The IFF Glossary for the Lattice of Theories

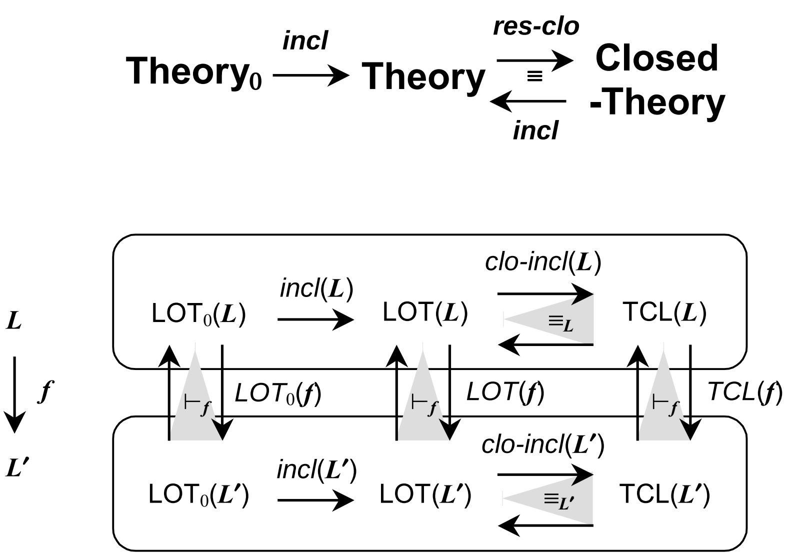

Figure 0: The Theory Picture

Basic Concepts

Languages

- An IFF *language* $\boldsymbol{L}$ (Figure 1) consists of a set of variables $var(\boldsymbol{L})$, a set of entity types $ent(\boldsymbol{L})$ and a set of relation types $rel(\boldsymbol{L})$ with a reference (sort) function $*_{\boldsymbol{L}} = refer(\boldsymbol{L}) : var(\boldsymbol{L}) \rightarrow ent(\boldsymbol{L})$ that sorts the variables, and an arity function $\#_{\boldsymbol{L}} = arity(\boldsymbol{L}) : rel(\boldsymbol{L}) \rightarrow \wp var(\boldsymbol{L})$ that associates a variables (name) with each argument of a relation type, and a signature function (equivalent to arity) $\partial_{\boldsymbol{L}} = sign(\boldsymbol{L}) : rel(\boldsymbol{L}) \rightarrow sign(refer(\boldsymbol{L}))$ that type the arguments for relations by associating a named entity type with each argument. Every IFF language inductively generates a set of expressions $expr(\boldsymbol{L})$ in the usual way. There are also recursively defined arity and signature functions $arity(\boldsymbol{L}) : expr(\boldsymbol{L}) \rightarrow \wp var(\boldsymbol{L})$ and $sign(\boldsymbol{L}) : expr(\boldsymbol{L}) \rightarrow sign(refer(\boldsymbol{L}))$ defined on expressions, which extend the relation arity and signature functions.

$$\begin{tikzcd} rel(\boldsymbol{L}) \arrow[rr, "\#_{\boldsymbol{L}}"] \arrow[dr, "\partial_{\boldsymbol{L}}"'] & & \wp var(\boldsymbol{L}) \arrow[dl, leftrightarrow, "\cong"] & var(\boldsymbol{L}) \arrow[d, "*_{\boldsymbol{L}}"'] \\ & sign(*_{\boldsymbol{L}}) & & ent(\boldsymbol{L}) \end{tikzcd}$$

Figure 1: Type Language

$$\begin{tikzcd} var(\boldsymbol{L}_1) \arrow[r, "var(\boldsymbol{f})"] \arrow[d, "refer(\boldsymbol{L}_1)"'] \arrow[dr, phantom, "refer(\boldsymbol{f})"] & var(\boldsymbol{L}_2) \arrow[d, "refer(\boldsymbol{L}_2)"] \\ ent(\boldsymbol{L}_1) \arrow[r, "ent(\boldsymbol{f})"'] & ent(\boldsymbol{L}_2) \end{tikzcd} \qquad \begin{tikzcd} rel(\boldsymbol{L}_1) \arrow[r, "rel(\boldsymbol{f})"] \arrow[d, "sign(\boldsymbol{L}_1)"'] \arrow[dr, phantom, "sign(\boldsymbol{f})"] & rel(\boldsymbol{L}_2) \arrow[d, "sign(\boldsymbol{L}_2)"] \\ sign(refer(\boldsymbol{L}_1)) \arrow[r, "sign(refer(\boldsymbol{f}))"'] & sign(refer(\boldsymbol{L}_2)) \end{tikzcd}$$

Figure 2: Type Language Morphism

- An IFF *language morphism* $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$ (Figure 2) consists of functions $var(\boldsymbol{f}) : var(\boldsymbol{L}_1) \rightarrow var(\boldsymbol{L}_2)$, $ent(\boldsymbol{f}) : ent(\boldsymbol{L}_1) \rightarrow ent(\boldsymbol{L}_2)$ and $rel(\boldsymbol{f}) : rel(\boldsymbol{L}_1) \rightarrow rel(\boldsymbol{L}_2)$ that map the variables, entity types and relation types of $\boldsymbol{L}_1$ to those of $\boldsymbol{L}_2$, preserving reference, arity and signature: $refer(\boldsymbol{L}_1) \cdot ent(\boldsymbol{f}) = var(\boldsymbol{f}) \cdot refer(\boldsymbol{L}_2)$, $arity(\boldsymbol{L}_1) \cdot \wp var(\boldsymbol{f}) = rel(\boldsymbol{f}) \cdot arity(\boldsymbol{L}_2)$, and $sign(\boldsymbol{L}_1) \cdot sign(\boldsymbol{f}) = rel(\boldsymbol{f}) \cdot sign(\boldsymbol{L}_2)$. Every IFF language morphism recursively generates a function on expressions $expr(\boldsymbol{f}) : expr(\boldsymbol{L}_1) \rightarrow expr(\boldsymbol{L}_2)$ in the usual way, and this preserves arity and signature of expressions: $arity(\boldsymbol{L}_1) \cdot \wp var(\boldsymbol{f}) = expr(\boldsymbol{f}) \cdot arity(\boldsymbol{L}_2)$ and $sign(\boldsymbol{L}_1) \cdot sign(\boldsymbol{f}) = expr(\boldsymbol{f}) \cdot sign(\boldsymbol{L}_2)$. The language component of a theory is called the *base* language of the theory. This terminology is actually appropriate, since we picture the set of theories over a language as a fiber on a stalk over the base. The notion of base extends to morphisms, and hence forms a functor from the category of theories to the category of languages. This functor is a bifibration: over any language morphism there is an (inverse image, direct image) adjoint pair of monotonic functions between the lattices of theories over the source and target languages. We can use these monotonic functions to move theories back and forth. In particular, the direct image can be used to compute the colimit of a diagram of theories, and the adjoint pair defines the notion of mapping closure that is of some significance for mapping theories.

Theories

- An IFF *theory* $\boldsymbol{T}$ consists of an IFF language $\boldsymbol{L} = base(\boldsymbol{T})$ and a subset of expressions $axm(\boldsymbol{T}) \subseteq expr(\boldsymbol{L})$ of that language. An $\boldsymbol{L}$-model $\boldsymbol{M}$ *satisfies* (is a model of) an $\boldsymbol{L}$-theory $\boldsymbol{T}$, symbolized $\boldsymbol{M} \models \boldsymbol{T}$, when $\boldsymbol{M}$ satisfies every axiom $\varphi \in axm(\boldsymbol{T})$. The set of $\boldsymbol{L}$-expressions satisfied by all models of $\boldsymbol{T}$ is denoted $thm(\boldsymbol{T})$ and called the set of *theorems* of $\boldsymbol{T}$. An $\boldsymbol{L}$-theory $\boldsymbol{T}$ logically *entails* an $\boldsymbol{L}$-expression φ, symbolized $\boldsymbol{T} \vdash \varphi$, when phi is a theorem: $\varphi \in thm(\boldsymbol{T})$. Obviously, a theory entails all of its axioms and any axiom is a theorem: $axm(\boldsymbol{T}) \subseteq thm(\boldsymbol{T})$. The *closure* of a theory $\boldsymbol{T}$ is the theory *clo*($\boldsymbol{T}$) having the same underlying language, but whose axioms are the theorems of $\boldsymbol{T}$: *axm*(*clo*($\boldsymbol{T}$)) = *thm*($\boldsymbol{T}$). A theory is *closed* when it is its own closure. The $\boldsymbol{L}$-theory *max-th*($\boldsymbol{L}$)($\boldsymbol{M}$) associated with an $\boldsymbol{L}$-model $\boldsymbol{M}$ consists of all $\boldsymbol{L}$-expressions satisfied by the model $\boldsymbol{M}$. It is the set-theoretically largest theory that the model satisfies. A *model theory* is any theory of the form *max-th*($\boldsymbol{L}$)($\boldsymbol{M}$) for some $\boldsymbol{L}$-model $\boldsymbol{M}$. An $\boldsymbol{L}$-theory $\boldsymbol{T}$ is *consistent* when there is an $\boldsymbol{L}$-model $\boldsymbol{M}$ that satisfies $\boldsymbol{T}$; that is, when there is a model theory below $\boldsymbol{T}$, *max-th*($\boldsymbol{L}$)($\boldsymbol{M}$) $\supseteq \boldsymbol{T}$.

- An IFF *theory morphism* $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ (Figure 3) consists of a language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, for $\boldsymbol{f}$ = *base*($\boldsymbol{g}$), $\boldsymbol{L}_1$ = *base*($\boldsymbol{T}_1$) and $\boldsymbol{L}_2$ = *base*($\boldsymbol{T}_2$), that maps source axioms to target theorems; thus, it satisfies any of the equivalent conditions:

  $\mathit{axm}(\boldsymbol{T}_2) \vdash_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{axm}(\boldsymbol{T}_1))$
  <u>iff</u> $\mathit{thm}(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{axm}(\boldsymbol{T}_1))$
  <u>iff</u> $\mathit{inv}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_2)) \supseteq_{\boldsymbol{L}1} \mathit{axm}(\boldsymbol{T}_1))$
  <u>iff</u> $\mathit{inv}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_2)) \supseteq_{\boldsymbol{L}1} \mathit{thm}(\boldsymbol{T}_1))$
  <u>iff</u> $\mathit{thm}(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_1))$
  <u>iff</u> $\mathit{axm}(\boldsymbol{T}_2) \vdash_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_1))$.

$$\boldsymbol{T}_1 \xrightarrow{\boldsymbol{f}} \boldsymbol{T}_2$$
$$\mathit{base}(\boldsymbol{T}_1) \xrightarrow{\mathit{base}(\boldsymbol{f})} \mathit{base}(\boldsymbol{T}_2)$$

Figure 3: Theory Morphism

  Let $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$ be any morphism of type languages. Let $\boldsymbol{T}_2 \in$ *fiber*($\boldsymbol{L}_2$) be any target theory. Then *inv*($\boldsymbol{f}$)(*thm*($\boldsymbol{T}_2$)) is the entailment-smallest source theory for all theory morphisms $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ over $\boldsymbol{f}$; that is, *inv*($\boldsymbol{f}$)(*thm*($\boldsymbol{T}_2$)) is the entailment-meet of the source theories for all theory morphisms $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ over $\boldsymbol{f}$. Dually, let $\boldsymbol{T}_1 \in$ *fiber*($\boldsymbol{L}_1$) be any source theory. Then *dir*($\boldsymbol{f}$)(*axm*($\boldsymbol{T}_1$)) is the entailment-largest target theory for all theory morphisms $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ over $\boldsymbol{f}$; that is, *dir*($\boldsymbol{f}$)(*axm*($\boldsymbol{T}_1$)) is the entailment-join of the target theories for all theory morphisms $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ over $\boldsymbol{f}$.

Theory Fibers

- Any language $\boldsymbol{L}$ determines a "lattice of theories" $\mathit{fiber}^{\vdash}(\boldsymbol{L})$, where each member theory has base (underlying language) $\boldsymbol{L}$. This is the *fiber* for the underlying base functor

  base : Theory → Language.

  The category of theories is partitioned by the category of languages, where each block of the partition corresponding to a language $\boldsymbol{L}$ is to the lattice of theories:

  $\mathrm{Theory} = \sum_{\boldsymbol{L} \in \mathrm{Language}} \mathit{fiber}^{\vdash}(\boldsymbol{L})$.

  Hence, the category of theories is the partition sum of the lattices of theories, and each lattice of theories lies strictly within the category of theories. Dropping the common base language, we finally define $\mathit{fiber}^{\vdash}(\boldsymbol{L}) = \langle \wp \mathit{expr}(\boldsymbol{L}), \vdash \rangle$ to be the power set of $\boldsymbol{L}$-expressions ordered by entailment: a theory $\boldsymbol{T}_2$ entails a theory $\boldsymbol{T}_1$, symbolized by $\boldsymbol{T}_2 \vdash \boldsymbol{T}_1$, when $\mathit{clo}(\boldsymbol{T}_2) \supseteq \boldsymbol{T}_1$. This is a complete preorder with meets being unions followed by closure and joins being intersections. There is also a complete poset *fiber*($\boldsymbol{L}$) = $\langle \wp \mathit{expr}(\boldsymbol{L}), \supseteq \rangle$ using reverse subset inclusion, that is a suborder of the lattice of theories.

- Let $\boldsymbol{f} : \boldsymbol{L}_0 \rightarrow \boldsymbol{L}_1$ be any morphism of languages. We apply the exists-substitution-forall adjunction[1] for functions to the expression function: $\boldsymbol{f}$ determines a *fiber* adjoint pair of monotonic functions

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[1] For any function $\boldsymbol{h} : \boldsymbol{A} \rightarrow \boldsymbol{B}$, there is an adjoint pair of monotonic functions

$$\langle \exists \boldsymbol{h}, \boldsymbol{h}^{-1} \rangle : \langle \wp \boldsymbol{A}, \subseteq_{\boldsymbol{A}} \rangle \rightarrow \langle \wp \boldsymbol{B}, \subseteq_{\boldsymbol{B}} \rangle$$

between the power orders of the source and target sets, symbolized by $\exists \boldsymbol{h} \dashv \boldsymbol{h}^{-1}$. The left adjoint, the *exist*(ential) operator (monotonic function)

$$\exists \boldsymbol{h} : \langle \wp \boldsymbol{A}, \subseteq_{\boldsymbol{A}} \rangle \rightarrow \langle \wp \boldsymbol{B}, \subseteq_{\boldsymbol{B}} \rangle,$$

maps a subset $X \in \boldsymbol{A}$ to the $\boldsymbol{B}$-subset

$$\exists \boldsymbol{h}(X) = \{b \in \boldsymbol{B} \mid \exists a \in \boldsymbol{A}, (\boldsymbol{h}(a) = b) \;\&\; (a \in X)\}$$
$$= \{b \in \boldsymbol{B} \mid \exists a \in X, \boldsymbol{h}(a) = b\},$$

and the right adjoint, the *subst*(itution) operator (monotonic function)

$$\boldsymbol{h}^{-1} : \langle \wp \boldsymbol{B}, \subseteq_{\boldsymbol{B}} \rangle \rightarrow \langle \wp \boldsymbol{A}, \subseteq_{\boldsymbol{A}} \rangle$$

maps a subset $Y \in \boldsymbol{B}$ to the $\boldsymbol{A}$-subset

$fiber(\boldsymbol{f}) = \langle inv(\boldsymbol{f}), dir(\boldsymbol{f})\rangle : fiber(\boldsymbol{L}_1) \rightarrow fiber(\boldsymbol{L}_0)$,

symbolized as $inv(\boldsymbol{f}) \dashv dir(\boldsymbol{f})$, that map between the lattices of theories at the target and source languages. The (existential) *direct image* operator is defined to be the opposite of the exist(ential) operator applied to the expression function: $dir(\boldsymbol{f}) = \exists expr(\boldsymbol{f})^{\mathrm{op}} : fiber(\boldsymbol{L}_0) \rightarrow fiber(\boldsymbol{L}_1)$ and the *inverse image* operator is defined to be the opposite of the subst(itution) operator applied to the expression function: $inv(\boldsymbol{f}) = expr(\boldsymbol{f})^{-1\mathrm{op}} : fiber(\boldsymbol{L}_1) \rightarrow fiber(\boldsymbol{L}_0)$. Being left adjoint, the inverse image operator preserves joins (intersections). Being right adjoint, the (existential) direct image operator preserves meets (unions). The fibered movement of theories is compositional (functorial). Given two composable language morphisms $\boldsymbol{f}_1 : \boldsymbol{L}_0 \rightarrow \boldsymbol{L}_1$ and $\boldsymbol{f}_2 : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the fiber composition
$fiber(\boldsymbol{f}_2) \circ fiber(\boldsymbol{f}_1) : fiber(\boldsymbol{L}_2) \rightarrow fiber(\boldsymbol{L}_0)$ is defined by componentwise composition
$fiber(\boldsymbol{f}_2) \circ fiber(\boldsymbol{f}_1) = fiber(\boldsymbol{f}_1 \cdot \boldsymbol{f}_2) = \langle inv(\boldsymbol{f}_2) \cdot inv(\boldsymbol{f}_1), dir(\boldsymbol{f}_1) \cdot dir(\boldsymbol{f}_2)\rangle$.

