INFO-F-409
Learning dynamics

Extensive form games, subgame-perfect Nash equilibria
and game theoretical extensions

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Summary

- What? Why?
- Rational choice
- Strategic games
- Nash Equilibrium
- Best response
- Dominance
- Mixed strategies
- Mixed-strategy Nash Equilibria
- Support finding
- Lemke-Howson algorithm

Extensive form games

Strategic games assume that each decision maker chooses her actions once and for all

Fragment fromThe Big-Bang Theory (2008)

Hence it does not take into account the **sequential structure of decision making**

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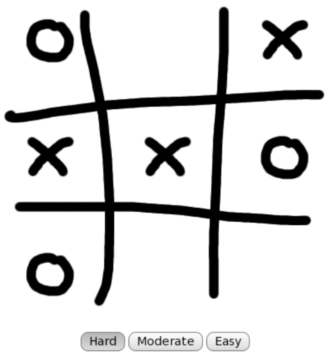
Strategic games assume that each decision maker chooses her actions once and for all

Fragment fromThe Big-Bang Theory (2008)

Hence it does not take into account the **sequential structure of decision making**

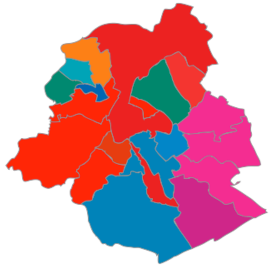
Example



See also Techniques in Artificial Intelligence

Example

In an **Entry game** there are two players:
A (the incumbent) and B (the challenger)



So B may decide to challenge (or to stay out) and if B challenges A may either allow entry or fight against entry

Example

In an **Entry game** there are two players:
A (the incumbent) and B (the challenger)

So B may decide to challenge (or to stay out) and if B challenges A may either allow entry or fight against entry

Some terminology

A **history** is the sequence of actions taken by the players up to some decision point

A **terminal history** is a history that contains the action choices of all the players up until the point where the payoff is distributed

The Entry game has 3 **terminal histories**

1. (Challenge, Allow entry),
2. (Challenge, Fight entry) and
3. (Stay out)

A **sub-history** is a history that contains part of a terminal history

Some terminology

One can assign to every sub-history which is not the complete history (= proper sub-history) a player using a **player function** *P(proper sub-history)* :

(Challenge) and ∅ are proper sub-histories of (Challenge, Allow entry) and (Challenge,Fight entry)

Thus *P(Challenge)* indicates that player A (the incumbent) acts after that point

Thus *P(∅)* indicates that player B (the challenger) acts after that point (which is the start of the game)

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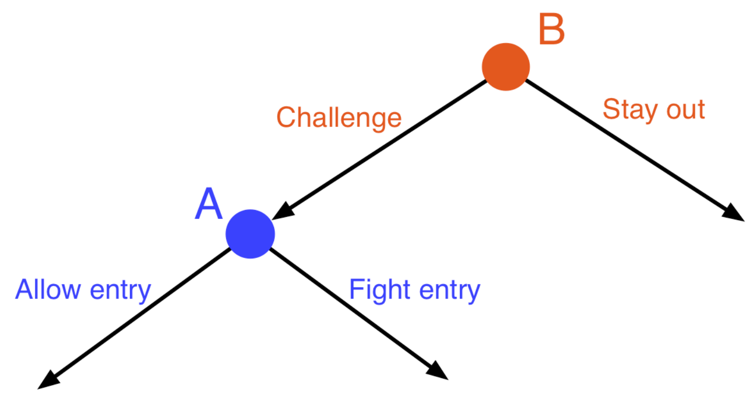
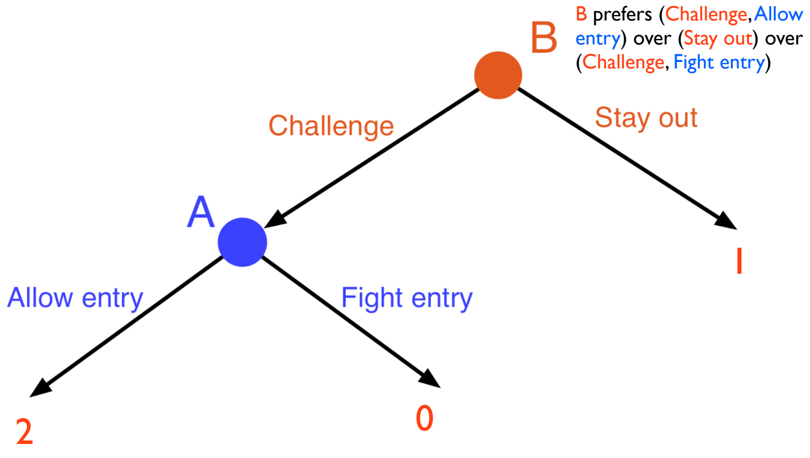
Tree representation



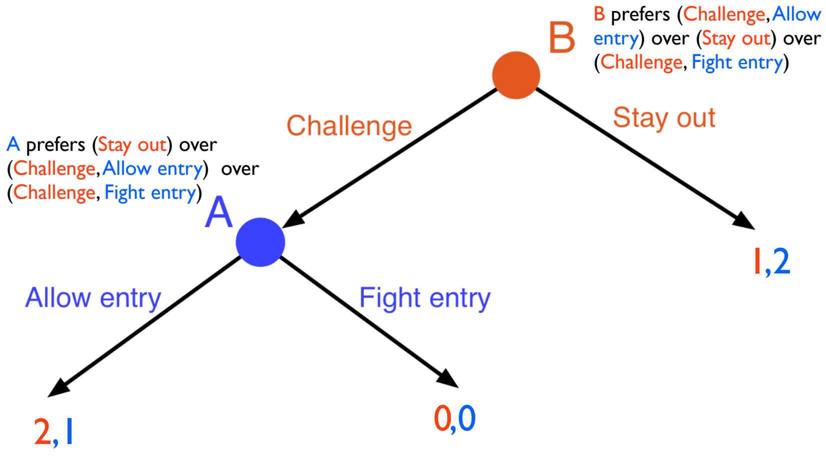
9-1

Tree representation



9-2

Tree representation



9-3

Extensive game

Definition :

