exit of a 20-cm-diameter pipe with a friction factor of f =0.02 could be replaced by an equivalent pipe length of L_e = 10*m*.

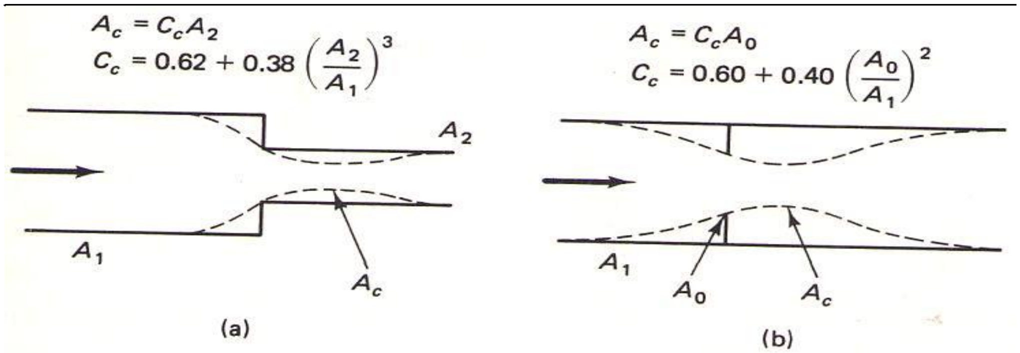


Figure 43 Vena contractas in contraction and orifices: (a) sudden contraction; (b) concentric orifice

Finally, a comment should be made concerning the magnitude of the minor losses. In piping systems involving intermediate lengths (i.e., 50 diameters) of pipe between minor losses, the minor losses may be of the same order of magnitude as the frictional losses; for short lengths the minor losses may be substantially greater than the frictional losses; and for long lengths (e.g. 1000 diameters) of pipe, the minor losses are usually neglected.

EXAMPLE 3.7

If the flow rate through a pipe of diameter d = 5cm, is 0.04 m^3/s, find the difference in elevation *H* of the two reservoirs

SOLUTION

The energy equation written for a control volume that contains the two reservoirs surfaces, where $V_1 = V_2$ *and* $p_1 = p_2 = 0$, is

$$0 = Z_2 - Z_1 + h_L$$

Thus, letting $Z_2 - Z_1 = H$ we have

The average velocity, Reynolds number, and relative roughness are

$$V = \frac{0.04}{\pi \, x \, 0.05^2} = 5.09 \, m/s$$

$$Re = \frac{5.09 \, x \, 0.1}{10^{-6}} = 5.09 \, x \, 10^5$$

$$\frac{e}{D} = \frac{0.046}{100} = 0.00046$$

From Moody diagram we find that

$$f = 0.0173$$

Using the loss coefficients from table 6 for screwed 4-in. elements we have

$$H = (0.5 + 5.7 + 2 \, x \, 0.64 + 1.0) \frac{5.09^2}{2 \, x \, 9.8} + 0.0173 \frac{50}{0.1} \frac{5.09^2}{2 \, x \, 9.8}$$

$$= 11.2 + 11.4 = 22.6m$$

EXAMPLE 5.8

Approximate the loss coefficient for the sudden contraction $A_1/A_2 = 2$ by neglecting the losses in the contracting portion up to the vena contracta and assuming that all the losses occur in the expansion from the vena contracta to A_2 (see Fig. 43). Compare with that given in Table 6

SOLUTION

The head loss from the vena contracta to area A_2 is (see Table 6)

$$h_L = \left(1 - \frac{A_c}{A_2}\right)^2 \frac{V_c{}^2}{2g}$$

From continuity equation

$$V_c = \frac{A_2}{A_C} V_2$$

$$h_L = \left(1 - \frac{A_c}{A_2}\right)^2 \left(\frac{A_2}{A_C}\right)^2 \frac{{V_2}^2}{2g}$$

So

$$K = \left(1 - \frac{A_c}{A_2}\right)^2 \left(\frac{A_2}{A_C}\right)^2$$

Using the expression C_c given in fig 43, we have

$$\frac{A_c}{A_2} = C_c$$

$$= 0.62 + 0.38 \left(\frac{1}{2}\right)^3 = 0.67$$

Finally,

$$K = (1 - 0.67)^2 \frac{1}{0.67^2} = 0.24$$

This compare favorably with the value of 0.25 in Table 6.

