Colimits in a Nutshell

Colimits are extensively used in the IFF to represent various notions† in the "lattice of theories" and the "library of modules" that have been deemed necessary to support a modularized framework for the SUO. In this small document, we abstract from, paraphrase and rearrange some of the discussion of colimits in Mac Lane's book *Categories for the Working Mathematician*.

Let $\boldsymbol{A}$ be a category. In the IFF, $\boldsymbol{A}$ could be any of the categories: Set, Hypergraph, Language, Theory, Model or Logic. In the following discussion the ambient category $\boldsymbol{A}$ remains fixed until the last paragraph where it is allowed to vary. Table 1 describes the kinds of (increasingly more general) colimits possible.

Table 1: Kinds of Colimits

Colimit Functor	Morphisms	Comments
$col_{\boldsymbol{G}} : \lvert\boldsymbol{A}\rvert^{\boldsymbol{G}} \to \boldsymbol{A}$	morphism $\alpha : \boldsymbol{D}' \Rightarrow \boldsymbol{D}$	$\boldsymbol{A}$ and $\boldsymbol{G}$ are fixed
$col : \langle \mathrm{Gph} \downarrow \lvert\boldsymbol{A}\rvert \rangle \to \boldsymbol{A}$	transformation $\boldsymbol{H} : (\boldsymbol{G}', \boldsymbol{D}') \to (\boldsymbol{G}, \boldsymbol{D})$	$\boldsymbol{A}$ is fixed, $\boldsymbol{G}$ varies
$col : \langle \mathrm{Gph} \Downarrow \lvert\boldsymbol{A}\rvert \rangle \to \boldsymbol{A}$	colax transformation $\mathsf{H} = (\boldsymbol{H}, \alpha) : (\boldsymbol{G}', \boldsymbol{D}') \to (\boldsymbol{G}, \boldsymbol{D})$	$\boldsymbol{A}$ is fixed, $\boldsymbol{G}$ varies

