Question #1

Solution 1. *For part [A]: For part (i):*

when $\mathbf{A}$ is an orthogoal matrix, we have by definition:

$$\begin{aligned}
&\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = I \\
&\Longrightarrow |\mathbf{A}\mathbf{A}^T| = |\mathbf{A}| \cdot |\mathbf{A}^T| = |\mathbf{A}| \cdot |\mathbf{A}| = |\mathbf{A}|^2 = 1 \\
&\Longrightarrow |\mathbf{A}| = 1 \ or \ -1
\end{aligned}$$

For part (ii):

when $\mathbf{A}$ is idempotent, we have:

$$\begin{aligned}
&|\mathbf{A}^2| = |\mathbf{A}|^2 = |\mathbf{A}| \\
&\Longrightarrow |\mathbf{A}|(|\mathbf{A}| - 1) = 0 \\
&\Longrightarrow |\mathbf{A}| = 0 \ or \ |\mathbf{A}| = 1
\end{aligned}$$

For part [B]:

since $\mathbf{A}$ is symmetric, there is orthogonal matrix $\mathbf{C}$ and diagonal matrix $\mathbf{D}$ such that

$$\mathbf{A} = \mathbf{C}\mathbf{D}\mathbf{C}^T$$

Here

$$\mathbf{D} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

with $\mathbf{I}_r$ as the identity matrix of dimension r, and r is the rank of matrix $\mathbf{A}$.

Then we can rewrite $\mathbf{D}$ as

$$\mathbf{D} = \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \end{bmatrix}$$

So

$$\mathbf{A} = \mathbf{C} \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \end{bmatrix} \mathbf{C}^T$$

If we define

$$\mathbf{B} = \mathbf{C} \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0} \end{bmatrix}$$

then

$$\mathbf{A} = \mathbf{B}\mathbf{B}^T$$

and

$$\mathbf{B}^T\mathbf{B} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0} \end{bmatrix} = \mathbf{I}_r$$

Thus completed the proof.

Question #2:

Solution 2. *For part [A]:*

For textbook problem 2.31*:*

we want to show that given:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

The inverse of $\mathbf{A}$ *is:*

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}^{-1} \end{bmatrix}$$

with

$$\mathbf{B} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$$

Let's prove this by verifying that the mutiplication of two matrices above is identity matrix.

$$\begin{aligned} &\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_{11}(\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}) + \mathbf{A}_{12}(-\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}), & \mathbf{A}_{11}(-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}) + \mathbf{A}_{12}\mathbf{B}^{-1} \\ \mathbf{A}_{21}(\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}) + \mathbf{A}_{22}(-\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}), & \mathbf{A}_{21}(-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}) + \mathbf{A}_{22}\mathbf{B}^{-1} \end{bmatrix} \end{aligned}$$

we will call the result of the multiplication matrix above as:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

and we are going to check that $\mathbf{C} = \mathbf{I}$.

Denote $\mathbf{I}_{11}$ *as the identity matrix of the same dimension as* $\mathbf{A}_{11}$ *and* $\mathbf{I}_{22}$ *as the identity matrix of the same dimension as* $\mathbf{A}_{22}$.

We have:

$$\begin{aligned} \mathbf{C}_{11} &= \mathbf{A}_{11}(\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}) + \mathbf{A}_{12}(-\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}) \\ &= \mathbf{I}_{11} + \mathbf{I}_{11}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} - \mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} \\ &= \mathbf{I}_{11} \end{aligned}$$

$$\begin{aligned} \mathbf{C}_{12} &= \mathbf{A}_{11}(-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}) + \mathbf{A}_{12}\mathbf{B}^{-1} \\ &= -\mathbf{I}_{11}\mathbf{A}_{12}\mathbf{B}^{-1} + \mathbf{A}_{12}\mathbf{B}^{-1} \\ &= \mathbf{0} \end{aligned}$$

$$
\begin{aligned}
\mathbf{C}_{21} &= \mathbf{A}_{21}(\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}) + \mathbf{A}_{22}(-\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}) \\
&= \mathbf{A}_{21}\mathbf{A}_{11}^{-1} + \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} - \mathbf{A}_{22}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} \\
&= \mathbf{A}_{21}\mathbf{A}_{11}^{-1} + (\mathbf{A}_{22} - B)\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} - \mathbf{A}_{22}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} \\
&= \mathbf{A}_{21}\mathbf{A}_{11}^{-1} + \mathbf{A}_{22}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1} - \mathbf{A}_{22}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} \\
&= 0
\end{aligned}
$$

For the third $'='$ above, we used the equation that:

$$
\begin{aligned}
&\mathbf{B} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\
&\Longrightarrow \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} = \mathbf{A}_{22} - \mathbf{B}
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
\mathbf{C}_{22} &= \mathbf{A}_{21}(-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}) + \mathbf{A}_{22}\mathbf{B}^{-1} \\
&= -\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1} + \mathbf{A}_{22}\mathbf{B}^{-1} \\
&= -(\mathbf{A}_{22} - \mathbf{B})\mathbf{B}^{-1} + \mathbf{A}_{22}\mathbf{B}^{-1} \\
&= -\mathbf{A}_{22}\mathbf{B}^{-1} + \mathbf{I}_{22} + \mathbf{A}_{22}\mathbf{B}^{-1} \\
&= \mathbf{I}_{22}
\end{aligned}
$$

Hence we have showed that

$$
\begin{aligned}
\mathbf{C} &= \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{22} \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}
$$

So the given form is indeed $\mathbf{A}^{-1}$.

For problem 2.32:

We want to show that given the partition matrix:

$$
\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{12}' & a_{22} \end{bmatrix}
$$

the inverse is

$$
\mathbf{A}^{-1} = \frac{1}{b}\begin{bmatrix} b\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{a}_{12} \\ -\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} & 1 \end{bmatrix}
$$

where $b = a_{22} - \mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12}$.

This is indeed just a special case of problem 2.30, we could have said that the proof is exactly the same as 2.30 and skip it. but since it is required as a homework, we present it as following, while using the same notation $\mathbf{C}$, $\mathbf{I}_{11}$ and $\mathbf{I}_{22}$ as in the proof of 2.30.

We want to verify the multiplication of two given matrices is identity matrix.

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{12}' & a_{22} \end{bmatrix} \cdot \frac{1}{b}\begin{bmatrix} b\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{a}_{12} \\ -\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} & 1 \end{bmatrix} \\
= &\begin{bmatrix} \mathbf{A}_{11}(\mathbf{A}_{11}^{-1} + \frac{1}{b}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}) - \frac{1}{b}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}, & -\frac{1}{b}\mathbf{A}_{11}\mathbf{A}_{11}^{-1}\mathbf{a}_{12} + \frac{1}{b}\mathbf{a}_{12} \\ \mathbf{a}_{12}'(\mathbf{A}_{11}^{-1} + \frac{1}{b}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}) - \frac{1}{b}a_{22}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}, & -\frac{1}{b}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12} + \frac{1}{b}a_{22} \end{bmatrix}
\end{aligned}
$$

we call the resulting matrix as $\mathbf{C}$ *:*

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & c_{22} \end{bmatrix}$$

and we show that $\mathbf{C} = \mathbf{I}$.

