Homework 2

Extended Bridge to CS, Spring 2025

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Question 5:

a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e

   Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of the argument and label each line of the proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

   (b)

$$\begin{array}{l} p \to (q \land r) \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Line #	Statement	Justification or Reasoning
1	$p \to (q \land r)$	Hypothesis 1
2	$\neg q$	Hypothesis 2
3	$\neg q \lor \neg r$	Addition on Line#2
4	$\neg(q \land r)$	De Morgan's law on Line#3
5	$\neg p$	Modus tollens on Line#1 and #4

   (e)

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \neg q \\ \hline \therefore r \end{array}$$

Line #	Statement	Justification or Reasoning
1	$p \lor q$	Hypothesis 1
2	$\neg p \lor r$	Hypothesis 2
3	$\neg q$	Hypothesis 3
4	p	Disjunctive syllogism on Line#1 and #3
5	$\neg\neg p$	Double negation law on Line#4
6	r	Disjunctive syllogism on Line#2 and Line#5

2. Exercise 1.12.3, section c

   Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

(c) One of the rules of inference is disjunctive syllogism:

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Prove that disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides disjunctive syllogism. (Hint: use one of the conditional identities from the laws of propositional logic).

Line #	Statement	Justification or Reasoning
1	$p \lor q$	Hypothesis 1
2	$\neg p$	Hypothesis 2
3	$\neg\neg p \lor q$	Double negation law on Line#1
4	$\neg p \to q$	Conditional identity on Line#3
5	q	Modus ponens on Line#2 and Line#4

3. Exercise 1.12.5, sections c, d

   Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

   (c)

   I will buy a new car and a new house only if I get a job.
   I am not going to get a job.

   $\therefore$ I will not buy a new car.

   $p =$ I will buy a new car

   $q =$ I will buy a new house

   $r =$ I will get a job

$$\begin{array}{l} (p \land q) \to r \\ \neg r \\ \hline \therefore \neg p \end{array}$$

   The conclusion derived from Modus Tollens on $(p \land q) \to r$ and $\neg r$ is $\neg(p \land q)$ as shown below.

$$\begin{array}{l} (p \land q) \to r \\ \neg r \\ \hline \therefore \neg(p \land q) \end{array}$$

   The argument is invalid because you need an additional premise like "q is true" to conclude.

   (d)

   I will buy a new car and a new house only if I get a job.
   I am not going to get a job.
   I will buy a new house.

   $\therefore$ I will not buy a new car.

$p =$ I will buy a new car

$q =$ I will buy a new house

$r =$ I will get a job

$$\begin{array}{l} (p \land q) \to r \\ \neg r \\ q \\ \hline \therefore \neg p \end{array}$$

The argument is valid as shown below.

Line #	Statement	Justification or Reasoning
1	$(p \land q) \to r$	Hypothesis 1
2	$\neg r$	Hypothesis 2
3	q	Hypothesis 3
4	$\neg(p \land q)$	Modus tollens on Line#1 and Line#2
5	$\neg p \land \neg q$	De Morgan's law on Line#4
6	$\neg\neg q$	Double negation law on Line#3
7	$\neg p$	Disjunctive syllogism on Line#5 and Line#6

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

   Show that the given argument is invalid by giving values for the predicates P and Q over the domain $\{a, b\}$.

   (b)

$$\begin{array}{l} \exists x(P(x) \lor Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

	P	Q
a	F	F
b	F	T

   In the domain set $\{a, b\}$, the two hypotheses, $\exists x(P(x) \lor Q(x))$ and $\exists x \neg Q(x)$ are both true for the values for P and Q on elements a and b given in the table. However, the conclusion $\exists x P(x)$ is false. Therefore, the argument is invalid.

2. Exercise 1.13.5, sections d, e

   Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.

The domain for each problem is the set of students in a class.

(d)

$$\begin{array}{l} \text{Every student who missed class got a detention.} \\ \textit{Penelope} \text{ is a student in the class.} \\ \textit{Penelope} \text{ did not miss class.} \\ \hline \therefore \textit{Penelope} \text{ did not get a detention.} \end{array}$$

U = set of students in a class

$M(x)$ = "x missed class"

$D(x)$ = "x got a detention"

$$\begin{array}{l} h1 : \forall x(M(x) \to D(x)) \\ h2 : \textit{Penelope} \text{ is an element} \\ h3 : \neg M(\textit{Penelope}) \\ \hline \therefore \neg D(\textit{Penelope}) \end{array}$$

The argument is valid as shown below.