Diagrams and Diagram Morphisms

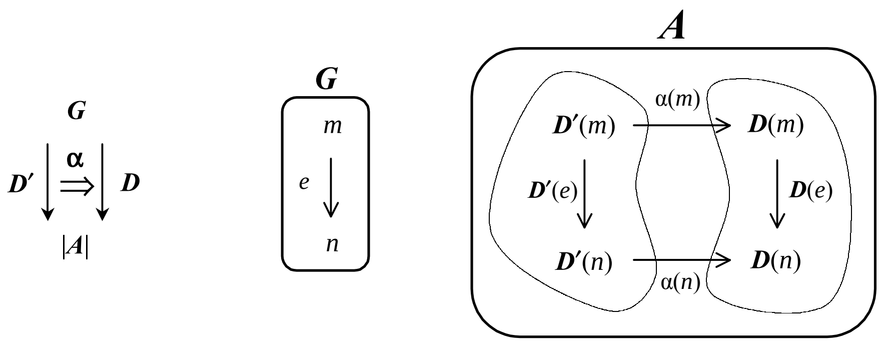


Figure 1: Diagrams and diagram morphisms

- A $\boldsymbol{G}$-indexed (small) *diagram* $\boldsymbol{D} : \boldsymbol{G} \to \lvert\boldsymbol{A}\rvert$ in the ambient category $\boldsymbol{A}$ is conceptually a graph morphism from the (small) graph $\boldsymbol{G}$ used for indexing to the (large) underlying graph of the category $\boldsymbol{A}$. It has the following components: an *index* or *shape* $\boldsymbol{G}$, a function from $\boldsymbol{G}$-nodes $n \in \boldsymbol{G}$ to $\boldsymbol{A}$-*objects* $\boldsymbol{D}(n)$, and a function from $\boldsymbol{G}$-edges $e : m \to n$ to $\boldsymbol{A}$-*morphisms* $\boldsymbol{D}(e) : \boldsymbol{D}(m) \to \boldsymbol{D}(n)$ that preserves source and target. The diagram $\boldsymbol{D}$ gives us a picture of $\boldsymbol{G}$ in $\boldsymbol{A}$. It is an object of $\boldsymbol{A}^{\lvert\boldsymbol{G}\rvert}$, the category of $\boldsymbol{G}$-indexed diagrams and diagram morphisms in $\boldsymbol{A}$.
- A *diagram morphism* $\alpha : \boldsymbol{D}' \Rightarrow \boldsymbol{D}$, from *source* diagram $\boldsymbol{D}' : \boldsymbol{G} \to \boldsymbol{A}$ to *target* diagram $\boldsymbol{D} : \boldsymbol{G} \to \boldsymbol{A}$, has a function that assigns to each node $n \in \boldsymbol{G}$ a *component* $\boldsymbol{A}$-morphism $\alpha(n) : \boldsymbol{D}'(n) \to \boldsymbol{D}(n)$, where every edge $e : m \to n$ in $\boldsymbol{G}$ yields a commutative diagram $\alpha(m) \cdot \boldsymbol{D}(e) = \boldsymbol{D}'(e) \cdot \alpha(n)$. When this holds we say that $\alpha(n) : \boldsymbol{D}'(n) \to \boldsymbol{D}(n)$ is *natural* in n. If we think of the diagram $\boldsymbol{D}$ as giving a picture of $\boldsymbol{G}$ in $\boldsymbol{A}$, then a diagram morphism α is a set of $\boldsymbol{A}$-morphisms that translate the picture $\boldsymbol{D}'$ to the picture $\boldsymbol{D}$.

† Such notions include forming the sum, quotient or meet of theories (modules), forming the sum of diagrams of theories (libraries of modules), axiomatizing the notions of monocosmic and polycosmic diagrams of theories, etc.

Colimits

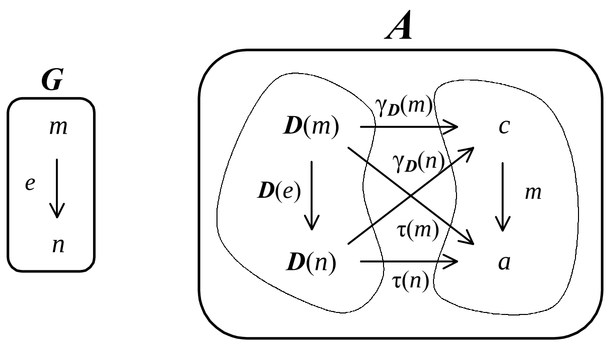


Figure 2: Cocones and colimiting cocones

- The *delta* functor $\Delta_{\boldsymbol{G}} : \boldsymbol{A} \rightarrow |\boldsymbol{A}|^{\boldsymbol{G}}$ indexes the constant $\boldsymbol{G}$-shaped diagrams. It sends each object $a \in \boldsymbol{A}$ to the constant diagram $\Delta_{\boldsymbol{G}}(a) : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$, which has the value a at every node $n \in \boldsymbol{G}$ and the value id_a at every edge e of $\boldsymbol{G}$. The delta functor sends each morphism $g : a_1 \rightarrow a_2$ of $\boldsymbol{A}$ to the constant diagram morphism $\Delta_{\boldsymbol{G}}(g) : \Delta_{\boldsymbol{G}}(a_1) \Rightarrow \Delta_{\boldsymbol{G}}(a_2) : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$, which has the same value g at each node $n \in \boldsymbol{G}$.
- A *cocone* $\tau : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(a)$ is a diagram morphism from a *base* diagram $\boldsymbol{D}$ to (the constant diagram $\Delta_{\boldsymbol{G}}(a)$ for) an *opvertex* $\boldsymbol{A}$-object a. As a diagram morphism τ has a function that assigns to each node $n \in \boldsymbol{G}$ a *component* $\boldsymbol{A}$-morphism $\tau(n) : \boldsymbol{D}(n) \rightarrow a$ that is natural in n; that is, every edge $e : m \rightarrow n$ in $\boldsymbol{G}$ yields a commutative diagram $\tau(m) = \boldsymbol{D}(e) \cdot \tau(n)$; in words, "the components commute with the base diagram".
- A *colimiting cocone* $\gamma_{\boldsymbol{D}} : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(c)$ is a cocone which is universal: for any cocone $\tau : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(a)$ from a base diagram $\boldsymbol{D}$ to an opvertex $\boldsymbol{A}$-object a, there is a unique *mediator* $\boldsymbol{A}$-morphism $m : c \rightarrow a$ that satisfies the diagram morphism composition identity $\tau = \gamma_{\boldsymbol{D}} \bullet \Delta_{\boldsymbol{G}}(m)$. The opvertex $\boldsymbol{A}$-object c is called "the" *colimit* and is symbolized as $c = \mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})$; it is unique up to $\boldsymbol{A}$-isomorphism. For each node $n \in \boldsymbol{G}$ the $\boldsymbol{A}$-morphism $\gamma_{\boldsymbol{D}}(n) : \boldsymbol{D}(n) \rightarrow \mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})$ is called the n^{th} *colimit injection*. Since this is natural in n, every edge $e : m \rightarrow n$ in $\boldsymbol{G}$ yields a commutative diagram $\gamma_{\boldsymbol{D}}(m) = \boldsymbol{D}(e) \cdot \gamma(n)$. The category $\boldsymbol{A}$ is (small) *cocomplete* when it has a colimit for all (small) diagrams. We assume colimits have been selected.

