Maximum Power Transfer

Introduction:

- This theorem describes the condition for maximum power transfer from an active network to an external load .
- Very often we come across various real time circuits that works based on maximum power transfer theorem.
- For effective way of connecting source to load, an impedance matching transformer is used. In case of transmission lines, the distortion and reflections are avoided by making source and load impedances to be matched to the characteristic impedance of the line.
- In case of solar photovoltaic (PV) systems, Maximum Power Point Tracking (MPPT) is achieved with incremental conductance method (ICM) in which the load resistance must be equal to the output resistance of the PV panel and Solar Cell.
- So there are several cases or applications that use maximum power transfer theorem for effectively connecting the source to a load. This theorem can be applied for both DC and AC circuits.

- A network that contains linear impedances and one or more voltage or current sources can be reduced to a Thevenin equivalent circuit as shown before.
- When a load is connected to the terminals of this equivalent circuit, power is transferred from the source to the load.
- A Thevenin equivalent circuit is shown in Figure 8 with source internal impedance, $Z_{th} = (R_{th} + jX_{th})\Omega$ and complex load $Z_L = (R_L + jX_L)\Omega$.

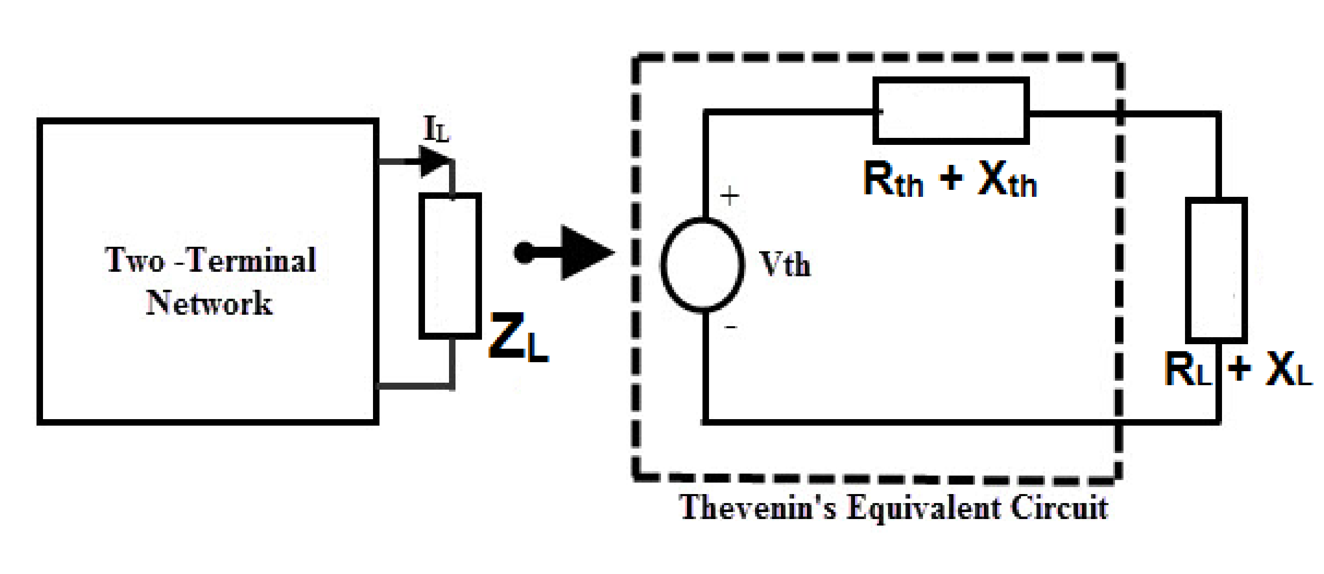


Fig. 8: Thevenin Equivalent Circuit

- **The maximum power transferred depends on the following conditions.**
- **Condition 1:**
- Let the load consist of a pure variable resistance $\boldsymbol{R_L}$ (i.e. let $\boldsymbol{X_L=0}$). Then current I_L in the load is given by:

$$I_L = \frac{V_{th}}{(R_{th}+R_L)+jX_{th}} \qquad 2.1$$

- And the magnitude of current,

$$|I_L| = \frac{V_{th}}{\sqrt{[(R_{th}+R_L)^2+X_{th}{}^2]}} \qquad 2.2$$

- The active power P delivered to load R_L is given by

$$P = |I_L|^2 R_L = \frac{V_{th}{}^2 R_L}{(R_{th}+R_L)^2+X_{th}{}^2} \qquad 2.3$$

