Spontaneous symmetry breaking

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Goal: describe the SM. This is a relativistic gauge QFT featuring scalars, fermions and vector fields in three different phases:

- Coulomb phase.
- Higgs phase (SSB) (*)
- Confined phase (QCD at low energy).

$\Big\}$ Today

(*) Spontaneous symmetry breaking

Quantum Mechanics

Consider a system described by a Hamiltonian H and a Hilbert space $\mathcal{H}$. The unitary operator U implements

$U^{-1} = U^{\dagger}$ $\qquad$ $U : \mathcal{H} \to \mathcal{H}$.

a symmetry iff:

$$H = U H U^\dagger \qquad ([H, U] = 0)$$

This is useful because:

- Degeneracy: consider two states

  $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$

  related by a symmetry transformation:

  $$|\psi_2\rangle = U|\psi_1\rangle$$

- Claim: $|\psi_2\rangle$ and $|\psi_1\rangle$ have the same energy (are degenerate):

- Proof: let $H|\psi_1\rangle = E_1|\psi_1\rangle$,

  $$H|\psi_2\rangle = E_2|\psi_2\rangle$$

$$E_2|\psi_2\rangle = H|\psi_2\rangle \underset{\substack{\uparrow \\ H = UHU^\dagger}}{=} UHU^\dagger|\psi_2\rangle = UH\underbrace{U^\dagger U}_{\mathbb{1}}|\psi_1\rangle =$$

$$= UH|\psi_1\rangle = E_1 U|\psi_1\rangle = E_1|\psi_2\rangle$$

The equality holds for $|\psi_2\rangle \neq 0$ iff $E_1 = E_2$ $\square$

- Selection rules:

$[U, H]=0 \to$ can choose a basis that diagonalises both operators:

$$U|\psi\rangle = u|\psi\rangle\,; \quad H|\psi\rangle = E|\psi\rangle$$

- Claim: time evolution from an eigenstate of U cannot map to states with different U-eigenvalues.

- Proof: in Schrödinger picture:

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle \to |\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$$

If $U|\psi_0\rangle = u|\psi_0\rangle$, then:

$$U|\psi(t)\rangle = U e^{-iHt}|\psi_0\rangle \underset{[U,H]=0}{\overset{\uparrow}{=}} e^{-iHt}U|\psi_0\rangle = u|\psi(t)\rangle$$

$\square$

$\therefore$ Matrix elements involving states with different U-eigenvalue are <u>forbidden by symmetry</u>: a selection rule.

(Small breaking of symmetries due to new physics allows for these transitions with small probability. Observing such transitions is a signal of new physics. eg: proton decay)

- Conservation laws:

A symmetry is <u>continuous</u> if U can be written as:

$$U = U(\theta) = e^{i\theta Q} \quad ; \quad \theta \in \mathbb{R}$$

If so, Q is hermitian ($Q = Q^{\dagger}$) and therefore a QM observable. Note

$$[U, H] = 0 \rightarrow [Q, H] = 0$$

So the eigenvalues of Q are also conserved under time evolution →

→ a conserved quantity.

Small breaking of a symmetry:

$$H = H_0 + \lambda \cdot H_b \quad ; \quad \lambda \ll 1$$

Symmetric under parity: $\vec{x} \to -\vec{x}$ (for H_0)

not symmetric under parity (for H_b)

This is an explicit breaking of the symmetry & if $\lambda \ll 1$ we do perturbation theory: Diagonalise H_0 & treat H_b perturbatively. Consider $|+\rangle$ $|-\rangle$ States even/odd under parity.

$\langle - | e^{-iHt} | + \rangle = O(\lambda)$ because λ breaks the symmetry explicitly.

(More info: Weinberg's QM book).

Field theory

Consider canonical quantization:

quantize fluctuations of fields with respect to a given background (typically spacetime independent). (*)

We quantize in interaction picture, morally "many simple harmonic oscillators".

A symmetry in field theory is a transformation on fields (internal) and/or spacetime coordinates (external) that leave the action invariant.

(*) $\varphi = \underbrace{\langle \varphi \rangle}_{} + \delta\varphi$ ← fluctuations to quantize

$\langle \varphi \rangle \neq \langle \varphi \rangle_{(x)}$

Other cases: Burgess' EFT book.

Consider an internal symmetry:

$$\varphi \to u \varphi u^{\dagger} = \tilde{\varphi}$$

Recall $\varphi \sim a_{\vec{k}} + a^{\dagger}_{\vec{k}}$

$$\therefore \; \tilde{a}_{\vec{k}} = u a_{\vec{k}} u^{\dagger}$$

Define the state

$$|\tilde{k}\rangle \equiv \tilde{a}^{\dagger}_{\vec{k}} |0\rangle$$

If $|k\rangle \equiv a^{\dagger}_{\vec{k}} |0\rangle$ satisfies $|\tilde{k}\rangle = u|k\rangle$

the above (QM) discussion applies.