An **extensive game** with perfect information consists of :

- a set of players
- a set of terminal histories with the property that none of these histories is a proper sub-history of another
- A player function that assigns a player to every proper sub-history that can be derived from the terminal histories
- For each player, preferences over the set of terminal histories

Perfect Information

- **What**:
  - Players know the node they are in
  - They know all the prior choices, including those of other agents
- What happens when agents have only incomplete knowledge of the actions taken by others or no longer remember their past actions?
- Games with *Imperfect Information* (see later)

More terminology

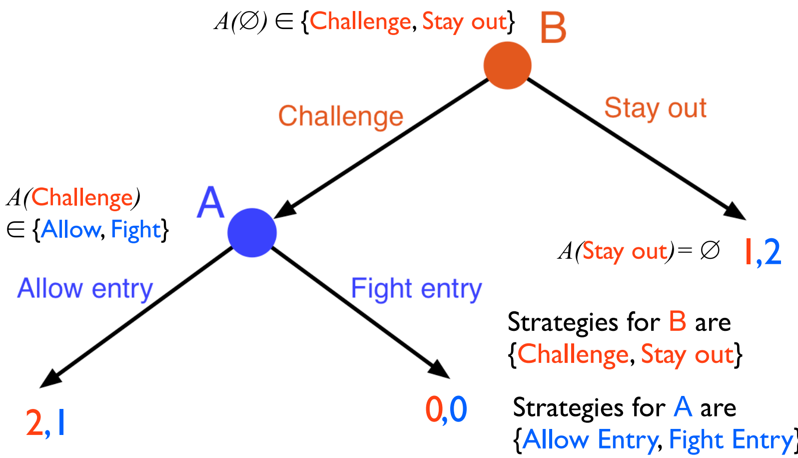
If all terminal histories are **finite**, then the game has a **finite horizon**

If a game has a finite horizon and finitely many terminal histories then the game is called **finite**

Definition :

A **strategy** of a player i in an extensive game with perfect information is a function that assigns to **each history** h after which it is player i's turn to move ($P(h)=i$, where P is the player function) an action in $A(h)$, i.e. the set of available actions after h

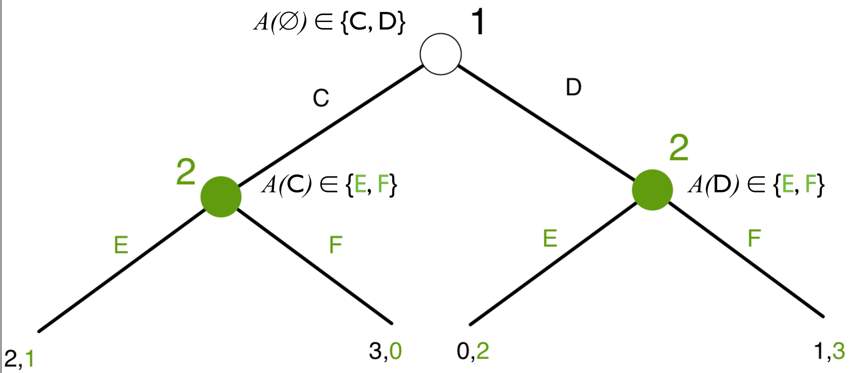
Strategies



Strategies for B are {Challenge, Stay out}

Strategies for A are {Allow Entry, Fight Entry}

Strategies



Strategies for 1 are {C, D}
Strategies for 2 are {CE, CF, DE, DF}

Equilibrium

One way to determine the steady states is to transform the extensive game into a strategic game and determine the Nash equilibria in that way

	Allow	Fight
Challenge	1 2	0 0
Stay out	2 1	2 1

Equilibrium

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Equilibrium

One way to determine the steady states is to transform the extensive game into a strategic game and determine the Nash equilibria in that way

	Allow	Fight
Challenge	2, 1	0, 0
Stay out	1, 2	1, 2

?

This NE seems to assume that player B knew that player A was going to *fight* and therefore selected *No challenge*

This is at odds with the extensive model where the action *Stay out* will never be observed when player B choses *not to challenge* A

Equilibrium

This NE ignores the sequential structure of the game since it treats strategies as choices that are made once and for all

a different notion of equilibrium is required to describe the steady state of an extensive game

The idea is that this equilibrium **requires each player's strategy to be optimal**, given the other players' strategies, not only at the start **but at every possible history**

To reach this new definition, we first need to define the notion of a sub-game

Sub-game

Definition :

Let Γ be an extensive game with perfect information, with player function P. For any nonterminal history h of Γ, the subgame $\Gamma(h)$ following the history h is the following extensive game

Players The players in Γ

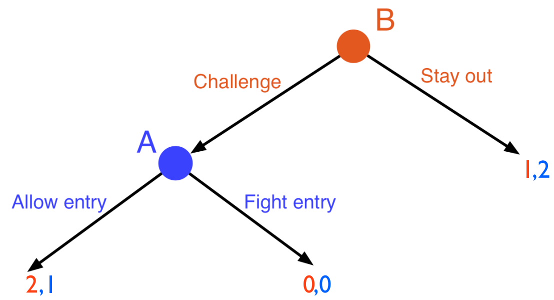
Terminal histories The set of all sequences h' of actions such that (h,h') is a terminal history of Γ

Player function The player function $P(h,h')$ is assigned to each proper sub-history h' of a terminal history

Preferences each player prefers h' to h'' if and only if she prefers (h,h') to (h,h'') in Γ

