CHAPTER 5

MECHANICAL SPRINGS

Chapter Outline

(A) LEAF SPRINGS

1) Semi-Elliptical Leaf Springs
2) Quarter-Elliptical Leaf Springs

(B) HELICAL SPRINGS

1) Closed-coiled Helical Springs
2) Open-coiled Helical Springs

Introduction

Springs are energy-absorbing elements whose function is to store energy and release it slowly or rapidly depending on the particular application.

Types of Springs

(1) Carriage springs or leaf springs

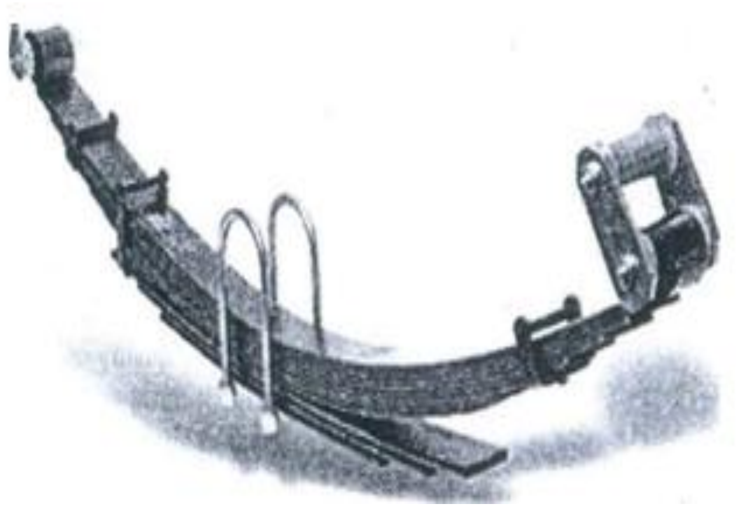
a) semi-elliptical types (*i.e.*, simply supported at its ends subjected to central load)
b) quarter-elliptical types (*i.e.*, cantilever)

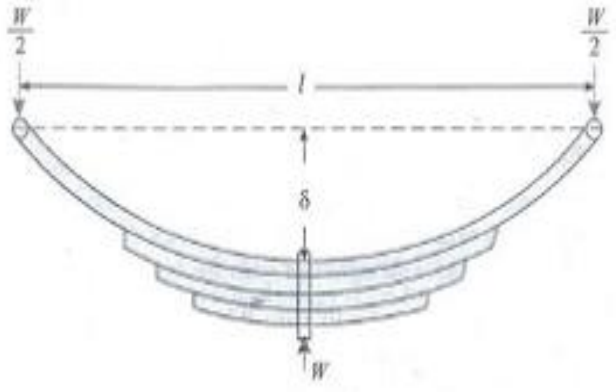
(2) Helical springs

a) Closely-coiled helical springs and
b) Open-coiled helical springs

LEAF SPRINGS

Semi-Elliptic Leaf Springs





Let

l = Span of the spring,

t = Thickness of plates,

b = Width of the plates,

n = Number of plates,

W = Load acting on the spring

σ = Maximum bending stress developed in the plates

δ = Original deflection of the top spring and

R = Radius of the spring

Semi-Elliptic Leaf Springs

The bending moment, at the centre of the span due to this load $M = \frac{Wl}{4} \cdots\cdots \text{(i)}$

Moment resisted by one plate $M_i = \frac{\sigma . I}{y} == \frac{\sigma . bt^2}{6}$

The total moment resisted by *n* plates, $M = M_i \times n = \frac{n\sigma bt^2}{6} \cdots\cdots \text{(ii)}$

Equating (i) and (ii), $\frac{Wl}{4} = \frac{n\sigma bt^2}{6}$

$$\Rightarrow \sigma = \frac{3Wl}{2nbt^2}$$

From the geometry, the central deflection, $\delta = \frac{l^2}{8R} \cdots\cdots \text{(iii)}$

Semi-Elliptic Leaf Springs

For a bending beam, $$\frac{\sigma}{y} = \frac{E}{R}$$

$$\Rightarrow R = \frac{E.y}{\sigma} = \frac{Et}{2\sigma}$$

Substituting this value of *R* in equation *(iii)*,

$$\delta = \frac{\sigma l^2}{4Et}$$

Substituting the value of σ in the above equation

$$\delta = \left(\frac{3Wl}{2nbt^2}\right)\left(\frac{l^2}{4Et}\right) = \frac{3Wl^3}{8Enbt^3}$$

Example 1

A laminated spring 1 m long is made up of plates each 50 mm wide and 10 mm thick. If the bending stress in the plates is limited to 100 MPa, how many plates are required to enable the spring to carry a central point load of 2 kN. If modulus of elasticity for the spring material is 200 GPa, what is the deflection under the load?

