CHAPTER 3

BENDING AND TORSION UNDER PLASTIC CONDITIONS

MOMENT UNDER PLASTIC CONDITION

INTRODUCTION

Until now we have learnt that when designing engineering structures, the allowable stresses should not exceed the elastic limit of the material (in tension or compression). This is known as designing based on elastic theory or elastic design.

The allowable stress ($\sigma_{allowable}$) is taken to be the yield stress of the material divided by a convenient factor of safety (usually based on design codes or experience).

Beyond the elastic limit, plastic deformation occurs.

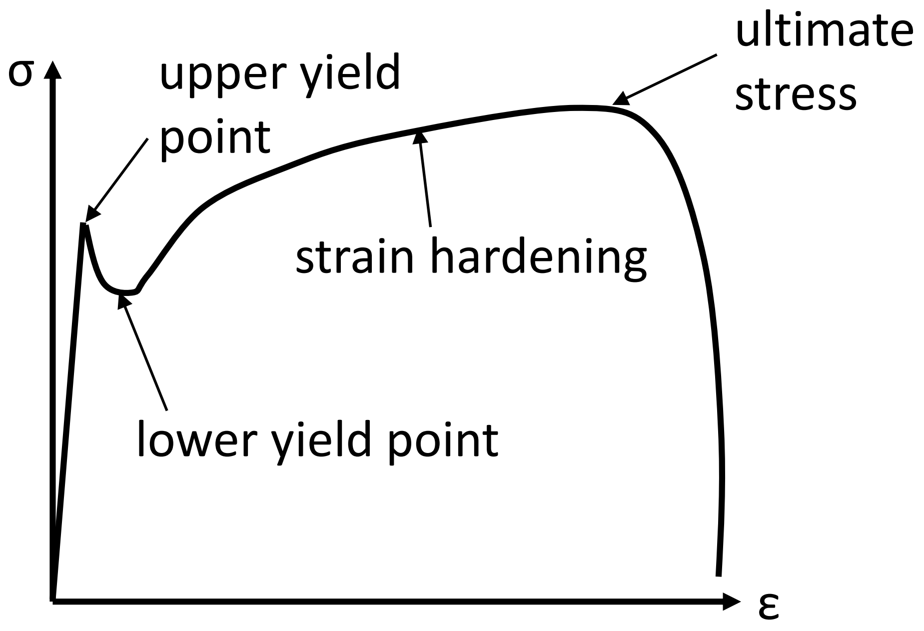
INTRODUCTION

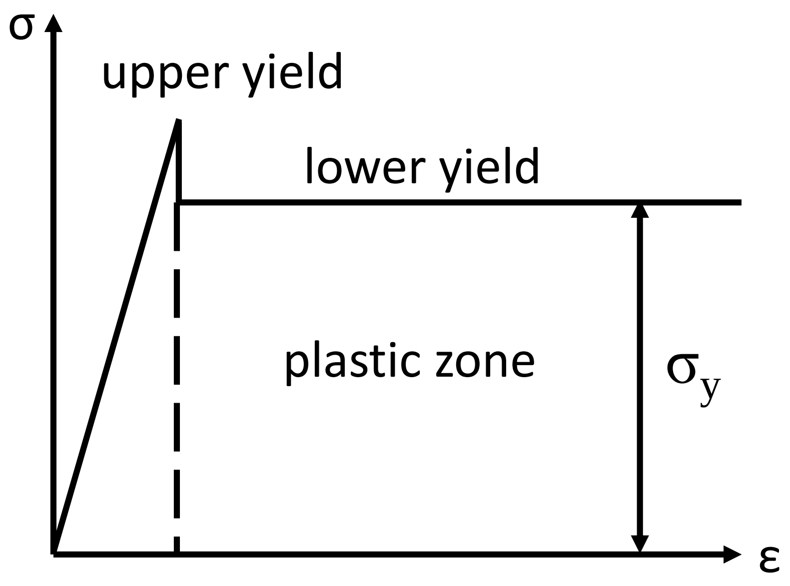
Failure of actual structures is a complex phenomenon because plastic yielding/deformation begins at the extreme fibres and proceeds towards the neutral axis.

This calls for the extension of elastic theory to account for the plastic deformation of structural materials.

