Homework 5

Extended Bridge to CS, Spring 2025

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Question 3:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 4.1.3, sections b, c

   Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

   (b) $f(x) = \frac{1}{x^2-4}$

   When x = 2 (or x = -2), a function doesn't map any elements from $\mathbb{R}$ to $\mathbb{R}$. Therefore, $f(x)$ is not a function.

   (c) $f(x) = \sqrt{x^2}$

   For every x, there is exactly one $f(x)$. Therefore, $f(x)$ is a function. The range of the function is [0, ∞].

2. b) Exercise 4.1.5, sections b, d, h, i, l

   Express the range of each function using roster notation.

   (b) Let $A = \{2, 3, 4, 5\}$. $f : A \to \mathbb{Z}$, such that $f(x) = x^2$.

   The range of the function is {4, 9, 16, 25}.

   (d) $f : \{0, 1\}^5 \to \mathbb{Z}$. For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1s that occur in x. For example $f(01101) = 3$, because there are three 1s in the string "01101".

   A 5-bit binary string can have anywhere from 0 to 5 ones. Therefore, the range of the function is {0, 1, 2, 3, 4, 5}.

   (h) Let $A = \{1, 2, 3\}$. $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$.

   The range of the function is {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)}.

   (i) Let $A = \{1, 2, 3\}$. $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$.

   The set should include all possible pairs $(x, y+1)$ where x and y are elements of A. Therefore, the range of the function is {(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}.

   (l) Let $A = \{1, 2, 3\}$. $f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - \{1\}$.

   $f(\emptyset) = \emptyset - \{1\} = \emptyset$ / $f(\{1\}) = \{1\} - \{1\} = \emptyset$ / $f(\{2\}) = \{2\} - \{1\} = \{2\}$ / $f(\{3\}) = \{3\} - \{1\} = \{3\}$ / $f(\{1, 2\}) = \{1, 2\} - \{1\} = \{2\}$ / $f(\{1, 3\}) = \{1, 3\} - \{1\} = \{3\}$ / $f(\{2, 3\}) = \{2, 3\} - \{1\} = \{2, 3\}$ / $f(\{1, 2, 3\}) = \{1, 2, 3\} - \{1\} = \{2, 3\}$

   Therefore, the range of the function is $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$.

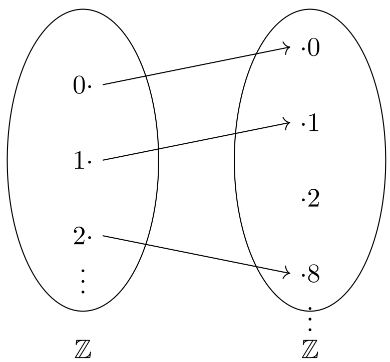
Question 4:

I. Solve the following questions from the Discrete Math zyBook:

1. a. Exercise 4.2.2, sections c, g, k

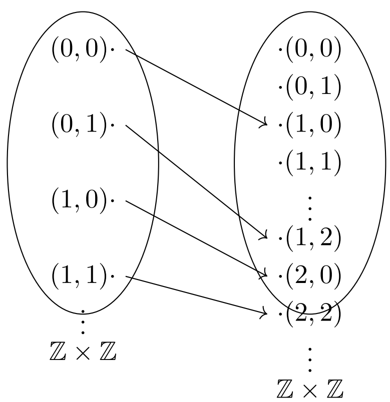
   For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

   (c) $h : \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3$



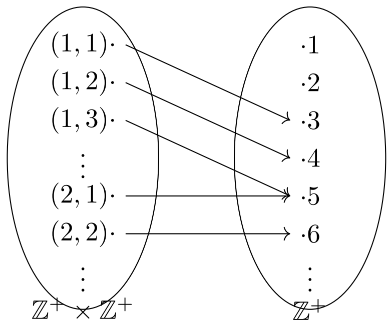
   There does not exist $x \in \mathbb{Z}$ such that $h(x) = 2$ as shown above. Therefore, the function is one-to-one but not onto.

   (g) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 1, 2y)$



   There does not exist a pair of $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(x, y) =$ (0, 0), (0, 1) or (1, 1) as shown above. Therefore, the function is one-to-one but not onto.

   (k) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x, y) = 2^x + y$.

