
HEAT TRANSFER PROCESSES

Lecture notes prepared by

Zsuzsanna Momade

KWAME NKUMAH UNIVERSITY OF
SCIENCE AND TECHNOLOGY, KUMASI
DEPARTMENT OF CHEMICAL ENGINEERING

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TABLE OF CONTENTS (CHECK!!)

REFERENCES, SUGGESTED READING LIST	4
1 FUNDAMENTAL CONCEPTS	5
1.1 First law of thermodynamics	6
1.2 Second law of thermodynamics.....	8
1.3 Temperature.....	9
1.4 The modes of heat transfer	10
1.5 Fourier's law of heat conduction	11
1.6 Interphase transport — Newton's law of cooling	13
1.7 Radiation heat transport	16
1.8 Combined heat transfer mechanism.....	24
2 THE GENERAL CONDUCTION EQUATION.....	25
2.1 Fourier's law of heat conduction.....	25
2.2 The general heat conduction equation.....	26
2.3 Boundary and initial conditions	29
3 STEADY-STATE HEAT CONDUCTION IN ONE DIMENSION.....	30
A) Plane geometry systems	30
3.1 Heat conduction in a plane wall	30
3.2 Composite walls.....	31
3.3 Overall heat transfer coefficient.....	32
B) Cylindrical geometry systems	33
3.4 Heat conduction in a cylindrical wall	33
3.5 Composite cylindrical walls.....	34
3.6 Overall heat transfer coefficient	35
3.7 Critical thickness of insulation.....	36
3.8 Internal heat generation in plane wall	38
4 UNSTEADY – STATE HEAT CONDUCTION	40
4.1 Systems with negligible internal resistance—Lumped capacitance model	41
4.2 Systems with negligible surface resistance	44
4.3 Systems with finite internal and surface resistances	47
4.4 Semi - infinite solid	48
4.5 Bodies of finite size.....	53
5 CONVECTION	56
5.1 Equation of continuity.....	57
5.2 Equation of motion (equation of conservation of momentum).....	57
5.3 Equation of thermal energy	58
5.4 Application of equations of change to laminar flow	59
5.5 Application of equations of change to turbulent flow.....	62
5.6 Boundary layer flow	63

6	CONVECTION HEAT TRANSFER IN CLOSED CONDUIT	74
6.1	Laminar flow, entrance region.....	74
6.2	Heat transfer in laminar flow — Constant heat flux at the wall	77
6.3	Heat transfer in laminar flow — Constant wall temperature	80
6.4	The combined-entry-length problem for laminar flow	82
6.5	Heat transfer in turbulent flow in tubes	83
7	HEAT EXCHANGERS	87
7.1	Classification of heat exchangers	87
7.2	Temperature distribution	88
7.3	Double-pipe heat exchanger	88
7.4	Shell-and-tube heat exchanger	90
7.5	Energy balance	95
7.6	Logarithmic mean temperature difference (LMTD)	96
7.7	Temperature correction factor F_T	97
7.8	Overall heat transfer coefficient	99
7.9	Fouling factors	100
7.10	Viscosity correction factor ϕ_v	102
7.11	Heat transfer correlations for the shell-side	103
8	CONDENSATION HEAT TRANSFER.....	104
8.1	Vertical surfaces — Laminar flow.....	105
8.2	Vertical surfaces — Turbulent flow	111
8.3	Horizontal surfaces	111
APPENDIX	113
	Cylindrical and spherical coordinate systems	113
	Heat exchangers.....	116
	Conversion table	118

REFERENCES, SUGGESTED READING LIST

It is recommended you use **any** good text you are able to understand. There are several in publications. You can find them in the library or as e-Book.

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1 FUNDAMENTAL CONCEPTS

Transport phenomena deals with the transport or movement of a given property by molecular movement through a system. The transported property can be the mass, the momentum, and thermal energy (heat).

We have 2 types of equations:

- (1) **Balance equations** or conservation equations of mass (equation of continuity), momentum (equation of motion), energy, and chemical component. These various conservation equations are called the equation of change, because they describe the change of density (ρ), velocity (v), temperature (T) and concentration (c_i) with respect to time and position in the system.
- (2) **Phenomenological equations (rate equations)**
The phenomenological method of describing a natural phenomena (process) ignores the microscopic structure of a substance and considers it as a continuous medium (continuum). It relates the properties characterising the phenomena. The phenomenological equations are the rate - driving force relationships necessary to write the balance equations. The coefficients are determined directly by experiments.

$\text{rate of transport process} = \frac{\text{driving force}}{\text{resistance}} \quad \text{or} \quad \text{rate} = \text{coefficient} \times \text{driving force}$
--

Heat transfer is energy in transit, which occurs as a result of temperature gradient or difference. This temperature difference is thought of as a driving force that causes heat to flow. The concept of heat transfer and temperature, the key words in the discipline of heat transfer, are two of the most basic concepts of thermodynamics.

System: a region in space containing a quantity of matter, which is separated from its surroundings by a boundary. **Closed system** (no-flow system): no exchange of matter with the surrounding; only heat and work cross the boundary. **Open system** (flow system): there is matter exchange with the surrounding in addition to heat and work.

Work (W): is **energy transfer**, which only appears at the boundary when a system changes its state due to the movement of a part of the boundary under the action of a force.

Sign convention:

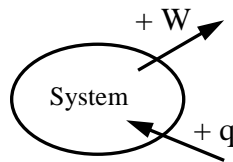
- + the work is done **by** the system on the surroundings: the work exits the system,
- the work is done **on** the system by the surroundings.

(Although there cannot be said to be any work in a system either before or after the change has taken place, work may be said to “flow” or “be transferred” across the boundary.)

Heat (Q): is **energy transfer**, which only appears at the boundary when a system changes its state due to a difference in temperature between the system and its surrounding. Heat, like work, is a transient quantity, which only appears at the boundary while a change is taking place within the system. (Although there cannot be said to be any heat in a system before or after a change of state, loosely speaking heat may be said to “flow” or “be transferred” across the boundary. Strictly speaking it is energy which is transferred, but to say “the heat transferred” is a shorthand way of saying “the energy transferred by virtue of a temperature difference”).)

Sign convention: + if heat flows **into** a system from the surrounding
 - if heat flows **from** the system to the surrounding.

So energy contained in a system might be transferred to the surrounding either by work done by the system or by heat transferred from the system. (E.g. equivalent reduction in energy is accomplished if 100 J of heat is transferred from a system or if 100 J of work is performed by a system.) Although heat and work are similar in that they both represent energy crossing a boundary, they differ in sign convention. Positive heat transfer adds energy to a system; whereas positive work subtracts energy from a system.



Heat transfer is the science that seeks to predict the energy transfer that may take place between material bodies as a result of a temperature difference. Thermodynamics teaches that this energy transfer is defined as heat. The science of heat transfer seeks not only to explain how heat energy may be transferred, but also to **predict the rate** at which the exchange will take place under certain specified conditions. The fact that a heat-transfer rate is the desired objective of an analysis points out the difference between heat transfer and thermodynamics. Thermodynamics deals with systems in equilibrium; it may be used to predict the amount of energy required to change a system from one equilibrium state to another; it may not be used to predict how fast a change will take place since the system is not in equilibrium during the process. Heat transfer supplements the first and second laws of thermodynamics by providing additional experimental rules that may be used to establish energy-transfer rates.

1.1 First law of thermodynamics

It is the principle of conservation of energy. It is an axiom.

The first law of thermodynamics says that there exists a property of a closed system (U) such that a change in its value is equal to the difference between the heat supplied and the work done during any change of state:

$$\sum_1^2 (\delta Q - \delta W) = U_2 - U_1 \quad \text{where } U = \text{internal energy, J}$$

Internal energy (U or $U/m = u$) includes chemical energy and energy associated with the molecules and atoms. Internal energy is the energy associated with the molecular motion of a substance like energy associated with translation, rotation and vibration of the molecules, atoms, electrons, protons and neutrons, chemical energy due to bonding between atoms and between subatomic particles, surface energy, energy stored in an electric condenser or in a battery ... The effect of an increase in internal energy is manifested macroscopically as an increase in temperature.

Writing Q and W for the quantities of heat and work crossing the boundary during the change of state: $Q - W = U_2 - U_1$ **non-flow energy equation**

By words: any quantity of heat supplied to a closed system must equal the increase of internal energy plus the work done by the system.

The internal energy of a closed system remains unchanged.

For isolated systems $Q = 0$, $W = 0$ therefore $\Delta U = 0$.

For irreversible non-flow processes, the energy equation can only be applied in integrated form:

$$Q - W = \Delta U$$

For reversible processes, the energy equation may also be applied in the differential form:

$$\left. \begin{array}{l} \delta Q - \delta W = dU \\ \delta W = p dV \end{array} \right\} \quad \delta Q - p dV = dU$$

For reversible, constant pressure processes (closed system):

$$p = \text{constant} \Rightarrow p dV = d(pV) \Rightarrow \delta Q - d(pV) = dU$$

$$\delta Q = d(U + pV) = dH \quad \text{where } H = \text{enthalpy, J}$$

$$\text{or} \quad Q = \Delta H$$

Steady-flow energy equation (for open system)

$$Q - W = (H_2 - H_1) + \frac{1}{2} m (v_2^2 - v_1^2) + mg (z_2 - z_1)$$

The potential energy term is either zero or negligible compared with the other terms.

The 1st law of thermodynamics does not make any distinction between heat transfer and work transfer: to it they are both energy “interactions” (non-properties) that must be distinguished from the energy change (property).

1.2 Second law of thermodynamics

It is an axiom.

The 1st law says: the net work can never be greater than the heat supplied.

The 2nd law says: it must always be less. Therefore, if a system is to undergo a cycle and produce work, it must operate between at least 2 reservoirs of different temperature.

As a consequence, work is a more valuable form of energy transfer than heat: heat can never be transformed continuously and completely into work, whereas work can always be transformed continuously and completely into heat.

The 2nd law leads to the definition of entropy:

There exists a property of a closed system (S) such that a change in its value is equal to $\int_1^2 \frac{dQ}{T}$ for any **reversible** process undergone by the system between state 1 and state 2.

Mathematically:

$$\int_1^2 \left(\frac{dQ}{T} \right)_{rev} = S_2 - S_1 \quad \text{where} \quad S = \text{entropy, J/K}$$

or in differential form: $\left(\frac{dQ}{T} \right)_{rev} = dS$

The entropy of a reversible isolated closed system remains constant.

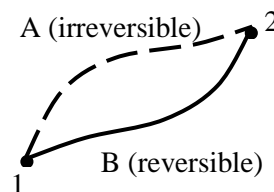
(For reversible adiabatic process $dQ = 0$ but $dS = dQ/T \Rightarrow dS = 0 \Rightarrow S = \text{constant}$)

The entropy of an irreversible isolated closed system increases:

$$\oint \left(\frac{dQ}{T} \right)_{irreversible} < 0$$

⇓

$$\int_1^2 \left(\frac{dQ}{T} \right)_{A,irreversible} + \underbrace{\int_2^1 \left(\frac{dQ}{T} \right)_{B,reversible}}_{= S_1 - S_2} < 0$$



The cycle as a whole is irreversible

$$\int_1^2 \left(\frac{dQ}{T} \right)_{A,irreversible} < S_2 - S_1 = \Delta S$$

That is the entropy change of a closed system during an irreversible process is greater than the $\int \frac{\delta Q}{T}$ evaluated for that process.

$$\Rightarrow \int_1^2 \frac{\delta Q}{T} \leq \Delta S \quad \text{and} \quad S_1 - S_2 \leq 0 \quad \Rightarrow \quad S_2 \geq S_1.$$

That is the entropy of an isolated system during a process always increases or, in the limiting case of a reversible process, remains constant. In other words, it never decreases.

The proper distinction between heat transfer and work transfer is made by the 2nd law of thermodynamics. There is always an entropy transfer associated with the heat transfer. A work transfer carries zero entropy change.

The 2nd law only provides definite quantitative statement about reversible processes. Only for such processes is it possible to predict the work and heat transfers crossing the boundary of a system. For irreversible processes the law merely provides statements of trend (that the entropy of an isolated irreversible system must increase), and quantitative prediction of energy transfer cannot be made. Therefore empirical relations (such as Newton's law of viscosity, Fourier's law of heat conduction, and Fick's law of diffusion) are used. They all take the form of proportionality between a quantity transferred in an irreversible process and the property gradient. These "laws" are approximate empirical relations, whose accuracy is manifested by the fact that the "proportionality constants" are not in fact constants but vary with the conditions of the experiment (temperature, pressure, or their gradient, ...)

1.3 Temperature

Although we are familiar with temperature as a measure of "hotness" or "coldness," it is not easy to give an exact definition for it. Temperature is a measure of molecular activity. However, we need to define the quantities in terms of macroscopic observations; therefore, a definition of temperature using molecular measurements is not useful. Thus, we must proceed without actually defining temperature. Instead, we discuss equality of temperatures.

Temperature is the system property that determines whether the system is in thermal equilibrium with another system. In thermal equilibrium of system A and B: $T_A = T_B$. The temperature of a system is measured by placing the system in "contact" with a special system (a test system) called thermometer. Temperature scales have 2 reference points.

Celsius scale

0°C: Ice-point of water (ice and air-saturated water in equilibrium at standard atmospheric pressure)

100°C: Boiling point of water (liquid water in equilibrium with its own vapor at standard atmospheric pressure).

Fahrenheit scale

32°F: Ice-point of water
212°F : Boiling point of water } 180 units between them

$$T(^{\circ}\text{C}) = 5/9 [T(^{\circ}\text{F}) - 32]$$

Kelvin scale (thermodynamic scale or absolute scale)

273.16 K: Triple-point of water

(it is only slightly above that of the ice-point 273.15 K)

0 K: **Absolute zero** (a hypothetical temperature where all thermal motion ceases)

The unit of thermodynamic temperature is $1/273.16 = 1 \text{ K}$.

This odd unit makes $T_{\text{bp}} - T_{\text{ice-point}} \approx 100 \text{ K}$ and

$\Delta T (^{\circ}\text{C}) = \Delta T (\text{K})$

 the temperature difference in Kelvin is the same as it is in °C.

$$0^{\circ}\text{C} = 273.15 \text{ K} \quad \text{and} \quad T (\text{K}) = T (^{\circ}\text{C}) + 273.15$$

1.4 The modes of heat transfer

There are 3 basic modes of heat transport:

- (1) Conduction
- (2) Convection
- (3) Radiation

Conduction (Fourier's law)

When a temperature gradient exists in a body, there is an energy transfer from the high-temperature region to the low-temperature region by conduction. Heat is transported on a molecular scale with no movement of macroscopic portions of matter relative to one another. Conduction can take place through solids, liquids, and gases.

In fluids: The kinetic energy of the molecules is associated with the property we call temperature. In high-temperature regions, molecules have higher kinetic energies than those in a low-temperature region. The molecules in the high-temperature region transfer their energy, through collisions, to molecules in the low-temperature region.

In solids: Conduction is due to motion of free electrons in metals, lattice waves (vibrations of the crystal lattice structure) in nonmetals, magnetic excitations, and electromagnetic radiation.

Convection (Newton's law)

It is the mode of energy transfer between a solid surface and the adjacent fluid that is in motion, and it involves the combined effects of conduction and fluid motion. The faster the fluid motion, the greater the convection heat transfer. In the absence of any bulk fluid motion, heat transfer between a solid surface and the adjacent fluid is by pure conduction. The presence of bulk motion of the fluid enhances the heat transfer between the solid surface and the fluid.

Convection is the heat transport due to the bulk motion of the fluid. The process occurs through the movement of macroscopic volume elements of the fluid in space from a region of one temperature to that of another. Convection is only possible in a fluid medium.

2 types of convection are distinguished:

- (a) free or natural convection — The fluid moves because of density difference resulting from the temperature difference in the fluid.
- (b) forced convection — The fluid is forced to flow past a solid surface.

Sensible heat is the amount of energy change of a single-phase fluid due to its temperature change. It is calculated as

$$Q = m c_p (T_{fl\ in} - T_{fl\ out}), \text{ J}$$

Specific heat capacity is defined as the amount of heat required to raise the temperature of a unit mass of a substance by one degree. Its unit is J/kg K. The specific heat capacity for constant pressure is c_p . It varies with temperature.

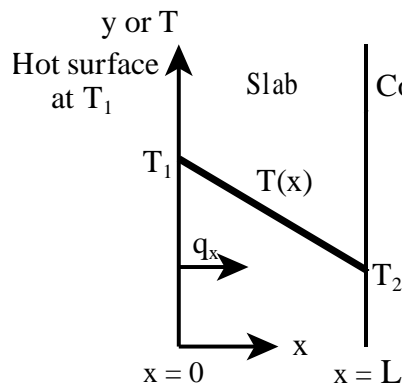
Radiation

It is the transport of energy by electromagnetic radiation having a defined range of wavelength of 0.1-100 μm as a result of the changes in the electronic configurations of the atoms or molecules.. All substances emit radiant heat but the **net** flow of heat is from the high- to low-temperature region. So the cooler substance will absorb more radiant energy than it emits. The radiation from all bodies depends on temperature (increases with temperature). No physical medium is needed for the propagation of radiation.

Often all 3 forms of heat transport are involved simultaneously. It is then usual to calculate the rate of heat transport by each mode separately and adding the separate effects to provide an estimate of the total rate of heat transport. In a number of cases, one mode of heat transport is dominant.

1.5 Fourier's law of heat conduction

Fourier's law of heat conduction is based on the empirical observation of one-dimensional steady heat flow through a solid. (**Steady heat flow** means the temperature at any point does not vary with time; **one-dimensional** means the temperature is uniform over surfaces perpendicular to the direction of heat flow.)



The surface area of slab is A

Heat is conducted through the slab in only x direction

$$\frac{q_x}{A} = -k \frac{dT}{dx}$$

Fourier's law (1822)

where q_x = heat flow rate in the x direction, J/s = W
 A = area normal to the heat flow direction, m²
 dT/dx = temperature gradient or slope of the temperature curve
 k = thermal conductivity, W/mK
 q_x/A = heat flux (heat flow rate per unit area), W/m²

As x increases, the temperature decreases, so the right hand side of the equation is a positive quantity. The – sign ensures that the equation properly indicates that the heat flow is in the direction of temperature fall.

Fourier's law of heat conduction is an experimentally observed law and serves as a definition of the property of substances called the thermal conductivity.

Solving Fourier's equation for the slab:

$$\int_0^L \frac{q_x}{A} dx = \int_{T_1}^{T_2} -k dT$$



at $x = 0$ $T = T_1$
 at $x = L$ $T = T_2$
 constant heat flux ($q/A = q''$)
 constant k

$$\frac{q_x}{A} \int_0^L dx = -k \int_{T_1}^{T_2} dT$$

$$\frac{q_x}{A} = k \frac{T_1 - T_2}{L}$$

Thermal conductivity k

Thermal conductivity is a property of a material. Its numerical value is an indication of how fast heat is conducted through the material. k is a function of temperature, but not a very strong one. For small ranges of temperature, k can be considered constant. For large temperature ranges

$$k = k_o + bT \quad \text{where} \quad \begin{array}{l} k_o = \text{thermal conductivity at } 0^\circ\text{C} \\ b = \text{empirical constant} \end{array}$$

k varies over a wide range. It is highest for metals and lowest for finely powdered materials. The thermal conductivities of most liquids are rather small, except for metallic liquids.

A few examples of thermal conductivity

Material	k, W/mK
Metals	50 - 400
Alloys	10 - 120
Water	0.598 (at 20°C)
Air	0.0251 (at 20°C)
Insulators	0.04 – 0.2

While the rate at which heat is transported in a body (q) is dependent on the thermal conductivity (k) and the temperature gradient ($\Delta T/\Delta x$), the rise in temperature that this heat will produce will vary with the specific heat (c_p) and the density (ρ) of the body.

1.6 Interphase transport — Newton's law of cooling

Interphase heat transport or convective heat transfer occurs when a fluid acts as a carrier or “conveyor belt” for the energy that it draws from (or delivers to) a solid wall. The characteristics of the flow affect greatly the heat transfer rate between the wall and the stream. The temperature gradient at the wall is dependent on the rate at which the fluid carries the heat away; a high velocity produces a large temperature gradient.

