We will start with Maxwell's eq^n of EM theory. The eq^{n}s can be read, in terms of electric field $\vec{E}$ & magnetic field $\vec{B}$, as:

① $\vec{\nabla} \cdot \vec{B}(\vec{x},t) = 0$

② $\vec{\nabla} \times \vec{E}(\vec{x},t) = -\dfrac{\partial \vec{B}(\vec{x},t)}{\partial t}$

③ $\vec{\nabla} \times \vec{B}(\vec{x},t) = \mu_0 \vec{J}(\vec{x},t) + \dfrac{1}{c^2}\dfrac{\partial \vec{E}(\vec{x},t)}{\partial t}$

④ $\vec{\nabla} \cdot \vec{E}(\vec{x},t) = \dfrac{\rho(\vec{x},t)}{\epsilon_0}$

Eq^n ① tells us that $\vec{B}$ can be written as $\vec{B} = \vec{\nabla} \times \vec{A}(\vec{x},t)$; $\vec{A}$ is called vector potential of magnetic field.

With this, Eq ② becomes:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{\nabla} \times \vec{A}}{\partial t}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \varphi(\vec{x},t)$$

↳ φ is called scalar potential of electric field.

So, in terms of the scalar & vector potentials φ & $\vec{A}$, $\vec{E}$ & $\vec{B}$ field can be expressed as

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi$$
$$\& \quad \vec{B} = \vec{\nabla}\times\vec{A}.$$

Note that the choice of $\vec{A}$ is not unique.

If $\vec{A}$ gives rise to field $\vec{B}$, then $\vec{A}' = \vec{A} + \vec{\nabla} f(\vec{r},t)$ also provides the same $\vec{B}$ field. $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \underbrace{\vec{\nabla}\times(\vec{\nabla} f)}_{=0}$
$= \vec{B}.$

We get the same $\vec{E}$ if we use
$$\varphi' = \varphi - \frac{\partial f}{\partial t}. \text{ along with } \vec{A}'$$

So, the choices of $\vec{A}$ & φ are not unique. Since the scalar & vector potentials $\vec{A}'$ & φ', which are related by some scalar f^n f as

$$\left.\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla} f \\ \varphi' &= \varphi - \frac{\partial f}{\partial t}. \end{aligned}\right\} \quad \cdots\cdots \quad (5)$$

gives rise to the same $\vec{E}$ & $\vec{B}$ field. Hence, they provide equivalent description of the system. In other words, the above transformations of the $\vec{A}$ & φ potentials leaves the system invariant. These transformations is called "gauge transformation" and the symmetry is called "gauge symmetry".

Let us now rewrite the other two eq^{n}s

Eq (3) becomes.

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{\partial}{\partial t}\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi\right) = \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} + \frac{\partial^2}{\partial t^2}\vec{A} + \frac{\partial}{\partial t}\vec{\nabla}\varphi = \mu_0\vec{J}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} + \vec{\nabla}\left(\frac{\partial\varphi}{\partial t} + \vec{\nabla}.\vec{A}\right) = \mu_0\vec{J} \quad \text{-----} (6)$$

Eqn. (4) becomes

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}.$$

$$\Rightarrow \vec{\nabla}.\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi\right) = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow -\nabla^2\varphi - \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi - \frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t} + \vec{\nabla}.\vec{A}\right) = \frac{\rho}{\epsilon_0}. \quad \text{----} (7)$$

These two eqns can be written in a compact notation, if we rewrite the whole set of eqns in a four-vector notation. In the four vector notation, we write the following:

$$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} \;\&\; x_\mu = \eta_{\mu\nu}x^\nu = \begin{pmatrix} t \\ -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix}$$

$$\partial_\mu = \frac{\partial}{\quad} = \begin{pmatrix} \partial_t \\ \vec{\nabla} \end{pmatrix} \;\&\; \partial^\mu = \begin{pmatrix} \partial_t \\ \vec{\nabla} \end{pmatrix}.$$

$$A^\mu = \begin{pmatrix} \varphi \\ \vec{A} \end{pmatrix} \quad \& \quad A_\mu = \begin{pmatrix} \varphi \\ -\vec{A} \end{pmatrix}$$

Combining the scalar & vector potential into four vector form

$$J^\mu = \begin{pmatrix} \rho_0/\epsilon_0 \\ \mu_0 \vec{J} \end{pmatrix} \quad \& \quad J_\mu = \begin{pmatrix} \rho_0/\epsilon_0 \\ -\mu_0 \vec{J} \end{pmatrix}$$

Four current.

Eq (6) & (7) compact form becomes.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = J^\nu$$

$$\Rightarrow \partial_\mu \underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{\equiv F^{\mu\nu}} = J^\nu \quad \text{---- (8)}$$

Let us define the bracketed term as $F^{\mu\nu}$.

