SUPPLEMENTARY LECTURE NOTES

System of non-linear equations

Objectives

At the end of this section you will learn how to solve non-linear systems using

1. functional fixed-point iteration,
2. the Gauss-Seidel(Extra Reading) and
3. Newton's methods.

Introduction

Solutions $x = x_0$ to equations of the form $f(x) = 0$ are often required where it is impossible or infeasible to find an analytical expression for the vector ***x***. If the scalar function *f* depends on *n* independent variables $x_1, x_2, \ldots, x_n$, then the solution $\boldsymbol{x}_0$ will describe a surface in *n* – 1 dimensional space. Alternatively we may consider the vector function ***f***(***x***) = **0**, the solutions of which typically collapse to particular values of ***x***. For this course we restrict our attention to *n* independent variables $x_1, x_2, \ldots, x_n$ and seek solutions to $F(\mathrm{x}) = 0$ where *F* is vector valued.

Fixed Point Method for Functions of Several Variables

The general of a system of nonlinear equations is

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 0 \\
f_2(x_1, x_2, \ldots, x_n) &= 0 \\
&\vdots \\
f_n(x_1, x_2, \ldots, x_n) &= 0
\end{aligned}
$$

where each function f_i maps n-dimensional space, R^n, into the real line *R*. The above system can be defined alternatively by defining the function $F(\mathrm{x}) = 0$, where

$F : R^n \to R^n$, $\mathrm{x} = (x_1, x_2, \ldots, x_n)$ and

$F(x_1, x_2, \ldots, x_n) = \left(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n) \right)$.

1.1 Functional or Fixed Point Iteration

Suppose a nonlinear system of the form $F(\mathrm{x}) = 0$ has been transformed into an equivalent fixed point problem $G(\mathrm{x}) = \mathrm{x}$. The functional or fixed point iteration process applied to *G* is as follows:

1. Select $\mathrm{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \right)$.
2. Generate the sequence of vectors $\mathrm{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)} \right)$ by

   $\mathrm{x}^{(k)} = \mathrm{G}\left(\mathrm{x}^{(k-1)} \right)$ for each $i = 1, 2, 3, \ldots$ or, component-wise,

$$
\begin{aligned}
x_1^{(k)} &= g_1\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right)\\
x_2^{(k)} &= g_2\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right)\\
&\vdots \qquad \vdots\\
x_n^{(k)} &= g_n\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right)
\end{aligned}
$$

The following theorem provides conditions for the iterative process to converge.

Theorem

Let $D=\{(x_1,x_2,\ldots,x_n): a_i \le x_i \le b_i, \text{ for each } i=1,2,\ldots,n\}$, for some collection of constants $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$. Suppose G is a continuous function with continuous first partial derivatives from $D\subset R^n$ into R^n with the property that $G(\mathrm{x})\in D$ whenever $\mathrm{x}\in D$. If a constant $K<1$ exists with

$$\left|\frac{\partial g_i(\mathrm{x})}{\partial x_j}\right| \le \frac{K}{n} \quad \text{whenever} \quad \mathrm{x}\in D$$

for each $j=1,2,\ldots,n$ and each component function g_i, then the sequence $\left\{\mathrm{x}^{(k)}\right\}_{k=0}^{\infty}$ defined by $\mathrm{x}^{(k)}=\mathrm{G}\left(\mathrm{x}^{(k-1)}\right)$ for each $i=1,2,3,\ldots$ converges to the unique fixed point $\mathrm{p}\in D$, for any $\mathrm{x}^{(0)}$ in D, and $\left\|\mathrm{x}^{(j)}-\mathrm{p}\right\|_\infty \le \dfrac{K^j}{1-K}\left\|\mathrm{x}^{(1)}-\mathrm{x}^{(0)}\right\|_\infty$.

