Linear Algebra I Exam Solutions 2007 — F. J. Wright

Note Title 03/01/2007

Section A

A1 AB does not exist. (1)

$$BA = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 2x+4y \end{pmatrix}$$ (1)

$$A^T C = (x \ \ y)\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$
$$= (x+2y, \ 3x+4y, \ 5x+6y)$$ (2)

$$B^T C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \end{pmatrix}$$ (2)

CB^T does not exist (1)

A2 By Laplace expansion about the middle column, $|A| = -2(1-2) = 2$. (1)

The matrix of minors of A is $\begin{pmatrix} -2 & 0 & 2 \\ -4 & -1 & 2 \\ 0 & -1 & 0 \end{pmatrix}$. (2)

Hence $A^{-1} = \dfrac{1}{2}\begin{pmatrix} -2 & 4 & 0 \\ 0 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix}$. (2)

Check:

$$A^{-1}A = \frac{1}{2}\begin{pmatrix} -2 & 4 & 0 \\ 0 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(2)

A3 The first row of A is the sum of the last two, so the rank of A is less than 3. Any two rows are clearly linearly independent, since none is a scalar multiple of another. Hence rank $(A) = 2$. 2

The vector x is 3-dimensional, so the solution space of $Ax = 0$ has dimension $3 - 2 = 1$. 2

The augmented matrix $(A, b) = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & -1 & 2 & 1 \end{pmatrix}$.

It is still the case that the first row is the sum of the last two but any two rows are linearly independent, so rank $(A, b) = 2$. 2

Since rank (A) = rank (A, b), the equation $Ax = b$ has a solution. 1

A4(a) $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
$(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$
$a(x, y, z) = (ax, ay, az) \ \forall\, a \in \mathbb{R}$ 3

(b) $0 \in U$ and
$u + v \in U \quad \forall\, u, v \in U$ and
$ku \in U \quad \forall\, k \in \mathbb{K},\ u \in U$. 3

(c) $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ 1

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A5(a) $\sum_i k_i v_i, \quad k_i \in \mathbb{K}, \quad v_i \in V$ 1

(b) $S \subset V$ is a spanning set for V if every vector in V is a linear combination of vectors in S. 1

(c) Suppose

$(x,y,z) = a(1,2,3) + b(2,3,1) + c(3,1,2) + d(1,1,1)$. 1

Then
$$\left.\begin{aligned} x &= a + 2b + 3c + d \quad (1) \\ y &= 2a + 3b + c + d \\ z &= 3a + b + 2c + d \end{aligned}\right\} \quad 1$$

Eliminate a: $y - 2x = -b - 5c - d \quad (2)$.

$z - 3x = -5b - 7c - 2d$

Eliminate b: $(z - 3x) - 5(y - 2x) = 18c + 3d$

$\Rightarrow 18c = 7x - 5y + z - 3d$

From (2), $18b = -5(18c) - 18d - 18(y - 2x)$

$= -35x + 25y - 5z + 15d - 18d - 18y + 36x$

$= x + 7y - 5z - 3d$

From (1), $18a = 18x - 2(18b) - 3(18c) - 18d$

$= 18x - 2x - 14y + 10z + 6d$

$\quad - 21x + 15y - 3z + 9d - 18d$

$= -5x + y + 7z - 3d$

Hence, for any values of x, y, z and d, a, b, c can be found, so any $(x,y,z) \in \mathbb{R}^3$ is a linear combination of the vectors in the set.

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A6 (a) S is linearly dependent if $\sum_i k_i v_i = 0$, $v_i \in S$, has a solution for the $k_i \in \mathbb{K}$ other than $k_i = 0 \; \forall i$ ②

(b) Suppose S is linearly dependent and $k_j \neq 0$. Then $v_j = \frac{1}{k_j} \sum_{i \neq j} k_i v_i$. ②

Suppose $v_j = \sum_{i \neq j} k_i v_i$. Then $\sum_i k_i v_i = 0$ where $k_j = -1$. ②

(c) $(1,2,3) + (2,3,1) + (3,1,2) = (6,6,6) = 6(1,1,1)$. i.e. $(1,1,1)$ is a linear combination of the other vectors in the set, so by part (b) the set is linearly dependent. ①

A7 (a) $\alpha(u+v) = \alpha(u) + \alpha(v) \; \forall u, v \in U$ and $\alpha(ku) = k\alpha(u) \; \forall u \in U, k \in \mathbb{K}$. ②

(b) $\beta\alpha : U \to T$, $(\beta\alpha)(u) = \beta(\alpha(u)) \; \forall u \in U$ provided $V = S$ ②

(c) $(\beta\alpha)(u+v) = \beta(\alpha(u+v))$
$= \beta(\alpha(u) + \alpha(v))$ by linearity of α
$= \beta(\alpha(u)) + \beta(\alpha(v))$ by linearity of β
$= (\beta\alpha)(u) + (\beta\alpha)(v)$.

