Let us remind ourselves that the free field Lagrangian for massless vector field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

$\mathcal{L}$ can also be written as

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$$

$$= -\frac{1}{4}\left[\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha - \partial_\beta A_\alpha \partial^\alpha A^\beta + \partial_\beta A_\alpha \partial^\beta A^\alpha\right]$$

$$= -\frac{1}{2}\left[\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha\right]$$

$$= -\frac{1}{4}\eta^{\rho\alpha}\eta^{\sigma\beta}\left[\partial_\alpha A_\beta \partial_\rho A_\sigma - \partial_\alpha A_\beta \partial_\sigma A_\rho\right]$$

Let us now try to quantize the theory through canonical quantization. For that, we need to find the canonical momenta conjugate to the field $A_\mu$

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \qquad \boxed{\mathcal{L} = -\frac{1}{4}\eta^{\alpha\rho}\eta^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma}}$$

conjugate momenta

$$= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$$

$$= -\frac{1}{4}\eta^{\alpha\rho}\eta^{\beta\sigma}\left(\delta^0_\alpha\delta^\mu_\beta - \delta^0_\beta\delta^\mu_\alpha\right)F_{\rho\sigma}$$
$$-\frac{1}{4}\eta^{\alpha\rho}\eta^{\beta\sigma}F_{\alpha\beta}\left(\delta^0_\rho\delta^\mu_\sigma - \delta^0_\sigma\delta^\mu_\rho\right)$$
$$= -\frac{1}{4}\left(\eta^{0\rho}\eta^{\mu\sigma} - \eta^{\mu\rho}\eta^{0\sigma}\right)F_{\rho\sigma}$$
$$-\frac{1}{4}F_{\alpha\beta}\left(\eta^{0\alpha}\eta^{\beta\mu} - \eta^{0\beta}\eta^{\alpha\mu}\right)$$
$$= -\frac{1}{4}\left(F^{0\mu} - F^{\mu 0}\right) - \frac{1}{4}\left(F^{0\mu} - F^{\mu 0}\right)$$
$$= -F^{0\mu}$$

$$\Rightarrow \pi^i = -F^{0i} = E^i$$
$$\& \ \pi^0 = -F^{00} = 0$$

This we already encounter in the last lecture. This is the consequence of the absence of kinetic term of $A^0$. Therefore, it is difficult to quantize the theory with with a consistent commutation relation between $A^0$ & $\pi^0$ since $[A^0, \pi^0] = 0$.

However, it is still possible to quantize the theory by setting $A^0 = 0$ and by taking relevant $A^i$ for the quantization.

Taking ...

(This exercise has been given as a problem in Coulomb gauge. I will not discuss it further here.)

This way of quantization lacks the manifest Lorentz covariance since some component of $A^\mu$ has special significance; they are not treated on equal footing. Thus, our effort of rewriting the Maxwell's theory in Lorentz covariance form goes in vain. Fortunately there is a way out.

Let us first look at the gauge choice we will making, namely Lorentz gauge.

The condition on $A_\mu$ as $\boxed{\partial^\mu A_\mu = 0}$

As before, if, e.g. $\partial^\mu A_\mu \neq 0$ but equal $f(x)$ then we can make a transformation

$$A'_\mu = A_\mu + \partial_\mu \lambda \quad \text{s.t.}$$

$$\boxed{\partial^\mu \partial_\mu \lambda = 0} \Rightarrow \partial^\mu A'_\mu = 0 .$$

So we can always make such choices

So, we can always

s.t. $\partial^\mu A_\mu = 0$. This is called Lorentz gauge because it has Lorentz covariant form.

