14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

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In Example 6 in Section 14.7, we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$—which expressed the side condition that the surface area was 12 m^2.

In this section, we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

14.8
Lagrange Multipliers

In this section, we will learn about:

Lagrange multipliers for two and three variables, and given one and two constraints.

LAGRANGE MULTIPLIERS

It's easier to explain the geometric basis of Lagrange's method for functions of two variables.

LAGRANGE MULTIPLIERS—TWO VARIABLES

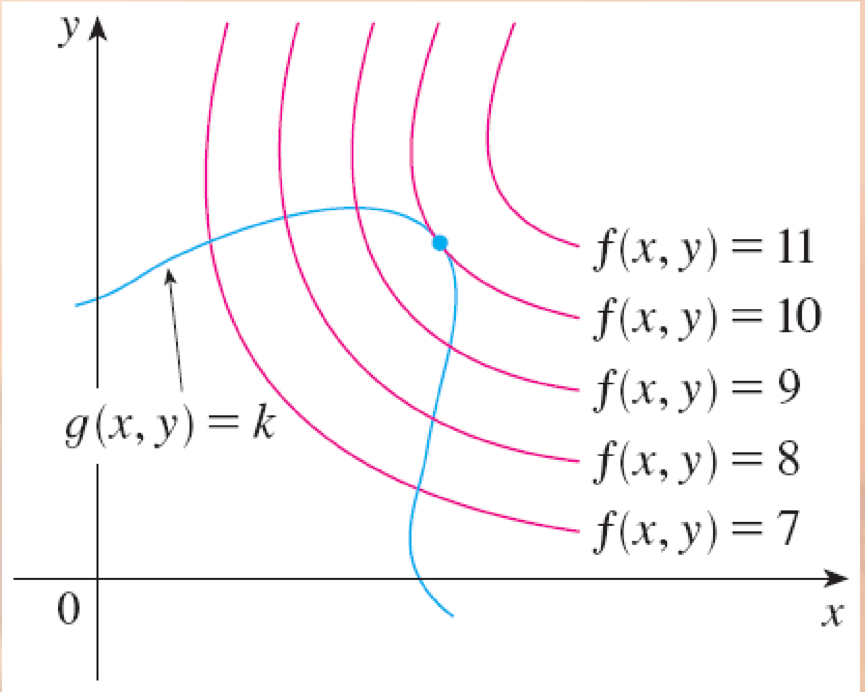
So, we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$.

- In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

LAGRANGE MULTIPLIERS—TWO VARIABLES

The figure shows this curve together with several level curves of *f*.

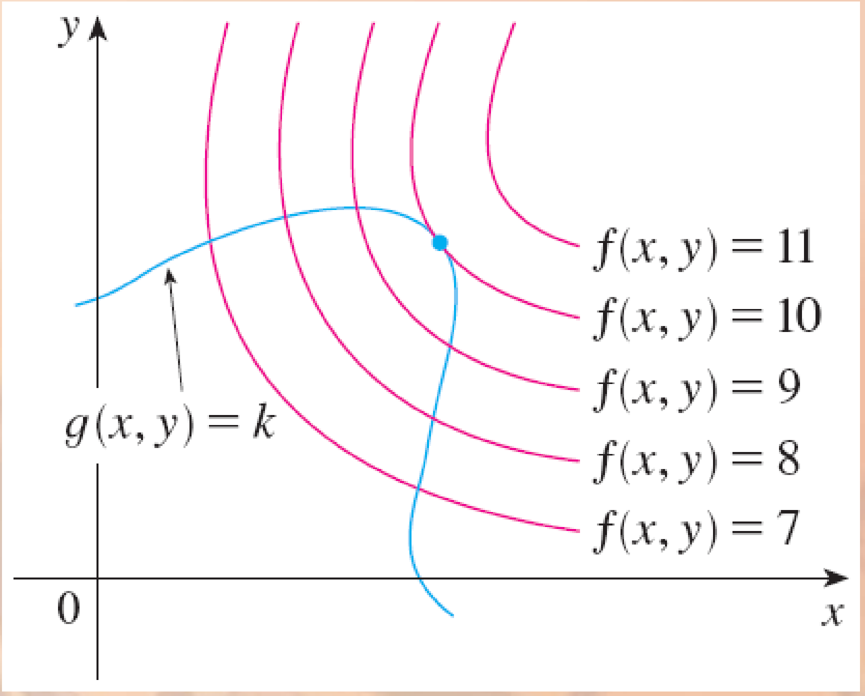
- These have the equations $f(x, y) = c$, where c = 7, 8, 9, 10, 11



