The Truthful Connection between the IFF and Institutions

This document describes the truthful connection between the Information Flow Framework (IFF) and the notion of an institution. In particular, it connects John Sowa's lattice of theories (LOT) with institutions. Of course, the title of this document is a bit tongue-in-cheek, since the phrase "truthful connection" has a double double entendre. The word "truthful" may mean revealed truth, but it also refers to the truth classification of Barwise and Seligman (and its associated truth concept lattice). The word "connection" may and does refer to some kind of linkage, but it also has the intended more specific meaning of Galois connection or adjunction in general, and in this case the categorical (triple) equivalence between classifications and concept lattices in more particular, and the truth categorical equivalence in most particular. Table 1 lists some of the correspondences between elements of the IFF Ontology (meta) Ontology and institutions. The institution definitions were taken from the paper: Joseph Goguen and Grigore Roşu, Institution Morphisms, written for a festschrift in honor of the retirement of Rod Burstall. Joseph Goguen's Theory of Institutions web page has further links to papers on institutions. The discussion is continued below the table.

Table 1: Correspondences between the IFF and Institutions

Information Flow Framework		Institutions	
Category of languages – Language		**Category of signatures – Sign**	
Language	$\boldsymbol{L}$	**Signature**	Sig
Language Morphism	$\boldsymbol{f} : \boldsymbol{L}_1 \to \boldsymbol{L}_2$	**Signature Morphism**	$\varphi : \Sigma_1 \to \Sigma_2$
Expression Functor	$expr(\boldsymbol{f}) : expr(\boldsymbol{L}_1) \to expr(\boldsymbol{L}_2)$	**Sentence Functor**	$\mathbf{Sen}(\varphi) : \mathbf{Sen}(\Sigma_1) \to \mathbf{Sen}(\Sigma_2)$
$expr$: Language $\to$ Language		$\mathbf{Sen} : \mathbf{Sign} \to \mathbf{Set}$	
Category of theories – Theory		**Category of specifications** – **Spec**	
Theory	$\boldsymbol{T} = \langle base(\boldsymbol{T}), axm(\boldsymbol{T})\rangle$	**Specification**	$S = \langle \Sigma, A\rangle$
	$\boldsymbol{L} = base(\boldsymbol{T})$ $axm(\boldsymbol{T}) \subseteq \wp expr(\boldsymbol{L})$		Σ is a signature A is a set of Σ-sentences
	closure $clo(\boldsymbol{T}) = \langle \boldsymbol{L}, thm(\boldsymbol{T})\rangle$		
Theory Morphism	$\boldsymbol{f} : \boldsymbol{T}_1 \to \boldsymbol{T}_2$	**Specification Morphism**	$\varphi : S_1 \to S_2$
	$\boldsymbol{f} : \boldsymbol{L}_1 \to \boldsymbol{L}_2$ $\wp expr(\boldsymbol{f})(axm(\boldsymbol{T}_1)) \subseteq thm(\boldsymbol{T}_2)$		$\varphi : \Sigma_1 \to \Sigma_2$, maps Σ_1-axioms to Σ_2-theorems
Model Fiber Functor – mod : Language $\to \mathbf{Cat}^{\mathrm{op}}$		**Model Functor – Mod** : $\mathbf{Sign} \to \mathbf{Cat}^{\mathrm{op}}$	
Model	$\boldsymbol{M} = \langle ent(\boldsymbol{M}), rel(\boldsymbol{M})\rangle$	**Model**	m
Truth Framework		**Institution**	$\mathbb{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models\rangle$
truth classification – $truth(\boldsymbol{L}) = \langle mod(\boldsymbol{L}), expr(\boldsymbol{L}), \models_{\boldsymbol{L}}\rangle$ $\boldsymbol{M} \models_{\boldsymbol{L}} e$ means model $\boldsymbol{M}$ satisfies expression e		**satisfaction relation**: $\models_\Sigma \subseteq \lvert\mathbf{Mod}(\Sigma)\rvert \times \mathbf{Sen}(\Sigma)$	
truth infomorphism – $truth(\boldsymbol{f}) = \langle mod(\boldsymbol{f}), expr(\boldsymbol{f})\rangle : truth(\boldsymbol{L}_1) \rightleftarrows truth(\boldsymbol{L}_2)$ $mod(\boldsymbol{f})(\boldsymbol{M}_2) \models_{\boldsymbol{L}1} e_1$ <u>iff</u> $\boldsymbol{M}_2 \models_{\boldsymbol{L}2} expr(\boldsymbol{f})(e_1)$, for all models $\boldsymbol{M}_2 \in mod(\boldsymbol{L}_2)$, expressions $e_1 \in expr(\boldsymbol{L}_1)$		**satisfaction condition**: $\mathbf{Mod}(\varphi)(m_2) \models_{\Sigma 1} e_1$ <u>iff</u> $m_2 \models_{\Sigma 2} \mathbf{Sen}(\varphi)(e_1)$, for all models $m_2 \in \mathbf{Mod}(\Sigma_2)$, sentences $e_1 \in \mathbf{Sen}(\Sigma_1)$	

