14

PARTIAL DERIVATIVES

MATH 252: CALCULUS OF SEVERAL VARIABLES

Emmanuel de-Graft Johnson Owusu-Ansah(PhD)

Department of Mathematics

Faculty of Physical and Computational Sciences

College of Science

14.5

The Chain Rule

In this section, we will learn about:

The Chain Rule and its application in implicit differentiation.

THE CHAIN RULE

Recall that the Chain Rule for functions of a single variable gives the following rule for differentiating a composite function.

THE CHAIN RULE

Equation 1

If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t,

and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

THE CHAIN RULE

For functions of more than one variable, the Chain Rule has several versions.

- Each gives a rule for differentiating a composite function.

THE CHAIN RULE

The first version (Theorem 2) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t.

- This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t.

We assume that *f* is differentiable (Definition 7 in Section 14.4).

- Recall that this is the case when f_x and f_y are continuous (Theorem 8 in Section 14.4).

THE CHAIN RULE (CASE 1) **Theorem 2**

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t.

- Then, z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

A change of Δt in t produces changes of Δx in x and Δy in y.

- These, in turn, produce a change of Δz in z.

THE CHAIN RULE (CASE 1) **Proof**

Then, from Definition 7 in Section 14.4, we have:

$$\Delta z = \frac{\partial f}{\partial y}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

- If the functions ε_1 and ε_2 are not defined at (0, 0), we can define them to be 0 there.

Dividing both sides of this equation by Δt, we have:

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then

$$\Delta x = g(t + \Delta t) - g(t) \to 0$$

as g is differentiable and thus continuous.

Similarly, $\Delta y \to 0$.

- This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Thus,

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}\\ &= \frac{\partial f}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}+\frac{\partial f}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}+\left(\lim_{\Delta t\to 0}\varepsilon_1\right)\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\\ &\qquad +\left(\lim_{\Delta t\to 0}\varepsilon_2\right)\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+0\cdot\frac{dx}{dt}+0\cdot\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}\end{aligned}$$

THE CHAIN RULE (CASE 1)

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

- The Chain Rule gives:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It’s not necessary to substitute the expressions for *x* and *y* in terms of *t*.

- We simply observe that, when *t* = 0, we have *x* = sin 0 = 0 and *y* = cos 0 = 1.

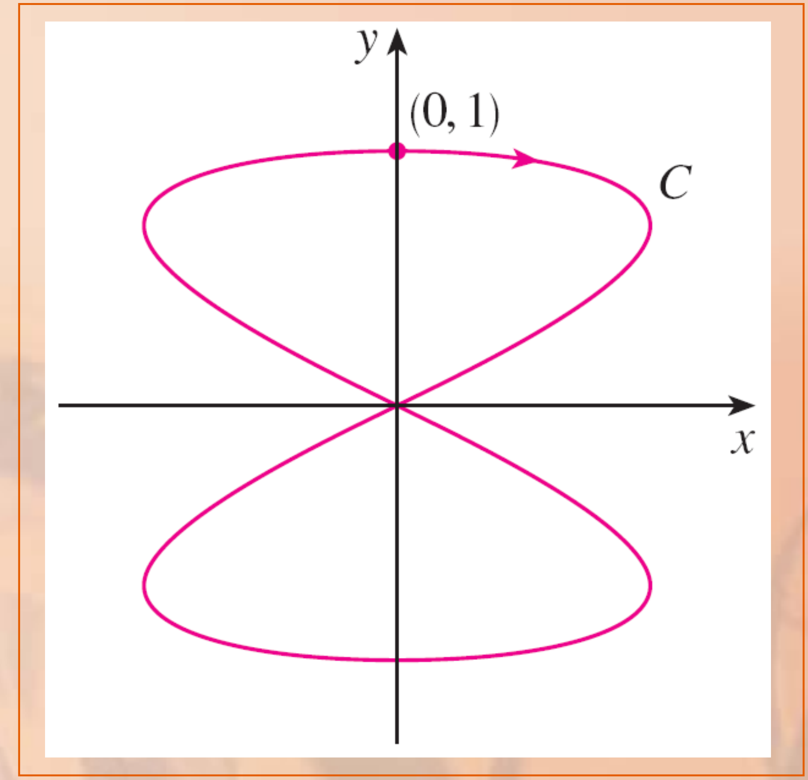
- Thus,

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$

THE CHAIN RULE (CASE 1)