LAGRANGE MULTIPLIERS—TWO VARIABLES

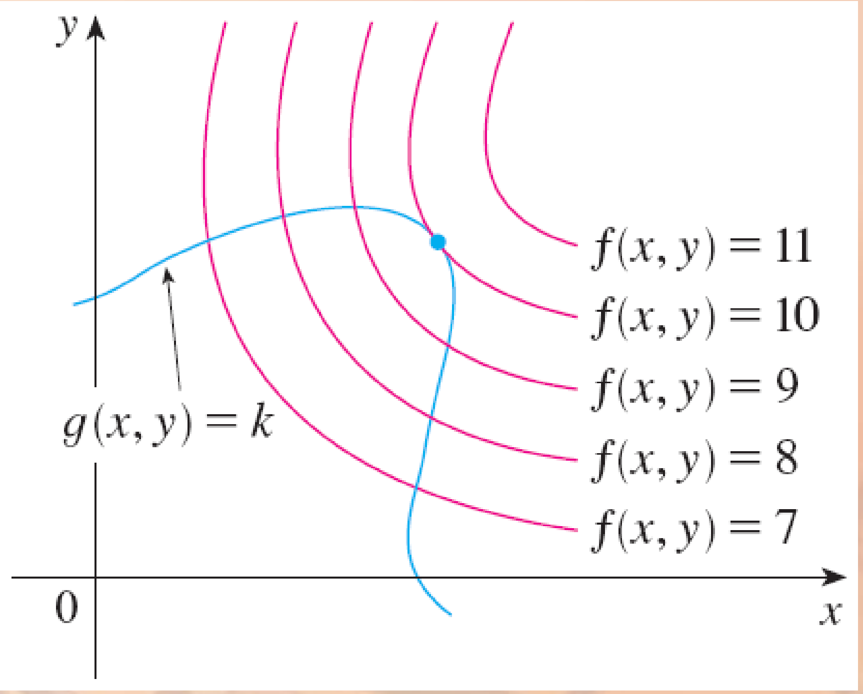
To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find:

- The largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.



It appears that this happens when these curves just touch each other—that is, when they have a common tangent line.

- Otherwise, the value of *c* could be increased further.



LAGRANGE MULTIPLIERS—TWO VARIABLES

This means that the normal lines at the point (x_0, y_0) where they touch are identical.

- So the gradient vectors are parallel.
- That is,

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some scalar λ.

LAGRANGE MULTIPLIERS—THREE VARIABLES

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

- Thus, the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$.

LAGRANGE MULTIPLIERS—THREE VARIABLES

Instead of the level curves in the previous figure, we consider the level surfaces $f(x, y, z) = c$.

- We argue that, if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$.
- So, the corresponding gradient vectors are parallel.

LAGRANGE MULTIPLIERS—THREE VARIABLES

This intuitive argument can be made precise as follows.

Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S.

- Then, let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P.

LAGRANGE MULTIPLIERS—THREE VARIABLES

If t_0 is the parameter value corresponding to the point *P*, then

$$\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$$

The composite function

$$h(t) = f(x(t), y(t), z(t))$$

represents the values that *f* takes on the curve *C*.

LAGRANGE MULTIPLIERS—THREE VARIABLES

f has an extreme value at (x_0, y_0, z_0).

