



We shall give two proofs of this result, usually known as Cauchy's integral theorem for a triangular path. The first of these proofs, given next, is based on an additional assumption that the derivative  $f'(z)$  is continuous in  $D$ .

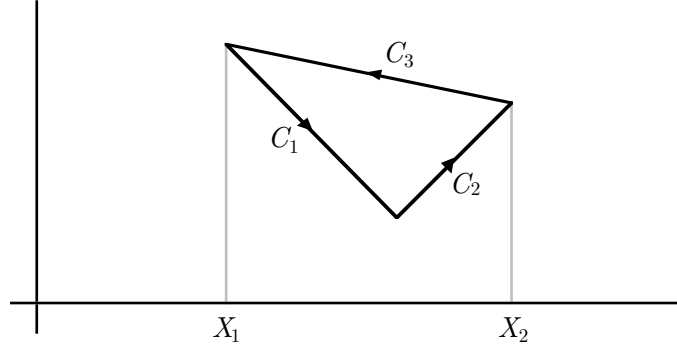
PROOF OF THEOREM 5A. Write  $f(z) = u(x, y) + iv(x, y)$ , where  $u$  and  $v$  are real valued. Since  $f'$  exists and is continuous, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold, and the four partial derivatives are continuous. On the other hand, we can write

$$\int_C f(z) dz = \int_C (u + iv)(dx + idy) = \int_C (u dx - v dy) + i \int_C (v dx + u dy).$$

Suppose that  $C = C_1 \cup C_2 \cup C_3$ , a union of the three straight directed edges.



Consider the integral

$$\int_C u dx.$$

We can write

$$\int_C u dx = \int_{C_1} u dx + \int_{C_2} u dx + \int_{C_3} u dx.$$

For each of the three integrals on the right hand side,  $y$  can be represented as a linear function of  $x$ , unless the edge is vertical, in which case the integral vanishes. Suppose that the projection of the triangle  $T$  on the  $x$ -axis is the line segment  $X_1 \leq x \leq X_2$ . Suppose also that the vertical line with abscissa  $x$  intersects the triangle in  $h_1(x)$  and  $h_2(x)$ , where  $h_1(x) \leq h_2(x)$  (in the diagram,  $h_1(x)$  describes  $C_1$  and  $C_2$ , while  $h_2(x)$  describes  $C_3$ ). Then

$$\begin{aligned} \int_C u dx &= \int_{X_1}^{X_2} u(x, h_1(x)) dx + \int_{X_2}^{X_1} u(x, h_2(x)) dx = - \int_{X_1}^{X_2} (u(x, h_2(x)) - u(x, h_1(x))) dx \\ &= - \int_{X_1}^{X_2} \left( \int_{h_1(x)}^{h_2(x)} \frac{\partial u}{\partial y}(x, y) dy \right) dx = - \int_T \frac{\partial u}{\partial y} dx dy. \end{aligned}$$

Note that the third equality above follows from the continuity of  $\partial u / \partial y$ . Similarly

$$\int_C v dy = \int_T \frac{\partial v}{\partial x} dx dy.$$

Hence

$$\int_C (u \, dx - v \, dy) = - \int_T \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \, dx \, dy = 0.$$

We can also show that

$$\int_C v \, dx = - \int_T \frac{\partial v}{\partial y} \, dx \, dy \quad \text{and} \quad \int_C u \, dy = \int_T \frac{\partial u}{\partial x} \, dx \, dy,$$

so that

$$\int_C (v \, dx + u \, dy) = \int_T \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \, dy = 0.$$

The result follows.  $\bigcirc$

## 5.2. Analytic Functions in a Star Domain

In this section, we shall use Theorem 5A to establish the existence of an indefinite integral and the Cauchy integral theorem for analytic functions in a certain class of domains.

**DEFINITION.** A domain  $D \subseteq \mathbb{C}$  is called a star domain if there exists a point  $z_0 \in D$  such that for every point  $z \in D$ , the line segment joining  $z$  and  $z_0$  also lies in  $D$ . In this case, the point  $z_0$  is called a star centre of the domain  $D$ .

