PRINCETON UNIVERSITY
Department of Economics
ECO468/FIN 568
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Homework Assignment # 4

The model we consider here is a two asset static version of Hong and Sraer (2011). Please show all work. Please **do not** take formulae from the lecture notes and set $N = 2$.

Consider a two period model ($t = 1, 2$). There are two risky asset and a risk-free asset. The risk-free asset is in perfectly elastic supply and pays interest rate $r > 0$. Each asset $i = 1, 2$ delivers a divident at date 2 of $d_{i,2}$. The dividend can be decomposed into its systematic and idiosyncratic components:

$$d_{i,2} = w_i z + \varepsilon_{i,2}$$

where $z \sim N(\bar{z}, \sigma_z^2)$, $\varepsilon_{i,t} \overset{i.i.d.}{\sim} N(0, \sigma_\varepsilon^2)$, $cov(z, \varepsilon_{i,t}) = 0\ \forall i$. Here w_i is the cashflow beta of asset i and are assumed to be non-negative. Each asset has a supply $s_i > 0$ and we assume without loss of generality that

$$\frac{w_1}{s_1} < \frac{w_2}{s_2}$$

We also assume that the supply of risky assets is normalized to 1 $(s_1 + s_2 = 1)$, and that the value-weighted average w in the economy is 1 $\left(\frac{s_1 w_1 + s_2 w_2}{s_1 + s_2} = 1\right)$.

The population of investors is divided into two groups (H and L) with mass $\frac{1}{2}$. At date 1, agents in group H (the optimists) (respectively L, the pessimists) believe that the average aggregate factor is $\bar{z} + \lambda$ (respectively $\bar{z} - \lambda$), where $\lambda > 0$. Agents continue to hold the same belief about the variance and normality of the aggregate factor.

We assume that agents face short-sale costs: shorting x units of the risky asset costs $\frac{c}{1+r}x^2$.

We assume that agents have some initial date 1 wealth W_1^k where $k \in \{\text{H,L}\}$. Agents are myopic mean-variance investors with utility function

$$U\left(W_t^k\right) = E_t^k\left[W_{t+1}^k\right] - \frac{1}{2\gamma} Var_t^k\left(W_{t+1}^k\right)$$

This homework will walk you through solving this model. Let $\mu_{i,t}^k$ be agent k's demand for asset i at time t, and $P_{i,t}$ be the price of asset i at time t.

(a) Assume that the optimists will long both assets, and the pessimists will long asset 1 and short asset 2. Write down the FOCs for each agent's optimization problem at time 1.

(b) Let $\sigma_c^2 = \sigma_\varepsilon^2 + \gamma c$. Let $S = w_1 \mu_{1,1}^L + w_2 \mu_{2,1}^L$. Using the FOCs obtained in the previous part, together with the market clearing conditions, solve for S in terms of exogenous parameters only (not in terms of demand, prices, etc...). Also, express your answers in terms of σ_c^2 (instead of c), so your answers should not have c floating around.

(c) Using the expression for S obtained in the previous part, together with the FOCs, solve for $\mu_{i,1}^L$ for $i = 1, 2$. That is, solve for the pessimist's demand for both assets. Then use the market clearing conditions to get the optimist's asset demand $\left(\mu_{i,1}^H \text{ for } i = 1, 2\right)$. Express your answers in terms of exogenous parameters only.

(d) Solve for the asset prices $P_{i,1}$ for $i = 1, 2$ in terms of exogenous parameters only (no c, only σ_c^2).

(e) Let $\theta = \frac{\sigma_\varepsilon^2}{\sigma_c^2} = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \gamma c}$. Express the above prices in terms of θ and other exogenous parameters. Your answer should no long depend on σ_c^2. What is the range of θ? Interpret θ.

(f) We have assumed that group H will long both assets, and group L will long asset 1 and short asset 2 in equilibrium. Find conditions on $2\gamma\lambda$ under which this assumption holds.

Hint: There are three conditions we need to verify. First, we need $\mu_{1,1}^L > 0$ and $\mu_{2,1}^L < 0$. This will give an inequality of the form $u_2 < 2\gamma\lambda < u_1$. Then you need two more conditions: $\mu_{1,1}^H > 0$, and $\mu_{2,1}^H > 0$. These two conditions will give two inequalities in the form $2\gamma\lambda < \eta_1$, $2\gamma\lambda < \eta_2$. Note that η_1, η_2, u_1,and u_2 are expressed only in terms of exogenous parameters (no prices and other demands). Finally, we combine all these inequalities to say that

$$u_2 < 2\gamma\lambda < \max\left(\eta_1, \eta_2, u_1\right)$$

(g) Define the return to asset $j = 1, 2$ to be $R_j = d_{j,2} - (1+r) P_{j,1}$, and the return to the market portfolio to be $R_M = s_1 d_{1,2} + s_2 d_{2,2} - s_1 P_{1,1} - s_2 P_{2,1}$. Define

$$\beta_j = \frac{cov\left(R_j, R_M\right)}{Var\left(R_M\right)}$$

Compute β_1 and β_2. Note that the covariance and variance is computed using the objective measure.

(h) Compute the expected return for asset 1 and 2. Express the expected return in terms of β_j, θ, and other exogenous parameters (no σ_c^2).

(i) Now suppose that we run a cross-sectional regression for R_j on β_j and a constant. What is the regression coefficient $\hat{\mu}$ for β_j? What happens to $\hat{\mu}$ as c goes from 0 to ∞? Show that $\hat{\mu}$ is decreasing with aggregate disagreement λ. Show that this effect is stronger for larger shorting costs c (i.e., show that $\hat{\mu}$ decreases at a faster rate with respect to λ as c gets larger). Interpret the results intuitively.