Self Assessment 1-5

1. Calculate the maximum average velocity V with which 20 ^{o}C water can flow in a pipe in the laminar state if the critical Reynolds number ($\mathrm{Re} = VD/\nu$) at which transition occurs is 2000; the pipe diameter is:
   a) 2 m (b) 2 cm (c) 2mm

Answer tips

(a) 1007 m/s (b) 100,700 m/s (c) 1,007,000 m/s

SESSION 2-3: Viscous resistance to bearings and viscosity determination

2-3.1 Viscous resistance to bearings

The effect of viscosity on flow and its effect on head losses have been examined in the preceding sections of this unit. A laminar-flow case of great practical importance is the hydrodynamic theory of lubrication. Simple aspects of this theory are developed in this section

2-3.1.1 Journal bearings

Large forces are developed in small clearances when the surfaces are slightly inclined and one is in motion so that fluid is “wedged” into the decreasing space. The slipper bearing, which operates on this principle, is illustrated in Fig. 44. The journal bearing (Fig. 45) develops its force by the same action, except that the surfaces are curved.

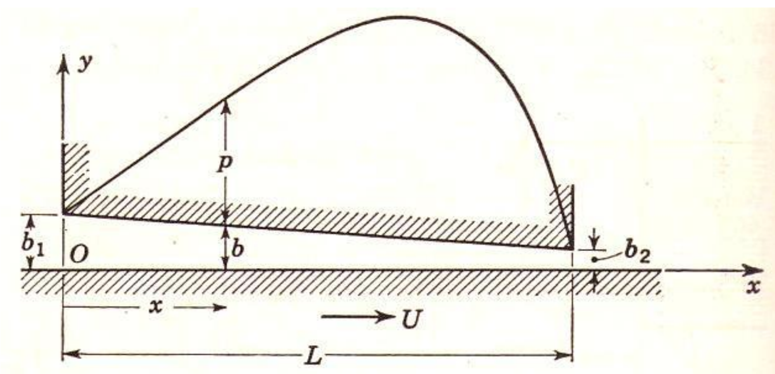


Figure 44 Sliding bearing

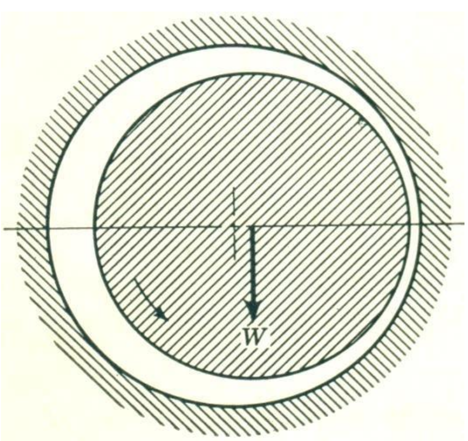


Figure 45 Journal bearing

The laminar flow equations may be used to develop theory of lubrication. The assumption is made that there is no flow out of the ends of the bearing, normal to the plane of fig. 44. Starting with the equation $\frac{dp}{dx} = \frac{d\tau}{dy}$ (3.5.1)

which relates pressure drop and shear stress, the equation for the force P that the bearing will support is worked out, and the drag on the bearing is computed.

Substituting Newton's second law of viscosity into eqn. (3.5.1) produces

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2} \qquad (3.5.2)$$

Since the inclination of the upper portion of the bearing (Fig. 44) is very slight, it is assumed that the velocity distribution is the same as if the plates were parallel and that p is independent of y. Integrating eqn. (3.5.2) twice with respect to y, with dp/dx constant, produces

$$\frac{dp}{dx}\int dy = \mu \int \frac{d^2u}{dy^2} + A$$

Or

$$\frac{dp}{dx} y = \mu \frac{du}{dy} + A$$

and the second time

$$\frac{dp}{dx}\int ydy = \mu \int \frac{du}{dy} dy + A\int dy + B$$

Or

$$\frac{dp}{dx}\frac{y^2}{2} = \mu u + Ay + B$$

The constants of integration A, B are determined from the conditions u = 0, y = b; u = U, y = 0.

$$\frac{dp}{dx}\frac{b^2}{2} = Ab + B \qquad \mu U + B = 0$$

Eliminating A and B and solving for u results in

$$u = \frac{y}{2\mu}\frac{dp}{dx}(y - b) + U\left(1 - \frac{y}{b}\right) \qquad (3.5.3)$$

