Notation	Meaning
T	True
F	False
$\oplus$	XOR
$\downarrow$	NOR
$\uparrow$	NAND
$\blacksquare$	Indicates the end of proof/solution
$\overset{d}{\equiv}$	Equivalent by definition
$\mathbb{E}$	Set of all even integers
$\mathbb{O}$	Set of all odd integers
$x * y$	$(x, y) \in R$
$x \not* y$	$(x, y) \notin R$

1.

(i) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = (X \vee Y)$	$Q = \neg(X \wedge Y)$	$P \to Q$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

(ii) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = (X \vee Y)$	$Q = (\neg X \wedge \neg Y)$	$P \to Q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

(iii) The formula $P \to Q$ is a **tautology**. Note that the second part of Q $(\neg X \vee X)$ is also a tautology.

X	Y	$P = (X \to Y)$	$Q = (\neg X \vee Y) \wedge (\neg X \vee X)$	$P \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(iv) The formula $P \to Q$ is a **tautology**.

X	Y	$P = (X \to \neg Y)$	$Q = (Y \to \neg X)$	$P \to Q$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

(v) The formula $P \to Q$ is a **tautology**.

X	Y	Z	$P = (X \wedge (Y \vee Z))$	$Q = ((X \vee Y) \wedge (X \vee Z))$	$P \to Q$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

(vi) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = (X \to Y)$	$Q = (\neg X \to \neg Y)$	$P \to Q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(vii) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = (X \to Y)$	$Q = \neg(Y \to X)$	$P \to Q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

(viii) The formula $P \to Q$ is a **tautology**.

X	Y	Z	$P = ((Y \to Z) \wedge (X \to Y))$	$Q = (X \to Z)$	$P \to Q$
T	T	T	T	F	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

2.

(a) There exist **16** different binary logical connectives. There are 4 possible combinations of T and F that a binary connective can take in. Each combination can result in 2 outcomes (T or F), therefore $2^4 = 16$.

(b) This section will make use of the following identities:

1. $X \land Y \equiv \neg(\neg X \lor \neg Y)$
2. $X \lor Y \equiv \neg(\neg X \land \neg Y)$
3. $X \rightarrow Y \equiv \neg X \lor Y$

(i) Consider the definition of XOR:

$$X \oplus Y \equiv (X \lor Y) \land \neg(X \land Y) \tag{1}$$

Let us transform equation (1) as follows:

$$\begin{aligned}(X \lor Y) \land \neg(X \land Y) &\equiv (X \lor Y) \land (\neg X \lor \neg Y)\\ &\equiv \neg(\neg X \land \neg Y) \land \neg(X \land Y)\end{aligned}$$

■

(ii) Let us transform equation (1) as follows:

$$\begin{aligned}(X \lor Y) \land \neg(X \land Y) &\equiv (X \lor Y) \land (\neg X \lor \neg Y)\\ &\equiv \neg(\neg(X \lor Y) \lor \neg(\neg X \lor Y))\end{aligned} \tag{2}$$

■

(iii) Let us transform equation (2) as follows:

$$\begin{aligned}\neg(\neg(X \lor Y) \lor \neg(\neg X \lor Y)) &\equiv \neg(\neg(\neg X \rightarrow Y) \lor \neg(X \rightarrow \neg Y))\\ &\equiv \neg((\neg X \rightarrow Y) \rightarrow \neg(X \rightarrow \neg Y))\end{aligned}$$

■

(c) By definition of NAND:

$$X \land Y \equiv \neg(X \uparrow Y)$$

Consider the truth table for NAND:

X	Y	$X \uparrow Y$
T	T	F
T	F	T
F	T	T
F	F	T

Notice that $X \uparrow X \equiv \neg X$. Hence:

$$\begin{aligned} \neg(X \uparrow Y) &\equiv (X \uparrow Y) \uparrow (X \uparrow Y) \Leftrightarrow \\ X \wedge Y &\equiv (X \uparrow Y) \uparrow (X \uparrow Y) \end{aligned}$$

