Metamodeling Facilities in the IFF

The IFF Representation of the MOF and the MDA

Introduction

The MOF standard has developed great intuitions about metamodeling structures. However, intuitions are not enough. We also need formalization. Formalization means (1) the development of a theory, in this case a categorical theory, and (2) the axiomatization of the appropriate elements of that theory. This paper gives a further categorical development of the intuitions and theory started in the paper Metamodeling Facilities, by Kenneth Bacławski, Mieczysław Kokar, Jeffrey Smith.

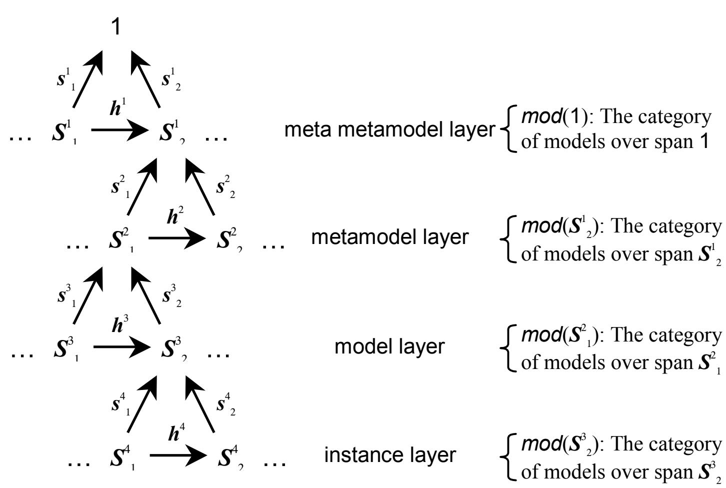


Figure 1: The MOF Model Structure

The theoretical step is very essential, since it gives a mathematical underpinning of intuitions. From experience, the axiomatization step is very important, since it usually turns up problems that have not surfaced in either the intuitive or the theoretical development. In the theoretical and axiomatic development, we need to be very methodical, proceeding in careful steps, and always checking with the intuitions. The procedure for this development is in two parts. First, we develop the central notions in the unordered case. After this unordered case has been settled, we will extend the theory to the ordered case. The central notions in the unordered case are *spans*, *models*, and *relational interpretations*. We also discuss the adjunction between models and relational interpretations.

It is reasonably clear from the unordered case that the notion of a MOF stack (Figure 1) is somewhat fragile and illusionary. One should start the level indexing at the metalevel above the MOF level, and then proceed downward. At level 0 there is only one span the terminal span – it has one source node, one target node and one edge. The 1st level, the next level down, is the level of the MOF span. However, any number of other spans could serve as a starting point. A model at level n+1 can be seen as a model at level n by composition. A model at level n can be seen as a model at level n+1 by using identity[*]. Hence, a model exists at any nonzero level of the model structure.

Unordered Structures

Models

Relations and Relation Morphisms

Definition

A (*typed*) *binary relation* $\boldsymbol{R} = \langle \mathit{set}_0(\boldsymbol{R}), \mathit{set}_1(\boldsymbol{R}), \mathit{ext}(\boldsymbol{R})\rangle$ consists of two component sets $\mathit{set}_0(\boldsymbol{R})$ and $\mathit{set}_1(\boldsymbol{R})$, and a subset of the component Cartesian product $\mathit{ext}(\boldsymbol{R}) \subseteq \mathit{set}_0(\boldsymbol{R}) \times \mathit{set}_1(\boldsymbol{R})$.

$$\begin{tikzcd}
\check{\boldsymbol{R}} \arrow[d, "\boldsymbol{g}"'] \\
\boldsymbol{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathit{set}_0(\check{\boldsymbol{R}}) \arrow[d, "\mathit{ftn}_0(\boldsymbol{g})"] & \mathit{set}_1(\check{\boldsymbol{R}}) \arrow[d, "\mathit{ftn}_1(\boldsymbol{g})"] & \mathit{ext}(\check{\boldsymbol{R}}) \arrow[r, "\mathit{incl}(\check{\boldsymbol{R}})"] \arrow[d, "\mathit{ext}(\boldsymbol{g})"'] & \mathit{set}_0(\check{\boldsymbol{R}}) \times \mathit{set}_1(\check{\boldsymbol{R}}) \arrow[d, "\mathit{ftn}_0(\boldsymbol{g}) \times \mathit{ftn}_1(\boldsymbol{g})"] \\
\mathit{set}_0(\boldsymbol{R}) & \mathit{set}_1(\boldsymbol{R}) & \mathit{ext}(\boldsymbol{R}) \arrow[r, "\mathit{incl}(\boldsymbol{R})"'] & \mathit{set}_0(\boldsymbol{R}) \times \mathit{set}_1(\boldsymbol{R})
\end{tikzcd}$$

Figure 2a: Relation Morphism – abstract

Figure 2b: Relation Morphism – details

Definition

A (*vertical*) *morphism* of binary relations $\boldsymbol{g} = \langle \mathit{ftn}_0(\boldsymbol{g}), \mathit{ftn}_1(\boldsymbol{g})\rangle : \check{\boldsymbol{R}} \rightarrow \boldsymbol{R}$ (Figure 2a) from source relation $\check{\boldsymbol{R}}$ to target relation $\boldsymbol{R}$ consists of a pair of component functions

$\mathit{ftn}_0(\boldsymbol{g}) : \mathit{set}_0(\check{\boldsymbol{R}}) \rightarrow \mathit{set}_0(\boldsymbol{R})$, and

$\mathit{ftn}_1(\boldsymbol{g}) : \mathit{set}_1(\check{\boldsymbol{R}}) \rightarrow \mathit{set}_1(\boldsymbol{R})$,

[*] I have some misgivings about the structure being erected here. The basic connection across the layers of the MOF stack is that of "instance-of". The M_0 layer contains instances such as "Susan Smith", the fish in her fish bowl, the closest star beyond the sun, etc. The M_1 layer contains models (corresponding to the theories of knowledge representation and logic) containing types such as "Person", "Organism", "Physical Object", etc. The central relationship between the M_0 and M_1 layers (and between the M_1 and M_2 layers) is that of classification: "Susan Smith" is a "Person", that star mentioned before is a "Star", etc. The sort functions and span morphisms used here are limited and functional versions of general classifications (Barwise and Seligman, 1997). In general, an object can be of any number of types: "Susan Smith" is an "Employee", and "Susan Smith" is a "Registered Voter". In addition, composition of the sort functions may not be meaningful. Such composition is not encouraged by the structure of the theory of Information Flow, as it is here. However, in the theory of Information Flow you do have other flexibilities that may be of some use – you can flip a classification, and still get a meaningful classification; you can use identity classifications, if desired; and of central importance, you can connect classifications in a meaningful manner through infomorphisms. The notion of an infomorphism extends the notion of a model morphism by allowing the type level to vary in a contravariant fashion. Another flexibility, as discussed in the Appendix, may relate to the spiral metamodel idea of Desmond D'Souza.

whose Cartesian product function $ftn_0(\boldsymbol{g}) \times ftn_1(\boldsymbol{g}) : set_0(\check{\boldsymbol{R}}) \times set_1(\check{\boldsymbol{R}}) \rightarrow set_0(\boldsymbol{R}) \times set_1(\boldsymbol{R})$ preserves extent (Figure 2b): $\wp(ftn_0(\boldsymbol{g}) \times ftn_1(\boldsymbol{g}))(ext(\check{\boldsymbol{R}})) \subseteq ext(\boldsymbol{R})$; that is, $(ftn_0(\boldsymbol{g})(a_0), ftn_1(\boldsymbol{g})(a_1)) \in ext(\boldsymbol{R})$ for all pairs $(a_0, a_1) \in ext(\check{\boldsymbol{R}})$. This constraint means that the Cartesian product function $ftn_0(\boldsymbol{g}) \times ftn_1(\boldsymbol{g})$ has a source and target restriction $ext(\boldsymbol{g}) : ext(\check{\boldsymbol{R}}) \rightarrow ext(\boldsymbol{R})$.

Composition of vertical morphisms is defined componentwise, and preservation of extent can be checked. Composition is associative, and identities are defined componentwise. Types binary relations and vertical morphisms of relation form the vertical category of the double category of relations $\mathsf{Rel}^{\mathrm{vert}}$. Objects of $\mathsf{Rel}^{\mathrm{vert}}$ are relations, which are the horizontal morphisms in Rel. Morphisms of $\mathsf{Rel}^{\mathrm{vert}}$ are vertical morphisms, which are the squares of Rel. For simplicity below, we use the notation Rel for $\mathsf{Rel}^{\mathrm{vert}}$.

The Context of Spans

IFF Code

spn

The collection of spans and span morphisms form a category Span.

```
(1) (CAT$category span)
    (= (CAT$object span)      spn.obj$span)
    (= (CAT$morphism span)    spn.mor$morphism)
    (= (CAT$source span)      spn.mor$source)
    (= (CAT$target span)      spn.mor$target)
    (= (CAT$composable span)  spn.mor$composable)
    (= (CAT$composition span) spn.mor$composition)
    (= (CAT$identity span)    spn.mor$identity)
```

Definition

A *span* $\boldsymbol{S} = \langle node_0(\boldsymbol{S}), node_1(\boldsymbol{S}), edge(\boldsymbol{S}), src(\boldsymbol{S}), tgt(\boldsymbol{S})\rangle$ (Figure 3) consists of two functions that share a common source. In more detail, a span consists of

$$\boldsymbol{S}$$

Figure 3a: Span – abstract

$$node_0(\boldsymbol{S}) \xleftarrow[src(\boldsymbol{S})]{} edge(\boldsymbol{S}) \xrightarrow[tgt(\boldsymbol{S})]{} node_1(\boldsymbol{S})$$

Figure 3b: Span – details

- a set of source nodes $node_0(\boldsymbol{S})$,
- a set of target nodes $node_1(\boldsymbol{S})$,
- a set of edges $edge(\boldsymbol{S})$,
- a source node function $src(\boldsymbol{S}) : edge(\boldsymbol{S}) \rightarrow node_0(\boldsymbol{S})$, and
- a target node function $tgt(\boldsymbol{S}) : edge(\boldsymbol{S}) \rightarrow node_1(\boldsymbol{S})$.

