Solutions to Homework 9

Section 12.7 # 12: Let D be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Setup integrals in cylindrical coordinates which compute the volume of D.

Solution:

The intersection of the paraboloid and the cone is a circle. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2+y^2} = r$ (assuming r is non-negative), $2 - r^2 = r$, which implies that $r^2 + r - 2 = (r+2)(r-1) = 0$. Since $r \geq 0$, $r = 1$. Therefore, $z = 1$. So, the intersection of these surfaces is a circle of radius 1 in the plane $z = 1$.

(a) Use $dV = r\,dz\,dr\,d\theta$.

The cone is the lower bound for z and the paraboloid is the upper bound for z, as is clear from a sketch of the figure. The projection (i.e. the shadow) of the region onto the xy-plane is the circle of radius 1 centered at the origin. Therefore,

$$\iiint_D dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r\,dz\,dr\,d\theta.$$

(b) Use $dV = r\,dr\,dz\,d\theta$.

The region is not simple in the r-direction. The lower bound for r is zero, but the upper bound is sometimes the cone $z = r$ and sometimes the paraboloid $z = 2 - r^2$. The plane $z = 1$ divides D into two r-simple regions. Therefore,

$$\iiint_D dV = \int_0^{2\pi} \int_0^1 \int_0^z r\,dr\,dz\,d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta.$$

(c) Use $dV = r\,d\theta\,dz\,dr$.

There is no restriction on θ as this region is rotationally symmetric. However, z is still constrained by the cone and the parabola. Therefore,

$$\iiint_D dV = \int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r\,d\theta\,dz\,dr.$$

Section 12.7 # 18: Let D be the region enclosed by the cylinders $r = \cos\theta$ and $r = 2\cos\theta$ and by the planes $z = 0$ and $z = 3 - y$. Set up an iterated integral which computes $\iiint_D f(r,\theta,z)\, dz\, r\, dr\, d\theta$.

Solution: Since $0 \le z \le 3 - y$, it follows that $0 \le z \le 3 - r\sin\theta$ in cylindrical coordinates. The projection of D onto the xy-plane is the region between the circles given in polar coordinates by $r = \cos\theta$ and $r = 2\cos\theta$. The first circle is inside the second, and these two circles intersect when $\theta = -\pi/2, \pi/2$. This can be seen from a sketch or by solving the equation $\cos\theta = 2\cos\theta$; gather like terms to obtain $0 = \cos\theta$. Therefore,

$$\iiint_D f(r,\theta,z)\, dV = \int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} \int_0^{3-r\sin\theta} f(r,\theta,z)\, r\, dz\, dr\, d\theta.$$

Note: You cannot double the integral and integtrate over $0 \le \theta \le \pi/2$. Doubling is only permissible if the function $f(r,\theta,z)$ is even with respect to the variable θ.

Section 12.7 # 32:a Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up triple integrals which compute the volume of D.

Solution:

(a) Use $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

If the point P lies in the region D, then varying its ρ-coordinate keeps P inside D so long as $0 \le \rho \le \sec\phi$. The upper bound is determined by the plane $z = 1$, which has equation $z = \rho\cos\phi = 1$ in spherical coordinates; solving for ρ yields $\rho = \sec\phi$.

Ignoring ρ (projecting onto $\rho = 1$ for instance), one see that the variable ϕ varies from 0 to $\pi/4$. Finally, since the figure is rotationlly symmetric, θ varies from 0 to 2π. Therefore,

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

(b) Use $dV = \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta$.

This integral is tricky to set up. If P lies in the region D, then varying its ϕ-coordinate keeps P inside D so long as either its distance from the origin is less than or equal to one and $0 \leq \phi \leq \pi/4$, or its distance from the origin is greater than or equal to one and its ϕ-coordinate is bounded below by the restriction that $z = 1$ and above by $\pi/4$. In other words, this region is not ϕ-simple: two integrals are required. For the second integral, the condition $z = 1$ implies that $\rho \cos\phi = 1$ so that $\sec^{-1}\rho \leq \phi \leq \pi/4$. Ignoring, ϕ, then the z coordinate can vary from 1 to $\sqrt{2}$; the upper bound is determined by the maximum distance from the origin to a point inside the region D, which is realized by a point which lies on the intersection of the cone with the plane $z = 1$.

