CM12004

Discrete mathematics problem sheet

Issued 9 October 2023, due 17 November 2023.

This problem sheet forms 15% of the assessment for this unit.
Each of eight questions carries the same weight.

1. For each pair of Boolean formulae P and Q state whether or not the formula $P \to Q$ is a tautology, or identically false, or neither. Prove your answer.
(i) $P = X \vee Y$, $Q = \neg(X \wedge Y)$,
(ii) $P = X \vee Y$, $Q = \neg X \wedge \neg Y$,
(iii) $P = X \to Y$, $Q = (\neg X \vee Y) \wedge (\neg X \vee X)$,
(iv) $P = X \to \neg Y$, $Q = Y \to \neg X$,
(v) $P = X \wedge (Y \vee Z)$, $Q = (X \vee Y) \wedge (X \vee Z)$,
(vi) $P = X \to Y$, $Q = \neg X \to \neg Y$,
(vii) $P = X \to Y$, $Q = \neg(Y \to X)$,
(viii) $P = (X \to Y) \wedge (Y \to Z)$, $Q = X \to Z$.

2.
(a) How many different binary logical connectives do there exist?
(b) Express the "exclusive or" (XOR) via (i) $\neg$, $\wedge$, (ii) $\neg$, $\vee$, (iii) $\neg$, $\to$.
(c) The "not and" (NAND) logical connective is defined as:

$$X \text{ NAND } Y \equiv \neg(X \wedge Y).$$

Express each of connectives $\wedge$, $\vee$, $\to$, $\neg$ via only NAND.
(d) The "not or" (NOR) logical connective is defined as:

$$X \text{ NOR } Y \equiv \neg(X \vee Y).$$

Express each of connectives $\wedge$, $\vee$, $\to$, $\neg$ via only NOR.

3.

(a) For each of the following statements answer whether it is true or false.
(i) $\forall x \in \mathbf{Z} \exists y \in \mathbf{Z} (x^2 < y + 1)$

(ii) $\exists x \in \mathbf{Z} \forall y \in \mathbf{Z} (x^2 < y + 1)$

(iii) $\exists y \in \mathbf{Z} \forall x \in \mathbf{Z} (x^2 < y + 1)$

(iv) $\forall x \in \mathbf{Z} \exists y \in \mathbf{Z} ((x < y) \to (x^2 < y^2))$

(b)

(i) Prove that the statement $\forall x \in \mathbf{Z} \forall y \in \mathbf{Z}(x^2 < y+1)$ is false by giving an example of integers x and y disproving it.

(ii) Prove that the statement $\exists x \in \mathbf{Z} \exists y \in \mathbf{Z}(x^2 < y+1)$ is true by giving an example of integers x and y proving it.

(iii) Prove that the statement $\forall y \in \mathbf{Z} \exists x \in \mathbf{Z}(x^2 < y+1)$ is false by giving an example of an integer y disproving it.

(iv) Prove that the statement $\exists x \in \mathbf{Z} \forall y \in \mathbf{Z}((x < y) \to (x^2 < y^2))$ is true by giving an example of an integer x proving it.

4.
(a) Determine the cardinalities of the following sets:

$$2^{\emptyset},\ 2^{\{0\}},\ 2^{\{0\}\cup\{1\}},\ 2^{\{0\}\cap\{1\}},\ 2^{\{\emptyset,0,1\}},\ 2^{2^{2^{\{0,1\}}}}.$$

(b) Let $A = \{1, 2, \ldots, n\}$. Determine the cardinalities of the following sets:

(i) $\{(x, S) | x \in S,\ S \in 2^A\}$

(ii) $\{(S, T) | S \in 2^A,\ T \in 2^A,\ S \cap T = \emptyset\}$.
Hint: Calculate the number of triples of pair-wise disjoint sets $(S, T, A \setminus (S \cup T))$.

5.

(i) How many maps are there from $\{1, 2\}$ to $\{1, 2, 3\}$?
(ii) How many injective maps are there from $\{1, 2\}$ to $\{1, 2, 3\}$?
(iii) How many bijective maps there are from $\{1, 2\}$ to $\{1, 2, 3\}$?
(iv) How many maps are there from $\{1, 2, 3\}$ to $\{2, 3\}$?
(v) How many surjective maps are there from $\{1, 2, 3\}$ to $\{2, 3\}$?

6. (a) Prove by induction that

$$\sum_{1 \leq i \leq n} (2i - 1) = n^2.$$

(b) Prove by induction that

$$\sum_{1 \leq i \leq n} i^2 = \frac{1}{6} n(n+1)(2n+1).$$

(c) What is the formula for the sum

$$\sum_{3 \leq i \leq n-2} i^2$$

(provided that $n \geq 5$)?

(d) Prove by induction that $7^n - 1$ is divisible by 6;

(e) Prove by induction that $2n + 1 \leq 2^n$ for all integer $n \geq 3$.

7. Each of the following defines a relation on $\mathbf{Z}$. In each case determine if the relation is reflexive, symmetric, or transitive. Justify your answers.

(i) $x + y$ is an odd integer;
(ii) $x + y$ is an even integer;
(iii) xy is an odd integer;
(iv) $x + xy$ is an even integer.

8. Consider the subset relation $\subset$ on the set $A = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.
(a) Is $\subset$ a partial order on A, a strict partial order on A, or neither? Justify your answer.
(b) Is $\subset$ a total order on A? Justify your answer.
(c) Decide whether there are maximal elements, and whether there are minimal elements, in A with respect to $\subset$. If such elements exist, list them all.