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Course: Computer Engineering

Assignment

1. a. Cholesky factorization can be used to solve the linear system of equations if the coefficient matrix of the system is Symmetric Positive Definite.

   b. $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

   the matrix will be symmetric if it is equal to its transpose

   $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

   Hence the matrix is symmetric

   Checking if the determinants are equal to zero

   $A_1 = |2| = 2$ $\qquad$ $A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2(2) + 1(-1) = 4 - 1 = 3$

   $A_3 = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 2(4 - 1) + 1(-2 + 3) + 3(1 - 6) = -8$

   $A_3 = -8$ which is lesser than 0 hence the matrix is not positive definite

   Since the matrix is not positive definite, the Cholesky decomposition cannot be performed on it.

2. A matrix (A) is Symmetric positive definite if its determinants is greater than zero and it is also equal to its transpose

First I will check if $A=A^T$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Which means that the matrix is symmetric

Checking for the determinants

$A_1 = |1| = 1$ $\qquad$ $A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$

$$A_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (12 - 9) - (6 - 3) + (3 - 2) = 1$$

$$A_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 10 \\ 1 & 10 & 20 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 & 4 \\ 1 & 6 & 10 \\ 1 & 10 & 20 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}$$

$= 1[2(120 - 100) - 3(60 - 40) + 4(30 - 24)] - 1[1(120 - 100) - 3(20 - 10) + 4(10 - 6)] + 1[1(60 - 40) - 2(20 - 10) + 4(4 - 3)] - 1[1(30 - 24) - 2(10 - 6) + 3(4 - 3)]$

$= 1(40 - 60 + 24) - (20 - 30 + 16) + (20 - 20 + 4) - (16 - 8 + 3)$

$A_4 = 1$

Since all the As are equal to 1which is greater than zero, it implies that the matrix is positive definite.

Hence the Pascal Matrix is Symmetric Positive Definite.

3.

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}, \; b = \begin{bmatrix} 24 \\ 30 \\ -24 \end{bmatrix}, \; X^{(0)} = [1 \quad 1 \quad 1]^T \;, w=1.25$$

$$X_i^{(k+1)} = \frac{w}{aii}\left(b_i - \sum_{j=i}^{j-1} aij\ X_j^{(k+1)} - \sum_{j=i+1}^{n} aij\ X_j^{(k)}\right) + (1-w)X_j^{(k)}$$

$X_1^{(1)} = \frac{1.25}{4}(24 - 3(1) - 0(1)) + (1 - 1.25)(1) = 6.3125$

$X_2^{(1)} = \frac{1.25}{4}(30 - 3(6.3125) + 1(1)) + (1 - 1.25)(1) = \frac{901}{256}$

$X_3^{(1)} = \frac{1.25}{4}(-24 - 0(6.3125) + 1(\frac{901}{256})) + (1 - 1.25)(1) = -\frac{27239}{4096}$

$X^{(1)} = \left[6.3125 \quad \frac{901}{256} \quad -\frac{27239}{4096}\right]$

Error margin(E) = $\left\| \frac{X_i^{(k+1)} - X_i^{(k)}}{X_i^{(k+1)}} \right\|$

$E_1^{(1)} = \left\| \frac{6.3125-1}{6.3125} \right\| = 0.8415$

$E_2^{(1)} = \left\| \frac{\frac{901}{256}-1}{\frac{901}{256}} \right\| = 0.71587$

$E_3^{(1)} = \left\| \frac{-\frac{27239}{4096}-1}{-\frac{27239}{4096}} \right\| = 1.15037$

$X_1^{(2)} = \frac{1.25}{4}(24 - 3(\frac{901}{256}) - 0(-\frac{27239}{4096})) + (1 - 1.25)(6.3125) = 2.6223$

$X_2^{(2)} = \frac{1.25}{4}(30 - 3(2.6223) + 1(-\frac{27239}{4096})) + (1 - 1.25)(\frac{901}{256}) = 3.95854$

$X_3^{(2)} = \frac{1.25}{4}(-24 - 0(2.6223) + 1(3.95854)) + (1 - 1.25)(-\frac{27239}{4096}) = -4.60042$

$X^{(2)} = [2.6223 \quad 3.95854 \quad -4.60042]$

$$E_1^{(2)} = \left\| \frac{2.6223 - 6.3125}{2.6223} \right\| = 1.40724$$

$$E_2^{(1)} = \left\| \frac{3.95854 - \frac{901}{256}}{3.95854} \right\| = 0.11090$$

$$E_3^{(1)} = \left\| \frac{-4.60042 + \frac{27239}{4096}}{-4.60042} \right\| = 0.44555$$

$X_1^{(3)} = \frac{1.25}{4} (24 – 3(3.95854) -0(-4.60042)) + (1 - 1.25) (2.6223)= 3.13329$

$X_2^{(3)} = \frac{1.25}{4} (30 – 3(3.13329) +1(-4.60042)) + (1 - 1.25) (3.95854) = 4.01027$

$X_3^{(3)} = \frac{1.25}{4} (-24 – 0(3.13329) +1(4.01027)) + (1 - 1.25)(-4.60042) = -5.09669$

$X^{(3)} = [3.13329 \quad 4.01027 \quad - 5.09669]$

$$E_1^{(2)} = \left\| \frac{3.13329 - 2.6223}{3.13329} \right\| = 0.1630$$

$$E_2^{(1)} = \left\| \frac{4.01027 - 3.95854}{4.01027} \right\| = 0.11090$$

$$E_3^{(1)} = \left\| \frac{-5.09669 + 4.60042}{-5.09669} \right\| = 0.09737$$

Since the error margin is approaching zero, it implies that the values of X can be approximated to $X^{(3)}$

