$\boxed{1}$ Define the operator ∇ (pronounced "del") by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Notice that the gradient ∇f (or also grad f) is just ∇ applied to f.

(a) We define the *divergence* of a vector field $\mathbf{F}$, written div $\mathbf{F}$ or $\nabla \cdot \mathbf{F}$, as the dot product of del with $\mathbf{F}$. So if $\mathbf{F} = \langle P, Q, R\rangle$, then

$$\text{div }\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle P, Q, R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Notice that div $\mathbf{F}$ is a scalar.

Find div $\mathbf{F}$ for each of the following vector fields:

(i) $\mathbf{F} = \langle xy, yz, xz\rangle$ (ii) $\mathbf{F} = \langle yz, xz, xy\rangle$

(iii) $\mathbf{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right\rangle$ (iv) $\mathbf{F} = \text{grad } f$, where f is a function with
       where $r = \sqrt{x^2 + y^2 + z^2}$ continuous second derivatives

(b) We define the *curl* of a vector field $\mathbf{F}$, written curl $\mathbf{F}$ or $\nabla \times \mathbf{F}$, as the cross product of del with $\mathbf{F}$. So if $\mathbf{F} = \langle P, Q, R\rangle$, then

$$\text{curl }\mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \langle P, Q, R\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Notice that curl $\mathbf{F}$ is a vector.

Find curl $\mathbf{F}$ for each of the following vector fields:

(i) $\mathbf{F} = \langle xy, yz, xz\rangle$ (ii) $\mathbf{F} = \langle yz, xz, xy\rangle$

(iii) $\mathbf{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right\rangle$ (iv) $\mathbf{F} = \text{grad } f$, where f is a function with
       where $r = \sqrt{x^2 + y^2 + z^2}$ continuous second derivatives

2 Check the appropriate box ("Vector", "Scalar", or "Nonsense") for each quantity.

Quantity	Vector	Scalar	Nonsense	Quantity	Vector	Scalar	Nonsense
$\text{curl}(\nabla f)$	☐	☐	☐	$\text{curl}(\text{curl } f)$	☐	☐	☐
$\nabla \cdot (\nabla \times \mathbf{F})$	☐	☐	☐	$\text{grad}(\text{div } f)$	☐	☐	☐
$\text{div}(\text{curl } f)$	☐	☐	☐	$\text{div}(\text{grad } \mathbf{F})$	☐	☐	☐
$\text{curl}(\text{curl } \mathbf{F})$	☐	☐	☐	$\nabla \cdot (\nabla f)$	☐	☐	☐
$\nabla(\nabla \cdot \mathbf{F})$	☐	☐	☐	$\text{div}(\text{div } \mathbf{F})$	☐	☐	☐

3 (a) Suppose we have a vector $\langle P, Q \rangle$ that we extend to a vector in space: $\mathbf{F} = \langle P(x,y), Q(x,y), 0 \rangle$. Find $\text{curl } \mathbf{F}$.

(b) When is the vector field $\mathbf{F}$ in part (a) conservative? Use the curl in your answer.

(c) If $\mathbf{F} = \nabla f = \langle f_x, f_y, f_z \rangle$ is conservative, then is $\text{curl } \mathbf{F} = \mathbf{0}$? (See Problem 1(b)(iv).)

In fact, we can identify conservative vector fields with the curl:

> **Theorem:** Let $\mathbf{F}$ be a vector field defined on all of $\mathbf{R}^3$. If the component functions of $\mathbf{F}$ all have continuous derivatives and $\text{curl } \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

4 Show that, if $\mathbf{F}$ is a vector field on $\mathbf{R}^3$ with components that have continuous second-order derivatives, then $\text{div curl } \mathbf{F} = 0$. (This is less useful for us right now than $\text{curl grad } f = \mathbf{0}$, but it's not a difficult computation.)

5 We can re-write Green's theorem in vector form (we get the formula on the left, below). The formula on the right can be thought of as a version of Green's theorem that uses the normal component rather than the tangential component:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA \qquad \text{and} \qquad \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA.$$

(If $\mathbf{r} = \langle x(t), y(t) \rangle$, then $\mathbf{n} = \frac{\langle y'(t), -x'(t) \rangle}{|\langle y'(t), -x'(t) \rangle|}$ is the outward unit normal.) For each of the following vector fields $\mathbf{F}$ and paths C, compute the integrals above using these vector forms of Green's theorem:

(a) $\mathbf{F} = \langle -y, x \rangle$ and C is the unit circle, positively oriented.

(b) $\mathbf{F} = \langle 2x, 2y \rangle$ and C is the unit square (with corners at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$), positively oriented.

(While we'll never really look at the normal version of Green's theorem again, one of our big integral theorems next week can be thought of as a higher-dimensional version of it.)

