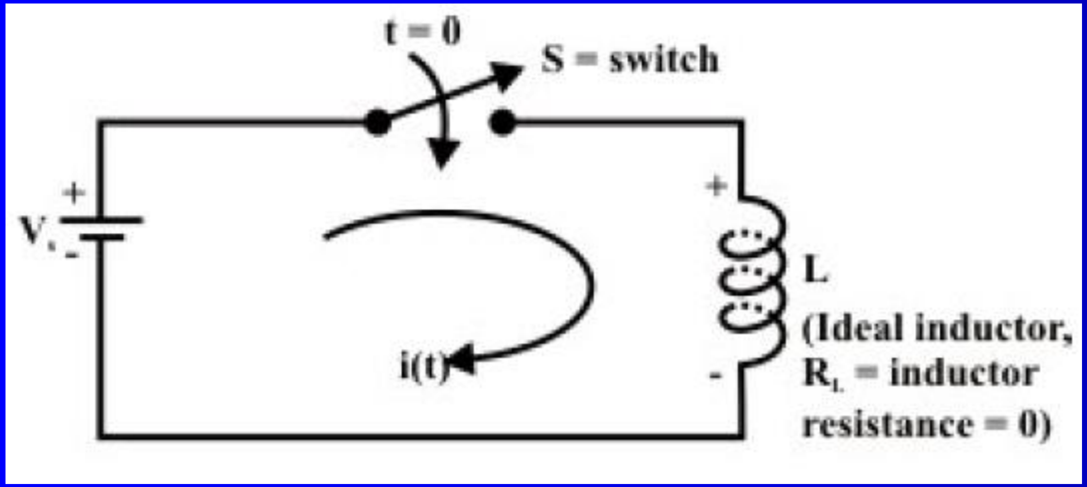
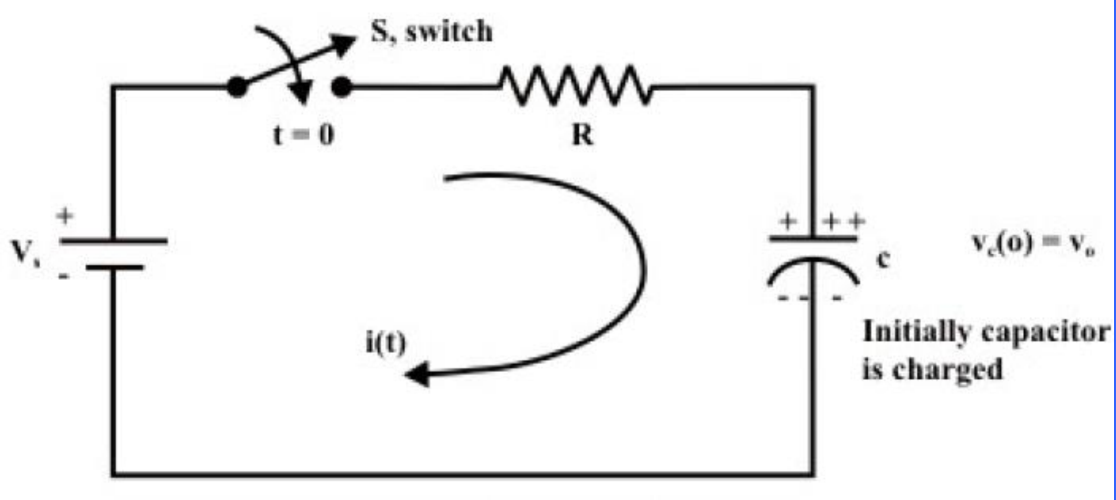
R-L and R-C Circuits

INTRODUCTION:

- When non-linear elements such as inductors and capacitors are introduced into a circuit, the behaviour is not instantaneous as it would be with resistors.
- A change of state will disrupt the circuit and the non-linear elements require time to respond to the change.
- Some responses can cause jumps in the voltage and current which may be damaging to the circuit.
- Accounting for the transient response with circuit design can prevent circuits from acting in an undesirable fashion.