Example 1

$$
\begin{aligned}
3x_1-\cos(x_2x_3) \qquad\qquad -\tfrac{1}{2} &= 0\\
x_1^2-81(x_2+0.1)^2+\sin x_3+1.06 &= 0\\
e^{-x_1x_2} \qquad\qquad +20x_3 + \tfrac{10\pi-3}{3} &= 0
\end{aligned}
$$

$$
\begin{aligned}
f_1(x_1,x_2,\ldots,x_n) &= 3x_1-\cos(x_2x_3)-\tfrac{1}{2}\\
f_2(x_1,x_2,\ldots,x_n) &= x_1^2-81(x_2+0.1)^2+\sin x_3+1.06\\
f_3(x_1,x_2,\ldots,x_n) &= e^{-x_1x_2}+20x_3 + \tfrac{10\pi-3}{3}
\end{aligned}
$$

$$
\begin{aligned}
F(x_1,x_2,\ldots,x_n) &= \left(f_1(x_1,x_2,\ldots,x_n),\, f_2(x_1,x_2,\ldots,x_n),\ldots,\, f_n(x_1,x_2,\ldots,x_n)\right)\\
&= \left(3x_1-\cos(x_2x_3)-\tfrac{1}{2},\, x_1^2-81(x_2+0.1)^2+\sin x_3+1.06, e^{-x_1x_2}+20x_3 + \tfrac{10\pi-3}{3}\right)
\end{aligned}
$$

Example 2

$$
\begin{aligned}
3x_1-\cos(x_2x_3) \qquad\qquad -\tfrac{1}{2} &= 0\\
x_1^2-81(x_2+0.1)^2+\sin x_3+1.06 &= 0\\
e^{-x_1x_2} \qquad\qquad +20x_3 + \tfrac{10\pi-3}{3} &= 0
\end{aligned}
$$

If the ith equation is solved for x_i, the system can be changed into the fixed point problem

$$x_1 = \tfrac{1}{3}\cos(x_2x_3) + \tfrac{1}{6},$$
$$x_2 = \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1,$$
$$x_3 = -\tfrac{1}{20}e^{-x_1x_2} - \tfrac{10\pi-3}{60}.$$

Let $G: R^3 \to R^3$ be defined by $G(\mathrm{x}) = \left(g_1(\mathrm{x}), g_2(\mathrm{x}), g_3(\mathrm{x})\right)$ where

$$g_1(x_1,x_2,x_3) = \tfrac{1}{3}\cos(x_2x_3) + \tfrac{1}{6},$$
$$g_2(x_1,x_2,x_3) = \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1,$$
$$g_3(x_1,x_2,x_3) = -\tfrac{1}{20}e^{-x_1x_2} - \tfrac{10\pi-3}{60}.$$

$$|g_1(x_1,x_2,x_3)| \le \tfrac{1}{3}|\cos(x_2x_3)| + \tfrac{1}{6} \le \tfrac{1}{2},$$
$$|g_2(x_1,x_2,x_3)| = \left|\tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1\right|$$
$$\le \tfrac{1}{9}\sqrt{1+\sin 1 + 1.06} - 0.1 < 0.90,$$
$$|g_3(x_1,x_2,x_3)| = \tfrac{1}{20}e^{-x_1x_2} + \tfrac{10\pi-3}{60}$$
$$\le \tfrac{1}{20}e + \tfrac{10\pi-3}{60} < 0.61$$

.

so $-1 \le g_i(x_1,x_2,x_3) \le 1$, for each $i = 1,2,3$. Thus, $G(\mathrm{x}) \in D$ whenever $\mathrm{x} \in D$.
Finding bounds on the partial derivatives on *D* gives the following:

$$\left|\frac{\partial g_1}{\partial x_1}\right| = 0, \quad \left|\frac{\partial g_1}{\partial x_2}\right| \le \tfrac{1}{3}|x_3||\sin(x_2x_3)| \le \tfrac{1}{3}\sin 1 = 0.281, \quad \left|\frac{\partial g_1}{\partial x_3}\right| \le \tfrac{1}{3}|x_2||\sin(x_2x_3)| \le \tfrac{1}{3}\sin 1 = 0.281,$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = \frac{|x_1|}{9\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{9\sqrt{0.218}} < 0.238, \quad \left|\frac{\partial g_2}{\partial x_2}\right| \le 0,$$
$$\left|\frac{\partial g_2}{\partial x_3}\right| \le \frac{|\cos x_3|}{18\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{18\sqrt{0.218}} < 0.119,$$

