INSTITUTE OF DISTANCE LEARNING

KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI, GHANA

ME 355
STRENGTH OF MATERIALS II

UNIT 2

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Introduction

COLUMNS AND STRUTS

Learning Objectives

- After reading this unit you should be able to:
- Define a strut
- Derive the Euler's critical load equations for the various ended-joints
- Compute the critical load using the Euler's load formulae
- Derive the equation of critical load using the other methods
- Compute the critical load using the other methods

EULER CRIPPLING LOAD FORMULA

Definition of Strut

- A structural member, subjected to an axial compressive force, is called a strut.

- A strut may be
  - horizontal,
  - inclined or
  - even vertical.

- A vertical strut, used in buildings or frames is called a *column*.

- A strut is subjected to some compressive force, then the compressive stress induced,

$$\sigma = \frac{p}{A}$$

Assumptions in the Euler's Column Theory

The following simplifying assumptions are made in the Euler's column theory:

- Initially the column is perfectly straight and the load applied is truly axial.
- The cross-section of the column is uniform throughout its length.
- The column material is perfectly elastic, homogeneous and isotropic and thus obeys Hooke's law.
- The length of column is very large as compared to its cross-sectional dimensions.
- The shortening of column, due to direct compression (being very small) is neglected.
- The failure of column occurs due to buckling alone.

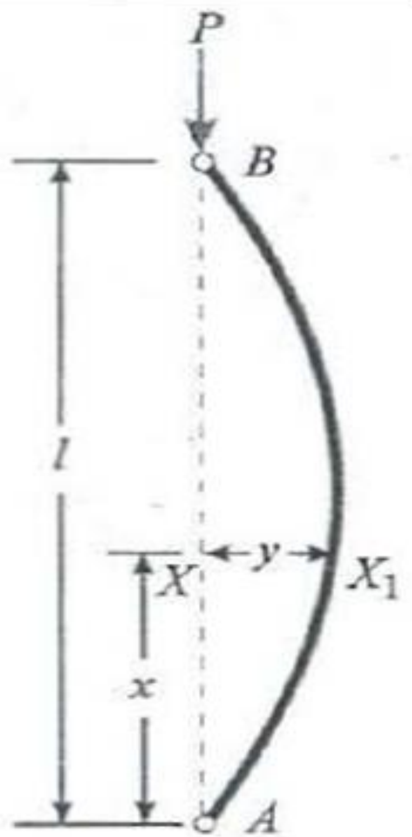
Types of End Conditions of Columns

- In actual practice, there are a number of end conditions, for columns.
- But, we shall study the Euler's column theory on the following four types of end conditions:

1. Both ends hinged
2. Both ends fixed
3. One end is fixed and the other hinged, and
4. One end is fixed and the other free.

- Now we shall discuss the value of critical load for all the above mentioned type of and conditions of columns one by one.

Case 1: Column/Strut – Both Ends Hinged



Now consider any section X, at a distance x from A.

Let P Critical load on the column

y Deflection of the column at X

Moment due to the critical load P, $M = -Py$

Differential Equation

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

where $\alpha^2 = P/EI$

Solution

$$y = A\sin\alpha x + B\cos\alpha x$$

Boundary Condition

At x==0; y==0; $\therefore B = 0$

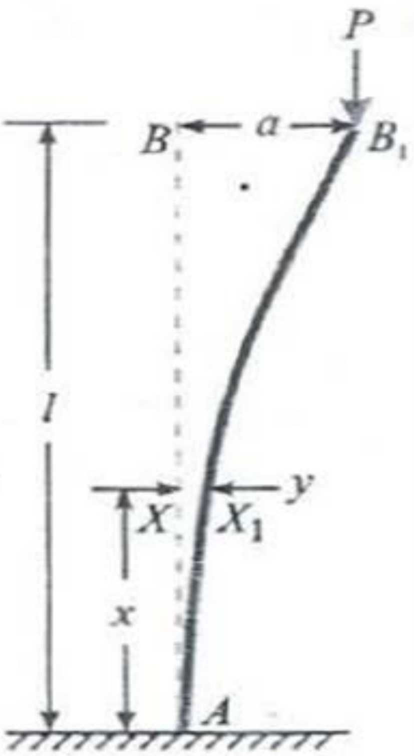
At x==*l*; y==0; $\therefore A\sin\alpha l = 0$

Since *A* ≠ 0, then $\sin\alpha l = 0$, therefore $\alpha l = \pi$

$$\alpha^2 = \pi^2/l^2 = P/EI$$

Hence, the Euler load $P_e = \pi^2 EI/l^2$

Case 2: Column/Strut-One End Fixed; Other Free



Now consider any section X, at a distance x from A.