- To determine the value of R_L for maximum power transferred to the load, P is differentiated with respect to R_L and then equated to zero.
- Using the quotient rule of differentiation,

$$\frac{dP}{dR_L} = V_{th}{}^2\left\{\frac{[(R_{th}+R_L)^2+X_{th}{}^2](1)-(R_L)(2)(R_{th}+R_L)}{[(R_{th}+R_L)^2+X_{th}{}^2]^2}\right\} = 0 \qquad 2.4$$

- *Hence*

$$(R_{th}+R_L)^2+X_{th}^2-2R_L(R_{th}+R_L)=0$$

- i.e., $R_{th}^2+2R_{th}R_L+R_L^2+X_{th}^2-2R_LR_{th}-2R_L^2=0$
- from which,

$$R_{th}^2+X_{th}^2=R_L^2$$

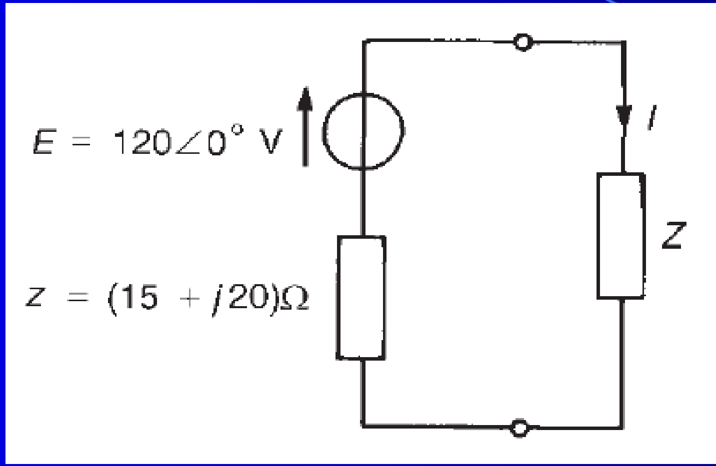
- or

$$R_L=\sqrt{(R_{th}^2+X_{th}^2)}=|Z_L| \qquad 2.5$$

- Thus, with a **variable purely resistive load**, the maximum power is delivered to the load if the load resistance R_L is made equal to the **magnitude of the source impedance**.

- Example 8:
- For the circuit shown in Fig. 9, the load impedance Z is a pure resistance, R. Determine

  (a) the value of R for maximum power to be transferred from the source to the load, and (b) the value of the maximum power delivered to R.



- Solution:

(a) From condition1, maximum power transfer occurs when $R = |z|$,

- i.e., when

$$\boldsymbol{R = |15 + j20| = \sqrt{(15^2 + 20^2)} = 25\Omega}$$

(b) Current I flowing in the load is given by $I = E/Z_T$, where the total circuit impedance $Z_T = Z + R = 15 + \ j20 + 25 = (40 + j20)\Omega$ or $44.72\angle 26.57^o\Omega$

- Hence

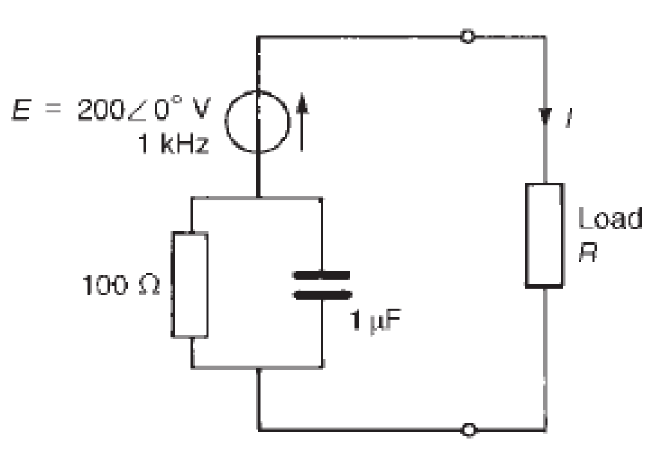
$$\boldsymbol{I = \frac{120\angle 0^o}{44.72\angle 26.57^o} = 2.683\angle - 26.57^o\ A}$$

- Thus maximum power delivered, $\boldsymbol{P = I^2R = (2.683)^2(25) = 180\ W}$

- **Example 9:**
- For the network shown in Fig. 10, determine

(a) the value of the load resistance R required for maximum power transfer, and

(b) the value of the maximum power transferred.



