Notation	Meaning
T	True
F	False
$\oplus$	XOR
$\downarrow$	NOR
$\uparrow$	NAND
$\mathbb{E}$	Set of all even integers
$\mathbb{O}$	Set of all odd integers
$\blacksquare$	Indicates the end of proof/solution
$\overset{d}{\equiv}$	Equivalent by definition

1.

(i) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = X \vee Y$	$Q = \neg(X \wedge Y)$	$P \to Q$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

(ii) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = X \vee Y$	$Q = \neg X \wedge \neg Y$	$P \to Q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

(iii) The formula $P \to Q$ is a **tautology**. Note that the second part of Q $(\neg X \vee X)$ is also a tautology.

X	Y	$P = X \to Y$	$Q = (\neg X \vee Y) \wedge (\neg X \vee X)$	$P \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(iv) The formula $P \to Q$ is a **tautology**.

X	Y	$P = X \to \neg Y$	$Q = Y \to \neg X$	$P \to Q$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

(v) The formula $P \to Q$ is a **tautology**.

X	Y	Z	$P = X \wedge (Y \vee Z)$	$Q = (X \vee Y) \wedge (X \vee Z)$	$P \to Q$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

(vi) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = X \to Y$	$Q = \neg X \to \neg Y$	$P \to Q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(vii) The formula $P \to Q$ is **neither** a tautology nor identically false.

X	Y	$P = X \to Y$	$Q = \neg(Y \to X)$	$P \to Q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

(viii) The formula $P \to Q$ is a **tautology**.

X	Y	Z	$P = (Y \to Z) \wedge (X \to Y)$	$Q = X \to Z$	$P \to Q$
T	T	T	T	F	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

2.

(a) There exist **16** different binary logical connectives. There are 4 possible combinations of T and F that a binary connective can take in. Each combination can result in 2 outcomes (T or F), therefore $2^4 = 16$.

(b) This section will make use of the following identities:

1. $X \wedge Y \equiv \neg(\neg X \vee \neg Y)$

2. $X \lor Y \equiv \neg(\neg X \land \neg Y)$
3. $X \to Y \equiv \neg X \lor Y$

(i) Consider the definition of XOR:

$$X \oplus Y \equiv (X \lor Y) \land \neg(X \land Y) \tag{1}$$

Let us transform equation (1) as follows:

$$\begin{aligned}(X \lor Y) \land \neg(X \land Y) &\equiv (X \lor Y) \land (\neg X \lor \neg Y)\\ &\equiv \neg(\neg X \land \neg Y) \land \neg(X \land Y)\end{aligned}$$

■

(ii) Let us transform equation (1) as follows:

$$\begin{aligned}(X \lor Y) \land \neg(X \land Y) &\equiv (X \lor Y) \land (\neg X \lor \neg Y)\\ &\equiv \neg(\neg(X \lor Y) \lor \neg(\neg X \lor Y)) \qquad (2)\end{aligned}$$

■

(iii) Let us transform equation (2) as follows:

$$\begin{aligned}\neg(\neg(X \lor Y) \lor \neg(\neg X \lor Y)) &\equiv \neg(\neg(\neg X \to Y) \lor \neg(X \to \neg Y))\\ &\equiv \neg((\neg X \to Y) \to \neg(X \to \neg Y))\end{aligned}$$

■

(c) By definition of NAND:

$$X \land Y \equiv \neg(X \uparrow Y)$$

Consider the truth table for NAND:

X	Y	$X \uparrow Y$
T	T	F
T	F	T
F	T	T
F	F	T

Notice that $X \uparrow X \equiv \neg X$. Hence:

$$\begin{aligned}&\neg(X \uparrow Y) \equiv (X \uparrow Y) \uparrow (X \uparrow Y) \Leftrightarrow\\ &X \land Y \equiv (X \uparrow Y) \uparrow (X \uparrow Y)\end{aligned}$$

Similarly, using the second and third identities, the remaining two connectives can be expressed as follows:

$$\begin{aligned}X \lor Y &\equiv \neg(\neg X \land \neg Y)\\ &\equiv \neg((X \uparrow X) \land (Y \uparrow Y))\\ &\overset{d}{\equiv} (X \uparrow X) \uparrow (Y \uparrow Y)\\ X \to Y &\equiv \neg X \lor Y\\ &\equiv \neg(X \land \neg Y)\\ &\overset{d}{\equiv} X \uparrow \neg Y\\ &\equiv X \uparrow (Y \uparrow Y)\end{aligned}$$

■

(d) By definition of NOR:

$$X \lor Y \equiv \neg(X \downarrow Y)$$

Consider the truth table for NOR:

X	Y	$X \downarrow Y$
T	T	F
T	F	F
F	T	F
F	F	T

Notice that $X \downarrow X \equiv \neg X$. Hence:

$$\begin{aligned} &\neg(X \downarrow Y) \equiv (X \downarrow Y) \downarrow (X \downarrow Y) \Leftrightarrow \\ &X \lor Y \equiv (X \downarrow Y) \downarrow (X \downarrow Y) \end{aligned}$$

Similarly, using the second and third identities, the remaining two connectives can be expressed as follows:

$$\begin{aligned} X \land Y &\equiv \neg(\neg X \lor \neg Y) \\ &\overset{d}{\equiv} \neg X \downarrow \neg Y \\ &\equiv (X \downarrow X) \downarrow (Y \downarrow Y) \\ X \to Y &\equiv \neg X \lor Y \\ &\equiv \neg\neg(\neg X \lor Y) \\ &\overset{d}{\equiv} \neg(\neg X \downarrow Y) \\ &\equiv ((X \downarrow X) \downarrow Y) \downarrow ((X \downarrow X) \downarrow Y) \end{aligned}$$

■

3.