So, it follows that *h* has an extreme value at $t_{0.}$

- Thus, $h'(t_0) = 0$.

LAGRANGE MULTIPLIERS—THREE VARIABLES

However, if *f* is differentiable, we can use the Chain Rule to write:

$$
\begin{aligned}
0 &= h'(t_0) \\
&= f_x(x_0, y_0, z_0)\,x'(t_0) + f_y(x_0, y_0, z_0)\,y'(t_0) \\
&\qquad + f_z(x_0, y_0, z_0)\,z'(t_0) \\
&= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0)
\end{aligned}
$$

This shows that the gradient vector $\nabla f\left(x_0, y_0, z_0\right)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve *C*.

LAGRANGE MULTIPLIERS—THREE VARIABLES

However, we already know from Section 14.6 that the gradient vector of g, $\nabla g\left(x_0, y_0, z_0\right)$, is also orthogonal to **r**’(t_0) for every such curve.

- See Equation 18 from Section 6.
- This means that the gradient vectors $\nabla f\left(x_0, y_0, z_0\right)$ and $\nabla g\left(x_0, y_0, z_0\right)$ must be parallel.

Equation 1

Therefore, if $\nabla g\left(x_0, y_0, z_0\right) \neq 0$, there is a number λ such that:

$$\nabla f\left(x_0, y_0, z_0\right) = \lambda \nabla g\left(x_0, y_0, z_0\right)$$

- The number λ in the equation is called a Lagrange multiplier.

- The procedure based on Equation 1 is as follows.

LAGRANGE MULTIPLIERS—METHOD

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$], we proceed as follows.

LAGRANGE MULTIPLIERS—METHOD

a. Find all values of *x*, *y*, *z*, and *λ* such that

$$\nabla f\left(x, y, z\right) = \lambda \nabla g\left(x, y, z\right)$$

and

$$g\left(x, y, z\right) = k$$

b. Evaluate *f* at all the points (*x*, *y*, *z*) that result from step a.

- The largest of these values is the maximum value of *f.*
- The smallest is the minimum value of *f*.

LAGRANGE'S METHOD

In deriving Lagrange's method, we assumed that $\nabla g \neq 0$.

- In each of our examples, you can check that $\nabla g \neq 0$ at all points where $g(x, y, z) = k$.

LAGRANGE'S METHOD

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of its components, then the equations in step a become:

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k$$

- This is a system of four equations in the four unknowns x, y, z, and λ.
- However, it is not necessary to find explicit values for λ.

LAGRANGE'S METHOD

For functions of two variables,

the method of Lagrange multipliers is

similar to the method just described.

LAGRANGE'S METHOD

To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that:

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

- This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

LAGRANGE'S METHOD

Our first illustration of Lagrange's method is to reconsider the problem given in Example 6 in Section 14.7

Example 1

A rectangular box without a lid is to be made from 12 m^2 of cardboard.

- Find the maximum volume of such a box.

Example 1

As in Example 6 in Section 14.7, we let x, y, and z be the length, width, and height, respectively, of the box in meters.

- Then, we wish to maximize $V = xyz$ subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Example 1

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that:

$$\nabla V = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 12$$

This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

The equations become:

$$yz = \lambda(2z + y)$$

$$xz = \lambda(2z + x)$$

$$xy = \lambda(2x + 2y)$$

$$2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations.

- Sometimes, some ingenuity is required.

Example 1

In this example, you might notice that if we multiply Equation 2 by x, Equation 3 by y, and Equation 4 by z, then left sides of the equations will be identical.

Doing so, we have:

$$xyz = \lambda(2xz + xy)$$

$$xyz = \lambda(2yz + xy)$$

$$xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from Equations 2, 3, and 4.

This would contradict Equation 5.

Therefore, from Equations 6 and 7,

we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$.

