Klein-Gordon equation:

For nonrelativistic free particle: $E = \vec{p}^{\,2}/2m$

Substitutions: $E \to i\frac{\partial}{\partial t}$, $p_i \to i\frac{\partial}{\partial x^i}$ lead to Schrödinger eqn:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\psi$$, admits plane wave sol $\psi = e^{-iEt + i\vec{p}\cdot\vec{x}}$

For relativistic free particle: $E = \sqrt{\vec{p}^{\,2} + m^2}$

Above substitutions would lead to square-root of a differential operator!

One route: take $E^2 = \vec{p}^{\,2} + m^2$, above substitutions lead to

$$-\frac{\partial^2\phi}{\partial t^2} = -\vec{\nabla}^2\phi + m^2\phi$$ ← Klein-Gordon equation

Admit plane wave solutions $\phi = e^{-iEt + i\vec{p}\cdot\vec{x}}$ with eigen energy $E = \pm\sqrt{\vec{p}^{\,2} + m^2}$. can be arbitrarily large −ve. unbounded from below.

* bounded −ve energies appear, e.g. square well potential.

∴ Interpretation of $\phi(t,\vec{x})$ as wave function of a particle has problem

However, consistent interpretation in second quantized formulation, regard ϕ as a quantum field.

Klein-Gordon Field $\phi(x)$: Real Lorentz scalar

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

Klein-Gordon equation

EOM $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$, $\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$, $\boxed{(\Box + m^2)\phi = 0}$, $\Box \equiv \partial_\mu\partial^\mu$

$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(x)$ conjugate momentum

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L} = \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

$H(t) = \int d^3\vec{x}\ \mathcal{H}(t,\vec{x})$, however time invariant/conserved.

Try, $\phi(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}\left(a(\vec{k})e^{-ik\cdot x} + b(\vec{k})e^{ik\cdot x}\right)$, $k^0 = \omega_{\vec{k}}$, some function

Reality: $\phi(x)^* = \phi(x) \Rightarrow a(\vec{k})^* = b(\vec{k})$

$$(\Box + m^2)\phi(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}(-k^2 + m^2)\left(a(\vec{k})e^{-ik\cdot x} + a(\vec{k})^* e^{ik\cdot x}\right) = 0.$$

$\Rightarrow -k^2 + m^2 = 0 \Rightarrow -\omega_{\vec{k}}^2 + \vec{k}^2 + m^2 = 0 \Rightarrow \boxed{\omega_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}}$

+ is a conventional choice

$k = (\omega_{\vec{k}}, \vec{k})$, exponents: $-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}$, $i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}$

$\omega_{\vec{k}} \to -\omega_{\vec{k}}$, we have: $i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}$, $-i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}$, change: $\vec{k} \to -\vec{k}$, $d^3\vec{k} \to -d^3\vec{k}$,

we have: $\omega_{\vec{k}} = -\omega_{-\vec{k}}$; $i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}$, $-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}$,

$$\phi(x) \longrightarrow \iiint_{-\infty}^{+\infty} \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \Big(\underbrace{b(\vec{k})^*}_{\parallel \atop a(\vec{k})} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} + \underbrace{b(\vec{k})}_{\parallel \atop a(\vec{k})^*} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} \Big)$$

Roles of a, a^* are switched

We take: $\phi(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right)$, a, a^* are promoted to operators $a, a^\dagger$

$$\therefore \Pi(x) = \dot{\phi}(x) = -i \int d^3\vec{k} \sqrt{\frac{\omega_{\vec{k}}}{2(2\pi)^3}} \left(a(\vec{k}) e^{-ik\cdot x} - a^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

Ex.5.