There is another alternate reason to choose particular gauge. For that, let us rewrite the eqn of motion for the free field:

$$\partial_\alpha F^{\alpha\beta} = 0$$

$$\Rightarrow \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = 0$$

$$\Rightarrow \partial_\alpha \partial^\alpha \eta^{\beta\rho} A_\rho - \partial_\alpha \partial^\beta \eta^{\alpha\rho} A_\rho = 0$$

$$\Rightarrow \underbrace{\left(\eta^{\beta\rho} \Box - \partial^\beta \partial^\rho\right)} A_\rho = 0.$$

This operator is a singular operator and is not invertible. So, the Green's fn can't be defined. This can be seen if we try to mode expand $A_\rho$ as

$$A_\rho = \int \frac{d^4p}{(2\pi)^4} \xi_\rho(p)\, e^{ip.x}$$

then the operator in the momentum space becomes:

$$O_{\beta\rho} \equiv \eta_{\beta\rho}\Box - \partial_\beta \partial_\rho$$

$$\rightarrow \quad \eta_{\beta\rho}(-p^2) + p_\beta p_\rho$$

[we have lowered the indices for writing convenience

explicitly:

$$\boxed{p^2 = p_0^2 - p_1^2 - p_2^2 + p_3^2} \quad \begin{pmatrix} -p^2 + p_0^2 & p_0 p_1 & p_0 p_2 & p_0 p_3 \\ p_1 p_0 & p^2 - p_1^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_0 & p_2 p_1 & p^2 - p_2 & p_2 p_3 \\ p_3 p_0 & p_3 p_1 & p_3 p_2 & p^2 - p_3 \end{pmatrix}$$

One can easily check that this is a singular matrix, i.e. $\det(O_{\beta\rho}) = 0$.

Thankfully the two problems ① $\pi^0 = 0$ & $\det(O_{\beta\rho}) = 0$ can be avoided in a single go by modifying the Lagrangian in the Lorentz gauge

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} \partial_\alpha A^\alpha \partial_\beta A^\beta$$

This, although at this point does not represent the same Lagrangian of Maxwell's theory, the eq$^{\underline{n}}$ of motion stays the same in the Lorentz gauge.

Eq$^{\underline{n}}$ of motion: (with extra term)

$$\ldots^{\alpha\beta} \ldots \partial^{\beta}(\partial_\alpha A^{\alpha}) = 0.$$

$$\partial_\alpha F^{\alpha\beta} + \partial^\beta (\partial_\alpha A^\alpha)$$

$$\Rightarrow \quad \partial_\alpha \partial^\alpha A^\beta - \cancel{\partial_\alpha \partial^\beta A^\alpha} + \cancel{\partial^\beta \partial_\alpha A^\alpha} = 0$$

$$\Rightarrow \quad \boxed{\Box A^\beta = 0}$$

→ This is similar to scalar fields but for four $\beta = 0, 1, 2, 3$ fields.

This eq$\underline{\underline{n}}$. of motion is same eq$\underline{\underline{n}}$. of motion for $\mathcal{L}_{\text{Maxwell}}$ in presence of Lorentz gauge.

Now, the momentum conjugate becomes:

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$$

$$= -F^{0\mu} - \eta^{0\mu}\,\partial_\beta A^\beta .$$

$$\pi^{0} = -\partial_\beta A^\beta$$

$$\pi^{i} = -F^{0i}$$

We also follow the similar prescription for canonical quantization as done for scalar fields. So, first, we promote the fields $A_\mu$ & their momentum conjugate to operators with the following commutation relations:

$$[A_\mu(\vec{x}), A_\nu(\vec{y})] = [\pi^\mu(\vec{x}), \pi^\nu(\vec{x})] = 0$$

$$\& \; [A_\mu(\vec{x}), \pi_\nu(\vec{y})] = i\eta_{\mu\nu}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

Now, with the usual mode expansion

$$A_\mu(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{\lambda=0}^{3} \epsilon^{\lambda}_{\mu}(\vec{p})\left[a^{\lambda}_{\vec{p}}\, e^{ip.x} + a^{\lambda\,\dagger}_{\vec{p}}\, e^{-ip.x}\right]$$

$\epsilon^{\lambda}(\vec{p})$ are basis vectors for polarization

$\epsilon^{\lambda}(p)$ are ...

as introduced in the last lecture. Notice that there are a total of four vectors instead of 2. We we see that how these extra two will not affect our physics. For now, let us go ahead with these 4 polarization vectors.