The derivative in Example 1 can be interpreted as:

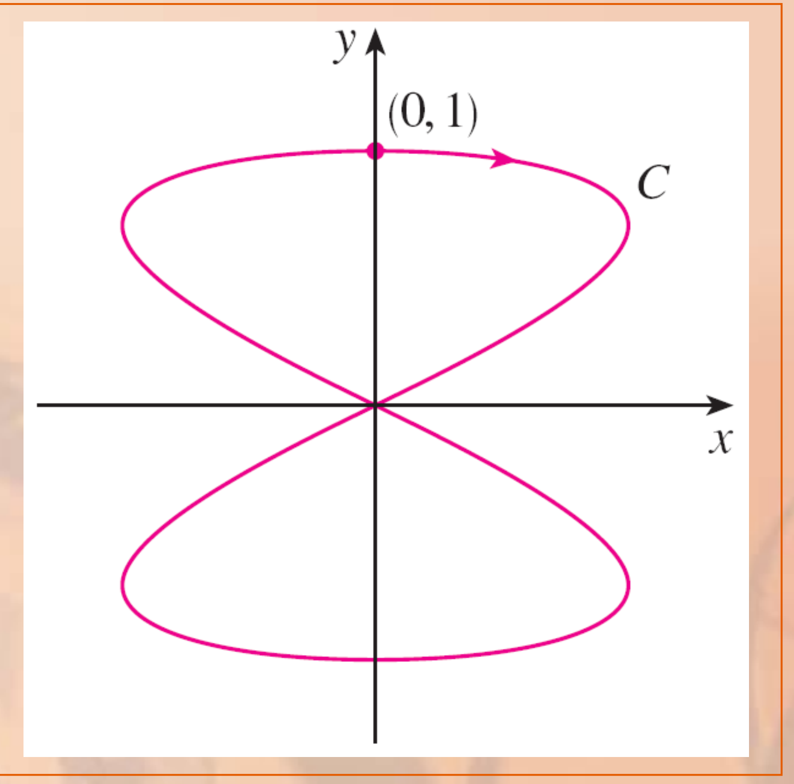
- The rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$



THE CHAIN RULE (CASE 1)

In particular, when $t = 0$,

- The point (x, y) is $(0, 1)$.
- $dz/dt = 6$ is the rate of increase as we move along the curve C through $(0, 1)$.



THE CHAIN RULE (CASE 1)

If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y), then

- The composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C
- The derivative dz/dt represents the rate at which the temperature changes along C.

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation

$$PV = 8.31T$$

Example 2

Find the rate at which the pressure is changing when:

- The temperature is 300 K and increasing at a rate of 0.1 K/s.
- The volume is 100 L and increasing at a rate of 0.2 L/s.

If t represents the time elapsed in seconds, then, at the given instant, we have:

- $T = 300$
- $dT/dt = 0.1$
- $V = 100$
- $dV/dt = 0.2$

Since $P = 8.31\dfrac{T}{V}$, the Chain Rule gives:

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.13T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2)$$

$$= -0.04155$$

- The pressure is decreasing at about 0.042 kPa/s.

THE CHAIN RULE (CASE 1)

We now consider the situation where $z = f(x, y)$, but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$.

- Then, z is indirectly a function of s and t, and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$.

THE CHAIN RULE (CASE 1)

Recall that, in computing $\partial z/\partial t$, we hold s fixed and compute the ordinary derivative of z with respect to t.

- So, we can apply Theorem 2 to obtain:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

THE CHAIN RULE (CASE 1)

A similar argument holds for $\partial z/\partial s$.

So, we have proved the following version of the Chain Rule.

THE CHAIN RULE (CASE 2) **Theorem 3**

Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t.

- Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

- Applying Case 2 of the Chain Rule, we get the following results.

THE CHAIN RULE (CASE 2)

Example 3

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)$$

THE CHAIN RULE (CASE 2)

Example 3

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)$$

Case 2 of the Chain Rule contains three types of variables:

- s and t are independent variables.
- x and y are called intermediate variables.
- z is the dependent variable.

