Calculus Tutorial

1 Derivatives

Derivative of function $f(x)$ is another function denoted by $\frac{df}{dx}$ or $f'(x)$. Since the derivative is a function, one can also compute derivative of the derivative

$$\frac{d}{dx}\left(\frac{df}{dx}\right)$$

which is called the second derivative and is denoted by either

$$\frac{d^2 f}{dx^2} \quad \text{or} \quad f''(x).$$

In this section we will learn how to compute derivatives of various functions. Toward that end we will first recall derivatives of some simple functions, summarized in the table below.

Original function, $f(x)$	**Derivative,** $f'(x)$
c	0
x	1
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$
$\exp x$	$\exp x$
$\ln x$	$1/x$

Table 1: Some common functions and their derivatives.

Example 1. Compute the derivative of $f(x) = x^3$.
Solution: From the table we get

$$f(x)' = 3x^{3-1} = 3x^2.$$

Example 2. Compute the derivative of $f(x) = \frac{1}{x}$.
Solution: From the table we get

$$f(x)' = \left(\frac{1}{x}\right)' \equiv \left(x^{-1}\right)' = -x^{-2} \equiv -\frac{1}{x^2}.$$

Example 3. Compute the derivative of $f(x) = \sqrt{x}$.
Solution: From the table we get

$$f(x)' = \left(\sqrt{x}\right)' \equiv \left(x^{1/2}\right)' = \frac{1}{2}x^{-1/2} \equiv \frac{1}{2\sqrt{x}}.$$

Properties of derivatives

Obviously, there are many more functions which are not listed in the table above. Nevertheless, with a few simple rules, based on properties of derivatives, you will be able to compute derivatives of more complicated functions.

Linearity

This property consists of two parts:

a) "Constant is not affected by differentiation"

$$[Cf(x)]' = Cf'(x).$$

b) "Derivative of a sum equals sum of derivatives"

$$[f(x) + g(x)]' = f'(x) + g'(x).$$

Usually, mathematicians combine the two properties into one:

$$[af(x) + bg(x)]' = af'(x) + bg'(x),$$

where a and b are constants, whereas $f(x)$ and $g(x)$ are functions. The expression inside the brackets on the lefthand side is referred to as *linear combination*. Thus the linearity property can be stated as "Derivative of a linear combination equals linear combination of derivatives."

Example 4. Compute the derivative of $f(x) = 5\ln x + 3\cos x$.
Solution: Using the linearity property and the table, we obtain

$$f(x)' = \frac{5}{x} - 3\sin x.$$

Leibnitz (product) rule

This property applies to a product of two functions

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

Exercise 1. Using the Leibnitz rule and the table, verify the first part of the linearity property, that is show that $[cf(x)]' = cf'(x)$.

Example 5. Compute the derivative of $h(x) = x^3 e^x$.
Solution: First of all, we notice that $h(x)$ can be treated as a product of two functions $f(x) = x^3$ and $g(x) = e^x$. Using the product rule and the table we obtain

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 3x^2 e^x + x^3 e^x \\ &= x^2 e^x (3 + x) . \end{aligned}$$

Chain rule

This property applies to *composite functions*, that is a "function of a function", $f(g(x))$. The rule reads

$$[f(g(x))]' = f'(g(x))g'(x).$$

Notice that the chain rule is *almost* saying that "derivative of a function of a function equals derivative of the outer function times derivative of the inner function". The 'almost' is there to emphasize the fact that the argument of the derivative of the outer function is not merely x, but the inner function $g(x)$. In fact, it sounds more complicated then it is. To illustrate how the rule works, consider an example.

Example 6. Compute the derivative of $h(x) = \sin(2x)$.
Solution: Let us view $h(x)$ as a composite function $f(g(x))$, where $f(y) = \sin(y)$ and $g(x) = 2x$. We use y for the argument of the outer function to avoid confusion, as we shall substitute $g(x)$ instead of y at the end. Using the chain rule and the table we obtain

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= \cos(2x)(2x)' \\ &= 2\cos(2x). \end{aligned}$$

If you compare this result with the table, you can come up with the following 'rule of thumb': "If the inner function is a linear function of x, that is $g(x) = ax + b$, the derivative of the composite function acquires an additional prefactor a".

