TE 262: Electromagnetic Fields

Examiner: M. S. Ellis, PhD
smellis.coe@knust.edu.gh
057 904 7591

Syllabus

- Introduction to electromagnetics: circuit theory, concept of electromagnetic theory, electromagnetic field quantities, universal EM constants and SI units
- Vector Analysis and Orthogonal Coordinate systems : Gradient, divergence, curl, divergence theorem, stokes theorem, null identities, Helmholtz's theorem
- Electrostatics: Postulates of Electrostatics, Solutions of electrostatics boundary value problems (Poisson, Laplace equations), Scalar electric potential, Dielectric material
- Magnetostatics: Postulates of Electrostatics, Magnetostatics boundary conditions, magnetic vector potential, Poissons equation, magnetic materials
- Electromagnetism: Electric fields, conductors, insulators, capacitance, magnetic field in free space, magnetic effects on iron.
- Maxwell's equation: Differential and integral forms
- EM Waves theory: EM waves in a homogeneous medium, uniform plane waves, skin effect, Poynting vector

Reference material

- Cheng, David, K, *Fundamentals of Engineering Electromagnetics*, Addison-Wesley, 1993

- Hayt, W. H. and Buck, J.A., *Engineering Electromagnetics*, 7th Edition, New York, McGraw-Hill, 2004

- Yi Huang and Kevin Boyle, *Antenna from theory to practice*, John Wiley, 2008

1. The Electromagnetic Model

Overview of electromagnetics

- Electromagnetics is the study of the electric and magnetic phenomena caused by *electric* charges at rest or in motion

- Existence of electric charges was discovered by Greek astronomer and philosopher Thales of Miletus

- He noticed that an amber rod, after being rubbed with silk or wool attracted small bits of paper

- The word electron was derived from the Greek word for amber which is elektron, from which electronics, electricity…etc stems from

- An electric field has two sources: a positive charge and a negative charge

- Moving electric charges produce current which gives rise to a magnetic field

- A field is a spatial distribution of a quantity which may or may not be a function of time

- Therefore a time-varying electric field is accompanied by a magnetic field, and vice versa

- A time-varying electric field coupled with a time-varying magnetic field results in a electromagnetic field

Why is the concept of electromagnetics essential ?

1. **The concept of fields and waves are essential in the explanation of actions at a distance**

- For example, how does an object fall toward the earth surface if there are no elastic strings connecting a free-falling object and the earth ??
- This can be explained by the existence of a gravitational field
- Similarly, the possibilities of satellite communication can be explained only by postulating the existence of electric and magnetic fields

- A mobile phone on transmit sends a message-carrying current at a certain frequency from the source to the antenna ends

- From circuit-theory point of view, the end of the antenna is connected to an open space (open circuit). Hence no current will flow

How is mobile communication possible then ??

- Electromagnetic concepts must be used to explain this phenomenon, since circuit-theory is limited in this regard

Three essential steps are involved in building a theory or model

Step 1: Define some basic quantities relevant to the subject of study

Step 2: Specify the rules of operation (mathematics) of these quantities

Step 3: Postulate some fundamental relations or Laws

For instance in Circuit theory (Circuit model):

Step 1 involves defining basic quantities like resistors, capacitors, inductors, voltages and currents

Step 2 involves defining algebraic rules, differential equations, and Laplace transformations.

Step 3 involves creating laws like Kirchhoff's voltage and current laws

Similarly, in electromagnetic model:

Step 1 will be defining the basic quantities of electromagnetics

Step 2 will be defining the rules of operation. This includes vector algebra and vector calculus

Step 3 will be presenting the fundamental postulates/laws of electromagnetics

- Quantities of an EM model can be divided into source and field quantities

- Source is electric charges at rest or in motion

- Electric charge is the fundamental property of matter and exists in positive or negative integrals of the charge of an electron, *e*

$$e = 1.60 \times 10^{-19} \text{ Coulomb (C)}$$

"Electric charge can neither be created nor destroyed but can be redistributed from one place to another" – ***principle of conservation of electric charge.***

It must be satisfied at all times and under all conditions

Source quantities are:

Volume charge density, surface charge density, and line charge density

$$\rho_v = \lim_{\Delta v \to 0} \Delta q/\Delta v \quad (C/m^3)$$ *Volume charge density*

$$\rho_s = \lim_{\Delta s \to 0} \Delta q/\Delta s \quad (C/m^2)$$ *Surface charge density*

$$\rho_l = \lim_{\Delta l \to 0} \Delta q/\Delta l \quad (C/m)$$ *Line charge density*

Charge densities are point functions because they vary from point to point

Current is the rate of change of charge with respect to time, that is

$$I = dq/dt \quad (C/s \text{ or } A)$$

Current must flow through a fine area, hence it is ***NOT*** *a point function*

- There are **4 fundamental vector field quantities** in electromagnetics:
- *Electric field intensity E, electric flux density D, magnetic flux density B, and magnetic field intensity H*

- Electric field intensity E, is the only vector needed in discussing electrostatics in free space

- Electric flux density (Electric displacement) D, is useful in the study of electric field in a material (medium)

- Magnetic flux density B, is the only vector used in discussing magneto-statics (steady electric currents) in free space

- Magnetic field Intensity H, is useful in the study of magnetic field in material media

TABLE 1-1 FUNDAMENTAL ELECTROMAGNETIC FIELD QUANTITIES

Symbols and Units for Field Quantities	Field Quantity	Symbol	Unit
Electric	Electric field intensity	**E**	V/m
	Electric flux density (Electric displacement)	**D**	C/m^2
Magnetic	Magnetic flux density	**B**	T
	Magnetic field intensity	**H**	A/m

In static, steady or stationary cases (when there is no time variation), the E field quantities E and D and magnetic quantities B and H form two separate vector pairs

In time-dependent cases, electric and magnetic field quantities are coupled. Meaning, a time varying E and D will create B and H, and vice versa

SI units

- In electromagnetics, four main SI units are used:

TABLE 1-2 FUNDAMENTAL SI UNITS

Quantity	*Unit*	*Abbreviation*
Length	meter	m
Mass	kilogram	kg
Time	second	s
Current	ampere	A

- The units in Table 1-1 are derived units that can be expressed in terms of the fundamentals unit; meters, kilograms, seconds, and amperes.

Universal constants

- In the electromagnetic model, there are 3 universal constants in addition to the field quantities, and SI units
- They relate to properties of free space (vacuum)
- These constants are: speed of an EM wave ***c***, permittivity of free space $\boldsymbol{\varepsilon_0}$, and permeability of free space $\boldsymbol{\mu_0}$

TABLE **1-3** UNIVERSAL CONSTANTS IN SI UNITS

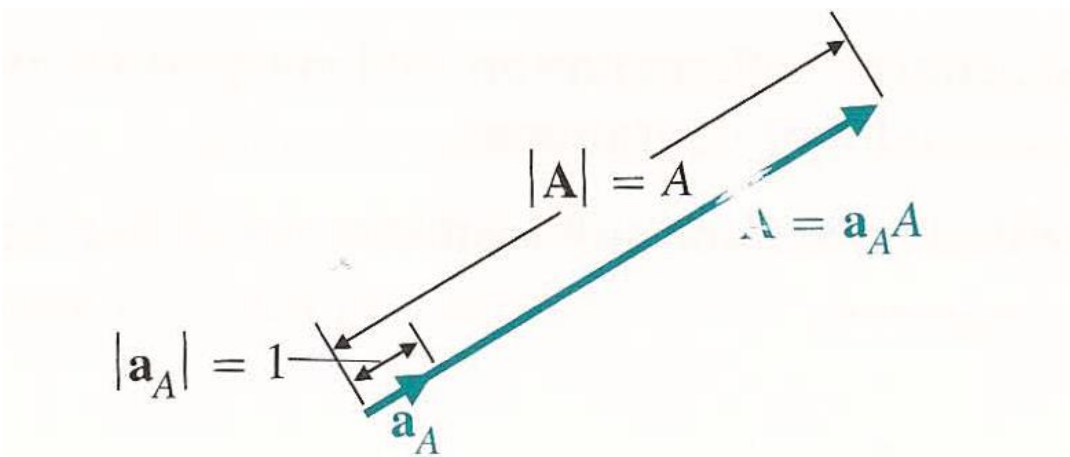
Universal Constants	Symbol	Value	Unit
Velocity of light in free space	c	3×10^8	m/s
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	H/m
Permittivity of free space	ϵ_0	$\frac{1}{36\pi} \times 10^{-9}$	F/m

2. VECTOR ANALYSIS

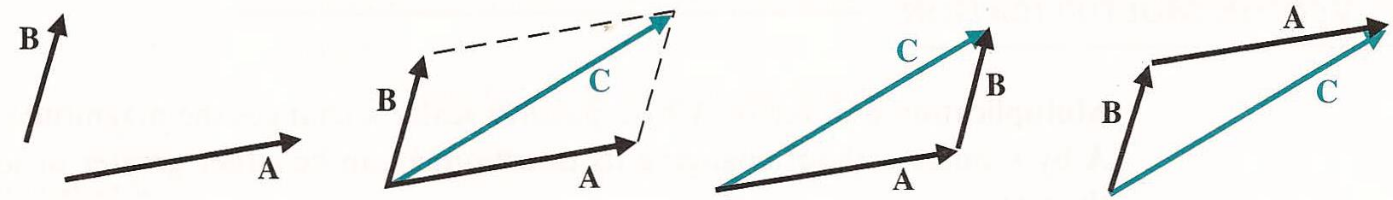
- Step 2 of creating the EM model was defining the rules of operation
- Many electromagnetic quantities are vectors so we must be able to handle them with ease
- Either by *addition*, *subtraction* or *multiplication*
- Certain differential operators and theorems are required to properly handle EM vectors*: gradient, divergence* and *curl operators* as well as *Stokes' theorems*
- In order to express results in a 3-Dimensional space, a suitable coordinate system must be chosen
- Here, the three most common orthogonal coordinate systems will be used: Cartesian, Cylindrical, and Spherical coordinate systems

Vector addition and subtraction

- A vector has magnitude and direction. A vector A can be written as **A** = $a_A A$
- Where A = **|A|** which is a scalar
- a_A = **A** / **|A|** is the unit vector having a magnitude of unity and dimensionless .



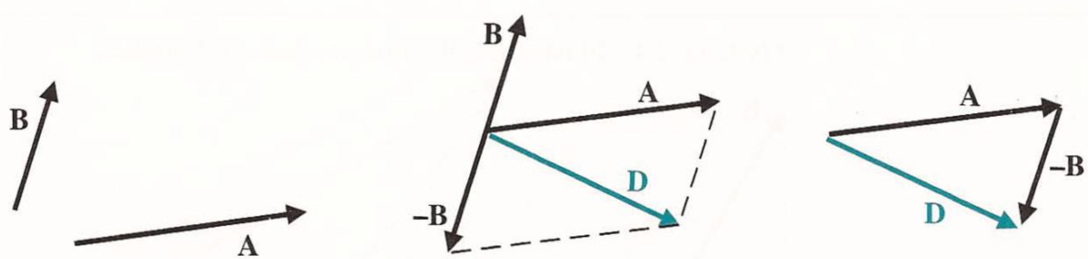
- Vector addition and subtraction can be done in two ways: *Parallelogram rule* and *head-to-tail*



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule, **A** + **B**. (d) Head-to-tail rule, **B** + **A**.

FIGURE 2-2 Vector addition, **C** = **A** + **B** = **B** + **A**.

FIGURE 2-3 Vector subtraction, **D** = **A** − **B** = **A** + (−**B**).

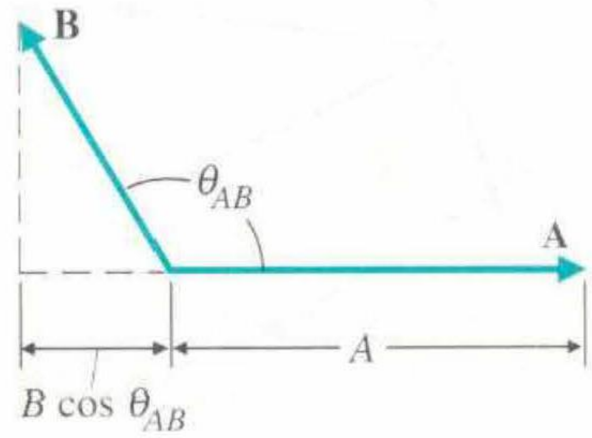


(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule.

Vector multiplication

- There are two ways of multiplying a vector: Scalar (dot) product and Vector (cross) product

FIGURE 2-4 Illustrating the dot product of **A** and **B**.



