

## LECTURE SEVEN

### Newton's Divided Difference Interpolation

*After reading this lecture notes, you should be able to:*

1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation, and*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

#### **What is interpolation?**

Many times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ , .....,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n+1$  data values with  $f(x)$  passing through the  $n+1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called *interpolation*.

Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

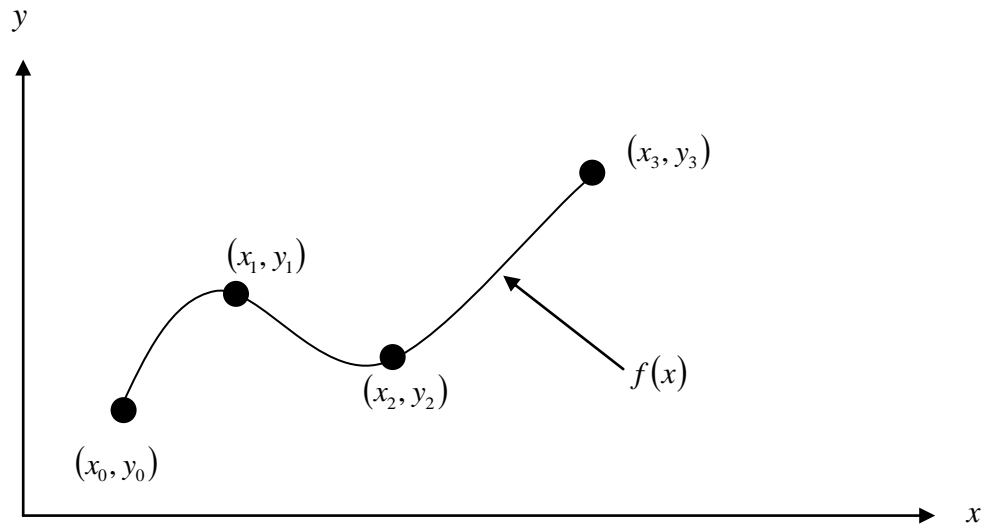
- (A) evaluate,
- (B) differentiate, and
- (C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n+1$  points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We will discuss Newton's divided difference polynomial method in this lecture.

### Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.



**Figure 1** Interpolation of discrete data.

#### Linear Interpolation

Given  $(x_0, y_0)$  and  $(x_1, y_1)$ , fit a linear interpolant through the data. Noting  $y = f(x)$  and  $y_1 = f(x_1)$ , assume the linear interpolant  $f_1(x)$  is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at  $x = x_0$ ,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at  $x = x_1$ ,

$$\begin{aligned} f_1(x_1) &= f(x_1) = b_0 + b_1(x_1 - x_0) \\ &= f(x_0) + b_1(x_1 - x_0) \end{aligned}$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

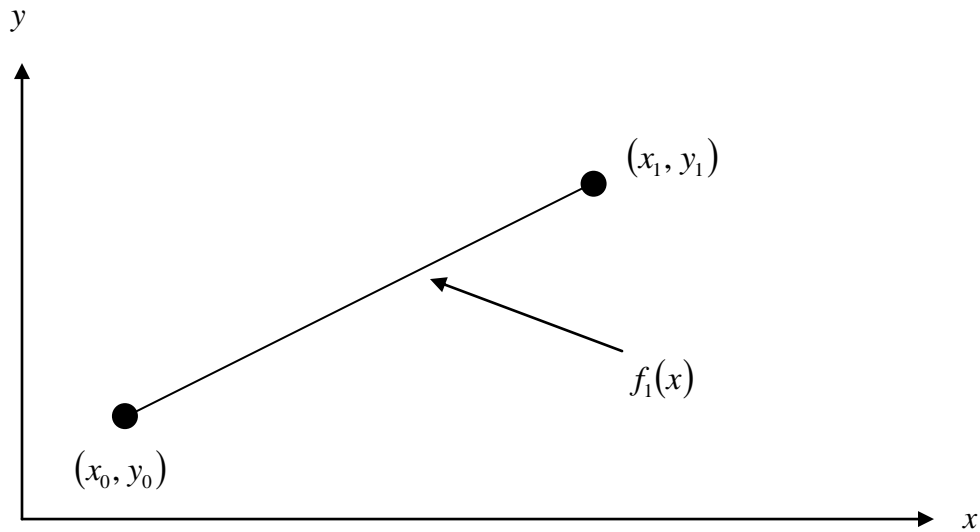
So

$$\begin{aligned} b_0 &= f(x_0) \\ b_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \end{aligned}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$



**Figure 2** Linear interpolation.

### Example 1

The upward velocity of a rocket is given as a function of time in Table 1 (Figure 3).

**Table 1** Velocity as a function of time.

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at  $t = 16$  seconds using first order polynomial interpolation by Newton's divided difference polynomial method.

