Discrete mathematics problem sheet

Issued 9 October 2023, due 17 November 2023.

This problem sheet forms 15% of the assessment for this unit.
Each of eight questions carries the same weight.

1. For each pair of Boolean formulae P and Q state whether or not the formula $P \to Q$ is a tautology, or identically false, or neither. Prove your answer.
(i) $P = X \vee Y$, $Q = \neg(X \wedge Y)$,
(ii) $P = X \vee Y$, $Q = \neg X \wedge \neg Y$,
(iii) $P = X \to Y$, $Q = (\neg X \vee Y) \wedge (\neg X \vee X)$,
(iv) $P = X \to \neg Y$, $Q = Y \to \neg X$,
(v) $P = X \wedge (Y \vee Z)$, $Q = (X \vee Y) \wedge (X \vee Z)$,
(vi) $P = X \to Y$, $Q = \neg X \to \neg Y$,
(vii) $P = X \to Y$, $Q = \neg(Y \to X)$,
(viii) $P = (X \to Y) \wedge (Y \to Z)$, $Q = X \to Z$.

2.
(a) How many different binary logical connectives do there exist?
(b) Express the "exclusive or" (XOR) via (i) $\neg$, $\wedge$, (ii) $\neg$, $\vee$, (iii) $\neg$, $\to$.
(c) The "not and" (NAND) logical connective is defined as:

$$X \text{ NAND } Y \equiv \neg(X \wedge Y).$$

Express each of connectives $\wedge$, $\vee$, $\to$, $\neg$ via only NAND.
(d) The "not or" (NOR) logical connective is defined as:

$$X \text{ NOR } Y \equiv \neg(X \vee Y).$$

Express each of connectives $\wedge$, $\vee$, $\to$, $\neg$ via only NOR.

3.

(a) For each of the following statements answer whether it is true or false.
(i) $\forall x \in \mathbf{Z} \exists y \in \mathbf{Z}(x^2 < y + 1)$

(ii) $\exists x \in \mathbf{Z} \forall y \in \mathbf{Z}(x^2 < y + 1)$

(iii) $\exists y \in \mathbf{Z} \forall x \in \mathbf{Z}(x^2 < y + 1)$

(iv) $\forall x \in \mathbf{Z} \exists y \in \mathbf{Z}((x < y) \to (x^2 < y^2))$

Q3 i) True. Let $y = x^2$: $x^2 < x^2 + 1 \Leftrightarrow 1 > 0$
ii) False, counter example $y = -1$: $x^2 < 0$ $\varnothing$
iii) False, suppose such y exists, then if $x = (y+2)$, then $y^2 + 2y + 4 - y - 1 < 0 \Leftrightarrow y^2 + 3y + 3 < 0$, which is always false $\Rightarrow$ contradiction $\Rightarrow$ proved.
iv) True, let $y = 2x$: $x < 2x \to x^2 < 4x^2$
$x > 0 \to 3x^2 > 0$
If $x > 0$ then $x^2 > 0$ is true.

Q1 i) Neither

X	Y	$X \vee Y$ (P)	$\neg(X \wedge Y)$ (Q)	$P \to Q$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

ii) Neither

X	Y	$P = X \vee Y$	$Q = \neg X \wedge \neg Y$	$P \to Q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

iii) Tautology

X	Y	$P = X \to Y$	$Q = (\neg X \vee Y) \wedge (\neg X \vee X)$ taut.	$P \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

iv) Tautology

X	Y	$P = X \to \neg Y$	$Q = Y \to \neg X$	$P \to Q$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

v) Tautology

X	Y	Z	$P = X \wedge (Y \vee Z)$	$Q = (X \vee Y) \wedge (X \vee Z)$	$P \to Q$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

vi) Neither

X	Y	$P = X \to Y$	$Q = \neg X \to \neg Y$	$P \to Q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

vii) Neither

X	Y	$P = X \to Y$	$Q = \neg(Y \to X)$	$P \to Q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

viii) Tautology

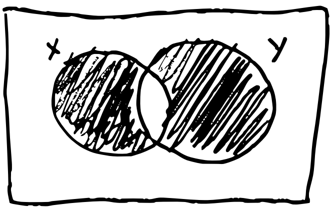
X	Y	Z	$P = (X \to Y) \wedge (Y \to Z)$	$Q = X \to Z$	$P \to Q$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Q2 a) 16, 4 combinations of T and F, where each combination can result in 2 outcomes (T or F) $\Rightarrow 2^4 = 16$

