Homework 11

Extended Bridge to CS, Spring 2025

Akihito Chinda

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Question 5:

1. a. Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder).

   Hint: you may want to use the formula: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

   Solution:

   Proof: By induction on n

   Let $P(n)$ be the proposition that 3 divide $n^3 + 2n$ (leaving no remainder) for any positive integer n.

   Base case / Basis step:

   When $n = 1$,

   $$n^3 + 2n = 1^3 + 2 \times 1 = 3$$

   $$\therefore \text{When } n = 1, P(1) \text{ is true.}$$

   Induction step:

   Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k, that is $k^3 + 2k$ is divisible by 3.

   Now we have to show that $P(k+1)$ is also true, that is $(k+1)^3 + 2(k+1)$ is divisible by 3.

   $$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + (3k^2 + 3k + 3) \\ &= (k^3 + 2k) + 3(k^2 + k + 1)\end{aligned}$$

   From induction hypothesis, we assume 3 divide $(k^3 + 2k)$, and since $(k^2 + k + 1)$ is a positive integer, 3 divide $(k+1)^3 + 2(k+1)$.

   Therefore, if $P(n)$ is true for $n = k$, we show that $P(n)$ is also true for $n = k + 1$.

2. b. Use strong induction to prove that any positive integer n $(n \geq 2)$ can be written as a product of primes.

   Solution:

   Proof: By induction on n

   Let $P(n)$ be the proposition that any positive integer n $(n \geq 2)$ can be written as a product of primes.

Base case / Basis step:

When $n = 2$, then it is a product of primes (just itself) since 2 is a prime number.

Therefore, when $n = 2$, P(2) is true.

Induction step:

Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k ($k \geq 2$), that is k can be written as a product of primes.

Now we have to show that $P(k+1)$ is also true, that is $k+1$ can be written as a product of primes.

Case 1: If $k+1$ is a prime number, then it is a product of primes (just itself) since $k+1$ is a prime number.

Case 2: If $k+1$ is NOT a prime number, then it can be written as a product of two integers a and b such that:

$$k + 1 = a \cdot b, \text{where } 2 \leq a \leq b < k + 1.$$

From induction hypothesis, we assume both a and b can be written as a product of primes since any positive integer k ($k \geq 2$) can be written as a product of primes.

Therefore, if $P(n)$ is true for $n = k$, we show that $P(n)$ is also true for $n = k + 1$.

Question 6:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 7.4.1, sections a-g

   Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that $P(3)$ is true.

Solution:

$$\sum_{j=1}^{n} j^2 = \sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\frac{n(n+1)(2n+1)}{6} = \frac{3(3+1)(2 \cdot 3 + 1)}{6} = 14$$

Therefore, when $n = 3$, $P(3)$ is true.

(b) Express $P(k)$.

Solution:

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(c) Express $P(k+1)$.

Solution:

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

(d) In an inductive proof that for every positive integer n,
$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
what must be proven in the base case?

Solution:When $n = 1$, $P(1)$ is true.

(e) In an inductive proof that for every positive integer n,
$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
what must be proven in the inductive step?

Solution:If $P(k)$ is true, then $P(k+1)$ is also true.

(f) What is the inductive hypothesis in the inductive step from the previous answer?

Solution:$P(k)$ is true for arbitrary positive integer k ($k \geq 1$).

(g) Prove by induction that for any positive integer n,
$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Proof: By induction on n

Let $P(n)$ be the proposition that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

for any positive integer n.

Base case / Basis step:

When $n = 1$,

$$\sum_{j=1}^{n} j^2 = \sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$$

$$\therefore \text{When } n = 1, P(1) \text{ is true.}$$

Induction step:

Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k.

Now we have to show that $P(k+1)$ is also true.

From induction hypothesis, we have

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 && \text{[By definition]} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{[Using induction hypothesis]} \\
&= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right) \\
&= (k+1)\left(\frac{k(2k+1) + 6(k+1)}{6}\right) \\
&= (k+1)\left(\frac{2k^2 + 7k + 6}{6}\right) \\
&= (k+1)\left(\frac{(k+2)(2k+3)}{6}\right) \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Since this matches the right-hand side of $P(k+1)$, the formula holds for $k+1$.

Therefore, if $P(n)$ is true for $n = k$, we show that $P(n)$ is also true for $n = k+1$.

2. b) Exercise 7.4.3, section c

   Hint: you may want to use the following fact: $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$

   (c) Prove that for $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

   Solution:

   Proof: By induction on n

   Let $P(n)$ be the proposition that:

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

   for any positive integer $n \geq 1$.

   Base case / Basis step:

   When $n = 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} = \sum_{j=1}^{1} \frac{1}{j^2} = \frac{1}{1^2} = 1$$

$$2 - \frac{1}{n} = 2 - \frac{1}{1} = 1$$

$$\therefore \text{When } n = 1, P(1) \text{ is true.}$$

   Induction step:

   Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k.

   Now we have to show that $P(k+1)$ is also true.

   From induction hypothesis, we have

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} && \text{[By definition]} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{[Using induction hypothesis]} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} && \text{[Using the fact: } \frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}\text{]} \\
&\leq 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} \\
&\leq 2 - \frac{k}{k(k+1)} \\
&\leq 2 - \frac{1}{k+1}
\end{aligned}$$

   Therefore, if $P(n)$ is true for $n = k$, we show that $P(n)$ is also true for $n = k+1$.

3. c) Exercise 7.5.1, section a

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Solution:

Proof: By induction on n

Let $P(n)$ be the proposition that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Base case / Basis step:

When $n = 1$,

$$3^{2n} - 1 = 3^{2 \cdot 1} - 1 = 8$$

$$\therefore \text{When } n = 1, P(1) \text{ is true.}$$

Induction step:

Assume the induction hypothesis that $P(k)$ is true for arbitrary positive integer k ($k \geq 1$), that is 4 evenly divides $3^{2k} - 1$.

Now we have to show that $P(k+1)$ is also true, that is 4 evenly divides $3^{2(k+1)} - 1$.

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^{2k} \cdot 3^2 - 1 \\ &= 3^{2k} \cdot 9 - 9 + 8 \\ &= 9(3^{2k} - 1) + 8 \\ &= 9(4m) + 8 \\ &= 36m + 8 \\ &= 4(9m + 2) \end{aligned}$$

From induction hypothesis, we assume 4 evenly divides $3^{2k} - 1$, and since $9m + 2$ is a positive integer, 4 evenly divides $3^{2(k+1)} - 1$.

Therefore, if $P(n)$ is true for $n = k$, we show that $P(n)$ is also true for $n = k + 1$.