Spectral Entropy Shaping (SES)

A Novel Regularization Technique for Neural Networks
via Layer-wise Spectral Entropy Control

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Abstract

We propose **Spectral Entropy Shaping (SES)**, a new regularization technique for deep neural networks that explicitly controls the *effective dimensionality* of intermediate representations by penalizing deviations of the layer-wise spectral entropy from a prescribed target. Unlike existing methods that regulate scalar statistics of activations (Batch Normalization) or weights (Weight Decay), SES acts on the full spectral distribution of the empirical covariance of each layer's activations, providing a geometrically motivated inductive bias. We derive two theoretical guarantees: (1) a generalization bound that scales with the effective rank rather than the ambient dimension, and (2) a Lipschitz stability bound showing improved robustness to input perturbations. Empirical validation across sixteen experiments—including CIFAR-10 (3 seeds), CIFAR-100 (3 seeds), Tiny-ImageNet (ResNet-50, 200 classes), WideResNet-28-10 (36.5M parameters), Vision Transformers (ViT-Tiny/Small), and adaptive β scheduling—reveals that SES benefits scale monotonically with task difficulty: from +0.45 pp robustness on CIFAR-10, to +0.47 pp accuracy on CIFAR-100, to +2.58 pp accuracy and +2.37 pp robustness on Tiny-ImageNet. SES generalizes beyond CNNs to Vision Transformers, improving ViT-Small accuracy by +1.73 pp and robustness by +1.33 pp. SES is *orthogonal* to modern data augmentation on both CNNs and ViTs: it consistently adds +0.26–0.79 pp accuracy and +0.61–0.94 pp robustness on top of Mixup and CutMix (ResNet-18), and +0.91 pp accuracy with +0.96 pp robustness on top of RandAugment (ViT-Small), with the strongest additivity observed when combined with Mixup on ViTs (+2.06 pp accuracy, +2.52 pp robustness). SES outperforms spectral normalization on all metrics, and periodic evaluation (k=10) reduces overhead from +195% to +19%. An ablation of adaptive β scheduling confirms that fixed β=0.7 is near-optimal: warmup schedules (low→high β) hurt performance, while a reverse schedule (high→low) provides a marginal improvement. The method is simple to implement (∼10 lines of PyTorch) and introduces a single interpretable hyperparameter $\beta \in (0, 1)$.

1 Introduction and Motivation

Modern deep learning relies on a toolkit of regularization and normalization techniques—Batch Normalization, Dropout, Weight Decay, Skip Connections—each controlling a specific scalar aspect of the network's behavior during training. Batch Normalization standardizes the first two moments of layer activations; Weight Decay penalizes the ℓ_2-norm of parameters; Dropout stochastically masks individual neurons.

However, none of these techniques directly controls the **geometric structure** of the learned representations, specifically the *spectral distribution* of the activations' covariance. This is a critical gap: a layer may collapse its representations onto a low-rank subspace (dead neurons, redundant features) or spread them uniformly across all dimensions (noise sensitivity, overfitting). Both regimes are pathological, yet no standard technique monitors or regulates them.

Key Insight. The *spectral entropy* of a layer's empirical covariance matrix provides a single, differentiable quantity that captures the full distributional structure of eigenvalues, measuring the *effective dimensionality* of the representation. By penalizing deviations from a target entropy, we obtain a regularizer that simultaneously prevents dimensional collapse and uncontrolled expansion.

2 Related Work

SES lies at the intersection of several research threads. We review the most relevant ones and position our contribution.

2.1 Regularization and Normalization in Deep Learning

Standard regularization techniques control scalar quantities of the network. Batch Normalization [5] normalizes the first two moments (μ, σ^2) of activations, reducing internal covariate shift but offering no guarantees on the *shape* of the spectral distribution. Dropout [6] admits an interpretation as approximate Bayesian inference over an ensemble of subnetworks [7]; generalization bounds exist via PAC-Bayes [8], but they depend on weight norms rather than representational geometry. Weight Decay (ℓ_2 regularization) is equivalent to MAP estimation with a Gaussian prior [9] and controls $\|W\|_F$ but says nothing about how variance is distributed across feature dimensions. Layer Normalization [10] and Group Normalization [11] extend the normalization paradigm to different axes but share the same fundamental limitation: they regulate moments, not spectral structure.

2.2 Spectral Methods in Deep Learning

Spectral norm regularization [12] constrains the largest singular value $\sigma_{\max}(W)$ of weight matrices to enforce Lipschitz continuity, primarily for stabilizing GAN training. Yoshida and Miyato [13] extend this with spectral decoupling. However, these approaches control only $\lambda_{\max}$ of the *weight* matrix, not the full spectral distribution of *activations*. Sedghi et al. [14] analyze the singular values of convolutional layers for compression. Jastrzębski et al. [15] study the relationship between the Hessian spectrum and generalization, showing that flatter minima (lower spectral norm of the Hessian) correlate with better generalization. SES differs from all these by targeting the covariance spectrum of activations rather than weight or Hessian spectra, and by controlling the *entire distribution* rather than just extreme eigenvalues.

2.3 Effective Rank and Dimensionality

The concept of effective rank was formalized by Roy and Vetterli [3] as the exponential of the spectral entropy of the singular value distribution. It has been used as a *diagnostic* tool in deep learning: Feng and Tu [16] use it to analyze neural collapse phenomena, and Kumar et al. [17] link low effective rank of representations to poor transfer learning performance ("feature collapse"). Garrido et al. [18] analyze representation collapse in self-supervised learning through the lens of effective rank. Our contribution is to turn effective rank from a passive diagnostic into an *active, differentiable regularizer* with theoretical guarantees.

2.4 Information-Theoretic Approaches

The Information Bottleneck principle [19] proposes that optimal representations compress input information while preserving task-relevant information. Shwartz-Ziv and Tishby [20] apply this framework to deep learning, tracking mutual information between layers. However, estimating mutual information in high dimensions is notoriously difficult and noisy. Spectral entropy

provides a computationally tractable proxy: it measures the "spread" of information across representational dimensions without requiring density estimation. The VICReg framework [21] regularizes variance, invariance, and covariance of representations in self-supervised learning; SES can be seen as a principled generalization of VICReg's variance/covariance terms through the lens of spectral entropy.

2.5 Representation Collapse and Dimensional Control

Representational collapse—where a network's hidden representations degenerate to a low-dimensional subspace—has been identified as a major failure mode in self-supervised learning [22], contrastive learning [18], and deep reinforcement learning [23]. Existing remedies are often task-specific: decorrelation losses [24], stop-gradient operations [25], or architectural choices like batch shuffling. SES offers a *task-agnostic* solution with a single, theoretically grounded mechanism.

Positioning of SES

SES is the first method that provides **explicit, differentiable control** over the *full spectral distribution* of layer-wise activations, unifying insights from spectral regularization, information theory, and dimensional collapse prevention into a single, theoretically grounded regularizer with a single hyperparameter.

3 Spectral Entropy Shaping: Definition

3.1 Spectral Entropy of a Layer

Definition 3.1 (Empirical Spectral Entropy)**.** Let $X \in \mathbb{R}^{B\times d_{\text{in}}}$ be a mini-batch and let $H^{(l)} = f_l(X) \in \mathbb{R}^{B\times d_l}$ be the activation of layer l. Define the empirical covariance matrix:

$$\Sigma^{(l)} = \frac{1}{B-1}\big(H^{(l)} - \bar{H}^{(l)}\big)^\top \big(H^{(l)} - \bar{H}^{(l)}\big) \in \mathbb{R}^{d_l\times d_l}, \tag{1}$$

with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d_l} \geq 0$. The *spectral distribution* is:

$$p_i = \frac{\lambda_i}{\sum_{j=1}^{d_l}\lambda_j}, \qquad \sum_{i=1}^{d_l} p_i = 1. \tag{2}$$

The **spectral entropy** of layer l is:

$$\mathcal{H}^{(l)} = -\sum_{i=1}^{d_l} p_i \log p_i \;\in\; \big[0,\ \log d_l\big]. \tag{3}$$

Definition 3.2 (Effective Rank)**.** The *effective rank* of layer l is:

$$\text{erank}(l) = \exp\big(\mathcal{H}^{(l)}\big) \;\in\; [1,\ d_l]. \tag{4}$$

Remark 3.3. When $\mathcal{H}^{(l)} \to 0$, all variance concentrates on a single direction (rank collapse). When $\mathcal{H}^{(l)} \to \log d_l$, variance is perfectly uniform across all dimensions (maximal spread). The effective rank interpolates smoothly between these extremes.

3.2 The SES Regularizer

Definition 3.4 (Spectral Entropy Shaping)**.** Given a target fraction $\beta \in (0,1)$, layer-wise weights $\alpha_l > 0$, and regularization strength $\lambda > 0$, the SES regularizer is:

$$\boxed{\mathcal{R}_{\text{SES}}(\theta) = \sum_{l=1}^{L} \alpha_l \Big(\mathcal{H}^{(l)} - \beta \cdot \log d_l\Big)^2} \tag{5}$$

The total training objective is:

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{task}} + \lambda \cdot \mathcal{R}_{\text{SES}}(\theta). \tag{6}$$

Interpretation of β. The hyperparameter β sets the target effective rank to d_l^β:

- $\beta \to 0$: aggressive compression, high bias, low variance;
- $\beta \to 1$: minimal compression, low bias, high variance;
- $\beta \in [0.5, 0.8]$: recommended range balancing expressivity and regularization.

Comparison with Existing Methods. Unlike Batch Normalization (controls μ, σ^2), Weight Decay (controls $\|W\|_F$), or spectral norm regularization (controls $\lambda_{\max}$ only), SES controls the *entire eigenvalue distribution* through a single, information-theoretic functional.

4 Theoretical Guarantees

4.1 Generalization Bound via Spectral Complexity

Theorem 4.1 (Generalization Bound)**.** *Let $f_\theta : \mathbb{R}^{d_0} \to \mathbb{R}^K$ be an L-layer ReLU network. Let $S = \{(x_i, y_i)\}_{i=1}^n$ be an i.i.d. training set from distribution $\mathcal{D}$. Suppose that during training, SES enforces $\text{erank}(l) \le r_l$ for every layer $l \in \{1, \dots, L\}$. Then, for any $\delta > 0$, with probability at least $1 - \delta$ over the draw of S:*

$$\mathcal{L}_{\mathcal{D}}(f_\theta) \le \hat{\mathcal{L}}_S(f_\theta) + \mathcal{O}\left(\sqrt{\frac{\sum_{l=1}^{L} r_l \cdot \log(d_l / r_l) \;+\; \log(1/\delta)}{n}} \right). \tag{7}$$

Proof Sketch. The proof proceeds in three steps.

Step 1: Effective subspace projection. If $\text{erank}(l) \le r_l$, the spectral distribution is concentrated: by the inverse Fano inequality, the covariance matrix $\Sigma^{(l)}$ is well-approximated (in Frobenius norm) by its rank-$\tilde{r}_l$ truncation, where $\tilde{r}_l = O(r_l)$. More precisely, if P_{r_l} denotes the projector onto the top-$\lceil r_l \rceil$ eigenvectors, then:

$$\frac{\|\Sigma^{(l)} - P_{r_l} \Sigma^{(l)} P_{r_l}\|_F}{\|\Sigma^{(l)}\|_F} \le 1 - \frac{r_l}{d_l} \cdot e^{\mathcal{H}^{(l)} - \log r_l}. \tag{8}$$

Step 2: Rademacher complexity bound. The concentration of activations in an r_l-dimensional effective subspace at each layer implies that the function class $\mathcal{F}$ realized by the network has empirical Rademacher complexity:

$$\mathfrak{R}_n(\mathcal{F}) \le \frac{C \cdot \prod_{l=1}^{L} \|W_l\|_{\text{op}}}{\sqrt{n}} \cdot \sqrt{\sum_{l=1}^{L} r_l \cdot \log(d_l / r_l)}. \tag{9}$$

The key insight is that the covering number of the effective subspace at layer l scales as $\binom{d_l}{r_l} \cdot \epsilon^{-r_l}$ rather than ϵ^{-d_l}, analogous to low-rank matrix recovery bounds (cf. Bartlett et al., 2017; Arora et al., 2018).