- The response of a circuit (containing resistances, inductances, capacitors and switches) due to sudden application of voltage or current is called **transient response**.
- The most common instance of a transient response in a circuit occurs when a switch is turned on or off in a circuit containing resistance and capacitor/inductor or containing all three passive elements as illustrated below.





DC TRANSIENTS IN R-L CIRCUITS

INTRODUCTION:

- An R-L circuit is one that contains a resistor (R) and an inductor (L)
- Fig. 3.3 shows an R-L circuit where the passive elements are connected through a switch to a DC voltage, **V_S**
- Our problem is to study the growth of current in the circuit through two stages, namely; (i) DC transient response (ii) Steady state response
- Transient response is the state of the circuit just after the switch is turned on and this elapses some few micro or milliseconds
- Steady state response is the state after the transient response as time goes to infinity

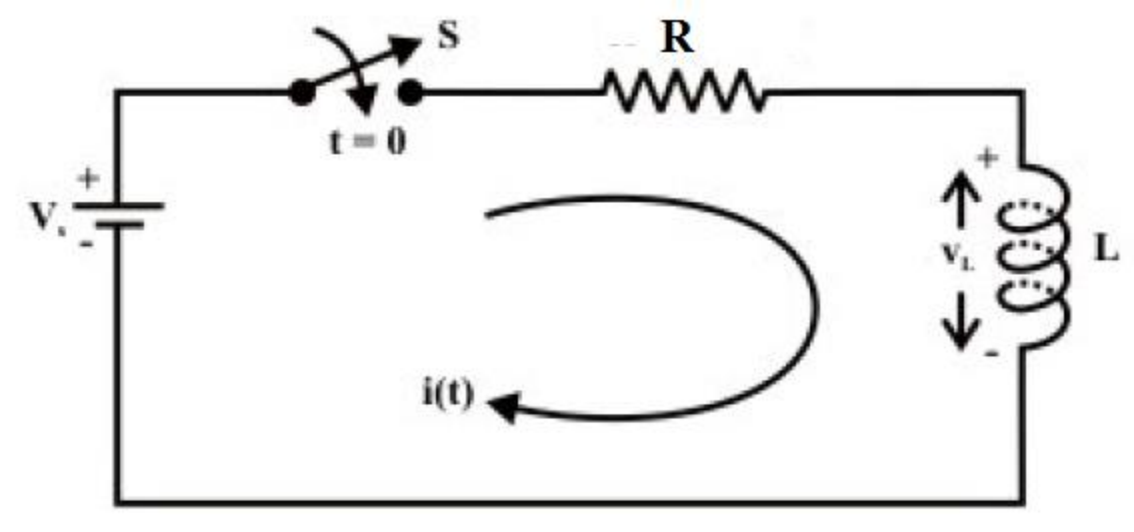


Fig. 3.3: R-L circuit

- To find the current expression (response) for the circuit shown in Fig. 3.3, we can write the KVL equation around the circuit

$$\boldsymbol{V_S = Ri(t) + L\frac{di(t)}{dt}} \qquad (\mathbf{3.1})$$

– Where V_S is the applied voltage or forcing function, R is the external resistance and L is the inductance.

- Equation (3.1) is a standard first order differential equation and it's solution is of the form:

$$\boldsymbol{i(t) = i_n(t) + i_f(t)} \qquad (\mathbf{3.2})$$

- Where $i_n(t)$ is the complementary solution/natural solution. It is sometimes called as transient response of the system.
- The second part $i_f(t)$ is the particular solution or steady state response of the system due to the input signal V_S

- Solving the differential equation (3.1)
- Rearrange the equation into a form that is easier to integrate

$$\frac{di(t)}{dt} = \frac{R}{L}\left(\frac{V_S}{R} - i(t)\right) \qquad (3.3)$$

- Divide by the term in brackets, and integrate.

$$\int \frac{di(t)}{dt} \frac{1}{i(t) - V_S/R} dt = -\frac{R}{L} \int dt \qquad (3.4)$$