$$\left|\frac{\partial g_3}{\partial x_1}\right| = \frac{|x_2|}{20}e^{-x_1x_2} \le \tfrac{1}{20}e = 0.14, \quad \left|\frac{\partial g_3}{\partial x_2}\right| = \frac{|x_1|}{20}e^{-x_1x_2}$$

Since the partial derivatives are bounded on *D*, the above Theorem implies that these functions are continuous on *D*. Consequently, *G* is continuous on *D*. Moreover, for every $x \in D$

$\left|\dfrac{\partial g_i(\mathrm{x})}{\partial x_j}\right| \le 0.281$ for each $i = 1,2,3$ and $j = 1,2,3,$ and the condition in the second part of Theorem 9.7 holds for $K = 0.843$. It can be shown that $\partial g_i(\mathrm{x})/\partial x_j$ for each $i = 1,2,3$ and $j = 1,2,3$ is continuous on *D*. Consequently, *G* has a unique fixed point on *D* and the nonlinear system has a solution in *D*.

Example 3

$$\begin{aligned} 3x_1 - \cos(x_2 x_3) \qquad\qquad & - \tfrac{1}{2} &= 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 & + 1.06 &= 0 \\ e^{-x_1 x_2} \qquad\qquad + 20x_3 & + \tfrac{10\pi - 3}{3} &= 0 \end{aligned}$$

Solution

If the i[th] equation is solved for x_i, the system can be changed into the fixed point problem as

$$\begin{aligned} x_1 &= \tfrac{1}{3}\cos(x_2 x_3) + \tfrac{1}{6}, \\ x_2 &= \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ x_3 &= -\tfrac{1}{20} e^{-x_1 x_2} - \tfrac{10\pi - 3}{60}. \end{aligned}$$

and write the iterative process as

$$\begin{aligned} x_1^{(k)} &= \tfrac{1}{3}\cos\left(x_2^{(k-1)} x_3^{(k-1)}\right) + \tfrac{1}{6}, \\ x_2^{(k)} &= \tfrac{1}{9}\sqrt{\left(x_1^{(k-1)}\right)^2 + \sin x_3^{(k-1)} + 1.06} - 0.1, \\ x_3^{(k)} &= -\tfrac{1}{20} e^{-x_1^{(k-1)} x_2^{(k-1)}} - \tfrac{10\pi - 3}{60}. \end{aligned}$$

$$\left\|\mathrm{x}^{(k)} - \mathrm{x}^{(k-1)}\right\|_\infty < 10^{-5}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\|\mathrm{x}^{(k)} - \mathrm{x}^{(k-1)}\right\|_\infty$
0	0.10000000	0.10000000	–0.10000000	--
1	0.49998333	0.00944115	–0.52310127	0.423
2	0.49999593	0.00002557	–0.52336331	9.4×10^{-3}
3	0.50000000	0.00001234	–0.52359814	2.3×10^{-4}
4	0.50000000	0.00000003	–0.52359847	1.2×10^{-5}
5	0.50000000	0.00000002	–0.52359877	3.1×10^{-7}

$$\left\|\mathrm{x}^{(5)}-\mathrm{p}\right\|_\infty \le \frac{(0.843)^5}{1-0.843}0.423 < 1.15$$

$$\mathrm{x}^{(3)} = \left(0.50000000, 1.234\times10^{-5}, -0.52359814\right)$$

$$\left\|\mathrm{x}^{(5)}-\mathrm{p}\right\|_\infty \le \frac{(0.843)^5}{1-0.843}(1.20\times10^{-5}) < 5.5\times10^{-5}$$

$$\mathrm{p} = \left(0.5, 0, -\tfrac{\pi}{6}\right) \approx \left(0.5, 0, -0.5235987757\right)$$

$$\left\|\mathrm{x}^{(5)}-\mathrm{p}\right\|_\infty \le 2\times10^{-8}$$

$$\begin{aligned} x_1^{(k)} &= \tfrac{1}{3}\cos\left(x_2^{(k-1)}x_3^{(k-1)}\right)+\tfrac{1}{6}, \\ x_2^{(k)} &= \tfrac{1}{9}\sqrt{\left(x_1^{(k)}\right)^2+\sin x_3^{(k-1)}+1.06}-0.1, \\ x_3^{(k)} &= -\tfrac{1}{20}e^{-x_1^{(k)}x_2^{(k)}}-\tfrac{10\pi-3}{60}. \end{aligned}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\|\mathrm{x}^{(k)}-\mathrm{x}^{(k-1)}\right\|_\infty$
0	0.10000000	0.10000000	–0.10000000	--
1	0.49998333	0.02222979	–0.52304613	0.423
2	0.49997747	0.00002815	–0.52359807	2.2×10^{-2}
3	0.50000000	0.00000004	–0.52359877	2.8×10^{-5}
4	0.50000000	0.00000000	–0.52359877	3.8×10^{-8}

Newton's Method

In order to construct an algorithm that led to an appropriate fixed-point method

$$A(\mathrm{x}) = \begin{pmatrix} a_{11}(\mathrm{x}) & a_{12}(\mathrm{x}) & \cdots & a_{1n}(\mathrm{x}) \\ a_{21}(\mathrm{x}) & a_{22}(\mathrm{x}) & \cdots & a_{2n}(\mathrm{x}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\mathrm{x}) & a_{n2}(\mathrm{x}) & \cdots & a_{nn}(\mathrm{x}) \end{pmatrix}$$

where each of the entries $a_{ij}(\mathrm{x})$ is a function from $R^n \to R$. The procedure requires that $A(\mathrm{x})$ be found so that $G(\mathrm{x}) = \mathrm{x} - A(\mathrm{x})^{-1}F(\mathrm{x})$ gives quadratic convergence to the solution $F(\mathrm{x}) = 0$, provided that $A(\mathrm{x})$ is nonsingular at the fixed point.

Theorem

Suppose p is a solution of $G(\mathrm{x})=\mathrm{x}$ for some function $G=(g_1,g_2,\ldots,g_n)$, mapping R^n into R^n. If a number $\delta>0$ exists with the property that

i) $\partial g_i/\partial x_j$ is continuous on $N_\delta=\{\mathrm{x}:\|\mathrm{x}-\mathrm{p}\|<\delta\}$ for each $i=1,2,\ldots,n$ and $j=1,2,\ldots,n$,

ii) $\partial^2 g_i(\mathrm{x})/\partial x_j\,\partial x_k$ is continuous, and $\left\|\partial^2 g_i(\mathrm{x})/\partial x_j\,\partial x_k\right\|\le M$ for some constant *M* whenever $\mathrm{x}\in N_\delta$ for each $i=1,2,\ldots,n,\ \ j=1,2,\ldots,n$ and $k=1,2,\ldots,n$,

iii) $\partial g_i(\mathrm{p})/\partial x_j=0$ for each $i=1,2,\ldots,n$ and $j=1,2,\ldots,n$,

then the sequence generated by $\mathrm{x}^{(k)}=G\left(\mathrm{x}^{(k-1)}\right)$ converges quadratically to p for any choice of $\mathrm{x}^{(0)}\in N_\delta$ and $\left\|\mathrm{x}^{(k)}-\mathrm{p}\right\|_\infty\le\dfrac{n^2M}{2}\left\|\mathrm{x}^{(k-1)}-\mathrm{p}\right\|_\infty^2$ for each $k\ge1$.

