LECTURE ONE
Gaussian Elimination

After reading this lecture, you should be able to:

1. *solve a set of simultaneous linear equations using Naive Gaussian elimination,*
2. *learn the pitfalls of the Naive Gaussian elimination method,*
3. *understand the effect of round-off error when solving a set of linear equations with the Naive Gaussian elimination method,*

How is a set of equations solved numerically?

One of the most popular techniques for solving simultaneous linear equations is the Gaussian elimination method. The approach is designed to solve a general set of n equations and n unknowns

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
&\quad . \qquad\qquad . \\
&\quad . \qquad\qquad . \\
&\quad . \qquad\qquad . \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n
\end{aligned}$$

Gaussian elimination consists of two steps

1. Forward Elimination of Unknowns: In this step, the unknown is eliminated in each equation starting with the first equation. This way, the equations are *reduced* to one equation and one unknown in each equation.
2. Back Substitution: In this step, starting from the last equation, each of the unknowns is found.

Forward Elimination of Unknowns:

In the first step of forward elimination, the first unknown, x_1 is eliminated from all rows below the first row. The first equation is selected as the pivot equation to eliminate x_1. So, to eliminate x_1 in the second equation, one divides the first equation by a_{11} (hence called the pivot element) and then multiplies it by a_{21}. This is the same as multiplying the first equation by a_{21}/a_{11} to give

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Now, this equation can be subtracted from the second equation to give

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

where

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$
$$\vdots$$
$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$

This procedure of eliminating x_1, is now repeated for the third equation to the n^{th} equation to reduce the set of equations as

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \\
&a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3 \\
&. \qquad . \qquad . \\
&. \qquad . \qquad . \\
&. \qquad . \qquad . \\
&a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n
\end{aligned}$$

This is the end of the first step of forward elimination. Now for the second step of forward elimination, we start with the second equation as the pivot equation and a'_{22} as the pivot element. So, to eliminate x_2 in the third equation, one divides the second equation by a'_{22} (the pivot element) and then multiply it by a'_{32}. This is the same as multiplying the second equation by a'_{32}/a'_{22} and subtracting it from the third equation. This makes the coefficient of x_2 zero in the third equation. The same procedure is now repeated for the fourth equation till the n^{th} equation to give

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \\
&a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3 \\
&. \qquad . \\
&. \qquad . \\
&. \qquad . \\
&a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n
\end{aligned}$$

The next steps of forward elimination are conducted by using the third equation as a pivot equation and so on. That is, there will be a total of $n-1$ steps of forward elimination. At the end of $n-1$ steps of forward elimination, we get a set of equations that look like

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
. \qquad & . \\
. \qquad & . \\
. \qquad & . \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}$$

Back Substitution:

Now the equations are solved starting from the last equation as it has only one unknown.

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Then the second last equation, that is the $(n-1)^{\text{th}}$ equation, has two unknowns: x_n and x_{n-1}, but x_n is already known. This reduces the $(n-1)^{\text{th}}$ equation also to one unknown. Back substitution hence can be represented for all equations by the formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{for } i = n-1, n-2, \ldots, 1$$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Example 1

The upward velocity of a rocket is given at three different times in Table 1.

Table 1 Velocity vs. time data.

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3, \qquad 5 \le t \le 12$$

The coefficients $a_1, a_2,$ and a_3 for the above expression are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Find the values of $a_1, a_2,$ and a_3 using the Naive Gaussian elimination method. Find the velocity at $t = 6, 7.5, 9, 11$ seconds.

Solution

Forward Elimination of Unknowns

Since there are three equations, there will be two steps of forward elimination of unknowns.

<u>First step</u>

Divide Row 1 by 25 and then multiply it by 64, that is, multiply Row 1 by $64/25 = 2.56$.

$$([25 \quad 5 \quad 1] \qquad\qquad [106.8]) \times 2.56 \text{ gives Row 1 as}$$

$$[64 \quad 12.8 \quad 2.56] \qquad [273.408]$$

Subtract the result from Row 2

$$\begin{array}{rrrrr} & [64 & 8 & 1] & [177.2] \\ - & [64 & 12.8 & 2.56] & [273.408] \\ \hline & 0 & -4.8 & -1.56 & -96.208 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 279.2 \end{bmatrix}$$

Divide Row 1 by 25 and then multiply it by 144, that is, multiply Row 1 by $144/25 = 5.76$.