Derive: $H = \frac{1}{2} \int d^3\vec{k}\, \omega_{\vec{k}} \left(a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right)$, clearly time invariant

Impose canonical commutation relations:

$$[\phi(x^0, \vec{x}), \Pi(x^0, \vec{y})] = i\,\delta^3(\vec{x} - \vec{y}),$$

$$[\phi(x^0, \vec{x}), \phi(x^0, \vec{y})] = 0 = [\Pi(x^0, \vec{x}), \Pi(x^0, \vec{y})].$$

Invert them to

$$[a(\vec{k}), a(\vec{k}')] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')], \quad [a(\vec{k}), a^\dagger(\vec{k}')] = \delta^3(\vec{k} - \vec{k}')$$

Ex.6. Derive

An useful relation: $\int d^3\vec{x}\, e^{\pm i\vec{x}\cdot(\vec{k} - \vec{k}')} = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$,

basically, for each dimension, $\int_{-\infty}^{+\infty} dx\, e^{\pm i x\cdot(k-k')} = (2\pi)\,\delta(k - k')$.

$$\int_{-L}^{+L} dx\, e^{\pm i x\cdot(k-k')} = 2\,\frac{\sin(k-k')L}{(k-k')}, \quad \int_{-\infty}^{+\infty} dk\, \frac{\sin kL}{k} = \pi \longleftarrow \text{independent of } L$$

$$H = \int d^3\vec{k}\, \omega_{\vec{k}}\, a^\dagger(\vec{k})a(\vec{k}) + \frac{1}{2}\int d^3\vec{k}\, \omega_{\vec{k}}\, \delta^3(\vec{0})$$

$$\delta^3(\vec{0}) = \delta^3(\vec{k}-\vec{k}) = \frac{1}{(2\pi)^3}\underbrace{\int d^3\vec{x}\, e^{i\vec{x}\cdot(\vec{k}-\vec{k})}}_{=\text{Vol}}$$

Define vacuum $|0\rangle$ s.t. $a(\vec{k})|0\rangle = 0 \ \forall \vec{k}$

$$H|0\rangle = \underbrace{\frac{1}{2}\int d^3\vec{k}\, \omega_{\vec{k}}\, \delta^3(0)}_{=E_0}|0\rangle$$

, vacuum energy density $= \dfrac{E_0}{\text{Vol}}$

$$\sim \int_0^\infty dk\, k^2 \sqrt{k^2+m^2} \longrightarrow \infty$$

Interested in differences in energy eigen values $\longrightarrow$ Remove the infinite constant by hand.

$$H = \int d^3\vec{k}\, \omega_{\vec{k}}\, a^\dagger(\vec{k})a(\vec{k}).$$

Ex. 7. Check- $[H, a^\dagger(\vec{p})] = \omega_{\vec{p}}\, a^\dagger(\vec{p})$, $[H, a(\vec{p})] = -\omega_{\vec{p}}\, a(\vec{p})$

$$[\underbrace{\tfrac{1}{2}\int d^3\vec{k}\, \omega_{\vec{k}}\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right)}_{H_{\text{Old}}}, a^\dagger(\vec{p})] = \frac{1}{2}\int d^3\vec{k}\, \omega_{\vec{k}}\left([a(\vec{k})a^\dagger(\vec{k}), a^\dagger(\vec{p})] + [a^\dagger(\vec{k})a(\vec{k}), a^\dagger(\vec{p})]\right)$$

$$[a(\vec{k})a^\dagger(\vec{k}), a^\dagger(\vec{p})] = [a(\vec{k}), a^\dagger(\vec{p})]a^\dagger(\vec{k}) = \delta^3(\vec{k}-\vec{p})a^\dagger(\vec{k})$$

$$[a^\dagger(\vec{k})a(\vec{k}), a^\dagger(\vec{p})] = a^\dagger(\vec{k})[a(\vec{k}), a^\dagger(\vec{p})] = \delta^3(\vec{k}-\vec{p})a^\dagger(\vec{k})$$

$[AB, C] = A[B, C] + [A, C]B$

Excited states: $|\vec{p}_1, \ldots, \vec{p}_N\rangle \equiv a^\dagger(\vec{p}_1)\ldots a^\dagger(\vec{p}_N)|0\rangle$

$H|\vec{p}_1, \ldots, \vec{p}_N\rangle = (\omega_{\vec{p}_1} + \cdots + \omega_{\vec{p}_N})|\vec{p}_1, \ldots, \vec{p}_N\rangle$

↑ energy of N free particles with momenta $\vec{p}_1, \ldots, \vec{p}_N$.

