Example sheet 1

1. If the exact answer is A and the computed answer is $\tilde{A}$, find the absolute and relative error when

$$\begin{array}{lll} \text{a)} & A = 10.147, & \tilde{A} = 10.159 \\ \text{b)} & A = 0.0047, & \tilde{A} = 0.0045 \\ \text{c)} & A = 0.671 \times 10^{12}, & \tilde{A} = 0.669 \times 10^{12}. \end{array}$$

2. Let $a = 0.471 \times 10^{-2}$ and $b = -0.185 \times 10^{-4}$. Use 3 digit floating point arithmetic to compute $a + b$, $a - b$, $a * b$ and a/b. Find the rounding error in each case.

   Let $c = 0.869 \times 10^4$. Show that the computed value of $a + c$ is equal to c.

3. Use the formula $\sin(x) \approx x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ to evaluate $\sin(0.3)$. Find, numerically, the truncation error.

4. Using MATLAB and the formula $\dfrac{\sin(x+h) - \sin(x)}{h}$ evaluate the derivative of $\sin(x)$ (i.e. $\cos(x)$) at $x = 0.5$ using values of $h = 10^{-p}$, $p = 1, 2, \cdots$. Explain the results you obtain.

5. Given that

$$\sqrt{a} = \lim_{n \to \infty} \{x_n\}$$

   where

$$x_0 = \frac{a}{2}, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

   find a fraction $\dfrac{p}{q}$, $p < 10^6$ that approximates $\sqrt{2}$ with a truncation error less than 10^{-10}.

6. The binary representations of 5 and $0 \cdot 5$ are 101 and $0 \cdot 1$ respectively where the $\cdot$ in $0 \cdot 5$ is a decimal point and in $0 \cdot 1$ is a binary point. Find the binary representations of

$$7,\ 27,\ 0 \cdot 125,\ 14 \cdot 75,\ 0 \cdot 3.$$

7. Find the six digit binary floating point representations of

$$7,\ 27,\ 0 \cdot 125,\ 14 \cdot 75,\ 0 \cdot 3.$$

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Example sheet 2

1. Use the bisection method to approximate $\sqrt{3}$ to 2 decimal places. Use $f(x) = x^2 - 3$ with $f(0) = -3$ and $f(2) = 1$ as the starting point.

2. Show that each of the following iterations have fixed points $x = \pm\sqrt{3}$

$$\begin{aligned}
&\text{a) } x_{i+1} = \frac{3}{x_i} \\
&\text{b) } x_{i+1} = x_i + (x_i)^2 - 3 \\
&\text{c) } x_{i+1} = x_i + 0.25\left((x_i)^2 - 3\right) \\
&\text{d) } x_{i+1} = x_i - 0.5\left((x_i)^2 - 3\right) \\
&\text{e) } x_{i+1} = \frac{(2x_i - 3)}{(2 - x_i)}.
\end{aligned}$$

   Determine which of these iterations converge to $+\sqrt{3}$ and which to $-\sqrt{3}$. Compute ϕ' at the fixed points and relate these values to the convergence properties of the iteration.

3. Sketch the cubic polynomial

$$p(x) = 4x^3 - 10x^2 + 2x + 5$$

   to get a rough estimate of its roots. Use the Newton Raphson method to approximate each root to 4 decimal places.

4. Use Newton's method to find the intersection of the curves

$$\begin{aligned}
x^2 + 3y^2 - 1 &= 0 \\
(x-2)^2 + (y-1)^2 - 4 &= 0.
\end{aligned}$$

5. Describe a Jacobi and a Gauss Seidel type of iteration for solving the pair of equations,

$$\begin{aligned}
e^{xy} + x^2 y &= 6 \\
xy \sin(4 - x) + 3y^2 &= 1.
\end{aligned}$$

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Example sheet 3

1. Solve the lower triangular system of equations

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -2 \\ -x_1 & + & 4x_2 & & & = & 6 \\ 2x_1 & + & 3x_2 & - & 2x_3 & = & 3\ . \end{array}$$

2. Solve the system of equations:-

$$\text{a)}\quad \begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 4 \\ -x_1 & + & 4x_2 & + & x_3 & = & -7 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 5 \end{array}$$

$$\text{b)}\quad \begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 0 \\ -x_1 & + & 4x_2 & + & x_3 & = & 3 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 1 \end{array}$$

by Gaussian elimination and back substitution without interchanging the equations.