<u>Scalar or dot product</u>

- **A . B =** $AB\cos\Theta_{AB}$ where A and B are the magnitudes of vectors **A** and **B** and Θ_{AB} is the smaller angle between A and B and is less than 180 degrees

- Dot product is commutative
  A.B = B.A

- **A.A** = A^2 or $A = \sqrt{(\mathbf{A.A})}$

Vector or Cross product

- $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n AB\sin\Theta_{AB}$ where *A* and *B* are the magnitudes of vectors **A** and **B** and Θ_{AB} is the smaller angle between A and B and is less than 180 degrees

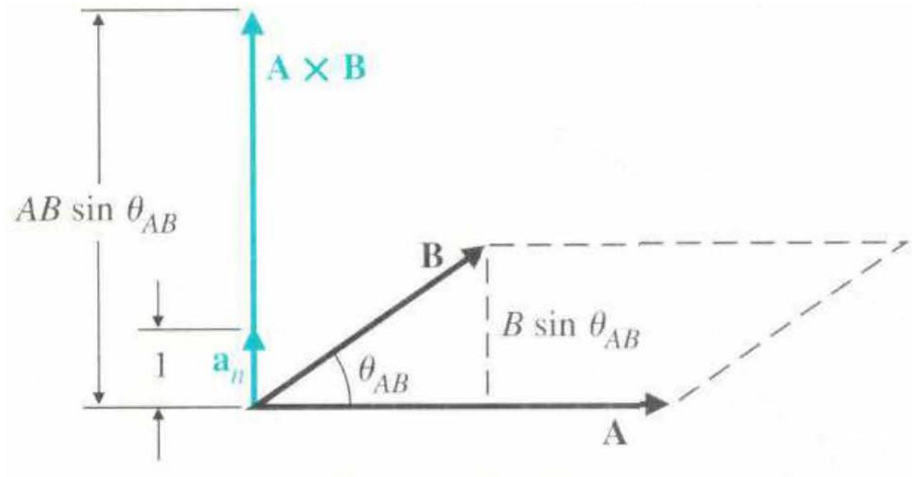
- $\mathbf{a}_n$ is a unit vector normal to the plane containing **A** and **B**

- The direction of follows that of the thumb of the right hand when the fingers rotate from **A** to **B** through angle Θ_{AB}

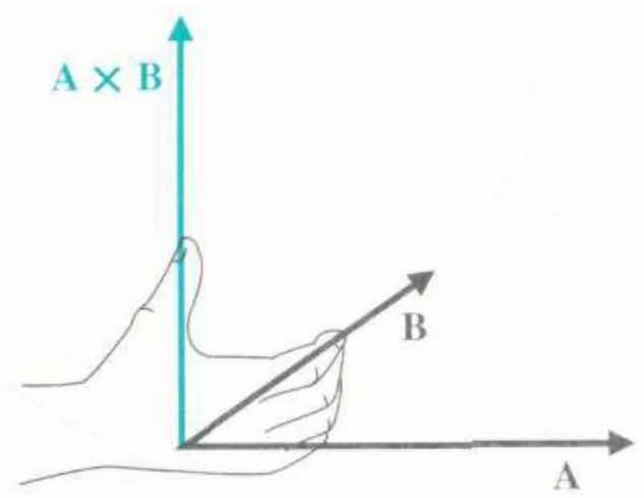
- Dot product is not commutative

  $\mathbf{B} \times \mathbf{A} = -\,\mathbf{A} \times \mathbf{B}$

- The cross product of **A x B** results in another vector obtained by the right-hand rule and whose magnitude is equal to the area of a parallelogram



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n AB \sin\theta_{AB}$.



(b) The right-hand rule.

Orthogonal Coordinate Systems

- Law of electromagnetics are invariant with coordinate systems. However solutions of practical problems requires that the law be expressed in a coordinate system that the problem is specified in

- In a 3-Dimentional space, a point can be located as the intersection of three surfaces

- These points may be either lengths or angles

- When these three surfaces are mutually perpendicular to one another, we have an ***orthogonal coordinate system***

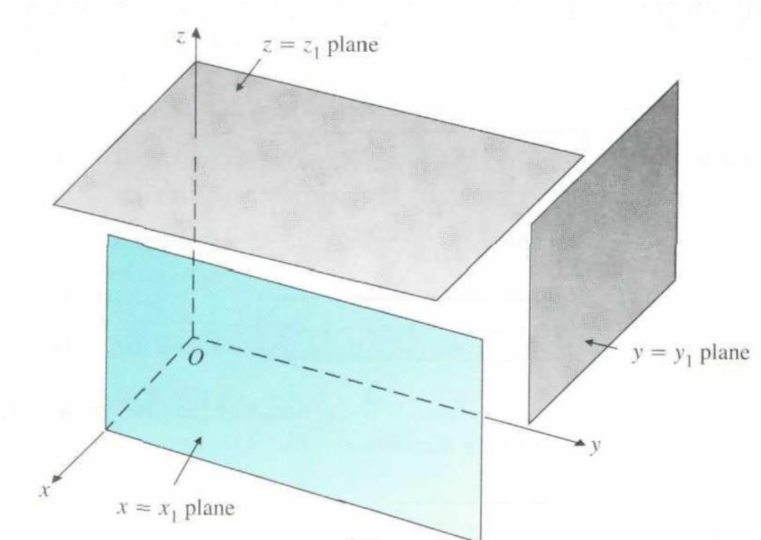
Many orthogonal coordinate systems exist; most common ones are

- *Cartesian (rectangle) coordinates*
- *Cylindrical coordinates*
- *Spherical coordinates*

Cartesian Coordinates

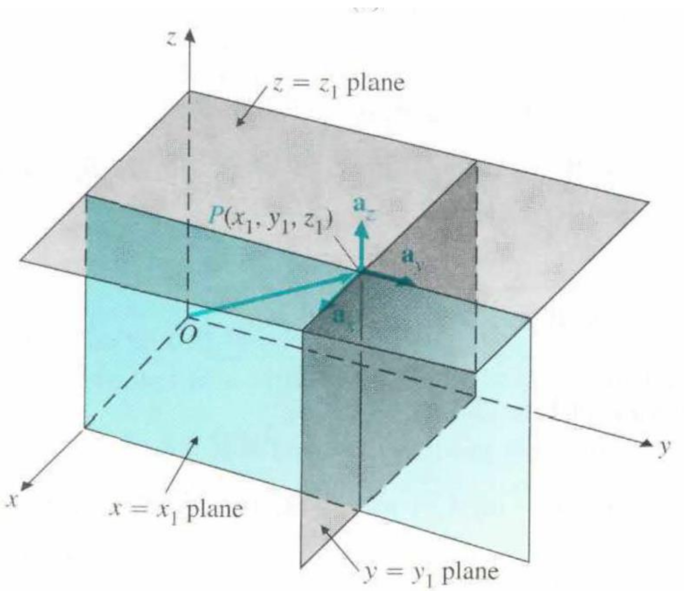
- A point P in Cartesian coordinates is represented by P (x, y, z)

- The three mutually perpendicular unit vectors a_x, a_y, a_z are called base vectors



(a)

$a_x \times a_y = a_z$	$a_x \cdot a_y = 0$	$a_x \cdot a_x = 1$
$a_y \times a_z = a_x$	$a_y \cdot a_z = 0$	$a_y \cdot a_y = 1$
$a_z \times a_x = a_y$	$a_x \cdot a_z = 0$	$a_z \cdot a_z = 1$



(b)

$$A = a_x A_x + a_y A_y + a_z A_z$$

Vector A in Cartesian coordinates

$$dl = a_x d_x + a_y d_y + a_z d_z$$

Vector differential length in Cartesian coordinates

$$dv = dxdydz$$

A differential volume in Cartesian coordinates

$$A.B = A_x B_x + A_y B_y + A_z B_z$$

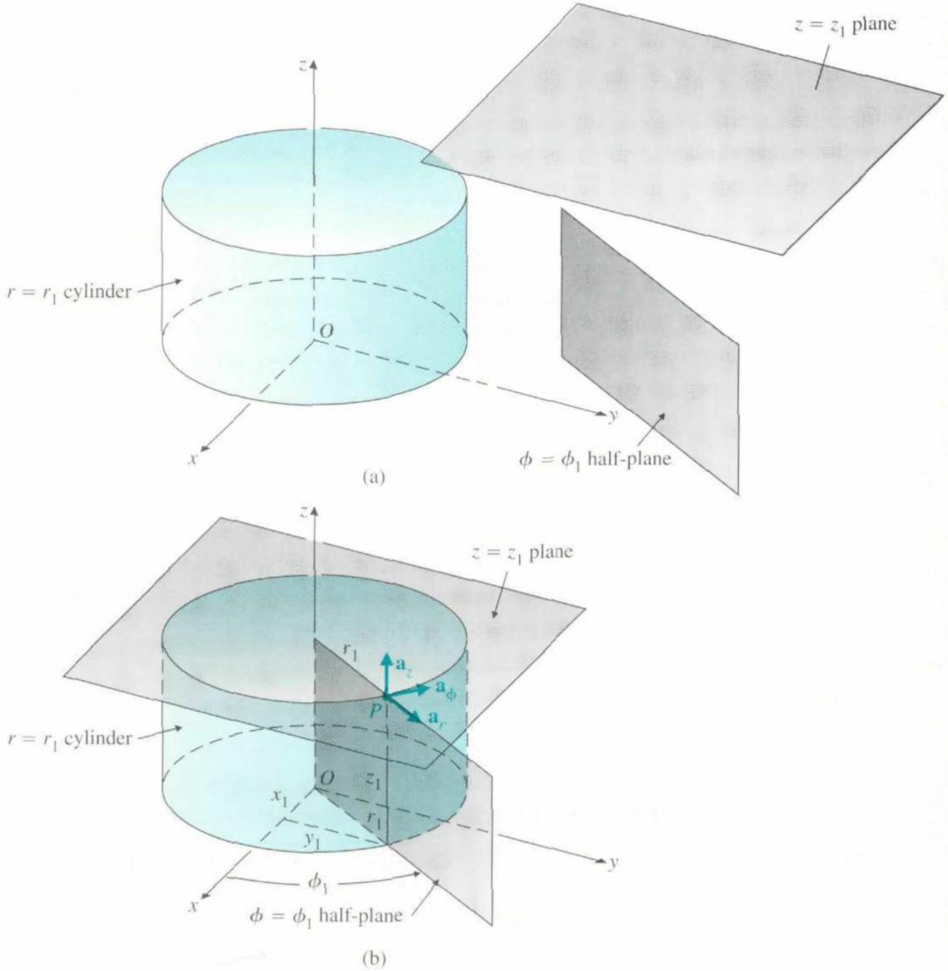
Scalar product of A and B in Cartesian coordinates

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Vector product of A and B in Cartesian coordinates

Cylindrical Coordinates

- A point P in cylindrical coordinates is given by P (r, ϕ, z)
- $a_r \times a_\phi = a_z$
- $a_\phi \times a_z = a_r$
- $a_z \times a_r = a_\phi$



$dl = a_r dr + a_\phi r d\phi + a_z dz$ → Vector differential length in cylindrical coordinates

$dv = r dr d\phi dz$ → volume in cylindrical coordinates

$A = a_r A_r + a_\phi A_\phi + a_z A_z$ → Vector A in cylindrical coordinates

- Transformation of the location of a point in ***cylindrical*** to ***Cartesian coordinates***

P(r,ϕ,z) to P(x,y,z):

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

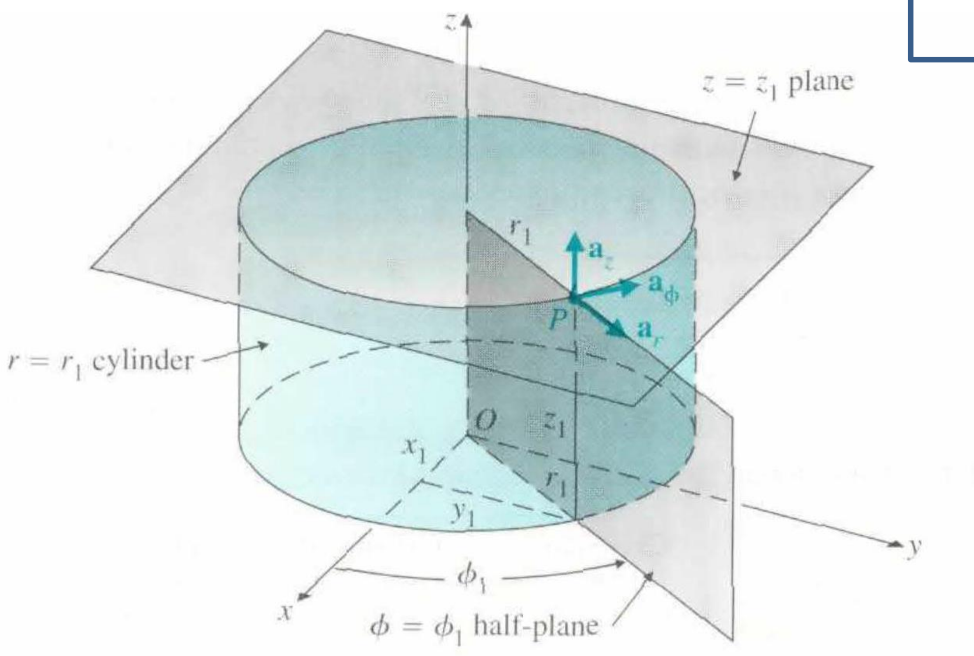
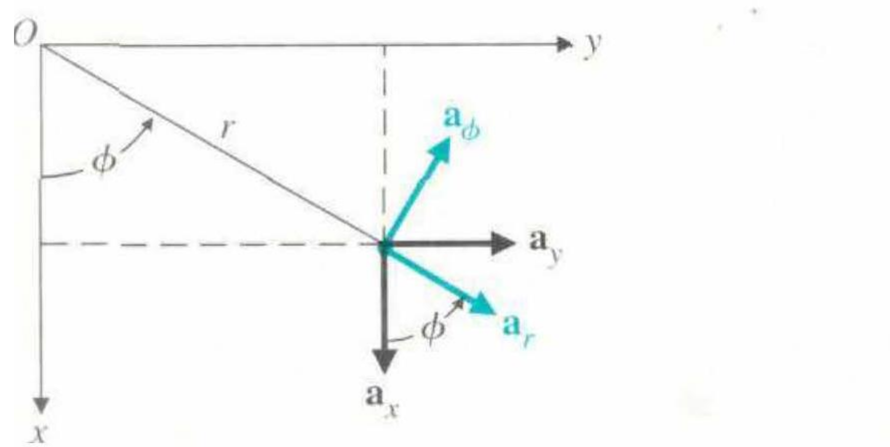


FIGURE 2-12 Relations among $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_r$, and $\mathbf{a}_\phi$.