### Solution

For linear interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0)$$

Since we want to find the velocity at  $t = 16$ , and we are using a first order polynomial, we need to choose the two data points that are closest to  $t = 16$  that also bracket  $t = 16$  to evaluate it. The two points are  $t = 15$  and  $t = 20$ .

Then

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$

gives

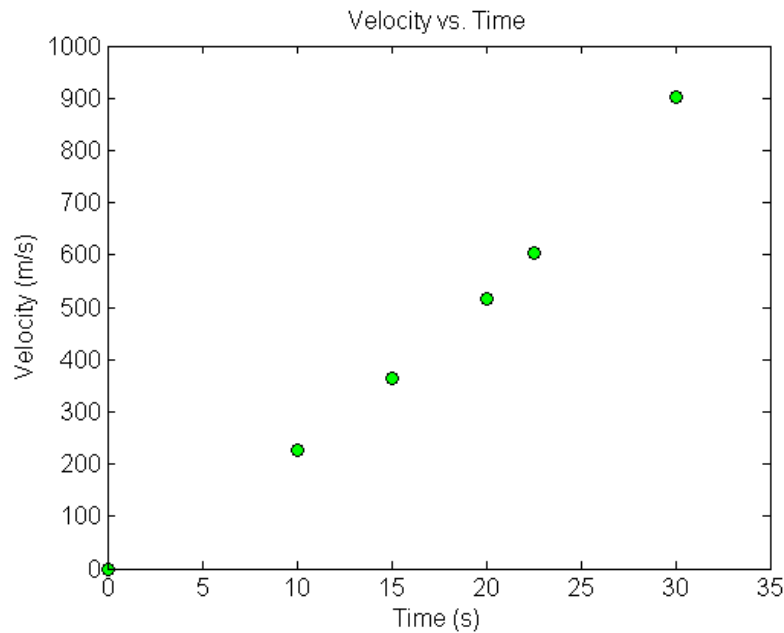
$$b_0 = v(t_0)$$

$$= 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0}$$

$$= \frac{517.35 - 362.78}{20 - 15}$$

$$= 30.914$$



**Figure 3** Graph of velocity vs. time data for the rocket example.

Hence

$$\begin{aligned} v(t) &= b_0 + b_1(t - t_0) \\ &= 362.78 + 30.914(t - 15), \quad 15 \leq t \leq 20 \end{aligned}$$

At  $t = 16$ ,

$$\begin{aligned} v(16) &= 362.78 + 30.914(16 - 15) \\ &= 393.69 \text{ m/s} \end{aligned}$$

If we expand

$$v(t) = 362.78 + 30.914(t - 15), \quad 15 \leq t \leq 20$$

we get

$$v(t) = -100.93 + 30.914t, \quad 15 \leq t \leq 20$$

and this is the same expression as obtained in the direct method.

Quadratic Interpolation

Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , fit a quadratic interpolant through the data. Noting  $y = f(x)$ ,  $y_0 = f(x_0)$ ,  $y_1 = f(x_1)$ , and  $y_2 = f(x_2)$ , assume the quadratic interpolant  $f_2(x)$  is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At  $x = x_0$ ,

$$\begin{aligned} f_2(x_0) &= f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) \\ &= b_0 \end{aligned}$$

$$b_0 = f(x_0)$$

At  $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At  $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

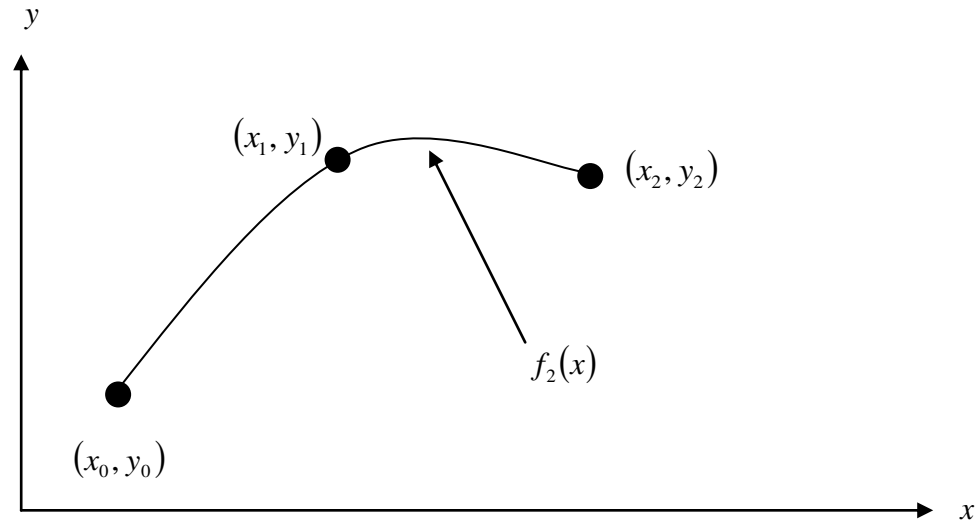
Giving

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$\begin{aligned} &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1) \end{aligned}$$



**Figure 4** Quadratic interpolation.

### Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

**Table 2** Velocity as a function of time.

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at  $t = 16$  seconds using second order polynomial interpolation using Newton's divided difference polynomial method.