Notice that Theorem 3 has one term for each intermediate variable.

- Each term resembles the one-dimensional Chain Rule in Equation 1.

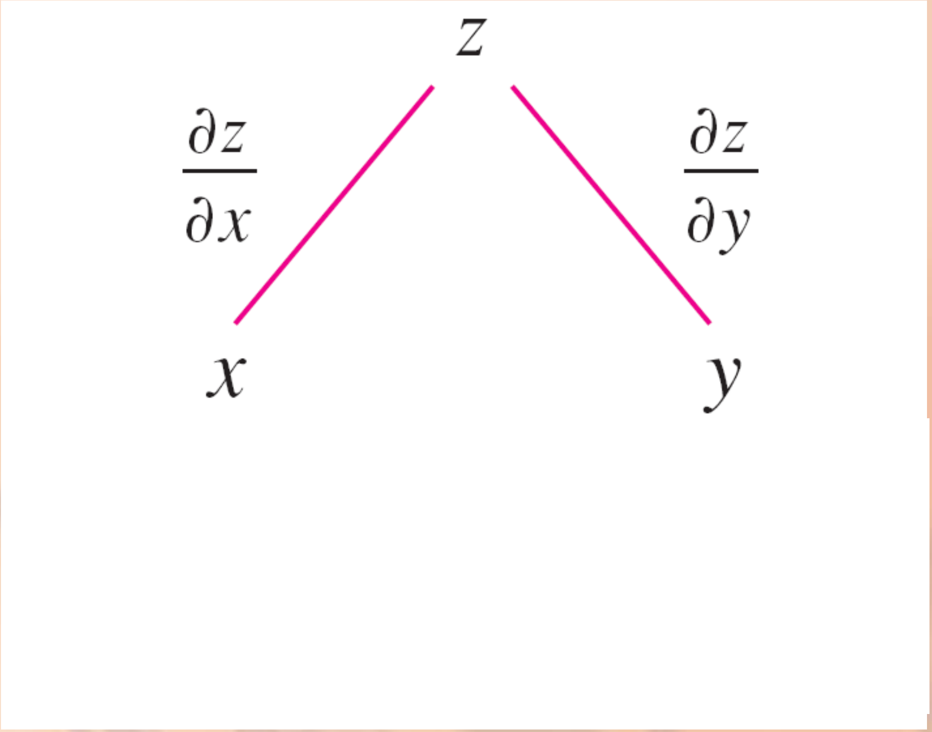
To remember the Chain Rule,

it's helpful to draw a tree diagram,

as follows.

TREE DIAGRAM

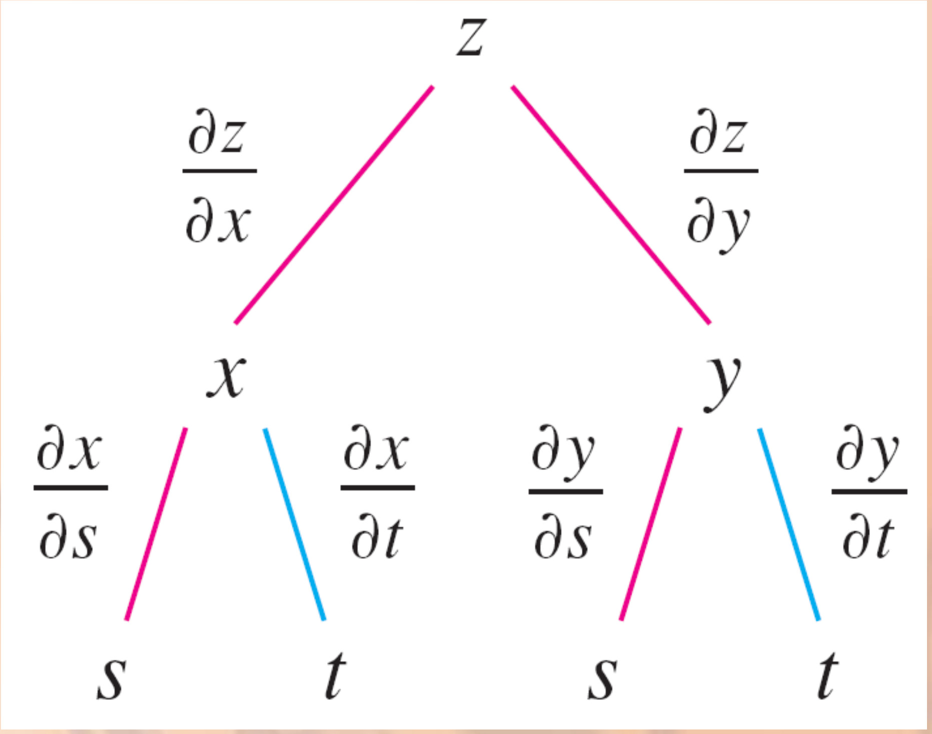
We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y.



