Dirac equation: $E = \sqrt{\vec{p}^{\,2} + m^2}$

linearize: $E = \gamma^0\gamma^i p^i + m\gamma^0$, $\gamma^\mu = \{\gamma^0, \gamma^i\}$ are some matrices

$$E^2 = (\gamma^0\gamma^i p^i + m\gamma^0)\cdot(\gamma^0\gamma^j p^j + m\gamma^0)$$
$$= \gamma^0\gamma^i\gamma^0\gamma^j p^i p^j + m(\gamma^0\gamma^i\gamma^0 + \gamma^{0\,2}\gamma^i)p^i + m^2\gamma^{0\,2}$$

$\gamma^{0\,2} = \mathbb{1}$, $\gamma^0\gamma^i\gamma^0 + \gamma^i = 0$, $\gamma^0\gamma^i\gamma^0\gamma^j + \gamma^0\gamma^j\gamma^0\gamma^i = 2\delta^{ij}$

$\therefore \gamma^i\gamma^0 + \gamma^0\gamma^i = 0$, $-\gamma^0(\gamma^0\gamma^i\gamma^0)\gamma^0\gamma^j - \gamma^j\gamma^i = 2\delta^{ij}$

$$\Rightarrow \gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij}$$

$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}} \Rightarrow \gamma^{i\,2} = -\mathbb{1}$

$Tr(\gamma^i) = -Tr(\gamma^0\gamma^i\gamma^0) = -Tr(\gamma^0\gamma^0\gamma^i) = -Tr(\gamma^i) \Rightarrow Tr(\gamma^i) = 0$.

$\gamma^0 = -\gamma^i\gamma^0(\gamma^i)^{-1} \Rightarrow Tr(\gamma^0) = 0$.

As in red box. All other relations written above follow.

Dirac γ-matrices form Clifford algebra.

Hermiticity: $(\gamma^0\gamma^i)^\dagger p^i + m\gamma^{0\dagger} = \gamma^0\gamma^i p^i + m\gamma^0$ for any p^i

$\Rightarrow \gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger}\gamma^{0\dagger} = \gamma^0\gamma^i \Rightarrow \gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0 = -\gamma^i$

Combining, $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

$(\det \gamma^0)^2 = 1$, $(\det \gamma^i)^2 = -1$, γ^0 eigen values are ± 1, γ^i eigen values are $\pm i$

Tr is sum of eigen values.

$\mathrm{Tr} = 0 \Rightarrow$ # of +ve eigen values = # of -ve eigen values.

$\therefore \gamma^\mu$ are even-dimensional matrices.

In 4 spacetime dimensions we need 4 anticommuting γ-matrices.

γ^μ cannot be realized as 2×2 matrices, can be realized as 4×4 matrices.

In lower space-time dimensions, 2×2 γ-matrices are realizable.

Substitute $E \longrightarrow i\frac{\partial}{\partial t}$, $p^i \longrightarrow -i\frac{\partial}{\partial x^i}$,

$$i\frac{\partial}{\partial t}\psi = \gamma^0\gamma^i\left(-i\frac{\partial}{\partial x^i}\right)\psi + m\gamma^0\psi \Rightarrow i\gamma^0\frac{\partial}{\partial t}\psi + i\gamma^i\frac{\partial}{\partial x^i}\psi - m\psi = 0.$$

$$\Rightarrow \boxed{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0} \Rightarrow (i\cancel{\partial} - m)\psi(x) = 0, \quad \cancel{\partial} \equiv \gamma^\mu\partial_\mu$$

$\cancel{p} \equiv \eta_{\mu\nu}\gamma^\mu p^\nu$

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_4(x) \end{pmatrix} \longleftarrow \text{multicomponent}$$

Next, construct a classical field theory whose field equations give Dirac equation.