Solution:

(a) This problem is an example of condition 1, where maximum power transfer is achieved when $\boldsymbol{R_L = |Z_{th}|}$. Source impedance Z_{th} is composed of a 100 Ω resistance in parallel with a 1 uF capacitor.

Capacitive reactance, $\boldsymbol{X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(1000)(1\,x\,10^{-6})} = 159.15\ \Omega}$

Hence source impedance,

$$\boldsymbol{Z_{th} = \frac{(100)(-j159.15)}{(100 - j159.15)} = \frac{15915\angle-90^o}{187.96\angle-57.86^o} = 84.67\angle-32.14^o}$$

- Thus the value of load resistance for maximum power transfer is **84.67 Ω**.

(b) With $\mathbf{Z_{th}}$ = $\boldsymbol{84.67\angle-32.14^o}$ Ω and **R = 84.67 Ω** for maximum power transfer, the total circuit impedance,

$$\boldsymbol{Z_T = 71.69 + 84.67 - j45.04 = (156.36 - j45.04)\Omega \quad or \quad 162.72\angle-16.07^o\ \Omega}$$

- Current flowing in the load,

$$I = \frac{E}{Z_T} = \frac{200\angle 0^o}{162.72\angle -16.07^o} = 1.23\angle 16.07^o \quad A$$

- Thus the maximum power transferred,

$$P = I^2 R = (1.23)^2(84.67) = 128\ W$$

Condition 2:

- Let both the load and the source impedance be purely resistive (i.e., let X_{th}=X_L=0).
- From equation (2.5), it may be seen that the maximum power is transferred when $R_L = R_{th}$.
- Under this condition, the maximum power delivered to the load is

$$P_{max} = \frac{{V_{th}}^2}{4R_L}$$

- Below figure shows a curve of power delivered to the load with respect to the load resistance.

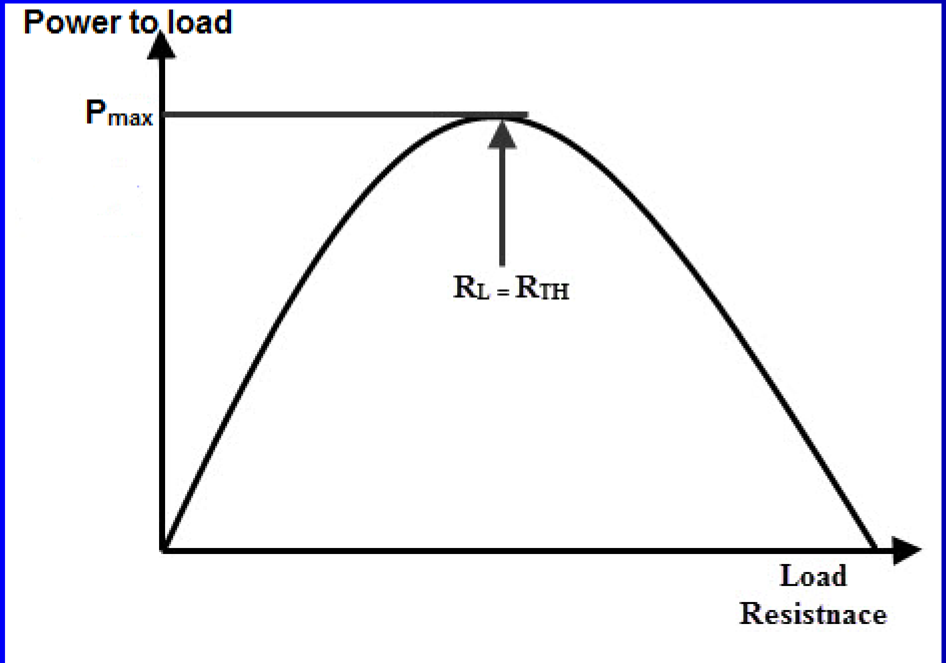


Fig 11: Load power vs load resistance

Note that the power delivered is zero when the load resistance is zero as there is no voltage drop across the load during this condition. Also, the power will be maximum, when the load resistance is equal to the internal resistance of the circuit (or Thevenin's equivalent resistance). Again, the power is zero as the load resistance reaches to infinity as there is no current flow through the load.