- However, $z \neq 0$ (since $z = 0$ would give $V = 0$).
- Thus, $x = y$.

Example 1

From Equations 7 and 8,

we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$.

- Thus, since $x \neq 0$, $y = 2z$.

If we now put $x = y = 2z$ in Equation 5, we get:

$$4z^2 + 4z^2 + 4z^2 = 12$$

- Since x, y, and z are all positive, we therefore have $z = 1$, and so $x = 2$ and $y = 2$.
- This agrees with our answer in Section 14.7

LAGRANGE’S METHOD

Another method for solving the system of equations 2–5 is to solve each of Equations 2, 3, and 4 for λ and then to equate the resulting expressions.

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

- We are asked for the extreme values of f subject to the constraint

$$g(x, y) = x^2 + y^2 = 1$$

Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$.

These can be written as:

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$g(x, y) = 1$$

They can also be written as:

$$2x = 2x\lambda$$

$$4y = 2y\lambda$$

$$x^2 + y^2 = 1$$

From Equation 9, we have

$$x = 0 \text{ or } \lambda = 1$$

- If $x = 0$, then Equation 11 gives $y = \pm 1$.
- If $\lambda = 1$, then $y = 0$ from Equation 10; so, then Equation 11 gives $x = \pm 1$.

Therefore, f has possible extreme values at the points

$$(0, 1), (0, -1), (1, 0), (-1, 0)$$

- Evaluating f at these four points, we find that:

$f(0, 1) = 2 \quad f(0, -1) = 2 \quad f(1, 0) = 1 \quad f(-1, 0) = 1$

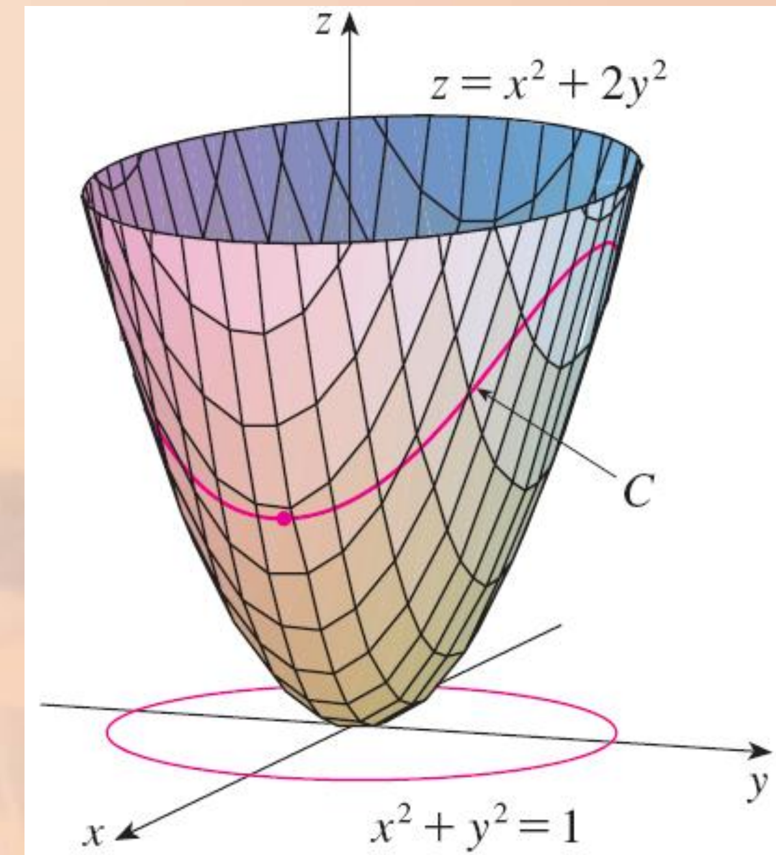
Therefore, the maximum value of f on the circle $x^2 + y^2 = 1$ is:

$$f(0, \pm 1) = 2$$

The minimum value is:

