Question 1.

Solution 1. *For part (a):*

We want to derive $Var(\hat{\beta}_1)$:

Notice that in the simple linear regression model, we have estimate for β_1 as:

$$\hat{\beta}_1 = \frac{\sum_x \sum_t xy_{xt} - n\bar{x}_{..}\bar{y}_{..}}{ss_{xx}}$$

So we can compute the variance as follows:

$$\begin{aligned}
Var(\hat{\beta}_1) &= Var\Big(\frac{\sum_x \sum_t xy_{xt} - n\bar{x}_{..}\bar{y}_{..}}{ss_{xx}}\Big) \\
&= \frac{1}{(ss_{xx})^2}\Big[\sum_x \sum_t x^2\, Var(y_{xt}) + n^2\bar{x}_{..}^2 \cdot Var(\bar{y}_{..}) - 2\sum_x \sum_t nx\bar{x}_{..}\, Cov(y_{xt}, \bar{y}_{..})\Big] \\
&= \frac{1}{(ss_{xx})^2}\Big[\sum_x \sum_t x^2\sigma^2 + n^2\bar{x}_{..}^2 \cdot \frac{1}{n}\sigma^2 - 2\cancel{n}\bar{x}_{..}\sum_x \sum_t x \cdot \frac{\cancel{1}}{\cancel{n}}\sigma^2\Big] \\
&= \frac{1}{(ss_{xx})^2}\Big[\sigma^2\sum_x r_x x^2 + \sigma^2 n\bar{x}_{..}^2 - 2\sigma^2\bar{x}_{..} \cdot n\bar{x}_{..}\Big] \quad (\ since\ \sum_x \sum_t x = n\bar{x}_{..}) \\
&= \frac{\sigma^2}{(ss_{xx})^2}\Big[\sum_x r_x x^2 - n\bar{x}_{..}^2\Big] \\
&= \frac{\sigma^2}{(ss_{xx})^2}\Big[\sum_x r_x x^2 - \sum_x r_x\bar{x}_{..}^2\Big] \\
&= \frac{\sigma^2}{(ss_{xx})^2}\sum_x r_x(x - \bar{x}_{..})^2
\end{aligned}$$

Let's justfiy the last $' ='$:

Notice that:

$$2\sum_x r_x x\bar{x}_{..} = 2\bar{x}_{..}\sum_x r_x x = 2\bar{x}_{..} \cdot x_{..} = 2n\bar{x}_{..}^2 = 2\sum_x r_x\bar{x}_{..}^2$$

So we have:

$$\begin{aligned}
\sum_x r_x x^2 - \sum_x r_x\bar{x}_{..}^2 &= \sum_x r_x x^2 - 2\sum_x r_x\bar{x}_{..}^2 + \sum_x r_x\bar{x}_{..}^2 \\
&= \sum_x r_x x^2 - 2\sum_x r_x x\bar{x}_{..} + \sum_x r_x\bar{x}_{..}^2 \\
&= \sum_x r_x(x^2 - 2x\bar{x}_{..} + \bar{x}_{..}^2) \\
&= \sum_x r_x(x - \bar{x}_{..})^2 = ss_{xx}
\end{aligned}$$

So we have:

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{(ss_{xx})^2} \cdot ss_{xx} = \frac{\sigma^2}{ss_{xx}}$$

For part (b):

Since $\hat{\beta}_0 = \bar{y}_{..} - \hat{\beta}_1 \bar{x}_{..}$, *we have:*

$$\begin{aligned} Var\ (\hat{\beta}_0) &= Var(\bar{y}_{..} - \hat{\beta}_1 \bar{x}_{..}) \\ &= Var(\bar{y}_{..}) + Var(\hat{\beta}_1 \bar{x}_{..}) - 2Cov(\bar{y}_{..}, \hat{\beta}_1 \bar{x}_{..}) \\ &= \frac{\sigma^2}{n} + \bar{x}_{..}^2 \cdot \frac{\sigma^2}{ss_{xx}} - 2\bar{x}_{..} Cov(\bar{y}_{..}, \hat{\beta}_1) \end{aligned}$$

Notice that:

$$\begin{aligned} Cov(\bar{y}_{..}, \hat{\beta}_1) &= Cov\Big(\bar{y}_{..}, \frac{\sum_x \sum_t x y_{xt} - n\bar{x}_{..}\bar{y}_{..}}{ss_{xx}}\Big) \\ &= \frac{1}{ss_{xx}} \Big\{ \sum_x \sum_t x\, Cov(\bar{y}_{..}, y_{xt}) - n\bar{x}_{..} Var(\bar{y}_{..}) \Big\} \\ &= \frac{1}{ss_{xx}} \Big\{ \sum_x \sum_t x \cdot \frac{\sigma^2}{n} - n\bar{x}_{..} \cdot \frac{\sigma^2}{n} \Big\} \\ &= \frac{1}{ss_{xx}} \Big\{ \sigma^2 \bar{x}_{..} - \sigma^2 \bar{x}_{..} \Big\} \\ &= 0 \end{aligned}$$

So we have:

$$Var\ (\hat{\beta}_0) = \frac{\sigma^2}{n} + \bar{x}_{..}^2 \cdot \frac{\sigma^2}{ss_{xx}} = \sigma^2 \Big(\frac{1}{n} + \frac{\bar{x}_{..}^2}{ss_{xx}} \Big)$$

For part (c):

We have:

$$\begin{aligned} Var\Big(\hat{Y}_{x_a t}\Big) &= Var\Big(\hat{\beta}_0 + \hat{\beta}_1 x_a\Big) = Var\Big(\bar{y}_{..} - \hat{\beta}_1 \bar{x}_{..} + \hat{\beta}_1 x_a\Big) \\ &= Var\Big(\bar{y}_{..} + (x_a - \bar{x}_{..})\hat{\beta}_1\Big) \\ &= Var\Big(\bar{y}_{..}\Big) + (x_a - \bar{x}_{..})^2 Var(\hat{\beta}_1) + 2Cov\Big(\bar{y}_{..}, (x_a - \bar{x}_{..})\hat{\beta}_1\Big) \\ &= \frac{\sigma^2}{n} + (x_a - \bar{x}_{..})^2 \frac{\sigma^2}{ss_{xx}} + 2(x_a - \bar{x}_{..}) \cdot \underbrace{Cov(\bar{y}_{..}, \hat{\beta}_1)}_{=0 \text{ as shown in part (b)}} \\ &= \Big(\frac{1}{n} + \frac{(x_a - \bar{x}_{..})^2}{ss_{xx}} \Big) \sigma^2 \end{aligned}$$