Since $G(\mathrm{x})=\mathrm{x}-A(\mathrm{x})^{-1}F(\mathrm{x})$, $g_i(\mathrm{x})=x_i-\sum_{j=1}^{n}b_{ij}(\mathrm{x})\,f_j(\mathrm{x})$;

so
$$\frac{\partial g_i(\mathrm{x})}{\partial x_k}=\begin{cases}1-\displaystyle\sum_{j=1}^{n}\left(b_{ij}(\mathrm{x})\frac{\partial f_j}{\partial x_k}(\mathrm{x})+\frac{\partial b_{ij}}{\partial x_k}(\mathrm{x})\,f_j(\mathrm{x})\right), & \text{if } i=k,\\ -\displaystyle\sum_{j=1}^{n}\left(b_{ij}(\mathrm{x})\frac{\partial f_j}{\partial x_k}(\mathrm{x})+\frac{\partial b_{ij}}{\partial x_k}(\mathrm{x})\,f_j(\mathrm{x})\right), & \text{if } i\ne k.\end{cases}$$

$$0=1-\sum_{j=1}^{n}b_{ij}(\mathrm{p})\frac{\partial f_j}{\partial x_k}(\mathrm{p})$$

so $\displaystyle\sum_{j=1}^{n}b_{ij}(\mathrm{p})\frac{\partial f_j}{\partial x_k}(\mathrm{p})=1$

$$0=-\sum_{j=1}^{n}b_{ij}(\mathrm{p})\frac{\partial f_j}{\partial x_k}(\mathrm{p})$$

So $\displaystyle\sum_{j=1}^{n}b_{ij}(\mathrm{p})\frac{\partial f_j}{\partial x_k}(\mathrm{p})=0$

$$J(\mathrm{x})=\begin{pmatrix}\frac{\partial f_1(\mathrm{x})}{\partial x_1} & \frac{\partial f_1(\mathrm{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathrm{x})}{\partial x_n}\\ \frac{\partial f_2(\mathrm{x})}{\partial x_1} & \frac{\partial f_2(\mathrm{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathrm{x})}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_n(\mathrm{x})}{\partial x_1} & \frac{\partial f_n(\mathrm{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathrm{x})}{\partial x_n}\end{pmatrix}$$

$$A(\mathrm{p})^{-1}J(\mathrm{p})=I$$

so
$$J(\mathrm{p})=A(\mathrm{p})$$
$$G(\mathrm{x})=\mathrm{x}-J(\mathrm{x})^{-1}F(\mathrm{x})$$
$$\mathrm{x}^{(k)}=G\left(\mathrm{x}^{(k\text{-}1)}\right)=\mathrm{x}^{(k\text{-}1)}-J\left(\mathrm{x}^{(k\text{-}1)}\right)^{-1}F\left(\mathrm{x}^{(k\text{-}1)}\right)$$

Example 1

Solve the nonlinear system

$$\begin{aligned}3x_1-\cos(x_2x_3) \qquad\qquad -\tfrac{1}{2} &= 0\\ x_1^2-81(x_2+0.1)^2+\sin x_3+1.06 &= 0\\ e^{-x_1x_2} \qquad\qquad +20x_3+\tfrac{10\pi-3}{3} &= 0\end{aligned}$$

Solution

The Jacobian matrix for the system is given by

$$J(x_1,x_2,x_3)=\begin{pmatrix}3 & x_3\sin x_2x_3 & x_2\sin x_2x_3\\ 2x_1 & -162(x_2+0.1) & \cos x_3\\ -x_2e^{-x_1x_2} & -x_1e^{-x_1x_2} & 20\end{pmatrix}$$

and

$$\begin{pmatrix}x_1^{(k)}\\ x_2^{(k)}\\ x_3^{(k)}\end{pmatrix}=\begin{pmatrix}x_1^{(k-1)}\\ x_2^{(k-1)}\\ x_3^{(k-1)}\end{pmatrix}+\begin{pmatrix}h_1^{(k-1)}\\ h_2^{(k-1)}\\ h_3^{(k-1)}\end{pmatrix}$$

where

$$\begin{pmatrix}h_1^{(k-1)}\\ h_2^{(k-1)}\\ h_3^{(k-1)}\end{pmatrix}=-\left[J\left(x_1^{(k-1)},x_2^{(k-1)},x_3^{(k-1)}\right)\right]^{-1}F\left(x_1^{(k-1)},x_2^{(k-1)},x_3^{(k-1)}\right).$$