$$f(\pm 1, 0) = 1$$

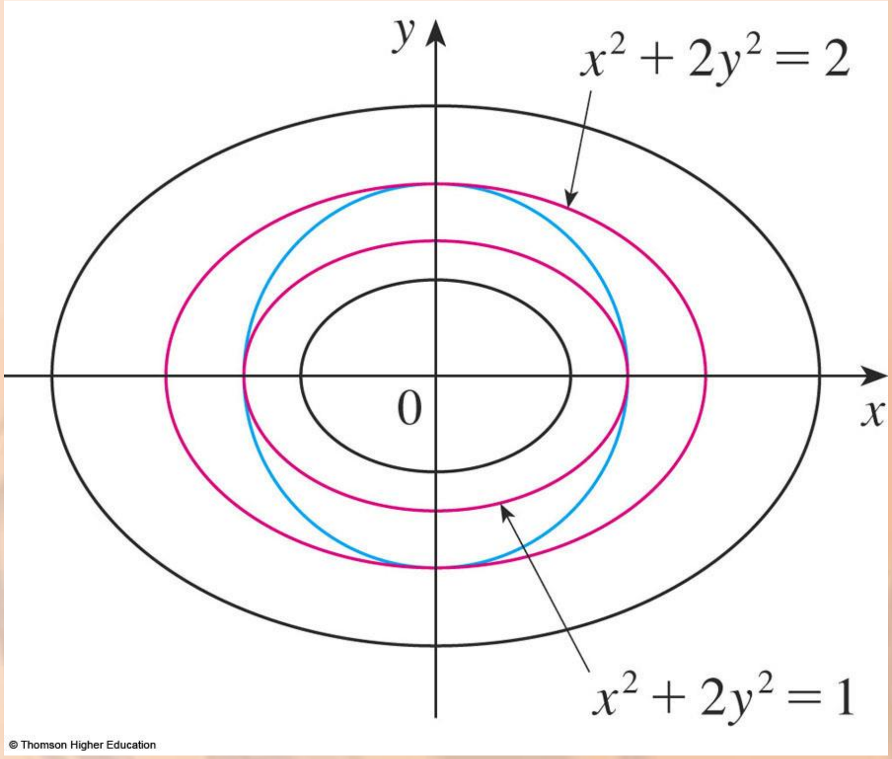
Checking with the figure, we see that these values look reasonable.



LAGRANGE'S METHOD

The geometry behind the use of Lagrange multipliers in Example 2 is shown here.

- The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$



Find the extreme values of

$f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$

- According to the procedure in Equation 9 in Section 14.7, we compare the values of f at the critical points with values at the points on the boundary.

Since $f_x = 2x$ and $f_y = 4y$, the only critical point is (0, 0).

- We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore, the maximum value of f on the disk $x^2 + y^2 \leq 1$ is:

$$f(0, \pm 1) = 2$$

The minimum value is:

$$f(0, 0) = 0$$

Example 4

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point (3, 1, –1).

The distance from a point (x, y, z) to the point (3, 1, –1) is:

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

- However, the algebra is simpler if we instead maximize and minimize the square of the distance:

$$\begin{aligned} d^2 &= f(x, y, z) \\ &= (x-3)^2 + (y-1)^2 + (z+1)^2 \end{aligned}$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2$$
$$= 4$$

According to the method of Lagrange multipliers, we solve:

$$\nabla f = \lambda \nabla g, \; g = 4$$

That gives:

$$2(x - 3) = 2x\lambda$$

$$2(y - 1) = 2y\lambda$$

$$2(z + 1) = 2z\lambda$$

$$x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from Equations 12, 13, and 14, and then substitute these values into Equation 15.

From Equation 12, we have:

$$x - 3 = x\lambda \quad \text{or} \quad x(1 - \lambda) = 3 \quad \text{or} \quad x = \frac{3}{1-\lambda}$$

- Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from Equation 12.