Similarly, using the second and third identities, the remaining two connectives can be expressed as follows:

$$\begin{aligned} X \vee Y &\equiv \neg(\neg X \wedge \neg Y) \\ &\equiv \neg((X \uparrow X) \wedge (Y \uparrow Y)) \\ &\stackrel{d}{\equiv} (X \uparrow X) \uparrow (Y \uparrow Y) \\ X \to Y &\equiv \neg X \vee Y \\ &\equiv \neg(X \wedge \neg Y) \\ &\stackrel{d}{\equiv} X \uparrow \neg Y \\ &\equiv X \uparrow (Y \uparrow Y) \end{aligned}$$

■

(d) By definition of NOR:

$$X \vee Y \equiv \neg(X \downarrow Y)$$

Consider the truth table for NOR:

X	Y	$X \downarrow Y$
T	T	F
T	F	F
F	T	F
F	F	T

Notice that $X \downarrow X \equiv \neg X$. Hence:

$$\begin{aligned} \neg(X \downarrow Y) &\equiv (X \downarrow Y) \downarrow (X \downarrow Y) \Leftrightarrow \\ X \vee Y &\equiv (X \downarrow Y) \downarrow (X \downarrow Y) \end{aligned}$$

Similarly, using the second and third identities, the remaining two connectives can be expressed as follows:

$$\begin{aligned} X \wedge Y &\equiv \neg(\neg X \vee \neg Y) \\ &\stackrel{d}{\equiv} \neg X \downarrow \neg Y \\ &\equiv (X \downarrow X) \downarrow (Y \downarrow Y) \\ X \to Y &\equiv \neg X \vee Y \\ &\equiv \neg\neg(\neg X \vee Y) \\ &\stackrel{d}{\equiv} \neg(\neg X \downarrow Y) \\ &\equiv ((X \downarrow X) \downarrow Y) \downarrow ((X \downarrow X) \downarrow Y) \end{aligned}$$

■

3.

(a)

(i) Let $y = x^2$, then the given predicate transforms into a tautology:

$$x^2 < x^2 + 1 \implies 0 < 1$$

Hence, the given statement is **true**.

(ii) The given statement is **false**. Consider a counterexample. Let $y = -1$, then:

$$x^2 < -1 + 1 \implies x^2 < 0$$

Since no such integer x exists, the above statement is false.

(iii) The given statement is **false**. Suppose such y exists. Let $x = y + 2$, then:

$$\begin{aligned}(y+2)^2 &< y + 1\\ y^2 + 4y + 4 &< y + 1\\ y^2 + 3y + 3 &< 0\end{aligned}$$

However, the quadratic $y^2 + 3y + 3$ is always positive for all integer numbers y, therefore the assumption leads to a contradiction, and no such y exists.

(iv) Let $y = 2x$, then:

$$\begin{gathered}(x < 2x) \to (x^2 < 4x^2)\\ (x > 0) \to (x^2 > 0)\end{gathered}$$

Since the above statement is a tautology, the given statement is **true**.

(b)

(i) Let $x = -1, y = -1$, then:

$$(-1)^2 < -1 + 1 \Leftrightarrow 1 < 0 \text{ is false.}$$

■

(ii) Let $x = 2, y = 2$, then:

$$2^2 < 4 + 1 \Leftrightarrow 4 < 5 \text{ is true.}$$

■

(iii) Let $y = -1$, then:

$$x^2 < -1 + 1 \Leftrightarrow x^2 < 0 \text{ is identically false.}$$

■

(iv) Let $x = 0$, then:

$$(y > 0) \to (y^2 > 0) \text{ is a tautology.}$$

■

4.