Thus, at the k^{th} step, the linear system

$$\begin{pmatrix} 3 & x_3^{(k-1)}\sin x_2^{(k-1)}x_3^{(k-1)} & x_2\sin x_2^{(k-1)}x_3^{(k-1)} \\ 2x_1^{(k-1)} & -162(x_2^{(k-1)}+0.1) & \cos x_3^{(k-1)} \\ -x_2^{(k-1)}e^{-x_1^{(k-1)}x_2^{(k-1)}} & -x_1^{(k-1)}e^{-x_1^{(k-1)}x_2^{(k-1)}} & 20 \end{pmatrix}\begin{pmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{pmatrix}$$
$$= \begin{pmatrix} 3x_1^{(k-1)} - \cos(x_2^{(k-1)}x_3^{(k-1)}) - \frac{1}{2} \\ \left(x_1^{(k-1)}\right)^2 - 81(x_2^{(k-1)}+0.1)^2 + \sin x_3^{(k-1)} + 1.06 \\ e^{-x_1^{(k-1)}x_2^{(k-1)}} + 20x_3^{(k-1)} + \frac{10\pi-3}{3} \end{pmatrix}$$

must be solved. The results obtained using the above iterative procedure is as shown below

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\lVert \mathrm{x}^{(k)} - \mathrm{x}^{(k-1)} \right\rVert_\infty$
0	0.10000000	0.10000000	–0.10000000	--
1	0.50003702	0.01946686	–0.52152047	0.422
2	0.50004593	0.00158859	–0.52355711	1.79×10^{-2}
3	0.50000034	0.00001244	–0.52359845	1.58×10^{-3}
4	0.50000000	0.00000000	–0.52359877	1.24×10^{-5}
5	0.50000000	0.00000000	–0.52359877	0

$$\mathrm{p} = \left(0.5, 0, -\tfrac{\pi}{6}\right) \approx \left(0.5, 0, -0.5235987757\right)$$

Examples

1. The nonlinear system

$$\underline{\mathrm{F}}(x,y) = \begin{bmatrix} x^2 - 10x + y^2 + 8 \\ xy^2 + x - 10y + 8 \end{bmatrix} = 0$$

can be transformed into the fixed point problem

$$\mathrm{G}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x^2+y^2+8}{10} \\ \frac{xy^2+x+8}{10} \end{bmatrix}$$

(a) Starting with the initial estimates $x_0 = y_0 = 0$, apply functional iteration to G to approximate the solution to an accuracy of 10^{-5}.

(b) Does Gauss-Seidel Method accelerate convergence?

Solution

(a)

i	x	y	g1(x,y)	g2(x,y)	Tol
0	0	0	0.8	0.8	
1	0.8	0.8	0.928	0.9312	0.8
2	0.928	0.9312	0.972832	0.97327	0.1312
3	0.972832	0.97327	0.989366	0.989435	0.044832
4	0.989366	0.989435	0.995783	0.995794	0.016534
5	0.995783	0.995794	0.998319	0.998321	0.006417
6	0.998319	0.998321	0.999328	0.999329	0.002536
7	0.999328	0.999329	0.999732	0.999732	0.00101
8	0.999732	0.999732	0.999893	0.999893	0.000403
9	0.999893	0.999893	0.999957	0.999957	0.000161
10	0.999957	0.999957	0.999983	0.999983	6.44E-05
11	0.999983	0.999983	0.999993	0.999993	2.58E-05
12	0.999993	0.999993	0.999997	0.999997	1.03E-05
13	0.999997	0.999997	0.999999	0.999999	4.12E-06
14	0.999999	0.999999	1	1	1.65E-06

