Guanlin Zhang
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Question 1.

Solution 1. *For part (a):*

When $n = 1$, the degree of freedom for ssE is $N - ab = ab \cdot n - ab = ab - ab = 0$. Thus σ^2 can not be estimated.

For part (b):

We want to show that $E[MSE] = \sigma^2$. We have the following computation:

$$
\begin{aligned}
E[MSE] &= E\Big[\frac{1}{e}\sum_{h=1}^{e} SSc_h\Big] = \frac{1}{e}\sum_{h=1}^{e} E\Big[\frac{(\sum\sum d_{ij}y_{ij})^2}{\sum\sum d_{ij}^2}\Big] \\
&= \frac{1}{e}\cdot\frac{1}{\sum\sum d_{ij}^2}\sum_{h=1}^{e} E\Big[(\sum_i\sum_j d_{ij}y_{ij})^2\Big] \\
&= \frac{1}{e}\cdot\frac{1}{\sum\sum d_{ij}^2}\sum_{h=1}^{e}\Big\{\Big(E[\sum_i\sum_j d_{ij}y_{ij}]\Big)^2 + Var\Big(\sum_i\sum_j d_{ij}y_{ij}\Big)\Big\}
\end{aligned}
$$

Recall from the restriction of the complete two way anova model:

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0$$

and the restriction of our contrast coeffecients:

$$\sum_i d_{ij} = \sum_j d_{ij} = 0 \text{ for any } i, j$$

So the first expectation above is:

$$
\begin{aligned}
E[\sum_i\sum_j d_{ij}y_{ij}] &= \sum_i\sum_j d_{ij}\Big(\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}\Big) \\
&= \mu\sum_i\cancel{\sum_j d_{ij}} + \cancel{\sum_i \alpha_i \sum_j d_{ij}} + \cancel{\sum_j \beta_j \sum_i d_{ij}} + \sum_i\sum_j d_{ij}(\alpha\beta)_{ij} \\
&= 0 + 0 + 0 + \sum_i\sum_j d_{ij}(\alpha\beta)_{ij} \\
&= 0
\end{aligned}
$$

The last step above needs $\sum_i\sum_j d_{ij}(\alpha\beta)_{ij} = 0$ but this is no problem here, because for the first e orthogonal contrasts we already know in advance that they are likely to be negligible.

On the other hand, let's look at the second part of the $E[MSE]$:

$$Var(\sum_i\sum_j d_{ij}y_{ij}) = \sum\sum d_{ij}^2\, Var(y_{ij}) = \sum_i\sum_j d_{ij}^2\sigma^2 = \sigma^2\sum_i\sum_j d_{ij}^2$$

So put the information above together we have:

$$\begin{aligned} E[MSE] &= \frac{1}{e} \cdot \frac{1}{\sum_i \sum_j d_{ij}^2} \sum_{h=1}^{e} \left\{ \sigma^2 \sum_i \sum_j d_{ij}^2 \right\} \\ &= \frac{\sigma^2 \cdot e \cdot \cancel{\sum_i \sum_j d_{ij}^2}}{e \cdot \cancel{\sum_i \sum_j d_{ij}^2}} \\ &= \sigma^2 \end{aligned}$$

Thus we have proved our intended result.

For part (c):

our test statistic here is:

$$\frac{SSAB_m/m}{SSE/e} = \frac{SSAB_m/m}{MSE}$$

with $m = (a-1)(b-1) - e$. *When under* H_0^{AB}, *the above test statistic follows* $F_{m,e}$ *distribution so the decision rule is:*

$$\begin{cases} \text{reject } H_0^{AB} & \text{if } \frac{SSAB_m/m}{SSE/e} > F_{m,e,\alpha} \\ \text{retain} H_0^{AB} & \text{otherwise} \end{cases}$$