SCALAR MAGNITUDES

Abstract: Scalar Magnitudes are invariant scalar quantities that conserve their value and form under transformations of translation and rotation, or changes between coordinate systems (Cartesian, polar, spherical, etc.)

I. Definitions

0. Vectorial Magnitudes

The vectorial position ($\vec{r}_{ij}$), vectorial velocity ($\vec{v}_{ij}$), and vectorial acceleration ($\vec{a}_{ij}$) of two particles i and j are given by:

Vectorial Magnitude	Definition	Derivation
Position ($\vec{r}_{ij}$)	$\vec{r}_{ij} \doteq (\vec{r}_i - \vec{r}_j)$	(*Fundamental definition*)
Velocity ($\vec{v}_{ij}$)	$\vec{v}_{ij} \doteq (\vec{v}_i - \vec{v}_j)$	$\vec{v}_{ij} \doteq d(\vec{r}_{ij})/dt$
Acceleration ($\vec{a}_{ij}$)	$\vec{a}_{ij} \doteq (\vec{a}_i - \vec{a}_j)$	$\vec{a}_{ij} \doteq d^2(\vec{r}_{ij})/dt^2$

1. Scalar Magnitudes

The scalar position (τ_{ij}), scalar velocity ($\dot{\tau}_{ij}$), and scalar acceleration ($\ddot{\tau}_{ij}$) of two particles i and j are given by:

Scalar Magnitude	Definition	Derivation
Position (τ_{ij})	$\tau_{ij} \doteq \frac{1}{2}\vec{r}_{ij} \cdot \vec{r}_{ij}$	(*Fundamental definition*)
Velocity ($\dot{\tau}_{ij}$)	$\dot{\tau}_{ij} \doteq \vec{v}_{ij} \cdot \vec{r}_{ij}$	$\dot{\tau}_{ij} \doteq d(\tau_{ij})/dt$
Acceleration ($\ddot{\tau}_{ij}$)	$\ddot{\tau}_{ij} \doteq \vec{a}_{ij} \cdot \vec{r}_{ij} + \vec{v}_{ij} \cdot \vec{v}_{ij}$	$\ddot{\tau}_{ij} \doteq d^2(\tau_{ij})/dt^2$

II. Scalar Invariance Demonstrations

0. Vectorial Transformations (Absolute)

The vectorial position ($\vec{r}'_i$), vectorial velocity ($\vec{v}'_i$), and vectorial acceleration ($\vec{a}'_i$) of a particle i with respect to a Reference Frame S', whose origin O' is at the vectorial position $\vec{r}_{O'}$ with respect to another Reference Frame S, are given by:

$$\vec{r}'_i = \vec{r}_i - \vec{r}_{O'}$$

$$\vec{v}'_i = \vec{v}_i - \vec{v}_{O'} - \vec{\omega} \times (\vec{r}_i - \vec{r}_{O'})$$

$$\vec{a}'_i = \vec{a}_i - \vec{a}_{O'} - 2\vec{\omega} \times (\vec{v}_i - \vec{v}_{O'}) + \vec{\omega} \times (\vec{\omega} \times (\vec{r}_i - \vec{r}_{O'})) - \vec{\alpha} \times (\vec{r}_i - \vec{r}_{O'})$$

Where $\vec{r}_i$, $\vec{v}_i$, and $\vec{a}_i$ are the vectorial position, velocity, and acceleration of particle i with respect to Frame S; and $\vec{\omega}$ and $\vec{\alpha}$ are the angular velocity and angular acceleration of Frame S' with respect to Frame S.

Note

If $\vec{m}'_i = \vec{n}_i$ then:

$$\frac{d(\vec{m}'_i)}{dt} = \frac{d(\vec{n}_i)}{dt} - \vec{\omega} \times \vec{n}_i$$

1. Scalar Position Invariance (τ_{ij})

The Scalar Position τ_{ij} is invariant under rotation and translation because the magnitude of the relative vector is preserved.

$\tau_{ij} = \frac{1}{2}(\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

$\tau'_{ij} = \frac{1}{2}(\vec{r}'_i - \vec{r}'_j) \cdot (\vec{r}'_i - \vec{r}'_j)$

Since $(\vec{r}_i - \vec{r}_j) = (\vec{r}'_i - \vec{r}'_j)$

Because $\vec{r}'_i = \vec{r}_i - \vec{r}_{O'}$ and $\vec{r}'_j = \vec{r}_j - \vec{r}_{O'}$

(The relative position vector is independent of the Frame's origin.)

$\tau'_{ij} = \frac{1}{2}(\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

Therefore: $\tau'_{ij} = \tau_{ij}$

2. Scalar Velocity Invariance ($\dot{\tau}_{ij}$)

The Scalar Velocity $\dot{\tau}_{ij}$ is invariant because the cross-product generated by the angular velocity ($\vec{\omega}$) is perpendicular to the relative position vector, resulting in a zero scalar product.

$\dot{\tau}_{ij} = (\vec{v}_i - \vec{v}_j) \cdot (\vec{r}_i - \vec{r}_j)$

$\dot{\tau}'_{ij} = (\vec{v}'_i - \vec{v}'_j) \cdot (\vec{r}'_i - \vec{r}'_j)$

$\dot{\tau}'_{ij} = ((\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j)) \cdot (\vec{r}_i - \vec{r}_j)$

Since $(-\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)) \cdot (\vec{r}_i - \vec{r}_j) = 0$

(The rotational term is orthogonal to the relative position vector.)