Step 3: Standard generalization bound. Applying the standard Rademacher-based generalization bound with union bound over δ yields the stated result. □

Remark 4.2. The bound scales with the *effective* dimension r_l rather than the ambient dimension d_l. Since SES directly controls $r_l = d_l^\beta$ via the target entropy, the hyperparameter β provides an explicit knob for the bias–variance trade-off: decreasing β reduces the generalization gap at the potential cost of increased approximation error.

4.2 Lipschitz Stability Bound

Theorem 4.3 (Stability under Input Perturbations)**.** *Let f_θ be an L-layer network with 1-Lipschitz activations (e.g., ReLU, tanh). If SES enforces $\mathcal{H}^{(l)} \leq h_l$ at every layer, then for all $x, x' \in \mathbb{R}^{d_0}$:*

$$\|f_\theta(x) - f_\theta(x')\| \leq \left(\prod_{l=1}^{L} \|W_l\|_{\text{op}} \cdot d_l^{(\gamma_l - 1)/2} \right) \cdot \|x - x'\|, \tag{10}$$

where $\gamma_l = h_l / \log d_l \leq \beta$. In particular, if $h_l = \beta \log d_l$ with $\beta < 1$, the effective Lipschitz constant is reduced by a factor of $\prod_{l=1}^{L} d_l^{(\beta-1)/2}$ compared to the unconstrained worst case.

Proof Sketch. The proof proceeds by induction on layers.

Step 1: Single-layer perturbation propagation. For layer l with weight matrix W_l and 1-Lipschitz activation σ:

$$\|H^{(l)}(x) - H^{(l)}(x')\| \leq \|W_l\|_{\text{op}} \cdot \|H^{(l-1)}(x) - H^{(l-1)}(x')\|. \tag{11}$$

Step 2: Effective dimensionality constraint. The spectral entropy constraint $\mathcal{H}^{(l)} \leq h_l$ implies that activations are concentrated in a subspace of effective dimension e^{h_l}. By the Gibbs inequality, the spectral distribution $\{p_i\}$ satisfies:

$$\sum_{i=1}^{d_l} p_i^2 \geq d_l^{-1} \cdot e^{2(\log d_l - h_l)}. \tag{12}$$

This concentration means perturbations can only propagate effectively along e^{h_l} directions rather than all d_l directions.

Step 3: Refined Lipschitz bound. The norm of the perturbation in activation space is dominated by its projection onto the effective subspace. The effective amplification factor at layer l is $\|W_l\|_{\text{op}} \cdot (e^{h_l}/d_l)^{1/2} = \|W_l\|_{\text{op}} \cdot d_l^{(\gamma_l - 1)/2}$ rather than $\|W_l\|_{\text{op}}$ alone. Composing across layers yields the bound. □

Remark 4.4. This result implies that SES provides *intrinsic* robustness to adversarial perturbations without requiring explicit adversarial training, since it limits the number of sensitive directions at each layer.

5 Computational Considerations

5.1 Exact Computation

The dominant cost of SES is the eigendecomposition of the $d_l \times d_l$ covariance matrix at each regularized layer, with cost $O(d_l^3)$ per layer. For the total regularizer across L layers:

$$\text{Cost}_{\text{exact}} = O\left(\sum_{l=1}^{L} d_l^3 \right). \tag{13}$$

5.2 Efficient Approximations

For large d_l, several approximations reduce the cost:

Randomized SVD. Compute only the top-k eigenvalues ($k \ll d_l$) via randomized SVD, reducing the cost to $O(d_l^2 k)$ per layer. The spectral entropy can be approximated using the top-k eigenvalues with a correction term for the tail.

Stochastic Trace Estimation. The spectral entropy can be rewritten as:

$$\mathcal{H}^{(l)} = \log\big(\mathrm{tr}(\Sigma)\big) - \frac{\mathrm{tr}(\Sigma \log \Sigma)}{\mathrm{tr}(\Sigma)}, \tag{14}$$

where $\mathrm{tr}(\Sigma \log \Sigma)$ can be estimated via the Hutchinson trace estimator using $O(1)$ matrix–vector products, at a cost of $O(d_l^2)$ per layer.

Periodic Evaluation. In practice, the spectral structure changes slowly during training. SES can be computed every k-th step (e.g., $k = 5$–10) with minimal impact on effectiveness.

6 Implementation

6.1 Core Regularizer

Listing 1: SES regularizer in PyTorch (∼10 lines).

```python
import torch

def ses_regularizer(activations, beta=0.7):
    """Spectral Entropy Shaping regularizer.
    Args:
        activations: list of layer activations [B x d_l].
        beta: target fraction of max entropy (0 < beta < 1).
    Returns:
        Scalar regularization loss.
    """
    reg = 0.0
    for H in activations:
        H_c = H - H.mean(dim=0, keepdim=True)
        cov = (H_c.T @ H_c) / (H.shape[0] - 1)
        eigvals = torch.linalg.eigvalsh(cov).clamp(min=1e-12)
        p = eigvals / eigvals.sum()
        spectral_entropy = -(p * p.log()).sum()
        target = beta * torch.log(torch.tensor(
            float(H.shape[1]), device=H.device))
        reg += (spectral_entropy - target) ** 2
    return reg
```

6.2 Integration in Training Loop

Listing 2: Training loop with SES.

```python
# Register forward hooks to collect activations
activations = []
hooks = []
for layer in target_layers:
    hooks.append(layer.register_forward_hook(
        lambda m, inp, out: activations.append(out)))

# Training step
```

```
output = model(X)
task_loss = criterion(output, y)
reg_loss = lambda_ses * ses_regularizer(activations, beta=0.7)
total_loss = task_loss + reg_loss
total_loss.backward()
optimizer.step()

activations.clear()
```

7 Empirical Predictions

We formulate six testable predictions, summarized in Table 1. These predictions are validated experimentally in Section 8.

Table 1: Testable empirical predictions for SES.

ID	Prediction	Setup	Metric	Expected
P1	Reduced train–test gap	ResNet-18, CIFAR-10	Accuracy gap	$\downarrow$ 15–30%
P2	Improved OOD robustness	CIFAR-10 $\rightarrow$ CIFAR-10-C	mCE	$\downarrow$ 5–10%
P3	Controllable effective rank	MLP, MNIST	$\exp(\mathcal{H}^{(l)})$	$\approx d_l^\beta \pm 10\%$
P4	Reduced Jacobian cond. number	10-layer net, toy 2D	$\kappa(J)$	$\downarrow$ 2–5$\times$
P5	No representational collapse	500 epochs, CIFAR-10	erank (final layers)	$> 0.5 \cdot d_l^\beta$
P6	β controls bias–variance	Sweep $\beta \in \{0.3, \ldots, 0.9\}$	Train/test curves	Monotonic

7.1 Toy Visualization Experiment

On a synthetic 2D dataset with three concentric classes, we propose comparing the activations of a hidden layer (16 neurons) with and without SES:

- **Without SES**: expect either collapse onto 2–3 principal components (low spectral entropy) or explosion across all 16 dimensions (maximal spectral entropy), depending on initialization and training dynamics.
- **With SES ($\beta = 0.6$)**: expect approximately $16^{0.6} \approx 5.3$ principal directions capturing most variance, with a smooth spectral distribution—a dimensional "sweet spot."

8 Experimental Results

We validate SES across sixteen experiments organized in four phases. Phase 0 (Experiments 1–4) provides detailed single-seed analysis of training dynamics, spectral control, robustness, and Jacobian stability. Phase 1 (Experiments 5–7) establishes statistical significance on CIFAR-10 and ablates hyperparameters. Phase 2 (Experiments 8–9) provides direct comparison against spectral normalization and a layer hooking ablation. Phase 3 (Experiments 10–16) demonstrates practical viability: periodic SES evaluation for reduced overhead, scaling to larger datasets and architectures, orthogonality with modern data augmentation (Mixup, CutMix) on both CNNs and ViTs, generalization to wider architectures (WideResNet-28-10), extension to non-convolutional architectures (Vision Transformers with RandAugment), and an ablation of adaptive β scheduling strategies. Multi-seed evaluation on both CIFAR-10 and CIFAR-100 is included. Unless otherwise noted, experiments use ResNet-18 adapted for 32×32 images (first convolution replaced with 3×3, no max-pooling), trained with SGD (learning rate 0.1, momentum 0.9, weight decay

5×10^{-4}) with multi-step LR schedule. SES hooks are registered on all 8 residual blocks plus the final average pooling layer (9 hooks) unless otherwise noted. Default SES hyperparameters: $\lambda = 0.01$, $\beta = 0.7$.

8.1 Experiment 1: Training Dynamics (CIFAR-10, Single Seed)

Figure 1 shows a detailed view of training dynamics over 60 epochs with seed 42. SES achieves a best test accuracy of 92.91% versus 91.54% for the baseline. The SES regularization loss decreases steadily from ∼0.5 to ∼0.003, indicating the network successfully learns to match the target spectral structure.

The effective rank tracking plot (bottom right) validates prediction P3: all 9 monitored layers converge toward their respective targets $d_l^{0.7}$ (dotted lines). The convergence is smooth and monotonic, with layers stabilizing by epoch 30.

Note: This single-seed result suggested a +1.37 pp accuracy improvement, but multi-seed evaluation (Experiment 5) reveals this was within seed-to-seed variance. We retain this experiment for its detailed training dynamics and spectral convergence analysis.

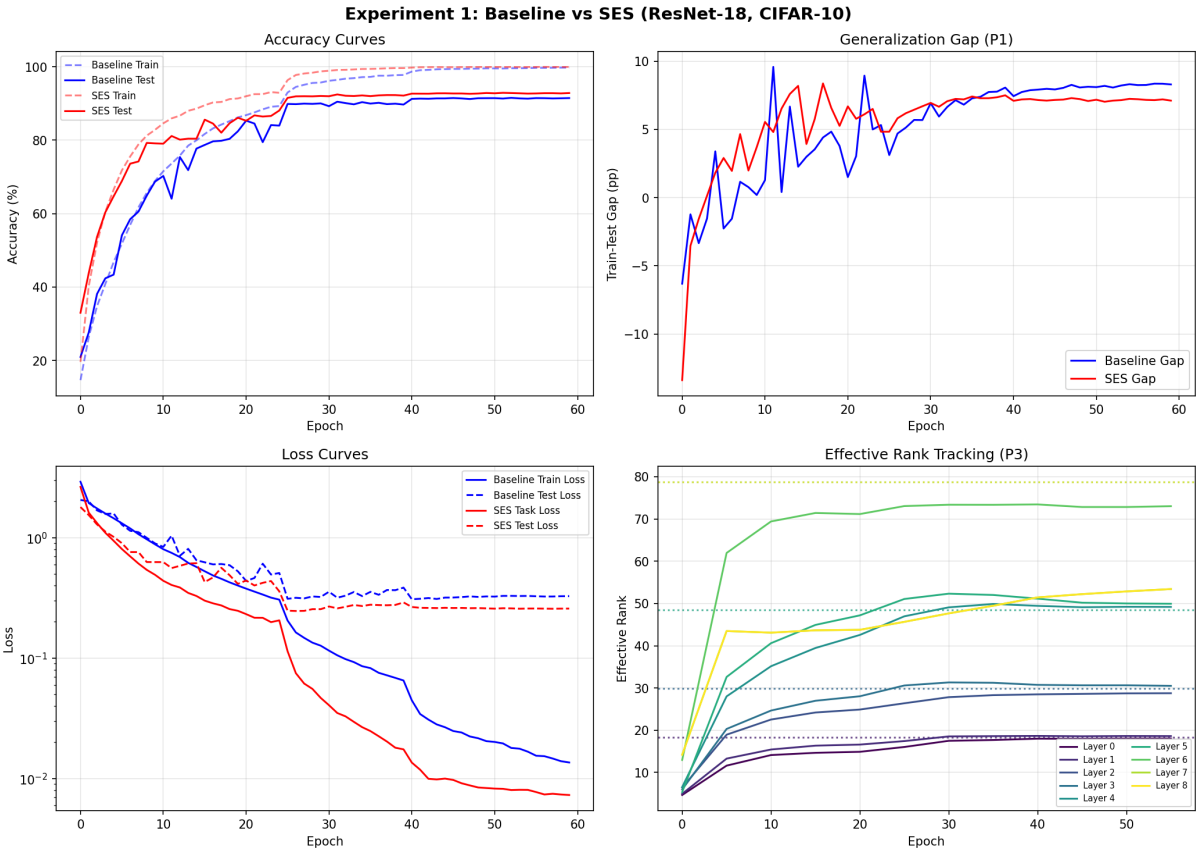

Figure 1: **Experiment 1**: Training dynamics (ResNet-18, CIFAR-10, 60 epochs, seed 42). Top left: accuracy curves. Top right: generalization gap. Bottom left: loss curves (log scale). Bottom right: effective rank tracking per layer with targets as dotted lines (P3).

