Mathematics 629 — Homework 2

Alex Kim

Due: Friday, February 13, 2026

1. Give an example of two σ-algebras whose union is not a σ-algebra.

Answer. Let $X = \{1, 2, 3\}$ be a set with $M_1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $M_2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$. Notice that

$$M_1 \cup M_2 = \{\emptyset, \{1\}, \{0\}, \{2, 3\}, \{1, 3\}\, X\}$$

which violates property 3 of the definition of an algebra on X since, for example, $\emptyset^C = M_1 \cup M_2 \notin M_1 \cup M_2$. □

2. Prove: An algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing union (that is, if E_j is a sequence of sets in $\mathcal{A}$ with $E_j \subset E_{j+1}$, then $\bigcup_j E_j \in \mathcal{A}$).

Answer. First, we will show $\mathcal{A}$ is a σ-algebra while closed under countable, increasing unions. □

3. A family of subsets of X is called a *ring* if the following axioms hold:

- If $E_1, \ldots, E_n \in \mathcal{R}$ then $\bigcup_{j=1}^n E_j \in \mathcal{R}$.
- If $E \in \mathcal{R}$ and $F \in \mathcal{R}$ then $E \setminus F \in \mathcal{R}$.

A ring that is closed under countable unions is called a σ-ring. Prove:

(i) Rings are closed under finite intersections. σ-rings are closed under countable intersections.

Answer. □

(ii) A ring is an algebra if and only if $X \in \mathcal{R}$.

Answer. □

(iii) If $\mathcal{R}$ is a σ-ring then the collection $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

Answer. □

(iv) If $\mathcal{R}$ is a σ-ring then the collection $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Answer. □

4. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on the real line.

(i) Prove that $\mathcal{B}_{\mathbb{R}}$ is generated by the closed and bounded intervals, i.e. sets of the form $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$.

Answer. □

(ii) Prove that $\mathcal{B}_{\mathbb{R}}$ is generated by the collection of sets of the form $(-\infty, a)$ with $a \in \mathbb{R}$.

Answer. □

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that for all $E, F \in \mathcal{M}$,

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Answer. □

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. The symmetric difference of two sets E and F is given by $E \triangle F := (E \setminus F) \cup (F \setminus E)$. Prove:

(i) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$ then $\mu(E) = \mu(F)$.

Answer. □

(ii) Define a relation on $\mathcal{M}$ by saying that $E \sim F$ if and only if $\mu(E \triangle F) = 0$. Show that $\sim$ is an equivalence relation.

Answer. □

(iii) For E, F define $\rho(E, F) = \mu(E \triangle F)$. Prove a triangle inequality

$$\rho(E, G) \leq \rho(E, F) + \rho(F, G)$$

for all E, F, G. Argue that ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Answer. □