When is that true:

$$|\tilde{k}\rangle = \tilde{a}^{\dagger}_{\vec{k}}|0\rangle = u a^{\dagger}_{\vec{k}} \underbrace{u^{\dagger}|0\rangle}_{|0\rangle} \overset{?\checkmark}{=} u|k\rangle$$

The equality is true iff

$\rightarrow$ $u|0\rangle = |0\rangle$ (Symmetry is manifest)

Otherwise the symmetry is <u>spontaneously broken</u>.

The statement $U|0\rangle \neq |0\rangle$ is the condition of SSB. But this is not the most useful way to look at things (for our current purposes), because we have been studying

$$\varphi \to e^{i\alpha^a T^a} \varphi \qquad (1)$$

which is not of the form

$$\varphi \to U \varphi U^{+}. \qquad (2)$$

Goal: make a statement about SSB in terms of T^a (the generators of the symmetry).

(1) = (2) & expand to first order writing $U = e^{iQ^a}$, find:

$$\boxed{[\varphi, Q] = i\alpha^a T^a \varphi} \qquad (3)$$

Now take the vacuum expectation value of (3):

$$\langle 0|T^a\varphi|0\rangle \equiv \langle T^a\varphi\rangle = \langle 0|\varphi Q - Q\varphi|0\rangle$$

Note the SSB condition in terms of Q is $Q|0\rangle \neq 0$ $\left(U|0\rangle = e^{iQ}|0\rangle\right)$

$$\therefore \boxed{SSB \leftrightarrow \langle T^a\varphi\rangle \neq 0}$$

- Example: the $U(1)$ system.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi^*\partial^\mu\varphi \underbrace{-\frac{\lambda}{4}\left(\varphi^*\varphi - v^2\right)^2}_{V(\varphi^*\varphi)}$$

$\lambda > 0$, $v \in \mathbb{R}$, φ is a complex scalar field.

This theory has a global $U(1)$

$$\varphi \to e^{i\alpha}\varphi \; ; \quad \alpha \in \mathbb{R}$$

but the symmetry is spontaneously

broken if $v \neq 0$.

- Identify the background: ← (ie: solve the eom)

$$\Box \bar{\varphi} = -V'(\bar{\varphi}) \approx (\bar{\varphi}^* \bar{\varphi} - v^2)\varphi \Leftarrow$$

A homogeneous static solution $\Box \bar{\varphi} = 0$

so $|\varphi|^2 = v^2$!! (4)

- Consider fluctuations: fluctuations to quantize

$$\to \quad \varphi = \langle \varphi \rangle + \delta\varphi = v + \frac{1}{\sqrt{2}}(\underline{\varphi_1} + i\,\underline{\varphi_2})$$

To derive the Lagrangian for (φ_1, φ_2):

$$\partial_\mu \varphi = \frac{1}{\sqrt{2}}\partial_\mu \varphi_1 + i\frac{1}{\sqrt{2}}\partial_\mu \varphi_2$$

$$\varphi^* \varphi - v^2 = \frac{2v\varphi_1}{\sqrt{2}} + \frac{v^2}{2}(\varphi_1^2 + \varphi_2^2)$$

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int}$$

$$\mathcal{L}_{kin} = -\frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{\lambda v^2}{2}\underline{\varphi_1^2}$$

So have a massless and a massive $m^2 = \lambda v^2$ real scalars.

$\mathcal{L}_{int}$ = exercise!

<u>Note</u> If $v=0$ the symmetry is manifest and the states are degenerate. If $v \neq 0$ SSB & the degeneracy is <u>lifted</u>.

To see this recall

$$\langle T^a \varphi \rangle \neq 0 \qquad (SSB)$$

where $\varphi \to e^{i\alpha^a T^a} \varphi$

In our case,

$$e^{i\alpha^a T^a} = e^{i\alpha} \to T^a = 1$$

$\langle \varphi \rangle = v$ by eq. (4), so for $v \neq 0$ there is SSB.

