# Homework 7

Extended Bridge to CS, Spring 2025

Akihito Chinda

June 4, 2025

**Question 3**:

1. a. Solve Exercise 8.2.2, section b from the Discrete Math zyBook.

   Give complete proofs for the growth rates of the polynomials below. Provide specific values for $c$ and $n_0$ and prove algebraically that the functions satisfy the definitions for $O$ and $\Omega$.

   (b) $f(n) = n^3 + 3n^2 + 4$. Prove that $f$ is $\Theta(n^3)$.

   **Solution**:

   Proof: For $f = \Theta(n^3)$, we take $c_1$, $c_2$ and $n_0$ such that for all n $\geq n_0$.

   $$c_2 \cdot n^3 \leq n^3 + 3n^2 + 4 \leq c_1 \cdot n^3$$

   If we take $c_1 = 8$, $c_2 = 1$ and $n_0 = 1$, then for all $n \geq n_0$ we have:

   $$n^3 \leq n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4 \leq 4n^3 + 4 \leq 8n^3$$
   $$\therefore n^3 \leq n^3 + 3n^2 + 4 \leq 8n^3$$

   for all n $\geq 1$. Therefore:

   $$f = \Theta(n^3)$$

2. b. Solve Exercise 8.3.5, sections a-e from the Discrete Math zyBook

   The algorithm below makes some changes to an input sequence of numbers.

   MysteryAlgorithm

   Input: $a_1, a_2, \ldots, a_n$
   $n$, the length of the sequence.
   $p$, a number.
   Output: ??

   $i := 1$
   $j := n$

   While $(i < j)$
   While $(i < j$ and $a_i < p)$
   $i := i + 1$

End-while
While ($i < j$ and $a_j \geq p$)
$j := j - 1$
End-while
If ($i < j$), swap $a_i$ and $a_j$
End-while

Return($a_1, a_2, \ldots, a_n$)

(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with $p = 0$)

**Solution**:

All numbers less than $P$ are on the left, and all numbers greater than or equal to $P$ are on the right by the algorithm, using two pointers $i$ which starts from the left and $j$ which starts from the right.

The algorithm example step by step:

Input list: [3, -2, 5, -7, 1, -1]
$i = 1$
$j(n) = 6$
$p = 0$

- $a[i]$ = 3 is not $< 0$, so stop. $a[j]$ = -1 is $< 0$, so stop.
  Then swap $a[i]$ and $a[j]$: [**-1**, -2, 5, -7, 1, **3**] and increment $i \to 2$, decrement $j \to 5$ after swapped elements that are now assumed to be in the correct partition.
- $a[i]$ = -2 is $< 0$, then $i \to 3$. $a[j]$ = 1 is $\geq 0$, then $j \to 4$.
- $a[i]$ = 5 is not $< 0$, so stop. $a[j]$ = -7 is $< 0$, so stop.
  Then swap $a[i]$ and $a[j]$: [-1, -2, **-7**, **5**, 1, 3] and increment $i \to 4$, decrement $j \to 3$. Now $i \geq j$, so the loop ends.

(b) What is the total number of times that the lines "$i := i + 1$" or "$j := j - 1$" are executed on a sequence of length $n$? Does the answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

**Solution**:

The total number of increments and decrements combined is at most n because each index is visited at most once by $i$ and once by $j$.

Therefore, the sum of the number of times $i := i + 1$ and $j := j - 1$ is at most $n - 1$ because the loop stops when $i \geq j$, depending on just the length of the sequence.

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

**Solution**:

The number of swaps definitely depends on the values in the sequence because it is bounded by how many such misplaced pairs exist.

To maximize the number of swaps: Arrange the list so that every element on the left half is $\geq p$, and every element on the right half is $< p$.

Since each swap fixes two elements, the maximum number of swaps is at most $\lfloor \frac{n}{2} \rfloor$.

To minimize the number of swaps: Arrange the list so that all elements are already correctly partitioned, all $< p$ elements are on the left, and all $\geq p$ elements are on the right.

Since no swaps needed, the minimum number of swaps is 0.

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using $\Omega$) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that $i$ is incremented or $j$ is decremented).

**Solution**:

From 2(b) and 2(c), even in the best case (e.g., the list is already partitioned), the algorithm still needs to scan through the list to verify that no swaps are needed. Therefore, the asymptotic lower bound will be $\Omega(n)$.