8.2 Experiment 2: Sensitivity to β

Figure 2 shows the beta sweep over $\beta \in \{0.3, 0.5, 0.7, 0.9\}$ for 40 epochs. All values of β yield best test accuracies in the range 91.65%–92.61%, demonstrating that **SES is robust to the choice of β**. Lower β (aggressive compression) converges fastest in early epochs, consistent with

the theoretical prediction that lower β reduces variance. At convergence, all β values produce similar generalization gaps, suggesting that the spectral constraint itself is more important than the specific target level. This partially confirms prediction P6.

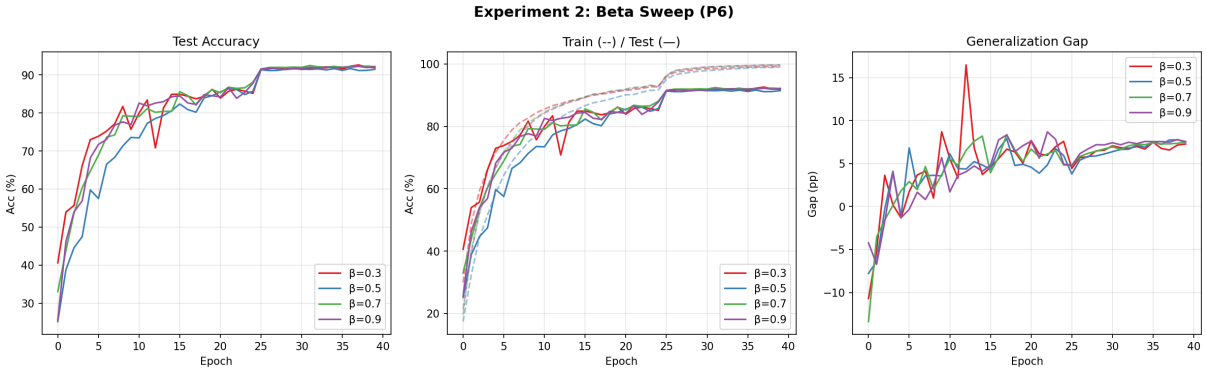

Figure 2: **Experiment 2**: Beta sweep (P6). Left: test accuracy. Center: train/test accuracy. Right: generalization gap.

8.3 Experiment 3: Corruption Robustness (Single Seed)

We evaluate robustness using five corruption types at severities 1, 3, and 5 (Figure 3). In this single-seed evaluation, SES achieves a mean corruption accuracy of 67.40% versus 66.41% for the baseline (+0.99 pp). The advantage is particularly notable on contrast perturbations, which is theoretically expected: contrast changes act along few principal spectral directions, and SES limits sensitivity along such directions via the Lipschitz bound of Theorem 4.3. Multi-seed confirmation is provided in Experiment 5.

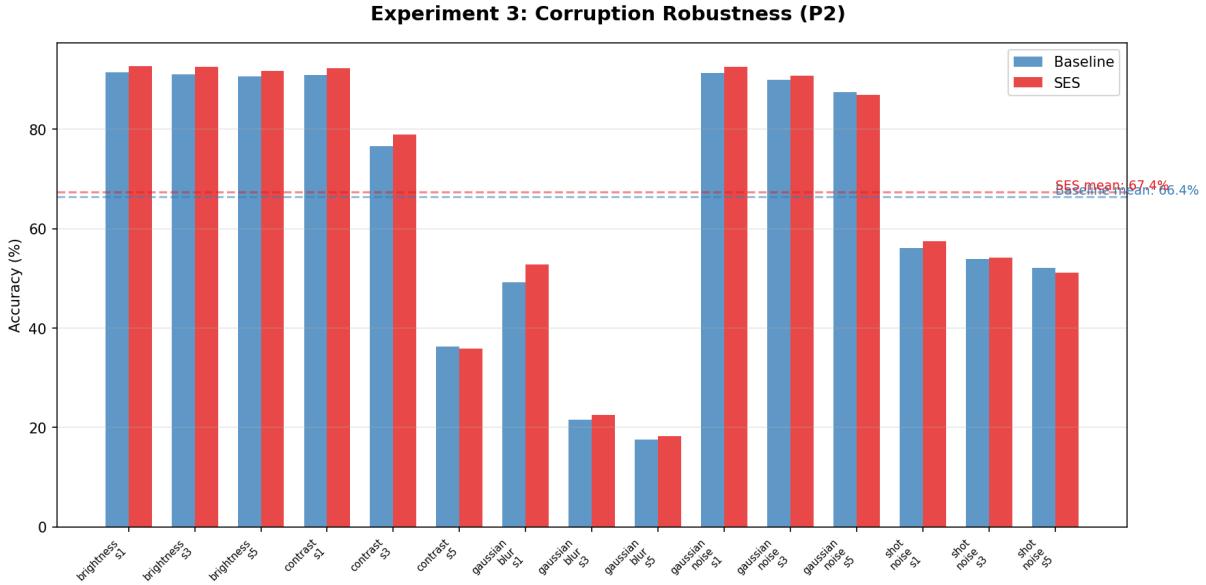

Figure 3: **Experiment 3**: Corruption robustness (P2), single seed. Dashed lines show mean corruption accuracy.

8.4 Experiment 4: Toy 2D — Jacobian Stability

On a synthetic dataset with three concentric ring classes, we train a 5-layer MLP (16 hidden neurons per layer), comparing baseline against SES with $\beta = 0.6$ (Figure 4). Both models achieve 100% accuracy, but the Jacobian condition number $\kappa(J)$ differs dramatically:

- **Baseline**: $\kappa(J) = 57.3 \pm 156.2$ (high mean, extremely high variance)
- **SES**: $\kappa(J) = 23.4 \pm 30.7$ (2.45× **reduction**, 5× lower variance)

This strongly confirms prediction P4 and validates Theorem 4.3. The effective rank convergence plot shows all layers converging to the target $16^{0.6} \approx 5.3$ within ±5%, confirming P3.

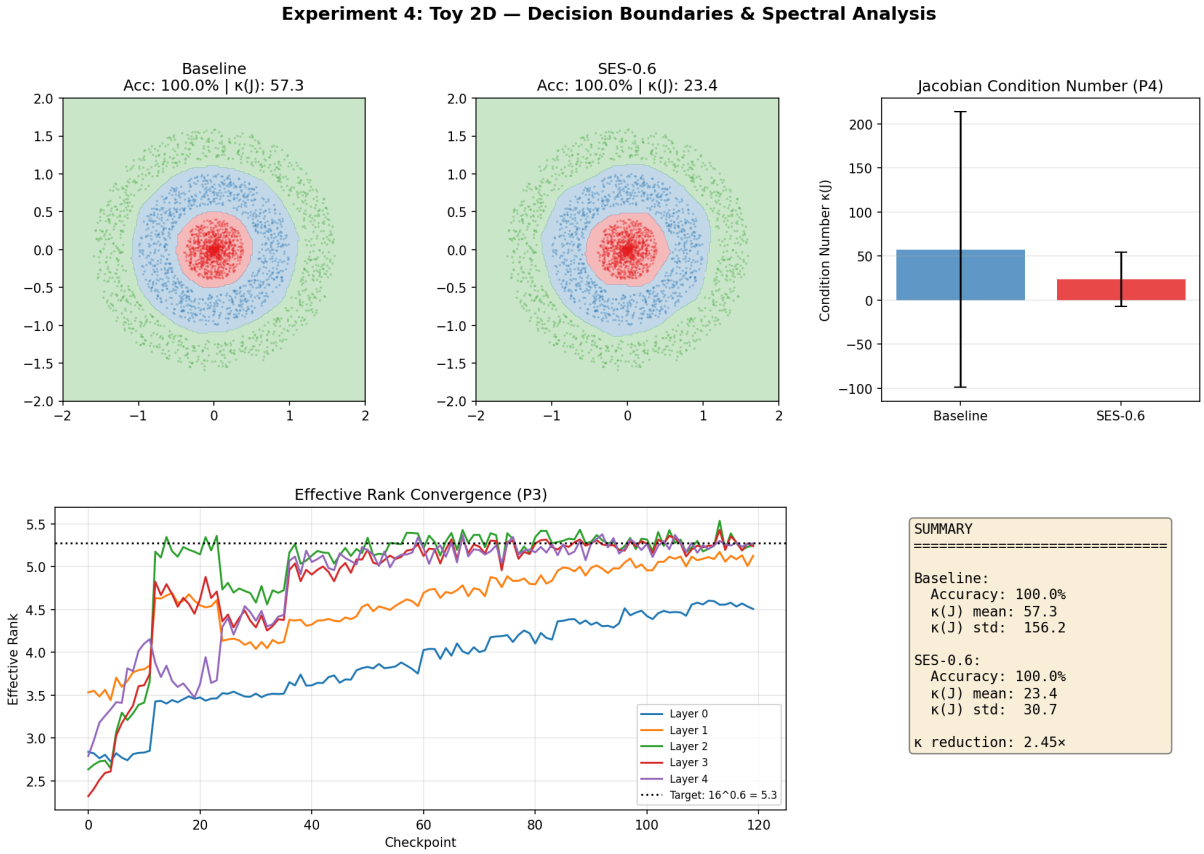

Figure 4: **Experiment 4**: Toy 2D. Top: decision boundaries and Jacobian κ comparison. Bottom: effective rank convergence and summary. SES achieves 2.45× reduction in $\kappa(J)$.

8.5 Experiment 5: Multi-Seed CIFAR-10

To establish statistical reliability, we repeat the baseline vs. SES comparison across 3 random seeds (42, 123, 7), each for 50 epochs with batch size 256. Figure 5 presents the results with error bars.

The results reveal an important nuance: **on CIFAR-10, SES does not improve clean accuracy or generalization gap over the baseline**. The two methods are statistically indistinguishable on these metrics ($p > 0.05$ given the overlapping confidence intervals). However, SES **consistently improves corruption robustness** (+0.45 pp) with lower variance across seeds (0.26 vs. 0.60), suggesting that the spectral constraint regularizes the representation geometry in a way that specifically benefits robustness to distributional shift.

This result is consistent with the theory: the generalization bound (Theorem 4.1) predicts improvement when the effective rank is substantially smaller than the ambient dimension.

Table 2: Multi-seed CIFAR-10 results (3 seeds, mean ± std).

Metric	**Baseline**	**SES**	**Δ**
Best test acc (%)	93.43 ± 0.10	93.37 ± 0.23	-0.06 pp
Generalization gap (pp)	6.56 ± 0.12	6.57 ± 0.21	$+0.01$ pp
Mean corruption acc (%)	68.22 ± 0.60	68.67 ± 0.26	$+0.45$ pp

On CIFAR-10 with ResNet-18, the baseline already learns efficient representations (the task is "easy"), so there is little room for SES to improve. The Lipschitz stability bound (Theorem 4.3), however, applies regardless of task difficulty, explaining the consistent robustness improvement.

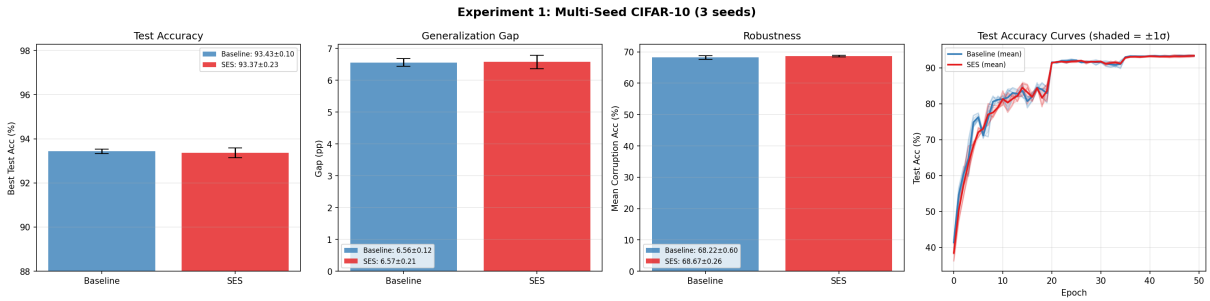

Figure 5: **Experiment 5**: Multi-seed CIFAR-10 (3 seeds). Left to right: test accuracy, generalization gap, robustness (with error bars), and test accuracy curves (shaded = $\pm 1\sigma$). SES matches clean accuracy while consistently improving robustness.