(a) This section will make use of the following theorem: $|2^A| = 2^{|A|}$.

$$\begin{aligned}
&\left|2^{\emptyset}\right| = 2^{|\emptyset|} = 2^0 = 1\\
&\left|2^{\{0\}}\right| = 2^1 = 2\\
&\left|2^{\{0\}\cup\{1\}}\right| = \left|2^{\{0,1\}}\right| = 2^2 = 4\\
&\left|2^{\{0\}\cap\{1\}}\right| = \left|2^{\emptyset}\right| = 1\\
&\left|2^{\{\emptyset,0,1\}}\right| = 2^3 = 8\\
&\left|2^{2^{2^{\{0,1\}}}}\right| = 2^{\left|2^{2^{\{0,1\}}}\right|} = 2^{2^{\left|2^{\{0,1\}}\right|}} = 2^{2^4} = 2^{16} = 65536
\end{aligned}$$

(b)

(i) Let $B = \{(x, S)|x \in S, S \in 2^A\}$. Consider $P \subset 2^A$, which contains all subsets of $A = \{1, 2, 3, \dots, n\}$ with cardinality 2:

$$P = \{\{1, 2\}, \{1, 3\}, \dots, \{1, n\}, \{2, 3\}, \{2, 4\}, \dots, \{n - 1, n\}\}$$

The cardinality of P is $\binom{n}{2}$, representing the number of ways to choose pairs from n elements. For an element of P, such as $S_1 = \{1, 2\}$, two pairs are contributed to B: $(1, S_1)$ and $(2, S_1)$. Similarly, each element of P contributes exactly two pairs to B. Thus, with $\binom{n}{2}$ elements in P, the total contribution to B is $2\binom{n}{2}$.
Likewise, all subsets of A with three elements each, contribute $3\binom{n}{3}$ pairs to B, those with four elements contribute $4\binom{n}{4}$ pairs, etc. Thus, generalizing the patter, the cardinality of B can be calculated as:

$$|B| = \sum_{x=0}^{n} x\binom{n}{x} \tag{3}$$

Notice that the sum starts at 0, since it corresponds to the empty set. When $S = \emptyset$, the statement $x \in S$ is not valid, hence no pair is contributed to B.

Now consider the binomial theorem:

$$(1 + a)^n = \sum_{x=0}^{n} a^x\binom{n}{x}$$

Differentiating with respect to a gives:

$$n(1 + a)^{n-1} = \sum_{x=0}^{n} xa^{x-1}\binom{n}{x}$$

Let $a = 1$, then:

$$n2^{n-1} = \sum_{x=0}^{n} x\binom{n}{x}$$

The right hand side of the above equation is equal to (3), hence the cardinality of B is:

$$|B| = n2^{n-1}$$

$\blacksquare$

(ii) Let $B = \{(S, T) | S \in 2^A, T \in 2^A, S \cap T = \emptyset\}$.

Consider the smallest value of $n = 2$:

$$A = \{1, 2\}$$
$$2^A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Now consider all possible pairs (U, k), where $U \in 2^A$ and $k \in \mathbb{N}$ represents the number of distinct subsets in 2^A with which U can be paired, such that U and each of these subsets have an empty intersection:

1. $(\emptyset, 4) \Leftrightarrow B = \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), \dots\}$
2. $(\{1\}, 2) \Leftrightarrow B = \{\dots, (\{1\}, \emptyset), (\{1\}, \{2\}), \dots\}$
3. $(\{2\}, 2) \Leftrightarrow B = \{\dots, (\{2\}, \emptyset), (\{2\}, \{1\}), \dots\}$
4. $(\{1, 2\}, 1) \Leftrightarrow B = \{\dots, (\{1, 2\}, \emptyset)\}$

Let $n = 3$, then:

$$A = \{1, 2, 3\}$$
$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

The pairs (U, k) are:

1. $(\emptyset, 8)$
2. $(\{1\}, 4)$
3. $(\{2\}, 4)$
4. $(\{3\}, 4)$
5. $(\{1, 2\}, 2)$
6. $(\{1, 3\}, 2)$
7. $(\{2, 3\}, 2)$
8. $(\{1, 2, 3\}, 1)$

Observe the emerging common pattern. Since each element can always form at least one pair, the total number of cases to be considered equals to $|2^A| = 2^{|A|}$ ($2^2 = 4$ when $n = 2$, and $2^3 = 8$ when $n = 3$). Furthermore, notice that k goes down by the factor of 2 with each extra element in U. Lastly, there are $\binom{n}{|U|}$ cases which are considered for each cardinality of U.

Thus, to calculate the cardinality of B, we have to find the sum of the product of k (the number of pairs for a given cardinality of U) and the number of possible ways to choose $|U|$ elements from the collection of n elements:

$$|B| = \sum_{|U|=0}^{n} 2^{|U|} \binom{n}{n - |U|}$$

Let us denote $|U|$ as x, and simplify the above sum using the fact that the number of ways to choose i elements from j elements equals to the number of ways to exclude $j-i$ elements from j elements:

$$|B| = \sum_{x=0}^{n} 2^x \binom{n}{x} \tag{4}$$

Now consider the binomial theorem:

$$(a+b)^n = \sum_{x=0}^{n} a^{n-x} b^x \binom{n}{x}$$

Let $a = 1$ and $b = 2$:

$$(1+2)^n = \sum_{x=0}^{n} 1^{n-x} 2^x \binom{n}{x}$$
$$3^n = \sum_{x=0}^{n} 2^x \binom{n}{x}$$

Hence (4) equals to 3^n, leading to the final answer:

$$|B| = 3^n$$

$\blacksquare$

Remark. The result can be interpreted intuitively as being analogous to the power set. Another way to calculate the cardinality of B is to find the number of triples of pair-wise disjoint sets $(S, T, A \backslash (S \cup T))$, as indicated by the given hint. In this case, there are three options for each element of A: it is either in T, S or it is excluded. Since there are n elements and each one has three choices, the total number of such triples is $3 \times 3 \times \cdots \times 3$ n times, i.e. 3^n.

5.

(i) For each of the two arguments in the domain, there are three choices of images in the range, therefore there are $3^2 = \mathbf{9}$ maps for the given sets.

(ii) The are **6** injective maps for the given sets.

Function	Image of 1	Image of 2
f_1	1	2
f_2	1	3
f_3	2	1
f_4	2	3
f_5	3	1
f_6	3	2

(iii) There are **0** bijective maps for the given sets, since the number of elements in the domain does not match the number of elements in the range.

(iv) Similar to (i), there are $2^3 = \mathbf{8}$ maps for the given sets.

(v) There are **6** surjective maps for the given sets.

Function	Image of 1	Image of 2	Image of 3
f_1	1	2	1
f_2	1	2	2
f_3	2	1	1
f_4	2	1	2
f_5	1	1	2
f_6	2	2	1

6.

This section will make use of the following structure:

Proposition $P(n)$

1. Base case.
2. Inductive hypothesis.
3. Inductive step.
4. Conclusion.

(a)

$$P(n) = \sum_{1 \leq i \leq n} (2i - 1) = n^2, \forall n \in \mathbb{N}$$

1. Let $n = 1$, then:

$$\begin{aligned} 2 \times 1 - 1 &= 1^2 \\ 1 &= 1 \end{aligned}$$

Hence, P(1) is true.

2. Let us assume that $P(k)$ is true, where $k \in \mathbb{N}$ is an arbitrary fixed number.
3. Let n = k + 1, then:

$$\begin{aligned} \sum_{1 \leq i \leq k+1} (2i - 1) &= (k + 1)^2 \\ \sum_{1 \leq i \leq k} (2i - 1) + 2(k + 1) - 1 &= (k + 1)^2 \end{aligned}$$

Using the inductive hypothesis:

$$\begin{aligned} k^2 + 2k + 2 - 1 &= k^2 + 2k + 1 \\ k^2 + 2k + 1 &= k^2 + 2k + 1 \end{aligned}$$

Hence $P(k + 1)$ is true.