It is not important to consider the wort-case input because the total number of pointer movements is at most $n - 1$, and this still gives us a worst-case time complexity of $\Omega(n)$.

(e) Give a matching upper bound (using $O$-notation) for the time complexity of the algorithm.

**Solution**:

From 2(b) and 2(c), the maximum runtime for the loops will be $n-1 + \lfloor \frac{n}{2} \rfloor$ in total. Therefore, the asymptotic upper bound will be $O(n)$.

**Question 4**:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 5.1.2, sections b, c

   Consider the following definitions for sets of characters:

   - Digits = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   - Letters = $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$
   - Special characters = $\{*, \&, \$, \#\}$

   Compute the number of passwords that satisfy the given constraints.

   (b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

   **Solution**:

   Total numbers of characters: 10 + 26 + 4 = 40
   Strings of length 7: $40 \times 7 = 40^7$
   Strings of length 8: $40 \times 8 = 40^8$
   Strings of length 9: $40 \times 9 = 40^9$

   Therefore: $40^7 + 40^8 + 40^9$

   (c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

   **Solution**:

   Total numbers of first characters: 10 + 4 = 14
   Strings of length 7: $14 \times 40 \times 6 = 14 \times 40^6$
   Strings of length 8: $14 \times 40 \times 7 = 14 \times 40^7$
   Strings of length 9: $14 \times 40 \times 8 = 14 \times 40^8$

   Therefore: $14 \times (40^6 + 40^7 + 40^8)$

2. b) Exercise 5.3.2, section a

   (a) How many strings are there over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" counts, and the strings "abbbcbabcb" and "aacbcbabcb" do not count.

   **Solution**:

   Total numbers of first characters: 3
   Total numbers of second- to tenth-characters: $2 \times 9 = 2^9$
   Therefore: $3 \times 2^9$ = 1,536.

3. c) Exercise 5.3.3, sections b, c

   License plate numbers in a certain state consist of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters ($A$ through $Z$), and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

(b) How many license plate numbers are possible if no digit appears more than once?

**Solution**:

Total numbers of digits: P(10,3) = 10 × 9 × 8 = 720
Total numbers of letters: 26 × 4 = $26^4$
Therefore: 720 × $26^4$ = 329,022,720.

(c) How many license plate numbers are possible if no digit or letter appears more than once?

**Solution**:

Total numbers of digits: P(10,3) = 10 × 9 × 8 = 720
Total numbers of letters: P(26,4) = 26 × 25 × 24 × 23 = 358,800
Therefore: 720 × 358,800 = 258,336,000.

4. d) Exercise 5.2.3, sections a, b

Let $B = \{0, 1\}$. $B^n$ is the set of binary strings with $n$ bits. Define the set $E_n$ to be the set of binary strings with $n$ bits that have an even number of 1s. Zero is an even number, so a string with zero 1s (that is, a string that is all 0s) has an even number of 1s.

(a) Show a bijection between $B^9$ and $E_{10}$. Explain why the function is a bijection.

**Solution**:

Define the function: $f : B^9 \to E_{10}$. The output of $f$ is obtained by adding a 0 to the end of the string if it has even number of 1s and add a 1 to the end of the string if it has odd number of 1s. For example, f (100000000) = 10000000001.

Since the function is both one-to-one and onto, and $f$ is bijection of one to one correspondence, it has a well-defined inverse. The inverse $f^{-1}$ reverses the correspondence given by as follows.

$$f^{-1} : E_{10} \to B^9$$

(b) What is $|E_{10}|$?

**Solution**:

From 4(a), there is a bijection between $B^9$ and $E_{10}$. Therefore, $|E_{10}| = |B^9| = 2^9 = 512$.

**Question 5**:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 5.4.2, sections a, b

   At a university in the United States, all phone numbers are seven digits long and start with either 824 or 825.

   (a) How many different phone numbers are possible?

   **Solution**:

   Total numbers of first three digits: 2
   Total numbers of fourth- to seventh-digit: $10 \times 4 = 10^4$
   Therefore: $2 \times 10^4 = 20{,}000$.

   (b) How many different phone numbers are there in which the last four digits are all different?

   **Solution**:

   Total numbers of first three digits: 2
   Total numbers of fourth- to seventh-digit: $P(10,4) = 10 \times 9 \times 8 \times 7 = 5{,}040$
   Therefore: $2 \times 5{,}040 = 10{,}080$.