Dirac field $\psi(x)$: $S[\psi,\bar\psi] = \int d^4x \, \underbrace{\bar\psi(x)(i\gamma^\mu\partial_\mu - m)\psi(x)}_{= \mathcal{L}}$

Vary $\bar\psi$: $\dfrac{\partial\mathcal{L}}{\partial\bar\psi} = (i\gamma^\mu\partial_\mu - m)\psi(x) = 0.$ ——— (a)

Vary ψ: $\dfrac{\partial\mathcal{L}}{\partial\psi} = -m(-\bar\psi(x)) = m\bar\psi(x)$

$\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i(-\bar\psi(x))\gamma^\mu = -i\bar\psi(x)\gamma^\mu$

results of diff. w.r.t. Grassmann variables

ψ EOM: $-i(\partial_\mu\bar\psi)\gamma^\mu = m\bar\psi \Rightarrow -i(\partial_\mu\bar\psi)\gamma^\mu - m\bar\psi = 0.$

why? Indentify, $\bar\psi = \psi^\dagger\gamma^0 \longrightarrow -i\,\partial_\mu\psi^\dagger\gamma^0\gamma^\mu - m\psi^\dagger\gamma^0 = 0.$

$\Rightarrow -i\,\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - m\psi \;\; = 0.$

$\Rightarrow -i\,\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - m\psi^\dagger = 0$ ← nothing but complex conjugation of eqn (a).

Reality of action:

$\mathcal{L}^\dagger = -i(\partial_\mu\psi^\dagger)\gamma^{\mu\dagger}\bar\psi^\dagger - m\psi^\dagger\bar\psi^\dagger = -i(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^{0\dagger}\psi - m\psi^\dagger\gamma^{0\dagger}\psi$

$= -i(\partial_\mu\bar\psi)\gamma^\mu\psi - m\bar\psi\psi = -i\partial_\mu(\bar\psi\gamma^\mu\psi) + i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi$

$= \mathcal{L} - i\partial_\mu(\bar\psi\gamma^\mu\psi)$

spacetime derivatives, not w.r.t. any Grassmann variables

Classical description of Fermi/Dirac fields can be given in terms of anticommuting variables.

$\psi_\alpha, \bar{\psi}_\beta, \partial_\mu \psi_\alpha, \partial_\nu \bar{\psi}_\beta \longrightarrow$ Complex Grassmann variables

Suitable for imposing quantization rules on anticommutators of canonical conjugate quantities $\xrightarrow[\text{leads to}]{}$ Pauli exclusion principle

Grassmann variables: $\theta_1, \ldots, \theta_n$ s.t.

1. $\theta_i \theta_j + \theta_j \theta_i = 0 \Rightarrow \theta_i^2 = 0$

2. $F(\theta_1, \ldots, \theta_n)$ a function is defined by the polynomial

$$F(\theta_1, \cdots, \theta_n) = C^{(0)} + C^{(1)}_i \theta_i + C^{(2)}_{ij} \theta_i \theta_j + \cdots + C^{(n)}_{i_1 \cdots i_n} \theta_{i_1} \cdots \theta_{i_n}$$

$C^{(k)}_{i_1 \cdots i_k} \longrightarrow$ totally antisymmetric, ordinary complex numbers

E.g. $n=1$, $F(\theta) = C_0 + C_1 \theta_1$,

$n=2$, $F(\theta_1, \theta_2) = C_0 + C_1 \theta_1 + C_2 \theta_2 + C_{12} \theta_1 \theta_2$.

3. Differentiation: the variable of differentiation must be brought to the left, take derivative.

E.g. $\frac{\partial}{\partial \theta_2} F(\theta_1, \theta_2) = C_2 - C_{12} \theta_1$.

Forbids existence of terms quadratic in derivatives in the $\mathcal{L}$

leads to $\longrightarrow$ EOM, first order in derivatives $\qquad (\partial_0 \Psi_\alpha)^2, (\partial_i \Psi_\alpha)^2$ etc.

Lorentz covariance, Spin

What transformation property should we assign to Dirac field s.t. the Dirac equation is Lorentz covariant?

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \Psi(x) \longrightarrow \Psi'(x') = L(\Lambda)\Psi(x)$$

$$(i\gamma^\nu \partial_\nu - m)\psi(x) = \left(i\gamma^\nu \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial}{\partial x'^\mu} - m\right) L^\dagger \psi'(x')$$

$$= i\gamma^\nu \Lambda^\mu{}_\nu L^\dagger \partial'_\mu \Psi'(x') - m L^\dagger \psi'(x')$$

$$= L^\dagger \Big(i \underbrace{L \gamma^\nu L^\dagger \Lambda^\mu{}_\nu}_{= \gamma^\mu} \partial'_\mu - m \Big) \psi'(x')$$

$$= L^{-1} (i\gamma^\mu \partial'_\mu - m)\psi'(x')$$

Condition on $L(\Lambda)$ that we get, $\boxed{\Lambda^\mu{}_\nu L \gamma^\nu L^\dagger = \gamma^\mu}$ ——— (a)

$$L^\dagger \cdot (a) \cdot L \Rightarrow \Lambda^\mu{}_\nu \gamma^\nu = L^\dagger \gamma^\mu L$$