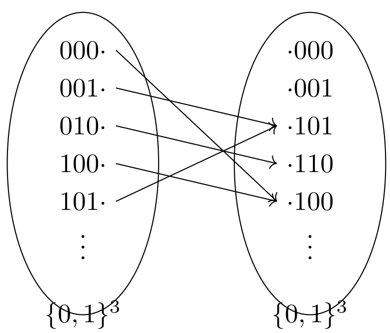


$(1,3) \neq (2,1)$ but $f(1,3) = f(2,1)$ and there does not exist a pair of $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $f(x, y) = 1$ or 2 as shown above. Therefore, the function is neither one-to-one nor onto.

2. b. Exercise 4.2.4, sections b, c, d, g

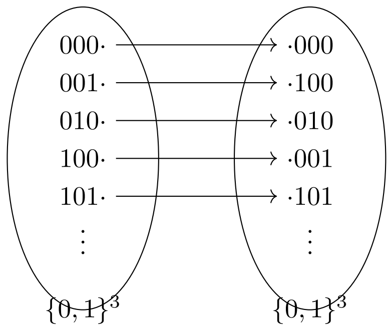
For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(b) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.



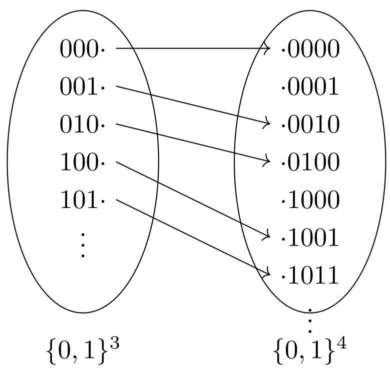
$000 \neq 100$ or $001 \neq 101$ but $f(000) = f(100)$ or $f(001) = f(101)$ and there does not exist a string $x \in \{0, 1\}^3$ such that $f(x) = 000$ or 001 as shown above. Therefore, the function is neither one-to-one nor onto.

(c) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.



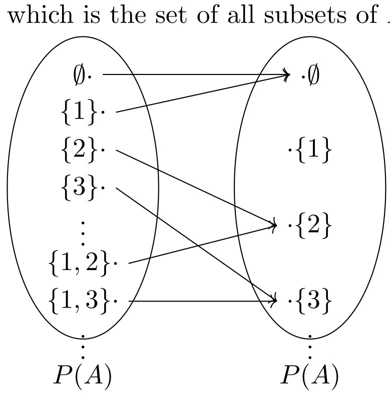
A function maps all elements from $\{0,1\}^3$ to $\{0,1\}^3$ respectively without any duplicates. Therefore, the function is one-to-one and onto.

(d) $f : \{0,1\}^3 \to \{0,1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.



There does not exist a string $x \in \{0,1\}^3$ such that $f(x) = 0001$ or 1000 as shown above. Therefore, the function is one-to-one but not onto.

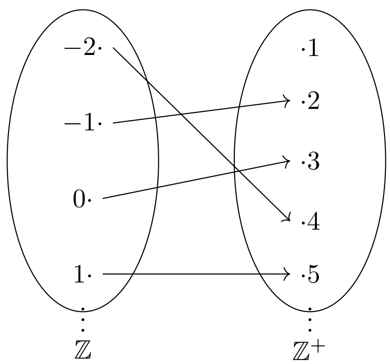
(g) Let A be defined to be the set $\{1,2,3,4,5,6,7,8\}$ and let $B = \{1\}$. $f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.



$\{2\} \neq \{1,2\}$ or $\{3\} \neq \{1,3\}$ but $f(\{2\}) = f(\{1,2\})$ or $f(\{3\}) = f(\{1,3\})$ and there does not exist $P(A)$ such that for $X \subseteq A$, $f(X) = \{1\}$ as shown above. Therefore, the function is neither one-to-one nor onto.

II. Give an example of a function from the set of integers to the set of positive integers that is:
a. one-to-one, but not onto.

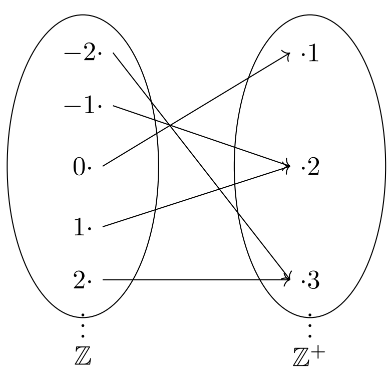
$$f : \mathbb{Z} \to \mathbb{Z}^+, \quad f(x) = \begin{cases} 2x+3 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$



There does not exist $\mathbb{Z} \to \mathbb{Z}^+$ such that $f(x) = 1$ as shown above. Therefore, the function is one-to-one but not onto.

b. onto, but not one-to-one.