A *directed graph* is a span over a common set of nodes $node_0(\boldsymbol{S}) = node_1(\boldsymbol{S})$.

IFF Code

spn.obj

The following is an IFF representation for the elements of a (small) span. This would appear in the IFF Lower Core (meta) Ontology. Axiom (1) defines the span class. The axioms (2–6) model the span structure of Figure 3b. The edge function in (2) is used to specify the edge set of the span, parameterized by an element in the span class. This is a set-theoretic class function, since it takes a span as a parameter. This parametric technique is used throughout the IFF. The source and target node sets are specified in axioms (3, 4). The source and target functions in axioms (5, 6) assign to edges the nodes that they connect.

```
(1) (SET$class span)
    (SET$class object)
    (= object span)
    (KIF$subcollection span SPN$span)

(2) (SET.FTN$function edge)
    (= (SET$source edge) span)
    (= (SET$target edge) set$set)
    (KIF$restriction edge SPN$morphism)

(3) (SET.FTN$function node1)
    (= (SET$source node1) span)
    (= (SET$target node1) set$set)
```

```
    (KIF$restriction node1 SPN$object1)

(4) (SET.FTN$function node2)
    (= (SET$source node2) span)
    (= (SET$target node2) set$set)
    (KIF$restriction node2 SPN$object2)

(5) (SET.FTN$function source)
    (= (SET$source source) span)
    (= (SET$target source) set.ftn$function)
    (= (SET$composition source set.ftn$source) edge)
    (= (SET$composition source set.ftn$target) node1)
    (KIF$restriction source SPN$source)

(6) (SET.FTN$function target)
    (= (SET$source target) span)
    (= (SET$target target) set.ftn$function)
    (= (SET$composition target set.ftn$source) edge)
    (= (SET$composition target set.ftn$target) node2)
    (KIF$restriction target SPN$target)
```

The terminal object in Span is the span $1 = \langle 1, 1, 1, id, id\rangle$ with one source node, one target node and one edge. Terminality means that for any span $\boldsymbol{S}$, there is a unique span morphism $!_{\boldsymbol{S}} : \boldsymbol{S} \rightarrow 1$. This consists of the unique functions $!_{node0(\boldsymbol{S})} : node_0(\boldsymbol{S}) \rightarrow 1$, $!_{node1(\boldsymbol{S})} : node_1(\boldsymbol{S}) \rightarrow 1$ and $!_{edge(\boldsymbol{S})} : edge(\boldsymbol{S}) \rightarrow 1$. More generally, this category is complete and cocomplete.

```
(7) (span terminal-span)
    (= (node0 terminal-span) set.lim$terminal)
    (= (node1 terminal-span) set.lim$terminal)
    (= (edge terminal-span) set.lim$terminal)
    (= (source terminal-span) (set.ftn$identity set.lim$terminal))
    (= (target terminal-span) (set.ftn$identity set.lim$terminal))
```

$$Element_2(\Omega) \xleftarrow{domain_2(\Omega)} Property_2(\Omega) \xrightarrow{range_2(\Omega)} Element_2(\Omega)+Literal$$

Figure 4: The MOF Metamodel

Example

The notion of a MOF metamodel, as discussed in the paper (Bacławski, Kokar and Smith), is an example of a span. Here we only consider the underlying set of this ordered structure. The later extension of the theory to ordered structures would complete the picture. An *MOF metamodel* Ω is a quintuple

$$\Omega = \langle Element_2(\Omega), Literal, Property_2(\Omega), domain_2(\Omega), range_2(\Omega)\rangle$$

such that:

1. $Element_2(\Omega)$ and $Property_2(\Omega)$ are partial orders,
2. *Literal* is a literal type structure,
3. $domain_2(\Omega)$ is a monotonic function
   $$domain_2(\Omega) : Property_2(\Omega) \rightarrow Element_2(\Omega),$$
4. $range_2(\Omega)$ is a monotonic function
   $$range_2(\Omega) : Property_2(\Omega) \rightarrow Element_2(\Omega) + Literal.$$

Ignoring order, a MOF metamodel is a span (Figure 4), where $node_0(\Omega) = Element_2(\Omega)$, $node_1(\Omega) = Element_2(\Omega) + Literal$, $edge(\Omega) = Property_2(\Omega)$, $src(\Omega) = domain_2(\Omega)$, and $tgt(\Omega) = range_2(\Omega)$.

IFF Code

`mof.obj`

The following is a possible IFF representation for the elements of a metamodel.

```
(set$set literal)

(SET$class metamodel)
(SET$subclass metamodel span)

(SET.FTN$function element2)
(= (SET.FTN$source element2) metamodel)
(= (SET.FTN$target element2) set$set)

(SET.FTN$function property2)
(= (SET.FTN$source property2) metamodel)
(= (SET.FTN$target property2) set$set)

(SET.FTN$function pair)
(= (SET.FTN$source pair) metamodel)
(= (SET.FTN$target pair) set.col.coprd2$pair)
(forall (?m (metamodel ?m))
    (and (= (set1 (pair ?m)) (element2 ?m))
         (= (set2 (pair ?m)) literal)))

(forall (?m (metamodel ?m))
    (and (= (node0 ?m) (element2 ?m))
         (= (node1 ?m) (set.col.coprd2$binary-coproduct (pair ?m)))
         (= (edge ?m) (property2 ?m))
         (= (source ?m) (domain2 ?m))
         (= (target ?m) (range2 ?m))))
```

$$\begin{tikzcd} \hat{\boldsymbol{S}} \arrow[d, "\boldsymbol{h}"'] \\ \boldsymbol{S} \end{tikzcd}$$

$$\begin{tikzcd} \mathit{node}_0(\hat{\boldsymbol{S}}) \arrow[d, "\mathit{node}_0(\boldsymbol{h})"'] & \mathit{edge}(\hat{\boldsymbol{S}}) \arrow[l, "\mathit{src}(\hat{\boldsymbol{S}})"'] \arrow[r, "\mathit{tgt}(\hat{\boldsymbol{S}})"] \arrow[d, "\mathit{edge}(\boldsymbol{h})"'] & \mathit{node}_1(\hat{\boldsymbol{S}}) \arrow[d, "\mathit{node}_1(\boldsymbol{h})"'] \\ \mathit{node}_0(\boldsymbol{S}) & \mathit{edge}(\boldsymbol{S}) \arrow[l, "\mathit{src}(\boldsymbol{S})"] \arrow[r, "\mathit{tgt}(\boldsymbol{S})"'] & \mathit{node}_1(\boldsymbol{S}) \end{tikzcd}$$

Figure 5a: Span Morphism – abstract

Figure 5b: Span Morphism – details

Definition

A *span morphism* $\boldsymbol{h} = \langle \mathit{node}_0(\boldsymbol{h}), \mathit{node}_1(\boldsymbol{h}), \mathit{edge}(\boldsymbol{h}) \rangle : \hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}$ (Figure 5a) consists of

- an *source* span $\mathit{src}(\boldsymbol{h}) = \hat{\boldsymbol{S}}$,
- a *target* span $\mathit{tgt}(\boldsymbol{h}) = \boldsymbol{S}$,
- a *source node* function $\mathit{node}_0(\boldsymbol{h}) : \mathit{node}_0(\hat{\boldsymbol{S}}) \rightarrow \mathit{node}_0(\boldsymbol{S})$,
- a *target node* function $\mathit{node}_1(\boldsymbol{h}) : \mathit{node}_1(\hat{\boldsymbol{S}}) \rightarrow \mathit{node}_1(\boldsymbol{S})$, and
- an *edge* function $\mathit{edge}(\boldsymbol{I}) : \mathit{edge}(\hat{\boldsymbol{S}}) \rightarrow \mathit{edge}(\boldsymbol{S})$,

which make the two diagrams in Figure 5b commutative. The latter constraints assert that interpretations preserve source and target. For any span $\boldsymbol{S}$ a *sub-span* $\hat{\boldsymbol{S}} \subseteq \boldsymbol{S}$ is a span whose node and edge sets are subsets of those of $\boldsymbol{S}$. Any sub-span $\hat{\boldsymbol{S}} \subseteq \boldsymbol{S}$ determines an inclusion span morphism $\mathit{incl}_{\hat{\boldsymbol{S}}, \boldsymbol{S}} : \hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}$, whose components are the subset inclusion functions.

Span morphisms can be composed in a componentwise fashion. This composition operator is associative. Any span determines an identity span morphism, whose component functions are identities. Composition of any span morphism with the identity span morphism at the source or target span returns that span morphism. With these considerations, spans and span morphisms form a category Span.

