The Namespace of Institutions

Basics

Table 1: Basic Terminology

	category	functor	natural transformation
	`SET` `CAT`	`inclusion` `object morphism`	`source target`
	`CLS`	`instance type`	

We need some terminology in the axiomatization of institutions that is potentially confusing. Hopefully the discussion below and particularly the diagram in Figure 2 will help clarify the meaning of this terminology.

As a review, a *quasicategory* A is a category whose object collection $obj(\mathsf{A}) = |\mathsf{A}|$ or morphism collection $mor(\mathsf{A})$ is a generic collection, not just a set-theoretic class. A *quasifunctor* $F : \mathsf{A} \rightarrow \mathsf{B}$ is a functor between quasicategories – its object function or morphism function is a generic collection function, not just a set-theoretic class function. A *quasi-natural transformation* $\alpha : F \rightarrow G : \mathsf{A} \rightarrow \mathsf{B}$ is a natural transformation between quasifunctors – its component function is a generic collection function, not just a set-theoretic class function.

$$\begin{tikzcd}
\mathsf{CAT} \arrow[d, shift right=3, "mor"'] \arrow[d, shift left=3, "obj"] \arrow[d, phantom, "\overset{src}{\Rightarrow}\underset{tgt}{\Rightarrow}"] \\
\mathsf{SET}
\end{tikzcd}$$

Figure 1a: Object and Morphism Functors

$$\begin{tikzcd}
\mathbf{A} \arrow[d, "F"'] & mor(\mathbf{A}) \arrow[r, shift left, "src(\mathbf{A})"] \arrow[r, shift right, "tgt(\mathbf{A})"'] \arrow[d, "mor(F)"'] & obj(\mathbf{A}) \arrow[d, "obj(F)"] \\
\mathbf{B} & mor(\mathbf{B}) \arrow[r, shift left, "src(\mathbf{B})"] \arrow[r, shift right, "tgt(\mathbf{B})"'] & obj(\mathbf{B})
\end{tikzcd}$$

Figure 1b: Source and target naturality

- Two particular canonical quasicategories come directly to mind (Figure 1a): the category SET of classes and class functions is a quasicategory, and the category CAT of categories and functors is a quasicategory. By including natural transformations, the latter is actually a 2-category, which has morphisms (of another kind) between its primary morphisms. Any ordered category is a somewhat trivial example of a 2-category.

```
(1)  (CAT2$quasicategory SET)
     (= (CAT2$object SET) SET$class)
     (= (CAT2$morphism SET) SET.FTN$function)

(2)  (CAT2$quasicategory CAT)
     (= (CAT2$object CAT) CAT$category)
     (= (CAT2$morphism CAT) FUNC$functor)
```

- There is an inclusion quasifunctor $incl : \mathsf{Set} \rightarrow \mathsf{SET}$ from the category Set of sets and their functions to the quasicategory SET of classes and their functions.

```
(3) (FUNC2$quasifunctor inclusion)
    (= (FUNC2$source inclusion) set$set)
    (= (FUNC2$target inclusion) SET)
    (= (FUNC2$object inclusion)
       (SET.FTN$inclusion [set.obj$object SET$class]))
    (= (FUNC2$morphism inclusion)
       (SET.FTN$inclusion [set.mor$morphism SET.FTN$function]))
```

- There are (Figure 1a) two canonical quasifunctors between the quasicategories CAT and SET. The *object* quasifunctor $obj = |\text{-}| : \mathrm{CAT} \rightarrow \mathrm{SET}$ that maps a category (that is, whose object collection function $obj(obj) : obj(\mathrm{CAT}) \rightarrow obj(\mathrm{SET})$ maps a category) to its class of objects and that maps a functor (that is, whose morphism collection function $mor(obj) : mor(\mathrm{CAT}) \rightarrow mor(\mathrm{SET})$ maps a functor) to its object class function. The *morphism* quasifunctor $mor : \mathrm{CAT} \rightarrow \mathrm{SET}$ that maps a category (that is, whose object collection function $obj(mor) : obj(\mathrm{CAT}) \rightarrow obj(\mathrm{SET})$ maps a category) to its class of morphisms and that maps a functor (that is, whose morphism collection function $mor(mor) : mor(\mathrm{CAT}) \rightarrow mor(\mathrm{SET})$ maps a functor) to its morphism class function. Functoriality of the object and morphism quasifunctors is expressed by the facts that functor composition $\boldsymbol{F} \circ \boldsymbol{G} : \mathbf{A} \rightarrow \mathbf{C}$ of two composable functors $\boldsymbol{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\boldsymbol{G} : \mathbf{B} \rightarrow \mathbf{C}$ is defined in terms of component function composition $obj(\boldsymbol{F} \circ \boldsymbol{G}) = obj(\boldsymbol{F}) \cdot obj(\boldsymbol{G})$ and $mor(\boldsymbol{F} \circ \boldsymbol{G}) = mor(\boldsymbol{F}) \cdot mor(\boldsymbol{G})$, and functor identity $id_{\boldsymbol{A}} : \mathbf{A} \rightarrow \mathbf{A}$ for a category $\mathbf{A}$ is defined in terms of component function identity $obj(id_{\boldsymbol{A}}) = id_{obj(\boldsymbol{A})}$ and $mor(id_{\boldsymbol{A}}) = id_{mor(\boldsymbol{A})}$.

```
(4) (FUNC2$quasifunctor object)
    (= (FUNC2$source object) CAT)
    (= (FUNC2$target object) SET)
    (= (FUNC2$object object) CAT$object)
    (= (FUNC2$morphism object) FUNC$object)

(5) (FUNC2$quasifunctor morphism)
    (= (FUNC2$source morphism) CAT)
    (= (FUNC2$target morphism) SET)
    (= (FUNC2$object morphism) CAT$object)
    (= (FUNC2$morphism morphism) FUNC$object)
```

