18.06 - Spring 2005 - Problem Set 8

Solution to the Challenge Problem

Challenge Problem: Consider the 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix}$$

Determine the entries a, b, c, d, e, f so that:

- the top left 1×1 block is a matrix with eigenvalue 2;
- the top left 2×2 block is a matrix with eigenvalues 3 and -3;
- the top left 3×3 block is a matrix with eigenvalues 0, 1 and -2.

Solution. Let A_i denote the top left $i \times i$ block of A. The matrix A_1 is the matrix (a). Since a is the only eigenvalue of this matrix, we conclude that $a = 2$.

We now move on to determining the entries of the matrix A_2, the top left 2×2 block of A:

$$A_2 = \begin{pmatrix} 2 & b \\ 1 & d \end{pmatrix}$$

Since the sum of the eigenvalues of A_2 is 0 by hypothesis, and it is also equal to the trace of A_2, we obtain that $2 + d = 0$, or $d = -2$. Moreover, the product of the eigenvalues of A_2 is -9 by hypothesis, and it is equal to the determinant of A_2. Thus we have

$$-9 = 2d - b = -4 - b$$

and we deduce that $b = 5$ and therefore

$$A_2 = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$$

Finally, consider $A = A_3$. Again, the sum of the eigenvalues of A is -1 and it is also equal to the trace of A. We deduce that $f = -1$. We still need to determine the entries c and e of A, and we have

$$A = \begin{pmatrix} 2 & 5 & c \\ 1 & -2 & e \\ 0 & 1 & -1 \end{pmatrix}$$

The characteristic polynomial of this matrix is

$$-\lambda^3 - \lambda^2 + (e + 9)\lambda + c - 2e + 9$$

We know that the roots of this polynomial must be 0, 1 and -2. Setting $\lambda = 0$ and $\lambda = 1$ we obtain

$$\begin{aligned} c - 2e + 9 &= 0 \\ -1 - 1 + (e + 9) + c - 2e + 9 &= 0 \end{aligned}$$

which are equivalent to

$$\begin{aligned} c - 2e &= -9 \\ c - e &= -16 \end{aligned}$$

Thus $c = -7$ and $e = 9$ and we conclude

$$A = \begin{pmatrix} 2 & 5 & -7 \\ 1 & -2 & -9 \\ 0 & 1 & -1 \end{pmatrix}$$