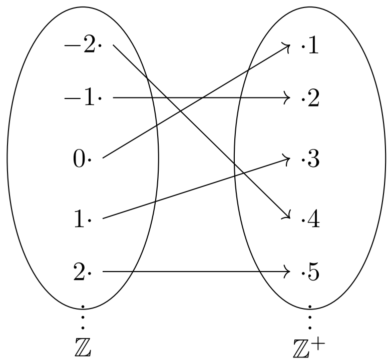
$$f : \mathbb{Z} \to \mathbb{Z}^+, \quad f(x) = |x| + 1$$



$-2 \neq 2$ or $-1 \neq 1$ but $f(-2) = f(2)$ or $f(-1) = f(1)$ as shown above. Therefore, the function is onto but not one-to-one.

c. one-to-one and onto.

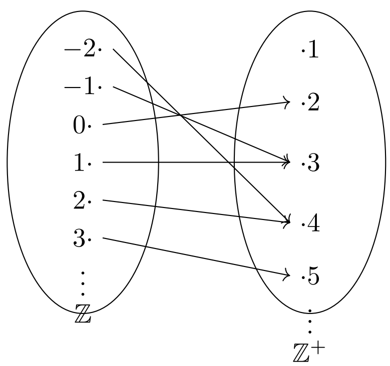
$$f : \mathbb{Z} \to \mathbb{Z}^+, \quad f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$



A function maps all elements from $\mathbb{Z}$ to $\mathbb{Z}^+$ respectively without any duplicates. Therefore, the function is one-to-one and onto.

d. neither one-to-one nor onto

$$f : \mathbb{Z} \to \mathbb{Z}^+, \quad f(x) = |x| + 2$$



$-2 \neq 2$ or $-1 \neq 1$ but $f(-2) = f(2)$ or $f(-1) = f(1)$ and there does not exist $\mathbb{Z} \to \mathbb{Z}^+$ such that $f(x) = 1$ as shown above. Therefore, the function is neither one-to-one nor onto.

Question 5:

Solve the following questions from the Discrete Math zyBook:

1. a) Exercise 4.3.2, sections c, d, g, i

   For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

   (c) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 2x + 3$

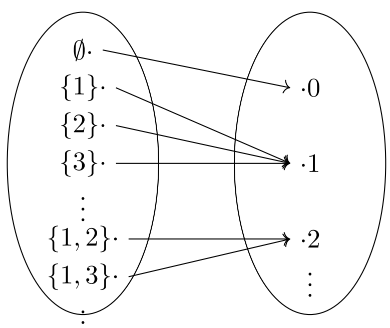
   Since the function is both one-to-one and onto, and f is bijection of one to one correspondence, it has a well-defined inverse.

$$
\begin{aligned}
y &= 2x + 3 \\
x &= \frac{y-3}{2} \\
f^{-1}(y) &= \frac{y-3}{2}
\end{aligned}
$$

   Therefore, the inverse function f^{-1} is:

$$f^{-1} : \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \frac{x-3}{2}$$

   (d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ $f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set A, $P(A)$ denotes the power set of A, which is the set of all subsets of A.



$P(A)$ $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

   $\{1\} \neq \{2\}$ or $\{1, 2\} \neq \{1, 3\}$ but $f\{1\} = f\{2\}$ or $f\{1, 2\} = f\{1, 3\}$ as shown above. Therefore, the function is not one-to-one and doesn't have a well-defined inverse.

   (g) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.

   Since the function is both one-to-one and onto, and f is bijection of one to one correspondence, it has a well-defined inverse. The inverse f^{-1} reverses the correspondence given by as follows.

$$f^{-1} : \{0, 1\}^3 \to \{0, 1\}^3$$

(i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 5, y - 2)$

Since the function is both one-to-one and onto, and f is bijection of one to one correspondence, it has a well-defined inverse.

$$\begin{aligned}(a, b) &= (x + 5, y - 2)\\ x &= a - 5\\ y &= b + 2\\ f^{-1}(a, b) &= (a - 5, b + 2)\end{aligned}$$

Therefore, the inverse function f^{-1} is:

$$f^{-1} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x - 5, y + 2)$$

2. b) Exercise 4.4.8, sections c, d

   The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:

   - $f(x) = 2x + 3$
   - $g(x) = 5x + 7$
   - $h(x) = x^2 + 1$

   Give an explicit formula for each function given below.