$$
\begin{tikzcd}
\boldsymbol{D} \arrow[r, Rightarrow, "\gamma_{\boldsymbol{D}}"] \arrow[dr, Rightarrow, "\tau"'] & \Delta_{\boldsymbol{G}}(\mathit{col}(\boldsymbol{D})) \arrow[d, Rightarrow, "\Delta_{\boldsymbol{G}}(m)"] & \mathit{col}_{\boldsymbol{G}}(\alpha) \arrow[d, "m"] \\
& \Delta_{\boldsymbol{G}}(a) & a
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{D}' \arrow[r, Rightarrow, "\gamma'_{\boldsymbol{D}'}"] \arrow[d, Rightarrow, "\alpha"'] & \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D}')) \arrow[d, Rightarrow, "\Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\alpha))"] \\
\boldsymbol{D} \arrow[r, Rightarrow, "\gamma_{\boldsymbol{D}}"'] & \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D}))
\end{tikzcd}
$$

Figure 3: The colimit functor $\mathit{col}_{\boldsymbol{G}} : |\boldsymbol{A}|^{\boldsymbol{G}} \rightarrow \boldsymbol{A}$ on the category $\boldsymbol{A}^{|\boldsymbol{G}|}$ of $\boldsymbol{G}$-indexed diagrams and diagram morphisms

- If $\boldsymbol{G}$-indexed diagrams $\boldsymbol{D}', \boldsymbol{D} : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$ in $\boldsymbol{A}$ have colimiting cocones $\gamma'_{\boldsymbol{D}'} : \boldsymbol{D}' \Rightarrow \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D}'))$ and $\gamma_{\boldsymbol{D}} : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D}))$, then any diagram morphism $\alpha : \boldsymbol{D}' \Rightarrow \boldsymbol{D}$ uniquely determines an $\boldsymbol{A}$-morphism $\mathit{col}_{\boldsymbol{G}}(\alpha) : \mathit{col}_{\boldsymbol{G}}(\boldsymbol{D}') \rightarrow \mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})$ satisfying the commutative diagram $\gamma'_{\boldsymbol{D}'} \bullet \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\alpha)) = \alpha \bullet \gamma_{\boldsymbol{D}}$. Hence, if $\boldsymbol{A}$ is (small) cocomplete, and assuming unique colimits have been selected, then the colimit operator is a functor $\mathit{col}_{\boldsymbol{G}} : |\boldsymbol{A}|^{\boldsymbol{G}} \rightarrow \boldsymbol{A}$, which is left adjoint
$$\mathit{col}_{\boldsymbol{G}} \dashv \Delta_{\boldsymbol{G}}$$
to the delta functor $\Delta_{\boldsymbol{G}} : \boldsymbol{A} \rightarrow |\boldsymbol{A}|^{\boldsymbol{G}}$ with the colimiting cocone $\gamma_{\boldsymbol{D}} : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D}))$ as the $\boldsymbol{D}^{\text{th}}$ component of the unit natural transformation.