6 Here are sketches of a few vector fields $\mathbf{F} = \langle P(x,y), Q(x,y), 0 \rangle$ (they're drawn in the plane, but they're defined in all of space). Can you tell which one has zero curl? Zero divergence?

$$\mathbf{F} = \langle -y, x, 0 \rangle \qquad\qquad \mathbf{F} = \langle y, x, 0 \rangle \qquad\qquad \mathbf{F} = \langle 2x, 2y, 0 \rangle$$
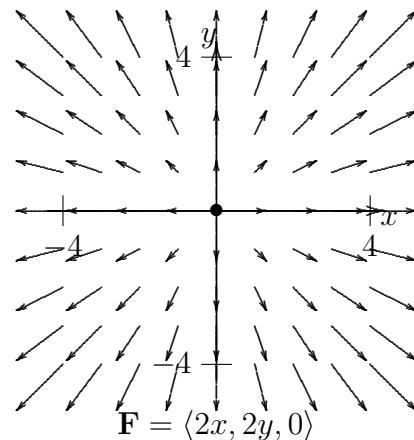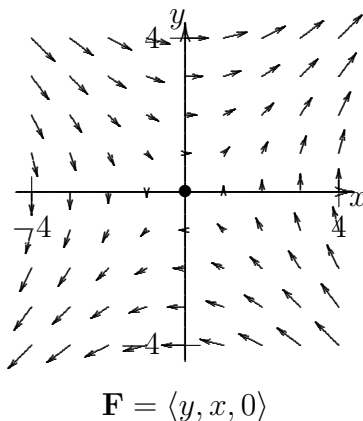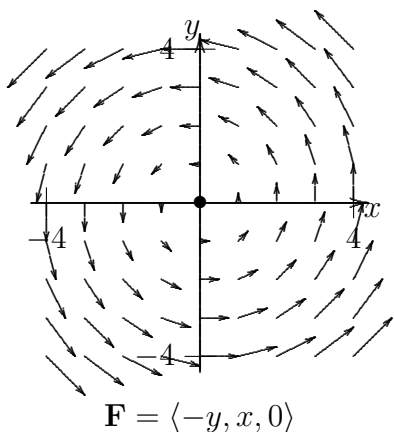
Curl and Divergence – Answers and Solutions

1 (a) (i) $\text{div}\,\mathbf{F} = x + y + z$

(ii) $\text{div}\,\mathbf{F} = 0$

(iii) $\text{div}\,\mathbf{F} = \frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{x^2+y^2}{r^3} = \frac{2(x^2+y^2+z^2)}{r^3} = \frac{2}{r}$

(iv) $\text{div}\,\mathbf{F} = \text{div}(\text{grad}\,f) = f_{xx} + f_{yy} + f_{zz}$. This is the Laplace operator applied to f.

(b) (i) $\text{curl}\,\mathbf{F} = \langle -y, -z, -x \rangle$

(ii) $\text{curl}\,\mathbf{F} = \mathbf{0}$

(iii) $\text{curl}\,\mathbf{F} = \mathbf{0}$

(iv) $\text{curl}\,\mathbf{F} = \text{curl}(\text{grad}\,f) = \mathbf{0}$.
Two quick comments:

- Notice that the vector from part (ii) is actually an example of this, since $\mathbf{F} = \text{grad}(xyz) = \langle yz, xz, xy \rangle$. The vector for part (iii) is $\mathbf{F} = \text{grad}(\sqrt{x^2 + y^2 + z^2})$, so it is another example as well.

- We'll see later on in this worksheet that $\text{curl}\,\mathbf{F} = \mathbf{0}$ precisely when $\mathbf{F} = \text{grad}\,f$ (when we make some continuity assumptions about the derivatives of the components of $\mathbf{F}$.

2 Here are some answers, although half are blank as they match questions from the homework:

Quantity	Vector	Scalar	Nonsense	Quantity	Vector	Scalar	Nonsense
$\text{curl}(\nabla f)$	X	☐	☐	$\text{curl}(\text{curl}\,f)$	☐	☐	X
$\nabla \cdot (\nabla \times \mathbf{F})$	☐	X	☐	$\text{grad}(\text{div}\,f)$	☐	☐	☐
$\text{div}(\text{curl}\,f)$	☐	☐	X	$\text{div}(\text{grad}\,\mathbf{F})$	☐	☐	X
$\text{curl}(\text{curl}\,\mathbf{F})$	☐	☐	☐	$\nabla \cdot (\nabla f)$	☐	☐	☐
$\nabla(\nabla \cdot \mathbf{F})$	☐	☐	☐	$\text{div}(\text{div}\,\mathbf{F})$	☐	☐	☐

3 (a) $\text{curl}\,\mathbf{F} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \langle 0, 0, Q_x - P_y \rangle$

(b) A planar vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative when $Q_x - P_y = 0$. This means there's a function $f(x, y)$ with $P = f_x$ and $Q = f_y$. We also get $f_z = 0$ since f is a function of only x and y, so $\mathbf{F} = \langle P(x, y), Q(x, y), 0 \rangle$ is conservative under this same condition. From part (a), this means that $\mathbf{F}$ is conservative when $\text{curl}\,\mathbf{F} = \mathbf{0}$.

