Why QFT?

Quantum mechanics can deal with a system of N number of particles, where N is fixed.

In nature, there are systems where number of particles is not conserved. For example -

* High energy collision of particles which can produce new particles beside the colliding ones.
* A gas of photons

- outside the scope of QM

Quantum field theory is <u>a framework</u> where one can <u>model</u> such systems of many particles.

QFT dealing with relativistic particles does not give rise to negative energy states. (QM of single relativistic particle has such states, in case familiar.)

Notations and conventions

Consider the relation $\underbrace{E}_{\text{energy}} = \underbrace{\hbar}_{\text{reduced Planck constant}} \underbrace{\omega}_{\text{angular frequency}}$

$[\hbar] = [E/\omega] = ML^2T^{-2} \cdot T = ML^2T^{-1}$. ——— (1)

Speed of light $= c$, $[c] = LT^{-1}$. ——— (2)

Units: $c = \hbar = 1$ (dimensionless)

$\therefore$ (1) & (2) $\Rightarrow$ $[\text{length}] = [\text{time}] = [\text{mass}]^{-1}$

Space-time coordinates: $x^\mu = (\underset{= ct}{x^0}, \underbrace{x^1, x^2, x^3}_{= \vec{x}}) = (ct, \vec{x}) = (t, \vec{x})$

4-momentum: $p^\mu = (\underset{= E/c}{p^0}, \underbrace{p^1, p^2, p^3}_{= \vec{p}}) = (\frac{E}{c}, \vec{p}) = (E, \vec{p})$

Minkowski metric: mostly negative

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Einstein's summation convention

lowering index: $x_\mu = \eta_{\mu\nu} x^\nu$ (repeated indices are summed over)

$$= (x^0, -x^1, -x^2, -x^3)$$

Similarly, $p_\mu = (p^0, -p^1, -p^2, -p^3)$

Scalar product: $A \cdot B \equiv A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu$

$\therefore \; x^2 = x^\mu x_\mu = t^2 - \vec{x}^2$,

$x \cdot p = Et - \vec{x} \cdot \vec{p}$, $p^2 = E^2 - \vec{p}^2$ $\left(= \frac{E^2}{c^2} - \vec{p} \right)$

Minkowski

Euclidean

(if factors of c are restored)

Fields: $\phi(x) = \phi(t, \vec{x})$, $A_\mu(x)$, etc

Scalar　　　　Vector

Action functional: $S[\phi, A_\mu] = \int d^4x \, \mathcal{L}(\phi, A_\mu, \partial_\mu \phi, \partial_\mu A_\nu)$

Lagrangian density

Shorthand - $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$$= \int dt \underbrace{\int d^3\vec{x} \, \mathcal{L}}_{= L}$$

L Lagrangian

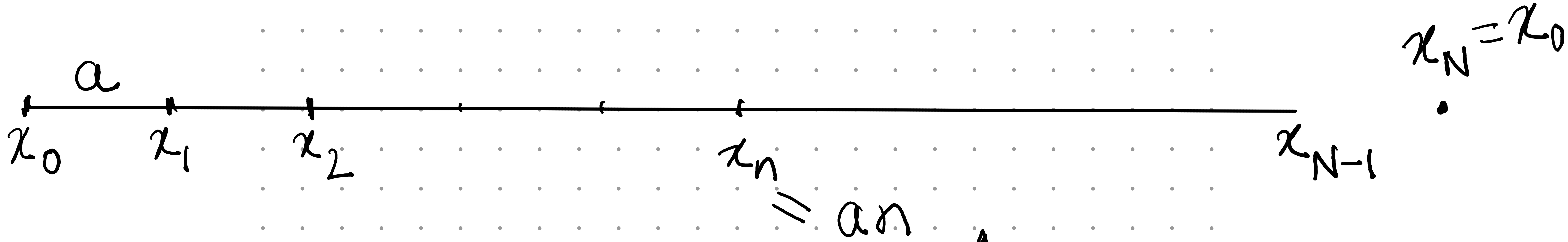
$$= \int dt \, L$$

Classical field configurations (ϕ_{cl}, A^μ_{cl}) are those which extremize $S[\phi, A_\mu]$.