Sub-game

The Entry game has 2 sub-games



Sub-game

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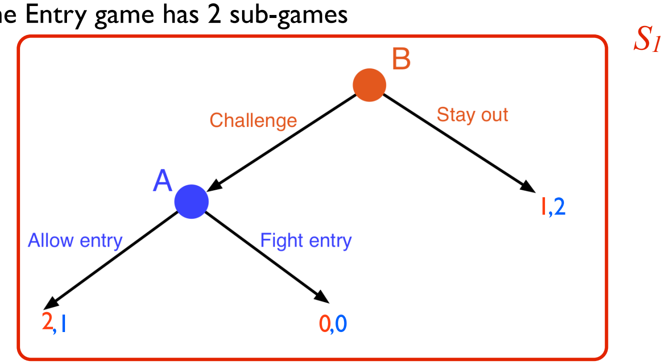
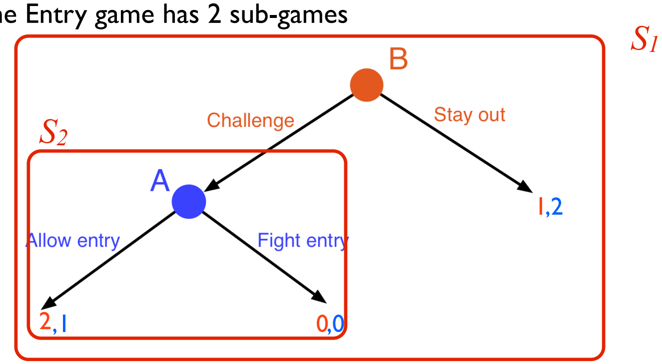


18-2

Sub-game

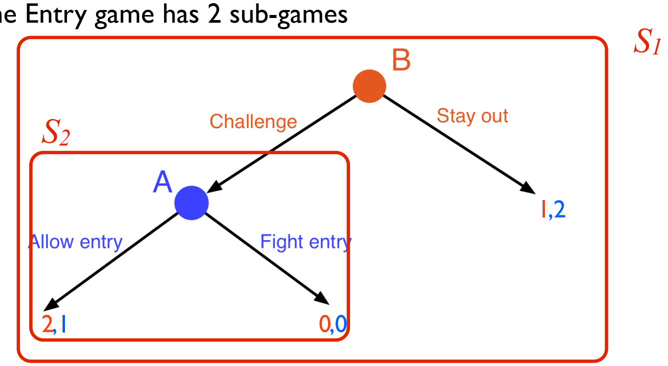
The Entry game has 2 sub-games



18-3

Sub-game

The Entry game has 2 sub-games

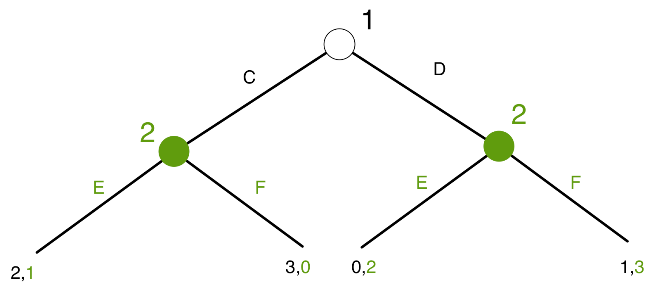


S_2 is a **proper sub-game**

18-4

Sub-game

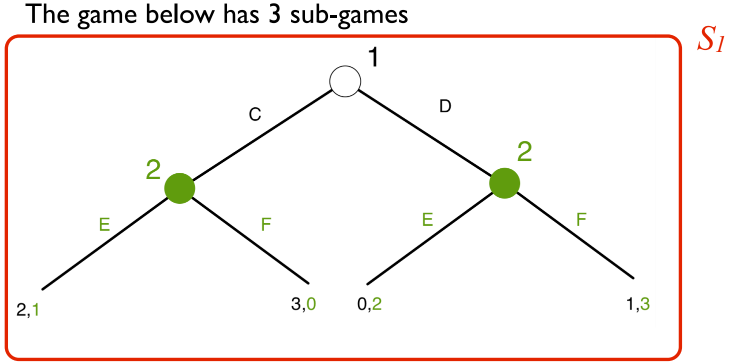
The game below has 3 sub-games



19-1

Sub-game

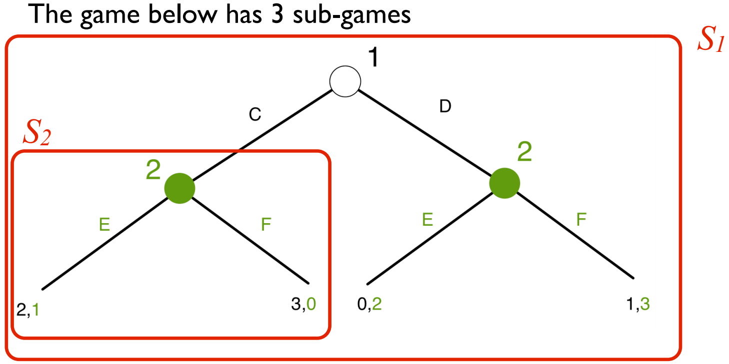
The game below has 3 sub-games



19-2

Sub-game

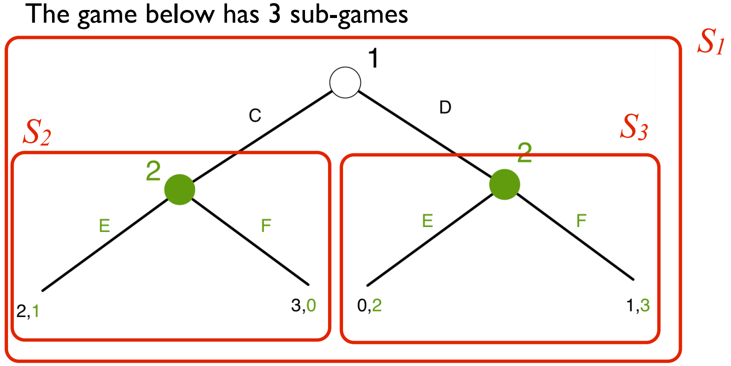
The game below has 3 sub-games



19-3

Sub-game

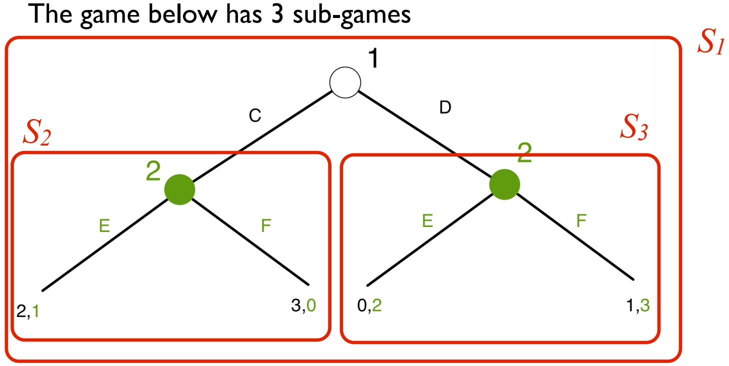
The game below has 3 sub-games



19-4

Sub-game

The game below has 3 sub-games



S_2 and S_3 are **proper sub-games**

19-5

Sub-game perfect equilibrium

Definition :