(c) Yep, we've already seen that $\text{curl}\,\text{grad}\,f = \mathbf{0}$.

4 Here's a quick detailed computation:

$$\text{div}\,\text{curl}\,\mathbf{F} = \text{div}\left(\nabla \times \langle P, Q, R \rangle \right)$$

$$= \nabla \cdot \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k} \right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= \left(\frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z} \right) + \left(\frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x} \right) + \left(\frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y} \right)$$

$$= (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}).$$

(a) We do this both ways, of course. First let's parameterize C by $\mathbf{r}(t) = \langle x, y \rangle = \langle \cos(t), \sin(t) \rangle$, so $dx = -\sin(t)\, dt$ and $dy = \cos(t)\, dt$. Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

The double integral is also straightforward: writing $\mathbf{F} = \langle -y, x, 0 \rangle$ as a vector field in space, we get $\operatorname{curl} \mathbf{F} = \langle 0, 0, 2 \rangle$. Thus, since $\mathbf{k} = \langle 0, 0, 1 \rangle$, we get

$$\iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA = \iint_D 2 \, dA = 2 \cdot \operatorname{Area}(D) = 2\pi.$$

The normal component version requires us to have a unit outward-pointing normal $\mathbf{n}$. For the unit circle, the unit normal at the point (x, y) is simply $\mathbf{n} = \langle x, y \rangle$. (One way to see this is to parameterize the circle as $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \cos(t), \sin(t) \rangle$, so by our formula $\mathbf{n}$ is the unit vector in the direction $\langle y'(t), -x'(t) \rangle = \langle \cos(t), \sin(t) \rangle$. But this is a unit vector already, and in fact it is the vector $\langle x, y \rangle$.) Thus $\mathbf{F} \cdot \mathbf{n} = \langle -y, x \rangle \cdot \langle x, y \rangle = 0$, so the line integral is zero. Note also that $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$, so the double integral is also zero.

(b) Again we do this both ways. We'll parameterize C in four parts (all with $0 \le t \le 1$):

$$C_1 : \quad \mathbf{r}(t) = \langle x, y \rangle = \langle t, 0 \rangle \qquad\qquad C_2 : \quad \mathbf{r}(t) = \langle x, y \rangle = \langle 1, t \rangle$$
$$C_3 : \quad \mathbf{r}(t) = \langle x, y \rangle = \langle 1 - t, 1 \rangle \qquad\qquad C_4 : \quad \mathbf{r}(t) = \langle x, y \rangle = \langle 0, 1 - t \rangle$$

From this we get four integrals:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \langle 2x, 2y \rangle \cdot \langle dx, dy \rangle = \int_0^1 \langle 2t, 0 \rangle \cdot \langle dt, 0 \rangle = \int_0^1 2t \, dt = 1$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \langle 2x, 2y \rangle \cdot \langle dx, dy \rangle = \int_0^1 \langle 2, 2t \rangle \cdot \langle 0, dt \rangle = \int_0^1 2t \, dt = 1$$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \langle 2x, 2y \rangle \cdot \langle dx, dy \rangle = \int_0^1 \langle 2(1 - t), 2 \rangle \cdot \langle -dt, 0 \rangle = \int_0^1 2(t - 1) \, dt = -1$$

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_4} \langle 2x, 2y \rangle \cdot \langle dx, dy \rangle = \int_0^1 \langle 0, 2(1 - t) \rangle \cdot \langle 0, -dt \rangle = \int_0^1 2(t - 1) \, dt = -1.$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 1 + 1 - 1 - 1 = 0.$$

The double integral is considerably easier: writing $\mathbf{F} = \langle 2x, 2y, 0 \rangle$ as a vector field in space, we compute $\operatorname{curl} \mathbf{F} = \mathbf{0}$, so the integrand in the double integral is zero. Thus the double integral is zero as well.