Diagrams of Theories

- In the IFF, a *diagram of theories* (and theory morphisms) of shape $\boldsymbol{G}$ for some graph $\boldsymbol{G}$ is a graph morphism

  $\boldsymbol{T} : \boldsymbol{G} \rightarrow |\mathsf{Theory}|$

  from $\boldsymbol{G}$ to the (set-theoretically large) graph $|\mathsf{Theory}|$ whose nodes are theories and whose edges are theory morphisms. Thus, a diagram of theories consists of a function $obj(\boldsymbol{T}) : node(\boldsymbol{G}) \rightarrow obj(\mathsf{Theory})$ from nodes to theories, and a function $mor(\boldsymbol{T}) : edge(\boldsymbol{G}) \rightarrow mor(\mathsf{Theory})$ from edges to theory morphisms that preserves source and target. In more detail, a diagram of theories consists of a collection of theories $obj(\boldsymbol{T}) = \{\boldsymbol{T}_n\}$ indexed by the nodes n of $\boldsymbol{G}$ and a collection of theory morphisms $mor(\boldsymbol{T}) = \{\boldsymbol{T}_e : \boldsymbol{T}_m \rightarrow \boldsymbol{T}_n\}$ indexed by edges $e : m \rightarrow n$ of $\boldsymbol{G}$. The diagram $\boldsymbol{T}$ is *discrete* when the shape graph is discrete (that is, when $\boldsymbol{G}$ has only nodes but no edges). Hence, a discrete diagram is just a "tuple" of theories $obj(\boldsymbol{T}) = \{\boldsymbol{T}_n\}$ with no connecting theory morphisms. Language diagrams can be defined in a similar fashion, and every theory diagram $\boldsymbol{T} : \boldsymbol{G} \rightarrow |\mathsf{Theory}|$ has a base language diagram

  $\boldsymbol{L} = base(\boldsymbol{T}) : \boldsymbol{G} \rightarrow |\mathsf{Language}|$,

  where $obj(\boldsymbol{L}) = \{base(\boldsymbol{T}_n)\}$ and $mor(\boldsymbol{L}) = \{base(\boldsymbol{T}_e) : base(\boldsymbol{T}_m) \rightarrow base(\boldsymbol{T}_n)\}$. A diagram of theories $\boldsymbol{T} : \boldsymbol{G} \rightarrow |\mathsf{Theory}|$ is *homogeneous* when all of the indexed theories have the same underlying base language: $base(\boldsymbol{T}_n) = \boldsymbol{L}$.

- A *theory diagram morphism*

  $\alpha : \boldsymbol{T}_1 \Rightarrow \boldsymbol{T}_2 : \boldsymbol{G} \rightarrow |\mathsf{Theory}|$

  has source theory diagram $\boldsymbol{T}_1$, target theory diagram $\boldsymbol{T}_2$, shape graph $\boldsymbol{G}$, and a collection $\{\alpha(n) : \boldsymbol{T}_1(n) \rightarrow \boldsymbol{T}_2(n) \mid n \in node(\boldsymbol{G})\}$ of component theory morphisms, where every edge $e : j \rightarrow k$ in

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$\boldsymbol{h}^{-1}(Y) = \{a \in \boldsymbol{A} \mid \boldsymbol{h}(a) \in Y\}$.

There is also the adjoint pair of monotonic functions

$\langle \boldsymbol{h}^{-1}, \forall\boldsymbol{h}\rangle : \langle \wp\boldsymbol{B}, \subseteq_{\boldsymbol{B}}\rangle \rightarrow \langle \wp\boldsymbol{A}, \subseteq_{\boldsymbol{A}}\rangle$,

where the right adjoint, symbolized by $\boldsymbol{h}^{-1} \dashv \forall\boldsymbol{h}$, is the *forall* (universal) operator (monotonic function) that is logically dual to the exist(ential) operator

$\forall\boldsymbol{h} : \langle \wp\boldsymbol{A}, \subseteq_{\boldsymbol{A}}\rangle \rightarrow \langle \wp\boldsymbol{B}, \subseteq_{\boldsymbol{B}}\rangle$,

maps a subset $X \in \boldsymbol{A}$ to the $\boldsymbol{B}$-subset

$\forall\boldsymbol{h}(X) = \{b \in \boldsymbol{B} \mid \forall a \in \boldsymbol{A}, (\boldsymbol{h}(a) = b) \rightarrow (a \in X)\}$.

These adjunctions are symbolized as $\exists\boldsymbol{h} \dashv \boldsymbol{h}^{-1} \dashv \forall\boldsymbol{h}$. The *closure* operator (monotonic function)

$\overline{\boldsymbol{h}} : \langle \wp\boldsymbol{A}, \subseteq_{\boldsymbol{A}}\rangle \rightarrow \langle \wp\boldsymbol{A}, \subseteq_{\boldsymbol{A}}\rangle$

on $\langle \wp\boldsymbol{A}, \subseteq_{\boldsymbol{A}}\rangle$ maps a subset $X \in \boldsymbol{A}$ to the more inclusive $\boldsymbol{A}$-subset

$\overline{\boldsymbol{h}}(X) = \{a \in \boldsymbol{A} \mid \boldsymbol{h}(a) \in \exists\boldsymbol{h}(X)\}$
$= \{a \in \boldsymbol{A} \mid \exists a' \in X, \boldsymbol{h}(a') = \boldsymbol{h}(a)\}$
$= \cup \{[a] \mid a \in \boldsymbol{A}\}$,

where $[a]$ is the equivalence class for the element a, and the equivalence relation is the kernel equivalence $\sim_{\boldsymbol{h}}$ induced by the function $\boldsymbol{h}$: $a \sim_{\boldsymbol{h}} a'$ when $\boldsymbol{h}(a) = \boldsymbol{h}(a')$. We can also define a dual *interior* operator.

$\boldsymbol{G}$ yields a commutative diagram $\alpha(j) \cdot \boldsymbol{T}_2(e) = \boldsymbol{T}_1(e) \cdot \alpha(k)$. When this holds we say that $\alpha(n) : \boldsymbol{T}_1(n) \Rightarrow \boldsymbol{T}_2(n)$ is *natural* in n. The diagram $\boldsymbol{T}$ gives us a picture of $\boldsymbol{G}$ in Theory. For the same reason, a theory diagram morphism α is a set of theory morphisms that translate the picture $\boldsymbol{T}_1$ to the picture $\boldsymbol{T}_2$.

A theory diagram is an object of $\mathsf{Theory}^{\boldsymbol{G}}$, the category of $\boldsymbol{G}$-indexed theory diagrams and theory diagram morphisms. The *delta* functor $\Delta_{\boldsymbol{G}} : \mathsf{Theory} \rightarrow \mathsf{Theory}^{\boldsymbol{G}}$ indexes the constant $\boldsymbol{G}$-shaped diagrams. It sends each theory $\boldsymbol{T}$ to the constant diagram $\Delta_{\boldsymbol{G}}(\boldsymbol{T}) : \boldsymbol{G} \rightarrow |\mathsf{Theory}|$, which has the value $\boldsymbol{T}$ at every node $n \in \boldsymbol{G}$ and the value $id_{\boldsymbol{T}}$ at every edge e of $\boldsymbol{G}$. The delta functor sends each theory morphism $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ to the constant diagram morphism $\Delta_{\boldsymbol{G}}(\boldsymbol{g}) : \Delta_{\boldsymbol{G}}(\boldsymbol{T}_1) \Rightarrow \Delta_{\boldsymbol{G}}(\boldsymbol{T}_2) : \boldsymbol{G} \rightarrow |\mathsf{Theory}|$, which has the same value $\boldsymbol{g}$ at each node $n \in \boldsymbol{G}$.

A *cocone* $\tau : \boldsymbol{T} \Rightarrow \Delta_{\boldsymbol{G}}(\check{\boldsymbol{T}})$ is a theory diagram morphism from a *base* diagram $\boldsymbol{T}$ to (the constant diagram $\Delta_{\boldsymbol{G}}(a)$ for) an *opvertex* theory $\check{\boldsymbol{T}}$. As a diagram morphism τ has a function that assigns to each node $n \in \boldsymbol{G}$ a *component* theory morphism $\tau(n) : \boldsymbol{T}(n) \rightarrow \check{\boldsymbol{T}}$ that is natural in n; that is, every edge $e : j \rightarrow k$ in $\boldsymbol{G}$ yields a commutative diagram $\tau(j) = \boldsymbol{T}(e) \cdot \tau(k)$; in words, "the components commute with the base diagram".

Language diagram morphisms can be defined in a similar fashion, and every theory diagram morphism α has a *base* language morphism

$$\beta = base(\alpha) : base(\boldsymbol{T}_1) \Rightarrow base(\boldsymbol{T}_2) : \boldsymbol{G} \rightarrow |\mathsf{Language}|,$$

consisting of the language morphisms $\{base(\alpha(n)) : base(\boldsymbol{T}_1(n)) \rightarrow base(\boldsymbol{T}_2(n)) \mid n \in node(\boldsymbol{G})\}$.

Diagram Fibers

○ Any language diagram $\boldsymbol{L} : \boldsymbol{G} \rightarrow |\mathsf{Language}|$ determines a "lattice of theory diagrams" $fiber^{\vdash}(\boldsymbol{L})$, where each member theory diagram has base (underlying language diagram) $\boldsymbol{L}$. This is also known as the *fiber* for the underlying base functor

$$base^{\boldsymbol{G}} : \mathsf{Theory}^{\boldsymbol{G}} \rightarrow \mathsf{Language}^{\boldsymbol{G}}.$$

The category of $\boldsymbol{G}$-shaped theory diagrams is partitioned by the category of $\boldsymbol{G}$-shaped language diagrams, where each block of the partition corresponding to a language diagram $\boldsymbol{L}$ is to the lattice of theory diagrams:

$$\mathsf{Theory}^{\boldsymbol{G}} = \textstyle\sum_{\boldsymbol{L} \in \mathsf{Language}\boldsymbol{G}} fiber^{\vdash}(\boldsymbol{L}).$$

The "lattice of theory diagrams" is the complete preorder $\langle fiber^{\vdash}(\boldsymbol{L}), \vdash\rangle$, where a theory diagram $\boldsymbol{T}_2$ with $obj(\boldsymbol{T}_2) = \{\boldsymbol{T}_{2n}\}$ and $mor(\boldsymbol{T}_2) = \{\boldsymbol{T}_{2e} : \boldsymbol{T}_{2m} \rightarrow \boldsymbol{T}_{2n}\}$ entails a theory diagram $\boldsymbol{T}_1$ with $obj(\boldsymbol{T}_1) = \{\boldsymbol{T}_{1n}\}$ and $mor(\boldsymbol{T}_1) = \{\boldsymbol{T}_{1e} : \boldsymbol{T}_{1m} \rightarrow \boldsymbol{T}_{1n}\}$, symbolized by $\boldsymbol{T}_2 \vdash \boldsymbol{T}_1$, when $clo(\boldsymbol{T}_{2n}) \supseteq \boldsymbol{T}_{1n}$ for all nodes n of $\boldsymbol{G}$.

○ Theories can be moved along any language diagram morphism. Let $\alpha : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2 : \boldsymbol{G} \rightarrow |\mathsf{Language}|$ be any morphism of language diagrams, consisting of the collection $\{\alpha(n) : \boldsymbol{L}_1(n) \rightarrow \boldsymbol{L}_2(n) \mid n \in node(\boldsymbol{G})\}$ of component language morphisms, where every edge $e : j \rightarrow k$ in $\boldsymbol{G}$ yields a commutative diagram $\alpha(j) \cdot \boldsymbol{L}_2(e) = \boldsymbol{L}_1(e) \cdot \alpha(k)$. We apply fiber movement of theories componentwise: α determines a *fiber* adjoint pair of monotonic functions,

$$fiber(\alpha) = \langle inv(\alpha), dir(\alpha)\rangle : fiber(\boldsymbol{L}_1) \rightarrow fiber(\boldsymbol{L}_0)$$

symbolized as $inv(\alpha) \dashv dir(\alpha)$, that map between the lattices of theories at the target and source language diagrams. The (existential) *direct image* operator $dir(\alpha) : fiber(\boldsymbol{L}_1) \rightarrow fiber(\boldsymbol{L}_2)$ is defined componentwise to be the opposite of the exist(ential) operator applied to the n^{th} expression function: $dir(\alpha)(\boldsymbol{T}_1)(n) = dir(\alpha(n))(\boldsymbol{T}_1(n)) = \exists expr(\alpha(n))^{\text{op}}(\boldsymbol{T}_1(n))$, and the *inverse image* operator $inv(\alpha) : fiber(\boldsymbol{L}_2) \rightarrow fiber(\boldsymbol{L}_1)$ is defined componentwise to be the opposite of the subst(itution) operator applied to the n^{th} expression function: $inv(\alpha)(\boldsymbol{T}_2)(n) = inv(\alpha(n))(\boldsymbol{T}_2(n)) = expr(\alpha(n))^{-1\text{op}}(\boldsymbol{T}_2(n))$. As a special case, in the theory colimit construction, theories are first moved along the colimit injection cocone of the base language diagram resulting in a homogeneous diagram of theories over the language colimit (and then are combined by forming the meet.

Objects

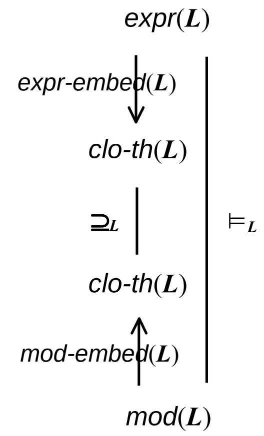

Figure 4: Truth Concept Lattice

Figure 4 illustrates the truth concept lattice over a language $\boldsymbol{L}$. In the IFF, the truth concept lattice over a language $\boldsymbol{L}$ is the concept lattice generated by and associated with the truth classification over $\boldsymbol{L}$. It is represented by the following components.

- The complete lattice of closed $\boldsymbol{L}$-theories *clo-th*($\boldsymbol{L}$) ordered by reverse subset inclusion $\supseteq_{\boldsymbol{L}}$. In FCA terminology, a closed $\boldsymbol{L}$-theory is a truth formal concept.
- The class of $\boldsymbol{L}$-models *mod*($\boldsymbol{L}$) and model embedding function

  mod-embed($\boldsymbol{L}$) : *mod*($\boldsymbol{L}$) → *clo-th*($\boldsymbol{L}$)

  that maps a $\boldsymbol{L}$-model to the closed $\boldsymbol{L}$-theory of all $\boldsymbol{L}$-expressions that it satisfies. In FCA terminology, an $\boldsymbol{L}$-model is a truth formal object.
- The set of $\boldsymbol{L}$-expressions *expr*($\boldsymbol{L}$) and expression embedding function

  expr-embed($\boldsymbol{L}_1$) : *expr*($\boldsymbol{L}$) → *clo-th*($\boldsymbol{L}$)

  that maps an $\boldsymbol{L}$-expression to the closed $\boldsymbol{L}$-theory of all the other $\boldsymbol{L}$-expressions that it entails. In FCA terminology, an $\boldsymbol{L}$-expression is a truth formal attribute.

The binary relation associated with the model embedding function *mod-embed*($\boldsymbol{L}$) is the theory satisfaction relation *th-sat*($\boldsymbol{L}$) = $\iota_{\boldsymbol{L}}$:

th-sat($\boldsymbol{L}$)($\boldsymbol{M}$, $\boldsymbol{T}$) iff *mod-embed*($\boldsymbol{L}$)($\boldsymbol{M}$) $\leq_{\boldsymbol{L}}$ $\boldsymbol{T}$

for any model $\boldsymbol{M} \in$ *mod*($\boldsymbol{L}$) and any closed theory $\boldsymbol{T} \in$ *clo-th*($\boldsymbol{L}$).