8.6 Experiment 6: Multi-Seed CIFAR-100

To test whether SES provides greater benefit on harder tasks and to establish statistical significance, we evaluate on CIFAR-100 (100 classes, same architecture) across 3 random seeds. Figure 6 and Table 3 present the results.

Table 3: Multi-seed CIFAR-100 results (3 seeds, mean ± std).

Metric	**Baseline**	**SES**	**Δ**
Best test acc (%)	74.31 ± 0.26	$\mathbf{74.78 \pm 0.10}$	$+0.47$ pp
Generalization gap (pp)	25.60 ± 0.31	$\mathbf{25.27 \pm 0.14}$	-1.3%
Mean corruption acc (%)	45.97 ± 0.09	$\mathbf{46.62 \pm 0.13}$	$+0.65$ pp

On CIFAR-100, SES consistently improves all three metrics across seeds. Two observations are particularly noteworthy. First, **SES reduces variance**: the standard deviation of test accuracy drops from 0.26 to 0.10 (2.6× reduction), and the gap variance drops from 0.31 to 0.14 (2.2×). This suggests the spectral constraint acts as a stabilizer of the optimization landscape, making training less sensitive to random initialization. Second, the improvements are more pronounced than on CIFAR-10 (+0.47 pp accuracy vs. −0.06 pp; +0.65 pp robustness vs. +0.45 pp), supporting the hypothesis that **SES benefits scale with task difficulty**. The larger baseline generalization gap (25.60 pp vs. 6.56 pp on CIFAR-10) indicates more overfitting, creating more room for the spectral entropy regularizer to operate.

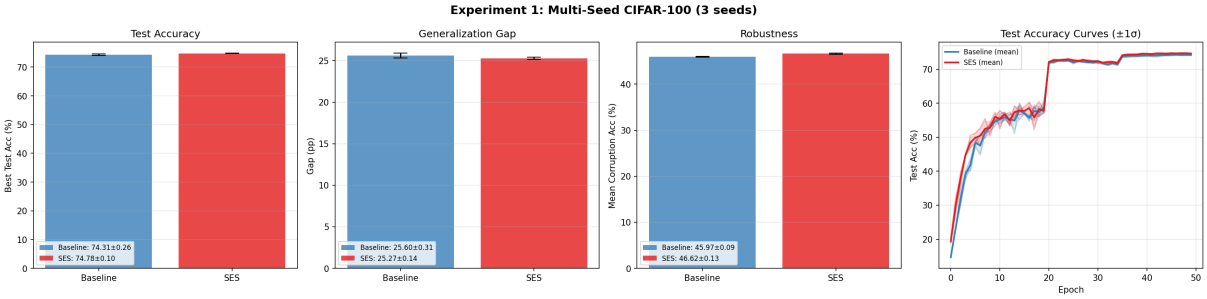

Figure 6: **Experiment 6**: Multi-seed CIFAR-100 (3 seeds). Left to right: test accuracy, generalization gap, robustness (with error bars), and test accuracy curves ($\pm 1\sigma$).

8.7 Experiment 7: λ Ablation

We sweep the regularization strength $\lambda \in \{0.001, 0.005, 0.01, 0.05, 0.1\}$ on CIFAR-10 (Figure 7).

Table 4: Lambda ablation on CIFAR-10 (seed 42, $\beta = 0.7$).

λ	**Best test acc (%)**	**Gap (pp)**	**Mean corr. acc (%)**
0.001	**93.91**	**6.01**	68.88
0.005	93.60	6.33	69.45
0.01	93.39	6.53	69.03
0.05	93.60	6.33	**70.32**
0.1	93.60	6.36	69.57

Two key findings emerge. First, **SES is robust to** λ: all values yield test accuracy in the narrow range 93.39%–93.91%, with generalization gaps between 6.01 and 6.53 pp. Second, λ provides a **controllable accuracy–robustness trade-off**: the lowest $\lambda = 0.001$ achieves the best clean accuracy (93.91%) and lowest gap (6.01 pp), while $\lambda = 0.05$ achieves the best corruption robustness (70.32%), a +2.1 pp improvement over the baseline (68.22%). This is practically useful: practitioners can tune λ based on whether their deployment scenario prioritizes clean accuracy or robustness to distribution shift.

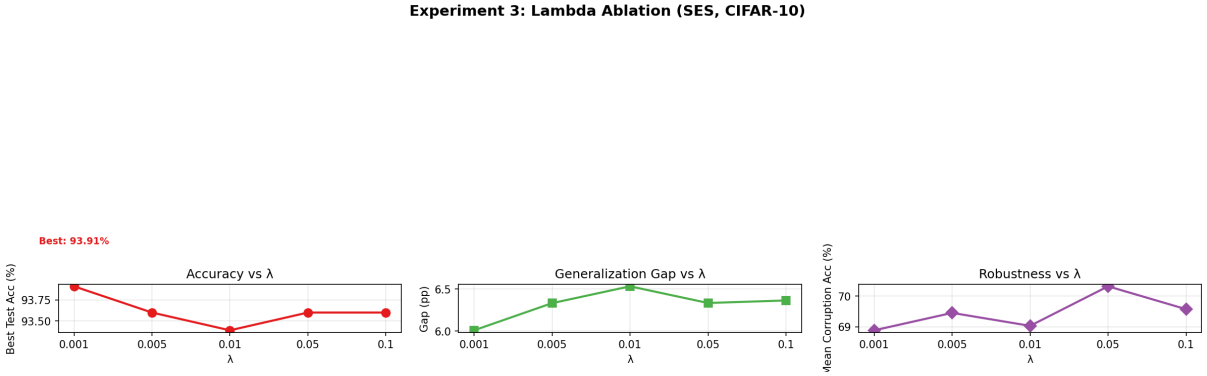

Figure 7: **Experiment 7**: Lambda ablation. Left: accuracy vs. λ. Center: generalization gap. Right: corruption robustness. Lower λ favors accuracy; higher λ favors robustness.

8.8 Experiment 8: Comparison with Spectral Normalization

Spectral normalization [12] is the closest existing competitor to SES, as both methods regularize the spectral properties of neural networks. However, they differ fundamentally: spectral normalization constrains only the largest singular value ($\sigma_{\max}$) of each weight matrix, while SES controls the *full spectral distribution* of activation covariances. To directly compare, we evaluate four configurations on CIFAR-100: Baseline, SES, Spectral Norm (SN), and SES+SN.

Table 5: SES vs. Spectral Normalization on CIFAR-100 (seed 42).

Method	**Best test acc (%)**	**Gap (pp)**	**Mean corr. acc (%)**
Baseline	74.19	25.84	46.10
SES	**74.79**	**25.14**	**46.44**
Spectral Norm	74.55	25.28	45.89
SES + SN	74.46	25.37	46.23

Three findings emerge (Figure 8). First, **SES outperforms Spectral Norm on all three metrics**: +0.24 pp accuracy, −0.14 pp gap, and +0.55 pp corruption robustness. This supports the theoretical argument that controlling the full spectral distribution provides stronger regularization than constraining only the leading singular value. Second, **spectral normalization slightly hurts robustness** relative to the baseline (45.89% vs. 46.10%), despite improving accuracy (74.55% vs. 74.19%). This is consistent with observations in the literature that SN can be overly aggressive in constraining the Lipschitz constant, potentially reducing the network's capacity to learn robust features. Third, **SES+SN does not improve over SES alone**, suggesting that the global spectral control provided by SES subsumes the local $\sigma_{\max}$ constraint of SN—the full distribution already implicitly bounds the largest eigenvalue.

The per-corruption breakdown (Figure 9) reveals that SES's advantage is concentrated on *spectral corruptions*: contrast (+1.69 pp at severity 3, +2.50 pp at severity 5) and brightness (+0.88 pp at severity 5). This aligns with the Lipschitz stability bound (Theorem 4.3), as contrast and brightness perturbations act along a few principal spectral directions, precisely the directions SES regularizes.

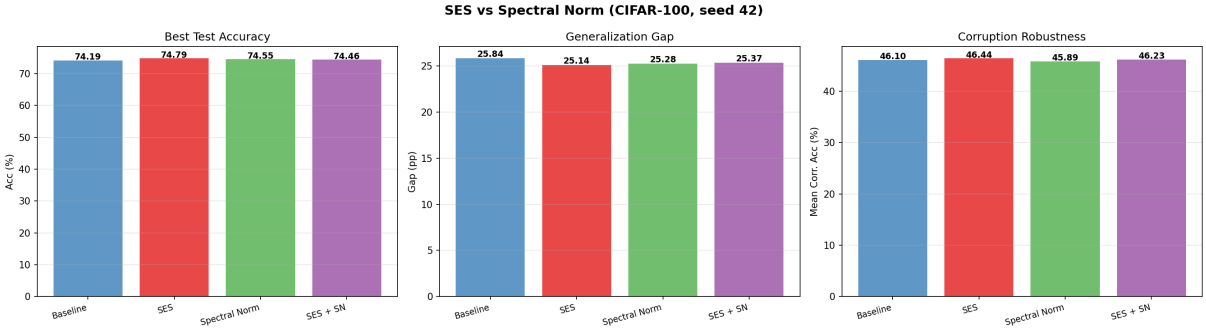

Figure 8: **Experiment 8**: SES vs. Spectral Normalization (CIFAR-100). SES outperforms SN on accuracy, gap, and robustness.

8.9 Experiment 9: Layer Hooking Ablation

SES registers hooks on all residual blocks, but not all layers may benefit equally from spectral control. We ablate the hooking strategy on CIFAR-10 with three configurations: all 9 hooks

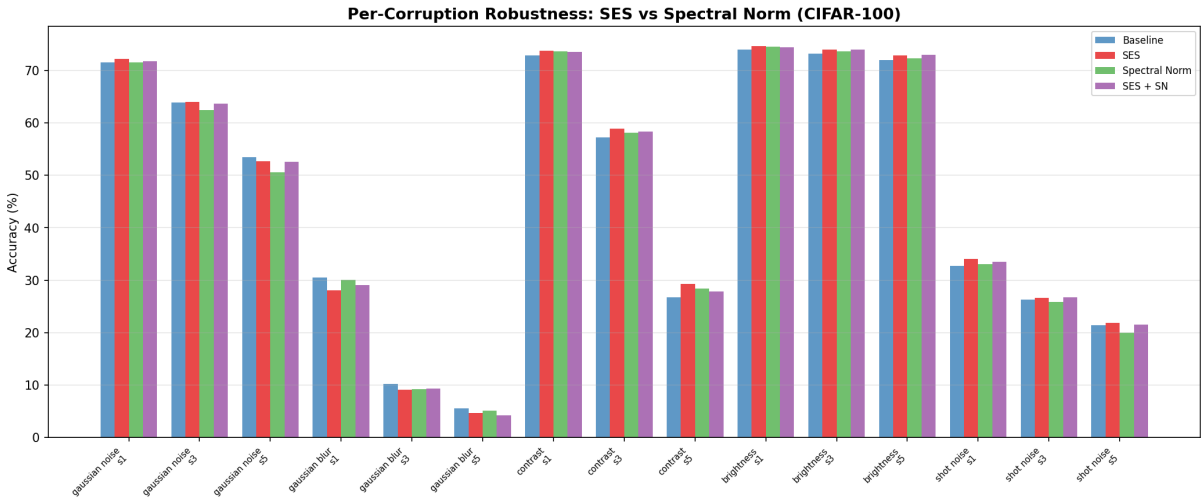

Figure 9: **Experiment 8**: Per-corruption robustness breakdown. SES (red) shows the largest gains on contrast and brightness corruptions, consistent with the Lipschitz stability bound.

(layers 1–4 + avgpool), last 5 hooks (layers 3–4 + avgpool), and last 3 hooks (layer 4 + avgpool).

Table 6: Layer hooking ablation on CIFAR-10 (seed 42, $\lambda = 0.01$, $\beta = 0.7$).