Consider 2-space-time dimensions, a single scalar field $\phi(t,x)$ with $\mathcal{L}$ given as

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}m^2\phi^2$$

<u>Discretize spacial dimension x</u>:



1d periodic lattice

$$L = \int dx\, \mathcal{L} \approx \sum_{n=0}^{N-1} \frac{a}{2}\left[\dot{\phi}_n^2 - \left(\frac{\phi_{n+1} - \phi_n}{a}\right)^2 - m^2\phi_n^2\right]$$

Now,

$$\sum_n (\phi_{n+1} - \phi_n)^2 = \sum_n (\phi_{n+1}^2 + \phi_n^2 - 2\phi_n\phi_{n+1}) = 2\sum_n (\phi_n^2 - \phi_n\phi_{n+1})$$

$$\therefore L \approx \sum_n \frac{a}{2}\left[\dot{\phi}_n^2 - \left(m^2 + \frac{2}{a^2}\right)\phi_n^2 + \frac{2}{a^2}\phi_n\phi_{n+1}\right]$$

$$L = \sum_{n=0}^{N-1} \frac{a}{2}\left[\dot{\phi}_n^2 - \left(m^2+\frac{2}{a^2}\right)\phi_n^2 + \frac{2}{a^2}\underbrace{\phi_n\phi_{n+1}}\right]$$

$\Rightarrow$ N d.o.f. ϕ_n , $n=0,1,\dots,N-1$; "interacting"

<u>EOM</u>: $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}_n}\right] = \frac{\partial L}{\partial \phi_n}$, $n=0,1,\dots,N-1$

Kronecker delta

$$\Rightarrow \ddot{\phi}_n = -\left(m^2+\frac{2}{a^2}\right)\phi_n + \frac{1}{a^2}\left(\phi_{n+1}+\phi_{n-1}\right) \quad \left(\because \frac{\partial \phi_r}{\partial \phi_s} = \delta_{r,s}\right)$$

(1)

<u>Trial solⁿ</u>:

$$\phi_n = \sum_{k=0}^{N-1} \mathcal{N}_k\left(a_k e^{\frac{2\pi i k n}{N}} e^{-i\omega_k t} + b_k\, e^{-\frac{2\pi i k n}{N}} e^{+i\omega_k t}\right)$$

Discrete Fourier Expansion

Reality: $\phi_n^* = \phi_n \Rightarrow a_k^* = b_k$ (taking $\mathcal{N}_k$ real)

$$(1) \Rightarrow -\omega_k^2 = -\left(m^2+\frac{2}{a^2}\right) + \frac{1}{a^2}\left(e^{\frac{2\pi i k}{N}} + e^{-\frac{2\pi i k}{N}}\right)$$

$$\Rightarrow \boxed{\omega_k^2 = m^2 + \frac{2}{a^2}\left(1 - 2\cos\frac{2\pi k}{N}\right)}$$

To get ω_k take the +ve square root

$$e^{\pm\frac{2\pi i k(n+N)}{N}} = e^{\pm\left(\frac{2\pi i k n}{N} + 2\pi i k n\right)} = e^{\pm\frac{2\pi i k n}{N}}, \quad k, n \in \mathbb{Z}.$$

$$\cos\frac{2\pi(N-k)}{N} = \cos\left(2\pi - \frac{2\pi k}{N}\right) = \cos\frac{2\pi k}{N}$$

$$\sum_{n=0}^{N-1} e^{\frac{2\pi i q n}{N}} = 1 + e^{\frac{2\pi i q}{N}} + e^{\frac{4\pi i q}{N}} + \cdots + e^{\frac{2\pi i q(N-1)}{N}}$$

$$= \begin{cases} 1+1+\cdots \ N \text{ times}, & q = 0, \pm N, \pm 2N, \pm 3N, \ldots \\ \dfrac{1-e^{2\pi i q}}{1-e^{\frac{2\pi i q}{N}}}, & q = \text{any other integer} \end{cases}$$

$$= \begin{cases} N, & q = 0, \pm N, \pm 2N, \ldots \\ 0, & q = \text{any other integer} \quad \left(e^{\frac{2\pi i q}{N}} \neq 1,\ e^{2\pi i q} = 1\right) \end{cases}$$