Let P Critical load on the column

y Deflection of the column at X

Moment due to the critical load P,

$$M = P(a - y) = -P(y - a)$$

Differential Equation

$$\frac{d^2y}{dx^2} + \alpha^2 y = a$$

Solution

$$y = A\sin\alpha x + B\cos\alpha x + a$$

$$\frac{dy}{dx} = A\alpha\cos\alpha x - B\alpha\sin\alpha x$$

Boundary Condition

$$x = 0; \quad y = 0; \quad B + a = 0 \Rightarrow B = -a$$

$$x = 0; \quad \frac{dy}{dx} = 0; \quad \therefore A\alpha = 0$$

$$\alpha \neq 0; \quad A = 0$$

Therefore

$$y = -a\cos\alpha x + a$$

Boundary Condition

$$x = l; \quad y = a$$

$$\therefore a = a - a\cos\alpha l$$

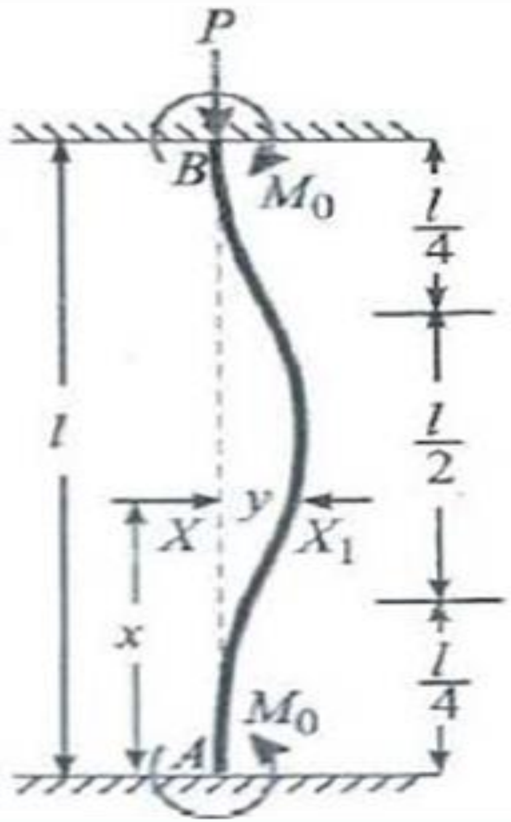
$$\Rightarrow 1 = 1 - \cos\alpha l$$

$$\therefore \alpha l = \pi/2$$

Hence, the Euler load

$$P_e = \pi^2 EI \Big/ 4l^2$$

Case 3: Column/Strut – Both Ends Fixed



Now consider any section *X*, at a distance *x* from *A*.

Let *P* Critical load on the column

y Deflection of the column at *X*

Moment due to the critical load *P*

$$M = -Py + M_0$$

Differential Equation

$$\frac{d^2y}{dx^2} + \alpha^2 y = M_0$$

Solution

$$y = A\sin\alpha x + B\cos\alpha x + M_0 / EI\alpha^2$$

Boundary Condition

$$x = 0; \quad y = 0;$$

$$\therefore B = -M_0 / EI\alpha^2 = -M_0 / P$$

$$x = 0; \quad \frac{dy}{dx} = 0; \quad \therefore A\alpha = 0$$

$$\alpha \neq 0; \quad A = 0$$

Therefore

$$y = \left(\frac{M_0}{P}\right)(1 - \cos\alpha x)$$

Boundary Condition

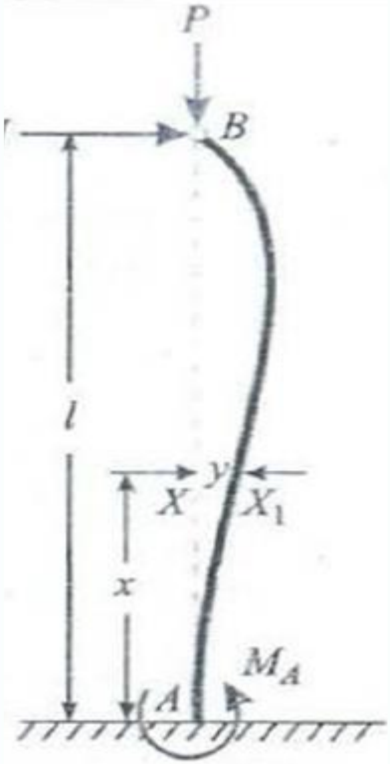
$$x = l; \quad y = 0; \quad \therefore \cos\alpha l = 1$$

$$\alpha l = 2\pi$$

Hence, the Euler load

$$P_e = 4\pi^2 EI / l^2$$

Case 4: Column/Strut – One End Fixed; Other Hinged



Now consider any section *X*, at a distance *x* from *A*.

Let P Critical load on the column

y Deflection of the column at *X*

Moment due to the critical load *P*,

$$M = -Py + Hx$$

Differential Equation $$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{Hx}{EI}$$

Solution

$$\therefore y = A\sin\alpha x + B\cos\alpha x + (H/P)x$$

Boundary Condition

$$x = 0; \quad y = 0; \quad \therefore B = 0$$

$$x = l; \quad y = 0; \quad \frac{dy}{dx} = 0 \therefore \tan \alpha l = \alpha l = 4.493$$

$$\Rightarrow \alpha = 4.493/l$$

Therefore

$$\alpha^2 = \frac{P}{EI} \Rightarrow P = \alpha^2 EI = \frac{2.047\pi^2 EI}{L^2}$$

Hence, the Euler load

$$P_e = 2.07\pi^2 EI / l^2$$

Euler's Formula and Equivalent length of a Column

$$P_E = \pi^2 EI / L_e^2$$

- General equation for Euler 's formula
- where L_e is the equivalent or effective length of column.

Table 1: The equivalent lengths (L) for the given end conditions

S.No.	End conditions	Relation between equivalent length (L_e) and actual length (l)	Crippling load (P)
1.	Both ends hinged	$L_e = l$	$P = \frac{\pi^2 EI}{(l)^2} = \frac{\pi^2 EI}{l^2}$
2.	One end fixed and the other free	$L_e = 2l$	$P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$
3.	Both ends fixed	$L_e = \frac{l}{2}$	$P = \frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2} = \frac{4\pi^2 EI}{l^2}$
4.	One end fixed and the other hinged	$L_e = \frac{l}{\sqrt{2}}$	$P = \frac{\pi^2 EI}{\left(\frac{l}{\sqrt{2}}\right)^2} = \frac{2\pi^2 EI}{l^2}$

Slenderness Ratio

Euler's formula for the crippling load $$P_E = \pi^2 EI \big/ L_e^2 \ldots(i)$$

Let $I = Ak^2$

$$P_E = \pi^2 E\left(Ak^2\right) \big/ L_e^2 = \pi^2 EA \big/ \left(L_e / k\right)^2$$

Slenderness Ratio $$L_e \big/ k$$

Limitation of Euler's Formula

Euler·s formula for the crippling load $$P_E = \pi^2 EA \big/ (L_e/k)^2$$

Euler's crippling stress $$\sigma_E = \frac{P}{A} = \pi^2 E \big/ (L_e/k)^2$$

- Now let us consider a mild steel column having a crushing stress of 320 MPa or 320 N/mm^2 and Young's modulus of 200 GPa or 200 x l 0^3 N/mm^2.