3. Find the solution of the system of equations represented by the augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 4 & 0 \\ -1 & 4 & 1 & -7 & 3 \\ 2 & 1 & 5 & 5 & 1 \end{bmatrix}$$

by Gaussian elimination using augmented matrix notation.

4. Factorize the matrix A where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 4 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

into the product of a lower and upper triangular matrix for which

   a) the diagonal elements of the lower triangular matrix are 1

   b) the diagonal elements of the upper triangular matrix are 1.

5. In 'Jordan elimination' for solving a system of n equations, the augmented matrix is reduced, by row operations, to a matrix in which the first $n \times n$ block is the identity matrix. (See, for example, a brief mention in the course book, Burden and Faires, in which this method is called the Gauss Jordan method.)

   Use Jordan elimination to solve the system of equations represented by the augmented matrix in example (3) above.

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Example sheet 4

1. Solve the system of equations

$$\begin{array}{rcrcrcrcr} 4x_1 & + & 4x_2 & + & x_3 & + & 4x_4 & = & 12 \\ 2x_1 & + & 5x_2 & + & 7x_3 & + & 4x_4 & = & 1 \\ 10x_1 & + & 5x_2 & - & 5x_3 & & & = & 25 \\ -2x_1 & - & 2x_2 & + & x_3 & - & 3x_4 & = & -10 \end{array}$$

   by Gaussian elimination with partial pivoting.

2. Find the number of multiplications and divisions in solving an upper triangular system of n equations.

3. Find a lower triangular matrix D such that

$$DD^T = A = \begin{bmatrix} 16 & -4 & 4 \\ -4 & 5 & 3 \\ 4 & 3 & 14 \end{bmatrix}.$$

   Hence find the solution of the set of equations

$$\begin{array}{rcrcrcr} 16x_1 & - & 4x_2 & + & 4x_3 & = & 24 \\ -4x_1 & + & 5x_2 & + & 3x_3 & = & -6 \\ 4x_1 & + & 3x_2 & + & 14x_3 & = & 15 \end{array}$$

   by Cholesky's method. Find the determinant of D and A and show that the total number of multiplications needed to solve a system of n equations is $\frac{1}{6}n^3 + O\left(n^2\right)$.

4. Using the MATLAB functions `hilb` and `cond` determine that the 5 by 5 Hilbert matrix is illconditioned.

   (Note: the MATLAB command `H=hilb(5)` will generate the 5 by 5 Hilbert matrix H.)

5. Solve the system of equations in example (3) by

   i) a Jacobi iteration

   ii) a Gauss Seidel iteration

   by writing the set of equations as

$$x_1 = \frac{1}{16}\left(24 + 4x_2 - 4x_3\right)$$
$$x_2 = \frac{1}{5}\left(-6 + 4x_1 - 3x_3\right)$$
$$x_3 = \frac{1}{14}\left(15 - 4x_1 - 3x_2\right)$$

   using the initial guess $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$.

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Example sheet 5

1. Write down the Lagrange interpolation polynomial $L(x)$ that interpolates

x	$f(x)$
1.2	2.847
1.8	1.680
2.5	0.039

and use it to estimate $f(1.5)$ and $f(2.0)$.

2. Write down the linear function $L(x)$ that passes through the points

$$(x_0, f(x_0)) \text{ and } (x_1, f(x_1)).$$

Assuming that $f(x)$ is continuously differentiable and setting $x_1 = x_0 + \epsilon$, find $H(x)$ given by

$$H(x) = \lim_{\epsilon \to 0} \{L(x)\}.$$

Show that

a) $H(x_0) = f(x_0)$

b) $H'(x_0) = f'(x_0)$.