4. The proposition $P(k+1)$ has been proven to be true for some arbitrary $k \in \mathbb{N}$ under the assumption that $P(k)$ is true. Since $P(1)$ has also been shown to be true, by the principles of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

$\blacksquare$

(b)

$$P(n) = \sum_{1 \leq i \leq n} i^2 = \frac{1}{6}n(n+1)(2n+1), \forall n \in \mathbb{N}$$

1. Let $n = 1$, then:

$$\begin{aligned} 1^2 &= \frac{1}{6}(1)(1+1)(2(1)+1) \\ 1 &= 1 \end{aligned}$$

Hence, P(1) is true.

2. Let us assume that $P(k)$ is true, where $k \in \mathbb{N}$ is an arbitrary fixed number.
3. Let n = k + 1, then:

$$\begin{aligned} \sum_{1 \leq i \leq k+1} i^2 &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ \sum_{1 \leq i \leq k} i^2 + (k+1)^2 &= \frac{1}{6}(2k^3 + 9k^2 + 13k + 6) \end{aligned}$$

Using the inductive hypothesis:

$$\begin{aligned} \frac{1}{6}k(k+1)(2k+1) + k^2 + 2k + 1 &= \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1 \\ \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1 &= \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1 \end{aligned}$$

Hence $P(k+1)$ is true.

4. The proposition $P(k+1)$ has been proven to be true for some arbitrary $k \in \mathbb{N}$ under the assumption that $P(k)$ is true. Since $P(1)$ has also been shown to be true, by the principles of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

$\blacksquare$

(c)

$$\sum_{3 \leq i \leq n-2} i^2 = \sum_{1 \leq i \leq n-2} i^2 - \sum_{1 \leq i \leq 2} i^2 = \sum_{1 \leq i \leq n-2} i^2 - 5$$

Using the result from part (b), the equation above transforms into:

$$\begin{aligned} &\frac{1}{6}(n-2)(n-1)(2n-3) - 5 = \\ &\frac{1}{6}(2n^3 - 9n^2 + 13n - 6) - 5 = \\ &\frac{1}{3}n^3 - \frac{3}{2}n^2 + \frac{13}{6}n - 6, \forall n \in \mathbb{N}, n \geq 5 \end{aligned}$$

$\blacksquare$

(d)

$$P(n) : 7^n - 1 \text{ is divisible by 6 for all } n \in \mathbb{N}_0.$$

1. Let $n = 0$, then:

$$7^0 - 1 = 1 - 1 = 0$$

Since 0 is divisible by 6, P(0) is true.

2. Let us assume that $P(k)$ is true, where $k \in \mathbb{N}_0$ is an arbitrary fixed number. Therefore, $P(k)$ can be expressed as $P(k) = 6m$, where $m \in \mathbb{N}_0$.

3. Consider $P(k+1) - P(k)$:

$$\begin{aligned} 7^{k+1} - 1 - 7^k + 1 &= \\ 7^k(7-1) &= \\ 6(7^k) \end{aligned}$$

Thus $P(k+1)$ can be expressed as follows:

$$P(k+1) = 6(7^k) + P(k)$$

Using the inductive hypothesis:

$$P(k+1) = 6(7^k) + 6m = 6(7^k + m) \text{ is divisible by 6.}$$

Hence $P(k+1)$ is true.

4. The proposition $P(k+1)$ has been proven to be true for some arbitrary $k \in \mathbb{N}_0$ under the assumption that $P(k)$ is true. Since $P(0)$ has also been shown to be true, by the principles of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}_0$.