- There are (Figure 1a) two quasi-natural transformations between the quasifunctors mor and obj. For any category $\mathbf{A}$, the *source* quasi-natural transformation $src = \partial_0 : mor \rightarrow obj : \mathrm{CAT} \rightarrow \mathrm{SET}$ has the source class function $src(\mathbf{A}) : mor(\mathbf{A}) \rightarrow obj(\mathbf{A})$ for the category $\mathbf{A}$ as its $\mathbf{A}^{\text{th}}$ component; the *target* quasi-natural transformation $tgt = \partial_1 : mor \rightarrow obj : \mathrm{CAT} \rightarrow \mathrm{SET}$ has the target class function $tgt(\mathbf{A}) : mor(\mathbf{A}) \rightarrow obj(\mathbf{A})$ for the category $\mathbf{A}$ as its $\mathbf{A}^{\text{th}}$ component. Naturality of the source and target quasi-natural transformations is expressed (Figure 1b) by the facts that a functor $\boldsymbol{F} : \mathbf{A} \rightarrow \mathbf{B}$ preserves source and target.

```
(6) (NAT2$quasi-natural-transformation source)
    (= (NAT2$source-category source) CAT)
    (= (NAT2$target-category source) SET)
    (= (NAT2$source-functor source) morphism)
    (= (NAT2$target-functor source) object)
    (= (NAT2$component source) CAT$source)

(7) (NAT2$quasi-natural-transformation target)
    (= (NAT2$source-category target) CAT)
    (= (NAT2$target-category target) SET)
    (= (NAT2$source-functor target) morphism)
    (= (NAT2$target-functor target) object)
    (= (NAT2$component target) CAT$target)
```

$mor(\mathbf{F}) : mor(\mathbf{A}) \to mor(\mathbf{B})$
$obj(\mathbf{F}) : obj(\mathbf{A}) \to obj(\mathbf{B})$

$\mathbf{F} : \mathbf{A} \to \mathbf{B}$

$$
\begin{tikzcd}
mor(\mathsf{A}) \arrow[r, "mor(F)"] \arrow[d, shift right, "src(\mathsf{A})"'] \arrow[d, shift left, "tgt(\mathsf{A})"] & mor(\mathsf{CAT}) \arrow[r, shift left, "mor(mor)"] \arrow[r, shift right, "mor(obj)"'] \arrow[d, shift right, "src(\mathsf{CAT})"'] \arrow[d, shift left, "tgt(\mathsf{CAT})"] & mor(\mathsf{SET}) \arrow[d, shift right, "src(\mathsf{SET})"'] \arrow[d, shift left, "tgt(\mathsf{SET})"] \\
obj(\mathsf{A}) \arrow[r, "obj(F)"'] & obj(\mathsf{CAT}) \arrow[r, shift left, "obj(mor)"] \arrow[r, shift right, "obj(obj)"'] & obj(\mathsf{SET})
\end{tikzcd}
$$

$\mathbf{A}$ $\mathbf{B}$

$mor(\mathbf{A})$ $mor(\mathbf{B})$
$obj(\mathbf{A})$ $obj(\mathbf{B})$

$$
\begin{tikzcd}
\mathsf{A} \arrow[r, "F"] & \mathsf{CAT} \arrow[r, shift left, "mor"] \arrow[r, shift right, "obj"'] & \mathsf{SET}
\end{tikzcd}
$$

Figure 2: Object and morphism operators for quasicategories and quasifunctors

- A CAT-based quasifunctor $F : \mathsf{A} \to \mathsf{CAT}$ (Figure 2) has an associated object collection function $obj(F) : obj(\mathsf{A}) \to obj(\mathsf{CAT})$ that maps an object $a \in obj(\mathsf{A})$ to the category $F(a)$ with the object class denoted by

  $$obj(F(a)) = obj(obj(F)(a)) = obj(obj)(obj(F)(a)) = (obj(F) \cdot obj(obj))(a)) = obj(F \circ obj)(a),$$

  (the first term is the usual shorthand, the third term is strictly accurate, the fourth term uses function composition, and equality with the fifth term uses the object component definition of quasifunctor composition, and the morphism class denoted by

  $$mor(F(a)) = mor(obj(F)(a)) = obj(mor)(obj(F)(a)) = (obj(F) \cdot obj(mor))(a)) = obj(F \circ mor)(a).$$

- A CAT-based quasifunctor $F : \mathsf{A} \to \mathsf{CAT}$ (Figure 2) also has an associated morphism collection function $mor(F) : mor(\mathsf{A}) \to mor(\mathsf{CAT})$ that maps a morphism $m \in mor(\mathsf{A})$, with source/target $m : a_1 \to a_2$, to the functor $F(m) : F(a_1) \to F(a_2)$ with the object class function denoted by

  $$\begin{aligned}
  &obj(F(m)) : obj(F(a_1)) \to obj(F(a_2)) \\
  &= obj(mor(F)(m)) : obj(obj(F)(a_1)) \to obj(obj(F)(a_2)) \\
  &= mor(obj)(mor(F)(m)) : obj(obj)(obj(F)(a_1)) \to obj(obj)(obj(F)(a_2)) \\
  &= (mor(F) \cdot mor(obj))(m)) : (obj(F) \cdot obj(obj))(a_1)) \to (obj(F) \cdot obj(obj))(a_2)) \\
  &= mor(F \circ obj)(m)) : obj(F \circ obj)(a_1) \to obj(F \circ obj)(a_2),
  \end{aligned}$$

  and the morphism class function denoted by

  $$\begin{aligned}
  &mor(F(m)) : mor(F(a_1)) \to mor(F(a_2)) \\
  &= mor(mor(F)(m)) : mor(obj(F)(a_1)) \to mor(obj(F)(a_2)) \\
  &= mor(mor)(mor(F)(m)) : obj(mor)(obj(F)(a_1)) \to obj(mor)(obj(F)(a_2)) \\
  &= (mor(F) \cdot mor(mor))(m)) : (obj(F) \cdot obj(mor))(a_1)) \to (obj(F) \cdot obj(mor))(a_2)) \\
  &= mor(F \circ mor)(m) : obj(F \circ mor)(a_1) \to obj(F \circ mor)(a_2).
  \end{aligned}$$

  These mappings describe

  - the associated object composite quasifunctor $F \circ obj = F \circ |\text{-}| = |F| : \mathsf{A} \to \mathsf{CAT} \to \mathsf{SET}$ and
  - the associated morphism composite quasifunctor $F \circ mor : \mathsf{A} \to \mathsf{CAT} \to \mathsf{SET}$.