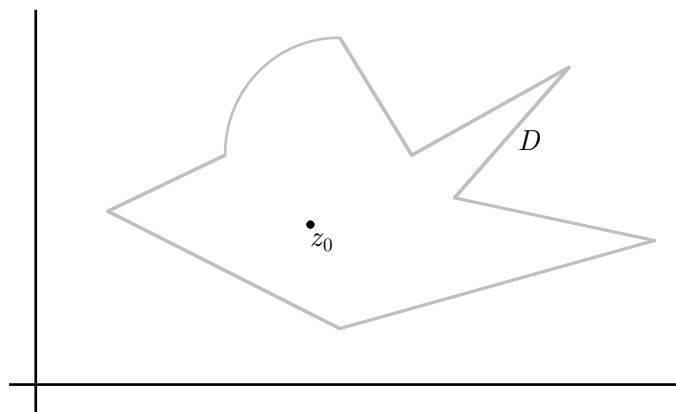
**EXAMPLE 5.2.1.** The disc  $\{z : |z| < 1\}$  is a star domain. Every point in this domain is a star centre.

**EXAMPLE 5.2.2.** The complex plane  $\mathbb{C}$  is a star domain. Again, every point in this domain is a star centre.

**EXAMPLE 5.2.3.** The complex plane  $\mathbb{C}$  with the non-negative real axis  $\{x + iy : x \geq 0, y = 0\}$  deleted is a star domain. Every point on the remaining part of the real axis is a star centre.

**EXAMPLE 5.2.4.** The set  $\{x + iy : |xy| < 1\}$  is a star domain. The point 0 is the only star centre.

**EXAMPLE 5.2.5.** The interior of the set shown below is a star domain, with a star centre  $z_0$  as shown.

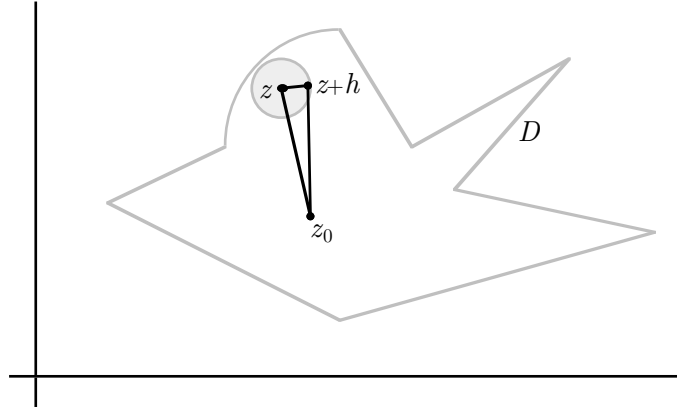


**THEOREM 5B.** Suppose that a function  $f$  is analytic in a star domain  $D$ . Then there exists a function  $F$ , analytic in  $D$  and such that  $F'(z) = f(z)$  for every  $z \in D$ .

PROOF. Suppose that  $z_0 \in D$  is a star centre. For every  $z \in D$ , define

$$(1) \quad F(z) = \int_{[z_0, z]} f(\zeta) d\zeta,$$

where, for every  $z_1, z_2 \in D$ ,  $[z_1, z_2]$  denotes the directed line segment from  $z_1$  to  $z_2$ . Since  $z \in D$ , there exists an  $\epsilon$ -neighbourhood of  $z$  which is contained in  $D$ . Furthermore, for every  $h \in \mathbb{C}$  satisfying  $|h| < \epsilon$ , the point  $z+h$  lies in this  $\epsilon$ -neighbourhood of  $z$ . It follows that the closed triangular region with vertices  $z_0, z$  and  $z+h$  lies in  $D$ .



By Theorem 5A, we have

$$\int_{[z_0, z]} f(\zeta) d\zeta + \int_{[z, z+h]} f(\zeta) d\zeta + \int_{[z+h, z_0]} f(\zeta) d\zeta = 0.$$

In other words,

$$\int_{[z_0, z+h]} f(\zeta) d\zeta - \int_{[z_0, z]} f(\zeta) d\zeta = \int_{[z, z+h]} f(\zeta) d\zeta.$$

It follows from (1) that

$$F(z+h) - F(z) = \int_{[z, z+h]} f(\zeta) d\zeta.$$

If  $h \neq 0$ , then

$$(2) \quad \frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{[z, z+h]} (f(\zeta) - f(z)) d\zeta.$$

Since the function  $f$  is continuous at  $z$ , it follows that given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|f(\zeta) - f(z)| < \epsilon$  whenever  $|\zeta - z| < \delta$ . This means that if  $|h| < \delta$ , then  $|f(\zeta) - f(z)| < \epsilon$  holds for every  $\zeta \in [z, z+h]$ . Theorem 4B now gives

$$(3) \quad \left| \int_{[z, z+h]} (f(\zeta) - f(z)) d\zeta \right| \leq \epsilon|h|.$$

Combining (2) and (3), we have

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \epsilon.$$

This gives

$$\lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

and completes the proof of the theorem.  $\circ$

If we examine our proof carefully, then it is not difficult to see that we have in fact established the following result.