The heat flow rate may be related to the temperature difference between the temperature at the interface (T_w) and that in the fluid (T_f):

$q = h A (T_w - T_f)$

Newton's law of cooling (1701)

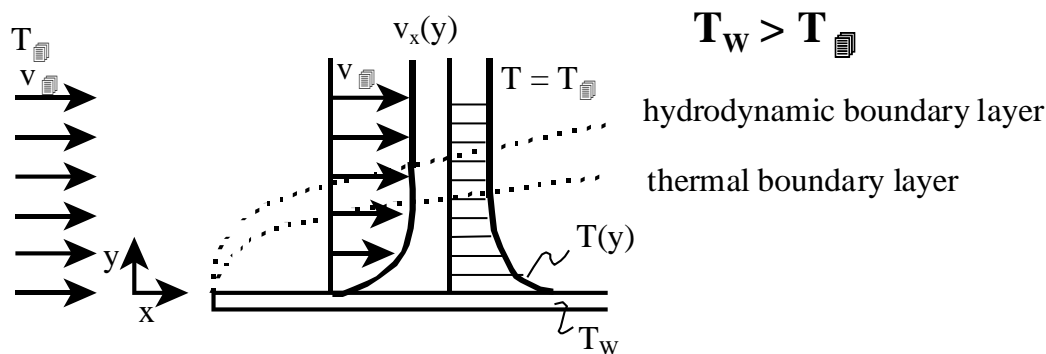
where

- q = heat flow rate, W
- A = characteristic area, m^2
- ΔT = characteristic temperature difference, $^\circ\text{C}$
- h = heat-transfer coefficient, $\text{W}/\text{m}^2\text{K}$
(or film transfer coefficient or surface conductivity)

Newton's law of cooling is not really a law but rather a **defining equation for h heat-transfer coefficient**. h is not defined for a specific situation until A and ΔT are stated.

If there is no bulk motion of the fluid, there is no convection, energy is transferred by conduction only.

An external flow configuration



A heated plate with a uniform wall temperature of T_w is immersed in a uniform fluid flow of velocity v_∞ and temperature T_∞ . At any location, the velocity at the wall is 0 and increases with increasing y to v_∞ free-stream velocity at some vertical distance away, known as the **hydrodynamic boundary layer**.

The temperature distribution $T(y)$ is also drawn. The temperature decreases from T_w at the wall to T_∞ free-stream temperature at some distance from the wall. This distance is called **thermal boundary layer** thickness and is highly dependent on the flow velocity.

Heat is transferred from the wall to the fluid.

$$q = h A (T_w - T_\infty) \quad \text{where } h = \text{average heat transfer coefficient, that is customarily assumed constant over the length of plate}$$

In many cases, empirical correlations are available to predict h , since it often cannot be predicted theoretically.

An internal flow configuration (tube flow)

$$q = h (D\pi L) (T_w - T_b) \quad \text{where } T_b = \text{bulk fluid temperature.}$$

The bulk fluid temperature is also called “cup mixing temperature” or “flow average temperature” or “mean temperature”.

When a fluid is being heated or cooled, the temperature will vary throughout the cross section of the stream. The bulk temperature is the temperature that would be attained if all the fluid flowing across the section in question were withdrawn and mixed adiabatically to a uniform temperature.

$T_w - T_b$ can be – the initial temperature difference,
– the arithmetic mean temperature difference

$$\frac{(T_{w1} - T_{b1}) + (T_{w2} - T_{b2})}{2}$$

– the logarithmic mean temperature difference

$$\frac{(T_{w1} - T_{b1}) - (T_{w2} - T_{b2})}{\ln \frac{T_{w1} - T_{b1}}{T_{w2} - T_{b2}}}$$

Heat transfer coefficient

The heat-transfer coefficient is not a constant characteristic of the fluid medium. It depends in a complicated way on

- fluid properties (μ , c_p , ρ , k),
- system geometry,
- flow velocity and velocity distribution,
- characteristic temperature difference (ΔT), and
- surface temperature distribution.

An analytical calculation of h may be made for some systems. For complex situations, it must be determined experimentally. For a particular situation, it can be obtained either by direct measurement or from existing empirical or semi-empirical correlations. These correlations are in the form of equations involving dimensionless numbers.

Typical values of the convective heat transfer coefficient
for various fluids

Fluid and condition	h, W/m²K
Air in natural convection	5 – 30
Liquids in natural convection	10 – 1 000
Superheated steam or air in forced convection	30 – 300
Oil in forced convection	60 – 1 800
Water in forced convection	300 – 18 000
Boiling water	3 000 – 60 000
Condensation of steam	6 000 – 120 000

1.7 Radiation heat transport

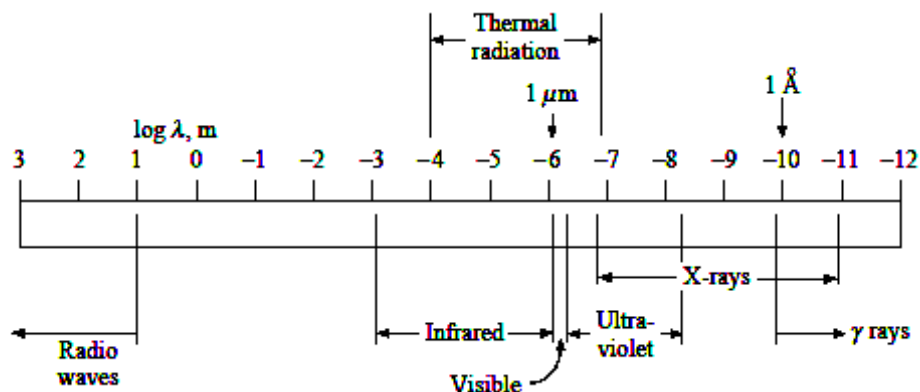
Fundamentals

Radiative heat transfer or thermal radiation is the science of transferring energy in the form of electromagnetic waves. Unlike heat conduction, electromagnetic waves do not require a medium for their propagation. Thermal radiation becomes the dominant mode of heat transfer in low-pressure (vacuum) and outer-space applications, and in high-temperature applications, such as combustion (fires, furnaces, rocket nozzles), nuclear reactions (solar emission, nuclear weapons), and others.

All materials above the temperature of absolute zero continuously emit and absorb electromagnetic waves, or photons, by changing their internal energy on a molecular level. Strength of emission and absorption of radiative energy depends on the temperature of the material, as well as on the wavelength λ , frequency ν “nu”, or wave number η , that characterizes the electromagnetic waves

$$\lambda = \frac{c}{\nu} = \frac{1}{\eta}$$

where λ = wavelength, m (usually measured in $\mu\text{m} = 10^{-6}\text{ m}$)
 c = speed of light = 3×10^8 m/s
 ν = frequency, Hz = 1/s
 η = wave number (usually measured in 1/cm)



The electromagnetic spectrum

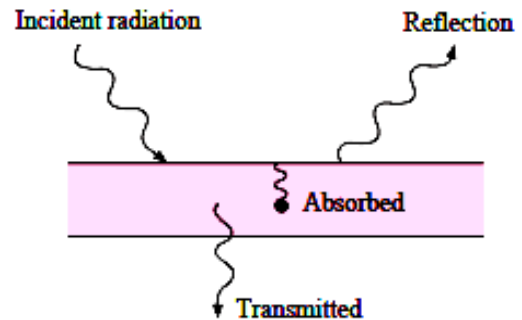
The wavelength of thermal radiation lies in the range of 0.1 - 100 μm . It includes visible radiation and infrared radiation. (Visible light has wavelength of 0.35-0.75 μm , infrared radiation has wavelength of 0.75-1000 μm (= 1 mm).)

The sun with an effective surface temperature of 5762 K emits most of its radiation at the extreme lower end of the spectrum between 0.1 - 4 μm having maximum intensity at 0.5 μm .

The radiations from a lamp filament are in the range of 1 - 10 μm . Objects at room temperature emit radiation in the 8 - 25 μm band.

Most solids and liquids have a continuous spectrum; they emit radiations of all wavelengths. Gases and vapours radiate energy only at certain bands of wavelength and hence are called selective emitters.

When thermal radiation falls upon a body, part is absorbed by the body in the form of heat, part is reflected back into space, and part may be transmitted through the body — this is negligible in process engineering. We define the **absorptivity** α as the fraction absorbed, **reflectivity** ρ as the fraction reflected, and the **transmissivity** τ as the fraction transmitted.



Thus

$$\alpha + \rho + \tau = 1$$

Most solid bodies do not transmit thermal radiation, so that for many applied problems the transmissivity may be taken as zero. Then

$$\alpha + \rho = 1$$

Blackbody

A **blackbody** is defined as one that absorbs all radiant energy at all wavelengths and reflects none: $\alpha = 1.0$. A black body also emits radiation, depending on its temperature: $\varepsilon = 1.0$, where ε is the **emissivity** — the ratio of the emissive power of a surface to that of a black body.

The concept of a blackbody is an idealization; that is, a perfect blackbody does not exist—all surfaces reflect radiation to some extent, however slight. Blackbody appears black to the eye; it appears black because it does not reflect any radiation. It is important to note that the “blackness” of a surface to thermal radiation can be quite deceiving insofar as visual observations are concerned. A surface coated with lampblack appears black to the eye and turns out to be black for the thermal-radiation spectrum. On the other hand, snow and ice appear quite bright to the eye but are essentially “black” for long-wavelength thermal radiation. Many white paints are also essentially black for long-wavelength radiation.

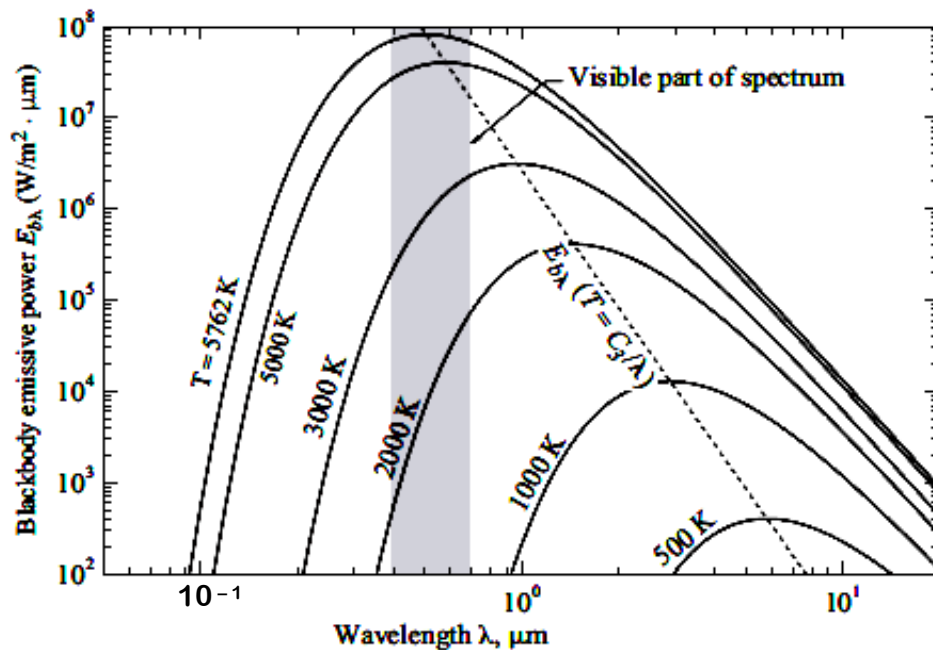
Actually no material with absorptivity $\alpha = 1$ and zero reflectivity exists. Even the blackest surfaces occurring in nature still have a reflectivity of about 1% (absorptivity of 99% $\Rightarrow \alpha = 0.99$).

For any real body the absorptivity varies considerably with frequency (α_v). A **gray body** is a hypothetical body for which α_v is a constant (< 1) over the entire frequency range and at all temperatures. That is a grey body always absorbs the same fraction of the incident radiation at all frequencies.

Emissive power

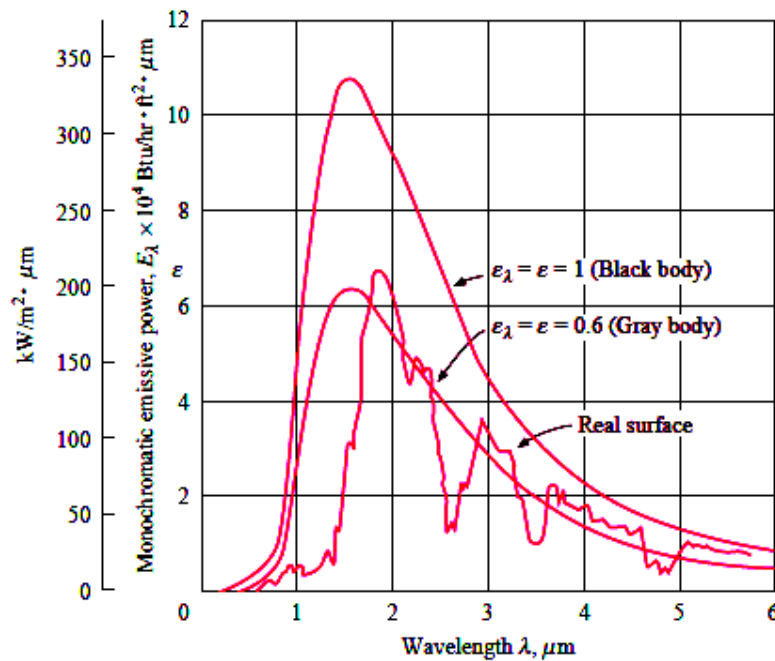
Every medium continuously emits electromagnetic radiation randomly into all directions at a rate depending on the local temperature and the properties of the material. The radiative heat flux emitted from a surface is called the **emissive power E** (or radiation heat flux emitted, measured in W/m^2).

A blackbody is not only a perfect absorber, but it is also a perfect emitter, that is, the emission from such a surface exceeds that of any other surface at the same temperature.



Blackbody emissive power spectrum in vacuum

Emissivity increases with increasing temperature for nearly all materials. The peak of the curve is shifted to the shorter wavelengths for the higher temperatures. The shift in the maximum point of the radiation curve explains the change in color of a body as it is heated. Since the band of wavelengths visible to the eye lies between about $0.3 - 0.7 \mu\text{m}$, only a very small portion of the radiant-energy spectrum at low temperatures is detected by the eye. As the body is heated, the maximum intensity is shifted to the shorter wavelengths, and the first visible sign of the increase in temperature of the body is a dark-red color. With further increase in temperature, the color appears as a bright red, then bright yellow, and finally white. The material also appears much brighter at higher temperatures because a larger portion of the total radiation falls within the visible range.



The relative radiation spectra from a blackbody at 1650°C (1923 K) and a corresponding ideal gray body with $\epsilon = 0.6$. Also shown is a curve indicating an approximate behavior for a real surface, which may differ considerably from that of either an ideal blackbody or an ideal gray body.

For analysis purposes, surfaces are usually considered as gray bodies, with emissivities taken as the integrated average value.

Comparison of emissive power of ideal blackbodies and gray bodies with that of a real surface

A blackbody emits thermal energy at a rate that is proportional to the 4-th power of its absolute temperature:

$$q_r = A \sigma T^4 \quad (\epsilon = 1.0) \quad \text{Stefan-Boltzmann equation}$$

where q_r = heat flow rate, W
 A = surface area of the body, m^2
 $\sigma = 5.676 \times 10^{-8} \text{ W/m}^2\text{K}^4$, Stefan-Boltzmann constant
 T = **absolute temperature** of black body, K

Real materials do not emit electromagnetic radiation ideally as blackbodies; they have an emissivity $\epsilon < 1.0$ and are called **gray bodies**. Their emissive power is reduced by ϵ

$$q_r = A \epsilon \sigma T^4 \quad (\epsilon < 1)$$

The emissivity of a material varies with temperature and the wavelength of the radiation: $\epsilon(T, \lambda)$.

Emissivity

The surface of a substance highly influences its radiation characteristics and therefore the amount of radiative heat the surface can absorb, transmit, reflect, and emit. Emissivity is a property that describes how radiant energy interacts with the surface of the material. The emissivity is determined entirely by the properties of the surface of the material, its temperature, and the wavelength of the radiation.

Some normal emissivity values of various materials at room temperature

Material	ϵ
Aluminium polished	0.04
rough	0.07
Asphalt	0.85 – 0.93
Concrete	0.88 – 0.93
Glass, window	0.90 – 0.95
Paint black	0.98
white	0.89 – 0.97
Red brick	0.93 – 0.96
Skin	0.95
Soil	0.93 – 0.96
Water	0.96
Wood	0.80 – 0.90

Clean, highly polished metallic surfaces have very low emissivities. The emissivity of metallic surfaces increases with increasing temperature.

Nonmetallic materials are characterized by high emissivities in the infrared region.

Smooth and rough surfaces have almost the same emissivity. This however holds only for **low temperature radiation**. The maximum of intensity of radiation shifts to greater wavelengths with decreasing temperature. Thus, radiation at low temperature is mainly **long-wave radiation**. If **short-wave radiation**, like that of the sun, strikes a white surface or ice, the absorptivity is much smaller than when it hits a black surface. The good reflection of sunlight by ice or white fabrics is well known. It is less known that the reflection of the same bodies is very small for long-wave radiation.

Kirchhoff's law states that a given temperature T_1 $\boxed{\alpha_1 = \epsilon_1}$ of a given surface.

That is Kirchhoff's law states that a gray body in thermodynamic equilibrium with its surrounding emits as much energy as it absorbs in each direction and at each wavelength.

Net heat transfer between 2 radiating surfaces

When 2 surfaces are at a given distance from each other, the resultant rate of heat transfer by radiation from one to the other may be determined.

Two long concentric cylinders and concentric spheres

Cylinder (1) with a surface area A_1 and temperature T_1 encloses cylinder (2) with a surface area A_2 and temperature T_2 ($T_1 > T_2$). The net radiation heat transfer from cylinder (1) to cylinder (2):

$$q = \frac{1}{\frac{1}{\epsilon_1} + \frac{A_1}{A_2} \left(\frac{1}{\epsilon_2} - 1 \right)} \sigma A_1 (T_1^4 - T_2^4)$$

Infinite parallel plates

If plate (1) is at higher temperature than plate (2), then the net heat flux from (1) to (2) is

$$q = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} \sigma A_1 (T_1^4 - T_2^4)$$

Radiation to a small object from surroundings

Let's examine a small gray object (area A_1 , temperature T_1) in a large enclosure at a higher temperature of T_2 so there is a net radiation to the small object.

The small body emits an amount of radiation of $A_1 \varepsilon_1 \sigma T_1^4$
It absorbs energy from the surroundings at T_2 by $A_1 \alpha_1 \sigma T_2^4$
and α_1 at $T_2 \approx \varepsilon_1$ at T_1

The net heat of absorption:

$$q_r = A_1 \varepsilon_1 \sigma (T_2^4 - T_1^4)$$

View factor (or shape factor)

In radiation heat transport it must be considered, that electromagnetic radiation travels in straight lines. F_{1-2} shape factor is defined as the fraction of total radiant energy that leaves surface 1 and arrives on surface 2. $F_{1-2} \leq 1$ and is a dimensionless factor. View factors for several geometries are available in many literatures.

$$q_r = A_1 \varepsilon_1 \sigma F_{1-2} (T_2^4 - T_1^4)$$

Radiation heat transfer coefficient

Analogously to Newton's law of cooling, a radiation heat transfer coefficient h_r can be defined as:

$$q_{\text{rad}} = A_1 h_r (T_2 - T_1)$$

q_{rad} = heat transfer rate by radiation, W

h_r = radiation heat transfer coefficient, W/m²K

When radiation heat transfer occurs from a surface, it is usually accompanied by convective heat transfer unless the surface is in vacuum:

$$q_{\text{conv}} = A_1 h_c (T_2 - T_1)$$

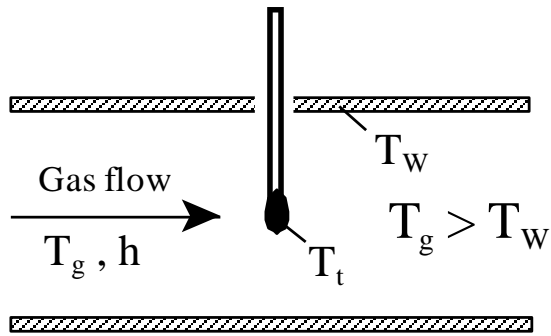
The total heat transfer is the sum of the two:

$$q = q_{\text{conv}} + q_{\text{rad}} = (h_c + h_r) A_1 (T_2 - T_1)$$

Note: the convective heat transfer coefficient is not strongly dependent on temperature; however, the radiation heat transfer coefficient is a very strong function of temperature.

Effect of radiation on temperature measurement

The temperature of a flowing fluid in a tube is usually measured by a thermometer or **thermocouple** put into a well, which is welded into the tube wall. When a thermometer is placed in a gas stream to measure temperature, the temperature indicated by the sensing element is determined by the overall energy balance on the element. Consider the element shown on the next sketch. Energy will be transferred by convection to the thermometer and then dissipated by radiation to the surroundings.



T_g = gas temperature
 T_t = temperature indicated by the thermometer
 T_w = surface temperature; the effective radiation surrounding temperature
 A = surface area of the element
 ε = emissivity of the thermometer surface

Thus the energy balance:

$$\underbrace{h A (T_g - T_t)}_{\text{heat gain by convection}} = \underbrace{\sigma A \varepsilon (T_t^4 - T_w^4)}_{\text{heat loss by radiation}}$$

This equation assumes that the surroundings are either very large or black. This equation shows that the temperature indicated by the thermometer is **not** the true gas temperature but some radiation-convection equilibrium temperature. Very large errors can result in temperature measurements if this energy balance is not properly taken into account. An estimate of the errors in temperature measurement should be made in each installation.