   (c) $f \circ h$

$$\begin{aligned}f \circ h &= f(h(x))\\ &= f(x^2 + 1)\\ &= 2 \cdot (x^2 + 1) + 3\\ &= 2x^2 + 5\end{aligned}$$

   (d) $h \circ f$

$$\begin{aligned}h \circ f &= h(f(x))\\ &= h(2x + 3)\\ &= (2x + 3)^2 + 1\\ &= 4x^2 + 12x + 10\end{aligned}$$

3. c) Exercise 4.4.2, sections b-d

   Consider three functions f, g, and h, with domain and target $\mathbb{Z}^+$. Let:

   $f(x) = x^2$ $\qquad$ $g(x) = 2^x$ $\qquad$ $h(x) = \lceil \frac{x}{5} \rceil$

(b) Evaluate $(f \circ h)(52)$

$$\begin{aligned}(f \circ h)(52) &= f(h(52)) \\ &= f\left(\left\lceil \frac{52}{5} \right\rceil\right) \\ &= f(11) \\ &= 11^2 \\ &= 121\end{aligned}$$

(c) Evaluate $(g \circ h \circ f)(4)$

$$\begin{aligned}(g \circ h \circ f)(4) &= g(h(f(4))) \\ &= g(h(16)) \\ &= g(4) \\ &= 16\end{aligned}$$

(d) Give a mathematical expression for $h \circ f$

$$\begin{aligned}h \circ f(x) &= h(f(x)) \\ &= h(x^2) \\ &= \left\lceil \frac{x^2}{5} \right\rceil\end{aligned}$$

4. d) Exercise 4.4.6, sections c-e

   Define the following functions f, g, and h:

   - $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
   - $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
   - $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

   (c) What is $(h \circ f)(010)$?

$$\begin{aligned}(h \circ f)(010) &= h(f(010)) \\ &= h(110) \\ &= 111\end{aligned}$$

   (d) What is the range of $h \circ f$?

   Since the range of f is $\{100, 101, 110, 111\}$ by definition, the range of $h \circ f$ will be $\{101, 111\}$.

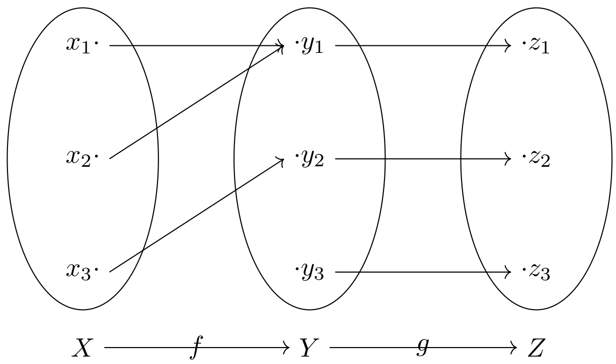
   (e) What is the range of $g \circ f$?

   Since the range of f is $\{100, 101, 110, 111\}$ by definition, the range of $h \circ f$ will be $\{001, 101, 011, 111\}$.

5. e) Extra Credit: Exercise 4.4.4, sections c, d

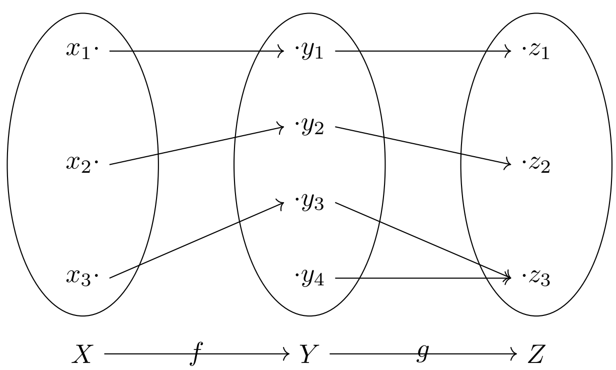
Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify the answer. If the answer is "yes," give a specific example for f and g.



Since g is a function, input y can only map to one element $g(y)$ and the function $g \circ f$ will map to the same output $g(y)$ when of two different input $x1$ and $x2$ as shown above. Therefore, it is NOT possible that f is not one-to-one and $g \circ f$ is one-to-one.

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify the answer. If the answer is "yes," give a specific example for f and g.



Since f maps distinct elements of X to distinct elements of Y, and $g \circ f$ maps distinct elements of X to the same element in Z, $g \circ f$ is one-to-one as shown above. Therefore, it is possible that g is not one-to-one and $g \circ f$ is one-to-one.