Infinitesimal transformations, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$,

$$L(\Lambda) = \mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}.$$

$e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}$ around $\omega_{\mu\nu}\simeq 0$

$$(\delta^\mu{}_\nu + \omega^\mu{}_\nu)\gamma^\nu = \left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(\mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$$\Rightarrow \underbrace{\omega^\mu{}_\nu\gamma^\nu}_{\omega_{\rho\sigma}\eta^{\rho\mu}\gamma^\sigma} = \frac{i}{2}\omega_{\rho\sigma}[S^{\rho\sigma},\gamma^\mu]$$

$$\therefore \quad (\eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho) = i[S^{\rho\sigma},\gamma^\mu]$$

Ex. 9. Check, above condition is solved by $\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]}$.

$S^{\mu\nu}$ matrices also satisfy Lorentz algebra: (as it should, for $L(\Lambda)$ to be in Lorentz group)

$$[\underbrace{M^{\mu\nu}}_{S^{\mu\nu}}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}).$$

$$S^{\mu\nu\dagger} = -\frac{i}{4}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger})$$

Recall, $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^{i}$

$$\left.\begin{aligned}\therefore\ S^{0i\dagger} &= -\frac{i}{4}(\gamma^{i\dagger}\gamma^{0\dagger} - \gamma^{0\dagger}\gamma^{i\dagger}) = -\frac{i}{4}(-\gamma^i\gamma^0 + \gamma^0\gamma^i) = -S^{0i}\\ S^{ij\dagger} &= -\frac{i}{4}(\gamma^{j\dagger}\gamma^{i\dagger} - \gamma^{i\dagger}\gamma^{j\dagger}) = -\frac{i}{4}(\gamma^j\gamma^i - \gamma^i\gamma^j) = S^{ij}\end{aligned}\right\}\ i,j=1,2,3$$

Since S^{0i} are non-hermitian, $\underbrace{e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}}_{L(\Lambda)} \longrightarrow$ non-unitary.

$$(\eta^{\rho\mu}\gamma^{\sigma} - \eta^{\sigma\mu}\gamma^{\rho}) = i[S^{\rho\sigma}, \gamma^{\mu}]$$

$\Rightarrow [S^{ij}, \gamma^0] \propto \eta^{i0}\gamma^j - \eta^{j0}\gamma^i = 0$, as $\eta \rightarrow$ diagonal matrix

$$S^{0i}\gamma^0 + \gamma^0 S^{0i} \propto [\gamma^0, \gamma^i]\gamma^0 + \gamma^0[\gamma^0, \gamma^i]$$
$$= (\gamma^0\gamma^i - \gamma^i\gamma^0)\gamma^0 + \gamma^0(\gamma^0\gamma^i - \gamma^i\gamma^0)$$
$$= \gamma^0\gamma^i\gamma^0 - \gamma^i + \gamma^i - \gamma^0\gamma^i\gamma^0 = -\gamma^i - \gamma^i + \gamma^i + \gamma^i = 0.$$

$\therefore \gamma^0$ commutes with S^{ij} and anticommutes with S^{0i}.

Using which one can show, $\bar{\Psi}(x) \longrightarrow \bar{\Psi}(x) L(\Lambda)^{-1}$

<u>Arguments</u>: $\bar{\Psi}(x) = \psi(x)^{\dagger}\gamma^0 \longrightarrow \psi'(x')^{\dagger}\gamma^0 = \psi(x)^{\dagger} L(\Lambda)^{\dagger}\gamma^0$

$$L(\Lambda)^{\dagger}\gamma^0 \simeq (\mathbb{1} - \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu})^{\dagger}\gamma^0 = (\mathbb{1} + \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger})\gamma^0$$

$$\omega_{\mu\nu}S^{\mu\nu\dagger} \sim \omega_{0i}S^{0i\dagger} + \omega_{ij}S^{ij\dagger} = -\omega_{0i}S^{0i} + \omega_{ij}S^{ij}$$

$$\omega_{\mu\nu}S^{\mu\nu\dagger}\gamma^0 \sim (-\omega_{0i}S^{0i} + \omega_{ij}S^{ij})\gamma^0 = \gamma^0(\omega_{0i}S^{0i} + \omega_{ij}S^{ij})$$
$$= \omega_{\mu\nu}\gamma^0 S^{\mu\nu}$$

$$\therefore L(\Lambda)^{\dagger}\gamma^0 \simeq \gamma^0(\mathbb{1} + \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}) \simeq \gamma^0 L(\Lambda)^{-1}, \quad L(\Lambda)^{-1} = e^{+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$\bar{\Psi}\Psi$ (not $\psi^{\dagger}\psi$) is Lorentz invariant.

