Mathematics 629
Spring 2026
Homework Assignment 3

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Due: 2/20/2026

1. Prove that the Borel σ-algebra in the plane, $\mathcal{B}_{\mathbb{R}^2}$, is generated by the collection of sets of the form $V_{a,b} = \{(x_1, x_2) : x_1 \geq a, x_2 > b\}$, $a, b \in \mathbb{R}$.

   Proof. $\square$

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure, $m(E) > 0$. Show that for any $\beta < 1$ there is an open interval I such that $m(E \cap I) > \beta\, m(I)$.

   Hint: Argue by contradiction and think about the definition of the outer measure that determines Borel-Lebesgue measure.

   Proof. $\square$

3. Let $E \subset \mathbb{R}$ be a set of positive Lebesgue measure. Let $N \in \mathbb{N}$. Show that E contains an arithmetic progression of length N, i.e. there is an $a > 0$ and a real number x so that $x, x + a, x + 2a, \ldots, x + (N - 1)a$ belong to E.

   Hint: Adapt the proof of Steinhaus' theorem.

   Proof. $\square$

4. Let E be the set of all real numbers which have the digit 7 missing in their decimal expansion. Show that E is a Lebesgue null set.

   Proof. $\square$