- Another useful quasicategory is the quasicategory $\mathsf{CLS} = \mathsf{CLASSIFICATION}$ of large classifications and infomorphisms. There are two component quasifunctors associated with CLS: the contravariant instance quasifunctor $inst : \mathsf{CLS} \to \mathsf{SET}^{op}$ and the covariant type quasifunctor $typ : \mathsf{CLS} \to \mathsf{SET}$.

```
(8) (CAT2$quasicategory CLS)
    (= (CAT2$object CLS) CLS$classification)
    (= (CAT2$morphism CLS) CLS.INFO$infomorphism)
```

```
 (9)  (FUNC2$quasifunctor instance)
      (= (FUNC2$source instance) CLS)
      (= (FUNC2$target instance) SET)
      (= (FUNC2$object instance) CLS$instance)
      (= (FUNC2$morphism instance) CLS.INFO$instance)

(10)  (FUNC2$quasifunctor type)
      (= (FUNC2$source type) CLS)
      (= (FUNC2$target type) SET)
      (= (FUNC2$object type) CLS$instance)
      (= (FUNC2$morphism type) CLS.INFO$instance)
```

The Context of Institutions

`INS`

Table 2: Institution Terminology

	category/collection/	**functor/function**	**other**

`INS`	`institution`
	`signature`
	`specification`
	`theory`
		`model`
		`sentence`
		`truth`
		`base theory-base`
		`inclusion closure`
			`eta`
			`kappa`

	`signature-object`
	`signature-morphism`
	`specification-object`
	`specification-morphism`
	`theory-object`
	`theory-morphism`
		`model-object`
		`model-morphism`
		`sentence-object`
		`sentence-morphism`
		`truth-classification`
		`truth-infomorphism`
		`base-object`
		`base-morphism`
		`closure-object`
		`closure-morphism`
			`eta-component`
			`satisfaction`

		`subset-closure`
		`axiom`
		`theorem`
		`specification-closure`
			`subset-satisfaction`
			`entailment`

`FBR`	`specification-fiber` `theory-fiber`		

Institutions formalize the intuitive notion of an abstract logical system. This includes syntax, semantics and satisfaction. For more information, see any paper listed on the Theory of Institutions web page, such as the paper "Institutions: Abstract Model Theory for Specification and Programming" by Goguen and Burstall.

$$
\begin{tikzcd}
\mathsf{Set} \arrow[r, "incl"] & \mathsf{SET} \\
\mathsf{Sign}(\mathbb{I}) \arrow[u, "sen(\mathbb{I})"] \arrow[r, "func(\mathbb{I})"] \arrow[d, "mod(\mathbb{I})"'] & \mathsf{CLS} \arrow[u, "typ"'] \arrow[d, "inst"] \\
\mathsf{CAT} \arrow[r, "obj = |\text{-}|"'] & \mathsf{SET}
\end{tikzcd}
$$

Figure 3: Institution

$$
\Sigma_1 \xrightarrow{\sigma} \Sigma_2
$$

$$
\begin{tikzcd}
sen(\mathbb{I})(\Sigma_1) \arrow[r, "sen(\mathbb{I})(\sigma)"] \arrow[d, dash, "\models_{\mathbb{I},\Sigma 1}"] & sen(\mathbb{I})(\Sigma_2) \arrow[d, dash, "\models_{\mathbb{I},\Sigma 2}"] \\
|mod(\mathbb{I})(\Sigma_1)| & |mod(\mathbb{I})(\Sigma_2)| \arrow[l]
\end{tikzcd}
$$

$$|mod(\mathbb{I})(\Sigma_2)| = obj(mod(\mathbb{I}) \circ obj)(\Sigma_2)$$

$$|mod(\mathbb{I})(\sigma)| = mor(mod(\mathbb{I}) \circ obj)(\sigma)$$

Figure 4: satisfaction condition

An *institution* or a *truth construction* $\mathbb{I} = \langle \mathsf{Sign}(\mathbb{I}), mod(\mathbb{I}), sen(\mathbb{I}), \models_{\mathbb{I}} \rangle$ (Figure 3) consists of

- a category $\mathsf{Sign}(\mathbb{I})$ of *signatures* and signature morphisms,
- a quasifunctor $mod(\mathbb{I}) : \mathsf{Sign}(\mathbb{I}) \rightarrow \mathsf{CAT}^{op}$ mapping a signature to its category of *models* and mapping a signature morphism to its (contravariant) model fiber functor,
- a functor $sen(\mathbb{I}) : \mathsf{Sign}(\mathbb{I}) \rightarrow \mathsf{Set}$ mapping a signature to its set of *sentences* and mapping a signature morphism to its sentence function, and
- a $|\mathsf{Sign}(\mathbb{I})|$-indexed *satisfaction* relation

$$\models_{\mathbb{I}} = \{\models_{\mathbb{I},\Sigma} \mid \Sigma \in |\mathsf{Sign}(\mathbb{I})|\} \text{ with } \models_{\mathbb{I},\Sigma} \subseteq |mod(\mathbb{I})(\Sigma)| \times sen(\mathbb{I})(\Sigma).$$

For any signature morphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$, an institution must satisfy the *satisfaction condition* (Figure 4)

$$|\mathit{mod}(\mathbb{I})(\sigma)|(m_2) \models_{\mathbb{I}, \Sigma 1} e_1 \ \underline{\text{iff}}\ m_2 \models_{\mathbb{I}, \Sigma 2} \mathit{sen}(\mathbb{I})(\sigma)(e_1)$$

for every source sentence $e_1 \in \mathit{sen}(\mathbb{I})(\Sigma_1)$ and every target model $m_2 \in |\mathit{mod}(\mathbb{I})(\Sigma_2)|$.

```
(1) (KIF$collection institution)

(2) (KIF$function signature)
    (= (KIF$source signature) institution)
    (= (KIF$target signature) CAT$category)

(3) (KIF$function model)
    (= (KIF$source model) institution)
    (= (KIF$target model) FUNC2$quasifunctor)
    (forall (?I (institution ?I))
        (and (= (FUNC2$source (model ?I)) (signature ?I))
             (= (FUNC2$target (model ?I)) CAT2$quasicategory)))

(4) (KIF$function sentence)
    (= (KIF$source sentence) institution)
    (= (KIF$target sentence) FUNC$functor)
    (forall (?I (institution ?I))
        (and (= (FUNC$source (sentence ?I)) (signature ?I))
             (= (FUNC$target (sentence ?I)) set$set)))
```