$(\beta\alpha)(ku) = \beta(\alpha(ku))$
$= \beta(k\alpha(u))$ by linearity of α
$= k\beta(\alpha(u))$ by linearity of β
$= k(\beta\alpha)(u)$. ③

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A8 (a) $\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i$

$\| x \| = \sqrt{\langle x, x \rangle}$ or $\sqrt{\sum_{i=1}^{3} x_i^2}$ 2

(b) $\{x_i\}$ is an orthonormal set of vectors if $\langle x_i, x_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\forall\, i, j$. 2

(c) $(x, y, z) \perp (1,1,1)$ if $x + y + z = 0$. (1)

So a choice for v is $(1, -1, 0)$. 1

$(x, y, z) \perp (1, -1, 0)$ if $x - y = 0$ (2)

A vector $\perp$ $(1,1,1)$ and $(1,-1,0)$ must satisfy (1) and (2) simultaneously.

(1) $\Rightarrow$ $y = x$ and (2) $\Rightarrow$ $z = -2x$.

So a choice for w is $(1, 1, -2)$. 1

Normalizing u, v, w gives the orthonormal set

$$\left\{ \frac{1}{\sqrt{3}}(1,1,1),\ \frac{1}{\sqrt{2}}(1,-1,0),\ \frac{1}{\sqrt{6}}(1,1,-2) \right\}$$

1

Section B

B1 (a) $\ker(\alpha) = \{u \in U \mid \alpha(u) = 0\}$ 1
$\operatorname{im}(\alpha) = \{\alpha(u) \mid u \in U\}$ 1

$\alpha(0) = 0 \Rightarrow 0 \in \ker(\alpha)$. 1
Let $u, v \in \ker(\alpha) \Rightarrow \alpha(u) = \alpha(v) = 0$.
Then $\alpha(u+v) = \alpha(u) + \alpha(v) = 0$
$\Rightarrow u + v \in \ker(\alpha)$. 1
Also let $k \in \mathbb{K}$. Then $\alpha(ku) = k\alpha(u) = 0$
$\Rightarrow ku \in \ker(\alpha)$. 1

$0 \in U$ and $\alpha(0) = 0 \Rightarrow 0 \in \operatorname{im}(\alpha)$. 1
Let $u, v \in \operatorname{im}(\alpha) \Rightarrow \exists\, u', v' \in U$
st $u = \alpha(u')$, $v = \alpha(v')$.
$u + v = \alpha(u') + \alpha(v') = \alpha(u' + v')$
$u' + v' \in U \Rightarrow u + v \in \operatorname{im}(\alpha)$. 1
Also let $k \in \mathbb{K}$. Then $ku = k\alpha(u') = \alpha(ku')$.
$ku' \in U \Rightarrow ku \in \operatorname{im}(\alpha)$. 1

(b) $S + T = \{s + t \mid s \in S, t \in T\}$ 1
$S \oplus T = S + T$ subject to $S \cap T = \{0\}$ 1

(c) $\ker(\alpha)$ and $\operatorname{im}(\alpha)$ are both vector subspaces of U. The two dimension theorems imply that $\dim U = \dim \ker(\alpha) + \dim \operatorname{im}(\alpha)$ and $\dim(\ker(\alpha) \oplus \operatorname{im}(\alpha)) = \dim \ker(\alpha) + \dim \operatorname{im}(\alpha)$ provided the sum is direct, i.e. $\ker(\alpha) \cap \operatorname{im}(\alpha) = \{0\}$.
Therefore $\dim U = \dim(\ker(\alpha) \oplus \operatorname{im}(\alpha))$.

But $\ker(\alpha) \oplus \operatorname{im}(\alpha)$ is a vector subspace of U, so $U = \ker(\alpha) \oplus \operatorname{im}(\alpha)$. 4

(d) $\ker(\alpha) = \{(x, y, z) \mid x + y = 0,\ y - z = 0,\ z + x = 0\}$
The constraints imply $y = -x$, $z = -x$, $z = y$.
Hence $\ker(\alpha) = \{(x, -x, -x) \mid x \in \mathbb{R}\}$ so a basis set for $\ker(\alpha)$ is $\{(1, -1, -1)\}$ 2

$$\operatorname{im}(\alpha) = \{(x + y, y - z, z + x) \mid x, y, z \in \mathbb{R}\}$$
$$= \{x(1, 0, 1) + y(1, 1, 0) + z(0, -1, 1) \mid x, y, z \in \mathbb{R}\}$$
$$= \langle (1, 0, 1), (1, 1, 0), (0, -1, 1) \rangle.$$
Hence a basis set for $\operatorname{im}(\alpha)$ is $\{(1, 0, 1), (1, 1, 0)\}$. 2