Spherical Coordinates

- A point P in cylindrical coordinates is given by P (R, Θ, ϕ)
- $a_R \times a_\Theta = a_\phi$
- $a_\Theta \times a_\phi = a_R$
- $a_\phi \times a_R = a_\Theta$

$A = a_R A_R + a_\Theta A_\Theta + a_\phi A_\phi$ Vector A in spherical coordinates

$dl = a_R\, dR + a_\Theta R\, d\Theta + a_\phi R \sin\Theta\, d\phi$ Vector differential length in spherical coordinates

$dv = R^2 \sin\Theta\, dR\, d\Theta\, d\phi$ A differential volume in spherical coordinates

- Transformation of a location of a point in spherical coordinates to Cartesian coordinates:

P(***R, Θ, ɸ***) to P(***x, y, z***)

$$X = R \sin\Theta \cos\phi$$

$$Y = R \sin\Theta \sin\phi$$

$$Z = R \cos\Theta$$

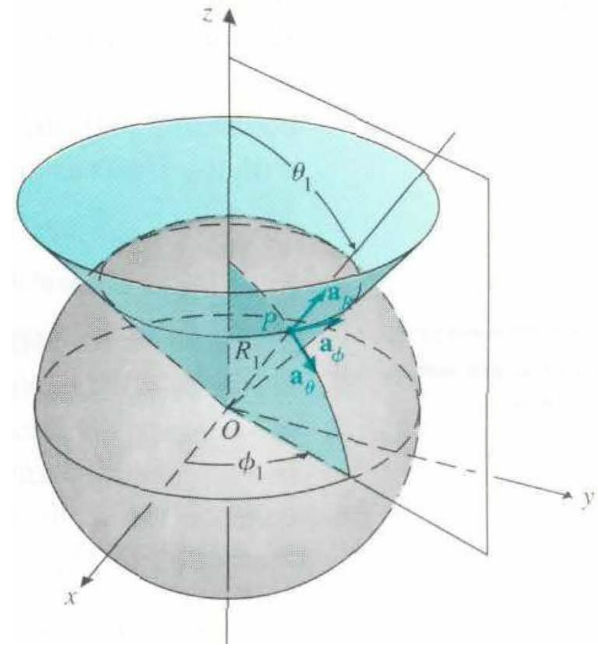


FIGURE 2-15 Showing interrelationship of space variables (x, y, z), (r, ϕ, z), and (R, θ, ϕ).

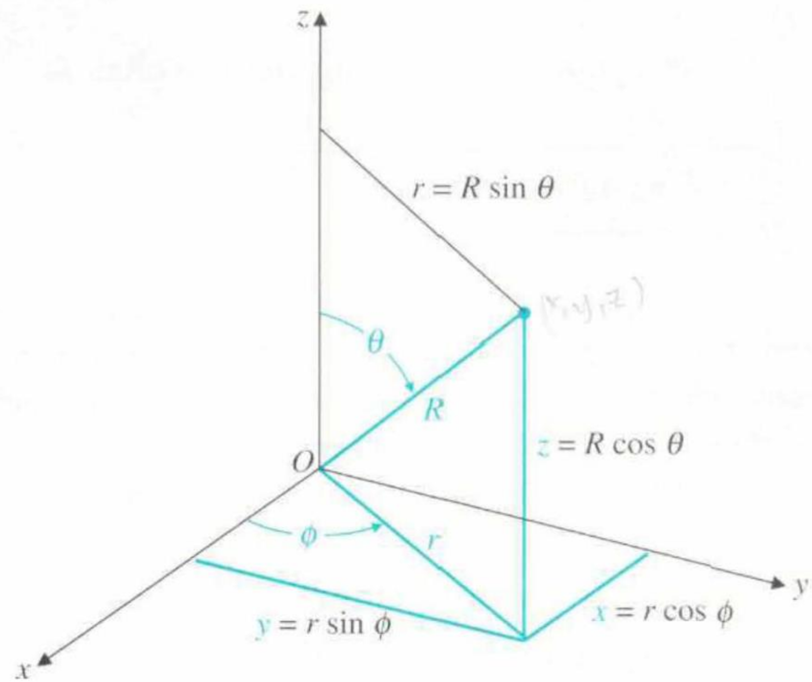
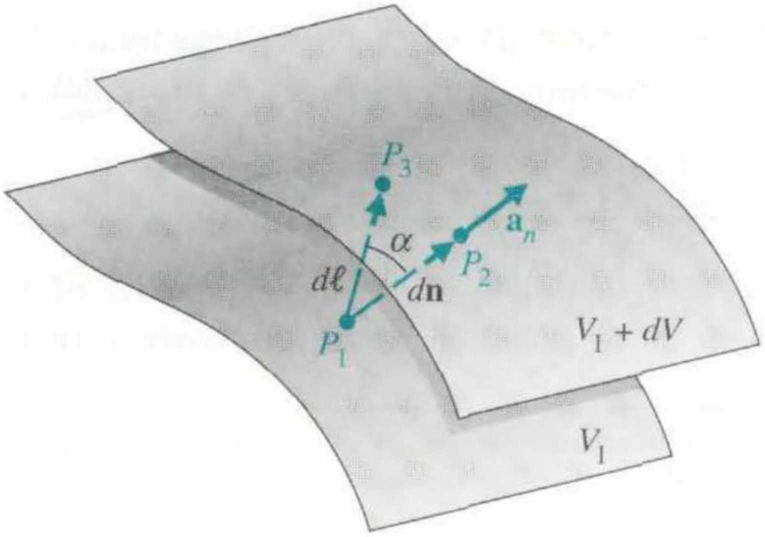


TABLE **2-1** THREE BASIC ORTHOGONAL COORDINATE SYSTEMS

		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
Base Vectors	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
	h_1	1	1	1
Metric Coefficients	h_2	1	r	R
	h_3	1	1	$R \sin \theta$
Differential Volume	dv	$dx\, dy\, dz$	$r\, dr\, d\phi\, dz$	$R^2 \sin \theta\, dR\, d\theta\, d\phi$

Gradient of a Scalar Field

- Electromagnetic quantities often depend on both time and position
- In general, fields may change as any of the variables in the coordinate system change
- It is necessary to describe the space rate of change of a scalar field at any given time, t
- Since the rate of change may be different in different directions, a vector is needed to define the space rate of change
- ***Gradient is defined as the vector that represents both the magnitude and direction of the maximum space rate of increase of a scalar***
- Gradient is denoted by the symbol ∇ (*nabla* operator or *del*) which can be expressed as shown in Cartesian coordinates
- In general orthogonal coordinates with coordinates (u_1, u_2, u_3) and metric coefficients (h_1, h_2, h_3), we can define gradient as :



Gradient of a scalar

$$\text{grad}\, V \triangleq \nabla V \triangleq \mathbf{a}_n \frac{dV}{dn}$$

$\nabla\mathbf{A}$ = Gradient of a Scalar field A

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$ **(Cartesian)**

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \frac{1}{\mathrm{r}} \mathbf{a}_\phi \frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$ **(Cylindrical)**

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \frac{1}{\mathrm{R}} \mathbf{a}_\theta \frac{\partial V}{\partial \theta} + \frac{1}{R sin\theta} \mathbf{a}_\phi \frac{\partial V}{\partial \phi}$$ **(Spherical)**

$$\nabla = (\mathbf{a}_{u1} \frac{\partial}{h_1 \partial u_1} + \mathbf{a}_{u2} \frac{\partial}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3 \partial u_3})$$ **(General)**

Divergence of a Vector Field

- In studying vector fields, it's convenient to represent field variations by directed field lines called flux lines
- The flux lines are directed lines or curves that indicate at each point the direction of the vector field
- The nature of the flux lines can dictate how strong or weak the field strength is at a certain point compared to other points
- The density of the flux lines at a certain point indicates stronger field strength
- Divergence of a vector field **A** is defined as the net outward flux of **A** per unit volume as the volume about the point tends to zero

$$\text{div}\,\mathbf{A} = \nabla.\mathbf{A} \triangleq \lim_{v \to 0} \frac{\oint \mathbf{A}.\,ds}{\Delta v}$$

•Numerator is a surface integral

•Integral is a double integral
Small circle on integral sign indicates that the integral is to be carried out over the entire surface S enclosing the volume

•The integrand $\boldsymbol{ds}$ is a vector with magnitude ds and direction $\mathbf{a}_n$ pointing outward from the enclosed volume

•The enclosed surface integral represents the net outward flux of the vector field **A**

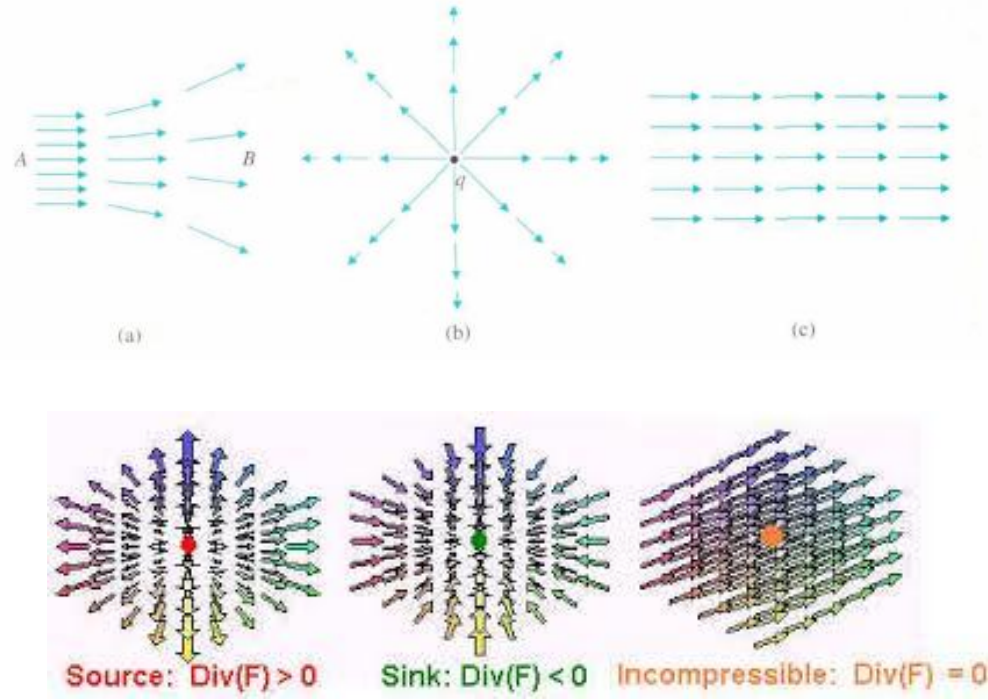
- The figure shows directed flux lines

- In Fig(a), the field in region A is stronger than in region B because of higher density of equal-length directed lines in A

- In Fig (b), decreasing arrow length away from point ***q*** indicates decreasing stronger radial field closest to ***q***

- A net positive divergence represents the presence of a source and a net negative divergence indicates the presence of a sink

- In Fig (c) there is an equal amount of inward and outward flux going through any closed volume containing no source or sink. Therefore resulting in No or Zero divergence