Diagrams and Diagram Transformations

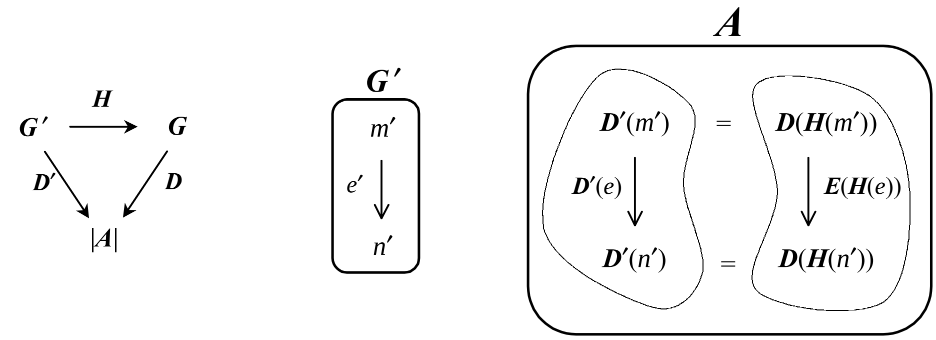


Figure 4: Diagrams and diagram transformations

- A *diagram transformation* $\boldsymbol{H} : (\boldsymbol{G}', \boldsymbol{D}') \rightarrow (\boldsymbol{G}, \boldsymbol{D})$, from *source* diagram $\boldsymbol{D}' : \boldsymbol{G}' \rightarrow \boldsymbol{A}$ to *target* diagram $\boldsymbol{D} : \boldsymbol{G} \rightarrow \boldsymbol{A}$, is a graph morphism $\boldsymbol{H} : \boldsymbol{G}' \rightarrow \boldsymbol{G}$ that satisfies the commutative diagram of composition of graph morphisms: $\boldsymbol{D}' = \boldsymbol{H} \circ \boldsymbol{D}$. Being a general graph morphism, $\boldsymbol{H}$ may map two $\boldsymbol{G}'$-nodes onto the same $\boldsymbol{G}$-node or it may not map any $\boldsymbol{G}'$-nodes to some $\boldsymbol{G}$–nodes (same for edges). An important special case of $\boldsymbol{H}$ is the inclusion of a subgraph $\boldsymbol{G}' \subseteq \boldsymbol{G}$.
- Assume that

  $\gamma'_{\boldsymbol{D}'} : \boldsymbol{D}' \Rightarrow \Delta_{\boldsymbol{G}'}(\mathit{col}_{\boldsymbol{G}'}(\boldsymbol{D}')) : \boldsymbol{G}' \rightarrow |\boldsymbol{A}|$ and

  $\gamma_{\boldsymbol{D}} : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})) : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$

  are the colimiting cocones of $\boldsymbol{D}'$ and $\boldsymbol{D}$, respectively. Pre-composing diagrams and diagram morphisms with $\boldsymbol{H} : \boldsymbol{G}' \rightarrow \boldsymbol{G}$ defines an *inverse image* operator. Inverse image gives the cocone

  $\boldsymbol{H} \circ \gamma_{\boldsymbol{D}} : \boldsymbol{D}' = \boldsymbol{H} \circ \boldsymbol{D} \Rightarrow \boldsymbol{H} \circ \Delta_{\boldsymbol{G}}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})) = \Delta_{\boldsymbol{G}'}(\mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})) : \boldsymbol{G}' \rightarrow |\boldsymbol{A}|$.