the strategy profile s^* in an extensive game with perfect information is a **sub-game perfect equilibrium** if, for every player i, every history h after which it is player i's turn to move ($P(h)=i$), and every strategy r_i of player i, the terminal history $O_h(s^*)$ generated by s^* after the history h is at least as good according to player i's preferences as the terminal $O_h(r_i, s_{-i}^*)$ generated by the strategy profile (r_i,s_{-i}^*) in which player i chooses r_i while every other player j chooses s_j^*

Sub-game perfect equilibrium

Definition :

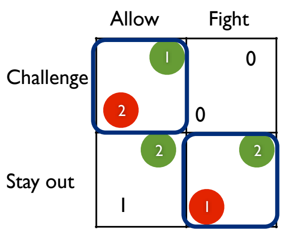
Equivalently, for every player i and every history h after which it is player i's turn to move,

$u_i(O_h(s^*)) \geq u_i(O_h(r_i,s_{-i}^*))$ for every strategy r_i of player i

where u_i is a payoff function that represents the player i's preferences and $O_h(s)$ is the terminal history consisting of h followed by the sequence of actions generated by s after h

Every sub-game perfect equilibrium is a Nash equilibrium

Example



Best response analysis of the strategic game claims that there are two NE

Are they also subgame perfect ?

Take the NE s^*=(challenge,allow): *Sub-game $S1$*

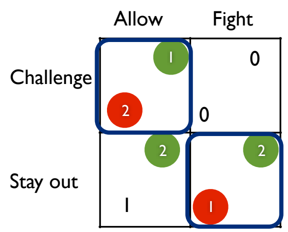
Player B (i=B) moves at $h=\varnothing \rightarrow O_h(s^*)$ =(challenge, allow)

So now we check the payoffs for every action r_B of B, given $O_h(s^*)$

r_B*= Challenge → $u_B(O_h(s^*)) = 2$

r_B= Stay out → $u_B(O_h(r_B,s_{-B}^*)) = 1$

Example



Best response analysis of the strategic game claims that there are two NE

Are they also subgame perfect ?

Take the NE s^*=(challenge,allow): *Sub-game $S2$*

Player A (i=A) moves at h=challenge→$O_h(s^*)$ =(challenge, allow)

So now we check the payoffs for every action r_A of A, given $O_h(s^*)$

r_A*=allow → $u_A(O_h(s^*)) = 1$

r_A=fight → $u_A(O_h(r_A, s_{-A}^*)) = 0$

Example

	Allow	Fight
Challenge	2, 1	0, 0
Stay out	1, 2	1, 2

Best response analysis of the strategic game claims that there are two NE

Are they also subgame perfect ?

Take the NE s^*=(challenge,allow): *Sub-game $S2$*

Player A (i=A) moves at h=challenge→$O_h(s^*)$ =(challenge, allow)

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r_A*=allow → $u_A(O_h(s^*)) = 1$

r_A=fight → $u_A(O_h(r_A, s_{-A}^*)) = 0$

So (challenge,allow) is a subgame perfect equilibirum

23-2

Example

	Allow	Fight
Challenge	2, 1	0, 0
Stay out	1, 2	1, 2

Best response analysis of the strategic game claims that there are two NE

Are they also subgame perfect ?

Take the NE s^*=(Stay out,fight): *Sub-game $S2$*

Player B (i=B) moves at $h=\varnothing$→$O_h(s^*)$ =(Stay out, fight)

So now we check the payoffs for every action r_B of B, given $O_h(s^*)$

r_B*=Stay out → $u_B(O_h(s^*)) = 1$

r_B= challenge → $u_B(O_h(r_B, s_{-B}^*)) = 0$

24

Example

	Allow	Fight
Challenge	2, 1	0, 0
Stay out	1, 2	1, 2

Best response analysis of the strategic game claims that there are two NE

Are they also subgame perfect ?

Take the NE s^*=(stay out,fight): *Sub-game $S2$*

Player A (i=A) moves at h=**challenge**→$O_h(s^*)$ =(**challenge**, fight)

So now we check the payoffs for every action r_A of A, given $O_h(s^*)$

r_A*=fight → $u_A(O_h(s^*)) = 0$

r_A=allow → $u_i(O_h(r_a, s_{-A}^*)) = 1$

25-1

Example

	Allow	Fight
Challenge	2, 1	0, 0
Stay out	1, 2	1, 2

Best response analysis of the strategic game claims that there are two NE

Are they also subgame perfect ?

Take the NE s^*=(stay out,fight): *Sub-game $S2$*

Player A (i=A) moves at h=**challenge**→$O_h(s^*)$ =(**challenge**, fight)

So now we check the payoffs for every action r_A of A, given $O_h(s^*)$

r_A*=fight → $u_A(O_h(s^*)) = 0$

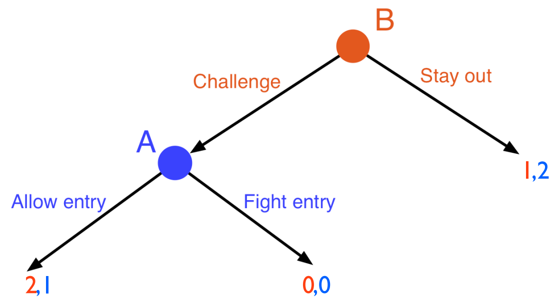
r_A=allow → $u_i(O_h(r_a, s_{-A}^*)) = 1$

So (no challenge, fight) is a **NOT** subgame perfect equilibirum

25-2

Finding the SPE

In a game with a finite horizon, the set of subgame perfect equilibria can be found more easily by using a procedure that extends the **backward induction** process :



