Mathematics 629 — Homework 2

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1. Give an example of two σ-algebras whose union is not a σ-algebra.

Example. Let $X = \{1, 2, 3\}$ be a set with $M_1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $M_2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$. Notice that

$$M_1 \cup M_2 = \{\emptyset, \{1\}, \{0\}, \{2, 3\}, \{1, 3\}, X\}$$

which violates property 3 of the definition of an algebra on X since, for example, $\emptyset^C = M_1 \cup M_2 \notin M_1 \cup M_2$. ☐

2. Prove: An algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing union (that is, if E_j is a sequence of sets in $\mathcal{A}$ with $E_j \subset E_{j+1}$, then $\bigcup_j E_j \in \mathcal{A}$).

Proof. First, assume $\mathcal{A}$ is a σ-algebra on some set X. Since $\{E_j\} \in \mathcal{A}$ is an increaseing sequence of sets for all $j \in \mathbb{N}$, and because $E_{j-1} \subset E_j$, it follows that $\bigcup_j E_j \in A$ by definition of σ-algebras. Next, let $\{E_j\}$ be any sequence of sets in $\mathcal{A}$ and let $F_n = \bigcup_{j=1}^n E_j$ for some $n \in \mathbb{N}$. Since $\{E_j\}$ is any sequence, then $F_n \subset F_{n+1}$, thus $\{F_n\}$ is increasing. By definition of $\mathcal{A}$ as an algebra, $F_n \in A$, and $\bigcup_n F_n \in \mathcal{A}$ because $\mathcal{A}$ is assumed to be closed under countable increasing unions. Thus by definition, $\mathcal{A}$ is a σ-algebra because $\bigcup_n F_n = \bigcup_j E_j \in \mathcal{A}$. ☐

3. A family of subsets of X is called a *ring* if the following axioms hold:

- If $E_1, \ldots, E_n \in \mathcal{R}$ then $\bigcup_{j=1}^n E_j \in \mathcal{R}$.
- If $E \in \mathcal{R}$ and $F \in \mathcal{R}$ then $E \setminus F \in \mathcal{R}$.

A ring that is closed under countable unions is called a σ-ring. Prove:

(i) Rings are closed under finite intersections. σ-rings are closed under countable intersections.

Proof. Let $E, F \subset \mathcal{R}$ be subsets. Observe that $E \backslash (E \backslash F) = E \cap F \in \mathcal{R}$. Let $\{E_j\}^n$ be a finite collection of sets in $\mathcal{R}$. By the previous observations, $E_1 \cap E_2 \in \mathcal{R}$. Assume for some $k \geq 2$, $\bigcap_{j=1}^k E_j \in \mathcal{R}$. Then for $E_{k+1} \in \mathcal{R}$, $\bigcap_{j=1}^k E_j \cap E_{k+1} \in \mathcal{R}$ by the base case.

Suppose $\mathcal{R}$ is a σ-ring. Then for $E_1, E_2 \cdots \in \mathcal{R}$, by definition, $\bigcup_{j=1}^{\infty} E_j \in \mathcal{R}$. By the previous identity, the countable intersection of sets in $\mathcal{R}$, $\bigcap j = 1^{\infty} E_j = E_1 \backslash \bigcup j = 1^{\infty}(E_1 \backslash E_j)$. Since each $E_1 \backslash E_j \in \mathcal{R}$ by axiom 2, and $\mathcal{R}$ is closed under countable unions by definition of σ-rings, then $\bigcap j = 1^{\infty} E_j \in \mathcal{R}$ by axiom 2.

□

(ii) A ring is an algebra if and only if $X \in \mathcal{R}$.

Proof. First, suppose a ring $\mathcal{R}$ is an algebra. Then by definition, $\mathcal{R}$ is a collection of subsets over some set X where $X \in \mathcal{R}$. Then, suppose $X \in \mathcal{R}$ where $\mathcal{R}$ is a ring. By axiom 1 of rings, $\mathcal{R}$, for all $E, F \in \mathcal{R}$, $E \cup F \in \mathcal{R}$. Since $X \in \mathcal{R}$ from the assumption, then for all $E \in \mathcal{R}$, $X \backslash E \in \mathcal{R}$ by axiom 2 of rings. Thus, $\mathcal{R}$ is an algebra by definition. □

(iii) If $\mathcal{R}$ is a σ-ring then the collection $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

Proof. Let $\mathcal{R}$ be a σ-ring and define collection $C = \{E \subset X : E \in \mathcal{R} \text{ or } X \backslash E \in \mathcal{R}\}$. By axiom 2 of rings, then for all $E \in C$, $X \backslash E \in C$ also. Next, let $E_1, E_2, \cdots, E_n \in C$. Notice that since $E \backslash E = \emptyset \in \mathcal{R}$, so $\emptyset^c = X \in C$ by construction. To show C is closed under countable unions, if $\mathcal{R} \in C$, we are done. Otherwise, consider some $E_k \notin \mathcal{R}$. Since $\cap E_n^c \subset E_k^c$, then

$$\bigcap_n E_n^c = E_k^c \backslash (\bigcup_n E_k^c \cap E_n).$$

so if $E_n \in \mathcal{R}$, $E_n \cap E_k^c \in \mathcal{R}$ and if $E_n^c \in \mathcal{R}$, $E_n^c \cap E_k^c \in \mathcal{R}$ also. Thus, $\bigcap E_n^c = \bigcup E_n \in \mathcal{R}$, so $\mathcal{R}$ is a σ-algebra by definition.

□

(iv) If $\mathcal{R}$ is a σ-ring then the collection $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Proof. Let $\mathcal{R}$ be a σ-ring and define collection $C = \{E \subset X : E \cap F \in \mathcal{R}$ for all $F \in \mathcal{R}$. Let $E_1, E_2, \cdots E_n \in \mathcal{R}$ and $F \in \mathcal{R}$. Since $E \cap E = \emptyset \in C$, by axiom 2 of rings $X \in \mathcal{R}$, so $X \cap \emptyset = X \in C$ by construction. By a similar argument, $\bigcup_n E_n \in C$ since $\mathcal{R}$ is closed under countable unions. Notice that because $X \in C$, and $X = E \cup (X \backslash E)$ for all $E \in C$, then $X \backslash E \in C$ since C is closed under unions. Thus, C is a σ-algebra by definition. □

4. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on the real line.

(i) Prove that $\mathcal{B}_{\mathbb{R}}$ is generated by the closed and bounded intervals, i.e. sets of the form $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$.

(ii) Prove that $\mathcal{B}_{\mathbb{R}}$ is generated by the collection of sets of the form $(-\infty, a)$ with $a \in \mathbb{R}$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that for all $E, F \in \mathcal{M}$,

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Proof. $\square$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. The symmetric difference of two sets E and F is given by $E \triangle F := (E \setminus F) \cup (F \setminus E)$. Prove:

(i) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$ then $\mu(E) = \mu(F)$.

Proof. $\square$

(ii) Define a relation on $\mathcal{M}$ by saying that $E \sim F$ if and only if $\mu(E \triangle F) = 0$. Show that $\sim$ is an equivalence relation.

Proof. $\square$

(iii) For E, F define $\rho(E, F) = \mu(E \triangle F)$. Prove a triangle inequality

$$\rho(E, G) \leq \rho(E, F) + \rho(F, G)$$

for all E, F, G. Argue that ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Proof. $\square$