Solution

Given: Length (l) = 1 m= 1 x 10^3 mm; Width (b) =50 mm; Thickness (t) = 10 mm Bending stress (σ_b) = 100 MPa = 100 N/mm^2; Central point load (W) = 2 kN = 2 x 10^3 N and modulus of elasticity (E) = 200 GPa = 200 x 10^3 N/mm^2

No. of plates required in the spring

$$100 = \frac{3Wl}{2nbt^2} = \frac{3(2000)(1000)}{2n(50)(10)^2} = \frac{600}{n}$$

$$\Rightarrow n = \frac{600}{100} = 6$$

Example 1 (continued)

Deflection under the load

$$\delta = \frac{3Wl^3}{8Enbt^3}$$

$$= \frac{3(2000)(1000)^3}{8(200\times10^3)(6)(50)(10)^3} = 12.5\ mm$$

Example 2

A leaf spring is to be made of seven steel plates 65 mm wide and 6.5 mm thick. Calculate the length of the spring, so that it may carry a central load of 2.75 kN, the bending stress being limited to 160 MPa. Also calculate the deflection at the centre of the spring. Take E for the spring material as 200 GPa.

Solution

Given: No. of plates (n) = 7; Width (b) = 65 mm; Thickness (t) = 6.5 mm; Central load (W) = 2.75 kN = 2.75 x 10^3 N; Maximum bending stress (σ_b) = 160 MPa = 160 N/mm2 and modulus of elasticity for the spring material (E) = 200 GPa = 200 x 10^3 N/mm^2

Length of the spring

$$160 = \frac{3Wl}{2nbt^2} = \frac{3(2750)l}{2(7)(65)(6.5)^2} = 0.215l$$

$$\Rightarrow l = \frac{160}{0.215} = 744.2\ mm$$

Deflection at the centre of the spring

$$\delta = \frac{3Wl^3}{8Enbt^3}$$

$$= \frac{3(2750)(744.2)^3}{8(200x10^3)(7)(65)(6.5)^3} = 17\ mm$$

Example 3

A leaf spring 750 mm long is required to carry a central point load of 8 kN. If the central deflection is not to exceed 20 mm and the bending stress is not greater than 200 MPa, determine the thickness, width and number of plates. Also, compute the radius, to which the plates should be curved. Assume width of the plate equal to 12 times its thickness and E equal to 200 GPa

Solution

Given: Length (l) = 750 mm; Point load (W) = 8 kN = 8×10^3 N; Central deflection (δ) = 20 mm; Bending stress (σ_b) = 200 MPa == 200 N/mm^2; Width of plates (b) = 12t (where t is the thickness of the plates) and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2

Thickness of the plates

$$20 = \frac{\sigma l^2}{4Et} = \frac{(200)(750)^2}{4(200 \times 10^3)t} = \frac{140.6}{t}$$

$$\Rightarrow \; t = \frac{140.6}{20} = 7.0\, mm$$

Example 3 (continued)

Width of plate

$$b = 12t = 12(7) = 84.0\ mm$$

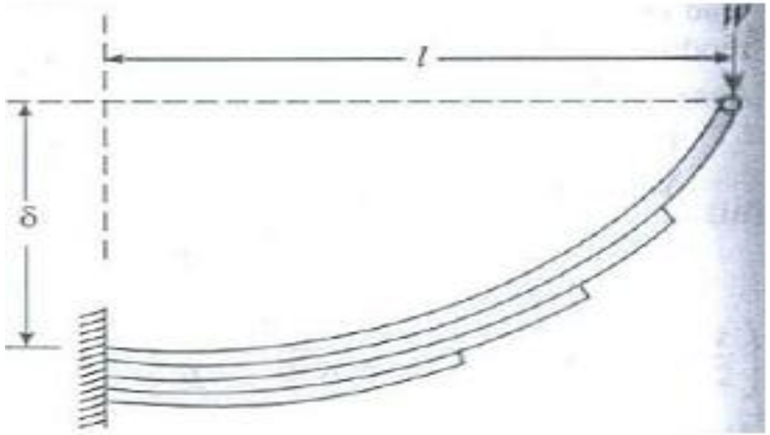
Number of plates

$$200 = \frac{3Wl}{2nbt^2} = \frac{3(8000)(750)}{2n(84)(7)^2} = \frac{2187}{n} \Rightarrow n = \frac{2187}{200} = 10.9 \approx 11$$