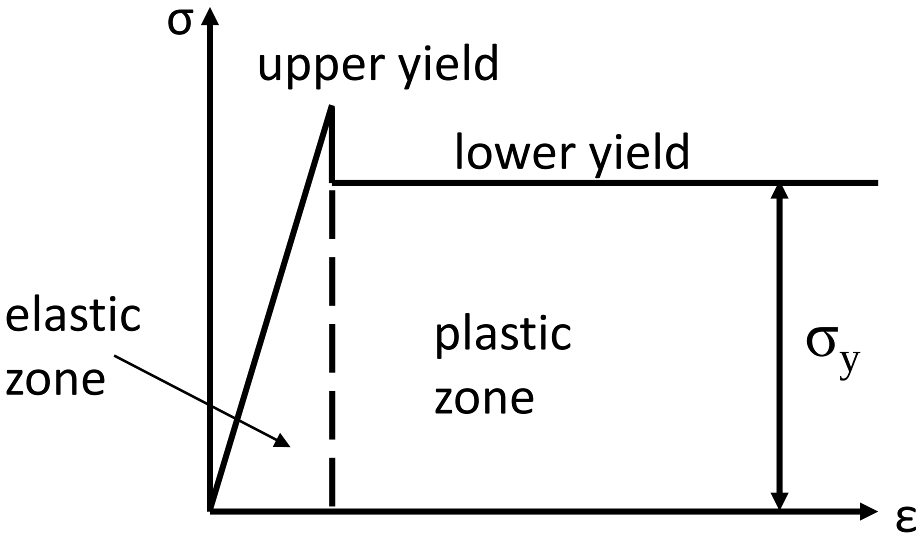
Elastic vs. Plastic Analysis

Consider a specimen of structural steel under a tensile test. The relationship between stress and strain is shown below.





Elastic vs. Plastic Analysis



In **elastic analysis** (e.g., slope-deflection of beams, failure of columns, etc.), the focus is on the elastic region. That is, design is based on the assumption that there is no yield anywhere in the structure.

In **plastic analysis**, the focus is on the both the elastic and plastic regions. That is, the design must account for plastic yielding or deformation of the structure.

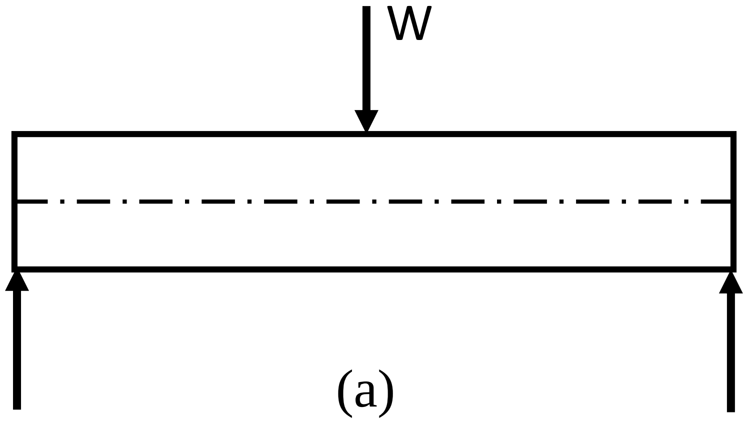
Advantages of Plastic Design:

(a) More suitable for steel structures that have more deformations

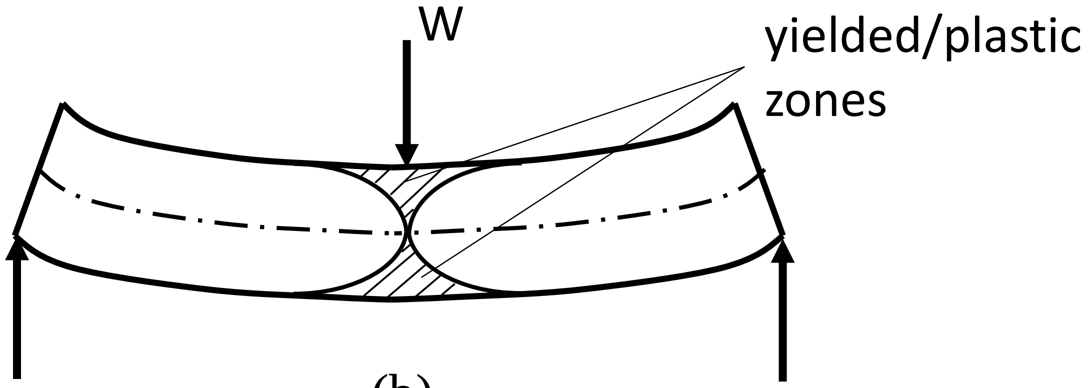
(b) Results in more economical structures

Plastic Theory of Bending

Consider a simply supported rectangular beam subjected to a point load, *W* (see figure a).



(a)



(b)

As the applied load is gradually increased, starting from zero, the beam is stressed first in a purely elastic manner until the elastic limit is reached.