Vectors in $\ker(\alpha) \cap \operatorname{im}(\alpha)$ must have the form
$a(1, -1, -1) = b(1, 0, 1) + c(1, 1, 0)$, $a, b, c \in \mathbb{R}$
Hence
$$\left.\begin{aligned} a &= b + c \\ -a &= c \\ -a &= b \end{aligned}\right\} \Rightarrow a = b = c = 0$$
Hence $\ker(\alpha) \cap \operatorname{im}(\alpha) = \{0\}$ and $\ker(\alpha) + \operatorname{im}(\alpha)$ is a direct sum. 2

$\ker(\alpha) \oplus \operatorname{im}(\alpha) = \langle (1, -1, -1), (1, 0, 1), (1, 1, 0) \rangle$
$(1, -1, -1) + (1, 0, 1) + (1, 1, 0) = 3(1, 0, 0)$. Hence
$$\ker(\alpha) \oplus \operatorname{im}(\alpha) = \langle (1, 0, 0), (1, 0, 1), (1, 1, 0) \rangle$$
$$= \langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle$$
$$= \mathbb{R}^3.$$ 2

Alternatively, show that
$\{(1, -1, -1), (1, 0, 1), (1, 1, 0)\}$ spans $\mathbb{R}^3$.

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B2(a) If $\alpha(u_j) = \sum_{i=1}^{n} a_{ij} v_i$ then A is the rectangular array of coefficients with $a_{ij} \in \mathbb{K}$ in row i and column j. (2)

(b)
$$\begin{aligned}
\gamma(u_1) &= \beta(\alpha(u_1)) = \beta(v_1 + 2v_2 + v_3)\\
&= \beta(v_1) + 2\beta(v_2) + \beta(v_3)\\
&= (w_1 + 2w_2) + 2(w_1 - w_2) + (-w_1 + w_2)\\
&= 2w_1 + w_2\\
\gamma(u_2) &= \beta(v_1 - v_3) = (w_1 + 2w_2) - (-w_1 + w_2)\\
&= 2w_1 + w_2\\
\gamma(u_3) &= \beta(-v_1 + v_2 - v_3)\\
&= -(w_1 + 2w_2) + (w_1 - w_2) - (-w_1 + w_2)\\
&= w_1 - 4w_2
\end{aligned}$$
(3)

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -4 \end{pmatrix}$.

(3)

(d) $C = BA$, the product of the matrices B & A.

$$BA = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -4 \end{pmatrix} = C.$$

(3)

(e)
$$\begin{aligned}
u &= xu_1 + yu_2 + zu_3\\
\gamma(u) &= x\gamma(u_1) + y\gamma(u_2) + z\gamma(u_3)\\
&= x(2w_1 + w_2) + y(2w_1 + w_2) + z(w_1 - 4w_2)\\
&= (2x + 2y + z)w_1 + (x + y - 4z)w_2
\end{aligned}$$

Thus $\gamma(x, y, z) = (2x + 2y + z, x + y - 4z)$ (3)

or $C\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 2y + z \\ x + y - 4z \end{pmatrix}$

(2)

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(f)

$$\begin{tikzcd}
U_{\mathcal{B}} \arrow[r, "A", "\alpha"'] \arrow[rr, bend left, "C"] \arrow[d, "P"'] & V_{\mathcal{C}} \arrow[r, "B", "\beta"'] \arrow[d, "Q"'] & W_{\mathcal{D}} \arrow[d, "R"] \\
U_{\mathcal{B}'} \arrow[r, "\alpha", "A'"'] \arrow[rr, bend right, "C'"'] & V_{\mathcal{C}'} \arrow[r, "\beta", "B'"'] & W_{\mathcal{D}'}
\end{tikzcd}$$

3

Let P, Q, R be the matrix representations of the identity maps on U, V, W between the bases shown. Then

$A' = QAP^{-1}$, $B' = RBQ^{-1}$, $C' = RCP^{-1}$.

$C = BA$ and $B'A' = RBQ^{-1} . QAP^{-1}$

$= RBAP^{-1} = RCP^{-1} = C'$.

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B3 (a) A basis set is a linearly independent spanning set. 2

The dimension is the number of elements in a basis set. 2

(b) The row rank is the dimension of the vector space spanned by the rows as vectors. 2

(c) (i) Interchange two rows.
(ii) Add a multiple of one row to another row.
(iii) Multiply a row by a non-zero scalar. 3

(d) (i) The order of the vectors in a spanning set is not significant. 1

(ii) Let $\{v_i\}$ be the set of row vectors and let k_i be scalars. Suppose $k v_j$ is added to v_i where k is a scalar. The linear combination

$$k_1 v + \dots k_i (v_i + k v_j) + \dots + k_j v_j + \dots$$

is the same as 4

$$k_1 v_1 + \dots + k_i v_i + \dots + (k_i k + k_j) v_j + \dots$$

Hence any linear combination of the new vectors is also a linear combination of the old ones, so both sets span the same space.