Flux lines of a vector field

$$\nabla \cdot \mathbf{A} = \text{divergence of a vector A}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ **(Cartesian)**

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial (A_\emptyset)}{\partial \emptyset} + \frac{\partial A_z}{\partial z}$$ **(Cylindrical)**

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{Rsin\theta}\frac{\partial (A_\theta sin\theta)}{\partial \theta} + \frac{1}{Rsin\theta}\frac{\partial A_\emptyset}{\partial \emptyset}$$ **(Spherical)**

$$\nabla . \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial \, (h_2 h_3 A_1)}{\partial u_1} + \frac{\partial \, (h_1 h_3 A_2)}{\partial u_2} + \frac{\partial \, (h_1 h_2 A_3)}{\partial u_3}\right]$$ **(General)**

Divergence Theorem

- Divergence of a vector field has been defined as the net outward flux per unit volume
- Thus ***the volume integral of the divergence of the vector field equals the total outward flux of the vector through the surface that bounds the volume***. This is known as the ***divergence theorem***
- This theory applies to any volume ***V*** that is bounded by surface ***S***
- The direction of ***ds*** is always that of the outward normal, perpendicular to the surface *ds* and directed away from the volume
- Divergence theorem is very important in vector analysis since it converts a volume integral of vector divergence to a closed surface integral of the vector and vice versa
- They are usually double and triple integrals but a single integral is used for simplicity

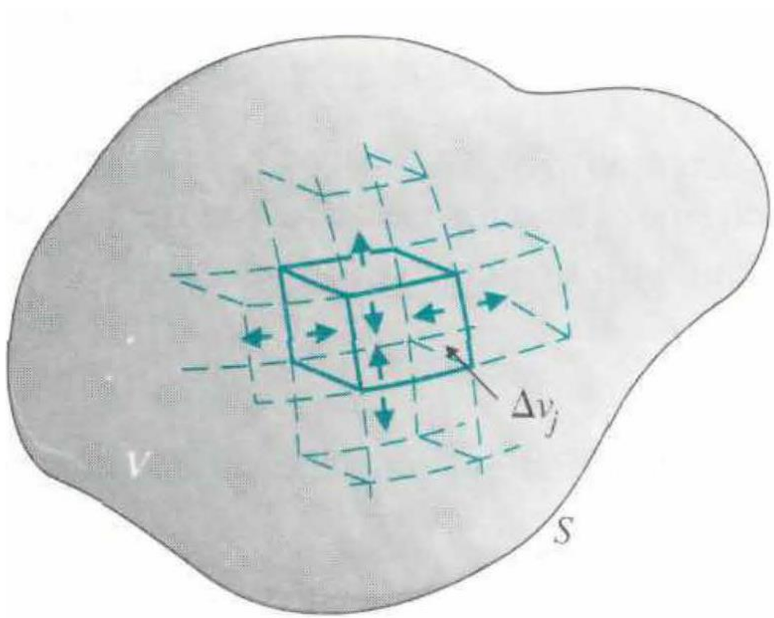
$$\nabla . \mathbf{A} \triangleq \lim_{v \to 0} \frac{\oint \mathbf{A} . ds}{\Delta v}$$

Divergence definition

$$\boxed{\int_v \nabla \cdot \mathbf{A}\, dv = \oint_s \mathbf{A} \cdot ds}$$

Divergence theorem

Proof of divergence theorem



A volume V can be divided in several differential volumes ΔV_j as shown above

For a very small differential volume ΔV_j bounded by a surface s_j, the definition of divergence is given by

$$(\nabla \cdot \mathbf{A})_j \Delta v_j = \oint_{s_j} \mathbf{A} \cdot ds$$

The individual contributions of the differential volumes can be combined as

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{s_j} \mathbf{A} \cdot ds \right]$$

The left side of the equation is simplified as

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \int_v \nabla \cdot \mathbf{A} \, dv$$

The right side of the equation is simplified as since net contribution is due only to that of the external surface ***S***

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{s_j} \mathbf{A} \cdot ds \right] = \oint_s \mathbf{A} \cdot ds$$

The divergence theorem is therefore given as

$$\boxed{\int_v \nabla \cdot \mathbf{A} \, dv = \oint_s \mathbf{A} \cdot ds}$$

Curl of a Vector Field

- The net outward flux of a vector through a surface bounding a volume indicates the presence of a source
- This is known as flow source. Another kind of source is called a vortex source
- A vortex source causes circulation of a vector field around it
- The circulation of a vector field around a closed path as defined as scalar integral of a vector over a closed path as shown
- The physical meaning of circulation depends on what kind of vector A represents

Circulation of A around contour $C \triangleq \oint_c \mathbf{A} \cdot d\boldsymbol{l}$

Physical meaning of circulation

1. If A is a force acting on an object, its circulation will be the work done by the force in moving the object around the contour
2. If A represents an electric field intensity, E, the circulation will be an electromotive force around the closed path
3. Water whirling down a sink drain is an example of a vortex sink causing circulation of fluid velocity

$$Curl\,A = \nabla \times \mathbf{A} \triangleq \lim_{\Delta s \to 0} \frac{\left[a_n \oint_c \mathbf{A} \cdot d\boldsymbol{l}\right]_{max}}{\Delta s}$$

Mathematical definition of curl of a vector A

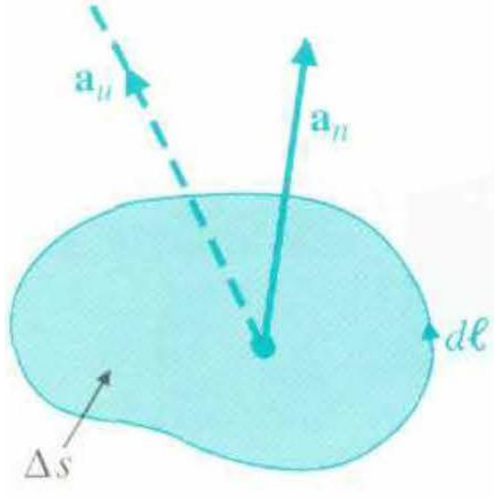
- In other words, *the curl of a vector A, is a vector whose magnitude is the maximum net circulation of A per unit area as the area tends to zero, and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum*

- Curl A is a vector point function

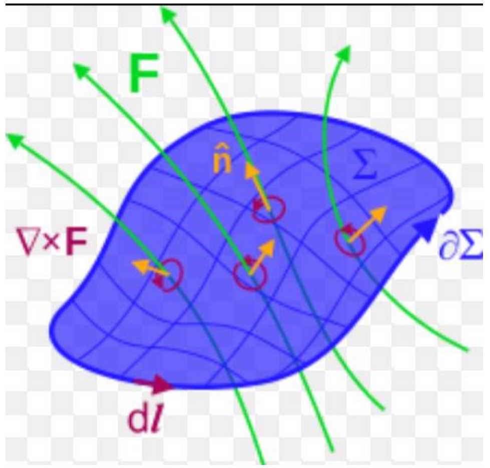
- Normal to an area can point in two opposite directions

- Using the right-hand rule such that the fingers follow the direction of **d*l*** and thumb points to the **a*n*** direction. This is illustrated above

- The component of curl A in any other direction aside the normal is given by curl of A dot the unit vector in that direction. This is illustrated above



Relation between **a*n*** and **d*l*** in defining curl



$$(\nabla \times \mathbf{A})_u \triangleq a_u \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta s_u \to 0} \frac{\oint_{c_u} \mathbf{A} \cdot d\mathbf{l}}{\Delta s_u}$$

$$Curl\ of\ Vector\ A = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad \textbf{(Cartesian)}$$

$$\nabla \times \mathbf{A} = \frac{\mathbf{1}}{\mathbf{r}} \begin{vmatrix} a_r & r a_\emptyset & a_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ A_r & r A_\emptyset & A_z \end{vmatrix} \quad \textbf{(Cylindrical)}$$

$$\nabla \times \mathbf{A} = \frac{\mathbf{1}}{\mathbf{R^2 sin\theta}} \begin{vmatrix} a_R & R a_\theta & R sin\theta a_\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_R & R A_\theta & R sin\theta A_\emptyset \end{vmatrix} \quad \textbf{(Spherical)}$$

$$\nabla \times \mathbf{A} = \frac{\mathbf{1}}{\mathbf{h_1 h_2 h_3}} \begin{vmatrix} a_{u_1} h_1 & a_{u_2} h_2 & a_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad \textbf{(General)}$$

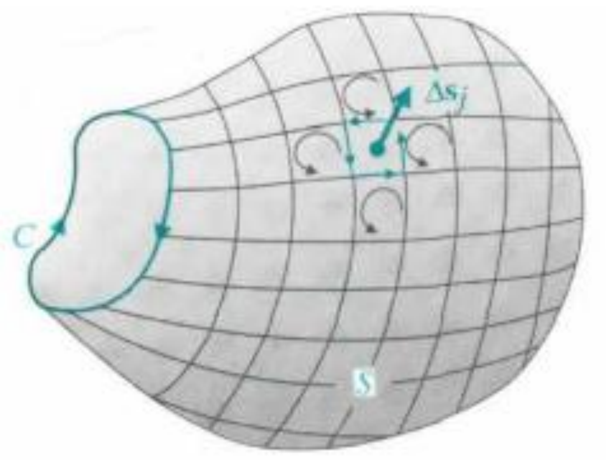
CURL

CARTESIAN $$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_z}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

CYLINDRICAL $$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right]\mathbf{a}_z$$

SPHERICAL $$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left[\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\mathbf{a}_\phi$$

Stokes' Theorem



Subdivided area: for proving Stokes' theorem

For a small differential area Δs_j bounded by a contour cj, the definition of curl (shown below)

$$Curl\,A = \nabla \times \mathbf{A} \triangleq \lim_{\Delta s \to 0} \frac{\left[a_n \oint_c \mathbf{A} \cdot dl\right]_{max}}{\Delta s}$$

becomes $(\nabla \times \mathbf{A}) \cdot (\Delta \boldsymbol{s}_j) = \oint_{c_j} \mathbf{A} \cdot dl$

This is achieved by taken the dot product of both sides of the curl equation with $\mathbf{a}_n \Delta s_j$ or $\Delta \boldsymbol{s}_j$.

For a given surface S, it can be subdivided into N small differential areas as seen in the figure. By adding the contributions of all the differential areas, we have

$$\lim_{\Delta s_j \to 0}\left[\sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot (\Delta \boldsymbol{s}_j)\right] = \lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \mathbf{A} \cdot dl\right)$$

Now, since the common part of the contours of two adjacent elements are in opposite direction, the net contribution of all the interior elements to the total line integral is zero, and the only contribution is from the external contour C bounding the entire area S, therefore

$$\lim_{\Delta s_j \to 0}\left[\sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot (\Delta \boldsymbol{s}_j)\right] = \int_s (\nabla \times \mathbf{A}) \cdot ds$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \mathbf{A} \cdot dl\right) = \oint_c \mathbf{A} \cdot dl$$

Combining the two equations gives us

$$\boxed{\int_s (\nabla \times \mathbf{A}) \cdot ds = \oint_c \mathbf{A} \cdot dl}$$

Two null identities

- The curl of the gradient of any scalar field is identity is identically zero
- A converse statement can also be made as: If a vector field is curl-free (or irrotational), then it can be expressed as the gradient of a scalar field

$$\boxed{\nabla \times (\nabla V) = 0}$$ IDENTITY I

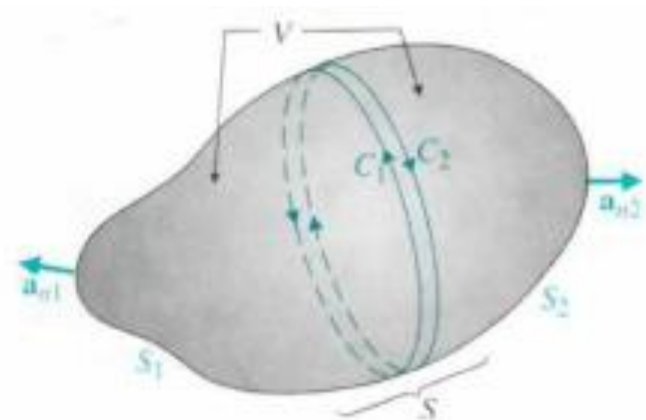
$$\int_S [\nabla \times (\nabla V)] \cdot ds = \oint_C (\nabla V) \cdot dl$$

$$\oint_C (\nabla V) \cdot dl = \oint_C dV = 0$$

USING STOKES' THEOREM

- The divergence of the curl of any vector field is identically zero
- A converse statement can also be made as: If a vector field is divergenceless, then it is solenoidal and can be expressed as the curl of another vector field

$$\boxed{\nabla \cdot (\nabla \times V) = 0}$$ IDENTITY II



Helmholtz's theorem

- Helmholtz's theorem states that: A vector field is determined if both its divergence and curl are specified everywhere
- Helmholtz's theorem is a basic element in the development of electromagnetism
- For each EM field study (electrostatics, time varying electromagnetic fields, etc), the divergence and curl will have to be specified