Example 10:

Determine the value of the load resistance R shown in Fig. 12 that gives maximum power dissipation and calculate the value of this power.

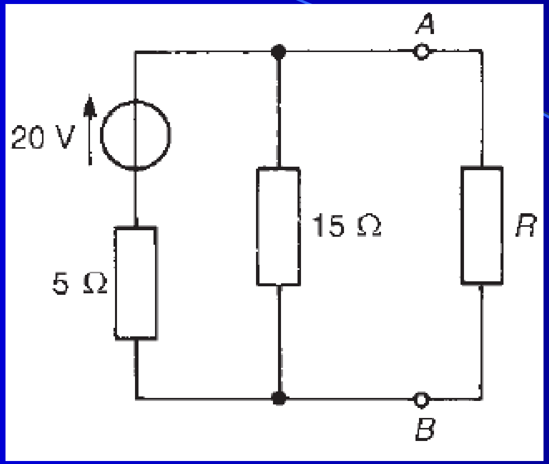


Fig. 12: See Example 10

Solution:

- Using the procedure of Thevenin's Theorem
- R is removed from the network as shown in Fig. 13
- Voltage across AB, $\boldsymbol{E = (15/(15+5))(20) = 15V}$
- Impedance looking-in at terminals AB with the 20V source removed is given by
- $\boldsymbol{R_{TH} = (5x15)/(5+15) = 3.75\ \Omega}$
- The equivalent Thevenin circuit supplying terminals AB is shown in Fig. 14. From condition (2), for maximum power transfer, $R = R_{TH}$, i.e. $\boldsymbol{R = 3.75\ \Omega}$

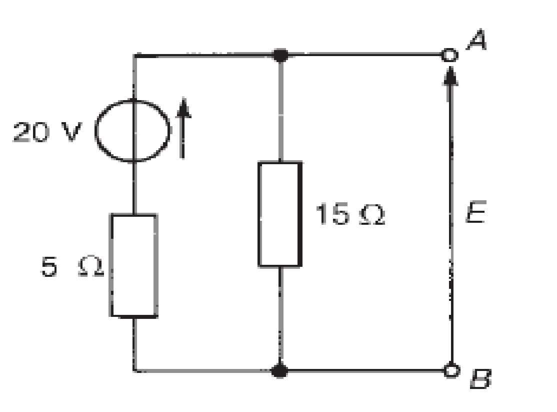


Fig. 13: Thevenin Equivalent Circuit

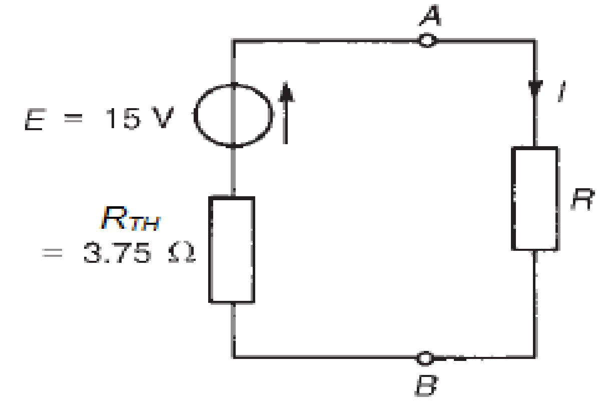


Fig. 14: Thevenin Equivalent Circuit

(a) Current

$$I = \frac{E}{R + R_{TH}} = \frac{15}{3.75 + 3.75} = 2\ A$$

(a) Maximum power dissipated in the load,

$$P = I^2 R = (2)^2(3.75) = 15\ W$$

Condition 3:

- Let the load Z_L have both variable resistance R_L and variable reactance X. From Fig. 8, current I_L is

$$I_L = \frac{V_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}$$

- And

$$|I_L| = \frac{V_{th}}{\sqrt{[(R_{th} + R_L)^2 + (X_{th} + X_L)^2]}}$$

- The active power P delivered to the load is given by

$$P = |I_L|^2 R = \frac{V_{th}{}^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