2. b) Exercise 5.5.3, sections a-g

   How many 10-bit strings satisfy each of the following restrictions?

   (a) No restrictions.

   **Solution**:$2^{10} = 1{,}024$.

   (b) The string starts with 001.

   **Solution**:$1 \times 1 \times 1 \times 2^7 = 128$.

   (c) The string starts with 001 or 10.

   **Solution**:

   Total numbers of the string that starts with 001: $= 1 \times 1 \times 1 \times 2^7 = 128$
   Total numbers of the string that starts with 10: $= 1 \times 1 \times 2^8 = 256$
   Therefore: $128 + 256 = 384$.

   (d) The first two bits are the same as the last two bits.

   **Solution**:$1 \times 1 \times 2^8 = 256$.

   (e) The string has exactly six 0s.

   **Solution**:$\binom{10}{6} = \frac{10!}{6! \cdot 4!} = 210$.

   (f) The string has exactly six 0s and the first bit is 1.

**Solution**:$1 \times \binom{9}{6} = \frac{9!}{6! \cdot 3!} = 84$.

(g) The first half of the string contains exactly one 1, and the second half contains exactly three 1s.

**Solution**:$\binom{5}{1}\binom{5}{3} = \frac{5!}{1!} \times \frac{5!}{3! \cdot 2!} = 50$.

3. c) Exercise 5.5.5, section a

   (a) 30 boys and 35 girls try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make the selection?

   **Solution**:$\binom{30}{10}\binom{35}{10} = \frac{30!}{10! \cdot 20!} \times \frac{35!}{10! \cdot 25!} = 30{,}045{,}015 \times 183{,}579{,}396 = 5{,}515{,}645{,}706{,}510{,}940$.

4. d) Exercise 5.5.8, sections c-f

   This question refers to a standard deck of playing cards. An explanation of playing cards appeads in the "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of five cards from a deck of 52 cards.

   (c) How many five-card hands are made entirely of hearts and diamonds?

   **Solution**:$\binom{26}{5} = \frac{26!}{5! \cdot 21!} = 65{,}780$.

   (d) How many five-card hands have four cards of the same rank?

   **Solution**:$\binom{13}{1}\binom{4}{4} \times \binom{12}{1}\binom{4}{1} = \frac{13!}{1!} \times \frac{4!}{4!} \times \frac{12!}{1!} \times \frac{4!}{1!} = 13 \times 1 \times 12 \times 4 = 624$.

   (e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, $\{\text{queen of hearts}, \text{queen of spades}, 8 \text{ of diamonds}, 8 \text{ of spades}, 8 \text{ of clubs}\}$. How many five-card hands contain a full house?

   **Solution**:$\binom{13}{1}\binom{4}{2} \times \binom{12}{1}\binom{4}{3} = \frac{13!}{1!} \times \frac{4!}{2! \cdot 2!} \times \frac{12!}{1!} \times \frac{4!}{3! \cdot 1!} = 13 \times 6 \times 12 \times 4 = 3{,}744$.

   (f) How many five-card hands do not have any two cards of the same rank?

   **Solution**:$\binom{13}{5} \times 4 \times 5 = \frac{13!}{5! \cdot 8!} \times 4^5 = 1{,}287 \times 1{,}024 = 1{,}317{,}888$.

5. e) Exercise 5.6.6, sections a, b

   The country has two major political parties, the Democrats and the Republicans. Suppose that the national senate consists of 100 members, 44 of whom are Democrats and 56 of whom are Republicans.

   (a) How many ways are there to select a committee of 10 senate members with the same number of Democrats and Republicans?

   **Solution**:$\binom{44}{5}\binom{56}{5} = \frac{44!}{5! \cdot 39!} \times \frac{56!}{5! \cdot 51!} = 1{,}086{,}008 \times 3{,}819{,}816 = 4{,}148{,}350{,}734{,}528$.

   (b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

   **Solution**:$\binom{44}{1}\binom{43}{1} \times \binom{56}{1}\binom{55}{1} = \text{P}(44,2) \times \text{P}(56,2) = 44 \times 43 \times 56 \times 55 = 5{,}827{,}360$.

**Question 6**:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 5.7.2, sections a, b

   A 5-card hand is drawn from a deck of standard playing cards.

   (a) How many 5-card hands have at least one club?