The heat transfer coefficient h for above heat balance can be estimated from:

$$Nu = 0.3 Re^{0.57} \quad \text{where} \quad Nu = hd/k = \text{Nusselt number}$$

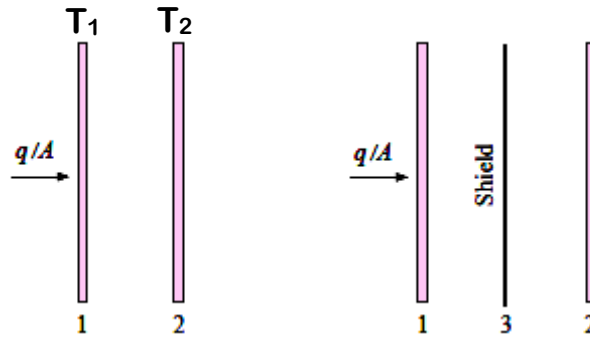
$$Re = \rho v d / \mu = \text{Reynolds number}$$

d = outer diameter of thermocouple well (house), m

Radiation shields

Radiation shields are frequently employed to alleviate this difficulty. Radiation shields are made of metals, which are highly reflective outside and blackened inside. They do not deliver or remove any heat from the overall system; they only place another resistance in the heat flow path so that the overall heat transfer is reduced. This insulation power depends upon the emissivity of the shield surface.

With the shield in place, the heat source exchanges heat directly with the shield. Then as the shield gains energy, it exchanges heat with the original receiver.



Radiation between parallel infinite planes with and without radiation shield

Since the shield does not deliver or remove heat from the system, the heat transfer between plate 1 and the shield must be precisely the same as that between the shield and plate 2, and this is the overall heat transfer.

$$q''_{1-3} = \frac{\sigma(T_1^4 - T_3^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1} \quad \text{and} \quad q''_{3-2} = \frac{\sigma(T_3^4 - T_2^4)}{\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1}$$

Eliminating T_3 and $q''_{1-3} = q''_{3-2} = q''$ results in the heat fluxes

$$q'' = \frac{\sigma(T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1\right) + \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1\right)} \quad \text{with shield}$$

$$q'' = \frac{\sigma(T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right)} \quad \text{without shield}$$

The ratio of radiant heat transfer rate with a shield to that obtained without shield:

$$\frac{q''_{shield}}{q''_{no\ shield}} = \frac{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1}{\frac{1}{\epsilon_1} + \frac{2}{\epsilon_2} + \frac{1}{\epsilon_3} - 2}$$

If the emissivities of all three surfaces are equal, that is, $\epsilon_1 = \epsilon_2 = \epsilon_3$, then

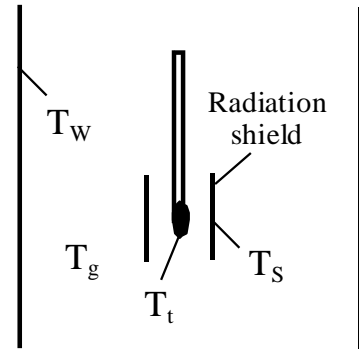
$$\frac{q''_{shield}}{q''_{no\ shield}} = \frac{1}{2}$$

and we obtain a simple relation for the unknown shield temperature T_3 :

$$T_3^4 = \frac{1}{2}(T_1^4 - T_2^4)$$

But since $\varepsilon_3 = \varepsilon_2$, the heat flow is just one-half of that which would be experienced if there were no shield present.

The radiation shield must be designed so that it protects the temperature-sensing element of the thermometer from the radiation coming from the walls but does not inhibit convective heat transfer at the thermometer surface. A short, hollow cylinder open on both ends is a desirable configuration.



Radiation from the sun

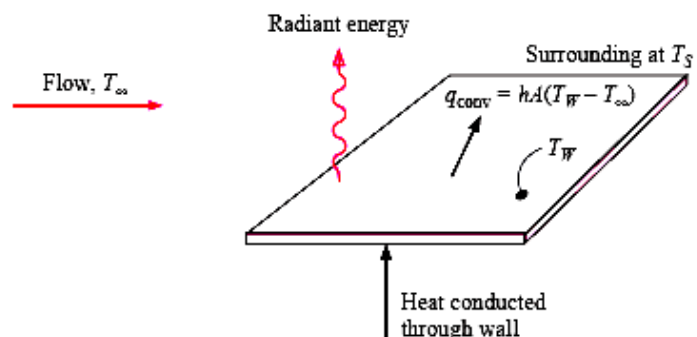
The sun radiates very nearly like a “black” circular disk with a temperature of 5600 K. As a consequence of the high temperature, the maximum radiation intensity is found at 0.5 μm wavelength and approximately $\frac{1}{3}$ of the radiation occurs in the visible wavelength range, the rest in the infra red up to $\sim 3 \mu\text{m}$.

In a yearly average the earth absorbs $\sim 43\%$ of the radiation coming from the sun (27% directly and 16% as diffuse sky radiation), 42% is reflected or refracted back to space from clouds and the air and reflected from the earth’s surface, and 15% is absorbed in the atmosphere.

The amount of solar radiation, which a surface absorbs, depends on its absorptivity. Solar radiation has short wavelength and the absorptivity for such radiation may be considerably different from the absorptivity for long wave radiation. For instance, white surface has considerably smaller absorptivity than the Al surface for solar radiation.

1.8 Combined heat transfer mechanism

In some cases, all the three modes of heat transfer are taken into account.



In that case the energy balance would give:

$$-k A \frac{dT}{dy} \Big|_{\text{wall}} = h A (T_w - T_{fl}) + A \varepsilon \sigma F_{1-2} (T_w^4 - T_s^4)$$

In many problems, however, it is useful to determine which mode is predominant and operate under that condition.

2 THE GENERAL CONDUCTION EQUATION

Symbols used: q = heat flow rate, W
 q'' = heat flux: heat flow rate per unit area normal to the heat flow direction, W/m²
 q''' = heat flow rate per unit volume, W/m³

2.1 Fourier's law of heat conduction

Fourier's law in 3 dimension: $q''_x = -k \frac{\partial T}{\partial x}$, $q''_y = -k \frac{\partial T}{\partial y}$, $q''_z = -k \frac{\partial T}{\partial z}$

Fourier's law in vector form: $\bar{q}'' = -k \nabla T$

The del operator ∇ in rectangular coordinates:

$$\nabla = \bar{e}_x \frac{\partial}{\partial x} + \bar{e}_y \frac{\partial}{\partial y} + \bar{e}_z \frac{\partial}{\partial z}$$

or $\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$ where \bar{e}_x , \bar{e}_y , \bar{e}_z are unit vectors in the x, y, and z direction.

∇ is a vector; it cannot stand alone but must operate on a scalar, vector or tensor function. It has dimensions of 1/L (1/m).

∇ in cylindrical coordinate: $\nabla = \bar{e}_r \frac{\partial}{\partial r} + \frac{\bar{e}_\Theta}{r} \frac{\partial}{\partial \Theta} + \bar{e}_z \frac{\partial}{\partial z}$

∇ in spherical coordinates: $\nabla = \bar{e}_r \frac{\partial}{\partial r} + \frac{\bar{e}_\Theta}{r} \frac{\partial}{\partial \Theta} + \frac{\bar{e}_\Phi}{r \sin \Theta} \frac{\partial}{\partial \Phi}$

Fourier's law in the different coordinate systems

Rectangular	Cylindrical	Spherical
$q''_x = -k \frac{\partial T}{\partial x}$	$q''_r = -k \frac{\partial T}{\partial r}$	$q''_r = -k \frac{\partial T}{\partial r}$
$q''_y = -k \frac{\partial T}{\partial y}$	$q''_\Theta = -\frac{k}{r} \frac{\partial T}{\partial \Theta}$	$q''_\Theta = -\frac{k}{r} \frac{\partial T}{\partial \Theta}$

$q_z'' = -k \frac{\partial T}{\partial z}$	$q_z'' = -k \frac{\partial T}{\partial z}$	$q_\Phi'' = -\frac{k}{r \sin \Theta} \frac{\partial T}{\partial \Phi}$
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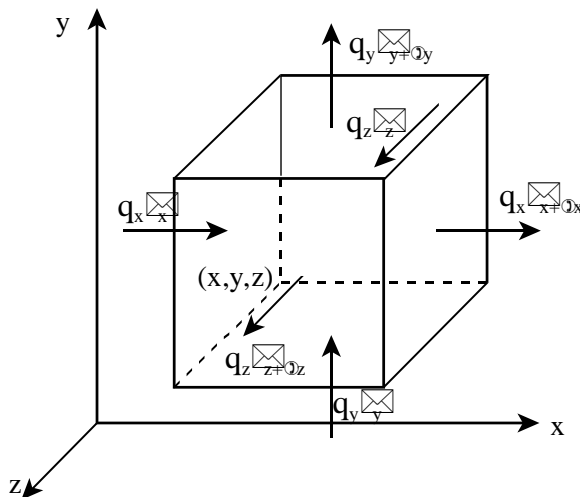
2.2 The general heat conduction equation

In the study of heat conduction, the aim is to determine an expression that describes the temperature distribution, which results from conditions existing on the boundary of the conducting material.

The statement of the law of conservation of energy:

$$\left\{ \begin{array}{c} \text{Rate of} \\ \text{thermal} \\ \text{energy} \\ \text{accumulation} \end{array} \right\} = \left\{ \begin{array}{c} \text{Rate of} \\ \text{thermal} \\ \text{energy} \\ \text{in} \end{array} \right\} - \left\{ \begin{array}{c} \text{Rate of} \\ \text{thermal} \\ \text{energy} \\ \text{out} \end{array} \right\} + \left\{ \begin{array}{c} \text{Rate of} \\ \text{thermal} \\ \text{energy} \\ \text{production} \end{array} \right\}$$

Consider a solid volume element with volume $\Delta x \Delta y \Delta z$ through which heat is flowing.



Assume:

- (1) ρ , k , c_p = constant — does not depend on T or direction (The material is homogeneous and **isotropic**. Isotropic material is one in which the physical property does not vary with direction.)
- (2) $v = 0$ (solid material has zero bulk velocity)

Thermal energy may be “produced” by:

- (a) the degradation of electrical energy (as a result of electric current passing through the material),
- (b) the conversion of chemical energy into heat (reaction heat of chemical reactions),
- (c) the degradation of mechanical energy (viscous dissipation),
- (d) nuclear reaction.

Thermal energy in or out:

- (1) by heat conduction — consider only this now,
- (2) by virtue of overall fluid motion (sensible heat).

Rate of thermal energy **in by conduction**

across surface at x $\Delta y \Delta z q_x''|_x$

across surface at y $\Delta x \Delta z q''_y|_y$

across surface at z $\Delta x \Delta y q''_z|_z$

Rate of thermal energy **out** by conduction

across surface at $x + \Delta x$ $\Delta y \Delta z q''_x|_{x + \Delta x}$

across surface at $y + \Delta y$ $\Delta x \Delta z q''_y|_{y + \Delta y}$

across surface at $z + \Delta z$ $\Delta x \Delta y q''_z|_{z + \Delta z}$

Rate of thermal energy production: $\Delta x \Delta y \Delta z q'''$

Rate of thermal energy accumulation: $\underbrace{\Delta x \Delta y \Delta z \rho}_{\text{mass}} \underbrace{\frac{\partial u}{\partial t}}_{\text{rate of change of internal energy per unit mass}}$

Substitute into the energy balance equation and divide by $\Delta x \Delta y \Delta z$

$$\rho \frac{\partial u}{\partial t} = \frac{q''_x|_x - q''_x|_{x+\Delta x}}{\Delta x} + \frac{q''_y|_y - q''_y|_{y+\Delta y}}{\Delta y} + \frac{q''_z|_z - q''_z|_{z+\Delta z}}{\Delta z} + q'''$$

Take the limits as $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, and $\Delta z \rightarrow 0$

$$\rho \frac{\partial u}{\partial t} = - \lim_{\Delta x \rightarrow 0} \frac{q''_x|_{x+\Delta x} - q''_x|_x}{\Delta x} - \lim_{\Delta y \rightarrow 0} \frac{q''_y|_{y+\Delta y} - q''_y|_y}{\Delta y} - \lim_{\Delta z \rightarrow 0} \frac{q''_z|_{z+\Delta z} - q''_z|_z}{\Delta z} + q'''$$

$$\rho \frac{\partial u}{\partial t} = - \left(\frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y} + \frac{\partial q''_z}{\partial z} \right) + q'''$$

Substitute Fourier's law in all direction and from thermodynamics: $du = dh = c_p dT$

[$u = h - pv$, differentiating gives $du = dh - p dv - v dp$, where v is the **specific volume**. Since in solids p and ρ are constants therefore $v = \text{constant}$ so $du = dh$.

From the definition of specific heat $dh = c_p dT$]

$$\boxed{\rho c_p \frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q'''}$$

net rate of heat conduction
into the volume element

This is the **general 3-dimensional unsteady-state conduction equation** in rectangular coordinate system. In vector notation:

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + q'''$$

where ∇^2 operator is called the **Laplacian**, in rectangular coordinates it is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Introducing the **thermal diffusivity** $\alpha \equiv \frac{k}{\rho c_p}$, m²/s

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T + \frac{q'''}{k} \quad \text{which is another form of the general 3-dimensional unsteady-state conduction equation.}$$

In practice, unsteady-state or transient heat conduction with heat generation occurs in the start-up and shut-down of continuous reactors, batch reactors, thermal run-away reactions, etc.

Simplified forms of heat conduction equation

No heat source:

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T \quad \text{or} \quad \frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad \textbf{Fourier's equation}$$

In practice, unsteady-state, or **transient** heat conduction with no heat generation occurs in many processes, e.g. batch heating and cooling operations; quenching of metallurgical products; pasteurization, or sterilization of foods in cans, bottles, etc.

Steady-state conditions with heat generation:

$$k \nabla^2 T + q''' = 0 \quad \text{or} \quad \nabla^2 T + \frac{q'''}{k} = 0 \quad \textbf{Poisson's equation}$$

In practice, steady-state heat conduction with heat generation occurs in the steady-state operation of continuous reactors (both chemical and nuclear reactors).

Steady-state conditions and no heat source:

$$\nabla^2 T = 0 \quad \textbf{Laplace equation}$$

In practice, steady-state heat conduction with no heat generation occurs in continuous heating and cooling processes such as in heat exchangers.

2.3 Boundary and initial conditions

Boundary and initial conditions are required to determine the integration constants. **Boundary conditions** are restrictions (specifications) applied at the physical boundaries of the system. **Initial conditions** are specifications applied at a specific time for unsteady problems. The number of boundary and initial conditions required is dependent on the order of the differential equation (for a 2nd order differential equation 2 conditions are required).

4 kinds of boundary conditions are commonly used:

(1) The temperature at a surface may be specified e.g. $T = T_w = \text{constant}$

(2) The heat flux at a surface may be specified e.g. $q'' = q_o'' = \text{constant}$
(This is equivalent to specify the temperature gradient at the surface.)

$$\frac{\partial T}{\partial x} = - \frac{q_o''}{k} = \text{constant}$$

(3) At a solid-fluid interface the heat flux may be related to the difference between the temperature at the interface and that in the fluid by Newton's law of cooling:

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = h (T|_{x=0} - T_{\infty})$$

This is the convective boundary condition.

(4) At solid-solid interfaces, the continuity of temperature and heat flux may be specified.

Initial conditions are commonly of 2 types:

- (1) An initial time condition: the temperature or heat flux is specified at the start,
- (2) Time goes infinity (for a number of problems when $t \rightarrow \infty$, steady-state is reached).

3 STEADY-STATE HEAT CONDUCTION

IN ONE DIMENSION

The temperature is a function of only 1 space variable.

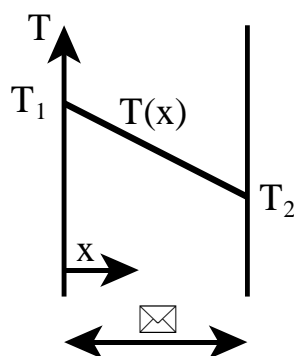
Objectives: – to determine the temperature distribution,
 – to find the heat transfer rate.

Procedure to solve problems:

- (1) Read the problem carefully, identify the given quantities and change them to the appropriate units,
- (2) Sketch,
- (3) Begin with the appropriate form of the general conduction equation,
- (4) Make assumptions to simplify the differential equation
- (5) Solve the differential equation by integration and using B.C. and I.C. to obtain the temperature distribution,
- (6) Find the heat transfer rate,
- (7) Comment.

A) Plane geometry systems

3.1 Heat conduction in a plane wall



The surface temperatures are T_1 and T_2 .
The thickness of the wall is δ .

Assumptions:

- $T = T(x)$ the temperature varies only in x direction
- steady-state
- $k, \rho, c_p = \text{constant}$
- there is no internal heat generation

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + q''' \Rightarrow \nabla^2 T = 0$$

In x -direction: $\frac{d^2 T}{dx^2} = 0 \Rightarrow \frac{dT}{dx} = C_1 \Rightarrow T = C_1 x + C_2$

where C_1 and C_2 are integration constants.

B.C.1 : at $x = 0$ $T = T_1 \Rightarrow T_1 = C_2$

B.C.2 : at $x = \delta$ $T = T_2 \Rightarrow T_2 = C_1\delta + T_1 \Rightarrow C_1 = (T_2 - T_1) / \delta$

$$T = \frac{T_2 - T_1}{\delta} x + T_1$$

the temperature distribution in the slab is **linear**

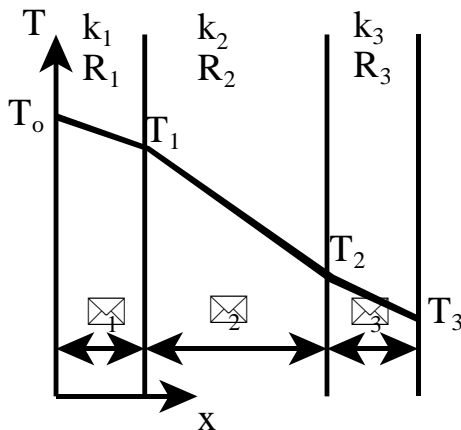
Fourier's law: $q_x = -kA \frac{dT}{dx} = -\frac{kA}{\delta} (T_2 - T_1) = \frac{kA}{\delta} (T_1 - T_2)$

Rate of flow = driving force / resistance

Rate of flow: q_x heat flow rate, W
 Driving force: $\Delta T = T_1 - T_2$ temperature difference, °C
 Resistance: $R = \delta / kA$ thermal resistance, K/W

$$q_x = \frac{\Delta T}{\delta / kA} = \frac{\Delta T}{R}$$

3.2 Composite walls (Materials in series)



A composite wall consist of 3 materials placed next to each other.

For each layer we can apply the solution for slab:

$$q_x = \frac{\Delta T}{R}$$

$\Delta T_1 = T_0 - T_1$	$R_1 = \delta_1 / k_1 A$	$q_{x1} = \Delta T_1 / R_1$	$\Delta T_1 = q_{x1} R_1$
$\Delta T_2 = T_1 - T_2$	$R_2 = \delta_2 / k_2 A$	$q_{x2} = \Delta T_2 / R_2$	$\Delta T_2 = q_{x2} R_2$
$\Delta T_3 = T_2 - T_3$	$R_3 = \delta_3 / k_3 A$	$q_{x3} = \Delta T_3 / R_3$	$\Delta T_3 = q_{x3} R_3$

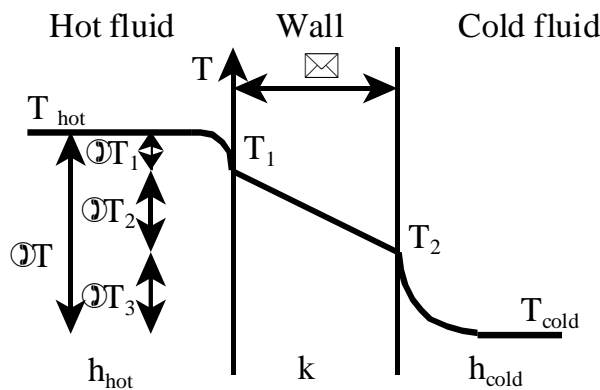
Since all the heat that passes through the first layer must pass through the second and the third one, $q_{x1} = q_{x2} = q_{x3} = q_x = \text{constant}$.

$$\Delta T = \Delta T_1 + \Delta T_2 + \Delta T_3 = q_x (R_1 + R_2 + R_3)$$

$$q_x = \frac{\Delta T}{R_1 + R_2 + R_3} = \frac{\text{overall temperature difference}}{\sum \text{thermal resistances}}$$

$$q_x = \frac{T_o - T_3}{\frac{\delta_1}{k_1 A} + \frac{\delta_2}{k_2 A} + \frac{\delta_3}{k_3 A}}$$

3.3 Overall heat transfer coefficient



Frequently it is required to calculate the heat flow through a wall separating two fluids of known temperature, when the surface temperatures of the walls are unknown. The fluid temperatures can be the bulk temperatures, when the fluids flow through ducts; or free-stream temperatures, when the fluid flows are external.