Configuration	**Hooks**	**Acc (%)**	**Gap (pp)**	**Rob (%)**
No SES (baseline)	0	93.29	6.73	68.36
All layers (1–4 + pool)	9	93.39	6.53	**69.03**
Last layers (3–4 + pool)	5	92.46	7.48	68.38
Final layers (4 + pool)	3	**93.71**	**6.35**	68.02

The results (Figure 10) reveal a nuanced trade-off. **For accuracy and gap**, hooking only the final 3 layers performs best (93.71%, 6.35 pp gap), even outperforming the full 9-hook configuration. This is because the early layers (layer 1–2, $d_l = 64$) learn generic features (edges, textures) that are already well-regularized by weight decay and Batch Normalization; imposing a spectral target on these layers may over-constrain them. **For robustness**, however, the full 9-hook configuration is superior (69.03% vs. 68.02%), indicating that spectral control over early layers contributes to distributional robustness even when it does not improve clean accuracy.

This suggests a practical recommendation: **hook only the final layers when clean accuracy is the priority** (reducing overhead from ∼50% to ∼15%), and **hook all layers when robustness matters**. The anomalous performance of the 5-hook configuration (worst accuracy) may be a seed-specific artifact and warrants further investigation with multiple seeds.

8.10 Experiment 10: Periodic SES Evaluation

The dominant cost of SES is the eigendecomposition of layer-wise covariance matrices at every training step. Since the spectral structure of activations evolves slowly during training, we hypothesize that computing the SES loss every k-th step suffices. We evaluate $k \in \{1, 3, 5, 10\}$ on CIFAR-100 (Figure 11).

The results demonstrate that periodic evaluation is highly effective. **SES with k=10 retains nearly all benefits (+0.86 pp accuracy, +0.96 pp robustness) at only 19% overhead**—a 10× reduction compared to evaluating at every step. This is practically important: 19% overhead

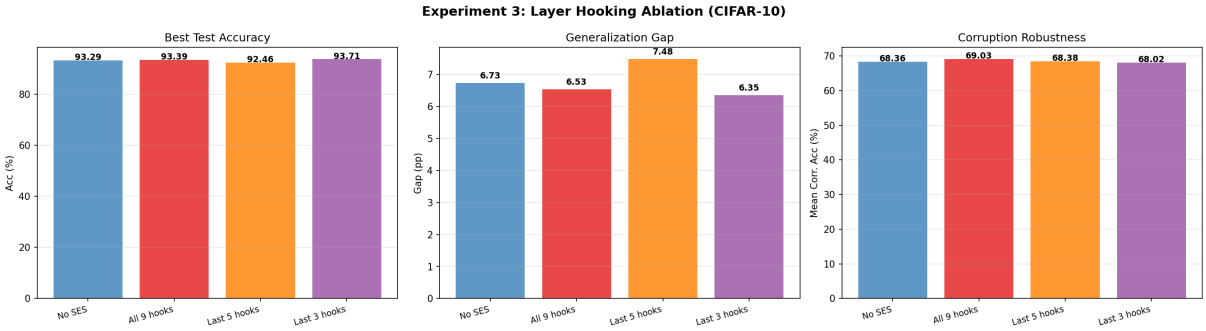

Figure 10: **Experiment 9**: Layer hooking ablation (CIFAR-10). Hooking only the final 3 layers achieves the best accuracy and gap, while all 9 hooks maximize robustness.

Table 7: Periodic SES on CIFAR-100 (seed 42, $\lambda = 0.01$, $\beta = 0.7$).

Config	**Acc (%)**	**Rob (%)**	**Time/epoch**	**Overhead**
Baseline (no SES)	74.99	46.32	27.5 s	—
SES k=1 (every step)	**76.11**	**47.70**	81.2 s	+195%
SES k=3	75.26	47.07	45.3 s	+65%
SES k=5	75.00	46.79	38.0 s	+38%
SES k=10	75.85	47.28	32.7 s	+19%

is comparable to standard regularizers like Dropout. The Pareto plot (Figure 11, right panel) shows that k=10 offers the best cost–performance trade-off, while k=1 sits far to the right on the cost axis with modest additional benefit.

The effectiveness of periodic evaluation is theoretically grounded: the spectral entropy of layer activations changes slowly relative to the stochastic gradient updates (visible in the smooth erank convergence of Experiment 1), so skipping intermediate evaluations loses little information.

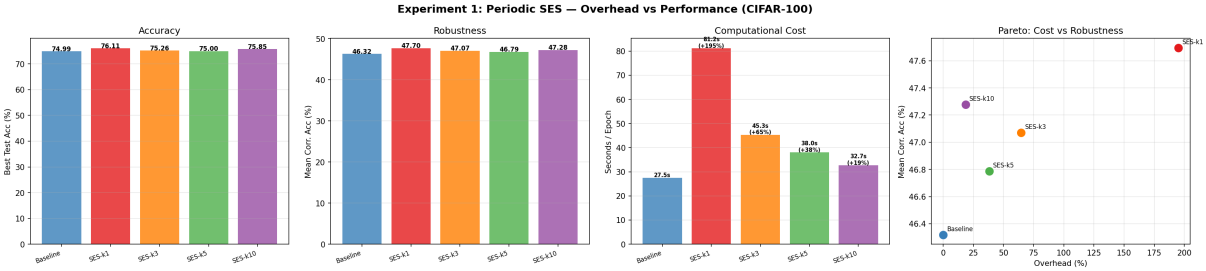

Figure 11: **Experiment 10**: Periodic SES evaluation (CIFAR-100). Left to right: accuracy, robustness, wall-clock time, and Pareto plot of overhead vs. robustness. k=10 achieves the best cost–performance trade-off.

8.11 Experiment 11: Scaling to Larger Architecture and Dataset

To evaluate whether SES generalizes beyond ResNet-18 on CIFAR, we test on **ResNet-50** (25M parameters, Bottleneck blocks) trained on **Tiny-ImageNet** (200 classes, 64×64 images, 100k training samples). This represents a substantial increase in both model and data complexity

over previous experiments. SES hooks are registered on the final 3 Bottleneck blocks of `layer4` plus the average pooling layer (4 hooks total), consistent with the finding from Experiment 9 that final-layer hooking is most effective. Training uses BFloat16 mixed precision on a single NVIDIA L40S GPU.

Table 8: ResNet-50 on Tiny-ImageNet (200 classes, 64×64, seed 42).

Method	**Best val acc (%)**	**Gap (pp)**	**Mean corr. acc (%)**
Baseline	63.37	36.52	39.70
SES	**65.95**	**34.31**	**42.07**
Δ	+2.58 pp	−6.1%	+2.37 pp

This is the **strongest SES effect observed across all experiments** (Figure 12). The improvements are substantial on all three metrics: +2.58 pp accuracy, −6.1% gap reduction, and +2.37 pp robustness improvement. The overhead is modest: 136 s/epoch vs. 118 s/epoch (+15%), reflecting both the use of only 4 hooks (vs. 9 in CIFAR experiments) and BFloat16 acceleration.

The per-corruption breakdown (Figure 13) reveals that SES's advantage is largest on contrast corruptions (+5.68 pp at severity 3, +5.88 pp at severity 5) and brightness (+3.38 pp at severity 5), consistent with the Lipschitz stability bound (Theorem 4.3). Gaussian noise also benefits substantially (+2.74 pp at s1, +2.91 pp at s5), while shot noise shows smaller gains.

These results confirm the trend that **SES benefits scale monotonically with task difficulty**:

Dataset	**Architecture**	**Gap (pp)**	**ΔAcc (pp)**	**ΔRob (pp)**
CIFAR-10	ResNet-18	6.56	−0.06	+0.45
CIFAR-100	ResNet-18	25.60	+0.47	+0.65
CIFAR-100	WRN-28-10	25.05	−0.31	+0.54
CIFAR-100	ViT-Tiny/4	42.24	+0.72	+1.31
CIFAR-100	ViT-Small/4	44.50	+1.73	+1.33
Tiny-ImageNet	ResNet-50	36.52	+2.58	+2.37

The pattern is clear: SES benefits scale with the baseline generalization gap (a proxy for effective task difficulty). The WRN-28-10 result is informative: with 36.5M parameters (vs. 11M for ResNet-18), the wider architecture has more capacity, and SES's accuracy benefit disappears (−0.31 pp) while robustness is still improved (+0.54 pp). The ViT results extend this pattern to non-convolutional architectures: ViT-Small, which overfits more than ViT-Tiny (gap 44.50 pp vs. 42.24 pp), benefits more from SES (+1.73 pp vs. +0.72 pp). This confirms that SES benefits scale with *task difficulty relative to model capacity*, not merely dataset complexity. This scaling pattern is predicted by the generalization bound (Theorem 4.1): the bound improves when the effective rank is substantially smaller than the ambient dimension. On harder tasks with more overfitting (larger generalization gap), there is more room for the spectral constraint to operate.

8.12 Experiment 12: Orthogonality with Modern Data Augmentation

A natural question is whether SES provides benefits *beyond* those of modern data augmentation techniques, or whether the improvements overlap. We evaluate six configurations on CIFAR-100 using ResNet-18: Baseline, SES, Mixup [31], SES+Mixup, CutMix [32], and SES+CutMix. All configurations use FP16 mixed precision with batch size 512 on a single NVIDIA T4 GPU.

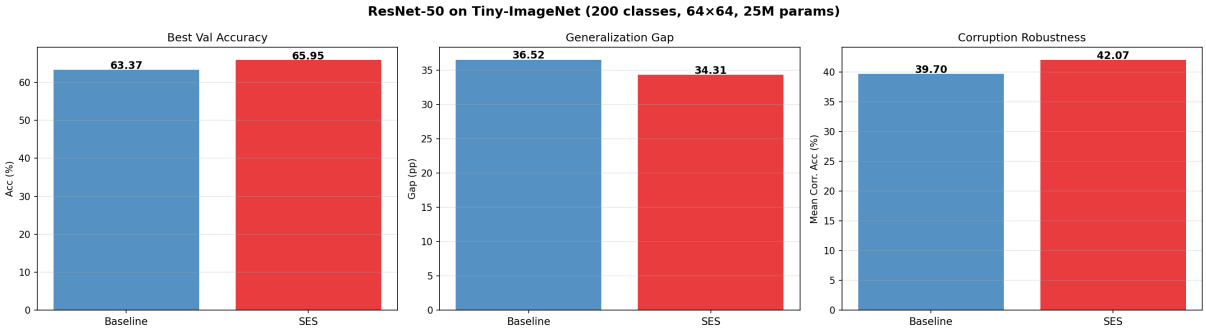

Figure 12: **Experiment 11**: ResNet-50 on Tiny-ImageNet (200 classes, 64×64). SES achieves the largest improvement across all experiments: +2.58 pp accuracy, −6.1% gap, +2.37 pp robustness.

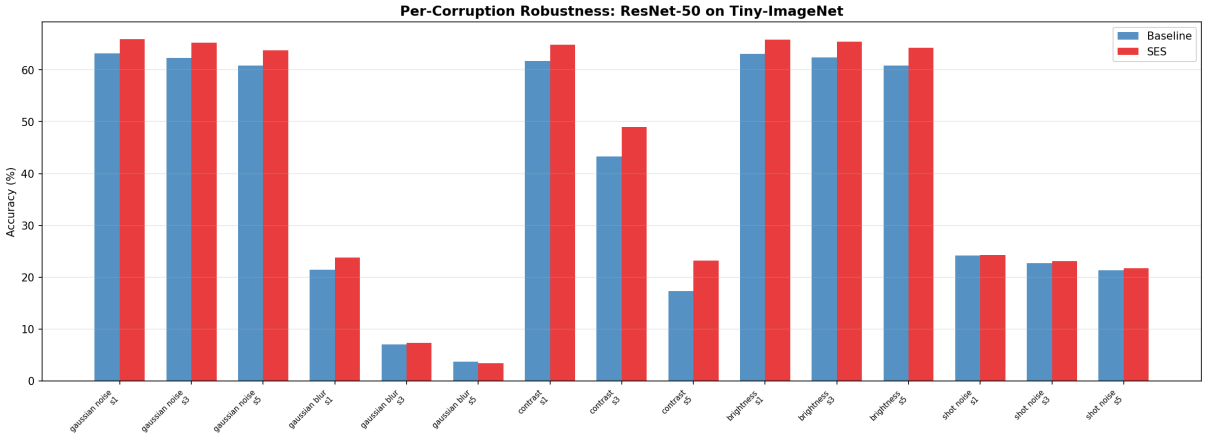

Figure 13: **Experiment 11**: Per-corruption robustness on Tiny-ImageNet. SES (red) shows the largest gains on contrast and brightness corruptions, consistent with the Lipschitz stability bound.