The binary relation associated with the expression embedding function *expr-embed*($\boldsymbol{L}$) is the expression membership relation *expr-mbr*($\boldsymbol{L}$) = $\tau_{\boldsymbol{L}}$:

expr-mbr($\boldsymbol{L}$)($\boldsymbol{T}$, φ) iff $\boldsymbol{T} \leq_{\boldsymbol{L}}$ *expr-embed*($\boldsymbol{L}$)(φ)

for any closed theory $\boldsymbol{T} \in$ *clo-th*($\boldsymbol{L}$) and any expression $\varphi \in$ *expr*($\boldsymbol{L}$).

Part of the fundamental theorem of FCA states the following: satisfaction *sat*($\boldsymbol{L}$) = $\vDash_{\boldsymbol{L}}$ (the truth classification) is the relational composition of model embedding, reverse closed theory inclusion and transpose expression embedding:

$\vDash_{\boldsymbol{L}} = \iota_{\boldsymbol{L}} \circ \leq_{\boldsymbol{L}} \circ \tau_{\boldsymbol{L}}$.

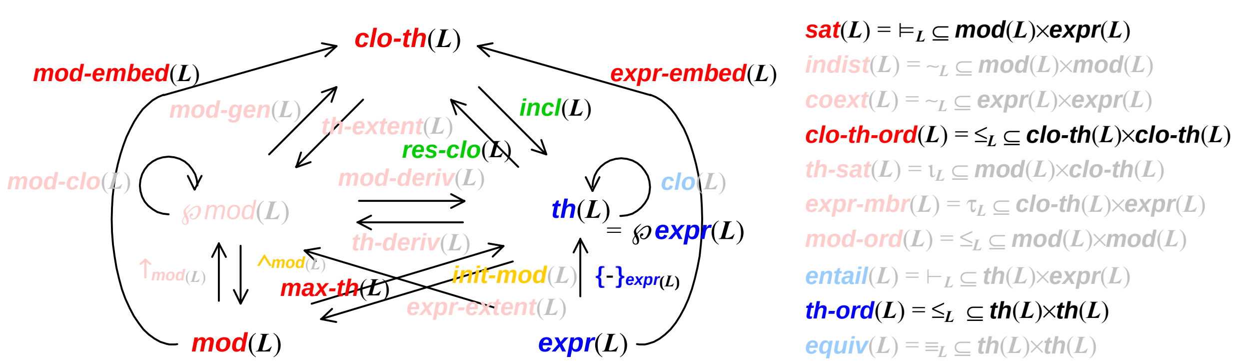

Figure 5: Sets, Functions and Relations in the Interpretative/Formal Aspect of the Lattice of Theories

Figure 5 graphically illustrates the terminology for most of the core classes/sets, relations and functions needed to specify the truth concept lattice and the lattice of theories for any fixed first order logic (FOL) type language ***L***. Those in **red** are only in the truth concept lattice namespace; those in **blue** are only in the lattice of theories namespace; and those in green connect the two. The emboldened terms are those important in the categorical representation for TCL and LOT. Of the five collections in the picture, the class of models and the set of expressions are basic, the set of theories is the power set of expressions, and the set of closed theories is derived from the set of theories and the closure operator. In addition, there are sixteen functions and ten binary relations. Hence, around thirty directly usable terms are used to specify the lattice of theories and truth concept lattice. The right side of the diagram is the formal aspect. Missing from Figure 5 are the tests for inconsistency and mutual consistency, and some helper terms that are used in the axiomatization. A final caveat is that the lattice of theories axiomatization is situated within the richer and more extensive category of theories axiomatization. Also missing from Figure 5 are the meet and join function and the top and bottom elements. These are represented by the join, meet, top and bottom commutative diagrams in Figure 6. Since inclusion is left adjoint to restricted-closure, inclusion should preserve joins, mapping joins of closed theories (intersection) to joins of theories (intersection), and restricted-closure should preserve meets, mapping meets of theories (union) to meets of closed theories (union then closure). These two facts are represented in Figure 6.

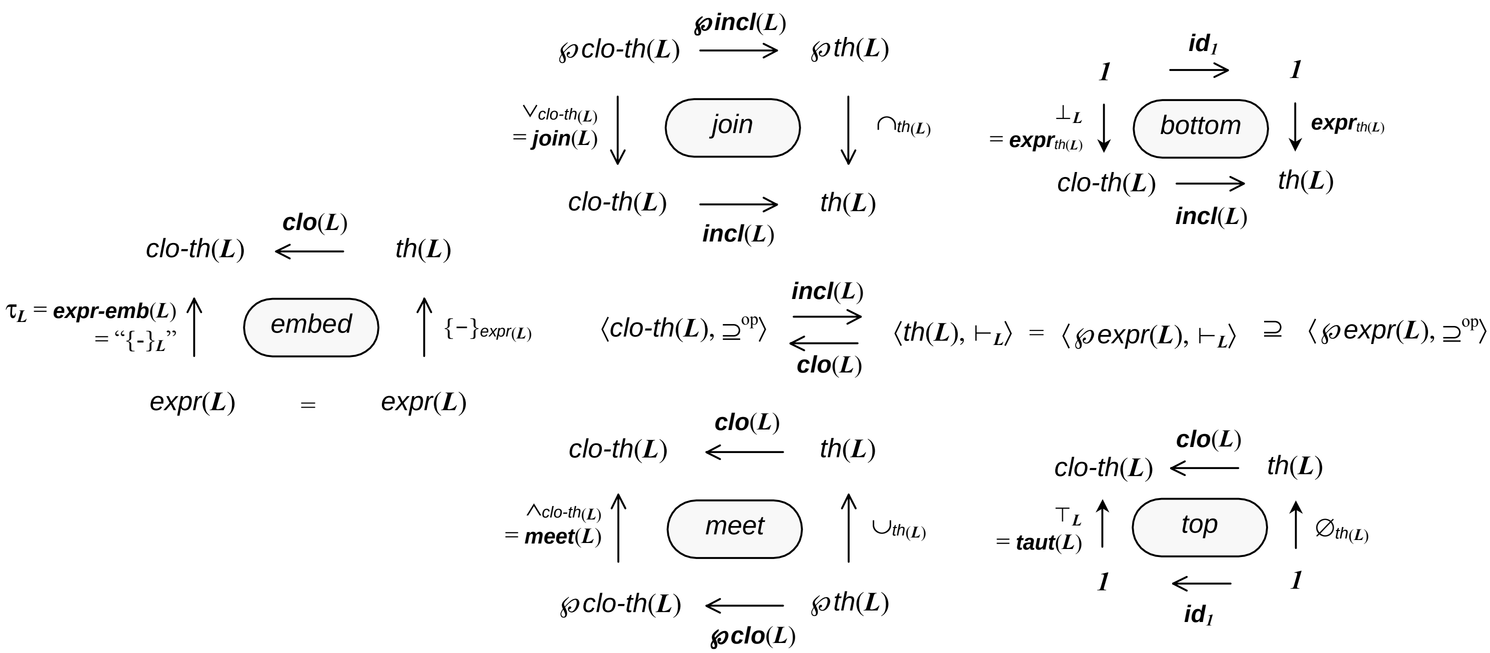

Figure 6: Join/Meet and Top/Bottom in the Lattice of Theories

Primitives	Comments
$\mathit{mod\text{-}deriv}(\boldsymbol{L}) : \wp\mathit{mod}(\boldsymbol{L}) \rightarrow \mathit{th}(\boldsymbol{L}) = \wp\mathit{expr}(\boldsymbol{L})$ $\mathit{th\text{-}deriv}(\boldsymbol{L}) : \wp\mathit{expr}(\boldsymbol{L}) = \mathit{th}(\boldsymbol{L}) \rightarrow \wp\mathit{mod}(\boldsymbol{L})$	These are the derivation functions defined on the truth classification. They are the principle means for defining the truth concept lattice.
$\mathit{incl}(\boldsymbol{L}) : \mathit{clo\text{-}th}(\boldsymbol{L}) \rightarrow \mathit{th}(\boldsymbol{L})$	This inclusion function is not truly primitive, since it is dependent upon the definition of closed theories.
$\{\text{-}\}_{\mathit{expr}(\boldsymbol{L})} : \mathit{expr}(\boldsymbol{L}) \rightarrow \wp\mathit{expr}(\boldsymbol{L}) = \mathit{th}(\boldsymbol{L})$ $\uparrow_{\mathit{mod}(\boldsymbol{L})} : \mathit{mod}(\boldsymbol{L}) \rightarrow \wp\mathit{mod}(\boldsymbol{L})$	These are the standard pointwise embeddings for sets and orders, respectively.
$\mathit{init\text{-}mod}(\boldsymbol{L}) : \mathit{th}(\boldsymbol{L}) \rightarrow \mathit{mod}(\boldsymbol{L})$	The initial model of a theory is the free model restricted to those tuples that satisfy the theory.

Function Identities	
$\mathit{mod\text{-}clo}(\boldsymbol{L}) = \mathit{mod\text{-}deriv}(\boldsymbol{L}) \cdot \mathit{th\text{-}deriv}(\boldsymbol{L})$ $\mathit{clo}(\boldsymbol{L}) = \mathit{th\text{-}deriv}(\boldsymbol{L}) \cdot \mathit{mod\text{-}deriv}(\boldsymbol{L})$	**Closure Operators:** These are induced by the derivation adjunction between subclasses of models and theories.
$\mathit{th\text{-}deriv}(\boldsymbol{L}) = \mathit{init\text{-}mod}(\boldsymbol{L}) \cdot \uparrow_{\mathit{mod}(\boldsymbol{L})}$ $\mathit{init\text{-}mod}(\boldsymbol{L}) = \mathit{th\text{-}deriv}(\boldsymbol{L}) \cdot \wedge_{\mathit{mod}(\boldsymbol{L})}$ $\mathit{max\text{-}th}(\boldsymbol{L}) = \uparrow_{\mathit{mod}(\boldsymbol{L})} \cdot \mathit{mod\text{-}deriv}(\boldsymbol{L})$ $\mathit{mod\text{-}deriv}(\boldsymbol{L}) = \wedge_{\mathit{mod}(\boldsymbol{L})} \cdot \mathit{max\text{-}th}(\boldsymbol{L})$	
$\mathit{res\text{-}clo}(\boldsymbol{L}) \cdot \mathit{incl}(\boldsymbol{L}) = \mathit{clo}(\boldsymbol{L})$ $\mathit{expr\text{-}extent}(\boldsymbol{L}) = \{\text{-}\}_{\mathit{expr}(\boldsymbol{L})} \cdot \mathit{res\text{-}clo}(\boldsymbol{L})$ $\mathit{mod\text{-}gen}(\boldsymbol{L}) \cdot \mathit{incl}(\boldsymbol{L}) = \mathit{mod\text{-}deriv}(\boldsymbol{L})$ $\mathit{th\text{-}extent}(\boldsymbol{L}) = \mathit{incl}(\boldsymbol{L}) \cdot \mathit{th\text{-}deriv}(\boldsymbol{L})$ $\mathit{mod\text{-}embed}(\boldsymbol{L}) = \uparrow_{\mathit{mod}(\boldsymbol{L})} \cdot \mathit{mod\text{-}gen}(\boldsymbol{L})$ $\mathit{expr\text{-}embed}(\boldsymbol{L}) = \{\text{-}\}_{\mathit{expr}(\boldsymbol{L})} \cdot \mathit{res\text{-}clo}(\boldsymbol{L})$ $\wedge_{\mathit{mod}(\boldsymbol{L})} = \mathit{mod\text{-}deriv}(\boldsymbol{L}) \cdot \mathit{init\text{-}mod}(\boldsymbol{L})$	
$\mathit{expr\text{-}extent}(\boldsymbol{L}) \cdot \mathit{mod\text{-}gen}(\boldsymbol{L}) = \mathit{expr\text{-}extent}(\boldsymbol{L})$	

Adjointness Constraints	
$\mathit{mod\text{-}gen}(\boldsymbol{L}) \cdot \mathit{th\text{-}extent}(\boldsymbol{L}) \supseteq \mathit{id}_{\wp\,\mathit{mod}(\boldsymbol{L})}$ $\mathit{th\text{-}extent}(\boldsymbol{L}) \cdot \mathit{mod\text{-}gen}(\boldsymbol{L}) = \mathit{id}_{\mathit{clo\text{-}th}(\boldsymbol{L})}$	**Generation-Extent Adjunction**
$\mathit{incl}(\boldsymbol{L}) \cdot \mathit{res\text{-}clo}(\boldsymbol{L}) = \mathit{id}_{\mathit{clo\text{-}th}(\boldsymbol{L})}$ $\mathit{res\text{-}clo}(\boldsymbol{L}) \cdot \mathit{incl}(\boldsymbol{L}) \supseteq(\equiv)\ \mathit{id}_{\mathit{th}(\boldsymbol{L})}$	**Theory Equivalence:** The lattice of theories is order-theoretically equivalent to the truth concept lattice.
$\mathit{mod\text{-}deriv}(\boldsymbol{L}) \cdot \mathit{th\text{-}deriv}(\boldsymbol{L}) \supseteq \mathit{id}_{\wp\,\mathit{mod}(\boldsymbol{L})}$ $\mathit{th\text{-}deriv}(\boldsymbol{L}) \cdot \mathit{mod\text{-}deriv}(\boldsymbol{L}) = \mathit{id}_{\mathit{th}(\boldsymbol{L})} = \mathit{id}_{\wp\,\mathit{expr}(\boldsymbol{L})}$	**Derivation Adjunction:** The derivation adjointness is basic in FCA. It is the composition of the generation-extent adjunction and the theory equivalence.
$\mathit{max\text{-}th}(\boldsymbol{L}) \cdot \mathit{init\text{-}mod}(\boldsymbol{L}) \equiv \mathit{id}_{\mathit{mod}(\boldsymbol{L})}$ $\mathit{init\text{-}mod}(\boldsymbol{L}) \cdot \mathit{max\text{-}th}(\boldsymbol{L}) \supseteq(\equiv)\ \mathit{id}_{\mathit{th}(\boldsymbol{L})} = \mathit{id}_{\wp\,\mathit{expr}(\boldsymbol{L})}$	**Theory-Model Equivalence:** It is the composition of the join-principal-ideal adjunction and the derivation adjunction. The initial model is the free model restricted by satisfaction.
$\wedge_{\mathit{mod}(\boldsymbol{L})} \cdot \uparrow_{\mathit{mod}(\boldsymbol{L})} \supseteq \mathit{id}_{\wp\,\mathit{mod}(\boldsymbol{L})}$ $\uparrow_{\mathit{mod}(\boldsymbol{L})} \cdot \wedge_{\mathit{mod}(\boldsymbol{L})} = \mathit{id}_{\mathit{mod}(\boldsymbol{L})}$	**Join-Principal-Ideal Adjunction:** The join operator is definable as the composition of model derivation and initial model.

Equivalence Relations	
$\mathit{indist}(\boldsymbol{L})$, induced by $\mathit{max\text{-}th}(\boldsymbol{L})$	Two models are indistinguishable when they have the same intent in the truth classification; that is, they generate the same theory.
$\mathit{coext}(\boldsymbol{L})$, induced by $\mathit{expr\text{-}extent}(\boldsymbol{L})$	Two expressions are coextensive when they have the same extent in the truth classification; that is, they are satisfied by the same models.

Orders	
$\mathit{clo\text{-}th\text{-}ord}(\boldsymbol{L})$	Reverse subset inclusion on closed theories.
$\mathit{th\text{-}ord}(\boldsymbol{L})$, induced by $\mathit{res\text{-}clo}(\boldsymbol{L})$	Entailment order, equivalent to the closed theory order.
$\mathit{mod\text{-}ord}(\boldsymbol{L})$, induced by $\mathit{mod\text{-}embed}(\boldsymbol{L})$	Specialization-generalization order on models.

Relational Identities	
$\mathit{th\text{-}sat}(\boldsymbol{L}) \circ \mathit{expr\text{-}mbr}(\boldsymbol{L}) = \mathit{sat}(\boldsymbol{L})$	This is a part of the basic theorem of FCA applied to the truth concept lattice.

Terminology

- $\boldsymbol{L}$

  IFF-KIF Code:

  ```
  language
  ```

  Meaning:

  The symbol $\boldsymbol{L}$ denotes a fixed FOL language.

Classes and Sets

- *mod*($\boldsymbol{L}$)

  IFF-KIF Code:

  (where '`?L`' denotes the fixed language being used.

  ```
  (model ?L)
  ```

  Meaning:

  The symbol *mod*($\boldsymbol{L}$) denotes the class of models for the language $\boldsymbol{L}$. $\boldsymbol{L}$-models in the IFF provide an interpretive semantics for object-level ontologies.