Newton's law Fourier's law Newton's law

Most of the temperature drop in the fluids occurs very near the wall in a relatively stagnant boundary layer (laminar sublayer) which adheres to the wall. In practice, the entire resistance to heat transfer is regarded as being in the laminar sublayer. Despite the fact, that heat flows across this layer by conduction, the mechanism is called as **convective** heat transfer. Newton's law of cooling applies. h_{hot} , h_{cold} , and k are known.

$$q = h_{hot} A (T_{hot} - T_1) \quad \Rightarrow \quad \Delta T_1 = T_{hot} - T_1 = q \frac{1}{h_{hot} A}$$

$$q = \frac{kA}{\delta} (T_1 - T_2) \quad \Rightarrow \quad \Delta T_2 = T_1 - T_2 = q \frac{\delta}{kA}$$

$$q = h_{cold} A (T_2 - T_{cold}) \quad \Rightarrow \quad \Delta T_3 = T_2 - T_{cold} = q \frac{1}{h_{cold} A}$$

$$T_{hot} - T_{cold} = \Delta T = \Delta T_1 + \Delta T_2 + \Delta T_3 = q \left(\frac{1}{h_{hot} A} + \frac{\delta}{k A} + \frac{1}{h_{cold} A} \right)$$

Therefore

$$q = \frac{T_{hot} - T_{cold}}{\frac{1}{h_{hot} A} + \frac{\delta}{k A} + \frac{1}{h_{cold} A}}$$

It is convenient to express the heat transfer rate in terms of an overall coefficient that accounts for both convection and conduction resistances:

$$q = U A (T_{hot} - T_{cold})$$

where U = overall heat transfer coefficient based on an area A , W/m^2K .

Comparing the last 2 equations:

$$U = \frac{1}{\frac{1}{h_{hot}} + \frac{\delta}{k} + \frac{1}{h_{cold}}} = \frac{1}{A(R_1 + R_2 + R_3)} \quad \text{for a plane wall}$$

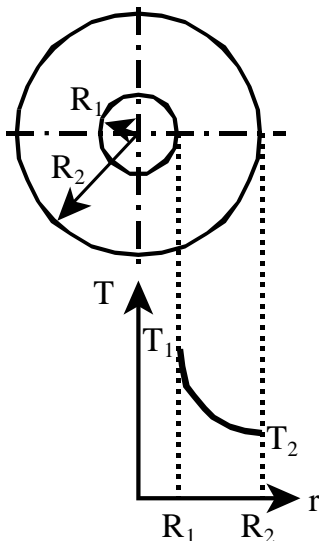
B) Cylindrical geometry systems

3.4 Heat conduction in a cylindrical wall

Derive an equation for heat flow through the wall of a hollow cylinder of outside radius R_2 , inside radius R_1 , length L , outside wall temperature T_2 , and inside wall temperature T_1 .

Assumptions:

- Steady state,
- No internal heat generation,
- $k, \rho, c_p = \text{constant}$, and
- $T = T(r)$ the temperature varies only with r (radial direction).



The general heat-conduction equation in cylindrical coordinates:

$$\cancel{\frac{1}{\alpha} \frac{\partial T}{\partial t}} = \cancel{\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)} + \cancel{\frac{1}{r^2} \frac{\partial^2 T}{\partial \Theta^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}} + \cancel{\frac{q'''}{k}}$$

After simplification and solving:

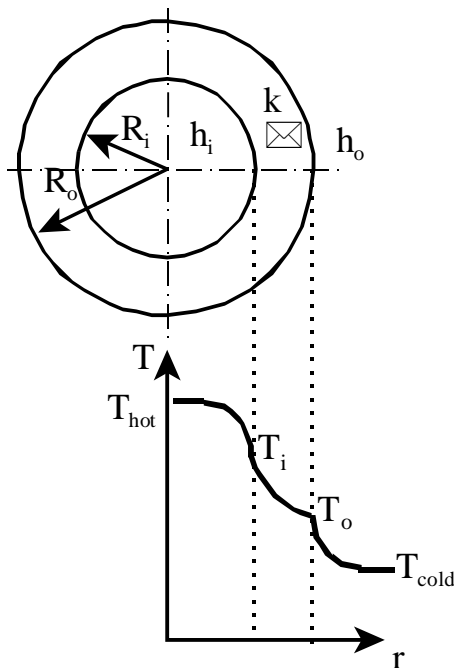
$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0 \Rightarrow \frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0 \Rightarrow r \frac{dT}{dr} = C_1$$

$$dT = \frac{C_1}{r} dr \Rightarrow T = C_1 \ln r + C_2$$

Temperature distribution	Location	Thermal resistance
$\frac{T - T_1}{T_2 - T_1} = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}$	$R_1 \leq r \leq R_2$	$R_{T1} = \frac{\ln(R_2/R_1)}{2\pi k_1 L}$
$\frac{T - T_2}{T_3 - T_2} = \frac{\ln(r/R_2)}{\ln(R_3/R_2)}$	$R_2 \leq r \leq R_3$	$R_{T2} = \frac{\ln(R_3/R_2)}{2\pi k_2 L}$
$\frac{T - T_3}{T_4 - T_3} = \frac{\ln(r/R_3)}{\ln(R_4/R_3)}$	$R_3 \leq r \leq R_4$	$R_{T3} = \frac{\ln(R_4/R_3)}{2\pi k_3 L}$

$$q_r = \frac{T_1 - T_4}{\sum_i R_{Ti}} = \frac{(T_1 - T_4)}{\frac{\ln(R_2/R_1)}{2\pi L k_1} + \frac{\ln(R_3/R_2)}{2\pi L k_2} + \frac{\ln(R_4/R_3)}{2\pi L k_3}}$$

3.6 Overall heat transfer coefficient



In many problems, fluids with different temperatures are flowing inside and around tubes and we need to determine the heat transfer rate knowing the bulk fluid temperatures.

L = length of tube

$\delta = R_o - R_i$ = wall thickness

The resistances:

$$R_{hot} = \frac{1}{h_i A_i} \quad A_i = 2\pi L R_i$$

$$R_{wall} = \frac{\ln(R_o/R_i)}{2\pi k L}$$

$$R_{cold} = \frac{1}{h_o A_o} \quad A_o = 2\pi L R_o$$

$$q_r = \frac{T_{hot} - T_{cold}}{\sum_i R_{Ti}} = \frac{T_{hot} - T_{cold}}{\frac{1}{h_i A_i} + \frac{\ln(R_o/R_i)}{2\pi k L} + \frac{1}{h_o A_o}} = \frac{2\pi L (T_{hot} - T_{cold})}{\frac{1}{h_i R_i} + \frac{\ln(R_o/R_i)}{k} + \frac{1}{h_o R_o}}$$

Often it is desirable to express this equation in terms of an overall heat transfer coefficient U :

$$q_r = UA(T_{\text{hot}} - T_{\text{cold}})$$

The question is which area to select for A . It is customary to select the outside surface area (A_o) of the cylinder (pipe), but sometimes the inside surface area (A_i) is chosen.

Comparing the last 2 equations, the overall heat transfer coefficient U_o based on A_o :

$$U_o = \frac{1}{A_o \Sigma R_T} = \frac{1}{\frac{1}{h_i} \frac{R_o}{R_i} + \frac{R_o \ln(R_o / R_i)}{k} + \frac{1}{h_o}}$$

The overall heat transfer coefficient U_i based on A_i :

$$U_i = \frac{1}{A_i \Sigma R_T} = \frac{1}{\frac{1}{h_i} + \frac{R_i \ln(R_o / R_i)}{k} + \frac{1}{h_o} \frac{R_i}{R_o}}$$

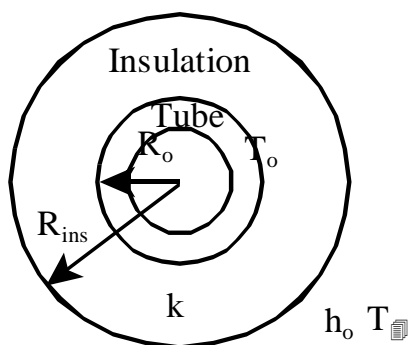
3.7 Critical thickness of insulation

In industry, insulating a pipe is a common practice because it is an inexpensive method of retarding heat losses. In some cases, however, adding insulation causes an increase in heat loss.

Adding insulation causes two effects:

- increases the resistance to conduction, and
- decreases the resistance to convection, because the outer surface area for convection increases.

Thus the heat loss from a cylindrical or spherical surface may increase depending on which effect dominates.



T_o = outside tube wall temperature

R_o = outside tube radius

T_∞ = ambient fluid temperature

The heat loss is: $q = \frac{T_o - T_\infty}{R_T}$

For no insulation, the heat loss is: $q = \frac{T_o - T_\infty}{\frac{1}{h_o A_o}} \Leftrightarrow q = h_o A_o (T_o - T_\infty)$

If insulation is added, the overall resistance is the sum of the “cylindrical shell” resistance of the insulation and the external convective heat transfer resistance:

$$R_T = \frac{1}{h_o (2\pi R_{ins} L)} + \frac{\ln (R_{ins} / R_o)}{2\pi k L}$$

and the heat transfer is:

$$q = \frac{T_o - T_\infty}{\frac{\ln (R_{ins} / R_o)}{2\pi k L} + \frac{1}{h_o (2\pi R_{ins} L)}} \Rightarrow \frac{q}{L} = \frac{2\pi (T_o - T_\infty)}{\frac{\ln (R_{ins} / R_o)}{k} + \frac{1}{h_o R_{ins}}}$$

The heat loss per unit length (q/L) reaches a maximum or the thermal resistance (R_T) a minimum value where

$$\frac{\partial R_T}{\partial R_{ins}} = 0 \quad \text{or} \quad \frac{\partial (q/L)}{\partial R_{ins}} = 0$$

Therefore

$$\frac{\partial R_T}{\partial R_{ins}} = \frac{1}{2\pi k L} \frac{1}{R_{ins}} - \frac{1}{h_o 2\pi L} \frac{1}{R_{ins}^2} = 0$$

Solving for the critical insulation radius for the cylinder $R_{ins \text{ crit}}$:

$$\frac{1}{2\pi L} \frac{1}{R_{ins \text{ crit}}} \left(\frac{1}{k} - \frac{1}{h_o R_{ins \text{ crit}}} \right) = 0 \Rightarrow \frac{1}{k} = \frac{1}{h_o R_{ins \text{ crit}}}$$

$$\boxed{R_{ins \text{ crit}} = \frac{k}{h_o}}$$

The critical insulation radius corresponds to maximum heat transferred.

If $R_{ins} < R_{ins \text{ crit}}$, then adding insulation increases the heat transferred.

In case of a “thick” bare cylinder: $R_o > R_{ins \text{ crit}}$ the addition of insulation always increases the insulation effect.

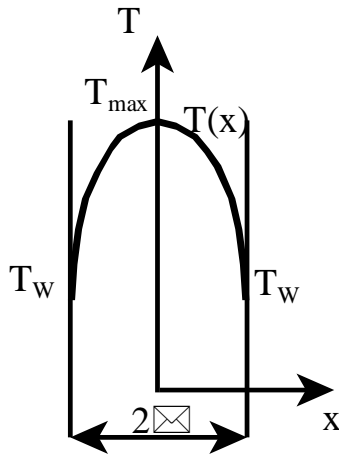
In case of a “thin” bare cylinder: $R_o < R_{ins \text{ crit}}$ the addition of the first layer of insulation decreases the overall thermal resistance R_T , so enhances heat transfer, not insulating until $R_{ins} > R_{ins \text{ crit}}$.

(E.g. A wire insulated with a covering of not too low thermal conductivity may run cooler, for a given current, than the same wire if bare. A practical application is the

problem of insulating electrical wires where the objective would be the provision of adequate electrical insulation at the same time providing for maximum wire cooling.)

The critical insulation radius for a sphere is: $R_{ins\ crit} = 2 (k/h_o)$

3.8 Internal heat generation in plane wall



Heat is generated in a slab. The walls of the slab are kept at the same T_w constant wall temperature. We place the coordinate system in the center of the wall ($x = 0$ is at the center).

Cases of this type occur in exothermic chemical reactions such as the combustion of fuel in the fuel bed of a boiler furnace, setting of cement etc. Another example is an electrical coil wherein heat is produced and dissipated by thermal conduction. The rate of heat produced homogeneously in the electric wires per unit volume of the coil may be calculated from Joule's law.

The height of the coil is assumed to be such that only heat conducted in the radial direction must be considered. If the curvature of the coil is negligible, then the coil may be replaced by a plane with heat sources uniformly distributed inside the plate.

Assumptions:

- there is uniform constant heat generation in the wall (q''' , W/m^3),
- one dimensional heat flow $T = T(x)$,
- $k, \rho, c_p = \text{constant}$,
- steady-state.

The general heat conduction equation in rectangular coordinates:

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k}$$

After simplification: $\frac{d^2 T}{dx^2} = - \frac{q'''}{k}$ Poisson's equation

Integrating twice yields a **parabolic** temperature profile:

$$T = - \frac{q'''}{2k} x^2 + C_1 x + C_2$$

B.C.1: at $x = \delta$ $T = T_w$

B.C.2: at $x = -\delta$ $T = T_w$

Applying the boundary conditions to the temperature profile:

$$\left. \begin{array}{l} \text{B.C.1: } T_W = -\frac{q'''\delta^2}{2k} + C_1\delta + C_2 \\ \text{B.C.2: } T_W = -\frac{q'''\delta^2}{2k} - C_1\delta + C_2 \end{array} \right\} \quad C_2 = T_W + \frac{q'''\delta^2}{2k}, \quad C_1 = 0$$

Substituting for C_1 and C_2 gives the temperature distribution in the wall:

$$T = -\frac{q'''}{2k}x^2 + T_W + \frac{q'''\delta^2}{2k} \quad \text{or} \quad T - T_W = \frac{q'''}{2k}(\delta^2 - x^2)$$

$$T - T_W = \frac{q'''\delta^2}{2k} \left[1 - \left(\frac{x}{\delta} \right)^2 \right]$$

The distribution is symmetric about $x = 0$ so at $x = 0$ $dT/dx = 0$ and maximum temperature exists at $x = 0$:

$$T_{\max} = T(0) = T_W + \frac{q'''\delta^2}{2k} \quad \Rightarrow \quad T_{\max} - T_W = \frac{q'''\delta^2}{2k}$$

The temperature distribution can be expressed with the maximum temperature:

$$\frac{T - T_W}{T_{\max} - T_W} = 1 - \left(\frac{x}{\delta} \right)^2$$

The total heat loss from the 2 faces at steady-state is equal to the total heat generated:

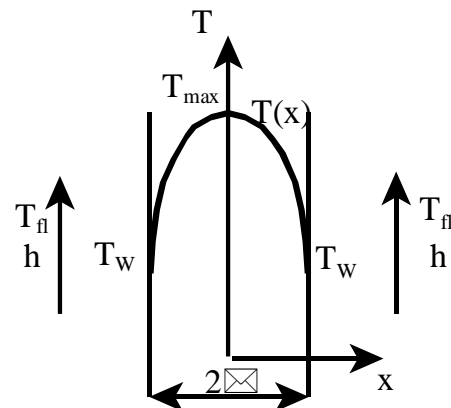
$$q_{\text{total}} = q'''(2A\delta)$$

Above solution requires the knowledge of the surface temperature T_W . However, in some cases each of the exposed surfaces the slab is bounded by a circulating fluid of known temperature T_{fl} and the heat transfer coefficient for each surface is h . In that case, we apply the conductive boundary condition on the surface:

$$q'' = -k \frac{dT}{dx} \Big|_{x=\pm\delta} = h(T_W - T_{fl})$$

Then the excess temperature ($T - T_W$) is:

$$T - T_W = \frac{q'''}{2k}(\delta^2 - x^2) + \frac{q'''\delta}{h}$$



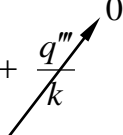
4 UNSTEADY – STATE HEAT CONDUCTION

In transient-conduction problems (also called unsteady-state problems), temperature varies with location within the system and with time. Many industrial heat conduction problems are unsteady state. Examples: heat flow through a building wall during the daily 24-hour heating and cooling cycle; annealing of casting, cooling of ingots, burning of bricks, heating or cooling batch reactors, thermal processing of foods using steam, refrigeration of fresh food products etc.

The aim is to find

- the heat-transfer rate,
- the temperature history, or
- the time needed to attain a certain temperature.

The general heat conduction equation:

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T + \frac{q'''}{k}$$


Assumptions:

- no internal heat generation ($q'''/k = 0$),
- constant k , ρ , c_p .

In 2-dimension:
$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

We need

- 2 boundary condition for x ,
- 2 boundary condition for y , and
- 1 initial condition for t .

For relatively simple geometries, the above differential equation can be solved, but for more complicated systems we need numerical or graphical techniques.

There are 2 types of resistances to heat transport:

- internal thermal resistance: δ / kA , and
- surface or film resistance: $1 / hA$.

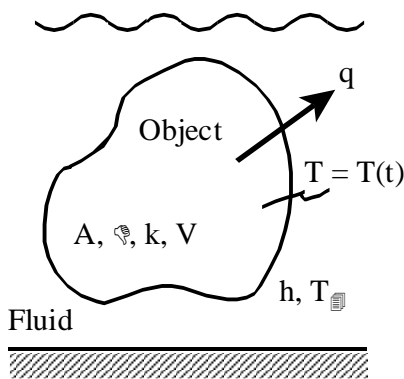
We can identify 3 types of problems that arise for a transient problem. For some problems, the internal resistance is negligible compared to the surface resistance. For other problems, the surface resistance is negligible. For still others both are important.

4.1 Systems with negligible internal resistance —

Lumped capacitance model

Consider an object at an initial temperature of T_i that is being cooled (or heated) by a fluid stream at a temperature of T_∞ . There is a heat loss (or gain) by convection (interface transfer) to the surroundings. If the thermal conductivity of the object is high, the internal resistance (δ/kA) to the heat flow is small then:

- the temperature profile within the object is uniform and everywhere equal to the surface temperature,
- at any instant in time there are no temperature gradients within the object, only a temperature difference between the surface and the free stream,
- the heat transferred between the object and the fluid is controlled by the surface resistance (i.e. the value of the heat transfer coefficient h).



This heat-transfer process is called Newtonian cooling (or heating). Because the temperature does not change (significantly) within the solid, we can formulate a model by performing an energy balance on the object.

$$q = \underbrace{-h A (T - T_\infty)}_{\text{heat leaving the body (convection heat loss)}} = \underbrace{\rho c_p V \frac{dT}{dt}}_{\text{change in internal energy}}$$

To simplify the differential equation introduce dimensionless temperature Θ :

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} \Rightarrow T = (T_i - T_\infty)\Theta + T_\infty \Rightarrow \frac{dT}{dt} = (T_i - T_\infty) \frac{d\Theta}{dt}$$

Then the heat flow rate equation becomes:

$$-h A (T - T_\infty) = \rho c_p V (T_i - T_\infty) \frac{d\Theta}{dt}$$

$$-\frac{T - T_\infty}{T_i - T_\infty} = \frac{\rho c_p V}{h A} \frac{d\Theta}{dt} \Rightarrow -\frac{d\Theta}{\Theta} = \frac{h A}{\rho c_p V} dt$$

$$-\int_1^\Theta \frac{d\Theta}{\Theta} = \frac{h A}{\rho c_p V} \int_0^t dt \quad \text{since I.C.: at } t = 0 \quad T = T_i \quad \text{or} \quad \Theta = 1$$

$$-\ln \Theta = \frac{h A}{\rho c_p V} t$$

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{h A}{\rho c_p V} t\right)$$

the body – fluid temperature difference decays exponentially

This equation describes the time – temperature history of the solid object. It is called **lumped capacitance equation**.

$$\begin{aligned} \rho c_p V & \text{ — lumped capacitance, J/K} \\ \rho c_p V / hA & \text{ — the characteristic time of decay, or thermal time constant, or} \\ & \text{decay time constant, s} \end{aligned}$$

Observations from the lumped capacitance equation:

- The temperature of a body approaches the ambient temperature T_∞ exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on.
- A large value of the time constant indicates that the body approaches the ambient temperature in a short time.
- It takes longer time for the body to reach equilibrium with the surrounding fluid when its lumped capacitance is large and/or its product hA is small.