**THEOREM 5C.** *Suppose that a function  $f$  is continuous in a star domain  $D$ . Suppose further that*

$$\int_C f(z) \, dz = 0$$

*for every closed triangular contour  $C$  lying in  $D$ . Then there exists a function  $F$ , analytic in  $D$  and such that  $F'(z) = f(z)$  for every  $z \in D$ .*

We can also deduce the Cauchy integral theorem for a star domain.

**THEOREM 5D.** *Suppose that a function  $f$  is analytic in a star domain  $D$ . Suppose further that  $C$  is a closed contour lying in  $D$ . Then*

$$\int_C f(z) \, dz = 0.$$

**PROOF.** By Theorem 5B, there exists a function  $F$ , analytic in  $D$  and such that  $F'(z) = f(z)$  for every  $z \in D$ . The result now follows from Remark (1) immediately after Theorem 4A.  $\circ$

**EXAMPLE 5.2.6.** Consider the contour integral

$$\int_{|z|=3} \frac{e^z + \sin z}{z^2 - 16} \, dz,$$

where the contour of integration is the circle centred at 0 and with radius 3, followed in the positive (anticlockwise) direction. Note that the function in question is analytic in the disc  $D = \{z : |z| < 4\}$ , clearly a star domain. It follows from Theorem 5D that the integral is 0.

**EXAMPLE 5.2.7.** Suppose that  $0 < r < R$ . Consider the contour integral

$$\int_{|z|=r} \frac{R+z}{(R-z)z} \, dz,$$

where the contour of integration is the circle centred at 0 and with radius  $r$ , followed in the positive (anticlockwise) direction. For every  $z \in \mathbb{C}$ , note that using partial fractions, we have

$$\frac{R+z}{(R-z)z} = \frac{1}{z} + \frac{2}{R-z}.$$

It follows that

$$\int_{|z|=r} \frac{R+z}{(R-z)z} \, dz = \int_{|z|=r} \frac{1}{z} \, dz + \int_{|z|=r} \frac{2}{R-z} \, dz.$$

Next, note that the function

$$\frac{2}{R-z}$$

is analytic in the star domain  $D = \{z : |z| < R\}$ . It follows from Theorem 5D that the last integral is 0, so that

$$(4) \quad \int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_{|z|=r} \frac{1}{z} dz = 2\pi i,$$

in view of Example 4.4.2. On the other hand, the contour can be described by  $z = re^{i\theta}$ , where  $\theta \in [0, 2\pi]$ . This formal substitution leads to the expression  $dz = ire^{i\theta} d\theta = iz d\theta$  and

$$\int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_0^{2\pi} \frac{R+re^{i\theta}}{R-re^{i\theta}} i d\theta.$$

Next, note that

$$\frac{R+re^{i\theta}}{R-re^{i\theta}} = \frac{(R+re^{i\theta})(R-re^{-i\theta})}{(R-re^{i\theta})(R-re^{-i\theta})} = \frac{R^2-r^2+2iRr\sin\theta}{R^2-2Rr\cos\theta+r^2},$$

so that

$$(5) \quad \int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_0^{2\pi} \frac{R^2-r^2+2iRr\sin\theta}{R^2-2Rr\cos\theta+r^2} i d\theta.$$

Combining (4) and (5) and equating imaginary parts, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos\theta+r^2} d\theta = 1.$$

### 5.3. Nested Triangles

In this section, we shall give a second proof of Theorem 5A, without the additional assumption that the derivative  $f'(z)$  is continuous in  $D$ . This proof is based on the following well-known result in real analysis: Suppose that

$$a_1 \leq a_2 \leq a_3 \leq \dots \quad \text{and} \quad b_1 \geq b_2 \geq b_3 \geq \dots$$

Suppose further that  $a_k \leq b_k$  for every  $k \in \mathbb{N}$ , and that  $b_k - a_k \rightarrow 0$  as  $k \rightarrow \infty$ . Then there exists a unique number  $\ell \in \mathbb{R}$  such that  $a_k \rightarrow \ell$  and  $b_k \rightarrow \ell$  as  $k \rightarrow \infty$ . This is a special case of the Cantor intersection theorem. In other words, if the intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \dots$$