At the elastic limit, the stresses in the upper and lower edges of the central section will have reached the yield stress, σ_y.

Further increase of the load will cause the yield zones to spread and move towards the neutral axis until the beam rotates freely about the load in a manner similar to a hinge. At this stage, a **plastic hinge** is said to have developed.

Assumptions in the Plastic Theory

- The material exhibits marked yield, and can undergo considerable strain during plastic deformation without further increase in stress.

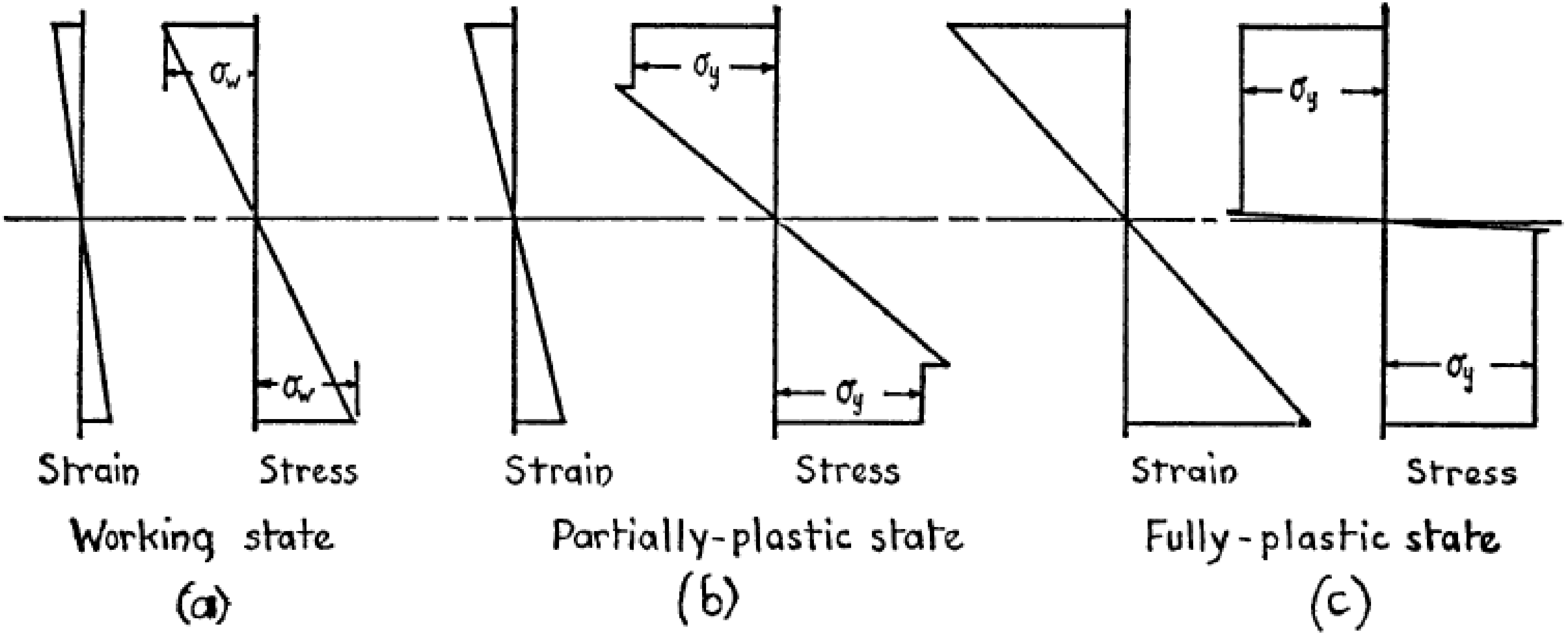
- The yield stress is taken to be the lower yield and is the same in tension and compression

- Transverse cross-sections remain plane, so that strain is proportional to the distance from the neutral axis, though in the plastic region stress will be constant, and not proportional to strain.

- When a plastic hinge has developed at any cross-section the moment of resistance at that point remains constant until collapse of the whole structure takes place due to the formation of the required number of further plastic hinges at other points.

Moment of Resistance at the Plastic Hinges

Our objective is to determine the moment of resistance or the moment required to produce the plastic hinge.