(b)

i	x	y	g1(x,y)	g2(x,y)	Tol
0	0	0	0.8	0.88	
1	0.8	0.88	0.94144	0.967049	0.88
2	0.94144	0.967049	0.982149	0.990064	0.14144
3	0.982149	0.990064	0.994484	0.99693	0.040709
4	0.994484	0.99693	0.998287	0.999045	0.012335
5	0.998287	0.999045	0.999467	0.999703	0.003803
6	0.999467	0.999703	0.999834	0.999907	0.00118
7	0.999834	0.999907	0.999948	0.999971	0.000367
8	0.999948	0.999971	0.999984	0.999991	0.000114
9	0.999984	0.999991	0.999995	0.999997	3.56E-05
10	0.999995	0.999997	0.999998	0.999999	1.11E-05
11	0.999998	0.999999	1	1	3.46E-06

(c)

From (a) and (b) it is seen that Gauss-Seidel Method accelerate convergence.

2. Convert the nonlinear system

$$\begin{aligned} 3x - \cos(yz) - \tfrac{1}{2} &= 0 \\ x^2 - 81\left(y + \tfrac{1}{10}\right)^2 + \sin z + 1.06 &= 0 \\ e^{-xy} + 20z \ + \tfrac{10\pi - 3}{3} &= 0 \end{aligned}$$

to a fixed point problem and use both functional iteration and the Gauss-Seidel variant of functional iteration to approximate the root to within 10^{-5} in the l_∞ norm, starting the initial estimate $x_0 = y_0 = 0.1$ and $z_0 = -0.1$. [Note the exact root is $\left(\tfrac{1}{2}, 0, -\tfrac{\pi}{6}\right)^T$]

Solution

$$x = \frac{\cos(yz) + \frac{1}{2}}{3}, \quad y = \sqrt{\frac{x^2 + \sin z + 1.06}{81}} - \frac{1}{10}, \quad z = \frac{-e^{-xy} - \frac{10\pi - 3}{3}}{20}$$

For $k = 1, 2, \ldots$, the functional iteration and the Gauss-Seidel variant of functional iteration are given as

$$x^{(k)} = \frac{\cos(y^{(k-1)} z^{(k-1)}) + \frac{1}{2}}{3}$$

$$y^{(k)} = \sqrt{\frac{(x^{(k-1)})^2 + \sin z^{(k-1)} + 1.06}{81}} - \frac{1}{10} \quad \text{and}$$

$$z^{(k)} = \frac{-e^{-x^{(k-1)} y^{(k-1)}} - \frac{10\pi - 3}{3}}{20}$$

$$x^{(k)} = \frac{\cos(y^{(k-1)} z^{(k-1)}) + \frac{1}{2}}{3}$$

$$y^{(k)} = \sqrt{\frac{(x^{(k)})^2 + \sin z^{(k-1)} + 1.06}{81}} - \frac{1}{10} \quad \text{respectively for the above fixed point}$$

$$z^{(k)} = \frac{-e^{-x^{(k)} y^{(k)}} - \frac{10\pi - 3}{3}}{20}$$

problem.

Using the functional iteration we have the following table:

i	x	y	z	g1(x,y,z)	g2(x,y,z)	g3(x,y,z)	Tol
0	0.1	0.1	-0.1	0.499983	0.009441	-0.5231013	
1	0.499983	0.009441	-0.523101	0.499996	2.56E-05	-0.5233633	0.399983
2	0.499996	2.56E-05	-0.523363	0.5	1.23E-05	-0.5235981	0.009154
3	0.5	1.23E-05	-0.523598	0.5	3.42E-08	-0.5235985	4.07E-06
4	0.5	3.42E-08	-0.523598	0.5	1.65E-08	-0.5235988	1.2E-05
5	0.5	1.65E-08	-0.523599	0.5	4.57E-11	-0.5235988	6.95E-12
6	0.5	4.57E-11	-0.523599	0.5	2.2E-11	-0.5235988	1.6E-08
7	0.5	2.2E-11	-0.523599	0.5	6.1E-14	-0.5235988	0
8	0.5	6.1E-14	-0.523599	0.5	2.94E-14	-0.5235988	2.14E-11
9	0.5	2.94E-14	-0.523599	0.5	0	-0.5235988	0
10	0.5	0	-0.523599	0.5	0	-0.5235988	2.87E-14