The radius of plates

$$R = \frac{Et}{2\sigma} = \frac{\left(200x10^3\right)(7)}{2(200)} = 3500\,mm$$

Quarter-Elliptical Leaf Spring



Quarter-elliptical spring

Let	l	Length of the spring,
	t	Thickness of the plates,
	b	Width of the plates,
	n	Number of plates,
	W	Load acting at the free end of the spring
	δ	Original deflection of the spring.

- Bending moment at the fixed end of the leaf $\quad M = Wl \ldots\text{(i)}$
- Moment resisted by one plate $\quad M_i = \dfrac{\sigma.I}{y} = \dfrac{\sigma.bt^2}{6}$
- The total moment resisted by n plates,

$$M = M_i.n = \frac{n\sigma bt^2}{6} \ldots(ii)$$

Quarter-Elliptical Leaf Spring

Equating (i) and (ii)

$$Wl = \frac{n\sigma bt^2}{6}$$

$$\Rightarrow \sigma = \frac{6Wl}{nbt^2}$$

From the geometry

$$\delta = \frac{l^2}{2R} \ldots (iii)$$

But

$$R = \frac{Et}{2\sigma}$$

Therefore

$$\delta = \frac{l^2}{2\left(Et/2\sigma\right)} = \frac{\sigma l^2}{Et}$$

Hence

$$\delta = \frac{\sigma l^2}{Et} = \frac{6Wl^3}{Enbt^3}$$

Example 4

A quarter-elliptic leaf spring 800 mm long is subjected to a point load of 10 kN. If the bending stress and deflection is not to exceed 320 MPa and 80 mm respectively, find the suitable size and number of plates required by taking the width as 8 times the thickness. Take E as 200 GPa.

Solution

Given: Length (I) = 800 mm; Point load (W) = 10 kN = 10×10^3 N; Bending stress (σ_b) = 320 MPa = 320 N/mm^2; Deflection (δ) = 80 mm; Plate width b = 8t (where t is the thickness of the plates) and modulus of elasticity (E) = 200 GPa = 200 x 10^3 N/mm^2

Thickness of the plates

Let *t* denote the thickness of the plates in mm, and *n* the number of the plates

bending stress $$320 = \frac{6Wl}{nbt^2} = \frac{6\left(10\times10^3\right)\left(800\right)}{nbt^2} = \frac{48\times10^6}{nbt^2} \ldots(i)$$

Example 4 (continued)

For deflection

$$80 = \frac{6Wl^3}{Enbt^3} = \frac{6\left(10\times10^3\right)\left(800\right)^3}{\left(2\times10^3\right)nbt^3} = \frac{153.6\times10^6}{nbt^3} \quad \ldots(ii)$$

Dividing equation (ii) by (i),

$$\frac{80}{320} = \left[\frac{153.6x10^6}{nbt^3}\right]\left[\frac{nbt^2}{48x10^6}\right] = \frac{3.2}{t} \Rightarrow t = \frac{3.2\left(320\right)}{80} = 13\,mm$$

Width of plates $b = 8t = 8\left(13\right) = 104\,mm$

N*umber of plates required*

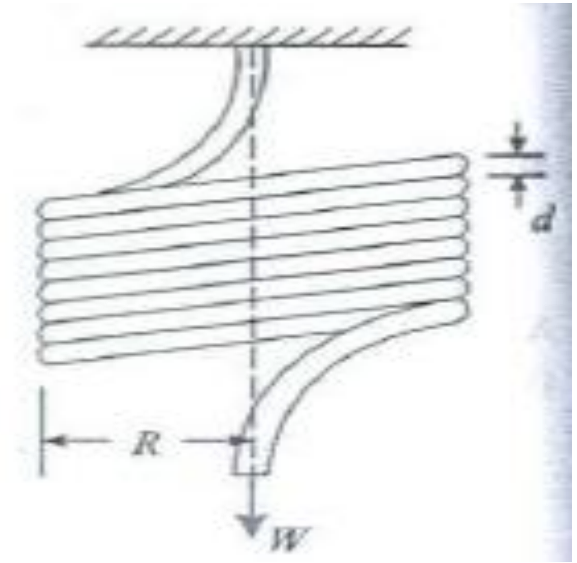
$$320 = \frac{48\times10^6}{nbt^2}$$

$$\Rightarrow\ n = \frac{48\times10^6}{\left(320\right)\left(104\right)\left(13\right)} = 8.5 \approx 9$$