Field Classifications

1. A static electric field in a charged-free region

$$\nabla \cdot \mathbf{F} = 0 \text{ and } \nabla \times \mathbf{F} = 0$$

Solenoidal and irrotational

2. A steady magnetic field in a current—carrying conductor

$$\nabla \cdot \mathbf{F} = 0 \text{ and } \nabla \times \mathbf{F} \neq 0$$

Solenoidal but not irrotational

3. A static electric field in a charged region

$$\nabla \cdot \mathbf{F} \neq 0 \text{ and } \nabla \times \mathbf{F} = 0$$

Not solenoidal but Irrotational

4. An electric field in a charged medium with a time-varying magnetic field

$$\nabla \cdot \mathbf{F} \neq 0 \text{ and } \nabla \times \mathbf{F} \neq 0$$

Neither Solenoidal nor irrotational

TUTORIALS

1. Given a Vector $\mathbf{A} = -\mathbf{a}_x + 2\mathbf{a}_y - 2\mathbf{a}_z$. Find

(a) Its magnitude $|A|$

(b) The expression of the unit vector a_A in the direction of A

(c) The angle that A makes with the z-axis

2. Given $A = 5\mathbf{a}_x – 2\mathbf{a}_y + \mathbf{a}_z$, and $B = -3\mathbf{a}_x + 4\mathbf{a}_z$. Find

(a) $A \cdot B$

(b) $A \times B$

3. Given a vector **B** = 2$\mathbf{a}_x$ – 6$\mathbf{a}_y$ + 3$\mathbf{a}_z$. Find

(a) The magnitude of **B**

(b) The expression of $\mathbf{a}_B$

(c) The angles that **B** makes with the x, y, and z axis

4. Assuming that a cloud of electrons confined in a region between two spheres of radii 2cm and 5cm has a charge density of:

$$\frac{3\times 10^{-8}}{R^4}\cos^2 \phi \quad (\mathrm{C/m^3})$$

Find the total charge contained in the region

5. Given a vector field $\mathbf{A} = r\,\mathbf{a}_r + z\,\mathbf{a}_z$

- Find the total outward flux over a circular cylinder around the z-axis with a radius 2 and a height 4 centered at origin
- Find the divergence of A ($\nabla . \mathbf{A}$)
- Verify the divergence theorem

6. Prove the two null identities mathematically

$$\boxed{\nabla \times (\nabla V) = 0}$$ IDENTITY I

$$\boxed{\nabla \cdot (\nabla \times V) = 0}$$ IDENTITY II

3. Static Electric Fields

- Static electric fields are caused by stationary electric charges

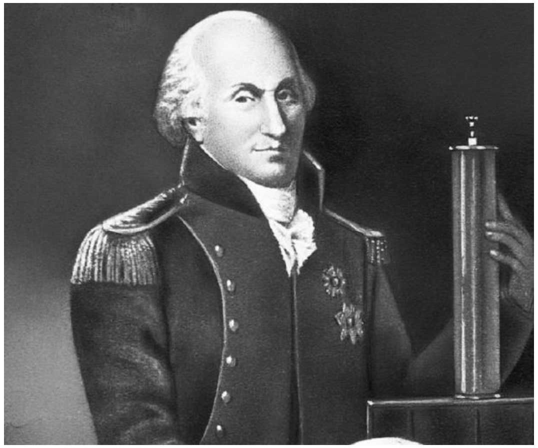
- When one walks over a carpet in a dry room, the rubbing of shoes soles on the carpet produces charges on our body (e.g. fingertips)

- The charges can jump across the air to a doorknob when ones touches a metal doorknob

- This potential difference generated may be a several thousand volts which may cause a slight shock

- Electrostatics is the study of the effects of electric charges at rest. Here the electric fields do not change with time

- Electrostatics is the simplest situation in electromagnetics but it has a lot of importance, i.e. it explains phenomenon such as lightening and some industrial applications such as ink-jet printers, oscilloscopes, etc.



Augustin de Coulomb

Karl Friedrich Gauss

Fundamental postulates of Electrostatics in Free Space

- As discussed previously, the only vector quantities considered in electrostatics is the Electric field intensity, E
- Also among the universal constants, only the permittivity of free space ε_0, is considered
- Electric field intensity, E is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists
- According to Helmholtz's theorem, we have to specify the *divergence* and *curl* of E in electrostatics
- The 2 postulates are simple n concise and can be used to derive any laws in electrostatics
- They can be written in differential form or integral form

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathrm{V/m})$$

$$\mathbf{F} = q\mathbf{E} \quad (\mathrm{N})$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon_0} \quad (in\ free\ space)$$

$$\nabla \times \mathbf{E} = 0$$

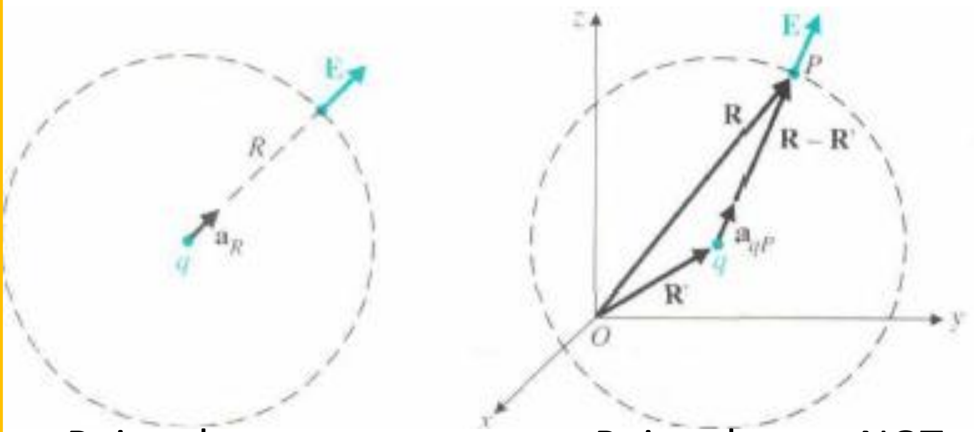
Fundamental postulates of electrostatics in free space (differential form)

$$\oint_s \mathbf{E} \cdot ds = \frac{Q}{\varepsilon_0}$$

$$\oint_c \mathbf{E} \cdot dl = 0 \quad (in\ free\ space)$$

Fundamental postulates of electrostatics in free space (integral form)

Coulomb's Law



Point charge at the origin

Point charge NOT at the origin

Fig.1 Electric field intensity due to a point charge

$$\oint_s \boldsymbol{E} \cdot ds = \oint_s (\mathbf{a}_R E_R) \cdot \mathbf{a}_R ds = \frac{q}{\varepsilon_0}$$

$$E_R \oint_s ds = E_R(4\pi R^2) = \frac{q}{\varepsilon_0}$$

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\varepsilon_0 R^2} \quad (\mathrm{V/m})$$

Electric field intensity of a point charge at the origin

$$\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\varepsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \quad (\mathrm{V/m})$$

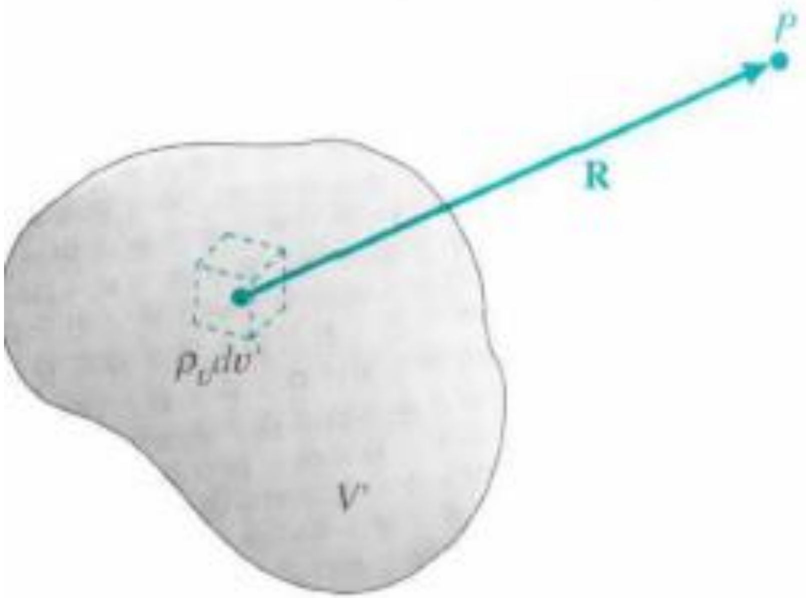
Electric field intensity of a point charge at an arbitrary location

When a point charge q2 is placed in the field of another point charge q1, a force F12 is experienced by q2 due to the electric field intensity E12 of q1 at q2. This force is given as:

$$\mathbf{F}_{12} = q_2 \mathbf{E}_{12} = \mathbf{a}_{12} \frac{q_1 q_2}{4\pi\varepsilon_0 {R_{12}}^2} \quad (\mathrm{N})$$

This is the Coulomb's law which states that: ***the force between two point charges is proportional to the product of the charges and inversely proportional to the square of the distance of separation***

Electric field due to a continuous distribution of charge



Electric field caused by a continuous distribution of charge can be obtained by integrating the contribution of an element charge over the charge distribution

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{v'} \mathbf{a}_R \frac{\rho_v}{R^2}\, dv' \quad \text{(V/m)}$$

Electric field intensity of a volume distribution of charge

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{s'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \quad \text{(V/m)}$$

Electric field intensity of a surface distribution of charge

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_l}{R^2}\, dl' \quad \text{(V/m)}$$

Electric field intensity of a line distribution of charge

$$\mathbf{E} = \mathbf{a}_r \frac{\rho_l}{2\pi\varepsilon_0 r} \quad \text{(V/m)}$$

Electric field intensity due to an infinite straight line charge of uniform density

Electric Potential

Electric potential **V** can be defined as

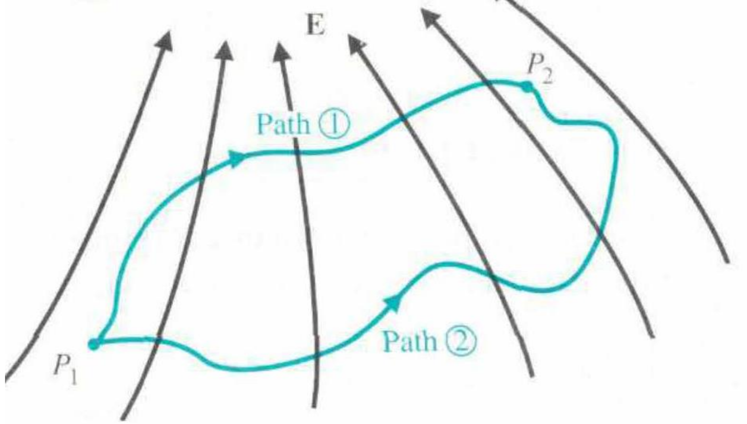
$$\mathbf{E} = -\nabla V$$

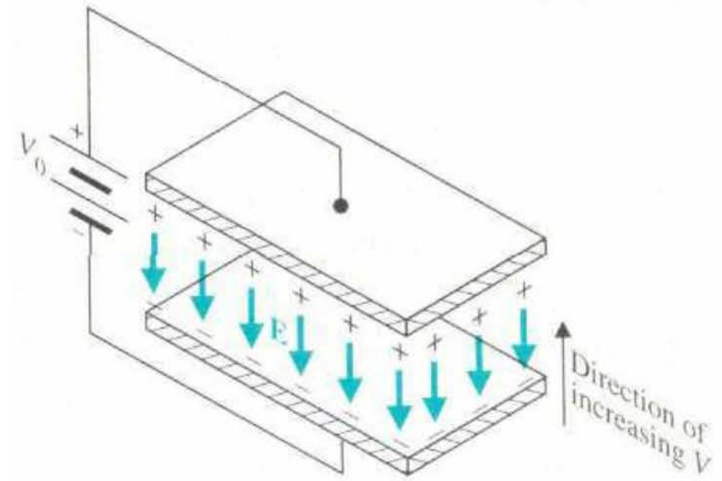
Electric potential is related to the work done in carrying a charge from one point to another. Therefore in moving a unit charge from point P1 to point P2 in a electric field, work must be done against the field and its equal to:

$$\frac{W}{q} = -\int_{p1}^{p2} \mathbf{E}.dl \quad \text{J/C or V}$$

$$V_2 - V_1 = -\int_{p1}^{p2} \mathbf{E} \cdot dl \quad (\mathrm{V})$$

Therefore the work done in moving a unit charge from P1 to P2 is equal to the Electrostatic potential difference between P1 and P2