IFF Code

spn.mor

The following is an IFF representation for the elements of a span morphism. The source and target functions assign spans to span morphisms. The node1, node2 and edge functions are the main components of a span morphism. All five functions (source, target, node1, node2 and edge) are set-theoretic class functions, since they have the span morphism as a parameter.

```
(1) (SET$class morphism)
    (KIF$subcollection span-morphism SPN.MOR$morphism)

(2) (SET.FTN$function source)
    (= (SET$source source) morphism)
    (= (SET$target source) spn.obj$span)
    (KIF$restriction source SPN.MOR$source)

(3) (SET.FTN$function target)
    (= (SET$source target) morphism)
    (= (SET$target target) spn.obj$span)

(4) (SET.FTN$function node1)
    (= (SET$source node1) morphism)
    (= (SET$target node1) set.ftn$function)
    (= (SET$composition [node1 set.ftn$source])
       (SET$composition [source spn.obj$node1]))
    (= (SET$composition [node1 set.ftn$target])
       (SET$composition [target spn.obj$node1]))

(5) (SET.FTN$function node2)
    (= (SET$source node2) span-morphism)
    (= (SET$target node2) set.ftn$function)
    (= (SET$composition [node2 set.ftn$source])
       (SET$composition [source spn.obj$node2]))
    (= (SET$composition [node2 set.ftn$target])
       (SET$composition [target spn.obj$node2]))

(6) (SET.FTN$function edge)
    (= (SET$source edge) span-morphism)
    (= (SET$target edge) set.ftn$function)
    (= (SET$composition [edge set.ftn$source])
       (SET$composition [source spn.obj$edge]))
    (= (SET$composition [edge set.ftn$target])
       (SET$composition [target spn.obj$edge]))
```

Two span morphisms are composable when the target of the first is equal to the source of the second. The composition $\hat{\boldsymbol{h}} \cdot \boldsymbol{h} : \check{\boldsymbol{S}} \rightarrow \boldsymbol{S}$ of two composable span morphisms $\hat{\boldsymbol{h}} : \check{\boldsymbol{S}} \rightarrow \hat{\boldsymbol{S}}$ and $\boldsymbol{h} : \hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}$ is defined in terms of the composition of their component functions.

```
(7) (SET.LIM.PBK$opspan composable-opspan)
    (= (SET.LIM.PBK$class1 composable-opspan) morphism)
    (= (SET.LIM.PBK$class2 composable-opspan) morphism)
    (= (SET.LIM.PBK$opvertex composable-opspan) spn.obj$span)
    (= (SET.LIM.PBK$first composable-opspan) target)
    (= (SET.LIM.PBK$second composable-opspan) source)

(8) (REL$relation composable)
    (= (REL$class1 composable) morphism)
    (= (REL$class2 composable) morphism)
    (= (REL$extent composable) (SET.LIM.PBK$pullback composable-opspan))
```

```
 (9) (SET.FTN$function composition)
     (= (SET.FTN$source composition) (SET.LIM.PBK$pullback composable-opspan))
     (= (SET.FTN$target composition) morphism)
     (forall (?h1 (morphism ?h1)
              ?h2 (morphism ?h2)
              (composable ?h1 ?h2))
         (and (= (source (composition [?h1 ?h2])) (source ?h1))
              (= (target (composition [?h1 ?h2])) (target ?h2))
              (= (node0 (composition [?h1 ?h2]))
                 (set.ftn$composition [(node0 ?h1) (node0 ?h2)]))
              (= (node1 (composition [?h1 ?h2]))
                 (set.ftn$composition [(node1 ?h1) (node1 ?h2)]))
              (= (edge (composition [?h1 ?h2]))
                 (set.ftn$composition [(edge ?h1) (edge ?h2)]))))
```

Composition satisfies the usual associative law.

```
(10) (forall (?h1 (morphism ?h1)
              ?h2 (morphism ?h2)
              ?h3 (morphism ?h3)
              (composable ?h1 ?h2) (composable ?h2 ?h3))
         (= (composition [?h1 (composition [?h2 ?h3])])
            (composition [(composition [?h1 ?h2]) ?h3])))
```

For any span $\boldsymbol{S}$, there is an identity spangraph morphism.

```
(11) (SET.FTN$function identity)
     (= (SET.FTN$source identity) spn.obj$span)
     (= (SET.FTN$target identity) morphism)
     (forall (?s (spn.obj$span ?s))
         (and (= (source (identity ?s)) ?s)
              (= (target (identity ?s)) ?s)
              (= (node0 (identity ?s))
                 (set.ftn$identity (spn.obj$node0 ?s)))
              (= (node1 (identity ?s))
                 (set.ftn$identity (spn.obj$node1 ?s)))
              (= (edge (identity ?s))
                 (set.ftn$identity (spn.obj$edge ?s)))))
```

The identity satisfies the usual identity laws with respect to composition.

```
(12) (forall (?h (morphism ?h))
         (and (= (composition [(identity (source ?h)) ?h]) ?h)
              (= (composition [?h (identity (target ?h))]) ?h)))
```

A span morphism $\boldsymbol{h} : \hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}$ is an isomorphism when all component functions are bijections.

```
(13) (SET$class isomorphism)
     (SET$subclass isomorphism morphism)
     (forall (?h (morphism ?h))
         (<=> (isomorphism ?h)
              (and (set.ftn$bijection (node0 ?h))
                   (set.ftn$bijection (node1 ?h))
                   (set.ftn$bijection (edge ?h)))))
```

The Reflection of Span in Relation

$rel \dashv spn$
$\eta, 1$

$$\begin{tikzcd} S \arrow[r, "\eta_L = \langle id,\, id,\, \nabla_S \rangle"] \arrow[dr, "g"'] & spn(rel(S)) \arrow[d, "spn(h)"] \\ & spn(R) \end{tikzcd} \qquad \begin{tikzcd} rel(S) \arrow[d, "h"] \\ R \end{tikzcd}$$

Diagram 1a: Universal Morphism

$$\begin{tikzcd} R \arrow[r, phantom, "="] & rel(spn(R)) \\ & rel(S) \arrow[ul, "h"] \arrow[u, "rel(g)"'] \end{tikzcd} \qquad \begin{tikzcd} spn(R) \\ S \arrow[u, "g"'] \end{tikzcd}$$

Diagram 1b: Couniversal Morphism

Any span $S = \langle node_0(S), node_1(S), edge(S), src(S), tgt(S) \rangle$ determines a binary relation $rel(S) = \langle node_0(S), node_1(S), ext(rel(S)) \rangle$, whose extent $ext(rel(S)) = \{(src(S)(e), tgt(S)(e)) \mid e \in edge(S)\}$ is the set of pairs connected by an edge. Any span morphism $h = \langle node_0(h), node_1(h), edge(h) \rangle : \hat{S} \rightarrow S$ determines a morphism of relations $rel(h) = \langle node_0(h), node_1(h) \rangle : rel(\hat{S}) \rightarrow rel(S)$, since preservation of source and target implies preservation of extent. Any binary relation $R = \langle set_0(R), set_1(R), ext(R) \rangle$ determines a span $spn(R) = \langle set_0(R), set_1(R), ext(R), src(R), tgt(R) \rangle$, where $src(R)$ and $tgt(R)$ are the component projection functions defined by inclusion of extent composed with the Cartesian projection functions. Any relation morphism $g = \langle set_0(g), set_1(g) \rangle : \check{R} \rightarrow R$ determines a span morphism $spn(h) = \langle set_0(g), set_1(g), ext(g) \rangle : span(\check{R}) \rightarrow span(R)$. The relation operator is a functor rel : Span $\rightarrow$ Rel, and the span operator is a functor spn : Rel $\rightarrow$ Span. In a sense, the relation functor "collapses" a span to a binary relation – two edges with the same source and target nodes are identified. In fact, this is realized by the *collapsing function* $\nabla_S : edge(S) \rightarrow ext(rel(S))$, defined by $\nabla_S(e) = (src(S)(e), tgt(S)(e))$. It is easy to check that $rel(span(R)) = R$ and $rel(span(g)) = g$. In the other direction, there is a canonical span morphism $\eta_S = \langle id, id, \nabla_S \rangle : S \rightarrow span(rel(S))$, whose node functions are the identities and whose edge function is the collapsing function. These relation and span inverse passages $\langle rel, spn, \eta, 1 \rangle$: Span $\rightarrow$ Rel form an adjunction (actually a reflection) – the category Rel is a reflective subcategory of the category Span with the relation functor rel : Span $\rightarrow$ Rel acting as the reflector.

Models and Model Morphisms

Let $\boldsymbol{A}$ be a fixed span, which is regarded as a relative MOF metamodel.

Definition

An $\boldsymbol{A}$*-model* $(\boldsymbol{S}, \boldsymbol{s})$ consists of a span $\boldsymbol{S}$ and a span morphism $\boldsymbol{s} : \boldsymbol{S} \rightarrow \boldsymbol{A}$. This is regarded as a relative MOF model of the relative metamodel $\boldsymbol{A}$. The span morphism s is called the sort morphism of the model.

$$
\begin{tikzcd}
\Sigma \arrow[d, "\sigma"'] \\
\Omega
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathit{Element}_1(\Sigma) \arrow[d, "\mathit{sort\text{-}e}_{12}(\Sigma)"'] \arrow[r, phantom, "(\leq)" near end, shift right=6] & \mathit{Property}_1(\Sigma) \arrow[l, "\mathit{domain}_1(\Sigma)"'] \arrow[r, "\mathit{range}_1(\Sigma)"] \arrow[d, "\mathit{sort\text{-}p}_{12}(\Sigma)"] \arrow[r, phantom, "(\geq)" near start, shift right=6] & \mathit{Element}_1(\Sigma) + \mathit{Literal} \arrow[d, "\mathit{sort}_{12}\text{-}e(\sigma)+id_{\mathit{Literal}}"] \\
\mathit{Element}_2(\Omega) & \mathit{Property}_2(\Omega) \arrow[l, "\mathit{domain}_2(\Omega)"] \arrow[r, "\mathit{range}_2(\Omega)"'] & \mathit{Element}_2(\Sigma) + \mathit{Literal}
\end{tikzcd}
$$

Figure 6a: The MOF Model – abstract

Figure 6b: The MOF Model – details

Example

The notion of a MOF model, as discussed in the paper (Bacławski, Kokar and Smith), is an example of a model. Here we only consider the underlying set of this ordered structure. The later extension of the theory to ordered structures would complete the picture. An *MOF model* Σ (Figure 6) based on a MOF metamodel Ω (that is, a model Σ of Ω) is a septuple

$$\Sigma = \langle \Omega, \mathit{Element}_1(\Sigma), \mathit{Property}_1(\Sigma), \mathit{sort\text{-}e}_{12}(\Sigma), \mathit{sort\text{-}p}_{12}(\Sigma), \mathit{domain}_1(\Sigma), \mathit{range}_1(\Sigma) \rangle$$

such that:

1. $\Omega = \langle \mathit{Element}_2(\Omega), \mathit{Literal}, \mathit{Property}_2(\Omega), \mathit{domain}_2(\Omega), \mathit{range}_2(\Omega) \rangle$ is a MOF metamodel,
2. $\mathit{Element}_1(\Sigma)$ and $\mathit{Property}_1(\Sigma)$ are partial orders,
3. *Literal* is a literal type structure,
4. $\mathit{sort\text{-}e}_{12}(\Sigma)$ is a monotonic function
   $\mathit{sort\text{-}e}_{12}(\Sigma) : \mathit{Element}_1(\Sigma) \rightarrow \mathit{Element}_2(\Sigma)$,
5. $\mathit{sort\text{-}p}_{12}(\Sigma)$ is a monotonic function
   $\mathit{sort\text{-}p}_{12}(\Sigma) : \mathit{Property}_1(\Sigma) \rightarrow \mathit{Property}_2(\Sigma)$,
6. $\mathit{domain}_1(\Sigma)$ is a monotonic function
   $\mathit{domain}_1(\Sigma) : \mathit{Property}_1(\Sigma) \rightarrow \mathit{Element}_1(\Sigma)$,
7. $\mathit{range}_1(\Sigma)$ is a monotonic function
   $\mathit{range}_1(\Sigma) : \mathit{Property}_1(\Sigma) \rightarrow \mathit{Element}_1(\Sigma) + \mathit{Literal}$.
8. (Sort compatibility conditions) The two diagrams in Figure 6b are (partially) commutative.

Ignoring order, which means treating a partial order as the identity order, a MOF model is an Ω-model (Σ, σ) (Figure 4), where $\Sigma = \langle \mathit{Element}_1(\Sigma), \mathit{Element}_1(\Sigma) + \mathit{Literal}, \mathit{Property}_1(\Sigma), \mathit{domain}_1(\Sigma), \mathit{range}_1(\Sigma) \rangle$ is a span, $\Omega = \langle \mathit{Element}_1(\Omega), \mathit{Element}_1(\Omega) + \mathit{Literal}, \mathit{Property}_1(\Omega), \mathit{domain}_1(\Omega), \mathit{range}_1(\Omega) \rangle$ is a span, $\sigma = \langle \mathit{sort\text{-}e}_{12}(\Sigma), \mathit{sort}_{12}\text{-}e(\sigma)+id_{\mathit{Literal}}, \mathit{sort\text{-}p}_{12}(\Sigma) \rangle : \Sigma \rightarrow \Omega$ is a span morphism.

Definition

A *morphism of* $\boldsymbol{A}$*-models* $\boldsymbol{h} : (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}) \rightarrow (\boldsymbol{S}, \boldsymbol{s})$ (Figure 7) is a span morphism $\boldsymbol{h} : \hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}$ that preserves model sorts, in the sense that $\boldsymbol{h} \cdot \boldsymbol{s} = \hat{\boldsymbol{s}}$.

The composition of model morphisms is defined in terms of composition of their span morphisms. Sort preservation is easy to check. This composition operator is associative. Any model determines an identity model morphism, whose span morphism is the identity at the model's span. Composition of any model morphism with the identity model morphism at the source or target model returns that model morphism. With these considerations, models and model morphisms form a category $\mathit{mod}(\boldsymbol{A})$ called the *category of models* for metamodel $\boldsymbol{A}$. The category of $\boldsymbol{A}$-models is know in category theory as $\mathit{mod}(\boldsymbol{A}) = \langle \mathsf{Span}, \boldsymbol{A} \rangle$, the category of *spans over* $\boldsymbol{A}$.

$$\begin{tikzcd} \hat{\boldsymbol{S}} \arrow[rr, "\boldsymbol{h}"] \arrow[dr, "\hat{\boldsymbol{s}}"'] & & \boldsymbol{S} \arrow[dl, "\boldsymbol{s}"] \\ & \boldsymbol{A} & \end{tikzcd}$$

Figure 7: Model Morphism

The terminal object in $\mathit{mod}(\boldsymbol{A})$ is the metamodel-as-model $1_{\boldsymbol{A}} = (\boldsymbol{A}, \mathit{id}_{\boldsymbol{A}})$ with identity sort span morphism. Terminality means that for any model $(\boldsymbol{S}, \boldsymbol{s})$, there is a unique model morphism $!_{(\boldsymbol{S},\,\boldsymbol{s})} : (\boldsymbol{S}, \boldsymbol{s}) \rightarrow 1_{\boldsymbol{A}}$. The span morphism for $!_{(\boldsymbol{S},\,\boldsymbol{s})}$ is the sort span morphism $\boldsymbol{s} : \boldsymbol{S} \rightarrow \boldsymbol{A}$. Check: the category of $\boldsymbol{A}$-models $\mathit{mod}(\boldsymbol{A})$ is complete and cocomplete.

The model architecture is concentrated in a diagram of functors (Figure 8). Any span morphism $\boldsymbol{a} : \boldsymbol{A} \rightarrow \bar{\boldsymbol{A}}$ determines a *model* functor $\mathit{mod}(\boldsymbol{a}) : \mathit{mod}(\boldsymbol{A}) \rightarrow \mathit{mod}(\bar{\boldsymbol{A}})$ that maps an $\boldsymbol{A}$-model $(\boldsymbol{S}, \boldsymbol{s})$ to the $\bar{\boldsymbol{A}}$-model $(\boldsymbol{S}, \boldsymbol{s} \cdot \boldsymbol{a})$ and maps a morphism of $\boldsymbol{A}$-models $\boldsymbol{h} : (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}) \rightarrow (\boldsymbol{S}, \boldsymbol{s})$ to the morphism of $\bar{\boldsymbol{A}}$-models $\boldsymbol{h} : (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}} \cdot \boldsymbol{a})) \rightarrow (\boldsymbol{S}, \boldsymbol{s} \cdot \boldsymbol{a})$. This model operator is functorial – it determines a quasi-functor

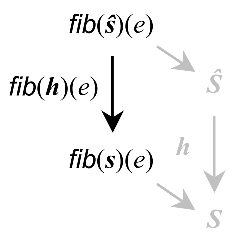
$$\mathit{mod} : \mathsf{Span} \rightarrow \mathsf{CAT}$$

$$\begin{tikzcd} \mathit{mod}(\boldsymbol{A}) \arrow[d, "\mathit{mod}(\boldsymbol{a})"'] \arrow[dr, "\mathit{proj}_{\boldsymbol{A}}"] \arrow[drr, "\mathit{mod}(!_{\boldsymbol{A}})"] & & \\ \mathit{mod}(\bar{\boldsymbol{A}}) \arrow[r, "\mathit{proj}_{\bar{\boldsymbol{A}}}"] \arrow[rr, bend right, "\mathit{mod}(!_{\bar{\boldsymbol{A}}})"'] & \mathsf{Span} \arrow[r, phantom, "\cong"] & \mathit{mod}(1) \end{tikzcd}$$

Figure 8: MOF Functors

from the category of spans to the quasi-category of categories. There is an $\boldsymbol{A}$-*projection* functor $\mathit{proj}_{\boldsymbol{A}} : \mathit{mod}(\boldsymbol{A}) \rightarrow \mathsf{Span}$ that maps a model to its span, ignoring the sort span morphism. Any span $\boldsymbol{S}$ can be regarded as the 1-model $!_{\boldsymbol{S}} : \boldsymbol{S} \rightarrow 1$, and any span morphism $\boldsymbol{h} : \hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}$ can be regarded as a 1-model morphism $\boldsymbol{h} : (\hat{\boldsymbol{S}}, !_{\hat{\boldsymbol{S}}}) \rightarrow (\boldsymbol{S}, !_{\boldsymbol{S}})$. This gives an *embedding* functor $\mathit{embed} : \mathsf{Span} \rightarrow \mathit{mod}(1)$ that is inverse to the 1-projection functor $\mathit{proj}_1 : \mathit{mod}(1) \rightarrow \mathsf{Span}$. The unique span morphism $!_{\boldsymbol{A}} : \boldsymbol{A} \rightarrow 1$, determines the terminal model functor $\mathit{mod}(!_{\boldsymbol{A}}) : \mathit{mod}(\boldsymbol{A}) \rightarrow \mathit{mod}(1)$, which is isomorphism to the $\boldsymbol{A}$-*projection* functor $\mathit{proj}_{\boldsymbol{A}} : \mathit{mod}(\boldsymbol{A}) \rightarrow \mathsf{Span}$: $\mathit{mod}(!_{\boldsymbol{A}}) = \mathit{proj}_{\boldsymbol{A}} \circ \mathit{embed}$ and $\mathit{proj}_{\boldsymbol{A}} = \mathit{mod}(!_{\boldsymbol{A}}) \circ \mathit{proj}_1$.

