Homework 8

Extended Bridge to CS, Spring 2025

Akihito Chinda

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Question 7:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 6.1.5, sections b-d

   A five-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

   (b) What is the probability that the hand is a three of a kind? A three of a kind has three cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, {4 ♠, 4♦, 4♣, J♠, 8♥} is a three of a kind hand.

   Solution:

   Total probability: C(52,5)
   Total numbers of a three of a kind: C(13,1) × C(4,3)
   Total numbers of the other two cards: C(12,2) × 4 × 4 (The order does not matter here)
   Therefore:

   $$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2} \cdot 4^2}{\binom{52}{5}} \approx 2.11\%$$

   (c) What is the probability that all five cards have the same suit?

   Solution:

   Total probability: C(52,5)
   Total numbers of all five cards which have the same suit: C(13,5) × C(4,1)
   Therefore:

   $$\frac{\binom{13}{5}\binom{4}{1}}{\binom{52}{5}} \approx 0.198\%$$

   (d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4 ♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

   Solution:

Total probability: C(52,5)
Total numbers of a two of a kind: C(13,1) × C(4,2)
Total numbers of the other three cards: C(12,3) × 4 × 4 × 4 (The order does not matter here)
Therefore:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\cdot 4^3}{\binom{52}{5}} \approx 42.26\%$$

2. b) Exercise 6.2.4, sections a-d

A five-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(a) The hand has at least one club card.

Solution:

Total probability: C(52,5)
Total numbers of the hand which has NO club card: C(39,5)
Therefore:

$$1 - \frac{\binom{39}{5}}{\binom{52}{5}} \approx 77.85\%$$

(b) The hand has at least two cards with the same rank.

Solution:

Total probability: C(52,5)
Total numbers of the hand which has NO two cards with the same rank: C(13,5) × 4 × 4 × 4 × 4 × 4 (The order does not matter here)
Therefore:

$$1 - \frac{\binom{13}{5}\cdot 4^5}{\binom{52}{5}} \approx 49.29\%$$

(c) The hand has exactly one club card or exactly one spade card.

Solution:

Total probability: C(52,5)
Total numbers of the hand which has exactly one club card or exactly one spade card: C(13,1) × C(39,4) + C(13,1) × C(39,4)
Total numbers of the hand which has exactly one club card AND exactly one spade card: C(13,1) × C(13,1) × C(26,3)
Therefore:

$$\frac{\binom{13}{1}\binom{39}{4}+\binom{13}{1}\binom{39}{4}-\binom{13}{1}\binom{13}{1}\binom{26}{3}}{\binom{52}{5}} \approx 65.38\%$$

(d) The hand has at least one club card or at least one spade card.

Solution:

Total probability: C(52,5)
Total numbers of the hand which has NO club card or NO spade card: C(26,5)
Therefore:

$$1-\frac{\binom{26}{5}}{\binom{52}{5}} \approx 97.47\%$$

Question 8:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 6.3.2, sections a-e

   The letters $\{a, b, c, d, e, f, g\}$ are put in a random order. Each permutation is equally likely. Define the following events:

   - A: The letter b falls in the middle (with three before it and three after it)
   - B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "$agbdcef$" is an outcome in this event.
   - C: The letters "def" occur together in that order (for example, "$gdefbca$")

   (a) Calculate the probability of each individual event. That is, calculate $p(A)$, $p(B)$, and $p(C)$.

   Solution:

   $p(A) = \frac{1 \times P(6,6)}{P(7,7)} = \frac{1\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \frac{1}{7}$

   $p(B) = \frac{C(7,2)\times P(5,5)}{P(7,7)} = \frac{21\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \frac{1}{2}$

   $p(C) = \frac{5\times P(4,4)}{P(7,7)} = \frac{5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \frac{1}{42}$

   (b) What is $p(A|C)$?

   Solution:

   $p(A|C) = \frac{|A\cap C|}{|C|} = \frac{2\times P(3,3)}{5\times P(4,4)} = \frac{2\times3\times2\times1}{5\times4\times3\times2\times1} = \frac{1}{10}$

   (c) What is $p(B|C)$?

   Solution:

   $p(B|C) = \frac{|B\cap C|}{|C|} = \frac{C(5,2)\times P(3,3)}{5\times P(4,4)} = \frac{10\times3\times2\times1}{5\times4\times3\times2\times1} = \frac{1}{2}$

   (d) What is $p(A|B)$?

   Solution:

   $p(A|B) = \frac{|A\cap B|}{|B|} = \frac{1\times3\times P(5,5)}{C(7,2)\times P(5,5)} = \frac{1\times3\times5\times4\times3\times2\times1}{21\times5\times4\times3\times2\times1} = \frac{1}{7}$

   (e) Which pairs of events among A, B, and C are independent?

   Solution:

   For two of events X and Y are independent, then we have $p(X|Y) = p(X)$. From 1(a) to 1(d), we have

   $p(A|B) = p(A) = \frac{1}{7}$

   $p(B|C) = p(B) = \frac{1}{2}$

   $p(A|C) \neq p(A)$

   for event A, B and C. Therefore, (A, B) and (B, C) are independent.

