Problem Sheet 2

1. The Weyl representation of the Clifford algebra is given by:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1}$$

where σ^i are the Pauli matrices. Show that if $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then:

$$\left[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu\right] = 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda. \tag{2}$$

Prove the Following Identities

(a) $\mathrm{Tr}(\gamma^\mu) = 0$

(b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

(c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$

(d) $\left(\gamma^5\right)^2 = 1$

(e) $\mathrm{Tr}(\gamma^5) = 0$

2. The Fourier decomposition of the Dirac operator $\psi(\vec{x})$ and the conjugate field $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^s_{\vec{p}} u^s(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right] \\ \psi^\dagger(\vec{x}) &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{s\dagger}_{\vec{p}} u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}} v^s(\vec{p})^\dagger e^{+i\vec{p}\cdot\vec{x}} \right] \end{aligned} \tag{3}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \left\{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}} \right\} &= (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) \\ \left\{ c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}} \right\} &= (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) \end{aligned}$$

with all other anti-commutators vanishing,

$$\{b^r_{\vec{p}}, b^s_{\vec{q}}\} = \{c^r_{\vec{p}}, c^s_{\vec{q}}\} = \left\{ b^r_{\vec{p}}, c^{s\dagger}_{\vec{q}} \right\} = \{b^r_{\vec{p}}, c^s_{\vec{q}}\} = \ldots = 0$$

Show that these imply that the field and it conjugate momenta satisfy the anticommutation relations,

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \left\{\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\right\} = 0 \\ \left\{\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\right\} &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{4}$$

Show that the quantum Hamiltonian

$$H = \int d^3x \bar{\psi}\left(-i\gamma^i\partial_i + m\right)\psi \tag{5}$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_{s=1}^{2} \left[b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s + c_{\vec{p}}^{s\dagger} c_{\vec{p}}^s\right] \tag{6}$$

3. Solve the exercises mentioned in lectures 3 and 4.

4. **Optional problem:** The purpose of this question is to give you a glimpse into the spin-statistics theorem. This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. Here we'll see what goes wrong if you try to quantize a spin 1/2 field as a boson. Let us start with the field decomposition but this time we choose bosonic commutation relations for the annihilation and creation operators,

$$\begin{aligned} \left[b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\right] &= (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}) \\ \left[c_{\vec{p}}^r, c_{\vec{q}}^{s\dagger}\right] &= -(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}) \end{aligned} \tag{7}$$

with all other commutators vanishing. Note the strange minus sign for the c operators. Now, show that these are equivalent to the commutation relations,

$$\begin{aligned} [\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] &= \left[\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\right] = 0 \\ \left[\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\right] &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{8}$$

Now repeat the calculations similar to the problem above to show that, after normal ordering, the Hamitonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_{s=1}^{2} \left[b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s - c_{\vec{p}}^{s\dagger} c_{\vec{p}}^s\right] \tag{9}$$

This Hamiltonian is not bounded below: you can lower the energy indefinitely by creating more and more c particles. This is the reason a theory of bosonic spin 1/2 particles is sick.

5. **Optional but recommended:** The Dirac equation for a free Dirac field is given by:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0.$$

(a) Using the Weyl representation for the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

(where σ^i are the Pauli matrices) write down the following Dirac (matrix) equation explicitly:

$$i\gamma^0 \frac{\partial}{\partial t}\psi + i\gamma^i \frac{\partial}{\partial x^i}\psi = m\psi.$$

This can be rewritten in Feynman slash notation as:

$$(i\not\partial - m)\psi(x) = 0.$$

(b) Assume a plane wave solution $\psi(x) = u(p)e^{-ip\cdot x}$, substituting in Dirac equation show that:

$$(\not p - m)u(p) = 0,$$

where $p^\mu = (\sqrt{\vec{p}^2 + m^2},\ \vec{p})$ is the four-momentum and $\not p = \gamma^\mu p_\mu$. In Weyl representation, express above equation as:

$$(\gamma^\mu p_\mu - m)\, u(p) = \begin{pmatrix} -m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -m \end{pmatrix} u(p) = 0\,,$$
$$\sigma^\mu = \left(\mathbb{1}, \sigma^1, \sigma^2, \sigma^3\right)\,, \quad \bar{\sigma}^\mu = \left(\mathbb{1}, -\sigma^1, -\sigma^2, -\sigma^3\right)\,.$$

(c) Take trial solutions as

$$u_r(p) = \begin{pmatrix} a p_\mu \sigma^\mu \chi_r \\ b\chi_r \end{pmatrix}\,, \quad \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,.$$

Show that the two constants are related as $b = ma$. Then in order to normalize $u_r(p)$ as discussed in lecture, a can be fixed.

(d) Check that there are two independent solutions $v_r(p)e^{+ip\cdot x}$, $r = 1, 2$ given by

$$v_r(p) = \begin{pmatrix} a p_\mu \sigma^\mu \chi_r \\ -b\chi_r \end{pmatrix}\,, \quad \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,$$

which satisfy $(\not p + m)v_r(p) = 0$ for $b = ma$ and $p^\mu = (\sqrt{\vec{p}^2 + m^2},\ \vec{p})$. Again a can be fixed requiring normalizations for v_r.

Deadline: 14/09/2024
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