Model Fibers



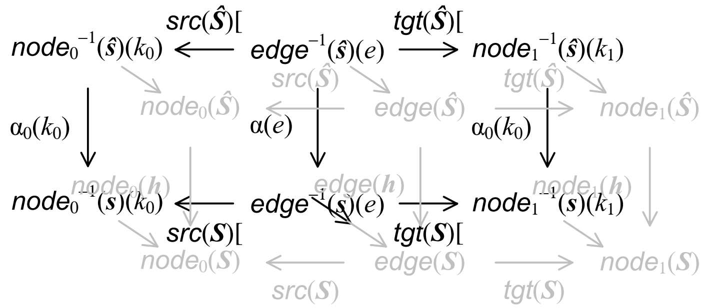


Figure 9a: Fiber Span Morphism – abstract

Figure 9b: Fiber Span Morphism – details

Let $(\boldsymbol{S}, \boldsymbol{s})$ be any model consisting of a span $\boldsymbol{S}$ and a span morphism $\boldsymbol{s} : \boldsymbol{S} \to \boldsymbol{A}$. A node of the target span $\boldsymbol{A}$ determines a set function fiber

$node_0^{-1}(\boldsymbol{s})(k_0) = \{n_0 \mid n_0 \in node_0(\boldsymbol{S}), node_0(\boldsymbol{s})(n_0) = k_0\} \subseteq node_0(\boldsymbol{S})$ and

$node_1^{-1}(\boldsymbol{s})(k_1) = \{n_1 \mid n_1 \in node_1(\boldsymbol{S}), node_1(\boldsymbol{s})(n_1) = k_1\} \subseteq node_1(\boldsymbol{S})$

for source and target nodes $k_0 \in node_0(\boldsymbol{A})$ and $k_1 \in node_1(\boldsymbol{A})$. An edge $e \in edge(\boldsymbol{A})$ with source node $src(\boldsymbol{A})(e) = k_0 \in node_0(\boldsymbol{A})$ and target node $tgt(\boldsymbol{A})(e) = k_1 \in node_1(\boldsymbol{A})$ determines a *fiber* span

$fib((\boldsymbol{S}, \boldsymbol{s}))(d) = \langle node_0^{-1}(\boldsymbol{s})(k_0), node_1^{-1}(\boldsymbol{s})(k_1), edge^{-1}(\boldsymbol{s})(e), src(\boldsymbol{S})[, tgt(\boldsymbol{S})[\rangle$,

where the edge set is the set function fiber

$edge^{-1}(\boldsymbol{s})(e) = \{\bar{e} \mid \bar{e} \in edge(\boldsymbol{S}), edge(\boldsymbol{s})(\bar{e}) = e\} \subseteq edge(\boldsymbol{S})$.

The source and target functions of $fbr((\boldsymbol{S}, \boldsymbol{s}))(e)$,

$src(\boldsymbol{S})[: edge^{-1}(\boldsymbol{s})(e) \to node_0^{-1}(\boldsymbol{s})(n_0)$ and

$tgt(\boldsymbol{S})[: edge^{-1}(\boldsymbol{s})(e) \to node_1^{-1}(\boldsymbol{s})(n_1)$,

are the restrictions of the source and target functions of $\boldsymbol{S}$. These are well defined, since span morphisms preserve source and target. This fiber span is a sub-span $fib((\boldsymbol{S}, \boldsymbol{s}))(e) \subseteq \boldsymbol{S}$ of the source span.

Let $\boldsymbol{h} : (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}) \to (\boldsymbol{S}, \boldsymbol{s})$ be a model morphism, where $\boldsymbol{h} : \hat{\boldsymbol{S}} \to \boldsymbol{S}$ and $\boldsymbol{s} : \boldsymbol{S} \to \boldsymbol{A}$ are two composable span morphisms and $\hat{\boldsymbol{s}} : \hat{\boldsymbol{S}} \to \boldsymbol{A}$ is their composition. How are the node and edge fibers of $\hat{\boldsymbol{s}}$ and $\boldsymbol{s}$ related? Let $k_0 \in node_0(\boldsymbol{A})$ be a source node of $\boldsymbol{A}$. Since $node_0(\hat{\boldsymbol{s}}) = node_0(\boldsymbol{h} \cdot \boldsymbol{s}) = node_0(\boldsymbol{h}) \cdot node_0(\boldsymbol{s})$, we have $node_0^{-1}(\hat{\boldsymbol{s}})(k_0) = node_0(\boldsymbol{h})^{-1}(node_0^{-1}(\boldsymbol{s})(k_0))$, and hence $node_0(\boldsymbol{h})$ restricts to a function

$\alpha_0(k_0) : node_0^{-1}(\hat{\boldsymbol{s}})(k_0) \to node_0^{-1}(\boldsymbol{s})(k_0)$.

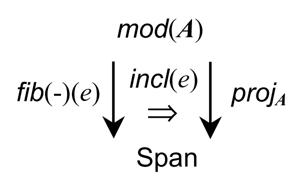


Figure 10: Fiber Functor

The same is true for the target node components. Let $e \in edge(\boldsymbol{A})$ be an edge of $\boldsymbol{A}$ with source node $src(\boldsymbol{A})(e) = k_0 \in node_0(\boldsymbol{A})$ and target node $tgt(\boldsymbol{A})(e) = k_1 \in node_1(\boldsymbol{A})$. Since $edge(\hat{\boldsymbol{s}}) = edge(\boldsymbol{h} \cdot \boldsymbol{s}) = edge(\boldsymbol{h}) \cdot edge(\boldsymbol{s})$, we have $edge^{-1}(\hat{\boldsymbol{s}})(e) = edge(\boldsymbol{h})^{-1}(edge^{-1}(\boldsymbol{s})(e))$, and hence $edge(\boldsymbol{h})$ restricts (Figure 9b) to a function

$\alpha(e) : edge^{-1}(\hat{\boldsymbol{s}})(e) \to edge^{-1}(\boldsymbol{s})(e)$.

Therefore, an edge $e \in edge(\boldsymbol{A})$ determines a span morphism (Figure 9) between fiber spans

$fib(\boldsymbol{h})(e) : fib((\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}))(e) \to fib((\boldsymbol{S}, \boldsymbol{s}))(e)$.

This fiber operator is functorial (Figure 8) in the sense that if $\hat{\boldsymbol{h}} : (\check{\boldsymbol{S}}, \check{\boldsymbol{s}}) \to (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}})$ and $\boldsymbol{h} : (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}) \to (\boldsymbol{S}, \boldsymbol{s})$ are two composable model morphisms, then $fib(\hat{\boldsymbol{h}} \cdot \boldsymbol{h})(e) = fib(\hat{\boldsymbol{h}})(e) \cdot fib(\boldsymbol{h})(e) : fib((\check{\boldsymbol{S}}, \check{\boldsymbol{s}}))(e) \to fib((\boldsymbol{S}, \boldsymbol{s}))(e)$, and for any model $(\boldsymbol{S}, \boldsymbol{s})$, $fib(id_{(\boldsymbol{S}, \boldsymbol{s})})(e) = id_{fib((\boldsymbol{S}, \boldsymbol{s}))(e)}$.

Interpretations

Interpretations and Interpretation Morphisms

$$\begin{tikzcd} \boldsymbol{A} \arrow[d, "\boldsymbol{I}"'] \\ |\mathsf{Rel}| \end{tikzcd}$$

$$\begin{tikzcd} \mathit{node}_0(\boldsymbol{A}) \arrow[d, "\mathit{node}_0(\boldsymbol{I})"'] & \mathit{edge}(\boldsymbol{A}) \arrow[l, "\mathit{src}(\boldsymbol{A})"'] \arrow[r, "\mathit{tgt}(\boldsymbol{A})"] \arrow[d, "\mathit{edge}(\boldsymbol{I})"'] & \mathit{node}_1(\boldsymbol{A}) \arrow[d, "\mathit{node}_1(\boldsymbol{I})"'] \\ \mathit{obj}(\mathsf{Rel}) & \mathit{mor}(\mathsf{Rel}) \arrow[l, "\mathit{src}(\mathsf{Rel})"] \arrow[r, "\mathit{tgt}(\mathsf{Rel})"'] & \mathit{obj}(\mathsf{Rel}) \end{tikzcd}$$

Figure 11a: Interpretation – abstract

Figure 11b: Interpretation – details

Given is a fixed span $\boldsymbol{A}$, a *relational interpretation* $\boldsymbol{I} : \boldsymbol{A} \rightarrow |\mathsf{Rel}|$ (Figure 11a) is a (large) span morphism from $\boldsymbol{A}$ a (small) indexing span to $|\mathsf{Rel}|$ the (large) underlying graph of the vertical category of relations. More explicitly, a relational interpretation consists of

- an *index* span $\mathit{ind}(\boldsymbol{I}) = \boldsymbol{A}$,
- a *source node* function $\mathit{node}_0(\boldsymbol{I}) : \mathit{node}_0(\boldsymbol{A}) \rightarrow \mathit{obj}(\mathsf{Rel})$,
- a *target node* function $\mathit{node}_1(\boldsymbol{I}) : \mathit{node}_1(\boldsymbol{A}) \rightarrow \mathit{obj}(\mathsf{Rel})$, and
- an *edge* function $\mathit{edge}(\boldsymbol{I}) : \mathit{edge}(\boldsymbol{A}) \rightarrow \mathit{mor}(\mathsf{Rel})$,

which make the two diagrams in Figure 11b commutative. The latter constraints assert that interpretations preserve source and target. Note that all three functions are large functions, since the targets are set-theoretic classes. We can view the node and edge functions as arities or indexed collections $\{\mathit{node}_0(\boldsymbol{I})(n_0) \mid n_0 \in \mathit{node}_0(\boldsymbol{A})\}$, $\{\mathit{node}_1(\boldsymbol{I})(n_1) \mid n_1 \in \mathit{node}_1(\boldsymbol{A})\}$ and $\{\mathit{edge}(\boldsymbol{I})(e) \mid e \in \mathit{edge}(\boldsymbol{A})\}$. A *relational diagram* is a relational interpretation indexed by a directed graph.