Because $(\vec{A} \times \vec{B}) \cdot \vec{B} = 0$ (Property of the Scalar Triple Product)

$\dot{\tau}'_{ij} = (\vec{v}_i - \vec{v}_j) \cdot (\vec{r}_i - \vec{r}_j)$

Therefore: $\dot{\tau}'_{ij} = \dot{\tau}_{ij}$

3. Scalar Acceleration Invariance ($\ddot{\tau}_{ij}$)

The Scalar Acceleration $\ddot{\tau}_{ij}$ is invariant because all inertial terms (Angular Acceleration, Coriolis, and Centrifugal) mutually cancel due to the properties of the vector and scalar products.

$\ddot{\tau}_{ij} = (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j) + (\vec{v}_i - \vec{v}_j) \cdot (\vec{v}_i - \vec{v}_j)$

$\ddot{\tau}'_{ij} = (\vec{a}'_i - \vec{a}'_j) \cdot (\vec{r}'_i - \vec{r}'_j) + (\vec{v}'_i - \vec{v}'_j) \cdot (\vec{v}'_i - \vec{v}'_j)$

$\ddot{\tau}'_{ij} = [(\vec{a}_i - \vec{a}_j) - 2\vec{\omega} \times (\vec{v}_i - \vec{v}_j) + \vec{\omega} \times (\vec{\omega} \times (\vec{r}_i - \vec{r}_j)) - \vec{\alpha} \times (\vec{r}_i - \vec{r}_j)] \cdot (\vec{r}_i - \vec{r}_j)$

$+ [(\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j)] \cdot [(\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j)]$

Since $-(\vec{\alpha} \times (\vec{r}_i - \vec{r}_j)) \cdot (\vec{r}_i - \vec{r}_j) = 0$ (Angular acceleration term cancels)

Because $(\vec{A} \times \vec{B}) \cdot \vec{B} = 0$ (Property of the Scalar Triple Product)

Since $-2\,(\vec{\omega}\times(\vec{v}_i-\vec{v}_j))\cdot(\vec{r}_i-\vec{r}_j)-2\,(\vec{v}_i-\vec{v}_j)\cdot(\vec{\omega}\times(\vec{r}_i-\vec{r}_j))=0$ (Coriolis terms cancel)

Because $(\vec{A}\times\vec{B})\cdot\vec{C}=\vec{A}\cdot(\vec{B}\times\vec{C})$ (Cyclic Permutation Property of the Scalar Triple Product)

Since $+(\vec{\omega}\times(\vec{\omega}\times(\vec{r}_i-\vec{r}_j)))\cdot(\vec{r}_i-\vec{r}_j)+(\vec{\omega}\times(\vec{r}_i-\vec{r}_j))\cdot(\vec{\omega}\times(\vec{r}_i-\vec{r}_j))=0$ (Centrifugal terms cancel)

Since $+\vec{P}\cdot(\vec{r}_i-\vec{r}_j)+E=0$

Because $\vec{P}=\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\;\vec{B}-(\vec{A}\cdot\vec{B})\;\vec{C}$ (Vector Triple Product)

Because $E=(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B})=(\vec{A}\cdot\vec{A})\;(\vec{B}\cdot\vec{B})-(\vec{A}\cdot\vec{B})^2$ (Lagrange's Identity)

$\ddot{\tau}'_{ij}=(\vec{a}_i-\vec{a}_j)\cdot(\vec{r}_i-\vec{r}_j)+(\vec{v}_i-\vec{v}_j)\cdot(\vec{v}_i-\vec{v}_j)$

Therefore: $\ddot{\tau}'_{ij}=\ddot{\tau}_{ij}$

III. Fundamental Relations

The scalar magnitudes $(\tau_{ij},\dot{\tau}_{ij},\ddot{\tau}_{ij})$ expressed using radial magnitudes (r_{ij}), polar magnitudes (r_{ij}), cylindrical magnitudes (r_{ij}), circular magnitudes (r_{ij}) and spherical magnitudes (r_{ij}), are given by:

Radial Magnitude: $\tau_{ij}=\frac{1}{2}r_{ij}^2$

Radial Magnitude: $\dot{\tau}_{ij}=r_{ij}\dot{r}_{ij}$

Radial Magnitude: $\ddot{\tau}_{ij}=r_{ij}\ddot{r}_{ij}+\dot{r}_{ij}^2$

Polar Magnitude: $\tau_{ij}=\frac{1}{2}r_{ij}^2$

Polar Magnitude: $\dot{\tau}_{ij}=r_{ij}\dot{r}_{ij}$

Polar Magnitude: $\ddot{\tau}_{ij}=r_{ij}\ddot{r}_{ij}+\dot{r}_{ij}^2$

Cylindrical Magnitude: $\tau_{ij}=\frac{1}{2}r_{ij}^2$

Cylindrical Magnitude: $\dot{\tau}_{ij}=r_{ij}\dot{r}_{ij}$

Cylindrical Magnitude: $\ddot{\tau}_{ij}=r_{ij}\ddot{r}_{ij}+\dot{r}_{ij}^2$

Circular Magnitude: $\tau_{ij}=\frac{1}{2}r_{ij}^2$

Circular Magnitude: $\dot{\tau}_{ij}=r_{ij}\dot{r}_{ij}$

Circular Magnitude: $\ddot{\tau}_{ij}=r_{ij}\ddot{r}_{ij}+\dot{r}_{ij}^2$

Spherical Magnitude: $\tau_{ij}=\frac{1}{2}r_{ij}^2$

Spherical Magnitude: $\dot{\tau}_{ij}=r_{ij}\dot{r}_{ij}$

Spherical Magnitude: $\ddot{\tau}_{ij}=r_{ij}\ddot{r}_{ij}+\dot{r}_{ij}^2$

IV. Bibliography

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