3. Write down the divided difference table for e^x using the values

x	e^x
0.0	1.00000
0.4	1.49182
0.9	2.45960
1.5	4.48169
1.8	6.04965.

Estimate $e^{1.2}$ using

a) cubic interpolation with $x_0 = 0.0$

b) cubic interpolation with $x_0 = 0.4$.

Which gives the better estimate?

Estimate $e^{1.2}$ using a quartic interpolation with $x_0 = 0.0$.

Course 157

Example sheet 6

1. Complete the finite difference table for the following function

x	$f(x)$	x	$f(x)$
0.0	−0.540	2.0	6.100
0.5	1.086	2.5	7.760
1.0	2.750	3.0	9.407
1.5	4.428	3.5	11.043.

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0.0	−0.540				
		1.626			
0.5	1.086		0.038		
		1.664			
1.0	2.750		0.014		
		1.678			
1.5	4.428				
2.0	6.100				
		1.660			
2.5	7.760		−0.013		
		1.647			
3.0	9.407		−0.011		
		1.636			
3.5	11.043				

With $x_0 = 1.0$, identify the values,

$$\text{a) } E^3 f_0 \quad \text{b) } E^{-2} f_2 \quad \text{c) } \Delta^3 f_{-1} \quad \text{d) } \nabla^2 f_3.$$

Choosing suitable x_0 in each case, use Newton's forward formula to estimate $f(0.2)$ and $f(1.8)$ and use Newton's backward formula to estimate $f(1.8)$ and $f(3.25)$.

2. Complete the finite difference table below for the polynomial $f(x) = x^3 - x + 2$ with values of x from 0 to 1.2 in steps $h = 0.2$ working to three decimal places. What is surprising about this table?

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0.0	2.000				
		-0.192			
0.2	1.808				
		-0.144			
0.4	1.664				
		-0.048			
0.6	1.616				
		0.096			
0.8	1.712				
		0.288			
1.0	2.000				
		0.528			
1.2	2.528				

Prove that the column of first differences of a polynomial of degree $n \leq 3$ represents exactly a polynomial of degree $n-1$.

Calculate the finite difference table for $f(x)$ for the same values of x but with the function values rounded to two decimal places.

3. The function values in a difference table represent a polynomial of degree $n \leq 3$ with rounding errors $\varepsilon_i = (-1)^i \varepsilon$ in f_i. Show how rounding errors build up in the difference table.

Course 157

Example sheet 7

1. Write down the linear approximation $L_1(x)$ to $f(x)$ that is exact at x_0 and x_1. Use it to approximate the first derivative of $f(x)$ at x_0 and x_1 and use the error term in Lagrange interpolation to estimate the error in these two approximations.

2. Use the Trapezium Rule and Simpson's Rule to estimate

$$\int_1^2 e^x dx$$

and compare your answers with the exact value.

Now use the Composite Trapezium and Simpson's Rules with 2, 4 and 8 strips to estimate this integral. Tabulate the error in each case. What is the rate of decrease of the error with respect to the strip width?

3. The error in Simpson's rule for approximating the integral $I = \int_a^b f(x)\,dx$ is,

$$-\frac{(b-a)^5}{2880} f^{(4)}(t_s), \qquad a < t_s < b$$

By assuming that the fourth derivative of $f(x)$ is continuous in $[a, b]$ show that the error in the *Composite* Simpson's Rule for integrating $f(x)$ from a to b is

$$-\frac{(b-a)}{180} h^4 f^{(4)}(\theta) \qquad a < \theta < b.$$

4. Show that the quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

is exact for polynomials of degree 3 when the multipliers w_1 and w_2 and sample points x_1 and x_2 are chosen such that

$$w_1 = 1,\ w_2 = 1,\ x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}}.$$

By assuming that the error term is of the form

$$Cf^{(q)}(\xi)$$

find q and C.