   **Solution**:$\binom{52}{5} - \binom{39}{5} = \frac{52!}{5!\cdot 47!} - \frac{39!}{5!\cdot 34!}$ = 2,598,960 - 575,757 = 2,023,203.

   (b) How many 5-card hands have at least two cards with the same rank?

   **Solution**:$\binom{52}{5} - \binom{13}{5} \times 4 \times 5 = \frac{52!}{5!\cdot 47!} - \frac{13!}{5!\cdot 8!} \times 4^5$ = 2,598,960 - 1,287 $\times$ 1,024 = 2,598,960 - 1,317,888 = 1,281,072.

2. b) Exercise 5.8.4, sections a, b

   Twenty different comic books are distributed to five kids.

   (a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all twenty are given out)?

   **Solution**:$5^{20}$.

   Since each of the 20 books has 5 independent choices, the total number of ways to assign them is:

   $$5 \times 5 \times 5 \times \cdots \times 5 \text{ (20 times)} = 5^{20}$$

   (b) How many ways are there to distribute the comic books if they are divided evenly so that four go to each kid?

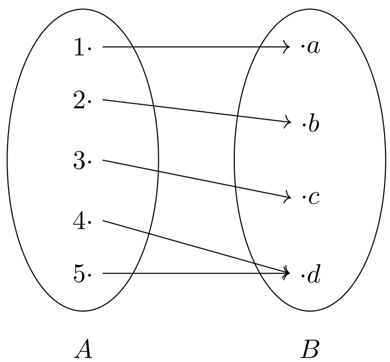
   **Solution**:$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{20!}{4!\cdot 16!} \times \frac{16!}{4!\cdot 12!} \times \frac{12!}{4!\cdot 8!} \times \frac{8!}{4!\cdot 4!} \times \frac{4!}{4!} = \frac{20!}{4!\cdot 4!\cdot 4!\cdot 4!\cdot 4!}$ = 305,540,235,000.

**Question 7**:

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4

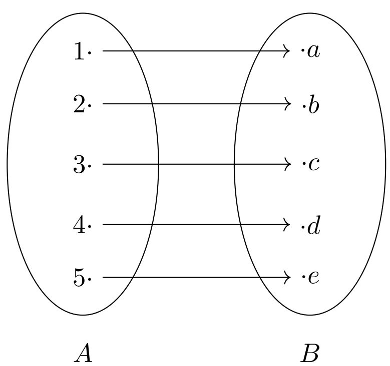
**Solution**:0.




There is no one-to-one function from a set $A$ with 5 elements to a set $B$ with 4 elements as shown above. Therefore, total number of one-to-one functions is 0.

b) 5

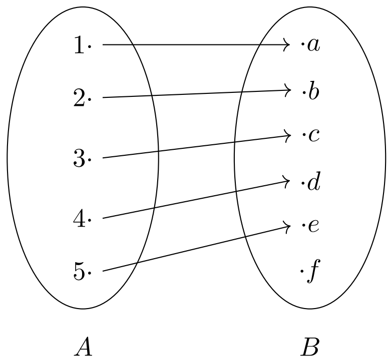
**Solution**:120.




There is a one-to-one function from a set $A$ with 5 elements to a set $B$ with 5 elements in which each of them can be mapped to exactly one element in the domain as shown above. Therefore, total number of one-to-one functions is P(5,5) $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

c) 6

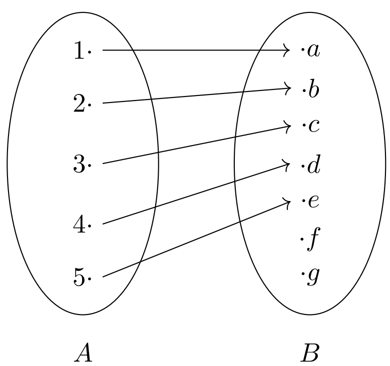
**Solution**:720.




There is a one-to-one function from a set $A$ with 5 elements to a set $B$ with 6 elements in which each of them can be mapped to exactly one element in the domain as shown above. Therefore, total number of one-to-one functions is P(6,5) $= 6 \times 5 \times 4 \times 3 \times 2 = 720$.

d) 7

**Solution**:2520.




There is a one-to-one function from a set $A$ with 5 elements to a set $B$ with 7 elements in which each of them can be mapped to exactly one element in the domain as shown above. Therefore, total number of one-to-one functions is P(7,5) $= 7 \times 6 \times 5 \times 4 \times 3 =$ 2,520.