SUPPLEMENTARY LECTURE NOTES

Gaussian Elimination with Pivoting Strategies

After this lecture, you should be able to:

1. *learn how to modify the Naive Gaussian elimination method to the Gaussian elimination with partial pivoting method to avoid pitfalls of the former method,*
2. *find the determinant of a square matrix using Gaussian elimination, and*
3. *understand the relationship between the determinant of a coefficient matrix and the solution of simultaneous linear equations.*

What are some techniques for improving the Naive Gaussian elimination method?

As seen in the previous lecture (Example 3), round off errors were large when five significant digits were used as opposed to six significant digits. One method of decreasing the round-off error would be to use more significant digits, that is, use double or quad precision for representing the numbers. However, this would not avoid possible division by zero errors in the Naive Gaussian elimination method. To avoid division by zero as well as reduce (not eliminate) round-off error, Gaussian elimination with partial pivoting is the method of choice.

How does Gaussian elimination with partial pivoting differ from Naive Gaussian elimination?

The two methods are the same, except in the beginning of each step of forward elimination, a row switching is done based on the following criterion. If there are n equations, then there are $n-1$ forward elimination steps. At the beginning of the kth step of forward elimination, one finds the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots, |a_{nk}|$$

Then if the maximum of these values is $|a_{pk}|$ in the pth row, $k \le p \le n$, then switch rows p and k.

The other steps of forward elimination are the same as the Naive Gaussian elimination method. The back substitution steps stay exactly the same as the Naive Gaussian elimination method.

Example 4

In the previous two examples, we used Naive Gaussian elimination to solve

$$20x_1 + 15x_2 + 10x_3 = 45$$
$$-3x_1 - 2.249x_2 + 7x_3 = 1.751$$
$$5x_1 + x_2 + 3x_3 = 9$$

using five and six significant digits with chopping in the calculations. Using five significant digits with chopping, the solution found was

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

This is different from the exact solution of

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the solution using Gaussian elimination with partial pivoting using five significant digits with chopping in your calculations.

Solution

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Forward Elimination of Unknowns

Now for the first step of forward elimination, the absolute value of the first column elements below Row 1 is

$$|20|, |-3|, |5|$$

or

20, 3, 5

So the largest absolute value is in the Row 1. So as per Gaussian elimination with partial pivoting, the switch is between Row 1 and Row 1 to give

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Divide Row 1 by 20 and then multiply it by –3, that is, multiply Row 1 by $-3/20 = -0.15$.

$([20 \quad 15 \quad 10] \quad [45]) \times -0.15$ gives Row 1 as

$$\begin{bmatrix}-3 & -2.25 & -1.5\end{bmatrix} \quad \begin{bmatrix}-6.75\end{bmatrix}$$

Subtract the result from Row 2

$$\begin{array}{rcccc} & \begin{bmatrix}-3 & -2.249 & 7\end{bmatrix} & \begin{bmatrix}1.751\end{bmatrix} \\ - & \begin{bmatrix}-3 & -2.25 & -1.5\end{bmatrix} & \begin{bmatrix}-6.75\end{bmatrix} \\ \hline & 0 \quad 0.001 \quad 8.5 & 8.501 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix}20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix}45 \\ 8.501 \\ 9\end{bmatrix}$$

Divide Row 1 by 20 and then multiply it by 5, that is, multiply Row 1 by $5/20 = 0.25$.

$(\begin{bmatrix}20 & 15 & 10\end{bmatrix} \quad \begin{bmatrix}45\end{bmatrix}) \times 0.25$ gives Row 1 as

$$\begin{bmatrix}5 & 3.75 & 2.5\end{bmatrix} \quad \begin{bmatrix}11.25\end{bmatrix}$$

Subtract the result from Row 3

$$\begin{array}{rcccc} & \begin{bmatrix}5 & 1 & 3\end{bmatrix} & \begin{bmatrix}9\end{bmatrix} \\ - & \begin{bmatrix}5 & 3.75 & 2.5\end{bmatrix} & \begin{bmatrix}11.25\end{bmatrix} \\ \hline & 0 \quad -2.75 \quad 0.5 & -2.25 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix}20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & -2.75 & 0.5\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix}45 \\ 8.501 \\ -2.25\end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, the absolute value of the second column elements below Row 1 is

$$|0.001|, \ |-2.75|$$

or

0.001, 2.75

So the largest absolute value is in Row 3. So Row 2 is switched with Row 3 to give

$$\begin{bmatrix}20 & 15 & 10 \\ 0 & -2.75 & 0.5 \\ 0 & 0.001 & 8.5\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix}7 \\ -2.25 \\ 8.501\end{bmatrix}$$

Divide Row 2 by –2.75 and then multiply it by 0.001, that is, multiply Row 2 by $0.001/-2.75 = -0.00036363$.