We have:

$$\begin{aligned}
\mathbf{C}_{11} &= \mathbf{A}_{11}(\mathbf{A}_{11}^{-1} + \frac{1}{b}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}) - \frac{1}{b}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} \\
&= \mathbf{I}_{11} + \frac{1}{b}\mathbf{I}_{11}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} - \frac{1}{b}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} \\
&= \mathbf{I}_{11}
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_{12} &= -\frac{1}{b}\mathbf{A}_{11}\mathbf{A}_{11}^{-1}\mathbf{a}_{12} + \frac{1}{b}\mathbf{a}_{12} \\
&= -\frac{1}{b}\mathbf{I}_{11}\mathbf{a}_{12} + \frac{1}{b}\mathbf{a}_{12} \\
&= \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_{21} &= \mathbf{a}_{12}'(\mathbf{A}_{11}^{-1} + \frac{1}{b}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}) - \frac{1}{b}a_{22}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} \\
&= \mathbf{a}_{12}\mathbf{A}_{11}^{-1} + \frac{1}{b}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} - \frac{1}{b}\mathbf{a}_{22}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} \\
&= \mathbf{a}_{12}'\mathbf{A}_{11}^{-1} + \frac{1}{b}(a_{22} - b)\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} - \frac{1}{b}a_{22}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} \\
&= \mathbf{a}_{12}'\mathbf{A}_{11}^{-1} + \frac{1}{b}a_{22}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} - \mathbf{a}_{12}'\mathbf{A}_{11}^{-1} - \frac{1}{b}a_{22}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} \\
&= \mathbf{0}
\end{aligned}$$

and finally

$$\begin{aligned}
c_{22} &= -\frac{1}{b}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12} + \frac{1}{b}\mathbf{a}_{22} \\
&= -\frac{1}{b}(a_{22} - b) + \frac{1}{b}a_{22} \\
&= -\frac{1}{b}a_{22} + 1 + \frac{1}{b}a_{22} \\
&= 1
\end{aligned}$$

So putting thins together, we got :

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \mathbf{I}$$

which verifies the given form is indeed $\mathbf{A}^{-1}$.

For problem 2.33*:*

assuming all the non-singularity condition holds, we want to check

$$(\mathbf{B}+\mathbf{c}\mathbf{c}')^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}$$

We have:

$$\begin{aligned}
&(\mathbf{B}+\mathbf{c}\mathbf{c}')\cdot\left[\mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}\right]\\
&= \mathbf{B}\mathbf{B}^{-1} - \mathbf{B}\cdot\frac{\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}} + \mathbf{c}\mathbf{c}'\mathbf{B}^{-1} - \mathbf{c}\mathbf{c}'\cdot\frac{\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}\\
&= \mathbf{I} - \frac{\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}} + \mathbf{c}\mathbf{c}'\mathbf{B}^{-1} - \frac{\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}\\
&= \mathbf{I} + \frac{-\mathbf{c}\mathbf{c}'\mathbf{B}^{-1} + \mathbf{c}\mathbf{c}'\mathbf{B}^{-1} + \mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c} - \mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}
\end{aligned}$$

The first two terms on the numerator are cancelled obviously. For the 3rd and 4th terms, notice that $\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}$ *is a quadratic form, and hence a scalar, so we have:*

$$\begin{aligned}
\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c} &= (\mathbf{c}'\mathbf{B}^{-1}\mathbf{c})\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\\
\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1} &= (\mathbf{c}'\mathbf{B}^{-1}\mathbf{c})\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}
\end{aligned}$$

Hence the numerator becomes $\mathbf{0}$*, and thus we have:*

$$(\mathbf{B}+\mathbf{c}\mathbf{c}')\cdot\left[\mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1+\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}\right] = \mathbf{I}+\mathbf{0} = \mathbf{I}$$

this completes the proof.

For part [B]:

Given partition matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

We want to prove that:

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}^{-1} \end{bmatrix}$$

We derive this proof with Gauss Elimination, and we define $*$ *to be the left product between matrice, namely:*

$$\mathbf{A} * \mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

So we have:

$$\left[\begin{array}{cc|cc} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{I}_{11} & \\ \mathbf{A}_{21} & \mathbf{A}_{22} & & \mathbf{I}_{22} \end{array}\right] \overset{row\ 1*-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}+row\ 2}{\Longrightarrow} \left[\begin{array}{cc|cc} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{I}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} & -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}_{22} \end{array}\right]$$
$$= \left[\begin{array}{cc|cc} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{I}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}_{22} \end{array}\right]$$
$$\overset{row\ 2*-\mathbf{A}_{12}\mathbf{B}^{-1}+\ row\ 1}{\Longrightarrow} \left[\begin{array}{cc|cc} \mathbf{A}_{11} & \mathbf{0} & \mathbf{I}_{11}+\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{12}\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B} & -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}_{22} \end{array}\right]$$
$$\overset{row\ 1\times\mathbf{A}_{11}^{-1},row\ 2\times\mathbf{B}^{-1}}{\Longrightarrow} \left[\begin{array}{cc|cc} \mathbf{I}_{11} & & \mathbf{A}_{11}+\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{B}^{-1} \\ & \mathbf{I}_{22} & -\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}^{-1} \end{array}\right]$$

Thus completed the proof.

Now for the case when:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}'_{12} & a_{22} \end{bmatrix}$$

the proof is the same, just need to replace $\mathbf{A}_{12}$ *by* $\mathbf{a}_{12}$ *and* $\mathbf{A}_{21}$ *by* $\mathbf{a}'_{12}$ *and also we could use the fact that* a_{22} *is a scalar so the inverse is just the reciprocal.*

Finally, for the case when

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}$$

we just need to plug in $\mathbf{A}_{12} = \mathbf{0}$ *and* $\mathbf{A}_{21} = \mathbf{0}$ *into the general result, and we get our conclusion that*

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1}+\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{bmatrix}$$

Question #3.

Solution 3. *For part [A]:*

For part (i):

Notice that $\mathbf{x}^T\mathbf{x}$ *is a scalar so it can switch order with others when doing matrix multiplication. So we have:*

$$\begin{aligned} \mathbf{Hx} &= (\mathbf{I}-\mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T)\cdot\mathbf{x} = \left[I-(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}\mathbf{x}^T\right]\cdot\mathbf{x} \\ &= \mathbf{x}-(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}(\mathbf{x}^T\mathbf{x}) \\ &= x-(\mathbf{x}^T\mathbf{x})^{-1}(\mathbf{x}^T\mathbf{x})\cdot\mathbf{x} \\ &= \mathbf{x}-\mathbf{x} = \mathbf{0} \end{aligned}$$

Hence $\mathbf{x}$ *is an eigen vector of* $\mathbf{H}$ *under the eigenvalue* $\lambda = 0$ *For part (ii):*

Suppose that $\mathbf{v}$ *is orthogonal to* $\mathbf{x}$*, this means* $\mathbf{x}^T\mathbf{v} = 0$*, so we have:*

$$\begin{aligned}
\mathbf{H}\mathbf{v} &= [\mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T]\mathbf{v} \\
&= \mathbf{v} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\underbrace{(\mathbf{x}^T\mathbf{v})}_{=0} \\
&= \mathbf{v} - \mathbf{0} = \mathbf{v}
\end{aligned}$$

So $\mathbf{v}$ *is an eigenvector of* $\mathbf{H}$ *under the eigenvalue* $\lambda = 1$*.*

For part (iii):