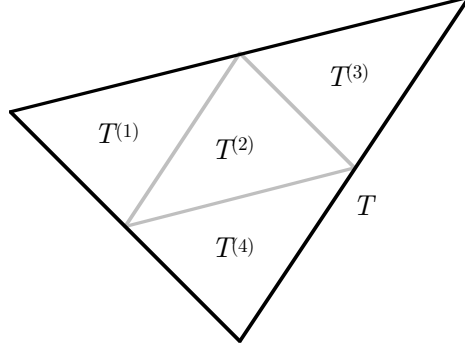
are nested, so that each contains all subsequent ones, and if their lengths decrease to 0, then the intervals collapse to a unique point.

We shall now prove Theorem 5A by the method of bisection.

Suppose that a function  $f$  is analytic in a domain  $D$ . Suppose further that the closed triangular region  $T$  lies in  $D$ , and that  $C$  denotes the boundary of  $T$  in the positive (anticlockwise) direction. Write

$$I(T) = \int_C f(z) dz.$$

We now divide  $T$  into four triangular regions by joining the midpoints of the three sides of  $T$  as shown in the diagram.



Suppose that the four triangular regions so obtained are denoted by  $T^{(j)}$ , where  $j = 1, 2, 3, 4$ , with boundaries  $C^{(j)}$  in the positive (anticlockwise) direction. Then since integrals over the common sides cancel each other, we have

$$I(T) = I(T^{(1)}) + I(T^{(2)}) + I(T^{(3)}) + I(T^{(4)}),$$

where for  $j = 1, 2, 3, 4$ ,

$$I(T^{(j)}) = \int_{C^{(j)}} f(z) dz.$$

Since the maximum is never less than the average, at least one of these four triangular regions  $T^{(j)}$  must satisfy

$$(6) \quad |I(T^{(j)})| \geq \frac{1}{4} |I(T)|.$$

We denote this triangular region by  $T_1$ , with the convention that if more than one of the four triangular regions  $T^{(j)}$  satisfies (6), then we choose one under some fixed rule. This process can now be repeated indefinitely, so that we obtain a sequence of nested triangles

$$T = T_0 \supseteq T_1 \supseteq T_2 \supseteq T_3 \supseteq \dots \supseteq T_k \supseteq \dots$$

with the property

$$|I(T_k)| \geq \frac{1}{4} |I(T_{k-1})|,$$

so that

$$(7) \quad |I(T_k)| \geq 4^{-k} |I(T)|.$$

Note now that the sequence of nested triangular regions must collapse to a point  $z^* \in D$ . Suppose now that  $\epsilon > 0$  is chosen. Since  $D$  is open and the function  $f$  is analytic at  $z^*$ , there exists a  $\delta$ -neighbourhood  $\{z : |z - z^*| < \delta\}$  of  $z^*$ , contained in  $D$  and such that

$$(8) \quad \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon$$

whenever  $|z - z^*| < \delta$ . Furthermore, we can choose  $k$  so large that

$$(9) \quad T_k \subset \{z : |z - z^*| < \delta\}.$$

Note that since

$$\int_{C_k} dz = 0 \quad \text{and} \quad \int_{C_k} z dz = 0,$$

we have

$$I(T_k) = \int_{C_k} f(z) dz = \int_{C_k} (f(z) - f(z^*) - (z - z^*)f'(z^*)) dz.$$

In view of (8) and (9), we have

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| \leq \epsilon |z - z^*| \leq \epsilon d_k,$$

where  $d_k$  denotes the diameter of  $T_k$ . It follows from Theorem 4B that

$$(10) \quad |I(T_k)| \leq \epsilon d_k L_k,$$

where  $L_k$  denotes the perimeter of  $T_k$ . Observe now that

$$(11) \quad d_k = 2^{-k}d \quad \text{and} \quad L_k = 2^{-k}L,$$

where  $d$  and  $L$  denote respectively the diameter and perimeter of  $T$ . Combining (7), (10) and (11), we obtain