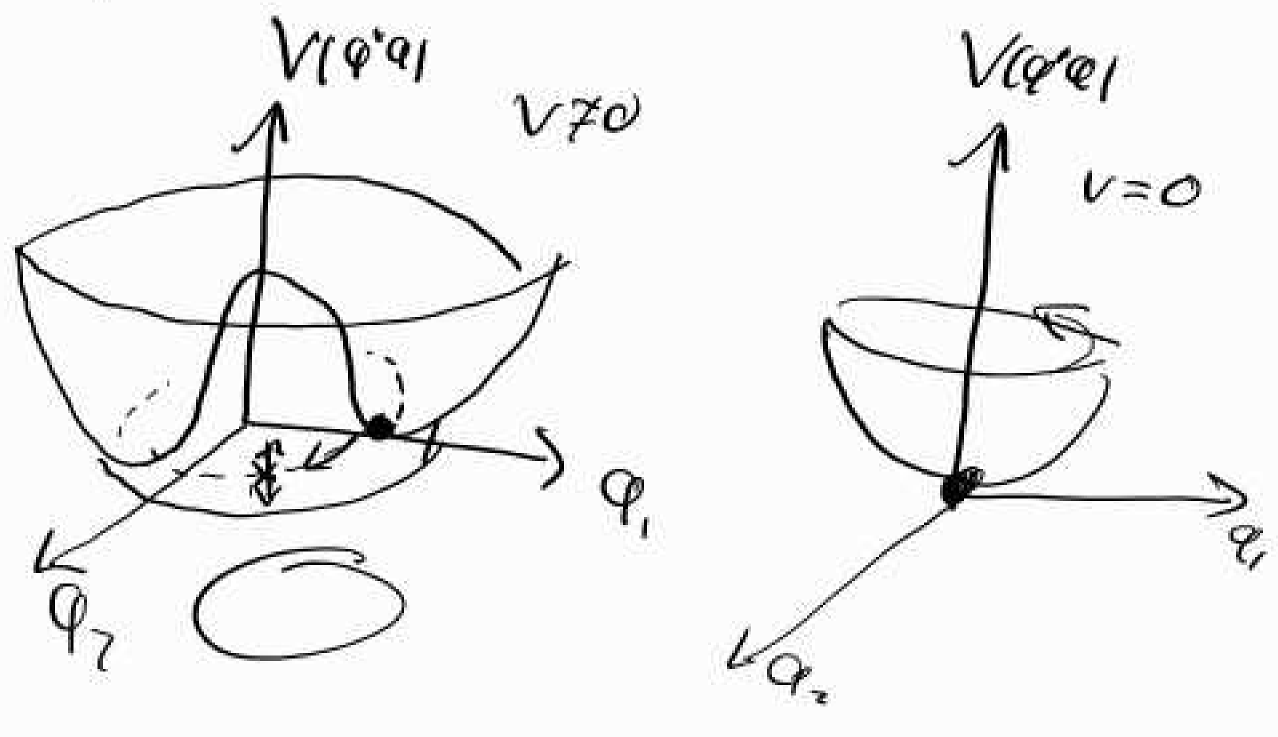
So the usual QM intuition doesn't hold. Does this mean SSB symmetries are not important? A: <u>NO</u>

Goldstone's theorem:

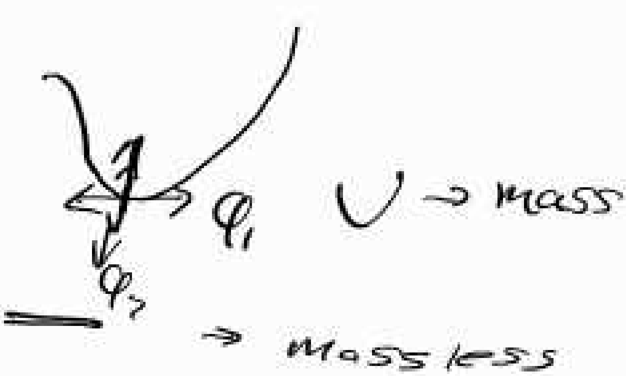
Any system with SSB of a continuous global symmetry has a massless state (Goldstone boson)

- Proof: exercise!

U(1) potential:



$$m_\varphi^2 = \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\langle\varphi\rangle}$$



Spontaneous breaking of gauge syms

SSB of gauge symmetries leads to the Higgs mechanism, a key ingredient in the SM.

Consider the Abelian-Higgs model:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\varphi^* D^\mu\varphi - V(\varphi^*\varphi)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu = \partial_\mu + ieA_\mu$$

$$V(\varphi^*\varphi) = -\frac{\lambda}{4}|\varphi^*\varphi - v^2|^2.$$

The system enjoys an $U(1)$ gauge symmetry: infinitesimally:

$$\varphi \to \varphi + i\alpha\varphi$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha \quad ; \quad e > 0$$

Dof counting: 1 massless vector $\to$ 2 dof

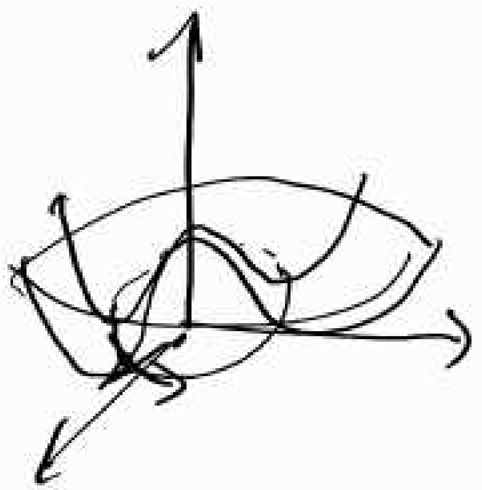
degree of freedom

1 complex scalar $\to$ $\underbrace{2 \text{ dof}}_{4 \text{ dof}}$

• Identify the background:

$$\langle |\varphi|^2 \rangle = v^2$$

• Quantize fluctuations:



$$\varphi = \left(v + \frac{\eta(x)}{\sqrt{2}}\right) e^{i\frac{\xi(x)}{\sqrt{2}v}}$$

(Numerical constants for normalization)

Exercise: check that the symmetry transformation only acts on ξ and ξ would be the Goldstone boson in the global U(1) system.