- If X_L is adjusted such that $X_L = -X_{th}$, then the value of power is a maximum.
- If $X_L = -X_{th}$, then

$$P = \frac{V_{th}{}^2 R_L}{(R_{th} + R_L)^2}$$

- **For maximum value,**

$$\frac{dP}{dR_L} = V_{th}{}^2 \left\{ \frac{(R_{th} + R_L)^2(1) - (R_L)(2)(R_{th} + R_L)}{(R_{th} + R_L)^4} \right\} = 0$$

- Hence

$$(R_{th} + R_L)^2 - 2R_L(R_{th} + R_L) = 0$$

- i.e.,

$$R_{th}{}^2 + 2R_LR_{th} + R_{th}{}^2 - 2R_{th}R_L - 2R^2 = 0$$

- From which,

$$R_{th}{}^2 - R_L{}^2 = 0 \qquad and \qquad R_L = R_{th}$$

- Thus with the load impedance Z consisting of variable resistance R and variable reactance X, maximum power is delivered to the load when

$$X_L = -X_{th} \qquad and \qquad R_L = R_{th}$$

- i.e., when

$$R_L + jX_L = R_{th} - jX_{th}$$

- Hence maximum power is delivered to the load when the **load impedance** is the **complex conjugate** of the **source impedance**.

Example 11:

Determine, for the network shown in Fig. 15,

(a) The values of R and X that will result in maximum power being transferred across terminals AB, and

(b) The value of the maximum power.

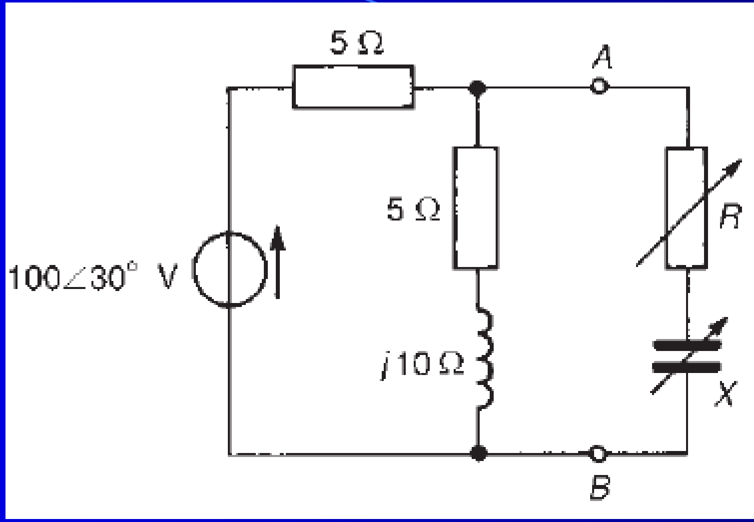


Fig. 15: See Example 11

Solution:

(a) Using the procedure for Thevenin's theorem:

(i) Resistance R and reactance X are removed from the network as shown in Fig. 16

(ii) Voltage across AB,

$$E = \left(\frac{5 + j10}{5 + j10 + 5}\right)(100\angle 30^o) = \frac{(11.18\angle 63.43^o)(100\angle 30^o)}{14.14\angle 45^o} = 79.07\angle 48.43^o\ V$$

(iii) With the $100\angle 30^o$ V source removed, the impedance, Z_{TH}, looking in at terminals AB is given by:

$$Z_{TH} = \frac{(5)(5 + j10)}{(5 + 5 + j10)} = \frac{(5)(11.18\angle 63.43^o)}{(14.14\angle 45^o)} = 3.953\angle 18.43^o\ \Omega\ \ or\ \ (3.75 + j1.25)\Omega$$

(iv) The equivalent Thevenin circuit is shown in Fig. 17. From condition 3, maximum power transfer is achieved when $X = X_{th}$ and $R = R_{th}$, i.e., in this case when $X = -1.25\Omega$ and $R = 3.75\ \Omega$

(iv) The equivalent Thevenin circuit is shown in Fig. 17. From condition 3, maximum power transfer is achieved when $X = X_{th}$ and $R = R_{th}$, i.e., in this case when $X = -1.25\Omega$ and $R = 3.75\ \Omega$

(b) Current

$$I = \frac{E}{Z_L + Z_{TH}} = \frac{79.07\angle 48.43^o}{(3.75 + j1.25) + (3.75 - j1.25)} = \frac{79.07\angle 48.43^o}{7.5} = 10.54\angle 48.43^o\ A$$