$([25 \quad 5 \quad 1] \qquad [106.8]) \times 5.76$ gives Row 1 as

$$[144 \quad 28.8 \quad 5.76] \qquad [615.168]$$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [144 & 12 & 1] & [279.2] \\ - & [144 & 28.8 & 5.76] & [615.168] \\ \hline & 0 & -16.8 & -4.76 & -335.968 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ -335.968 \end{bmatrix}$$

<u>Second step</u>

We now divide Row 2 by –4.8 and then multiply by –16.8, that is, multiply Row 2 by $-16.8/-4.8 = 3.5$.

$([0 \quad -4.8 \quad -1.56] \qquad [-96.208]) \times 3.5$ gives Row 2 as

$$[0 \quad -16.8 \quad -5.46] \qquad [-336.728]$$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [0 & -16.8 & -4.76] & [-335.968] \\ - & [0 & -16.8 & -5.46] & [-336.728] \\ \hline & 0 & 0 & 0.7 & 0.76 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Back substitution

From the third equation

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$=1.08571$$

Substituting the value of a_3 in the second equation,

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$= \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$= 19.6905$$

Substituting the value of a_2 and a_3 in the first equation,

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25}$$

$$= 0.290472$$

Hence the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then

$$v(t) = a_1 t^2 + a_2 t + a_3$$

$$= 0.290472t^2 + 19.6905t + 1.08571, \quad 5 \le t \le 12$$

Since we want to find the velocity at $t = 6, 7.5, 9$ and 11 seconds, we could simply substitute each value of t in $v(t) = 0.290472t^2 + 19.6905t + 1.08571$ and find the corresponding velocity. For example, at $t = 6$

$$v(6) = 0.290472(6)^2 + 19.6905(6) + 1.08571$$

$$= 129.686 \text{ m/s}$$

However we could also find all the needed values of velocity at t = 6, 7.5, 9, 11 seconds using matrix multiplication.

$$v(t) = \begin{bmatrix} 0.290472 & 19.6905 & 1.08571 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

So if we want to find $v(6), v(7.5), v(9), v(11),$ it is given by

$$\begin{bmatrix} v(6) & v(7.5) & v(9) & v(11) \end{bmatrix} = \begin{bmatrix} 0.290472 & 19.6905 & 1.08571 \end{bmatrix} \begin{bmatrix} 6^2 & 7.5^2 & 9^2 & 11^2 \\ 6 & 7.5 & 9 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.290472 & 19.6905 & 1.08571 \end{bmatrix} \begin{bmatrix} 36 & 56.25 & 81 & 121 \\ 6 & 7.5 & 9 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 129.686 & 165.104 & 201.828 & 252.828 \end{bmatrix}$$

$v(6) = 129.686$ m/s
$v(7.5) = 165.104$ m/s
$v(9) = 201.828$ m/s
$v(11) = 252.828$ m/s

Example 2

Use Naive Gaussian elimination to solve

$$20x_1 + 15x_2 + 10x_3 = 45$$
$$-3x_1 - 2.249x_2 + 7x_3 = 1.751$$
$$5x_1 + x_2 + 3x_3 = 9$$

Use six significant digits with chopping in your calculations.

Solution

Working in the matrix form

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Forward Elimination of Unknowns

First step

Divide Row 1 by 20 and then multiply it by –3, that is, multiply Row 1 by $-3/20 = -0.15$.

$(\begin{bmatrix} 20 & 15 & 10 \end{bmatrix} \quad \begin{bmatrix} 45 \end{bmatrix}) \times -0.15$ gives Row 1 as

$\begin{bmatrix} -3 & -2.25 & -1.5 \end{bmatrix} \qquad \begin{bmatrix} -6.75 \end{bmatrix}$

Subtract the result from Row 2

$$\begin{array}{rrrrr} & [-3 & -2.249 & 7] & [1.751] \\ - & [-3 & -2.25 & -1.5] & [-6.75] \\ \hline & 0 & 0.001 & 8.5 & 8.501 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 9 \end{bmatrix}$$

Divide Row 1 by 20 and then multiply it by 5, that is, multiply Row 1 by $5/20 = 0.25$

$(\begin{bmatrix} 20 & 15 & 10 \end{bmatrix} \quad \begin{bmatrix} 45 \end{bmatrix}) \times 0.25$ gives Row 1 as

$\begin{bmatrix} 5 & 3.75 & 2.5 \end{bmatrix} \qquad \begin{bmatrix} 11.25 \end{bmatrix}$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [5 & 1 & 3] & [9] \\ - & [5 & 3.75 & 2.5] & [11.25] \\ \hline & 0 & -2.75 & 0.5 & -2.25 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & -2.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ -2.25 \end{bmatrix}$$

<u>Second step</u>

Now for the second step of forward elimination, we will use Row 2 as the pivot equation and eliminate Row 3: Column 2.