26-1

Finding the SPE

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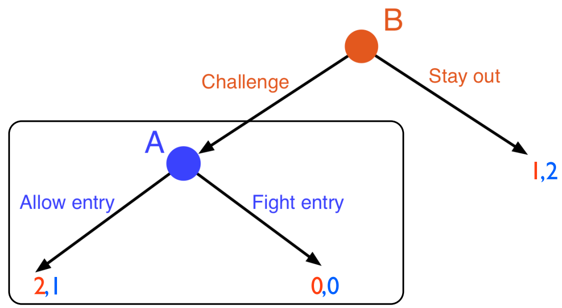

26-2

Finding the SPE

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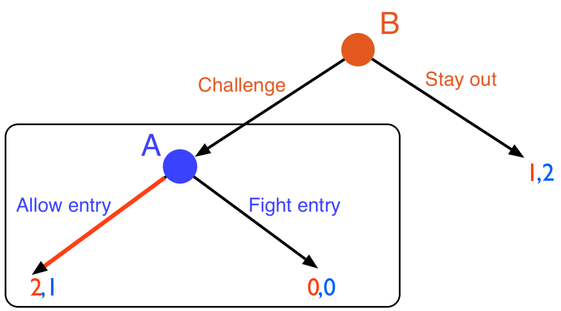

26-3

Finding the SPE

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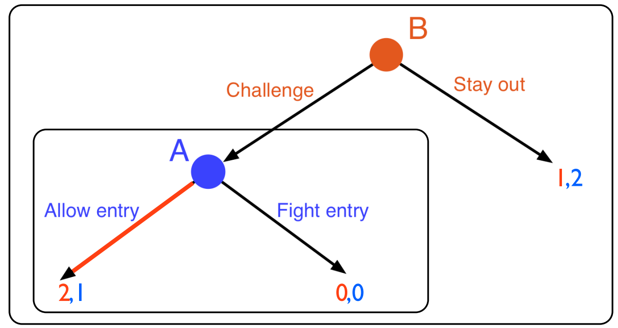

26-4

Finding the SPE

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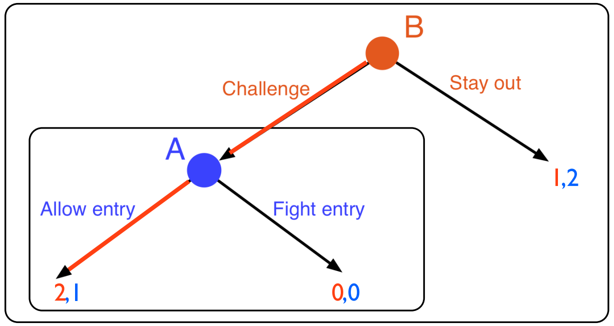

26-5

Finding the SPE

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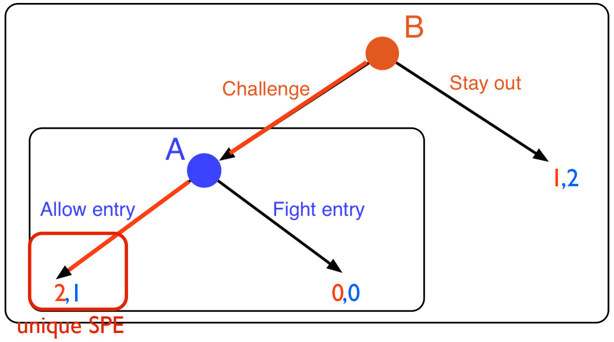
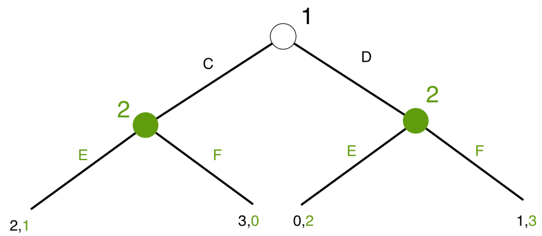