(iii) The linear combination $\dots k_i (k v_i) \dots$ is the same as $\dots (k_i k) v_i \dots$, so multiplying a vector by a non-zero scalar can be compensated by adjusting the coefficients and so does not change the space spanned. 2

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(e) Put the vectors into a matrix and use elementary row operations to reduce it to echelon form. The resulting nonzero vectors form a basis for the row space.

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 3 \\ 2 & -1 & 3 & 5 & 0 \\ 2 & 5 & -5 & 0 & -4 \\ -1 & 3 & 1 & -4 & 3 \\ 3 & 1 & 7 & 6 & 3 \end{bmatrix} \xrightarrow{\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \\ R_5 \to R_5 - 3R_1 \end{array}} \begin{bmatrix} 1 & 2 & 4 & 1 & 3 \\ 0 & -5 & -5 & 3 & -6 \\ 0 & 1 & -13 & -2 & -10 \\ 0 & 5 & 5 & -3 & 6 \\ 0 & -5 & -5 & 3 & -6 \end{bmatrix}$$

There is no need to take this any further. It is now obvious the $R_4 = -R_2$ and $R_5 = R_2$ and that the first three rows are linearly independent. Hence a suitable basis set is

$$\{(1,2,4,1,3),(0,-5,-5,3,-6),(0,1,-13,-2,-10)\}$$

6

B4 (a) An orthogonal matrix R satisfies
$RR^T = R^T R = I$.
The standard inner product in $\mathbb{R}^n$ is
$\langle x, y \rangle = x^T y$.
Then $\langle Rx, Ry \rangle = (Rx)^T (Ry) = x^T R^T R y$
$= x^T y = \langle x, y \rangle$.

2

3

(b) Let A be a real symmetric matrix with eigenvectors x, y and corresponding eigenvalues λ, μ, so $Ax = \lambda x$, $Ay = \mu y$.
Then $y^T A x = \lambda y^T x$ and $x^T A y = \mu x^T y$.
Transposing the second equation, using $A^T = A$, and subtracting gives $0 = (\lambda - \mu) y^T x$.
Distinct eigenvalues means $\lambda \neq \mu$, hence $y^T x = 0$ and the eigenvectors are orthogonal.

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(c) If an $n \times n$ real symmetric matrix A has n distinct eigenvalues λ_i then it has n corresponding orthogonal eigenvectors x_i.
Normalize the eigenvectors and put them into the columns of an $n \times n$ matrix R, which is therefore an orthogonal matrix.
$Ax_i = \lambda x_i \Rightarrow AR = RA'$ where A' is a diagonal matrix with λ_i in sequence on the principal diagonal. Then
$R^T A R = A'$ is diagonal since $R^{-1} = R^T$.
A' is the matrix A under a change of basis corresponding to the orthogonal matrix R.

4

(d) $\begin{vmatrix} -4-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(\lambda^2 + 5\lambda + 4 - 4) = 0$

$\Rightarrow (3-\lambda)\lambda(\lambda+5) = 0$

$\Rightarrow \underline{\lambda = 3, 0, -5}$ 3

(e) Find the eigenvector corresponding to each eigenvalue and normalize it.

$\lambda = 3$: $\begin{pmatrix} -7 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\Rightarrow \left.\begin{array}{r} -7x + 2y = 0 \\ x - 2y = 0 \end{array}\right\} \Rightarrow x = y = 0, \; z = \text{anything}.$

Hence a normal eigenvector is $(0, 0, 1)$. 1

$\lambda = 0$: $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\Rightarrow \left.\begin{array}{r} 2x - y = 0 \\ 3z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} y = 2x \\ z = 0 \end{array}$

Hence an eigenvector is $(1, 2, 0)$ 1
and a normal eigenvector is $(1, 2, 0)/\sqrt{5}$

$\lambda = -5$: $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\Rightarrow \left.\begin{array}{r} x + 2y = 0 \\ 8z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = -2y \\ z = 0 \end{array}$

Hence an eigenvector is $(-2, 1, 0)$ 1
and a normal eigenvector is $(-2, 1, 0)/\sqrt{5}$.

Hence $R = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & 1 \\ \sqrt{5} & 0 & 0 \end{pmatrix}$ 1, $A' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$ 1

<u>Optional check</u> — no marks:

$$R^T A R = \frac{1}{5}\begin{pmatrix} 0 & 0 & \sqrt{5} \\ 1 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & 1 \\ \sqrt{5} & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 0 & 0 & \sqrt{5} \\ 1 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 10 \\ 0 & 0 & -5 \\ 3\sqrt{5} & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -25 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix} \checkmark$$