BIOS -900 Fall 2017 Take-Home Midterm Exam

Name:____

Date assigned: 10/11/2017;   Due Date: 10/18/2017 (by 11:59 pm Blackboard clock time)

Instructions:

1. To receive full credit, show all work. Please make your work readable.
2. Use $\alpha = 0.05$ unless otherwise mentioned.
3. Total points for this exam are **50**.
4. Do not forget to write your name and page numbers on the homework.
5. Do not turn in irrelevant SAS output.
6. You cannot discuss this exam with anyone other than the instructor.

Question # 1: **(14 points)**

[A] Find the singular value decomposition (SVD) of the matrix $\mathbf{A}$. Show work-out similar to the suggested reading material on the bonus question of HW#2. Then verify your answers using software.

$$\mathbf{A} = \begin{bmatrix} 10 & -5 \\ 2 & -11 \\ 6 & -8 \end{bmatrix}$$

[B] Let $\mathbf{A}$ be a *n* x *n* matrix of rank *r.* Consider its SVD given by $\mathbf{M}\begin{bmatrix} \mathbf{W}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^T$ where $\mathbf{W}_r$ is *r* x *r* diagonal matrix of containing the non-zero singular values of $\mathbf{A}$. Also, $\mathbf{M}$ and $\mathbf{Q}$ are *n* x *n* matrices with orthogonal columns.

[i] Show that $\mathbf{G} = \mathbf{Q}\begin{bmatrix} \mathbf{W}_r^{\,\mathbf{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{M}^T$ is a generalized inverse of $\mathbf{A}$.

[ii] Show that $\mathbf{AG}$ is symmetric matrix.

Question # 2: **(8 points)**

Prove equation 7.57 from your textbook.

Also find $\mathrm{Var}(R^2)$ when $\beta_1 = \beta_2 = \cdots = \beta_k = 0$. Hence find $\mathrm{E}\left(R_{adj}^2\right)$ and $\mathrm{Var}\left(R_{adj}^2\right)$

Hint: In addition to material from Chapter 5 and Chapter 7, you may have to look into the Appendix section (Table of Common Distributions) of your Theory I textbook (Statistical Inference – by George Casella, Robert L. Berger).

Question # 3: **(8 points)**

The normal equations for a linear regression model are given by $(\mathbf{X}^T\mathbf{X})\widehat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{y}$ and setting the partial derivatives equal to 0 allows us to obtain the estimates of $\boldsymbol{\beta}$.
In this problem you have to prove that $(\mathbf{X}^T\mathbf{X})^{(-)}$ is a generalized inverse of $(\mathbf{X}^T\mathbf{X})$ <u>if and only if</u> **(iff)** $(\mathbf{X}^T\mathbf{X})^{(-)}\mathbf{X}^T\mathbf{y}$ is a solution of the normal equations.

Question # 4: **(12 points)**

Let $\mathbf{y} \sim \mathbb{N}_n(\mathbf{O}, a\mathbf{I}_n + b\mathbf{J}_n)$. Let $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$ and $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n}$

[i] For what values of a and b is $\bar{y}^2$ distributed as χ^2 random variable?

[ii] For what values of a and b is $\hat{\sigma}^2$ distributed as χ^2 random variable?

[iii] For fixed a and b, for what values of c and d are $cn\bar{y}^2$ and $\mathbf{y}^T(\mathbf{I} - d\mathbf{J})\mathbf{y}$ distributed as χ^2 random variables?

Question # 5: **(8 points)**

Solve Exercise 6.14 from your textbook. You cannot use R functions or SAS PROCs like GLM or REG to directly obtain the answers. You may use PROC IML to show all intermediary steps.

☺ GOOD LUCK ☺