- *expr*($\boldsymbol{L}$)

  IFF-KIF Code:

  ```
  (expression ?L)
  ```

  Meaning:

  The symbol *expr*($\boldsymbol{L}$) denotes the set of expressions for the language $\boldsymbol{L}$. The set of $\boldsymbol{L}$-expressions is built up recursively in the IFF.

- *th*($\boldsymbol{L}$)

  IFF-KIF Code:

  ```
  (theory ?L)
  ```

  Meaning:

  The symbol *th*($\boldsymbol{L}$) denotes the set of theories for the language $\boldsymbol{L}$. For a fixed language $\boldsymbol{L}$, in the IFF the set of theories is identical (not just isomorphic) to the set of all subsets of expressions of $\boldsymbol{L}$: *th*($\boldsymbol{L}$) = $\wp$*expr*($\boldsymbol{L}$). Any expression entailed by a theory $\boldsymbol{T}$ is called a theorem of $\boldsymbol{T}$.

- *clo-th*($\boldsymbol{L}$)

  IFF-KIF Code:

  ```
  (closed-theory ?L)
  ```

  Meaning:

  The symbol *clo-th*($\boldsymbol{L}$) denotes the set of closed theories for the language $\boldsymbol{L}$. A theory is closed when it equals its closure: $\boldsymbol{T}$ = *clo*($\boldsymbol{L}$)($\boldsymbol{T}$). This means that the axiom set of $\boldsymbol{T}$ equals the theorem set of $\boldsymbol{T}$: *axm*($\boldsymbol{T}$) = *thm*($\boldsymbol{T}$). The set of closed theories represent precisely the set of formal concepts of the truth concept lattice.

Binary Relations

- *sat*($\boldsymbol{L}$) = $\models_{\boldsymbol{L}}$

  IFF-KIF Code:

  ```
  (satisfaction ?L)
  ```

  Meaning:

The symbol *sat*($\boldsymbol{L}$) = $\vDash_{\boldsymbol{L}}$ denotes the binary satisfaction relation between models and expressions. It is also known as the truth classification relation. An $\boldsymbol{L}$-model $\boldsymbol{M}$ satisfies (or is a model of) an $\boldsymbol{L}$-expression e, symbolized by

$\boldsymbol{M} \vDash_{\boldsymbol{L}} e$

when all tuples of $\boldsymbol{M}$ satisfy e. Satisfaction is the most basic relation of semantics.

- *indist*($\boldsymbol{L}$) = $\sim_{\boldsymbol{L}}$

IFF-KIF Code:

```
(indistinguishable ?L)
```

Meaning:

The symbol *indist*($\boldsymbol{L}$) = $\sim_{\boldsymbol{L}}$ denotes the binary indistinguishable relation between two models.
For any type language $\boldsymbol{L}$, two $\boldsymbol{L}$-models $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ are indistinguishable

$\boldsymbol{M}_1 \sim_{\boldsymbol{L}} \boldsymbol{M}_2$

when they have the same formal intent (maximal theory): *max-th*($\boldsymbol{M}_1$) = *max-th*($\boldsymbol{M}_2$).

- *coext*($\boldsymbol{L}$) = $\sim_{\boldsymbol{L}}$

IFF-KIF Code:

```
(coextensive ?L)
```

Meaning:

The symbol *coext*($\boldsymbol{L}$) = $\sim_{\boldsymbol{L}}$ denotes the binary coextensive relation between two expressions.
For any type language $\boldsymbol{L}$, two $\boldsymbol{L}$-expressions e_1, e_2 are coextensive

$e_1 \sim_{\boldsymbol{L}} e_2$

when they have the same interpretive extent (expression intent): *expr-intent*(e_1) = *expr-intent*(e_2).

- *entail*($\boldsymbol{L}$) = $\vdash_{\boldsymbol{L}}$

IFF-KIF Code:

```
(entailment ?L)
```

Meaning:

The symbol *entail*($\boldsymbol{L}$) = $\vdash_{\boldsymbol{L}}$ denotes the binary entailment relation between theories and expressions. The principal semantic (and also proof-theoretic) relation on theories is the binary entailment relation between a theory and an expression. A theory T entails an expression e, symbolized

$\boldsymbol{T} \vdash_{\boldsymbol{L}} e$

when any model for (the axioms of) $\boldsymbol{T}$ is also a model for e.

- *clo-th-ord*($\boldsymbol{L}$) = $\leq_{\boldsymbol{L}}$

IFF-KIF Code:

```
(closed-theory-order ?L)
```

Meaning:

The symbol *clo-th-ord*($\boldsymbol{L}$) = $\leq_{\boldsymbol{L}}$ denotes the order between two closed theories. For any type language $\boldsymbol{L}$, two closed $\boldsymbol{L}$-theories $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ are ordered

$\boldsymbol{T}_1 \leq_{\boldsymbol{L}} \boldsymbol{T}_2$

when any $\boldsymbol{T}_2$-theorem is a $\boldsymbol{T}_1$-theorem; that is, when the second theory is a subset of the first theory: $\boldsymbol{T}_2 \subseteq_{\boldsymbol{L}} \boldsymbol{T}_1$. Then $\boldsymbol{T}_1$ is said to be more specialized that $\boldsymbol{T}_2$, and $\boldsymbol{T}_2$ is said to be more generalized that $\boldsymbol{T}_1$. This is a true generalization-specialization hierarchy.

- *mod-ord*($\boldsymbol{L}$) = $\leq_{\boldsymbol{L}}$

IFF-KIF Code:

```
(model-order ?L)
```

Meaning:

The symbol *mod-ord*($\boldsymbol{L}$) = $\leq_{\boldsymbol{L}}$ denotes the order between two models. For any type language $\boldsymbol{L}$, two $\boldsymbol{L}$-models $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ are ordered

$\boldsymbol{M}_1 \leq_{\boldsymbol{L}} \boldsymbol{M}_2$

when the intents (maximal theories) of the two models are covariantly ordered: *max-th*($\boldsymbol{M}_1$) $\leq_{\boldsymbol{L}}$ *max-th*($\boldsymbol{M}_2$).

- *th-ord*($\boldsymbol{L}$) = $\leq_{\boldsymbol{L}}$

  IFF-KIF Code:

  ```
  (theory-order ?L)
  ```

  Meaning:

  The symbol *th-ord*($\boldsymbol{L}$) = $\leq_{\boldsymbol{L}}$ denotes the order between two theories. For any type language $\boldsymbol{L}$, two $\boldsymbol{L}$-theories $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ are ordered

  $\boldsymbol{T}_1 \leq_{\boldsymbol{L}} \boldsymbol{T}_2$

  when any $\boldsymbol{T}_2$-axiom is a $\boldsymbol{T}_1$-theorem; that is, when the first theory entails every axiom of the second theory: $\boldsymbol{T}_1 \vdash_{\boldsymbol{L}} \boldsymbol{e}_2$ for every $\boldsymbol{e}_2 \in \boldsymbol{T}_2$.; that is, when the closures of the two theories are covariantly ordered: *clo*($\boldsymbol{T}_1$) $\leq_{\boldsymbol{L}}$ *clo*($\boldsymbol{T}_2$). Then $\boldsymbol{T}_1$ is said to be more specialized that $\boldsymbol{T}_2$, and $\boldsymbol{T}_2$ is said to be more generalized that $\boldsymbol{T}_1$. This is a true generalization-specialization hierarchy.

- *th-sat*($\boldsymbol{L}$) = $\iota_{\boldsymbol{L}}$ = $\vDash_{\boldsymbol{L}}$

  IFF-KIF Code:

  ```
  (theory-satisfaction ?L)
  ```

  Meaning:

  The symbol *th-sat*($\boldsymbol{L}$) = $\iota_{\boldsymbol{L}}$ = $\vDash_{\boldsymbol{L}}$ denotes the binary satisfaction relation between models and closed theories. An $\boldsymbol{L}$-model $\boldsymbol{M}$ satisfies (or is a model of) a closed $\boldsymbol{L}$-theory $\boldsymbol{T}$, symbolized by

  $\boldsymbol{M} \vDash_{\boldsymbol{L}} \boldsymbol{T}$

  when it satisfies every theorem of $\boldsymbol{T}$. This means that $\boldsymbol{M}$ is in the truth extent of $\boldsymbol{T}$; that is, $\boldsymbol{M} \in$ *extent*$_{\boldsymbol{L}}$($\boldsymbol{T}$). This relation, which is also called iota, is induced by the model embedding function and the truth concept lattice order. Theory satisfaction is closed on the right with respect to the truth concept lattice order (reverse inclusion between closed theories).

- *expr-mbr*($\boldsymbol{L}$) = $\tau_{\boldsymbol{L}}$

  IFF-KIF Code:

  ```
  (expression-membership ?L)
  ```

  Meaning:

  The symbol *expr-mbr*($\boldsymbol{L}$) = $\tau_{\boldsymbol{L}}$ denotes the (opposite) binary membership relation between closed theories and expressions. This relation, which is also called tau, is induced by the expression embedding function and the truth concept lattice order. Theory satisfaction is closed on the right with respect to the truth concept lattice order (reverse inclusion between closed theories).

Functions

- *clo*($\boldsymbol{L}$) : *th*($\boldsymbol{L}$) $\rightarrow$ *th*($\boldsymbol{L}$)

  IFF-KIF Code:

  ```
  (closure ?L)
  ```

  Meaning:

  The symbol *clo*($\boldsymbol{L}$) denotes the closure function on $\boldsymbol{L}$-theories. The closure of an $\boldsymbol{L}$-theory $\boldsymbol{T}$ is the set-theoretically larger theory consisting of all theorems of $\boldsymbol{T}$.

- *max-th*($\boldsymbol{L}$)

  IFF-KIF Code:
  ```
  (maximal-theory ?L)
  ```
  Meaning:

  The symbol *max-th*($\boldsymbol{L}$) denotes the maximal theory function from $\boldsymbol{L}$-models to $\boldsymbol{L}$-theories. The maximal theory of a model is the set of all expressions satisfied by the model: this is the 12-fiber function of the satisfaction relation. The maximal theory is closed. It is the largest theory that the model satisfies. In the lattice of theories, it is the meet of all these theories.

- *init-mod*($\boldsymbol{L}$)

  IFF-KIF Code:
  ```
  (initial-model ?L)
  ```
  Meaning:

  The symbol *init-mod*($\boldsymbol{L}$) denotes the initial model function from $\boldsymbol{L}$-theories to $\boldsymbol{L}$-models. This is left adjoint to the maximal theory function. The initial model of a theory is the restriction of the free model over underling language to the tuples that satisfy the theory. Since it is the initial model in *th-deriv*($\boldsymbol{L}$), the subcategory of all models that satisfy the theory, it is the join of all models that satisfy that theory.

- *expr-intent*($\boldsymbol{L}$)

  IFF-KIF Code:
  ```
  (expression-intent ?L)
  ```
  Meaning:

  The symbol *expr-intent*($\boldsymbol{L}$) denotes the expression intent function from $\boldsymbol{L}$-expressions to subclasses of $\boldsymbol{L}$-models. The intent of an expression is the class of all models that satisfy the expression. It is the composition of the expression singleton function and the theory derivation function.

- *mod-deriv*($\boldsymbol{L}$)

  IFF-KIF Code:
  ```
  (model-derivation ?L)
  ```
  Meaning:

  The symbol *mod-deriv*($\boldsymbol{L}$) denotes the model derivation function from subclasses of $\boldsymbol{L}$-models to theories (subsets of $\boldsymbol{L}$-expressions). The derivation of a subclass of models is the set-theoretically largest theory (set of all expressions) that they all satisfy.

- *th-deriv*($\boldsymbol{L}$)

  IFF-KIF Code:
  ```
  (theory-derivation ?L)
  ```
  Meaning:

  The symbol *th-deriv*($\boldsymbol{L}$) denotes the theory derivation function from theories (subsets of $\boldsymbol{L}$-expressions) to subclasses of $\boldsymbol{L}$-models. The derivation of a theory is the subclass of models that satisfy the theory.

- *mod-clo*($\boldsymbol{L}$)

  IFF-KIF Code:
  ```
  (model-closure ?L)
  ```
  Meaning:

  The symbol *mod-clo*($\boldsymbol{L}$), which denotes the model closure function on subclasses of $\boldsymbol{L}$-models, is the composition of the model derivation function followed by the theory derivation function. A model $\boldsymbol{M}$ is in the closure of a subclass of models `M` when $\boldsymbol{M}$ satisfies every expression that all the models in `M` satisfy.

- $\mathit{mod\text{-}gen}(\boldsymbol{L})$

  IFF-KIF Code:

  ```
  (model-generation ?L)
  ```

  Meaning:

  The symbol $\mathit{mod\text{-}gen}(\boldsymbol{L})$ denotes the model generation function from subclasses of $\boldsymbol{L}$-models to closed theories (closed subsets of $\boldsymbol{L}$-expressions). The (formal truth concept) generation of a subclass of models is the derivation of that subclass. The model derivation function returns closed theories, and hence factors through the set of closed theories – it is the composition of model generation followed by the inclusion of the set of closed theories into the set of theories

  $$\mathit{mod\text{-}deriv}(\boldsymbol{L}) = \mathit{mod\text{-}gen}(\boldsymbol{L}) \cdot \mathit{incl}(\boldsymbol{L}).$$

Closure Properties

- **Closure:** For any language $\boldsymbol{L}$, the closure function

  $$\mathit{clo}(\boldsymbol{L}) : \mathit{th}(\boldsymbol{L}) \rightarrow \mathit{th}(\boldsymbol{L})$$

  is, well, a closure function; that is, it is a monotonic function $\mathit{clo}(\boldsymbol{L}) : \langle \mathit{th}(\boldsymbol{L}), \subseteq_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \subseteq_{\boldsymbol{L}} \rangle$,

  $$\boldsymbol{T} \subseteq_{\boldsymbol{L}} \check{\boldsymbol{T}} \ \underline{\text{implies}}\ \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}) \subseteq_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\check{\boldsymbol{T}}),$$

  for any pair of theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in \mathit{th}(\boldsymbol{L})$; identity bounding and idempotent

  $$\boldsymbol{T} \subseteq_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}) \text{ and}$$
  $$\mathit{clo}(\boldsymbol{L})(\mathit{clo}(\boldsymbol{L})(\boldsymbol{T})) = \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}),$$

  for any theory $\boldsymbol{T} \in \mathit{th}(\boldsymbol{L})$.

- **Closure monotonicity1:** For any language $\boldsymbol{L}$, the closure is a monotonic function

  $$\mathit{clo}(\boldsymbol{L}) : \langle \mathit{th}(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$$

  satisfying the monotonicity condition

  $$\boldsymbol{T} \vdash_{\boldsymbol{L}} \check{\boldsymbol{T}} \ \underline{\text{implies}}\ \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\check{\boldsymbol{T}}),$$

  for any pair of theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in \mathit{th}(\boldsymbol{L}_1)$.

  Proof. This is true, since closure is monotonic and idempotent.

- **Closure Monotonicity2:** For any language $\boldsymbol{L}$, the closure is a monotonic function

  $$\mathit{clo}(\boldsymbol{L}) : \langle \mathit{th}(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle$$

  satisfying the monotonicity condition,

  $$\boldsymbol{T} \supseteq_{\boldsymbol{L}} \check{\boldsymbol{T}} \ \underline{\text{implies}}\ \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}) \vdash_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\check{\boldsymbol{T}}),$$

  for any pair of theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in \mathit{th}(\boldsymbol{L})$.

  Proof. This is true, since closure is monotonic and idempotent.

- **Closure Self-Adjointness:** For any language $\boldsymbol{L}$, the closure function is a self-adjoint monotonic function

  $$\langle \mathit{clo}(\boldsymbol{L}), \mathit{clo}(\boldsymbol{L}) \rangle : \langle \mathit{th}(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$$

  satisfying the adjointness condition

  $$\mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}} \boldsymbol{T}_1 \ \underline{\text{iff}}\ \boldsymbol{T}_2 \vdash_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_1),$$

  for any two theories $\boldsymbol{T}_2, \boldsymbol{T}_1 \in \mathit{th}(\boldsymbol{L})$.

  Proof. This result can be directly proven using the closure properties above.

  $\mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}} \boldsymbol{T}_1$

  $\underline{\text{iff}}$ $\mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_1)$ by monotonicity, idempotency and identity bounding of closure

  $\underline{\text{iff}}$ $\boldsymbol{T}_2 \vdash_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_1)$.