■

(e)

$$P(n) : 2n + 1 \leq 2^n, \forall n \in \mathbb{N}, n \geq 3$$

1. Let $n = 3$, then:

$$\begin{aligned} 2(3) + 1 &\leq 2^3 \\ 7 &\leq 8 \end{aligned}$$

Hence, P(3) is true.

2. Let us assume that $P(k)$ is true, where $k \in \mathbb{N}$ is an arbitrary fixed number.

3. Consider $P(k)$:

$$\begin{aligned} 2k + 1 &\leq 2^k \quad \Big| \times 2 \\ 4k + 2 &\leq 2^{k+1} \end{aligned}$$

Now consider the statement $2k + 3 \leq 4k + 2$:

$$\begin{aligned} 2k + 3 &\leq 4k + 2 \\ 2k &\geq 1 \\ k &\geq \frac{1}{2} \end{aligned}$$

Thus, the above statement is true for all $k \in \mathbb{N}, k \geq 3$. Therefore, using the principle of transitivity:

$$\begin{aligned} 2k + 3 &\leq 2^{k+1} \\ 2(k + 1) + 1 &\leq 2^{k+1} \end{aligned}$$

Hence $P(k + 1)$ is true.

4. The proposition $P(k+1)$ has been proven to be true for some arbitrary $k \in \mathbb{N}$ under the assumption that $P(k)$ is true. Since $P(3)$ has also been shown to be true, by the principles of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}, n \geq 3$.

$\blacksquare$

7.

This section will make use of the following structure:

Relation R on $\mathbb{Z}$.

1. Check if R is reflexive (for any $x \in \mathbb{Z}, x * x$).
2. Check if R is symmetric (if $x * y$, then $y * x$).
3. Check if R is transitive (if $x * y$ and $y * z$, then $x * z$).

Furthermore, the following facts will be utilized:

1. If $a \in \mathbb{E}$ and $a = b + c$, then it must be the case that either both $b, c \in \mathbb{E}$ or both $b, c \in \mathbb{O}$.
2. If $a \in \mathbb{E}$ and $a = bc$, then it must be the case that either $b \in \mathbb{E}$ or $c \in \mathbb{E}$.

(i)

$$R = \{x * y \,|\, (x + y) \in \mathbb{O}\}$$

1. $\forall x \in \mathbb{Z}(x + x = 2x \in \mathbb{E}) \implies x \not* x \implies R$ is **irreflexive**.
2. If $(x + y) \in \mathbb{O}$, then $x + y = (y + x) \in \mathbb{O} \implies y * x \implies R$ is **symmetric**.
3. Suppose $x \in \mathbb{E}$ and $(x + y) \in \mathbb{O}$. Moreover, suppose $z \in \mathbb{E}$ and $(y + z) \in \mathbb{O}$. Then $(x + z) \in \mathbb{E} \implies x \not* z \implies R$ is **intransitive**.

(ii)

$$R = \{x * y \,|\, (x + y) \in \mathbb{E}\}$$

1. $\forall x \in \mathbb{Z}(x + x = 2x \in \mathbb{E}) \implies x * x \implies R$ is **reflexive**.
2. If $(x + y) \in \mathbb{E}$, then $x + y = (y + x) \in \mathbb{E} \implies y * x \implies R$ is **symmetric**.
3. Consider the case when $x, y \in \mathbb{O}$. Then $(x + y) \in \mathbb{E}$, and for $(y + z) \in \mathbb{E}$ to be true, $z \in \mathbb{O}$ must be true. Therefore, $(x + z) \in \mathbb{E} \implies x * z$.

   Now consider an alternative case when $x, y \in \mathbb{E}$. Then $(x+y) \in \mathbb{E}$, and for $(y+z) \in \mathbb{E}$ to be true, $z \in \mathbb{E}$ Therefore, $(x + z) \in \mathbb{E} \implies x * z$.

   Since both cases have shown that $x * z$, R is **transitive**.

