Mathematics 629
Spring 2026
Homework Assignment 1

Alexander Kim

Due: 2/2/2026

1. Define $f(0) = 0$, and for $0 < x \leq 1$,

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is irrational} \\ \frac{x}{2} - q^{-1/4} & \text{if } x \text{ is rational and } x = p/q \text{ in lowest terms.} \end{cases}$$

   (a) Determine the set of discontinuities of f on $[0, 1]$.

   When x_i is irrational, it is defined as $\frac{x_i}{2} \in (0, 1]$. Thus f is continuous if x is irrational.
   When x_j is rational, $x = \frac{p}{q}$ in lowest terms such that $f(x) = \frac{p}{2q} - \frac{1}{q^4}$. By approaching x_j through the irrationals, $f(x_{j-1}) = \frac{x_{j-1}}{2}$. However, when x_j is rational, $\frac{1}{q^4} > 0$, so $f(x_j) \neq \lim_{x_i \to x_j} \frac{x_j}{2q} - \frac{1}{q^4}$. Thus, the set of discontinuities of f on $[0, 1]$ is $\mathbb{Q} \cap (0, 1]$.

   (b) Determine whether f is Riemann integrable on $[0, 1]$ and if so determine the value of the integral.

   Since the set of discontinuities of f on $[0, 1]$ is countable, we can use the trick from the proof of Lemma 2.6 to construct a set of intervals to show the set of discontinuities is content zero.
   Let $D_f = \{0 = d_1, d_2, \cdots d_n = 1\}$ be the set of discontinuities as defined in (a). Then for every $\epsilon > 0$, we can construct a countable family of intervals

$$\{I_j\}_{j=1}^{\infty}$$

   for $d_j \in D_f$ such that

$$I_j = (d_j - \frac{\epsilon}{2^{j+2}}, d_j + \frac{\epsilon}{2^{j+2}}).$$

   Then, $\sum_{j=1}^{\infty} l(I_j) = \sum_{j=1}^{\infty} (d_j - d_j + \frac{\epsilon}{2^{j+2}} + \frac{\epsilon}{2^{j+2}}) = \frac{\epsilon}{8} < \epsilon$. Thus, D_f is a Lebesgue nullset, so f is Riemann Integrable by Theorem 4.1.

   Since the set of discontinuities is Lebesgue null, $\int_0^1 f(x) = \int_0^1 \frac{x}{2} = \frac{x^2}{4}|_0^1 = \frac{1}{4}$.

2. Show: The set $\mathcal{R}[a,b]$ of real valued Riemann integrable functions on $[a,b]$ form an algebra, i.e. they form a vector space with respect to the usual addition and scalar multiplication, and if f and g are in $\mathcal{R}[a,b]$ then pointwise product $fg : x \mapsto f(x)g(x)$ is in $\mathcal{R}[a,b]$.

   Proof. Let $f, g \in \mathcal{R}[a,b]$ and let $x \in [a,b]$. Then in the case that f and g are Riemann integrable at x, $f(x) + g(x) \in \mathbb{R}$. In the case that at least one of f or g are not continuous at x, we will show the set of discontinuities is a Lebesgue nullset. Because f and g are both continuous, D_f and D_g sets of discontinuities of f and g are Lebesgue nullsets by Theorem 4.1. Then by Lemma 2.2 we know D_{f+g} is a Lebesgue nullset since $D_{f+g} \in D_f \cup D_G$, thus $f + g \in \mathcal{R}[a,b]$.

   Next, we will show f, g are closed under scalar multiplication. Let $c \in \mathbb{R}$ and $y \in [a,b]$. Similarly to showing closure under addition, if f is Riemann integrable at y, we are done since $cf(y) \in \mathbb{R}$. If $y \in D_f$, cf is Riemann Integrable since the contents of D_f don't change, thus it is still a Lebesgue nullset.

   Lastly, we will verify the pointwise product of $f, g \in \mathcal{R}[a,b]$ is closed. Let $z \in [a,b]$. Like the previous operations, if f, g are Riemann integrable at z, then $f(z)g(z) \in \mathbb{R}$ so fg is also Riemann integrable. Suppose $z \in D_{fg}$. Then either $z \in D_f$ and/or $z \in D_g$ which shows $D_{fg} \subseteq D_f \cup D_g$. Since countable unions of Lebesgue subsets are Lebesgue nullsets, then D_{fg} is also a Lebesgue nullset. Thus, $\mathcal{R}[a,b]$ satisfies closure under addition, scalar multiplication and pointwise product.