Morphisms

A truth concept morphism is a morphism between truth concept lattices. It is naturally defined as a concept morphism (see the IFF Upper Classification (meta) Ontology (IFF-UCLS) for the axiomatization of concept morphism) between truth concept lattices. We are especially interested in the truth concept morphisms generated by language morphisms. In the IFF, a truth concept morphism over a language morphism $\boldsymbol{f}$ is the concept morphism generated by and associated with the truth infomorphism over $\boldsymbol{f}$ – for any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, there is (1) a truth infomorphism from the truth classification over $\boldsymbol{L}_1$ to the truth classification over $\boldsymbol{L}_2$, and hence (2) a truth concept morphism from the truth concept lattice over $\boldsymbol{L}_1$ to the truth concept lattice over $\boldsymbol{L}_2$. Figure 7 illustrates the truth concept morphism over a language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$. It is represented by the following components.

$$
\begin{tikzcd}
expr(\boldsymbol{L}_1) \arrow[r, "\boldsymbol{expr(f)}"] \arrow[d, "\boldsymbol{expr\text{-}embed}(\boldsymbol{L}_1)"] \arrow[ddd, dash, "\models_{L1}"'] & expr(\boldsymbol{L}_2) \arrow[d, "\boldsymbol{expr\text{-}embed}(\boldsymbol{L}_2)"'] \arrow[ddd, dash, "\models_{L2}"] \\
clo\text{-}th(\boldsymbol{L}_1) \arrow[r, "(\wp \boldsymbol{expr(f)}(\text{-}))''"] \arrow[d, dash, "\supseteq_{L1}"] & clo\text{-}th(\boldsymbol{L}_2) \arrow[d, dash, "\supseteq_{L2}"'] \\
clo\text{-}th(\boldsymbol{L}_1) & clo\text{-}th(\boldsymbol{L}_2) \arrow[l, "\boldsymbol{expr(f)}^{-1}"] \\
mod(\boldsymbol{L}_1) \arrow[u, "\boldsymbol{mod\text{-}embed}(\boldsymbol{L}_1)"] & mod(\boldsymbol{L}_2) \arrow[l, "\boldsymbol{mod(f)}"'] \arrow[u, "\boldsymbol{mod\text{-}embed}(\boldsymbol{L}_2)"']
\end{tikzcd}
$$

Figure 7: Truth Concept Morphism

- The complete lattice morphism consisting of the pair of adjoint monotonic functions
  - the (left adjoint) inverse image expression function
    $expr(\boldsymbol{f})^{-1} : clo\text{-}th(\boldsymbol{L}_2) \rightarrow clo\text{-}th(\boldsymbol{L}_1)$, and
  - the (right adjoint) closure of direct image expression function
    $(\wp expr(\boldsymbol{f})(\text{-}))'' : clo\text{-}th(\boldsymbol{L}_1) \rightarrow clo\text{-}th(\boldsymbol{L}_2)$.

  These satisfy the conditions
  - [direct monotonicity]
    If $\boldsymbol{T}_1 \subseteq_{L1} \check{\boldsymbol{T}}_1$ then $(\wp expr(\boldsymbol{f})(\boldsymbol{T}_1))'' \subseteq_{L2} (\wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1))''$,
    for any pair of closed source theories $\boldsymbol{T}_1, \check{\boldsymbol{T}}_1 \in th(\boldsymbol{L}_1)$.
  - [inverse monotonicity]
    If $\boldsymbol{T}_2 \subseteq_{L2} \check{\boldsymbol{T}}_2$ then $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) \subseteq_{L1} expr(\boldsymbol{f})^{-1}(\check{\boldsymbol{T}}_2)$,
    for any pair of closed target theories $\boldsymbol{T}_2, \check{\boldsymbol{T}}_2 \in th(\boldsymbol{L}_2)$.
  - [adjointness]
    $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) \supseteq_{L1} \boldsymbol{T}_1$ <u>iff</u> $\boldsymbol{T}_2 \supseteq_{L2} (\wp expr(\boldsymbol{f})(\boldsymbol{T}_1))''$,
    for any closed target theory $\boldsymbol{T}_2 \in th(\boldsymbol{L}_2)$ and closed source theory $\boldsymbol{T}_1 \in th(\boldsymbol{L}_1)$.
- The model fiber function

  $mod(\boldsymbol{f}) : mod(\boldsymbol{L}_2) \rightarrow mod(\boldsymbol{L}_1)$,

  which satisfies the condition (commutative diagram)

  $mod(\boldsymbol{f}) \cdot mod\text{-}embed(\boldsymbol{L}_1) = mod\text{-}embed(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1}$.
- The expression function

  $expr(\boldsymbol{f}) : expr(\boldsymbol{L}_1) \rightarrow expr(\boldsymbol{L}_2)$,

  which satisfies the condition (commutative diagram)

  $expr(\boldsymbol{f}) \cdot expr\text{-}embed(\boldsymbol{L}_2) = expr\text{-}embed(\boldsymbol{L}_1) \cdot (\wp expr(\boldsymbol{f})(\text{-}))''$.

Part of an extension of the fundamental theorem of FCA to morphisms states the following: the fundamental condition for the truth infomorphism is implied by (1) the adjointness condition for the (inverse image, direct image) pair of monotonic functions (2) the model fiber condition and (3) the expression condition:

For closed theories $\boldsymbol{T}_1, \boldsymbol{\hat{T}}_1 \in clo\text{-}th(\boldsymbol{L}_1)$ and $\boldsymbol{T}_2, \boldsymbol{\hat{T}}_2 \in clo\text{-}th(\boldsymbol{L}_2)$

$mod(\boldsymbol{f})(\boldsymbol{M}_2) \models_{L1} \varphi_1$

<u>iff</u> $mod\text{-}embed(\boldsymbol{L}_1)(mod(\boldsymbol{f})(\boldsymbol{M}_2)) \leq_{L1} \boldsymbol{T}_1 \leq_{L1} \boldsymbol{\hat{T}}_1 \leq_{L} expr\text{-}embed(\boldsymbol{L}_1)(\varphi_1)$

$\underline{\text{iff}}$ $expr(\boldsymbol{f})^{-1}(mod\text{-}embed(\boldsymbol{L}_2)(\boldsymbol{M}_2)) \leq_{L1} \boldsymbol{T}_1 \leq_{L1} \boldsymbol{Ŧ}_1 \leq_{L} expr\text{-}embed(\boldsymbol{L}_1)(\varphi_1)$

[$\underline{\text{only if}}$, where $\boldsymbol{T}_2 = (\wp expr(\boldsymbol{f})(\boldsymbol{T}_1))''$ and $\boldsymbol{Ŧ}_2 = \wp expr(\boldsymbol{f})(\boldsymbol{Ŧ}_1))''$]

[$\underline{\text{if}}$, where $\boldsymbol{T}_1 = expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2)$ and $\boldsymbol{Ŧ}_1 = expr(\boldsymbol{f})^{-1}(\boldsymbol{Ŧ}_1)$]

$mod\text{-}embed(\boldsymbol{L}_2)(\boldsymbol{M}_2) \leq_{L2} \boldsymbol{T}_2 \leq_{L} \boldsymbol{Ŧ}_2 \leq_{L} (\wp expr(\boldsymbol{f})(expr\text{-}embed(\boldsymbol{L}_1)(\varphi_1))''$

$\underline{\text{iff}}$ $mod\text{-}embed(\boldsymbol{L}_2)(\boldsymbol{M}_2) \leq_{L2} \boldsymbol{T}_2 \leq_{L2} \boldsymbol{Ŧ}_2 \leq_{L2} expr\text{-}embed(\boldsymbol{L}_2)(expr(\boldsymbol{f})(\varphi_1))$

$\underline{\text{iff}}$ $\boldsymbol{M}_2 \models_{L1} expr(\boldsymbol{f})(\varphi_1)$.

For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, Figure 8 illustrates two kinds of morphisms: the *truth concept morphism* from the truth classification over $\boldsymbol{L}_1$ to the truth classification over $\boldsymbol{L}_2$ is pictured in the back plane of Figure 8, and the *lattice morphism of theories* from the lattice of theories over $\boldsymbol{L}_1$ to the lattice of theories over $\boldsymbol{L}_2$ is pictured in the front plane of Figure 8. These lattices are the (weak) lattices of theories with the reverse subset order (the opposite powerset order for the set of expressions), and hence the lattice morphisms are weak – they do not involve preserving any entailment orders or using any closure operations. We describe the constraints that these morphisms must satisfy. We also include proofs for the constraints that are not obvious. We list these in three parts: the TCL constraints, the LOT constraints, and connecting constraints. However, the one common fundamental infomorphism constraint is listed first.

- **Infomorphism condition:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$,

  $mod(\boldsymbol{f})(\boldsymbol{A}_2) \models \varphi_1$ $\underline{\text{iff}}$ $\boldsymbol{A}_2 \models expr(\boldsymbol{f})(\varphi_1)$

  for all target models $\boldsymbol{A}_2 \in mod(\boldsymbol{L}_2)$ and all source expressions $\varphi_1 \in expr(\boldsymbol{L}_1)$.

Proof. Let $\boldsymbol{A}_2 \in mod(\boldsymbol{L}_2)$ be any model in the target fiber and $mod(\boldsymbol{f})(\boldsymbol{A}_2) = \boldsymbol{A}_1 \in mod(\boldsymbol{L}_1)$ be the model in the source fiber that is its image under the fiber function $mod(\boldsymbol{f})$. Here, $tuple(\boldsymbol{A}_1) = tuple(\boldsymbol{A}_2)$ and the extent of any relation type $\rho_1 \in rel(\boldsymbol{L}_1)$ in model $\boldsymbol{A}_1$ is the same as the extent of the relation type $rel(\boldsymbol{f})(\rho_1) \in rel(\boldsymbol{L}_2)$ in model $\boldsymbol{A}_2$. By induction this means that the extent of any expression $\varphi_1 \in rel(expr(\boldsymbol{L}_1))$ in model $expr(\boldsymbol{A}_1)$ is the same as the extent of the expression $expr(\boldsymbol{f})(\varphi_1) \in rel(expr(\boldsymbol{L}_2))$ in model $expr(\boldsymbol{A}_2)$. Now, $\boldsymbol{A}_1 \models \varphi_1$ iff the extent of φ_1 is $tuple(\boldsymbol{A})$, all abstract tuples of $\boldsymbol{A}_1$. And $\boldsymbol{A}_2 \models expr(\boldsymbol{f})(\varphi_1)$ iff the extent of $expr(\boldsymbol{f})(\varphi_1)$ is $tuple(\boldsymbol{A}_2)$, all abstract tuples of $\boldsymbol{A}_2$. But, φ_1 and $expr(\boldsymbol{f})(\varphi_1)$ have the same extent.

TCL Morphism Constraints

Assuming the infomorphism condition, all the other TCL constraints come from the general theory axiomatized in the IFF-UCLS.

- **Inverse monotonicity:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the inverse-image is a monotonic function

  $expr(\boldsymbol{f})^{-1} : \langle clo\text{-}th(\boldsymbol{L}_2), \supseteq_{L2}\rangle \rightarrow \langle clo\text{-}th(\boldsymbol{L}_1), \supseteq_{L1}\rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T}_2 \subseteq_{L2} \check{\boldsymbol{T}}_2$ $\underline{\text{implies}}$ $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) \subseteq_{L1} expr(\boldsymbol{f})^{-1}(\check{\boldsymbol{T}}_2)$,

  for any pair of closed target theories $\boldsymbol{T}_2, \check{\boldsymbol{T}}_2 \in clo\text{-}th(\boldsymbol{L}_2)$.

- **Direct monotonicity:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the direct-image-closure is a monotonic function

  $(\wp expr(\boldsymbol{f})(\text{-}))'' : \langle clo\text{-}th(\boldsymbol{L}_1), \supseteq_{L1}\rangle \rightarrow \langle clo\text{-}th(\boldsymbol{L}_2), \supseteq_{L2}\rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T}_1 \subseteq_{L1} \check{\boldsymbol{T}}_1$ $\underline{\text{implies}}$ $(\wp expr(\boldsymbol{f})(\boldsymbol{T}_1))'' \subseteq_{L2} (\wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1))''$,

  for any pair of closed source theories $\boldsymbol{T}_1, \check{\boldsymbol{T}}_1 \in clo\text{-}th(\boldsymbol{L}_1)$.

- **Inverse-Direct Adjointness:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the inverse-image monotonic function is left adjoint to the direct-image-closure monotonic function

  $\langle expr(\boldsymbol{f})^{-1}, (\wp expr(\boldsymbol{f})(\text{-}))''\rangle : \langle clo\text{-}th(\boldsymbol{L}_2), \supseteq_{L2}\rangle \rightarrow \langle clo\text{-}th(\boldsymbol{L}_1), \supseteq_{L1}\rangle$,

  satisfying the adjointness condition

  $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) \supseteq_{L1} \boldsymbol{T}_1$ $\underline{\text{iff}}$ $\boldsymbol{T}_2 \supseteq_{L2} (\wp expr(\boldsymbol{f})(\boldsymbol{T}_1))''$, for any closed target theory $\boldsymbol{T}_2 \in$ *clo-th*($\boldsymbol{L}_2$) and closed source theory $\boldsymbol{T}_1 \in$ *clo-th*($\boldsymbol{L}_1$).

Proof. This result can be directly proven. However, it is a special case of the general theory in the IFF-UCLS, as we now discuss. Formal truth concepts are represented by their intent, closed theories. In the IFF-UCLS, the left adjoint of the concept morphism of the truth infomorphism of a type language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$ is defined to be the inverse image of the expression function $expr(\boldsymbol{f})^{-1}(\text{-})$ for all closed theories $\boldsymbol{T}_2 \in$ *clo-th*($\boldsymbol{L}_2$), and the right adjoint is defined to be the closure of the direct image of the expression function $(\wp expr(\boldsymbol{f})(\text{-}))''$ for all closed theories $\boldsymbol{T}_1 \in$ *clo-th*($\boldsymbol{L}_1$).

- **Model embedding condition:**

  $$mod(\boldsymbol{f}) \cdot mod\text{-}embed(\boldsymbol{L}_1) = mod\text{-}embed(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1};$$

  pointwise, this becomes $mod(\boldsymbol{f})(\boldsymbol{A}_2) \models \varphi_1$ iff $\boldsymbol{A}_2 \models expr(\boldsymbol{f})(\varphi_1)$ for all target models $\boldsymbol{A}_2 \in mod(\boldsymbol{L}_2)$ and all source expressions $\varphi_1 \in expr(\boldsymbol{L}_1)$, the infomorphism condition.