### Solution

For quadratic interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

Since we want to find the velocity at  $t = 16$ , and we are using a second order polynomial, we need to choose the three data points that are closest to  $t = 16$  that also bracket  $t = 16$  to evaluate it. The three points are  $t_0 = 10$ ,  $t_1 = 15$ , and  $t_2 = 20$ .

Then

$$t_0 = 10, v(t_0) = 227.04$$

$$t_1 = 15, v(t_1) = 362.78$$

$$t_2 = 20, v(t_2) = 517.35$$

gives

$$\begin{aligned}
b_0 &= v(t_0) \\
&= 227.04 \\
b_1 &= \frac{v(t_1) - v(t_0)}{t_1 - t_0} \\
&= \frac{362.78 - 227.04}{15 - 10} \\
&= 27.148 \\
b_2 &= \frac{\frac{v(t_2) - v(t_1)}{t_2 - t_1} - \frac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} \\
&= \frac{\frac{517.35 - 362.78}{20 - 15} - \frac{362.78 - 227.04}{15 - 10}}{20 - 10} \\
&= \frac{30.914 - 27.148}{10} \\
&= 0.37660
\end{aligned}$$

Hence

$$\begin{aligned}
v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) \\
&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \quad 10 \leq t \leq 20
\end{aligned}$$

At  $t = 16$ ,

$$\begin{aligned}
v(16) &= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \\
&= 392.19 \text{ m/s}
\end{aligned}$$

If we expand

$$v(t) = 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \quad 10 \leq t \leq 20$$

we get

$$v(t) = 12.05 + 17.733t + 0.37660t^2, \quad 10 \leq t \leq 20$$

This is the same expression obtained by the direct method.

### General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$\begin{aligned}
b_0 &= f(x_0) \\
b_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
b_2 &= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}
\end{aligned}$$

Note that  $b_0$ ,  $b_1$ , and  $b_2$  are finite divided differences.  $b_0$ ,  $b_1$ , and  $b_2$  are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$\begin{aligned} f[x_2, x_1, x_0] &= \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} \\ &= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \end{aligned}$$

where  $f[x_0]$ ,  $f[x_1, x_0]$ , and  $f[x_2, x_1, x_0]$  are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for  $n + 1$  data points,  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

where

$$\begin{aligned} b_0 &= f[x_0] \\ b_1 &= f[x_1, x_0] \\ b_2 &= f[x_2, x_1, x_0] \\ &\vdots \\ b_{n-1} &= f[x_{n-1}, x_{n-2}, \dots, x_0] \\ b_n &= f[x_n, x_{n-1}, \dots, x_0] \end{aligned}$$

where the definition of the  $m^{\text{th}}$  divided difference is

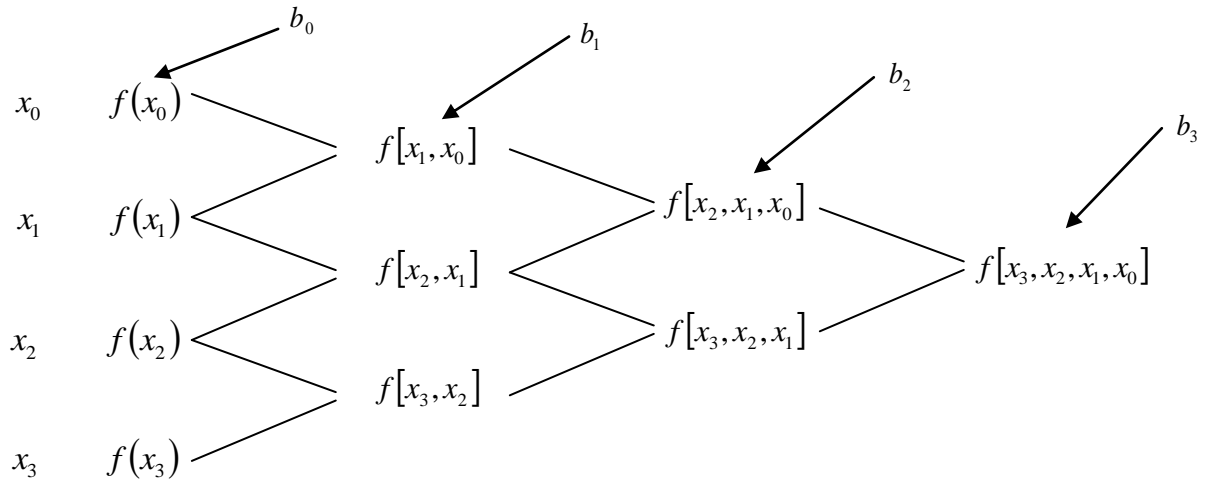
$$\begin{aligned} b_m &= f[x_m, \dots, x_0] \\ &= \frac{f[x_m, \dots, x_1] - f[x_{m-1}, \dots, x_0]}{x_m - x_0} \end{aligned}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ ,

