

Mathematics 629  
Spring 2026  
Homework Assignment 3

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Due: 2/20/2026

1. Prove that the Borel  $\sigma$ -algebra in the plane,  $\mathcal{B}_{\mathbb{R}^2}$ , is generated by the collection of sets of the form  $V_{a,b} = \{(x_1, x_2) : x_1 \geq a, x_2 > b\}$ ,  $a, b \in \mathbb{R}$ .

*Proof.* Since  $V_{a,b}$  is constructed such that  $(x_1, x_2) \in V_{a,b}$  for  $x_1 \geq a, x_2 > b$  given  $a, b \in \mathbb{R}$ , then  $\{[a, \infty); a \in \mathbb{R}\}$  generates  $\mathcal{B}_{\mathbb{R}}$  by proposition 1.35. Similarly  $\{(b, \infty); b \in \mathbb{R}\}$  generates  $\mathcal{B}_{\mathbb{R}}$ . Thus,  $\mathfrak{M}(V_{a,b}) \supset \mathcal{B}_{\mathbb{R}^2}$  because  $V_{a,b} = [a, \infty) \times (b, \infty) \in \mathcal{B}_{\mathbb{R}^2}$  for all  $a, b \in \mathbb{R}$ . Next, we will show  $\mathcal{B}_{\mathbb{R}^2} \subset \mathfrak{M}(V_{a,b})$ . By definition of a plane, for all  $(a_i, b_i)$  and  $(a_j, b_j)$ ,  $(a_i, b_i) \times (a_j, b_j) \in \mathcal{B}_{\mathbb{R}^2}$ . Notice that for  $a_i, b_i \in \mathbb{R}$ , since  $[a_i, \infty) \setminus (b_i, \infty) = [a_i, \infty) \cap (b_i, \infty)^c \in \mathfrak{M}(V_{a,b})$  by definition of  $\sigma$ -algebras, then  $[a_i, \infty) \setminus (b_i, \infty) = (a_i, b_i) \in \mathfrak{M}(V_{a,b})$  also. Thus,  $(a_i, b_i) \times (a_j, b_j) \in \mathfrak{M}(V_{a,b})$  which shows  $\mathcal{B}_{\mathbb{R}^2} \subset \mathfrak{M}(V_{a,b})$ .  $\square$

2. Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set of positive Lebesgue measure,  $m(E) > 0$ . Show that for any  $\beta < 1$  there is an open interval  $I$  such that  $m(E \cap I) > \beta m(I)$ .

*Hint: Argue by contradiction and think about the definition of the outer measure that determines Borel-Lebesgue measure.*

*Proof.*  $\square$

3. Let  $E \subset \mathbb{R}$  be a set of positive Lebesgue measure. Let  $N \in \mathbb{N}$ . Show that  $E$  contains an arithmetic progression of length  $N$ , i.e. there is an  $a > 0$  and a real number  $x$  so that  $x, x + a, x + 2a, \dots, x + (N - 1)a$  belong to  $E$ .

*Hint: Adapt the proof of Steinhaus' theorem.*

*Proof.* Without loss of generality, let  $E \subset [0, 1]$ . By theorem 1.64, let  $E$  be compact such that for  $a > 0$  there is  $U \supset E$  such that  $m(U) < (1 + a)m(E)$ . Let  $\epsilon = \frac{1}{2} \text{dist}(E, U^c) > 0$  and  $|kt| < \frac{\epsilon}{(N-1)}$ . Then, for all  $x \in E$ ,  $x + kt \in E$  also. We want to show  $E \cap (E - t) \cap (E - 2t) \cap \dots \cap (E - Nt) = \emptyset$  by showing the measure of intersections is greater than 0. By the construction of  $\epsilon$ ,  $(E - kt) \cap U^c = \emptyset$  if  $t < \epsilon$ , thus

$$m(U \setminus (E - kt)) = m(U) - m(E - kt) = m(u) - m(E) < (1 + a)m(E) - m(E) \leq am(E).$$

By subadditivity,

$$m\left(\bigcup_{k=0}^{N-1}(E - kt)\right) \geq m(U) - \sum_{k=0}^{N-1} m(U \setminus (E - kt))$$

From the previous inequality showing  $m(U \setminus (E - kt)) \leq am(E)$  for  $N$  terms then,

$$m\left(\bigcup_{k=1}^{N-1}(E - kt)\right) \geq m(U) - Nam(E) \geq (N - 1)m(E) > 0.$$

if we specify that  $a < \frac{1}{N-1}$ .

□

4. Let  $E$  be the set of all real numbers which have the digit 7 missing in their decimal expansion. Show that  $E$  is a Lebesgue null set.

*Proof.* Let  $S = [0, 1)$ . Then it suffices to show  $\mu(E \cap S) = 0 < \epsilon$  for all  $\epsilon > 0$  since  $\mu(E) = \mu(\bigcup_{i \in \mathbb{Z}}(S + i))$  by countable additivity. Next, let  $P_0$  be a partition of  $[0, 1)$  containing sets of the form  $[\frac{k}{10}, \frac{k+1}{10})$  for  $0 \leq k \leq 9$ . Since  $\mu(P_0) = \frac{9}{10}$ , notice that forming a similar partition  $P_j$  on every  $p \in P_0$  has measure  $\frac{9}{10}\mu(p)$  so

$$\mu\left(\bigcup_{j=0}^9 P_j\right) = \left(\frac{9}{10}\right)^2.$$

Then for  $n \in \mathbb{N}$  decimal places of  $x \in S$ ,

$$\mu(E) = \left(\frac{9}{10}\right)^n \rightarrow 0$$

as a geometric sequence.

□