Moment of Resistance at the Plastic Hinges

Rectangular Section:

The total moment of resistance (Fig b)

$$M = M_{elastic} + M_{plastic} = \tfrac{1}{6}\sigma_y b(d-2h)^2 + \sigma_y bh(d-h)$$

In an elastic range, $h = 0$, (Fig a)

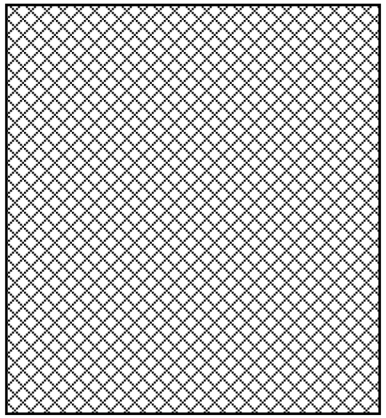
$$M_y = \sigma_y\left(\frac{I}{y}\right) = \sigma_y\left(\frac{bd^2}{6}\right)$$

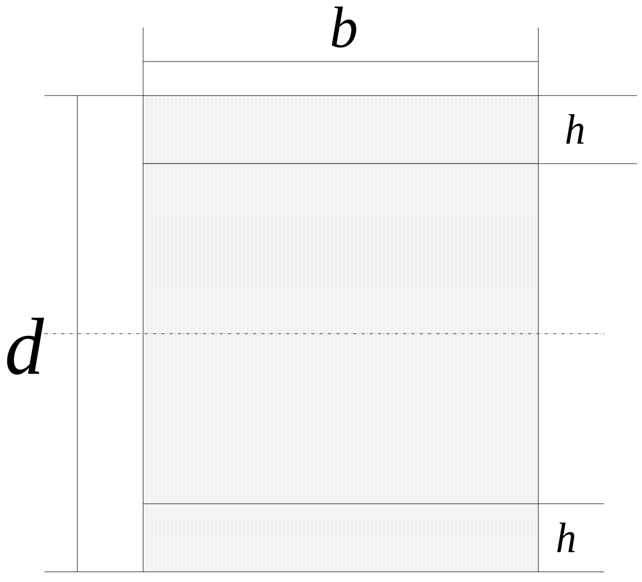
For fully plastic, $h = d/2$

$$M_p = \frac{3}{2}\left(\frac{\sigma_y bd^2}{6}\right) = \tfrac{1}{4}\sigma_y bd^2 = 1.5M_y$$

The ratio $M_p \backslash M_y$ is called the *shape factor* , S

$$s = \frac{M_p}{M_y} = 1.5$$





Example 1

A steel bar of rectangular section 72 mm by 30 mm is used as a simply supported beam on a span of l.2 m and loaded at mid-span. If the yield stress is 280 N/mm^2 and the long edges of the section are vertical, find the load when yielding first occurs.
Assuming that a further increase in load causes yielding to spread inwards towards the neutral axis, with the stress in the yielded part remaining at 280 N/mm^2, find the load required to cause yielding for a depth of 12 mm at the top and bottom of the section at mid-span, and find the length of beam over which yielding has occurred.

Solution

If W_y is the load at first yield, then:

$$M_y = \left(\frac{bd^2}{6}\right)\sigma_y = \frac{(30)(72)^2(280)}{6} \quad = 7257600\,\text{Nmm}...\text{(i)}$$

$$M_y = 300W_y \text{ ...(ii)}$$

From (i) and (ii) $\qquad W_y = 24.2 \text{ kN}$

Example 1 (continued)

The outer 12 mm on each side of the neutral axis being under constant stress of 280 N/mm^2 with no drop of stress at yield.

The moment of resistance

$$M = 300W = 9273600 \Rightarrow W = 31kN$$

First yield

$$W_y(0.3) = 0.3(24.2) = 7.26kN$$

$$M = \tfrac{1}{6}\sigma_y b(d-2h)^2 + \sigma_y bh(d-h)$$

$$= \tfrac{1}{6}(280)(30)(48)^2 + (280)(30)(12)(60)$$

$$= 3225600 \quad + 6048000$$

$$= 9273600$$

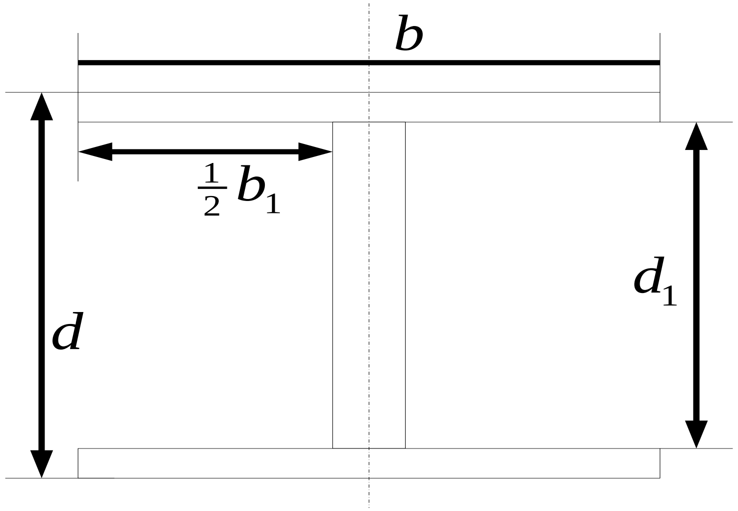
Distance x from either end under the central load W