Ex.10. Prove Lorentz inv. of the Dirac Lagrangian $\mathcal{L}$.

Solutions of Dirac eqn

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0.$$

$$\psi(x) = f(k)\, e^{-ik\cdot x}, \quad k = (k^0, \vec{k}), \quad (i\gamma^\mu(-ik_\mu) - m) f(k) = 0.$$

$$\Rightarrow (\not{k} - m) f(k) = 0, \quad \not{k} \equiv \eta_{\mu\nu}\gamma^\mu k^\nu$$

For non-trivial solution, $\det(\not{k} - m) = 0$ —— (a)

Alt, $(\not{k} + m)(\not{k} - m) f(k) = 0 \Rightarrow (\gamma^\mu\gamma^\nu k_\mu k_\nu - m^2) f(k) = 0.$

$$\Rightarrow \Big(\tfrac{1}{2}\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{=\,2\eta^{\mu\nu}} k_\mu k_\nu - m^2\Big) f(k) = 0.$$

Anticommutator $\{\ ,\ \}$ s.t. $\{A, B\} = AB + BA$

a number

$$\Rightarrow (k^2 - m^2) f(k) = 0$$

$$\Rightarrow k^0 = \pm\sqrt{\vec{k}^2 + m^2}$$

We have squared by hand, so are both the sol^{n}s satisfy eqn (a)? YES

Arguments: go to rest frame, $(\gamma^0 k^0 - m) f(k) = 0$

$$\Rightarrow \gamma^0 f(k) = \frac{m}{k^0} f(k)$$

However, $\gamma^0 \rightarrow 4\times4$ matrix

eigen values of $\gamma^0 = \pm 1$ (as $\gamma^{0\,2} = \mathbb{1}$)

$\Rightarrow$ each eigen value $\longrightarrow$ two independent eigen vectors.
(doubly degenerate)

$$\therefore \psi(x) \propto u_s(\vec{k})\, e^{-ik\cdot x}, \quad v_s(\vec{k})\, e^{+ik\cdot x}, \quad s = 1, 2, \quad k = (\omega_{\vec{k}}, \vec{k}), \quad \omega_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}$$

Dirac eqn $\Rightarrow$ $(\not{k}-m)\,u_s(\vec{k})=0\,,\qquad (\not{k}+m)\,v_s(\vec{k})=0\,.$

$$\bar{u}_s(\vec{k})(\not{k}-m)=0,\quad \bar{v}_s(\vec{k})(\not{k}+m)=0,\quad \bar{u}_s=u_s^\dagger\gamma^0,\quad \bar{v}_s=v_s^\dagger\gamma^0.$$

Normalizations: $$u_r^\dagger(\vec{k})\,u_s(\vec{k}) = v_r^\dagger(\vec{k})\,v_s(\vec{k}) = 2\omega_{\vec{k}}\,\delta_{rs}$$
$$v_r^\dagger(\vec{k})\,u_s(-\vec{k}) = u_r^\dagger(\vec{k})\,v_s(-\vec{k}) = 0\,.$$

Spin sums: $$\sum_s u_s(\vec{k})\bar{u}_s(\vec{k}) = \not{k}+m\,,\quad \sum_s v_s(\vec{k})\bar{v}_s(\vec{k}) = \not{k}-m,$$ for above normalizations

check
$$\sum_s u_s(\vec{k})u_s^\dagger(\vec{k}) = (\not{k}+m)\gamma^0 = \gamma^0\gamma^0k^0+\gamma^i\gamma^0k_i+m\gamma^0$$
$$=\gamma^0(\not{\tilde{k}}+m),\quad \tilde{k}=(\underbrace{k^0=\sqrt{\vec{k}^2+m^2}}_{=\,\omega_{\vec{k}}},-\vec{k})$$

$$\sum_s u_s(\vec{k})\underbrace{u_s^\dagger(\vec{k})v_r(-\vec{k})}_{=\,0} = \gamma^0\underbrace{(\not{\tilde{k}}+m)v_r(-\vec{k})}_{=\,0}$$