   Since $\mathcal{R}[a,b]$ is composed of real valued functions, there exists an additive identity function, multiplicative identity function, additive inverses for all $f \in \mathcal{R}[a,b]$ as well as distributivity of scalar multiplication and the pointwise product. By definition, $\mathcal{R}[a,b]$ is an algebra.

   □

3. Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Show that the set of discontinuities of f is a Lebesgue null set (and hence f is Riemann integrable).

   Proof. Let D_f denote the set of discontinuities of f. Notice that if we express the length of the distance of a discontinuity as $j_k = \lim_{x \to d_i^+} f(x) - \lim_{x -\to d_i^-} f(x)$, then the maximum of the sum of lengths is

$$S_f = \sum_{k=2}^{m} j_k <= f(b) - f(a)$$

   because f is an increasing function. Since $[a,b]$ is compact and f is increasing, then S_f has an upper bound which means that D_f is countable because an infinite series of positive elements is unbounded. Thus we can construct countable intervals $\{I_i\}_{i=1}^{\infty}$ such that for every $d_i \in D_f$ and $\epsilon > 0$,

$$I_i = (d_i - \frac{\epsilon}{2^{i+2}}, d_i + \frac{\epsilon}{2^{i+2}}).$$

Notice that

$$\sum_{i=1}^{\infty} l(I_i) = \sum_{i=1}^{\infty}(d_i - d_i + \frac{\epsilon}{2^{i+2}} + \frac{\epsilon}{2^{i+2}}) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon.$$

Hence, D_f is a Lebesgue nullset by definition.

□

4. Let $f : [a, b] \to \mathbb{R}$ be a nonnegative Riemann integrable function such that $\int f = 0$. Show that $\{x : f(x) \neq 0\}$ is a Lebesgue null set.

Proof. Let $f : [a, b] \to \mathbb{R}$ is Riemann Integrable and nonnegative such that $\int f = 0$. Since f is nonnegative, then $\{x : f(x) \neq 0\} = \{x : f(x) > 0\}$. Let $E_n = \{x \in [a, b] : f(x) \geq \frac{1}{n}\}$. Then, $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$. Since this is a countable union of sets, by Lemma 2.2 it suffices to show E_n is a Lebesgue nullset. Suppose to the contrary that E_n is not a Lebesgue nullset. Next, construct a partition $P = \{a = x_i, x_2 \cdots x_m = b\}$ such that $J = \{i : [x_{i-1}, x_i] \cap E_n \neq \emptyset\}$ that covers E_n. Then for all $\epsilon > 0$,

$$\sum_{i \in J}(x_i - x_{i-1}) \geq \epsilon$$

since E_n is not a Lebesgue nullset. Notice that for all E_n, $supf(x) \geq \frac{1}{n}$ which shows

$$U(f, P) = \sum_{i=1}^{m} supf(x)(x_i - x_{i-1}) \geq \sum_{i=1}^{m} \frac{1}{n}(x_i - x_{i-1}) \geq \frac{\epsilon}{n} > 0.$$

This contradicts the assumption that $\int f = 0$, hence $\{x : f(x) \neq 0\}$ is a Lebesgue null set as a result of Lemma 2.2. □

5. The intersection of σ-algebras is a σ-algebra (important, and easy to check). This does not work for unions:

   Give an example of two σ-algebras whose union is not a σ-algebra.

Example 1. Let M_1 be a σ-algebra on $\mathbb{N}$ with a set $A = \{1, 2, 3\}$ where $A^c = \{n \in \mathbb{N} : n > 3\}$. Then $M_1 = \{\emptyset, \mathbb{N}, A, A^c$.
Let M_2 be a σ-algebra on $\mathbb{N}$ such that $B = \{4, 5, 6\}$ and $B^c = \{n \in \mathbb{N} : n \neq 4, 5, 6\}$.
Then the union $M_1 \cup M_2 = \{\emptyset, \mathbb{N}, A, A^c, B, B^c\}$. However, since $A \cup B = \{1, 2, 3, 4, 5, 6\} \notin M_1 \cup M_2$, then $M_1 \cup M_2$ is not a σ-algebra.