$$\tfrac{1}{2}Wx = 7.26 \Rightarrow x = 0.468m$$

The length of the beam: $1.2 - 2x = 0.264m$

Moment of Resistance at the Plastic Hinges

I-Section



In an elastic range, (Fig a)

$$M_y = \sigma_y \left(\frac{bd^3}{12} - \frac{b_1 d_1^3}{12} \right) \left(\frac{2}{d} \right)$$

For fully plastic, (Fig c

$$M_p = \sigma_y \left(\frac{bd^2}{4} - \frac{b_1 d_1^2}{4} \right)$$

The ratio $M_p \backslash M_y$ is called the *shape factor* , S

$$S = \frac{M_p}{M_y} = 1.5d \left(\frac{bd^2 - b_1 d_1^2}{bd^3 - b_1 d_1^3} \right)$$

The shape factor will vary slightly with the proportions of flange to web, an average value being about 1.15, as illustrated by the example below.

Example 2

A 300 mm by 125 mm I-beam has flanges 13 mm thick and web 8.5 mm thick. Calculate the shape factor and the moment of resistance in the fully plastic state. Take σ_y =250 N/mm^2 and I_x =85 x 10^6 mm^4

Solution

At fist yield,

$$M_y = (I/y)\sigma_y = \left(85.10^6/150\right)(250) = 141x10^6\ Nmm = 141kNm$$

In the fully plastic,

$$M_p = \sigma_y\left(\frac{bd^2}{4} - \frac{b_1 d_1^2}{4}\right) = \frac{250}{4}\left[(125)(300)^2 - (116.5)(274)^2\right] = 156kNm$$

The shape factor

$$S = \frac{M_p}{M_y} = \frac{156}{141} = 1.11$$

Moment of Resistance at the Plastic Hinges

General Cross Section

If A is the total area of cross-section, then it is clear that for pure bending in the fully plastic state the "neutral axis" must divide the area into equal halves.

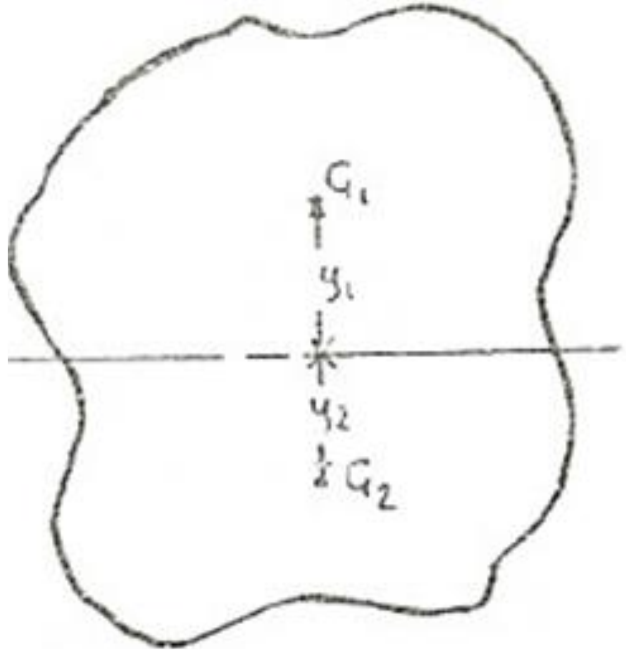
If the centroids of these halves are G_1 and G_2 at a distance $y_1 + y_2$ apart, then

$$M_p = \left(\tfrac{1}{2}\sigma_y A\right)\left(y_1 + y_2\right)$$

But at first yield $M_y = Z\sigma_y$

where Z is the section modulus.