Exercise 2. Compute the derivative of the function in the last example, using the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$ and the Leibnitz rule. Make sure that your result agrees with the previous one.

Exercise 3. Compute the derivative of $h(x) = \ln(cx)$, where c is a constant, first using the 'rule of thumb' above, and then using the property of logarithms ($\ln(ab) = \ln(a) + \ln(b)$) and the linearity property of derivatives. Make sure that the two results agree.

Example 7. Compute the derivative of $h(x) = \tan(x^2)$.
Solution: Let us view $h(x)$ as a composite function $f(g(x))$, where $f(y) = \tan(y)$ and $g(x) = x^2$. Using the chain rule and the table we obtain

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{\cos^2(x^2)}(x^2)' \\ &= \frac{2x}{\cos^2(x^2)}. \end{aligned}$$

Exercise 4. Compute the derivative of $h(x) = \ln(\cos x)$.

Example 8. Derivative of an inverse. Compute the derivative of $h(x) = \frac{1}{g(x)}$.
Solution: Let us view $h(x)$ as a composite function $f(g(x))$, where $f(y) = \frac{1}{y}$. Using the result of the first exercise, the chain rule and the table we obtain

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= -\frac{g'(x)}{g^2(x)}. \end{aligned}$$

Exercise 5. Compute the derivative of $\cot(x)$ using the example above, the identity $\cot(x) = \frac{1}{\tan(x)}$ and the table.

Example 9. Quotient rule. Compute the derivative of $h(x) = \frac{f(x)}{g(x)}$.
Solution: Let us view $h(x)$ as a product of two functions $f(x)$ and $\frac{1}{g(x)}$. Using the Leibnitz rule and the result of the previous example we obtain

$$\begin{aligned} h'(x) &= \frac{f'}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' \\ &= \frac{f'}{g(x)} - \frac{f(x)g'(x)}{g^2(x)}, \end{aligned}$$

which usually written as

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Exercise 6. Compute the derivative of $\tan(x)$ using the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule. Make sure that your result agrees with the table.

Exercise 7. Compute the derivative of $\cot(x)$ using the identity $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and the quotient rule. Make sure that your result agrees with Example 5.

Example 10. General exponential function. Compute the derivative of $h(x) = a^x$.
Solution: Using the identity $a \equiv e^{\ln a}$, the function $h(x)$ can be rewritten as $e^{x \ln a}$, which can be recognized a composite function $f(g(x))$ with $f(y) = e^y$ and $g(x) = x \ln a$. Using the 'rule of thumb' version of the chain rule, we obtain

$$\begin{aligned} h'(x) &= \left(e^{x \ln a}\right)' = (\ln a) e^{x \ln a} \\ &= (\ln a) a^x. \end{aligned}$$

Thus the derivative of a general exponential function equals the same exponential function multiplied by the natural logarithm of the base.

Inverse function

The fraction-like Leibnitz notation for derivatives ($\frac{df}{dx}$) is not unmotivated. In fact, one can often treat derivatives as fractions. In particular, the derivatives of mutually inverse functions, e.g. $y = x^2$ and $x = \sqrt{y}$, are inverses of each other

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

For instance, for the two functions above $\frac{dy}{dx} = 2x$ and $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$ (see Exercise 3). Now using that $y = x^2$, we see that the second derivative is nothing else but $\frac{1}{2x}$, which is obviously inverse to the first derivative.

Exercise 8. Show that the rule holds for the two mutually inverse functions $y = x^3$ and $x = y^{1/3}$.

Exercise 9. Show that the rule holds for the two mutually inverse functions $y = e^x$ and $x = \ln y$.

Example 10. Inverse trig-functions. Compute the derivative of $y(x) = \arcsin x$.
Solution: It straightforward to compute the derivative of the inverse function $x = \sin y$

$$\frac{dx}{dy} = \cos y.$$

What remains to do is to express $\cos y$ on the righthand side in terms of x:

$$\cos y \equiv \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Finally,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1 - x^2}}.$$

The last result is quite often given in tables.

Exercise 10. Compute the derivative of $y(x) = \arccos x$. As a quick test for your answer, make use of the identity $\arcsin x + \arccos x = \pi/2$.

Exercise 11. Compute the derivative of $y(x) = \arctan x$.