Problem Sheet 1

Solve any three problems.

1. A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$ where $\eta_{\mu\nu}$ is the Minkowski metric and x^μ is any 4-vector. Show that this implies that

$$\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \,. \tag{1}$$

Use this result to show that an infinitesimal transformation around identity of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{2}$$

($\omega^\mu{}_\nu$ is infinitesimally small) is a Lorentz tranformation when $\omega_{\mu\nu}$ is antisymmetric: i.e. $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

2. The Lagrangian density for a complex scalar field $\Phi(x) = \phi_1(x) + i\phi_2(x)$ is given by

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{\lambda}{2} \left(\Phi^* \Phi\right)^2 \,. \tag{3}$$

Write down the Euler-Lagrange equations of motion (EOM). Verify that the Lagrangian is invariant under the infinitesimal transformations

$$\delta\Phi = i\alpha\Phi \quad , \quad \delta\Phi^* = -i\alpha\Phi^* \,, \tag{4}$$

which is an infinitesimal version of $\Phi \to e^{i\alpha}\Phi$.

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ϕ.

It is also instructive to write $\mathcal{L}$ in terms of the two real scalar fields ϕ_1, ϕ_2. Can you justify the numerical factors appeared in the Lagrangian?

3. Verify Wick's theorem for

   (a) The product of three scalars:

$$\begin{aligned} T\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right) = {}&: \phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right) : + \phi\left(x_1\right)\Delta_F\left(x_2 - x_3\right) \\ &+ \phi\left(x_2\right)\Delta_F\left(x_3 - x_1\right) + \phi\left(x_3\right)\Delta_F\left(x_1 - x_2\right) \end{aligned} \tag{5}$$

   (b) The product of four scalars:

$$T\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\right) \,. \tag{6}$$

4. The Fourier decompositions of a real scalar field and its conjugate momentum are (note that they are at $t = 0$, as compared to the lectures)

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right] \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right] \end{aligned} \tag{7}$$

with (equal time at $t = 0$) commutation relationships

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{8}$$

implying

$$\left[a_{\vec{p}}, a_{\vec{q}}\right] = \left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\right] = 0 \quad \text{and} \quad \left[a_{\vec{p}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{9}$$

   (a) Generalize above analysis for the complex scalar field whose lagrangian is given in problem 2.
   (b) Compute the Feynman propagator for the complex scalar field defined by $\langle 0 | T\left(\Phi(x)\Phi^\dagger(y)\right) | 0 \rangle$.
   (c) Use the mode expansion of the field and derive the propagator in momentum space.
   (d) Lastly, compute the time-ordered product $T\left(\Phi(x)\Phi^\dagger(y)\Phi(z)\right)$.

5. Solve the exercises mentioned in the lectures 1 and 2.

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