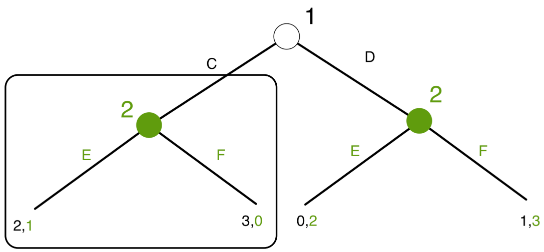
26-6

Finding the SPE



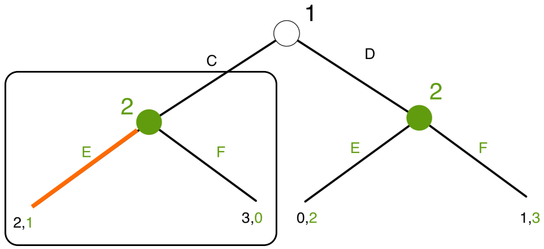
27-1

Finding the SPE



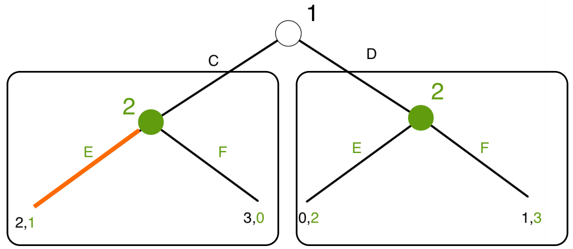
27-2

Finding the SPE



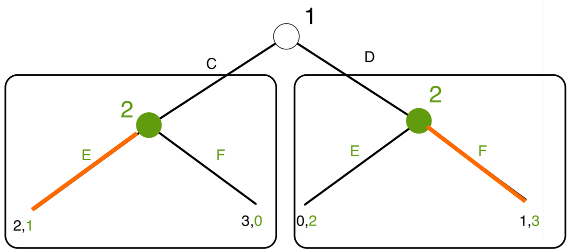
27-3

Finding the SPE



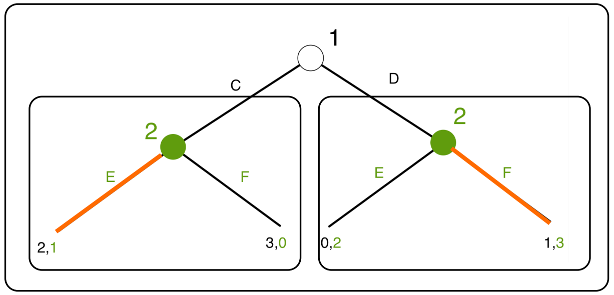
27-4

Finding the SPE



27-5

Finding the SPE



27-6

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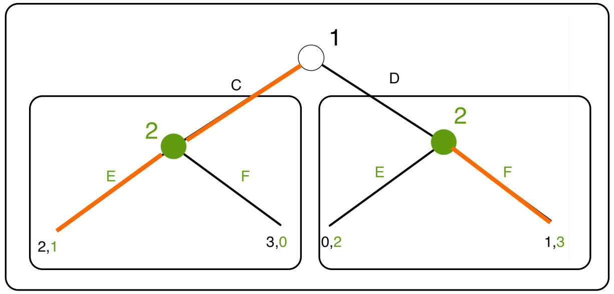
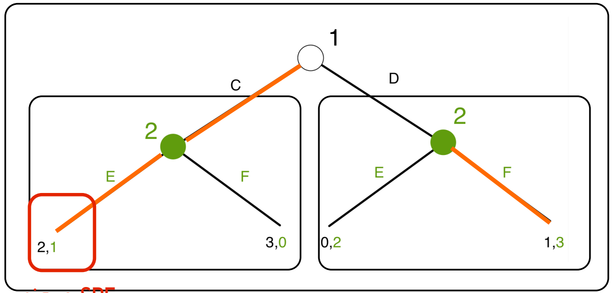


27-7

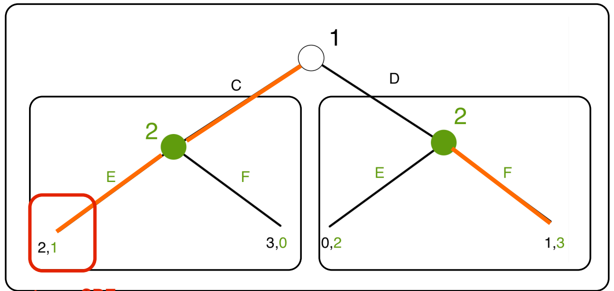
Finding the SPE



unique SPE

27-8

Finding the SPE



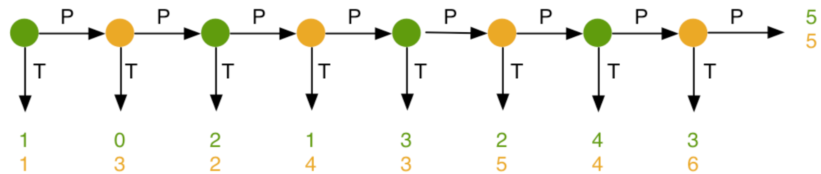
unique SPE

Backward induction **cannot be applied to every extensive game** with perfect information : e.g. games with infinitely long histories ,...

27-9

The Centipede game

Originally invented by Rosenthal in 1982 (with 100 moves)



An eight-moves centipede game

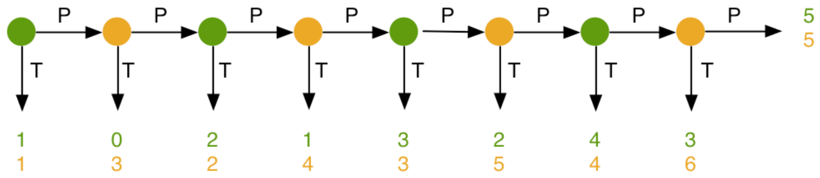
Note that passing (P) the money always means that you may receive less than currently possible

What is the Sub-game perfect Nash equilibrium here?

28

The Centipede game

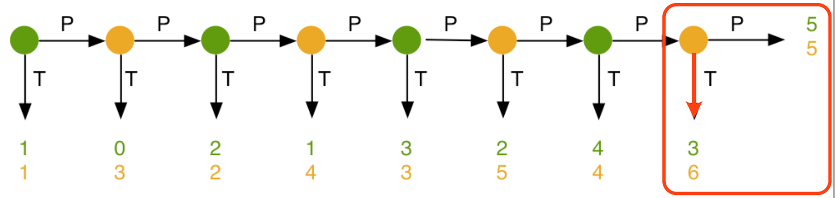
Originally invented by Rosenthal in 1982 (with 100 moves)



29-1

The Centipede game

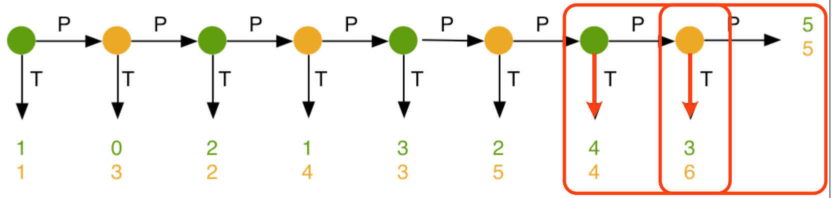
Originally invented by Rosenthal in 1982 (with 100 moves)



29-2

The Centipede game

Originally invented by Rosenthal in 1982 (with 100 moves)



29-3

The Centipede game

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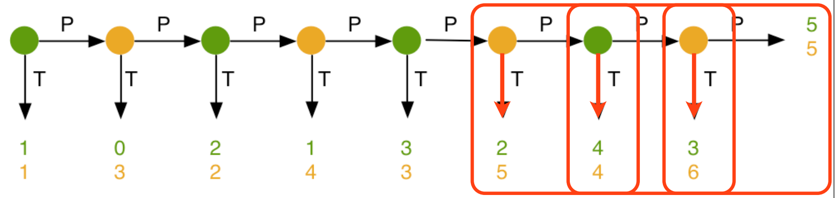