- Thus the maximum power transferred,

$$P = I^2R = (10.543)^2(3.75) = 417\ W$$

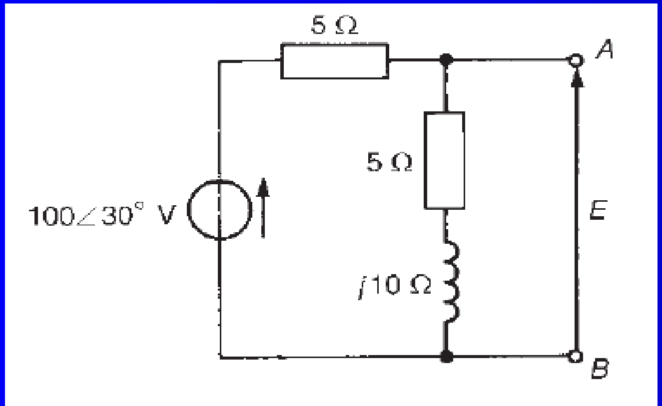


Fig. 16: Finding Thevenin voltage and impedance

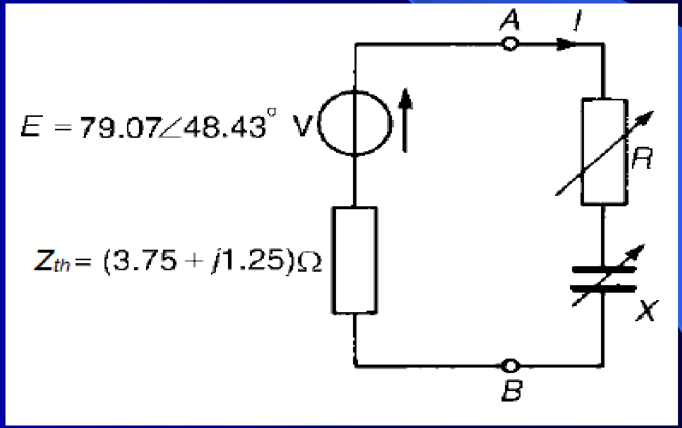


Fig. 17: Thevenin equivalent circuit

Condition 4:

- Let the load impedance Z have variable resistance R and fixed reactance X. From Fig. 8., the magnitude of current,

$$|I| = \frac{V_{th}}{\sqrt{[(R_{th}+R_L)^2+(X_{th}+X_L)^2]}}$$

- And the power dissipated in the load,

$$P = \frac{{V_{th}}^2 R_L}{(R_{th}+R_L)^2+(X_{th}+X_L)^2}$$

$$\frac{dP}{dR} = {V_{th}}^2 \left\{ \frac{[(R_{th}+R_L)^2+(X_{th}+X_L)^2](1)-(R_L)(2)(R_{th}+R_L)}{[(R_{th}+R_L)^2+(X_{th}+X_L)^2]^2} \right\} = 0 \qquad for\ a\ max.value$$

- Hence

$$(R_{th}+R_L)^2+(X_{th}+X_L)^2-2R_L(R_{th}+R_L) = 0$$

$${R_{th}}^2 + 2R_{th}R_L + {R_L}^2 + (X_{th}+X_L)^2 - 2R_LR_{th} - 2R^2 = 0$$

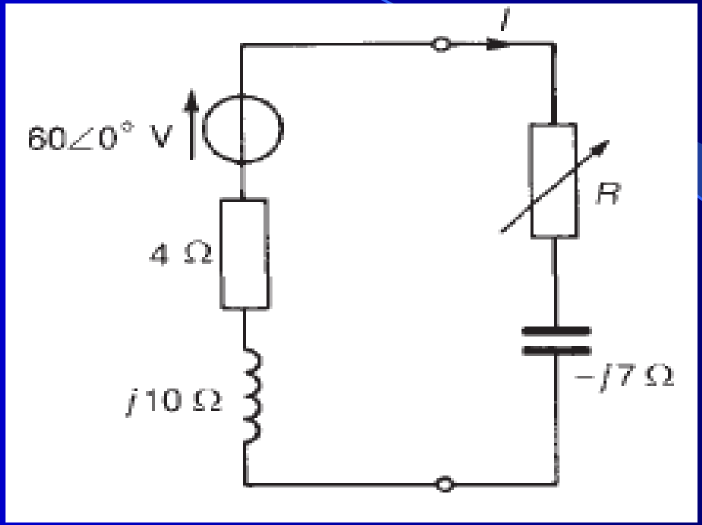
- From which,

$${R_L}^2 = {R_{th}}^2 + (X_{th}+X_L)^2 \qquad and \qquad R = \sqrt{[{R_{th}}^2 + (X_{th}+X_L)^2]}$$