The normal component version requires us to have a unit outward-pointing normal $\mathbf{n}$. These are simple to find for each component (using the above parameterization):

$$C_1 : \quad \mathbf{n} = -\mathbf{j} = \langle 0, -1 \rangle \qquad\qquad C_2 : \quad \mathbf{n} = \mathbf{i} = \langle 1, 0 \rangle,$$
$$C_3 : \quad \mathbf{n} = \mathbf{j} = \langle 0, 1 \rangle, \qquad\qquad C_4 : \quad \mathbf{n} = -\mathbf{i} = \langle -1, 0 \rangle.$$

Thus the integrals are all (note that $ds = dt$ in each case)

$$\int_{C_1} \mathbf{F} \cdot \mathbf{n} \; ds = \int_0^1 \langle 2t, 0 \rangle \cdot \langle 0, -1 \rangle \; dt = 0$$

$$\int_{C_2} \mathbf{F} \cdot \mathbf{n} \; ds = \int_0^1 \langle 2, 2t \rangle \cdot \langle 1, 0 \rangle \; dt = \int_0^1 2 \; dt = 2$$

$$\int_{C_3} \mathbf{F} \cdot \mathbf{n} \; ds = \int_0^1 \langle 2(1-t), 2 \rangle \cdot \langle 0, 1 \rangle \; dt = \int_0^1 2 \; dt = 2$$

$$\int_{C_4} \mathbf{F} \cdot \mathbf{n} \; ds = \int_0^1 \langle 0, 2(1-t) \rangle \cdot \langle -1, 0 \rangle = 0.$$

Thus

$$\int_C \mathbf{F} \cdot \mathbf{n} \; ds = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0 + 2 + 2 + 0 = 4.$$

The double integral is again much simpler: we compute $\operatorname{div} \mathbf{F} = 2 + 2 = 4$, so the double integral is just 4 times the area of the square. Thus the double integral has value 4 as well.

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6 │ The idea here is that we can do this two ways: first, we can compute the curl and divergence of the given vector fields:

(a) $\operatorname{div} \mathbf{F} = 0$
      $\operatorname{curl} \mathbf{F} = \langle 0, 0, 2 \rangle$

(b) $\operatorname{div} \mathbf{F} = 0$
      $\operatorname{curl} \mathbf{F} = \mathbf{0}$

(c) $\operatorname{div} \mathbf{F} = 4$
      $\operatorname{curl} \mathbf{F} = \mathbf{0}$

Thus we see that the first vector field is the only one with a non-zero curl, and that the last vector field is similarly the only one with a non-zero divergence.

Here's a way to see this from the graph. We'll use the two Green's theorems from the previous problem:

$$\oint_C \mathbf{F} \cdot \mathbf{T} \; ds = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \; dA$$

$$\oint_C \mathbf{F} \cdot \mathbf{n} \; ds = \iint_D \operatorname{div} \mathbf{F} \; dA.$$

(Here we've written the first one slightly differently, but it's the same. Since $d\mathbf{r} = \mathbf{r}'(t) \; dt$ and $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)| \; dt$, we get $d\mathbf{r} = \mathbf{T} \; ds$.)

Let's start with the curl and the first of Green's theorems. Let's integrate around a small circle C centered at the origin. We'll choose it so small that $\operatorname{curl} \mathbf{F}$ is essentially constant on the tiny little disk D bounded by C. By this argument, the double integral is roughly

$$\iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \; dA \approx \iint_D (\operatorname{curl} \mathbf{F}(0,0,0)) \cdot \mathbf{k} \; dA = \Big((\operatorname{curl} \mathbf{F}(0,0,0)) \cdot \mathbf{k} \Big) \cdot \operatorname{Area}(D).$$

On the other hand, the line integral is the integral of $\mathbf{F} \cdot \mathbf{T}$. This is the tangential component of $\mathbf{F}$, and in particular

$\mathbf{F} \cdot \mathbf{T} > 0$ means $\mathbf{F}$ and $\mathbf{T}$ are mostly in the same direction,

$\mathbf{F} \cdot \mathbf{T} < 0$ means $\mathbf{F}$ and $\mathbf{T}$ are mostly in opposing directions, and

In the case of the first sketch, it's clear that $\mathbf{F}$ is mostly parallel to the tangent vector to the small circle C, so $\mathbf{F} \cdot \mathbf{T} > 0$. Thus $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds > 0$, so by Green's theorem,

$$(\operatorname{curl} \mathbf{F}(0, 0, 0)) \cdot \mathbf{k} > 0.$$

The moral: the curl of $\mathbf{F}$ is non-zero means that there is some kind of rotation in the vector field. We'll see much more of this later.

We could do something similar with the divergence. Let's cut straight to the chase: the (outward-pointing) normal $\mathbf{n}$ produces a positive divergence at the origin. This gives us what we call a *source* (when $\operatorname{div} \mathbf{F} > 0$) at the origin; if the vector fields were pointing in we'd get a sink (and $\operatorname{div} \mathbf{F} < 0$).