$$\begin{tikzcd} \boldsymbol{A} \arrow[d, bend right, "\hat{\boldsymbol{I}}"'] \arrow[d, bend left, "\boldsymbol{I}"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ |\mathsf{Rel}| \end{tikzcd}$$

$$\begin{tikzcd} \mathit{edge}(\hat{\boldsymbol{I}})(e) \arrow[d, "\mathit{edge}(\alpha)(e)"'] \\ \mathit{edge}(\boldsymbol{I})(e) \end{tikzcd}$$

$$\begin{tikzcd} \mathit{node}_0(\hat{\boldsymbol{I}})(n_0) \arrow[d, "\alpha_0(n_0)"'] & \mathit{ext}(\mathit{edge}(\hat{\boldsymbol{I}})(e)) \arrow[l, "\mathit{src}(\mathsf{Rel})"'] \arrow[r, "\mathit{tgt}(\mathsf{Rel})"] \arrow[d, "\alpha(e)"'] & \mathit{node}_1(\hat{\boldsymbol{I}})(n_1) \arrow[d, "\alpha_1(n_1)"'] \\ \mathit{node}_0(\boldsymbol{I})(n_0) & \mathit{ext}(\mathit{edge}(\boldsymbol{I})(e)) \arrow[l, "\mathit{src}(\mathsf{Rel})"] \arrow[r, "\mathit{tgt}(\mathsf{Rel})"'] & \mathit{node}_1(\boldsymbol{I})(n_1) \end{tikzcd}$$

Figure 12a: Interpretation Morphism – abstract

Figure 12b: Interpretation Morphism – details

Let $\boldsymbol{A}$ be a fixed span for indexing. A *relational interpretation morphism* $\alpha : \hat{\boldsymbol{I}} \Rightarrow \boldsymbol{I}$ (Figure 12a) is a similar to a natural transformation between functors. In detail, a morphism of relational interpretations consists of a edge-indexed collection of vertical morphisms of relations

$\{\alpha(e) : \mathit{ext}(\mathit{edge}(\hat{\boldsymbol{I}})(e)) \rightarrow \mathit{ext}(\mathit{edge}(\boldsymbol{I})(e)) \mid e \in \mathit{edge}(\boldsymbol{A})\}$.

In further detail, a morphism of relational interpretations consists of three collections of functions

$\{\alpha_0(n_0) : \mathit{node}_0(\hat{\boldsymbol{I}})(n_0) \rightarrow \mathit{node}_0(\boldsymbol{I})(n_0) \mid n_0 \in \mathit{node}_0(\boldsymbol{A})\}$ indexed by the set of source nodes,

$\{\alpha_1(n_1) : \mathit{node}_1(\hat{\boldsymbol{I}})(n_1) \rightarrow \mathit{node}_1(\boldsymbol{I})(n_1) \mid n_1 \in \mathit{node}_1(\boldsymbol{A})\}$ indexed by the set of target nodes,

$\{\alpha(e) : \mathit{ext}(\mathit{edge}(\hat{\boldsymbol{I}})(e)) \rightarrow \mathit{ext}(\mathit{edge}(\boldsymbol{I})(e)) \mid e \in \mathit{edge}(\boldsymbol{A})\}$ indexed by the set of edges, which satisfy the two commutative diagram in Figure 12b. That is, a relational interpretation morphism preserves relational source and target: for each edge $e \in \mathit{edge}(\boldsymbol{A})$ with source node $n_0 \in \mathit{node}_0(\boldsymbol{A})$ and target node $n_1 \in \mathit{node}_1(\boldsymbol{A})$, $\alpha(e) \cdot \mathit{src}(\mathrm{Rel}) = \mathit{src}(\mathrm{Rel}) \cdot \alpha_0(n_0)$ and $\alpha(e) \cdot \mathit{tgt}(\mathrm{Rel}) = \mathit{tgt}(\mathrm{Rel}) \cdot \alpha_0(n_1)$. This effectively states that when the Cartesian product of the two node functions $\alpha_0(n_0)$ and $\alpha_1(n_0)$ is restricted at the source to the extent of $\mathit{edge}(\hat{\boldsymbol{I}})(e)$, it restricts at the target to the extent of $\mathit{edge}(\boldsymbol{I})(e)$, and that the function $\alpha(e)$ is this restriction. In the notation of vertical morphisms of relations, $\alpha_0(n_0) = \mathit{ftn}_0(\mathit{edge}(\alpha)(e))$, $\alpha_1(n_1) = \mathit{ftn}_1(\mathit{edge}(\alpha)(e))$ and $\alpha(e) = \mathit{ext}(\mathit{edge}(\alpha)(e))$.

Interpretation morphisms can be composed by composing their node and edge functions. Source and target preservation is easy to check. This composition operator is associative. Any interpretation determines an identity interpretation morphism, whose node and edge functions are identities. Composition of any interpretation morphism with the identity interpretation morphism at the source or target interpretations returns that interpretation morphism. With these considerations, interpretations and interpretation morphisms form a category $\mathrm{Interp}(\boldsymbol{A})$ parameterized by the fixed index span $\boldsymbol{A}$.

Models to Interpretations

A span $\boldsymbol{S}$ determines a 1-interpretation $\mathit{int}(\boldsymbol{S}) : 1 \rightarrow |\mathsf{Rel}|$ of trivial index 1, where the node function $\mathit{node}_0(\mathit{int}(\boldsymbol{S})) : 1 \rightarrow \mathit{obj}(\mathsf{Rel})$ maps the single node in 1 to the set $\mathit{node}_0(\boldsymbol{S})$, the node function $\mathit{node}_1(\mathit{int}(\boldsymbol{S})) : 1 \rightarrow \mathit{obj}(\mathsf{Rel})$ maps the single node in 1 to the set $\mathit{node}_1(\boldsymbol{S})$, and the edge function $\mathit{edge}(\mathit{int}(\boldsymbol{S})) : 1 \rightarrow \mathit{mor}(\mathsf{Rel})$ maps the single edge in 1 to the relation $\mathit{rel}(\boldsymbol{S})$. Note that a span $\boldsymbol{S}$ is considered a 1-model $\boldsymbol{a} : \boldsymbol{S} \rightarrow 1$.

$$\begin{tikzcd} \boldsymbol{S} \arrow[d, "\boldsymbol{s}"'] \\ \boldsymbol{A} \arrow[d, "\mathit{int}((\boldsymbol{S}, \boldsymbol{s}))"'] \\ |\mathsf{Rel}| \end{tikzcd}$$

Figure 13: The Interpretation of a Model

More generally, an $\boldsymbol{A}$-model $(\boldsymbol{S}, \boldsymbol{s})$ consisting of a span $\boldsymbol{S}$ and a span morphism $\boldsymbol{s} : \boldsymbol{S} \rightarrow \boldsymbol{A}$ determines an $\boldsymbol{A}$-interpretation $\mathit{int}((\boldsymbol{S}, \boldsymbol{s})) : \boldsymbol{A} \rightarrow |\mathsf{Rel}|$ (Figure 13). The node functions of $\mathit{int}((\boldsymbol{S}, \boldsymbol{s}))$ are defined as the set function fibers of the node functions of $\boldsymbol{s}$,

$$\mathit{node}_0(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(n_0) = \mathit{node}_0^{-1}(\boldsymbol{s})(n_0)$$

$$\mathit{node}_1(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(n_1) = \mathit{node}_1^{-1}(\boldsymbol{s})(n_1)$$

for source node $n_0 \in \mathit{node}_0(\boldsymbol{A})$ and target node $n_1 \in \mathit{node}_1(\boldsymbol{A})$ in the target span. The edge function of $\mathit{int}((\boldsymbol{S}, \boldsymbol{s}))$ is the composition of the edge fiber span and relational operators

$$\mathit{edge}(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(e) = \mathit{rel}(\mathit{fib}((\boldsymbol{S}, \boldsymbol{s}))(e))$$

for edge $e \in \mathit{edge}(\boldsymbol{A})$.

$$\begin{tikzcd} \hat{\boldsymbol{S}} \arrow[rr, "\boldsymbol{h}"] \arrow[dr, "\hat{\boldsymbol{s}}"'] & & \boldsymbol{S} \arrow[dl, "\boldsymbol{s}"] \\ & \boldsymbol{A} \arrow[d, bend right, "\mathit{int}((\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}))"'] \arrow[d, bend left, "\mathit{int}((\boldsymbol{S}, \boldsymbol{s}))"] \arrow[d, phantom, "\overset{\mathit{int}(\boldsymbol{h})}{\Rightarrow}"] & \\ & |\mathsf{Rel}| & \end{tikzcd}$$

Figure 14: The Interpretation of a Model Morphism

A morphism of models $\boldsymbol{h} : (\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}) \rightarrow (\boldsymbol{S}, \boldsymbol{s})$ determines a morphism of interpretations $\mathit{int}(\boldsymbol{h}) : \mathit{int}((\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}})) \Rightarrow \mathit{int}((\boldsymbol{S}, \boldsymbol{s}))$ (Figure 14). For any edge $e \in \mathit{edge}(\boldsymbol{A})$, just apply the relational operator to the span morphism $\mathit{fib}(\boldsymbol{h})(e) : \mathit{fib}((\hat{\boldsymbol{S}}, \hat{\boldsymbol{s}}))(e) \rightarrow \mathit{fib}((\boldsymbol{S}, \boldsymbol{s}))(e)$ getting a vertical morphism of relations $\mathit{edge}(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(e) = \mathit{rel}(\mathit{fib}(\boldsymbol{s})(e))$, where

$$\mathit{edge}(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(e) : \mathit{edge}(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(e) = \mathit{rel}(\mathit{fib}(\hat{\boldsymbol{s}})(e)) \rightarrow \mathit{rel}(\mathit{fib}((\boldsymbol{S}, \boldsymbol{s}))(e)) = \mathit{edge}(\mathit{int}((\boldsymbol{S}, \boldsymbol{s})))(e).$$