$$320 = \pi^2 E \big/ (L_e/k)^2 = \pi^2 (200x10^3) \big/ (L_e/k)^2 \Rightarrow \frac{L_e}{k} = 78.5 \approx 80$$

- Thus, if the slenderness ratio is less than 80 the Euler's formula is not valid for a mild steel column

- *Example 2-1: A straight bar of alloy, 1 m long and 12·5 mm by 4·8 mm in section, is mounted in a strut-testing machine and loaded axially until it buckles. Assuming the Euler formula to apply, estimate the maximum central deflection before the material attains its yield point of 280 N/mm². E = 72,000 N/mm².*

Solution

There will be no deflection at all until the Euler load is reached, i.e.

$$load = \left(\frac{\pi}{l}\right)^2 EI = \left(\frac{\pi}{1000}\right)(72000)\left[\frac{(12.5)(4.8^3)}{12}\right] = 82N$$

Maximum bending moment

$$P\delta = 82\delta$$

Maximum stress is the sum of direct and bending stresses at the centre

Maximum bending stress

$$\sigma_m = \frac{My}{I_x}$$

$$280 = \frac{82}{(12.5)(4.8)} + \frac{82\delta(6)}{(12.5)(4.8^2)} = 1.37 + 1.71\delta$$

$$\Rightarrow \delta = 163mm$$

- *Example 2-2: A uniform bar of cross-sectional area A and flexural stiffness EI is heated so that its temperature varies linearly from ½t at one end to t at the other end. One end is pin-jointed to a rigid foundation; the other end is pin-jointed so that it can slide in the direction of the length of the bar, the thermal expansion of which is resisted by a compression spring of stiffness k. If there is no load in the spring when t =0, obtain an expression for the stress in the bar when it is heated and show that it buckles in flexure when*

$$t = \frac{4\pi^2 I}{3\alpha l^2 A}\left(1 + \frac{EA}{kl}\right)$$

where a=coefficient of linear thermal expansion.

Solution

The average temperature along the bar is *¾t,*

The thermal expansion of the bar i $\frac{3}{4}\alpha l t$

The compression produced in the bar Pl/AE

The compression of the spring is *P/k*

Net expansion of bar
=compression of spring

$$\frac{3}{4}\alpha lt - \frac{Pl}{AE} = \frac{P}{k}$$

Hence

$$P = \frac{\tfrac{3}{4}\alpha lt}{l/AE + 1/k}$$

But $P = \pi^2 EI \big/ l^2$

Therefore $$\frac{\pi^2 EI}{l^2} = \frac{\tfrac{3}{4}\alpha lt}{l/AE + 1/k}$$

$$\Rightarrow t = \frac{4\pi^2 I}{3\alpha l^2 A}\left(1 + \frac{AE}{kl}\right)$$

Stress in bar

$$\sigma = \frac{P}{A} = \frac{\tfrac{3}{4}\alpha lt}{l/E + A/k}$$

- *Example 2-3: A steel rod 5 m long and of 40 mm diameter is used as a column, with one end fixed and the other free. Determine the crippling load by Euler's formula. Take E as 200 GPa*

Solution

Given : Length (*l*) = 5 m = 5 x 10^3 mm; Diameter of column (*d*) = 40 mm and modulus of elasticity (*E*) = 200 GPa = 200 x 10^3 N/mm^2

Moment of inertia of the column section

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64}(40)^4 = 40{,}000\pi \ mm^4$$

Since the column is fixed at one end and free at the other, therefore equivalent length of the column

$$L_e = 2l = 2(5000) = 10{,}000\ mm$$

Therefore, Euler's crippling load

$$P_E = \pi^2 EI \big/ L_e^2 = \frac{\pi^2 (200x10^3)(40000\pi)}{(10000)^2} = 2480\ N$$

- *Example 2-4: A hollow alloy tube 4 m long with external and internal diameters of 40 mm and 25 mm respectively was found to extend 4.8 mm under a tensile load of 60 kN. Find the buckling load for the tube with both ends pinned. Also find the safe load on the tube, taking a factor of safety as 5.*

Solution

Given: Length *l* = 4 m ; External diameter of column (D) = 40 mm; Internal diameter of column *(d)* = 25 mm ; Deflection (δl) = 4.8 mm ; Tensile load = 60 kN = 60 x 10^3 N and factor of safety = 5.

Area of the tube

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(40^2 - 25^2\right) = 765.8\ mm^2$$

Moment of inertia of the tube

$$I = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{\pi}{4}\left(40^4 - 25^4\right) = 106{,}500\ mm^4$$

Strain in the alloy tube

$$\varepsilon = \frac{\delta l}{l} = \frac{4.8}{4000} = 0.0012$$

The modulus of elasticity for the alloy

$$E = \frac{P}{A\varepsilon} = \frac{60000}{(765.8)(0.0012)} = 65{,}290\ N/mm^2$$

$$L_e = l = 4{,}000\ mm$$

Since the column is pinned at its both ends, therefore equivalent length of the column

Euler's buckling load

$$P_E = \pi^2 EI \big/ L_e^2 = \frac{\pi^2 (65290)(106500)}{(4000)^2} = 4290\ N$$