Line #	Statement	Justification or Reasoning
1	$\forall x(M(x) \to D(x))$	Hypothesis 1
2	*Penelope* is an element	Hypothesis 2
3	$\neg M(\textit{Penelope})$	Hypothesis 3
4	$M(\textit{Penelope}) \to D(\textit{Penelope})$	Universal instantiation on Line#1
5	$\neg D$	Modus tollens on Line#3 and Line#4

(e)

$$\begin{array}{l} \text{Every student who missed class or got a detention did not get an A.} \\ \textit{Penelope} \text{ is a student in the class.} \\ \textit{Penelope} \text{ got an A.} \\ \hline \therefore \textit{Penelope} \text{ did not get a detention.} \end{array}$$

U = set of students in a class

$M(x)$ = "x missed class"

$D(x)$ = "x got a detention"

$A(x)$ = "x got an A"

$$\begin{array}{l} h1 : \forall x((M(x) \lor D(x)) \to \neg A(x)) \\ h2 : \textit{Penelope} \text{ is an element} \\ h3 : A(\textit{Penelope}) \\ \hline \therefore \neg D(\textit{Penelope}) \end{array}$$

The argument is valid as shown below.

Line #	Statement	Justification or Reasoning
1	$\forall x((M(x) \vee D(x)) \to \neg A(x))$	Hypothesis 1
2	$Penelope$ is an element	Hypothesis 2
3	$A(Penelope)$	Hypothesis 3
4	$(M(Penelope) \vee D(Penelope)) \to \neg A(Penelope)$	Universal instantiation on Line#1
5	$\neg\neg A(Penelope)$	Double negation law on Line#3
6	$\neg(M(Penelope) \vee D(Penelope))$	Modus tollens on Line#4 and Line#5
7	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's law on Line#6
8	$\neg D(Penelope)$	Simplification on Line#7

Table 1: Rules of inference known to be valid arguments and for quantified statements

Rule of inference	Name
p $p \to q$ $\therefore q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	Modus tollens
p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification
p q $\therefore p \wedge q$	Conjunction
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

Rule of inference	Name
c is an element (arbitrary or particular) $\forall x P(x)$ $\therefore P(c)$	Universal instantiation
c is an arbitrary element $P(c)$ $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore$ (c is a particular element) $\wedge P(c)$	Existential instantiation
c is an element (arbitrary or particular) $P(c)$ $\therefore \exists x P(x)$	Existential generalization

Question 6:

Solve Exercise 2.4.1, section d; Exercise 2.4.3, section b, from the Discrete Math zyBook:

1. Exercise 2.4.1, section d

   Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

   Prove each of the following statements using a direct proof.

   (d) The product of two odd integers is an odd integer.

   U = set of integers

   $O(xy)$ = "xy is odd"

   (a) Let x and y be two odd integers.

   (b) We show that(x is odd) $\land$ (y is odd) $\rightarrow$ (xy is odd)

      i. Since x is an odd integer, $x = 2k + 1$ for some integer k.

      ii. Since y is an odd integer, $y = 2n + 1$ for some integer n.

      iii.

$$\begin{aligned}\therefore xy &= (2k+1)(2n+1)\\ &= 4kn + 2k + 2n + 1\\ &= 2(2kn + k + n) + 1\end{aligned}$$

      $\therefore$ Since k and n are integers, $(2kn + k + n)$ will be an integer.
      $\therefore$ $2(2kn + k + n) + 1$ is also an odd integer.

   Therefore, the product of two odd integers is an odd integer.

2. Exercise 2.4.3, section b

   Prove each of the following statements using a direct proof.

   (b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

   (a) Assume that x is a real number and $x \leq 3$.

   (b) We show that $12 - 7x + x^2 \geq 0$.

      i. Since $12 - 7x + x^2 \geq 0$,

$$\equiv (x-3)(x-4) \geq 0$$

      ii. Since $x \leq 3$,

$$\equiv x - 3 \leq 0$$

iii. Since $x \leq 3$,

$$\equiv x - 4 \leq -1$$

$\therefore$ Since both $(x-3)$ and $(x-4)$ are less than or equal to 0,
$(x-3)(x-4)$ will be greater than or equal to 0.
$\therefore$ $12 - 7x + x^2$ is also greater or equal to 0.

Therefore, if x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Question 7:

Solve Exercise 2.5.1, section d; Exercise 2.5.4, sections a, b; Exercise 2.5.5, section c, from the Discrete Math zyBook:

1. Exercise 2.5.1, section d

   Prove each statement by contrapositive

   (d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

   (a) Assume that for every integer n, $n^2 - 2n + 7$ is even.
   (b) We show that n is odd.
   (c) We use proof by contrapositive, we then assume that $n^2 - 2n + 7$ is odd and n is even.
       i. Since n is an even integer, $n = 2k$ for some integer k.
       ii.

$$\begin{aligned} \therefore n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \end{aligned}$$

   $\therefore$ Since k is an integer, $(2k^2 - 2k + 3)$ will be an integer.
   $\therefore 2(2k^2 - 2k + 3) + 1$ is also an odd integer.