  By definition of the colimit, there is a unique $\boldsymbol{A}$-morphism $s : \mathit{col}_{\boldsymbol{G}'}(\boldsymbol{D}') \rightarrow \mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})$ that satisfies the commutative diagram: $\gamma'_{\boldsymbol{D}'} \bullet \Delta_{\boldsymbol{G}'}(s) = \boldsymbol{H} \circ \gamma_{\boldsymbol{D}}$. Graph-diagram pairs such as $(\boldsymbol{G}, \boldsymbol{D})$ and diagram transformations such as $\boldsymbol{H} : (\boldsymbol{G}', \boldsymbol{D}') \rightarrow (\boldsymbol{G}, \boldsymbol{D})$ constitute the comma category $\langle \mathrm{Gph} \downarrow |\boldsymbol{A}| \rangle$ of graphs over

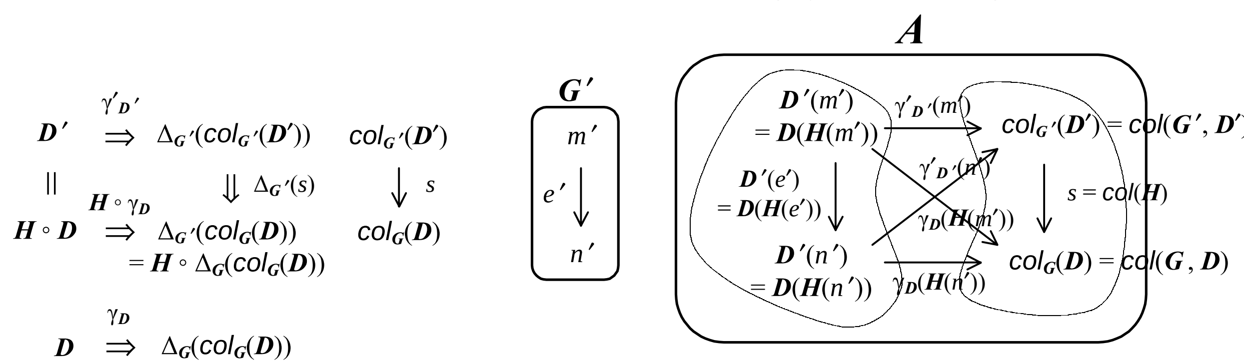


Figure 5: The colimit functor $\mathit{col} : \langle \mathrm{Gph} \downarrow \boldsymbol{A} \rangle \rightarrow \boldsymbol{A}$ **on the comma category** $\langle \mathrm{Gph} \downarrow \boldsymbol{A} \rangle$ **of diagrams and diagram transformations**

the underlying graph of category $\boldsymbol{A}$. Define the colimit operator on objects by $\mathit{col}((\boldsymbol{G}, \boldsymbol{D})) = \mathit{col}_{\boldsymbol{G}}(\boldsymbol{D})$ and on morphisms by $\mathit{col}(\boldsymbol{H}) = s$. Then the colimit operator is seen as a functor

$\mathit{col} : \langle \mathrm{Gph} \downarrow |\boldsymbol{A}| \rangle \rightarrow \boldsymbol{A}$.

Diagrams and Colax Diagram Transformations

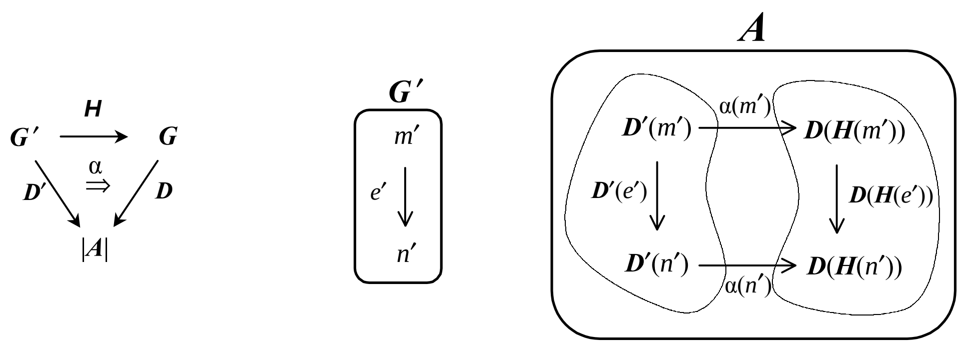


Figure 6: Diagrams and colax diagram transformations

○ A *colax diagram transformation* $\mathsf{H} = (\boldsymbol{H}, \alpha) : (\boldsymbol{G}', \boldsymbol{D}') \to (\boldsymbol{G}, \boldsymbol{D})$, from *source* diagram $\boldsymbol{D}' : \boldsymbol{G}' \to |\boldsymbol{A}|$ to *target* diagram $\boldsymbol{D} : \boldsymbol{G} \to |\boldsymbol{A}|$, consists of a graph morphism $\boldsymbol{H} : \boldsymbol{G}' \to \boldsymbol{G}$ and a diagram morphism $\alpha : \boldsymbol{D}' \Rightarrow \boldsymbol{H} \circ \boldsymbol{D} : \boldsymbol{G}' \to |\boldsymbol{A}|$.