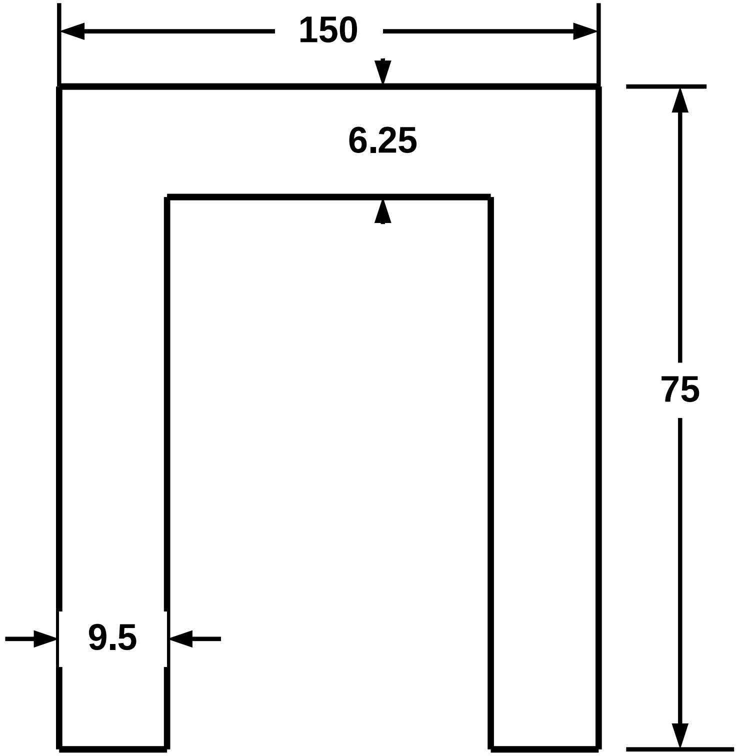
Hence $$S = \frac{M_p}{M_y} = \frac{A\left(y_1 + y_2\right)}{2Z}$$

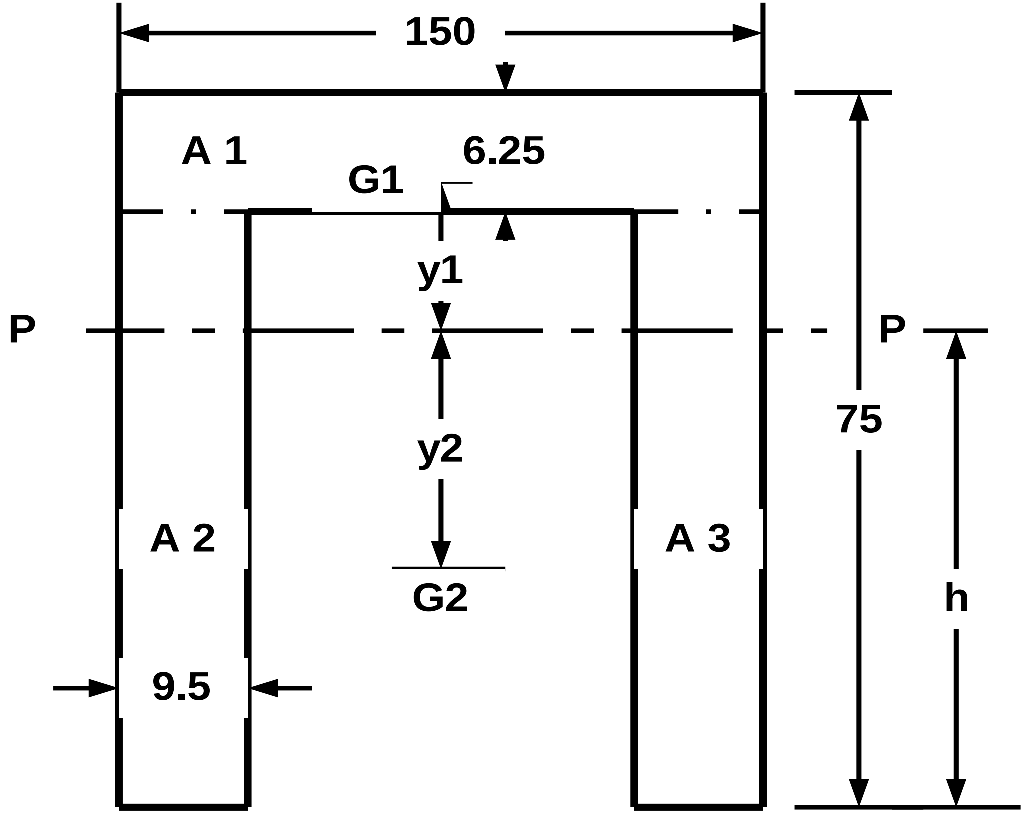


Example 3

Find the shape factor for a 150 mm by 75 mm channel in pure bending with the plane of bending perpendicular to the web of the channel. Take, Z =12,000 mm^3.