- **Expression embedding condition:**

  $$expr(\boldsymbol{f}) \cdot expr\text{-}embed(\boldsymbol{L}_2) = expr\text{-}embed(\boldsymbol{L}_1) \cdot (\wp expr(\boldsymbol{f})(\text{-}))''.$$

Connecting the TCL and Weak LOT morphisms

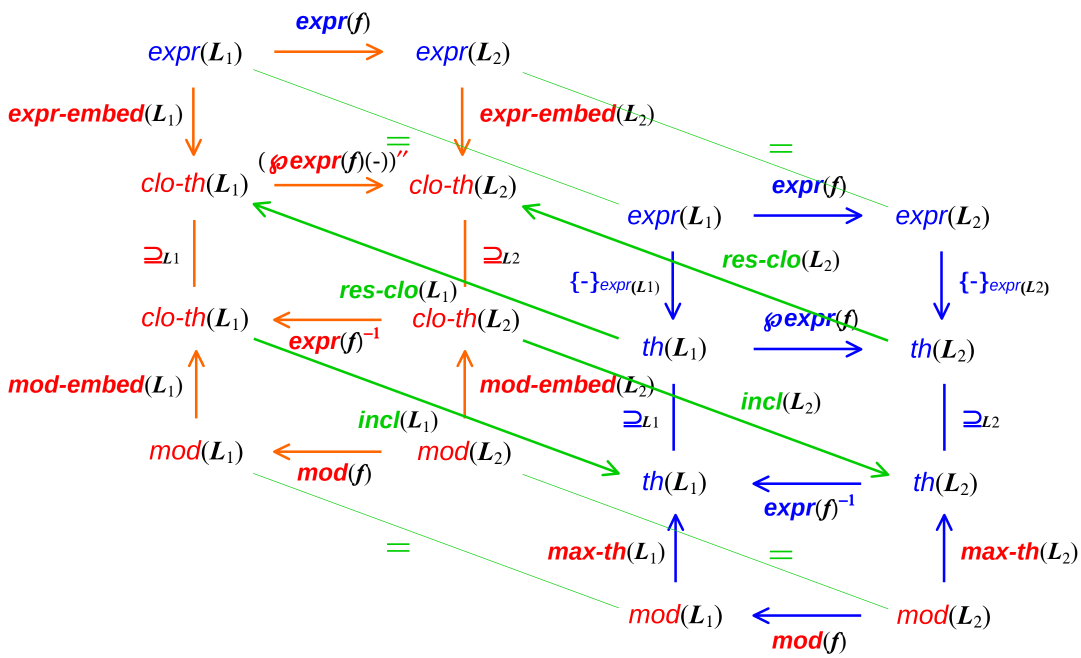

Figure 8: Connecting the TCL and weak LOT Morphisms

Weak LOT Morphism Constraints

- **Direct monotonicity:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the direct-image is a monotonic function

  $\wp expr(\boldsymbol{f}) : \langle th(\boldsymbol{L}_1), \sqsupseteq_{L1} \rangle \rightarrow \langle th(\boldsymbol{L}_2), \sqsupseteq_{L2} \rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T}_1 \sqsupseteq_{L1} \check{\boldsymbol{T}}_1$ <u>implies</u> $\wp expr(\boldsymbol{f})(\boldsymbol{T}_1) \sqsupseteq_{L2} \wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1)$,

  for any pair of source theories $\boldsymbol{T}_1, \check{\boldsymbol{T}}_1 \in th(\boldsymbol{L}_1)$.
- **Inverse monotonicity:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the inverse-image is a monotonic function

  $expr(\boldsymbol{f})^{-1} : \langle th(\boldsymbol{L}_2), \sqsupseteq_{L2} \rangle \rightarrow \langle th(\boldsymbol{L}_1), \sqsupseteq_{L1} \rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T}_2 \sqsupseteq_{L2} \check{\boldsymbol{T}}_2$ <u>implies</u> $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) \sqsupseteq_{L1} expr(\boldsymbol{f})^{-1}(\check{\boldsymbol{T}}_2)$,

  for any pair of target theories $\boldsymbol{T}_2, \check{\boldsymbol{T}}_2 \in th(\boldsymbol{L}_2)$.
- **Inverse-Direct Adjointness:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the inverse-image monotonic function is left adjoint to the direct-image monotonic function

  $\langle expr(\boldsymbol{f})^{-1}, \wp expr(\boldsymbol{f}) \rangle : \langle th(\boldsymbol{L}_2), \sqsupseteq_{L2} \rangle \rightarrow \langle th(\boldsymbol{L}_1), \sqsupseteq_{L1} \rangle$,

  satisfying the adjointness condition

  $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) \sqsupseteq \boldsymbol{T}_1$ <u>iff</u> $\boldsymbol{T}_2 \sqsupseteq \wp expr(\boldsymbol{f})(\boldsymbol{T}_1)$

  for all target theories $\boldsymbol{T}_2 \in th(\boldsymbol{L}_2)$ and all source theories $\boldsymbol{T}_1 \in th(\boldsymbol{L}_1)$.
- **Model embedding condition:**

$mod(\boldsymbol{f}) \cdot max\text{-}th(\boldsymbol{L}_1) = max\text{-}th(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1}$.

- **Expression embedding condition:**

  $expr(\boldsymbol{f}) \cdot \{\text{-}\}_{expr(\boldsymbol{L}2)} = \{\text{-}\}_{expr(\boldsymbol{L}1)} \cdot \wp expr(\boldsymbol{f})(\text{-})$.

TCL – Weak LOT Morphism Connection Constraints

- **Inclusion monotonicity:** For any language $\boldsymbol{L}$, the inclusion is a monotonic function

  $incl(\boldsymbol{L}) : \langle clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T} \supseteq_{\boldsymbol{L}} \check{\boldsymbol{T}}$ <u>implies</u> $incl(\boldsymbol{L})(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} incl(\boldsymbol{L})(\check{\boldsymbol{T}})$,

  for any pair of closed theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in clo\text{-}th(\boldsymbol{L}_1)$.

  Note. Since the closure of a closed theory is itself by idempotency of closure, this inclusion monotonic function is the composition of the inclusion monotonic function

  $incl(\boldsymbol{L}) : \langle clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle th(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle$

  connecting TCL and strong LOT morphisms, and the closure function

  $clo(\boldsymbol{L}) : \langle th(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle \rightarrow \langle th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$,

  connecting strong and weak LOT morphisms.

- **Restricted Closure monotonicity:** For any language $\boldsymbol{L}$, the restricted-closure is a monotonic function

  $res\text{-}clo(\boldsymbol{L}) : \langle th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T} \supseteq_{\boldsymbol{L}} \check{\boldsymbol{T}}$ <u>implies</u> $res\text{-}clo(\boldsymbol{L})(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} res\text{-}clo(\boldsymbol{L})(\check{\boldsymbol{T}})$,

  for any pair of theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in th(\boldsymbol{L}_1)$.

  Proof. This is true, since closure is monotonic.

  Note. It is a near triviality that this restricted-closure monotonic function is the composition of the identity monotonic function

  $id(\boldsymbol{L}) : \langle th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle th(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle$

  connecting weak and strong LOT morphisms, and the restricted-closure monotonic function

  $res\text{-}clo(\boldsymbol{L}) : \langle th(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle \rightarrow \langle clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$

  connecting strong LOT and TCL morphisms.

- **Inclusion-Closure Adjointness:** For any language $\boldsymbol{L}$, the inclusion monotonic function is left adjoint to the restricted-closure monotonic function

  $\langle incl(\boldsymbol{L}), res\text{-}clo(\boldsymbol{L}) \rangle : \langle clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$,

  satisfying the adjointness condition

  $incl(\boldsymbol{L})(\check{\boldsymbol{T}}) \supseteq_{\boldsymbol{L}} \boldsymbol{T}$ <u>iff</u> $\check{\boldsymbol{T}} \supseteq_{\boldsymbol{L}} res\text{-}clo(\boldsymbol{L})(\boldsymbol{T})$

  for any closed theory $\check{\boldsymbol{T}} \in clo\text{-}th(\boldsymbol{L})$ and any theory $\boldsymbol{T} \in th(\boldsymbol{L})$.

- **Inverse image commutes with inclusion:**

  $incl(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1} = expr(\boldsymbol{f})^{-1} \cdot incl(\boldsymbol{L}_1)$;

  that is, $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2)'' = expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2)$ for all closed theory $\boldsymbol{T}_2 \in clo\text{-}th(\boldsymbol{L}_2)$.

  Comment: This states that inverse image preserves closed theories.

  Proof. Any closed theory $\boldsymbol{T}_2 \in clo\text{-}th(\boldsymbol{L}_2)$ can be written as the intersection $\boldsymbol{T}_2 = \cap\{\boldsymbol{M}_2' \mid \boldsymbol{M}_2 \models \boldsymbol{T}_2\}$ using the shorthand $\boldsymbol{M}_2' = max\text{-}th(\boldsymbol{L}_2)(\boldsymbol{M}_2)$. Hence, the inverse image is $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2) =$ $expr(\boldsymbol{f})^{-1}(\cap\{\boldsymbol{M}_2' \mid \boldsymbol{M}_2 \models \boldsymbol{T}_2\}) = \cap\{expr(\boldsymbol{f})^{-1}(\boldsymbol{M}_2') \mid \boldsymbol{M}_2 \models \boldsymbol{T}_2\} = \cap\{mod(\boldsymbol{f})(\boldsymbol{M}_2)' \mid \boldsymbol{M}_2 \models \boldsymbol{T}_2\}$, which is a closed source theory.

- **Direct image commutes with restricted closure:**

  $\wp expr(\boldsymbol{f}) \cdot res\text{-}clo(\boldsymbol{L}_2) = res\text{-}clo(\boldsymbol{L}_1) \cdot (\wp expr(\boldsymbol{f})(\text{-}))''$.

Proof: Let $\boldsymbol{T}_1 \in \mathit{th}(\boldsymbol{L}_1)$ be any theory in the source LOT. We want to prove the following identity:

$$\mathit{res\text{-}clo}(\boldsymbol{L}_2)(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1)) = (\wp\mathit{expr}(\boldsymbol{f})(\mathit{res\text{-}clo}(\boldsymbol{L}_1)(\boldsymbol{T}_1))'';$$

that is, $(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1))'' = (\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1''))''$ or $\boldsymbol{T}_2'' = \check{\boldsymbol{T}}_2''$, where $\boldsymbol{T}_2 = \wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1)$ and $\check{\boldsymbol{T}}_2 = (\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1'')$. Clearly, (1) the inclusion $\boldsymbol{T}_2'' \subseteq \check{\boldsymbol{T}}_2''$ holds. Now, $\boldsymbol{T}_1'' = \cap\{\boldsymbol{T}_{1,n} \in \mathit{clo\text{-}th}(\boldsymbol{L}_1) \mid \boldsymbol{T}_1 \subseteq \boldsymbol{T}_{1,n}\}$, where the $\boldsymbol{T}_{1,n}$ are all of the source theories that contain $\boldsymbol{T}_1$. Let $Ŧ_2$ be any closed target theory that contains $\boldsymbol{T}_2$, $\boldsymbol{T}_2 \subseteq Ŧ_2$. Since $\boldsymbol{T}_1 \subseteq \mathit{expr}(\boldsymbol{f})^{-1}(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1))$, the inverse image $\mathit{expr}(\boldsymbol{f})^{-1}(Ŧ_2)$ is a closed source theory that contains $\boldsymbol{T}_1$, $\boldsymbol{T}_1 \subseteq \mathit{expr}(\boldsymbol{f})^{-1}(Ŧ_2)$. Hence, it is one of the $\boldsymbol{T}_{1,n}$. Since, $\wp\mathit{expr}(\boldsymbol{f})(\mathit{expr}(\boldsymbol{f})^{-1}(Ŧ_2)) \subseteq Ŧ_2$, the direct image of this is contained in $Ŧ_2$, $\boldsymbol{T}_2 = \wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1) \subseteq \wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n}) = \wp\mathit{expr}(\boldsymbol{f})(\mathit{expr}(\boldsymbol{f})^{-1}(Ŧ_2)) \subseteq Ŧ_2$. And hence, $(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n}))''$ is contained in $Ŧ_2$, $\boldsymbol{T}_2 \subseteq (\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n}))'' \subseteq Ŧ_2$. Thus, $\cap_n\{(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n}))''\}$ is contained in $\boldsymbol{T}_2''$, $\cap_n\{(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n}))''\} \subseteq \boldsymbol{T}_2''$. But $\check{\boldsymbol{T}}_2 = \wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_1'') = \wp\mathit{expr}(\boldsymbol{f})(\cap_n\{\boldsymbol{T}_{1,n}\}) \subseteq \cap_n\{\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n})\} \subseteq \cap_n\{(\wp\mathit{expr}(\boldsymbol{f})(\boldsymbol{T}_{1,n}))''\} \subseteq \boldsymbol{T}_2''$, and hence (2) the inclusion $\check{\boldsymbol{T}}_2'' \subseteq \boldsymbol{T}_2''$. Putting (1) and (2) together, $\check{\boldsymbol{T}}_2'' \subseteq \boldsymbol{T}_2''$.

$$\begin{tikzcd}[column sep=large, row sep=large]
\boldsymbol{T}_1'' \quad \mathit{clo\text{-}th}(\boldsymbol{L}_1) \arrow[r, "\check{\boldsymbol{T}}_2 \quad (\wp\mathit{expr}(\boldsymbol{f})(\text{-}))''"] & \mathit{clo\text{-}th}(\boldsymbol{L}_2) \quad \check{\boldsymbol{T}}_2'' = \boldsymbol{T}_2'' \\
\boldsymbol{T}_1 \quad \mathit{th}(\boldsymbol{L}_1) \arrow[u, "\mathit{res\text{-}clo}(\boldsymbol{L}_1)"] \arrow[r, "\wp\mathit{expr}(\boldsymbol{f})"'] & \mathit{th}(\boldsymbol{L}_2) \quad \boldsymbol{T}_2 \arrow[u, "\mathit{res\text{-}clo}(\boldsymbol{L}_2)"']
\end{tikzcd}$$

Connecting the TCL and the Strong LOT Morphisms

Extending the Category of Theories

The inclusion and restricted-closure functions

$incl(\boldsymbol{L}) : clo\text{-}th(\boldsymbol{L}) \rightarrow th(\boldsymbol{L})$

$res\text{-}clo(\boldsymbol{L}) : th(\boldsymbol{L}) \rightarrow clo\text{-}th(\boldsymbol{L})$

are adjoint pairs of monotonic functions in two different ways

1. $\langle incl(\boldsymbol{L}), res\text{-}clo(\boldsymbol{L})\rangle : (clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}}) \rightarrow (th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}})$

and

2. $\langle incl(\boldsymbol{L}), res\text{-}clo(\boldsymbol{L})\rangle : (clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}}) \rightarrow (th(\boldsymbol{L}), \vdash_{\boldsymbol{L}})$.

The orders here are the complete (actually concept) lattice of closed theories $(clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}})$ with reverse subset order, the complete partial order of theories $(th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}})$ with reverse subset order, and the complete preorder of theories $(th(\boldsymbol{L}), \vdash_{\boldsymbol{L}})$ with the entailment order $\vdash_{\boldsymbol{L}} \subseteq th(\boldsymbol{L}) \times th(\boldsymbol{L})$: $\boldsymbol{T} \vdash_{\boldsymbol{L}} \boldsymbol{T'}$ when $clo(\boldsymbol{T}) \supseteq \boldsymbol{T'}$. We have already discussed the monotonic and adjointness constraints for the first adjunction. The monotonic constraints for the second adjunction are trivial; the adjointness constraint for the second adjunction is as follows.

- **Adjointness:**

  $incl(\boldsymbol{L})(\boldsymbol{T}_2) \vdash_{\boldsymbol{L}} \boldsymbol{T}_1$ <u>iff</u> $\boldsymbol{T}_2 \supseteq_{\boldsymbol{L}} res\text{-}clo(\boldsymbol{L})(\boldsymbol{T}_1)$

  for all closed theories $\boldsymbol{T}_2 \in clo\text{-}th(\boldsymbol{L})$ and all theories $\boldsymbol{T}_1 \in th(\boldsymbol{L})$. Both sides reduce to $\boldsymbol{T}_2 \supseteq \boldsymbol{T}_1$.

- **Inclusion monotonicity:**

  $\boldsymbol{T} \supseteq_{\boldsymbol{L}} \boldsymbol{T'}$ <u>iff</u> $clo(\boldsymbol{T}) \supseteq \boldsymbol{T'}$ <u>iff</u> $\boldsymbol{T} \vdash_{\boldsymbol{L}} \boldsymbol{T'}$ <u>implies</u> $incl(\boldsymbol{L})(\boldsymbol{T}) \vdash_{\boldsymbol{L}} incl(\boldsymbol{L})(\boldsymbol{T'})$.

  for any pair of closed theories $\boldsymbol{T}, \boldsymbol{T'} \in th(\boldsymbol{L})$. This is obvious true.

- **Restricted closure monotonicity:**

  $\boldsymbol{T} \vdash_{\boldsymbol{L}} \boldsymbol{T'}$ <u>iff</u> $clo(\boldsymbol{T}) \supseteq \boldsymbol{T'}$ <u>implies</u> $clo(\boldsymbol{L})(\boldsymbol{T}) \supseteq clo(\boldsymbol{L})(\boldsymbol{T'})$ <u>iff</u> $res\text{-}clo(\boldsymbol{L})(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} res\text{-}clo(\boldsymbol{L})(\boldsymbol{T'})$,

  for any pair of theories $\boldsymbol{T}, \boldsymbol{T'} \in th(\boldsymbol{L})$.

Because of the identities

- $res\text{-}clo(\boldsymbol{L})(incl(\boldsymbol{L})(\boldsymbol{T})) = \boldsymbol{T}$ for any closed theory $\boldsymbol{T}$, "the closure of a closed theory is itself".
- $incl(\boldsymbol{L})(res\text{-}clo(\boldsymbol{L})(\boldsymbol{T})) \equiv \boldsymbol{T}$ for any theory $\boldsymbol{T}$, "any theory is equivalent to its closure".

the second adjunction is an order-theoretic equivalence between the complete lattice of closed theories $(clo\text{-}th(\boldsymbol{L}), \supseteq_{\boldsymbol{L}})$ and the complete preorder of theories $(th(\boldsymbol{L}), \vdash_{\boldsymbol{L}})$.

Currently there is the category of theories Theory and the category of closed theories Closed-Theory. One idea for an extension to the IFF axiomatization is to define a category on theories that is categorically equivalent to the category of closed theories. This can obviously be done by defining theory morphisms with the following less restrictive constraint.

Definition: A *morphism* of theories $\boldsymbol{f} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ is a language morphism $base(\boldsymbol{f}) : base(\boldsymbol{T}_1) \rightarrow base(\boldsymbol{T}_2)$ that maps source axioms to target theorems: $base(\boldsymbol{f})(axm(\boldsymbol{T}_1)) \subseteq thm(\boldsymbol{T}_2)$.