To work out $\mathcal{L}$ of fluctuations:

$$D_\mu \varphi = \left(\frac{\partial_\mu \eta}{\sqrt{2}} + i\left(v + \frac{\eta(x)}{\sqrt{2}}\right)\left(\frac{\partial_\mu \xi}{\sqrt{2}v} + eA_\mu\right)\right) e^{\frac{i\xi}{\sqrt{2}v}}$$

$$D_\mu \varphi^* D^\mu \varphi = \frac{1}{2}\partial_\mu \eta \, \partial^\mu \eta +$$

$$+ \frac{1}{2}\left(\partial_\mu \xi + \sqrt{2}evA_\mu\right)^2 \left(1 + \frac{\eta(x)}{\sqrt{2}v}\right)$$

$$V(\varphi^*\varphi) = \frac{\lambda}{4}\left(\frac{\eta(x)^2}{2} + \sqrt{2}v\,\eta(x)\right)^2$$

Note: $\xi(x)$ only appears in the combination

$$\partial_\mu \xi + \sqrt{2}evA_\mu \longrightarrow A_\mu$$

and so can be redefined as A_μ by an appropriate choice of gauge (do it!!)

(unitary gauge)

The Lagrangian thus reads:

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int}$$

$$\mathcal{L}_{kin} = \frac{1}{2}\partial^\mu\eta\,\partial_\mu\eta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}v^2\eta^2$$

$$- e^2v^2A_\mu A^\mu \quad !!!$$

$$\mathcal{L}_{int} = -\frac{1}{16}\eta^4_{(x)} - \frac{1}{4}\sqrt{2}\,\eta^3_{(x)} - e^2v^2A_\mu A^\mu\left(\frac{\sqrt{2}}{v}\eta + \frac{\eta^2}{2v^2}\right)$$

This is the theory of a real scalar coupled to a massive vector field, with no obvious symmetries.

DoF: $\underset{\substack{\uparrow \\ \text{massive} \\ \text{vector}}}{3} + \underset{\substack{\uparrow \\ \text{real} \\ \text{scalar}}}{1} = 4$

Higgs mechanism: upon SSB of a gauge symmetry, the would-be Goldstone boson becomes the longitudinal component of the gauge vector.

"The vector field <u>eats up</u> the Goldstone boson"

<u>Note</u> this happened in a given gauge but the physics is gauge invariant, this gauge is useful to identify the dof but the computations can be done in any gauge: Goldstone-equivalence theorem (Peskin-Schroeder ch. 21).

Non-abelian breaking

Non-abelian groups have several generators & syms can be partially broken to subgroups of the original sym. group.

↑

a group contained in a larger group. eg: rotations along axis $SO(2)$ are a subgroup of rotations in 3D $SO(3)$.

If the transformation is generated by T^a:

$$\varphi \to e^{i\alpha^a T^a} \varphi$$

$$\varphi_i \to \varphi_i + i\alpha^a (\underline{\underline{T}}^a)_{ij} \varphi_j$$

then the subgroup generated by T^a so that $\langle T^a \varphi \rangle = 0$ is preserved.

- $SO(3) \to SO(2)$.

$\to \mathcal{L} = \frac{-1}{2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) - D_\mu \varphi^\dagger \cdot D^\mu \varphi - V(\varphi^\dagger \varphi)$

Now $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is a triplet of real scalars, and the generators are

$$(T^i)_{jk} = -i\, \underbrace{\varepsilon_{ijk}}_{\text{Levi-Civita}}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g A_\mu^b A_\nu^c f^{abc}$$

$$\vec{F}_{\mu\nu} = F_{\mu\nu}^a T^a$$

$$D_\mu \varphi_i = \partial_\mu \varphi_i - i e A_\mu^a (T^a)_{ij} \varphi_j$$

Dof: recall the defining rep. of $SO(3)$

has 3 generators. $\to$ 3 massless

vectors $\to$ 6 degrees of freedom $\Big\}$ 9 dof.

Three real scalars $\to$

· Background: $\langle \varphi^+ \cdot \varphi \rangle = v$

Without loss of generality:

$$\varphi = e^{\frac{i}{v}(\eta^1 T^1 + \eta^2 T^2)} \begin{pmatrix} 0 \\ 0 \\ v+\eta \end{pmatrix}$$

Choose unitary gauge:

$$\varphi \to e^{-\frac{i}{v}(\eta^1 T^1 + \eta^2 T^2)} \varphi = \begin{pmatrix} 0 \\ 0 \\ v+\eta \end{pmatrix}$$

The kinetic part of the resulting Lagrangian is:

$$\mathcal{L}_{kin} = \frac{-1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} \partial_\mu \eta \, \partial^\mu \eta +$$

$$- \frac{g^2 v^2}{2} \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu} \right) - \frac{m^2}{2} \eta^2_{(x)}$$

Which features a massive real scalar (Higgs) η, two massive vectors A^1, A^2 & a massless vector A^3.

