Ch 9. Question 2.

Solution 1. *For part (a):*

$$\begin{aligned} E[Y_{it}] &= E\Big[\mu + \tau_i + \beta(x_{it} - \bar{x}_{..}) + \epsilon_{it}\Big] \\ &= \mu + \tau_i + \beta(x_{it} - \bar{x}_{..}) \end{aligned}$$

For part (b):

we have:

$$\begin{aligned} sp^*_{xY} &= \sum_i \sum_t (x_{it} - \bar{x}_{i.})(Y_{it} - \bar{Y}_{i.}) \\ &= \sum_i \sum_t \Big[(x_{it} - \bar{x}_{i.})Y_{it} - (x_{it} - \bar{x}_{i.})\bar{Y}_{i.}\Big] \\ &= \sum_i \sum_t (x_{it} - \bar{x}_{i.})Y_{it} - \sum_i \sum_t (x_{it} - \bar{x}_{i.})\bar{Y}_{i.} \end{aligned}$$

Notice that

$$\sum_i \sum_t (x_{it} - \bar{x}_{i.})\bar{Y}_{i.} = \sum_i \bar{Y}_{i.} \sum_t (x_{it} - \bar{x}_{i.}) = \sum_i \bar{Y}_{i.} \cancel{(r_i\bar{x}_{i.} - r_i\bar{x}_{i.})} = 0$$

So we have

$$sp^*_{xY} = \sum_i \sum_t (x_{it} - \bar{x}_{i.})Y_{it}$$

For part (c):

we have:

$$\begin{aligned} E[\hat{\beta}] &= E\Big[\frac{sp^*_{xY}}{sp^*_{xx}}\Big] = \frac{1}{sp^*_{xx}} E[sp^*_{xY}] = \frac{E\Big[\sum_i \sum_t (x_{it} - \bar{x}_{i.})Y_{it}\Big]}{\sum_i \sum_t (x_{it} - \bar{x}_{i.})^2} \\ &= \frac{\sum_i \sum_t (x_{it} - \bar{x}_{i.})E[Y_{it}]}{\sum_i \sum_t (x_{it} - \bar{x}_{i.})^2} = \frac{\sum_i \sum_t (x_{it} - \bar{x}_{i.})\Big[\mu + \tau_i + \beta(x_{it} - \bar{x}_{..})\Big]}{\sum_i \sum_t (x_{it} - \bar{x}_{i.})^2} \\ &= \frac{\mu \cancel{\sum_i \sum_t (x_{it} - \bar{x}_{i.})} + \sum_i \tau_i \cancel{\sum_t (x_{it} - \bar{x}_{i.})} + \beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})(x_{it} - \bar{x}_{..})}{\sum_i \sum_t (x_{it} - \bar{x}_{i.})^2} \end{aligned}$$

Notice that for the remaining term on the numerator, we have:

$$\begin{aligned} &\beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})(x_{it} - \bar{x}_{..}) = \beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})\Big[(x_{it} - \bar{x}_{i.}) + (\bar{x}_{i.} - \bar{x}_{..})\Big] \\ &= \beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})^2 + \beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})(\bar{x}_{i.} - \bar{x}_{..}) \\ &= \beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})^2 + \beta \sum_i (\bar{x}_{i.} - \bar{x}_{..}) \cancel{\sum_t (x_{it} - \bar{x}_{i.})} \\ &= \beta \sum_i \sum_t (x_{it} - \bar{x}_{i.})^2 \end{aligned}$$

So we have

$$E[\hat{\beta}] = \frac{\beta \sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})^2}{\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})^2} = \beta$$

For part (d):

We have:

$$Var(\hat{\beta}) = E[\hat{\beta}^2] - \beta^2 = \frac{1}{(ss^*_{xx})^2} E\Big[\Big(sp^*_{xY}\Big)^2\Big] - \beta^2$$

Notice that:

$$\begin{aligned}
E\Big[\Big(sp^*_{xY}\Big)^2\Big] &= E\Big[\Big(\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})Y_{it}\Big)^2\Big] \\
&= Var\Big(\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})Y_{it}\Big) + \Big(E\Big[\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})Y_{it}\Big]\Big)^2 \\
&= \sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})^2 \, Var(Y_{it}) + \Big(\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})E[Y_{it}]\Big)^2 \\
&= \sigma^2 \sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})^2 + \Big(\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})(\mu + \tau_i + \beta(x_{it} - \bar{x}_{\cdot\cdot}))\Big)^2 \\
&= \sigma^2 ss^*_{xx} + \Big[\mu \cancel{\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})} + \sum_i \tau_i \cancel{\sum_t (x_{it} - \bar{x}_{i\cdot})} + \beta \sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})(x_{it} - \bar{x}_{\cdot\cdot})\Big]^2 \\
&= \sigma^2 ss^*_{xx} + \Big[\beta \sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})\Big((x_{it} - \bar{x}_{i\cdot}) + (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})\Big)\Big]^2 \\
&= \sigma^2 ss^*_{xx} + \beta^2 \Big[\sum_i \sum_t (x_{it} - \bar{x}_{i\cdot})^2 + \sum_i (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot}) \cancel{\sum_t (x_{it} - \bar{x}_{i\cdot})}\Big]^2 \\
&= \sigma^2 ss^*_{xx} + \beta^2 (ss^*_{xx})^2
\end{aligned}$$