Examining the exponent of the lumped-capacitance equation:

The ratio of volume to surface area for an object is often used as a **characteristic length** L_{ch} .

$$\frac{hA}{\rho c_p V} t = \frac{h}{k} \frac{V}{A} \frac{k}{\rho c_p} \frac{A^2}{V^2} t = \frac{h L_{ch}}{k} \frac{\alpha t}{L_{ch}^2} = Bi \cdot Fo$$

where $\boxed{Bi \equiv \frac{h L_{ch}}{k}}$ **Biot number**, dimensionless

The physical meaning of Biot number:

$$Bi = \frac{\text{internal thermal resistance}}{\text{surface resistance}} = \frac{L/kA}{1/hA} = \frac{hL}{k}$$

or $Bi = \frac{\text{convection at the surface of the body}}{\text{conduction within the body}} = \frac{h \Delta T}{(k/L) \Delta T} = \frac{hL}{k}$

A small Biot number represents small resistance to heat conduction, and thus small temperature gradients within the body. The lumped system analysis is exact when $Bi = 0$ and approximate when $Bi > 0$. Of course, the smaller the Bi number, the more accurate the lumped system analysis.

$\boxed{Fo \equiv \frac{\alpha t}{L_{ch}^2}}$ **Fourier number**, dimensionless

Fourier number is a dimensionless time parameter. It represents the ratio of heat transfer by conduction to the energy storage rate within the material.

Thus, the lumped-capacitance equation becomes:

$$\frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp(-Bi \cdot Fo)$$

The assumption of negligible internal resistance is reasonably accurate when **Bi < 0.1**.

This is the **criterion** for the lumped-capacitance equation to apply. However, uncertainties in the knowledge of the heat transfer coefficient of $\pm 25\%$ are quite common, so that the condition $Bi < 0.1$ should allow for some leeway in application.

Characteristic length for the lumped capacitance equation

Body	Characteristic length $L_{ch} = V/A$
Plane wall	$\frac{AL}{A} = L$
Sphere	$\frac{(4/3)\pi R^3}{4\pi R^2} = \frac{R}{3}$
Long cylinder ($L \gg 2D$)	$\frac{R^2\pi L}{2\pi RL} = \frac{R}{2}$
Long square rod	$\frac{x^2L}{4xL} = \frac{x}{4}$

(x is the thickness of the rod)

At any time, the instantaneous rate of heat transfer $q(t)$ in Watt from the solid body is:

$$q(t) = hA(T - T_{\infty}) \quad \text{where} \quad T = \text{instantaneous temperature}$$

the instantaneous temperature from the lumped-capacitance equation is:

$$T - T_{\infty} = (T_i - T_{\infty}) \exp\left(-\frac{hA}{\rho c_p V} t\right)$$

$$q(t) = hA(T_i - T_{\infty}) e^{-\frac{hA}{\rho c_p V} t}$$

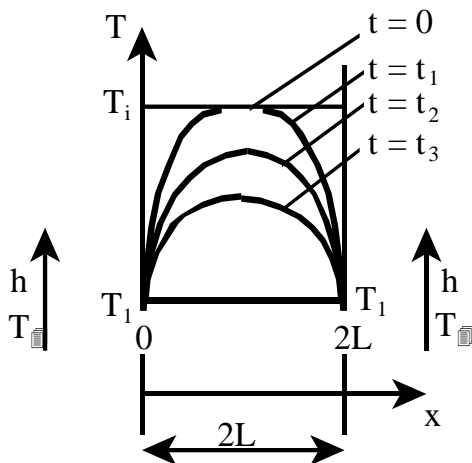
The total amount of heat Q in Jules transferred from the solid from time $t = 0$ to $t = t$ is:

$$Q = \int_0^t q(t) dt = \int_0^t h A (T_i - T_\infty) e^{-\frac{h A}{\rho c_p V} t} dt \Rightarrow$$

$$Q = \rho c_p V (T_i - T_\infty) (1 - e^{-\frac{h A}{\rho c_p V} t})$$

In many transient heat transfer problems, the Biot number is larger than 0.1, and lumped system cannot be assumed. In these cases, the temperature within the body changes appreciably from point to point as well as with time.

4.2 Systems with negligible surface resistance



Consider an infinite plate of thickness $2L$ at an initial uniform temperature T_i . At some time zero the left and right surfaces of the plate are suddenly cooled to T_1 by a fluid at temperature T_∞ . The thickness of the plate is $2L$.

The aim is to predict the temperature profile in the plate with time, and the heat transfer rate. (Fourier solved this problem by developing the method of Fourier series.)

If the heat transfer coefficient h is very large then the film resistance ($1/hA$) is negligible so the surface temperature, once exposed, will assume the same temperature as the fluid:

$$T_1 = T_\infty .$$

$T_1 = T_\infty$ assumption holds when **Bi** > 1.

The descriptive differential equation: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

The boundary and initial conditions are:

B.C.1:	at $x = 0$	$T = T_1$
or	at $x = 2L$	$T = T_1$
B.C.2:	at $x = L$	$\partial T / \partial x = 0$
I.C.:	at $t = 0$	$T = T_i$ for all x

Change to dimensionless quantities:

$$\Theta = \frac{T - T_1}{T_i - T_1}, \quad \xi = \frac{x}{L}, \quad \tau = \frac{t}{t_{ref}}$$

Θ = dimensionless temperature,
 ξ = dimensionless x coordinate ("xi"),
 τ = dimensionless time,
 t_{ref} = some unspecified reference time.

The dimensionless differential equation is then:

$$\frac{\partial^2 \Theta}{\partial \xi^2} = \frac{L^2}{\alpha t_{ref}} \frac{\partial \Theta}{\partial \tau}, \quad \text{where} \quad \frac{L^2}{\alpha t_{ref}} = \frac{1}{Fo}$$

The new boundary and initial conditions are:

B.C.1:	at $\xi = 0$	$\Theta = 0$
B.C.2:	at $\xi = 1$	$\Theta = 0$
I.C.:	at $\tau = 0$	$\Theta = 1$

The Fourier number appears in the dimensionless differential equation therefore $\Theta = \Theta(Bi, Fo, x/L)$ the temperature distribution is a function of Fourier and Biot numbers and x/L .

The analytical solution got by the separation of variables method:

$$\Theta = \frac{T - T_1}{T_i - T_1} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left[\left(- \frac{n\pi}{2L} \right)^2 \alpha t \right] \cdot \sin \frac{n\pi x}{2L} \quad n = 1, 3, 5, \dots$$

This equation is for the case where the left and right faces of an infinite wall are maintained at the same temperature T_1 .

The centerline of the plate is a line of symmetry and therefore analogous to an insulated or an adiabatic surface. The temperature history at the midplane ($x = L$) is of practical interest and is a function of Fourier number:

$$\frac{T_c - T_1}{T_i - T_1} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left[\left(- \frac{n\pi}{2L} \right)^2 \alpha t \right] \cdot \sin \frac{n\pi}{2} \quad n = 1, 3, 5, \dots$$

For values of Fo much greater than 0.1, only the first term in the series need be used in the solution (except at points very close to the boundaries). This makes possible a graphical presentation. On a chart (Fig. 6.6) knowing Fo number, the dimensionless temperature for the center plane can be read for different shapes.

An **infinite plate** is a plate whose thickness is small in relation to the other dimensions. An **infinite cylinder** is a cylinder where the diameter is small compared to the length.

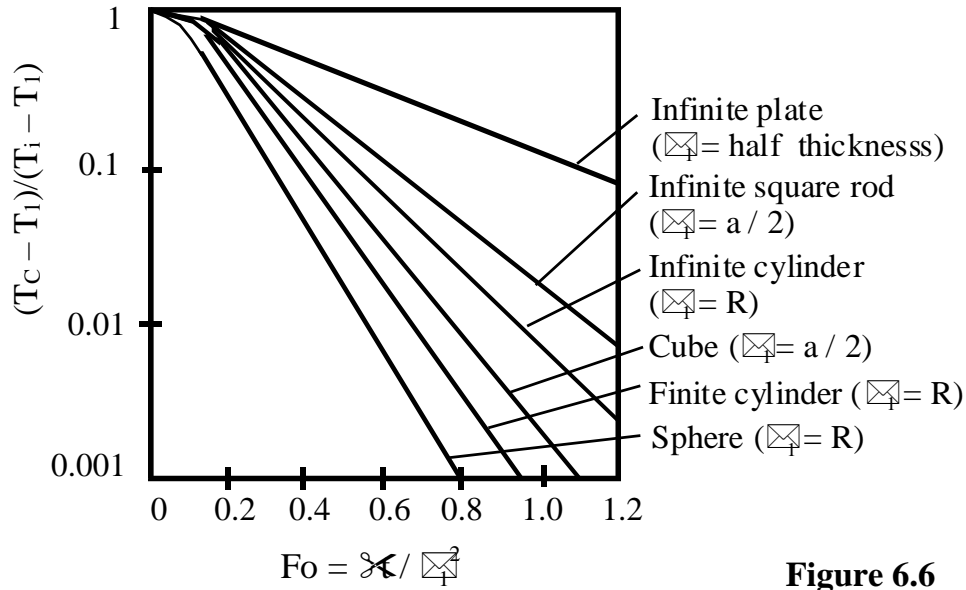


Figure 6.6

The instantaneous heat flow rate (q) and the amount of heat transferred (Q) for the infinite plate are:

$$q = -k A \left. \frac{dT}{dx} \right|_{\text{surface}} = \frac{4kA}{2L} (T_1 - T_i) \sum_{n=1}^{\infty} \exp \left[\left(-\frac{n\pi}{2L} \right)^2 \alpha t \right] \cdot \cos \frac{n\pi x}{2L}$$

$$Q = \int_0^t q dt = \frac{kAL}{\alpha \pi^2} (T_1 - T_i) \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ 1 - \exp \left[\left(-\frac{n\pi}{2L} \right)^2 \alpha t \right] \cdot \cos \frac{n\pi x}{2L} \right\}$$

On a chart (Fig. 6.7) q (in Watt) and Q (in Joule) can be read as a function of Fo .

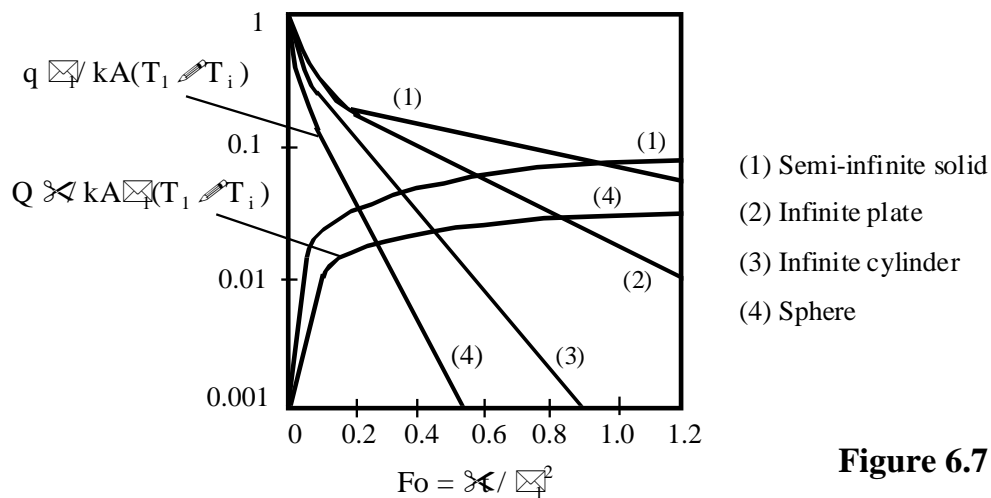


Figure 6.7

A similar analysis can be formulated for an infinitely long cylinder, a semi-infinite solid, a sphere, and other types of common geometries. The solutions are conveniently summarized graphically on the above charts.

4.3 Systems with finite internal and surface resistances

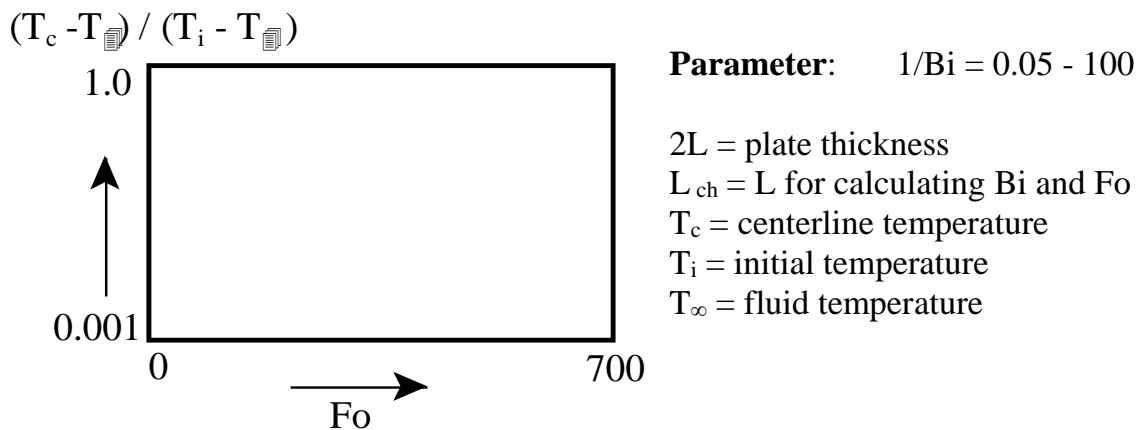
Both internal and film resistances are significant: $0.1 < Bi < 1$

Analytical solutions for a variety of problems are given in the literature. The solutions are complex in form. Calculations made from such solutions are presented in graphical form by Heisler (1947) and Gröber (1961). The calculations were performed by truncating the infinite series solutions for the problems into a few terms. This restricts the applicability of the charts to values of the $Fo > 0.2$.

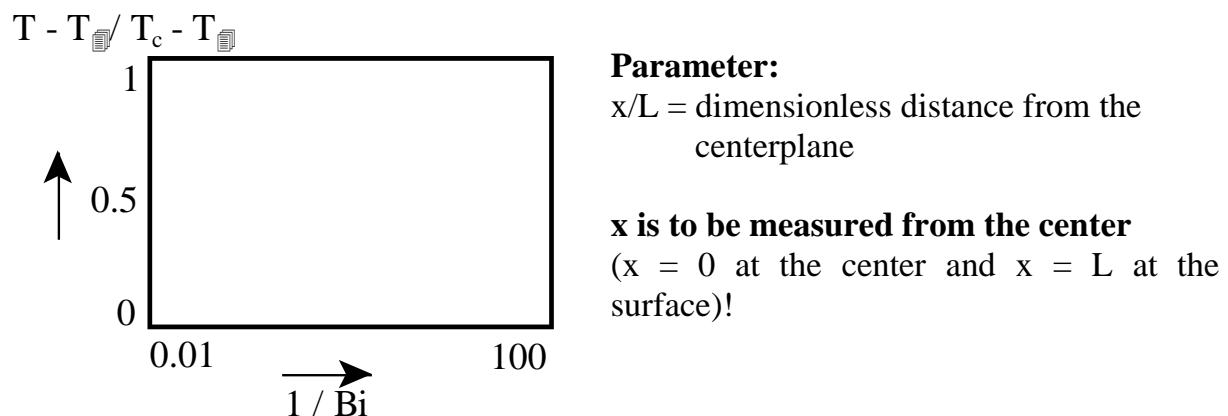
Heisler charts

- (1) The **dimensionless temperature history at the center** of an infinite plate of thickness $2L$ is given in Fig. 3-6.

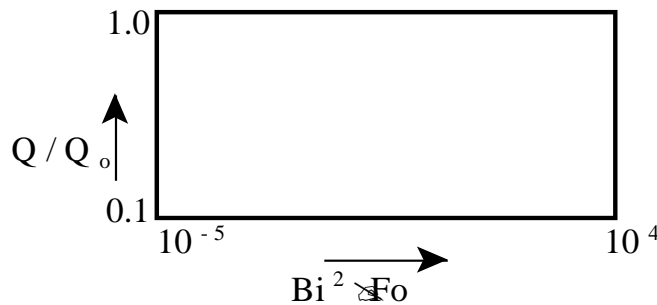
This chart can be used to calculate the midplane temperature from Fourier number, which in turn can be used in the next chart to determine the temperature at any other x location within the plate.



- (2) The **dimensionless temperature distribution** in an infinite plate at any x location within the plate is given in Fig. 3-9.



Gröber charts



Parameter: $Bi = 0.001 - 50$

Q/Q_o = dimensionless heat flow ratio

(see Figure 6.8c)

Q = the total amount of energy (J) that has passed through the wall up to any time t

Q_o = the initial internal energy of the wall relative to the fluid temperature T_∞ (it is equal to the maximum internal energy change that can occur).

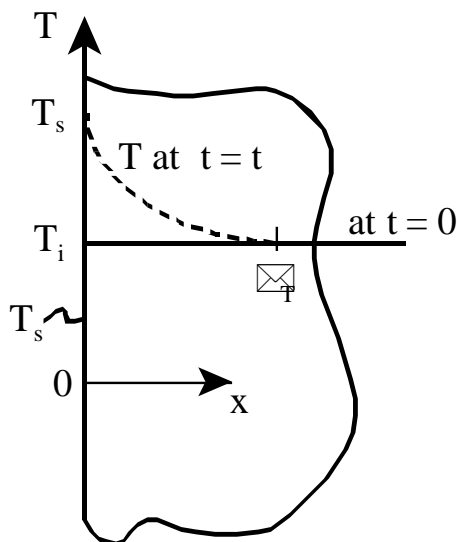
$$Q_o = mc_p(T_i - T_\infty) = \rho V c_p (T_i - T_\infty)$$

Similar charts for infinite cylinder ($L/R \geq 10$ at least; $L_{ch} = R$) and sphere ($L_{ch} = R$) are also available.

Note:

- The characteristic length (L_{ch}) used in the lumped-capacity method is different from the characteristic length used in the Heisler and Gröber charts.
- For the negligible surface resistance case, the heat transfer coefficient is large (i.e. $h \rightarrow \infty$). For this case $1/Bi = k / hL_{ch} = 0$. This corresponds to the case of specified surface temperature. This is also represented on the charts.
- If $1/Bi$ is too small to give accurate reading from the Heisler chart, we should use the 1st term of the series solution equation for calculation.

4.4 Semi - infinite solid



A solid body occupying the space from $x = 0$ to $x = \infty$ is initially at temperature T_i . At time $t=0$, the surface at $x = 0$ is suddenly raised to temperature T_s and maintained at that temperature for $t > 0 \Rightarrow$ **constant wall temperature**

The solution to this problem appeared in Fig. 6.7 for the case of negligible surface resistance (h is high $\Rightarrow T_s = T_\infty$). Let us examine the analytical solution to the same problem.

Assumptions:

- heat conduction occurs only in x direction,
- no heat generation,
- constant physical properties.

The conduction equation:
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

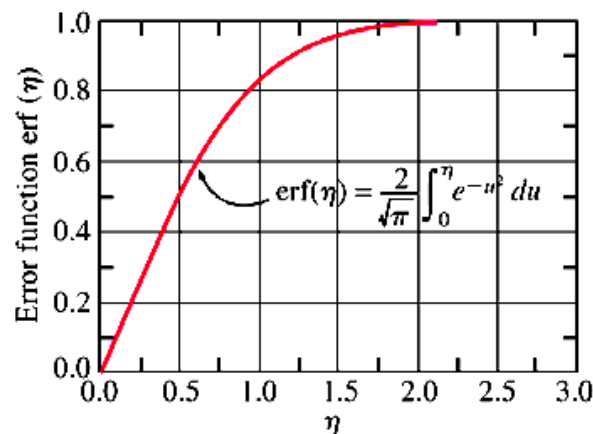
The boundary conditions for constant surface temperature:

$$\begin{array}{lll} \text{B.C.1:} & \text{at } x = 0 & T = T_s \\ \text{B.C.2:} & \text{at } x = \infty & T = T_i \\ \text{I.C.:} & \text{at } t \leq 0 & T = T_i \text{ for all } x \end{array}$$

Solution:
$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^{\lambda} e^{-\lambda^2} d\lambda \quad \text{where} \quad \lambda = \frac{x}{2\sqrt{\alpha t}}$$

This integral has no closed form. The right hand side is called **Gauss error function** or probability integral and denoted by $\text{erf}(\lambda)$. The integral is a function of its upper limit. Its values are available in tables.