The discharge Q must be the same at each cross section. By integrating over a typical section, again with dp /dx constant,

$$Q = \int_0^b u dy = \frac{Ub}{2} - \frac{b^3}{12\mu}\frac{dp}{dx} \qquad (3.5.4)$$

Now since Q cannot vary with x, b may be expressed in terms of x, $b = b_1 - ax$, in which $a = (b_1 - b_2)/L$ and the equation is integrated with respect to x to determine the pressure distribution. Solving eqn. (3.5.4) for *dp/dx* produces

$$\frac{dp}{dx} = \frac{6\mu U}{(b_1 - ax)^2} - \frac{12\mu Q}{(b_1 - ax)^3} \qquad (3.5.5)$$

By integrating,

$$\int \frac{dp}{dx}dx = 6\mu U \int \frac{dx}{(b_1 - ax)^2} - 12\mu Q \int \frac{dx}{(b_1 - ax)^3} + C$$

$$p = \frac{6\mu U}{\alpha(b_1 - ax)} - \frac{12\mu Q}{\alpha(b_1 - ax)^2} + C$$

In this equation Q and C are unknowns. Since the pressure must be the same, say zero, at the ends of the bearing, namely, $p = o, x = 0$; $p = 0, x = L$, the constants may be determined,

$$Q = \frac{Ub_1b_2}{b_1 + b_2} \qquad C = \frac{6\mu U}{\alpha(b_1 + b_2)}$$

With these values inserted, the equation for pressure distribution becomes

$$p = \frac{6\mu Ux(b - b_2)}{b^2(b_1 + b_2)} \qquad (3.5.6)$$

This equation shows that p is positive between *x = 0* and *x = L* if $b = b_2$. It is plotted in Fig. 44 to show the distribution of pressure throughout the bearing. With this one-dimensional method of analysis the very slight change in pressure along a vertical line *x = constant* is neglected.

The total force *P* that the bearing will sustain, per unit width, is

$$P = \int_0^L p dx = \frac{6\mu U}{b_1 + b_2} \int_0^L \frac{x(b - b_2) dx}{b^2}$$

After substituting the value of b in terms of x and performing the integration,

$$P = \frac{6\mu U L^2}{(b_1 - b_2)} \left(In \frac{b_1}{b_2} - 2 \frac{b_1 - b_2}{b_1 + b_2} \right)$$

b_2 □ b_2 □ b_1 □ b_2 □

The drag force D required to move the lower surface at speed U is expressed by

$$D = \int_0^L \tau$$

By evaluating *du/dy* from eqn. (3.5.7), for y = 0,

With this value in the integral, along with the value of dp/dx from equ. (3.5.5),

$$D = \int_0^L \frac{2\mu UL}{b_1 - b_2} \left(2In \frac{b_1}{b_2} - 3 \frac{b_1 - b_2}{b_1 + b_2} \right) \qquad (3.5.8)$$

The maximum load P is computed with eqn. (3.5.7) when $b_1 = 2.2 b_2$. With this ratio,

$$P = 0.16 \frac{\mu U L^2}{b_2^2} \qquad D = 0.75 \frac{\mu UL}{b_2} \qquad (3.5.9)$$

The ratio of load to drag for optimum load is

$$\frac{P}{D} = 0.21 \frac{L}{b_2} \qquad (3.5.10)$$

Which can be large since b_2 can be very small.

EXAMPLE 3.9

A vertical turbine shaft carries a load 0f 80,000lb on a thrust bearing consisting of 16 flat rocker plates, 3 in. by 9 in., arranged with their long dimensions radial from the shaft and with their centers on a circle of radius 1.5 ft. The shaft turns at 120 rpm; μ=0.002 $lb\text{-sec}/ft^2$. If the plates take the angle for maximum load, neglecting effects of curvature of path and radial lubricant flow, find (a) the clearance between rocker plate and fixed plate; (b) the torque loss due to the bearing.