$\hat{N} \equiv \int d^3\vec{k}\, a^\dagger(\vec{k})a(\vec{k})$

should count the number of particles in a state.

$a^\dagger(\vec{p}_i), a^\dagger(\vec{p}_j)$ commute $\Rightarrow |\vec{p}_1, \ldots, \vec{p}_N\rangle$ has symmetry under interchange of labels $\vec{p}_i \longleftrightarrow \vec{p}_j$

single mode $\vec{p}$ can be populated $a(\vec{p})^n \longleftarrow$ Bose symmetry. $\longleftarrow$

1-particle states: $|\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle$

Normalized to $\rightarrow \langle\vec{p}|\vec{p}'\rangle = \langle 0|a(\vec{p})a^\dagger(\vec{p}')|0\rangle = \delta^3(\vec{p}-\vec{p}')$, $\langle 0|0\rangle = 1$.

not Lorentz invariant normalization

Indentity on 1-particle states

$$\mathbb{1} = \int d^3\vec{k}\, |\vec{k}\rangle\langle\vec{k}|, \text{ e.g. } \mathbb{1}|\vec{p}\rangle = \int d^3\vec{k}\, |\vec{k}\rangle\, \delta^3(\vec{k}-\vec{p}) = |\vec{p}\rangle$$

Lorentz inv.

$$\int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} = \int \underbrace{d^4k\, \delta(k^2-m^2)\,\Theta(k^0)}_{\text{Lorentz inv.}}$$

$$\left(= \int d^3\vec{k}\, dk^0\, \Theta(k^0) \left(\frac{\delta(k^0 - \sqrt{\vec{k}^2+m^2})}{|2k^0|} + \frac{\delta(k^0+\sqrt{\vec{k}^2+m^2})}{|2k^0|} \right)\right)$$

For $k^2 = m^2$, i.e. timelike k, Lorentz transformations connected to identity preserve sign of k^0.

$$\therefore\ \mathbb{1} = \int \frac{d^3\vec{k}}{2\omega_{\vec{k}}}\, 2\omega_{\vec{k}}\, |\vec{k}\rangle\langle\vec{k}| = \int \frac{d^3\vec{k}}{2\omega_{\vec{k}}}\, |k\rangle\langle k|,$$

$$|k\rangle = \sqrt{2\omega_{\vec{k}}}\, |\vec{k}\rangle = \sqrt{2\omega_{\vec{k}}}\, a^\dagger(\vec{k})|0\rangle$$

$$\Rightarrow \langle k|k'\rangle = 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{k}')$$ $\longleftarrow$ Lorentz inv. normalization

Matrix elements: $\Delta_+(x_1,x_2) \equiv \langle 0|\phi(x_1)\phi(x_2)|0\rangle$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{k}\, d^3\vec{k}'\, \frac{1}{\sqrt{2\omega_{\vec{k}}\, 2\omega_{\vec{k}'}}} \underbrace{\langle 0|a(\vec{k})\, a^\dagger(\vec{k}')|0\rangle}_{=\ \delta^3(\vec{k}-\vec{k}')} e^{-ik\cdot x_1 + ik'\cdot x_2}$$

$k = (\omega_{\vec{k}}, \vec{k})$, $k' = (\omega_{\vec{k}'}, \vec{k}')$

$x_1 = (x_1^0, \vec{x}_1)$, $x_2 = (x_2^0, \vec{x}_2)$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega_{\vec{k}}}\, e^{-ik\cdot(x_1-x_2)}$$

$$= \frac{1}{(2\pi)^4}\int d^4k\, 2\pi\,\delta(k^2-m^2)\,\theta(k^0)\, e^{-ik\cdot(x_1-x_2)}$$

Lorentz inv.: $\Delta_+(\Lambda x_1, \Lambda x_2) = \Delta_+(x_1,x_2)$ (Change of integration variable $k = \Lambda\tilde{k}$ would work)

Time ordered product: $T(\phi(x_1)\phi(x_2)\cdots\phi(x_n)) = \phi(x_{i_1})\phi(x_{i_2})\cdots\phi(x_{i_n})$

$(i_1,\ldots,i_n)$ = a permutation of $(1,2,\ldots,n)$ s.t. $x_{i_1}^0 \geq x_{i_2}^0 \geq \cdots \geq x_{i_n}^0$