29-4

The Centipede game

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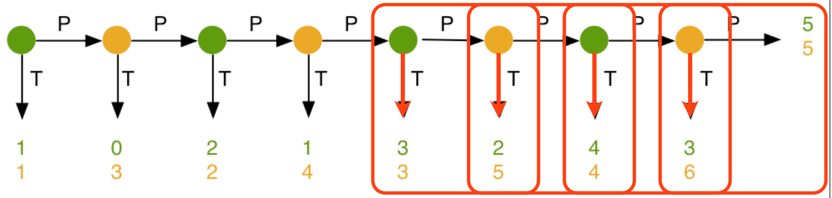

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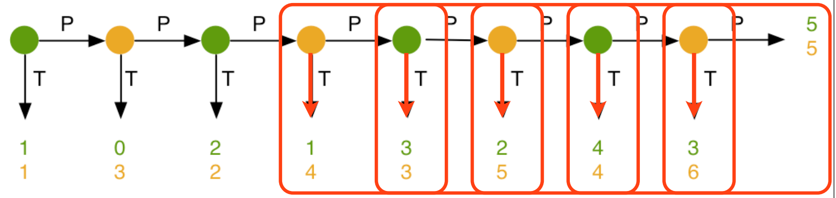

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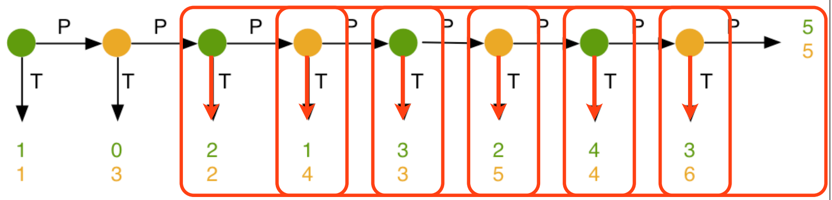
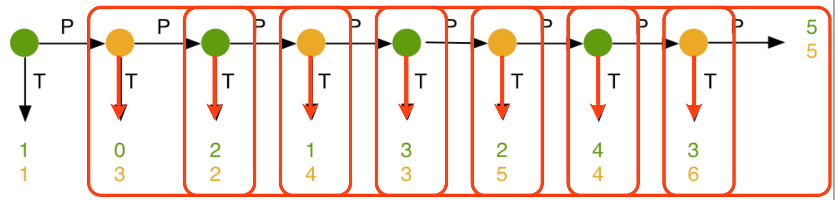


The Centipede game

Originally invented by Rosenthal in 1982 (with 100 moves)



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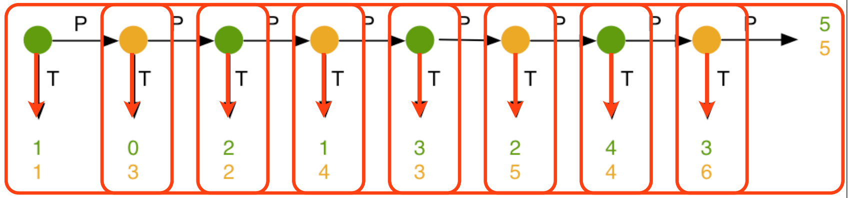

29-9

The Centipede game

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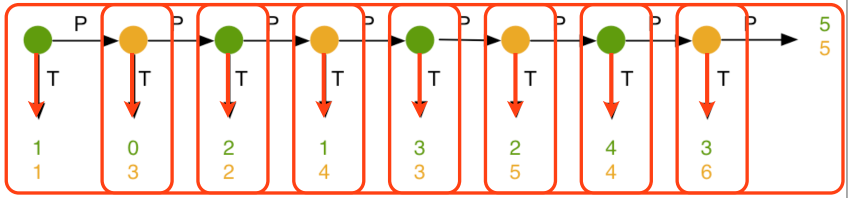

So the rational choice is to take immediately the money

Yet experiments show different results (see McKelvey and Palfrey, 1992 and Nagel and Tang 1998)

29-10

The Centipede game

TABLE IIA

PROPORTION OF OBSERVATIONS AT EACH TERMINAL NODE

	Session		N	f_1	f_2	f_3	f_4	f_5	f_6	f_7
Four Move	1	(PCC)	100	.06	.26	.44	.20	.04		
	2	(PCC)	81	.10	.38	.40	.11	.01		
	3	(CIT)	100	.06	.43	.28	.14	.09		
	Total	1–3	281	.071	.356	.370	.153	.049		
High Payoff	4	(High-CIT)	100	.150	.370	.320	.110	.050		
Six Move	5	(CIT)	100	.02	.09	.39	.28	.20	.01	.01
	6	(PCC)	81	.00	.02	.04	.46	.35	.11	.02
	7	(PCC)	100	.00	.07	.14	.43	.23	.12	.01
	Total	5–7	281	.007	.064	.199	.384	.253	.078	.014

from McKelvey and Palfrey (1992) An Experimental Study of the Centipede Game. *Econometrica* 60(4):803-836

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The pirate puzzle

Opening scene of The Dark Knight

31-1

The pirate puzzle

Opening scene of The Dark Knight

The pirate puzzle

Assume 5 pirates (A, B, C, D and E) who have found a treasure of 100 gold coins.

They split up the money according to an ancient pirate code that depends on the ferocity of the pirates.

Assume that pirate E is the fiercest, then D, then C ...
So the strict ferocity ranking is E>D>C>B>A

The code works as follows;

1. The fiercest pirate proposes a split of the gold
2. All the pirates including the fiercest vote whether to accept the split
3. If 50% or more agree, the split happens
4. Otherwise the pirate who proposed the split gets thrown overboard and is eaten by the sharks
5. The process is repeated with the next fiercest pirate, until a proposal is accepted

The pirate puzzle

What proposal should the fiercest pirate make ?

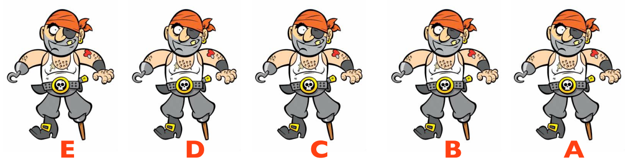
Should he (or she) give away most of the loot?