HELICAL SPRINGS

Closely-Coiled Helical Springs

Subjected to an Axial Load



Let

d = Diameter of the spring wire,
R = Mean radius of the spring coil
n = No. of turns of coils,
G = Modulus of rigidity for the spring material,
W = Axial load on the spring
τ = Maximum shear stress induced in the wire due to twisting,
θ = Angle of twist in the spring wire
δ = Deflection of the spring, as a result of the axial load

Twisting moment $T = WR \ldots (i)$

$$T = \frac{\pi}{16}.\tau.d^3 \ldots (ii)$$

Equating (i) and (ii) $W.R = \frac{\pi}{16}.\tau.d^3$

Closely-Coiled Helical Springs

From geometry $l = 2\pi R.n$

Torsion of circular shafts $\frac{T}{J} = \frac{G\theta}{l}$

This implies

$$\theta = \frac{Tl}{JG} = \frac{WR.2\pi Rn}{\frac{\pi}{32} \times d^4 G} = \frac{64WR^2 n}{Gd^4}$$

Deflection of the spring

$$\delta = R\theta = \frac{64WR^3 n}{Gd^4}$$

Energy stored in the spring

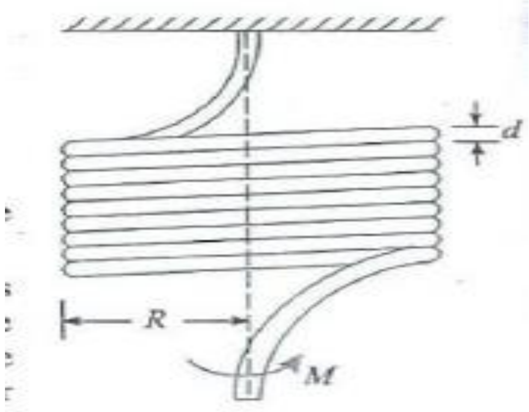
$$U = \tfrac{1}{2} W\delta$$

Stiffness of the spring

$$s = \frac{W}{\delta} = \frac{Gd^4}{64R^3 n}$$

Closely-Coiled Helical Springs

Subjected to an Axial Twist



Let	d	Diameter of the spring wire,
	R	Mean radius of the spring coil,
	n	No. of turns of coils,
	e	Modulus of rigidity for the spring material
	M	Moment or axial twist applied on the spring.

Length of the spring, $$l = 2\pi Rn = 2\pi R'n' ...(i)$$

Therefore,

$$\frac{1}{R} = \frac{2\pi n}{l} \qquad \frac{1}{R'} = \frac{2\pi n'}{l}$$

$$\frac{M}{I} = E \ x \ Change \quad of \ curvature$$

$$= E\left(\frac{1}{R'} - \frac{1}{R}\right) = E\left[\frac{2\pi n'}{l} - \frac{2\pi n}{l}\right]$$

$$= \frac{2\pi E}{l}(n' - n)$$

Closely-Coiled Helical Springs

Therefore $$2\pi(n'-n)=\frac{Ml}{EI}\ldots(ii)$$

The total angle of bend $$\phi = 2\pi(n'-n)=\frac{Ml}{EI}$$ $$\frac{d\phi}{dl}=\frac{M}{EI}$$

The energy stored in the spring,

$$U=\tfrac{1}{2}M.\phi$$

Example 5

A close-coiled helical spring is required to carry a load of 150 N. If the mean coil diameter is to be 8 times that of the wire, calculate these diameters. Take maximum shear stress as 100 MPa.