- The integral becomes,

$$In\left(i(t) - \frac{V_S}{R}\right) = -\frac{t}{L/R} + D \qquad (3.5)$$

where D is the constant of integration

- Remove the natural log and solve for the inductor current

$$i(t) = \frac{V_S}{R} + e^D e^{-\frac{t}{L/R}} \qquad (3.6)$$

- **Let** $e^D = A$, $\boldsymbol{a\ constant}$
- **Hence**

$$i(t) = \frac{V_S}{R} + Ae^{-\frac{t}{L/R}} \qquad (7)$$

- The constant A is revealed at time t=0 when the switch is just closed. From equation (3.7)

$$i(0) - \frac{V_S}{R} = A \qquad (3.8)$$

- Now if the current before the switch is closed $i(0^-)$ is zero, then $i(0) = 0$
- It means the constant A is,

$$-\frac{V_S}{R} = A \qquad (3.9)$$

- As the time goes to infinity, the steady state response from equation (3.7) is found

$$i(\infty) = \lim_{t\to\infty} i(t) = \frac{V_S}{R} \qquad (3.10)$$

- The time constant τ is,

$$\tau = \frac{L}{R}$$

- Therefore the complete response of the current through an inductor connected to a DC voltage is

$$i(t) = i_f(t) + i_n(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{\frac{-t}{\tau}} = \frac{V_S}{R}\left(1 - e^{\frac{-t}{\tau}}\right) \qquad (3.11)$$

- The table shows how the current **i(t)** builds up in a R-L circuit

Actual time (t) in sec	Growth of current in inductor
$t=0$	$i(0)=0$
$t=\tau\left(=\frac{L}{R}\right)$	$i(\tau)=0.632\times\frac{V_s}{R}$
$t=2\tau$	$i(2\tau)=0.865\times\frac{V_s}{R}$
$t=3\tau$	$i(3\tau)=0.950\times\frac{V_s}{R}$
$t=4\tau$	$i(4\tau)=0.982\times\frac{V_s}{R}$
$t=5\tau$	$i(5\tau)=0.993\times\frac{V_s}{R}$

- Note: Theoretically at time **$t\rightarrow\infty$** the current in inductor reaches its steady state value but in practice the inductor reaches 99.3% of its steady state value at time **$t=5\tau$ (sec)**

- The expression for voltage across the resistance R is

$$V_R = i(t)R = \frac{V_S}{R}\left(1 - e^{\frac{-t}{\tau}}\right)R \qquad (3.12)$$

$$= V_S\left(1 - e^{\frac{-t}{\tau}}\right)$$

- The expression for voltage across the inductor is

$$V_L(t) = V_S - V_S\left(1 - e^{\frac{-t}{\tau}}\right) = V_s e^{\frac{-t}{\tau}} \qquad (3.13)$$