Divide Row 2 by 0.001 and then multiply it by –2.75, that is, multiply Row 2 by $-2.75/0.001 = -2750$.

$$([0 \quad 0.001 \quad 8.5] \quad [8.501]) \times -2750 \text{ gives Row 2 as}$$

$$[0 \quad -2.75 \quad -23375] \qquad [-23377.75]$$

Rewriting within 6 significant digits with chopping

$$[0 \quad -2.75 \quad -23375] \qquad [-23377.7]$$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [0 & -2.75 & 0.5] & [-2.25] \\ - & [0 & -2.75 & -23375] & [-23377.7] \\ \hline & 0 & 0 & 23375.5 & 23375.45 \end{array}$$

Rewriting within 6 significant digits with chopping

$$[0 \quad 0 \quad 23375.5] \qquad [-23375.4]$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & 0 & 23375.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 23375.4 \end{bmatrix}$$

This is the end of the forward elimination steps.

Back substitution

We can now solve the above equations by back substitution. From the third equation,

$$23375.5x_3 = 23375.4$$

$$x_3 = \frac{23375.4}{23375.5}$$

$$= 0.999995$$

Substituting the value of x_3 in the second equation

$$0.001x_2 + 8.5x_3 = 8.501$$

$$x_2 = \frac{8.501 - 8.5x_3}{0.001}$$

$$= \frac{8.501 - 8.5 \times 0.999995}{0.001}$$

$$= \frac{8.501 - 8.49995}{0.001}$$
$$= \frac{0.00105}{0.001}$$
$$= 1.05$$

Substituting the value of x_3 and x_2 in the first equation,

$$20x_1 + 15x_2 + 10x_3 = 45$$
$$x_1 = \frac{45 - 15\,x_2 - 10x_3}{20}$$
$$= \frac{45 - 15 \times 1.05 - 10 \times 0.999995}{20}$$
$$= \frac{45 - 15.75 - 9.99995}{20}$$
$$= \frac{29.25 - 9.99995}{20}$$
$$= \frac{19.2500}{20}$$
$$= 0.9625$$

Hence the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Compare this with the exact solution of

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Are there any pitfalls of the Naive Gaussian elimination method?

Yes, there are two pitfalls of the Naive Gaussian elimination method.

Division by zero: It is possible for division by zero to occur during the beginning of the $n-1$ steps of forward elimination.

For example

$$5x_2 + 6x_3 = 11$$
$$4x_1 + 5x_2 + 7x_3 = 16$$
$$9x_1 + 2x_2 + 3x_3 = 15$$

will result in division by zero in the first step of forward elimination as the coefficient of x_1 in the first equation is zero as is evident when we write the equations in matrix form.

$$\begin{bmatrix} 0 & 5 & 6 \\ 4 & 5 & 7 \\ 9 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 16 \\ 15 \end{bmatrix}$$

But what about the equations below: Is division by zero a problem?

$$5x_1 + 6x_2 + 7x_3 = 18$$
$$10x_1 + 12x_2 + 3x_3 = 25$$
$$20x_1 + 17x_2 + 19x_3 = 56$$

Written in matrix form,

$$\begin{bmatrix} 5 & 6 & 7 \\ 10 & 12 & 3 \\ 20 & 17 & 19 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 25 \\ 56 \end{bmatrix}$$

there is no issue of division by zero in the first step of forward elimination. The pivot element is the coefficient of x_1 in the first equation, 5, and that is a non-zero number. However, at the end of the first step of forward elimination, we get the following equations in matrix form

$$\begin{bmatrix} 5 & 6 & 7 \\ 0 & 0 & -11 \\ 0 & -7 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -11 \\ -16 \end{bmatrix}$$

Now at the beginning of the 2nd step of forward elimination, the coefficient of x_2 in Equation 2 would be used as the pivot element. That element is zero and hence would create the division by zero problem.