$(\begin{bmatrix}0 & -2.75 & 0.5\end{bmatrix} \quad \begin{bmatrix}-2.25\end{bmatrix}) \times -0.00036363$ gives Row 2 as

$$\begin{bmatrix}0 & 0.00099998 & -0.00018182\end{bmatrix} \quad \begin{bmatrix}0.00081816\end{bmatrix}$$

Subtract the result from Row 3

$$\begin{array}{rcccc} & \begin{bmatrix}0 & 0.001 & 8.5\end{bmatrix} & \begin{bmatrix}8.501\end{bmatrix} \\ - & \begin{bmatrix}0 & 0.00099998 & -0.00018182\end{bmatrix} & \begin{bmatrix}0.00081816\end{bmatrix} \\ \hline & 0 \quad 0 \quad 8.50018182 & 8.50018184 \end{array}$$

Rewriting within 5 significant digits with chopping

$$\begin{bmatrix} 0 & 0 & 8.5001 \end{bmatrix} \qquad \begin{bmatrix} 8.5001 \end{bmatrix}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & -2.75 & 0.5 \\ 0 & 0 & 8.5001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ -2.25 \\ 8.5001 \end{bmatrix}$$

Back substitution

$$8.5001x_3 = 8.5001$$

$$x_3 = \frac{8.5001}{8.5001}$$

$$=1$$

Substituting the value of x_3 in Row 2

$$-2.75x_2 + 0.5x_3 = -2.25$$

$$x_2 = \frac{-2.25 - 0.5x_2}{-2.75}$$

$$= \frac{-2.25 - 0.5 \times 1}{-2.75}$$

$$= \frac{-2.25 - 0.5}{-2.75}$$

$$= \frac{-2.75}{-2.75}$$

$$=1$$

Substituting the value of x_3 and x_2 in Row 1

$$20x_1 + 15x_2 + 10x_3 = 45$$

$$x_1 = \frac{45 - 15x_2 - 10x_3}{20}$$

$$= \frac{45 - 15 \times 1 - 10 \times 1}{20}$$

$$= \frac{45 - 15 - 10}{20}$$

$$= \frac{30 - 10}{20}$$

$$= \frac{20}{20}$$

$$=1$$

So the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This, in fact, is the exact solution. By coincidence only, in this case, the round-off error is fully removed.

Can we use Naive Gaussian elimination methods to find the determinant of a square matrix?

One of the more efficient ways to find the determinant of a square matrix is by taking advantage of the following two theorems on a determinant of matrices coupled with Naive Gaussian elimination.

Theorem 1:

Let $[A]$ be a $n \times n$ matrix. Then, if $[B]$ is a $n \times n$ matrix that results from adding or subtracting a multiple of one row to another row, then $\det(A) = \det(B)$ (The same is true for column operations also).

Theorem 2:

Let $[A]$ be a $n \times n$ matrix that is upper triangular, lower triangular or diagonal, then

$$\det(A) = a_{11} \times a_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn}$$

$$= \prod_{i=1}^{n} a_{ii}$$

This implies that if we apply the forward elimination steps of the Naive Gaussian elimination method, the determinant of the matrix stays the same according to Theorem 1. Then since at the end of the forward elimination steps, the resulting matrix is upper triangular, the determinant will be given by Theorem 2.

Example 5

Find the determinant of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Remember in Example 1, we conducted the steps of forward elimination of unknowns using the Naive Gaussian elimination method on $[A]$ to give

$$[B] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

According to Theorem 2

$$\begin{aligned}\det(A) &= \det(B)\\ &= 25\times(-4.8)\times 0.7\\ &= -84.00\end{aligned}$$

What if I cannot find the determinant of the matrix using the Naive Gaussian elimination method, for example, if I get division by zero problems during the Naive Gaussian elimination method?

Well, you can apply Gaussian elimination with partial pivoting. However, the determinant of the resulting upper triangular matrix may differ by a sign. The following theorem applies in addition to the previous two to find the determinant of a square matrix.

Theorem 3:

Let $[A]$ be a $n\times n$ matrix. Then, if $[B]$ is a matrix that results from switching one row with another row, then $\det(B) = -\det(A)$.

Example 6

Find the determinant of

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}$$

Solution

The end of the forward elimination steps of Gaussian elimination with partial pivoting, we would obtain

$$[B] = \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix}$$

$$\begin{aligned}\det(B) &= 10\times 2.5\times 6.002\\ &= 150.05\end{aligned}$$

Since rows were switched once during the forward elimination steps of Gaussian elimination with partial pivoting,

$$\begin{aligned}\det(A) &= -\det(B)\\ &= -150.05\end{aligned}$$

Example 7

Prove

$$\det(A) = \frac{1}{\det\left(A^{-1}\right)}$$

Solution

$$[A][A]^{-1} = [I]$$
$$\det\left(A\,A^{-1}\right) = \det\left(I\right)$$
$$\det\left(A\right)\det\left(A^{-1}\right) = 1$$
$$\det\left(A\right) = \frac{1}{\det\left(A^{-1}\right)}$$

If $[A]$ is a $n \times n$ matrix and $\det(A) \neq 0$, what other statements are equivalent to it?

1. $[A]$ is invertible.
2. $[A]^{-1}$ exists.
3. $[A][X] = [C]$ has a unique solution.
4. $[A][X] = [0]$ solution is $[X] = [\vec{0}]$.
5. $[A][A]^{-1} = [I] = [A]^{-1}[A]$.