For ease of reference, we decompose the components of an institution into object and morphism classes/functions; that is, we name the object and morphism subcomponents of the signature, model and sentence components.

```
(2.1) (KIF$function signature-object)
      (= (KIF$source signature-object) institution)
      (= (KIF$target signature-object) SET$class)
      (forall (?I (institution ?I))
          (= (signature-object ?I) (CAT$object (signature ?I))))

(2.2) (KIF$function signature-morphism)
      (= (KIF$source signature-morphism) institution)
      (= (KIF$target signature-morphism) SET$class)
      (forall (?I (institution ?I))
           (= (signature-morphism ?I) (CAT$morphism (signature ?I))))

(3.1) (KIF$function model-object)
      (= (KIF$source model-object) institution)
      (= (KIF$target model-object) KIF$function)
      (forall (?I (institution ?I))
          (and (= (KIF$source (model-object ?I)) (signature-object ?I))
               (= (KIF$target (model-object ?I)) CAT$category)
               (= (model-object ?I) (FUNC2$object (model ?I)))))

(3.2) (KIF$function model-morphism)
      (= (KIF$source model-morphism) institution)
      (= (KIF$target model-morphism) KIF$function)
      (forall (?I (institution ?I))
          (and (= (KIF$source (model-morphism ?I)) (signature-morphism ?I))
               (= (KIF$target (model-morphism ?I)) FUNC$functor)
               (= (model-morphism ?I) (FUNC2$morphism (model ?I)))))

(4.1) (KIF$function sentence-object)
      (= (KIF$source sentence-object) institution)
      (= (KIF$target sentence-object) SET.FTN$function)
      (forall (?I (institution ?I))
          (and (= (SET.FTN$source (sentence-object ?I)) (signature-object ?I))
               (= (SET.FTN$target (sentence-object ?I)) set.obj$object)
               (= (sentence-object ?I) (FUNC$object (sentence ?I)))))

(4.2) (KIF$function sentence-morphism)
      (= (KIF$source sentence-morphism) institution)
```

```
      (= (KIF$target sentence-morphism) SET.FTN$function)
      (forall (?I (institution ?I))
          (and (= (SET.FTN$source (sentence-morphism ?I)) (signature-morphism ?I))
               (= (SET.FTN$target (sentence-morphism ?I)) set.mor$morphism)
               (= (sentence-morphism ?I) (FUNC$morphism (sentence ?I)))))
```

We continue with the specification of the satisfaction component of an institution and the axiomatic expression of the satisfaction condition.

```
 (5) (KIF$function satisfaction)
     (= (KIF$source satisfaction) institution)
     (= (KIF$target satisfaction) KIF$function)
     (forall (?I (institution ?I))
         (and (= (KIF$source (satisfaction ?I)) (signature-object ?I))
              (= (KIF$target (satisfaction ?I)) REL$relation)
              (forall (?S ((signature-object ?I) ?S))
                  (and (= (REL$class1 ((satisfaction ?I) ?S))
                          (CAT$object ((model-object ?I) ?S)))
                       (= (REL$class2 ((satisfaction ?I) ?S))
                          ((sentence-object ?I) ?S))))))

 (6) (forall (?I (institution ?I)
              ?s ((signature-morphism ?I) ?s)
              ?e1 (((sentence-object ?I) ((CAT$source (signature ?I)) ?s)) ?e1)
              ?m2 ((CAT$object ((model-object ?I) ((CAT$target (signature ?I)) ?s))) ?m2))
         (<=> (((satisfaction ?I) ?S) ((FUNC$object ((model-morphism ?I) ?s)) ?m2) ?e1)
              (((satisfaction ?I) ?S) ?m2 (((sentence-morphism ?I) ?s) ?e1))))
```

Clearly, the satisfaction condition is the fundamental condition for (large) infomorphisms. Hence, any institution has an associated truth quasifunctor

$$\mathit{truth}(\mathbb{I}) : \mathsf{Sign}(\mathbb{I}) \rightarrow \mathsf{CLS}.$$

In the passage from the institution or truth structure to the truth quasifunctor, the morphic aspect of models is forgotten. We use the object class quasifunctor $\mathit{obj} = |\text{-}| : \mathsf{CAT} \rightarrow \mathsf{SET}$ to do this forgetting – it maps a category to its object class and maps a functor to its object class function. This means that composition of the truth quasifunctor with the component instance contravariant quasifunctor $\mathit{inst} : \mathsf{CLS} \rightarrow \mathsf{SET}^{\mathrm{op}}$ gives the model-object quasifunctor $\mathit{mod}(\mathbb{I}) \circ \mathit{obj} : \mathsf{A} \rightarrow \mathsf{CAT}^{\mathrm{op}} \rightarrow \mathsf{SET}^{\mathrm{op}}$. In addition, composition of the truth quasifunctor with the component type quasifunctor $\mathit{typ} : \mathsf{CLS} \rightarrow \mathsf{SET}$ gives the sentence quasifunctor $\mathit{sen}(\mathbb{I}) \circ \mathit{incl} : \mathsf{A} \rightarrow \mathsf{Set} \rightarrow \mathsf{SET}$, where the category of sets is included into the quasicategory of classes. The object component of $\mathit{truth}(\mathbb{I})$ maps signatures to the satisfaction relation regarded as a large classification. The morphism component of $\mathit{truth}(\mathbb{I})$ maps signature morphisms to the infomorphism between the satisfaction classifications of its source and target, with the type component being the expression function, the instance component being the model fiber function (object component of the contravariant model functor), and the satisfaction condition being the fundamental condition for the infomorphism.

```
(7) (KIF$function truth)
     (= (KIF$source truth) institution)
     (= (KIF$target truth) FUNC2$quasifunctor)
     (forall (?I (institution ?I))
         (and (= (FUNC2$source (truth ?I)) (signature ?I))
              (= (FUNC2$target (truth ?I)) CLS$CLS)
              (= (FUNC2$composition [(truth ?I) CLS$instance])
                 (FUNC2$composition [(model ?I) FUNC2$object]))
              (= (FUNC2$composition [(truth ?I) CLS$type])
                 (FUNC2$composition [(sentence ?I) CAT2$inclusion]))
              (= (FUNC2$object (truth ?I)) (truth-classification ?I))
              (= (FUNC2$morphism (truth ?I)) (truth-infomorphism ?I))))

 (7.1) (KIF$function truth-classification)
       (= (KIF$source truth-classification) institution)
       (= (KIF$target truth-classification) KIF$function)
       (forall (?I (institution ?I))
```

```
        (and (= (KIF$source (truth-classification ?I)) (signature-object ?I))
             (= (KIF$target (truth-classification ?I)) CLS$classification)
             (forall (?S ((signature-object ?I) ?S))
                 (and (= (CLS$instance ((truth-classification ?I) ?S))
                         (CAT$object ((model-object ?I) ?S)))
                      (= (CLS$type ((truth-classification ?I) ?S))
                         ((sentence-object ?I) ?S))
                      (= ((truth-classification ?I) ?S)
                         ((satisfaction ?I) ?S))))))

(7.2) (KIF$function truth-infomorphism)
      (= (KIF$source truth-infomorphism) institution)
      (= (KIF$target truth-infomorphism) KIF$function)
      (forall (?I (institution ?I))
          (and (= (KIF$source (truth-infomorphism ?I)) (signature-morphism ?I))
               (= (KIF$target (truth-infomorphism ?I)) CLS.INFO$infomorphism)
               (forall (?s ((signature-morphism ?I) ?s))
                   (and (= (CLS.INFO$instance ((truth-infomorphism ?I) ?s))
                           (FUNC$object ((model-morphism ?I) ?s)))
                        (= (CLS.INFO$type ((truth-infomorphism ?I) ?s))
                           ((sentence-morphism ?I) ?s))))))
```