Solution





Example 3 (continued)

Area of the channel

$$A_2 = A_3 = 9.5(75 - 6.25) = 653\,mm^2$$

$$A_1 = 150(6.25) = 937.5\ mm^2$$

$$A_1 + 2A_2 = 9.5(75 - 6.25) = 937.5 + 2(653) = 2243\,mm^2$$

Neutral axis that divides the area into equal halves

$$\tfrac{1}{2}A = \tfrac{1}{2}(2243) = 1121.5\ mm^2$$

Since, $A_1 < \tfrac{1}{2}A$

Then, the "neutral axis will pass through A_2 and A_3 labeled PP as shown in the Fig

Let the centroids of these halves be G_1 and G_2 at a distance $y_1 + y_2$ apart.

Therefore,

$$2(9.5)h = 1121.5$$

$$\Rightarrow h = 59mm$$

$$y_1 = h/2 = 29.5mm$$

Example 3 (continued)

$$y_2 == \frac{150(6.25)(16-3.125)+(2)(9.5)(9.75)(4.875)}{150(6.25)+2(9.5)(9.75)} = 11.8mm$$

The shape factor

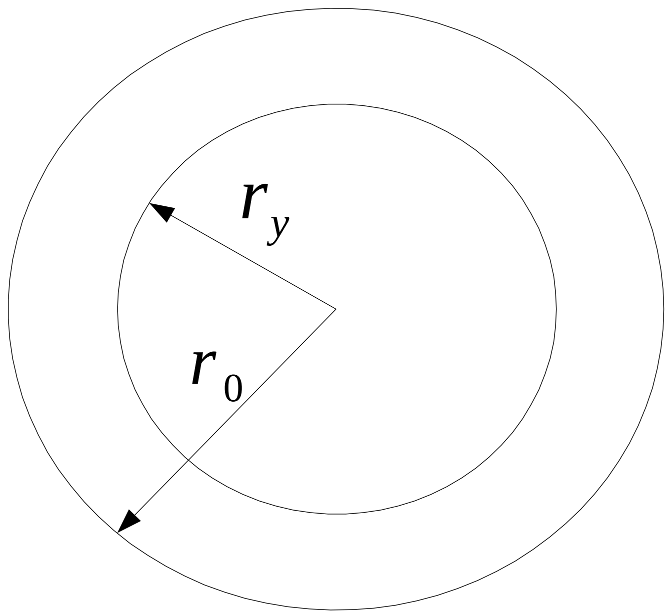
$$S = \frac{A(y_1 + y_2)}{2Z} = \frac{2243(41.76)}{2(21000)} = 2.23$$