To show that $\mathbf{H}$ *is idempotent, we compute as follows:*

$$\begin{aligned}
\mathbf{H}^2 &= \left[\mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\right] \cdot \left[\mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\right] \\
&= \mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T + \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T \\
&= \mathbf{I} - 2\mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T + \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T \\
&= \mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T \\
&= \mathbf{H}
\end{aligned}$$

so $\mathbf{H}$ *is verified to be idempotent.*

For part (B):

We check that $\mathbf{A}$ *is idempotent:*

$$\mathbf{A}^2 = \begin{bmatrix} \frac{1-\cos(\theta)}{2} & \frac{\sin(\theta)}{2} \\ \frac{\sin(\theta)}{2} & \frac{1+\cos(\theta)}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1-\cos(\theta)}{2} & \frac{\sin(\theta)}{2} \\ \frac{\sin(\theta)}{2} & \frac{1+\cos(\theta)}{2} \end{bmatrix}$$

we have:

$$\begin{aligned}
\mathbf{A}^2_{11} &= \frac{1-\cos(\theta)}{2} \cdot \frac{1-\cos(\theta)}{2} + \frac{\sin(\theta)}{2} \cdot \frac{\sin(\theta)}{2} \\
&= \frac{1 - 2\cos\theta + \cos^2(\theta)}{4} + \frac{\sin^2(\theta)}{4} \\
&= \frac{1 - 2\cos\theta + 1}{4} \\
&= \frac{2(1-\cos(\theta))}{4} \\
&= \frac{1-\cos(\theta)}{2} \\
&= \mathbf{A}_{11}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^2_{12} &= \frac{1-\cos(\theta)}{2} \cdot \frac{\sin(\theta)}{2} + \frac{\sin(\theta)}{2} \cdot \frac{1+\cos(\theta)}{2} \\
&= \frac{\sin(\theta) - \cos(\theta)\sin(\theta)}{4} + \frac{\sin(\theta) + \sin(\theta)\cos(\theta)}{4} \\
&= \frac{2\sin(\theta)}{4} = \frac{\sin(\theta)}{2} = \mathbf{A}_{12}
\end{aligned}$$

Also, since $\mathbf{A}$ *is symmetric, so* $\mathbf{A}^2$ *is also symmetric, thus we have:*

$$\mathbf{A}^2_{21} = \mathbf{A}^2_{12} = \mathbf{A}_{12} = \mathbf{A}_{21}$$

Finally,

$$\begin{aligned}
\mathbf{A}^2_{22} &= \frac{\sin(\theta)}{2} \cdot \frac{\sin(\theta)}{2} + \frac{1+\cos(\theta)}{2} \cdot \frac{1+\cos(\theta)}{2} \\
&= \frac{\sin^2(\theta)}{4} - \frac{1+2\cos(\theta)+\cos^2(\theta)}{4} \\
&= \frac{2+2\cos(\theta)}{4} \\
&= \frac{1+\cos(\theta)}{2} = \mathbf{A}_{22}
\end{aligned}$$

So together we have verified that $\mathbf{A}^2 = \mathbf{A}$ *and is idempotent.*

Question #4.

Solution 4. *For part [A]:*

We run the following SAS code and observe information from log:

```
data hw1q4;
input x1 x2 x3;
datalines;
3 10 4
2 11 5
6 15 8
6 9 9
8 5 10
;
run;
```

```
1      dm 'log; clear; output; clear;';
2
3      data hw1q4;
4      input x1 x2 x3;
5      datalines;

NOTE: The data set WORK.HW1Q4 has 5 observations and 3 variables.
NOTE: DATA statement used (Total process time):
      real time           0.18 seconds
      cpu time            0.07 seconds
```

Since there is 5 *observation with* 3 *variables, the size of the matrix is* 5 *by* 3*, and the size of the transpose is then* 3 *by* 5*.*

For part [B]:

We run the following code and output from SAS:

```
proc means data = hw1q4;
run;
```

The MEANS Procedure

Variable	N	Mean	Std Dev	Minimum	Maximum
x1	5	5.0000000	2.4494897	2.0000000	8.0000000
x2	5	10.0000000	3.6055513	5.0000000	15.0000000
x3	5	7.2000000	2.5884358	4.0000000	10.0000000

So we have:

$$\bar{X} = (5, 10, 7.2)^T$$

For part [C]:

We first use proc corr to find the covariance matrix S and correlation matrix R:

```
proc corr data = hw1q4 cov;
run;
```

Covariance Matrix, DF = 4

	x1	x2	x3
x1	6.00000000	-3.50000000	6.00000000
x2	-3.50000000	13.00000000	-3.50000000
x3	6.00000000	-3.50000000	6.70000000

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	x1	x2	x3
x1	1.00000	-0.39630 0.5090	0.94632 0.0148
x2	-0.39630 0.5090	1.00000	-0.37502 0.5339
x3	0.94632 0.0148	-0.37502 0.5339	1.00000

We then use proc iml to check if we have the same answer. So under proc iml, we use the following commands:

First read in the data we created:

```
use work.hw1q4;
read all var _NUM_ into x;
nm = {x1 x2 x3};
```

then we run the cov and corr function and print it. we also pasted the code togethere here that is need to find the inverse of S for part [D], so in part [D] we will ony past the output.

```
/*using SAS default functions for cov, corr*/
cov=cov(x);
corr = corr(x);
covinv = inv(cov(x));
id = cov*covinv;
print x;
print cov[rowname=nm colname=nm label= "Covariance Matrix"];
print corr[rowname=nm colname=nm label="Correlation Matrix"];
print covinv[rowname=nm colname=nm label="Covariance Inverse Matrix"];
print id[rowname=nm colname=nm label="Cov multiply invers cov"];
```

we got the following out put for S and R:

Covariance Matrix

	X1	X2	X3
X1	6	-3.5	6
X2	-3.5	13	-3.5
X3	6	-3.5	6.7

Correlation Matrix

	X1	X2	X3
X1	1	-0.396297	0.9463204
X2	-0.396297	1	-0.375024
X3	0.9463204	-0.375024	1

As expected both proc corr and proc iml gave the same result. for part [D]:

we use the inv() function under proc iml and print out both the inverse function of covariance matrix and the multiplication between covariance matrix and its inverse:

```
Covariance Inverse Matrix
           X1        X2        X3

X1  1.6262901 0.0532319 -1.428571
X2  0.0532319 0.0912548         0
X3  -1.428571         0 1.4285714

   Cov multiply invers cov
           X1        X2        X3

X1          1         0         0
X2  -8.88E-16         1         0
X3          0         0         1
```

As expected that the multiplication between covariance and its inverse is identity matrix, although there is apprently some rounding errors on some of the entries.

For part [E]:

So we have:

$$\bar{\mathbf{X}}'\mathbf{S}^{-1} = (5, 10, 7.2) \cdot \begin{bmatrix} 1.6262901 & 0.0532319 & -1.428571 \\ 0.0532319 & 0.0912548 & 0 \\ -1.428571 & 0 & 1.4285714 \end{bmatrix}$$

to compute by hand, we have:

$$\bar{\mathbf{X}}'\mathbf{S}_1^{-1} = 5 \times 1.6262901 + 10 \times 0.0532319 + 7.2 \times -1.428571 = -1.621942$$
$$\bar{\mathbf{X}}'\mathbf{S}_2^{-1} = 5 \times 0.0532319 + 10 \times 0.0912548 + 7.2 \times 0 = 1.178707$$
$$\bar{\mathbf{X}}'\mathbf{S}_3^{-1} = 5 \times -1.428571 + 10 \times 0 + 7.2 \times 1.4285714 = 3.142859$$

So we have:

$$\bar{\mathbf{X}}'\mathbf{S}^{-1} = (-1.621942, 1.178707, 3.142859)$$

Now to verify with SAS, we compute with the following code and print it:

```
stdx = mean(x)*inv(cov(x));
```

and our output is:

```
              stdx

-1.621945 1.1787072 3.1428571
```

and it matches with our computing results by hand.