TREE DIAGRAM

Then, we draw branches from *x* and *y* to the independent variables *s* and *t*.

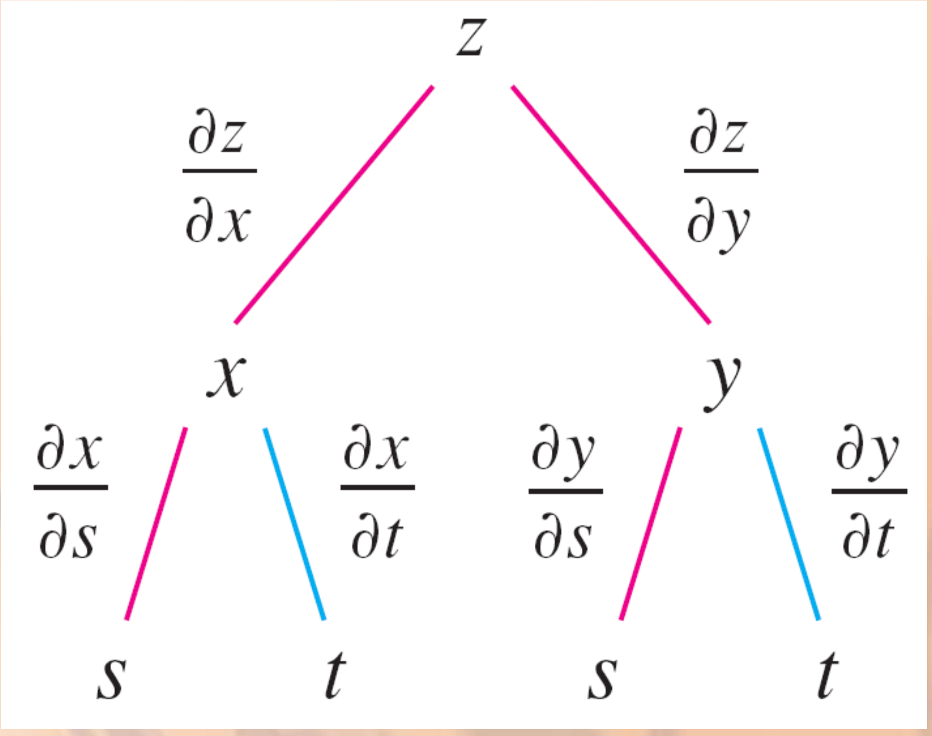
- On each branch, we write the corresponding partial derivative.



TREE DIAGRAM

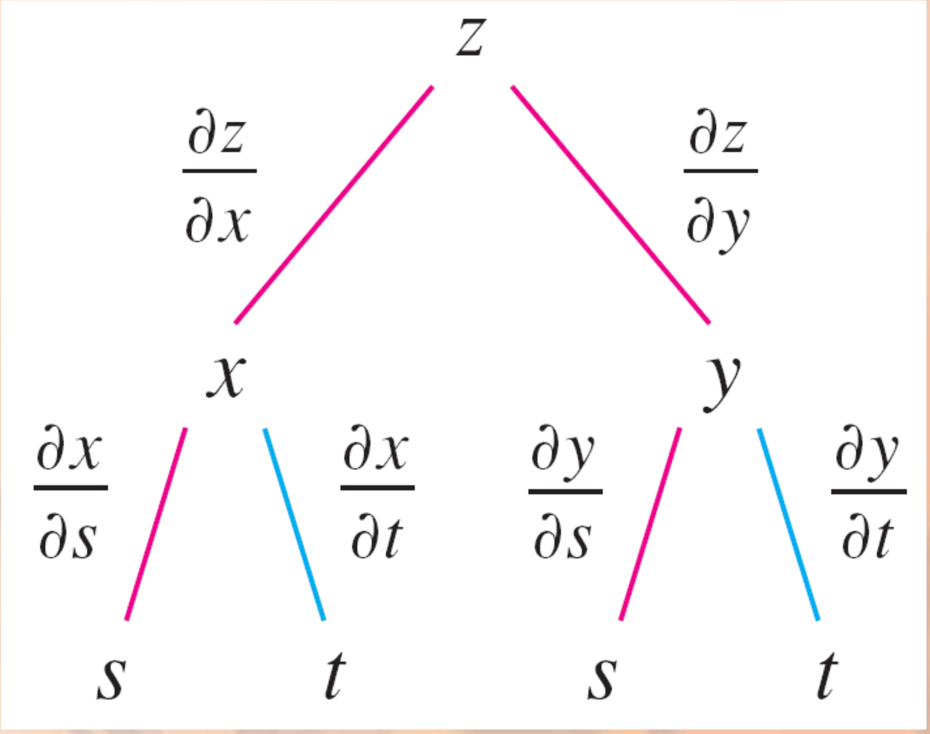
To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$



TREE DIAGRAM

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.



THE CHAIN RULE

Now, we consider the general situation in which a dependent variable *u* is a function of *n* intermediate variables $x_1, \ldots, x_{n.}$

Each of this is, in turn, a function of *m* independent variables $t_1, \ldots, t_m$.

Notice that there are *n* terms—one for each intermediate variable.

The proof is similar to that of Case 1.

THE CHAIN RULE (GEN. VERSION) **Theorem 4**

Suppose u is a differentiable function of the n variables x_1, x_2, …, x_n and each x_j is a differentiable function of the m variables $t_1, t_2 \, . \, . \, . \, , t_m$.

THE CHAIN RULE (GEN. VERSION) **Theorem 4**

Then, *u* is a function of $t_1, t_2, \ldots, t_m$

and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

THE CHAIN RULE (GEN. VERSION) **Example 4**

Write out the Chain Rule for the case

where $w = f(x, y, z, t)$

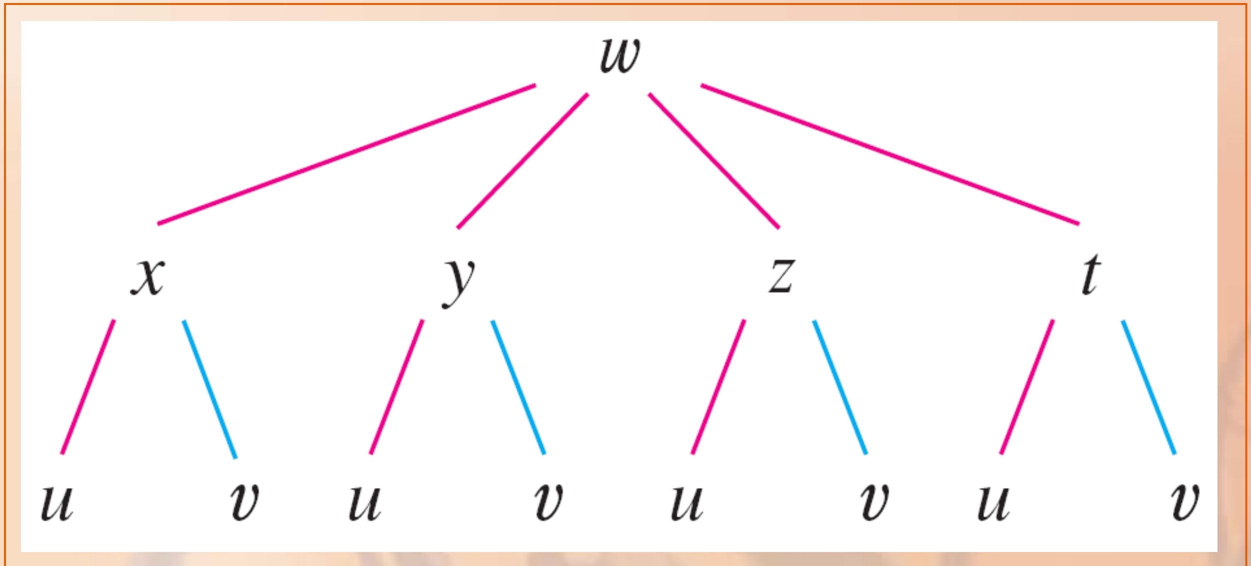
and

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ t = t(u, v)$$

- We apply Theorem 4 with $n = 4$ and $m = 2$.