This $F^{\mu\nu}$ term is call Electromagnetic field strength tensor. Let us now explicitly find its form.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{---- (9)}$$

$$F^{00} = 0 = F^{ii} \quad (\text{no sum}).$$

$$F^{0i} = -F^{i0} = \partial^0 A^i - \partial^i A^0 = \frac{\partial A^i}{\partial t} + \frac{\partial \varphi}{\partial x^i} = -E^i$$

$$F^{ij} = -F^{ji} = \partial^i A^j - \partial^j A^i = -\frac{\partial A^j}{\partial x^i} + \frac{\partial A^i}{\partial x^j}$$

$$F^{12} = -F^{21} = -\frac{\partial A^2}{\partial x^1} + \frac{\partial A^1}{\partial x^2} = -(\vec{\nabla} \times \vec{A})^3 = -B^3$$

$$F^{23} = -F^{32} = -\frac{\partial A^3}{\partial x^2} + \frac{\partial A^2}{\partial x^3} = -(\vec{\nabla} \times \vec{A})^1 = -B^1$$

$$F^{13} = -F^{31} = -\frac{\partial A^3}{\partial x^1} + \frac{\partial A^1}{\partial x^3} = (\vec{\nabla} \times \vec{A})^2 = B^2$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & B^1 & -B^2 & 0 \end{pmatrix}$$

$$\& \quad F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

Note that $F^{\mu\nu}$ & $F_{\mu\nu}$ can be expressed in terms of physically measurable $\vec{E}$ & $\vec{B}$ fields. This means $F^{\mu\nu}$ & $F_{\mu\nu}$ are invariant under gauge tranformation. This can be seen easily:

By noting that the previously introduced gauge transformations in Eq. 5 can be expressed as $A^\mu \to A'^\mu = A^\mu - \partial^\mu f$.

$$\begin{aligned} F^{\mu\nu} \to F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu \\ &= \partial^\mu A^\nu - \partial^\mu \partial^\nu f - \partial^\nu A^\mu + \partial^\nu \partial^\mu f \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}. \end{aligned}$$

Let us now formulate the Lagrangian

which gives rise to the eqn. (8)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu} \quad \cdots\cdots (10)$$

Ex. Find the Euler-Lagrange eqn. of motion for the Lagrangian given in eqn. (10)

The Lagrangian given in eqn. (10) gives two of the maxwell's eqns (3) & (4). How do we get eqns (1) & (2) ?
The answer is that the four-vector field $A_\mu(\vec{x}, t)$ is constructed out of eqns (1) & (2). Therefore these two eqns are satisfied automatically. Or, other words, there is an identity by which these two eqns can be obtained. The identity is called Bianchi identity. For the field strength tensor defined in eqn. (9) satisfies

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\mu\nu} = 0 \quad \cdots (11)$$

Ex. Show that the eqns (1) & (2) can be obtained from the Bianchi Identity.

<u>Conserved Current in the theory:</u>

The Euler-Lagrange eqn becomes

$$\partial_\mu F^{\mu\nu} = j^\nu .$$

$$\Rightarrow \quad \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu$$

But, the LHS is zero since $\partial_\nu \partial_\mu$ is symmetric & $F^{\mu\nu}$ is antisymmetric. This implies $\partial_\nu j^\nu = 0$.

$$\Rightarrow \quad \partial_0 \left(\frac{\rho}{\epsilon_0}\right) + \vec{\nabla} (\mu_0 \vec{J}) = 0$$

$$\Rightarrow \quad \dot{\rho} + \mu_0 \epsilon_0 \, \vec{\nabla} \cdot \vec{J} = 0$$

In our unit choice, $\mu_0 \epsilon_0 = \frac{1}{c^2} = 1$. So, we recover the electromagnetic continuity eq^n:

$$\dot{\rho} + \vec{\nabla} \cdot \vec{J} = 0 .$$

Gauge Invariance:

Let us now come back to the gauge invariance.

Let us first write down the free field Lagrangian by expanding it out:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$= -\frac{1}{4} \left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right] \times 2$$

- Let us collect the terms in the Lagrangian with time derivative:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{4} F^{0i} F_{0i} - \frac{1}{4} F^{i0} F_{i0} - \frac{1}{4} F^{ij} F_{ij}$$

$$\rightarrow -\frac{1}{4} F^{0i} F_{0i} \times 2$$

$$= -\frac{1}{2} (\partial^0 A^i - \partial^i A^0)(\partial_0 A_i - \partial_i A_0)$$

$$\rightarrow +\frac{1}{2} \partial^0 A^i \partial_0 A^i = \frac{1}{2} \dot{A}^i \dot{A}^i .$$

Notice that the Lagrangian does not have $\dot{A}^0$ term. Meaning that $\dot{A}^0$ remains unevolved.