The electric potential V of a point at a distance R from a point charge ***q*** referred at infinity is given by

$$V = \frac{q}{4\pi\varepsilon_0 R} \quad (\mathrm{V})$$

Material Media in Static electric field

In previous slide, the electric field of stationary charge distribution has been discussed in Free space. Here, we'll look at static field behavior in a material medium

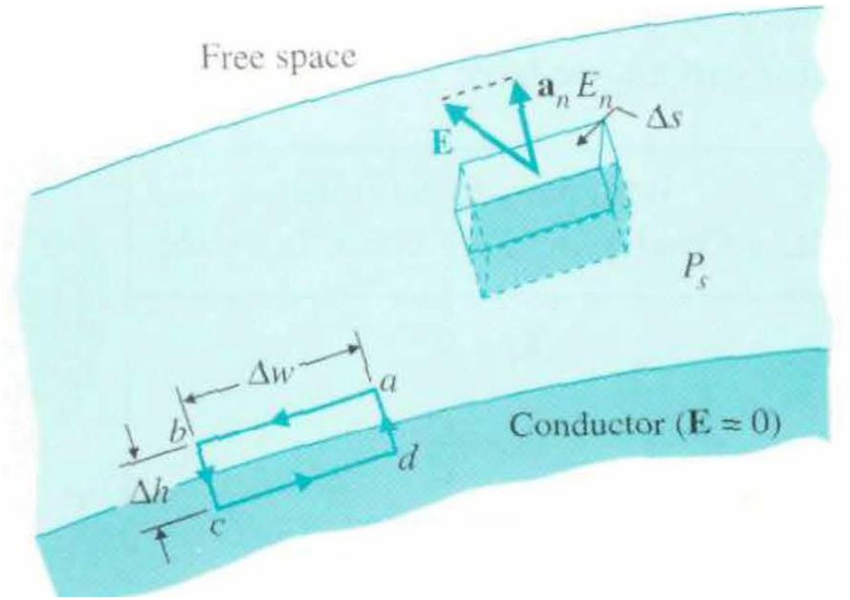
The 3 main types of materials are ***conductors, semi-conductors and insulators (dielectrics)***

INSIDE A CONDUCTOR

$$\rho_v = 0$$

$$\mathbf{E} = \mathbf{0}$$

Inside a conductor, both free charge and electric field intensity vanish under static conditions



The tangential components of the electric field intensity E at the surface of the conductor is should be zero as well. Here, the E-field on a conductor is the normal components

Boundary conditions at a conductor-Free Space interface

$$E_t = 0$$

$$E_n = \frac{\rho_s}{\varepsilon_0}$$

Dielectrics in Static Electric Field

- Usually dielectrics have neutral macroscopic molecules

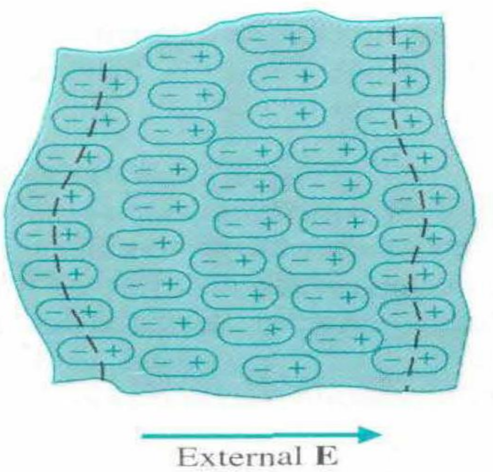
- But the presence of an external electric field can cause a force to be exerted on each charge particle and results in the small movements of positive and negative charges in opposite directions

- This creates electric dipoles and therefore causes the dielectric material to have polarity

- This induced electric dipoles will modify the electric field both inside and outside the dielectric material

- With the applied electric field on a dielectric, the dipoles tend to align in a manner shown above. Otherwise, they are random

To analyze the effect of the induced dipoles, we define the ***Polarization Vector, P,*** ***which is the volume density of the electric dipole moment***



Cross section of Polarized dielectric medium

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n \quad (C/m^2)$$

Equivalent polarization surface charge density

$$\rho_{pv} = -\nabla \cdot \mathbf{P} \quad (C/m^3)$$

Equivalent polarization volume charge density

The electric field intensity E gives rise to a volume charge density in free space. Now a given source in a dielectric will also produce a polarization volume charge density. Therefore the electric field intensity E, for a given source distribution (free charges) in a dielectric will be

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho_v + \rho_{pv})$$

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_v$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2})$$

D is the **electric flux density** or **electric displacement**

$$\nabla \cdot \mathbf{D} = \rho_v \qquad (\mathrm{C/m^3})$$

GAUSS LAW

Prove that Gauss Law can be written as:

$$\oint_s \mathbf{D}.ds = Q \quad (\mathrm{C})$$

Gauss Law states that ***the total outward flux over any closed surface is equal to the total free charge enclosed in that surface***

IS GAUSS LAW APPLICABLE TO FREE SPACE OR DIELECTRIC MEDIUM, or both ??

When the dielectric properties of the medium are linear and isotropic, the polarization, P is directly proportional to the electric field intensity, E, as shown

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

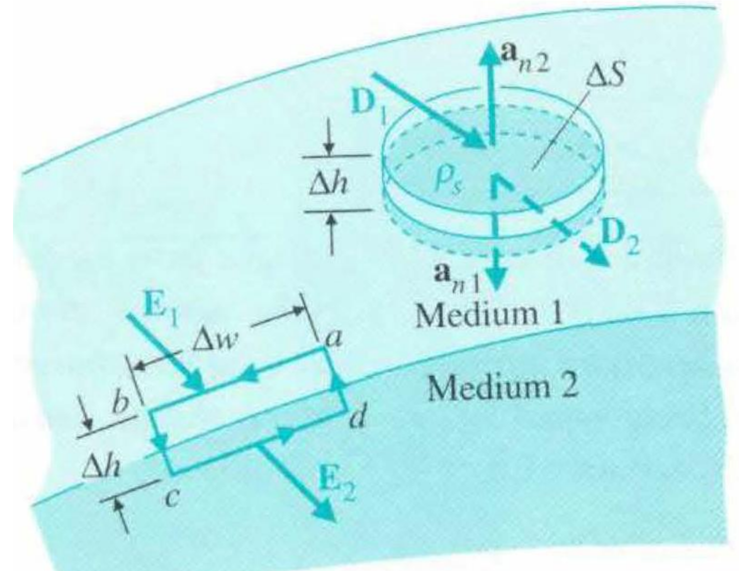
$$\mathbf{D} = \varepsilon_0 (1 + \chi_e)\mathbf{E}$$

$$= \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \qquad (\mathrm{C/m^2})$$

Where ε_r is known as **relative permittivity** or dielectric constant of the medium and χ_e is called **electric susceptibility**

Boundary Conditions for Electrostatic Fields

- Electromagnetic problems usually involve different media
- Therefore knowledge of field relations are the media interfaces in important
- For example, how does E and D change after leaving water into air and vice versa?
- We know boundary conditions of a conductor-free space
- Lets consider two general media



Interface between two media

$$E_{1t} = E_{2t} \quad (\mathrm{V/m})$$

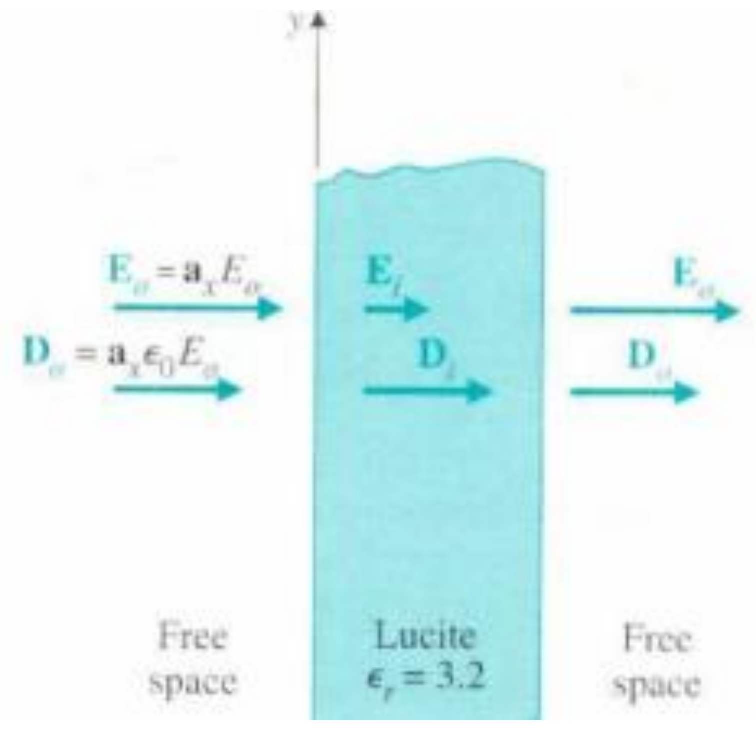
$$a_{n2} \cdot (D_1 - D_2) = \rho_s$$

$$D_{1n} - D_{2n} = \rho_s \quad (\mathrm{C/m^2})$$

BOUNDARY CONDITIONS FOR ELECTROSTATIC FIELDS

1. State and explain the boundary conditions that must be satisfied by the electric potential at an interface between perfect dielectric constants e_{r1} and e_{r2}

2. Find E_i,[fig. below



Capacitance

•It is clear that a conductor in a static field is equipotential

•Suppose the potential, V, is increasing due to increase in Q

•Increase in Q will in turn increase the surface charge density everywhere by the same factor without affecting the charge distribution

•Therefore we can say that the potential of an isolated conductor is directly proportional to the total charge on it

•The constant of proportionality C is called the capacitance of the isolated conductor, in Coulomb/Volt or Farad

$$Q \propto V$$

$$Q = CV$$

$$\boxed{C = \frac{Q}{V_{12}} = \varepsilon\frac{A}{d}}$$

Capacitance of a parallel-plate capacitor

$$\boxed{C = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}}$$

Capacitance of cylindrical capacitor

1. Assume the earth to be a large conducting sphere (radius = 6370 km) surrounded by air. Find its capacitance referring to infinity

- **Read on Electrostatic Energy. Page 120 – 125 in the Course Book (David K. Cheng)**

Poisson's and Laplace Equations

- Techniques of determining electric field intensity E, electric potential V, electric flux density D for a given charge distribution is now known

- In this section, we'll discuss certain methods of solving electrostatic boundary problems when the exact charge distribution isn't known everywhere

- Here, we use Laplace and Poisson equations

- Simply put, Laplace equation are used when there are no electrical charges and Poisson is used when electrical charges exist

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_v}{\varepsilon_0}$$

$$\mathbf{E} = -\boldsymbol{\nabla V}$$

$$\nabla \cdot \nabla \boldsymbol{V} = \nabla^2 \boldsymbol{V} = -\frac{\rho_v}{\varepsilon_0}$$

$$\nabla^2 \boldsymbol{V} = -\frac{\rho_v}{\varepsilon_0}$$

Poisson's Equation

$\boldsymbol{\nabla^2}$ *(del square)is the Laplacian operator, whichs* stands for "*divergence of the gradient*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon_0} \qquad (\mathrm{V/m^2})$$

Poisson's equation in Cartesian Coordinates

At a point in a medium where there is no free charg the Poisson equations reduces to

$$\nabla^2 \boldsymbol{V} = \mathbf{0}$$

Laplace's equation

1. The radius of the inner conductor of a long coaxial cable is a. The inner radius of the outer conductor is b. If the inner and outer conductors are kept at potentials ***V_0*** and ***0*** respectively, determine the electric potential, ***V***, and Electric Field

4. Steady Electric Currents

- We have considered static electric fields in *Section* ***3*** and the boundary conditions associated with static electric fields

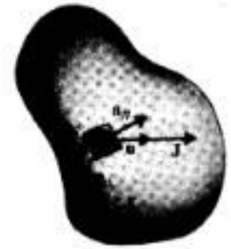
- We'll now consider charges in motion. These charges constitute current flow

- 2 main types of currents: Conduction current and Convection current

- Conduction current: Drift motion of electrons in conductors and semiconductors. Their Average drift velocities are low, (about 0.001 m/s)
- Convection current: Motion of electrons or ions in a vacuum or in rarefied gas

Current density and Ohm’s Law: *convection current*

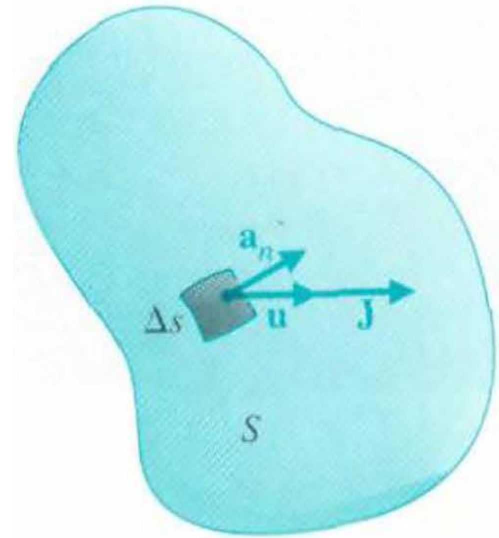
$$\Delta Q = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s\, \Delta t \qquad \text{(C)}.$$



Total current over the differential surface Δs:

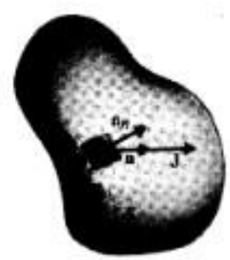
$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = Nq\mathbf{u} \cdot \Delta\mathbf{s} \qquad \text{(A)}.$$

Current density: $\mathbf{J} = Nq\mathbf{u} \qquad (\mathrm{A/m^2}), \Longrightarrow \Delta I = \mathbf{J} \cdot \Delta\mathbf{s}.$



The total current I flowing through an arbitrary surface S:

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.}$$



Noting that the charge density $\rho = Nq$, we have

$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\mathrm{A/m^2}),} \Longleftarrow \mathbf{J} = Nq\mathbf{u}$$

Consider a steady motion of one kind of charge carriers, each of charges ***q*** over a differential surface ***ΔS***, with velocity ***U***. Then the total amount of charges ***ΔQ*** passing this surface in a time interval ***Δt*** is:

$$\Delta Q = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s \Delta t \quad \text{(C)}$$

Current density and Ohm's Law: *conduction current*

$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}).$$

- ☐ Drift motion of charge carriers under electric field
- ☐ Atoms remain neutral (ρ=0)

For most of metallic conductors in which the average electron drift is proportional to electric field, we write

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\mathrm{m/s}),$$

Electron mobility μ_e:

Cu: $3.2 \times 10^{-3}\ (\mathrm{m^2/V \cdot s})$.