Ex1 Check- (1) $\phi_{n+N} = \phi_n$, (2) $\omega_k = \omega_{N-k}$,

(3) $$\sum_{n=0}^{N-1} e^{\frac{2\pi i q n}{N}} = \begin{cases} N & , q = 0, \pm N, \pm 2N, \ldots \\ 0 & , q = \text{any other integer}. \end{cases}$$

Quantization: Promote a_k, a_k^* to operators $a_k, a_k^\dagger$

$$\phi_n(t) = \sum_{k=0}^{N-1} N_k \left(a_k e^{\frac{2\pi i k n}{N}} e^{-i\omega_k t} + a_k^\dagger e^{-\frac{2\pi i k n}{N}} e^{+i\omega_k t} \right) \qquad \text{(a)}$$

Conjugate momentum: $p_n = \frac{\partial L}{\partial \dot{\phi}_n} = a \dot{\phi}_n$

$$\Rightarrow p_n(t) = i a \sum_{k=0}^{N-1} \omega_k N_k \left(-a_k e^{\frac{2\pi i k n}{N} - i\omega_k t} + a_k^\dagger e^{-\frac{2\pi i k n}{N} + i\omega_k t} \right) \qquad \text{(b)}$$

* Dirac's prescription on equal-time commutators-

$$[\phi_n(t), \phi_m(t)] = 0 = [p_n(t), p_m(t)], \quad [\phi_n(t), p_m(t)] = i\,\delta_{n,m}$$

We want to express $a_k, a_k^\dagger$ in terms of ϕ_n, p_n; $a_k, a_k^\dagger$ are time-independent, thus can be computed at any time. Choose $t = 0$.

$$\phi_n(0) = \sum_{k=0}^{N-1} N_k \left(a_k e^{\frac{2\pi i k n}{N}} + a_k^\dagger e^{-\frac{2\pi i k n}{N}} \right) \quad \text{———} \quad (1)$$

$$p_n(0) = i a \sum_{k=0}^{N-1} \omega_k N_k \left(-a_k e^{\frac{2\pi i k n}{N}} + a_k^\dagger e^{-\frac{2\pi i k n}{N}} \right) \quad \text{———} \quad (2)$$

For $q = 0, 1, \ldots, N-1$

$$\sum_{n=0}^{N-1} \phi_n(0) e^{-\frac{2\pi i q n}{N}} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} N_k a_k e^{\frac{2\pi i (k-q) n}{N}} + N_k a_k^\dagger e^{-\frac{2\pi i (k+q) n}{N}}$$

$$= \sum_{k=0}^{N-1} N_k a_k N \delta_{k,q} + N_k a_k^\dagger N \left(\delta_{k+q,0} + \delta_{k+q,N} \right)$$

$$= N N_q a_q + N \sum_{k=0}^{N-1} N_k a_k^\dagger \left(\delta_{k+q,0} + \delta_{k+q,N} \right)$$

result depends on the value of q.

Similarly,

$$\sum_n p_n(0) e^{-\frac{2\pi i q n}{N}} = -i a N \omega_q N_q a_q + i a N \sum_k \omega_k N_k a_k^\dagger \left(\delta_{k+q,0} + \delta_{k+q,N} \right)$$

$$\sum_n \phi_n(0)\, e^{\frac{2\pi i q n}{N}} = N\mathcal{N}_q a_q^\dagger + N\sum_{k=0}^{N-1} \mathcal{N}_k a_k \left(\delta_{k+q,0} + \delta_{k+q,N}\right)$$

$$\sum_n p_n(0)\, e^{\frac{2\pi i q n}{N}} = i a N \omega_q \mathcal{N}_q a_q^\dagger - i a N \sum_k \omega_k \mathcal{N}_k a_k \left(\delta_{k+q,0} + \delta_{k+q,N}\right)$$

Take dagger on previous formulae to obtain above formulae.