- Graphical representation of equations (3.13) – (3.15) are shown in Fig. 3.4

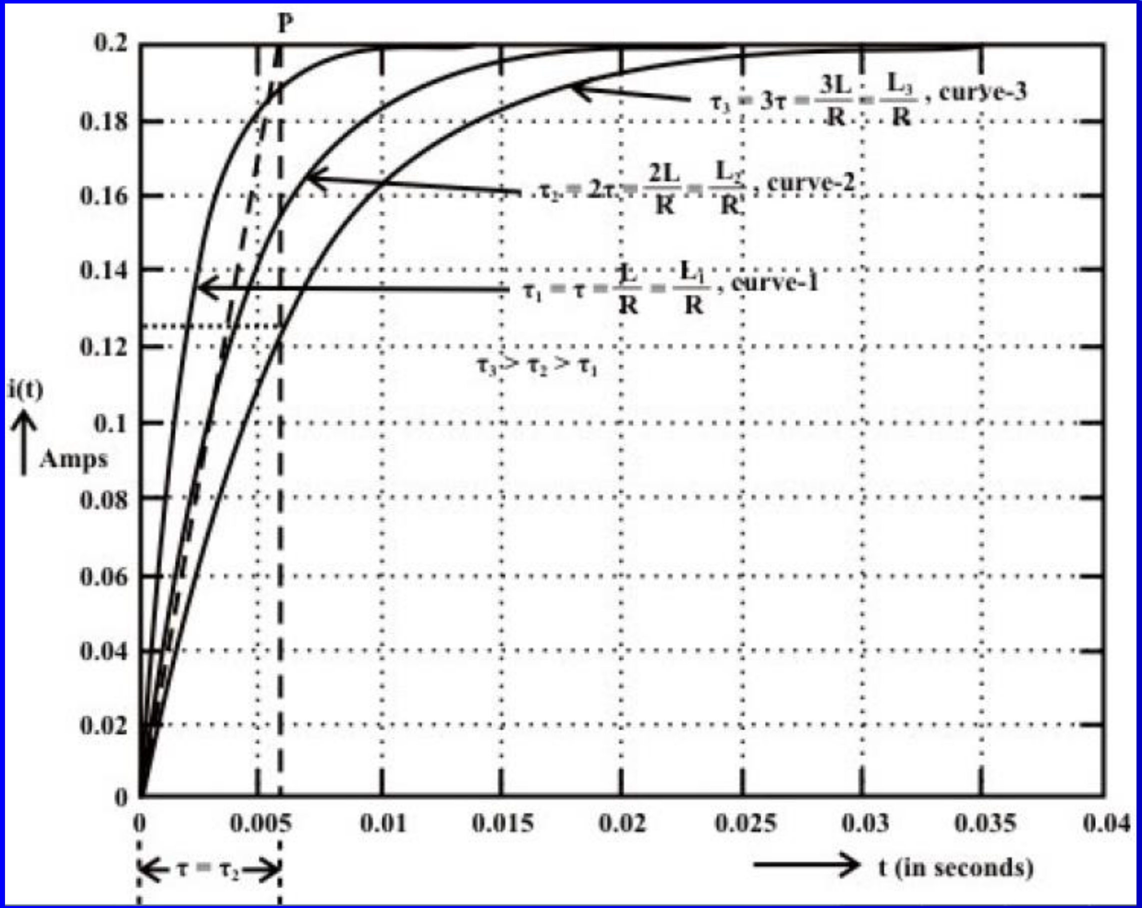


Fig. 3.4(a): Growth of current in R-L circuit (assumed initial current through inductor is zero)