Al: $1.4 \times 10^{-4}\ (\mathrm{m^2/V \cdot s})$

Then we have,

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}. \qquad \mathbf{u} = -\mu_e \mathbf{E}, \quad \mathbf{J} = \rho \mathbf{u}$$

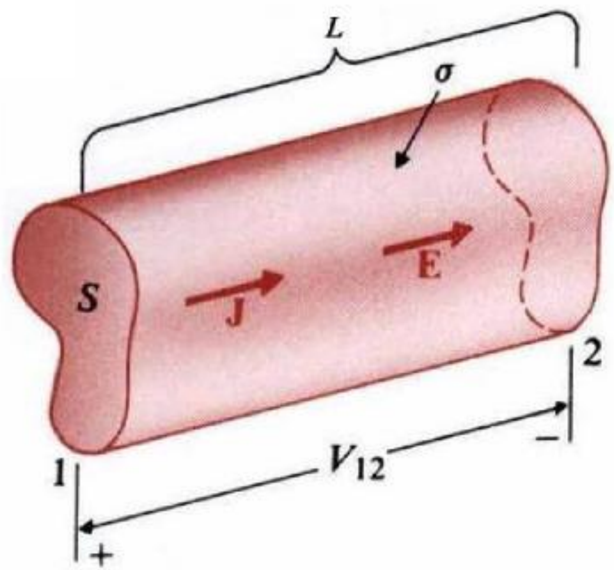
Where ρ_e=-Ne is the charge density of the drifting electrons. We rewrite it by

$$\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\mathrm{A/m^2}),}$$

For a current density of 7 (A/mm^2) in Copper with conductivity 5.8x10^{-7} S/m. Find the Electric Intensity and the Electron drift velocity

Ans. E = 0.121 V/m, u = 0.0000347 m/s

Resistance and Conductance



Voltage between terminals 1 and 2:

$$V_{12} = E\ell$$

The total current:

$$I = \int_s \mathbf{J} \cdot d\mathbf{s} = JS$$

Point for microscopic form of Ohm's law

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI,$$

Macroscopic Ohm's law

Resistance

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).}$$

Resistance of a straight homogeneous material of uniform cross section

Conductance $G = \dfrac{1}{R} = \delta \dfrac{S}{l} \qquad (S)\ or\ (\Omega^{-1})$

1. Three resistors having resistances 1 MΩ, 2 MΩ, and 4 MΩ are connected in parallel. Calculate the overall *conductance* and ***resistance***

Ans. (1) 1.75 µOhm, (2) 1.28 mm

2. Determine the dc-resistance of a 1km copper wire having a 1mm radius

- If an aluminium wire of the same length having the same resistance is used. What is the radius of the wire

- Analytically Explain the following equations related to **Steady Currents**

1. $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \qquad (\mathrm{A/m^3})$$

Continuity Equation

Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J}.\,ds = 0$
$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_S \frac{1}{\sigma}\mathbf{J}.\,dl = 0$

2. $$\nabla \cdot \mathbf{J} = 0$$

3. $$\sum_j I_j = 0 \qquad (\mathrm{A})$$

Boundary Conditions

$$J_{1n} = J_{2n} \qquad (\mathbf{A/m^2})$$

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

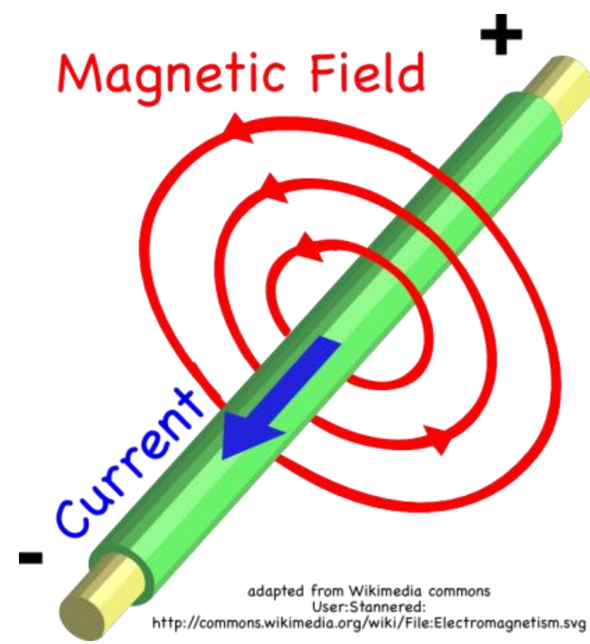
Two blocks of conducting material are in contact at the z=0 plane. At a point P in the interface, the current density is $J_1 = 10(3a_y + 4a_z)\,A/m^2$ in medium 1 of conductivity σ_1. Determine J_2 in medium 2 if $\sigma_2 = 2\sigma_1$

Ans. $J_2 = 20(3a_y + 4a_z)$ $\mathrm{A/m^2}$

5. Static Magnetic Fields

- We have discussed static fields where we discussed the vectors D and E, where $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ in a linear isotropic medium
- In this section, we'll proceed to discuss magnetism and introduce the vector quantities B and H
- Words like magnet, magnetism, magnetization come from the ancient city called Magnesia where the phenomenon of magnetism was discovered
- A magnetic field can be caused by a moving charge, or by current flow or by a permanent magnet

Permanent magnet



Current-carrying conductor

Electric intensity, E, is defined as the force on a stationary charge

$$\boldsymbol{F_e} = q\mathbf{E} \qquad (\mathbf{N})$$

When the test charge is moving in a magnetic field, with magnetic flux density B, it experiences the same force given as shown. Where **u** is the *velocity* of the moving charge in *meters/second*

$$\boldsymbol{F_m} = q\mathbf{u} \times \mathbf{B} \quad (\mathbf{N})$$

Then the total electromagnetic force (F_e + F_q) on a charge q is given as shown. **B** is measured in webers per sq. meter ($\boldsymbol{Wb/m^2}$)

$$\boxed{\boldsymbol{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad (\mathbf{N})}$$

Lorentz's Force equation

- Charges in motion produce a current. The current in turn, creates a magnetic field. If the current is steady, they are accompanied by static magnetic fields
- Therefore to understand the vector B, like Helmholtz's theory stated, we have to identify the divergence and curl of the B

Fundamental Postulates of Magnetostatics in Non-magnetic media (e.g. FREE SPACE)

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{J}$$

Differential form

$$\oint_S \mathbf{B} \cdot d\boldsymbol{s} = 0$$

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \mathbf{I}$$

Integral form

1. The divergence of the magnetic flux density is equal to zero (or vanishes)

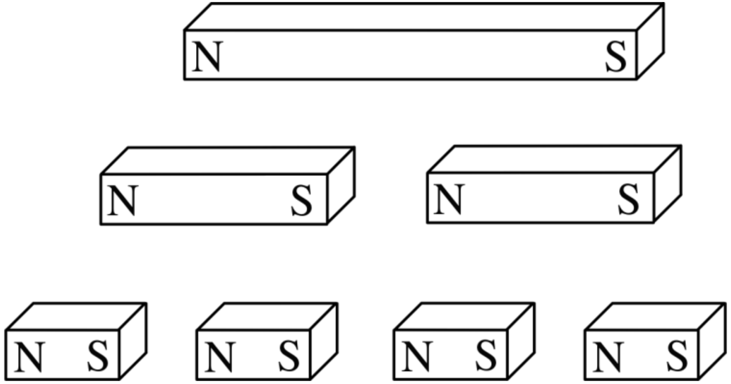
2. The curl of the static magnetic flux density in a non-magnetic medium is equal to the current density, **J**

3. Comparing the electrostatic divergence equation to the magnetostatic divergence equation, we can conclude that ***THERE ARE NO MAGNETIC FLOW SOURCES***

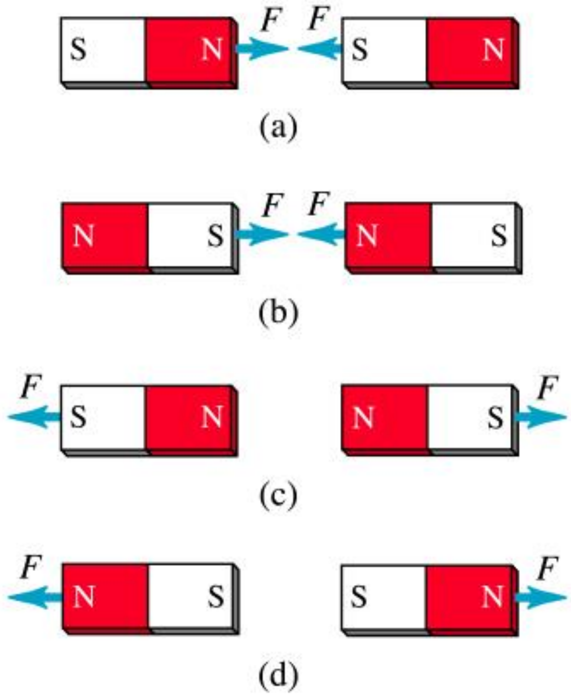
4. $\oint_S \mathbf{B} \cdot d\boldsymbol{s} = 0$ is also referred to as the law of conservation of magnetic flux, which is: The total outward magnetic flux through any closed surface is zero

The magnetic field

- Permanent magnets, like bar magnets, are usually designated into North and South poles
- However, an isolated north pole or an isolated south pole cannot exist
- For example, no matter how small a bar magnet is divided, there'll always be a north pole and a south pole
- This means, unlike electric charges, magnetic charges do not exist. It always exist in dipoles
- This will be further explained by Gauss' Law for magnetism (Maxwell's equations)



Successive division of a bar magnet



Attraction and repulsion of a magnet

Vector Magnetic Potential

The vector magnetic potential is given by

$$\mathbf{B} = \nabla \times \mathbf{V} \qquad \textbf{Tesla (T)}$$

To understand the magnetic potential V, we can have to define its divergence and curl according to Helmholtz

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{V} = \mu_0 \mathbf{J}$$

$$\boxed{\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}}$$

$$\nabla^2 \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \underbrace{\nabla \times \nabla \times \mathbf{V}}_{\mu_0 \mathbf{J}}$$

$$\nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} = \mu_0 \mathbf{J}$$

Using the Coulomb's conditions: $\nabla \cdot \mathbf{V} = \mathbf{0}$

$$\boxed{\nabla^2 \mathbf{V} = -\mu_0 \mathbf{J}}$$

Vector Poisson's equation

Static E-field potential

$$\mathbf{V} = \frac{1}{4\pi\varepsilon_0} \int_{v'} \frac{\rho_v}{R} dv'$$

$$\mathbf{V} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\mathbf{J}}{R} dv' \qquad (\mathrm{Wb/m})$$

Finding the vector magnetic potential from current density

$$\mathbf{J}\, dv' = JS\, d\boldsymbol{l}' = I\, d\boldsymbol{l}'$$

$$\mathbf{V} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{l}'}{R} \qquad (\mathrm{Wb/m})$$

Finding the vector magnetic potential from current in a closed circuit

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{l}' \times \boldsymbol{a_R}}{R^2} \qquad (\mathrm{T})$$