The truthful connection starts from the fundamental relation of satisfaction between model (-theoretic structure)s and expressions (in the IFF the term "expression" is used in place of the possibly more common term "formula", and for theoretical convenience the IFF uses expressions instead of their more common subset of sentences). For a fixed language $\boldsymbol{L}$, a model $\boldsymbol{M} \in \mathit{mod}(\boldsymbol{L})$ of $\boldsymbol{L}$ satisfies an expression $e \in \mathit{expr}(\boldsymbol{L})$, symbolized by

$$\boldsymbol{M} \models_{\boldsymbol{L}} e,$$

when all the tuples of the model $\boldsymbol{M}$ satisfy the expression e (recall that an IFF model $\boldsymbol{M}$ represents a context, since it specifies in its tuple set $\mathit{tuple}(\boldsymbol{M})$ exactly which tuples it uses – there might even be only a finite set of desired tuples). The triple of models, expressions and satisfaction forms the truth classification

$$\mathit{truth}(\boldsymbol{L}) = \langle \mathit{mod}(\boldsymbol{L}), \mathit{expr}(\boldsymbol{L}), \models_{\boldsymbol{L}} \rangle.$$

The fiber model class $\mathit{mod}(\boldsymbol{L})$ and the expression set $\mathit{expr}(\boldsymbol{L})$ are defined in the language namespace of the IFF Ontology (meta) Ontology (IFF-OO). The satisfaction relation is defined in the model namespace of IFF-OO. We ambiguously use the same symbol $\mathit{expr}(\boldsymbol{L})$ for both the set of expressions of $\boldsymbol{L}$ and the expression language that extends $\boldsymbol{L}$ whose set of relation symbols is the set of expressions of $\boldsymbol{L}$. Any model $\boldsymbol{M} \in \mathit{mod}(\boldsymbol{L})$ inductively extends to a model of the expression language $\mathit{expr}(\boldsymbol{M}) \in \mathit{mod}(\mathit{expr}(\boldsymbol{L}))$.

Next, we move from the somewhat static situation of the truth classification of an IFF language to the truth infomorphism over an IFF language morphism. For a fixed language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, there is an inductively defined expression function $\mathit{expr}(\boldsymbol{f}) : \mathit{expr}(\boldsymbol{L}_1) \rightarrow \mathit{expr}(\boldsymbol{L}_2)$ that maps an expression of the source language $e_1 \in \mathit{expr}(\boldsymbol{L}_1)$ to the expression of the target language $\mathit{expr}(\boldsymbol{f})(e_1) \in \mathit{expr}(\boldsymbol{L}_2)$ gotten by inductively applying the relevant components of $\boldsymbol{f}$ to all of the variables, functions and relation symbols in e_1. There is also a fiber model function $\mathit{mod}(\boldsymbol{f}) : \mathit{mod}(\boldsymbol{L}_2) \rightarrow \mathit{mod}(\boldsymbol{L}_1)$ that maps a model of the target language $\boldsymbol{M}_2 \in \mathit{mod}(\boldsymbol{L}_2)$ to the model of the source language $\mathit{mod}(\boldsymbol{f})(\boldsymbol{M}_2) \in \mathit{mod}(\boldsymbol{L}_1)$ gotten by forming the "type fiber image" of the model $\boldsymbol{M}_2$. It can be proven that these functions form an infomorphism

