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Ch 10. Question 7.

Solution 1. *For part (a):*

We know that the in a BIBD case, the model is:

$$Y_{hi} = \mu + \theta_h + \tau_i + \epsilon_{hi}$$

where $\epsilon_{hi} \sim N(0, \sigma^2)$ *i.i.d and* (h, i) *in the design, with* $1 \le h \le b$ *and* $1 \le i \le v$.

The least square estimate for any contrast $\sum_i c_i \tau_i$ *is:*

$$\sum_i c_i \hat{\tau}_i = \frac{k}{\lambda v} \sum c_i Q_i$$

where

$$Q_i = T_i - \frac{1}{k} \sum_{h=1}^{b} n_{hi} B_h \text{ adjusted sum on treatment } i$$
$$T_i = \sum_{h=1}^{b} n_{hi} y_{hi} \text{ unadjusted sum on treatment } i$$
$$B_h = \sum_{i=1}^{v} n_{hi} y_{hi} \text{ sum of observations in block } h$$

So particularly, we have for contrast $\tau_4 - \tau_6$, *there is least square estimate:*

$$\begin{aligned}\hat{\tau}_4 - \hat{\tau}_6 &= \frac{3}{1 \times 9}\Big(Q_4 - Q_6\Big) \\ &= \frac{1}{3}\Big[\Big(T_4 - \frac{1}{3}\sum_{h=1}^{12} n_{h4} B_h\Big) - \Big(T_6 - \frac{1}{3}\sum_{h=1}^{12} n_{h6} B_h\Big)\Big]\end{aligned}$$

So we have the expectation:

$$E[\hat{\tau}_4 - \hat{\tau}_6] = \frac{1}{3} E[T_4 - T_6] - \frac{1}{9} E[\sum_{h=1}^{12} (n_{h4} - n_{h6}) B_h]$$

Now follow Table 11.8 *on page* 355, *we have:*

$$T_4 = Y_{14} + Y_{24} + Y_{64} + Y_{84}$$
$$T_6 = Y_{36} + Y_{56} + Y_{66} + Y_{96}$$

Also we got:

$$n_{14} = 1, n_{16} = 0; n_{24} = 1, n_{26} = 0; n_{34} = 0, n_{36} = 1; n_{44} = 0, n_{46} = 0;$$
$$n_{54} = 0, n_{56} = 1; n_{64} = 1, n_{66} = 1; n_{74} = 0, n_{76} = 1, n_{84} = 1, n86 = 0;$$
$$n_{94} = 0, n_{96} = 1; n_{10,4} = 0, n_{10,6} = 0; n_{11,4} = 0, n_{11,6} = 0, n_{12,4} = 0, n_{12,6} = 0$$

So we got:

$$\sum_{h=1}^{12}(n_{h4}-n_{h6})B_h = B_1 + B_2 - B_3 - B_5 + B_8 - B_9$$

Specifically, we have:

$$\begin{aligned}
B_1 &= Y_{13} + Y_{14} + Y_{18}\\
B_2 &= Y_{22} + Y_{24} + Y_{29}\\
B_3 &= Y_{33} + Y_{36} + Y_{39}\\
B_5 &= Y_{52} + Y_{56} + Y_{57}\\
B_8 &= Y_{81} + Y_{84} + Y_{87}\\
B_9 &= Y_{91} + Y_{96} + Y_{98}
\end{aligned}$$

So

$$\begin{aligned}
E[\hat{\tau}_4 - \hat{\tau}_6] &= \frac{1}{3}E[T_4 - T_6] - \frac{1}{9}E[\sum_{h=1}^{12}(n_{h4}-n_{h6})B_h]\\
&= \frac{1}{3}E\Big[\Big(Y_{14} + Y_{24} + Y_{64} + Y_{84}\Big) - \Big(Y_{36} + Y_{56} + Y_{66} + Y_{96}\Big)\Big]\\
&\quad - \frac{1}{9}E\Big[B_1 + B_2 - B_3 - B_5 + B_8 - B_9\Big]\\
&= \frac{1}{3}\Big[\cancel{\Big(4\mu - 4\mu\Big)} + \Big(\theta_1 + \theta_2 + \theta_6 + \theta_8\Big) - \Big(\theta_3 + \theta_5 + \theta_6 + \theta_9\Big) + 4\tau_4 - 4\tau_6\Big]\\
&\quad - \frac{1}{9}\Big[\cancel{\Big(3\mu + 3\mu - 3\mu - 3\mu + 3\mu - 3\mu\Big)} + 3\Big(\theta_1 + \theta_2 - \theta_3 - \theta_5 + \theta_8 - \theta_9\Big)\\
&\quad + (\tau_3 + \tau_4 + \tau_8) + (\tau_2 + \tau_4 + \tau_9) - (\tau_3 + \tau_6 + \tau_9) - (\tau_2 + \tau_6 + \tau_7) + (\tau_1 + \tau_4 + \tau_7)\\
&\quad - (\tau_1 + \tau_6 + \tau_8)\Big]
\end{aligned}$$

Let's check the coefficients for each block:

$$\begin{aligned}
&\theta_1 : \frac{1}{3} + (-\frac{1}{3} \times 3) = 0 \quad \theta_2 : \frac{1}{3} + (-\frac{1}{9} \times 3) = 0 \quad \theta_3 : -\frac{1}{3} + (-\frac{1}{9}) \times (-3) = 0\\
&\theta_5 : -\frac{1}{3} + (-\frac{1}{9}) \times (-3) = 0 \quad \theta_6 : \frac{1}{3} - \frac{1}{3} = 0 \quad \theta_8 : \frac{1}{3} + (-\frac{1}{9}) \times 3 = 0\\
&\theta_9 : -\frac{1}{3} + (-\frac{1}{9}) \times (-3) = 0
\end{aligned}$$

Also we have the coefficients for each treatment:

$$\begin{aligned}
&\tau_1 : -\frac{1}{9} \times (1-1) = 0 \quad \tau_2 : -\frac{1}{9} \times (1-1) = 0 \quad \tau_3 : -\frac{1}{9} \times (1-1) = 0\\
&\tau_4 : \frac{4}{3} - \frac{1}{9} \times 3 = \frac{4}{3} - \frac{1}{3} = 1 \quad \tau_6 : -\frac{4}{3} - \frac{1}{9} \times (-3) = -\frac{4}{3} + \frac{1}{3} = -1\\
&\tau_7 : -\frac{1}{9} \times (-1+1) = 0 \quad \tau_8 : -\frac{1}{9} \times (1-1) = 0 \quad \tau_9 : (-\frac{1}{9}) \times (1-1) = 0
\end{aligned}$$

So in summary, we have:

$$E[\hat{\tau}_4 - \hat{\tau}_6] = \tau_4 - \tau_6$$

which means the least square estimate is unbiased.

For part (b):

For short solution, we can use the result from page 354 *that for any contrast* $\sum_i c_i \tau_i$ *in BIBD model, we have*

$$Var\Big(\sum_i c_i \hat{\tau}_i\Big) = \sum_i c_i^2 \Big(\frac{k}{\lambda v}\Big)\sigma^2$$

Then in our case we have:

$$Var\Big(\hat{\tau}_4 - \hat{\tau}_6\Big) = \frac{2k\sigma^2}{\lambda v}$$

If we really want to verify this in detail, we can use the formula developed in part (a), single out all the independent Y_{hi}*, and sum up their variance with coefficients. We should be able to get the answer matching the one above.*