Assignment 5 (MATH 215, Q1)

1. Evaluate the triple integral.

   (a) $\iiint_E xy\,dV$, where E is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

Solution. The plane containing the three points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$ has an equation $6x+3y+2z=6$. Thus,

$$\begin{aligned}\iiint_E xydV &= \int_0^1\int_0^{2-2x}\int_0^{(6-6x-3y)/2} xy\,dz\,dy\,dx\\ &= \int_0^1\int_0^{2-2x}\frac{1}{2}(6-6x-3y)xy\,dy\,dx\\ &= \int_0^1 2(-x^4+3x^3-3x^2+x)\,dx = \frac{1}{10}.\end{aligned}$$

   (b) $\iiint_E x\,dV$, where E is bounded by the paraboloid $x=4y^2+4z^2$ and the plane $x=4$.

Solution. We have $Q=\{(y,z): y^2+z^2\le 1\}$ and

$$\begin{aligned}\iiint_E xdV &= \iint_Q \Big[\int_{4y^2+4z^2}^4 x\,dx\Big]\,dA = \iint_Q \big[8-8(y^2+z^2)^2\big]\,dA\\ &= \int_0^{2\pi}\int_0^1 (8-8r^4)r\,dr\,d\theta = 2\pi\int_0^1 (8r-8r^5)\,dr = \frac{16\pi}{3}.\end{aligned}$$

2. (a) Find the volume of the region inside the cylinder $x^2+y^2=9$, lying above the xy-plane, and below the plane $z=y+3$.

Solution. We have $Q=\{(x,y): x^2+y^2\le 9\}$ and

$$\begin{aligned}V &= \iiint_E dV = \iint_Q\Big(\int_0^{y+3} dz\Big)\,dA = \iint_Q (y+3)\,dA\\ &= \int_0^{2\pi}\int_0^3 (r\sin\theta+3)\,r\,dr\,d\theta = \int_0^{2\pi}(9\sin\theta+27/2)\,d\theta = 27\pi.\end{aligned}$$

   (b) Find the volume of the region bounded by the paraboloids $z=x^2+y^2$ and $z=36-3x^2-3y^2$.

Solution. We have $Q=\{(x,y): x^2+y^2\le 9\}$ and

$$\begin{aligned}V &= \iiint_E dV = \iint_Q\int_{x^2+y^2}^{36-3x^2-3y^2} dz\,dA = \iint_Q (36-4x^2-4y^2)\,dA\\ &= \int_0^{2\pi}\int_0^3 (36-4r^2)r\,dr\,d\theta = 2\pi\int_0^3 (36r-4r^3)\,dr = 162\pi.\end{aligned}$$

3. Use cylindrical coordinates in the following problems.
   (a) Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

Solution. In cylindrical coordinates the region E is described by

$$0 \le r \le 3,\ 0 \le \theta \le 2\pi, \text{ and } 0 \le z \le 9 - r^2.$$

Thus,

$$\begin{aligned}\iiint_E \sqrt{x^2+y^2}\,dV &= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r^2(9-r^2)\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^3 (9r^2 - r^4)\,d\theta = \frac{324\pi}{5}.\end{aligned}$$

   (b) Evaluate the integral $\iiint_E x^2\,dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2 = 4x^2+4y^2$.

Solution. In cylindrical coordinates the region E is described by

$$0 \le r \le 1,\ 0 \le \theta \le 2\pi, \text{ and } 0 \le z \le 2r.$$

Thus,

$$\begin{aligned}\iiint_E x^2\,dV &= \int_0^{2\pi}\int_0^1\int_0^{2r} (r\cos\theta)^2\, r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 2r^4\,dr = \frac{2\pi}{5}.\end{aligned}$$

4. Use spherical coordinates in the following problems.
   (a) Evaluate $\iiint_E x\,e^{(x^2+y^2+z^2)^2}\,dV$, where E is the solid that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant

$$\{(x,y,z) : x \ge 0, y \ge 0, z \ge 0\}.$$

Solution. In spherical coordinates the region E is described by

$$1 \le \rho \le 2,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2.$$

Thus,

$$\begin{aligned}\iiint_E x\,e^{(x^2+y^2+z^2)^2}\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\sin\phi\cos\theta)\,e^{\rho^4}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\int_1^2 \rho^3 e^{\rho^4}\,d\rho \\ &= \frac{\pi}{16}(e^{16}-e).\end{aligned}$$

(b) Evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$$

Solution. The integral is equal to $\iiint_E z\sqrt{x^2+y^2+z^2}\,dV$, where

$$E = \{(\rho,\theta,\phi) : 0 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2\}.$$

Therefore, the integral is equal to

$$\begin{aligned}&\int_0^{\pi/2}\int_0^{2\pi}\int_0^3 (\rho\cos\phi)\,\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^3\rho^4\,d\rho = \frac{243\pi}{5}.\end{aligned}$$

5. (a) Find the center of mass of the solid S bounded by the paraboloid $z = 4x^2+4y^2$ and the plane $z = 1$ if S has constant density K.

Solution. In cylindrical coordinates the region E is described by

$$0 \le r \le 1/2,\ 0 \le \theta \le 2\pi, \text{ and } 4r^2 \le z \le 1$$

Thus, the mass of the solid is

$$M = \iiint_E K\,dV = \int_0^{2\pi}\int_0^{1/2}\int_{4r^2}^1 Kr\,dz\,dr\,d\theta = \frac{K\pi}{8}.$$

The moment about the xy-plane is

$$M_{xy} = \iiint_E zK\,dV = \int_0^{2\pi}\int_0^{1/2}\int_{4r^2}^1 Kz\,r\,dz\,dr\,d\theta = \frac{K\pi}{12}.$$

Similarly, the other two moments are $M_{xz} = M_{yz} = 0$. We have $M_{xy}/M = 2/3$. Hence, the center of mass is $(0,0,2/3)$.

(b) Find the mass of a ball given by $x^2+y^2+z^2 \le a^2$ if the density at any point is proportional to its distance from the z-axis.

Solution. In spherical coordinates the region E is described by

$$0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

The density is $k\rho\sin\phi$, where k is a constant. Hence, the mass is

$$M = \iiint_E k\rho\sin\phi\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^a (k\rho\sin\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{k\pi^2a^4}{4}.$$