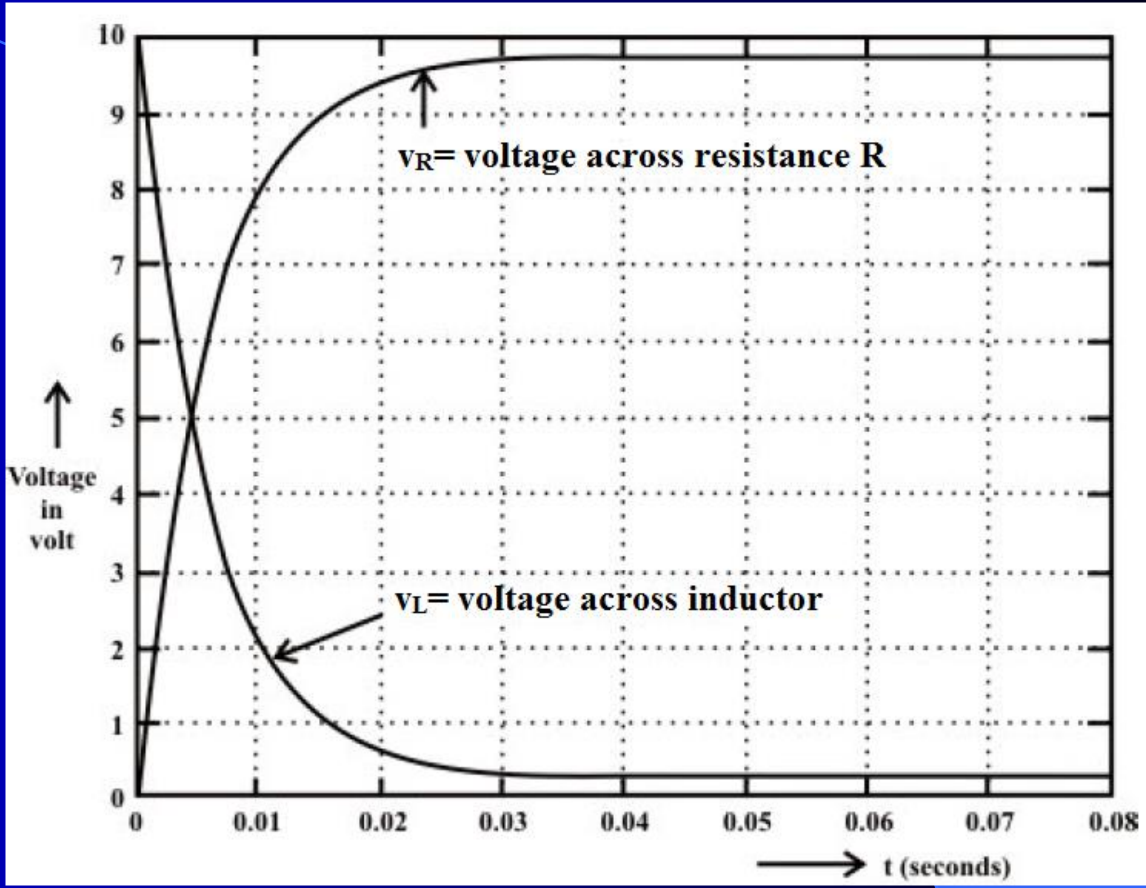


Fig. 3.4(b): Voltage response in different elements of R-L circuit (assumed i_o=0)

Finite Initial Current through an R-L Circuit:

- **Assume current flowing through the inductor just before closing the switch "S" (at $t = 0^-$) is $i(0^-) = i_o \neq 0$.**
- **Using equation (3.8), we get the value of $A = i(0) - A = i_o - \frac{V_s}{R}$**
- Using this value in equation (3.7), the expression for current flowing through the circuit is given by

$$i(t) = \frac{V_S}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \left(i_o e^{\frac{-t}{\tau}}\right) \qquad (3.14)$$

- The second part of the right hand side of the expression (3.14) indicates the current flowing to the circuit due to initial current i_o of inductor and the first part due to the input V_S applied to the circuit.
- This means that the complete response of the circuit is the algebraic sum of two outputs due to two inputs; namely (i) **due to input V_S** (ii) **due to initial current through the inductor**.
- This implies that the superposition theorem is also valid for such type of linear circuit.

Fig. 3.5 shows the response of inductor current when the circuit is excited with a constant voltage source V_S and the initial current through inductor is i_o