2. b) Exercise 6.3.6, sections b, c

   A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is $\frac{1}{3}$ and the probability of tails is $\frac{2}{3}$. The outcomes of the coin flips are mutually independent. What is the probability of each event?

   (b) The first 5 flips come up heads. The last 5 flips come up tails.

   Solution:$\left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5$

   (c) The first flip comes up heads. The rest of the flips come up tails.

   Solution:$\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^9$

3. c) Exercise 6.4.2, section a

   (a) Assume that Ariel has two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4, and 5. Ariel chooses a die at random and rolls it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die Ariel chose is the fair die? The outcomes of the rolls are mutually independent.

   Solution:

   By using Bayes' theorem, suppose that F and X are events from the same sample space and $p(F) \neq 0$ and $p(X) \neq 0$. Then

$$p(F \mid X) = \frac{p(X \mid F)p(F)}{p(X \mid F)p(F) + p(X|\overline{F})p(\overline{F})} = \frac{p(X \mid F)p(F)}{p(X)}$$

   From definition, we have

$$P(F) = \frac{1}{2}$$
$$P(\overline{F}) = \frac{1}{2}$$
$$P(X \mid F) = \left(\frac{1}{6}\right)^6$$
$$P(X \mid \overline{F}) = 0.15^4 \cdot 0.25^2$$

   Then plugging into Bayes' theorem gives us the probability that the die Ariel chose is the fair die.

   Therefore:

$$= \frac{\left(\frac{1}{6}\right)^6 \cdot \frac{1}{2}}{\left(\frac{1}{6}\right)^6 \cdot \frac{1}{2} + 0.15^4 \cdot 0.25^2 \cdot \frac{1}{2}} \approx 0.404$$

Question 9:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 6.5.2, sections a, b

   A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

   (a) What is the range of A?

   Solution:$A = \{0, 1, 2, 3, 4\}$

   (b) Give the distribution over the random variable A.

   Solution:

   For $p(A = k)$ for $k = 0, 1, 2, 3, 4$, we have

   $p(A = 0) = \frac{\binom{48}{5}}{\binom{52}{5}} \approx 0.6588$

   $p(A = 1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \approx 0.2995$

   $p(A = 2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \approx 0.0399$

   $p(A = 3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} \approx 0.0017$

   $p(A = 4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \approx 0.0000185$

   Therefore:

   $\{(0, 0.6588)(1, 0.2995)(2, 0.0399)(3, 0.0017)(4, 0.0000185)\}$

2. b) Exercise 6.6.1, section a

   (a) Two student council representatives are chosen at random from a group of seven girls and three boys. Let G be the random variable denoting the number of girls chosen. What is $E[G]$?

   Solution:

   For expected value of $E[G]$, we have

   $p(G = 0) = \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{1}{15}$

   $p(G = 1) = \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{7}{15}$

   $p(G = 2) = \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{7}{15}$

   Therefore:

   $E[G] = 0 \times \frac{1}{15} + 1 \times \frac{7}{15} + 2 \times \frac{7}{15} = 1.4$

3. c) Exercise 6.6.4, sections a, b

(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then $X = 25$. What is $E[X]$?

Solution:

$$\begin{cases} X : S \to R \\ X = \text{the square of the number that shows up on the die.} \end{cases}$$

I. Distribution of X

r	$p(X = r)$
$1^2 = 1$	$1/6$
$2^2 = 4$	$1/6$
$3^2 = 9$	$1/6$
$4^2 = 16$	$1/6$
$5^2 = 25$	$1/6$
$6^2 = 36$	$1/6$

II. Expected value of X

Therefore: $E[X] = 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} + 16 \times \frac{1}{6} + 25 \times \frac{1}{6} + 36 \times \frac{1}{6} = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and $Y = 4$. What is $E[Y]$?

Solution:

$$\begin{cases} Y : S \to R \\ Y = \text{the square of the number of heads.} \end{cases}$$

I. Distribution of Y

r	$p(Y = r)$
$0^2 = 0$	$1/8$
$1^2 = 1$	$3/8$
$2^2 = 4$	$3/8$
$3^2 = 9$	$1/8$

II. Expected value of Y

Therefore: $E[Y] = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 4 \times \frac{3}{8} + 9 \times \frac{1}{8} = \frac{3+12+9}{8} = \frac{24}{8} = 3$

4. d) Exercise 6.7.4, section a

(a) Ten students in a class hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each student. What is the expected number of students who get his or her own coat?

Solution:

Let X be the random variable denoting the number of children who get his or her own coat. Define an indicator variable X_i for each student:

- $X_i = 1$ if student i gets their own coat (Only 1 of them is the correct one for student$_i$)
- $X_i = 0$ otherwise (the chance of not getting the correct one)

For expected value of $E[X_i]$, we have
$p(X_i = 0) = \frac{9}{10}$
$p(X_i = 1) = \frac{1}{10}$
$\therefore\ E[X_i] = 0 \times \frac{9}{10} + 1 \times \frac{1}{10} = \frac{1}{10}$

By linearity of expectation, the total expected number of students who get their own coat $E[X]$ is the sum of each $E[X_i]$:

$$E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 10 \cdot \frac{1}{10} = 1$$

Therefore, the expected number of students who get their own coat is 1.