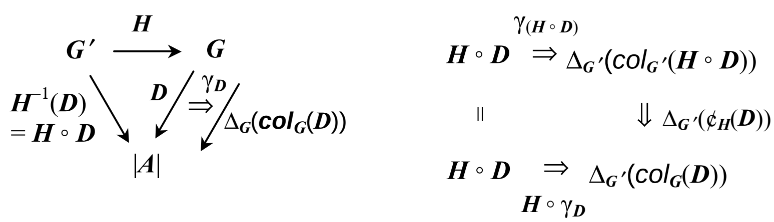
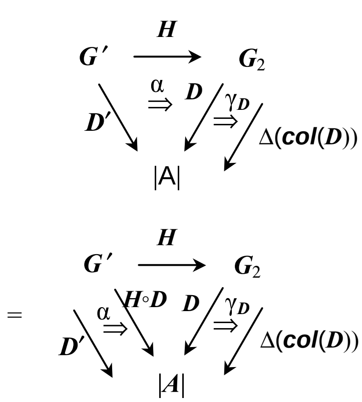


Figure 7: Colimit connection via inverse image

○ The following operator is needed in order to define colimits on colax diagram transformations. The colimit of a diagram is connected to the colimit of its inverse image along a graph morphism. For any graph morphism $\boldsymbol{H} : \boldsymbol{G}' \to \boldsymbol{G}$, the *connection* class function maps a diagram $\boldsymbol{D} : \boldsymbol{G} \to |\boldsymbol{A}|$ in the target fiber to a connection set function $¢_{\boldsymbol{H}}(\boldsymbol{D}) : col_{\boldsymbol{G}'}(\boldsymbol{H}^{-1}(\boldsymbol{D})) = col(\boldsymbol{H} \circ \boldsymbol{D}) \to col(\boldsymbol{D})$ from the colimit of the inverse image diagram $\boldsymbol{H}^{-1}(\boldsymbol{D}) = \boldsymbol{H} \circ \boldsymbol{D} : \boldsymbol{G}' \to |\boldsymbol{A}|$ to the colimit of $\boldsymbol{D}$. This function is defined as the comediator of a connection cocone.



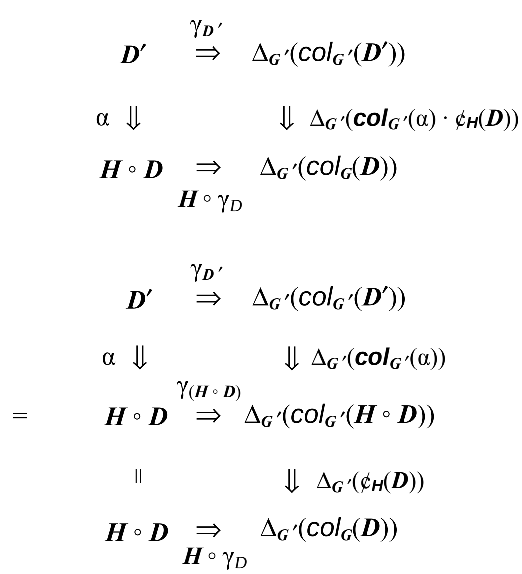


Figure 8: Colimits and colax transformations

○ We put together the previous two ideas, diagram morphism and diagram transformations. Let $\langle \mathsf{Gph} \Downarrow |\boldsymbol{A}| \rangle$ be the "super-comma" category of graph-diagram pairs and colax diagram transformations over the underlying graph of category $\boldsymbol{A}$. Then the colimit operator is seen as a functor