$$\mathit{truth}(\boldsymbol{f}) = \langle \mathit{mod}(\boldsymbol{f}), \mathit{expr}(\boldsymbol{f}) \rangle : \mathit{truth}(\boldsymbol{L}_1) \rightleftarrows \mathit{truth}(\boldsymbol{L}_2)$$

called the truth infomorphism, since they satisfy the fundamental condition

$$\mathit{mod}(\boldsymbol{f})(\boldsymbol{M}_2) \models_{\boldsymbol{L}1} e_1 \ \underline{\text{iff}}\ \boldsymbol{M}_2 \models_{\boldsymbol{L}2} \mathit{expr}(\boldsymbol{f})(e_1),$$

for all target models $\boldsymbol{M}_2 \in \mathit{mod}(\boldsymbol{L}_2)$ and all source expressions $e_1 \in \mathit{expr}(\boldsymbol{L}_1)$. The notion of a language interpretation $\alpha : \boldsymbol{L}_1 \rightharpoonup \boldsymbol{L}_2$, a useful extension of the notion of a language morphism, can also be used for the truth infomorphism; an interpretation maps relation symbols to expressions – it is a language morphism $\alpha : \boldsymbol{L}_1 \rightarrow \mathit{expr}(\boldsymbol{L}_2)$ from the source language $\boldsymbol{L}_1$ to $\mathit{expr}(\boldsymbol{L}_2)$ the expression language of the target language $\boldsymbol{L}_2$. An interpretation is a morphism in the Kleisli category of the expression monad.

The expression function, the model fiber function and the notion of an interpretation are axiomatically defined in the language namespace of IFF-OO. The truth classification and truth infomorphism constructions are axiomatized in the truth sub-namespace of the language namespace of IFF-OO. The latest versions of documentation for any IFF namespaces and IFF meta-ontologies are conveniently located in the IFF site map. The motivating example of truth classification and truth information, and in particular the fundamental condition of the truth infomorphisms that corresponds to the integrity requirement of a model management schema transformation framework, appeared originally in Jon Barwise and Jerry Seligman, *Information Flow: The Logic of Distributed Systems*, 1997, Cambridge Tracts in Theoretical Computer Science **44**, Cambridge University Press. The notion of truth, both the truth classification and the truth concept lattice, has been repeatedly discussed by the author on the SUO email list – see the following list of links.

2000 Oct 24, Reply: KIF & Naming Problems
2001 Mar 7, Reply: Proposed SUO Content Outline
2001 Mar 30, 3 fundamental kinds of SUO terms
2001 May 7, Reply: Powers That B
2001 May 22, Reply: Foundation Ontology
2001 Aug 16, Reply: Vote 2001-02: IFF Foundation Ontology
2001 Aug 28, The category of concept lattices as the lattice of all theories
2001 Sep 25, The IFF Foundation Ontology – version 1.5
2002 Jan 18, Reply1: Thoughts and judgments
2002 Jan 18, Reply2: Thoughts and judgments
2002 Jan 23, Reply1: Intension & Extension
2002 Feb 6, Reply: *Date 07 Feb 2002
2002 Feb 14, Reply2: Intension & Extension
2002 May 11, IFF interface (control portal)
2002 Jun 10, Reply: The lattice of theories + language-games
2003 Apr 28, Inferencing relevance of the IFF

Fact. *The* IFF Ontology (meta) Ontology (IFF-OO) *defines and axiomatizes an institution*

$$\mathbb{I}_{\mathrm{IFF}} = \langle \mathsf{Language}, \mathit{mod}, \mathit{expr}, \models \rangle$$

where $\mathsf{Language}$ *is the category of languages,* $\mathit{mod} : \mathsf{Language} \rightarrow \mathbf{Cat}^{\mathrm{op}}$ *is the model fiber functor,* $\mathit{expr} : \mathsf{Language} \rightarrow \mathsf{Set}$ *is the composition* $\mathit{expr} = \mathit{expr} \circ \mathit{rel}$ *of the expression endofunctor* $\mathit{expr} : \mathsf{Language} \rightarrow \mathsf{Language}$ *and the relation functor* $\mathit{rel} : \mathsf{Language} \rightarrow \mathsf{Set}$*, and* $\models$ *is the satisfaction relation between models and expressions.*