$$|I(T)| \leq \epsilon dL.$$

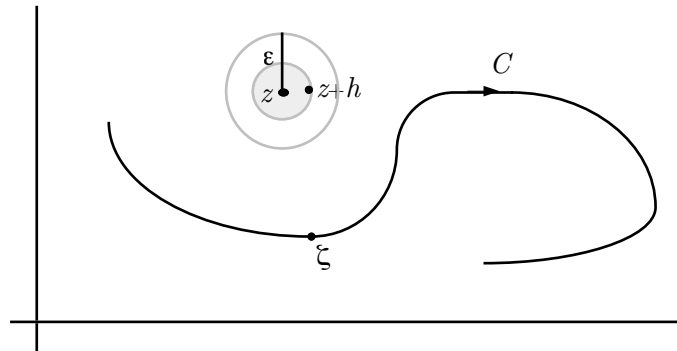
Since  $\epsilon > 0$  is arbitrary, we must have  $I(T) = 0$ . This completes the proof of Theorem 5A.

#### 5.4. Further Examples

EXAMPLE 5.4.1. Suppose that  $C$  is any contour. For any  $z \in \mathbb{C}$  not lying on  $C$ , consider the integral

$$I(z) = \int_C \frac{d\zeta}{\zeta - z}.$$

We shall show that the function  $I(z)$  is continuous at  $z$ . Since  $z \notin C$ , there exists  $\epsilon > 0$  such that the  $\epsilon$ -neighbourhood of  $z$  does not meet  $C$ . Suppose that  $h \in \mathbb{C}$  satisfies  $|h| < \epsilon/2$ .



Then

$$I(z+h) - I(z) = \int_C \left( \frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) d\zeta = h \int_C \frac{d\zeta}{(\zeta - z - h)(\zeta - z)}.$$



Note next that for any  $\zeta \in C$ , we have

$$|\zeta - z| > \epsilon \quad \text{and} \quad |\zeta - z - h| > \frac{\epsilon}{2},$$

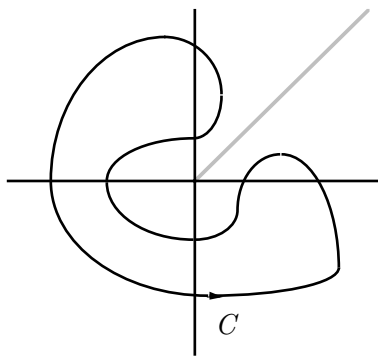
and so it follows from Theorem 4B that

$$|I(z+h) - I(z)| \leq \frac{2L|h|}{\epsilon^2},$$

where  $L$  is the length of  $C$ . This clearly tends to 0 as  $h \rightarrow 0$ .

The final example in this chapter exhibits the possibility of defining a continuous logarithm.

**EXAMPLE 5.4.2.** Consider the domain obtained by deleting from  $\mathbb{C}$  the origin 0 and a half-line starting from 0. This is a star domain in which the function  $1/z$  has a continuous derivative. Suppose that  $C$  is a closed contour that does not meet this half-line.



Then

$$\int_C \frac{d\zeta}{\zeta} = 0.$$

Furthermore, the integral

$$\int_{z_0}^z \frac{d\zeta}{\zeta}$$

is independent of the path joining  $z_0$  to  $z$  in this domain, and can therefore be used to define a continuous logarithm.

#### PROBLEMS FOR CHAPTER 5

1. Give an example to show that the conclusion of Theorem 5D may not hold if  $D$  is not a star domain.
2. Suppose that  $R > 0$  is fixed. By integrating the function  $(R-z)^{-1}$  over the circle  $C = \{z : |z| = r\}$ , where  $0 < r < R$ , and referring to Example 5.2.7, show that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R \cos \theta}{R^2 - 2Rr \cos \theta + r^2} d\theta = \frac{r}{R^2 - r^2}.$$

3. a) Suppose that  $C$  is the rectangle with vertices at  $\pm b$  and  $\pm b + ia$ , where  $a, b > 0$ . Explain why

$$\int_C e^{-z^2} dz = 0.$$

- b) Let  $C = C_1 \cup C_2 \cup C_3 \cup C_4$ , where  $C_1, C_2, C_3, C_4$  represent the four edges of  $C$  followed in the positive (anticlockwise) direction, with initial point  $z = -b$ . Show that

$$\left| \int_{C_2} e^{-z^2} dz \right| \leq e^{-b^2} \int_0^a e^{y^2} dy \quad \text{and} \quad \left| \int_{C_4} e^{-z^2} dz \right| \leq e^{-b^2} \int_0^a e^{y^2} dy.$$

- c) Explain why

$$\int_{-b}^b e^{-(x+ia)^2} dx - \int_{-b}^b e^{-x^2} dx \rightarrow 0 \quad \text{as } b \rightarrow \infty.$$

Deduce that the integral

$$\int_{-\infty}^{\infty} e^{-(x+ia)^2} dx$$

is independent of the choice of  $a > 0$ .