We can chose the normalization as

$$\boxed{\epsilon^{\lambda}\cdot\epsilon^{\lambda'} = \eta^{\lambda\lambda'}}$$

we chose Lorentz invariant normalization

We can further chose $\vec{\epsilon}^{\,1}\cdot\vec{p} = \vec{\epsilon}^{\,2}\cdot\vec{p} = 0$ to keep $\epsilon^1$ & $\epsilon^2$ transnverse as before & $\vec{\epsilon}^{\,3}$ to be along $\vec{p}$.

The momentum conjugate takes the form

$$\begin{aligned}\pi^0(\vec{x}) &= -\partial^{\alpha} A_{\alpha} \\ &= -\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{\lambda=0}^{3}(+i)\, p^{\mu}\epsilon^{\lambda}_{\mu}(\vec{p}) \\ &\qquad \left[a^{\lambda}_{\vec{p}}\, e^{ip\cdot x} - a^{\lambda}_{\vec{p}}\, e^{-ip\cdot x}\right]\end{aligned}$$

0

<u>choices</u> for $p = (|\vec{p}|, 0, 0, |\vec{p}|)$

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

we can make this choice & $p \cdot \epsilon^1 = p \cdot \epsilon^2 = 0$

$$p \cdot \epsilon^0 = |\vec{p}| = -p \cdot \epsilon^3$$

$$A_\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2|\vec{p}|}}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \left[ a_{\vec{p}}^\lambda e^{ip.x} + a_{\vec{p}}^{\lambda +} e^{-ip.x} \right]$$

remember the normalization introduced in the scalar field theory. This is with massless. $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2} = |\vec{p}|$.

Now, $\pi^0(x) = -\partial^\mu A_\mu$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\cancel{2}\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \left[ a_{\vec{p}}^\lambda (ip^\mu) e^{ip.x} + a_{\vec{p}}^{\lambda +} (-ip^\mu) e^{-ip.x} \right]$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} (+i) \sum_{\lambda=0}^{3} p.\epsilon^\lambda(\vec{p}) \left[ a_{\vec{p}}^\lambda e^{ip.x} - a_{\vec{p}}^{\lambda +} e^{-ip.x} \right]$$

Only term with $p^0 = |\vec{p}|$ survive since $p^i \to -p^i$ picks up a $-$ve sign for the integrand. when integration is done from $-\infty$ to $+\infty$ over $p^i$

So, $$\pi^0(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \sqrt{\frac{|\vec{p}|}{2}} (+i) \sum_{\lambda=0}^{3} (\epsilon^0)^\lambda(\vec{p}) \left[ a_{\vec{p}}^\lambda e^{ip.x} - a_{\vec{p}}^{\lambda +} e^{-ip.x} \right]$$

Similarly, $$\pi^i(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \sqrt{\frac{|\vec{p}|}{2}} (+i) \sum_{\lambda=0}^{3} (\epsilon^i)^\lambda(\vec{p}) \left[ a_{\vec{p}}^\lambda e^{ip.x} - a_{\vec{p}}^{\lambda +} e^{-ip.x} \right].$$

Now, for the creation & annihilation operators. the commutation relations becomes

$$[a_{\vec{p}}^\lambda, a_{\vec{q}}^{\lambda'}] = [a_{\vec{p}}^{\lambda +}, a_{\vec{q}}^{\lambda +}] = 0$$

$$\& \ [a_{\vec{p}}^\lambda, a_{\vec{q}}^{\lambda' +}] = -\eta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

~~Let us now~~ (2)

$$[a^0_{\vec{p}}, a^{0\dagger}_{\vec{q}}] = -(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$

& $$[a^i_{\vec{p}}, a^{i\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}).$$

The space part is fine but the time part has a -ve sign, which gives rise to -ve norm.