Solution

Given: Load (W) = 150 N; Diameter of coil (D) = 8d (where d is the diameter of the wire) or radius (R)= 4 d and maximum shear stress (τ) = 100 MPa = 100 N/mm^2

We know that relation for the twisting moment, $$W.R = \frac{\pi}{16}.\tau.d^3$$

This implies, $$150 \times 4d = \frac{\pi}{16} \times 100 \times d^3 \quad \therefore d^2 = \frac{150 \times 4 \times 16}{100\pi} = 30.6$$

Hence, $$d = \sqrt{30.6} = 5.53 \approx 6\ mm$$

and $$D = 8d = 8(6) = 48\ mm$$

Example 6

A closely. coiled helical spring of round steel wire 5 mm in diameter having 12 complete coils of 50 mm mean diameter is subjected to an axial load of 100 N. Find the deflection of the spring and the maximum shearing stress in the material. Modulus of rigidity (G) = 80 GPa

Solution

Given: Diameter of spring wire (d) = 5 mm; No. of coils (n) = 12; Diameter of spring (D) = 50 mm or radius (R) = 25 mm; Axial load (W) = 100 N and Modulus of rigidity (G) = 80 GPa = 80 x 10^3 N/mm^2

Deflection of the spring

$$\delta = \frac{64WR^3n}{Cd^4} = \frac{64(100)(25)^3(12)}{(80x10^3)(5)^4}$$

$$= 24\ mm$$

Maximum shearing stress in the material

$$W.R = \frac{\pi}{16}.\tau.d^3$$

$$100x.25 = \frac{\pi}{16}.x\tau x.(5)^3$$

$$\therefore \tau = \frac{2500}{24.54} = 101.9\ N/mm^2$$

Example 7

A closely-coiled helical spring is made lip of 10 nun diameter steel wire having 10 coils with 80 mm mean diameter. If the spring is subjected 10 an axial twist of 10 kNmm, determine the bending stress and increase ill the number of turns. Take E as 200 GPa

Solution

Given: Diameter of spring wire (d) = 10 mm; No. of coils (n) = 10; Diameter of coil (D) = 80 mm or radius (R) = 40 mm; Axial twist (M) = 10 kN -mm = 10 x 10^3 N-mm and Modulus of elasticity (E) = 200 GPa = 200 x 10^3 N/mm^2

Moment of inertia $I = \frac{\pi}{64} x d^4 = \frac{\pi}{64} x (10)^4 = 490.9\ mm^4$

Bending stress in file wire

$$\sigma = \frac{M}{I}.y = \frac{(10x10^3)}{490.9} x(5) = 101.9\ N/mm^2$$

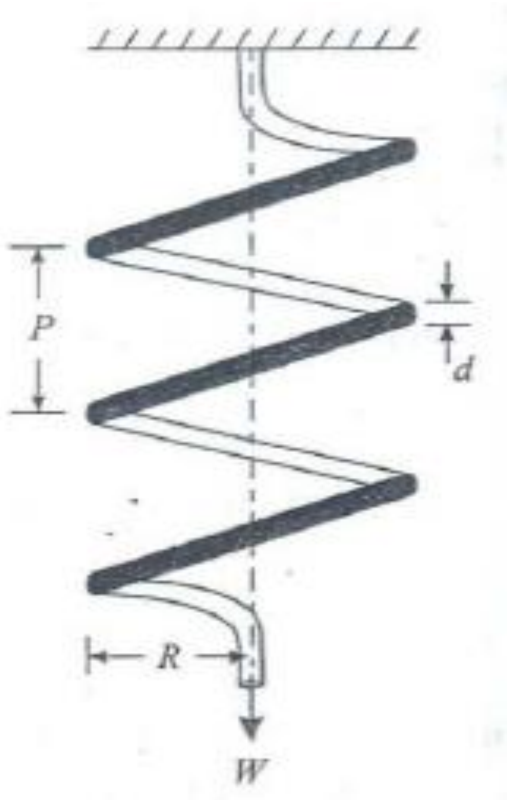
Example 7 (continued)

Increase in the number of turns

$$l = 2\pi Rn = 2\pi \times 40 \times 10 = 800\pi \ mm$$

$$(n' - n) = \frac{Ml}{EI} \times \frac{1}{2\pi} = \frac{(10 \times 10^3)(800\pi)}{(200 \times 10^3)(490.9)} \times \frac{1}{2\pi} = 0.04mm$$

Open-Coiled Helical Springs



Let

d	Diameter of the spring wire,
R	Mean radius of the spring coil,
P	Pitch of the spring coils,
n	No. of turns of coils,
G	Modulus of rigidity for the spring material
W	Axial load on the spring,
τ	Maximum shear stress induced in the spring wire due to loading,
σ_b	Bending stress induced in the spring wire due to bending,
δ	Deflection of the spring as a result of axial load and
α	Angle of helix.