4.

$$A = \begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix},\ b = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix},\ X^{(0)} = [1 \quad 3 \quad 5]^T$$

$$X_i^{(k+1)} = \frac{1}{aii}(b_i - \sum_{j=i}^{j-1} aij\ X_j^{(k+1)} - \sum_{j=i+1}^{n} aij\ X_j^{(k)})$$

$$X_1^{(1)} = \frac{1}{12}(2 - 7(3) - 3(5)) = -\frac{17}{6}$$

$$X_2^{(1)} = \frac{1}{5}(-5 - 1(-\frac{17}{6}) - 1(5)) = -\frac{43}{30}$$

$$X_3^{(1)} = -\frac{1}{11}(6 - 2(-\frac{17}{6}) - 7(-\frac{43}{30})) = -\frac{217}{110}$$

$$X^{(1)} = \left[-\frac{17}{6} \quad -\frac{43}{30} \quad -\frac{217}{110}\right]$$

$$X_1^{(2)} = \frac{1}{12}(2 - 7(-\frac{43}{30}) - 3(-\frac{217}{110})) = \frac{1481}{990}$$

$$X_2^{(2)} = \frac{1}{5}(-5 - 1(\frac{1481}{990}) - 1(-\frac{217}{110})) = -\frac{2239}{2475}$$

$$X_3^{(2)} = -\frac{1}{11}(6 - 2(\frac{1481}{990}) - 7(-\frac{2239}{2475})) = -\frac{7706}{9075}$$

$$X^{(2)} = \left[\frac{1481}{990} \quad -\frac{2239}{2475} \quad -\frac{7706}{9075}\right]$$

$$X_1^{(3)} = \frac{1}{12}(2 - 7(-\frac{2239}{2475}) - 3(-\frac{7706}{9075})) = \frac{296207}{326700}$$

$$X_2^{(3)} = \frac{1}{5}(-5 - 1(\frac{296207}{326700}) - 1(-\frac{7706}{9075})) = -\frac{2207123}{1633500}$$

$X_3^{(3)} = -\frac{1}{11}(6 - 2(\frac{296207}{326700}) - 7(-\frac{2207123}{1633500})) = -\frac{7429597}{5989500}$

$X^{(3)} = \begin{bmatrix} \frac{296207}{326700} & -\frac{2207123}{1633500} & -\frac{7429597}{5989500} \end{bmatrix}$

5.

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}$$

$M_1 = -\frac{2}{6} = -\frac{1}{3}$

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}$$

$M_2 = -\frac{1}{6}$

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{23}{6} & -\frac{7}{6} \\ -1 & 0 & -1 & 3 \end{bmatrix}$$

$M_3 = \frac{1}{6}$

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{23}{6} & -\frac{7}{6} \\ 0 & \frac{1}{3} & -\frac{5}{6} & \frac{19}{6} \end{bmatrix}$$

$M_4 = -(\frac{2}{3} \times \frac{3}{10}) = -\frac{1}{5}$

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{37}{10} & -\frac{11}{10} \\ 0 & \frac{1}{3} & -\frac{5}{6} & \frac{19}{6} \end{bmatrix}$$

$M_5 = -(\frac{1}{3} \times \frac{3}{10}) = -\frac{1}{10}$

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{37}{10} & -\frac{11}{30} \\ 0 & 0 & -\frac{9}{10} & \frac{16}{5} \end{bmatrix}$$

$M_6 = -(-\frac{9}{10} \times \frac{10}{37}) = -\frac{9}{37}$

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{37}{10} & -\frac{11}{10} \\ 0 & 0 & 0 & \frac{217}{74} \end{bmatrix}$$

Hence U = $\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{37}{10} & -\frac{11}{10} \\ 0 & 0 & 0 & \frac{217}{74} \end{bmatrix}$

L = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & 1 & 0 \\ -\frac{1}{6} & \frac{1}{10} & -\frac{9}{37} & 1 \end{bmatrix}$

Lz = b

⇨ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & 1 & 0 \\ -\frac{1}{6} & \frac{1}{10} & -\frac{9}{37} & 1 \end{bmatrix} \begin{bmatrix} z1 \\ z2 \\ z3 \\ z4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 11 \\ 20 \end{bmatrix}$

⇨ $Z_1 = 2$

⇨ $Z_2 = 3 - \frac{1}{3}(2) = \frac{7}{3}$

⇨ $Z_3 = 11 - \frac{1}{6}(2) - \frac{1}{5}(\frac{7}{3}) = \frac{51}{5}$

⇨ $Z_4 = 20 + \frac{1}{6}(2) - \frac{1}{10}(\frac{7}{3}) + \frac{1}{6}(\frac{51}{5}) = 22.5811$

Ux = z

⇨ $\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & \frac{10}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{37}{10} & -\frac{11}{10} \\ 0 & 0 & 0 & \frac{217}{74} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{7}{3} \\ \frac{51}{5} \\ 22.5811 \end{bmatrix}$

$X_4 = 22.5811 \div \frac{217}{74} = 7.7005$

$X_3 = (\frac{51}{5} - \frac{11}{10}(7.7005)) \div \frac{37}{10} = 5.0461$

$X_2 = (\frac{7}{3} - \frac{2}{3}(5.0461) + \frac{1}{3}(7.7005)) \div \frac{10}{3} = 0.4608$

$X_1 = (2 - 2\,(0.4608) - 1\,(5.0461) + \frac{1}{6}(7.7005) = -1.9447$