Problem Sheet 3

1. The Lagrangian for the electromagnetic vector potential A^μ, in presence of a source current j^μ, is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu, \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu$ is called electromagnetic field strength tensor.

   (a) Find the Euler-Lagrange equation corresponding to this Lagrangian.

   (b) The Euler-Lagrange equation will give rise to two of Maxwell's equations. Recast these equations into Maxwell's equations of electrodynamics.

2. The other two Maxwell's equations can be obtained from an identity present in the formulation of $F_{\mu\nu}$.

   (a) Show that the electromagnetic field strength tensor satisfies the Bianchi identity

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{2}$$

   (b) Obtain the two Maxwell's equations from the Bianchi identity.

3. In the lecture, we saw that the A_μ field has two degrees of freedom. In the Coulomb gauge, this is easy to see since we have the constraints

$$A_0 = 0 \qquad \& \qquad \vec{\nabla}\cdot\vec{A} = 0. \tag{3}$$

The second constraint, in the momentum space, gives the condition

$$\vec{p}\cdot\vec{\epsilon}(\vec{p}) = 0, \tag{4}$$

where $\vec{\epsilon}(\vec{p})$ is called the polarization vector. This means that only the components transverse to the momentum $\vec{p}$ will survive. Therefore, we can pick two orthonormal basis vectors $\vec{\epsilon}_a(\vec{p})$ $(a = 1, 2)$ satisfying Eq. (4) and

$$\vec{\epsilon}_a(\vec{p})\cdot\vec{\epsilon}_b(\vec{p}) = \delta_{ab}. \tag{5}$$

Let the momentum of photon be $\vec{p} = |\vec{p}|\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$. This momentum can be obtained from the momentum $\vec{p'} = \begin{pmatrix}0\\0\\|\vec{p}|\end{pmatrix}$ by a rotation

$$\mathbb{R}(\theta,\phi) = \begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}. \tag{6}$$

In the primed frame, one can choose the two polarization basis vectors as

$$\vec{\epsilon}_1'(\vec{p}') = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \& \qquad \vec{\epsilon}_2'(\vec{p}') = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{7}$$

(a) Obtain the two polarization vectors $\vec{\epsilon}_1(\vec{p})$ and $\vec{\epsilon}_2(\vec{p})$ in the unprimed frame by applying the rotation.

Let us define $\vec{\epsilon}_3(\vec{p}) = \hat{p}$. One can easily see that $\vec{\epsilon}_r(\vec{p})$ $(r = 1, 2, 3)$ forms a complete set of orthonormal basis for three-dimensional momentum space. Therefore, they should obey the completeness relation:

$$\sum_{r=1}^{3} \epsilon_r^i(\vec{p})\epsilon_r^j(\vec{p}) = \delta^{ij}. \tag{8}$$

(b) From this, obtain the completeness relation for the two transverse polarization vectors

$$\sum_{r=1}^{2} \epsilon_r^i(\vec{p})\epsilon_r^j(\vec{p}) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}. \tag{9}$$

4. The free field Lagrangian for Maxwell's theory is written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{10}$$

(a) Show that the conjugate momentum Π^0 and Π^i corresponding to the fields A_0 and A_i

$$\Pi^0 = 0 \qquad \& \qquad \Pi^i = -\dot{A}^i = E^i. \tag{11}$$

In the Coulomb gauge with $A^0 = 0$ and $\vec{\nabla} \cdot \vec{A} = 0$, the theory can be quantized in terms of the two transverse polarizations. The mode expansion can written as

$$\vec{A}(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2|\vec{p}|}} \sum_{i=1}^{2} \left[\vec{\epsilon}_r(\vec{p})\ a_{\vec{p}}^r\ e^{ip.x} + \vec{\epsilon}_r(\vec{p})\ a_{\vec{p}}^{r\dagger}\ e^{-ip.x} \right], \tag{12}$$

with $p^0 = |\vec{p}|$.

(b) Obtain an expression for the conjugate momentum $\vec{\Pi}$.

In this case, the consistent set of commutation relations are

$$[A_i(x), A_j(y)] = \left[\Pi^i(x), \Pi^j(y)\right] = 0, \tag{13}$$

$$[A_i(x), \Pi_j(y)] = i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta^{(3)}(\vec{x} - \vec{y}). \tag{14}$$

(c) Using the commutation relation on the fields, show that the commutation relations for the creation and annihilation operators follow

$$\left[a^r_{\vec{p}},\, a^s_{\vec{q}}\right] = \left[a^{r\dagger}_{\vec{p}},\, a^{s\dagger}_{\vec{q}}\right] = 0, \tag{15}$$

$$\left[a^r_{\vec{p}},\, a^{s\dagger}_{\vec{q}}\right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}). \tag{16}$$

(d) Obtain the normal ordered Hamiltonian

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3} |\vec{p}| \sum_{r=1}^{2} a^{r\dagger}_{\vec{p}} a^r_{\vec{p}}. \tag{17}$$

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