E.g., $T(\phi(x_1)\phi(x_2)) = \theta(x_1^0 - x_2^0)\,\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\,\phi(x_2)\phi(x_1)$

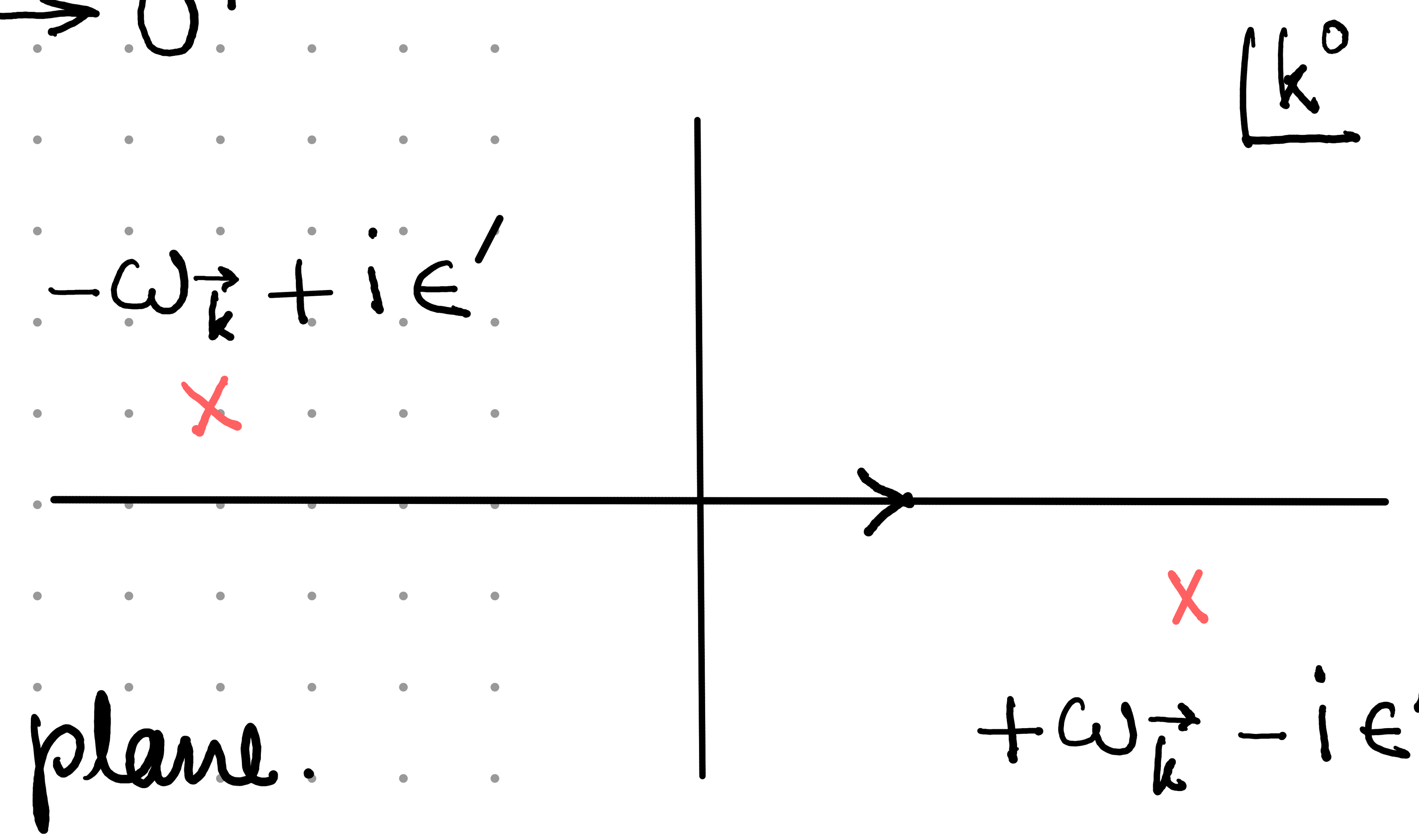
$\Delta_F(x_1,x_2) \equiv \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle = \Delta_F(x_2,x_1)$, $F \longleftarrow$ Feynman

$$\Delta_F(x_1,x_2) = \theta(x_1^0 - x_2^0)\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_{\vec{k}}}\, e^{-ik\cdot(x_1-x_2)} + \theta(x_2^0 - x_1^0)\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_{\vec{k}}}\, e^{-ik\cdot(x_2-x_1)}$$

