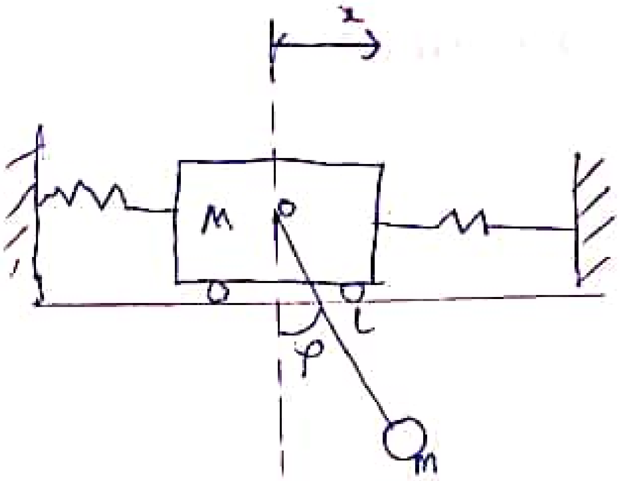
ASSIGNMENT 3

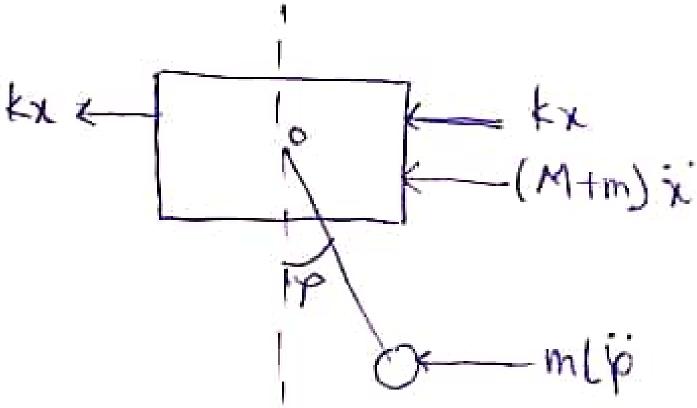
ME 363

INDEX NO: 9383417

Q1



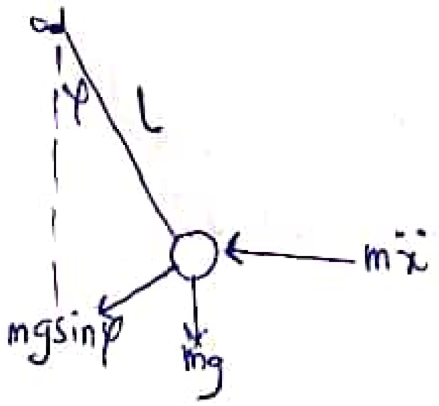
Free body diagrams



Finding Equations of motion for the system

$\Sigma F_x = 0$

$\Rightarrow \quad (M+m)\ddot{x} + mL\ddot{\varphi} + 2kx = 0 \quad \text{—— (1)}$



$$\sum M_o = I_o \ddot{\varphi}$$

$\Rightarrow$ $-mgl\sin\varphi - m\ddot{x}l\cos\varphi = ml^2\ddot{\varphi}$

Dividing through by ml

$\Rightarrow$ $-g\sin\varphi - \ddot{x}\cos\varphi = l\ddot{\varphi}$

$\Rightarrow$ $l\ddot{\varphi} + \ddot{x}\cos\varphi + g\sin\varphi = 0$

For small values of φ $[\sin\varphi = \varphi$ and $\cos\varphi \approx 1]$

$\therefore$ $\ddot{x} + l\ddot{\varphi} + g\varphi = 0$

Equations of motion:

$$(M+m)\ddot{x} + ml\ddot{\varphi} + 2kx = 0$$

$$\ddot{x} + l\ddot{\varphi} + g\varphi = 0$$

In Matrix form

$$\begin{bmatrix} M+m & ml \\ 1 & l \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\varphi} \end{Bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & g \end{bmatrix} \begin{Bmatrix} x \\ \varphi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\ddot{x} = -\omega_n^2 x$ $\quad \ddot{\varphi} = -\omega_n^2 \varphi$