Question 10:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 6.8.1, sections a-d

   The probability that a circuit board produced by a particular manufacturer has a defect is 1%. Assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

   (a) What is the probability that out of 100 circuit boards made exactly 2 have defects?

   Solution:

   Binomial distribution: $b(k; n, p) = \binom{n}{k} p^k q^{n-k}$, where $q = 1 - p$.

   Therefore: b(2; 100, 0.01) $= \binom{100}{2}(0.01)^2(0.99)^{98} \approx 0.185$

   (b) What is the probability that out of 100 circuit boards made at least 2 have defects?

   Solution:

   The probability that out of 100 circuit boards made 0 have defects:
   b(0; 100, 0.01) $= \binom{100}{0}(0.01)^0(0.99)^{100}$
   The probability that out of 100 circuit boards made 1 have defects:
   b(1; 100, 0.01) $= \binom{100}{1}(0.01)^1(0.99)^{99}$
   Therefore: $1 - (0.99)^{100} - \binom{100}{1}(0.01)^1(0.99)^{99} \approx 0.264$

   (c) What is the expected number of circuit boards with defects out of the 100 made?

   Solution:

   Let X be the random variable denoting the number of circuit boards with defects out of the 100 made. Define an indicator variable X_i for each defect:

   - $X_i = 1$ if board i is defective
   - $X_i = 0$ otherwise

   For expected value of $E[X_i]$, we have
   $p(X_i = 0) = \frac{99}{100}$
   $p(X_i = 1) = \frac{1}{100}$
   $\therefore E[X_i] = 0 \times \frac{99}{100} + 1 \times \frac{1}{100} = \frac{1}{100}$

   By linearity of expectation, the total expected number of circuit boards with defects out of the 100 made $E[X]$ is the sum of each $E[X_i]$:

   $$E\left[\sum_{i=1}^{100} X_i\right] = \sum_{i=1}^{100} E[X_i] = 100 \cdot \frac{1}{100} = 1$$

Therefore, the expected number of circuit boards with defects out of the 100 is 1.

(d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or both are free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do the answers compared to the situation in which each circuit board is made separately?

Solution:

The probability that No batch has defects:

b(0; 50, 0.01) $= \binom{50}{0}(0.01)^0(0.99)^{50}$

The probability that at least 1 batch has defects:

1 - $\binom{50}{0}(0.01)^0(0.99)^{50}$

Therefore, the probability that out of 100 circuit boards (50 batches) at least 2 have defects is: $1 - (0.99)^{50} \approx 0.395$

Let X be the random variable denoting the number of circuit boards with defects out of the 100 made and Y be the random variable denoting the number of batches with defects of the 100 made. Similarly from 1(c) for expected value of $E[Y]$, we have

$E[Y] = 50 \times \frac{1}{100} = \frac{1}{2}$

Therefore, the expected number of circuit boards with defects out of the 100 made is:

$E[X] = 2 \times E[Y] = 2 \times \frac{1}{2} = 1$

Even though both scenarios have the same expected number of defective boards (1), the batched scenario has a higher probability of seeing at least 2 defective boards.

This is because in the batched case, defects come in pairs — so a single defective batch already gives us 2 defective boards and the distribution is more concentrated in the batched case.

2. b) Exercise 6.8.3, section b

   A gambler has a coin that is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. The gambler selects one of the two coins in secret and flips that coin 10 times. An observer guesses which coin was selected as follows: if the number of heads is at least 4, the observer guesses that the fair coin was selected. If the number of heads is less than 4, the observer guesses that the biased coins was selected.

   (b) What is the probability that the observer reaches an incorrect conclusion if the coin is biased?

   Solution:

   The probability that the number of heads is 0:

   $p(H = 0) =$ b(0; 10, 0.3) $= \binom{10}{0}(0.3)^0(0.7)^{10}$

   The probability that the number of heads is at least 1:

$p(H = 1) = \text{b}(1;\, 10,\, 0.3) = \binom{10}{1}(0.3)^1(0.7)^9$
The probability that the number of heads is at least 2:
$p(H = 2) = \text{b}(2;\, 10,\, 0.3) = \binom{10}{2}(0.3)^2(0.7)^8$
The probability that the number of heads is at least 3:
$p(H = 3) = \text{b}(3;\, 10,\, 0.3) = \binom{10}{3}(0.3)^3(0.7)^7$

Therefore, the probability that the observer reaches an incorrect conclusion if the coin is biased (the number of heads is at least 4 and the biased coins was selected) is:
$1 - (0.7)^{10} - \binom{10}{1}(0.3)^1(0.7)^9 - \binom{10}{2}(0.3)^2(0.7)^8 - \binom{10}{3}(0.3)^3(0.7)^7 \approx 0.3504$