Gram-Schmidt in 9 Lines of MATLAB

The Gram-Schmidt algorithm starts with n independent vectors $a_1, \ldots, a_n$ (the columns of A). It produces n orthonormal vectors $q_1, \ldots, q_n$ (the columns of Q). To find q_j, start with a_j and subtract off its projections onto the previous q's—and then divide by the length of that vector v to produce a unit vector.

The inner products $q_i^{\mathrm{T}} a_j$ go into a square matrix R that satisfies $A = QR$. This R is upper triangular, because $q_i^{\mathrm{T}} a_j = 0$ when i is larger than j (later q's are orthogonal to earlier a's, that is the point of the algorithm).

Here is a 9-line MATLAB code to build Q and R from A. Start with $[m,n] =$ `size`(A); $Q =$ `zeros`(m,n); $R =$ `zeros`(n,n); to get the shapes correct.

```
for j=1:n                     % Gram-Schmidt orthogonalization
  v=A(:,j);                   % v begins as column j of A
  for i=1:j-1
    R(i,j)=Q(:,i)'*A(:,j);    % modify A(:,j) to v for more accuracy
    v=v-R(i,j)*Q(:,i);        % subtract the projection (q_i^T a_j)q_i = (q_i^T v)q_i
  end                         % v is now perpendicular to all of q_1,...,q_{j-1}
  R(j,j)=norm(v);
  Q(:,j)=v/R(j,j);            % normalize v to be the next unit vector q_j
end
```

If you undo the last step and the middle steps, you find column j:

$$R(j,j)q_j = (v \textit{ minus its projections}) = (\text{column } j \text{ of } A) - \sum_{i=1}^{j-1} R(i,j)q_i \,.$$

Moving the sum to the far left, this is column j in the multiplication $A = QR$.

That crucial change from a_j to v in line 4 gives "*modified Gram-Schmidt*." In exact arithmetic, the number $R(i,j) = q_i^{\mathrm{T}} a_j$ is the same as $q_i^{\mathrm{T}} v$. (The current v has subtracted from a_j its projections onto earlier $q_1, \ldots, q_{i-1}$. But the new q_i is orthogonal to those directions.) In real arithmetic this orthogonality is not perfect, and computations show a difference in Q. Everybody uses v at that step in the code.

EXAMPLE $\quad A$ is 2 by 2. The columns of Q, normalized by $\frac{1}{5}$, are q_1 and q_2:

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 0 & 2 \end{bmatrix} = QR\,.$$

Starting with the columns a_1 and a_2 of A, Gram-Schmidt normalizes a_1 to q_1 and subtracts from a_2 its projection in the direction of q_1. Here are the steps to the q's:

$$a_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad q_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad a_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad v = a_2 - (q_1^{\mathrm{T}} a_2) q_1 = \frac{1}{5}\begin{bmatrix} -6 \\ 8 \end{bmatrix} \quad q_2 = \frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Along the way, we divided by $\|a_1\| = 5$ and $\|v\| = 2$. Then 5 and 2 go on the diagonal of R, and $q_1^{\mathrm{T}} a_2 = -1$ is $R(1,2)$. This figure shows every vector: