MATHEMATICS FOR COMPUTATION 2024
PROBLEM SHEET 1

ISSUED 12 FEBRUARY 2024, DUE 11 MARCH 2024

Coursework forms 25% of the assessment for this unit and will be comprised of 2 problem sheets equally weighted. On this sheet, each question is equally weighted.

1. Solve the following system of linear equations using the Cramer's rule.

$$3X - Y = 8$$
$$-2X + Y + Z = 9$$
$$2X - Y + 4Z = -5.$$

The solution should be represented by simple fractions. Show your calculations of necessary determinants.

2. Solve the same system of equations by first inverting its matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}.$$

Compute the inverse of A using the calculation of the adjugate matrix A^*. The elements of A^{-1} should be represented by simple fractions. Show your working, including all intermediate steps.

3. Given a matrix

$$A = \begin{pmatrix} 1 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & 1 & -1 \end{pmatrix},$$

use Gaussian elimination to compute the determinant $\det(A)$ of A and to solve the system of linear equations $AX = b$, where $X = (X_1, X_2, X_3)^T$ is the vector of unknowns and $b = (1, 0, 1)^T$. The solution should be represented by simple fractions. Show your working.

4. Using Gaussian elimination, compute the rank of the following (4×4)-matrix

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ -2 & 4 & 2 & 4 \end{pmatrix}$$

Show your working.

+ **5.** Consider the following system of linear equations.

$$3X_1 - 2X_2 + 3X_3 - X_4 = 1$$
$$X_2 + X_4 = 3$$
$$X_1 + X_2 - 2X_3 + 4X_4 = 1$$

Using Gaussian elimination, show that it has at least one solution. Represent the general solution as an affine map from one vector space to another, in matrix/vector form. Find one specific solution. Show your working.

Q1

$$\underbrace{\begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix}}_{B}$$

According to Cramer's rule:

$$x = \frac{\det A_x}{\det A} \qquad y = \frac{\det A_y}{\det A} \qquad z = \frac{\det A_z}{\det A}$$

Where $A_x = \begin{pmatrix} \underbrace{\begin{matrix} 8 \\ 9 \\ -5 \end{matrix}}_{B} & \begin{matrix} -1 \\ 1 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 4 \end{matrix} \end{pmatrix}$, and similarly for A_y and A_z.

$$\det A = 3(1)(4) + 0 + 2(-1)(1) - 0 - 3(-1)(1) - 4(-2)(-1) =$$
$$= 5$$

$$\det A_x = 8(1)(4) + 0 + 1(-1)(-5) - 0 - 9(-1)(4) - 8(-1)(1) =$$
$$= 81$$

$$\det A_y = \det \begin{pmatrix} 3 & 8 & 0 \\ -2 & 9 & 1 \\ 2 & -5 & 4 \end{pmatrix} = 3(9)(4) + 0 + 2(8)(1) - 0 -$$
$$- 8(-2)(4) - 1(3)(-5) = 203$$

$$\det_z = \det\begin{pmatrix} 3 & -1 & 8 \\ -2 & 1 & 9 \\ 2 & -1 & -5 \end{pmatrix} = 3(1)(-5)+8(-2)(-1)+2(9)(-1)-$$

$$-2(1)(8)-(-2)(-1)(-5)-(9)(-1)(3)=4$$

$$x=\frac{81}{5},\ y=\frac{203}{5},\ z=\frac{4}{5}$$

Q2 $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix}$$

$$A^* = \begin{pmatrix} (-1)^{1+1}\det A_{11} & (-1)^{1+2}\det A_{21} & (-1)^{1+3}\det A_{31} \\ (-1)^{2+1}\det A_{12} & (-1)^{2+2}\det A_{22} & (-1)^{2+3}\det A_{32} \\ (-1)^{3+1}\det A_{13} & (-1)^{3+2}\det A_{23} & (-1)^{3+3}\det A_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} +(4(1)-(-1)(1)) & -((-1)(4)-0) & +((-1)(1)-0) \\ -((-2)(4)-2(1)) & +((3)(4)-0) & -((3)(1)-0) \\ +((-2)(-1)-2(1)) & -(3(-1)-(-1)(2)) & +(3(1)-(-2)(-1)) \end{pmatrix} =$$

$$= \begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

$\det A = 5$ (see Q1)

$$A^{-1} = \frac{1}{5}A^* \longrightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 4/5 & -1/5 \\ 2 & 12/5 & -3/5 \\ 0 & 1/5 & 1/5 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} 1(8) + 9\left(\frac{4}{5}\right) - \frac{1}{5}(-5) \\ 2(8) + 9\left(\frac{12}{5}\right) - \frac{3}{5}(-5) \\ 0 + \frac{1}{5}(9) + \frac{1}{5}(-5) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{81}{5} \\ \frac{203}{5} \\ \frac{4}{5} \end{pmatrix}$$

Q3 $A_E = \begin{pmatrix} 1 & 4 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 3 & 1 & -1 & 1 \end{pmatrix}$, where A_E meand extended A.

Choose pivot $a_{11} = 1$. Add row 3 and row 1 times -3:

$$A_E = \begin{pmatrix} 1 & 4 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 0 & -11 & -4 & -2 \end{pmatrix}$$

Add row 2 and row 1:

$$A_E = \begin{pmatrix} 1 & 4 & 1 & 1 \\ 0 & 6 & 3 & 1 \\ 0 & -11 & -4 & -2 \end{pmatrix}$$

Choose pivot $a_{22} = 6$. Add row 3 and row 2 times $11/6$:

$$A_E = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 4 & 1 \\ 0 & 6 & 3 \\ 0 & 0 & 3/2 \end{matrix}}_{A} & \begin{matrix} 1 \\ 1 \\ -1/6 \end{matrix} \end{pmatrix} \Rightarrow \det A = 1(6)\left(\frac{3}{2}\right) = 9$$

$$\begin{cases} x_1 + 4x_2 + x_3 = 1 \\ 6x_2 + 3x_3 = 1 \\ \frac{3}{2}x_3 = -\frac{1}{6} \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 + \frac{1}{9} - 4x_2 \\ x_2 = \left(1 - 3\left(-\frac{1}{9}\right)\right)\left(\frac{1}{6}\right) \\ x_3 = -\frac{1}{9} \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{2}{9} \\ x_2 = \frac{2}{9} \\ x_3 = \end{cases}$$

Q4 $A = \begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ -2 & 4 & 2 & 4 \end{pmatrix}$

Choose pivot $a_{11} = -1$. Add row 1 times 2 to row 2:

$$A = \begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 1 & 3 & 1 & 3 \\ -2 & 4 & 2 & 4 \end{pmatrix}$$

Add row 1 to row 3:

$$A = \begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 5 & 2 & 5 \\ -2 & 4 & 2 & 4 \end{pmatrix}$$

Add (row 1) x(-2) to row 4:

$$A = \begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Choose pivot $a_{22} = 5$. Add (row 2) x(-1) to row 3:

$$A = \begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(A) = 2$$

Q5 $A_E = \begin{pmatrix} 3 & -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 1 & 1 & -2 & 4 & 1 \end{pmatrix}$

Choose pivot $a_{11} = 3$. Add (row 1) x$(-\frac{1}{3})$ to row 3:

$$A_E = \begin{pmatrix} 3 & -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & \frac{5}{3} & -3 & \frac{13}{3} & \frac{2}{3} \end{pmatrix}$$

Choose pivot $a_{22} = 1$. Add $(\text{row } 2) \times \left(-\frac{5}{3}\right)$ to row 3:

$$A_E = \begin{pmatrix} 3 & -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -3 & \frac{8}{3} & -\frac{13}{3} \end{pmatrix}$$

$$\begin{cases} 3x_1 - 2x_2 + 3x_3 - x_4 = 1 \\ x_2 + x_4 = 3 \\ -3x_3 + \frac{8}{3}x_4 = -\frac{13}{3} \end{cases} \Leftrightarrow$$

Let $x_4 = 0$:

$$\begin{cases} 3x_1 - 2x_2 + 3x_3 = 1 \\ x_2 = 3 \\ x_3 = \left(-\frac{13}{3}\right)\left(-\frac{1}{3}\right) \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{8}{9} \\ x_2 = 3 \\ x_3 = \frac{13}{9} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = \frac{1}{3}\left(1 + 2(-x_4 + 3) - 3\left(\frac{8}{9}x_4 + \frac{13}{9}\right) + x_4\right) = -\frac{11}{9}x_4 + \frac{8}{9} \\ x_2 = -x_4 + 3 \\ x_3 = \frac{8}{9}x_4 + \frac{13}{9} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{11}{9} \\ -1 \\ \frac{8}{9} \end{pmatrix} x_4 + \begin{pmatrix} \frac{8}{9} \\ 3 \\ \frac{13}{9} \end{pmatrix}$$