Safe load for the tube

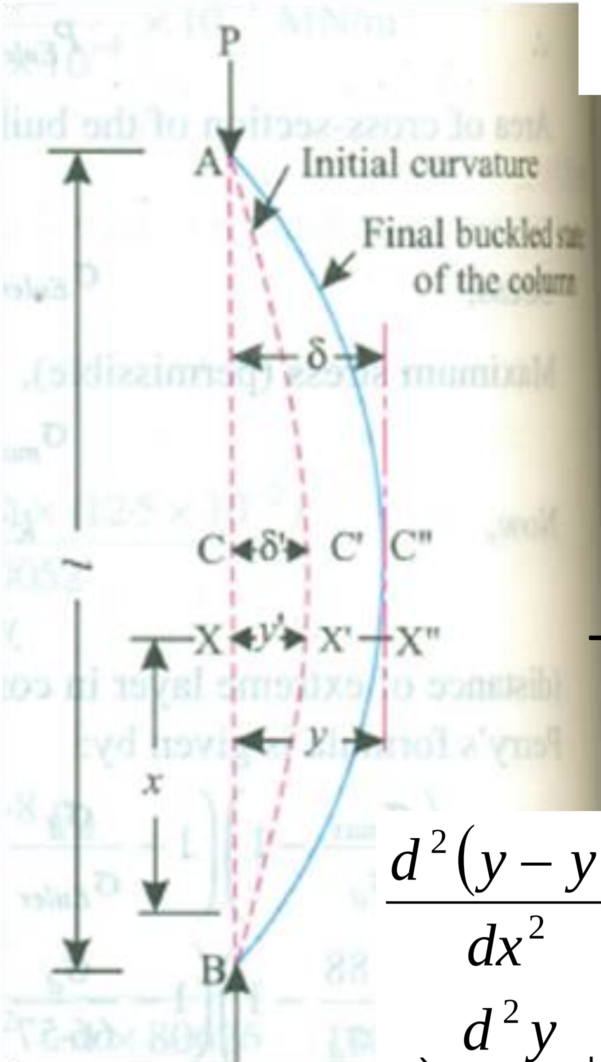
$$Safe\ load = \frac{Buckling\ load}{Factor\ of\ safety} = \frac{4290\ N}{5} = 858\ N$$

EMPIRICAL FORMULAE FOR COLUMNS

In this session, we shall study the other methods used to derive the critical load of a strut:

- Case 1: Perry-Robertson formula
- Case 2: Rankine's formula
- Case 3: Johnson's formula
- Case 4: Indian standard code and
- Case 5: Long columns subjected to eccentric loading

Case 1: Perry–Robertson's Formula



Initial deflection at a distance x from the end B

$$y' = \delta' . \sin \frac{\pi x}{l}$$

$$\frac{dy'}{dx} = \frac{\pi \delta'}{l} . \cos \frac{\pi x}{l}$$

$$\frac{d^2 y'}{dx^2} = -\frac{\pi^2 \delta'}{l^2} . \sin \frac{\pi x}{l}$$

The deflection at x changes from *y/* to *y*

$$\therefore \; EI \frac{d^2 (y - y')}{dx^2} = -Py$$

$$\frac{d^2 (y - y')}{dx^2} = -\frac{Py}{EI}$$

$$\Rightarrow \frac{d^2 y}{dx^2} + \frac{Py}{EI} = \frac{d^2 y'}{dx^2} = -\frac{\pi^2}{l^2} \delta' . \sin \frac{\pi x}{l}$$

Solution

$$y = C\delta' \sin\frac{\pi x}{l}$$

$$\frac{dy}{dx} = \frac{\pi}{l} C\delta' \cos\frac{\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{\pi}{l}\right)^2 C\delta' \sin\frac{\pi x}{l}$$

Inserting the values of y and dy/dx

$$C == \frac{P_E}{P_E - P}$$

Hence the equation to the deflected form of the column

$$y = \frac{P_E}{P_E - P}\delta'.\sin\frac{\pi x}{l}$$

The deflection will be maximum at the mid-point

$$y = \delta \Rightarrow \delta = \frac{P_E}{P_E - P}\delta'$$

Maximum bending moment

$$M = P\delta = \frac{P.P_E}{P_E - P}\delta'$$

Maximum compressive stress

$$\left[\frac{\sigma_{\max}}{\sigma_d} - 1\right]\left[1 - \frac{\sigma_d}{\sigma_E}\right] = \frac{\delta' y_c}{k^2}$$

- *Example 2-7: A steel strut has an outside diameter of 180mm and inside diameter of 120mm and is 6m long. It is hinged at both ends and is initially bent. Assuming the centre line of the strut as sinusoidal with maximum deviation of 9mm, determine the maximum stress developed due to an axial load of 150kN.take E=208 GPa*

Solution

Given: Outside diameter of the strut, (D) = 180 mm; Inside diameter of the strut, (d) = 120 mm; Length of the strut, (*l*) = 6 m =6 x 10^3 mm; Maximum deviation at the centre, ($\delta^/$) = 9 mm; Young's modulus, (E) = 208 GPa = 208 x 10^3 N/mm^2; Axial load, (P) = 150 kN = 150 x 10^3 N

Maximum stress developed

Area of cross-section

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(180^2 - 120^2\right) = 14.14x10^3\ mm^2$$

Moment of inertia $I = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{\pi}{64}\left(180^4 - 120^4\right) = 41.35x10^6\ mm^4$

Radius of gyration

$$k^2 = \frac{I}{A} = \frac{41.35x10^6\ mm^4}{14.14x10^3\ mm^2} = 2.924x10^3\ mm^2$$

Euler load, for pinned at both ends, $L_e = l = 6x10^3$ mm

$$P_E = \left(\frac{\pi}{L_e}\right)^2 EI = \left(\frac{\pi}{6x10^3}\right)^2 \left(208\,x10^3\right)\left(41.35x10^6\right) = 2.36x10^6\ N$$

Euler Stress

$$\sigma_E = \frac{P_E}{A} = \frac{2.36x10^6}{14.14x10^3} = 166.75\ N/mm^2$$

Direct stress

$$\sigma_d = \frac{P}{A} = \frac{150\,x10^3}{14.14x10^3} = 10.6\ N/mm^2$$

Distance of the extreme layer in compression from the neutral axis

$$y_c = \frac{D}{2} = \frac{180}{2} = 90\,mm$$

We know that

$$\left[\frac{\sigma_{\max}}{\sigma_d} - 1\right]\left[1 - \frac{\sigma_d}{\sigma_E}\right] = \frac{\delta' y_c}{k^2} \Rightarrow \left[\frac{\sigma_{\max}}{10.6} - 1\right]\left[1 - \frac{10.6}{166.75}\right] = \frac{9x90}{2.924x10^3}$$