$$\begin{aligned} f_3(x) &= f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\ &\quad + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2) \end{aligned}$$





**Figure 5** Table of divided differences for a cubic polynomial.

### Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

**Table 3** Velocity as a function of time.

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

- Determine the value of the velocity at  $t = 16$  seconds with third order polynomial interpolation using Newton's divided difference polynomial method.
- Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from  $t = 11$  s to  $t = 16$  s.
- Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at  $t = 16$  s.

### Solution

- For a third order polynomial, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Since we want to find the velocity at  $t = 16$ , and we are using a third order polynomial, we need to choose the four data points that are closest to  $t = 16$  that also bracket  $t = 16$  to evaluate it. The four data points are  $t_0 = 10$ ,  $t_1 = 15$ ,  $t_2 = 20$ , and  $t_3 = 22.5$ .

Then

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

gives

$$b_0 = v[t_0]$$

$$= v(t_0)$$

$$= 227.04$$

$$b_1 = v[t_1, t_0]$$

$$= \frac{v(t_1) - v(t_0)}{t_1 - t_0}$$

$$= \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = v[t_2, t_1, t_0]$$

$$= \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$

$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{517.35 - 362.78}{20 - 15}$$

$$= 30.914$$

$$v[t_1, t_0] = 27.148$$

$$b_2 = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$

$$= \frac{30.914 - 27.148}{20 - 10}$$

$$= 0.37660$$

$$b_3 = v[t_3, t_2, t_1, t_0]$$

$$= \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$

$$v[t_3, t_2] = \frac{v(t_3) - v(t_2)}{t_3 - t_2}$$

$$= \frac{602.97 - 517.35}{22.5 - 20}$$

$$= 34.248$$

$$\begin{aligned}
v[t_2, t_1] &= \frac{v(t_2) - v(t_1)}{t_2 - t_1} \\
&= \frac{517.35 - 362.78}{20 - 15} \\
&= 30.914 \\
v[t_3, t_2, t_1] &= \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1} \\
&= \frac{34.248 - 30.914}{22.5 - 15} \\
&= 0.44453 \\
v[t_2, t_1, t_0] &= 0.37660 \\
b_3 &= \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0} \\
&= \frac{0.44453 - 0.37660}{22.5 - 10} \\
&= 5.4347 \times 10^{-3}
\end{aligned}$$

Hence

$$\begin{aligned}
v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2) \\
&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) \\
&\quad + 5.5347 \times 10^{-3}(t - 10)(t - 15)(t - 20)
\end{aligned}$$

At  $t = 16$ ,

$$\begin{aligned}
v(16) &= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \\
&\quad + 5.5347 \times 10^{-3}(16 - 10)(16 - 15)(16 - 20) \\
&= 392.06 \text{ m/s}
\end{aligned}$$

b) The distance covered by the rocket between  $t = 11$  s and  $t = 16$  s can be calculated from the interpolating polynomial

$$\begin{aligned}
v(t) &= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) \\
&\quad + 5.5347 \times 10^{-3}(t - 10)(t - 15)(t - 20) \\
&= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3, \quad 10 \leq t \leq 22.5
\end{aligned}$$

Note that the polynomial is valid between  $t = 10$  and  $t = 22.5$  and hence includes the limits of  $t = 11$  and  $t = 16$ .

So

$$\begin{aligned}
s(16) - s(11) &= \int_{11}^{16} v(t) dt \\
&= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3) dt \\
&= \left[ -4.2541t + 21.265 \frac{t^2}{2} + 0.13204 \frac{t^3}{3} + 0.0054347 \frac{t^4}{4} \right]_{11}^{16}
\end{aligned}$$

$$= 1605 \text{ m}$$

c) The acceleration at  $t = 16$  is given by

$$a(16) = \left. \frac{d}{dt} v(t) \right|_{t=16}$$

$$a(t) = \frac{d}{dt} v(t)$$

$$= \frac{d}{dt} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)$$

$$= 21.265 + 0.26408t + 0.016304t^2$$

$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$

$$= 29.664 \text{ m/s}^2$$