So, $\vec{\nabla} \cdot \vec{E} = 0$

$$\Rightarrow \nabla^2 A_0 + \vec{\nabla} \cdot \dot{\vec{A}} = 0 .$$

$$\Rightarrow \nabla^2 A_0 = -\vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} .$$

$$A_0(\vec{x}) = \int d^3x' \, \frac{\nabla \cdot \frac{\partial \vec{A}}{\partial t}}{4\pi |\vec{x} - \vec{x}'|} .$$

So, $A_0(\vec{x})$ is not independent. It appears through $\vec{A}$.

* Secondly we can make use of the gauge freedom that we have & ~~fix one~~ remove one of the degrees of freedom.

We can always choose $\vec{A}$ s.t.

$$\vec{\nabla}\cdot\vec{A} = 0.$$

If $\vec{\nabla}\cdot\vec{A} = f \neq 0$

then $\vec{A}' = \vec{A} + \vec{\nabla} f$ will satisfy

$\vec{\nabla}\cdot\vec{A}' = 0$. & we can work with $\vec{A}'$.

This also says from the previous expression that $A_0 = 0$,

With this we can ~~specify~~ say that the photon field A_μ has two degrees of freedom.

So, If $\vec{A}$ is expanded

$$\vec{A} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \epsilon(\vec{p})\, e^{ip\cdot x}.$$

then $\vec{\nabla}\cdot\vec{A} \Rightarrow \vec{p}\cdot\epsilon(\vec{p})$. So, only transverse modes survive.

Polarization :

One can choose $\vec{\epsilon}_1(\vec{p})$ & $\vec{\epsilon}_2(\vec{p})$ be two basis vector along with $\epsilon(\vec{p})$ can be expressed.

$\vec{\epsilon}_1(\vec{p})$ & $\vec{\epsilon}_2(\vec{p})$ are two polarization vectors.

It can be chosen s.t.

$$\left[\begin{array}{l} \vec{\epsilon}_r(\vec{p}) \cdot \vec{\epsilon}_s(\vec{p}) = \delta_{rs} \text{ along with} \\ \vec{p} \cdot \vec{\epsilon}_r(\vec{p}) = 0 \end{array}\right.$$

QED Lagrangian.

Let us now couple A_μ ~~fied~~ field with a fermion, e.g. electron. The Lagrangian for a fermion (as introduced by Rahul) can be written as $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$.

If we now write

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e A_\mu \bar{\psi}\gamma^\mu\psi.$$

It appears that the Lagrangian is invariant under the following transformation.

$$A_\mu \to A'_\mu - \partial_\mu \lambda$$

$$\psi \to e^{ie\lambda}\psi.$$

To see this, let us take the second term

2nd term $\bar{\Psi}(i\not{\partial}-m)\Psi \to \bar{\Psi}'(i\not{\partial}-m)\Psi'$

$$\cancel{= \bar{\Psi} e^{ie\lambda(x)}\left[-e\gamma^\mu(\partial_\mu\lambda)+i\not{\partial}-m\right]}$$

$$= \bar{\Psi} e^{-ie\lambda(x)} \, e^{ie\lambda(x)}\left[-e\gamma^\mu(\partial_\mu\lambda)+i\not{\partial}-m\right]\Psi$$

$$= \bar{\Psi}(i\not{\partial}-m)\Psi \underbrace{- e\,\partial_\mu\lambda\,\bar{\Psi}\gamma^\mu\Psi}_{\hookrightarrow \text{ extra term}}.$$

third term

$$-eA_\mu\bar{\Psi}\gamma^\mu\Psi \to -eA'_\mu\bar{\Psi}'\gamma^\mu\Psi'$$

$$= -e(A_\mu-\partial_\mu\lambda)\,\bar{\Psi}e^{-ie\lambda(x)}\gamma^\mu e^{ie\lambda(x)}\Psi$$

$$= -eA_\mu\bar{\Psi}\gamma^\mu\Psi + \underbrace{e\,\partial_\mu\lambda\,\bar{\Psi}\gamma^\mu\Psi}_{\hookrightarrow \text{ extra term}}.$$

The extra terms in 2nd & 3rd terms precisely cancel each other & leave the Lagrangian invariant.

A more smarter way to do it is by defining covariant derivate as follows:

$$D_\mu\Psi = \partial_\mu\Psi + ieA_\mu\Psi.$$

then after the gauge transformation,

$$D_\mu\Psi \to D'_\mu\Psi' = \partial_\mu(e^{ie\lambda(x)}\Psi) + ie(A_\mu-\partial_\mu\lambda)e^{ie\lambda(x)}\Psi$$

$$= e^{ie\lambda(x)}\left[\cancel{ie(\partial_\mu\lambda)} + \partial_\mu + ieA_\mu - \cancel{ie\partial_\mu\lambda}\right]\Psi$$

$$= e^{ie\lambda(x)} D_\mu\Psi.$$

So, $D_\mu\Psi$ transforms as Ψ transforms & hence

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}(i\not{D}-m)\Psi$$ remains

invariant under gauge transformations.