Strong LOT Morphism Constraints

- **Monotonicity of direct-image:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the direct-image is a monotonic function

  $\wp expr(\boldsymbol{f}) : \langle th(\boldsymbol{L}_1), \vdash_{\boldsymbol{L}1} \rangle \rightarrow \langle th(\boldsymbol{L}_2), \vdash_{\boldsymbol{L}2} \rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T}_1 \vdash_{\boldsymbol{L}1} \check{\boldsymbol{T}}_1$ implies $\wp expr(\boldsymbol{f})(\boldsymbol{T}_1) \vdash_{\boldsymbol{L}2} \wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1)$,

  for any pair of source theories $\boldsymbol{T}_1, \check{\boldsymbol{T}}_1 \in th(\boldsymbol{L}_1)$.

  Proof.

  $\boldsymbol{T}_1 \vdash_{\boldsymbol{L}1} \check{\boldsymbol{T}}_1$

  iff $clo(\boldsymbol{L}_1)(\boldsymbol{T}_1) \supseteq_{\boldsymbol{L}1} \check{\boldsymbol{T}}_1$

  implies $clo(\boldsymbol{L}_2)(\wp expr(\boldsymbol{f})(clo(\boldsymbol{L}_1)(\boldsymbol{T}_1))) \supseteq_{\boldsymbol{L}2} clo(\boldsymbol{L}_2)(\wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1))$,

  iff $clo(\boldsymbol{L}_2)(\wp expr(\boldsymbol{f})(clo(\boldsymbol{L}_1)(\boldsymbol{T}_1))) \supseteq_{\boldsymbol{L}2} \wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1)$ since closure bounds identity,

  iff $clo(\boldsymbol{L}_2)(\wp expr(\boldsymbol{f})(\boldsymbol{T}_1)) \supseteq_{\boldsymbol{L}2} \wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1)$ which is the special property proven before,

  iff $\wp expr(\boldsymbol{f})(\boldsymbol{T}_1) \vdash_{\boldsymbol{L}2} \wp expr(\boldsymbol{f})(\check{\boldsymbol{T}}_1)$,

- **Monotonicity of closure-inverse-image:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the inverse-image is a monotonic function

  $expr(\boldsymbol{f})^{-1}((\text{-})'') : \langle th(\boldsymbol{L}_2), \vdash_{\boldsymbol{L}2} \rangle \rightarrow \langle th(\boldsymbol{L}_1), \vdash_{\boldsymbol{L}1} \rangle$,

  satisfying the monotonicity condition

  $\boldsymbol{T}_2 \vdash_{\boldsymbol{L}2} \check{\boldsymbol{T}}_2$ implies $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'') \vdash_{\boldsymbol{L}1} expr(\boldsymbol{f})^{-1}(\check{\boldsymbol{T}}_2'')$,

  for any pair of target theories $\boldsymbol{T}_2, \check{\boldsymbol{T}}_2 \in th(\boldsymbol{L}_2)$.

  Proof.

  $\boldsymbol{T}_2 \vdash_{\boldsymbol{L}2} \check{\boldsymbol{T}}_2$

  iff $clo(\boldsymbol{L}_2)(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \check{\boldsymbol{T}}_2$

  iff $clo(\boldsymbol{L}_2)(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} clo(\boldsymbol{L}_2)(\check{\boldsymbol{T}}_2)$, since closure is monotonic and idempotent

  implies $expr(\boldsymbol{f})^{-1}(clo(\boldsymbol{L}_1)(\boldsymbol{T}_1)) \supseteq_{\boldsymbol{L}1} expr(\boldsymbol{f})^{-1}(clo(\boldsymbol{L}_2)(\check{\boldsymbol{T}}_2))$, since inverse image is monotonic

  iff $clo(\boldsymbol{L}_1)(expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'')) \supseteq_{\boldsymbol{L}1} expr(\boldsymbol{f})^{-1}(\check{\boldsymbol{T}}_2'')$, since closure bounds identity

  iff $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'') \vdash_{\boldsymbol{L}1} expr(\boldsymbol{f})^{-1}(\check{\boldsymbol{T}}_2'')$.

- **Inverse-Direct Adjointness:** For any language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, the closure-inverse-image monotonic function is left adjoint to the direct-image monotonic function

  $\langle expr(\boldsymbol{f})^{-1}((\text{-})''), \wp expr(\boldsymbol{f}) \rangle : \langle th(\boldsymbol{L}_2), \vdash_{\boldsymbol{L}2} \rangle \rightarrow \langle th(\boldsymbol{L}_1), \vdash_{\boldsymbol{L}1} \rangle$,

  satisfying the adjointness condition

  $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'') \vdash_{\boldsymbol{L}1} \boldsymbol{T}_1$ iff $\boldsymbol{T}_2 \vdash_{\boldsymbol{L}2} \wp expr(\boldsymbol{f})(\boldsymbol{T}_1)$

  for all target theories $\boldsymbol{T}_2 \in th(\boldsymbol{L}_2)$ and all source theories $\boldsymbol{T}_1 \in th(\boldsymbol{L}_1)$.

  Proof.

  $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'') \vdash_{\boldsymbol{L}1} \boldsymbol{T}_1$

  iff $(expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2''))'' \supseteq_{\boldsymbol{L}1} \boldsymbol{T}_1$

  iff $(expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2''))'' \supseteq_{\boldsymbol{L}1} \boldsymbol{T}_1''$ by the monotonicity and idempotency of closure

  iff $\boldsymbol{T}_2'' \supseteq_{\boldsymbol{L}2} \wp expr(\boldsymbol{f})(\boldsymbol{T}_1'')$ by the TCL adjointness,

  for any target theory $\boldsymbol{T}_2 \in th(\boldsymbol{L}_2)$ and source theory $\boldsymbol{T}_1 \in th(\boldsymbol{L}_1)$.

- **Model embedding condition:**

  $mod(\boldsymbol{f}) \cdot max\text{-}th(\boldsymbol{L}_1) = max\text{-}th(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1}((\text{-})'')$;

  Proof. This follows from the similar condition for TCL morphisms, since (1) maximal theory is the same as model embedding and (2) the maximal theory is closed.

- **Expression embedding condition:**

  $expr(f) \cdot \{\text{-}\}_{expr(L2)} = \{\text{-}\}_{expr(L1)} \cdot \wp expr(f)$.

  Proof. True, since direct image preserves singletons.

TCL – Strong LOT Morphism Connection Constraints

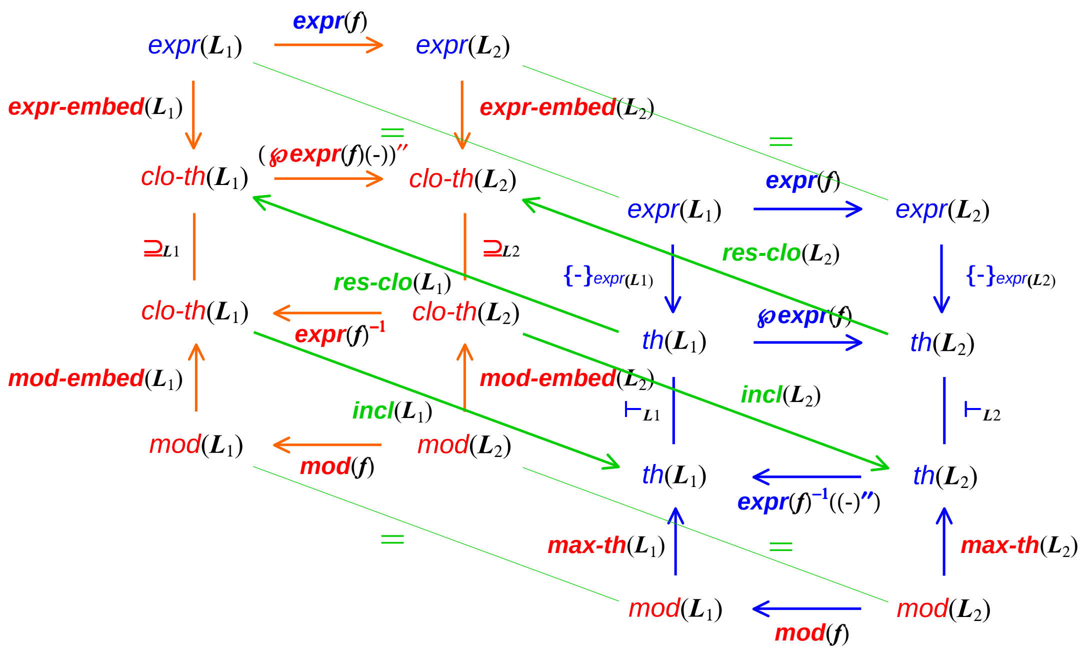

Figure 10: Connecting the TCL and strong LOT Morphisms

- **Inclusion monotonicity:** For any language L, the inclusion is a monotonic function

  $incl(L) : \langle clo\text{-}th(L), \sqsupseteq_L\rangle \rightarrow \langle th(L), \vdash_L\rangle$,

  satisfying the monotonicity condition

  $T \sqsupseteq_L \check{T}$ <u>implies</u> $incl(L)(T) \vdash_L incl(L)(\check{T})$,

  for any pair of closed theories $T, \check{T} \in clo\text{-}th(L_1)$.

  Proof. This is true, since the closure is the identity on closed theories.

- **Restricted Closure monotonicity:** For any language L, the restricted-closure is a monotonic function

  $res\text{-}clo(L) : \langle th(L), \vdash_L\rangle \rightarrow \langle clo\text{-}th(L), \sqsupseteq_L\rangle$,

  satisfying the monotonicity condition

  $T \vdash_L \check{T}$ <u>implies</u> $res\text{-}clo(L)(T) \sqsupseteq_L res\text{-}clo(L)(\check{T})$,

  for any pair of theories $T, \check{T} \in th(L_1)$.

  Proof. This is true, since closure is monotonic and idempotent.

- **Inclusion-Closure Adjointness:** For any language L, the inclusion monotonic function is left adjoint to the restricted-closure monotonic function

  $\langle incl(L), res\text{-}clo(L)\rangle : \langle clo\text{-}th(L), \sqsupseteq_L\rangle \rightarrow \langle th(L), \vdash_L\rangle$,

  satisfying the adjointness condition

  $incl(L)(\check{T}) \vdash_L T$ <u>iff</u> $\check{T} \sqsupseteq_L res\text{-}clo(L)(T)$

  for any closed theory $\check{T} \in clo\text{-}th(L)$ and any theory $T \in th(L)$.

Proof.

$incl(\boldsymbol{L})(\check{\boldsymbol{T}}) \vdash_{\boldsymbol{L}} \boldsymbol{T}$ iff $\check{\boldsymbol{T}} \vdash_{\boldsymbol{L}} \boldsymbol{T}$ iff $clo(\boldsymbol{L})(\check{\boldsymbol{T}}) \supseteq_{\boldsymbol{L}} \boldsymbol{T}$ iff $\check{\boldsymbol{T}} \supseteq_{\boldsymbol{L}} \boldsymbol{T}$ iff $\check{\boldsymbol{T}} \supseteq_{\boldsymbol{L}} clo(\boldsymbol{L})(\boldsymbol{T})$, since closure bounds identity.

- **Inverse image commutes with inclusion:**

  $$incl(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1}((\text{-})'') = expr(\boldsymbol{f})^{-1} \cdot incl(\boldsymbol{L}_1);$$

  that is, $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'') = expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2)$ for all closed theory $\boldsymbol{T}_2 \in clo\text{-}th(\boldsymbol{L}_2)$.

- **Direct image commutes with restricted closure:**

  $$\wp expr(\boldsymbol{f}) \cdot res\text{-}clo(\boldsymbol{L}_2) = res\text{-}clo(\boldsymbol{L}_1) \cdot (\wp expr(\boldsymbol{f})(\text{-}))''.$$

  Proof: Let $\boldsymbol{T}_1 \in th(\boldsymbol{L}_1)$ be any source theory. We want to prove the following identity:

  $$res\text{-}clo(\boldsymbol{L}_2)(\wp expr(\boldsymbol{f})(\boldsymbol{T}_1)) = (\wp expr(\boldsymbol{f})(res\text{-}clo(\boldsymbol{L}_1)(\boldsymbol{T}_1))'';$$

  However, this was proved before.

Equivalence

Inclusion monotonicity means that reverse subset inclusion for closed theories is compatible with entailment order for theories in general:

$\boldsymbol{T} \supseteq_{\boldsymbol{L}} \boldsymbol{T}'$ iff $\boldsymbol{T} \vdash_{\boldsymbol{L}} \boldsymbol{T}'$.

for any pair of closed theories $\boldsymbol{T}, \boldsymbol{T}' \in \mathit{th}(\boldsymbol{L})$.

A *morphism* of theories $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ has a language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, for $\boldsymbol{f} = \mathit{base}(\boldsymbol{g})$, $\boldsymbol{L}_1 = \mathit{base}(\boldsymbol{T}_1)$ and $\boldsymbol{L}_2 = \mathit{base}(\boldsymbol{T}_2)$, that maps source axioms to target theorems:

$\mathit{axm}(\boldsymbol{T}_2) \vdash_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{axm}(\boldsymbol{T}_1))$ <u>iff</u> $\mathit{thm}(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{axm}(\boldsymbol{T}_1))$

<u>iff</u> $\mathit{inv}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_2)) \supseteq_{\boldsymbol{L}1} \mathit{axm}(\boldsymbol{T}_1))$ <u>iff</u> $\mathit{inv}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_2)) \supseteq_{\boldsymbol{L}1} \mathit{thm}(\boldsymbol{T}_1))$

<u>iff</u> $\mathit{thm}(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_1))$ <u>iff</u> $\mathit{axm}(\boldsymbol{T}_2) \vdash_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_1))$.

A *morphism* of closed theories $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$ is a language morphism $\boldsymbol{f} : \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, for $\boldsymbol{f} = \mathit{base}(\boldsymbol{g})$, $\boldsymbol{L}_1 = \mathit{base}(\boldsymbol{T}_1)$ and $\boldsymbol{L}_2 = \mathit{base}(\boldsymbol{T}_2)$, that maps source axioms (= theorems) to target axioms (= theorems):

$\mathit{axm}(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{axm}(\boldsymbol{T}_1))$ <u>iff</u> $\mathit{thm}(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}2} \mathit{dir}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_1))$ <u>iff</u> $\mathit{nv}(\boldsymbol{f})(\mathit{thm}(\boldsymbol{T}_2)) \supseteq_{\boldsymbol{L}1} \mathit{thm}(\boldsymbol{T}_1))$.

Hence, the closed theories and their theory morphisms form a full subcategory

Closed-Theory $\subseteq$ Theory.

This inclusion is objectified with the inclusion functor

incl : Closed-Theory $\subseteq$ Theory.

The restricted-closure operator goes in the other direction. Does this form a functor? Consider a morphism of theories $\boldsymbol{g} : \boldsymbol{T}_1 \rightarrow \boldsymbol{T}_2$. Apply the restricted-closure operator to the theories. Since $\mathit{axm}(\mathit{clo}(\boldsymbol{T})) = \mathit{thm}(\mathit{clo}(\boldsymbol{T})) = \mathit{thm}(\boldsymbol{T})$, the underlying language morphism forms a theory morphism between the closures. Hence, there is a closure functor

res-clo : Theory $\subseteq$ Closed-Theory

that maps theories to their closure and preserves the underlying language morphism. Since the closure of a closed theory is itself, the functor composition of inclusion with closure is the identity functor on the closed theory subcategory: $\mathit{incl} \circ \mathit{res\text{-}clo} = \mathit{id}_{\text{Closed-Theory}}$. Consider the other direction. For any arbitrary theory $\boldsymbol{T}$ with underlying language $\boldsymbol{L}$, how do the two theories $\boldsymbol{T} = \langle \boldsymbol{L}, \mathit{axm}(\boldsymbol{T}) \rangle$ and $\mathit{res\text{-}clo}(\boldsymbol{T}) = \langle \boldsymbol{L}, \mathit{thm}(\boldsymbol{T}) \rangle$ relate to each other (in the entire theory category)? Taking $\boldsymbol{T}$ as source and $\mathit{res\text{-}clo}(\boldsymbol{T})$ as target the identity language morphisms $\mathit{id}_{\boldsymbol{L}}$ underlies a *connection* theory morphism $\eta_{\boldsymbol{T}} : \boldsymbol{T} \rightarrow \mathit{res\text{-}clo}(\mathit{incl}(\boldsymbol{T}))$, since

$\mathit{axm}(\mathit{clo}(\boldsymbol{T})) \vdash_{\boldsymbol{L}} \mathit{axm}(\boldsymbol{T})$ <u>iff</u> $\mathit{thm}(\mathit{clo}(\boldsymbol{T})) \supseteq_{\boldsymbol{L}} \mathit{axm}(\boldsymbol{T})$

<u>iff</u> $\mathit{thm}(\mathit{clo}(\boldsymbol{T})) \supseteq_{\boldsymbol{L}} \mathit{thm}(\boldsymbol{T})$ <u>iff</u> $\mathit{axm}(\mathit{clo}(\boldsymbol{T})) \vdash_{\boldsymbol{L}} \mathit{thm}(\boldsymbol{T})$.