Therefore

$$\left[\frac{\sigma_{\max}}{10.6} - 1\right] = \frac{0.277}{0.936} = 0.296$$

$$\Rightarrow \sigma_{\max} = 10.6x(1 + 0.296) = 13.74\ N/mm^2$$

Case 2: Rankine Formula

For very long struts the failure will occur through buckling as in Euler load

$$P_e = \frac{\pi^2 EI}{L_e^2}$$

For a very short columns failure is by crushing (or yielding)

$$P_c = A.\sigma_c = area\ x\ crushing\ stress$$

Rankine load for the failure of any length of strut

$$\frac{1}{P_R} = \frac{1}{P_c} + \frac{1}{P_e}$$

For a very short column, P_e is large

$$\frac{1}{P_e} \approx small$$

$$\frac{1}{P_R} \cong \frac{1}{P_c} \quad \therefore P_R \cong P_c$$

For a very long column P_e is small

$$\frac{1}{P_e} \approx large$$

$$\frac{1}{P_R} \cong \frac{1}{P_e} \quad \therefore P_R \cong P_e$$

- Rewriting
$$\frac{1}{P_R} = \frac{1}{P_c} + \frac{1}{P_e} = \frac{P_e + P_c}{P_c P_e}$$
- Thus
$$P_R = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{A\sigma_c}{1 + a\left(L_e / k\right)^2}$$

Where

- ❑ P_c Crushing load of the column material
- ❑ σ_c Crushing stress of the column material
- ❑ A Cross-sectional area of the column
- ❑ a Rankine's constant
- ❑ L_e Equivalent length of the column, and
- ❑ K Least radius of gyration

- The following table gives the values of crushing stress (σ_c) and Rankine's constant (a) for various materials:

S.No.	*Material*	σ_C *in MPa*	$a = \frac{\sigma_C}{\pi^2 E}$
1.	Mild Steel	320	$\frac{1}{7500}$
2.	Cast Iron	550	$\frac{1}{1600}$
3.	Wrought Iron	250	$\frac{1}{9000}$
4.	Timber	40	$\frac{1}{750}$

- **Note : The above values arc only for a column with both ends hinged. For other end conditions, the equivalent length should be used.**

- *Example 2-8. Find the Euler's crippling load for a hollow cylindrical steel column of 38 mm external diameter and 2.5 mm thick. Take length of the column as 2.3 m and hinged at its both ends. Take E = 205 GPa. Also determine crippling load by Rankine's formula using constants as 335 MPa and 1/7500*

Solution

Give: External diameter (*D*) = 38 mm; Thickness = 2.5 mm or inner diameter (*d*) = 38- (2 x 2.5) = 33 mm ; Length of the column (*1*) = 2.3 m = 2.3 x 10^3 mm; Yield stress (σ_c) = 335 MPa = 335 N/mm^2 and Rankine's constant (a) =1/7500

Since the column is hinged at its both ends, therefore effective length of the column, Le = l = 2.3 x 103 mm

Moment of inertia of the column section

$$I_{XX} = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{\pi}{64}\left[(38)^4 - (33)^4\right] = 14.05x10^3 \pi \; mm^4$$

Area of the column section

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left[(38)^2 - (33)^2\right] = 88.75\pi \ mm^2$$

The least radius of gyration

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{14.05x10^3 \pi}{88.75\pi}} = 12.6 \ mm$$

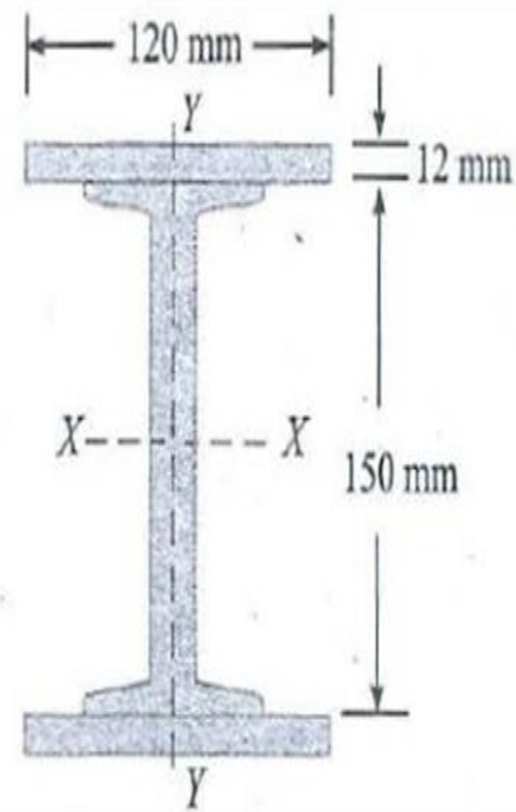
Euler's crippling load

$$P_E = \pi^2 EI \big/ L_e^2 = \frac{\pi^2 \left(205x10^3\right)\left(14.05x10^3 \pi\right)}{(2300)^2} = 16{,}880 \ N$$

Rankine's crippling load

$$P_R == \frac{A\sigma_c}{1 + a\left(L_e / k\right)^2} = \frac{(88.75\pi)(335)}{1 + \left(\frac{1}{7500}\right)\left(\frac{2300}{12.6}\right)^2} = 17{,}160 \ N$$

- *Example 2-9: Fig. 27 shows a built-up column consisting of 150 mm x 100 mm R. S. J. with 120 mm x 12 mm plate riveted to each flange. Calculate the safe load, the column can carry, if it is 4 m long having one end fixed and the other hinged with a factor of safety 3.5. Take the properties of the joist as Area= 2167 mm² I_{XX} = 8.391 x 10^6 mm⁴, I_{YY} = 0.948 x 10^6 mm⁴. Assume the yield stress as 315 MPa and Rankine's constant (a) = 1/7500*