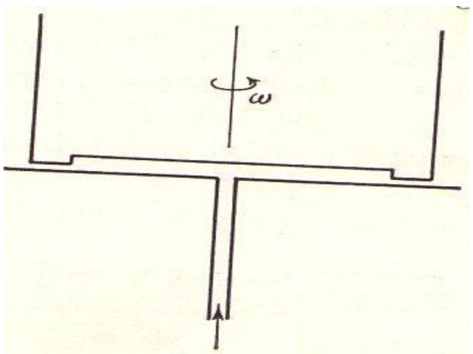


Figure 46 Hydrostatic lubrication by high-pressure pumping of oil

SOLUTION

(a) Since the motion is considered straight-line,

$$U = 1.5\,x\,\frac{120}{60}2\pi = 18.85\,ft/sec$$

$$L = 0.25ft$$

The load is 5000 lb for each plate, which is 5000/0.75 = 6667 lb for unit width. By solving for the clearanceb_2, from equ. 93.5.9),

$$b_2 = \sqrt{\frac{0.16\mu UL^2}{P}} = 0.4 \times 0.25\sqrt{\frac{0.002 \times 18.85}{6667}} = 2.38 \times 10^{-4}\,ft = 0.0029\,in.$$

(b) The drag due to one rocker plate is, per foot of width,

$$D = 0.75\frac{\mu UL}{b_2} = \frac{0.75 \times 0.002 \times 18.85 \times 0.25}{2.38 \times 10^{-4}} = 29.6lb$$

For a 9-in. plate, $D = 29.6 \text{ x } 0.75 \text{ x } 22.2 \ lb$. The torque loss due to the 16 rocker plates is

$$16 \text{ x } 22.2 \text{ x} 1.5 \text{ x} 533 \ \ ft\text{ - }lb$$

Another form of lubrication, called hydrostatic lubrication has many important applications. It involves the continuous pumping of high-pressure oil under a step bearing as illustrated in Fig. 46. The load may be lifted by the lubrication before rotation starts, which greatly reduces starting friction.

2-3.1.2 Collar bearings

Fig. 47 shows a section of an open collar bearing

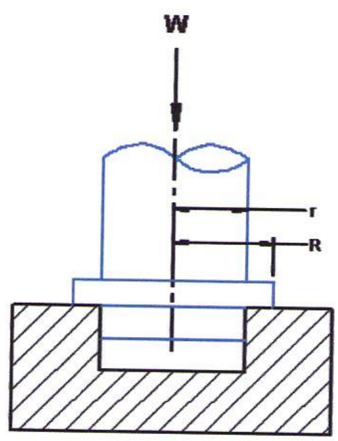


Figure 47 Open collar bearing

The torque transmitted to the lubrication fluid in Pound-inches is given as

$$T = 2/3fW\left[\frac{R^3-r^3}{R^2-r^2}\right] \qquad (3.5.11)$$

Power P Lost by Friction in Foot-pounds per Second

$$P = [(4n\pi)/(36)]fW\left[\frac{R^3-r^3}{R^2-r^2}\right]$$

Where:

f = Coefficient of friction; W = Load in pounds

T = Torque about the axis of the shaft; R = Radius in inches

r = Radius in inches; n = Revolutions per second

NB: The best material for collar bearing is high carbon chromium bearing steel, which features desired hardness, strong rolling fatigue resistance, wearing resistance and stable dimension.

2-3.2 Methods of determination of coefficient of viscosity

Viscosity can be thought of as the internal stickiness of a fluid. It is one of the properties that controls the amount of fluid that can be transported in a pipeline during a specific period of time. It accounts for the energy losses associated with the transport of fluids in ducts, channels, and pipes.

The rate of deformation of a fluid is directly linked to the viscosity of the fluid. For a given stress, a highly viscous fluid deforms at a slower rate than a fluid with a low viscosity. This section looks at methods of viscosity determination

2-3.2.1 Rotating cylinder method

For a simple field, in which u = u(y), we can define the Viscosity μ of the fluid by the relationship

$$\tau = \mu\frac{du}{dy} \text{.................} (3.7.1)$$

Where τ the shear stress and u is velocity in the x- direction. The units of τ are N/m^2, and of μ are $\frac{N.s}{m^2}$. The quantity du/dy is a velocity gradient and can be interpreted as a strain rate.