Interpretations to Models: Case and Indication

A case *source element* of an interpretation $\boldsymbol{I} : \boldsymbol{A} \rightarrow |\mathsf{Rel}|$ is an element in the coproduct (disjoint union)

$$\mathit{elmt}_0(\boldsymbol{I}) = \prod\{\mathit{node}_0(\boldsymbol{I})(n_0) \mid n_0 \in \mathit{node}_0(\boldsymbol{A})\}$$
$$= \prod_{n0 \in \mathit{node}0(\boldsymbol{S})} \mathit{node}_0(\boldsymbol{I})(n_0) = \{(n_0, a_0) \mid n_0 \in \mathit{node}_0(\boldsymbol{A}),\ a_0 \in \mathit{node}_0(\boldsymbol{I})(n_0)\}.$$

For every indexing source node $n_0 \in \mathit{node}_0(\boldsymbol{A})$, there is a coproduct *injection* function

$$\mathit{inj}_0(\boldsymbol{I})(n_0) : \mathit{node}_0(\boldsymbol{I})(n_0) \rightarrow \mathit{elmt}_0(\boldsymbol{I})$$

defined by $\mathit{inj}_0(\boldsymbol{I})(n_0)(a_0) = (n_0, a_0)$ for $a_0 \in \mathit{node}_0(\boldsymbol{I})(n_0)$. There is also a source node *indication* function

$$\mathit{node}_0\text{-}\mathit{indic}(\boldsymbol{I}) : \mathit{elmt}_0(\boldsymbol{I}) \rightarrow \mathit{node}_0(\boldsymbol{A}),$$

which projects the element (n_0, a_0) to the indexing node n_0. A case *target element* is defined similarly.

$$\begin{tikzcd} \mathit{case}(\boldsymbol{I}) \arrow[d, "\mathit{indic}(\boldsymbol{I})"'] \\ \boldsymbol{A} \arrow[d, "\boldsymbol{I}"'] \\ |\mathsf{Rel}| \end{tikzcd}$$

Figure 15: The Model of an Interpretation

A case *link* of an interpretation $\boldsymbol{I} : \boldsymbol{A} \rightarrow |\mathsf{Rel}|$ is an element in the coproduct (disjoint union)

$$\mathit{link}(\boldsymbol{I}) = \prod\{\mathit{edge}(\boldsymbol{I})(e) \mid e \in \mathit{edge}(\boldsymbol{A})\}$$
$$= \prod_{e \in \mathit{edge}(\boldsymbol{S})} \mathit{edge}(\boldsymbol{I})(e) = \{(e, r) \mid e \in \mathit{edge}(\boldsymbol{A}),\ r \in \mathit{edge}(\boldsymbol{I})(e)\}$$
$$= \{(e, a_0, a_1) \mid e \in \mathit{edge}(\boldsymbol{A}),\ a_0 \in \mathit{node}_0(\boldsymbol{I})(\mathit{src}(\boldsymbol{I})(e)),\ a_1 \in \mathit{node}_1(\boldsymbol{I})(\mathit{tgt}(\boldsymbol{A})(e)),\ (a_0, a_1) \in \mathit{edge}(\boldsymbol{I})(e)\}.$$

For every indexing edge $e \in \mathit{edge}(\boldsymbol{A})$, there is a *coproduct injection* function

$$\mathit{inj}(\boldsymbol{I})(e) : \mathit{edge}(\boldsymbol{I})(e) \rightarrow \mathit{link}(\boldsymbol{I})$$

defined by $\mathit{inj}(\boldsymbol{I})(e)((a_0, a_1)) = (e, a_0, a_1)$ for $(a_0, a_1) \in \mathit{edge}(\boldsymbol{I})(e)$. There is also an edge *indication* function

$$\mathit{edge}\text{-}\mathit{indic}(\boldsymbol{I}) : \mathit{link}(\boldsymbol{I}) \rightarrow \mathit{edge}(\boldsymbol{A}),$$

which projects the link (e, a_0, a_1) to the indexing edge e.

$$\begin{tikzcd} \mathit{case}(\boldsymbol{I}) \arrow[d, "\mathit{indic}(\boldsymbol{I})"'] \\ \boldsymbol{S} \end{tikzcd}$$

Figure 16a: Indication Span Morphism – abstract

$$\begin{tikzcd} \mathit{elmt}_0(\boldsymbol{A}) \arrow[d, "\mathit{node}_0\text{-}\mathit{indic}(\boldsymbol{I})"'] & \mathit{link}(\boldsymbol{S}) \arrow[l, "\mathit{src}(\boldsymbol{I})"'] \arrow[r, "\mathit{tgt}(\boldsymbol{I})"] \arrow[d, "\mathit{edge}\text{-}\mathit{indic}(\boldsymbol{I})"'] & \mathit{elmt}_1(\boldsymbol{A}) \arrow[d, "\mathit{node}_1\text{-}\mathit{indic}(\boldsymbol{I})"'] \\ \mathit{node}_0(\boldsymbol{A}) & \mathit{edge}(\boldsymbol{A}) \arrow[l, "\mathit{src}(\boldsymbol{A})"] \arrow[r, "\mathit{tgt}(\boldsymbol{S})"'] & \mathit{node}_1(\boldsymbol{S}) \end{tikzcd}$$

Figure 16b: Indication Span Morphism – details

Combining the above case constructions, associated with an interpretation $\boldsymbol{I}$ is the *case* span $\mathit{case}(\boldsymbol{I}) = \langle \mathit{elmt}_0(\boldsymbol{I}), \mathit{elmt}_1(\boldsymbol{I}), \mathit{link}(\boldsymbol{I}), \mathit{src}(\boldsymbol{I}), \mathit{tgt}(\boldsymbol{I})\rangle$ (Figure 16), where for any edge with source node $\mathit{src}(\boldsymbol{S})(e) = n_0$ and target node $\mathit{tgt}(\boldsymbol{A})(e) = n_1$ the source function is defined by $\mathit{src}(\boldsymbol{I})((e, a_0, a_1)) = (n_0, a_0)$ and the target function is defined by $\mathit{tgt}(\boldsymbol{I})((e, a_0, a_1)) = (n_1, a_1)$. Also, associated with an interpretation $\boldsymbol{I}$ is the *indication* span morphism $\mathit{indic}(\boldsymbol{I}) : \mathit{case}(\boldsymbol{I}) \rightarrow \boldsymbol{A}$ (Figure 16), whose source node function is the source node indication function $\mathit{node}_0(\mathit{indic}(\boldsymbol{I})) = \mathit{node}_0\text{-}\mathit{indic}(\boldsymbol{I})$ (similarly for target nodes) and whose edge function is the edge indication function $\mathit{edge}(\mathit{indic}(\boldsymbol{I})) = \mathit{edge}\text{-}\mathit{indic}(\boldsymbol{I})$.

For every indexing source node $n_0 \in \mathit{node}_0(\boldsymbol{A})$, the source node indication fiber over n_0 is isomorphic to the source node set: $\{(n_0, a_0) \mid a_0 \in \mathit{node}_0(\boldsymbol{I})(n_0)\} \cong \mathit{node}_0(\boldsymbol{I})(n_0)$. Also, for every indexing edge $e \in \mathit{edge}(\boldsymbol{S})$, the edge indication fiber over e is isomorphic to the edge set: $\{(e, a_0, a_1) \mid (a_0, a_1) \in \mathit{edge}(\boldsymbol{I})(e)\} \cong \mathit{edge}(\boldsymbol{I})(e)$. These facts will be the basis for asserting that the interpretation of the model of an interpretation is isomorphic to the original interpretation; $\mathit{int}(\mathit{mod}(\boldsymbol{I})) \cong \boldsymbol{I}$. It is clear that by using the fibers of the node/edge functions, the information in the indication span morphism $\mathit{indic}(\boldsymbol{I})$ is conceptually equivalent to the information in the interpretation $\boldsymbol{I}$.

Ordered Structures

Ordered Spans and Lax Span Morphisms

. . . (to be continued) . . .

$$
\begin{tikzcd}
\mathit{case}(\hat{\boldsymbol{I}}) \arrow[rr, "\mathit{case}(\alpha)"] \arrow[dr, "\mathit{indic}(\hat{\boldsymbol{I}})"'] & & \mathit{case}(\check{\boldsymbol{I}}) \arrow[dl, "\mathit{indic}(\check{\boldsymbol{I}})"] \\
& \boldsymbol{S} \arrow[d, shift right=3, "\hat{\boldsymbol{I}}"'] \arrow[d, shift left=3, "\check{\boldsymbol{I}}"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \\
& |\mathrm{Rel}| &
\end{tikzcd}
$$

Figure 17: The Model Morphism of an Interpretation Morphism

$$
\begin{tikzcd}
\cdots & \hat{\boldsymbol{S}}_1 \arrow[rr, "\hat{\boldsymbol{h}}"] \arrow[d, "\boldsymbol{m}_1"'] & & \hat{\boldsymbol{S}}_2 \arrow[d, "\boldsymbol{m}_2"] & \cdots \\
\cdots & \boldsymbol{S}_1 \arrow[rr, "\boldsymbol{h}"] \arrow[dr, "\boldsymbol{I}_1"'] & & \boldsymbol{S}_2 \arrow[dl, "\boldsymbol{I}_2"] & \cdots \\
& & |\mathrm{Rel}| & &
\end{tikzcd}
$$

Figure 18: Interpretations and Models

Appendix: An Alternate Point of View

The central atomic notion in Information Flow is a *classification*. According to the theory of Information Flow, information presupposes a system of classification. Classifications have been important in library science for the last 2,000 years. The library science classification system most in accord with the philosophy and techniques of Information Flow is the Colon classification system invented by the library scientist Ranganathan. A classification is called a *formal context* in Formal Concept Analysis. A classification is identical to a binary relation in set theory. However, from a category-theoretic standpoint, the category of classifications is very different from the category of relations, since their morphisms are very different. A domain-neutral notion of classification is given by the following abstract mathematical definition. A *classification* $\boldsymbol{A} = \langle \mathit{inst}(\boldsymbol{A}), \mathit{typ}(\boldsymbol{A}), \models_{\boldsymbol{A}} \rangle$ (Figure 19a) consists of:

- a set $\mathit{inst}(\boldsymbol{A})$ of things to be classified, called the *instances* of $\boldsymbol{A}$,
- a set $\mathit{typ}(\boldsymbol{A})$ of things used to classify the instances, called the *types* of $\boldsymbol{A}$, and
- a binary relation, $\models_{\boldsymbol{A}}$, from $\mathit{inst}(\boldsymbol{A})$ to $\mathit{typ}(\boldsymbol{A})$, called the incidence or *classification* relation of $\boldsymbol{A}$,

$$\begin{array}{c} \mathit{typ}(\boldsymbol{A}) \\ \Big| \; \models_{\boldsymbol{A}} \\ \mathit{inst}(\boldsymbol{A}) \end{array}$$

Figure 19a: Classification – abstract

$\mathit{typ}(\boldsymbol{A})$ (columns), $\mathit{inst}(\boldsymbol{A})$ (rows)

×		×	×	
	×			×
		×		
	×		×	

Figure 19b: Classification – graphic

which is identified with the extent set of the binary relation.