Solution

Given: Length of the column (l)= 4 m = 4 x 10^3 mm; Factor of safety = 3.5; Yield stress (σ_c) = 315.MPa = 315 N/mm²; Area of joist= 2167 mm²; Moment of inertia, about *X-X* axis (I_{XX}) = 8.391 x 10^6 mm⁴; Moment of inertia about *Y-Y* axis (I_{YY}) = 0.948 x 10^6 mm⁴ and Rankine's constant *(a)* = 1/7500

Area of the column section, A = 2167 + (2 x 120 x 12) = 5047 mm^2

Moment of inertia of the column section

$$I_{XX} = \left(83.91x10^6\right) + 2\left[\frac{(120)(12)^3}{12} - (120)(12)(81)^2\right] = 27.32x10^6 \ mm^4$$

$$I_{YY} = \left(0.948x10^6\right) + 2\left[\frac{(12)(120)^3}{12}\right] = 4.404x10^6 \ mm^4$$

The least of two, I_{YY} = 4.404 x 10^6 mm^4

The least radius of gyration

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{4.404x10^6}{5047}} = 29.5\, mm$$

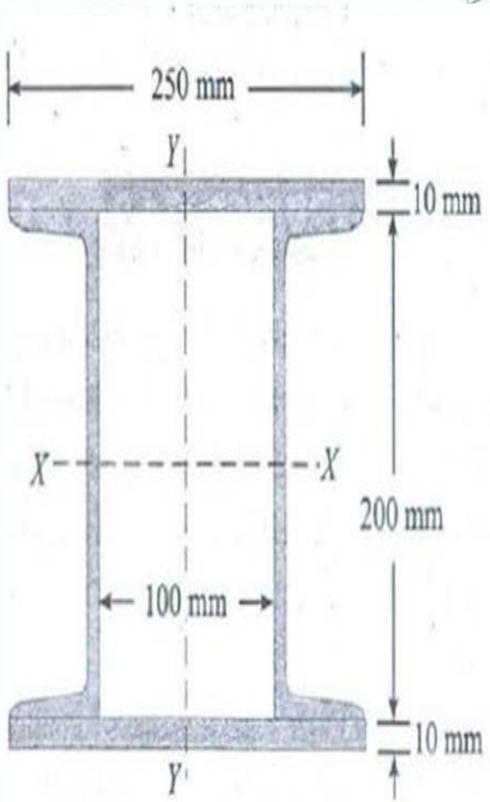
For fixed at one end and hinged at the other,

$$L_e = \frac{l}{\sqrt{2}} = \frac{4000}{\sqrt{2}} = 2.83x10^3 \ mm$$

Rankine's crippling load

$$P_R == \frac{A\sigma_c}{1 + a(L_e/k)^2} = \frac{(5047)(315)}{1 + \left(\frac{1}{7500}\right)\left(\frac{2830}{29.5}\right)^2} = 714\, kN$$

- *Example 2-10: A column is made up of two channels. ISJC 200 and two 250 mm x 10 mm flange plates as shown in Fig.28. Determine by Rankine's formula the safe load, the column of 6 m length, with both ends fixed, can carry with a factor of safety 4. The properties of one channel are Area = I 777 mm², I_{XX} = 11.612 X 106 mm⁴ and I_{YY} = 0.842 x 10⁶ mm⁴. Distance of centroid from back to web=. 19.7 mm. Take (σ_c) = 320 MPa and (a)= 1/7500*



Solution

Given : Length of the column (l)= 6 m = 6 X 10^3 mm ; Factor of safety= 4 ; Area of channel = 1777 mm²; Moment of inertia about *X-X* axis (I_{XX}) = 11.612 x 10^6 mm⁴; Moment of inertia about *Y-Y* axis (I_{YY}) = 0.842 x 10^6 mm⁴; Distance of centroid from the back of web= 19.7 mm; Crushing stress (σ_c) = 320 MPa = 320 N/mm² and Rankine's constant (a)= 1/7500

Area of the column section, A = 2 [1777 + (250 x 10)] = 8554 mm^2

Moment of inertia of the column section,

$$I_{XX} = \left(2x11.612x10^6\right) + 2\left[\frac{(250)(10)^3}{12} - (250)(10)(105)^2\right] = 78.391x10^6\ mm^4$$

$$I_{YY} = 2\left[\frac{(10)(250)^3}{12} + \left(0.846x10^6\right) + 1777x(50+19.7)^2\right] = 44.992x10^6\ mm^4$$

The least of two, I_{YY} = 44.992 x 10^6 mm^4

For fixed at both ends,

$$L_e = \frac{l}{2} = \frac{6000}{2} = 3x10^3\ mm$$

The least radius of gyration

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{44.992x10^6}{8554}} = 72.5\,mm$$

Rankine's crippling load

$$P_R == \frac{A\sigma_c}{1+a(L_e/k)^2} = \frac{(8554)(320)}{1+\left(\frac{1}{7500}\right)\left(\frac{3000}{72.5}\right)^2} = 2228.5\,kN$$

Safe load on the column

$$Safe\,load = \frac{Crippling\,load}{Factor\ of\ \ safety} = \frac{2228.5}{4} = 557.1\,kN$$

Case 3: Johnson's Formula for Columns

- Prof. Johnson, after a series of experiments and observations, proposed the following two formulae for columns:
  - Straight line formula
  - Parabolic formula

Johnson's Straight Line Formula for Columns

$$P = A\left[\sigma_C - n\left(\frac{L_e}{k}\right)\right]$$

where

P	Safe load on the column
A	Area of the column cross-section
σ_C	Allowable compressive stress in the column material
n	A constant, whose value depends upon the column material
$\frac{L_e}{k}$	Slenderness ratio

S. No.	Material	σ_C in MPa	n
1.	Mild Steel	320	0.0053
2.	Wrought Iron	250	0.0053
3.	Cast Iron	550	0.008

Johnson's Parabolic Formula for Columns

$$P = A\left[\sigma_C - r\left(\frac{L_e}{k}\right)^2\right]$$

- where
- P Safe load on the column
- A Area of the column cross-section
- σ_C Allowable compressive stress in the column material
- r A constant, whose value depends upon the column material and
- $\frac{L_e}{k}$ Slenderness ratio