Symmetry is only partially broken!

DoF: $1+3+3+2=9$ ✓

Check: $\langle T^a \varphi \rangle = 0$?

T^a in this case are the generators of angular momentum:

$$\langle T^1 \varphi \rangle = \langle L_x \varphi \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -iv \\ 0 \end{pmatrix} \neq 0$$

(broken along T^1)

$$\langle T^2\varphi\rangle = \langle L_y\varphi\rangle = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} iv \\ 0 \\ 0 \end{pmatrix} \neq 0$$

(broken along T^2).

$$\langle T^3\varphi\rangle = \langle L_z\varphi\rangle = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = 0 \quad !!!$$

Remaining $SO(2)$ symmetry.

$$F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \leftarrow$$

$\partial_\mu A_\nu$ ← 1 power of energy

$\frac{\partial}{\partial x} \to$ 1 power of energy

$$S = \int \underbrace{d^4x}_{-4 \text{ powers of energy}} \overbrace{\mathcal{L}}^{4 \text{ powers of energy}}$$

$F_{\mu\nu}F^{\mu\nu} \to$ 4 powers of energy

$F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \to$ 8 powers of energy by dimensional analysis,

$$\mathcal{O} = \frac{1}{\Lambda^4} F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$ 4 powers of energy

Eventually, S-matrix elements will read

$$P_{\mathcal{O}} \sim |\langle \psi_1 | \mathcal{O} | \psi_2 \rangle|^2 \to \left|\frac{E}{\Lambda}\right|^8$$

E energy of the process.

If $E \sim \Lambda$ out of business

$E \ll \Lambda$ can ignore that term

EFFECTIVE FIELD THEORY

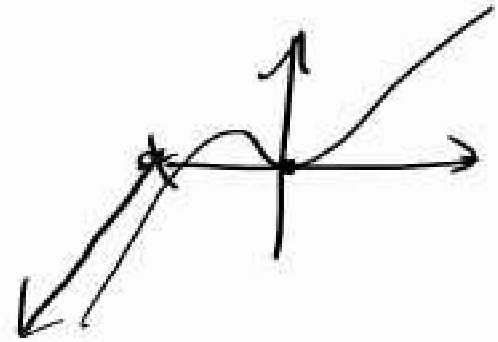
$F_{\mu\nu} F^{\mu\nu}$ ← dim. 4

Scalar: $|\partial\varphi|^2$ ← dim 4 $D\varphi$

$m^2\varphi^2$ ← dim 4

φ^4 ← dim 4 $\cancel{\varphi^6}$.

q^3



q^2, q^4

Lecture 2

SSB & chiral fermions

The SM is <u>chiral</u> in the sense that it treats differently left and right-handed fermions. The natural language to describe this is in terms of <u>Weyl spinors</u>.

In Rahul's lecture, we derived the Dirac equation by looking for a "relativistic Schrödinger equation".

→ Dirac spinors, which have nice properties under Lorentz transformations

Λ:

$$\psi(x) \xrightarrow{\Lambda} \underbrace{S(\Lambda)}_{\exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)} \psi(\Lambda^{-1}x)$$

$$S^{0i} = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad S^{ij} = \frac{\varepsilon^{ijk}}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

The block-diagonal form suggests that the Dirac spinor is composed of two, more fundamental objects: <u>Weyl spinors</u>.

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$$

where $\psi_{L,R}(x)$ are two-dimensional spinors.

Note that ψ_L, ψ_R have opposite behaviour under boosts S^{0i}, but same under rotations S^{ij}.

In addition, one can show that a parity operation $\vec{x} \to -\vec{x}$ maps $\psi_{L,R}(\vec{x}) \to \psi_{R,L}(-\vec{x})$.

In the modern understanding of QFT in terms of symmetries, (in this case Lorentz invariance) one would naturally find Weyl spinors as fundamental and then <u>build</u> the Dirac action as the most general renormalizable Lagrangian involving such objects.

In terms of Weyl variables:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi =$$

$$= i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R +$$

$$- m(\underbrace{\psi_L^\dagger \psi_R} + \underbrace{\psi_R^\dagger \psi_L})$$

where $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$; $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$.

- Chiral theories:

A gauge theory is chiral if the left and right-handed Weyl fermions transform in different ways under gauge transformations.

▶ Example: (warning: anomalous!)

Consider the $U(1)$ theory of a complex scalar, a charged LH spinor & an uncharged RH spinor. The most general renormalizable lagrangian with this field content and symmetry is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^+ \cdot D^\mu \varphi - V(\varphi^* \varphi)$$

$$+ i \bar{\psi}_L \sigma^\mu D_\mu \psi_L + i \psi_R^+ \bar{\sigma}^\mu \underline{\underline{\partial_\mu}} \psi_R +$$

Yukawa coupling

$$- (y \cdot \psi_L^+ \varphi \cdot \psi_R + \text{hermitian conjugate})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$D_\mu \varphi = \partial_\mu \varphi - ieA_\mu \, ; \quad D_\mu \psi_L = \partial_\mu \psi_L + ieA_\mu \psi_L$$

No covariant derivative for ψ_R!!