Let ~~us~~ see how.

$$|\vec{p},\lambda\rangle = a^{\lambda\dagger}_{\vec{p}}|0\rangle$$

↳ vacuum.

Creation of a particle of momentum $\vec{p}$ & ~~be~~ polarization $\lambda$.

then $$\langle\vec{p},0|\vec{q},0\rangle = \langle 0|\, a^0_{\vec{p}}\, a^{0\dagger}_{\vec{q}}|0\rangle$$
$$= \langle 0|\{a^{0\dagger}_{\vec{q}}\, a^0_{p} - (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\}|0\rangle$$
$$= -(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$$

-ve norm.

However it can be resolved by restricting our ~~bit~~ Hilbert space to only those states which satisfy $\partial_\mu A^{\mu(+)}(x)|\psi\rangle = 0$.

<u>Meaning</u> that we can form the Hilbert space from vacuum state $|0\rangle$ by $a^{\lambda\dagger}_{\vec{p}}$ any number of time. But only the subset which satisfy $\partial_\mu A^{\mu(+)}|\psi\rangle = 0$ will be physical state.

(3)

where $A_\mu^{+}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda a_{\vec{p}}^\lambda e^{-ip\cdot x}$.

The condition is called Gupta-Bleuler condition.

So, any physical ~~statate~~ state should satisfy $\partial_\mu A^{\mu(+)} |\psi\rangle = 0$

Let us see if $|0\rangle$ (vacuum) satisfy it or not.

$$\partial_\mu A^{\mu(+)} |0\rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda \, p^\mu \, a_{\vec{p}}^\lambda e^{+ip\cdot x} |0\rangle$$

annihilates.

So, for any other state $|\psi\rangle$

$$\partial_\mu A^{\mu(+)} |\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{\lambda=0}^{3} p^\mu \epsilon_\mu^\lambda a_{\vec{p}}^\lambda e^{+ip\cdot x} |\psi\rangle_{phys}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \left[ p\cdot\epsilon^0 a_{\vec{p}}^0 + p\cdot\epsilon^3 a_{\vec{p}}^3 \right] e^{ip\cdot x} |\psi\rangle_{phys}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \sqrt{\frac{|\vec{p}|}{2}} \left[ a_{\vec{p}}^0 - a_{\vec{p}}^3 \right] e^{ip\cdot x} |\psi\rangle_{phys}$$

$$\Rightarrow \left[ a_{\vec{p}}^0 - a_{\vec{p}}^3 \right] |\psi\rangle_{phys} = 0.$$

For any physical state $a_{\vec{p}}^0 |\psi\rangle_{phys.} = a_{\vec{p}}^3 |\psi\rangle_{phys.}$

Whenever a state contains a timelike polarization, it should also contain a longitudinal photons

This also satisfy.

$$ {}_{phys.}\langle \Psi' | \partial_\mu A^\mu | \Psi \rangle_{phys.} = 0 $$

Now, the ~~H~~ Hamiltonian

$$ :H: = \dot{A}_\mu \pi^\mu - \mathcal{L} $$

normal ordered

$$ \cancel{= \int \frac{d^3\vec{p}\, d^3\vec{p}'}{(2\pi)^6} \sqrt{\frac{|\vec{p}'||\vec{p}|}{4}} \Big[ } $$

$$ = \int \frac{d^3\vec{p}}{(2\pi)^3} |\vec{p}| \left( \sum_{i=1}^{3} a^{i\dagger}_{\vec{p}} a^{i}_{\vec{p}} - a^{0\dagger}_{\vec{p}} a^{0}_{\vec{p}} \right) $$

$\langle \Psi | H | \Psi \rangle \rightarrow$ ensures that longitudinal & ~~transverse~~ timelike are cancelled properly.

$\cancel{\langle \Psi | H | \Psi \rangle} =$

Propagator:



$$\langle 0 | T\, A_\mu(x)\, A_\nu(x) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

Ex. Following the steps Ratul pointed out, show the ~~pr~~ expression for the propagator.