MATHEMATICS FOR COMPUTATION 2024
PROBLEM SHEET 2
SOLUTIONS

ISSUED 11 MARCH 2024, DUE 22 APRIL 2024

Coursework forms 25% of the assessment for this unit and will be comprised of 2 problem sheets equally weighted. On this sheet, each question is equally weighted.

1.

(a) Write a linear equation in the form $aX + bY + cZ = d$ defining a plane in $\mathbb{R}^3$ passing through three points, $(0, 1, 2)$, $(5, -1, -1)$ and $(2, 0, 0)$.

Solution. Using the formula from lectures, we get the following equation:

$$\det \begin{pmatrix} x & y-1 & z-2 \\ 5 & -2 & -3 \\ 2 & -1 & -2 \end{pmatrix} = 0,$$

which reduces to the equation $x + 4y - z = 2$.

(b) Is there the unique plane passing through the following three points in $\mathbb{R}^3$: $(0, 0, 0)$, $(1, 1, 0)$, $(-2, -2, 0)$? If yes, prove it. Otherwise, how many planes? Justify your answer.

Solution. These points belong to the same straight line, span of vector $(1, 1, 0)$. There are infinitely many planes containing this line. For example, planes containing the line and passing through different points in the line defined by equation $x + z - 1 = 0$ are pair-wise distinct.

2. Prove that the following functions are continuous, using the ε/δ definition.

(i)

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \frac{1}{2}\, x^2$$

Solution. Standard stuff.

(ii)

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto -2\,|x|$$

Solution. Need to prove that f is continuous at each $c \in \mathbb{R}$.

First, let $c \neq 0$. Fix any $\varepsilon > 0$ and choose δ strictly less than $|c|$ and strictly less than $\frac{1}{2}\varepsilon$. Since $\delta < |c|$, for any x such that $|x - c| < \delta$, numbers x and c have the same sign. Hence, $||x| - |c|| = |x - c|$ when x, c are both positive, and

$||x|-|c|| = |-x+c| = |x-c|$ when x, c are both negative. Thus, in any case, if $|x-c| < \delta$, then $|f(x)-f(c)| = |-2|x|+2|c|| = 2||x|-|c|| = 2|x-c| < 2\delta < \varepsilon$.

Now let $c = 0$. Fix any $\varepsilon > 0$ and choose $\delta < \frac{1}{2}\varepsilon$. For any x such that $|x| < \delta$ we have: $|f(x)-f(0)| = |-2|x|| = 2|x| < 2\delta < \varepsilon$.

We have satisfied the definition of continuity.

3. Consider the function f defined by the following expression

$$f(x) = \frac{x^4}{\ln x} - \sin(e^{2x-1}) + e^{1/x} + 2.$$

(Reminder: $\ln x$ stands for $\log_e x$, where $e = 2.71828\ldots$)
What is the largest subset of $\mathbb{R}$ in which f is defined? Calculate the derivative $f'(x)$ of this function.

Solution. The largest domain for f is $\{x \in \mathbb{R} |\ x > 0,\ x \neq 1\}$, because $\ln x$ is defined on $\{x \in \mathbb{R} |\ x > 0\}$, and $\ln 1 = 0$.

$$f'(x) = \frac{4x^3 \ln x - x^3}{\ln^2 x} - 2\cos(e^{2x-1})e^{2x-1} - \frac{e^{1/x}}{x^2}.$$

4. Find the second derivative $f''(x)$ of the function

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \leq 0 \\ -x^3 & \text{if} \quad x > 0 \end{cases}.$$

Does this function have the third derivative? Justify your answers.

Solution. We have:

$$f'(x) = \begin{cases} 3x^2 & \text{if} \quad x \leq 0 \\ -3x^2 & \text{if} \quad x > 0 \end{cases}, \quad f''(x) = \begin{cases} 6x & \text{if} \quad x \leq 0 \\ -6x & \text{if} \quad x > 0 \end{cases}.$$

We see that $f''(x)$ is not differentiable at 0 (for the same reason as the function $|x|$ considered in class). Hence $f''(x)$ is not differentiable, and $f'''(x)$ does not exist.

5. For each of the following two matrices decide whether or not it has eigenvalues among real numbers. If yes, find all real eigenvalues and corresponding eigenvectors. If not, explain why not.

(i)

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution. The characteristic polynomial of the matrix is

$$\det \begin{pmatrix} 3-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} =$$

$$= (3-\lambda)^2(1-\lambda) - (3-\lambda) = (3-\lambda)(\lambda^2 - 4\lambda + 2).$$

It follows that the first eigenvalue $\lambda_1 = 3$ while the other two are solutions of the quadratic equation $\lambda^2 - 4\lambda + 2 = 0$. According to well-known formula for solving quadratic equations, the other two eigenvalues are both real numbers, $\lambda_2 = 2 + \sqrt{2}$ and $\lambda_3 = 2 - \sqrt{2}$.

Consider the eigenvalue λ_1. All corresponding real eigenvectors (x_1, x_2, x_3) satisfy the system of linear equations

$$x_1 + x_3 = 0$$
$$x_2 - 2x_3 = 0$$

hence are of the form $(a, -a, -a/2)$ for all non-zero real numbers a.

Consider the eigenvalue $\lambda_2 = 2+\sqrt{2}$. All corresponding real eigenvectors (x_1, x_2, x_3) satisfy the system of linear equations

$$(1-\sqrt{2})x_1 = 0$$
$$x_1 + (1-\sqrt{2})x_2 + x_3 = 0$$
$$x_2 + (-1-\sqrt{2})x_3 = 0$$

hence are of the form $(0, a, a/(1+\sqrt{2})$ for all non-zero real numbers a.

Similarly, for the eigenvalue λ_3, eigenvectors are of the form $(0, a, a/(\sqrt{2}-1)$ for all non-zero real numbers a.

(ii)

$$\begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix}$$

Solution. The characteristic polynomial of the matrix is

$$\det \begin{pmatrix} 1-\lambda & -1 \\ 5 & -\lambda \end{pmatrix} = \lambda^2 - \lambda + 5.$$

The formula for solving quadratic equations, applied to this polynomial, gives a negative number under the $\sqrt{}$ sign. Hence the characteristic polynomial does not have solutions in real numbers.