S. No.	Material	σ_C in MPa	r
1.	Mild Steel	320	0.000057
2.	Wrought Iron	250	0.000039
3.	Cast Iron	550	0.000016

Case 4: Indian Standard Code for Columns

- The Bureau of Indian Standards (I. S. I.) has also given a code for the safe stress in I. S. 226-19621 which states

$$\sigma_C = \sigma'_C = {\sigma_y}/{m}\left[1+0.20\sec\left(\frac{L_e}{k}\sqrt{\frac{mp'_c}{4E}}\right)\right]^{-1} \quad \text{for } \frac{L_e}{k} = 0 \text{ to } 160$$

$$\sigma_C = \sigma'_C\left(1.2-\frac{L_e}{800k}\right) \quad \text{for } \frac{L_e}{k} = 160 \text{ and above}$$

where

- σ_c Allowable axial compressive stress
- σ'_c A value obtained from the above secant formula
- σ_y The guaranteed minimum yield stress

- m Factor of safety taken as 1.68
- L_e/k Slenderness ratio with equivalent column length
- E Modulus of elasticity equal to 200 GPa

- The I. S. I. has also given a table in I. S. 800 -1962 which gives the values of σ_c for mild steel for slenderness ratio from 0 to 350.

- The value of σ_y *i.e.,* the guaranteed minimum yield stress for mild steel is taken as 260 MPa. This table is given below:

$\frac{L}{k}$	σ_C *in MPa*	$\frac{L_e}{k}$	σ_C *in MPa*	$\frac{L_e}{k}$	σ_C *in MPa*
0	125	90	92.8	180	33.6
10	124.6	100	84.0	190	30.0
20	123.9	110	75.3	200	27.0
30	122.4	120	67.1	210	24.3
40	120.3	130	59.7	220	21.9
50	117.2	140	53.1	230	19.9
60	113.0	150	47.4	240	18.1
70	102.5	160	42.3	300	10.9
80	100.7	170	37.7	350	3.6

- *Example 2-11: A hollow cylindrical steel tube of 38 mm external diameter and 2.5 mm thick is used as a column of 2.3 m. long with both ends hinged. Determine the safe load by I. S. code.*

Solution

Given: External diameter (*D*) = 38 mm; Thickness = 2.5 mm and length of column (*l*) = 3 m = 3 x 10^3 mm.

Area of the column section

$$A = \frac{\pi}{4}\left[D^2 - d^2\right] = \frac{\pi}{4}\left[(38)^2 - (33)^2\right] = 278.8\ mm^2$$

Moment of inertia of column section

$$I = \frac{\pi}{64}\left[D^4 - d^4\right] = \frac{\pi}{64}\left[(38)^4 - (33)^4\right] = 44.14x10^3\ mm^4$$

For hinged at both ends,

$$L_e = l = 2300\,mm$$

The least radius of gyration

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{44.14x10^3}{278.8}} = 12.6\ mm$$

Slenderness ratio

$$\frac{L_e}{k} = \frac{2300}{12.6} = 182.5$$

- From table, we find that allowable stress for slenderness ratio of 182.5 is 32.7 MPa or 32.7 N/mm^2.

- Therefore safe load on the column

$$P = \sigma_C . A = (32.7)(278.8) = 9117\ N$$

$$\sigma_C = \sigma'_C \left(1.2 - \frac{L_e}{800k} \right)$$

Case 5: Long Columns Subjected to Eccentric Loading

- In the previous articles, we have discussed the effect of loading on long columns.

- We have always referred the cases when the load acts axially on the column (*i.e.*, the line of action of the load coincides with the axis of the column).

- But in actual practice it is not always possible to have an axial load, on the column, and eccentric loading takes place.

- Here we shall discuss the effect of eccentric loading on the Rankine's and Euler's formulae for long columns.

Case 5_1: Rankine's formula

- Consider a long column subjected to an eccentric load.
- Let *P* Load on the column
- *A* Area of cross-sect ion
- *E* Eccentricity of the load
- Z Modulus of section,
- *y* Distance of the extreme fibre (on compressive side) from the axis of the column,
- *k* Least radius of gyration

- The maximum intensity of compressive stress

$$\sigma_{\max} = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + \frac{P.e}{Z}$$

- But $Z = \dfrac{I}{y_e} = \dfrac{Ak^2}{y_e}$

Therefore,

$$\sigma_{\max} = \frac{P}{A}\left(1+\frac{e.y_e}{k^2}\right)$$

The safe crushing load for the given column

$$\sigma_c = \frac{P}{A}\left(1+\frac{e.y_e}{k^2}\right)$$

$$\Rightarrow P = \frac{A\sigma_c}{\left(1+\frac{e.y_e}{k^2}\right)}$$

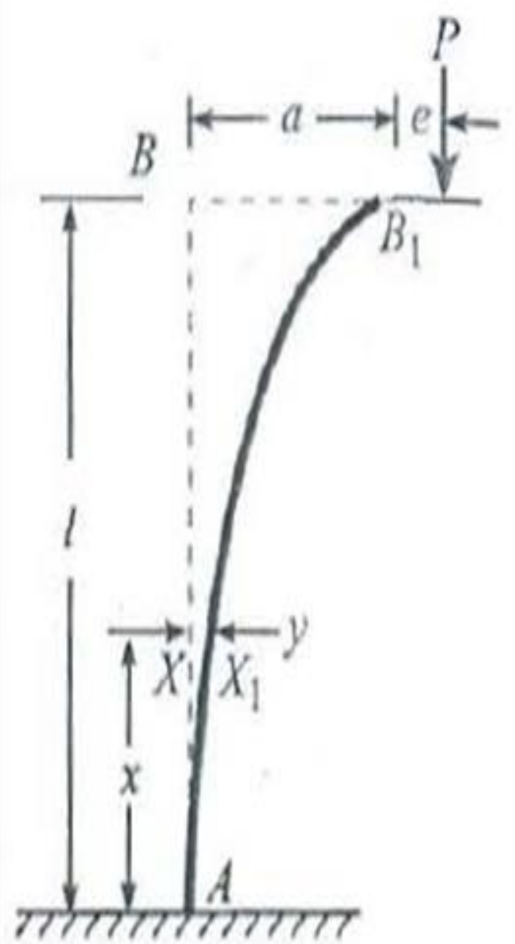
Rankine's formula for long columns and axial load is given by the relation