Ex. 2 Check above formulae explicitly.

$$\sum_n \left(a\omega_q \phi_n(0) + i p_n(0)\right) e^{-\frac{2\pi i q n}{N}} = 2aN\omega_q \mathcal{N}_q a_q$$

$$\sum_n \left(a\omega_q \phi_n(0) - i p_n(0)\right) e^{+\frac{2\pi i q n}{N}} = 2aN\omega_q \mathcal{N}_q a_q^\dagger$$

take dagger, ϕ_n, p_n are real

Commutators -

$$[a_k, a_q^\dagger] = \frac{1}{4a^2N^2\omega_k\omega_q\mathcal{N}_k\mathcal{N}_q} \sum_{n,n'} -ia\omega_k\, e^{\frac{2\pi i(qn'-kn)}{N}} \times \underbrace{[\phi_n(0), p_{n'}(0)]}_{= i\delta_{n,n'}} + ia\omega_q\, e^{\frac{2\pi i(qn'-kn)}{N}} \times [p_n(0), \phi_{n'}(0)]$$

$$= \frac{1}{4a^2N^2\omega_k\omega_q\mathcal{N}_k\mathcal{N}_q} \sum_n a\omega_k\, e^{\frac{2\pi i(q-k)n}{N}} + a\omega_q\, e^{\frac{2\pi i(q-k)n}{N}}$$

$$\Rightarrow [a_k, a_q^\dagger] = \frac{a(\omega_k + \omega_q)}{4a^2N^2\omega_k\omega_q \mathcal{N}_k \mathcal{N}_q} N\delta_{k,q} = \frac{1}{2aN\,\omega_k \mathcal{N}_k^2}\delta_{k,q}$$

Choice: $\mathcal{N}_k = \frac{1}{\sqrt{2aN\omega_k}} \Rightarrow \boxed{[a_k, a_q^\dagger] = \delta_{k,q}}$

$$[a_k, a_q] = \frac{1}{4a^2N^2\omega_k\omega_q\mathcal{N}_k\mathcal{N}_q}\sum_{n,n'} ia\omega_k e^{-\frac{2\pi i(kn+qn')}{N}} \overbrace{[\phi_n(0), p_{n'}(0)]}^{i\delta_{n,n'}} + ia\omega_q e^{-\frac{2\pi i(kn+qn')}{N}} [p_n(0), \phi_n(0)]$$

$$= 0.$$

Also, $[a_k^\dagger, a_q^\dagger] = 0.$

Hamiltonian: $H = \sum_n p_n\dot{\phi}_n - L = \frac{1}{2}\sum_n \frac{p_n^2}{a} + \frac{1}{a}(\phi_{n+1} - \phi_n)^2 + am^2\phi_n^2$

Next subsititute mode expansions of ϕ_n, p_n. Carry out $\sum_n$.

$$= \frac{1}{2}\sum_n \frac{p_n^2}{a} + a\left(m^2 + \frac{2}{a^2}\right)\phi_n^2 - \frac{2}{a}\phi_n\phi_{n+1}$$

$$H = \frac{1}{2}\sum_{k=0}^{N-1}\omega_k\left(a_k^\dagger a_k + a_k a_k^\dagger\right) = \sum_{k=0}^{N-1}\omega_k\left(a_k^\dagger a_k + \frac{1}{2}\mathbb{1}\right)$$

Collection of N non-interacting 1D harmonic oscillators with angular frequencies $\omega_0, \omega_1, \dots, \omega_{N-1}$. "free theory"

Ground state: $|0\rangle$ s.t. $a_k|0\rangle = 0 \quad \forall\, k = 0, 1, \dots, N-1$

"annihilators" a_k

$$H|0\rangle = \underbrace{\left(\frac{1}{2}\sum_{k=0}^{N-1}\omega_k\right)}_{\equiv C}|0\rangle = C|0\rangle$$