Claim: $\Delta_F(x_1-x_2) = \frac{1}{(2\pi)^4}\int d^4k\, \tilde{\Delta}_F(k)\, e^{-ik\cdot(x_1-x_2)}$,

$$\tilde{\Delta}_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}, \quad \epsilon \longrightarrow 0^+$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk^0\, e^{-ik^0(x_1^0-x_2^0)}\, \frac{i}{k^{0\,2} - \vec{k}^2 - m^2 + i\epsilon}$$



when $x_1^0 > x_2^0$, close the contour in lower half plane.

$$-i(x_1^0-x_2^0)\underbrace{(R\cos\theta + iR\sin\theta)}_{k^0} \sim R(x_1^0-x_2^0)\sin\theta < 0.$$

$$(-2\pi i)\frac{1}{2\pi}\cdot e^{-i\omega_{\vec{k}}(x_1^0-x_2^0)}\cdot \frac{i}{\omega_{\vec{k}}+\omega_{\vec{k}}} = \frac{e^{-i\omega_{\vec{k}}(x_1^0-x_2^0)}}{2\omega_{\vec{k}}}.$$

when $x_1^0 < x_2^0$, close the contour in upper half plane, apply residue theorem.

$$(\Box_{x_1}+m^2)\, i\Delta_F(x_1-x_2) = \frac{1}{(2\pi)^4}\int d^4k\, \frac{-k^2+m^2}{k^2-m^2+i\epsilon}\, i^2\, e^{-ik\cdot(x_1-x_2)}$$

$$= \delta^4(x_1-x_2)$$

$\therefore\ i\Delta_F(x_1-x_2) \rightarrow$ Green's funtion of Klein-Gordon operator $\Box + m^2$

micro

Causality: $\langle 0|[\phi(x_1),\phi(x_2)]|0\rangle = \Delta_+(x_1,x_2) - \Delta_+(x_2,x_1)$

$\Delta(x_1,x_2) \equiv$ vanishes for $(x_1-x_2)^2<0$.

$\Delta_+(x_1,x_2) = \Delta_+(x_1-x_2)$. For $(x_1-x_2)^2<0$, $x_1-x_2 \longrightarrow x_2-x_1$ by continuous Lorentz transformation

Lorentz inv.

Alt. $\Delta(x_1,x_2)$ is Lorentz inv., For $(x_1-x_2)^2<0$, go to a frame s.t.

$x_1^0 = x_2^0 \Rightarrow [\phi(x_1),\phi(x_2)] = 0$.

(one of our quantization rule)

Divergence: $\Delta_+(x_1,x_2) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}(t_1-t_2)} e^{+i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}$

not absolutely convergent

$d^3\vec{k} = 4\pi k^2 dk$, $k=|\vec{k}|$, $\int^{\infty} 4\pi k^2 dk \frac{1}{\sqrt{k^2+m^2}} \sim 4\pi \int^{\infty} k\, dk$

$\frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}(t_1-t_2-i\epsilon)} e^{+i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}$, damping factor $e^{-\epsilon\omega_{\vec{k}}}$

$\longrightarrow$ convergent, $\epsilon \longrightarrow 0^+$ limit is finite for $(x_1-x_2)^2 \neq 0$

analytic continuation for $\Delta_+(x_1,x_2)$.