For part [F]:

To compute by hand, we have:

$$\bar{\mathbf{X}}'\mathbf{S}^{-1}\bar{\mathbf{X}} = (-1.621942, 1.178707, 3.142859) \cdot (5, 10, 7.2)^T = 26.30594$$

To verify with SAS, under proc iml, we use the command

```
quadratic = mean(x)*inv(cov(x))*mean(x)`;
```

we print the result and it gave us:

```
quadratic

26.305921
```

there is some rouding error but we almost got the same results as the one computed by hand.

For part [G]:

The determinant of **S** *is hand computed as following:*

$$\begin{aligned}
|S| &= \begin{vmatrix} 6 & -3.5 & 6 \\ -3.5 & 13 & -3.5 \\ 6 & -3.5 & 6.7 \end{vmatrix} \\
&= 6\begin{vmatrix} 13 & -3.5 \\ -3.5 & 6.7 \end{vmatrix} + 3.5\begin{vmatrix} -3.5 & -3.5 \\ 6 & 6.7 \end{vmatrix} + 6\begin{vmatrix} -3.5 & 13 \\ 6 & -3.5 \end{vmatrix} \\
&= 6 \times (13 \times 6.7 - 3.5 \times 3.5) + 3.5 \times (-3.5 \times 6.7 + 3.5 \times 6) + 6 \times (3.5 \times 3.5 - 6 \times 13) \quad = 46.025
\end{aligned}$$

To verify with SAS, we use the follwoing code:

```
detcov = det(cov(x));
```

the output is following:

```
detcov

46.025
```

and the two answers do match. For part [H]:

The trace of S is the sum of its diagonal entries, so:

$$tr(S) = 6 + 13 + 6.7 = 25.7$$

Verify with following SAS code and output:

```
trace = trace(cov(x));
```

```
trace

25.7
```

the answers match.

For part [I]:

To find the eigenvalues and eigenvectors of $\mathbf{S}$*, under proc iml, we run the following code:*

```
call eigen(eigenval, eigenvec, cov(x));
```

print the output, we have:

```
Eigenvalues and eigenvectors
         eigenval

         17.638718
         7.7234384
         0.3378439
          eigenvec

0.4736926 0.4909418 0.7311576
 -0.72959   0.68375 0.0135675
0.4932682 0.5398722 -0.682074
```

The eigenvalues of $\mathbf{S}^2$ *are just square of the eigenvalues for* $\mathbf{S}$*. We got:*

```
eigen values of S^2
  eigenval_square

       311.12436
       59.651501
       0.1141385
```

(we checked with both directly looking for eigenvalues of $\mathbf{S}^2$ *and also by squaring the eigenvalues of* $\mathbf{S}$*, we got the same results.). For part [J]:*

we find the sum of eigenvalues under proc iml:

```
sum_eig = sum(eigval(cov(x)));
```

The output is as expected that it is the same as the trace we found before:

```
sum of eigenvalues of S
        sum_eig

          25.7
```

For part [K], we manually multiply all the eigenvalues. (I do not knwo how to do multiplication of all the elements inside one vector with SAS) We got:

$$17.638718 \times 7.7234384 \times 0.3378439 = 46.025$$

which is the same as the determinat of $\mathbf{S}$ *that we found before.*

For part [L]:

we do the pairwise dot product between eigen vectors under proc iml:

```
/*compute the dot product of eigen vectors*/
eigvec_12 = sum(eigenvec[, 1]#eigenvec[, 2]);
eigvec_13 = sum(eigenvec[, 1]#eigenvec[, 3]);
eigvec_23 = sum(eigenvec[, 2]#eigenvec[, 3]);
```

The output is:

```
pairwise dot product of eigenvectors:
                eigvec_12

                -3.33E-16
                eigvec_13

                5.551E-17
                eigvec_23

                        0
```

Not worry too much about rounding error, these eigen vectors are orthogonal pairwise.

Question #5:

Solution 5. *For part [A]:*

define $\mathbf{z} = \mathbf{y} - \mathbf{x}$*, so* $\mathbf{z}$ *would be the vector connecting from the end point of* $\mathbf{x}$ *to the end point of* $\mathbf{y}$ *and these three vectors form a triangle.*

In basic geometry we have this rule:

$$L_{\mathbf{z}}^2 = L_{\mathbf{x}}^2 + L_{\mathbf{y}}^2 - 2L_{\mathbf{x}}L_{\mathbf{y}}\cos\theta$$

on the other hand, we have:

$$\begin{aligned} L_{\mathbf{z}}^2 &= \mathbf{z}^T \cdot \mathbf{z} = (\mathbf{y} - \mathbf{x})^T \cdot (\mathbf{y} - \mathbf{x}) \\ &= L_{\mathbf{y}}^2 + L_{\mathbf{x}}^2 - 2\mathbf{x}^T\mathbf{y} \end{aligned}$$

put the two above together we can build the following equation:

$$\begin{aligned} & L_{\mathbf{x}}^2 + L_{\mathbf{y}}^2 - 2L_{\mathbf{x}}L_{\mathbf{y}}\cos\theta = L_{\mathbf{y}}^2 + L_{\mathbf{x}}^2 - 2\mathbf{x}^T\mathbf{y} \\ & \Longrightarrow -2L_{\mathbf{x}}L_{\mathbf{y}}\cos\theta = -2\mathbf{x}^T\mathbf{y} \\ & \Longrightarrow \cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{L_{\mathbf{x}}L_{\mathbf{y}}} \end{aligned}$$

For part [B]:

We use the equation from part [A]:

$$\begin{aligned} \cos(\theta) &= \frac{\mathbf{x}^T\mathbf{y}}{L_{\mathbf{x}}L_{\mathbf{y}}} = \frac{(1,3,2)\cdot(-2,1,-1)^T}{\sqrt{1+9+4}\cdot\sqrt{4+1+1}} = \frac{-2+3-2}{\sqrt{84}} \\ &= -\frac{1}{2\sqrt{21}} \end{aligned}$$

By using the acos function in R we got the angle as 1.680123 *radian, or we can transfer it to degree unit as* 96.26395 *degree.*

These two vectors are **NOT** *linearly independent, since they are not multiple of each other, or in other words, they do not point in either the same or opposite direction, or in other words, they do not expand into the same linear space, or in other words, their linear combination can not be* 0 *without having all the coeffecients being* 0*.*

Question #6.