$$P = \frac{\sigma_c .A}{1+a\left(\frac{L_e}{k^2}\right)^2}$$

It is thus obvious that if the effect of buckling is also to be taken into account, the safe axial load with eccentricity

$$P = \frac{A\sigma_c}{\left(1+\frac{e.y_e}{k^2}\right)\left[1+a\left(\frac{L_e}{k}\right)^2\right]}$$

Case 5_2: Euler's formula



- Now consider any section *X*, at a distance x from *A*.
- Let *P* Critical load on the column
- *e* Eccentricity of the load
- *y* Deflection of the column at *X*.

Moment due to load

$$M == P(a+e-y) = P(a+e) - Py$$

The differential equation

$$EI\, {d^2y}\big/{dx^2} = P(a+ey) - Py$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{P}{EI}(a+e)x$$

The general solution

$$y = A\sin\alpha x + B\cos\alpha x + (a+e)$$

$$\frac{dy}{dx} = \alpha A\cos\alpha x - \alpha B\sin\alpha x$$

Boundary condition

x = 0 and *y* = 0, then *B =-(a+* e)

x = 0 and dy/dx = 0 then $\alpha A = 0$

Thus A = 0

Hence

$$y == (a+e)[1-\cos\alpha x]$$

Boundary condition

x = l and *y =a*

Therefore

$$a = (a+e)[1-\cos\alpha l]$$

$$= a + e - (a+e)\cos\alpha l$$

$$\Rightarrow e = (a+e)\cos\alpha l$$

$$\therefore (a+e) = e\sec\alpha l$$

Maximum bending moment

$$M_{\max} = P.e.\sec\alpha l$$

Maximum compressive stress

$$\sigma = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + \frac{P.e.\sec\alpha l}{Z}$$

For one end fixed and other end free

$$L_e = 2l \Rightarrow l = \frac{L_e}{2}$$

But $\alpha = \sqrt{\frac{P}{EI}}$

$$l = \frac{L_e}{2}$$

Thus maximum bending moment

$$M_{\max} = P.e.\sec\left(\frac{L^2}{2}\sqrt{\frac{P}{EI}}\right)$$

The maximum compressive stress

$$\sigma == \frac{P}{A} + \frac{1}{Z}.P.e.\sec\left(\frac{L_e}{2}\sqrt{\frac{P}{EI}}\right)$$

- *Example 2-12: An alloy hollow circular column of 200 mm external and 160 mm internal diameter is 5 m long and fixed at both of its ends. It is subjected to a load of 120 kN at an eccentricity of 20 mm from the geometrical axis. Determine the maximum stress induced in the column section. Take E as 120 GPa.*

Solution

Given: External diameter *(D)* = 200 mm; Internal diameter (*d*) = 160 mm; Length *(l)* = 5 m = 5 x 10^3 mm; Load *(P)* = 120 kN = 120 x 10^3 N; Eccentricity *(e)* = 20 mm and modulus of elasticity (*E*) = 120 GPa = 120 x 10^3 N/mm^2

Area of the column section

$$A = \frac{\pi}{4}\left[D^2 - d^2\right] = \frac{\pi}{4}\left[(200)^2 - (160)^2\right] = 11.31x10^3\ mm^2$$

Moment of inertia of column section

$$I = \frac{\pi}{64}\left[D^4 - d^4\right] = \frac{\pi}{64}\left[(200)^4 - (160)^4\right] = 46.37x10^6\ mm^4$$

Modulus of section

$$Z = \frac{I}{D/2} = \frac{46.37x10^6}{200/2} = 463.7x10^3 \ mm^3$$

For both ends fixed $L_e = \frac{l}{2} = \frac{5000}{2} = 2500\,mm$

Thus

$$\left(\frac{L_e}{2}\sqrt{\frac{P}{EI}}\right) = \frac{2500}{2}\sqrt{\frac{120x10^3}{\left(120x10^3\right)\left(46.7x10^6\right)}} = 0.1836\,rad = 10.52^o$$

The maximum compressive stress

$$\sigma = \frac{P}{A} + \frac{1}{Z}.P.e.\sec\left(\frac{L_e}{2}\sqrt{\frac{P}{EI}}\right) = \frac{120x10^3}{11.31x10^3} + \frac{\left(120x10^3\right)\left(20\sec 10.52^0\right)}{463.7x10^3} = 15.87\ N/mm^2$$

Quiz 2
Time 40 minutes

Sample Questions

Problem·1: *Determine the ratio of the buckling strengths of two columns of circular cross-section one hollow and other solid when both are made of the same material, have the same length and cross-sectional area and end-conditions. The internal diameter of the hollow column is half of the external diameter.*

Problem·2: *Compare the crippling loads given by Rankine's and Euler's formulae for tubular strut 2·25 m long having outer and inner diameters of 37·5 mm and 32·5 mm loaded through pin-joint at both ends. Take: Yield stress as 315 MNlm^2, a =1/7500 and E = 200 GPa. If elastic limit for the material is taken as 200 MPa^2, then for what length of the strut Euler formula cease to apply?*

Problem 9: *From the following data, determine the diameter of the piston rod.*
Diameter of the engine cylinder = 0-3 m
Maximum effective steam pressure in the cylinder = 800 kNlm^2
Distance from piston to cross-head centre = 1·5 m.
Factor of safety = 4
Assume σ_c =330 MPa; a = 1/30000 for both ends fixed.