$C \leftarrow$ ground state energy

"creators" $a_k^\dagger$

Eigen states of H: $|k_1, k_2, \dots, k_r\rangle \equiv a_{k_1}^\dagger a_{k_2}^\dagger \dots a_{k_r}^\dagger|0\rangle$

$$H|k_1, \dots, k_r\rangle = E_{k_1,\dots,k_r}|k_1, \dots, k_r\rangle, \quad E_{k_1,\dots,k_r} = (\omega_{k_1} + \dots + \omega_{k_r}) + C$$

C will have no physical significance, as we will only be interested in differences in energy eigen values.

e.g. $H|k_1\rangle = C|k_1\rangle + \sum_{k=0}^{N-1}\omega_k a_k^\dagger \underbrace{a_k a_{k_1}^\dagger}_{= a_{k_1}^\dagger a_k + \delta_{k,k_1}}|0\rangle = C|0\rangle + \sum_{k=0}^{N-1}\delta_{k,k_1}\omega_k a_k^\dagger|0\rangle$

$\boxed{E_{k_1} = \omega_{k_1} + C.}$

Ex3 Check - (1) $[H, a_k^\dagger] = \omega_k a_k^\dagger$, $[H, a_k] = -\omega_k a_k$

(2) $H|k_1, \dots, k_r\rangle = \left[(\omega_{k_1} + \dots + \omega_{k_r}) + c\right]|k_1, \dots, k_r\rangle$

(3) <u>Number operator</u> : $\hat{N} \equiv \sum_{k=0}^{N-1} a_k^\dagger a_k$,

$\hat{N}|k_1, \dots, k_r\rangle = r\,|k_1, \dots, k_r\rangle$

↖ counts the number of creators in the state

$\underbrace{H(a_k^\dagger)^r|0\rangle}_{\sigma} = (r\omega_k + c)\, a_k^\dagger|0\rangle$, corollary of (2) when we set $k_1 = k_2 = \dots = k_r = k$

↑ excited states of individual oscillators

$|k_1, \dots, k_r\rangle \leftarrow$ in general many of the N oscillators are excited.

Classical theory of fields:

Set of fields: $\phi_i(x)$

Action: $S[\phi_i] = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu \phi_i)$, local

Vary $\phi_i(x)$: $\phi_i(x) \longrightarrow \phi_i'(x) = \phi_i(x) + \underbrace{\delta\phi_i(x)}_{\text{variations}}$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \underbrace{\delta(\partial_\mu \phi_i)}_{\text{commutes}} \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu(\delta\phi_i) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right) \delta\phi_i + \cancelto{0}{\int d^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right)}$$

$\delta\phi_i = 0$ at spacetime boundaries

$\delta S = 0$ for independent varitions $\delta\phi_i$

$$\Rightarrow \boxed{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] = \frac{\partial \mathcal{L}}{\partial \phi_i} \text{ for each } i.}$$

Euler-Lagrange equations/ Classical equations of motion

$$\partial_\mu \phi_i'(x) = \partial_\mu \phi_i(x) + \partial_\mu(\delta\phi_i(x)) \Rightarrow \delta(\partial_\mu \phi_i) = \partial_\mu(\delta\phi_i)$$

Lagrangian, $L(t) = \int d^3\vec{x}' \, \mathcal{L}\left(\phi_j(t,\vec{x}'), \partial_\mu \phi_j(t,\vec{x}')\right)$

Conjugate momenta: $\Pi_i(t,\vec{x}) \equiv \dfrac{\partial L}{\partial(\partial_t \phi_i(t,\vec{x}))} = \dfrac{\partial L}{\partial \dot{\phi}_i(t,\vec{x})}$

$$\Pi_i(t,\vec{x}) = \int d^3\vec{x}' \left(\frac{\partial \mathcal{L}}{\partial \phi_j(t,\vec{x}')} \underbrace{\frac{\partial \phi_j(t,\vec{x}')}{\partial \dot{\phi}_i(t,\vec{x})}}_{0} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j(t,\vec{x}')} \underbrace{\frac{\partial \dot{\phi}_j(t,\vec{x}')}{\partial \dot{\phi}_i(t,\vec{x})}}_{= \delta_{ij} \cdot \delta^3(\vec{x}'-\vec{x})} \right)$$

$$= \int d^3\vec{x}' \, \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j(t,\vec{x}')} \delta_{ij} \delta^3(\vec{x}'-\vec{x})$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(t,\vec{x})}$$