The error function



Some properties of the error function:

$$\text{erf}(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda} e^{-\lambda^2} d\lambda$$

$$\text{erf}(0) = 0, \quad \text{erf}(\infty) = 1, \quad \text{erf}(-\lambda) = -\text{erf}(\lambda)$$

$$\text{erfc}(\lambda) = 1 - \text{erf}(\lambda) \text{ complementary error function, and}$$

$$\frac{d \text{erf}(\lambda)}{d \lambda} = \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \quad \text{and} \quad \left. \frac{d \text{erf}(\lambda)}{d \lambda} \right|_{\lambda=0} = \frac{2}{\sqrt{\pi}}$$

Therefore
$$\frac{T - T_s}{T_i - T_s} = \operatorname{erf} \frac{x}{2\sqrt{\alpha t}} \quad \text{or} \quad \frac{T - T_i}{T_s - T_i} = 1 - \operatorname{erf} \frac{x}{2\sqrt{\alpha t}}$$

Gauss error function table

λ	$\operatorname{erf}(\lambda)$	λ	$\operatorname{erf}(\lambda)$
0	0	1.0	0.84270
0.01	0.01128	1.1	0.88020
0.02	0.02256	1.2	0.91031
0.04	0.04511	1.3	0.93400
0.06	0.06762	1.4	0.95228
0.08	0.09007	1.5	0.96610
0.1	0.11246	1.6	0.97634
0.15	0.16800	1.7	0.98379
0.2	0.22270	1.8	0.98909
0.3	0.32863	1.8214	0.99000
0.4	0.42839	1.9	0.99279
0.5	0.52050	2.0	0.99532
0.6	0.60386	2.2	0.99814
0.7	0.67780	2.5	0.99950
0.8	0.74210	2.8	0.99992
0.9	0.79691	3.0	0.99998

The error function reaches a value of 0.99 when the argument is about 2 (1.8214 precisely), therefore we can define a **thermal penetration thickness** δ_T as that distance x for which the temperature dropped to a value of $0.01(T_s - T_i)$. The thermal penetration thickness varies with time.

$$\frac{x}{2\sqrt{\alpha t}} \cong 2 \quad \Rightarrow \quad x = 4\sqrt{\alpha t} \equiv \delta_T \quad \text{thermal penetration thickness}$$

For distances $x > \delta_T$, the initial temperature is changed by less than 1% of the difference $(T_s - T_i)$.

A solid is considered semi-infinite, if $x > \delta_T$ or $\frac{x}{2\sqrt{\alpha t}} > 2$.

When δ_T is small with respect to the plate thickness L , then the semi-infinite solid solution can be used. When δ_T is of the order of magnitude of the plate thickness L , then the finite slab solution (rectangular parallelepipedon) has to be used.

The heat flux is
$$q'' = -k \frac{\partial T}{\partial x} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} e^{-\frac{x^2}{4\alpha t}}$$

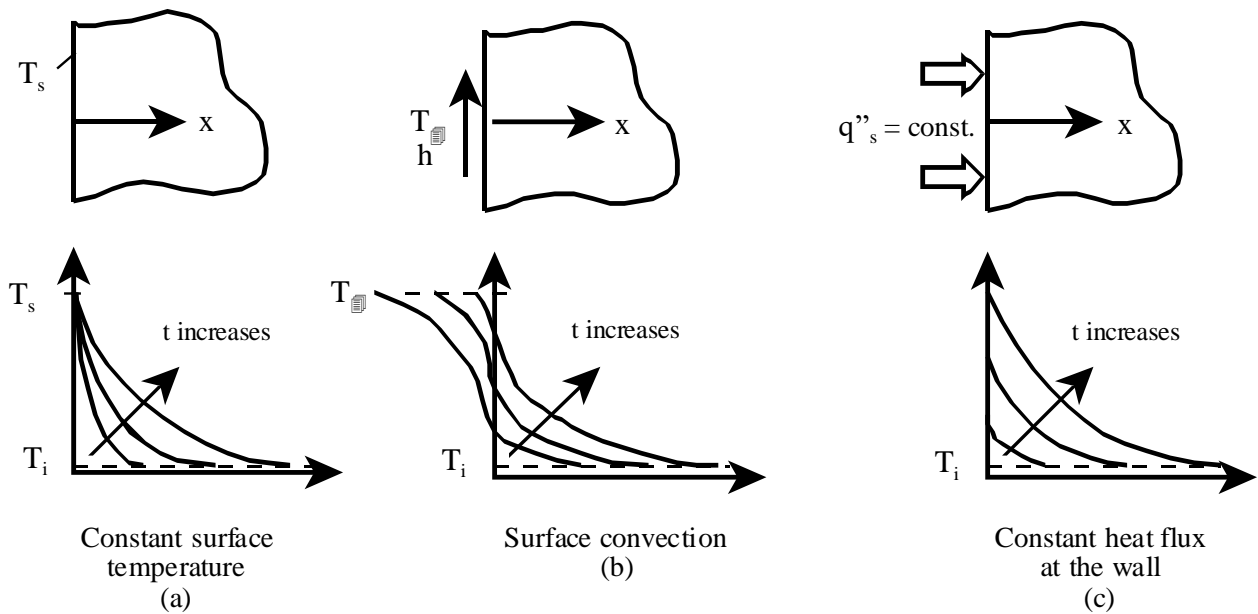
Then the heat flux at the surface is:

$$\underline{\underline{q_s'' = -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = \frac{2k(T_s - T_i)}{\sqrt{\pi}} \frac{1}{2\sqrt{\alpha t}} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}}}$$

The total amount of heat (Q) transferred from the solid from time $t = 0$ to $t = t$ is:

$$\underline{\underline{Q = \int_0^t q(t) dt = \frac{2kA(T_s - T_i)}{\sqrt{\frac{\pi \alpha}{t}}}}}$$

The problem can be redefined and the surface convection effect or constant heat flux at the surface could be accounted for. The next figure compares the solutions with 3 types of possible boundary conditions.

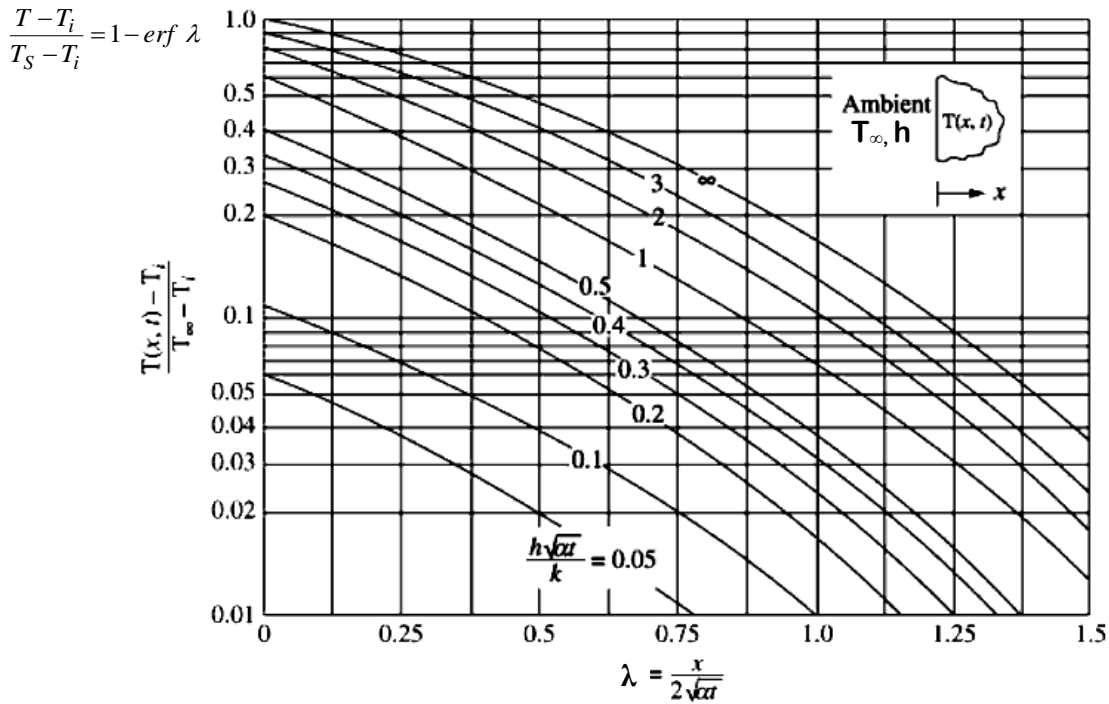


The solutions are:

$$(a) \quad \frac{T - T_s}{T_i - T_s} = \text{erf} \frac{x}{2\sqrt{\alpha t}}$$

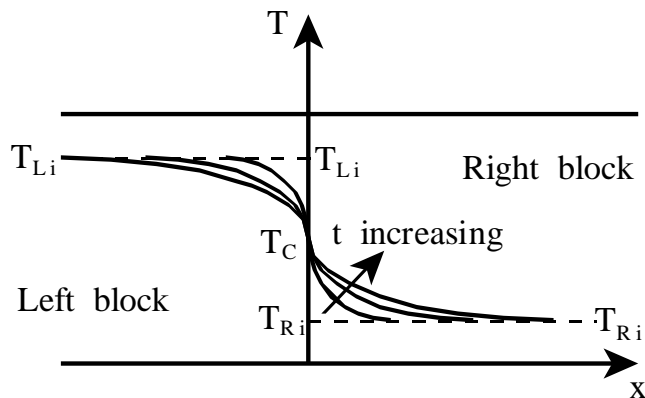
$$(b) \quad \frac{T - T_s}{T_i - T_s} = \text{erf} \frac{x}{2\sqrt{\alpha t}} + \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right) \left(1 - \text{erf} \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

$$(c) \quad T - T_i = \frac{2q_s'' \sqrt{\alpha t / \pi}}{k} e^{-\lambda^2} - \frac{q_s'' x}{k} (1 - \text{erf} \lambda)$$



Temperature history in a semi-infinite solid initially at a uniform temperature and suddenly exposed at the surface to a fluid at a constant temperature T_∞

One important application of the semi-infinite-body solution is in determining **contact temperatures**.



Consider two blocks of different materials and at different temperatures brought into contact. The initial temperatures are T_{Li} and T_{Ri} . For short times after contact, both blocks act as semi-infinite bodies. The contact plane suddenly attains some intermediate temperature T_c .

The heat flows into the right block and from the left block are:

$$q_R'' = \frac{k_R (T_c - T_{Ri})}{(\sqrt{\pi \alpha t})_R} \quad q_L'' = - \frac{k_L (T_c - T_{Li})}{(\sqrt{\pi \alpha t})_L}$$

At any time $q_R'' = q_L''$ so the contact temperature is:

$$T_c = \frac{T_{Li} (\sqrt{k \rho c_p})_L + T_{Ri} (\sqrt{k \rho c_p})_R}{(\sqrt{k \rho c_p})_L + (\sqrt{k \rho c_p})_R}$$

4.5 Bodies of finite size

Solutions to one-dimensional problems can be combined to obtain solutions to problems having finite geometries. If $\Theta_1(x,t)$ and $\Theta_2(y,t)$ are known solutions to 2 separate one-dimensional problems, their product gives the solution to the combined problem:

$$\Theta(x,y,t) = \Theta_1(x,t) \cdot \Theta_2(y,t)$$

For purposes of simplifying the notation, we shall use the following symbols:

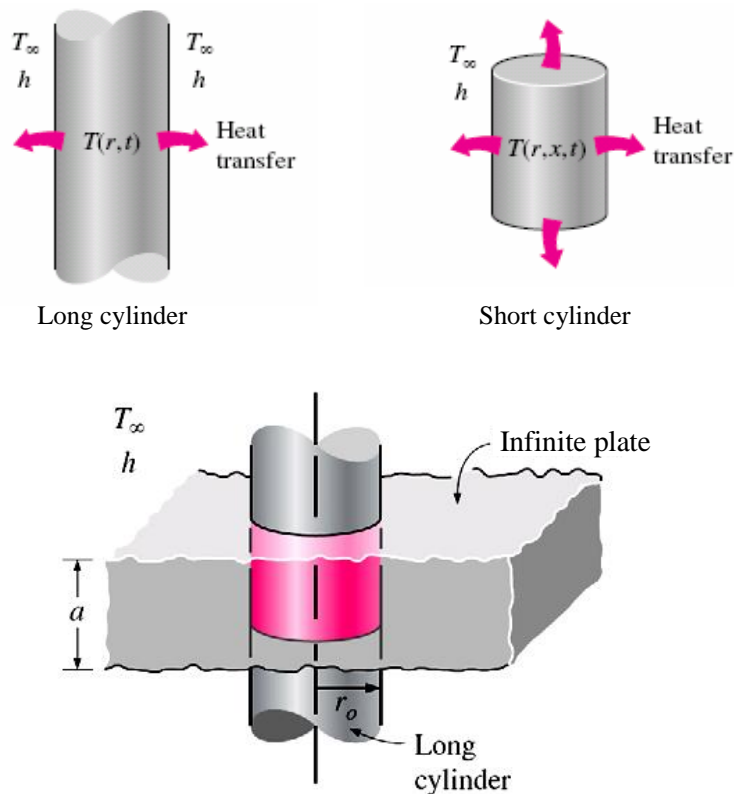
$$S(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \quad \text{the solution for semi-infinite solid } [\text{erf}(\lambda)]$$

$$P(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \quad \text{the solution for infinite plate [Fig. 3-6, 3-9]}$$

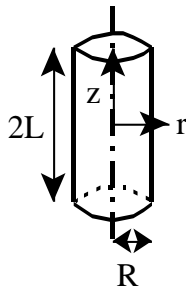
$$C(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} \quad \text{the solution for infinite cylinder [Fig. 3-7, 3-10]}$$

The following figures show how the solution to 2 one-dimensional problems are combined to obtain the solution to a two-dimensional problem. Other combinations can be made. In each case, the solid exists at an initial temperature T_i , and at time zero, it is subjected to a fluid temperature T_∞ with a heat transfer coefficient h .

Finite cylinder

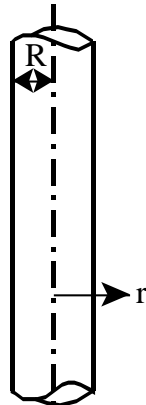


Short cylinder



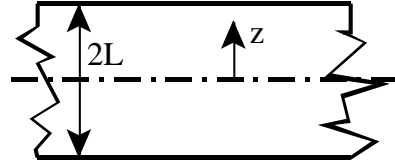
$\Theta(r, z, t)$

Long cylinder



$C(r, t)$

Infinite plate



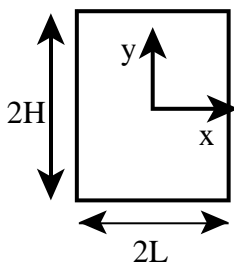
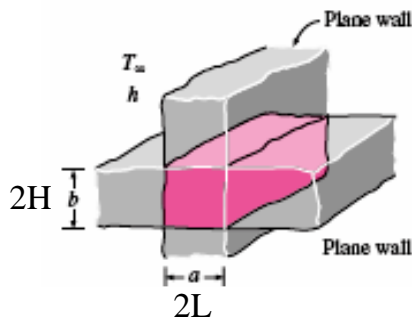
$P(z, t)$

The solution:

$$\Theta(r, z, t) = C(r, t) \cdot P(z, t)$$

$$\frac{T(r, x, t) - T_{\infty}}{T_i - T_{\infty}} \Big|_{\text{Short Cylinder}} = \frac{T(r, t) - T_{\infty}}{T_i - T_{\infty}} \Big|_{\text{Infinite Cylinder}} \cdot \frac{T(x, t) - T_{\infty}}{T_i - T_{\infty}} \Big|_{\text{Infinite Plate}}$$

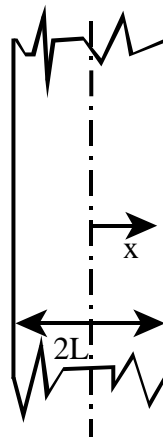
Infinite rectangular bar



$Q(x, y, t)$



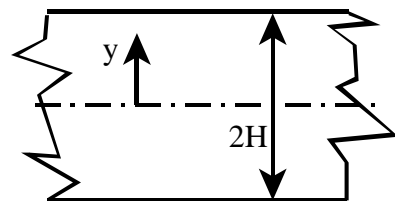
=



$P(x, t)$

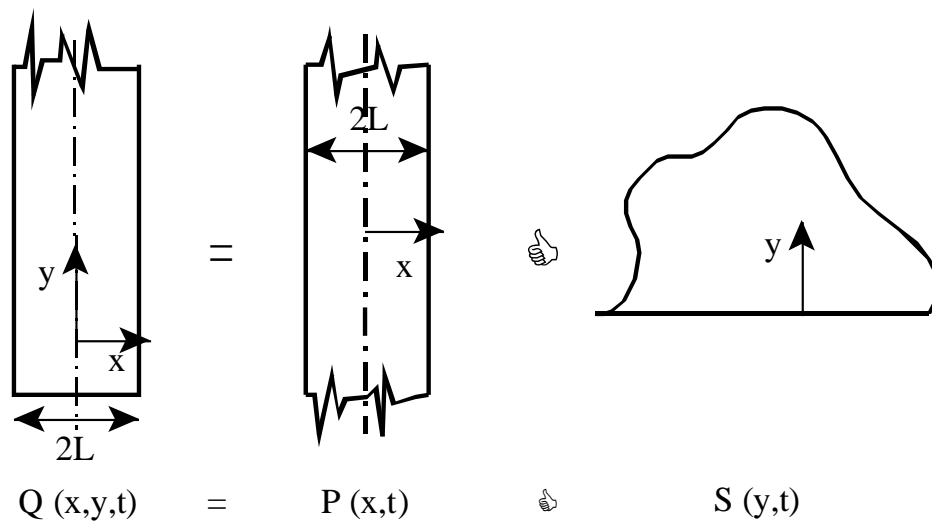


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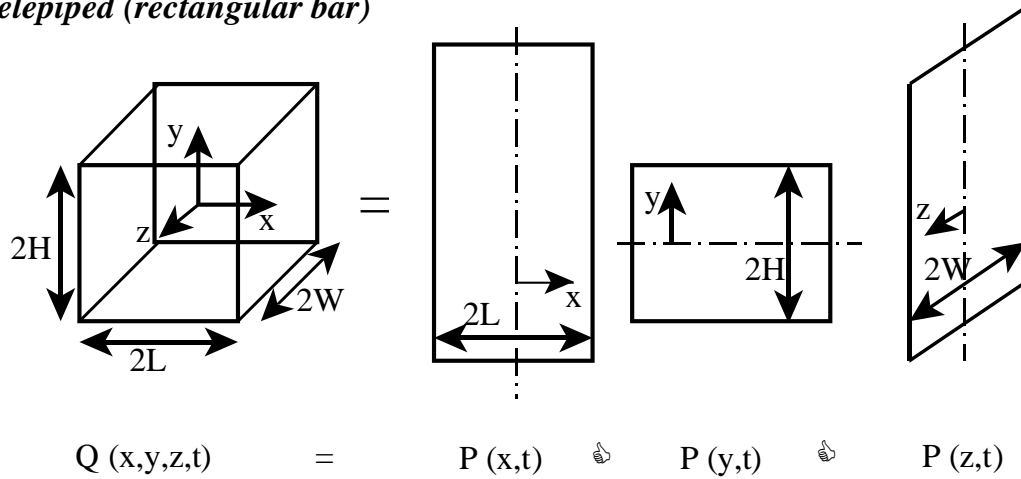


$P(y, t)$

Semi-infinite plate



Parallelepiped (rectangular bar)



This body can be viewed as the intersection of 3 infinite plates that are perpendicular.

5 CONVECTION

In most transfer processes heat transfer in fluids is accompanied by some form of fluid motion so that the heat transfer does not occur by conduction alone.

There are 2 types of heat transport involving fluids:

(1) Forced convection

The fluid flow is established by an external driving force (pressure difference—provided by a pump or gravitational force).

The method of solution:

First the velocity profile is found, then it is used to derive the temperature profile.

(2) Free (or natural) convection

The fluid flow is caused by the buoyant effect resulting from heating or cooling the fluid.

The method of solution:

The velocity and temperature profiles are intimately related and therefore determined together.

Whether the heat transfer mechanism is natural or forced convection, the fluid motion can be described by the equations of fluid mechanics. At low velocities, the flow is laminar through the system; at high velocities, it is laminar near the heating surface and turbulent at some distance away. All problems of convective heat transfer can be expressed in terms of the differential mass, energy, and momentum balance (equations of change). However, because of the mathematical difficulties, solutions exist only for simplified cases. Even high-speed computers using numerical methods could solve only relatively simple problems.

The final objective of an analysis is to determine the heat-transfer rate. We need information on the fluid properties and on the equations of motion for a fluid. Then we need to predict the value of the convection heat transfer coefficient (h) in different geometries and under different flow conditions in order to determine the rate of heat transfer.