There is gauge invariance under:

$$A_\mu \to A_\mu + \partial_\mu \alpha(x) \, ; \qquad \psi_R \to \psi_R$$

$$\psi_L \to e^{+ie\alpha(x)} \psi_L \, ; \quad \varphi \to e^{ie\alpha(x)} \varphi$$

Note a mass term for the fermions is forbidden, since

$$\psi_L^+ \psi_R \to e^{ie\alpha(x)} \psi_L^+ \psi_R$$

Does this change upon SSB? Yes!

Take the usual SSB potential & find (in unitary gauge):

$$\varphi = \left(v + \frac{\eta(x)}{\sqrt{2}}\right) e^{\frac{ih(x)}{\sqrt{2}v}} \xrightarrow[\text{unitary gauge}]{} v + \frac{\eta(x)}{\sqrt{2}}$$

The Yukawa couplings become:

$$y \psi_L^+ \varphi \psi_m \longrightarrow y v \cdot \psi_L^+ \psi_m + y \cdot \frac{\eta(x)}{\sqrt{2}} \psi_L^+ \psi_m$$

(similarly for the hermitian conjugate).

The fermions acquire a mass $m = y \cdot v$!!

(and a coupling to the Higgs).

The kinetic part of $\mathcal{L}$ is:

$$\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - v^2 e^2 A_\mu A^\mu + \rightarrow \text{massive vector}$$

$$+ \frac{1}{2} \partial_\mu \eta \, \partial^\mu \eta - \frac{\lambda v^2}{2} \cdot \eta(x)^2 \rightarrow \text{massive real scalar}$$

$$\left.\begin{aligned} &+ i \psi_L^+ \sigma^\mu D_\mu \psi_L + i \psi_m^+ \bar{\sigma}^\mu \partial_\mu \psi_m \\ &- y v (\psi_m^+ \psi_L + \psi_L^+ \psi_m) \end{aligned}\right\} \text{two massive Weyl spinors}$$

Anomalies

Anomalous symmetries are symmetries in the classical sense $\int d^4x \, \mathcal{L}(x)$ is invariant but are at the quantum level. That is:

$$\int \mathcal{D}_\psi \mathcal{D}_{\bar\psi}\, e^{i\int d^4x \mathcal{L}(x)}$$

the measure $\mathcal{D}_\psi$ transforms non trivially under the symmetry. Chiral theories tend to be anomalous; whether a chiral theory is anomalous depends on the charges of the fields. The SM is anomaly-free.
$SU(2)$ (Burgess-Moore's SM book)

Gauge anomalies render inconsistent theories (loss of unitarity) while global anomalies are useful in understanding the non-perturbative structure of QFTs (cf: Tong's notes on gauge theories & SUSY).

Confined phase & QCD

Quantum chromodynamics

So far: Coulomb phase & Higgs phase
$\downarrow$ unbroken $\qquad$ $\downarrow$ broken

There is one more phase featured in the SM: <u>confined</u> phase.

Facts:

- The only forces we observe at our energy scales are gravity & EM. That we don't observe weak forces at these scales is explained by SSB: mediators get masses → not excited at low energies.
- Attractive forces form bound states. Eg: planets in the solar system, atoms. (in the case of atoms to give electrically neutral states).
- QCD is a gauge theory with $SU(3)$ gauge group: 8 generators (aka gauge bosons), 3 charges (called colours).

- One can argue (cf: Burgess-Moore ch. 8) that the interaction energy of a quark system is negative when the combinations are colourless.

fermions charged under QCD

This suggests that QCD forms bound states, these are colourless → suggests that we only observe colourless objects (bound states) at low energies.

However, we can observe electromagnetic charges in isolation, but not coloured ones. Why?

$\boxed{\text{Confinement hypothesis}}$

"The only energy eigenstates of the QCD Hamiltonian with finite energy are color neutral"