Hamiltonian: $H(t) = \int d^3\vec{x} \sum_i \Pi_i(t,\vec{x}) \dot{\phi}_i(t,\vec{x}) - L(t)$

$$= \int d^3\vec{x} \underbrace{\left(\sum_i \Pi_i(t,\vec{x}) \dot{\phi}_i(t,\vec{x}) - \mathcal{L} \right)}_{\equiv \, \mathcal{H}}$$

$\mathcal{H} \leftarrow$ Hamiltonian density

Symmetry: $\phi(x) \to \phi'(x)$, $S[\phi] \to S[\phi'] = S[\phi]$

E.g. 1. $S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 \right)$

real $\phi(x) \to \phi'(x) = -\phi(x)$ is a symmetry. Discrete

E.g. 2. $S = \int d^4x \left[\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2 \right]$

$\Phi(x) \to \Phi'(x) = e^{i\alpha}\Phi(x)$ is a symmetry. Continuous + Global

complex

continuous parameter independent of x

Spacetime transformations

$$\left. \begin{aligned} & x \to x'(x) \\ & \phi(x) \to \phi'(x') = \mathcal{F}(\phi(x)) \end{aligned} \right\}$$

E.g. 1 Translations: $x^\mu \to x'^\mu = x^\mu + a^\mu$

$$\phi(x) \to \phi'(x') = \phi(x)$$ Scalar

$\therefore\ \phi'(x+a) = \phi(x) \Rightarrow \phi'(x) + a^\mu\partial_\mu\phi(x) = \phi(x)$

$\Rightarrow \delta\phi(x) \equiv \phi'(x) - \phi(x) = -a^\mu\partial_\mu\phi(x)$

E.g. 2. Lorentz transformations: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\phi(x) \to \phi'(x') = L(\Lambda)\phi(x)$$

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$$

$$\Rightarrow \eta_{\rho\sigma}(\delta^\rho_\mu + \omega^\rho{}_\mu)(\delta^\sigma_\nu + \omega^\sigma{}_\nu) = \eta_{\mu\nu}$$

$$\Rightarrow \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} = \eta_{\mu\nu} \Rightarrow \boxed{\omega_{\mu\nu} = -\omega_{\nu\mu}}$$

How many independent parameters?

$$L(\Lambda) \simeq \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

antisymmetric

$$\therefore \phi'(x^\mu + \omega^\mu{}_\nu x^\nu) = \phi(x^\mu) - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\phi(x^\mu)$$

$$\Rightarrow \phi'(x) + \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = \phi(x) - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\phi(x)$$

$$\Rightarrow \delta\phi(x) = -\omega_{\mu\nu}x^\nu\partial^\mu\phi(x) - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\phi(x)$$

$$= \frac{1}{2}\omega_{\mu\nu}\left[(x^\mu\partial^\nu - x^\nu\partial^\mu) - iS^{\mu\nu}\right]\phi(x)$$

Given an action we need to plug $\phi'(x)$ and test $S[\phi'] \stackrel{?}{=} S[\phi]$