Using the Gauss-Seidel variant of functional iteration we have the following table:

i	x	y	z	g1(x,y,z)	g2(x,y,z)	g3(x,y,z)	Tol
0	0.1	0.1	-0.1	0.499983	0.02223	-0.5230461	
1	0.499983	0.02223	-0.523046	0.499977	2.82E-05	-0.5235981	0.399983
2	0.499977	2.82E-05	-0.523598	0.5	3.76E-08	-0.5235988	0.02165
3	0.5	3.76E-08	-0.523599	0.5	5.03E-11	-0.5235988	2.74E-05
4	0.5	5.03E-11	-0.523599	0.5	6.72E-14	-0.5235988	3.66E-08
5	0.5	6.72E-14	-0.523599	0.5	0	-0.5235988	4.9E-11
6	0.5	0	-0.523599	0.5	0	-0.5235988	6.55E-14

3. Starting with the initial guess $x_0 = y_0 = 1.0$, use fixed point (functional iteration to approximate the solution to the system
$$2x^2 + y^2 = 4.32, \quad x^2 - y^2 = 0$$
by performing 5 iterations.
4. Consider the nonlinear system
$$2x + xy - 1 = 0$$
$$2y - xy + 1 = 0$$
which has a unique root $x = (1, -1)^T$. Starting with the initial estimate $x_0 = y_0 = 0$, compare the methods of functional iteration, Gauss-Seidel Newton when approximating the root of this system (perform 5 iterations in each case).
5. Use Newton's Method to approximate the solution of the nonlinear system
$$x^2 - 2x - y + \tfrac{1}{2} = 0$$
$$x^2 + 4y^2 - 4 = 0$$
starting with the initial estimate $\left(x_0, y_0\right) = \left(2, \tfrac{1}{4}\right)$ and computing 3 iterations.

Solution

Using the Newton's iterative method
$x^{(k)} = x^{(k-1)} - J^{-1}(x^{(k-1)})F(x^{(k-1)}), \quad \text{for } k = 1, \ldots$ with $x^{(0)} = \left(x_0, y_0\right) = \left(2, \tfrac{1}{4}\right)$, we have
$$x^{(1)} = x^{(0)} - J^{-1}(x^{(0)})F(x^{(0)}) = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8}\begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8}\begin{pmatrix} 0.75 \\ -0.50 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix}$$
$$x^{(2)} = x^{(1)} - J^{-1}(x^{(1)})F(x^{(1)}) = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \begin{pmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{pmatrix}^{-1} \begin{pmatrix} 0.00879 \\ -1.70312 \end{pmatrix}$$
$$= \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \frac{1}{8.34375}\begin{pmatrix} 2.5 & 1 \\ -3.8125 & 1.8125 \end{pmatrix}\begin{pmatrix} 0.00879 \\ -1.70312 \end{pmatrix} = \begin{pmatrix} 2.10773 \\ 0.68232 \end{pmatrix}$$
6. Use Newton's Method to approximate the two solutions of the nonlinear system
$$ye^x = 2, \quad x^2 + y^2 = 4$$
by computing 2 iterations for each of the given initial estimates
(a) $\left(x_0, y_0\right) = \left(-0.6, +3.7\right)$
(b) $\left(x_0, y_0\right) = \left(+1.9, +0.4\right)$
7. Use Newton's Method to approximate the solution of the nonlinear system
$$x^2 + y^2 + 0.6y - 0.16 = 0$$
$$x^2 - y^2 + x - 1.6y = 0$$
by computing 3 iterations with the initial estimate of $\left(x_0, y_0\right) = \left(0.6, 0.25\right)$.
Using the more accurate initial estimate of $\left(x_0, y_0\right) = \left(0.3, 0.1\right)$, repeat the process using the modified Newton's method whereby the Jacobian is evaluated and held constant for subsequent iterations. Compare the two results.