An institution $\mathbb{I} = \langle \mathsf{Sign}(\mathbb{I}), \mathit{mod}(\mathbb{I}), \mathit{sen}(\mathbb{I}), \models_{\mathbb{I}} \rangle$ has associated with it

- a $|\mathsf{Sign}|$-indexed *satisfaction* relation $\models_{\mathbb{I}} = \{\models_{\mathbb{I},\Sigma} \subseteq |\mathit{mod}(\mathbb{I})(\Sigma)| \times \wp\mathit{sen}(\mathbb{I})(\Sigma) \mid \Sigma \in |\mathsf{Sign}(\mathbb{I})|\}$ that extends satisfaction to subsets of sentences and is define by

  $$m \models_{\mathbb{I},\Sigma} E \ \underline{\text{iff}}\ (\forall_{e \in E})(m \models_{\mathbb{I},\Sigma} e)$$

  for any model $m \in |\mathit{mod}(\mathbb{I})(\Sigma)|$ for any subset of sentences $E \subseteq \mathit{sen}(\mathbb{I})(\Sigma)$.
- a $|\mathsf{Sign}|$-indexed *entailment* relation $\vdash_{\mathbb{I}} = \{\vdash_{\mathbb{I},\Sigma} \subseteq \wp\mathit{sen}(\mathbb{I})(\Sigma) \times \mathit{sen}(\mathbb{I})(\Sigma) \mid \Sigma \in |\mathsf{Sign}|\}$, defined by

  $$E \vdash_{\mathbb{I},\Sigma} e \ \underline{\text{iff}}\ (\forall_{m \in |\mathit{mod}(\mathbb{I})(\Sigma)|})(\, m \models_{\mathbb{I},\Sigma} E \text{ implies } m \models_{\mathbb{I},\Sigma} e)$$

  for any subset of sentences $E \subseteq \mathit{sen}(\mathbb{I})(\Sigma)$ and Σ-sentence $e \in \mathit{sen}(\mathbb{I})(\Sigma)$.
- a $|\mathsf{Sign}|$-indexed *subset closure* function $(\text{-})^{\bullet}_{\mathbb{I}} = \{(\text{-})^{\bullet}_{\mathbb{I},\Sigma} : \wp\mathit{sen}(\mathbb{I})(\Sigma) \rightarrow \wp\mathit{sen}(\mathbb{I})(\Sigma) \mid \Sigma \in |\mathsf{Sign}|\}$, defined by

  $$E^{\bullet}_{\mathbb{I},\Sigma} = \{e \in \mathit{sen}(\mathbb{I})(\Sigma) \mid E \vdash_{\mathbb{I},\Sigma} e\}$$

  for any subset of sentences $E \subseteq \mathit{sen}(\mathbb{I})(\Sigma)$. Subset closure is a closure operator.

```
(8) (KIF$function subset-satisfaction)
    (= (KIF$source subset-satisfaction) institution)
    (= (KIF$target subset-satisfaction) KIF$function)
    (forall (?I (institution ?I))
        (and (= (KIF$source (subset-satisfaction ?I)) (signature-object ?I))
             (= (KIF$target (subset-satisfaction ?I)) REL$relation)
             (forall (?S ((signature-object ?I) ?S))
                 (and (= (REL$class1 ((subset-satisfaction ?I) ?S))
                         (CAT$object ((model-object ?I) ?S)))
                      (= (REL$class2 ((subset-satisfaction ?I) ?S))
                         (set$power ((sentence-object ?I) ?S)))
                      (forall (?m ((CAT$object ((model-object ?I) ?S)) ?m)
                               ?E ((set$power ((sentence-object ?I) ?S)) ?E))
                          (<=> (((subset-satisfaction ?I) ?S) ?m ?E)
                               (forall (?e (?E ?e))
                                   (((satisfaction ?I) ?S) ?m ?e)))))))))

(9) (KIF$function entailment)
    (= (KIF$source entailment) institution)
    (= (KIF$target entailment) SET.FTN$function)
    (forall (?I (institution ?I))
        (and (= (SET.FTN$source (entailment ?I)) (signature-object ?I))
             (= (SET.FTN$target (entailment ?I)) rel$relation)
             (= (SET.FTN$composition [(entailment ?I) rel$set1])
                (SET.FTN$composition [(sentence-object ?I) set$power]))
             (= (SET.FTN$composition [(entailment ?I) rel$set2])
```

```
                (sentence-object ?I))
             (forall (?S ((signature-object ?I) ?S)
                      ?E ((set$power ((sentence-object ?I) ?S)) ?E)
                      ?e (((sentence-object ?I) ?S) ?e))
                 (<=> (((entailment ?I) ?S) ?E ?e)
                      (forall (?m ((CAT$object ((model-object ?I) ?S)) ?m))
                          (=> (((subset-satisfaction ?I) ?S) ?m ?E)
                              (((satisfaction ?I) ?S) ?m ?e)))))))

(10) (KIF$function subset-closure)
     (= (KIF$source subset-closure) institution)
     (= (KIF$target subset-closure) SET.FTN$function)
     (forall (?I (institution ?I))
         (and (= (SET.FTN$source (subset-closure ?I)) (signature-object ?I))
              (= (SET.FTN$target (subset-closure ?I)) set.mor$morphism)
              (= (SET.FTN$composition [(subset-closure ?I) set.mor$source])
                 (SET.FTN$composition [(sentence-object ?I) set$power]))
              (= (SET.FTN$composition [(subset-closure ?I) set.mor$target])
                 (SET.FTN$composition [(sentence-object ?I) set$power]))
              (forall (?S ((signature-object ?I) ?S))
                  (= ((subset-closure ?I) ?S)
                     (rel$fiber12 ((entailment ?I) ?S))))))

(11) (forall (?I (institution ?I)
             ?S ((signature-object ?I) ?S))
         (ord.mor$closure-operator ((subset-closure ?I) ?S)))
```

