Guanlin Zhang
Spring '17

Ch6 Question 19.

Solution 1. *For part (a):*

To verify the formula for computing ssE of the main effects model, since we know the least square estimate for $\mu + \alpha_i + \beta_j$ is $\bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot}$, so

$$
\begin{aligned}
ssE &= \sum\sum\sum (y_{ijt} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2 \\
&= \sum\sum\sum \Big(y_{ijt}^2 + \bar{y}_{i\cdot\cdot}^2 + \bar{y}_{\cdot j\cdot}^2 + \bar{y}_{\cdot\cdot\cdot}^2 - 2y_{ijt}\bar{y}_{i\cdot\cdot} - 2y_{ijt}\bar{y}_{\cdot j\cdot} + 2y_{ijt}\bar{y}_{\cdot\cdot\cdot} + 2\bar{y}_{i\cdot\cdot}\bar{y}_{\cdot j\cdot} \cdots \\
&\cdot -2\bar{y}_{i\cdot\cdot}\bar{y}_{\cdot\cdot\cdot} - 2\bar{y}_{\cdot j\cdot}\bar{y}_{\cdot\cdot\cdot} \Big) \\
&= \sum\sum\sum y_{ijt}^2 + bn\sum_i \bar{y}_{i\cdot\cdot}^2 + an\sum_j \bar{y}_{\cdot j\cdot}^2 + abn\bar{y}_{\cdot\cdot\cdot}^2 - 2bn\sum_i \bar{y}_{i\cdot\cdot}^2 - 2an\sum_j \bar{y}_{\cdot j\cdot}^2 \cdots \\
&\cdots + \cancel{2abn\bar{y}_{\cdot\cdot\cdot}^2} + \cancel{2abn\bar{y}_{\cdot\cdot\cdot}^2} - \cancel{2abn\bar{y}_{\cdot\cdot\cdot}^2} - \cancel{2abn\bar{y}_{\cdot\cdot\cdot}^2} \\
&= \sum\sum\sum y_{ijt}^2 - bn\sum_i \bar{y}_{i\cdot\cdot}^2 - an\sum_j \bar{y}_{\cdot j\cdot}^2 + abn\bar{y}_{\cdot\cdot\cdot}^2 \\
&= \sum\sum\sum y_{ijt}^2 - \frac{1}{bn}\sum_i y_{i\cdot\cdot}^2 - \frac{1}{an}\sum_j y_{\cdot j\cdot}^2 + \frac{1}{abn}y_{\cdot\cdot\cdot}^2 = (6.5.39)
\end{aligned}
$$

For part (b):

we realize that we abused the notation a little when solving part (a). To make it consistent with the conclusion asked to be proved, we now denote n to be the total sample size and r to be the group sample size with a and b be the number of treatment levels.

So from part (a), we can replace small y with big Y to indicate random variables, thus we get:

$$
\begin{aligned}
E[SSE] &= \sum\sum\sum E[Y_{ijt}^2] - \frac{1}{br}\sum_i E[Y_{i\cdot\cdot}^2] - \frac{1}{ar}\sum_j E[Y_{\cdot j\cdot}^2] + \frac{1}{n}E[Y_{\cdot\cdot\cdot}^2] \\
&= \sum\sum\sum \left\{ \Big(E[Y_{ijt}]\Big)^2 + \underbrace{Var(Y_{ijt})}_{=\sigma^2} \right\} - \frac{1}{br}\sum_i \left\{ \Big(E[Y_{i\cdot\cdot}]\Big)^2 + \underbrace{Var(Y_{i\cdot\cdot})}_{=br\sigma^2} \right\} \cdots \\
&\cdots - \frac{1}{ar}\sum_j \left\{ \Big(E[Y_{\cdot j\cdot}]\Big)^2 + \underbrace{Var(Y_{\cdot j\cdot})}_{=ar\sigma^2} \right\} + \frac{1}{n}\left\{ \Big(E[Y_{\cdot\cdot\cdot}]\Big)^2 + \underbrace{Var(Y_{\cdot\cdot\cdot})}_{=n\sigma^2} \right\}
\end{aligned}
$$

if we focus on all the variance terms, we find that we get

$$
\sum\sum\sum \sigma^2 - \frac{1}{\cancel{br}}\sum_i \cancel{br}\sigma^2 - \frac{1}{\cancel{ar}}\sum_j \cancel{ar}\sigma^2 + \sigma^2 = (n - a - b + 1)\sigma^2
$$

which is what we need. so all we need to do for the rest of the part is to show:

$$
\sum\sum\sum \Big(E[Y_{ijt}]\Big)^2 - \frac{1}{br}\sum_i \Big(E[Y_{i\cdot\cdot}]\Big)^2 - \frac{1}{ar}\sum_j \Big(E[Y_{\cdot j\cdot}]\Big)^2 + \frac{1}{n}\Big(E[Y_{\cdot\cdot\cdot}]\Big)^2 = 0
$$

This is easily done if we notice our model is $Y_{ijt} = \mu + \alpha_i + \beta_j + \epsilon_{ijt}$ with restrictions $\sum_i \alpha_i = 0$ and $\sum_j \beta_j = 0$