$\Rightarrow$ $$\begin{bmatrix} -(M+m)\omega_n^2 & -ml\omega_n^2 \\ -\omega_n^2 & l\omega_n^2 \end{bmatrix} \begin{Bmatrix} x \\ \varphi \end{Bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & g \end{bmatrix} \begin{Bmatrix} x \\ \varphi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \omega_n^2$

$$\Rightarrow \begin{bmatrix} 2k-(M+m)\lambda & -ml\lambda \\ -\lambda & g-l\lambda \end{bmatrix} \begin{Bmatrix} x \\ \varphi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$M = 100kg$, $m = 10kg$ $k = 2\times10^3 N/m$ $l = 2$

$$\begin{bmatrix} 4000-110\lambda & -20\lambda \\ -\lambda & 9.81-2\lambda \end{bmatrix} \begin{Bmatrix} x \\ \varphi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For natural frequencies

$$\begin{vmatrix} 4000-110\lambda & -20\lambda \\ -\lambda & 9.81-2\lambda \end{vmatrix} = 0$$

$$(4000-110\lambda)(9.81-2\lambda) - 20\lambda^2 = 0$$

$$39240 - 8000\lambda - 1079.1\lambda + 220\lambda^2 - 20\lambda^2 = 0$$

$$200\lambda^2 - 9079.1\lambda + 39240 = 0$$

$$\therefore \quad \lambda_1 = 40.558 \qquad \lambda_2 = 4.838$$

$\therefore$ Natural Frequencies

$$\omega_{n_1} = \sqrt{40.558} = 6.37 \text{ rad/s}$$

$$\omega_{n_2} = \sqrt{4.838} = 2.199 \text{ rad/s}$$

Finding mode shapes:

1st mode

$$\begin{pmatrix} 4000-110(40.56) & -20(40.56) \\ -40.56 & 9.81-2(40.56) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2/L \end{pmatrix}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-461.6 V_1 - 811.2 V_2/L$$

$$-461.6 V_1 = 811.2 V_2/L$$

$$L = 2$$

$$\Rightarrow \frac{V_1}{V_2} = -0.879$$

2nd Mode Shape

$$\begin{pmatrix} 4000-110(4.84) & -20(4.84) \\ -4.84 & 9.81-2(4.84) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2/L \end{pmatrix}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

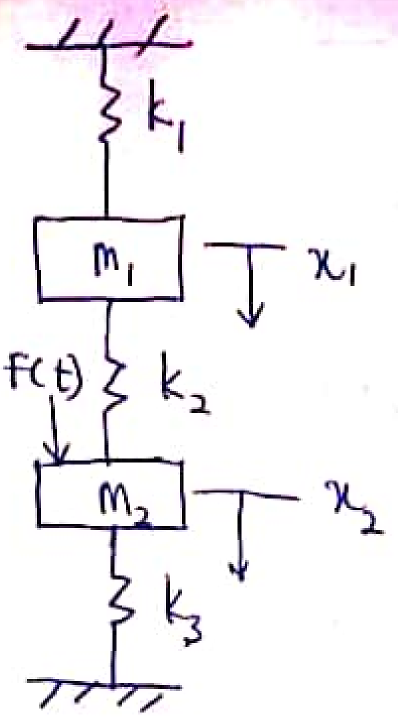
$$3467.6 V_1 - 96.8 V_2/L = 0$$

$$3467.6 V_1 = 96.8 V_2/L$$

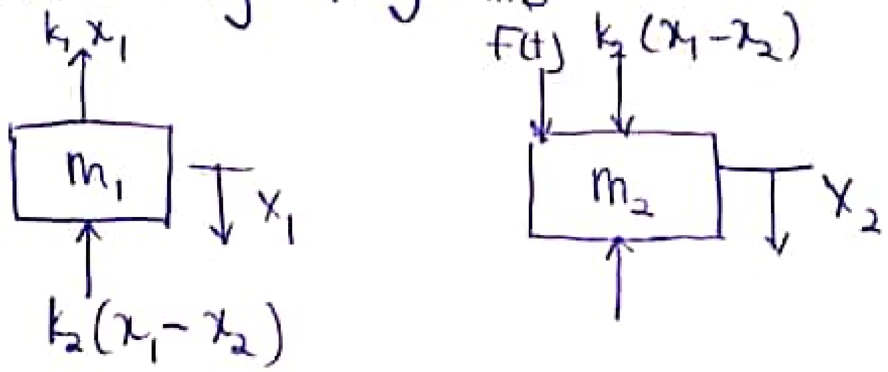
$$L = 2$$

$$\therefore \frac{V_1}{V_2} = 0.014$$

Q2



Free body diagrams



$$\downarrow \sum f_{m_1} = m_1\ddot{x}_1$$

$$-k_1x_1 - k_2x_1 + k_2x_2 = m_1\ddot{x}_1$$

$$\therefore \; m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \quad \text{——} (1)$$

$$\downarrow \sum f_{m_2} = m_2\ddot{x}_2$$

$$F(t) + k_2x_1 - k_2x_2 - k_3x_2 = m_2\ddot{x}_2$$

$$\therefore \; m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = F(t) \quad \text{——} (2)$$

In matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 15\times10^3 \end{Bmatrix} \cos 3t$$

$m_1 = 7kg$, $m_2 = 17kg$ $k_1 = 30kN/m$ $k_2 = 20kN/m$

$k_3 = 16kN/m$

$$\Rightarrow \begin{bmatrix} 7 & 0 \\ 0 & 17 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 50\times10^3 & -20\times10^3 \\ -20\times10^3 & 36\times10^3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 15\times10^3 \end{Bmatrix} \cos 3t$$

Finding Natural Frequencies

$$\begin{bmatrix} 7 & 0 \\ 0 & 17 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 50\times10^3 & -20\times10^3 \\ -20\times10^3 & 36\times10^3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\omega_n^2 = \lambda$

$$\begin{bmatrix} 50\times10^3 - 7\lambda & -20\times10^3 \\ -20\times10^3 & 36\times10^3 - 17\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 50000 - 7\lambda & -20000 \\ -20000 & 36000 - 17\lambda \end{vmatrix} = 0$$

$$18\times10^8 - 1102\times10^3\lambda + 119\lambda^2 - 4\times10^8 = 0$$

$$119\lambda^2 - 1102\times10^3\lambda + 14\times10^8 = 0$$

$\lambda_1 = 7740.643$, $\lambda_2 = 1519.862$

$$\therefore \omega_{n_1} = \sqrt{7740.643} = 87.98 \text{ rad/s}$$

$$\omega_{n_2} = \sqrt{1519.862} = 38.985 \text{ rad/s}$$

For steady State response

$$\begin{bmatrix} 7 & 0 \\ 0 & 17 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 50\times10^3 & -20\times10^3 \\ -20\times10^3 & 36\times10^3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 15\times10^3 \end{Bmatrix} \cos 3t$$

Let $x_i(t) = X_i \cos \omega t$

$$\ddot{x}_i(t) = -\omega^2 X_i \cos \omega t$$

$$\therefore \begin{bmatrix} 7 & 0 \\ 0 & 17 \end{bmatrix} \begin{Bmatrix} -\omega^2 X_1 \\ -\omega^2 X_2 \end{Bmatrix} \cos 3t + \begin{bmatrix} 50\times10^3 & -20\times10^3 \\ -20\times10^3 & 36\times10^3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \cos 3t$$

$$= \begin{Bmatrix} 0 \\ 15\times10^3 \end{Bmatrix} \cos 3t$$

$\omega = 3$

$$\Rightarrow \begin{bmatrix} 50\times10^3 - 9(7) & -20\times10^3 \\ -20\times10^3 & 36\times10^3 - 9(17) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \cos 3t = \begin{Bmatrix} 0 \\ 15\times10^3 \end{Bmatrix} \cos 3t$$

$$\Rightarrow \begin{bmatrix} 49937 & -20000 \\ -20000 & 35847 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 15\times10^3 \end{Bmatrix}$$

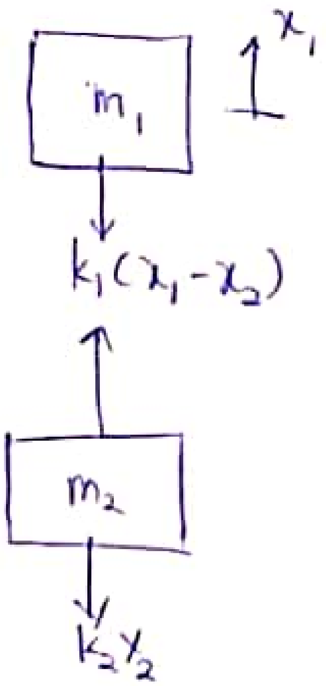
$$\therefore \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 49937 & -20000 \\ -20000 & 35847 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 15 \times 10^3 \end{Bmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.2158 \\ 0.5389 \end{pmatrix}$$

Steady state response

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{pmatrix} 0.2158 \\ 0.5389 \end{pmatrix} \cos 3t$$

Q 3.