Open-Coiled Helical Springs

Bending moment

Causes twisting of coils $T = WR\cos\alpha$

Causes bending of coils $M = WR\sin\alpha$

Length of the spring wire

$$l = 2\pi Rn\sec\alpha \ldots (i)$$

Twisting moment

$$WR\cos\alpha = \frac{\pi}{16}\times\tau\times d^3 \ldots (ii)$$

Bending stress

$$\sigma_b = \frac{M}{I}.y = \frac{WR\sin\alpha.\frac{d}{2}}{\frac{\pi}{64}\times d^4}$$

$$= \frac{32WR\sin\alpha}{\pi d^3} \ldots (iii)$$

Angle of twist

$$\theta = \frac{Tl}{JG} = \frac{WR\cos\alpha.l}{JG}$$

Angle of bend due to bending moment

$$\phi = \frac{Ml}{EI} = \frac{WR\sin\alpha.l}{EI}$$

Open-Coiled Helical Springs

The work done by the load in deflecting the spring, is equal to the strain energy of the spring.

Therefore, $\frac{1}{2}W\delta = \frac{1}{2}T\theta + \frac{1}{2}M\phi \Rightarrow W.\delta = T\theta + M\phi$

Hence, $\delta = WR^2 l = \frac{1}{W}\left[\frac{\cos^2\alpha}{JG} + \frac{\sin^2\alpha}{EI}\right]$

Or $$\delta = \frac{T\theta + M\phi}{W}$$

$$= \frac{1}{W}\left\{\left[\left(WR\cos\alpha\right)\left(\frac{WR\cos\alpha.l}{JG}\right)\right] + \left[\left(WR\sin\alpha\right)\left(\frac{WR\sin\alpha.l}{EI}\right)\right]\right\}$$

Open-Coiled Helical Springs

Now substituting the values of $l = 2\pi Rn\sec\alpha$ $J = \frac{\pi}{32}(d)^4$ $I = \frac{\pi}{64}(d)^4$

in the above equation

$$\delta = WR^2 \times 2\pi nR\sec\alpha = \left[\frac{\cos^2\alpha}{\frac{\pi}{32}d^4G} + \frac{\sin^2\alpha}{\frac{\pi}{64}d^4E}\right]$$

$$= \frac{64WR^3n\sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{\sin^2\alpha}{E}\right]$$

NOTE: If we substitute $\alpha = 0$ in the above equation, it gives deflection of a closed coiled spring

Example 8

An open coil helical spring made up of 10 nun diameter wire and of mean diameter of 100 mm has 12 coils and angle of helix being 15°. Determine the axial deflection and the intensities of bending and shear stresses under an axial load of 500 N. Take C as 80 GPa and E as 200 GPa.

Solution

Given: Diameter of wire (d) = 10 mm; Mean diameter of spring (D) = 100 mm or radius (R) == 50 mm; No. of coils (n) = 12; Angle of helix (α) = 15^o; Load (W) = 500 N; Modulus of rigidity (C) = 80 GPa = 80 x 10^3 N/mm^2 and modulus of elasticity (E) = 200 GPa = 200 x 10^3 N/mm^2

Deflection of the spring

$$\delta == \frac{64x500x(59)^3 x12\sec15^o}{(10)^4}\left[\frac{\cos^2 15^o}{80x10^3}+\frac{\sin^2 15^o}{200x10^3}\right]=61.3\,mm$$

Example 8 (continued)

Bending moment $M = WR\sin\alpha = 500x50\sin 15^o = 6470 \;\; N.mm$

Moment of inertia

$$I = \frac{\pi}{64}(d)^4 = \frac{\pi}{64}(10)^4 = 490.9 \; mm^4$$

The bending stress

$$\sigma_b = \frac{M}{I}.y = \frac{6470}{490.9} \times 5 = 65.9 \; N/mm^2$$

Shear stress induced in the wire

$$T = WR\cos\alpha = 500 \times 50 \cos 15^\circ = 24150 \; N.mm$$

We also know that twisting moment (T),

$$24150 = \frac{\pi}{16} \times \tau \times d^3 = \frac{\pi}{16} \times \tau \times (10)^3 = 196.4\tau \Rightarrow \tau = \frac{24150}{196.4} = 123 \; N/mm^2$$