$$\sum_s u_s(\vec{k})\underbrace{u_s^\dagger(\vec{k})u_r(\vec{k})}_{=\,2\omega_{\vec{k}}\delta_{rs}} = (k^0+\gamma^i\gamma^0k_i+m\gamma^0)u_r(\vec{k})$$

$$\Rightarrow\quad 2\omega_{\vec{k}}\,u_r(\vec{k}) = (\omega_{\vec{k}}-\gamma^0\gamma^ik_i+m\gamma^0)u_r(\vec{k}) \Rightarrow (\omega_{\vec{k}}+\gamma^0\gamma^ik_i-m\gamma^0)u_r(\vec{k})=0$$
$$\Rightarrow \gamma^0(\not{k}-m)u_r(\vec{k})=0.$$

Both sides of the spin sums act similarly on all the basis spinors.

Fourier decomposition

$$\psi(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \sum_{s=1,2} \left(a_s(\vec{k}) u_s(\vec{k}) e^{-ik.x} + b_s^*(\vec{k}) v_s(\vec{k}) e^{+ik.x} \right)$$

$$\overline{\psi}(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \sum_{s=1,2} \left(a_s^*(\vec{k}) \overline{u}_s(\vec{k}) e^{+ik.x} + b_s(\vec{k}) \overline{v}_s(\vec{k}) e^{-ik.x} \right)$$

Treat a_s, b_s, a_s^*, b_s^* as Grassmann variables.

Hence $\psi, \overline{\psi}$ → Grassmann

$$\mathcal{L} = i\overline{\psi} \underbrace{\not{\partial}}_{\gamma^\mu \partial_\mu} \psi - m\overline{\psi}\psi \,, \quad \Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = -i\overline{\psi}\gamma^0 = -i\psi^\dagger \,, \; \Pi_{\overline{\psi}} = 0.$$

Hamiltonian

$$H = \int d^3\vec{x} \, \mathcal{H} \,, \quad \mathcal{H} = -\Pi_\psi \dot{\psi} - \mathcal{L}$$
$$= i\psi^\dagger \dot{\psi} - i\overline{\psi}\gamma^\mu \partial_\mu \psi + m\overline{\psi}\psi$$
$$= -i\overline{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\psi + m\overline{\psi}\psi$$
$$= \overline{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$$

$$(-i\vec{\gamma}\cdot\vec{\nabla} + m) u_s(\vec{k}) e^{-ik.x} = (-i\vec{\gamma}\cdot(i\vec{k}) + m) u_s(\vec{k}) e^{-ik.x}$$
$$= (\vec{\gamma}\cdot\vec{k} + m) u_s(\vec{k}) e^{-ik.x}$$
$$= \omega_{\vec{k}} \gamma^0 u_s(\vec{k}) e^{-ik.x}$$

$k.x = \omega_{\vec{k}} t - \vec{k}.\vec{x}$

$(\not{k} - m) u_s(\vec{k}) = 0.$

$\Rightarrow \omega_{\vec{k}} \gamma^0 u_s(\vec{k}) = (\vec{\gamma}.\vec{k} + m) u_s(\vec{k})$

Similarly, $(-i\vec{\gamma}\cdot\vec{\nabla} + m) v_s(\vec{k}) e^{ik.x} = -\omega_{\vec{k}} \gamma^0 v_s(\vec{k}) e^{ik.x}$

Using the normalizations, $H=\int d^3\vec{k}\,\omega_{\vec{k}} \sum_{s=1,2}\left(a_s^*(\vec{k})a_s(\vec{k}) - b_s(\vec{k})b_s^*(\vec{k})\right)$

Now promote a_s, b_s, a_s^*, b_s^* to operators $\longrightarrow a_s, b_s, a_s^\dagger, b_s^\dagger$

In order to avoid −ve energy eigen values, usual normal ordering will not work. Solution, impose anticommutation relations:

$\{a_r(\vec{k}), a_s^\dagger(\vec{k}')\} = \delta_{rs}\delta^3(\vec{k}-\vec{k}') = \{b_r(\vec{k}), b_s^\dagger(\vec{k}')\}$, all other anticommutators $\{a,a\}, \{b,b\}, \{a,b\}, \{a,b^\dagger\}$ vanish.