<u>Alt</u>

$$S[\phi'] = \int d^4x \, \mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right)$$

$$= \int d^4x' \, \mathcal{L}\left(\phi'(x'), \partial'_\mu \phi'(x')\right)$$

$$= \int d^4x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x))\right)$$

$$x'^\mu(x) = x^\mu + \epsilon_i \frac{\delta x'^\mu}{\delta \epsilon_i}(x), \quad \mathcal{F}(\phi(x)) = \phi(x) + \epsilon_i \frac{\delta \mathcal{F}}{\delta \epsilon_i}(x)$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\left(\epsilon_i \frac{\delta x'^\mu}{\delta \epsilon_i}\right), \quad \frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial'_\mu\left(\epsilon_i \frac{\delta x'^\nu}{\delta \epsilon_i}\right)$$

inverse

$$= \delta^\nu_\mu - \frac{\partial x^\sigma}{\partial x'^\mu} \partial_\sigma\left(\epsilon_i \frac{\delta x'^\nu}{\delta \epsilon_i}\right)$$

$$\simeq \delta^\nu_\mu - \partial_\mu\left(\epsilon_i \frac{\delta x'^\nu}{\delta \epsilon_i}\right)$$

$$\left|\frac{\partial x'}{\partial x}\right| = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$$

$$\simeq 1 + \partial_\mu\left(\epsilon_i \frac{\delta x'^\mu}{\delta \epsilon_i}\right), \quad \left(\det(\mathbb{1} + X) \simeq 1 + \mathrm{Tr}(X), \; X \simeq 0\right)$$

$$\frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F} = \left(\delta^\nu_\mu - \partial_\mu\left(\epsilon_i \frac{\delta x'^\nu}{\delta \epsilon_i}\right)\right)\left(\partial_\nu \phi(x) + \partial_\nu\left(\epsilon_i \frac{\delta \mathcal{F}}{\delta \epsilon_i}\right)\right)$$

$$= \partial_\mu \phi + \partial_\mu\left(\epsilon_i \frac{\delta \mathcal{F}}{\delta \epsilon_i}\right) - \partial_\nu \phi \, \partial_\mu\left(\epsilon_i \frac{\delta x'^\nu}{\delta \epsilon_i}\right)$$

$$\therefore\ S[\phi'] = \int d^4x \left(1 + \partial_\mu\left(\epsilon_i \frac{\delta x'^\mu}{\delta \epsilon_i}\right)\right)$$

$$\times\, \mathcal{L}\left(\phi(x) + \epsilon_i \frac{\delta \mathcal{F}}{\delta \epsilon_i},\ \partial_\mu \phi + \partial_\mu\left(\epsilon_i \frac{\delta \mathcal{F}}{\delta \epsilon_i}\right) - \partial_\nu \phi\, \partial_\mu\left(\epsilon_i \frac{\delta x'^\nu}{\delta \epsilon_i}\right)\right)$$

$$= S[\phi] + \int d^4x\, (\cdots)_i\, \epsilon_i \ - \int d^4x\, j^\mu_i\, \partial_\mu \epsilon_i$$

$$-j^\mu_i = \frac{\delta x'^\mu}{\delta \epsilon_i} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\left(\frac{\delta \mathcal{F}}{\delta \epsilon_i} - \partial_\nu \phi \frac{\delta x'^\nu}{\delta \epsilon_i}\right)$$

$$\Rightarrow \boxed{j^\mu_i = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right) \frac{\delta x'^\nu}{\delta \epsilon_i} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \epsilon_i}}$$

Suppose the action has symmetry under global transformations,
$\Rightarrow (\cdots)_i$ vanishes identically for any field configuration.

For such an action, when $\epsilon_i(x)$, $\delta S = -\int d^4x\, j^\mu_i\, \partial_\mu \epsilon_i$

$$= \int d^4x \left(\partial_\mu j^\mu_i\right) \epsilon_i(x)$$

When field config. obeys classical EOM, action is stationary against any variation:

$$\delta S\big|_{\phi_{cl}} = 0 \Rightarrow \boxed{\partial_\mu j^\mu_i\big|_{\phi_{cl}} = 0}$$

Noether's theorem: For every continuous global symmetry of an action, $\exists$ a conserved current for classical field configuration.