- **Example 12:**
- In the network shown in Fig. 18, the load consists of a fixed capacitive reactance of 7 Ω and a variable resistance R. Determine

(a) The value of R for which the power transferred to the load is a maximum

(b) The value of the maximum power



Solution:

(a) From condition (4), maximum power transfer is achieved when

$$R = \sqrt{[R_{th}{}^2 + (X_{th} + X_L)^2]} = \sqrt{[4^2 + (10-7)^2]} = 5\ \Omega$$

(a) Current

$$I = \frac{60\angle 0^o}{(4+j10)+(5-j7)} = \frac{60\angle 0^o}{(9+j3)} = \frac{60\angle 0^o}{9.487\angle 18.43^o} = 6.324\angle -18.43^o\ A$$

Thus the maximum power transferred,

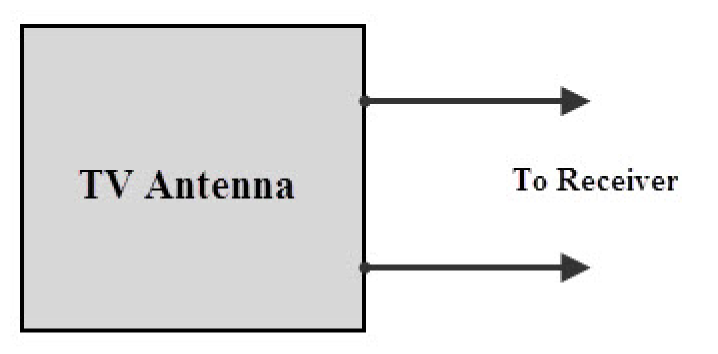
$$P = I^2R = (6.324)^2(5) = 200\ W$$

Summary:

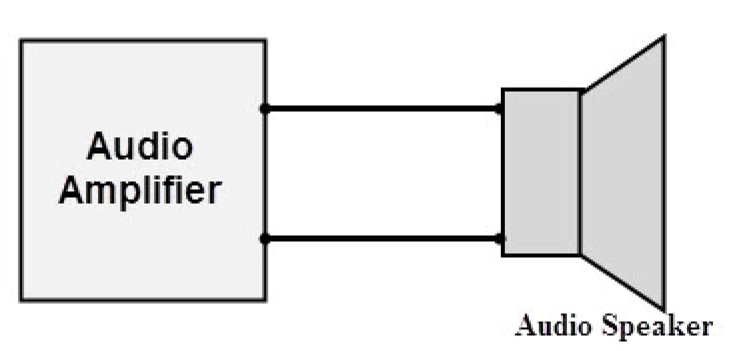
- With reference to Fig. 8:
- When the load is purely resistive (i.e., $X_L = 0$) and adjustable, maximum power transfer is achieved when $\boldsymbol{R_L = |Z_{th}| = \sqrt{R_{th}{}^2 + X_{th}{}^2}}$
- When both the load and the source impedance are purely resistive (i.e., $X_L = X_{th} = 0$), maximum power transfer is achieved when $\boldsymbol{R_L = R_{th}}$
- When the load resistance R_L and reactance X_L are both independently adjustable, maximum power transfer is achieved when the load impedance is the complex conjugate of the source impedance.
- When the load resistance R_L is adjustable with reactance X_L fixed, maximum power transfer is achieved when $\boldsymbol{R_L = \sqrt{[R_{th}{}^2 + (X_{th} + X_L)^2]}}$

APPLICATIONS OF MAXIMUM POWER TRANSFER THEOREM:

- In electronic circuits, especially in communication system the signal present at the receiving antenna is of low strength. In order to receive the maximum signal from the antenna, impedance of (TV) receiver and (TV) antenna should be matched.



- In an audio amplifier with audio speaker arrangement in public addressing systems, speaker resistance must be equal to the amplifier resistance in order to transfer maximum power from amplifier to the speaker.



- In case of a car engine starting system, starter motor resistance must be matched with internal resistance of the battery. If the battery if full and these resistances are matched, maximum power will be transferred to the motor to turn ON the engine.