In general, we have 2 types of flow situation:

- (1) External flows — they are not enclosed but have contact with a solid surface
- (2) Internal flows — they are enclosed completely by solid surfaces (flow in a pipe or duct)

5.1 Equation of continuity (equation of conservation of mass)

The equation of continuity in vector form is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{v} = 0 \quad \text{where} \quad \rho \bar{v} = \text{mass flux or mass velocity, kg/m}^2\text{s}$$

The equation of continuity in rectangular coordinates (Table 7.1) is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0$$

For incompressible fluid

$\rho = \text{constant} \Rightarrow \partial \rho / \partial t = 0$ so the equation of continuity reduces to

$$\nabla \cdot \bar{v} = 0 \quad \text{or} \quad \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0$$

5.2 Equation of motion (equation of conservation of momentum)

$$\left\{ \begin{array}{c} \text{Rate of} \\ \text{momentum} \\ \text{accumulation} \end{array} \right\} = \left\{ \begin{array}{c} \text{Rate of} \\ \text{momentum} \\ \text{in} \end{array} \right\} - \left\{ \begin{array}{c} \text{Rate of} \\ \text{momentum} \\ \text{out} \end{array} \right\} + \left\{ \begin{array}{c} \text{Sum of external} \\ \text{forces acting} \\ \text{on system} \end{array} \right\}$$

The equation of motion in vector form:

$$\frac{\partial \rho \bar{v}}{\partial t} = -\nabla \cdot \rho \bar{v} \bar{v} - \nabla \cdot \bar{\tau} - \nabla p + \rho \bar{g}$$

- 1st term: Rate of accumulation of momentum per unit volume
- 2nd term: Rate of momentum gain by convection per unit volume
- 3rd term: Rate of momentum gain by viscous transfer per unit volume
- 4th term: Pressure force on element per unit volume
- 5th term: Gravity force on element per unit volume

The x-component of the equation of motion in rectangular coordinate system:

$$\frac{\partial \rho v_x}{\partial t} = - \left(\frac{\partial}{\partial x} \rho v_x v_x + \frac{\partial}{\partial y} \rho v_y v_x + \frac{\partial}{\partial z} \rho v_z v_x \right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

For $\rho = \text{constant}$, $\mu = \text{constant}$ — Navier–Stokes equation (Table 7.2)

In vector form:
$$\rho \frac{\partial \bar{v}}{\partial t} = -\rho \nabla \cdot \bar{v} \bar{v} + \mu \nabla^2 \bar{v} - \nabla p + \rho \bar{g}$$

The x-component in rectangular coordinate system:

$$\rho \underbrace{\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)}_{\frac{D\bar{v}}{Dt}} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

5.3 Equation of thermal energy

$$\frac{\partial \rho u}{\partial t} = -\nabla \cdot \rho \bar{v} u - \nabla \cdot \bar{q} - \nabla \cdot p \bar{v} - \nabla \cdot (\bar{\tau} \cdot \bar{v})$$

- 1st term: Rate of increase of internal energy per unit volume
- 2nd term: Rate of internal energy input by convection per unit volume
- 3rd term: Rate of energy in by conduction per unit volume
- 4th term: Rate of work done on fluid by pressure forces per unit volume
- 5th term: Rate of work done of fluid by viscous forces per unit volume

It is more convenient to have the thermal energy equation in terms of fluid properties (μ , T , c_p).

If ρ , μ , and k are constants, then the thermal energy equation (Table 7.3) becomes:

$$\rho c_p \frac{\partial T}{\partial t} = -\rho c_p \bar{v} \cdot \nabla T - k \nabla^2 T - T \left(\frac{\partial p}{\partial T} \right)_\rho \nabla \cdot \bar{v} + \mu \Phi_v$$

where Φ_v = viscous dissipation function. It is usually negligible; it is needed only in special situations.

Simplification of the equation of thermal energy:

For ideal gas

$$\left(\frac{\partial p}{\partial T} \right)_\rho = \frac{p}{T} \quad \Rightarrow \quad \rho c_p \frac{\partial T}{\partial t} = -\rho c_p \bar{v} \cdot \nabla T + k \nabla^2 T - p \nabla \cdot \bar{v}$$

In rectangular coordinate system:

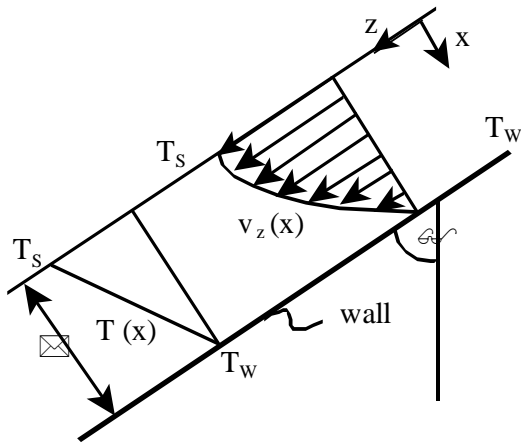
$$\rho c_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

For solids ($\rho = \text{constant}$)

$$\bar{v} = 0 \quad \Rightarrow \quad \rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T \quad \text{which is the general heat conduction equation}$$

5.4 Application of equations of change to laminar flow

Steady flow of a non-isothermal falling film



Consider a liquid flowing by gravity in steady laminar flow over an inclined plane surface. The free liquid surface is maintained at $T = T_s$ and the solid surface is maintained at $T = T_w$.

Assumptions:

- laminar flow,
- steady flow,
- constant physical properties (ρ , μ , k),
- the velocity and temperature profiles are fully developed and will not change with increasing z

For laminar flow: $v_x = v_y = 0$
 $v_z \neq 0$ and $v_z = v_z(x)$
 (v_z can depend on x , y , z , and t , but at steady flow v_z does not depend on t . The velocity profile is fully developed so v_z does not depend on z either. v_z does not vary in y direction.)

Equation of continuity: all terms are 0.

Equation of motion (Navier–Stokes equations):

$$\text{x-component: } 0 = -\frac{\partial p}{\partial x} + \rho g_x$$

$$\text{y-component: } 0 = -\frac{\partial p}{\partial y} + \rho g_y$$

$$\text{z-component: } 0 = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g_z$$

Pressure variations in the flow are not significant therefore the x- and y-components of the equations vanish, we are left with:

$$\begin{aligned} \mu \frac{d^2 v_z}{dx^2} &= -\rho g_z \\ \downarrow \quad g_z &= g \cos \beta \\ \mu \frac{d^2 v_z}{dx^2} &= -\rho g \cos \beta \end{aligned}$$

$$\begin{array}{l}
 \downarrow \quad \begin{array}{ll} \text{B.C.1: at } x = \delta & v_z = 0 \text{ (no slip condition)} \\ \text{B.C.2: at } x = 0 & \tau_{xz} = 0 \text{ (no viscous interaction between the air and the liquid)} \end{array} \\
 v_z = \frac{\rho g \delta^2 \cos \beta}{2\mu} \left[1 - \left(\frac{x}{\delta} \right)^2 \right] \quad \text{parabolic velocity profile (as we have seen it in the 1st semester)}
 \end{array}$$

Equation of energy:

The temperature can be a function of x, y, z, and t: T(x,y,z,t), but the temperature does not vary with z (4th assumption), the y dependence is not significant and because of steady-state, it does not depend on t. Therefore, the temperature depends only on x: T = T(x) only.

The energy equation reduces to:

$$0 = k \frac{d^2 T}{dx^2} + \mu \left(\frac{dv_z}{dx} \right)^2 \quad \begin{array}{ll} \text{B.C.1: at } x = 0 & T = T_s \\ \text{B.C.2: at } x = \delta & T = T_w \end{array}$$

From the velocity distribution:

$$\frac{dv_z}{dx} = \frac{\rho g \delta^2 \cos \beta}{2\mu} \left(-\frac{2x}{\delta^2} \right) = -\frac{\rho g \cos \beta x}{\mu}$$

Substituting this into the energy equation:

$$\frac{d^2 T}{dx^2} = -\frac{\mu}{k} \left(-\frac{\rho g \cos \beta x}{\mu} \right)^2 = -\frac{\rho^2 g^2 \cos^2 \beta x^2}{\mu k}$$

Integrating twice yields:

$$\begin{aligned}
 \frac{dT}{dx} &= -\frac{\rho^2 g^2 \cos^2 \beta}{3\mu k} x^3 + C_1 \\
 T &= -\frac{\rho^2 g^2 \cos^2 \beta}{12\mu k} x^4 + C_1 x + C_2
 \end{aligned}$$

C₁ and C₂ can be evaluated by the use of the boundary conditions. The temperature distribution is then:

$$\frac{T - T_s}{T_w - T_s} = \frac{\rho^2 g^2 \delta^4 \cos^2 \beta}{12\mu k (T_w - T_s)} \left(\frac{x}{\delta} - \frac{x^4}{\delta^4} \right) + \frac{x}{\delta}$$

If viscous dissipation is neglected:

$$0 = k \frac{d^2 T}{dx^2} \Rightarrow T = C_1 x + C_2 \quad \text{the temperature distribution is **linear**}$$

Using the boundary conditions, the temperature distribution becomes:

$$\frac{T - T_s}{T_w - T_s} = \frac{x}{\delta}$$

Once the temperature profile is known, other parameters can be calculated such as heat flux, maximum temperature rise, and average temperature rise.

Average temperature rise

Average temperature or average temperature rise is obtained in a manner analogous to average velocity. The temperature profile is integrated over the cross-sectional area, and the result is divided by the area itself. This gives the arithmetic average temperature.

$$T_{avg} = \frac{\iint T(x) dA}{\iint dA} \quad \text{or} \quad T_{avg} - T_{ref} = \frac{\iint [T(x) - T_{ref}] dA}{\iint dA}$$

where $T_{ref} = T_s$ will be.

Bulk temperature

Bulk temperature is also called “cup mixing temperature” or “flow average temperature”. It is the **cross-sectional weighted average** of the local fluid temperature T . The weighing factor is the longitudinal velocity v_z . ($v_z dA$ is the volumetric flow rate.)

$$T_{bulk}(z) = \frac{(v_z T)_{avg}}{v_{z, avg}} = \frac{\iint v_z T dA}{\iint v_z dA} \quad \text{or} \quad T_{bulk} - T_{ref} = \frac{\iint v_z(x) [T(x) - T_{ref}] dA}{\iint v_z(x) dA}$$

When a fluid is being heated or cooled, the temperature will vary throughout the cross section of the stream. The bulk temperature is the temperature that would be attained if all the fluid flowing across the section in question were withdrawn and mixed adiabatically to a uniform temperature.

The average temperature rise for the falling film is:

$$\frac{T_{avg} - T_s}{T_W - T_s} = \frac{\int_0^b \int_0^\delta \frac{x}{\delta} dx dy}{\int_0^b \int_0^\delta dx dy} = \frac{b \int_0^\delta \frac{x}{\delta} dx}{b\delta} = \frac{\left[\frac{x^2}{2\delta} \right]_0^\delta}{\delta} = \frac{\delta^2}{2\delta\delta} = \frac{1}{2}$$

$$\frac{T_{avg} - T_s}{T_W - T_s} = \frac{1}{2} \quad \Rightarrow \quad T_{avg} = \frac{T_W + T_s}{2}$$

The bulk temperature for the falling film is:

$$T_b - T_s = \frac{\int_0^b \int_0^\delta \frac{\rho g \delta^2 \cos \beta}{2\mu} \left(1 - \frac{x^2}{\delta^2} \right) (T_W - T_s) \frac{x}{\delta} dx dy}{\int_0^b \int_0^\delta \frac{\rho g \delta^2 \cos \beta}{2\mu} \left(1 - \frac{x^2}{\delta^2} \right) dx dy}$$

Then

$$\frac{T_b - T_s}{T_W - T_s} = \frac{b \int_0^\delta \left(\frac{x}{\delta} - \frac{x^3}{\delta^3} \right) dx}{b \int_0^\delta \left(1 - \frac{x^2}{\delta^2} \right) dx} = \frac{\left[\frac{x^2}{2\delta} - \frac{x^4}{4\delta^3} \right]_0^\delta}{\left[x - \frac{x^3}{3\delta^2} \right]_0^\delta} = \frac{3}{8}$$

$$\Rightarrow T_b = \frac{3}{8} T_w + \frac{5}{8} T_s$$

5.5 Application of equations of change to turbulent flow

The equations of change in laminar flow for certain simple geometrical situations can be solved and the temperature distribution can be determined analytically. Once the temperature profile is known, other parameters, such as heat flux or bulk temperature, can be calculated.

For turbulent flows, none of the velocities vanishes in the Navier–Stokes equations and all of the nonlinear terms remain. The instantaneous velocities, pressures, and temperatures fluctuate randomly about mean values. Therefore, it is customary to

“time smooth” the instantaneous velocity, pressure, and temperature to get mean values and time-average the equations of change.

For very simple geometry these equations can be solved, but for most of the situations cannot. Therefore, alternative solution methods have been devised such as the boundary-layer theory and the dimensional analysis.

5.6 Boundary layer flow

Consider the problem of the 2-dimensional steady flow of an incompressible Newtonian fluid with constant properties over a flat plate in laminar flow. The equations of change describing the flow are:

Equation of continuity: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

Equation of motion
x-component: $v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x}$

y-component: $v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y}$

Equation of energy: $v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$

Assumptions:

- $\rho, \mu, k, c_p = \text{constant}$
- steady flow
- laminar flow
- $v_x(x, y)$
- $v_y(x, y)$
- $v_z = 0$
- $T(x, y)$

Boundary conditions:

- B.C.1: at $y = 0$ $v_x = 0$
- B.C.2: at $y = \infty$ $v_x = v_\infty$
- B.C.3: at $y = 0$ $v_y = 0$
- B.C.4: at $y = \infty$ $\partial v_x / \partial y = 0$
(since if the function is constant, its first derivative is 0)

These equations are the exact equations, which were considerably simplified by Prandtl. He recognized in 1904 that the effect of the solid boundary on the flow is confined to a very thin layer of fluid immediately adjacent to the solid wall and can be considered negligible beyond it. This layer is called boundary layer.

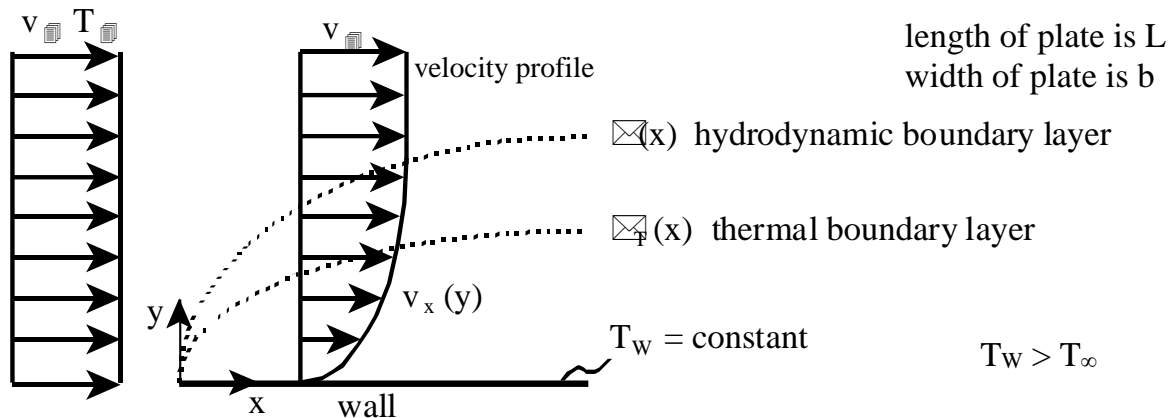
When a fluid flows over a surface the elements in contact with the surface will be brought to rest and the adjacent layers retarded by the viscous drag of the fluid. Thus, the velocity in the neighbourhood of the surface will change with distance perpendicular to the flow.

Suppose a fluid approaches the plate with uniform v_∞ approach velocity (or free-stream velocity) and T_∞ temperature. When the fluid reaches the surface, a velocity gradient is set up because of the viscous forces acting within the fluid. The fluid in

contact with the surface will be brought to rest and will gradually approach the free stream velocity (v_∞) at some distance from the surface.

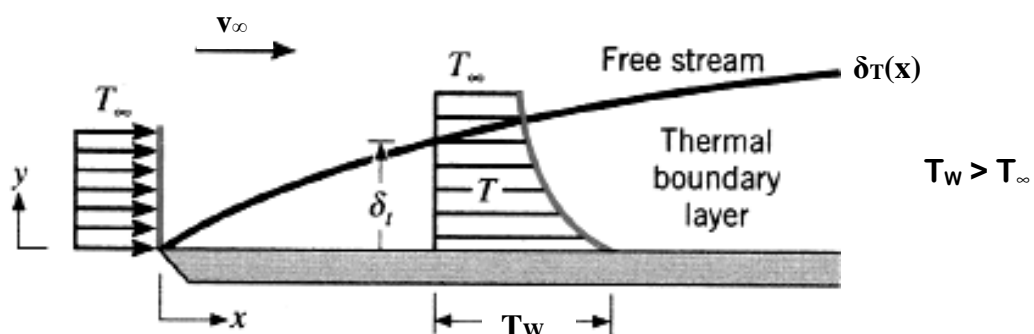
We divide the flow region into 2 parts:

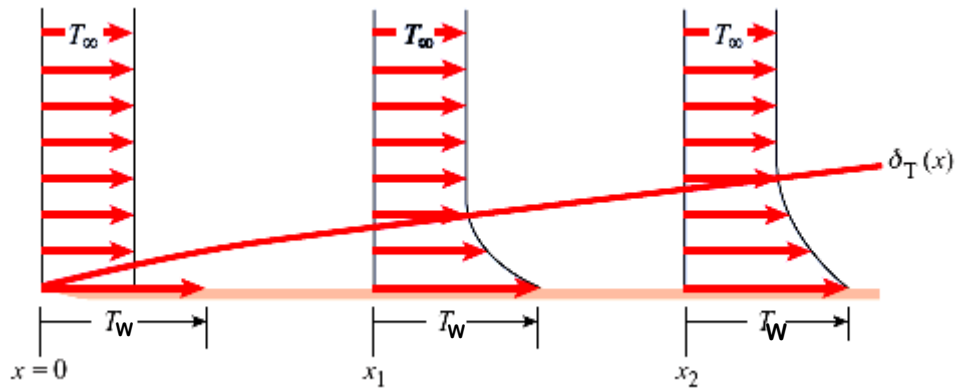
- (1) A non-viscous region away from any solid surfaces
(ideal fluid: $\mu = 0$, $\rho = \text{constant}$)
- (2) A boundary layer (the fluid adheres to the surface due to viscous effects)



The thickness of the layer in which the fluid is retarded becomes greater with distance in the direction of flow. This layer was termed boundary layer by Prandtl. The free stream velocity (v_∞) is approached asymptotically, and therefore the boundary layer strictly has no precise outer limit. However, it is convenient to define the boundary layer thickness such that the velocity at its outer edge equals 99% of the free stream velocity: the **hydrodynamic boundary layer** thickness δ is where $v_x \cong 0.99v_\infty$. Within the boundary layer the velocity in x direction varies only with y : $v_x = v_x(y)$.

Similar behaviour occurs with temperature. A **thermal boundary layer** develops when the fluid T_∞ free stream temperature is different from the wall temperature. Heat transfer occurs until both the fluid and solid surface assume the same temperature at the layer adjacent to the wall. In turn, the fluid molecules exchange heat energy with those in the adjoining layer, thus temperature gradients develop in the fluid. The region of fluid in which temperature gradients exist is the thermal boundary layer. Its thickness δ_T is the vertical distance from the surface where the temperature is $T \cong 0.99T_\infty$. The boundary layers grow from the leading edge with distance x . Typically, δ and δ_T are a few thousandths of a millimeter thick. Despite their size, both are important in determining drag forces and heat transfer rates.





The growing thermal boundary layer thickness $\delta_T(x)$ is shown along with temperature distributions superimposed at three x locations for a heated plate ($T_w > T_\infty$)

Heat transfer from the solid surface to the stationary fluid layer adjacent to the surface is by conduction. The local heat flux here can be expressed by Fourier's law of heat conduction:

$$q'' = k_{fluid} \left. \frac{dT}{dy} \right|_{y=0}$$

This heat is then transferred by convection away from the solid surface as a result of fluid motion. Thus, the surface heat flux equals the convective heat flux expressed by Newton's law of cooling:

$$q'' = h_x(T_w - T_\infty)$$

where h_x is the local heat transfer coefficient.

The mathematical simplifications for the equations of change:

- (1) The boundary layer is thin compared to the distance measured from the leading edge.
- (2) As a result of (1), $\partial^2 v_x / \partial x^2$ is much smaller than $\partial^2 v_x / \partial y^2$
- (3) $v_y \ll v_x$ (so all terms in the Navier–Stokes equation y-component which involve v_y may be neglected).
- (4) The pressure across the boundary layer is assumed to be constant: $\partial p / \partial y \cong 0$.
- (5) There is no pressure gradient or gravity force in x direction.
- (6) There is negligible heat conduction in the direction of flow: $\partial^2 T / \partial x^2 \ll \partial^2 T / \partial y^2$

The equations for the boundary layer become (approximate equations):

Equation of continuity: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

Equation of motion:
(x-component) $v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$

Equation of energy:
$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

~~Solutions for temperature profiles depend on the boundary conditions specified (heating could begin at the leading edge of the plate or somewhere downstream, q''_w or T_w can be constant).~~

B.C.1:	at	$y = 0$	$v_x = 0$	$T = T_w$ constant wall temperature
B.C.2:	at	$y = \infty$	$v_x = v_\infty$	$T = T_\infty$
B.C.3:	at	$y = 0$	$v_y = 0$	

There are 2 strategies for solving the boundary layer momentum equation for the velocity profile:

- (1) The exact solution by Blasius (1908),
- (2) The approximate solution from the integral method developed by von Kármán and Pohlhausen (1921).