The notation $a \models_{\boldsymbol{A}} \alpha$ is read "instance $a \in \mathit{inst}(\boldsymbol{A})$ is of type $\alpha \in \mathit{typ}(\boldsymbol{A})$ in $\boldsymbol{A}$". Classifications abound. Biologists classify organisms (instances) into categories (types). Linguists classify words (instances) by parts of speech (types). A classification $\boldsymbol{A}$ can be graphically represented as a Boolean matrix as in Figure 19b, where the incidence $a \models_{\boldsymbol{A}} \alpha$ is represented by an '×' in row a and column α for every instance $a \in \mathit{inst}(\boldsymbol{A})$ and type $\alpha \in \mathit{typ}(\boldsymbol{A})$. There are various operations, which can be defined on classifications. We discuss two in particular, apposition and subposition. These are both kinds of sums in fibers of the category of classifications and infomorphisms.

○ If two classifications $\boldsymbol{A}_1 = \langle \mathit{inst}(\boldsymbol{A}), \mathit{typ}(\boldsymbol{A}_1), \models_{\boldsymbol{A}1} \rangle$ and $\boldsymbol{A}_2 = \langle \mathit{inst}(\boldsymbol{A}), \mathit{typ}(\boldsymbol{A}_2), \models_{\boldsymbol{A}2} \rangle$ share a common set of instances $\mathit{inst}(\boldsymbol{A})$, the *apposition* classification

  $$\mathit{app}(\boldsymbol{A}_1, \boldsymbol{A}_2) = \langle \mathit{inst}(\boldsymbol{A}), \mathit{typ}(\boldsymbol{A}_1) + \mathit{typ}(\boldsymbol{A}_2), (\models_{\boldsymbol{A}1}, \models_{\boldsymbol{A}2}) \rangle$$

  (Figure 20) has the common set of instances $\mathit{inst}(\boldsymbol{A})$, the disjoint union of the set of types $\mathit{typ}(\boldsymbol{A}_1) + \mathit{typ}(\boldsymbol{A}_2)$, and the target sum incidence $(\models_{\boldsymbol{A}1}, \models_{\boldsymbol{A}2})$ gotten by pasting the Boolean matrices left to right.

$$\mathit{app}(\boldsymbol{A}_1, \boldsymbol{A}_2) \;=\; \boldsymbol{A}_1 \,\big|\, \boldsymbol{A}_2 \;=\; \begin{array}{cc} & \begin{array}{cc} \mathit{typ}(\boldsymbol{A}_1) & \mathit{typ}(\boldsymbol{A}_2) \end{array} \\ \mathit{inst}(\boldsymbol{A}) & \begin{array}{|c|c|} \hline \;\models_{\boldsymbol{A}1}\; & \;\models_{\boldsymbol{A}2}\; \\ \hline \end{array} \end{array}$$

Figure 20: Apposition

- If two classifications $\boldsymbol{A}_1 = \langle \mathit{inst}(\boldsymbol{A}_1), \mathit{typ}(\boldsymbol{A}), \models_{A1} \rangle$ and $\boldsymbol{A}_2 = \langle \mathit{inst}(\boldsymbol{A}_2), \mathit{typ}(\boldsymbol{A}), \models_{A2} \rangle$ share a common set of types $\mathit{typ}(\boldsymbol{A})$, the *subposition* classification

  $$\mathit{sub}(\boldsymbol{A}_1, \boldsymbol{A}_2) = \langle \mathit{inst}(\boldsymbol{A}_1) + \mathit{inst}(\boldsymbol{A}_2), \mathit{typ}(\boldsymbol{A}), [\models_{A1}, \models_{A2}] \rangle$$

  (Figure 21) has the disjoint union of the set of instances $\mathit{inst}(\boldsymbol{A}_1) + \mathit{inst}(\boldsymbol{A}_2)$, the common set of types $\mathit{typ}(\boldsymbol{A})$, and the source sum incidence $[\models_{A1}, \models_{A2}]$ gotten by pasting the Boolean matrices top to bottom.

$$\mathit{sub}(\boldsymbol{A}_1, \boldsymbol{A}_2) \quad = \quad \frac{\boldsymbol{A}_1}{\boldsymbol{A}_2} \quad = \quad \begin{array}{c|c|} & \mathit{typ}(\boldsymbol{A}) \\ \hline \mathit{inst}(\boldsymbol{A}_1) & \models_{A1} \\ \hline \mathit{inst}(\boldsymbol{A}_2) & \models_{A2} \\ \hline \end{array}$$

Figure 21: Subposition

Note that for four appropriately compatible classifications,

$$\mathit{sub}(\mathit{app}(\boldsymbol{A}_{11}, \boldsymbol{A}_{12}), \mathit{app}(\boldsymbol{A}_{21}, \boldsymbol{A}_{22})) = \mathit{app}(\mathit{sub}(\boldsymbol{A}_{11}, \boldsymbol{A}_{21}), \mathit{sub}(\boldsymbol{A}_{12}, \boldsymbol{A}_{22})).$$

$$\mathit{combine}(\boldsymbol{\mathcal{A}}) \quad = \quad \begin{array}{c|c|c|c|c|c|} & \mathit{typ}(\boldsymbol{A}_1) & \mathit{typ}(\boldsymbol{A}_2) & & \cdots & \mathit{typ}(\boldsymbol{A}_n) \\ \hline \mathit{inst}(\boldsymbol{A}_1) & \models_{A1} & \varnothing & \varnothing & \cdots & \varnothing \\ \hline \mathit{inst}(\boldsymbol{A}_2) & 1 & \models_{A2} & \varnothing & \cdots & \varnothing \\ \hline & \varnothing & 1 & \cdots & \cdots & \cdots \\ \hline \vdots & \cdots & \cdots & \cdots & \cdots & \varnothing \\ \hline \mathit{inst}(\boldsymbol{A}_n) & \varnothing & \varnothing & \cdots & 1 & \models_{An} \\ \hline \end{array}$$

Figure 22: Combination

The aim of this appendix is to discuss an Information Flow approach for mixing instances and types. Two ideas are important here. First, a classification does not restrict the content of the set of instances and types – they may have non-empty intersection. Second, a tower or stack of classifications can be coherently combined into a single classification. We use the two pasting operations of apposition and subposition in order to do this. Suppose that we have an tuple of classifications $\boldsymbol{\mathcal{A}} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n)$, where for any $1 \leq k \leq n-1$ we have $\mathit{typ}(\boldsymbol{A}_k) = \mathit{inst}(\boldsymbol{A}_{k+1})$. By using various apposition and subposition pasting operations, we can define the *combination* operation $\mathit{combine}(\boldsymbol{\mathcal{A}})$ as graphically illustrated in Figure 22. The instance set is the disjoint union $\mathit{inst}(\mathit{combine}(\boldsymbol{\mathcal{A}})) = \mathit{inst}(\boldsymbol{A}_1) + \mathit{inst}(\boldsymbol{A}_2) + \ldots + \mathit{inst}(\boldsymbol{A}_n)$. The type set is the disjoint union $\mathit{typ}(\mathit{combine}(\boldsymbol{\mathcal{A}})) = \mathit{typ}(\boldsymbol{A}_1) + \mathit{typ}(\boldsymbol{A}_2) + \ldots + \mathit{typ}(\boldsymbol{A}_n)$. The instance and type sets have the set $\mathit{inst}(\boldsymbol{A}_2) + \ldots + \mathit{inst}(\boldsymbol{A}_n) = \mathit{typ}(\boldsymbol{A}_1) + \ldots + \mathit{typ}(\boldsymbol{A}_{n-1})$ in common. The incidence matrix consists of the component classification matrices arrayed along the main diagonal, with identity matrices arrayed along the subdiagonal, and empty matrices everywhere else. Here the symbol $\varnothing$ denotes an empty classification matrix and the symbol 1 denotes an identity classification matrix. This combination operation may be a more meaningful way to combine a MOF-like tower of metamodels than composing sort functions, and can be compared with the spiral metamodel idea of Desmond D'Souza. It "collapses" a tower of classifications into one classification, but retains all the original information.

References

Bacławski, Kenneth, Kokar, Mieczysław and Smith, Jeffrey. *Metamodeling Facilities*. This was presented at OMG Technical Meeting, September 2001. It is a work in progress for UML 2.0 Math Framework and MOF 2.0 Transformation Proposal for OMG.

Barwise, Jon and Seligman, Jerry. 1997. *Information Flow: The Logic of Distributed Systems*. Cambridge Tracts in Theoretical Computer Science **44**. Cambridge University Press.