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# Numerical Analysis – Lecture 8

## 6.3 The error of polynomial interpolation

Let $[a,b]$ be a closed interval of $\mathbb{R}$. We denote by $C[a,b]$ the space of all continuous functions from $[a,b]$ to $\mathbb{R}$ and let $C^s[a,b]$, where $s$ is a positive integer, stand for the linear space of all functions in $C[a,b]$ that possess $s$ continuous derivatives.

**Theorem** Given $f \in C^{n+1}[a,b]$, let $p \in \mathbb{P}_n[x]$ interpolate the values $f(x_i)$, $i = 0,1,\ldots,n$, where $x_0,\ldots,x_n \in [a,b]$ are pairwise distinct. Then for every $x \in [a,b]$ there exists $\xi \in [a,b]$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i). \tag{6.1}$$

**Proof.** The formula (6.1) is true when $x = x_j$ for $j \in \{0,1,\ldots,n\}$, since both sides of the equation vanish. Let $x \in [a,b]$ be any other point and define

$$\phi(t) := [f(t) - p(t)] \prod_{i=0}^{n} (x - x_i) - [f(x) - p(x)] \prod_{i=0}^{n} (t - x_i), \qquad t \in [a,b].$$

*[Note: The variable in $\phi$ is $t$, whereas $x$ is a fixed parameter.]* Note that $\phi(x_j) = 0$, $j = 0,1,\ldots,n$, and $\phi(x) = 0$. Hence, $\phi$ has at least $n+2$ distinct zeros in $[a,b]$. Moreover, $\phi \in C^{n+1}[a,b]$.
We now apply the *Rolle theorem:* if the function $g \in C^1[a,b]$ vanishes at two distinct points in $[a,b]$ then its derivative vanishes at an intermediate point. We deduce that $\phi'$ vanishes at (at least) $n+1$ distinct points in $[a,b]$. Next, applying Rolle to $\phi'$, we conclude that $\phi''$ vanishes at $n$ points in $[a,b]$. In general, we prove by induction that $\phi^{(s)}$ vanishes at $n+2-s$ distinct points of $[a,b]$ for $s = 0,1,\ldots,n+1$. Letting $s = n+1$, we have $\phi^{(n+1)}(\xi) = 0$ for some $\xi \in [a,b]$. But $p^{(n+1)} \equiv 0$, $\mathrm{d}^{n+1} \prod_{i=0}^{n} (t - x_i)/\mathrm{d}t^{n+1} \equiv (n+1)!$, and we obtain (6.1). □

**Runge's example** We interpolate $f(x) = 1/(1+x^2)$, $x \in [-5,5]$, at the equally-spaced points $x_j = -5 + 10\frac{j}{n}$, $j = 0,1,\ldots,n$. Some of the errors for $n = 20$ are

| $x$ | $f(x) - p(x)$ | $\prod_{i=0}^{n}(x - x_i)$ |
|---|---|---|
| 0.75 | $3.2 \times 10^{-3}$ | $-2.5 \times 10^{6}$ |
| 1.75 | $7.7 \times 10^{-3}$ | $-6.6 \times 10^{6}$ |
| 2.75 | $3.6 \times 10^{-2}$ | $-4.1 \times 10^{7}$ |
| 3.75 | $5.1 \times 10^{-1}$ | $-7.6 \times 10^{8}$ |
| 4.75 | $4.0 \times 10^{+2}$ | $-7.3 \times 10^{10}$ |

The growth in the error is explained by the product term in (6.1) (the rightmost column of the table). Adding more interpolation points makes the largest error even worse. A remedy to this state of affairs is to cluster points toward the end of the range.

A considerably smaller error is attained for $x_j = 5\cos\frac{(n-j)\pi}{n}$, $j = 0,1,\ldots,n$ (so-called *Chebyshev points*). It is possible to prove that this choice of points minimizes the magnitude of $\max_{x \in [-5,5]} |\prod_{i=0}^{n} (x - x_i)|$.

## 6.4 Divided differences: a definition

Given pairwise-distinct points $x_0, x_1, \ldots, x_n \in [a,b]$, we let $p \in \mathbb{P}_n[x]$ interpolate $f \in C[a,b]$ there. The coefficient of $x^n$ in $p$ is called the *divided difference* and denoted by $f[x_0, x_1, \ldots, x_n]$. We say

that this divided difference is of *degree n*.
We can derive $f[x_0,\dots,x_n]$ from the Lagrange formula,

$$f[x_0,x_1,\dots,x_n]=\sum_{k=0}^{n} f(x_k)\prod_{\substack{\ell=0\\ \ell\neq k}}^{n}\frac{1}{x_k-x_\ell}. \tag{6.2}$$

**Theorem** Let $[\bar{a},\bar{b}]$ be the shortest interval that contains $x_0,x_1,\dots,x_n$ and let $f\in C^n[\bar{a},\bar{b}]$. Then there exists $\xi\in[\bar{a},\bar{b}]$ such that

$$f[x_0,x_1,\dots,x_n]=\tfrac{1}{n!}f^{(n)}(\xi). \tag{6.3}$$

**Proof.** Let $p$ be the interpolating polynomial. The error function $f-p$ has at least $n+1$ zeros in $[\bar{a},\bar{b}]$ and, applying Rolle's theorem $n$ times, it follows that $f^{(n)}-p^{(n)}$ vanishes at some $\xi\in[\bar{a},\bar{b}]$. But $p(x)=\frac{1}{n!}p^{(n)}(\xi)x^n+$ lower order terms (for any $\xi\in\mathbb{R}$), and we deduce (6.3). □

**Application** It is a consequence of the theorem that divided differences can be used to approximate derivatives.

## 6.5 Recurrence relations for divided differences

Our next topic is a useful way to calculate divided differences (and, ultimately, to deduce yet another means to construct an interpolating polynomial). We commence with the remark that $f[x_i]$ is the coefficient of $x^0$ in the polynomial of degree 0 (i.e., a constant) that interpolates $f(x_i)$, hence $f[x_i]=f(x_i)$.

**Theorem** Suppose that $x_0,x_1,\dots,x_{k+1}$ are pairwise distinct, where $k\geq 0$. Then

$$f[x_0,x_1,\dots,x_{k+1}]=\frac{f[x_1,x_2,\dots,x_{k+1}]-f[x_0,x_1,\dots,x_k]}{x_{k+1}-x_0}. \tag{6.4}$$

**Proof.** Let $p,q\in\mathbb{P}_k[x]$ be the polynomials that interpolate $f$ at

$$\{x_0,x_1,\dots,x_k\}\qquad\text{and}\qquad\{x_1,x_2,\dots,x_{k+1}\}$$

respectively and define

$$r(x):=\frac{(x-x_0)q(x)+(x_{k+1}-x)p(x)}{x_{k+1}-x_0}\in\mathbb{P}_{k+1}[x].$$

We readily verify that $r(x_i)=f(x_i)$, $i=0,1,\dots,k+1$. Hence $r$ is the $(k+1)$-degree interpolating polynomial and $f[x_0,\dots,x_{k+1}]$ is the coefficient of $x^{k+1}$ therein. The recurrence (6.4) follows from the definition of divided differences. □