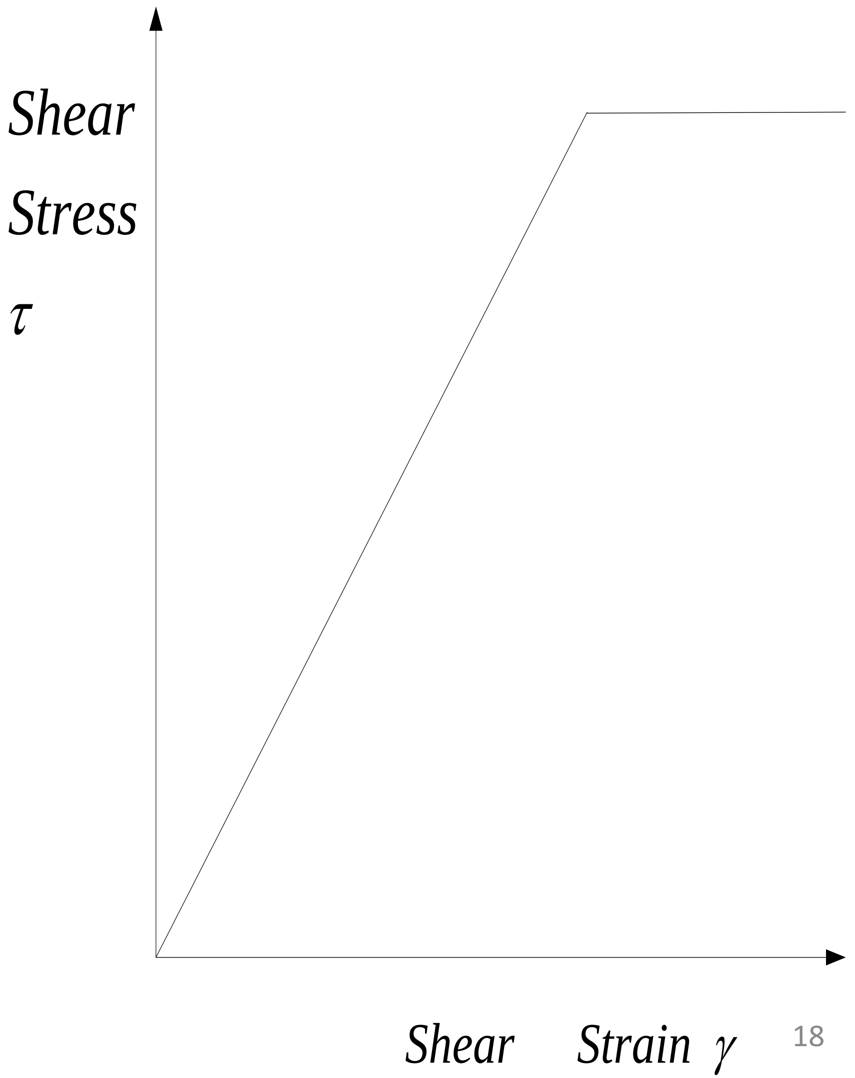
TORSION UNDER PLASTIC CONDITION

Assumptions in the Plastic Theory

A plane cross section of the shaft remains plane when in the plastic state.

Radius remains straight





Torque of Resistance at the Plastic Hinges

The total torque of resistance

$$T_1 = \frac{\tau_y \pi r_y^3}{2}$$

$$T_2 = \int_{r_y}^{r_0} 2\pi r \tau_y . r dr = \frac{2\pi}{3}\tau_y \left(r_0^3 - r_y^3\right)$$

$$T = T_1 + T_2 = \frac{2}{3}\pi\tau_y r_0^3 \left(1 - \frac{r_y^3}{4r_0^3}\right)$$

In an elastic range, $r_0 = r_y$,

$$T_y = \frac{\tau_y \pi r_y^3}{2}$$

For fully plastic, $r_y = 0$

$$T_p = \frac{2}{3}\pi\tau_y r_0^3$$

Torque of Resistance at the Plastic Hinges

When the fibres at the outer surface of the shaft are about to become plastic, the angle of twist of the shaft is given by

$$\theta_y = \frac{\tau_y L}{G r_0} \qquad \frac{\theta_y}{\theta} = \frac{r_y}{r_0} \qquad \theta = \frac{\tau_y L}{G r_y}$$

L the length of the shaft
G Modulus of Rigidity

The *shape factor* , S

$$s = \tau_p / \tau_y = 4/3$$

For elastoplastic condition

$$T = \frac{2}{3}\pi \tau_y r_0^3 \left(1 - \tfrac{1}{4}\left(\frac{\theta_y}{\theta}\right)^3 \right)$$

Example 4

A mild steel shaft 40 mm in diameter and 250 mm in length is subjected to an overload torque of 1800 Nm which caused shear yielding of the shaft, 120 MPa. Determine the radial depth to which plasticity has penetrated and the angle of twist. Take G = 80 GPa

Solution

The Torque $$T = \frac{2}{3}\pi\tau_y r_0^3\left(1-\tfrac{1}{4}\left(\frac{r_y}{r_0}\right)^3\right) \quad 1800 = \frac{2}{3}\pi\left(120.10^6\right)\left(0.02\right)^3\left[1-\frac{r_y^3}{4\left(0.02\right)^3}\right]$$

From which $$r_y = 15mm$$

Hence depth of plastic deformation is 5mm

The shear strain at r_y==15mm is

$$\gamma = \frac{r_y\theta}{L} = \frac{\tau_y}{G} = \frac{120\left(10^6\right)}{80\left(10^9\right)} = 1.5x10^{-3}$$

$$\Rightarrow \theta = \frac{\gamma L}{r_y} = \frac{\left(0.0015\right)\left(0.25\right)}{0.015} = 0.025rad = 1.43^0$$

Further Examples

1. (a) A rectangular section beam, 50 mm wide by 20 mm deep, is used as a simply supported beam over a span of 2 m with the dimension vertical. Determine the value of the central concentrated load which will produce initiation of yield at the outer fibres of the beam.
(b) If the central load is then increased by 10%, find the depth to which yielding will take place at the centre of the beam span.
(c) Over what length of beam will yielding then have taken place?
(d) What are the maximum deflections for each load case?
For steel σ_y in simple tension and compression = 225 MPa and E = 206.8 GPa.

2. A solid circular shaft of diameter 50 mm and length 300 mm is subjected to a gradually increasing torque *T*. The yield stress in shear for the shaft material is 120 MPa and, up to the yield point, the modulus of rigidity is 80 GPa.
(a) Determine the value of *T* and the associated angle of twist when the shaft material first yields
(b) If, after yielding, the stress is assumed to remain constant for any further increase in strain, determine the value of *T* when the angle of twist is increased to twice that at yield.