So we have:

$$\begin{aligned}
Var(\hat{\beta}) &= \frac{1}{(ss^*_{xx})^2}\Big(\sigma^2 ss^*_{xx} + \beta^2 (ss^*_{xx})^2\Big) - \beta^2 \\
&= \frac{\sigma^2}{ss^*_{xx}} + \beta^2 - \beta^2 \\
&= \frac{\sigma^2}{ss^*_{xx}}
\end{aligned}$$

On the other hand,

$$Cov\Big(\bar{Y}_{i\cdot}, \hat{\beta}\Big) = \frac{Var\Big(\bar{Y}_{i\cdot} + \hat{\beta}\Big) - Var(\bar{Y}_{i\cdot}) - Var(\hat{\beta})}{2}$$

We have:

$$Var(\hat{\beta}) = \frac{\sigma^2}{ss^*_{xx}}$$

and

$$Var(\bar{Y}_{i\cdot}) = Var\Big(\frac{1}{r_i}\sum_{t=1}^{r_i} Y_{it}\Big) = \frac{r_i\sigma^2}{r_i^2} = \frac{\sigma^2}{r_i}$$

also

$$\begin{aligned}
Var\Big(\bar{Y}_{i\cdot} + \hat{\beta}\Big) &= Var\Big(\bar{Y}_{i\cdot} + \frac{\sum_t (x_{it} - \bar{x}_{i\cdot})Y_{it}}{ss^*_{xx}} + \frac{\sum_{i'\neq i}\sum_t (x_{i't} - \bar{x}_{i'\cdot})Y_{i't}}{ss^*_{xx}}\Big) \\
&= Var\Big(\bar{Y}_{i\cdot} + \frac{\sum_t (x_{it} - \bar{x}_{i\cdot})Y_{it}}{ss^*_{xx}}\Big) + Var\Big(\frac{\sum_{i'\neq i}\sum_t (x_{i't} - \bar{x}_{i'\cdot})Y_{i't}}{ss^*_{xx}}\Big) \\
&= Var\Big(\sum_{t=1}^{r_i}(\frac{1}{r_i} + \frac{x_{it} - \bar{x}_{i\cdot}}{ss^*_{xx}})Y_{it}\Big) + \sigma^2 \cdot \frac{\sum_{i'\neq i}\sum_t (x_{i't} - \bar{x}_{i'\cdot})^2}{(ss^*_{xx})^2} \\
&= \sigma^2 \sum_{t=1}^{r_i}\Big(\frac{1}{r_i^2} + \frac{(x_{it} - \bar{x}_{i\cdot})^2}{(ss^*_{xx})^2} + \frac{2}{r_i}\cdot\frac{x_{it} - \bar{x}_{i\cdot}}{ss^*_{xx}}\Big) + \sigma^2 \cdot \frac{\sum_{i'\neq i}\sum_t (x_{i't} - \bar{x}_{i'\cdot})^2}{(ss^*_{xx})^2} \\
&= \sigma^2\Big[\frac{1}{r_i} + \frac{ss^*_{xx}}{(ss^*_{xx})^2} + 0\Big] = \frac{\sigma^2}{r_i} + \frac{\sigma^2}{ss^*_{xx}}
\end{aligned}$$

So we have

$$Var\Big(\bar{Y}_{i\cdot} + \hat{\beta}\Big) - Var(\bar{Y}_{i\cdot}) - Var(\hat{\beta}) = \frac{\sigma^2}{r_i} + \frac{\sigma^2}{ss^*_{xx}} - \frac{\sigma^2}{r_i} - \frac{\sigma^2}{ss^*_{xx}} = 0$$

hence

$$Cov(\bar{Y}_{i\cdot}, \hat{\beta}) = 0$$

For part (e):

we have:

$$\begin{aligned}
E[\hat{\mu} + \hat{\tau}_i] &= E\Big[\bar{Y}_{i\cdot} - \hat{\beta}(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})\Big] \\
&= E[\bar{Y}_{i\cdot}] - (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})E[\hat{\beta}] \\
&= \mu + \tau_i + \beta(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot}) - \beta(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot}) \\
&= \mu + \tau_i
\end{aligned}$$

For part (f):

Being estimable means we can write the estimate as function of the sample. Here we already know

$$\hat{\beta} = \frac{sp^*_{xx}}{ss^*_{xx}}$$

so β is estimable. Also notice

$$\hat{\mu} + \hat{\tau}_i = \bar{Y}_{i\cdot} - \hat{\beta}(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})$$

so $\mu + \tau_i$ is also estimable, as well as the linear combination of $\mu + \tau_i$ and β.