Blasius' exact solution

It is assumed that the velocity profile have similar shapes at various distances from the leading edge of the plate, that is they have the same functional dependence on y regardless of x location.

If δ is the value where $v_x/v_\infty = 0.99$, then

$$\delta = \frac{4.96x}{\sqrt{\text{Re}_x}} = 4.96 \sqrt{\frac{\nu x}{v_\infty}}$$

where $\text{Re}_x \equiv \frac{\rho v_\infty x}{\mu} = \frac{v_\infty x}{\nu}$ the Reynolds number based on the downstream distance x

That is δ , the thickness of the boundary layer is proportional to the square root of the kinematic viscosity ν "nu" and the square root of the downstream distance x . Some literature gives the constant 5.0 instead of 4.96.

Solution by the integral method

This method for solving the boundary layer momentum equation is approximate and much easier to apply to a wide range of problems than is any exact method of solution. We are not really interested in the details of the velocity or temperature profiles in the boundary layer, beyond learning their slopes at the wall. (These slopes give the shear stress and the heat flux at the wall.) Therefore, we integrate the boundary layer equations from the wall, $y = 0$, to the boundary layer thickness, $y = \delta$, to make ordinary differential equations of them. It turns out that these much simpler equations do not reveal anything new about the temperature and velocity profiles, but they do give accurate explicit equations for τ_w and q''_w .

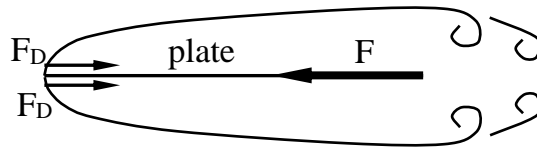
$$\delta = \frac{4.64x}{\sqrt{\text{Re}_x}}$$

The solution leads to the boundary layer thickness

The result is in close agreement with that obtained in the exact solution.

Shear stress at the wall

If we place a very thin plate into a water channel with a flow, a laminar boundary layer develops. This causes shearing force F_s or F_D at the surface of the plate. To keep the plate in place, a counter balance force F needs to be applied otherwise the plate moves downstream. Assuming the flow stays in the laminar regime, we can calculate the shear force with the Blasius equation.



The shear stress at the surface of the plate $\tau_w(x)$ maybe determined from the rate of fluid shear ($\partial v_x / \partial y$) in the boundary layer at $y = 0$:

$$\tau_w(x) = \mu \left. \frac{\partial v_x}{\partial y} \right|_{y=0} = 0.332 \frac{v_\infty \mu}{x} \sqrt{\text{Re}_x}$$

The total skin friction drag is obtained by integrating the wall shear stress over the plate area (with W as the plate width):

$$F_D = \int_0^L \tau_w W dx = 0.664 v_\infty \mu W \sqrt{\text{Re}_x}$$

It is customary to express drag data in terms of a drag coefficient C_D (skin friction coefficient), defined as the ratio of drag force per area to kinetic energy of the free stream:

$$C_D = \frac{F_D}{(WL) (\rho v_\infty^2 / 2)} = \frac{1.328}{\sqrt{\text{Re}_L}}$$

Heat and momentum transfer analogy

Let's define a new dimensionless group called **Prandtl number** as

$$\boxed{\text{Pr} \equiv \frac{c_p \mu}{k} = \frac{\nu}{\alpha}}$$

The physical significance of the Prandtl number of a fluid is that it represents the ratio of the transport coefficient for viscous momentum transport to the transport coefficient for heat conduction.

- When the Prandtl number is large, as in the case of most fluids, the ability of the fluid to transport momentum is greater than that to transport thermal energy. Under these conditions, the hydrodynamic boundary layer is greater than the thermal boundary layer: $\delta > \delta_T$
- For Prandtl numbers much smaller than unity, as in the case of liquid metals, the hydrodynamic boundary layer is relatively small and we may neglect the effect of the solid surface on the velocity distribution when we formulate the thermal energy balance; that is, we may ignore any velocity gradients and assume that all the fluid moves at the free stream velocity.
- When the Prandtl number is close to unity, as is the case with most gases, the hydrodynamic and thermal boundary layers practically coincide: $\delta = \delta_T$.

(a) For $Pr = 1$

The equation of energy is similar to the equation of motion. The boundary conditions are identical for the dimensionless velocity and dimensionless temperature. If $\alpha = \nu$, then the differential equations are identical, so each problem must have the same solution (i.e. the dimensionless velocity profile solution is the same as the dimensionless temperature profile solution and for any point in the flow system, the dimensionless velocity is equal to the dimensionless temperature):

$$\frac{v_x}{v_\infty} = \frac{T - T_W}{T_\infty - T_W}$$

This implies that in this case $\delta = \delta_T$.

In this case, we can immediately calculate the heat transfer coefficient from the heat flux at the wall from Blasius' exact solution:

$$\left. \frac{\partial v_x}{\partial y} \right|_{y=0} = 0.332 \frac{v_\infty}{x} \sqrt{Re_x} \Rightarrow \left. \frac{\partial T}{\partial y} \right|_{y=0} = (T_\infty - T_W) 0.332 \frac{1}{x} \sqrt{Re_x}$$

and $q_w'' = h_x(T_W - T_\infty) = -k \left. \frac{\partial T}{\partial y} \right|_{y=0}$ where h_x = local heat transfer coefficient

\therefore

$$h_x(T_W - T_\infty) = -k(T_\infty - T_W) 0.332 x^{-1} Re_x^{1/2}$$

\therefore

$$h_x = (k/x) 0.332 Re_x^{1/2}$$

It is customary to express the heat transfer coefficient in terms of dimensionless group Nu_x

$$\boxed{Nu_x \equiv \frac{h_x x}{k}} \quad \text{where} \quad Nu_x = \text{local Nusselt number (contains } h_x \text{ local heat transfer coefficient)}$$

Therefore $Nu_x = 0.332 Re_x^{1/2}$ for $Pr = 1$

(b) For $Pr \neq 1$

From the integral method of solution, Pohlhausen found that

$$\frac{\delta_T}{\delta} = \frac{1}{1.025} Pr^{-1/3} \cong Pr^{-1/3} \quad \text{for } 0.6 \leq Pr \leq 50$$

$\Rightarrow \delta > \delta_T$ for $Pr \geq 0.6$ that is for gases and all liquids (except liquid metals)

With this, the temperature gradient at the wall becomes:

$$\left. \frac{\partial T}{\partial y} \right|_{y=0} = (T_\infty - T_W) 0.332 Pr^{1/3} \frac{1}{x} \sqrt{Re_x}$$

Therefore, the heat flux at the wall, the local heat transfer coefficient, and the local Nusselt number is:

$$q_w'' = h_x (T_W - T_\infty) = -k \left. \frac{\partial T}{\partial y} \right|_{y=0} \quad \text{where } h_x = \text{local heat transfer coefficient}$$

$$h_x = (k/x) 0.332 Pr^{1/3} Re_x^{1/2}$$

$$Nu_x = 0.332 Pr^{1/3} Re_x^{1/2} \quad \text{for } Pr \geq 0.6$$

These equations give the local heat transfer coefficient (h_x) and the dimensionless local heat transfer coefficient (Nu_x) at a distance x from the leading edge of the plate. The local heat transfer coefficient varies along the length of the plate.

We can define an average heat transfer coefficient by integrating the local coefficient over the length of the plate. Therefore the average heat transfer coefficient h (or \bar{h}) for the entire surface is:

$$h = \frac{\int_0^L \int_0^b h_x dz dx}{\int_0^L \int_0^b dz dx} = \frac{b k 0.332 Pr^{1/3} \left(\frac{v_\infty}{\nu} \right)^{1/2} \int_0^L \frac{dx}{\sqrt{x}}}{bL} = \frac{k}{L} 0.332 Pr^{1/3} \left(\frac{v_\infty}{\nu} \right)^{1/2} [2x^{1/2}]_0^L$$

$$h = \frac{k}{L} 0.664 Pr^{1/3} Re_x^{1/2} \quad \text{and}$$

$$\underline{\underline{\bar{Nu}_L = 2 Nu_x = 0.664 Pr^{1/3} Re_L^{1/2}}}$$

for laminar flow ($Re_L < 5 \times 10^5$) and $Pr > 0.6$

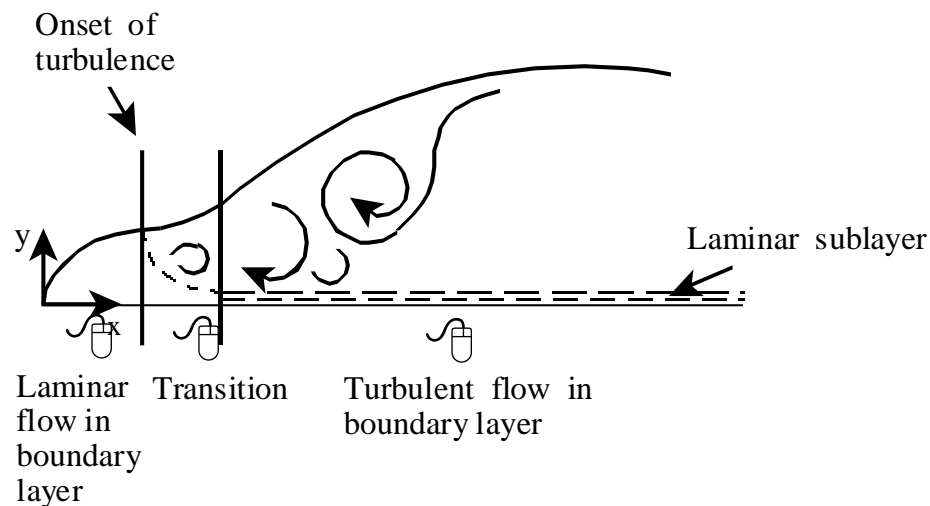
The result shows that $\bar{h} = 2h_{x=L}$.

Since the analysis was based on constant fluid properties, which, in real situations are not constant, they depend on the local temperature, fluid properties have to be evaluated at the average temperature of the fluid in the boundary layer:

$$T_f = \frac{T_W + T_\infty}{2}$$

where T_f = film temperature

Turbulent flow over a flat plate



As the boundary layer thickens, at distances farther from the leading edge, a point is reached where turbulence appears. The onset of turbulence is characterised by a sudden rapid increase in the thickness of the boundary layer. In the turbulent region, however, there is still a **laminar sublayer** near the surface. Transition usually occurs over a range between $2 \times 10^5 \leq Re_x \leq 3 \times 10^6$ and not at a single point. For purposes of calculation it is customarily assumed that transition occurs at **$Re_x = 5 \times 10^5$** . The velocity profiles in laminar and turbulent flows are different, the turbulent profile is flatter over a greater portion of the boundary layer due to mixing effect.

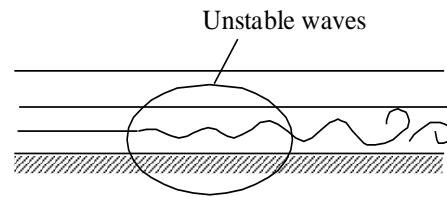
Fluid motion in the turbulent region is highly irregular, and the resultant velocity fluctuation enhances heat transfer. Thus, the local coefficient is higher in the turbulent region than in the laminar region.

How do turbulent boundary layers form?

Laminar boundary layers become unstable due to some disturbance. The disturbance forms unstable waves in the laminar boundary layer, which grows into eddies and a

turbulent boundary layer. Some common disturbances include pressure gradients, surface roughness, heat transfer, body forces, and free stream disturbances. The flow disturbances may grow to form a turbulent boundary layer.

The turbulent boundary layer is divided into 3 regions; they are the **viscous (or laminar) sublayer**, the **buffer layer**, and the **turbulent layer**. The viscous sublayer represents a very small portion in the turbulent boundary layer. The buffer layer represents ~15% and the turbulent



layer represents ~85% of the turbulent boundary layer. Each region has its own velocity profile.

It is not possible to solve the boundary layer equations to obtain an exact solution. However, the integral method gives very good results.

A reasonable velocity profile is: $\frac{v_x}{v_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$

The hydrodynamic boundary layer thickness is:

$$\delta = \frac{0.368 x}{\text{Re}_x^{1/5}} \Rightarrow \delta \sim x^{0.8} \quad \text{and} \quad \underline{\underline{\delta_T \propto \delta}}$$

and the local Nusselt number is:

$$\text{Nu}_x = 0.0287 \text{Re}_x^{0.8} \text{Pr}^{1/3} \quad \text{for} \quad \begin{array}{l} 5 \times 10^5 < \text{Re}_x < 5 \times 10^7 \\ 0.6 < \text{Pr} < 60 \quad \text{and} \\ \text{constant wall temperature} \end{array}$$

If the plate is long enough, the laminar portion can be neglected, and the flow can be treated as all turbulent. To check what type of flow you have, calculate the Reynolds number at the end of the plate.

Constant wall heat flux

In a number of practical problems q'' wall heat flux is constant rather than T_w wall temperature. For flow over a plate, the continuity, momentum, and energy equations have been solved for the constant wall heat flux problem to give the following exact solution:

$$\text{Nu}_x = \frac{h_x x}{k} = 0.453 \text{Re}_x^{1/2} \text{Pr}^{1/3} \quad \text{for laminar flow}$$

$$\text{Nu}_x = \frac{h_x x}{k} = 0.0308 \text{Re}_x^{0.8} \text{Pr}^{1/3} \quad \text{for turbulent flow}$$

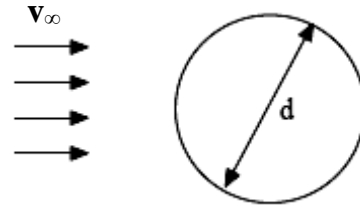
Procedure for calculations

- (1) Identify the flow geometry (flow over flat plate, cylinder, or sphere, or flow through pipe)
- (2) Evaluate fluid properties at film temperature $T_f = (T_w + T_\infty)/2$ unless otherwise noted.
- (3) Calculate Reynolds number and determine whether the flow is laminar or turbulent.
Transition criteria: for plates $Re = 5 \times 10^5$,
for cylinders and spheres $Re_D < 2 \times 10^5$
- (4) Decide whether a local or average heat transfer coefficient is required
- (5) Select the appropriate heat transfer correlation and calculate
- (6) Calculate heat flow rate or heat flux as required.

Constants for the Zukauskas correlation

$$Nu = C Re_D^m Pr^n \left(\frac{\mu}{\mu_w} \right)^{1/4}$$

Re_D	C	m
0.1 – 40	0.75	0.4
40 – 10^3	0.51	0.5
10^3 – 2×10^5	0.26	0.6
2×10^5 – 10^6	0.08	0.7



Cross flow around a cylinder or sphere

or

$$\overline{Nu}_D = 0.989 Re_D^{0.330} Pr^{1/3} \quad 0.4 < Re_D < 4$$

$$\overline{Nu}_D = 0.911 Re_D^{0.358} Pr^{1/3} \quad 4 < Re_D < 40$$

$$\overline{Nu}_D = 0.683 Re_D^{0.466} Pr^{1/3} \quad 40 < Re_D < 4 \times 10^3$$

$$\overline{Nu}_D = 0.193 Re_D^{0.618} Pr^{1/3} \quad 4 \times 10^3 < Re_D < 4 \times 10^4$$

$$\overline{Nu}_D = 0.027 Re_D^{0.805} Pr^{1/3} \quad 4 \times 10^4 < Re_D < 7.6 \times 10^5$$

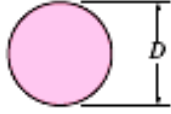


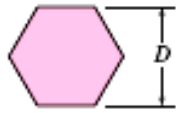
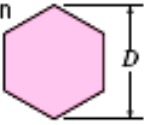
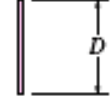

In general, transition to turbulence occurs at around $Re \approx 2 \times 10^5$ for cylinders and spheres.

Summary of heat transfer correlations for flow over flat plates, cylinders and spheres

Geometry	Flow pattern	Heat tr. coeff.	Correlation	Application range
Flat plate $T_w = \text{const.}$	Laminar $Re_x < 5 \times 10^5$	–	$\delta = 5 Re_x^{-1/2}$, $\delta_T = \frac{1}{1.025} Pr^{-1/3}$	
		Local	$Nu_x = 0.332 Pr^{1/3} Re_x^{1/2}$ (Blasius solution)	$0.6 < Pr < 50$
		Average	$Nu = 0.664 Pr^{1/3} Re_x^{1/2}$ (Blasius solution)	$0.6 < Pr < 50$
	Turbulent	–	$\delta = \frac{0.37}{Re_x^{1/5}}$	
		Local	$Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3}$	$5 \times 10^5 < Re < 10^7$ $0.6 \leq Pr \leq 60$
		Average	$Nu = 0.037 Re^{4/5} Pr^{1/3}$	$5 \times 10^5 < Re < 10^7$ $0.6 \leq Pr \leq 50$
			$Nu_x = \frac{0.3387 Re_x^{1/2} Pr^{1/3}}{[1 + (0.0468/Pr)^{2/3}]^{1/4}}$ (Churchill and Ozoe)	
	Mixed b.l. conditions	Average	$Nu = (0.037 Re^{4/5} - 871) Pr^{1/3}$	$5 \times 10^5 < Re_L < 10^8$ $0.6 \leq Pr \leq 60$
Flat plate $q'' = \text{const.}$	Laminar	Local	$Nu_x = 0.453 Re_x^{1/2} Pr^{1/3}$	$0.6 < Pr < 50$
	Turbulent	Local	$Nu_x = 0.0308 Re_x^{4/5} Pr^{1/3}$	
Cylinder	Cross flow around a cylinder	Average	$Nu = C Re_D^m Pr^n \left(\frac{\mu}{\mu_w} \right)^{1/4}$ (Zukauskas correlation) C and m are parameters; D = diameter of cylinder n = 0.37 for $Pr \leq 10$, n = 0.36 for $Pr > 10$. Properties evaluated at T_∞	$0.7 \leq Pr \leq 500$ $1 < Re_D < 2 \times 10^6$
			$Nu = (0.35 + 0.56 Re^{0.52}) Pr^{0.3}$ (Fand correlation)	$0.1 < Re < 10^5$

Cylinder			$Nu = 0.3 + \frac{0.62 Re^{1/2} Pr^{1/3}}{\left[1 + \left(\frac{0.4}{Pr}\right)^{2/3}\right]^{1/4}} \left[1 + \left(\frac{Re}{282000}\right)^{5/8}\right]^{4/5}$ <p>(Churchill and Bernstein)</p>	$100 < Re < 10^7$ $Pr > 0.2$ Fluid properties at $T_f = \frac{T_w + T_\infty}{2}$
Sphere	Cross flow around a sphere		$Nu = 0.37 Re_D^{0.54}$	$17 < Re < 70\,000$ $Pr \sim 0.7$ (gases)
		Average	$Nu = 2 + \left(0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}\right) Pr^{0.4} \left(\frac{\mu}{\mu_w}\right)^{0.14}$ <p>(Whitaker correlation)</p>	$3.5 < Re_D < 7.6 \times 10^4$ $0.7 < Pr < 380$ Fluid properties at T_∞
Freely falling liquid drops		Average	$Nu_D = 2 + 0.6 Re_D^{1/2} Pr^{1/3}$ (Ranz and Marshall correlation)	

Zukauskas's empirical correlations for average Nusselt number for forced convection over circular and non-circular cylinders in cross flow

Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$Nu = 0.989Re^{0.330} Pr^{1/3}$ $Nu = 0.911Re^{0.385} Pr^{1/3}$ $Nu = 0.683Re^{0.466} Pr^{1/3}$ $Nu = 0.193Re^{0.618} Pr^{1/3}$ $Nu = 0.027Re^{0.805} Pr^{1/3}$
Square 	Gas	5000–100,000	$Nu = 0.102Re^{0.675} Pr^{1/3}$
Square (tilted 45°) 	Gas	5000–100,000	$Nu = 0.246Re^{0.588} Pr^{1/3}$
Hexagon 	Gas	5000–100,000	$Nu = 0.153Re^{0.638} Pr^{1/3}$
Hexagon (tilted 45°) 	Gas	5000–19,500 19,500–100,000	$Nu = 0.160Re^{0.638} Pr^{1/3}$ $Nu = 0.0385Re^{0.782} Pr^{1/3}$
Vertical plate 	Gas	4000–15,000	$Nu = 0.228Re^{0.731} Pr^{1/3}$
Ellipse 	Gas	2500–15,000	$Nu = 0.248Re^{0.612} Pr^{1/3}$