MATHEMATICS FOR COMPUTATION 2024
PROBLEM SHEET 2

ISSUED 11 MARCH 2024, DUE 22 APRIL 2024

Coursework forms 25% of the assessment for this unit and will be comprised of 2 problem sheets equally weighted. On this sheet, each question is equally weighted.

+ **1.**

(a) Write a linear equation in the form $aX + bY + cZ = d$ defining a plane in $\mathbb{R}^3$ passing through three points, $(0, 1, 2)$, $(5, -1, -1)$ and $(2, 0, 0)$.

(b) Is there the unique plane passing through the following three points in $\mathbb{R}^3$: $(0, 0, 0)$, $(1, 1, 0)$, $(-2, -2, 0)$? If yes, prove it. Otherwise, how many planes? Justify your answer.

+ **2.** Prove that the following functions are continuous, using the ε/δ definition.

(i)

$$\begin{aligned} f :\ & \mathbb{R} \to \mathbb{R} \\ x & \mapsto \frac{1}{2}\, x^2 \end{aligned}$$

(ii)

$$\begin{aligned} f :\ & \mathbb{R} \to \mathbb{R} \\ x & \mapsto -2\, |x| \end{aligned}$$

+ **3.** Consider the function f defined by the following expression

$$f(x) = \frac{x^4}{\ln x} - \sin(e^{2x-1}) + e^{1/x} + 2.$$

(Reminder: $\ln x$ stands for $\log_e x$, where $e = 2.71828\ldots$)
What is the largest subset of $\mathbb{R}$ in which f is defined? Calculate the derivative $f'(x)$ of this function.

+ **4.** Find the second derivative $f''(x)$ of the function

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \le 0 \\ -x^3 & \text{if} \quad x > 0 \end{cases}.$$

Does this function have the third derivative? Justify your answers.

+ **5.** For each of the following two matrices decide whether or not it has eigenvalues among real numbers. If yes, find all real eigenvalues and corresponding eigenvectors. If not, explain why not.

(i)

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix}$$

Q1 a) $\vec{OA} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$, $\vec{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

$$\vec{r} = \vec{OC} + \mu \vec{AB} + \lambda \vec{AC}$$

$$\vec{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4-3 \\ -6+10 \\ -5+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

$$\vec{r} \cdot \vec{n} = -d$$

$$x + 4y - z + d = 0$$

Use $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$: $2 + 4(0) - 0 + d = 0 \iff d = -2$

$$\boxed{x + 4y - z - 2 = 0}$$

b) $\vec{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\vec{A} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{B} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$

$\vec{OA} \times \vec{OB} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2-(-2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ The vectors are collinear, i.e. there are infinitely many planes passing through the given points.

Q2 i) $f(x) = \frac{1}{2}x^2, x \in \mathbb{R}$

$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in \mathbb{R} \left(0 < |x-c| < \delta \rightarrow |f(x) - f(c)| < \varepsilon \right)$

Find δ such that if $|x-c| < \delta$, then $\left| \frac{x^2}{2} - \frac{c^2}{2} \right| < \varepsilon \; \forall x$.

$\frac{1}{2}|x^2 - c^2| = \frac{1}{2}|x-c||x+c| < \frac{1}{2}\delta|x+c|$

$\frac{1}{2}\delta|x+c| = \frac{1}{2}\delta|x-c+2c| \leq \frac{1}{2}\delta(|x-c| + |2c|) < \frac{1}{2}\delta^2 + \delta c$

By the triangular inequality.

$\frac{1}{2}\delta^2 + \delta c = \varepsilon$, Solving for δ:

$\delta = -c \pm \sqrt{c^2 + 4(\frac{1}{2})\varepsilon} = -c \pm \sqrt{c^2 + 2\varepsilon}$

Hence, for all ε there exists $\delta = -c \pm \sqrt{c^2 + 2\varepsilon}$

Either plus (1) or minus (2)

such that if $|x-c| < \delta$, then:

(1) $\left| \frac{x^2}{2} - \frac{c^2}{2} \right| < \frac{1}{2}\delta^2 + \delta c =$

$= \frac{1}{2}\left(-c + \sqrt{c^2 + 2\varepsilon}\right)^2 + c\left(-c + \sqrt{c^2 + 2\varepsilon}\right)$

$= \frac{1}{2}\left(c^2 - 2c\sqrt{c^2 + 2\varepsilon} + c^2 + 2\varepsilon\right) - c^2 + c\sqrt{c^2 + 2\varepsilon}$

$= \cancel{c^2} - \cancel{c\sqrt{c^2 + 2\varepsilon}} + \varepsilon \cancel{- c^2} + \cancel{c\sqrt{c^2 + 2\varepsilon}} = \varepsilon$ for all x.

(2) $\left| \frac{x^2}{2} - \frac{c^2}{2} \right| < \frac{1}{2}\delta^2 + \delta c =$

$= \frac{1}{2}\left(-c - \sqrt{c^2 + 2\varepsilon}\right)^2 + c\left(-c - \sqrt{c^2 - 2\varepsilon}\right)$

$= \frac{1}{2}\left(c^2 + 2c\sqrt{c^2 + 2\varepsilon} + c^2 + 2\varepsilon\right) - c^2 - c\sqrt{c^2 + 2\varepsilon}$

$= \cancel{c^2} + \cancel{c\sqrt{c^2 + 2\varepsilon}} + \varepsilon \cancel{- c^2} - \cancel{c\sqrt{c^2 + 2\varepsilon}} = \varepsilon$ for all x.