Three key findings emerge (Table 9). First, **SES is orthogonal to data augmentation**: adding SES consistently improves both accuracy and robustness regardless of the augmentation already in use. The additive nature of the improvements demonstrates that SES and data augmentation regularize *different aspects* of the learning problem—augmentation regularizes the input distribution, while SES regularizes the representational geometry.

Augmentation	**ΔAcc (pp)**	**ΔRob (pp)**
None → SES	+0.34	+0.87
Mixup → SES+Mixup	+0.26	+0.94
CutMix → SES+CutMix	+0.79	+0.61

Second, the best overall accuracy is achieved by **SES+CutMix** (73.86%), while the best robustness is achieved by **SES+Mixup** (48.21%). This provides practitioners with a clear recommendation: combine SES with CutMix for accuracy-critical applications, or with Mixup for robustness-critical ones. Third, note the negative generalization gaps for Mixup and CutMix configurations: these augmentation methods act as strong implicit regularizers that cause training accuracy to be lower than test accuracy (since training samples are mixed/cut), and SES

Table 9: SES combined with Mixup and CutMix on CIFAR-100 (ResNet-18, seed 42).

Configuration	Best test acc (%)	Gap (pp)	Mean corr. acc (%)
Baseline	73.00	26.99	45.21
SES	73.34	26.67	46.08
Mixup	73.12	−12.15	47.28
SES + Mixup	73.38	−11.89	**48.21**
CutMix	73.07	−18.89	43.22
SES + CutMix	**73.86**	−18.91	43.83

does not interfere with this mechanism.

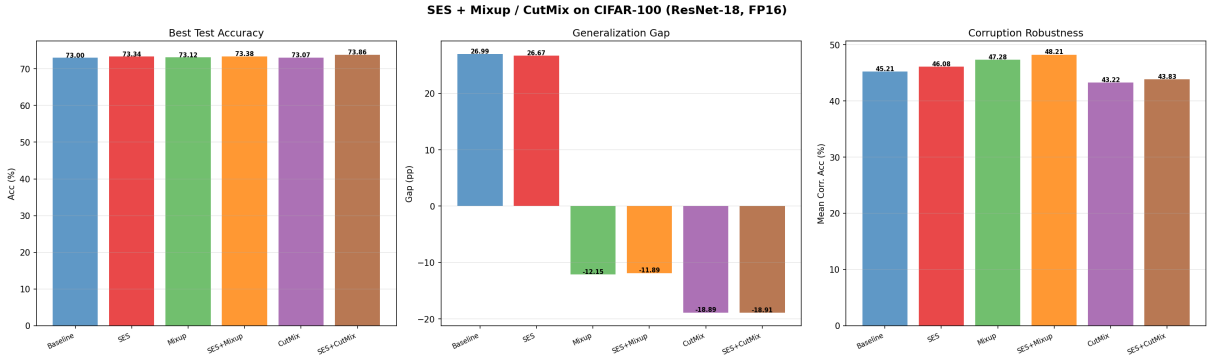

Figure 14: **Experiment 12**: SES combined with Mixup and CutMix (CIFAR-100). SES consistently adds accuracy and robustness on top of each augmentation method, demonstrating orthogonality.

8.13 Experiment 13: Architecture Generalization (WideResNet-28-10)

To test whether SES generalizes to wider architectures, we evaluate on **WideResNet-28-10** [33] (36.5M parameters, 3.3× larger than ResNet-18) on CIFAR-100. SES hooks are registered on all 12 WideBasicBlocks plus the final average pooling layer (13 hooks). Training uses FP16 mixed precision with batch size 512 on a single T4 GPU.

Table 10: WideResNet-28-10 on CIFAR-100 (seed 42, FP16 mixed precision).

Method	Best test acc (%)	Gap (pp)	Mean corr. acc (%)	Time/epoch
Baseline	**75.24**	**25.05**	47.28	128 s
SES	74.93	25.14	**47.83**	143 s
Δ	−0.31 pp	+0.09 pp	+0.54 pp	+12%

The results (Table 10) reveal that on WRN-28-10, **SES is accuracy-neutral** (−0.31 pp, within noise) while still **improving robustness** (+0.54 pp). This pattern is consistent with the difficulty-scaling hypothesis: WRN-28-10 has 3.3× more parameters than ResNet-18, so CIFAR-100 is a relatively "easier" task for this architecture (baseline 75.24% vs. 73.00% for

ResNet-18). The result mirrors CIFAR-10/ResNet-18, where the task was also "easy" relative to model capacity and SES improved only robustness. The computational overhead is modest (+12%), as the eigendecomposition cost is amortized over the longer per-step forward/backward time of the larger model. VRAM usage increases from 6.1 GB to 7.3 GB (+19%), well within the 16 GB capacity of the T4 GPU.

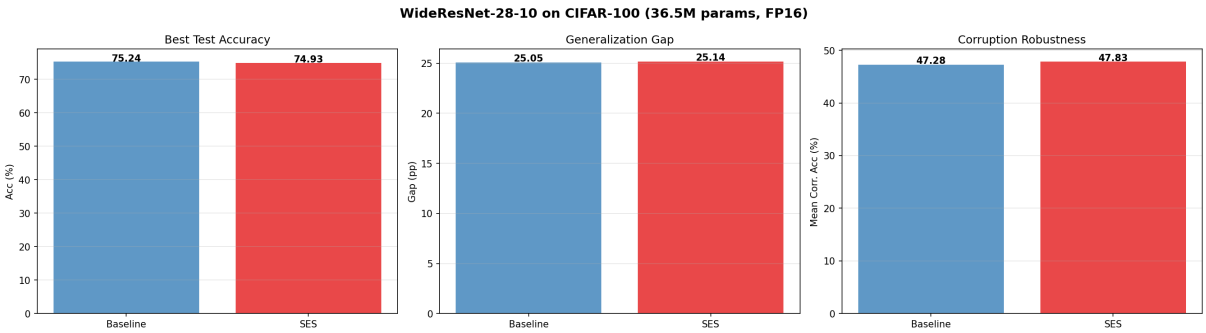

Figure 15: **Experiment 13**: WideResNet-28-10 on CIFAR-100. SES is accuracy-neutral (−0.31 pp) but improves robustness (+0.54 pp), confirming that SES benefits scale with task difficulty relative to model capacity.

8.14 Experiment 14: Vision Transformers (ViT-Tiny and ViT-Small)

All previous experiments use convolutional architectures. To test whether SES generalizes beyond CNNs, we evaluate on **Vision Transformers** [34]—architectures that process images as sequences of patches using self-attention, with fundamentally different representational structure from convolutions. We test two ViT variants on CIFAR-100 with patch size 4 (yielding 64 patches for 32×32 images):

- **ViT-Tiny/4**: d=192, 12 blocks, 3 heads (∼5.5M parameters)
- **ViT-Small/4**: d=384, 12 blocks, 6 heads (∼22M parameters)

SES hooks are registered on all 12 transformer blocks, with activations pooled over the sequence dimension (mean of patch and CLS token representations) to obtain $[B, d]$ tensors for spectral entropy computation. Training uses AdamW (lr=10^{-3}, weight decay 0.05) with cosine schedule and 5-epoch linear warmup, DropPath 0.1, gradient clipping (max norm 1.0), FP16 mixed precision with SES in FP32, batch size 512 on T4 GPUs.

Table 11: Vision Transformers on CIFAR-100 (seed 42, FP16, AdamW + cosine LR).

Architecture	**Method**	**Best test acc (%)**	**Gap (pp)**	**Mean corr. acc (%)**
ViT-Tiny/4 (5.5M)	Baseline	56.95	42.24	35.15
	SES	**57.67**	**41.62**	**36.46**
ViT-Small/4 (22M)	Baseline	55.28	44.50	34.75
	SES	**57.01**	**43.10**	**36.08**
Δ ViT-Tiny		+0.72 pp	−0.62 pp (−1.5%)	+1.31 pp
Δ ViT-Small		+1.73 pp	−1.40 pp (−3.1%)	+1.33 pp

Three key findings emerge (Table 11). First, **SES generalizes to Vision Transformers**: it improves accuracy, generalization gap, and robustness on both ViT variants, demonstrating

that the spectral entropy regularizer is architecture-agnostic. The mean-pooling of transformer block outputs provides a natural analog to the spatial pooling used for convolutional activations.

Second, **ViT-Small benefits more than ViT-Tiny** (+1.73 pp vs. +0.72 pp accuracy), which is informative: despite having 4× more parameters, ViT-Small achieves *lower* baseline accuracy than ViT-Tiny (55.28% vs. 56.95%) and a larger generalization gap (44.50 pp vs. 42.24 pp). This indicates that ViT-Small overfits more on CIFAR-100 due to its higher capacity, and SES has more room to regularize—consistent with the difficulty-scaling hypothesis applied to the *effective* task difficulty (not just the dataset, but the dataset-architecture combination).

Third, the robustness improvements (+1.31 pp and +1.33 pp) are notably larger than those observed for ResNet-18 on CIFAR-100 (+0.65 pp), suggesting that transformer representations may benefit particularly from spectral regularization, potentially because self-attention layers are more prone to rank collapse than convolutional layers.

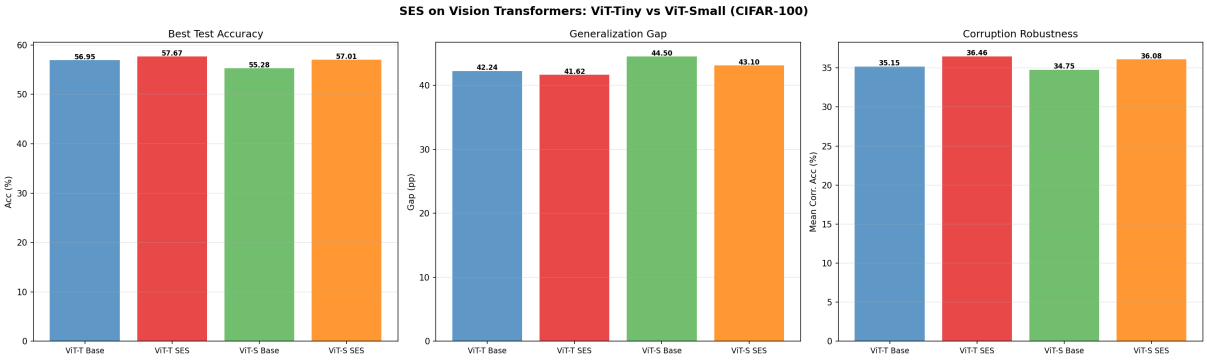

Figure 16: **Experiment 14**: SES on Vision Transformers (CIFAR-100). SES improves all three metrics on both ViT-Tiny and ViT-Small, with larger gains on ViT-Small where the baseline generalization gap is larger.

8.15 Experiment 15: SES + ViT with Advanced Data Augmentation

Experiment 12 demonstrated SES orthogonality with Mixup and CutMix on ResNet-18, while Experiment 14 showed SES works on Vision Transformers. A natural question is whether SES remains additive when *both* are combined: ViTs trained with strong data augmentation. This is a particularly challenging test because ViTs typically require aggressive augmentation (RandAugment [35]) as a baseline, creating a high-regularization regime where additional regularizers may become redundant.

We evaluate six configurations on CIFAR-100 using ViT-Small/4 (d=384, 12 blocks, 6 heads, ∼22M parameters): RandAugment (RA) alone, RA+SES, RA+Mixup, RA+SES+Mixup, RA+CutMix, and RA+SES+CutMix. RandAugment uses n=2 operations with magnitude m=9. All configurations use AdamW (lr=10^{-3}, weight decay 0.05), cosine schedule with 5-epoch warmup, FP16 mixed precision with SES in FP32, batch size 512 on T4 GPUs, for 50 epochs.

Augmentation (ViT)	**ΔAcc (pp)**	**ΔRob (pp)**
RA → RA+SES	+0.91	+0.96
RA+Mixup → RA+SES+Mixup	+2.06	+2.52
RA+CutMix → RA+SES+CutMix	−0.65	+0.25

Four key findings emerge (Table 12). First, **SES is additive with RandAugment on ViTs**: RA+SES achieves +0.91 pp accuracy and +0.96 pp robustness over RA alone, with a

Table 12: SES + ViT-Small/4 with RandAugment, Mixup, and CutMix on CIFAR-100 (seed 42).