$$col : \langle \mathsf{Gph} \Downarrow |\boldsymbol{A}| \rangle \rightarrow \boldsymbol{A}.$$

from the "super-comma" category to $\boldsymbol{A}$. On objects, the colimit operator maps a graph-diagram pair to its colimit. On morphisms, the colimit operator maps a colax diagram transformation $\mathsf{H} = (\boldsymbol{H}, \alpha) : (\boldsymbol{G}', \boldsymbol{D}') \rightarrow (\boldsymbol{G}, \boldsymbol{D})$ to the colimit $\boldsymbol{A}$-morphism $col(\mathsf{H}) = col_{\boldsymbol{G}'}(\alpha) \cdot ¢_{\boldsymbol{H}}(\boldsymbol{D}) : col((\boldsymbol{G}', \boldsymbol{D}')) = col_{\boldsymbol{G}'}(\boldsymbol{D}') \rightarrow col_{\boldsymbol{G}}(\boldsymbol{D}) = col((\boldsymbol{G}, \boldsymbol{D}))$. That is, the colimit of a colax diagram transformation is the composition of the colimit of the diagram morphism $\alpha : \boldsymbol{D}' \Rightarrow \boldsymbol{H} \circ \boldsymbol{D} : \boldsymbol{G}' \rightarrow |\boldsymbol{A}|$ followed by the connection of the target diagram $\boldsymbol{D} : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$. The colimit operator preserves composition and identities.

Letting the Ambient Category Vary

In the above discussion, the category $\boldsymbol{A}$ is considered an ambience, since all of the diagrams were situated within $\boldsymbol{A}$; that is, they all took values in $\boldsymbol{A}$. In this section, we allow the ambient category $\boldsymbol{A}$ to vary by considering a functor $\boldsymbol{F} : \boldsymbol{A} \rightarrow \boldsymbol{B}$. What effect does this have on the colimit operator?

$$\begin{array}{ccccc}
\boldsymbol{D} \circ |\boldsymbol{F}| & \overset{\tilde{\gamma}_{\boldsymbol{D}\circ|\boldsymbol{F}|}}{\Rightarrow} & \tilde{\Delta}_{\boldsymbol{G}}(col_{\boldsymbol{B},\boldsymbol{G}}(\boldsymbol{D} \circ |\boldsymbol{F}|)) & & col_{\boldsymbol{B},\boldsymbol{G}}(\boldsymbol{D} \circ |\boldsymbol{F}|) \\
\| & & \Downarrow \tilde{\Delta}_{\boldsymbol{G}}(s) & & \downarrow s \\
\boldsymbol{D} \circ |\boldsymbol{F}| & \overset{\gamma_{\boldsymbol{D}} \circ |\boldsymbol{F}|}{\Rightarrow} & \tilde{\Delta}_{\boldsymbol{G}}(\boldsymbol{F}(col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D}))) & & \boldsymbol{F}(col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D})) \\
 & & & & \\
\boldsymbol{D} & \overset{\gamma_{\boldsymbol{D}}}{\Rightarrow} & \Delta_{\boldsymbol{G}}(col_{\boldsymbol{G}}(\boldsymbol{D})) & &
\end{array}$$

