Mathematics 629
Spring 2026
Homework Assignment 3

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Due: 2/20/2026

1. Prove that the Borel σ-algebra in the plane, $\mathcal{B}_{\mathbb{R}^2}$, is generated by the collection of sets of the form $V_{a,b} = \{(x_1, x_2) : x_1 \geq a, x_2 > b\}$, $a, b \in \mathbb{R}$.

   Proof. Since $V_{a,b}$ is constructed such that $(x_1, x_2) \in V_{a,b}$ for $x_1 \geq a, x_2 > b$ given $a, b \in \mathbb{R}$, then $\{[a, \infty); a \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$ by proposition 1.35. Similarly $\{(b, \infty); b \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$. Thus, $\mathfrak{M}(V_{a,b}) \supset \mathcal{B}_{\mathbb{R}^2}$ because $V_{a,b} = [a, \infty) \times (b, \infty) \in \mathcal{B}_{\mathbb{R}^2}$ for all $a, b \in \mathbb{R}$. Next, we will show $\mathcal{B}_{\mathbb{R}^2} \subset \mathfrak{M}(V_{a,b})$. By definition of a plane, for all (a_i, b_i) and (a_j, b_j), $(a_i, b_i) \times (a_j, b_j) \in \mathcal{B}_{\mathbb{R}}$. Notice that for $a_i, b_i \in \mathbb{R}$, since $[a_i, \infty) \backslash (b_i, \infty) = [a_i, \infty) \cap (b_i, \infty)^c \in \mathfrak{M}(V_{a,b})$ by definition of σ-algebras, then $[a_i, \infty) \backslash (b_i, \infty) = (a_i, b_i) \in \mathfrak{M}(V_{a,b}$ also. Thus, $(a_i, b_i) \times (a_j, b_j) \in \mathfrak{M}(V_{a,b})$ which shows $\mathcal{B}_{\mathbb{R}^2} \subset \mathfrak{M}(V_{a,b})$. □

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure, $m(E) > 0$. Show that for any $\beta < 1$ there is an open interval I such that $m(E \cap I) > \beta\, m(I)$.

   Hint: Argue by contradiction and think about the definition of the outer measure that determines Borel-Lebesgue measure.

   Proof. Suppose to the contrary that $m(E \cap I) \leq \beta m(I)$. Since E is a Lebesgue measurable set, then there exists a collection of n-intervals $\{I_k\}^n$ such that $E \subset \bigcup_k I_k$. Furthermore, by definition of Lebesgue measure, $m(E) = inf_{E \subset \bigcup I_k} \sum_{k=1}^{n} length(I_k)$ or equivalently that $\sum_{k=1}^{n} length(I_k) < m(E) + \epsilon$ for $\epsilon > 0$. Then by subadditivity and substition,

   $$m(E) = m(E \cap \bigcup I_k) \leq \sum_{k=1}^{n} m(E \cap I_k) \leq \beta \sum_{k=1}^{n} m(I_k) < \beta(m(E) + \epsilon).$$

   Notice that while $\epsilon \to 0$, $m(E) < \beta(m(E) + \epsilon)$ holds only when $\beta \geq 1$ which is a contradiction. □

3. Let $E \subset \mathbb{R}$ be a set of positive Lebesgue measure. Let $N \in \mathbb{N}$. Show that E contains an arithmetic progression of length N, i.e. there is an $a > 0$ and a real number x so that $x, x + a, x + 2a, \ldots, x + (N - 1)a$ belong to E.

   Hint: Adapt the proof of Steinhaus' theorem.

Proof. Without loss of generality, let $E \subset [0,1]$. By theorem 1.64, let E be compact such that for $a > 0$ there is $U \supset E$ such that $m(U) < (1+a)m(E)$. Let $\epsilon = \frac{1}{2}\text{dist}(E, U^c) > 0$ and $|kt| < \frac{\epsilon}{(N-1)}$. Then, for all $x \in E$, $x + kt \in E$ also. We want to show $E \cap (E-t) \cap (E-2t) \cap \cdots \cap (E-Nt) = \emptyset$ by showing the measure of intersections is greater than 0. By the construction of ϵ, $(E-kt) \cap U^c = \emptyset$ if $t < \epsilon$, thus

$$m(U \backslash (E-kt)) = m(U) - m(E-kt) = m(u) - m(E) < (1+a)m(E) - m(E) \leq am(E).$$

By subadditivity,

$$m(\bigcup_{k=0}^{N-1}(E-kt)) \geq m(U) - \sum_{k=0}^{N-1}(U \backslash (E-kt))$$

From the previous inequality showing $m(U\backslash(E-kt)) \leq am(E)$ for N terms then,

$$m(\bigcup_{k=1}^{N-1}(E-kt)) \geq m(U) - Nam(E) \geq (N-1)m(E) > 0.$$

if we specify that $a < \frac{1}{N-1}$.

□

4. Let E be the set of all real numbers which have the digit 7 missing in their decimal expansion. Show that E is a Lebesgue null set.

Proof. Let $S = [0,1)$. Then it suffices to show $\mu(E \cap S) = 0 < \epsilon$ for all > 0 since $\mu(E) = \mu(\bigcup_{i \in \mathbb{Z}}(S+i))$ by countable additivity. Next, let P_0 be a partition of $[0,1)$ containing sets of the form $[\frac{k}{10}, \frac{k+1}{10})$ for $0 \leq k \leq 9$. Since $\mu(P_0) = \frac{9}{10}$, notice that forming a similar partition P_j on every $p \in P_0$ has measure $\frac{9}{10}\mu(p)$ so

$$\mu(\bigcup_{j=0}^{9} P_j) = (\frac{9}{10})^2.$$

Then for $n \in \mathbb{N}$ decimal places of $x \in S$,

$$\mu(E) = (\frac{9}{10})^n \to 0$$

as a geometric sequence.

□