So it is important to consider that the possibility of division by zero can occur at the beginning of any step of forward elimination.

Round-off error: The Naive Gaussian elimination method is prone to round-off errors. This is true when there are large numbers of equations as errors propagate. Also, if there is subtraction of numbers from each other, it may create large errors. See the example below.

Example 3

Remember Example 2 where we used Naive Gaussian elimination to solve

$$20x_1 + 15x_2 + 10x_3 = 45$$
$$-3x_1 - 2.249x_2 + 7x_3 = 1.751$$
$$5x_1 + x_2 + 3x_3 = 9$$

using six significant digits with chopping in your calculations? Repeat the problem, but now use five significant digits with chopping in your calculations.

Solution

Writing in the matrix form

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Forward Elimination of Unknowns

First step

Divide Row 1 by 20 and then multiply it by –3, that is, multiply Row 1 by $-3/20 = -0.15$.

$$([20 \quad 15 \quad 10] \quad [45]) \times -0.15 \text{ gives Row 1 as}$$

$$[-3 \quad -2.25 \quad -1.5] \qquad [-6.75]$$

Subtract the result from Row 2

$$\begin{array}{rrrrr} & [-3 & -2.249 & 7] & [1.751] \\ - & [-3 & -2.25 & -1.5] & [-6.75] \\ \hline & 0 & 0.001 & 8.5 & 8.501 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 9 \end{bmatrix}$$

Divide Row 1 by 20 and then multiply it by 5, that is, multiply Row 1 by $5/20 = 0.25$.

$$([20 \quad 15 \quad 10] \quad [45]) \times 0.25 \text{ gives Row 1 as}$$

$$[5 \quad 3.75 \quad 2.5] \qquad [11.25]$$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [5 & 1 & 3] & [9] \\ - & [5 & 3.75 & 2.5] & [11.25] \\ \hline & 0 & -2.75 & 0.5 & -2.25 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & -2.75 & 0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ -2.25 \end{bmatrix}$$

Second step

Now for the second step of forward elimination, we will use Row 2 as the pivot equation and eliminate Row 3: Column 2.

Divide Row 2 by 0.001 and then multiply it by –2.75, that is, multiply Row 2 by $-2.75/0.001 = -2750$.

$$([0 \quad 0.001 \quad 8.5] \quad [8.501]) \times -2750 \text{ gives Row 2 as}$$

$$[0 \quad -2.75 \quad -23375] \qquad [-23377.75]$$

Rewriting within 5 significant digits with chopping

$$[0 \quad -2.75 \quad -23375] \qquad [-23377]$$

Subtract the result from Row 3

$$\begin{array}{rcccc} & [0 & -2.75 & 0.5] & [-2.25] \\ - & [0 & -2.75 & -23375] & [-23377] \\ \hline & 0 & 0 & 23375 & 23374 \end{array}$$

Rewriting within 6 significant digits with chopping

$$[0 \quad 0 \quad 23375] \qquad [-23374]$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & 0 & 23375 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 23374 \end{bmatrix}$$

This is the end of the forward elimination steps.

Back substitution

We can now solve the above equations by back substitution. From the third equation,

$$23375x_3 = 23374$$

$$x_3 = \frac{23374}{23375}$$

$$= 0.99995$$

Substituting the value of x_3 in the second equation

$$0.001x_2 + 8.5x_3 = 8.501$$

$$x_2 = \frac{8.501 - 8.5x_3}{0.001}$$

$$= \frac{8.501 - 8.5 \times 0.99995}{0.001}$$

$$= \frac{8.501 - 8.499575}{0.001}$$

$$= \frac{8.501 - 8.4995}{0.001}$$

$$= \frac{0.0015}{0.001}$$

$$= 1.5$$

Substituting the value of x_3 and x_2 in the first equation,

$$20x_1 + 15x_2 + 10x_3 = 45$$

$$x_1 = \frac{45 - 15\,x_2 - 10x_3}{20}$$

$$= \frac{45 - 15 \times 1.5 - 10 \times 0.99995}{20}$$

$$= \frac{45 - 22.5 - 9.9995}{20}$$
$$= \frac{22.5 - 9.9995}{20}$$
$$= \frac{12.5005}{20}$$
$$= \frac{12.500}{20}$$
$$= 0.625$$

Hence the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Compare this with the exact solution of

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$