Figure 9: Colimits when the ambient category varies

Assume that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are (small) complete. As above, let $\Delta_{\boldsymbol{G}} : \boldsymbol{A} \rightarrow |\boldsymbol{A}|^{\boldsymbol{G}}$ symbolize the delta functor that indexes the constant $\boldsymbol{G}$-shaped diagrams in $\boldsymbol{A}$. And let $\tilde{\Delta}_{\boldsymbol{G}} : \boldsymbol{B} \rightarrow |\boldsymbol{B}|^{\boldsymbol{G}}$ symbolize the delta functor that indexes the constant $\boldsymbol{G}$-shaped diagrams in $\boldsymbol{B}$. Let $\boldsymbol{D} : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$ be a diagram in the source category $\boldsymbol{A}$, and let $\gamma_{\boldsymbol{D}} : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(c) : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$ denote the colimiting cocone of $\boldsymbol{D}$ with colimit $col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D}) = c$ in $\boldsymbol{A}$. Postcomposition with the underlying graph morphism of the functor $\boldsymbol{F}$ gives the diagram $\boldsymbol{D} \circ |\boldsymbol{F}| : \boldsymbol{G} \rightarrow |\boldsymbol{A}| \rightarrow |\boldsymbol{B}|$ in the target category $\boldsymbol{B}$ and the cocone $\gamma_{\boldsymbol{D}} \circ |\boldsymbol{F}| : \boldsymbol{D} \circ |\boldsymbol{F}| \Rightarrow \Delta_{\boldsymbol{G}}(c) \circ |\boldsymbol{F}| = \tilde{\Delta}_{\boldsymbol{G}}(\boldsymbol{F}(c)) : \boldsymbol{G} \rightarrow |\boldsymbol{B}|$ in $\boldsymbol{B}$. Let $\tilde{\gamma}_{\boldsymbol{D}\circ|\boldsymbol{F}|} : \boldsymbol{D} \circ |\boldsymbol{F}| \Rightarrow \tilde{\Delta}_{\boldsymbol{G}}(\tilde{c})$ denotes the colimiting cocone of $\boldsymbol{D} \circ |\boldsymbol{F}|$ with colimit $col_{\boldsymbol{B},\boldsymbol{G}}(\boldsymbol{D} \circ |\boldsymbol{F}|) = \tilde{c}$ in $\boldsymbol{B}$. How do these latter two cocones in $\boldsymbol{B}$ compare? By definition of the colimit, there is a unique $\boldsymbol{B}$-morphism $s : col_{\boldsymbol{B},\boldsymbol{G}}(\boldsymbol{D} \circ |\boldsymbol{F}|) \rightarrow \boldsymbol{F}(col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D}))$ that satisfies the commutative diagram: $\tilde{\gamma}_{\boldsymbol{D}\circ|\boldsymbol{F}|} \bullet \tilde{\Delta}_{\boldsymbol{G}}(s) = \gamma_{\boldsymbol{D}} \circ |\boldsymbol{F}|$.

The functor $\boldsymbol{F}$ is *cocontinuous* when it preserves all (small) colimits. Left adjoint functors are cocontinuous. Suppose that $\boldsymbol{F}$ is cocontinuous. By cocontinuity, if the functor $\boldsymbol{F}$ maps a $\boldsymbol{G}$-shaped $\boldsymbol{A}$-diagram $\boldsymbol{D}$ to the $\boldsymbol{G}$-shaped $\boldsymbol{B}$-diagram $\boldsymbol{E} = \boldsymbol{D} \circ |\boldsymbol{F}|$, then it maps the colimiting cocone $\gamma_{\boldsymbol{D}} : \boldsymbol{D} \Rightarrow \Delta_{\boldsymbol{G}}(c) : \boldsymbol{G} \rightarrow |\boldsymbol{A}|$ to the colimiting cocone $\gamma_{\boldsymbol{E}} : \boldsymbol{E} \Rightarrow \tilde{\Delta}_{\boldsymbol{G}}(\boldsymbol{F}(c)) : \boldsymbol{G} \rightarrow |\boldsymbol{B}|$, and in particular it maps the colimit $col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D})$ to the colimit $col_{\boldsymbol{B},\boldsymbol{G}}(\boldsymbol{E}) = \boldsymbol{F}(col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D}))$ with s the identity at this colimit and it maps the n^{th} colimit injection $\gamma_{\boldsymbol{D}}(n) : \boldsymbol{D}(n) \rightarrow col_{\boldsymbol{A},\boldsymbol{G}}(\boldsymbol{D})$ to the n^{th} colimit injection $\gamma_{\boldsymbol{E}}(n) : \boldsymbol{E}(n) \rightarrow col_{\boldsymbol{B},\boldsymbol{G}}(\boldsymbol{E})$. For example, the functor $base : \mathsf{Theory} \rightarrow \mathsf{Language}$ is left adjoint, and hence cocontinuous. Therefore, diagrams of theories map to base diagrams of languages and their colimiting cocones of theories map to the colimiting cocones of languages for their base diagrams.