Normal ordered Hamiltonian

$$\therefore \boxed{N(H) = \int d^3\vec{k}\,\omega_{\vec{k}} \sum_{s=1,2}\left(a_s^\dagger(\vec{k})a_s(\vec{k}) + b_s^\dagger(\vec{k})b_s(\vec{k})\right)}$$

$N(a_s^\dagger a_r) = a_s^\dagger a_r$, $N(a_r a_s^\dagger) = -a_s^\dagger a_r$, etc.

Ex. 11. Using above anticommutation relations show that

$$\{\psi_\alpha(x^0,\vec{x}), \psi_\beta^\dagger(x^0,\vec{y})\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

which is same as $\{\psi_\alpha(x^0,\vec{x}), (\Pi_\psi)_\beta(x^0,\vec{y})\} = -i\,\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$.

all other anticommutators vanish

Vacuum & excited states

$a_s(\vec{k})|0\rangle = 0 = b_s(\vec{k})|0\rangle$, $s=1,2$, $\forall \vec{k}$

One particle states: $a_s^\dagger(\vec{k})|0\rangle$, $b_s^\dagger(\vec{k})|0\rangle$

fermion   antifermion

Pauli exclusion principle

$a_s^\dagger(\vec{k})a_s^\dagger(\vec{k}) = 0 = b_s^\dagger(\vec{k})b_s^\dagger(\vec{k}) \Rightarrow$ 2 particles or 2 antiparticles with same spin and same momentum cannot be created.

$$H a_r^\dagger(\vec{p})|0\rangle = \omega_{\vec{k}}\, a_r^\dagger(\vec{p})|0\rangle, \quad H b_r^\dagger(\vec{p})|0\rangle = \omega_{\vec{k}}\, b_r^\dagger(\vec{p})|0\rangle$$

N(H) understood

some eigen states of Hamiltonian

U(1) charge, particle, antiparticle

a fixed real number

$\mathcal{L}$ has symmetry under $\psi(x) \to e^{-iq\theta}\psi(x)$

continuous real parameter

$$\bar{\psi}(x) \to e^{iq\theta}\bar{\psi}(x)$$

Infinitesimal $\theta \Rightarrow \delta_\theta\psi(x) = -iq\theta\psi(x), \quad \delta_\theta\bar{\psi}(x) = iq\theta\bar{\psi}(x)$

Noether current, $j^\mu = q\,\bar{\psi}(x)\gamma^\mu\psi(x)$

Noether charge:

$$Q = \int d^3\vec{x}\; j^0 = q\int d^3\vec{x}\;\psi^\dagger(x)\psi(x)$$

$$N(Q) = q\int d^3\vec{x}\, N(\psi^\dagger\psi) = q\int d^3\vec{k}\sum_{s=1,2}\left(a_s^\dagger(\vec{k})a_s(\vec{k}) - b_s^\dagger(\vec{k})b_s(\vec{k})\right)$$

fermion, antifermion have opposite charges:

$$Q|0\rangle = 0, \quad Q\,a_r^\dagger(\vec{p})|0\rangle = +q\,a_r^\dagger(\vec{p})|0\rangle, \quad Q\,b_r^\dagger(\vec{p})|0\rangle = -q\,b_r^\dagger(\vec{p})|0\rangle$$

N(Q) understood

Normal ordered Q and H so that vacuum has zero charge and zero energy.

Green's function, Feynman propagator

(-i) factor is taken out from Green function as $S_F(x_1-x_2)$ will be identified as Feynman propagator

$$\left(i\gamma^\mu \frac{\partial}{\partial x_1^\mu} - m\right)\left(-i S_F(x_1-x_2)\right) = \delta^4(x_1-x_2)\,\mathbb{1}_{4\times4}$$

$\uparrow$ 4×4 matrix

$$S_F(x_1-x_2) = \frac{1}{(2\pi)^4}\int d^4k\, \tilde{S}_F(k)\, e^{-ik\cdot(x_1-x_2)}$$, Fourier expansion