(a)

(i) Let $y = x^2$, then the given predicate transforms into a tautology:

$$x^2 < x^2 + 1 \implies 0 < 1$$

Hence, the given statement is **true**.

(ii) The given statement is **false**. Consider a counterexample. Let $y = -1$, then:

$$x^2 < -1 + 1 \implies x^2 < 0$$

Since no such integer x exists, the above statement is false.

(iii) The given statement is **false**. Suppose such y exists. Let $x = y + 2$, then:

$$\begin{aligned} (y+2)^2 &< y+1 \\ y^2 + 4y + 4 &< y + 1 \\ y^2 + 3y + 3 &< 0 \end{aligned}$$

However, the quadratic $y^2 + 3y + 3$ is always positive for all integer numbers y, therefore the assumption leads to a contradiction, and no such y exists.

(iv) Let $y = 2x$, then:

$$(x < 2x) \to (x^2 < 4x^2)$$
$$(x > 0) \to (x^2 > 0)$$

Since the above statement is a tautology, the given statement is **true**.

(b)

(i) Let $x = -1, y = -1$, then:

$$(-1)^2 < -1 + 1 \Leftrightarrow 1 < 0 \text{ is false.}$$

■

(ii) Let $x = 2, y = 2$, then:

$$2^2 < 4 + 1 \Leftrightarrow 4 < 5 \text{ is true.}$$

■

(iii) Let $y = -1$, then:

$$x^2 < -1 + 1 \Leftrightarrow x^2 < 0 \text{ is identically false.}$$

■

(iv) Let $x = 0$, then:

$$(y > 0) \to (y^2 > 0) \text{ is a tautology.}$$

■

4.

(a) This section will make use of the following theorem: $|2^A| = 2^{|A|}$.

$$\begin{aligned}
&|2^{\emptyset}| = 2^{|\emptyset|} = 2^0 = 1 \\
&|2^{\{0\}}| = 2^1 = 2 \\
&|2^{\{0\}\cup\{1\}}| = |2^{\{0,1\}}| = 2^2 = 4 \\
&|2^{\{0\}\cap\{1\}}| = |2^{\emptyset}| = 1 \\
&|2^{\{\emptyset,0,1\}}| = 2^3 = 8 \\
&\left|2^{2^{\{0,1\}}}\right| = 2^{\left|2^{\{0,1\}}\right|} = 2^{2^{\left|\{0,1\}\right|}} = 2^{2^4} = 2^{16} = 65536
\end{aligned}$$

(b)

(i) Let $B = \{(x, S)|x \in S, S \in 2^A\}$. Consider $P \subset 2^A$, which contains all subsets of $A = \{1, 2, 3, \ldots, n\}$ with cardinality 2:

$$P = \{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}, \{2, 3\}, \{2, 4\}, \ldots, \{n-1, n\}\}$$

The cardinality of P is $\binom{n}{2}$, representing the number of ways to choose pairs from n elements. For an element of P, such as $S_1 = \{1, 2\}$, two pairs are contributed to B: $(1, S_1)$ and $(2, S_1)$. Similarly, each element of P contributes exactly two pairs to B. Thus, with $\binom{n}{2}$ elements in P, the total contribution to B is $2\binom{n}{2}$.

Likewise, all subsets of A with three elements each, contribute $3\binom{n}{3}$ pairs to B, those with four elements contribute $4\binom{n}{4}$ pairs, etc. Thus, generalizing the patter, the cardinality of B can be calculated as:

$$|B| = \sum_{x=0}^{n} x\binom{n}{x} \tag{3}$$

Notice that the sum starts at 0, since it corresponds to the empty set. When $S = \emptyset$, the statement $x \in S$ is not valid, hence no pair is contributed to B.

Now consider the binomial theorem:

$$(1 + a)^n = \sum_{x=0}^{n} a^x\binom{n}{x}$$

Differentiating with respect to a gives:

$$n(1 + a)^{n-1} = \sum_{x=0}^{n} xa^{x-1}\binom{n}{x}$$

Let $a = 1$, then:

$$n2^{n-1} = \sum_{x=0}^{n} x\binom{n}{x}$$

The right hand side of the above equation is equal to (3), hence the cardinality of B is:

$$|B| = n2^{n-1}$$

■

5.

(i) For each of the two arguments in the domain, there are three choices of images in the range, therefore there are $3^2 = \mathbf{9}$ maps for the given sets.

(ii) The are **6** injective maps for the given sets.

Function	Image of 1	Image of 2
f_1	1	2
f_2	1	3
f_3	2	1
f_4	2	3
f_5	3	1
f_6	3	2

(iii) There are **0** bijective maps for the given sets, since the number of elements in the domain does not match the number of elements in the range.

(iv) Similar to (i), there are $2^3 = \mathbf{8}$ maps for the given sets.

(v) There are **6** surjective maps for the given sets.

Function	Image of 1	Image of 2	Image of 3
f_1	1	2	1
f_2	1	2	2
f_3	2	1	1
f_4	2	1	2
f_5	1	1	2
f_6	2	2	1

6.

7.

8.