Check in mathematica.
$\vec{k}\cdot\vec{r} = k\, r\cos\theta$, $d^3\vec{k} = 2\pi k^2 \sin\theta\, d\theta$,
$k >> m$

$$\phi(x) = \int [\vec{k}] \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(k) e^{ik\cdot x} \right), \quad [\vec{k}] \equiv \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}, \quad k = (\omega_{\vec{k}}, \vec{k})$$

$$= \phi^-(x) + \phi^+(x)$$

<u>Normal ordered product</u> : move all the annihilators to the right

E.g. $N(a(\vec{k}) a^\dagger(\vec{k})) = a^\dagger(\vec{k}) a(\vec{k})$

$$N(\phi_1 \phi_2) = N(\phi_1^- \phi_2^- + \phi_1^- \phi_2^+ + \phi_1^+ \phi_2^- + \phi_1^+ \phi_2^+), \quad \phi_1 = \phi(x_1), \ \phi_2 = \phi(x_2)$$

$$= \phi_1^- \phi_2^- + \phi_2^+ \phi_1^- + \phi_1^+ \phi_2^- + \phi_1^+ \phi_2^+ \quad = N(\phi_2 \phi_1)$$

$$\longrightarrow \langle 0 | N \hat{O} | 0 \rangle = 0.$$

$$N(\phi_1 \phi_2) = \phi_1 \phi_2 - \phi_1^- \phi_2^+ + \phi_2^+ \phi_1^- = \phi_1 \phi_2 - [\phi_1^-, \phi_2^+]$$

$$\Rightarrow \phi_1 \phi_2 = N(\phi_1 \phi_2) + \Delta_+(x_1, x_2) \mathbb{1}$$

$$\therefore T(\phi_1 \phi_2) = \theta(x_1^0 - x_2^0) \phi_1 \phi_2 + \theta(x_2^0 - x_1^0) \phi_2 \phi_1$$

$$= \left(\theta(x_1^0 - x_2^0) + \theta(x_2^0 - x_1^0) \right) N(\phi_1 \phi_2)$$
$$+ \left(\theta(x_1^0 - x_2^0) \Delta_+(x_1, x_2) + \theta(x_2^0 - x_1^0) \Delta_+(x_2, x_1) \right) \mathbb{1}$$

$$[\phi_1^-, \phi_2^+]$$

$$= \iint [\vec{k}_1][\vec{k}_2] \underbrace{[a(\vec{k}_1), a^\dagger(\vec{k}_2)]}_{= \delta^{(3)}(\vec{k}_1 - \vec{k}_2)} e^{-ik_1\cdot x_1 + ik_2 \cdot x_2}$$

$$= \int \frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_{\vec{k}_1}} e^{-ik_1\cdot(x_1 - x_2)}$$

$$= \Delta_+(x_1, x_2)$$

$$\Rightarrow T(\phi_1\phi_2) = N(\phi_1\phi_2) + \Delta_F(x_1,x_2)\mathbb{1}$$

<u>Wick's theorem</u>: Define Wick contraction as

$$N(\wick{\phi_1\cdots\c\phi_l\cdots\c\phi_m\cdots\phi_n}) = N(\phi_1\cdots\phi_{l-1}\phi_{l+1}\cdots\phi_{m-1}\phi_{m+1}\cdots\phi_n)\,\Delta_F(x_l,x_m)$$

Time ordered product is equal to the normal ordered product, plus all possible ways of contracting pairs of fields within it.

<u>E.g.</u>
$$\begin{aligned}
T(\phi_1\phi_2\phi_3\phi_4) = {} & N(\phi_1\phi_2\phi_3\phi_4) + N(\wick{\c\phi_1\c\phi_2\phi_3\phi_4}) + N(\wick{\c\phi_1\phi_2\c\phi_3\phi_4}) \\
& + N(\wick{\c\phi_1\phi_2\phi_3\c\phi_4}) + N(\wick{\phi_1\c\phi_2\c\phi_3\phi_4}) + N(\wick{\phi_1\c\phi_2\phi_3\c\phi_4}) \\
& + N(\wick{\phi_1\phi_2\c\phi_3\c\phi_4}) + N(\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}) + N(\wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}) \\
& + N(\wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4})
\end{aligned}$$

Easier to prove, than Wick's theorem
Ex.8.

$$\Rightarrow \langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \Delta_F(x_1,x_2)\Delta_F(\phi_3\phi_4) + \Delta_F(x_1,x_3)\Delta_F(x_2,x_4) + \Delta_F(x_1,x_4)\Delta_F(x_2,x_3)$$

correlator