Similarly, Equations 13 and 14 give:

$$y = \frac{1}{1-\lambda} \qquad z = -\frac{1}{1-\lambda}$$

So, from Equation 15, we have:

$$\frac{3^2}{(1-\lambda)^2}+\frac{1^2}{(1-\lambda)^2}+\frac{(-1)^2}{(1-\lambda)^2}=4$$

- This gives $(1 - \lambda)^2 = 11/4$, $1 - \lambda = \pm \sqrt{11}/2$.
- Thus,

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of *λ* then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},-\frac{2}{\sqrt{11}}\right) \text{ and } \left(-\frac{6}{\sqrt{11}},-\frac{2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)$$

- It's easy to see that f has a smaller value at the first of these points.

Thus, the closest point is:

$$\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$$

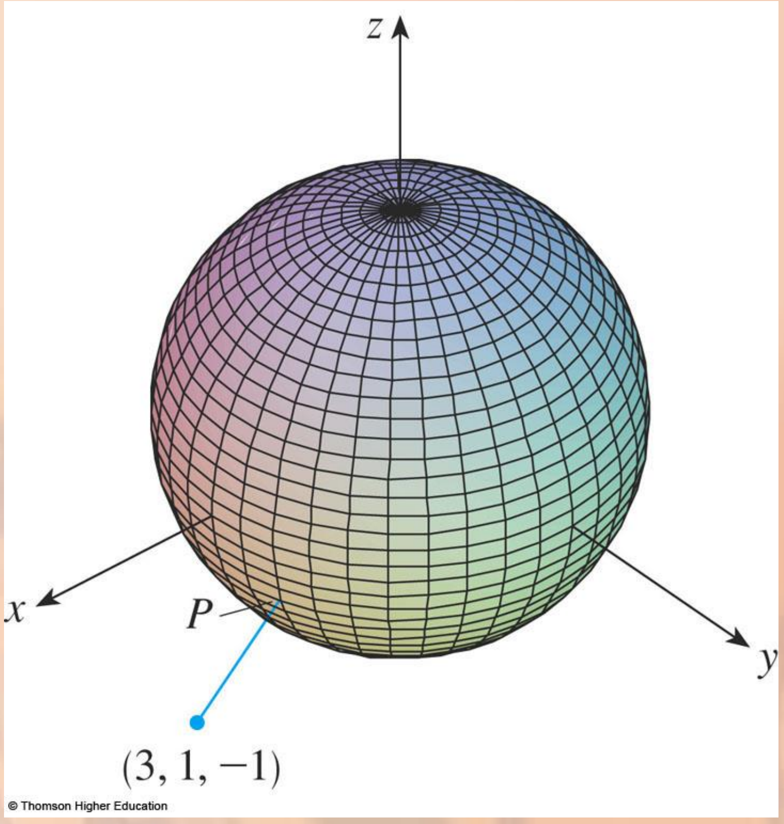
The farthest is:

$$\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$$

LAGRANGE'S METHOD

The figure shows the sphere and the nearest point in Example 4.

- Can you see how to find the coordinates of *P* without using calculus?