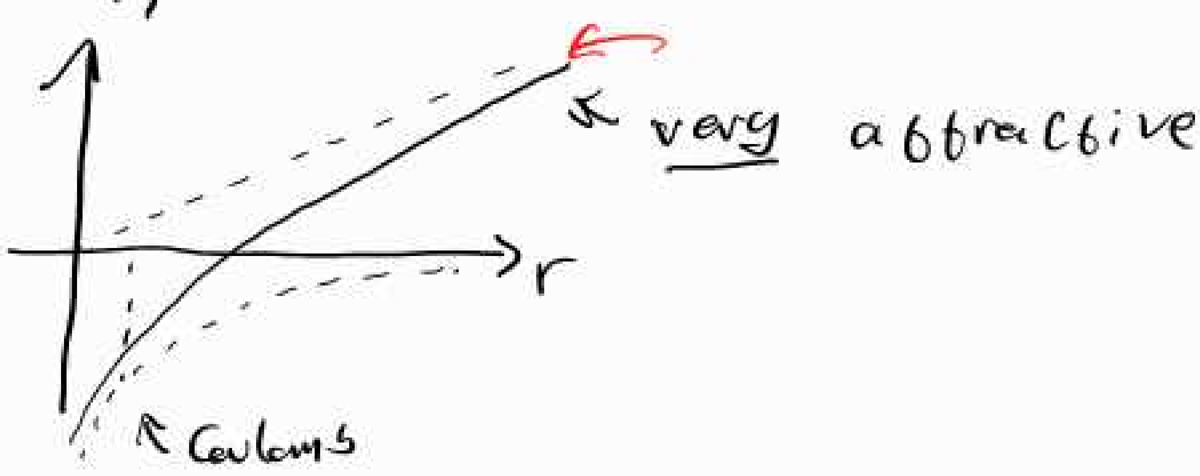
{Related: Yang-Mills existence & mass gap is a Millenium prize problem of the Clay math institute: prove that this finite energy is not zero → 10^6 $!

This means we cannot see coloured states in isolation. Heuristically:

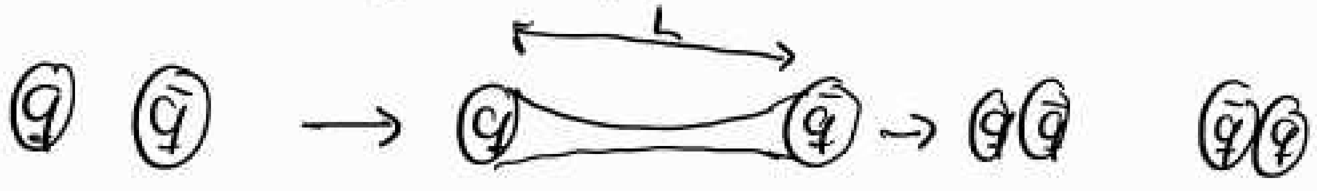
$$V(r) \sim -\frac{\alpha}{r} + \underbrace{\sigma \cdot r}_{???}$$

← Non-perturbative

Coulomb-like (perturbative)



In particular it costs a lot of energy to separate quarks. When this energy gets too large $q\bar{q}$ pairs are nucleated from the vacuum and screen the original pair.



$E \sim 2M_q - \frac{\alpha}{r}$ $\quad$ $E \sim 2M_q + \underline{\sigma} L$ $\quad$ $E \sim 2\left(2M_q - \frac{\alpha}{r}\right)$

How can we understand this σ in QFT?

Asymptotic freedom

- Main idea: coupling constants are <u>energy-dependent</u> (note $E \sim 1/x$ so length-dependent). While the EM coupling decreases with energy the QCD coupling <u>grows</u>. Confinement occurs when the system gets strongly coupled. $\rightarrow$ <u>confined phase</u>.

· The math: write the Yang-Mills action as

$$S = \int d^4x \frac{1}{2g^2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

In the path integral formulation:

$$\int \mathcal{D}_A \, e^{-i\int d^4x \frac{1}{2g^2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})}$$

~ fluctuations around a classical background + curve.

Recall the saddle point approximation:

$$\int dx \, e^{-N f(x)} \xrightarrow[N \gg 1]{} e^{-N f(x_0)} \int dx \, e^{-\frac{N}{2}(x-x_0)^2 f''(x_0)}$$

Saddle point $\leftrightarrow$ fluctuations along a classical background
$g \ll 1$

When $g \sim 1$ all configurations contribute & we don't know what happens! (except in some SUSY cases).

$$V(r) \sim \underset{\text{Coulomb (perturbative)}}{-\frac{\alpha}{r}} + \underbrace{\sigma r}_{\text{Non-perturbative}}$$

→ At some energy scale QCD becomes strongly coupled.

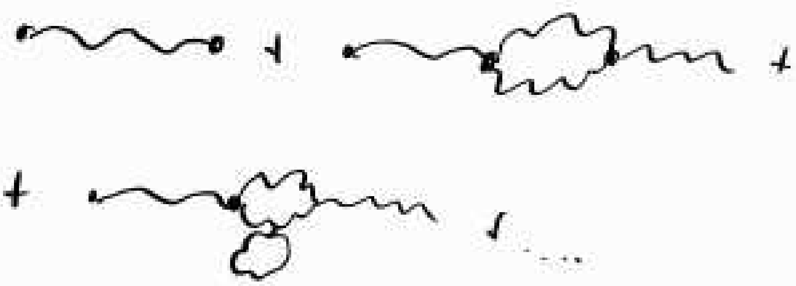
Running couplings

Coupling constants receive energy-dependent quantum corrections.