   Therefore, for every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

2. Exercise 2.5.4, sections a, b

   Prove each statement by contrapositive

   (a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

   (a) Assume that for every pair of real numbers x and y, $x^3 + xy^2 \leq x^2y + y^3$.
   (b) We show that $x \leq y$.
   (c) We use proof by contrapositive, we then assume that for every pair of real numbers x and y, $x \geq y$ and $x^3 + xy^2 \geq x^2y + y^3$.
       i. Since x and y are real numbers,

$$\begin{aligned} &\therefore x^2 \geq 0 \\ &\therefore y^2 \geq 0 \\ &\equiv x^2 + y^2 \geq 0 \end{aligned}$$

   $\therefore$ Since $(x^2 + y^2)$ is grater than or equal to 0, we multiply $(x^2 + y^2)$ on both sides of the inequity $x \geq y$.
   $\therefore x(x^2 + y^2) \geq y(x^2 + y^2)$
   $\therefore x^3 + xy^2 \geq x^2y + y^3$

   Therefore, for every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

(b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

(a) Assume that for every pair of real numbers x and y, $x + y > 20$.

(b) We show that $x > 10$ or $y > 10$.

(c) We use proof by contrapositive, we then assume that for every pair of real numbers x and y, $x < 10$ or $y < 10$ and $x + y < 20$.

i. Since x and y are real numbers,

$$\begin{aligned} &\therefore x < 10 \\ &\therefore y < 10 \\ &\equiv x + y < 20 \end{aligned}$$

ii. $\therefore$ Total

$$\begin{aligned} &= 9.9... + 9.9... \\ &\leq 20 \end{aligned}$$

Therefore, for every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

3. Exercise 2.5.5, section c

Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other.

(c) For every nonzero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

(a) Assume that for every nonzero real number x, x is irrational.

(b) We show that $\frac{1}{x}$ is also irrational.

(c) We use proof by contrapositive, we then assume that for every nonzero real number x, x is rational and $\frac{1}{x}$ is also rational.

i. Since x is nonzero real number, $\frac{1}{x} = \frac{a}{b}$ for some integers a and b.

$$\begin{aligned} &\therefore \frac{1}{x} = \frac{a}{b} \\ &\equiv x = \frac{b}{a} \end{aligned}$$

Therefore, for every nonzero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Question 8:

Solve Exercise 2.6.6, sections c, d, from the Discrete Math zyBook:

1. Exercise 2.6.6, sections c, d

   Give a proof for each statement.

   (c) The average of three real numbers is greater than or equal to at least one of the numbers.

   (a) Assume that the average of three real numbers is greater than or equal to at least one of the numbers.

   (b) We show that $\frac{x+y+z}{3} \geq x, y$ or z.

   (c) We use proof by contradiction, we then assume that the average of three real numbers is less than any of these numbers.

      i. Since three numbers are real numbers, the average $= \frac{x+y+z}{3}$ for some integers x, y and z.

$$\therefore \frac{x+y+z}{3} < x$$
$$\therefore \frac{x+y+z}{3} < y$$
$$\therefore \frac{x+y+z}{3} < z$$
$$\equiv x+y+z < x+y+z$$

      This is a contradiction as the average of three real numbers should be greater than or equal to at least one of the numbers.

   Therefore, the average of three real numbers is greater than or equal to at least one of the numbers.

   (d) There is no smallest integer.

   (a) Assume that there is no smallest integer.

   (b) We show that $x < x - 1$.

   (c) We use proof by contradiction, we then assume that there is a smallest integer.

      i. Since the number is an integer, the smallest integer $= x - 1 < x$ for some integers x.

$$\therefore x - 1 < x$$

      This is a contradiction as x should be the smallest number.

   Therefore, there is no smallest integer.

Question 9:

Solve Exercise 2.7.2, section b, from the Discrete Math zyBook:

1. Exercise 2.7.2, section b

   Prove each statement.

   (b) If integers x and y have the same parity, then $x + y$ is even.

   The parity of a number tells whether the number is odd or even. If x and y have the same parity, either both even or both are odd.

   (a) Assume that integers x and y have the same parity.

   (b) We show that $x + y$ is even.

      i. Case1: Since x and y are both even integers, $x = 2k$ and $y = 2n$ for some integer k and n.

      ii.

$$\begin{aligned} \therefore x + y &= 2k + 2n \\ &= 2(k + n) \end{aligned}$$

      $\therefore$ Since k and n are integers, $(k + n)$ will be an integer.
      $\therefore$ $2(k + n)$ is also an even integer.

      i. Case2: Since x and y are both odd integers, $x = 2k + 1$ and $y = 2n + 1$ for some integer k and n.

      ii.

$$\begin{aligned} \therefore x + y &= (2k + 1) + (2n + 1) \\ &= 2(k + n + 1) \end{aligned}$$

      $\therefore$ Since k and n are integers, $(k + n + 1)$ will be an integer.
      $\therefore$ $2(k + n + 1)$ is also an even integer.

   Therefore, if integers x and y have the same parity, then $x + y$ is even.