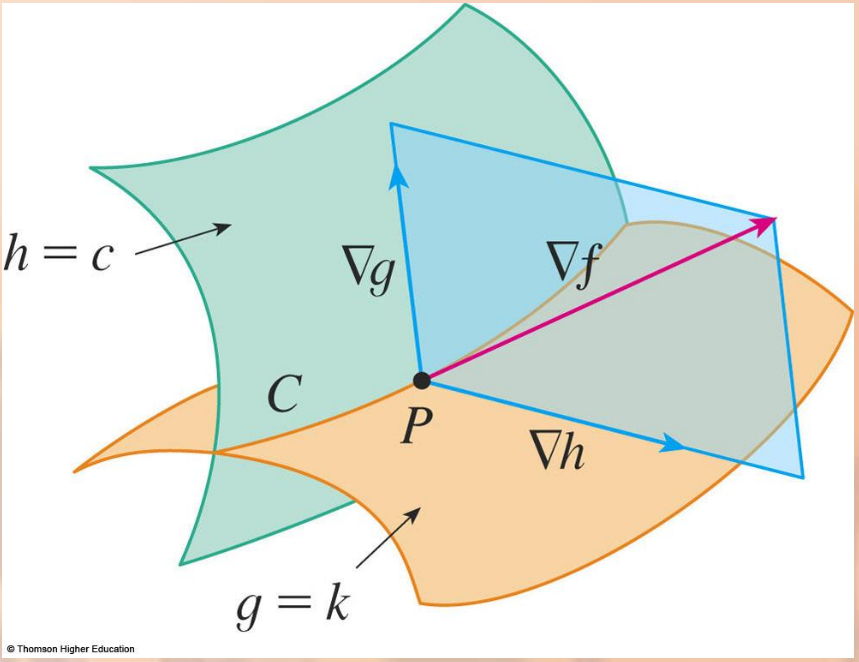


TWO CONSTRAINTS

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$.

Geometrically, this means:

- We are looking for the extreme values of *f* when (x, y, z) is restricted to lie on the curve of intersection *C* of the level surfaces
  $g(x, y, z) = k$ and
  $h(x, y, z) = c$.



TWO CONSTRAINTS

Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$.

We know from the beginning of this section that ∇f is orthogonal to C at P.

TWO CONSTRAINTS

However, we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$.

So, ∇g and ∇h are both orthogonal to C.

TWO CONSTRAINTS

This means that the gradient vector $\nabla f\left(x_0, y_0, z_0\right)$ is in the plane determined by $\nabla g\left(x_0, y_0, z_0\right)$ and $\nabla h\left(x_0, y_0, z_0\right)$

- We assume that these gradient vectors are not zero and not parallel.

 Equation 16

So, there are numbers *λ* and *μ*

(called Lagrange multipliers)

such that:

$$\nabla f\left(x_0, y_0, z_0\right) = \lambda \nabla g\left(x_0, y_0, z_0\right) + \mu \nabla h\left(x_0, y_0, z_0\right)$$

TWO CONSTRAINTS

In this case, Lagrange's method is to look for extreme values by solving five equations in the five unknowns

$$x, y, z, \lambda, \mu$$

TWO CONSTRAINTS

These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x \quad f_y = \lambda g_y + \mu h_y \quad f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k \qquad h(x, y, z) = c$$

TWO CONSTRAINTS

Example 5

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$

We maximize the given function subject to the constraints

$$g(x, y, z) = x - y + z = 1$$

$$h(x, y, z) = x^2 + y^2 = 1$$

The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$

So, we solve the equations

$$1 = \lambda + 2x\mu$$

$$2 = -\lambda + 2y\mu$$

$$3 = \lambda$$

$$x - y + z = 1$$

$$x^2 + y^2 = 1$$

Putting $\lambda = 3$ (from Equation 19) in Equation 17, we get $2x\mu = -2$.

Thus, $x = -1/\mu$.

- Similarly, Equation 18 gives $y = 5/(2\mu)$.

Substitution in Equation 21 then gives:

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

- Thus, $\mu^2 = \frac{29}{4}, \mu = \pm\sqrt{29}/2$

Then,

$$x = \mp 2/\sqrt{29}$$

$$y = \pm 5/\sqrt{29}$$

and, from Equation 20,

$$z = 1 - x + y$$

$$= 1 \pm 7/\sqrt{29}$$

The corresponding values of *f* are:

$$\mp\frac{2}{\sqrt{29}}+2\left(\pm\frac{5}{\sqrt{29}}\right)+3\left(1\pm\frac{7}{\sqrt{29}}\right)=3\pm\sqrt{29}$$

- Hence, the maximum value of *f* on the given curve is: $3+\sqrt{29}$

TWO CONSTRAINTS

The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse.

- Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.