The concept of viscosity and velocity gradients can also be illustrated by considering a fluid within the small gap between two concentric cylinders, as shown in Fig. 48. A torque is necessary to rotate the inner cylinder at constant speed while the outer cylinder remains stationary. This resistance to the rotation of the cylinder is due to viscosity. The only stress that exists to resist the applied torque for this simple flow is a shear stress, which is observed to depend directly on the velocity gradient; that is,

$$\tau = \mu | {}^{du}/_{dr} |$$

Where du/dr is the velocity gradient and u is the tangential velocity component, which depends only on r. For a small gap (h<<R), this gradient can be approximated by assuming a linear velocity distribution in the gap (**NB**: if the gap is not small relative to R, the velocity distribution will not be linear). Thus

$$\left|\frac{du}{dr}\right| = \frac{\omega r}{h}$$

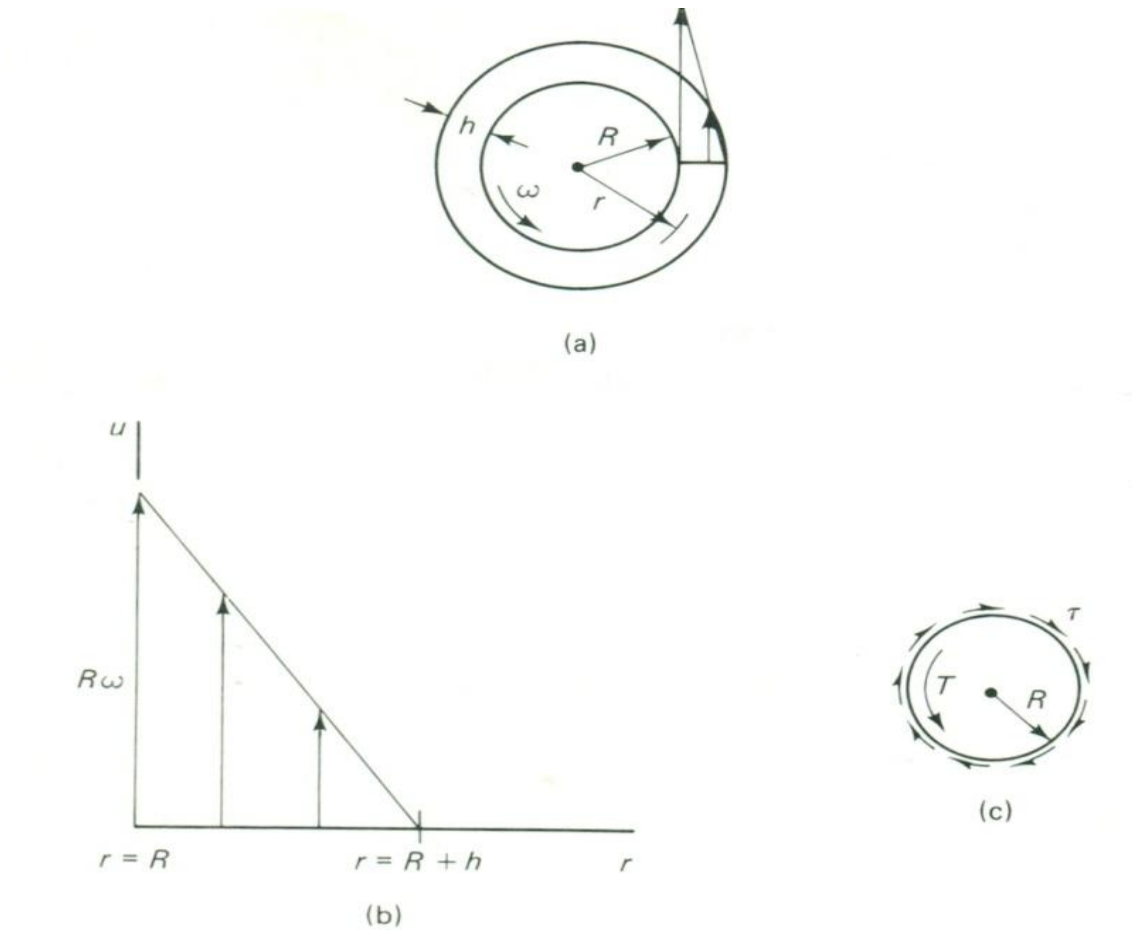


Figure 48 Fluid being sheared between cylinders with a single gap:(a)rotating inner cylinder; (b) velocity distribution; (c) the outer cylinder fixed and the inner cylinder is rotated

where h is the gap width. We can thus relate the applied torque T to the viscosity and other parameters by the equation

$$T = stress\ x\ area\ x\ moment\ arm$$

$$= r\ x 2\mu RL\ x\ R$$

$$\mu \frac{\omega R}{h} \times 2\pi RL \times R = \frac{2\pi R^3 \omega L \mu}{h}$$

Where we have neglected the shearing stress acting on the ends of the cylinder; L represents the length of the rotating cylinder. Note that the torque depends directly on viscosity, thus the cylinders could be used as a **viscometer,** a device that measures the viscosity of a fluid.