The above shows that

$$\iiint_D dV = \int_0^{2\pi} \int_0^1 \int_0^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\sec^{-1}(\rho)}^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta.$$

Section 12.7 # 34: Set up an integral in spherical coordinates which computes the volume of the region bounded below by the hemisphere $\rho = 1$, $z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$. Then compute the value of the integral.

Solution: Clearly $1 \leq z \leq 1 + \cos\phi$. A careful sketch of the figure reveals that $0 \leq \phi \leq \pi/2$. This can also be determined algebraically. The cardioid and the hemisphere meet when $1 = 1 + \cos\phi$, which implies that $\cos\phi = 0$. Thus,

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\pi/2} \int_1^{1+\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The innermost integral evaluates to

$$\frac{(1+\cos\phi)^3}{3} \sin\phi - \frac{\sin\phi}{3}.$$

An anti-derivative with respect to ϕ can be determined for the first summand by using the substitution $u = 1 + \cos\phi$, $du = -\sin\phi d\phi$. The second summand is easy to integrate. Answer: $11\pi/6$.

Section 12.8 # 2: Solve the system $u = x + 2y$, $v = x - y$ in terms of x and y and compute the Jacobian determinant $\partial(x, y)/\partial(u, v)$. Then sketch the image under the transformation $T(x, y) = (u, v)$ of the triangular region in the xy-plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$.

Solution: Solving the system for x and y results in $x = (1/3)(u + 2v)$ and $y = (1/3)(u - v)$. The transformed region is a triangle bounded by the line $v = 0$ (which corresponds to $x = y$), the line $u = 2$ (which corresponds to $x + 2y = 2$), and the line $u = v$ (which corresponds to $y = 0$). The Jacobian determinant is equal to $-1/3$.

Section 12.8 # 4: Solve the system $u = 2x - 3y$, $v = -x + y$ in terms of x and y and compute the Jacobian determinant $\partial(x, y)/\partial(u, v)$. Then sketch the image under the transformation $T(x, y) = (u, v)$ of the parallelogram in the xy-plane with boundary lines $x = -3$, $x = 0$, $y = x$, and $y = x + 1$.

Solution: Solving the system for x and y results in $x = -u - 3v$ and $y = -u - 2v$. The transformed region is again a parallelogram. It is bounded by the line $v = 0$ (corresponding to $y = x$), the line $v = 1$ (corresponding to $y = x + 1$), the line $u + 3v = 3$ (corresponding to $x = -3$), and the line $u + 3v = 0$ (corresponding to $x = 0$). The Jacobian determinant is equal to -1.

Section 12.8 # 17: Show that the Jacobian determinant of the transformation from Cartesian (ρ, ϕ, θ)-space to Cartesian (x, y, z)-space is $\rho^2 \sin\phi$.

Solution: The equations $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$ define a transformation from (ρ, ϕ, θ) into (x, y, z). The Jacobian matrix is equal to

$$\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

The determinant of the above matrix is most easily computed by using the last row since one of the terms is equal to zero. The Jacobian determinant is equal to

$$\cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + (-1)(-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

From here, evaluate the 2×2 determinants and simplify by gathering terms so as to apply $\cos^2\theta + \sin^2\theta = 1$ (twice) and $\cos^2\phi + \sin^2\phi = 1$ (once).

(Bonus) Compute the determinant of the following square tri-diagonal matrix assuming that the matrix has has 2012 rows:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

Clarification: the matrix has 1's along each of the middle three diagonals, and 0's in all other entries.

Solution: Let A_n be the tridiagonal matrix with n rows, and let $A_n(i,j)$ be the (i,j) minor. Computing the cofactors of the first row, one obtains

$$\det A_n = \det A_n(1,1) - \det A_n(1,2) = \det A_{n-1} - \det A_n(1,2)$$

To compute the determinant of $A_n(1,2)$, compute the cofactors of its first column:

$$\det A_n(1,2) = \det A_{n-2}$$

Therefore, $\det A_n = \det A_{n-1} - \det A_{n-2}$. Since, $\det A_1 = 1$, $\det A_2 = 0$, the sequence $\{\det A_n\}$ is for $n \geq 1$ is equal to the following:

$$1, 0, -1, -1, 0, 1, 1, 0, \dots,$$

which is periodic with period of length 6 and one period is equal to $1, 0, -1, -1, 0, 1$. Since $2012 \equiv 2 \mod 6$, one deduces that $\det A_{2012} = 0$.