Solution 6. *For part [A]:*

We want to prove theorem 2.12 (e) that if $\mathbf{A}$ *is any* $n \times n$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$*, then*

$$(i) \qquad |\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$$

$$(ii) \qquad tr(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$$

Theorem 2.12(*d*) *assumed the matrix is symmetric but theorem* 2.12(*e*) *assumes for general matrix. The spectrum decomposition result actually holds for general matrix than just symmetric, so let's feel free to use the result of* 2.12(*d*) *without assuming* $\mathbf{A}$ *being symmetric.*

Then we know that there is orthogonal matrix $\mathbf{C} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$*, each* $\mathbf{x}_i$ *being the unit eigen vector corresponding to the eigen value* λ_i*, such that*

$$\mathbf{A} = \mathbf{CDC}'$$

where $\mathbf{D} = diag(\lambda_1, \ldots, \lambda_n)$ *is the diagonal matrix whose diagnoal elements are the eigen values of* $\mathbf{A}$ *from* λ_1 *to* λ_n*.*

So when we take determinant on $\mathbf{A}$*, we have:*

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{CDC}'| = |\mathbf{C}| \cdot |\mathbf{D}| \cdot |\mathbf{C}'| \\ &= |\mathbf{C}|^2 \cdot \prod_{i=1}^{n} \lambda_i \\ &= \prod_{i=1}^{n} \lambda_i \end{aligned}$$

The second to the last $' =' $ *is due to the fact that for any orthogonal matrix, its determinant is either* 1 *or* -1 *and any transpose of a matrix has the same determinant as the original matrix.*

Now for the trace, we know that since $\mathbf{A} = \mathbf{CDC}'$*, we can denote the columns of orthogonal matrix* $\mathbf{C}$ *as* $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$*, and the entry at* (i, j) *as* c_{ij}*. Observe that for* a_{ii} *(the digonal entry* *of* $\mathbf{A}$ *at* (i,i)*), we have:*

$$\begin{aligned} a_{ii} &= \sum_{k,\ell} c_{ik} d_{k\ell} c'_{\ell i} \\ &= \sum_{k,\ell} c_{ik} c_{i\ell} d_{k\ell} \qquad (c'_{\ell i} = c_{i\ell}) \\ &= \sum_{k=\ell=1}^{n} c_{ik}^2 \lambda_k \qquad (\mathbf{D} \textit{ is diagonal}) \end{aligned}$$

So the trace of $\mathbf{A}$ *will be:*

$$\begin{aligned} tr(\mathbf{A}) = \sum a_{ii} &= \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_k c_{ik}^2 \\ &= \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{n} c_{ik}^2 \\ &= \sum_{k=1}^{n} \lambda_k \qquad (\sum_{i=1}^{n} c_{ik}^2 = 1 \textit{ since } \mathbf{C} \textit{ is orthogonal}) \end{aligned}$$

Thus finished the proof of 2.12(e).

For part $[B]$*:*

To computer the eigen values and eigen vectors of $\mathbf{A}$*, we do the following hand calculation:*

$$\begin{aligned} &|\lambda\mathbf{I} - \mathbf{A}| = 0 \Longrightarrow \begin{vmatrix} \lambda-13 & 4 & -2 \\ 4 & \lambda-13 & 2 \\ -2 & 2 & \lambda-10 \end{vmatrix} = 0 \\ &\Longrightarrow (\lambda-13)\begin{vmatrix} \lambda-13 & 2 \\ 2 & \lambda-10 \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ -2 & \lambda-10 \end{vmatrix} - 2\begin{vmatrix} 4 & \lambda-13 \\ -2 & 2 \end{vmatrix} = 0 \\ &\Longrightarrow (\lambda-13)\Big[(\lambda-13)(\lambda-10)-4\Big] - 4\Big[4(\lambda-10)+4\Big] - 2\Big[8+2(\lambda-13)\Big] = 0 \\ &\Longrightarrow (\lambda-13)(\lambda^2-23\lambda+126) - 4(4\lambda-36) - 2(2\lambda-18) = 0 \\ &\Longrightarrow \lambda^3 - 36\lambda^2 + 405\lambda - 1458 = 0 \\ &\Longrightarrow (\lambda-18)(\lambda-9)^2 = 0 \end{aligned}$$

So we have eigenvalues $\lambda_1 = \lambda_2 = 9$ *and* $\lambda_3 = 18$.

To find the unit eigen vectors, we solve the homogeneous linear equation, by performing Gauss elimination

When $\lambda_1 = \lambda_2 = 9$:

$$\lambda\mathbf{I} - \mathbf{A} = 9\mathbf{I} - A = \begin{bmatrix} -4 & 4 & -2 \\ 4 & -4 & 2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{Gauss elimination}} \begin{bmatrix} -2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so we have two free varialbe x_2 and x_3 and from the first row we get

$$\begin{aligned} & -2x_1 + 2x_2 - x_3 = 0 \\ & \Longrightarrow 2x_1 = 2x_2 - x_3 \\ & \Longrightarrow x_1 = x_2 - \frac{1}{2}x_3 \end{aligned}$$

So our solution vector is:

$$\begin{bmatrix} x_2 - \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So two of the eigen vectors for $\lambda = 9$ will be:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \textit{and} \qquad \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Since we aim to find the orthogonal matrix that give **A** *decomposition, we want to make the eigenvectors be a unitvector, so the two* **unit** *eigenvectors for $\lambda = 9$ would be:*

$$\mathbf{C}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{C}_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Similarly, when $\lambda = 18$, we have:

$$\lambda\mathbf{I} - \mathbf{A} = 18\mathbf{I} - \mathbf{A} = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$$

Use the same Gauss elimination pocess as above, we got the unit eigen vector as:

$$\mathbf{C}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}$$

It is easy to check that $\mathbf{C}_1, \mathbf{C}_2$ and $\mathbf{C}_3$ are pairwisely orthogonal to each other by taking dot product, and since these are unit eigen vectors, we can define:

$$\mathbf{C} = [\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3]$$

and hence $\mathbf{C}$ *is a* 3 *by* 3 *orthogonal matrix that gives* $\mathbf{A}$ *spectral decomposition. Actually we can easily compute:*

$$\mathbf{C}'\mathbf{A}\mathbf{C} = \begin{bmatrix} 9 & & \\ & 9 & \\ & & 18 \end{bmatrix}$$

Thus 2.12(*d*) *holds true.*

Question #7.

Solution 7. *For part* [*A*]*:*

Before I get to my solution, I would like to comment that, there might be a possible typo for the last element of row 2*. It would make much more sense to input* 5 *instead of* −5*, that way when we do Gauss elimiation, there will be 2 rows cancelled out and the rank is 3, and also the first 3 by 3 block of* $\mathbf{A}$ *would also have rank* 3 *and it is easy to apply* 2.8*b.*

Now if we do not change anything, the rank of $\mathbf{A}$ *would be* 4 *as we will show below, and also the first* 4 *by* 4 *submatrix of* $\mathbf{A}$ *will not be rank* 4*, then we will need to use the* 5 *step process as is suggested on the book.*

Now let's get to our solution:

First find the rank of $\mathbf{A}$ *by performing Gauss elimination:*

$$\begin{bmatrix} 3 & 7 & -3 & 6 & 4 \\ 1 & 4 & 0 & 2 & -5 \\ 2 & -1 & 1 & 4 & -9 \\ 0 & -5 & -3 & 0 & -11 \\ 0 & -9 & 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 4 & 0 & 2 & -5 \\ 3 & 7 & -3 & 6 & 4 \\ 2 & -1 & 1 & 4 & -9 \\ 0 & -5 & -3 & 0 & -11 \\ 0 & -9 & 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 4 & 0 & 2 & -5 \\ 0 & -5 & -3 & 0 & 19 \\ 0 & -9 & 1 & 0 & 1 \\ 0 & -5 & -3 & 0 & -11 \\ 0 & -9 & 1 & 0 & 1 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} 1 & 4 & 0 & 2 & -5 \\ 0 & -5 & -3 & 0 & 19 \\ 0 & 0 & -\frac{22}{5} & 0 & -\frac{166}{5} \\ 0 & 0 & 0 & 0 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix shows that the rank of $\mathbf{A}$ *is* 4*.(I checked with SAS and it is correct).*