The figure shows the tree diagram.

- We haven't written the derivatives on the branches.
- However, it's understood that, if a branch leads from *y* to *u*, the partial derivative for that branch is $\partial y/\partial u$.



THE CHAIN RULE (GEN. VERSION) **Example 4**

With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

THE CHAIN RULE (GEN. VERSION) **Example 5**

If $u = x^4y + y^2z^3$, where

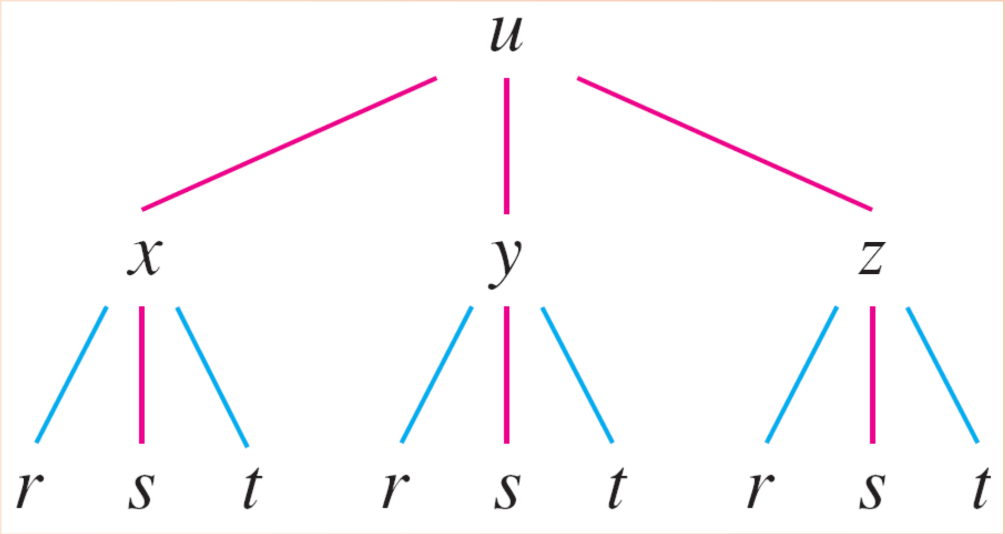
$$x = rse^t,\ y = rs^2e^{-t},\ z = r^2s \sin t$$

find the value of $\partial u/\partial s$ when

$$r = 2,\ s = 1,\ t = 0$$

THE CHAIN RULE (GEN. VERSION) **Example 5**

With the help of this tree diagram, we have:



$$\frac{\partial u}{\partial s}$$

$$= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)$$

THE CHAIN RULE (GEN. VERSION) **Example 5**

When $r = 2$, $s = 1$, and $t = 0$,

we have:

$$x = 2,\ y = 2,\ z = 0$$

Thus,

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0)$$

$$= 192$$

THE CHAIN RULE (GEN. VERSION) **Example 6**

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Let $x = s^2 – t^2$ and $y = t^2 – s^2$.