Now we apply the categorical equivalence between classifications and concept lattices. This equivalence provides a category theory for the basic theorem of formal concept analysis. Associated with any classification is an equivalent complete lattice of formal concepts, where a formal concept is a maximal pair of subcollections of instances and types that participate in the classification. Now we can always replace the canonical definition of a concept lattice with just its intent-ional part, and we do that in the case of truth for reasons of tractability. The result goes as follows. For any language $\boldsymbol{L}$, a theory $\boldsymbol{T} \in \mathit{th}(\boldsymbol{L})$ can be identified with a subcollection of expressions $\mathit{axm}(\boldsymbol{T}) \subseteq \mathit{expr}(\boldsymbol{L})$ called the axioms of that theory. An expression $e \in \mathit{expr}(\boldsymbol{L})$ is a member of the closure of a theory $\boldsymbol{T}$, $e \in \mathit{axm}(\mathit{clo}(\boldsymbol{T}))$, when any model that satisfies the axioms of the theory also satisfies e. Any theory is contained in its closure $\boldsymbol{T} \subseteq \mathit{clo}(\boldsymbol{T})$. A theory is close when it equals its closure. The truth concept lattice $\mathit{fiber}^{\blacksquare}(\boldsymbol{L})$ has the closed theories of $\boldsymbol{L}$ as its formal concepts with the reverse subset order. However, the truth concept lattice is not the usual notion of a "lattice of theories". Usually one does not restrict to just the closed theories, but instead uses all theories. It turns out that there are actually three adjointly connected versions of the notion "lattice of theories": $\mathit{fiber}(\boldsymbol{L})$, $\mathit{fiber}^{\blacksquare}(\boldsymbol{L})$, and $\mathit{fiber}^{\blacktriangleleft}(\boldsymbol{L})$. These depend on three different, but adjointly connected, notions of the category of theories: Theory, $\mathsf{Theory}^{\blacksquare}$ or $\mathsf{Theory}^{\blacktriangleleft}$. The simple category Theory of theories has theories as objects and morphisms that require the strong condition that a source axiom is mapped to a target axiom. The category $\mathsf{Theory}^{\blacksquare}$ of closed theories has closed theories as objects and morphisms that require a source axiom (= theorem) is mapped to a target axiom (= theorem). The entailment category $\mathsf{Theory}^{\blacktriangleleft}$ of theories has theories as objects and morphisms that require the weakened condition that a source axiom is mapped to a target theorem. In all three notions, the lattice of theories is identified with the fiber over a language $\boldsymbol{L}$, and a morphism in the fiber is identified with downward lattice order. The document Formal or Axiomatic Semantics in the IFF gives a very rigorous axiomatics for the semantics architecture concentrated in IFF theories. These three kinds of lattices of theories are three adjointly connected bifibrations along the base functor $\mathit{base} : \mathsf{Theory} \rightarrow \mathsf{Language}$. In summary, the lattice of theories for a particular language $\boldsymbol{L}$ is the complete lattice $\mathit{fiber}(\boldsymbol{L})$ of all theories whose base language is $\boldsymbol{L}$. However, there are actually three adjointly-connected categories of theories (open theories, closed theories and entailment-oriented morphisms), and hence three adjointly-connected lattices of theories bifibrations. The truth concept lattice is the lattice of theories bifibration for the category of closed theories.

The important connection missing in institution column of Table 1 is the recognition that the integrity requirement for institutions is the fundamental condition of an infomorphism (the truth infomorphism), and recognition of the categorical equivalence between the truth classifications and infomorphisms and the truth concept lattice and concept morphisms. As mentioned before, the categorical equivalence between classifications and concept lattices is based upon the basic theorem of formal concept analysis. The proof of this categorical equivalence (actually three connected categorical equivalences) appeared in Robert E. Kent, Distributed Conceptual Structures, (2002), in: *Proceedings of the 6th International Workshop on Relational Methods in Computer Science (RelMiCS 6)*, *Lecture Notes in Computer Science* **2561**, Springer, Berlin. This categorical equivalence is axiomatized in the IFF Upper Classification (meta) Ontology. The special case of the equivalence between the truth classification and the truth concept lattice is axiomatized in the current version of the language and theory namespaces of the IFF Ontology (meta) Ontology. The original version of this document was oriented toward the theory behind the Model Management Project of Phil Bernstein. However, it was recognized early on that the truthful connection is actually between the IFF and the theory of institutions.