$k_1 = 10^3 \, N/m$

$k_2 = 10^4 \, N/m$

$m_1 = 2000 kg$

$m_2 = 50 kg$

Equations of motion

$$\Sigma F_1 = m_1 \ddot{x}_1$$

$$-k_1(x_1 - x_2) = m_1 \ddot{x}_1$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 = 0 \quad \text{——} (1)$$

$$\Sigma F_2 = m_2 \ddot{x}_2$$

$$-k_2 x_2 + k_1(x_1 - x_2) = m_2 \ddot{x}_2$$

$$m_2 \ddot{x}_2 - k_1 x_1 + (k_1 + k_2) x_2 = 0$$

Given $k_1 = 10^3$, $k_2 = 10^4$, $m_1 = 2000$ and $m_2 = 50$

$$\Rightarrow \quad 2000\ddot{x}_1 + 1000 x_1 - 10000 x_2 = 0$$

$$50\ddot{x}_2 - 1000 x_1 + 11000 x_2 = 0$$

In matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1000 & -10000 \\ -1000 & 11000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\ddot{x} = -\omega_n^2 X$$

Let $\omega_n^2 = \lambda$

$$\ddot{x} = -\lambda X$$

$$\Rightarrow \quad \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.5 & -0.5 \\ -20 & 220 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{vmatrix} 0.5-\lambda & -0.5 \\ -20 & 220-\lambda \end{vmatrix} = 0$$

$$(0.5-\lambda)(220-\lambda) - 10 = 0$$

$$110 - 0.5\lambda - 220\lambda + \lambda^2 - 10 = 0$$

$$\lambda^2 - 220.5\lambda + 100 = 0$$

$$\lambda_1 = 0.454 \quad \text{and} \quad \lambda_2 = 220.046$$

For Natural frequencies

Since $\lambda = \omega_n^2$

$$\Rightarrow \quad \omega_{n_1} = \sqrt{\lambda_1} = \sqrt{0.454}$$

$$\therefore \quad \omega_{n_1} = 0.6737 \text{rad/s}$$

$$\omega_{n_2} = \sqrt{\lambda_2} = \sqrt{220.046}$$

$$\omega_{n_2} = 14.83 \text{rad/s}$$

Hence The natural frequencies are 0.674rad/s and 14.8 rad/s

11) Modal Matrix

$$P = \begin{bmatrix} 0.9999 & -0.1044 \\ -0.1044 & 0.9999 \end{bmatrix}$$

Initial conditions are all zero(0)

$$M = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}, \quad K = \begin{bmatrix} 1000 & -1000 \\ -1000 & 11000 \end{bmatrix}$$

$\lambda_1 = 0.4545$, $\omega_{n1} = 0.6741\,rad/s$

$\lambda_2 = 220.05$ $\quad \omega_{n2} = 14.83\,rad/s$

Give $\zeta_1 = 0.01$ and $\zeta_2 = 0.2$

$$\omega_{d_1} = \omega_{n_1}\sqrt{1-\zeta_1^2} = 0.6741\sqrt{1-0.01^2}$$

$$\omega_{d_1} = 0.6741\,rad/s$$

$$\omega_{d_2} = \omega_{n_2}\sqrt{1-\zeta_2^2} = 14.834\sqrt{1-0.2^2}$$

$$\omega_{d_2} = 14.534\,rad/s$$

$$F(t) = \begin{bmatrix} 0 \\ 10\sin 3t \end{bmatrix}$$

modal force vector:

$$P^T M^{-\frac{1}{2}} F(t) = \begin{bmatrix} 0.02036 \\ 1.4141 \end{bmatrix} \sin 3t$$

Scanned with CamScanner

Modal Equations

$$\ddot{r}_1 + 0.01348\dot{r}_1 + 0.454r_1 = 0.02036\sin 3t$$

$$\ddot{r}_2 + 5.9336\dot{r}_2 + 220.046r_2 = 1.414\sin 3t$$

Solutions to modal equations:

$$r_1(t) = -0.1088e^{-0.006741t}\sin(0.6741t + 1.0914\times10^{-4})$$
$$+ 0.002445\sin(3t - 0.004857)$$

$$r_2(t) = -0.07500e^{-2.9668t}\sin(14.534t + 1.3087)$$
$$+ (0.07586\sin(3t + 1.26947))$$

The solutions in physical coordinates:

$$X(t) = M^{-1/2}Pr(t)$$

The response of the body is

$$X_1(t) = -0.002433e^{-0.006741t}\sin(0.6471t - 1.091\times10^{-4})$$
$$+ 5.4665\times10^{-5}\sin(3t - 0.004857)$$
$$+ 2.4153\times10^{-5}e^{-2.9668}\sin(14.534t - 13087)$$
$$- 2.4430\times10^{-5}\sin(3t + 1.2694)$$