Problem Sheet 6

1. In this question you will prove Goldstone's theorem: any system with Spontaneous Symmetry Breaking (SSB) of a continuous global symmetry has a massless state (Goldstone boson). The proof has two steps.

   - Prove that SSB implies the existence of a state $|G\rangle \sim j^0(x)|0\rangle$ which is not the vacuum, i.e $\langle G|0\rangle = 0$. Here $j^0(x)$ is the time-component of the conserved current $j^\mu(x)$ predicted by Noether's theorem.

     (Hint: recall SSB requires $\langle 0|[Q,\phi]|0\rangle \neq 0$. Relate Q to j and insert a complete set of states between ϕ and j^0 to find that the matrix element must be nonzero).
   - Prove that this state is massless, using $\langle \partial_\mu j^\mu \rangle = 0$.

2. *Electroweak theory.* Consider a toy model of the Weinberg-Salam theory with gauge group $SU(2)\times U(1)$, a complex scalar duplet and two Dirac fermions.

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\phi)^\dagger\cdot D^\mu\phi + i\bar{\psi}\gamma^\mu D_\mu\psi \\ &-\frac{\lambda}{4}\left(\phi^\dagger\cdot\phi - v^2/2\right)^2 - \left(\Gamma_2\bar{\psi}\phi P_R\psi_2 + \Gamma_1\bar{\psi}\phi^c P_R\psi_1 + \text{h.c.}\right)\,,\end{aligned} \tag{1}$$

where:

$$\begin{aligned}&F_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}_\mu\times\mathbf{W}_\nu\,, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,\\ &\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad \phi^c = i\sigma^2\phi^*\,, \qquad P_{L,R} = (1\mp\gamma^5)/2\,,\\ &D_\mu\phi = \partial_\mu\phi + i(g\mathbf{W}_\mu\cdot\boldsymbol{\sigma}/2 + g'B_\mu Y)\phi\,,\\ &D_\mu\psi = \partial_\mu\psi + i(g\mathbf{W}_\mu\cdot\boldsymbol{\sigma}/2 + g'B_\mu y)P_L\psi + ig'B_\mu(y+Y\sigma_3)P_R\psi\,,\end{aligned} \tag{2}$$

Here boldface quantities indicate triplets, for instance

$$\mathbf{W}_\mu = \begin{pmatrix}W^1_\mu\\ W^2_\mu\\ W^3_\mu\end{pmatrix} \tag{3}$$

$\times$ is the standard vector product and $\cdot$ indicates the scalar product. The scalar and fermions are written as a 2x2 vector which indicates their $SU(2)$ transformations (and so are subject to the action of e.g. Pauli matrices σ). In addition, λ, v, Y, y, Γ_1 and Γ_2 are positive numbers (Higgs couplings, hypercharges and Yukawa couplings).

Show that this Lagrangian density is gauge invariant, and rule out the existence of bare mass terms on grounds of gauge invariance. Then show that, upon SSB, mass terms are produced for some of the fields. Identify these fields and compute their masses. In order to have standard kinetic terms, you will have to perform field redefinitions. For instance, you should find

$$m_W^2 W_\mu^\dagger W^\mu + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \,, \tag{4}$$

where

$$W_\mu = \frac{1}{\sqrt{2}}\left(W_\mu^1 - iW_\mu^2\right) \,, \qquad \cos(\theta_W) W_\mu^3 - \sin(\theta_W) B_\mu \,, \tag{5}$$

where the *Weinberg angle* is defined as

$$\cos(\theta_w) = \frac{g}{\sqrt{g^2+g'^2}} \,, \qquad \sin(\theta_W) = \frac{g'}{\sqrt{g^2+g'^2}} \,. \tag{6}$$

How is m_Z/m_W related to θ_W?

3. **Optional: electromagnetism.** In this question you are invited to explore how standard electromagnetism arises from breaking of the electroweak force. Upon solving the previous question, you should have found that a linear combination

$$A_\mu = \cos(\theta_W) W_\mu^3 + \sin(\theta_W) B_\mu \tag{7}$$

remains massless. Understand this in terms of an unbroken symmetry generated by a certain subgroup of $SU(2) \times U(1)$. That is, writing

$$U = e^{i\alpha^a \sigma^a/2 + i\beta Y} \,, \tag{8}$$

(and writing Y as proportional to the 2x2 identity matrix), identify a condition in α^a, β so that $\langle (i\alpha^a\sigma^a/2 + i\beta Y)\phi\rangle = 0$. What are the charges of the fermions under electromagnetism in terms of g and g'?

Deadline: 27/09/2024

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