EXAMPLE 3.10

A viscometer is constructed with two 30 cm-long concentric cylinders, one 20 cm in diameter and the other 20.2 cm in diameter. A torque of 0.13 N.m is required to rotate the inner cylinder at 400 rpm. Calculate the viscosity.

Solution

The applied torque is just balanced by a resisting torque due to the shear stresses (see fig 48(c)).

The viscosity is

$$\mu = \frac{Th}{2\pi R^3 \omega L}$$

$$= \frac{0.13(0.001)}{2\pi(0.1)^3 \left[400 \, {}^{(2\pi)}/_{60}\right]} = 0.00165 \; N.s/m^2$$

2-3.2.2 Falling sphere method

Fig. 49 shows the arrangement for the falling sphere viscometer. The length from which the sphere is dropped is L_{cyl} while the length where the sphere is speeding up to terminal velocity is L_o and the length over which the sphere is travelling at terminal velocity is L_t . The basic task involved in the determination of the viscosity of the fluid is to predict the total time for the sphere dropped from a height L_{cyl} to reach the bottom of the cylinder.

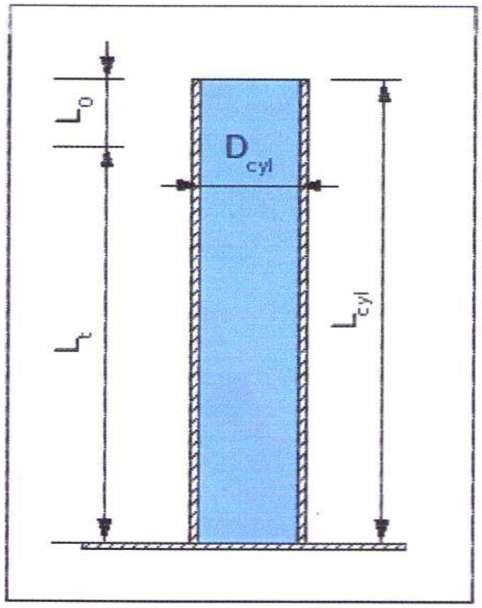


Figure 49 The falling sphere viscometer

Analysis:
Figure 50 shows a free body diagram of the sphere. In figure 50, F_D is the drag force exerted by the fluid as the sphere drops downward, F_W is the weight of the sphere and F_B is the buoyancy of the sphere.

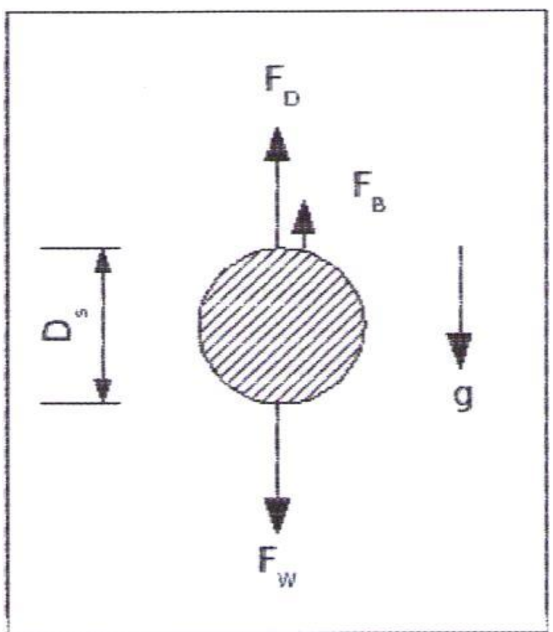


Figure 50 Free body diagram of a sphere falling

$F_W = mg = \frac{\pi}{6} {D_s}^3 \rho_s g$

$$F_D = C_D \frac{1}{2} \rho V^2 \left(\frac{\pi}{4} {D_s}^2\right) \qquad (3.7.5)$$