4. Suppose that a function  $f(z)$  is analytic in  $\{z : |z| < R\}$  and continuous in  $\{z : |z| \leq R\}$ , where  $R > 0$  is fixed. Suppose further that  $C$  denotes the circle  $\{z : |z| = R\}$ .

- a) Suppose that  $r < R$ . Explain why

$$\int_C f(z) dz = \int_0^{2\pi} f(Re^{i\theta}) Re^{i\theta} i d\theta - \int_0^{2\pi} f(re^{i\theta}) re^{i\theta} i d\theta.$$

- b) The function  $f(z)z$  is continuous in  $\{z : |z| \leq R\}$ , and so uniformly continuous in  $\{z : |z| \leq R\}$ . This implies that given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|f(Re^{i\theta})Re^{i\theta} - f(re^{i\theta})re^{i\theta}| < \epsilon$  whenever  $R - \delta < r < R$ . Use this to show that

$$\left| \int_C f(z) dz \right| < 2\pi\epsilon.$$

- c) Explain why it follows that

$$\int_C f(z) dz = 0.$$

- d) Explain also why this result does not follow directly from Theorem 5D.

5. Suppose that a function  $f(z)$  is continuous on a closed contour  $C$ . Suppose further that  $f(z)$  can be uniformly approximated with arbitrary precision by a polynomial; in other words, given any  $\epsilon > 0$ , there exists a polynomial  $P(z)$  such that  $|f(z) - P(z)| < \epsilon$  for every  $z \in C$ . Prove that

$$\int_C f(z) dz = 0.$$

6. Suppose that a function  $f(z)$  is analytic in  $\{z : |z| \leq 1\}$ . By considering a suitable integral over the unit circle  $\{z : |z| = 1\}$ , show that

$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \geq 1.$$

7. Suppose that  $C$  is a closed contour, and that  $D$  is a domain not containing any point of  $C$ . By noting Examples 4.4.2 and 5.4.1, show that the integral

$$n(C, z_0) = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}$$

is independent of the choice of  $z_0 \in D$ .

[REMARK: The value  $n(C, z_0)$  is called the winding number of the contour  $C$  round the point  $z_0$ , and measures the number of times the contour winds round the point  $z_0$ .]

8. Suppose that  $C$  is a contour  $z = r(\theta)e^{i\theta}$  for  $\theta \in [0, 2\pi]$ , where  $r(\theta) > 0$  for every  $\theta \in [0, 2\pi]$ . Suppose further that  $r(0) = r(2\pi)$ , so that  $C$  is a closed contour. Let  $D$  be the domain containing the origin  $z = 0$  and with boundary  $C$ .
- Show that  $D$  is a star domain with the origin  $z = 0$  as a star centre.
  - Suppose that  $z_0 \notin D \cup C$ . Explain why the half line  $L = \{\lambda z_0 : \lambda \in [1, \infty)\}$  satisfies  $L \cap C = \emptyset$ . Show also that  $\mathbb{C} \setminus L$  is a star domain with star centre  $z = 0$ .
  - Explain why

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - z_0} = \begin{cases} 0 & \text{if } z_0 \notin D \cup C, \\ 1 & \text{if } z_0 \in D. \end{cases}$$

[HINT: For the case  $z_0 \in D$ , refer to Problem 7 if necessary.]

- d) Suppose that  $P(z)$  is a polynomial with no roots on the contour  $C$ . By referring to Problem 1 in Chapter 3 if necessary, show that the number of roots of  $P(z)$  in  $D$  is given by

$$\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)} dz.$$

9. Suppose that  $P(z)$  is a polynomial of degree  $k$  and with distinct roots  $z_1, \dots, z_k$ . Suppose further that  $C$  is a closed contour which does not contain any of these roots. By referring to Problem 7 if necessary, show that

$$\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)} dz = n(C, z_1) + \dots + n(C, z_k).$$

10. Suppose that two star domains  $D_1$  and  $D_2$  both have the point  $z_0$  as star centre. Show that  $D_1 \cap D_2$  and  $D_1 \cup D_2$  are both star domains with star centre  $z_0$ .