For any institution $\mathbb{I} = \langle \mathsf{Sign}(\mathbb{I}), \mathit{mod}(\mathbb{I}), \mathit{sen}(\mathbb{I}), \models_{\mathbb{I}} \rangle$, a *specification* is a pair $S = \langle \Sigma, E \rangle$ where $\Sigma \in |\mathsf{Sign}(\mathbb{I})|$ is a signature and $E \subseteq \mathit{sen}(\mathbb{I})(\Sigma)$ is a subset of sentences. A *specification morphism* $\sigma : S_1 \rightarrow S_2$ from the source specification $S_1 = \langle \Sigma_1, E_1 \rangle$ to the target specification $S_2 = \langle \Sigma_2, E_2 \rangle$ is a signature morphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ that maps source axioms of target theorems: $\wp\sigma(E_1) \subseteq E_2^{\bullet}$. The category of specifications and specification morphisms of an institution $\mathbb{I}$ is denoted $\mathsf{Spec}(\mathbb{I})$. There is an *axiom* class function $\mathit{axm}(\mathbb{I}) : |\mathsf{Spec}(\mathbb{I})| \rightarrow |\mathsf{Set}|$, where $\mathit{axm}(\mathbb{I})(S) \subseteq \mathit{sen}(\mathbb{I})(\mathit{base}(\mathbb{I})(S))$ for every specification $S \in |\mathsf{Spec}(\mathbb{I})|$, and a *theorem* class function $\mathit{thm}(\mathbb{I}) : |\mathsf{Spec}(\mathbb{I})| \rightarrow |\mathsf{Set}|$, which pointwise is the closure of the axiom of a specification: $\mathit{thm}(\mathbb{I})(S) = (\mathit{axm}(\mathbb{I})(S))^{\bullet}_{\mathbb{I},\, \mathit{base}(\mathbb{I})(S)} \subseteq \mathit{sen}(\mathbb{I})(\mathit{base}(\mathbb{I})(S))$ for specification $S \in |\mathsf{Spec}(\mathbb{I})|$. A *theory* $T = \langle \Sigma, E \rangle$ is a specification with $E = E^{\bullet}$. The specification closure $(\text{-})^{\bullet}_{\mathbb{I}} : |\mathsf{Spec}(\mathbb{I})| \rightarrow |\mathsf{Spec}(\mathbb{I})|$ is the function that maps a specification to its closure theory. The full subcategory of theories in $\mathsf{Spec}(\mathbb{I})$ is denoted $\mathsf{Th}(\mathbb{I})$ (Figure 5). An institution $\mathbb{I}$ has associated with it a functor $\mathit{base}(\mathbb{I}) : \mathsf{Spec}(\mathbb{I}) \rightarrow \mathsf{Sign}(\mathbb{I})$ mapping a specification to its *base* or underlying signature and mapping a specification morphism to its base signature morphism.