(iii)

$$R = \{x * y \,|\, xy \in \mathbb{O}\}$$

1. Let $x = 2$, then $x^2 = 4 \in \mathbb{E} \implies x \not* x \implies R$ is **irreflexive**.
2. If $xy \in \mathbb{O}$, then $xy = yx \in \mathbb{O} \implies y * x \implies R$ is **symmetric**.
3. If $xy, yz \in \mathbb{O}$, then $x, y, z \in \mathbb{O}$ must be true. Therefore $xz \in \mathbb{O} \implies x * z \implies R$ is **transitive**.

(iv)

$$R = \{x * y \,|\, (x + xy) \in \mathbb{E}\}$$

1. Consider $x \in \mathbb{E}$: $x + x^2 = x(1 + x) \in \mathbb{E} \implies x * x$.

   Now consider $x \in \mathbb{O}$: $x + x^2 = x(1 + x) \in \mathbb{E}$ because $(1 + x) \in \mathbb{E} \implies x * x$.

   Since both cases have shown that $x * x$, R is **reflexive**.
2. Consider $x \in \mathbb{E}$: if $(x + xy) \in \mathbb{E}$ is true, then $y \in \mathbb{E}$ must also be true. Therefore $y + yx = y(1 + x) \in \mathbb{E} \implies y * x$.

   Now consider $x \in \mathbb{O}$: if $(x+xy) \in \mathbb{E}$ is true, then $y \in \mathbb{O}$ must also be true. Therefore $y + yx = y(1 + x) \in \mathbb{E}$ because $(1 + x) \in \mathbb{E} \implies y * x$.

   Since both cases have shown that $y * x$, R is **symmetric**.
3. Consider the case when $x, y \in \mathbb{E}$. Then $x + xy = x(1 + y) \in \mathbb{E}, y + yz = y(1 + z) \in \mathbb{E}$ and $x + xz = x(1 + z) \in \mathbb{E} \implies x * z$.

   Moreover, consider the case when $x, y \in \mathbb{O}$. Then $x + xy = x(1 + y) \in \mathbb{E}$, and for $y + yz = y(1 + z) \in \mathbb{E}$ to be true, $z \in \mathbb{O}$ must be true. Therefore $x + xz = x(1 + z) \in \mathbb{E} \implies x * z$.

   Furthermore, consider the case when $x \in \mathbb{E}, y \in \mathbb{O}$. Then $x + xy = x(1 + y) \in \mathbb{E}$, and for $y + yz = y(1 + z) \in \mathbb{E}$ to be true, $z \in \mathbb{O}$ must be true. Therefore $x + xz = x(1 + z) \in \mathbb{E} \implies x * z$.

   Lastly, consider the case when $x \in \mathbb{O}, y \in \mathbb{E}$. Then $x+xy = x(1+y) \in \mathbb{O} \implies x \not* y$, therefore this case is invalid.

   Since none of the cases have disproved that $x * z$, R is **transitive**.

8.

Consider the following relation R on the set A:

$$R = \{X * Y \mid X \subset Y\}$$

(a) Since every set is a subset of itself, $X * X \implies R$ is reflexive. This implies that R is not asymmetric, and consequently that it is **not strict partial order**.

Furthermore, if $X \subset Y$ and $Y \subset X$, then X must be equal to Y, therefore R is antisymmetric.

Lastly, if $X \subset Y$ and $Y \subset Z$, then $X \subset Z \implies X * Z$, hence R is transitive.

Since R has been shown to be reflexive, antisymmetric and transitive, it is **partial order**.

(b) Consider $\{b\}, \{c\} \in A$. Since neither $\{b\} \subset \{c\}$, nor $\{c\} \subset \{b\}$ is true, R is **not total order**.

(c) $\{b\}, \{c\} \in A$ are the **minimal elements** because there are no elements in A that are the subsets of these sets.

$\{a, b, c\} \in A$ is the **maximal element** because there are no elements in A that contain this set.