$x \wedge y \equiv \neg(\neg x \vee \neg y)$
$x \vee y \equiv \neg(\neg x \wedge \neg y)$
$x \to y = \neg x \vee y$

b)
i)

x	y	x XOR y	$(x \vee y) \wedge \neg(x \wedge y)$ $\equiv$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F



$\equiv (x \vee y) \wedge (\neg x \vee \neg y) \equiv \neg(\neg x \wedge \neg y) \wedge \neg(x \wedge y)$

ii) $\equiv (x \vee y) \wedge (\neg x \vee \neg y) \equiv$

$\equiv \neg(\neg(x \vee y) \vee \neg(\neg x \vee \neg y))$ $\boxed{\equiv}$

iii) $\boxed{\equiv}$ $\neg(\neg(\neg x \to y) \vee \neg(x \to \neg y)) \equiv$

$\equiv \neg((\neg x \to y) \to \neg(x \to \neg y))$

c)

x	y	x NAND $y = \neg(x \wedge y)$
T	T	F
T	F	T
F	T	T
F	F	T

Notice that

x	x NAND x	$\neg x$
T	F	F
F	T	T

$\Rightarrow \neg x \equiv x$ NAND x

$x \wedge y \equiv \neg(x \text{ NAND } y) \equiv (x \text{ NAND } y) \text{ NAND } (x \text{ NAND } y)$

$x \vee y \equiv \neg(\neg x \wedge \neg y) \equiv \neg(\underbrace{(x \text{ NAND } x)}_{\text{"}x\text{"}} \wedge \underbrace{(y \text{ NAND } y)}_{\text{"}y\text{"}}) \equiv$

$\equiv (x \text{ NAND } x) \text{ NAND } (y \text{ NAND } y)$ by definition of NAND.

$x \to y \equiv \neg x \vee y \equiv \neg(x \wedge \neg y) \equiv x \text{ NAND } \neg y \equiv$

$\equiv (x) \text{ NAND } (y \text{ NAND } y)$

d)

x	y	x NOR $y \equiv \neg(x \vee y)$
T	T	F
T	F	F
F	T	F
F	F	T

Notice that $\neg x \equiv x$ NOR x

$x \vee y \equiv \neg(x \text{ NOR } y) \equiv (x \text{ NOR } y) \text{ NOR } (x \text{ NOR } y)$

$x \wedge y \equiv \neg(\neg x \vee \neg y) \overset{def.}{\equiv} \neg x \text{ NOR } \neg y \equiv (x \text{ NOR } x) \text{ NOR } (y \text{ NOR } y)$

$x \to y \equiv \neg x \vee y \equiv \neg\neg(\neg x \vee y) \equiv$

$\equiv \neg(\neg x \text{ NOR } y) \equiv ((x \text{ NOR } x) \text{ NOR } y) \text{ NOR } ((x \text{ NOR } x) \text{ NOR } y)$

(b) $x=-1$ $y=-1$ $1<0$ is false

(i) Prove that the statement $\forall x \in \mathbf{Z} \forall y \in \mathbf{Z}(x^2 < y+1)$ is false by giving an example of integers x and y disproving it.

(ii) Prove that the statement $\exists x \in \mathbf{Z} \exists y \in \mathbf{Z}(x^2 < y+1)$ is true by giving an example of integers x and y proving it. $x=2$ $y=4$ $4<5$ is true

(iii) Prove that the statement $\forall y \in \mathbf{Z} \exists x \in \mathbf{Z}(x^2 < y+1)$ is false by giving an example of an integer y disproving it. $y=-1$ $x^2<0$ is false

(iv) Prove that the statement $\exists x \in \mathbf{Z} \forall y \in \mathbf{Z}((x<y) \to (x^2 < y^2))$ is true by giving an example of an integer x proving it. $x=0$ If $y>0$ then $y^2>0$ is true.

4.

+ (a) Determine the cardinalities of the following sets:

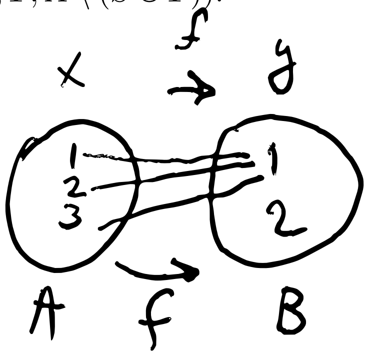
$$2^{\emptyset},\ 2^{\{0\}},\ 2^{\{0\}\cup\{1\}},\ 2^{\{0\}\cap\{1\}},\ 2^{\{\emptyset,0,1\}},\ 2^{2^{2^{\{0,1\}}}}.$$

(b) Let $A = \{1, 2, \ldots, n\}$. Determine the cardinalities of the following sets:

± (i) $\{(x, S) | x \in S,\ S \in 2^A\}$

+ (ii) $\{(S, T) | S \in 2^A,\ T \in 2^A,\ S \cap T = \emptyset\}$.
Hint: Calculate the number of triples of pair-wise disjoint sets $(S, T, A \setminus (S \cup T))$.

+ **5.**



(i) How many maps are there from $\{1,2\}$ to $\{1,2,3\}$?
(ii) How many injective maps are there from $\{1,2\}$ to $\{1,2,3\}$?
(iii) How many bijective maps there are from $\{1,2\}$ to $\{1,2,3\}$?
(iv) How many maps are there from $\{1,2,3\}$ to $\{2,3\}$?
(v) How many surjective maps are there from $\{1,2,3\}$ to $\{2,3\}$?

6. (a) Prove by induction that

$$\sum_{1\le i\le n} (2i-1) = n^2. \quad n \in \mathbb{N}$$

(b) Prove by induction that

$$\sum_{1\le i\le n} i^2 = \frac{1}{6}n(n+1)(2n+1). \quad n \in \mathbb{N}$$

(c) What is the formula for the sum $n \ge 5$ $\quad 2(n-2) = 2n-4$

$$\sum_{3\le i\le n-2} i^2 =$$

$$= \sum_{1\le i\le n-2} i^2 - \sum_{1\le i\le 2} i^2 = \sum_{1\le i\le n-2} i^2 - 1^2 - 2^2 = \frac{1}{6}(n-2)(n-1)(2n-3) - 5 =$$

$$= \frac{1}{6}(n^2-3n+2)(2n-3) - 5 = \frac{1}{6}(2n^3-3n^2-6n^2+9n+4n-6) - 5 =$$

$$= \frac{1}{3}n^3 - \frac{3}{2}n^2 + \frac{13}{6}n - 6$$

i) $\{(x,S) \mid x \in S, S \in 2^A\} = B \quad A = \{1, 2, \dots n\}$

$2^A = \{\emptyset, \{1\}, \{2\}, \dots \{n\}, \{1,2\}, \{1,3\} \dots \{1, 2, \dots n\}\}$

E.g. $P = \{\underset{S_1}{\{1,2\}}, \dots \underset{S_n}{\{1,n\}} \dots$ all combinations of pairs$\}$

$S = \{\emptyset\}$

$x \in S \quad x = \emptyset$?

$$\left.\begin{array}{l}(1, S_1)\\(2, S_1)\end{array}\right] 2$$

$\vdots$

$$\left.\begin{array}{l}(1, S_n)\\(2, S_n)\end{array}\right] 2$$

two elements for each combination from $|P| = \binom{n}{2} \Rightarrow 2\binom{n}{2}$ pairs of (x,S)

Similarly, there are $3\binom{n}{3}$ pairs when $|S_i| = 3$. However, for $|S_i| = 1$ the number of elements is $1\binom{n}{1} - 1 = n-1$, since the $\emptyset$ has to be accounted for.* Therefore, the pattern is:

$$|B| = -1 + \sum_{x=1}^{n} x\binom{n}{x} \quad (1)$$

*: If $S = \emptyset$, then there is no x such that $x \in S$.

Consider the Binomial theorem:

$$(1+a)^n = \sum_{x=0}^{n} \binom{n}{x} a^x$$

Differentiating w.r.t. a:

$$n(1+a)^{n-1} = \sum_{x=0}^{n} \binom{n}{x} x a^{x-1} = \sum_{x=1}^{n} \binom{n}{x} x a^{x-1}$$, since when $x=0$ the term cancels out. Now consider $a=1$:

$$n(1+1)^{n-1} = \sum_{x=1}^{n} \binom{n}{x} x$$ is the same as a part of (1), hence $-1 + \sum_{x=1}^{n} \binom{n}{x} x = -1 + n2^{n-1}$.

6) ii) $A = \{1, 2, 3, \dots, n\}$

$B = \{(S,T) \mid S \in 2^A, T \in 2^A, S \cap T = \emptyset\}$

$(S, T, A \setminus (S \cup T))$

② $A = \{1,2\}$: $2^A = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$(\emptyset, 4)$ Nº of possible elements $|2^A| = 2^2 = 4$

$(\{1\}, 2)$

$(\{2\}, 2)$ $\sum = 9 = 3^2$

$(\{1,2\}, 1)$

③ $A = \{1,2,3\}$: $|2^A| = 2^3 = 8$

$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$(\emptyset, 8)$ $(\{1,2\}, 2)$

$(\{1\}, 4)$ $(\{1,3\}, 2)$ $\sum = 27 = 3^3$

$(\{2\}, 4)$ $(\{2,3\}, 2)$

$(\{3\}, 4)$ $(\{1,2,3\}, 1)$

④ $A = \{1,2,3,4\}$ $|2^A| = 2^4 = 16$

$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\},$
$\{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\},$
$\{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$

$(\emptyset, 16) \times 1$ $\sum = 81 = 3^4$

$(\{x\}, 8) \times 4$ $(2^4)\binom{4}{0}$

$(\{x,y\}, 4) \times 6$ $(2^3)\binom{4}{1}$

$(\{x,y,z\}, 2) \times 4$ $(2^2)\binom{4}{2}$

$(\{x,y,z,w\}, 1) \times 1$ $(2^1)\binom{4}{3}$

$(2^0)\binom{4}{4}$

$$\sum_{x=0}^{4} 2^x \binom{4}{4-x} = 81$$

$$\sum_{x=0}^{n} 2^x \binom{n}{n-x} = \sum_{x=0}^{n} 2^x \binom{n}{x}$$

The Binomial Theorem:

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^{n-x} b^x$$

Let $a = 1$ and $b = 2$:

$$(1+2)^n = \sum_{x=0}^{n} \binom{n}{x} 1^{n-x} 2^x$$

$$\underline{\sum_{x=0}^{n} \binom{n}{x} 2^x = 3^n}$$

(Q4) a) $|2^A| = 2^{|A|}$

$|2^{\varnothing}| = 2^{|\varnothing|} = 2^{|\{\}|} = 2^0 = 1$

$|2^{\{0\}}| = 2^1 = 2$

$|2^{\{0\} \cup \{1\}}| = |2^{\{0,1\}}| = 2^2 = 4$

$|2^{\{0\} \cap \{1\}}| = |2^{\varnothing}| = 1$

$|2^{\{\varnothing,0,1\}}| = 2^3 = 8$

$|2^{2^{2^{\{0,1\}}}}| = 2^{|2^{2^{\{0,1\}}}|} = 2^{2^{|2^{\{0,1\}}|}} = 2^{2^4} = 2^{16} =$

$= 65536$

b) $A = \{1, 2, \dots, n\} \Rightarrow |A| = n$

i) $B = \{(x,S) \mid x \in S,\ S \in 2^A\}$

$\left[\begin{array}{l} X: \{1,2,\dots,2^n\} \\ S: \{1,2,\dots,2^n\} \end{array}\right.$

$|S| = |2^A| = 2^{|A|} = 2^n$

$$\left.\begin{array}{l} |(1,1)\ (1,2) \dots (1,2^n)| = 2^n \\ \vdots \\ (2^n,1), (2^n,2) \dots (2^n,2^n) \end{array}\right\} 2^n \Rightarrow 2^n(2^n) = 2^{2n}$$

total pairs

$|B| = 4^n$

ii) $C = \{(S,T) \mid S \in 2^A,\ T \in 2^A,\ S \cap T = \varnothing\}$

$S: \{1, 2, \dots, 2^n\}$

$T: \{1, 2, \dots, 2^n\}$

$$\left.\begin{array}{l} |X\ (1,2)\ (1,3) \dots (1,2^n)| = 2^n - 1 \\ (2,1)\ X\ (2,3) \dots (2,2^n) \\ (3,1)\ (3,2)\ X \dots (3,2^n) \\ \vdots \\ (2^n,1)(2^n,2)(2^n,3) \dots (2^n,2^{n-1})\ X \end{array}\right\} 2^n$$

$|C| = 2^n(2^n - 1) = 4^n - 2^n$

b) $A = \{1, 2, \dots, n\} \Rightarrow |A| = n$

$B = \{(x,S) \mid x \in S,\ S \in 2^A\}$

S is an element of $2^A = \{\varnothing, \{1\}, \{1,2\}, \{1,3\}, \dots, \{1,n\}, \dots\}$

$|2^A| = 2^n$

x is an element of S. E.g.:

- $S_1 = \{1,2,3\} \Rightarrow x=1$ or $x=2$ or $x=3 \Rightarrow$
$\Rightarrow B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), \dots\}$: $3+3+3 = 9$ pairs
- $S_2 = \{1,2\} \Rightarrow x=1$ or $x=2 \Rightarrow$
$\Rightarrow B = \{(1,1), (1,2), (2,1), (2,2), \dots\}$: $2^2 = 4$ pairs

However, $S_2 \subset S_1 \Rightarrow$ All pairs generated by S_2 have already been generated by $S_1 \Rightarrow$ Consider the largest possible $S = \{\varnothing, 1, 2, \dots, n\}$:

$$\left.\begin{array}{l} x = \varnothing : n+1 \text{ ways to make a pair} \\ \text{or} \\ x = 1 : n+1 \text{ —"—} \\ \text{or} \\ \dots \\ \text{or} \\ x = n-1 : n+1 \text{ —"—} \\ \text{or} \\ x = n : n+1 \text{ —"—} \end{array}\right\} n+1 \quad XS$$

There are $|S \times \{\varnothing, 1, \dots n\}|$ pairs:

$|\{\varnothing, 1, \dots n\} \times \{\varnothing, 1, \dots, n\}| =$

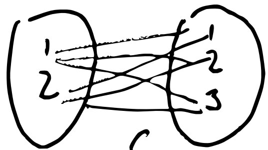
$= \underbrace{(n+1) + (n+1) + \dots + (n+1)}_{(n+1) \text{ times}} = (n+1)(n+1) = n^2 + 2n + 1$ pairs

$|B| = n^2 + 2n + 1$

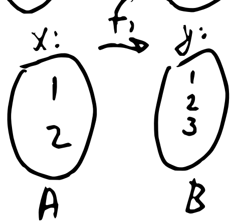
(Q5) i) $A = \{1, 2\}$ $|A| = 2$

$B = \{1, 2, 3\}$ $|B| = 3$

$3^2 = 9$ maps for $f: A \to B$



ii)



$f: A \to B$

1	2
1	2
1	3
2	1
2	3
3	1
3	2

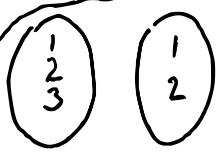
} x

} y => 6 maps

iii) 0, the number of elements in the domain doesn't match the number of elements in the range.

iv) $2^3 = 8$

v)



1	2	3
1	1	2
1	2	1
1	2	2
2	1	1
2	1	2
2	2	1

6, i.e. exclude two maps from № iv)

1	2	3
1	1	1
2	2	2

Q6 a) $\sum_{1 \le i \le n} (2i-1) = n^2$ (1)

where $n \in \mathbb{N}$

1) Base: Let $n=1$: $2(1)-1 = 1^2$

$1 = 1 \Rightarrow$ True

2) Inductive hypothesis: Assume (1) is True

3) Inductive step: let $n = n+1$:

$$\sum_{1 \le i \le n+1} (2i-1) = (n+1)^2$$

$$\sum_{1 \le i \le n} (2i-1) + 2(n+1) - 1 = (n+1)^2$$

$$n^2 + 2n + 2 - 1 = n^2 + 2n + 1$$

LHS = RHS $\rightarrow$ Q.E.D.

4) Don't need the conclusion?

A: No, but write it anyway for JM ♡.

b) $\sum_{1 \le i \le n} i^2 = \frac{1}{6} n(n+1)(2n+1) = S_n$ (1)

where $n \in \mathbb{N}$

1) Let $n=1$: $1^2 = \frac{1}{6}(1)(1+1)(2(1)+1)$

$1 = 1 \Rightarrow$ (1) is true for 1.

2) Assume that (1) is true for $n = k \in \mathbb{N}$.

$$S_k = \frac{1}{6} k(k+1)(2k+1)$$

3) Let $k = k+1$:

$$\text{LHS} = S_{k+1} = S_k + (k+1)^2 = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 =$$

$$= \frac{1}{6} k(2k^2 + 3k + 1) + k^2 + 2k + 1 = \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1$$

$$\text{RHS} = \frac{1}{6}(k+1)(k+2)(2k+3) =$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6) = \frac{1}{6}(2k^3 + 7k^2 + 6k + 2k^2 + 7k + 6) =$$

$$= \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1 = \text{LHS}$$

4) ...

(provided that $n \geq 5$)?

+(d) Prove by induction that $7^n - 1$ is divisible by 6;

+ (e) Prove by induction that $2n + 1 \leq 2^n$ for all integer $n \geq 3$.

+**7.** Each of the following defines a relation on $\mathbf{Z}$. In each case determine if the relation is reflexive, symmetric, or transitive. Justify your answers.

(i) $x + y$ is an odd integer;
(ii) $x + y$ is an even integer;
(iii) xy is an odd integer;
(iv) $x + xy$ is an even integer.

8. Consider the subset relation $\subset$ on the set $A = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.
(a) Is $\subset$ a partial order on A, a strict partial order on A, or neither? Justify your answer.
(b) Is $\subset$ a total order on A? Justify your answer.
(c) Decide whether there are maximal elements, and whether there are minimal elements, in A with respect to $\subset$. If such elements exist, list them all.

d) $\overset{(1)}{7^n - 1 = 6m}$, where $m, n \in \mathbb{Z}$, $n, m \geq 0$

1) Let $n = 0$: $7^0 - 1 = 1 - 1 = 0$

$\frac{0}{6} = 0 \Rightarrow (1)$ is true for 0.

2) Assume $7^k - 1 = 6L$ $\underset{n}{\overset{}{k}}, \underset{m}{L} \in \mathbb{Z}$, $k, L \geq 0$ is true.

3) Consider $P(k+1) - P(k) = 7^{k+1} - 1 - 7^k + 1 =$
$= 7^k(7-1) = 6(7^k) \Rightarrow P(k+1) = 6(7^k) + P(k) =$
$= 6(7^k) + 6L = 6(7^k + L)$ – divisible by 6

4) ...

e) $P(n): 2n+1 \leq 2^n \quad n \geq 3$

1) $P(3): 2(3)+1 \leq 2^3$

$7 \leq 8 \Rightarrow P(n)$ is true

2) Assume that $2k+1 \leq 2^k$, $k \geq 3$, $k \in \mathbb{N}$

3) $2k+1 \leq 2^k \mid \times 2$

$4k+2 \leq 2^{k+1}$

$4k+2 \geq 2k+3$ for all $k \in \mathbb{N}, k \geq 3$

$(2k \geq 1 \Leftrightarrow k \geq \frac{1}{2})$

Hence $2k+3 \leq 4k+2 \leq 2^{k+1}$

$2(k+1)+1 \leq 2^{k+1} \Rightarrow$ proved for $P(k+1)$

4) ...

Q7 i) $x+y$ is an odd int, $x, y \in \mathbb{Z}$

- $\forall x \in \mathbb{Z}$ ($x+x = 2x$ is an even int) $\Rightarrow$ Not Refl. X
- If $x+y$ is an odd int, then $y+x = x+y$ is also an odd int for $\forall x, y \in \mathbb{Z} \Rightarrow$ Symmetric ✓
- Suppose x is even and $x+y$ is an odd int. And suppose that z is even and $y+z$ is an odd int. Then $x+z$ would be even $\Rightarrow$ Not Trans. X

ii) $x+y$ is an even int.

- $\forall x \in \mathbb{Z}$ ($x+x = 2x$ is an even int) $\Rightarrow$ Refl. ✓
- $(x+y = 2n, n \in \mathbb{Z}) \rightarrow (y+x = x+y = 2n)$ is a tautology $\Rightarrow$ Sym. ✓
- ① $\left.\begin{array}{l} x = 2n+1 \\ y = 2m+1 \end{array}\right\} \Rightarrow x+y = 2(m+n+1)$, then z must be odd to get an even int: $z = 2k+1 \Rightarrow y+z = 2(k+m+1)$

$x+z = 2(k+n+1)$.

② $\left.\begin{array}{l} x = 2n \\ y = 2m \end{array}\right\} \Rightarrow x+y = 2(n+m)$, then z must be even to get an even int: $z = 2k \Rightarrow y+z = 2(k+m)$

$x+z = 2(n+k)$

①, ② $\Rightarrow$ Trans. ✓

Q7 iii) xy is an odd int.

- $x(x) = x^2$ is not an odd int for all $x \in \mathbb{Z}$ (e.g. $2^2 = 4$) $\Rightarrow$ Not Refl. X
- $(xy = 2n+1,\ n \in \mathbb{Z}) \Rightarrow (yx = xy = 2n+1)$ is a taut. $\Rightarrow$ Sym. ✓
- If xy and yz are both odd ints, then x, y, z are also odd ints $\Rightarrow xz$ is an odd int. $\Rightarrow$ $\Rightarrow$ Trans. ✓

iv) $x + xy$ is an even int.

- $x + x(x) = x + x^2$

① $x = 2n,\ n \in \mathbb{Z}$: $2n + 4n^2 = 2(n + 2n^2)$

② $x = 2n+1,\ n \in \mathbb{Z}$: $2n+1 + (2n+1)^2 = 2n+1 + 4n^2 + 4n + 1 =$
$= 4n^2 + 6n + 2 = 2(2n^2 + 3n + 1)$

①,② $\Rightarrow$ Refl. ✓

- ① If $x = 2n$, then y must equal to $2m$ for $x + xy$ to be an even int (odd+even ≠ even). Thus consider $y + yx = 2m + 2m(2n) = 2(m + 2mn)$

② If $x = 2n+1$, then $y = 2m+1$ for $x+xy$ to be even, then
$y + yx = 2m+1 + (2m+1)(2n+1) = 2m+1 + 4mn + 2m + 2n + 1 =$
$= 2(2m + 2mn + n + 1)$

①,② $\Rightarrow$ Sym. ✓

- If $\begin{matrix} x + xy = E \\ y + yz = E \end{matrix}$ then $x + xz = E$?

~~Consider~~ $x = 2n+1$: for $y+yz$ to be even, $z \overset{\overset{2k}{\shortparallel}}{}$ must ~~$y = 2m+1$~~
be even, therefore $x + xz = \cancel{2n+1} + (2n+1)(2k) =$
$= 2n+1 + 4nk + 2k = 2(n + 2nk + k) + 1 \Rightarrow$ Trans. X

Consider all 4 cases of x, y odd/even to conclude that it is transitive. ✓

Q8 $A = \{\{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$

Consider the relation $\subset$ on A.

a) Since every set is a subset of itself, R is reflexive. This implies that R is not asymmetric, and consequently not strict partial order.

If $X \subset Y$ and $Y \subset Z$, then $X \subset Z$, hence R is transitive.

If $X \subset Y$ and $Y \subset X$, then X must be equal to Y, therefore R is antisymmetric.

Thus, R satisfies all 3 properties to be called partial order.

b) Consider $\{b\}, \{c\} \in A$. Since neither $\{b\} \subset \{c\}$, nor $\{c\} \subset \{b\}$ is true, R is not total order.

d) $\{b\}$ and $\{c\}$ are the minimal elements, since there are no elements in A that are subsets of these elements.

$\{a,b,c\}$ is the maximal element because there are no sets in A that include this element.