BIOS -900 Homework #1

Date assigned: 09/04/2017
Due Date: 09/18/2017 by 11:59 pm (Blackboard clock time);

Instructions:

1. To receive full credit, show all work. Please make your work legible.
2. Total points for this homework are 100.
3. Do not forget to write your name on the homework.
4. Insert page numbers on all pages and also total # of pages submitted.
5. Homework can be typed or hand-written. Provide SAS code wherever necessary.
6. Use the BLACKBOARD drop box to turn in the homework (preferably as pdf) or bring it to class on 09/18/2017.

Question # 1: 12 points

[A] Use properties of symmetric matrices and properties of determinants to show:
{i} $|\mathbf{A}|$ is either 1 or -1 when **A** is an orthogonal matrix.
{ii} $|\mathbf{A}|$ is either 0 or -1 when **A** is an idempotent matrix.

[B] Read and understand Theorems 2.12 and 2.13 from your textbook. Then use any (one or more) of these theorems to show that: If **A** is idempotent and symmetric, then $\mathbf{A} = \mathbf{B}\mathbf{B}^T$ where $\mathbf{B}^T\mathbf{B} = \mathbf{I}$

Question # 2: 16 points

[A] Solve the following Exercise problems from your textbook:
{i} 2.31
{ii} 2.32
{iii} 2.33

[B] Prove Equation 2.50 from your textbook. Then show that Equation 2.51 and Equation 2.52 are special cases of Equation 2.50.

Question # 3: 12 points

[A] Let **x** be a vector of size nx1. Let $\mathbf{H} = \mathrm{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T$. Answer the following questions:
{i} Show that **x** is an eigenvector of **H**. What is the eigenvalue associated with **x**?
{ii} Show that any vector **v** orthogonal to **x** is also an eigenvector of **H**. What are the eigenvalues of **H**?
{iii} Show that **H** is idempotent.

[B] Show that the matrix $\mathbf{A} = \begin{bmatrix} \frac{1-cos(\theta)}{2} & \frac{sin(\theta)}{2} \\ \frac{sin(\theta)}{2} & \frac{1+cos(\theta)}{2} \end{bmatrix}$ is idempotent.

Question # 4: **12 points**

Consider the following data matrix of 3 variables X_1, X_2, and X_3 :

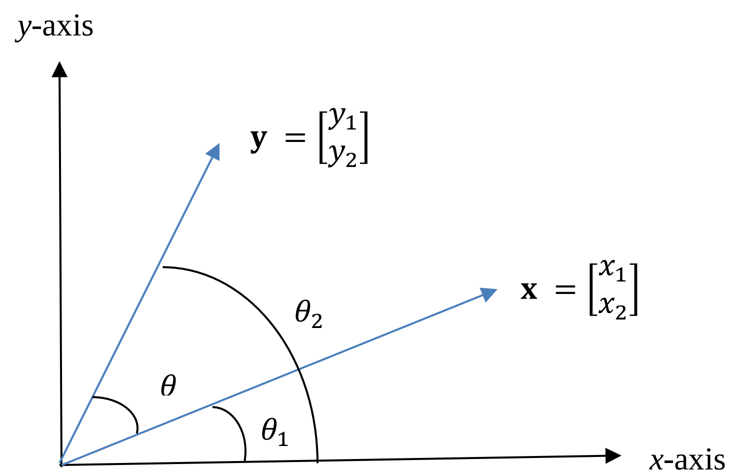
$$\begin{bmatrix} 3 & 10 & 4 \\ 2 & 11 & 5 \\ 6 & 15 & 8 \\ 6 & 9 & 9 \\ 8 & 5 & 10 \end{bmatrix}$$

Use SAS (or equivalent code in R) to answer the following questions:

[A] What is the size of the matrix and the size of the transpose of the matrix?
[B] Find the sample mean vector for the 3 variables: $\overline{\mathbf{X}}$ (a column vector).
[C] Find the sample covariance matrix **S** and the sample correlation matrix **R** using PROC CORR and PROC IML.
[D] Find $\mathbf{S}^{-1}$. Show that multiplying these two matrices results in an identity matrix.
[E] Compute $\overline{\mathbf{X}}'\mathbf{S}^{-1}$ by hand using the rules for multiplying matrices and verify results by SAS
[F] Post-multiply your answer in (E) by $\overline{\mathbf{X}}$ to obtain $\overline{\mathbf{X}}'\mathbf{S}^{-1}\,\overline{\mathbf{X}}$. Verify the results using SAS.
[G] Compute the determinant of **S** using SAS. Also show your hand calculations.
[H] Calculate the trace of **S** by hand and then verify the answer using SAS.
[I] Find eigenvalues and eigenvectors of **S** using SAS. What are the eigenvalues of $\mathbf{S}^2$?
[J] Show the sum of eigenvalues equals the trace of **S.**
[K] Show the product of eigenvalues equals the determinant of **S**.
[L] Show the eigenvectors of **S** are orthogonal.

Question # 5: **10 points**

Consider the following graphical representation for two vectors **x** and **y.**



Let L_x and L_y be the length of **x** and **y** respectively.

[A] Show that: $cos(\theta) = \frac{\mathbf{x}^T\mathbf{y}}{L_x L_y}$

[B] Find the angle between $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$. Are the two vectors linearly independent?

Question # 6: **12 points**

[A]
Theorem 2.12d in your textbook is about the Spectral Decomposition of a symmetric matrix. Read and understand this theorem carefully. Now use this theorem to prove Theorem 2.12e.

Hint: Start with $|\mathbf{A}| = |\mathbf{CDC}^T|$ keeping in mind properties of determinants.

[B]
Consider the matrix $\mathbf{A} = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}$.
Show hand calculations for eigenvalues and eigenvectors of **A**. For this matrix **A**, show how Theorem 2.12d holds true.

Question # 7: **14 points**

[A]
Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 7 & -3 & 6 & 4 \\ 1 & 4 & 0 & 2 & -5 \\ 2 & -1 & 1 & 4 & -9 \\ 0 & -5 & -3 & 0 & -11 \\ 0 & -9 & 1 & 0 & 1 \end{bmatrix}$

Use Theorem 2.8b to find the generalized inverse $\mathbf{A}^{(-)}$. Now use SAS to verify Equation 2.58.

[B]
For the matrix **A** defined above, verify Theorem 2.8c using SAS.

[C]
Using SAS, verify that the following equations are true:
{i} Equation 2.90 {ii} Equation 2.92 {iii} Equation 2.93 {iv} Equation 2.108

Question # 8: **12 points**

The 5x1 vector **X** has the following mean and variance-covariance matrices:

$$\boldsymbol{\mu}_{\mathbf{x}} = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 3 \\ 0 \end{bmatrix} \qquad \boldsymbol{\Sigma}_{\mathbf{x}} = \begin{bmatrix} 4 & -1 & 0.5 & -0.5 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 0.5 & 1 & 6 & 1 & -1 \\ -0.5 & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$. Also partition **X** as shown below.

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X_1} \\ \hline \mathbf{X_2} \end{bmatrix}$$

Now considering the linear combinations $\mathbf{AX_1}$ and $\mathbf{BX_2}$, find:

[A] $E(\mathbf{X_1})$
[B] $E(\mathbf{AX_1})$
[C] $Cov(\mathbf{X_1})$
[D] $Cov(\mathbf{AX_1})$
[E] $E(\mathbf{X_2})$
[F] $E(\mathbf{BX_2})$
[G] $Cov(\mathbf{X_2})$
[H] $Cov(\mathbf{BX_2})$
[I] $Cov(\mathbf{X_1}, \mathbf{X_2})$
[J] $Cov(\mathbf{AX_1}, \mathbf{BX_2})$
[K] $\mathbf{P}_\rho$
[L] rank(**B**)

Question # 9 (Bonus) **5 points**

Read Page #56 – 60 about Vector and Matrix Calculus from your textbook. Confirm in writing that you understood the proofs. Additionally search for the terms ‘Moore-Penrose’ inverse, one-sided inverse, and Drazin inverse on the internet and do some reading on these topics. Visit the website http://mathworld.wolfram.com/Matrix.html for additional reading and confirm that you spent at least one hour reading about various topics on this website.

GOOD LUCK ☺☺