Conserved charges $\quad Q_i = \int d^3\vec{x}\; j_i^0(t,\vec{x})$

$$\dot{Q}_i = \int d^3\vec{x}\, \partial_0 j_i^0 = -\int d^3\vec{x}\, \partial_a j_i^a = 0. \text{ Assuming } \vec{j}_i(t,\vec{x}) \text{ vanishes at spatial } \infty.$$

Conserved currents under (1) $x'^\mu = x^\mu + a^\mu$, $\phi'(x+a) = \phi(x)$ ← always

(2) $x' = \Lambda x$, $\phi'(\Lambda x) = \phi(x)$ ← Lorentz scalar

Ex. 4. Derive/find

(1) $T^\mu{}_\rho = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right) \delta^\nu_\rho$ ← energy momentum tensor

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L}, \quad P_\rho = \int d^3\vec{x}\; T^0{}_\rho, \quad P_0 = H$$

(2) $M^{\mu,\rho\sigma} = \frac{1}{2}\left(T^{\mu\rho} x^\sigma - T^{\mu\sigma} x^\rho\right)$, $J^{\rho\sigma} = \int d^3\vec{x}\; M^{0,\rho\sigma}$,

$J^i = \frac{1}{2}\epsilon^{ijk} J_{jk}$, $i = 1,2,3$ are the angular momenta.

Show that $\partial_\mu T^\mu{}_\rho = 0$, $\partial_\mu M^{\mu,\rho\sigma} = 0$. (Need to use Euler-Lagrange EOM)

$$M^{\mu,\rho\sigma} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu \mathcal{L}\right)\eta^{\nu[\rho}x^{\sigma]}$$

$$x'^\nu = x^\nu + \omega^\nu{}_\sigma x^\sigma = x^\nu + \eta^{\nu\rho}\omega_{\rho\sigma}x^\sigma$$

$$\frac{\delta x'^\nu}{\delta\omega_{\rho\sigma}} = \eta^{\nu[\rho}x^{\sigma]} = \frac{1}{2}\left(\eta^{\nu\rho}x^\sigma - \eta^{\nu\sigma}x^\rho\right)$$

$$M^{\mu,\rho\sigma} = T^\mu{}_\nu\, \eta^{\nu[\rho}x^{\sigma]} = \frac{1}{2}\left(T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho\right)$$

$$\partial_\mu\left(\tilde{M}^{\mu,\rho\sigma}\right)\omega_{\rho\sigma} = 0 \Rightarrow M^{\mu,\rho\sigma} = \tilde{M}^{\mu,[\rho\sigma]}$$

(antisymmetrization of p indices include a factor $\frac{1}{p!}$

$$\begin{aligned}\partial_\mu M^{\mu,\rho\sigma} &\propto \cancelto{0}{(\partial_\mu T^{\mu\rho})x^\sigma} + T^{\mu\rho}\delta^\sigma_\mu \\ &\quad - \cancelto{0}{(\partial_\mu T^{\mu\sigma})x^\rho} - T^{\mu\sigma}\delta^\rho_\mu \\ &= T^{\sigma\rho} - T^{\rho\sigma}\end{aligned}$$

(Is cannonical energy-momentum tensor symmetric?)

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}_{\nu} \mathcal{L}$$

$$\partial_\mu T^{\mu}{}_{\nu} = \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right) \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi - \partial_\nu \mathcal{L}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi - \partial_\nu \mathcal{L} \quad (\text{EOM used})$$

$$\mathcal{L} = K - V, \quad K(\partial_\mu \phi), \quad V(\phi)$$

$$\partial_\nu \mathcal{L} = \partial_\nu K - \partial_\nu V = K'^{\mu} \partial_\nu (\partial_\mu \phi) - V'(\phi) \partial_\nu \phi, \quad K'^{\mu} \equiv \frac{\partial K}{\partial(\partial_\mu \phi)}, \quad V'(\phi) \equiv \frac{\partial V}{\partial \phi}$$

$$\partial_\mu T^{\mu}{}_{\nu} = \cancel{-V'(\phi) \partial_\nu \phi} + \cancel{K'^{\mu} \partial_\mu \partial_\nu \phi} - \cancel{K'^{\mu} \partial_\nu \partial_\mu \phi} + \cancel{V'(\phi) \partial_\nu \phi}$$

$$= 0.$$