Q.E.D.

ii) $f(x) = -2|x|, x \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in \mathbb{R} \left(0 < |x-c| < \delta \rightarrow |f(x) - f(c)| < \varepsilon \right)$

Find δ such that if $|x-c| < \delta$, then $\big| -2|x| + 2|c| \big| < \varepsilon \ \forall x$.

$$\big| 2|c| - 2|x| \big| = 2 \big| |x| - |c| \big| \leq 2|x-c| < 2\delta$$

By the reverse triangular inequality

$$2\delta = \varepsilon \Leftrightarrow \delta = \frac{\varepsilon}{2}$$

Hence, for all ε there exists $\delta = \frac{\varepsilon}{2}$, such that if $|x-c| < \delta$, then:

$$\big| -2|x| + 2|c| \big| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon \text{ for all } x.$$

Q.E.D.

(Q3) $f(x) = \frac{x^4}{\ln x} - \sin(e^{2x-1}) + e^{1/x} + 2$

$x \in (0,1) \cup (1,\infty)$ since ln is defined on positive $\mathbb{R}$ numbers, and exclude 1 to account for ln x in the denominator.

$$f'(x) = \frac{4x^3 \ln x - \frac{x^4}{x}}{\ln^2 x} - \cos(e^{2x-1})(e^{2x-1})(2) - x^{-2}e^{1/x} =$$

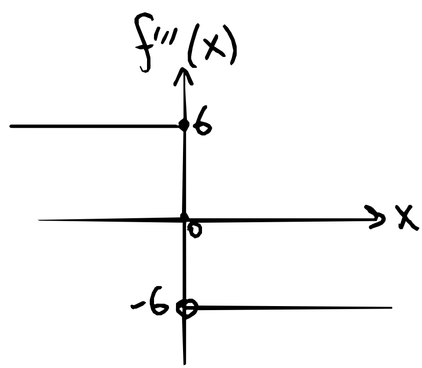
$$= \frac{4x^3}{\ln x} - \frac{x^3}{\ln^2 x} - 2e^{2x-1}\cos(e^{2x-1}) - \frac{e^{1/x}}{x^2}$$

(Q4)

$$f'(x) = \begin{cases} 3x^2, & x \le 0 \\ -3x^2, & x > 0 \end{cases}$$

$$f''(x) = \begin{cases} 6x, & x \le 0 \\ -6x, & x > 0 \end{cases}$$

$$f'''(x) = \begin{cases} 6, & x \le 0 \\ -6, & x > 0 \end{cases}$$



This function has a third derivative for all x except for x=0, where $f'''(x)$ is discontinuous.

Q5 i) $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} =$$

$$= (3-\lambda)\Big((3-\lambda)(1-\lambda) - 1\Big) =$$

$$= (3-\lambda)(2 - 4\lambda + \lambda^2) = (3-\lambda)(\lambda - 2 - \sqrt{2})(\lambda - 2 + \sqrt{2}) = 0 \Rightarrow$$

$\Rightarrow$ Eigenvalues are $\lambda = 3$, $\lambda = 2+\sqrt{2}$, $\lambda = 2-\sqrt{2}$

Eigenvectors $\vec{v}_i$ are:

① $\lambda = 3$: $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow \begin{cases} x + z = 0 \\ y - 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -z \\ y = 2z \end{cases}$

$\vec{V}_1 = \begin{pmatrix} -a \\ 2a \\ a \end{pmatrix}$ for $\forall a \in \mathbb{R}$, $a \neq 0$.

② $\lambda = 2+\sqrt{2}$: $\begin{pmatrix} 1-\sqrt{2} & 0 & 0 \\ 1 & 1-\sqrt{2} & 1 \\ 0 & 1 & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow \begin{cases} (1-\sqrt{2})x = 0 \\ x + (1-\sqrt{2})y + z = 0 \\ y - (1+\sqrt{2})z = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = 0 \\ z = (\sqrt{2}-1)y \\ y = (1+\sqrt{2})z \end{cases} \Rightarrow \vec{V}_2 = \begin{pmatrix} 0 \\ (\sqrt{2}+1)a \\ a \end{pmatrix}$ for $\forall a \in \mathbb{R}$, $a \neq 0$.

③ $\lambda = 2-\sqrt{2}$: $\begin{pmatrix} 1+\sqrt{2} & 0 & 0 \\ 1 & 1+\sqrt{2} & 1 \\ 0 & 1 & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow \begin{cases} (1+\sqrt{2})x = 0 \\ (1+\sqrt{2})y + z = 0 \\ y + (-1+\sqrt{2})z = 0 \end{cases}$

$\begin{cases} x = 0 \\ z = (-1-\sqrt{2})y \\ y = (1-\sqrt{2})z \end{cases} \Rightarrow \vec{V}_3 = \begin{pmatrix} 0 \\ (1-\sqrt{2})a \\ a \end{pmatrix}$ for $\forall a \in \mathbb{R}$, $a \neq 0$.

ii) $A = \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -1 \\ 5 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - 5(-1) =$$

$$= \lambda^2 - \lambda + 6 = 0$$

$\Delta = 1 - 4(1)(6) = -23 < 0 \Rightarrow \lambda \notin \mathbb{R} \Rightarrow$

$\Rightarrow$ No eigenvalues (and hence eigenvectors) exist among real numbers.