Now let's implement the 5 *step process on page* 35 *of the textbook.*

Step 1: Find any non-singular 4×4 *sub matrix* $\mathbf{C}$*.*

We found that $\mathbf{C}$ *happees to be the* 4 *by* 4 *upper right sub matrix of* $\mathbf{A}$*:*

$$\mathbf{C} = \begin{bmatrix} 7 & -3 & 6 & 4 \\ 4 & 0 & 2 & -5 \\ -1 & 1 & 4 & -9 \\ -5 & -3 & 0 & -11 \end{bmatrix}$$

Step 2: Find $\mathbf{C}^{-1}$ *and* $(\mathbf{C}^{-1})'$:

$$\mathbf{C}^{-1} = \begin{bmatrix} -0.008333 & 0.2125 & -0.09375 & -0.022917 \\ -0.108333 & 0.0125 & 0.15625 & -0.172917 \\ 0.1 & -0.175 & 0.1875 & -0.0375 \\ 0.0333333 & -0.1 & 0 & -0.033333 \end{bmatrix}$$

$$(\mathbf{C}^{-1})' = \begin{bmatrix} -0.008333 & -0.108333 & 0.1 & 0.0333333 \\ 0.2125 & 0.0125 & -0.175 & -0.1 \\ -0.09375 & 0.15625 & 0.1875 & 0 \\ -0.022917 & -0.172917 & -0.0375 & -0.033333 \end{bmatrix}$$

Step 3. *Replace the elements of* $\mathbf{C}$ *by the element of* $(\mathbf{C}^{-1})'$. *So we get:*

$$\begin{bmatrix} 3 & -0.008333 & -0.108333 & 0.1 & 0.0333333 \\ 1 & 0.2125 & 0.0125 & -0.175 & -0.1 \\ 2 & -0.09375 & 0.15625 & 0.1875 & 0 \\ 0 & -0.022917 & -0.172917 & -0.0375 & -0.033333 \\ 0 & -9 & 1 & 0 & 1 \end{bmatrix}$$

Step 4, *replace all other elements in* $\mathbf{A}$ *by zeros, so we get:*

$$\begin{bmatrix} 0 & -0.008333 & -0.108333 & 0.1 & 0.0333333 \\ 0 & 0.2125 & 0.0125 & -0.175 & -0.1 \\ 0 & -0.09375 & 0.15625 & 0.1875 & 0 \\ 0 & -0.022917 & -0.172917 & -0.0375 & -0.033333 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5, *transpose the resulting matrix, so we get:*

$$\mathbf{A}^{-} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.008333 & 0.2125 & -0.09375 & -0.022917 & 0 \\ -0.108333 & 0.0125 & 0.15625 & -0.172917 & 0 \\ 0.1 & -0.175 & 0.1875 & -0.0375 & 0 \\ 0.0333333 & -0.1 & 0 & -0.033333 & 0 \end{bmatrix}$$

Now we want to use SAS to verify equation 2.58, *which is:*

$$\mathbf{A}\mathbf{A}^{-}\mathbf{A} = \mathbf{A}$$

The following code input manually $\mathbf{A}$ *and* $\mathbf{A}^{-}$ *that we have found above, then we printout the result of* $\mathbf{A}\mathbf{A}^{-}\mathbf{A}$*(called checka in the code) to see if it is the same as* $\mathbf{A}$.

```
/*verify equation 2.58*/
a = {3 7 -3 6 4, 1 4 0 2 -5, 2 -1 1 4 -9, 0 -5 -3 0 -11, 0 -9 1 0 1};
aginv = {0 0 0 0 0,
         -0.008333 0.2125 -0.09375 -0.022917 0,
         -0.108333 0.0125 0.15625 -0.172917 0,
         0.1 -0.175 0.1875 -0.0375 0,
         0.0333333 -0.1 0 -0.033333 0};
  checka = a*aginv*a;
  print checka;
```

Output is:

checka

3.0000036	7.0000084	-3.000004	6.0000072	4.0000048
1.0000045	4.0000255	4.5E-6	2.000009	-4.999961
2.0000009	-0.999983	1.0000081	4.0000018	-8.999966
-6.9E-6	-5.000011	-2.99999	-0.000014	-11
-8.1E-6	-9.000034	0.9999991	-0.000016	0.9999562

It is clear to see that ignore the rounding error, the resulting matrix is the same as $\mathbf{A}$*, so the stpes worked.*

For part [B]:

Continue with the input we had from [A], we check in SAS that:

(i) $rank(\mathbf{A}^-\mathbf{A}) = rank(\mathbf{A}\mathbf{A}^-) = rank(\mathbf{A}) = r$ *(here* $r = 4$*).*

(ii) $(\mathbf{A}^-)'$ *is a generalized inverse of* $\mathbf{A}'$*; that is,* $(\mathbf{A}')^- = (\mathbf{A}^-)'$

(iii) $\mathbf{A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'\mathbf{A}$ *and* $\mathbf{A}' = \mathbf{A}'\mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$

(iv) $(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ *is a generalized inverse of* $\mathbf{A}$*, that is,* $\mathbf{A}^- = (\mathbf{A}'\mathbf{A})^-\mathbf{A}'$

(v) $\mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ *is symmetric, has rank* $= rm$ *and is invariant to the choice of* $(\mathbf{A}'\mathbf{A})^-$*; that is,* $\mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ *remains the same, no matter what value of* $(\mathbf{A}'\mathbf{A})^-$ *is used.*

We check one by one:

For (i):

Since proc iml does not have a default function to find the rank of matrices, we do it by finding the trace of multiplication between a matrix and its generalized inverse. This will adds up all the 1's in the diagonal which gives us the rank of the matrix.

Code:

```
/*verify 2.8c*/

/*part (i)*/

/*input A*/
a = {3 7 -3 6 4, 1 4 0 2 -5, 2 -1 1 4 -9, 0 -5 -3 0 -11, 0 -9 1 0 1};
/*getting generalized inverse*/
aginv = ginv(a);
/*getting rank of A*/
rankA = round(trace(ginv(a)*a));
/*getting rank of A inverse times A*/:
rankAinv_A = round(trace(ginv(ginv(a)*a)*(ginv(a)*a)));
/*getting rank of A times A inverse*/
rankA_Ainv = round(trace(ginv(a*ginv(a))*(a*ginv(a))));
/*print all three ranks to see if they are the same*/
print rankA rankAinv_A rankA_Ainv;
```

Output: (print all 3 ranks)

ranka	rankAinv_A	rankA_Ainv
4	4	4

we see that as expected all three ranks are same and equal to 4*.*

For (*ii*)*:*

Code:

```
/*part (ii)*/
A_trans = a`;
/*computing transpose of inverse*/
invA_trans = ginv(a)`;
/*computing inverse of transpose*/
tranA_inv = ginv(a`);
/*print both to see if they are the same*/
print invA_trans;
print tranA_inv;
```

Output: (print both matrices to see if they are the same)

invA_trans

0.04	-0.008333	-0.108333	0.08	0.0333333
0.0016667	0.0395833	0.05625	0.0033333	-0.033333
0.0391667	-0.007292	0.134375	0.0783333	-0.033333
-0.015	-0.022917	-0.172917	-0.03	-0.033333
0.0358333	-0.086458	0.021875	0.0716667	0.0333333

tranA_inv

0.04	-0.008333	-0.108333	0.08	0.0333333
0.0016667	0.0395833	0.05625	0.0033333	-0.033333
0.0391667	-0.007292	0.134375	0.0783333	-0.033333
-0.015	-0.022917	-0.172917	-0.03	-0.033333
0.0358333	-0.086458	0.021875	0.0716667	0.0333333

So we have checked that the two are indeed the same.