$\mathsf{Th}(\mathbb{I}) \subseteq \mathsf{Spec}(\mathbb{I})$

Figure 5: Full Subcategory

```
(12) (KIF$function specification)
     (= (KIF$source specification) institution)
     (= (KIF$target specification) CAT$category)

(12.1) (KIF$function specification-object)
       (= (KIF$source specification-object) institution)
       (= (KIF$target specification-object) SET$class)
       (forall (?I (institution ?I))
           (= (specification-object ?I) (CAT$object (specification ?I))))

(12.2) (KIF$function specification-morphism)
       (= (KIF$source specification-morphism) institution)
       (= (KIF$target specification-morphism) SET$class)
       (forall (?I (institution ?I))
           (and (= (specification-morphism ?I) (CAT$morphism (specification ?I)))
                (forall (?g ((specification-morphism ?I) ?g))
                    (set$subset
                        ((set.mor$direct-image ((sentence-morphism ?I) ((base-morphism ?I) ?g)))
                            ((axiom ?I) ((CAT$source (specification ?I)) ?g)))
                        ((theorem ?I) ((CAT$target (specification ?I)) ?g))))))
```

```
(13) (KIF$function base)
     (= (KIF$source base) institution)
     (= (KIF$target base) FUNC$functor)
     (forall (?I (institution ?I))
         (and (= (FUNC$source (base ?I)) (specification ?I))
              (= (FUNC$target (base ?I)) (signature ?I))
              (= (FUNC$object (base ?I)) (base-object ?I))
              (= (FUNC$morphism (base ?I)) (base-morphism ?I))))

(13.1) (KIF$function base-object)
       (= (KIF$source base-object) institution)
       (= (KIF$target base-object) SET.FTN$function)
       (forall (?I (institution ?I))
           (and (= (SET.FTN$source (base-object ?I)) (specification-object ?I))
                (= (SET.FTN$target (base-object ?I)) (signature-object ?I))))

(13.2) (KIF$function base-morphism)
       (= (KIF$source base-morphism) institution)
       (= (KIF$target base-morphism) SET.FTN$function)
       (forall (?I (institution ?I))
           (and (= (SET.FTN$source (base-morphism ?I)) (specification-morphism ?I))
                (= (SET.FTN$target (base-morphism ?I)) (signature-morphism ?I))))

(14) (KIF$function axiom)
     (= (KIF$source axiom) institution)
     (= (KIF$target axiom) SET.FTN$function)
     (forall (?I (institution ?I))
         (and (= (SET.FTN$source (axiom ?I)) (specification-object ?I))
              (= (SET.FTN$target (axiom ?I)) set.obj$object)
              (forall (?S ((CAT$object (specification ?I)) ?S))
                  (set$subset ((axiom ?I) ?S)
                              ((sentence ?I) ((base-object ?I) ?S))))))

(15) (KIF$function theorem)
     (= (KIF$source theorem) institution)
     (= (KIF$target theorem) SET.FTN$function)
     (forall (?I (institution ?I))
         (and (= (SET.FTN$source (theorem ?I)) (specification-object ?I))
              (= (SET.FTN$target (theorem ?I)) set.obj$object)
              (forall (?S ((specification-object ?I) ?S))
                  (= ((theorem ?I) ?S)
                     (((subset-closure ?I) ((base-object ?I) ?S))
                         ((axiom ?I) ?S))))))

(16) (KIF$function theory)
     (= (KIF$source theory) institution)
     (= (KIF$target theory) CAT$category)
     (forall (?I (institution ?I))
         (and (CAT$full-subcategory (theory ?I) (specification ?I))
              (= (CAT$object (theory ?I)) (theory-object ?I))))

(16.1) (KIF$function theory-object)
       (= (KIF$source theory-object) institution)
       (= (KIF$target theory-object) SET$class)
       (forall (?I (institution ?I))
           (and (SET$subclass (theory-object ?I) (specification-object ?I))
                (forall (?S ((specification-object ?I) ?S))
                    (<=> ((theory-object ?I) ?S)
                         (= ((axiom ?I) ?S) ((theorem ?I) ?S))))))

(16.2) (KIF$function theory-morphism)
       (= (KIF$source theory-morphism) institution)
       (= (KIF$target theory-morphism) SET$class)
       (forall (?I (institution ?I))
           (= (theory-morphism ?I) (CAT$morphism (theory ?I))))

(17) (KIF$function specification-closure)
     (= (KIF$source specification-closure) institution)
     (= (KIF$target specification-closure) SET.FTN$function)
```

```
     (forall (?I (institution ?I))
         (and (= (SET.FTN$source (specification-closure ?I)) (specification-object ?I))
              (= (SET.FTN$target (specification-closure ?I)) (specification-object ?I))
              (forall (?S ((specification-object ?I) ?S))
                  (= ((axiom ?I) ((specification-closure ?I) ?S))
                     (((subset-closure ?I) ((base-object ?I) ?S)) ((axiom ?I) ?S))))))
```

$$\begin{array}{c}\boldsymbol{\mathcal{F}}(\mathbb{I}) \dashv \boldsymbol{\mathcal{U}}(\mathbb{I})\\ \eta,\ 1\end{array}$$

$$\begin{tikzcd}
S \arrow[r, "\eta_{\mathbb{I},S} = id"] \arrow[dr, "s"'] & \boldsymbol{\mathcal{U}}(\mathbb{I})(\boldsymbol{\mathcal{F}}(\mathbb{I})(S)) = S^{\bullet} \arrow[d, "\boldsymbol{\mathcal{U}}(\mathbb{I})(t) = t"] & \boldsymbol{\mathcal{F}}(\mathbb{I})(S) = S^{\bullet} \arrow[d, "t"] \\
& \boldsymbol{\mathcal{U}}(\mathbb{I})(T) = T & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T = \boldsymbol{\mathcal{F}}(\mathbb{I})(\boldsymbol{\mathcal{U}}(\mathbb{I})(T)) & \boldsymbol{\mathcal{U}}(\mathbb{I})(T) = T \\
\boldsymbol{\mathcal{F}}(\mathbb{I})(S) = S^{\bullet} \arrow[u, "\boldsymbol{\mathcal{F}}(\mathbb{I})(s) = s"'] \arrow[u, shift left=6, "t"] & S \arrow[u, "s"']
\end{tikzcd}$$

Diagram 1a: Universal Morphism **Diagram 1b: Couniversal Morphism**

The *inclusion functor* $\boldsymbol{\mathcal{U}}(\mathbb{I}) : \mathsf{Th}(\mathbb{I}) \rightarrow \mathsf{Spec}(\mathbb{I})$ is an equivalence of categories: it has a left-adjoint-right-inverse *closure functor* $\boldsymbol{\mathcal{F}}(\mathbb{I}) : \mathsf{Spec}(\mathbb{I}) \rightarrow \mathsf{Th}(\mathbb{I})$ defined by $\boldsymbol{\mathcal{F}}(\mathbb{I})(S) = \boldsymbol{\mathcal{F}}(\mathbb{I})(\langle \Sigma, E \rangle) = \langle \Sigma, E^{\bullet} \rangle = S^{\bullet}$ on objects and identity on the base of morphisms; that is, $\boldsymbol{\mathcal{U}}(\mathbb{I}) \circ \boldsymbol{\mathcal{F}}(\mathbb{I}) = \mathit{id}$ and $\boldsymbol{\mathcal{F}}(\mathbb{I}) \dashv \boldsymbol{\mathcal{U}}(\mathbb{I})$. The unit natural transformation $\eta_{\mathbb{I}} : 1 \Rightarrow \boldsymbol{\mathcal{F}}(\mathbb{I}) \circ \boldsymbol{\mathcal{U}}(\mathbb{I})$ (Figure 6) has the specification <u>iso</u>morphism with identity base $\eta_{\mathbb{I},S} = id : S \Rightarrow S^{\bullet}$ as its S^{th} component, for any specification $S \in |\mathsf{Spec}(\mathbb{I})|$. The counit is the identity. Since this is an equivalence, $\boldsymbol{\mathcal{F}}(\mathbb{I})$ is also a right-adjoint-right-inverse to $\boldsymbol{\mathcal{U}}(\mathbb{I})$. Thus, $\mathsf{Th}(\mathbb{I})$ is both a reflective and coreflective subcategory of $\mathsf{Spec}(\mathbb{I})$. Equivalent categories are complete (cocomplete) together. We assert that $\kappa = \langle \boldsymbol{\mathcal{F}}(\mathbb{I}), \boldsymbol{\mathcal{U}}(\mathbb{I}), \eta, 1 \rangle : \mathsf{Spec}(\mathbb{I}) \rightarrow \mathsf{Th}(\mathbb{I})$ forms an adjunction (actually a reflection, or more particularly, an equivalence) (Diagram 1). Let us call this the *closure equivalence*.