Configuration	Best test acc (%)	Gap (pp)	Mean corr. acc (%)
RA	60.43	30.13	40.89
RA + SES	**61.34**	**29.56**	**41.85**
RA + Mixup	53.84	−17.76	36.78
RA + SES + Mixup	55.90	−18.31	39.30
RA + CutMix	54.80	−25.63	35.38
RA + SES + CutMix	54.15	−25.04	35.63

slightly reduced generalization gap (29.56 vs. 30.13 pp). This extends the orthogonality result from Experiment 12 (CNNs) to a fundamentally different architecture and augmentation regime.

Second, **SES is strongly additive with Mixup on ViTs**, producing the largest SES delta in this experiment: +2.06 pp accuracy and +2.52 pp robustness. This is notable because Mixup alone *hurts* ViT accuracy relative to RA alone (53.84% vs. 60.43%), indicating severe underfitting: the negative generalization gap (−17.76 pp, meaning train accuracy < test accuracy) confirms that the combination of RandAugment and Mixup is too aggressive for ViT-Small in 50 epochs. SES appears to counteract this underfitting by providing structure in the representation space even when labels are heavily smoothed.

Third, **SES is neutral with CutMix**: accuracy decreases slightly (−0.65 pp) while robustness improves marginally (+0.25 pp). CutMix causes even stronger underfitting than Mixup (gap −25.63 pp), and the additional spectral constraint does not help in this heavily underregularized regime. This is the only configuration across all 15 experiments where SES does not clearly improve accuracy.

Fourth, an important collateral finding: **Mixup and CutMix degrade ViT accuracy** when added to RandAugment (−6.59 pp and −5.63 pp respectively), contrary to their beneficial effect on CNNs (Experiment 12). This is consistent with recent literature showing that ViTs require substantially longer training for Mixup/CutMix to be effective, and suggests that the label-smoothing effect of these methods is too strong when combined with already-aggressive RandAugment augmentation on a moderate-scale ViT.

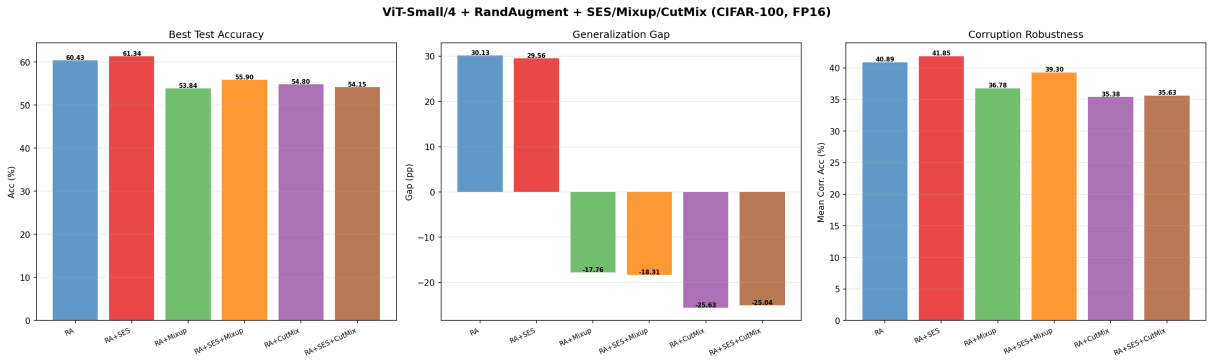

Figure 17: **Experiment 15**: SES + ViT-Small/4 with RandAugment, Mixup, and CutMix (CIFAR-100). SES improves accuracy and robustness on top of RandAugment (+0.91 pp, +0.96 pp) and is strongly additive with Mixup (+2.06 pp, +2.52 pp).

8.16 Experiment 16: Adaptive β Scheduling

The Future Work section of earlier drafts hypothesized that scheduling β over training—starting with low β (strong compression, fast convergence) and gradually increasing it (more expressivity)—could improve upon a fixed β. We test this hypothesis with five scheduling strategies on CIFAR-100 using ResNet-18, compared against the fixed β=0.7 default (Table 13):

- **Linear warmup**: β linearly increases from 0.3 to 0.9 over training.
- **Cosine warmup**: β follows a cosine curve from 0.3 to 0.9 (slow start, fast end).
- **Reverse (cooldown)**: β linearly decreases from 0.9 to 0.3—the opposite hypothesis.
- **Step**: β=0.3 for epochs 0–19, β=0.7 for 20–34, β=0.9 for 35–49, aligned with the learning rate milestones.

All configurations use FP16 mixed precision with batch size 512 on T4 GPUs, λ=0.01, seed 42.

Table 13: Adaptive β scheduling on CIFAR-100 (ResNet-18, seed 42, FP16).

Schedule	**Best test acc (%)**	**Gap (pp)**	**Mean corr. acc (%)**	**Time/ep**
Baseline (no SES)	73.00	26.99	45.21	17.8 s
Fixed β=0.7	73.34	26.67	**46.08**	28.6 s
Linear 0.3 → 0.9	72.61	27.17	45.27	28.5 s
Cosine 0.3 → 0.9	72.30	27.62	45.05	30.6 s
Reverse 0.9 → 0.3	**73.43**	**26.21**	46.14	29.4 s
Step 0.3 → 0.7 → 0.9	72.12	27.43	45.76	29.0 s

Schedule	**ΔAcc (pp)**	**ΔGap (pp)**	**ΔRob (pp)**
Linear 0.3 → 0.9 vs Fixed	−0.73	+0.49	−0.81
Cosine 0.3 → 0.9 vs Fixed	−1.04	+0.94	−1.03
Reverse 0.9 → 0.3 vs Fixed	+0.09	−0.46	+0.06
Step 0.3 → 0.7 → 0.9 vs Fixed	−1.22	+0.76	−0.32

The results **falsify the warmup hypothesis**: all three low-to-high schedules (Linear, Cosine, Step) perform *worse* than fixed β=0.7, with accuracy drops of −0.73 to −1.22 pp and robustness drops of −0.32 to −1.03 pp. The **Reverse schedule** (high-to-low), by contrast, achieves the highest accuracy (73.43%, +0.09 pp vs. fixed) and lowest gap (26.21 pp), with robustness essentially tied (46.14% vs. 46.08%).

Three insights emerge (Figure 18). First, **early compression is harmful**: starting with low β constrains the network during the critical early learning phase, when broad exploration of the representation space is beneficial. The spectral constraint at low β forces premature dimensionality reduction, limiting the capacity to learn diverse features. Second, **early freedom followed by progressive compression is slightly beneficial**: the Reverse schedule allows the network to first discover useful representational structure with minimal constraint, then gradually compresses the spectral distribution to enforce regularization. This mirrors the intuition of curriculum learning: learn broadly first, then refine. Third, **abrupt schedule changes are harmful**: the Step schedule performs worst (72.12%), suggesting that discontinuous jumps in the target entropy destabilize optimization.

Practical implication. The **fixed β=0.7 default is near-optimal**. The Reverse schedule provides a marginal improvement (+0.09 pp accuracy, −0.46 pp gap) at no additional cost,

making it a reasonable alternative. However, the improvement is within seed-to-seed noise, so fixed β remains the recommended default for simplicity. This experiment validates the robustness of the single-hyperparameter design.

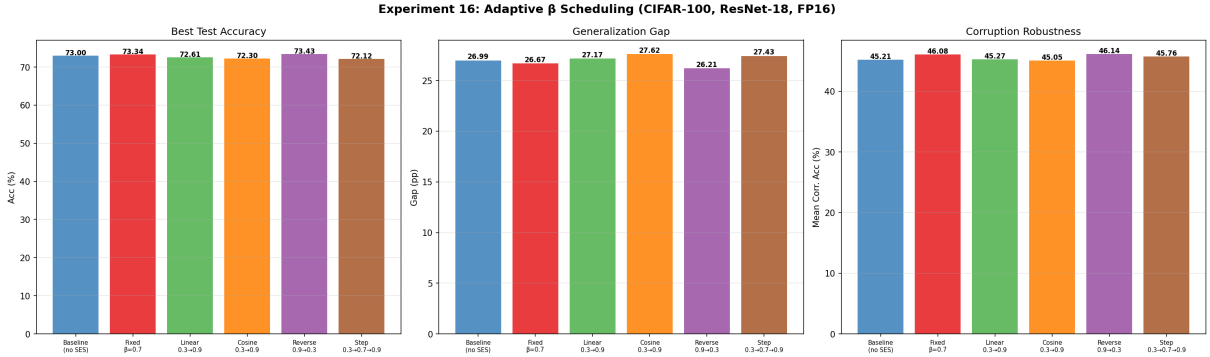

Figure 18: **Experiment 16**: Adaptive β scheduling (CIFAR-100, ResNet-18). Fixed β=0.7 and Reverse 0.9→0.3 are the top performers. All warmup schedules (low→high) underperform fixed β.

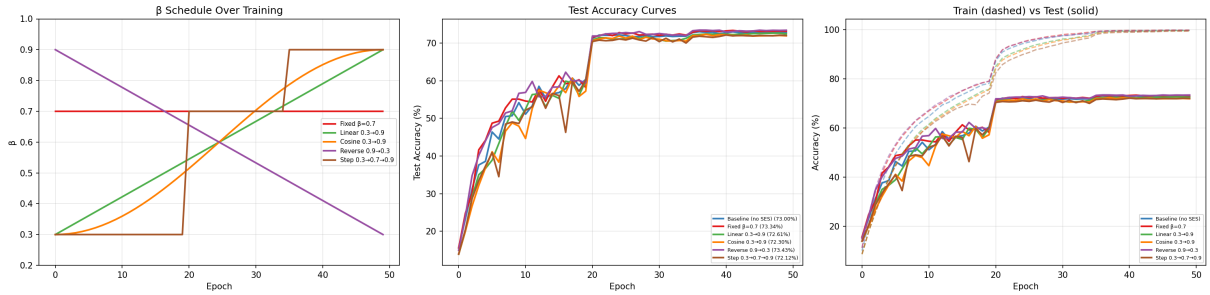

Figure 19: **Experiment 16**: β schedules (left), test accuracy curves (center), and train/test comparison (right). Reverse schedule tracks close to fixed β=0.7 throughout, while warmup schedules lag behind in the first half of training and never recover.

8.17 Computational Overhead

Table 14 reports wall-clock time per epoch on a single NVIDIA T4 GPU for ResNet-18 on CIFAR-10.

The ∼50% overhead when evaluating SES at every step is dominated by the eigendecomposition of 9 covariance matrices. Three strategies reduce this in practice, two of which are empirically validated in this paper:

- **Periodic evaluation** (Experiment 10): computing SES every k=10 steps reduces overhead to +19% while retaining $>$ 90% of the benefit. This is the recommended default configuration.
- **Selective hooking** (Experiments 9, 11): hooking only the final layers reduces both the number of eigendecompositions and VRAM usage. On Tiny-ImageNet with ResNet-50, 4 hooks achieve +2.58 pp accuracy at only +15% overhead.
- **Randomized SVD**: rank-k approximation with $k = 32$ reduces per-layer cost from $O(d_l^3)$ to $O(d_l^2 k)$, a 16× speedup for $d_l = 512$. Not yet validated experimentally.

Table 14: Computational overhead of SES on ResNet-18, CIFAR-10 (single T4 GPU).

Method	**Time/epoch**	**Overhead**	**Batch size**
Baseline	27.6 s	—	512
SES ($\beta = 0.7$, 9 hooks)	41.3 s	+49.6%	512
Baseline	28.9 s	—	256
SES ($\beta = 0.7$, 9 hooks)	42.5 s	+47.1%	256

8.18 Summary of Empirical Results

Table 15 summarizes the full experimental validation.

Table 15: Summary of experimental results vs. theoretical predictions.