$\nwarrow$ Fourier transform, a matrix

$$\therefore \underbrace{\left(i\not{\partial}_{x_1} - m\right)\frac{1}{(2\pi)^4}\int d^4k\, \tilde{S}_F(k)\, e^{-ik\cdot(x_1-x_2)}}_{=\frac{1}{(2\pi)^4}\int d^4k \underbrace{\left(i\gamma^\mu(-ik_\mu) - m\right)}_{=\not{k}-m}\tilde{S}_F(k)\, e^{-ik\cdot(x_1-x_2)}} = \frac{i}{(2\pi)^4}\int d^4k\, e^{-ik\cdot(x_1-x_2)}\,\mathbb{1}$$

$$\Rightarrow (\not{k}-m)\tilde{S}_F(k) = i\mathbb{1} \Rightarrow (\not{k}+m)(\not{k}-m)\tilde{S}_F(k) = i(\not{k}+m)$$

$$\Rightarrow (k^2-m^2)\tilde{S}_F(k) = i(\not{k}+m)$$

$$\Rightarrow \tilde{S}_F(k) = \frac{i(\not{k}+m)}{k^2-m^2}$$

which has poles along the real k^0 integration contour

Remedy: take Feynman prescription $m^2 \to m^2 - i\epsilon$, at end $\epsilon \to 0^+$

$$\boxed{\tilde{S}_F(k) = \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} = \frac{i}{\not{k}-m+i\epsilon'}}\ , \quad \epsilon, \epsilon' \longrightarrow 0^+$$

the last expression is written for simplicity

With this choice of $i\epsilon$, $S_{F\alpha\beta}(x_1-x_2) = \langle 0|T(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2))|0\rangle$ Ex.12

where time ordered product for fermionic fields is defined as

need to use spin sums

$$T(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)) = \begin{cases} \psi_\alpha(x_1)\bar{\psi}_\beta(x_2), & x_1^0 > x_2^0 \\ -\bar{\psi}_\beta(x_2)\psi_\alpha(x_1), & x_2^0 > x_1^0 . \end{cases}$$

$\langle 0|T(\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_m}(y_m)|0\rangle$ correlators vanish unless unless there are equal number of ψ's and $\bar{\psi}$'s, i.e. unless $n=m$

consequence of charge conservation

$$\Rightarrow \langle 0|T(\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2))|0\rangle = 0 = \langle 0|T(\bar{\psi}_{\beta_1}(x_1)\bar{\psi}_{\beta_2}(x_2))|0\rangle$$

Wicks theorem applies to free fermions, with the difference that a sign must be included infront of each term, according to the number of anticommutations required to bring the contracted fields next to each other

$$\Rightarrow \langle 0|T(\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\bar{\psi}_{\beta_1}(x_3)\bar{\psi}_{\beta_2}(x_4))|0\rangle$$

$$= -S_{F\alpha_1\beta_1}(x_1-x_3)S_{F\alpha_2\beta_2}(x_2-x_4) + S_{F\alpha_1\beta_2}(x_1-x_4)S_{F\alpha_2\beta_1}(x_2-x_3)$$

Weyl representation of Dirac γ-matrices

σ^i are the 2x2 Pauli matrices

$$\gamma^0 = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}, \quad i=1,2,3$$

$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ become block diagonal $\Rightarrow$ they form a reducible rep. of Lorentz algebra

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \; S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \; S^{23} = \frac{1}{2}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \; S^{31} = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

Angular momenta $\quad J_i = \frac{1}{2}\epsilon_{ijk}S^{jk}$

$$\therefore \; J_3 = \frac{1}{2}\left(\epsilon_{312}S^{12} + \epsilon_{321}S^{21}\right) = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

J_3 eigen values are $\pm\frac{1}{2}$, each doubly degenerate $\xRightarrow{\text{leads to}}$ two spin $\frac{1}{2}$ particles

---

General comments on Lorentz algebra, not restricted to Dirac spinor rep.

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}\right)$$

$J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$, $\quad K_i \equiv M^{0i}$.

generates 3d rotations

generate Lorentz boosts

$$[J_i, J_j] = i\,\epsilon_{ijk}J_k \quad \longleftarrow \text{ angular momentum algebra}$$

$$[J_i, K_j] = i\,\epsilon_{ijk}K_k$$

$$[K_i, K_j] = -i\,\epsilon_{ijk}J_k$$

Define: $X_i \equiv \frac{1}{2}(J_i - iK_i)$, $Y_i \equiv \frac{1}{2}(J_i + iK_i)$ whose physical significance is obscure

$[X_i, X_j] = i\,\epsilon_{ijk} X_k$, $[Y_i, Y_j] = i\,\epsilon_{ijk} Y_k$,

$[X_i, Y_j] = 0$.

$\Rightarrow$ two different SU(2) algebras.

And, $J_i = X_i + Y_i \longleftarrow$ addition of "angular momenta".