Recall the kinetic term of Yang-Mills (which gives the propagator):

$$\frac{1}{2g_0^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \to$$ 

This quantity gets quantum corrections:

$$\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \to$$ 

These contributions correct the coupling constant. One can show, at 1-loop for $SU(N_c)$ coupled to N_f fermions:

$$g^2(\mu) = \frac{g^2(\Lambda)}{1 - \frac{g^2(\Lambda)}{(4\pi)^2} \cdot \left(\frac{11}{3} N_c - \frac{2}{3} N_f\right) \log\left(\frac{\Lambda^2}{\mu^2}\right)}$$

With μ, Λ are energy scales.

That is, measure $g(\Lambda) \to$ at energy μ the coupling is different!

Whether at low energies the coupling is stronger depends on the sign of $\frac{11}{3} N_c - \frac{2}{3} N_f$ ($\equiv \beta$ - function)

In particular, $U(1)_{EM}$ gets weaker at low energies & $SU(3)_{QCD}$ gets <u>larger</u>.

Experimentally

$$g_s^2(\Lambda = 90 \text{ GeV}) \simeq 0'12 \cdot 4\pi$$

One can compute the scale at which QCD gets strongly coupled by taking $g(\Lambda_{QCD}) \to \infty$:

$$\Lambda_{QCD} \simeq 200 \text{ MeV}$$

Conclusion: QCD is well described at high energies by a weakly coupled gauge theory with $SU(3)$ gauge group and fermions. At Λ_{QCD} the description breaks down $\to$ non-perturbative treatments (e.g. lattice QCD).

However at $E \ll \Lambda_{QCD}$ there is an effective weakly coupled description: Chiral perturbation theory.

fluctuations along classical background

$$\int \mathcal{D}A\, e^{i\int \frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})} \simeq \underbrace{\sum_{A}}_{\text{classical sols of eom}} \int \mathcal{D}A\, e^{iS_0}$$

$$\langle \dots \rangle \sim \text{perturbative} + \sum_{\text{all other solutions}}$$

Does not admit a Taylor expansion $\rightarrow \underline{\underline{e^{-\frac{1}{g^2}}}}$

A non-perturbative effect:

$$\frac{1}{g^2(\Lambda)} - \frac{1}{g^2(\mu)} = \beta \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

When $g^2(\mu) \to \infty$

$$\mu\,(\equiv \Lambda_{QCD}) = \Lambda\, \underline{e^{-\frac{1}{\beta g^2(\Lambda)}}}$$

More on exact computations: SUSY localization (Mirror symmetry, Vafa et al)

The Standard Model

The SM is a relativistic, gauge chiral QFT with gauge group

$$G = SU(3)_c \times SU(2)_L \times U(1)_Y$$

generators: $\quad G_\mu^A \qquad W_\mu^a \qquad B_\mu$

Coupled to a complex scalar (Higgs field) which spontaneously break

$$SU(2)_L \times U(1)_Y \to U(1)_{EM}$$

& right & left handed fermions (quarks & leptons). We denote the charges as

$$\underset{SU(3)}{(A,}\ \underset{SU(2)}{a,}\ \underset{U(1)}{Y)}$$

Higgs: $(1, 2, \frac{1}{2})$

$$H = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix}$$ with no $SU(3)$ charge

(exercise: check that H_+ is charged under EM & H_0 is not)

$m_H \sim 125$ GeV

The fermionic is repeated in <u>three families</u> with = quantum numbers but ≠ Yukawa couplings (and so masses):

Quarks
$$\begin{cases} Q_L^i = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\} & ; \quad (3, 2, \frac{1}{6}) \\ u_R^i = \{ u_R, c_R, t_R \} & ; \quad (\bar{3}, 1, \frac{2}{3}) \\ d_R^i = \{ d_R, s_R, b_R \} & ; \quad (\bar{3}, 1, -\frac{1}{3}) \end{cases}$$

ie: u_R^i has opposite $SU(3)$ charge but completely different $SU(2) \times U(1)$ behaviour → chirality!

Leptons $\begin{cases} L_L^i = \left\{ \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \right\} ; (1, 2, -\frac{1}{2}) \\ e_R^i = \{ e_R, \mu_R, \tau_R \} \quad ; (1, 1, -1) \end{cases}$

We already know of BSM physics!

- Neutrinos have a mass and this is not allowed in the SM.

Simplest solution: RH neutrinos

$$\nu_R = \{ \nu_{eR}, \nu_{\mu R}, \nu_{\tau R} \} \; ; \; \underline{\underline{(1, 1, 0)}}$$

- Dark matter.
- Cosmological constant problem.
- $\boxed{\text{Gravity}}$ ... String theory?

Detail: θ terms in the Lagrangian

$$\mathcal{L} \supset \theta \, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Resources: in his website

- Fernando Quevedo & Andreas Schachner on the SM.
- Weinberg's books on QFT.
- Cliff Burgess' book on EFT.
- Burgess-Moore SM.
- Schwarz: QFT & SM.

Why trust a theory? Epistemology of fundamental physics

Baryogenesis: needs explanation.

Why there is more matter than antimatter in the Universe?

Sakharov's conditions:

- Out of equilibrium processes.
- ~~Flavor~~ Baryon number violation.
- CP violation.