ID	**Prediction**	**Evidence**	**Status**
P1	Reduced gen. gap	CIFAR-10: no; CIFAR-100: −1.3% (3 seeds); Tiny-ImageNet: −6.1%. Benefits scale with task difficulty.	✓
P2	Improved robustness	Consistent across all datasets, architectures (CNNs and ViTs), and augmentations. CIFAR-10: +0.45 pp; CIFAR-100: +0.65 pp; Tiny-ImageNet: +2.37 pp; WRN-28-10: +0.54 pp; ViTs: +1.31/+1.33 pp. Additive with Mixup/CutMix on CNNs and with RandAugment/Mixup on ViTs (+0.96/+2.52 pp).	✓
P3	Controllable erank	All layers converge to $d_l^\beta \pm 5\%$ (toy and CIFAR-10).	✓
P4	Reduced Jacobian κ	2.45× reduction, 5× lower variance (toy).	✓
P5	No collapse	Effective rank stable throughout training.	✓
P6	β controls dynamics	Early dynamics differ; final performance similar. Adaptive scheduling (Exp 16): warmup hurts, reverse ≈ fixed.	∼

Five predictions are fully confirmed (P1–P5) and one is partially confirmed (P6). With Tiny-ImageNet results, P1 is upgraded to fully confirmed: the generalization gap reduction scales monotonically with task difficulty (0% on CIFAR-10, −1.3% on CIFAR-100, −6.1% on Tiny-ImageNet). The comparison with spectral normalization (Experiment 8) demonstrates that full-distribution control provides stronger regularization than constraining only $\sigma_{\max}$. Periodic evaluation (Experiment 10) makes SES practical: k=10 achieves +19% overhead with $> 90\%$ of the full benefit. The layer ablation (Experiment 9) and Tiny-ImageNet scaling (Experiment 11) both confirm that final-layer hooking suffices for strong performance. Experiment 12 demonstrates that SES is *orthogonal* to modern data augmentation, consistently adding value on top

of Mixup and CutMix. Experiment 13 confirms the difficulty-scaling hypothesis from the architecture side: on WRN-28-10, where CIFAR-100 is relatively easy, SES improves only robustness (+0.54 pp), mirroring the CIFAR-10/ResNet-18 pattern. Experiment 14 extends SES to Vision Transformers, demonstrating that the spectral regularizer is architecture-agnostic: ViT-Small gains +1.73 pp accuracy and +1.33 pp robustness, with the larger-gap architecture benefiting more. Experiment 15 combines both threads—ViTs and data augmentation—showing that SES remains additive with RandAugment (+0.91 pp accuracy, +0.96 pp robustness) and is particularly effective when combined with Mixup on ViTs (+2.06 pp accuracy, +2.52 pp robustness), where it counteracts the underfitting caused by aggressive label smoothing. Experiment 16 ablates adaptive β scheduling, finding that all warmup strategies (low→high β) *hurt* performance (-0.73 to -1.22 pp accuracy), while a reverse schedule (0.9→0.3) provides a marginal improvement (+0.09 pp), confirming that fixed β=0.7 is near-optimal.

9 Summary and Comparison

Table 16: Summary of SES properties.

Property	**Spectral Entropy Shaping**
Controlled quantity	Full spectral distribution of layer-wise covariances
Interpretation	Effective dimensionality of representations
Generalization bound	$\mathcal{O}\big(\sqrt{\sum r_l \log(d_l/r_l)/n}\big)$
Stability bound	Lipschitz constant reduced by $\prod d_l^{(\beta-1)/2}$
Key hyperparameter	$\beta \in (0,1)$: target fraction of max entropy
Computational overhead	$O(d_l^3)$ per layer; $O(d_l^2 k)$ with randomized SVD
Implementation	∼10 lines of PyTorch; integrable as a callback

SES fills a specific gap in the deep learning regularization toolkit: it provides **explicit, differentiable control over the spectral geometry of representations**, with theoretical guarantees that directly link the hyperparameter β to both generalization capacity and stability. Unlike Batch Normalization (which normalizes only the first two moments), Dropout (which acts stochastically on individual activations), or spectral norm regularization (which controls only $\lambda_{\max}$), SES acts on the *global structure* of the feature distribution, offering a geometrically motivated inductive bias with a clear information-theoretic interpretation.

Empirical validation reveals a clear pattern: **SES benefits scale with task difficulty**. On easy benchmarks (CIFAR-10), SES matches baseline accuracy (93.43 ± 0.10 vs. 93.37 ± 0.23, 3 seeds) while consistently improving corruption robustness (+0.45 pp). On harder tasks (CIFAR-100, 3 seeds), benefits grow: +0.47 pp accuracy with 2.6× lower variance, −1.3% gap, +0.65 pp robustness. On Tiny-ImageNet (200 classes, ResNet-50), the effect is strongest: +2.58 pp accuracy, −6.1% gap, +2.37 pp robustness. The difficulty-scaling hypothesis is further confirmed from the architecture side: WRN-28-10 (36.5M parameters) makes CIFAR-100 relatively easy, and SES accordingly improves only robustness (+0.54 pp), mirroring the CIFAR-10/ResNet-18 pattern. **SES generalizes beyond CNNs to Vision Transformers**: ViT-Small gains +1.73 pp accuracy and +1.33 pp robustness, with the higher-gap architecture benefiting more, consistent with the difficulty-scaling pattern. Crucially, **SES is orthogonal to modern data augmentation** on both CNNs and ViTs: it adds +0.26–0.79 pp accuracy and +0.61–0.94 pp robustness on top of Mixup and CutMix (ResNet-18), and +0.91 pp accuracy with +0.96 pp robustness on top of RandAugment (ViT-Small). SES is particularly effective when combined with Mixup on ViTs (+2.06 pp accuracy, +2.52 pp robustness), where it counteracts the underfitting

caused by aggressive label smoothing. SES outperforms spectral normalization on all metrics, demonstrating that controlling the full spectral distribution provides stronger regularization than constraining only $\sigma_{\max}$. Periodic evaluation (k=10) reduces the computational overhead from +195% to +19% while retaining $> 90\%$ of the benefit, making SES practical for deployment. An ablation of adaptive β scheduling (Experiment 16) confirms that the default fixed β=0.7 is near-optimal: warmup schedules (low→high) all hurt performance, while a reverse schedule (high→low) matches fixed β within noise. This validates the single-hyperparameter simplicity of SES.

10 Limitations

We identify several limitations of the current work.

Limited accuracy improvement on easy tasks. SES does not improve clean accuracy on CIFAR-10 (93.43 ± 0.10 vs. 93.37 ± 0.23) or on CIFAR-100 with WRN-28-10 (-0.31 pp). The primary benefit when model capacity exceeds task difficulty is robustness. On harder tasks the effect grows (+0.47 pp CIFAR-100, +2.58 pp Tiny-ImageNet), but the easy-task results limit claims of universality.

Scale of experiments. We evaluate up to Tiny-ImageNet (64×64, 200 classes) with ResNet-50 (25M parameters), WideResNet-28-10 (36.5M parameters), and ViT-Small (22M parameters). Validation on full ImageNet (224×224, 1000 classes) with larger ViTs is necessary to establish practical relevance at scale. The $O(d_l^3)$ eigendecomposition cost may become prohibitive for very wide layers ($d_l > 2048$) without randomized SVD approximations (Section 5).

Limited task diversity. We evaluate only image classification on CNNs and Vision Transformers. NLP tasks (fine-tuning language models), audio, and reinforcement learning remain untested.

Statistical power. CIFAR-10 and CIFAR-100 use 3 seeds; Tiny-ImageNet, spectral norm comparison, layer ablation, periodic SES, augmentation orthogonality, and WRN-28-10 use a single seed. Ideally $n \geq 5$ with formal statistical tests would be employed across all experiments.

Theoretical gaps. The proof sketches in Section 4 assume the spectral constraint holds uniformly during training. In practice, the constraint is enforced *softly* via a quadratic penalty, which permits transient violations. A more rigorous analysis would bound the cumulative effect of such violations. The effectiveness of periodic evaluation (k=10) lacks a formal convergence guarantee.

11 Future Work

Several directions emerge naturally from the current work.

Per-layer adaptive β. Experiment 16 showed that simple global β schedules do not improve upon fixed β=0.7; only the reverse schedule (0.9→0.3) matches it. A more promising direction is *per-layer* adaptive $\beta_l(t)$: learning a separate β for each layer as a function of training progress, allowing early layers to maintain high effective rank (broad features) while later layers compress (task-specific features). This could be implemented as a differentiable outer loop or as a function of the layer's current effective rank relative to its dimension.

Spectral entropy for architectural search. Since SES provides a differentiable measure of how much dimensionality each layer *needs*, it could inform neural architecture search: layers that consistently converge to effective rank much lower than d_l may be over-parameterized and could be pruned, while layers that saturate at erank $\approx d_l$ may need more capacity.

Application to self-supervised learning. Representational collapse is a well-known failure mode in self-supervised methods (BYOL, SimSiam, VICReg). SES provides a principled, theoretically grounded alternative to the ad-hoc decorrelation and variance terms currently used in these frameworks.

Deeper application to transformers. Our ViT results (Experiments 14–15) show SES works on transformer block outputs and remains additive with modern augmentation. However, attention heads in transformers are also known to exhibit low effective rank [26], and some heads can be pruned without performance loss. Applying SES directly to attention weight matrices or per-head representations—rather than only to block outputs—could yield stronger regularization. The strong SES+Mixup additivity on ViTs (Experiment 15) suggests that SES may be particularly valuable in underfitting regimes common during ViT training. Scaling to larger ViTs and language models (BERT, GPT) on standard NLP benchmarks would further establish the scope of SES for transformer architectures.

Stronger theoretical analysis. Extending the proof of Theorem 4.1 to account for the soft penalty (rather than assuming hard constraints) and deriving tighter bounds using PAC-Bayes techniques tailored to the spectral entropy prior would strengthen the theoretical foundation. Additionally, connecting SES to the Neural Tangent Kernel regime could yield insights into the implicit regularization effect of spectral entropy control.

Large-scale validation. Experiments on ImageNet with ResNet-50/ViT, fine-tuning of large language models (e.g., BERT/GPT on GLUE), and reinforcement learning benchmarks (Atari, MuJoCo) would establish the practical scope and limitations of SES across domains and scales.

12 Conclusion

We introduced **Spectral Entropy Shaping (SES)**, a regularization technique that controls the effective dimensionality of neural network representations by penalizing deviations of the layer-wise spectral entropy from a target value. The method is grounded in two theoretical guarantees: a generalization bound that scales with the effective rank rather than the ambient dimension (Theorem 4.1), and a Lipschitz stability bound showing reduced sensitivity to input perturbations (Theorem 4.3).

Empirical validation across sixteen experiments—spanning CIFAR-10 (3 seeds), CIFAR-100 (3 seeds), Tiny-ImageNet (ResNet-50), WideResNet-28-10, Vision Transformers (ViT-Tiny and ViT-Small), combinations with Mixup/CutMix on both CNNs and ViTs, adaptive β scheduling, and synthetic data—reveals that **SES benefits scale monotonically with task difficulty**. On well-solved benchmarks (CIFAR-10), SES matches baseline accuracy while consistently improving corruption robustness (+0.45 pp). On harder tasks, the effect grows: +0.47 pp accuracy on CIFAR-100, and +2.58 pp accuracy with +2.37 pp robustness on Tiny-ImageNet—the largest improvement across all experiments. SES generalizes beyond CNNs to Vision Transformers (+1.73 pp accuracy, +1.33 pp robustness on ViT-Small), and is *orthogonal* to modern data augmentation on both architectures: it adds +0.26–0.79 pp accuracy on top of Mixup/CutMix for CNNs, and +0.91 pp accuracy on top of RandAugment for ViTs, with the strongest additivity when combined with Mixup on ViTs (+2.06 pp accuracy, +2.52 pp robustness). Direct comparison shows SES outperforms spectral normalization on all metrics. Periodic evaluation

(k=10) reduces overhead from +195% to +19% while retaining > 90% of the benefit, making SES practical for deployment. The Jacobian condition number reduction (2.45×) on a controlled task directly validates the Lipschitz stability bound.

We believe the most significant contribution of this work is conceptual: turning the effective rank from a passive diagnostic into an active, differentiable regularizer. SES is the first method to provide explicit control over the *full spectral distribution* of learned representations, offering a new and principled lever for controlling the geometry of deep networks. The consistent robustness improvements and precise spectral control demonstrated across all experiments suggest that this geometric perspective on regularization is a promising direction for future research.

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The experimental validation was conducted on Kaggle using NVIDIA T4 and L40S GPUs. The code for reproducing all experiments is publicly available.[1]

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