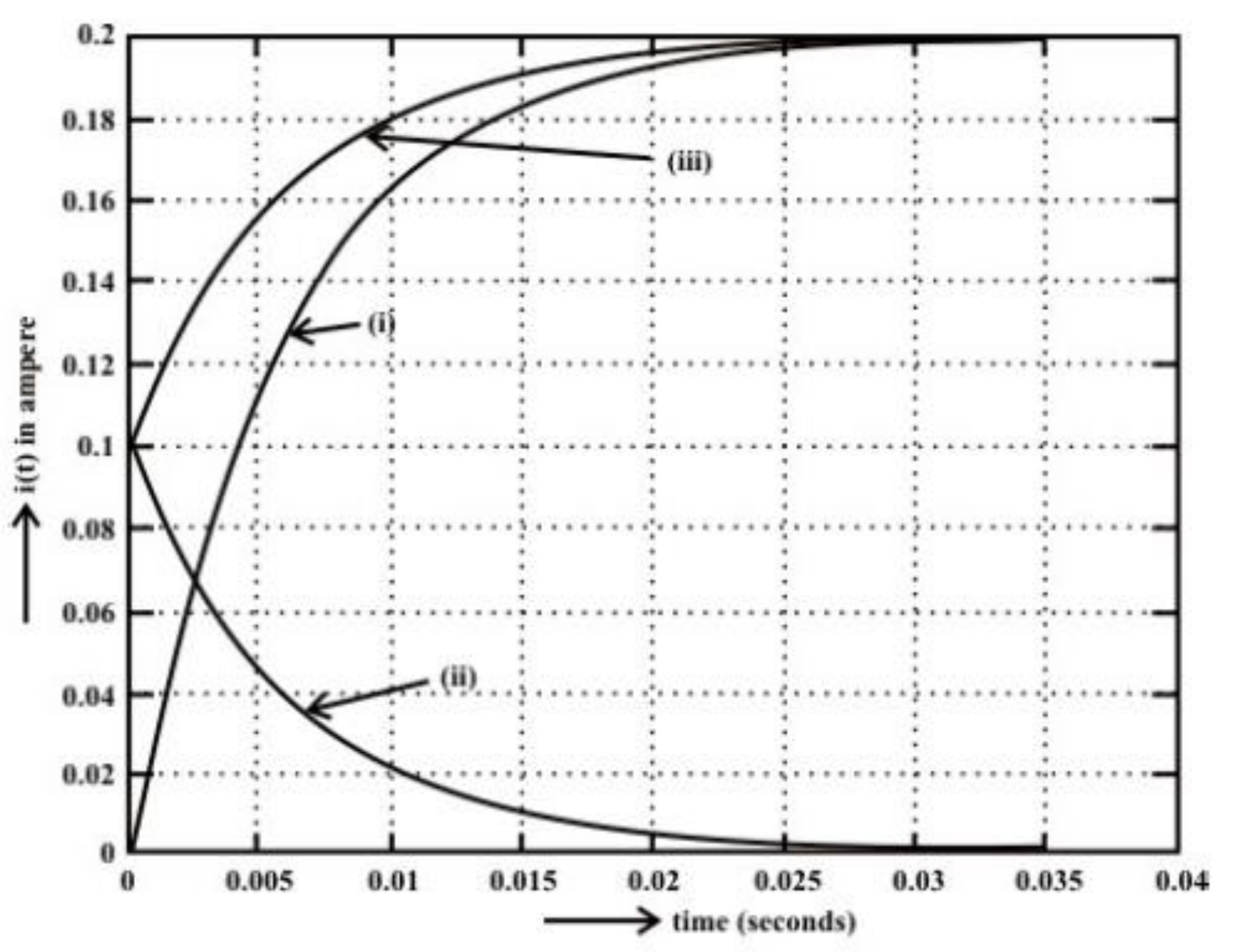


Fig. 3.5: Current through inductor due to (i) input V_S only, (ii) initial condition i_o only, (iii) combined effect of (i) and (ii)

Time Constant (τ) of R-L Circuit:

- It is the time required for any variable or signal (in our case either current (i(t)) or voltage (v_R or v_L)) to reach 63.2% of its final value.
- It is possible to write an exact mathematical expression to calculate the time constant (τ) of any first-order differential equation
- Let 't' be the time required to reach 63.2% of steady-state value of inductor current and the corresponding time 't' expression can be obtained as

$$i(t) = 0.632 * \frac{V_s}{R} = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

⇒ **This implies,** $0.632 = 1 - e^{-\frac{R}{L}t}$ => $0.368 = e^{-1} = e^{-\frac{R}{L}t}$

⇒ **Hence**

$$t = \frac{L}{R} = \tau\ (sec)$$

- The behavior of all circuit responses (for first-order differential equation) is fixed by a single time constant τ(for *R-L circuit*) and it provides information about the speed of response or in other words, it indicates how fast or slow the system response reaches its steady state from the instant of switching the circuit.
- Observe the equation (3.11) that the smaller the time constant (τ), the more rapidly the current increases and subsequently it reaches the steady state (or final value) quickly.
- On the other hand, a circuit with a larger time constant (τ) provides a slow response because it takes longer time to reach steady state.
- These facts are illustrated in Fig. 3.4(a).