MATHEMATICS FOR COMPUTATION 2024
PROBLEM SHEET 1
SOLUTIONS

1. Solve the following system of linear equations using the Cramer's rule.

$$3X - Y = 8$$

$$-2X + Y + Z = 9$$

$$2X - Y + 4Z = -5.$$

The solution should be represented by simple fractions. Show your working.

Solution. *The determinants are:* $\det A = 5, \det A_X = 81, \det A_Y = 203, \det A_Z = 4$. *Hence,* $x = 81/5,\ y = 203/5,\ z = 4/5$.

2. Solve the same system of equations by first inverting its matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}.$$

Compute the inverse of A using the calculation of the adjugate matrix A^*. The elements of A^{-1} should be represented by simple fractions. Show your working.

Solution.

$$A^{-1} = \begin{pmatrix} 1 & 4/5 & -1/5 \\ 2 & 12/5 & -3/5 \\ 0 & 1/5 & 1/5 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4/5 & -1/5 \\ 2 & 12/5 & -3/5 \\ 0 & 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} 81/5 \\ 203/5 \\ 4/5 \end{pmatrix}.$$

3. Given a matrix

$$A = \begin{pmatrix} 1 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & 1 & -1 \end{pmatrix},$$

use Gaussian elimination to compute the determinant $\det A$ of A and to solve the system of linear equations $AX = b$, where $X = (X_1, X_2, X_3)^T$ is the vector of unknowns and $b = (1, 0, 1)^T$. The solution should be represented by simple fractions. Show your working.

Solution. The sequence of matrices leading to a row echelon form:

$$\begin{pmatrix} 1 & 4 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 3 & 1 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 0 & -11 & -4 & -2 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 4 & 1 & 1 \\ 0 & 6 & 3 & 1 \\ 0 & -11 & -4 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & 1 & 1 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{6} \end{pmatrix}$$

The determinant $\det A$ equals the determinant of the last matrix with removed last column, which is the product of its diagonal elements, i.e., 9.

The solution of the system $AX = b$ is $x_1 = 2/9,\ x_2 = 2/9,\ x_3 = -1/9$.

4. Using Gaussian elimination, compute the rank of the following (4×4)-matrix

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ -2 & 4 & 2 & 4 \end{pmatrix}$$

Show your working.

Solution. The sequence of matrices leading to a row echelon form:

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ -2 & 4 & 2 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 5 & 2 & 5 \\ -2 & 4 & 2 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\operatorname{rank} A = 2$.

5. Consider the following system of linear equations.

$$3X_1 - 2X_2 + 3X_3 - X_4 = 1$$
$$X_2 + X_4 = 3$$
$$X_1 + X_2 - 2X_3 + 4X_4 = 1$$

Using Gaussian elimination, show that it has at least one solution. Represent the general solution as an affine map from one vector space to another, in matrix/vector form. Find one specific solution. Show your working.

Solution. Reduce the matrix of the system of equations to row echelon form:

$$\begin{pmatrix} 3 & -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 1 & 1 & -2 & 4 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & \frac{5}{3} & -3 & \frac{13}{3} & \frac{2}{3} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} -3 & -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -3 & \frac{8}{3} & -\frac{13}{3} \end{pmatrix}.$$

Since there are no rows consisting of all zeroes except the rightmost non-zero element, the system of equations does have a solution. The original system of equations has the same set of solutions as

$$3X_1 - 2X_2 + 3X_3 - X_4 = 1$$
$$X_2 + X_4 = 3$$

$$-3X_3 + \frac{8}{3}X_4 = -\frac{13}{3}.$$

In the last equation, expressing X_3 via X_4, we get $X_3 = \frac{8}{9}X_4 + \frac{13}{9}$. In the second equation, expressing X_2 via X_4, we get $X_2 = -X_4 + 3$. Substituting these expressions into the first equation, we get $X_1 = -\frac{11}{9}X_4 + \frac{8}{9}$. Collecting these results together, we get:

$$X_1 = -\frac{11}{9}X_4 + \frac{8}{9},\ X_2 = -X_4 + 3,\ X_3 = \frac{8}{9}X_4 + \frac{13}{9}.$$

It follows that the general solution can be represented as the following affine map:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\frac{11}{9} \\ -1 \\ \frac{8}{9} \end{pmatrix} \begin{pmatrix} X_4 \end{pmatrix} + \begin{pmatrix} \frac{8}{9} \\ 3 \\ \frac{13}{9} \end{pmatrix}. \tag{0.1}$$

One specific solution can be obtained by setting x_4 equal to a particular number. For example, taking $x_4 = 0$, we get the solution $(\frac{8}{9}, 3, \frac{13}{9}, 0)$.