For part (*iii*)*:*

A *was already entered in the preivous question. We compute the other 3 matrices in the code here and print them.*

Code:

```
/*part (iii)*/

/*computing a transpose*/
Atran = a`;

/*computing A(A'A)^-A'A*/
a1 = a*ginv((a`*a))*a`*a;

/*computing (A'A)(A'A)^-A'*/
a2 = (a`*a)*ginv(a`*a)*a`;

/*print to see if they match*/
print a;
print a1;
print Atran;
print a2;
```

Output:

a

3	7	-3	6	4
1	4	0	2	-5
2	-1	1	4	-9
0	-5	-3	0	-11
0	-9	1	0	1

a1

3	7	-3	6	4
1	4	-6.11E-16	2	-5
2	-1	1	4	-9
-1.58E-15	-5	-3	-3.16E-15	-11
-2.33E-15	-9	1	-4.66E-15	1

Atran

3	1	2	0	0
7	4	-1	-5	-9
-3	0	1	-3	1
6	2	4	0	0
4	-5	-9	-11	1

a2

3	1	2	1.242E-15	-1.56E-15
7	4	-1	-5	-9
-3	3.886E-16	1	-3	1
6	2	4	2.484E-15	-3.12E-15
4	-5	-9	-11	1

they match as expected if we disregard the rounding error.

For part (iv):

we compute each side separately and print them to see if thy are the same:

Code:

```
/*part (iv)*/

/*compute generalized inverse of A*/
Ainv = ginv(a);

/*compute (A'A)^-A'*/
Ainv_2 = ginv(Atran*a)*Atran;

print Ainv;
print Ainv_2;
```

Output:

```
                          Ainv

     0.04 0.0016667 0.0391667    -0.015 0.0358333
-0.008333 0.0395833 -0.007292 -0.022917 -0.086458
-0.108333   0.05625  0.134375 -0.172917  0.021875
     0.08 0.0033333 0.0783333     -0.03 0.0716667
0.0333333 -0.033333 -0.033333 -0.033333 0.0333333

                         Ainv_2

     0.04 0.0016667 0.0391667    -0.015 0.0358333
-0.008333 0.0395833 -0.007292 -0.022917 -0.086458
-0.108333   0.05625  0.134375 -0.172917  0.021875
     0.08 0.0033333 0.0783333     -0.03 0.0716667
0.0333333 -0.033333 -0.033333 -0.033333 0.0333333
```

and as expected they match.

For part (v):

It is hard to verify the invariancy of the matrix. However we check $\mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$ *that is symmetric by printing out both itself and its transpose. We also compute its rank to see it is* r *(in this case it is* 4*).*

Code:

```
/*part (v)*/

/*compute A(A'A)^-A' and its transpose*/
a3 = a*ginv(Atran*a)*Atran;
a4 = (a*ginv(Atran*a)*Atran)`;

/*compute the ran of A(A'A)^-A'*/
ranka3 = round(trace(ginv(a*ginv(Atran*a)*Atran)*a*ginv(Atran*a)*Atran));

print a3;
print a4;
print ranka3;
```

Output:

a3

1	2.22E-16	0	-3.89E-16	-4.16E-16
1.943E-16	0.3333333	0.3333333	0	-0.333333
-1.94E-16	0.3333333	0.8333333	5.551E-16	0.1666667
-1.94E-16	-2.22E-16	-3.89E-16	1	-6.25E-17
-5.55E-16	-0.333333	0.1666667	4.441E-16	0.8333333

a4

1	1.11E-16	-1.11E-16	2.776E-17	-4.44E-16
1.665E-16	0.3333333	0.3333333	-2.5E-16	-0.333333
5.551E-17	0.3333333	0.8333333	-6.11E-16	0.1666667
5.551E-17	5.551E-17	4.441E-16	1	3.331E-16
-2.84E-16	-0.333333	0.1666667	-1.32E-16	0.8333333

ranka3

4

There is a lot of rounding error in the output, most of the high negative power term should be treated as entry 0, thus the two matrices is indeed the same, which shows symmetry. and the rank is 4 as expected.

For part [C]:

part (i):

Verify equation 2.90, which is:

$$tr(\mathbf{A}'\mathbf{A}) = tr(\mathbf{A}\mathbf{A}')$$

Code:

```
/*question 7 part C*/

/*part (i)*/

/*compute the trace of A'A and AA'*/
trace1 = trace(Atran*A);
trace2 = trace(A*Atran);

print trace1;
print trace2;
```

Output:

trace1

506

trace2

506

part (ii):

Verify equation 2.92*, which is:*

$$tr(\mathbf{C}'\mathbf{A}\mathbf{C}) = tr(\mathbf{A})$$

Code:

```
/*part (ii)*/

/*creating orthogonal matrix C*/
call comport(q,r,p,piv,lindep,A);
C = q;
/*check to see if C is orthogonal*/
/*annotate this in the formal code*/
identity = C*C`;
print C;
print identity;

/*compute the trace of C'AC and A*/

a5= C`*a*C;

trace3 = trace(a5);
trace4 = trace(a);

print trace3;
print trace4;
```

Output: we print out the traces for both matrices

```
trace3

     9

trace4

     9
```

and as expected they are the same and equal to 9*.*

For part (iii):

Check equation 2.93*, which is:*

$$tr(\mathbf{A}^{-}\mathbf{A}) = tr(\mathbf{A}A^{-}) = r$$

Code:

```
/*part (iii)*/

/*compute the trace for A^-A and AA^-*/
a6 = a*ginv(a);
a7 = ginv(a)*a;

/*compute their traces*/
trace6 = trace(a6);
trace7 = trace(a7);

print trace6;
print trace7;
```

```
trace6

     4

trace7

     4
```

and they are both equal to the rank of $\mathbf{A}$*, which is* 4*.*

part (*iv*)*, check equation* 2.108*, which is:*

$$tr(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$$

Code:

```
/*part (iv)*/

/*compute sum of eigen values for A*/
trace8 = sum(eigval(a));

/*compute trace of A*/
traceA = trace(a);

print trace8;
print traceA;
```

```
trace8

     9

traceA

     9
```

and it is verified.

Thus completed the solution of questino 7.

Question #8.