Biot-Savart Law

For finding Magnetic flux density from Current in a closed circuit

Just like with static E fields, a magnetic material under the influence of a magnetic field will be affected

Atoms have nucleus and orbiting electrons.
The orbiting electrons produces currents and cause magnetic dipoles

In the absence of an external electric field, the magnetic dipoles have random orientation and no net magnet dipole moment. The applications of an external magnetic field introduces dipole moments due to the movement and spinning of the electrons

The magnetic material becomes polarized and introduces a magnetization vector, **M**

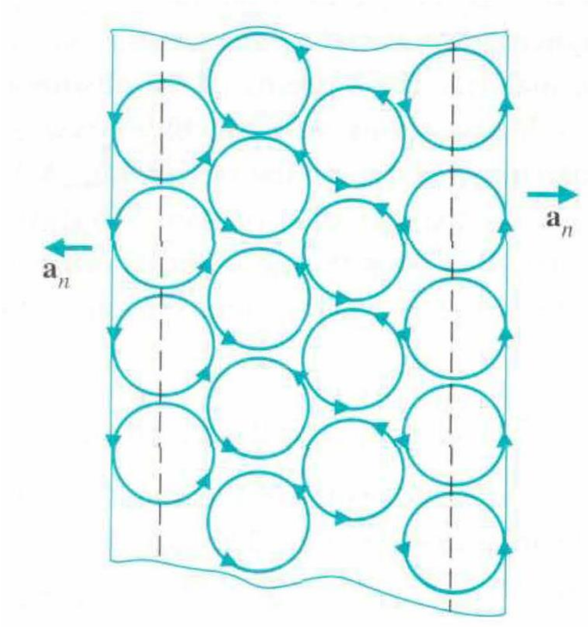
IF M is uniform inside a material, the currents in the dipoles will flow in opposite direction and cancel each other, leaving $\boldsymbol{J_{mv}}$ **= 0**, otherwise it is non-zero

$$\mathbf{J_{ms}} = \mathbf{M} \times \mathbf{a_n} \qquad (\mathrm{A/m})$$

Magnetization surface current density

$$\mathbf{J_{mv}} = \nabla \times \mathbf{M} \qquad (\mathrm{A/m^2})$$

Magnetization volume current density



$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J_{mv}})$$

Curl of magnetic flux density in the presence of a magnetized material

Magnetic field Intensity, H, and relative permeability, μ

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J_{mv}})$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \mathbf{J_{mv}} = \mathbf{J} + \nabla \times \mathbf{M}$$

$$\nabla \times \underbrace{\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right)}_{} = \mathbf{J}$$

$$\boxed{\mathbf{H} = \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) \qquad (\mathrm{A/m})}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \qquad (\mathrm{A/m^2})}$$

$$\int_s (\nabla \times \mathbf{H}) \cdot ds = \int_s \mathbf{J} \cdot ds$$

Using Stokes theorem

$$\boxed{\oint_c \mathbf{H} \cdot dl = I \quad (\mathrm{A})}$$

Ampere's circuital law for steady currents

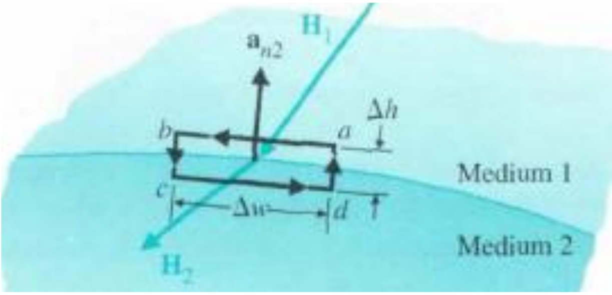
When the magnetic medium is linear and isotropic, the magnetization is proportional to the magnetic field intensity

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H}$$

$$= \mu_0\, \mu_r \mathbf{H}$$

$$\boxed{\mathbf{B} = \mu \mathbf{H} \qquad (\mathrm{Wb/m^2})}$$



$$\boxed{\begin{gathered} B_{1n} = B_{2n} \ (\mu_1 H_{1n} = \mu_2 H_{2n}) \\ \mathbf{H}_{1t} = \mathbf{H}_{2t} \end{gathered}}$$

Boundary Conditions

Magnetostatic Fields

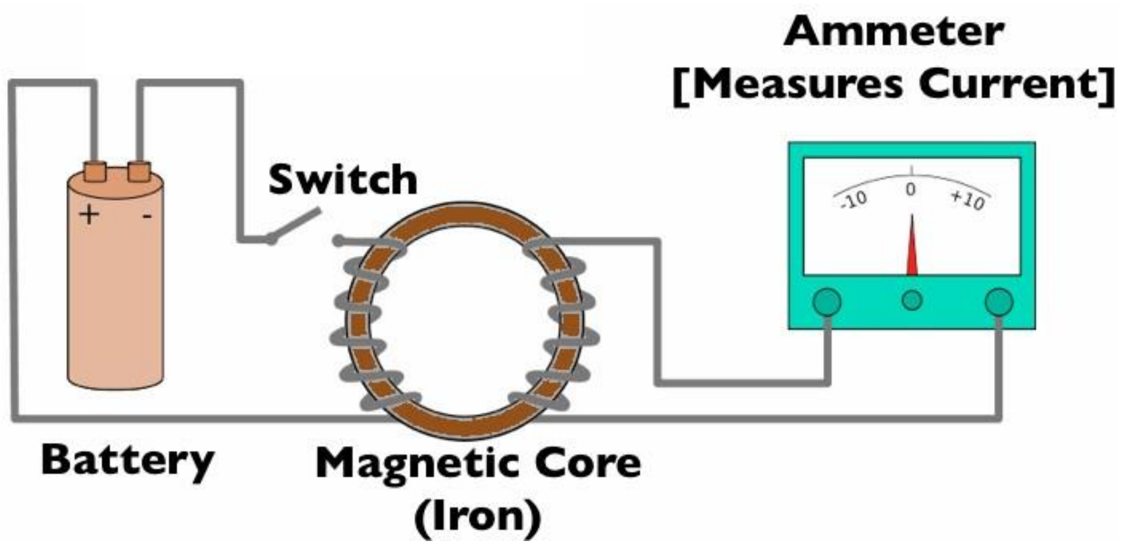
Static Electric Fields ***vs Static Magnetic Fields***

- PREPARE NOTES ON THIS PAPER FOR CLASS

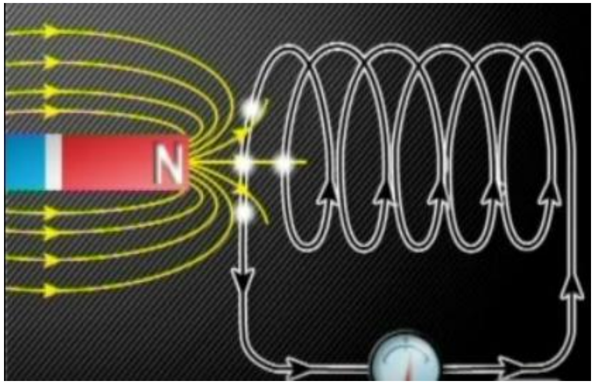
6. Time-Varying Fields *and Maxwell's Equations*

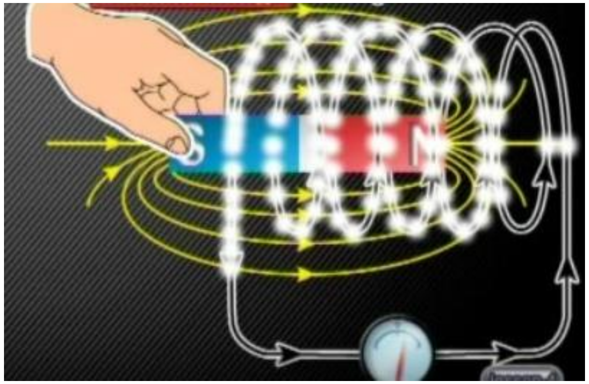
- Static models are simple but inadequate for explaining time-varying electromagnetic phenomena
- Static fields do not give rise to waves that propagates and carry energy
- Waves are the outcomes of EM action at a distance
- In time-varying fields, a model that relates the field vectors **E** and **D**, with **B** and **H** will be created
- In time-varying fields, the two divergence equations for static electric fields and static magnetic will be used, and their two curl equations will be modified so it is consistent with the law of conservation of charge
- The two modified curl equations and the two divergence equations are known as Maxwell's equations, which constitute the foundation of electromagnetics

FARADAY'S LAW



Faraday's Experimental setup





$$EMF = -\frac{d\phi}{dt} \qquad (\mathrm{V})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

FARADAY'S LAW

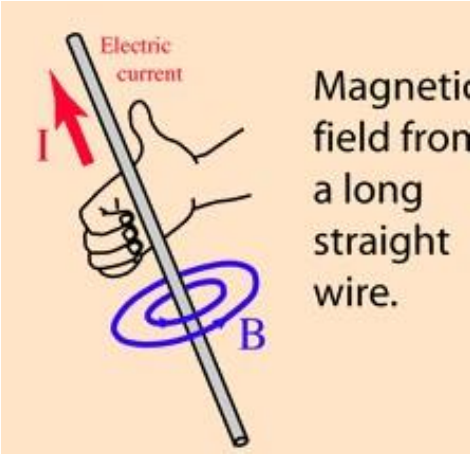
Interpretation of Faraday's law ??

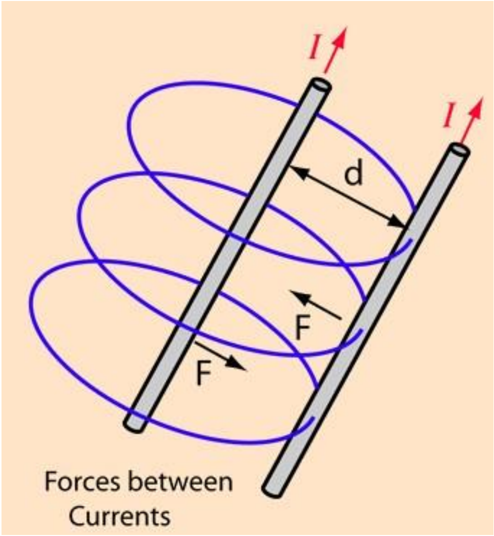
AMPERE'S LAW

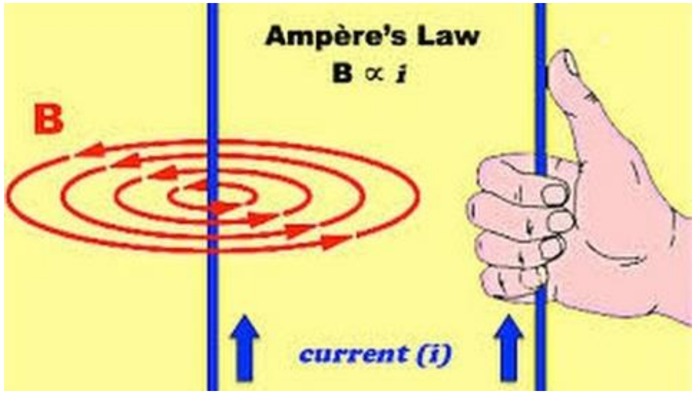
Ampere's Law relates the net magnetic field along a closed loop to the electric current flowing through the loop

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

Interpretation of Ampere's law ??







Maxwell's equations

- James Clerk Maxwell's [1837 – 1879] took a set of known experimental laws and unified them into a coherent set of equations known as Maxwell's equations
- Maxwell was one of the first to determine the speed of an EM wave was the same as the speed of light – and conclude that EM waves and visible light were really the same thing

Differential Form	Integral Form	
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_s \mathbf{D} \cdot d\boldsymbol{s} = \int_V \rho_v \, d\boldsymbol{v}$	**GAUSS'LAW**
$\nabla \cdot \mathbf{B} = 0$	$\oint_s \mathbf{B} \cdot d\boldsymbol{s} = 0$	**GAUSS'LAW for magnetism**
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_c \mathbf{E} \cdot dl = \int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{s}$	**FARADAY'S LAW**
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_c \mathbf{H} \cdot dl = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\boldsymbol{s}$	**AMPERE'S LAW**

- Relevance of Maxwell's third and Fourth Equations ??

- Boundary conditions of ***Time-Varying electric fields*** between two media

$$\boxed{\begin{gathered} \boldsymbol{E}_{1t} = \boldsymbol{E}_{2t} \quad (\mathrm{V/m}) \\ \boldsymbol{B}_{1n} = \boldsymbol{B}_{2n} \quad (\mathrm{T}) \\ \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\mathrm{A/m}) \\ \boldsymbol{D}_{1n} - \boldsymbol{D}_{2n} = \rho_S \quad (\mathrm{C/m^2}) \end{gathered}}$$

How do these boundary conditions change in lossless media ??

How do these boundary conditions change between a dielectric and a perfect conductor ??