- Then, $g(s, t) = f(x, y)$, and the Chain Rule gives:

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore,

$$t\frac{\partial g}{\partial s}+s\frac{\partial g}{\partial t}$$

$$=\left(2st\frac{\partial f}{\partial x}-2st\frac{\partial f}{\partial y}\right)+\left(-2st\frac{\partial f}{\partial x}+2st\frac{\partial f}{\partial y}\right)$$

$$=0$$

THE CHAIN RULE (GEN. VERSION) **Example 7**

If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find:

a. $\partial z/\partial r$

b. $\partial^2 z/\partial r^2$

The Chain Rule gives:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

THE CHAIN RULE (GEN. VERSION) E. g. 7 b—Equation 5

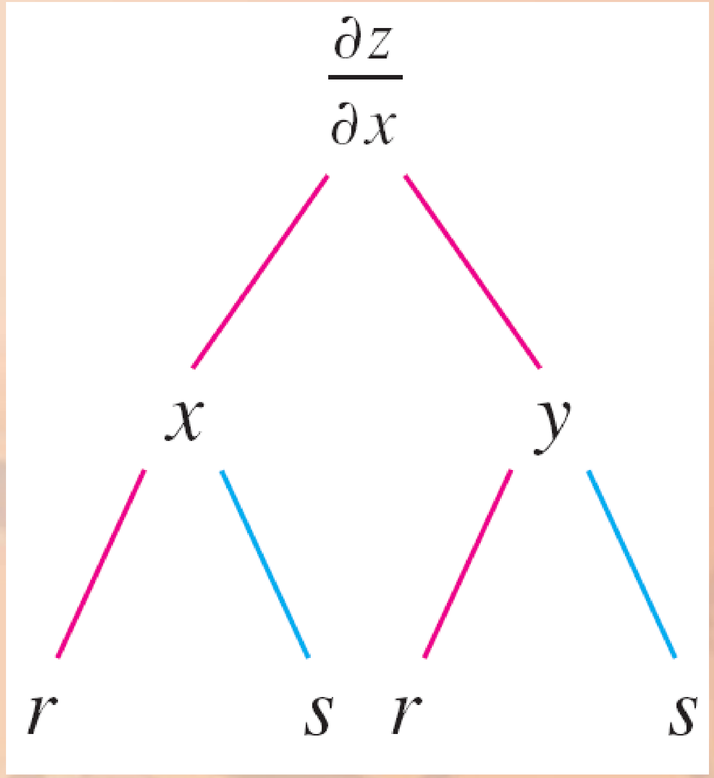
Applying the Product Rule to the expression in part a, we get:

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

THE CHAIN RULE (GEN. VERSION) **Example 7 b**

However, using the Chain Rule again, we have the following results.



THE CHAIN RULE (GEN. VERSION) **Example 7 b**

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}$$

$$=\frac{\partial^2 z}{\partial x^2}(2r)+\frac{\partial^2 z}{\partial y \partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}$$

$$=\frac{\partial^2 z}{\partial x \partial y}(2r)+\frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain the following result.

THE CHAIN RULE (GEN. VERSION) **Example 7 b**

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}\right)$$

$$+ 2s\left(2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

IMPLICIT DIFFERENTIATION

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 3.5 and 14.3

IMPLICIT DIFFERENTIATION

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x.

- That is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

IMPLICIT DIFFERENTIATION

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

- Since both x and y are functions of x, we obtain:

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

Equation 6

However, $dx/dx = 1$.

So, if $\partial F/\partial y \neq 0$, we solve for dy/dx and obtain:

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

IMPLICIT FUNCTION THEOREM

To get the equation, we assumed $F(x, y) = 0$ defines y implicitly as a function of x.

The Implicit Function Theorem, proved in advanced calculus, gives conditions under which this assumption is valid.

The theorem states the following.

- Suppose F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk.

- Then, the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

Find y' if $x^3 + y^3 = 6xy$.

- The given equation can be written as:

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

- So, Equation 6 gives:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

IMPLICIT DIFFERENTIATION

Now, we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

- This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f.

IMPLICIT DIFFERENTIATION

If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

However,

$$\frac{\partial}{\partial x}(x) = 1 \quad \text{and} \quad \frac{\partial}{\partial x}(y) = 0$$

So, that equation becomes:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in these equations.

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

- The formula for $\partial z/\partial y$ is obtained in a similar manner.

Again, a version of the Implicit Function Theorem gives conditions under which our assumption is valid.

This version states the following.

- Suppose F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere.

- Then, the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c), and this function is differentiable, with partial derivatives given by Equations 7.

Find $\partial z/\partial x$ and $\partial z/\partial y$ if

$$x^3 + y^3 + z^3 + 6xyz = 1$$

- Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$

Then, from Equations 7, we have:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

END