Problem Sheet 5

1. Prove the "Fundamental Theorem of Quantum Field Theory": if A is an $N \times N$ matrix, and J an N-component vector, then

$$\int \prod_{i=1}^{N} dx^i \exp\left(-\frac{1}{2}x^T A x + J^T x\right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \exp\left(+\frac{1}{2}J^T A^{-1} J\right) \quad (1)$$

Hint: diagonalize A to reduce the problem into a series of one-dimensional integrals.

2. **Optional problem: the quantum harmonic oscillator at finite temperature**

Here we will study the simple harmonic oscillator at finite temperature by evaluating some functional determinants. We will study the quantum harmonic oscillator with Hamiltonian and Lagrangian:

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 \qquad L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 \quad (2)$$

(a) The fundamental object in quantum statistical mechanics is the *partition function*, defined as

$$Z(\beta) = \mathrm{Tr}\exp(-\beta H) \quad (3)$$

where H is the Hamiltonian of the quantum mechanical system. Use properties of the path integral to convince yourself that the partition function can be computed by the following path integral over *compact* Euclidean time τ:

$$Z(\beta) = \int_{q(0)=q(\beta)} [\mathcal{D}q] \exp\left[-\int_0^\beta d\tau \left(\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{1}{2}\omega^2 q^2\right)\right] \quad (4)$$

where compact Euclidean time means that $\tau \in [0, \beta]$, and the field $q(\tau)$ satisfies *periodic* boundary conditions, i.e. $q(0) = q(\beta)$. Thus we are doing the functional integral over fields defined on a circle of length β.

(b) This functional integral can be evaluated using the "fundamental theorem" above to be the functional determinant

$$Z(\beta) \propto \left[\det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)\right]^{-\frac{1}{2}} \quad (5)$$

where I have omitted ω-independent factors of $(2\pi)^\infty$. Evaluate this determinant and compute the ω dependence of the partition function. (Getting the β dependence exactly right is tricky, but the ω dependence is unambiguous). You may find the identity

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right) \tag{6}$$

useful.

Compare your result to the usual partition function computed via canonical methods (i.e. by directly calculating the sum over the eigenstates of the simple harmonic oscillator in (3)).

3. Consider a real scalar field $\phi(x)$ with the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2. \tag{7}$$

The path-integral expression for the generating functional $Z[J]$ in the presence of an external source $J(x)$:

$$Z[J] = \int \mathcal{D}\phi \; e^{i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + J\phi\right]}. \tag{8}$$

Compute the Feynman propagator $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$ by evaluating the path-integral in the limit of $J(x) \to 0$. Use the result to derive the expression for the propagator in momentum space:

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}. \tag{9}$$

Using the generating functional above, compute

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle \tag{10}$$

4. **Optional problem:** We did not discuss Grassman integrals in this course; however they are the way to do path integrals for *fermions*, and are discussed in many places, e.g. Peskin and Schroeder Chapter 9, or in these lecture notes, Chapter 5. After reading that you can tackle this problem.

Consider a Dirac field $\psi(x)$ with the Lagrangian density:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{11}$$

The path-integral expression for the generating functional $Z[J, \bar{J}]$ in the presence of external sources $J(x)$ and $\bar{J}(x)$:

$$Z[J, \bar{J}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; e^{i \int d^4x \left[\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \bar{J}\psi + \bar{\psi}J\right]}. \tag{12}$$

Compute the Feynman propagator $\langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle$ by evaluating the path-integral in the limit of $J(x) \to 0$ and $\bar{J}(x) \to 0$. Use the result to derive the expression for the propagator in momentum space:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}, \tag{13}$$

where $\not{p} = \gamma^\mu p_\mu$.

Using the generating function above, compute

$$\langle 0|T\{\psi(x_1)\bar{\psi}(x_2)\psi(x_3)\bar{\psi}(x_4)\}|0\rangle \tag{14}$$

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