$$\begin{tikzcd}
S_1 = \langle \Sigma_1, E_1 \rangle \arrow[d, "s"'] & S_1 = \langle \Sigma_1, E_1 \rangle \arrow[r, "\eta_{\mathbb{I},S1} = id"] \arrow[d, "s"'] & \boldsymbol{\mathcal{U}}(\mathbb{I})(\boldsymbol{\mathcal{F}}(\mathbb{I})(S_1)) = \langle \Sigma_1, E_1^{\bullet} \rangle \arrow[d, "\boldsymbol{\mathcal{U}}(\mathbb{I})(\boldsymbol{\mathcal{F}}(\mathbb{I})(s)) = s"'] \\
S_2 = \langle \Sigma_2, E_2 \rangle & S_2 = \langle \Sigma_2, E_2 \rangle \arrow[r, "\eta_{\mathbb{I},S2} = id"'] & \boldsymbol{\mathcal{U}}(\mathbb{I})(\boldsymbol{\mathcal{F}}(\mathbb{I})(S_2)) = \langle \Sigma_2, E_2^{\bullet} \rangle
\end{tikzcd}$$

Figure 6: Naturality of $\eta_{\mathbb{I}}$

```
(18) (KIF$function inclusion)
     (= (KIF$source inclusion) institution)
     (= (KIF$target inclusion) FUNC$functor)
     (forall (?I (institution ?I))
         (and (= (FUNC$source (inclusion ?I)) (signature ?I))
              (= (FUNC$target (inclusion ?I)) (theory ?I))
              (= (inclusion ?I) (FUNC$inclusion (signature ?I) (theory ?I)))))

(19) (KIF$function theory-base)
     (= (KIF$source theory-base) institution)
     (= (KIF$target theory-base) FUNC$functor)
     (forall (?I (institution ?I))
         (and (= (FUNC$source (theory-base ?I)) (theory ?I))
              (= (FUNC$target (theory-base ?I)) (signature ?I))
              (= (theory-base ?I) (FUNC$composition [(inclusion ?I) (base ?I)]))))

(20) (KIF$function closure)
     (= (KIF$source closure) institution)
     (= (KIF$target closure) FUNC$functor)
     (forall (?I (institution ?I))
```

```
         (and (= (FUNC$source (closure ?I)) (signature ?I))
              (= (FUNC$target (closure ?I)) (theory ?I))
              (= (FUNC$object (closure ?I)) (closure-object ?I))
              (= (FUNC$morphism (closure ?I)) (closure-morphism ?I))))

(20.1) (KIF$function closure-object)
       (= (KIF$source closure-object) institution)
       (= (KIF$target closure-object) SET.FTN$function)
       (forall (?I (institution ?I))
           (and (= (SET.FTN$source (closure-object ?I)) (specification-object ?I))
                (= (SET.FTN$target (closure-object ?I)) (theory-object ?I))
                (SET.FTN$restriction (closure-object ?I) (specification-closure ?I))))

(20.2) (KIF$function closure-morphism)
       (= (KIF$source closure-morphism) institution)
       (= (KIF$target closure-morphism) SET.FTN$function)
       (forall (?I (institution ?I))
           (and (= (SET.FTN$source (closure-morphism ?I)) (specification-morphism ?I))
                (= (SET.FTN$target (closure-morphism ?I)) (theory-morphism ?I))
                (= (SET.FTN$composition [(closure-morphism ?I) (CAT$source (theory ?I))])
                   (SET.FTN$composition [(CAT$source (specification ?I)) (closure-object ?I)]))
                (= (SET.FTN$composition [(closure-morphism ?I) (CAT$target (theory ?I))])
                   (SET.FTN$composition [(CAT$target (specification ?I)) (closure-object ?I)]))
                (= (SET.FTN$composition [(closure-morphism ?I) (FUNC$morphism (theory-base ?I))])
                   (base-morphism ?I))))

(21) (KIF$function eta)
     (= (KIF$source eta) institution)
     (= (KIF$target eta) NAT$natural-isomorphism)
     (forall (?I (institution ?I))
         (and (= (NAT$source-category (eta ?I)) (specification ?I))
              (= (NAT$target-category (eta ?I)) (specification ?I))
              (= (NAT$source-functor (eta ?I))  (FUNC$identity (specification ?I)))
              (= (NAT$target-functor (eta ?I))  (FUNC$composition [(closure ?I) (inclusion ?I)]))
              (= (NAT$component (eta ?I)) (eta-component ?I))))

(21.1) (KIF$function eta-component)
       (= (KIF$source eta-component) institution)
       (= (KIF$target eta-component) SET.FTN$function)
       (forall (?I (institution ?I))
           (and (= (SET.FTN$source (eta-component ?I)) (specification-object ?I))
                (= (SET.FTN$target (eta-component ?I)) (CAT$morphism (specification ?I)))
                (= (SET.FTN$composition [(eta-component ?I) (CAT$source (specification ?I))])
                   (SET.FTN$identity (specification-object ?I)))
                (= (SET.FTN$composition [(eta-component ?I) (CAT$target (specification ?I))])
                   (specification-closure ?I))
                (= (SET.FTN$composition [(eta-component ?I) (base-morphism ?I)])
                   (CAT$identity (specification ?I)))))

(22) (KIF$function kappa)
     (= (KIF$source kappa) institution)
     (= (KIF$target kappa) ADJ$adjunction)
     (forall (?I (institution ?I))
         (and (= (ADJ$underlying-category (kappa ?I)) (specification ?I))
              (= (ADJ$free-category (kappa ?I)) (theory ?I))
              (= (ADJ$left-adjoint (kappa ?I)) (closure ?I))
              (= (ADJ$right-adjoint (kappa ?I)) (inclusion ?I))
              (= (ADJ$unit (kappa ?I)) (eta ?I))
              (= (ADJ$counit (kappa ?I)) (NAT$identity (FUNC$identity (theory ?I))))))
```

The Context of Institution Morphisms

`INS.MOR`

The Context of Institution Comorphisms

`INS.MOR.CO`