Taking $\mathit{clo}(\boldsymbol{T})$ as source and $\boldsymbol{T}$ as target the identity language morphisms $\mathit{id}_{\boldsymbol{L}}$ underlies a theory morphism $\eta_{\boldsymbol{T}}^{-1} : \mathit{clo}(\boldsymbol{T}) \rightarrow \boldsymbol{T}$, since

$\mathit{axm}(\boldsymbol{T}) \vdash_{\boldsymbol{L}} \mathit{axm}(\mathit{clo}(\boldsymbol{T}))$ <u>iff</u> $\mathit{thm}(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} \mathit{axm}(\mathit{clo}(\boldsymbol{T}))$

<u>iff</u> $\mathit{thm}(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} \mathit{thm}(\mathit{clo}(\boldsymbol{T})))$ <u>iff</u> $\mathit{axm}(\boldsymbol{T}) \vdash_{\boldsymbol{L}} \mathit{thm}(\mathit{clo}(\boldsymbol{T}))$.

$$\begin{tikzcd}
\boldsymbol{T}_1 \arrow[r, "\eta_{\boldsymbol{T}1}"] \arrow[d, "\boldsymbol{g}"'] & \mathit{res\text{-}clo}(\mathit{incl}(\boldsymbol{T}_1)) \arrow[d, "\mathit{res\text{-}clo}(\mathit{incl}(\boldsymbol{g}))"] \\
\boldsymbol{T}_2 \arrow[r, "\eta_{\boldsymbol{T}2}"'] & \mathit{res\text{-}clo}(\mathit{incl}(\boldsymbol{T}_2))
\end{tikzcd}$$

Figure 11: Naturality

Clearly, these two morphism are inverse to each other and $\boldsymbol{T}$ is isomorphic to its closure $\mathit{clo}(\boldsymbol{T})$ in the entire category of theories Theory: $\boldsymbol{T} \cong \mathit{clo}(\boldsymbol{T})$. The connection theory (iso)morphism $\eta_{\boldsymbol{T}} : \boldsymbol{T} \rightarrow \mathit{res\text{-}clo}(\mathit{incl}(\boldsymbol{T}))$, is the $\boldsymbol{T}^{\text{th}}$ component of a natural transformation $\eta : \mathit{id}_{\text{Theory}} \Rightarrow \mathit{res\text{-}clo} \circ \mathit{incl}$ (Figure 11). We want to show that the restricted-closure functor *res-clo* is left adjoint to the inclusion functor *incl* with unit $\eta : \mathit{id}_{\text{Theory}} \Rightarrow \mathit{res\text{-}clo} \circ \mathit{incl}$ and counit $1 : \mathit{incl} \circ \mathit{res\text{-}clo} \Rightarrow \mathit{id}_{\text{Closed-Theory}}$. In fact, this adjunction is an equivalence of categories.

$$\begin{gathered}\textit{res-clo} \dashv \textit{incl}\\ \eta, 1\end{gathered}$$

$$\begin{tikzcd}
\mathbf{T} \arrow[r, "\eta_{\mathbf{T}}"] \arrow[dr, "g"'] & \textit{incl}(\textit{res-clo}(\mathbf{T})) \arrow[d, "\textit{incl}(\mathbf{h})"] & \textit{res-clo}(\mathbf{T}) \arrow[d, "\mathbf{h}"] \\
& \textit{incl}(\mathbf{K}) & \mathbf{Ŧ}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Ŧ} \arrow[r, phantom, "="] & \textit{res-clo}(\textit{incl}(\mathbf{Ŧ})) & \textit{incl}(\mathbf{Ŧ}) \\
& \textit{res-clo}(\mathbf{L}) \arrow[ul, "\mathbf{h}"] \arrow[u, "\textit{res-clo}(\mathbf{g})"] & \mathbf{T} \arrow[u, "\mathbf{g}"']
\end{tikzcd}$$

Figure 12a: Universal Morphism **Figure 12b: Couniversal Morphism**

The Closure Reflection γ

We want to verify that $\gamma = \langle \textit{res-clo}, \textit{incl}, \eta, 1\rangle$: Theory → Closed-Theory forms an adjunction (actually an equivalence). This adjunction links the contexts of theories and closed theories by restricted closure – the category Closed-Theory is an equivalent subcategory of the category Theory with the restricted-closure *res-clo* : Theory → Closed-Theory as the reflector. To do this we need to verify that the closed theory inclusion functor

incl : Closed-Theory → Theory

is the right adjoint left inverse (rali) of the reflection $\gamma = \langle \textit{res-clo}, \textit{incl}, \eta, 1\rangle$: Theory → Closed-Theory. In other words, given any theory $\mathbf{T}$, we want to verify that the closure $\textit{clo}(\mathbf{T})$ is the free theory over $\mathbf{T}$. More abstractly, we want to verify that the closure theory and the connection theory morphism participate in an adjunction where

- the restricted-closure functor *res-clo* : Theory → Closed-Theory is the left adjoint,
- the inclusion functor *incl* : Closed-Theory → Theory is the right adjoint,
- for any theory $\mathbf{T}$ the connection theory morphism $\eta_{\mathbf{T}} : \mathbf{T} \rightarrow \textit{incl}(\textit{res-clo}(\mathbf{T}))$ with the identity underlying language morphism $\textit{base}(\eta_{\mathbf{T}}) = \textit{id}_{\textit{base}(\mathbf{T})}$ is the $\mathbf{T}^{\text{th}}$ component of the unit natural transformation $\eta : \textit{id}_{\text{Theory}} \Rightarrow \textit{res-clo} \circ \textit{incl}$, and
- the counit natural transformation is the identity $1_{\text{Closed-Theory}} = \textit{incl} \circ \textit{res-clo}$.

[**Universal morphism**] More concretely, we want to verify (Figure 12a) that for any theory $\mathbf{T}$, the pair $\langle \eta_{\mathbf{T}}, \textit{res-clo}(\mathbf{T})\rangle$ consisting the closure $\textit{clo}(\mathbf{T})$ and the connection theory morphism $\eta_{\mathbf{T}} : \mathbf{T} \rightarrow \textit{incl}(\textit{res-clo}(\mathbf{T}))$, forms a universal morphism from $\mathbf{T}$ to *incl*. To verify this, we need to show that for every pair $\langle \mathbf{g}, \mathbf{Ŧ}\rangle$ consisting of a closed theory $\mathbf{Ŧ}$ and a theory morphism $\mathbf{g} : \mathbf{T} \rightarrow \mathbf{Ŧ}$, there is a unique theory morphism (between closed theories) $\mathbf{h} : \textit{clo}(\mathbf{T}) \rightarrow \mathbf{Ŧ}$ with $\eta_{\mathbf{T}} \cdot \textit{incl}(\mathbf{h}) = \mathbf{g}$.

Let $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{Ł}$ denote the underlying base language morphism for the theory morphism $\mathbf{g} : \mathbf{T} \rightarrow \mathbf{Ŧ}$, with $\mathbf{f} = \textit{base}(\mathbf{g})$, $\mathbf{L} = \textit{base}(\mathbf{T})$ and $\mathbf{Ł} = \textit{base}(\mathbf{Ŧ})$. We want to show that $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{Ł}$ is the underlying base language morphism for a (unique) theory morphism $\mathbf{h} : \textit{clo}(\mathbf{T}) \rightarrow \mathbf{Ŧ}$. But based upon the equivalence

$$\textit{axm}(\mathbf{Ŧ}) \vdash_{\textit{base}(\mathbf{Ŧ})} \textit{dir}(\mathbf{f})(\textit{axm}(\mathbf{T})) \;\underline{\text{iff}}\; \textit{axm}(\mathbf{Ŧ}) \vdash_{\textit{base}(\mathbf{Ŧ})} \textit{dir}(\mathbf{f})(\textit{thm}(\mathbf{T}))$$

and the fact that $\textit{axm}(\textit{clo}(\mathbf{T})) = \textit{thm}(\mathbf{T})$, this is immediately seen to be true. The equality $\eta_{\mathbf{L}} \cdot \textit{incl}(\mathbf{h}) = \mathbf{g}$ is trivially true.

[**Couniversal morphism**] Alternately, we want to verify (Figure 12b) that for any closed theory $\mathbf{Ŧ}$, the pair $\langle \textit{incl}(\mathbf{Ŧ}), \textit{id}_{\mathbf{Ŧ}}\rangle$ consisting of the closed theory $\mathbf{Ŧ}$ included as a theory $\textit{incl}(\mathbf{Ŧ})$ and the identity theory morphism (between closed theories) $\textit{id}_{\mathbf{Ŧ}} : \textit{res-clo}(\textit{incl}(\mathbf{Ŧ})) \rightarrow \mathbf{Ŧ}$, forms a couniversal morphism to $\mathbf{Ŧ}$ from *res-clo*. To verify this, we need to show that for every pair $\langle \mathbf{T}, \mathbf{h}\rangle$ consisting of a theory $\mathbf{T}$ and a theory morphism (between closed theories) $\mathbf{h} : \textit{res-clo}(\mathbf{T}) \rightarrow \mathbf{Ŧ}$, there is a unique theory morphism $\mathbf{g} : \mathbf{T} \rightarrow \textit{incl}(\mathbf{Ŧ})$ with $\textit{res-clo}(\mathbf{g}) = \mathbf{h}$.

Let ***f*** : ***L*** → ***Ł*** denote the underlying base language morphism for the theory morphism (between closed theories) ***h*** : *clo*(***T***) → ***Ŧ***, with ***f*** = *base*(***h***), ***L*** = *base*(*clo*(***T***)) = *base*(***T***) and ***Ł*** = *base*(***Ŧ***). We want to show that ***f*** : ***L*** → ***Ł*** is the underlying base language morphism for a (unique) theory morphism ***g*** : ***T*** → ***Ŧ***. But (again) based upon the equivalence

axm(***Ŧ***) $\vdash_{base(Ŧ)}$ *dir*(***f***)(*axm*(***T***)) <u>iff</u> *axm*(***Ŧ***) $\vdash_{base(Ŧ)}$ *dir*(***f***)(*thm*(***T***))

and the fact that *axm*(*clo*(***T***)) = *thm*(***T***), this is immediately seen to be true. The equality *res-clo*(***g***) = ***h***. is trivially true.

Connecting the Strong LOT and Weak LOT Morphisms

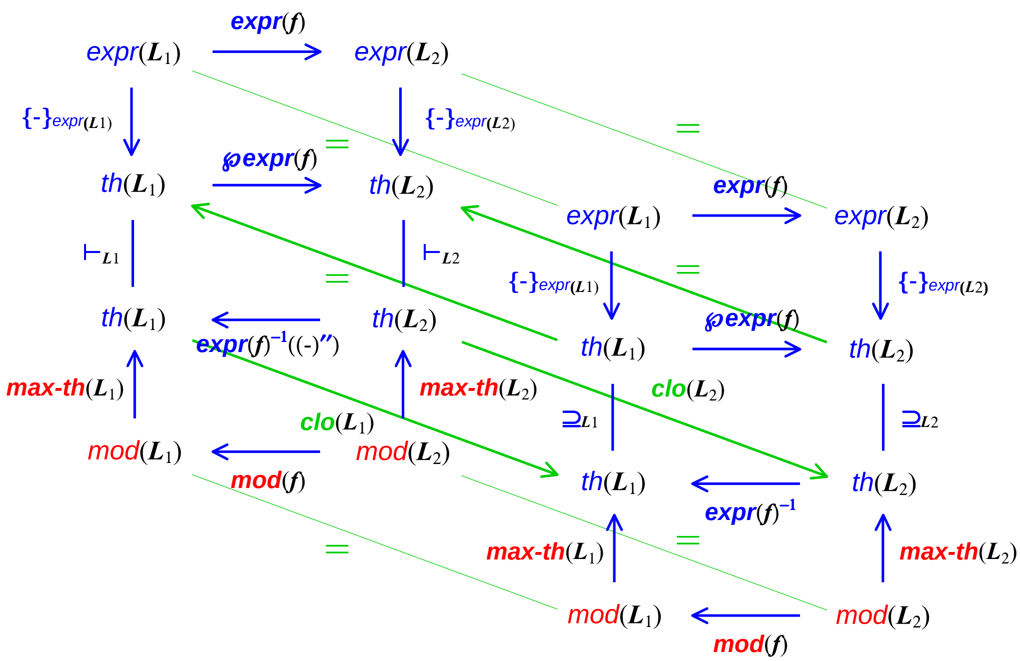

Figure 9: Connecting the weak and strong LOT Morphisms

Strong LOT - Weak LOT Morphism Connection Constraints

- **Closure monotonicity:** For any language $\boldsymbol{L}$, the closure is a monotonic function

  $$\mathit{clo}(\boldsymbol{L}) : \langle \mathit{th}(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$$

  satisfying the monotonicity condition

  $$\boldsymbol{T} \vdash_{\boldsymbol{L}} \check{\boldsymbol{T}} \ \underline{\text{implies}}\ \mathit{clo}(\boldsymbol{L})(\boldsymbol{T}) \supseteq_{\boldsymbol{L}} \mathit{clo}(\boldsymbol{L})(\check{\boldsymbol{T}}),$$

  for any pair of theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in \mathit{th}(\boldsymbol{L}_1)$.

  Proof. This is true, since closure is monotonic and idempotent.

- **Identity monotonicity:** For any language $\boldsymbol{L}$, the theory identity is a monotonic function

  $$\mathit{id}(\boldsymbol{L}) : \langle \mathit{th}(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle,$$

  satisfying the monotonicity condition

  $$\boldsymbol{T} \supseteq_{\boldsymbol{L}} \check{\boldsymbol{T}} \ \underline{\text{implies}}\ \boldsymbol{T} \vdash_{\boldsymbol{L}} \check{\boldsymbol{T}},$$

  for any pair of theories $\boldsymbol{T}, \check{\boldsymbol{T}} \in \mathit{th}(\boldsymbol{L}_1)$.

  Proof. This is true, since closure is identity bounding.

- **Closure-Identity Adjointness:** For any language $\boldsymbol{L}$, the closure monotonic function is left adjoint to the identity monotonic function

  $$\langle \mathit{clo}(\boldsymbol{L}), \mathit{id}(\boldsymbol{L}) \rangle : \langle \mathit{th}(\boldsymbol{L}), \vdash_{\boldsymbol{L}} \rangle \rightarrow \langle \mathit{th}(\boldsymbol{L}), \supseteq_{\boldsymbol{L}} \rangle$$

  satisfying the adjointness condition

  $$\mathit{clo}(\boldsymbol{L})(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}} \boldsymbol{T}_1 \ \underline{\text{iff}}\ \boldsymbol{T}_2 \vdash_{\boldsymbol{L}} \boldsymbol{T}_1,$$

  for any two theories $\boldsymbol{T}_2, \boldsymbol{T}_1 \in \mathit{th}(\boldsymbol{L})$.

  Proof. This result can be directly proven using the closure properties above.

$clo(\boldsymbol{L})(\boldsymbol{T}_2) \supseteq_{\boldsymbol{L}} \boldsymbol{T}_1$

<u>iff</u> $\boldsymbol{T}_2 \vdash_{\boldsymbol{L}} \boldsymbol{T}_1$

<u>iff</u> $\boldsymbol{T}_2 \vdash_{\boldsymbol{L}} id(\boldsymbol{L})(\boldsymbol{T}_1)$.

- **Inverse image commutes with closure:**

  $clo(\boldsymbol{L}_2) \cdot expr(\boldsymbol{f})^{-1} = expr(\boldsymbol{f})^{-1}((\text{-})'') \cdot clo(\boldsymbol{L}_1)$;

  that is, $expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2'') = (expr(\boldsymbol{f})^{-1}(\boldsymbol{T}_2''))''$ for all theory $\boldsymbol{T}_2 \in th(\boldsymbol{L}_2)$.

  Proof: This is true, since the inverse image preserves closure.

- **Direct image commutes with identity:**

  $\wp expr(\boldsymbol{f}) \cdot id(\boldsymbol{L}_2) = id(\boldsymbol{L}_1) \cdot \wp expr(\boldsymbol{f})$.