Solution 8. *For part [A]:*

$$E[\mathbf{X}_1] = E\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

For part [B]:

$$\mathbf{AX}_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

So

$$E[\mathbf{AX}_1] = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 \end{bmatrix} = \begin{bmatrix} 2-4 \\ 2+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

For part [C]:

$$Cov[\mathbf{X}_1] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

For part [D]:

$$Cov[\mathbf{AX}_1] = Cov\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} var(x_1 - x_2) & cov(x_1 - x_2, x_1 + x_2) \\ cov(x_1 - x_2, x_1 + x_2) & var(x_1 + x_2) \end{bmatrix}$$

We have:

$$\begin{aligned} var(x_1 - x_2) &= var(x_1) + var(x_2) - 2cov(x_1, x_2) \\ &= 4 + 3 - 2 \times (-1) = 9 \end{aligned}$$

$$\begin{aligned} cov(x_1 - x_2, x_1 + x_2) &= var(x_1) - var(x_2) \\ &= 4 - 3 = 1 \end{aligned}$$

$$\begin{aligned} var(x_1 + x_2) &= var(x_1) + var(x_2) + 2cov(x_1, x_2) \\ &= 4 + 3 - 2 = 5 \end{aligned}$$

So we have:

$$Cov[\mathbf{AX}_1] = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

For part [E]:

$$E[\mathbf{X}_2] = \begin{bmatrix} \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

For part [F]':

$$\mathbf{BX}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 + x_5 \\ x_3 + x_4 - 2x_5 \end{bmatrix}$$

So

$$E[\mathbf{BX}_2] = \begin{bmatrix} \mu_3 + \mu_4 + \mu_5 \\ \mu_3 + \mu_4 - 2\mu_5 \end{bmatrix} = \begin{bmatrix} -1 + 3 + 0 \\ -1 + 3 - 2 \times 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

For part [G]:

$$cov(\mathbf{X}_2) = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

For part [H]:

$$\begin{aligned} cov(\mathbf{BX}_2) &= cov \begin{bmatrix} x_3 + x_4 + x_5 \\ x_3 + x_4 - 2x_5 \end{bmatrix} \\ &= \begin{bmatrix} var(x_3 + x_4 + x_5) & cov(x_3 + x_4 + x_5, x_3 + x_4 - 2x_5) \\ cov(x_3 + x_4 + x_5, x_3 + x_4 - 2x_5) & var(x_3 + x_4 - 2x_5) \end{bmatrix} \end{aligned}$$

We have:

$$\begin{aligned} var(x_3 + x_4 + x_5) &= var(x_3) + var(x_4) + var(x_5) + 2cov(x_3, x_4) + 2cov(x_3, x_5) + 2cov(x_4, x_5) \\ &= 6 + 4 + 2 + 2 \times 1 + 2 \times (-1) + 2 \times 0 \\ &= 12 \end{aligned}$$

$$\begin{aligned} &cov(x_3 + x_4 + x_5, x_3 + x_4 - 2x_5) \\ &= var(x_3) + var(x_4) - 2var(x_5) + 2cov(x_3, x_4) - cov(x_3, x_5) - cov(x_4, x_5) \\ &= 6 + 4 - 2 \times 2 + 2 \times 1 - (-1) - 0 \\ &= 9 \end{aligned}$$

$$\begin{aligned} &var(x_3 + x_4 - 2x_5) \\ &= var(x_3) + var(x_4) + 4var(x_5) + 2cov(x_3, x_4) - 4cov(x_3, x_5) - 4cov(x_4, x_5) \\ &= 6 + 4 + 4 \times 2 + 2 \times 1 - 4 \times (-1) - 4 \times 0 \\ &= 24 \end{aligned}$$

So we have:

$$cov(\mathbf{BX}_2) = \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix}$$

For part [I]:

$$\begin{aligned} cov(\mathbf{X}_1, \mathbf{X}_2) &= cov(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}) \\ &= \begin{bmatrix} cov(x_1, x_3) & cov(x_1, x_4) & cov(x_1, x_5) \\ cov(x_2, x_3) & cov(x_2, x_4) & cov(x_2, x_5) \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & -0.5 & 0 \\ 1 & -1 & 0 \end{bmatrix} \end{aligned}$$

For part [J]:

$$\begin{aligned} cov(\mathbf{AX}_1, \mathbf{BX}_2) &= cov(\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}, \begin{bmatrix} x_3 + x_4 + x_5 \\ x_3 + x_4 - 2x_5 \end{bmatrix}) \\ &= \begin{bmatrix} cov(x_1 - x_2, x_3 + x_4 + x_5) & cov(x_1 - x_2, x_3 + x_4 - 2x_5) \\ cov(x_1 + x_2, x_3 + x_4 + x_5) & cov(x_1 + x_2, x_3 + x_4 - 2x_5) \end{bmatrix} \end{aligned}$$

We have:

$$\begin{aligned} & cov(x_1 - x_2, x_3 + x_4 + x_5) \\ &= cov(x_1, x_3) + cov(x_1, x_4) + cov(x_1, x_5) - cov(x_2, x_3) - cov(x_2, x_4) - cov(x_2, x_5) \\ &= 0.5 - 0.5 + 0 - 1 - (-1) - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} & cov(x_1 - x_2, x_3 + x_4 - 2x_5) \\ &= cov(x_1, x_3) + cov(x_1, x_4) - 2cov(x_1, x_5) - cov(x_2, x_3) - cov(x_2, x_4) + 2cov(x_2, x_5) \\ &= 0.5 - 0.5 - 2 \times 0 - 1 - (-1) + 2 \times 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} & cov(x_1 + x_2, x_3 + x_4 + x_5) \\ &= cov(x_1, x_3) + cov(x_1, x_4) + cov(x_1, x_5) + cov(x_2, x_3) + cov(x_2, x_4) + cov(x_2, x_5) \\ &= 0.5 - 0.5 + 0 + 1 - 1 + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} & cov(x_1 + x_2, x_3 + x_4 - 2x_5) \\ &= cov(x_1, x_3) + cov(x_1, x_4) - 2cov(x_1, x_5) + cov(x_2, x_3) + cov(x_2, x_4) - 2cov(x_2, x_5) \\ &= 0.5 - 0.5 - 0 + 1 - 1 - 0 \\ &= 0 \end{aligned}$$

So we have:

$$cov(\mathbf{AX}_1, \mathbf{BX}_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For part [K]:

$$\mathbf{P}_\rho = \begin{bmatrix} 1 & \frac{-1}{\sqrt{4}\sqrt{3}} & \frac{0.5}{\sqrt{4}\sqrt{6}} & -\frac{0.5}{\sqrt{4}\sqrt{4}} & 0 \\ \frac{-1}{\sqrt{4}\sqrt{3}} & 1 & \frac{1}{\sqrt{3}\sqrt{6}} & \frac{-1}{\sqrt{3}\sqrt{4}} & 0 \\ \frac{0.5}{\sqrt{6}\sqrt{4}} & \frac{1}{\sqrt{3}\sqrt{6}} & 1 & \frac{1}{\sqrt{6}\sqrt{4}} & \frac{-1}{\sqrt{6}\sqrt{2}} \\ \frac{-0.5}{\sqrt{4}\sqrt{4}} & \frac{-1}{\sqrt{3}\sqrt{4}} & \frac{1}{\sqrt{6}\sqrt{4}} & 1 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{6}\sqrt{2}} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2\sqrt{3}} & \frac{1}{4\sqrt{6}} & -\frac{1}{8} & 0 \\ -\frac{1}{2\sqrt{3}} & 1 & \frac{1}{3\sqrt{2}} & -\frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{4\sqrt{6}} & \frac{1}{3\sqrt{2}} & 1 & \frac{1}{2\sqrt{6}} & \frac{-1}{2\sqrt{3}} \\ -\frac{1}{8} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & 1 & 0 \\ 0 & 0 & \frac{-1}{2\sqrt{3}} & 0 & 1 \end{bmatrix}$$

For part [L]:

rank$(\mathbf{B}) = \mathbf{2}$ *since the first and second rows are not linearly independent.*

This comlete the solutin of quesiton #8.