$F_B = \frac{\pi}{6} {D_s}^3 \rho g$

C_D is the coefficient of drag which may be found from Fig. 22 of unit 3. The Reynolds number for the flow is given as

$$R_D = \frac{\rho V D_s}{\mu} \qquad (3.7.6)$$

If the sphere is falling at a constant speed (terminal velocity) then Newton's second law applied to Fig. 50 gives

$$F_D + F_B = F_W \qquad (3.7.7)$$

Equation (3.7.7) along with equations (3.7.6) and (3.7.5) in conjunction with C_D may be used to determine the terminal velocity V_T . If the sphere is falling with terminal velocity then for the distance L_T the time to fall this distance will be

$$t_T = \frac{L_T}{V_T}$$

EXAMPLE 3.11: Terminal Velocity Calculation

Find the terminal velocity of a steel ball bearing ($\rho = 7.80\ x\ 10^3\ kg/m^3$) with a diameter of 1.00cm in SAE 10 oil at 20^{0}C ($\rho = 860\ kg/m^3\ and\ \mu = 0.100\ \ kg/(m.s)$).

SOLUTION

First of all when the sphere is at terminal velocity the drag force plus the buoyant force equals the weight of the sphere. The weight of the sphere and the buoyant forces are

$$F_W = mg = \frac{\pi}{6} {D_s}^3 \rho_s g = \frac{\pi}{6}(0.0100)^3 \times 7.80 \times 10^3 \frac{kg}{m^3} \times \frac{9.81m}{s^2} = 4.01 \times 10^{-2}\ Newtons$$

$$F_B = \frac{\pi}{6} {D_s}^3 \rho g = \frac{\pi}{6}(0.0100m)^3 \times 860 \times 10^3 \frac{kg}{m^3} \times \frac{9.81m}{s^2} = 4.42 \times 10^{-2}\ Newtons$$

Combining F_w and F_B to get F_D from equation (3.7.7) gives

$$F_D = F_W - F_B = 3.56 \times 10^{-2}\ Newtons = C_D \frac{1}{2}(860kg/m^3){V_T}^2 \left(\frac{\pi}{4}(0.01m)^2\right)$$

Solving for $C_D v^2$gives

$$C_D V_T^2 = 1.06\, m^2/s^2$$

Now let's get R_D as far as we can

$$R_D = \frac{\rho V D_s}{\mu} = \frac{(860kg/m)^3(0.0100m)}{0.100(\frac{kg}{ms})} V_T = (8.60\frac{s}{m})V_T$$

We cannot go any further without the plot of drag coefficient versus Reynolds number. But with that graph we can guess values of V_T then calculate Re_D. This allows us to look up a value of C_D .Then we know that the combination $C_D V_T^2 = 1.06$. If the guessed value of V does not match up with this value then we need to re-guess V_T . It is left as an exercise to find the approximate value of V_T .

Self Assessment 2-5

2. Calculate the laminar entrance length in a 4-cm-diameter pipe if $2\text{x}10^{-4}\ m^3/s$ of water is flowing at:
   (a) 10 °C (b) 20 °C (c) 40 °C (d) 80 °C

Answer tips

(a) $1.265\text{x}10^{-5}\ m$ (b) $1.637\ \text{x}10^{-5}\ m$ (c) $2.504\,\text{x}10^{-5}\ m$ (d) $4.510\ \text{x}10^{-5}\ m$

Learning Track Activities

Unit Summary

1. A **developed flow** results when the velocity profile ceases to change in the flow direction. For laminar flow in a circular pipe with a uniform profile at the inlet, the entrance length is given by

$$\frac{L_E}{D} = 0.065\,\text{Re}$$

where the Reynolds number is based on the average velocity and the diameter.

For laminar flow between two parallel plates and a uniform profile at the inlet the entrance length is

$$\frac{L_E}{D} = 0.04Re \qquad Re = \frac{Vh}{v}$$

where h is the distance between the plates

2. The Darcy-Weisbach equation is given as $\frac{\Delta p}{\gamma} = h_L = f\frac{L}{D}\frac{V^2}{2g}$

where h_L is the head loss with dimension of length. The friction factor, f, is given as

$f = \frac{64}{Re}$ $\quad$ *For laminar* flow in a pipe

Key terms/ New Words in Unit

Developed flow, entrance region, entrance length, poiseulle flow, coquette flow, Moody
diagram, critical flow, relative roughness

Unit Assignments 5

Estimate the loss coefficient based on V_2 using the data of Fig. 3.4.4