The code works as follows;

1. The fiercest pirate proposes a split of the gold
2. All the pirates including the fiercest vote whether to accept the split
3. If 50% or more agree, the split happens
4. Otherwise the pirate who proposed the split gets thrown overboard and is eaten by the sharks
5. The process is repeated with the next fiercest pirate, until a proposal is accepted

The pirate puzzle

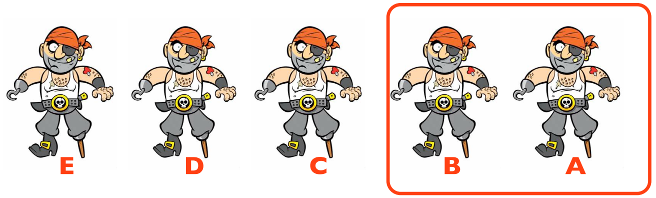
This problem can be solved by reasoning backwards:



The pirate puzzle

This problem can be solved by reasoning backwards:

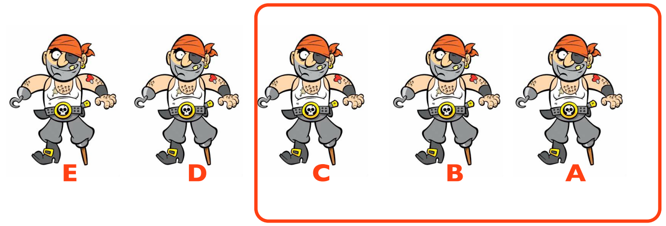
Let’s start with game where only 2 pirates remain : **A** and **B**



B: 100 - 0

The pirate puzzle

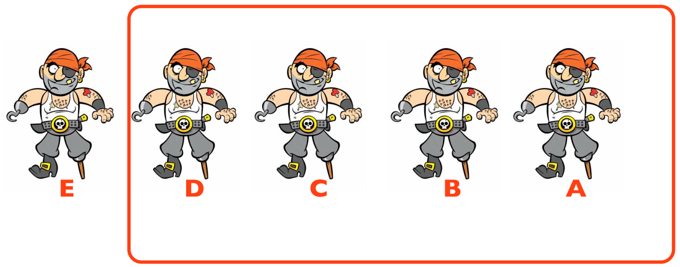
This problem can be solved by reasoning backwards:



C: 99 - 0 - 1

The pirate puzzle

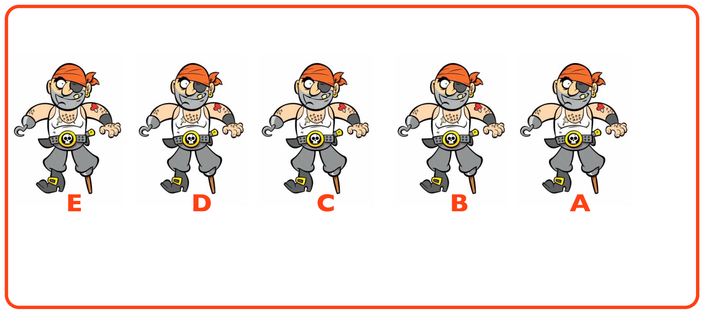
This problem can be solved by reasoning backwards:



D: 99 - 0 - 1- 0

The pirate puzzle

This problem can be solved by reasoning backwards:



E: 98 - 0 - 1- 0 -1

The pirate puzzle

So what happened in the opening scene ?

The joker **bribed** every weaker robber with the promise of a bigger share

reaching in the end the situation of the two pirates , allowing him to steal everything

Extensions

Simultaneous moves :

In some situations, after some sequence of actions of the players, the players may need to choose the next action simultaneously

Chance moves :

Sometimes, random events may occur that alter the sequence of actions

Bayesian games

Sometimes you lack information about the opponent

Simultaneous moves

Take for instance the following variant of the **battle of the sexes:**

First one player decides to stay home and watch television or to attend a concert.

When he or she decided to stay home, the game ends

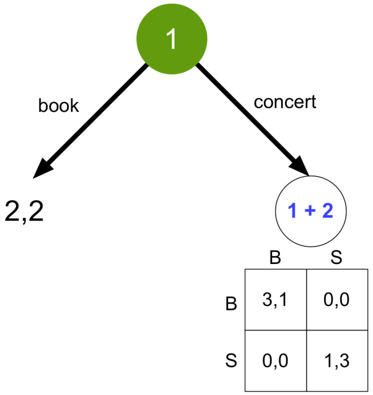
If he or she decides to attend the concert, then both players have to choose simultaneously which concert, Bach or Stravinsky

Simultaneous moves

This defines the following extensive form game:

- a set of players: *{1,2}*
- a set of terminal histories: *{Book, (Concert, (B,B)), (Concert, (B,S)), (Concert, (S,B)), (Concert, (S,S))}*
- A player function: $P(\emptyset) = \{1\}$, $P(Concert)=\{1,2\}$
- Preferences:
  - Player 1: *(Concert,(B,B)) > Book > (Concert,(S,S)) > (Concert,(S,B)) = (Concert,(B,S))*
  - Player 2: *(Concert,(S,S)) > Book > (Concert,(B,B)) > (Concert,(S,B)) = (Concert,(B,S))*

Simultaneous moves



How to find the equilibrium?

Actions of player 1: $A_1(\emptyset)=\{Concert, Book\}$, $A_1(Concert)=\{B, S\}$,
Actions of player 2: $A_2(Concert)=\{B, S\}$

Strategies player 1: *(Concert, B), (Concert, S), (Book, B)* and *(Book, S)*
Strategies player 2: *B* and *S*

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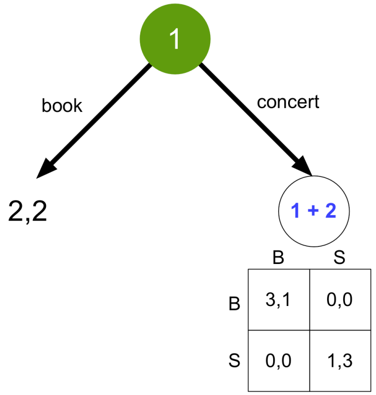
Simultaneous moves

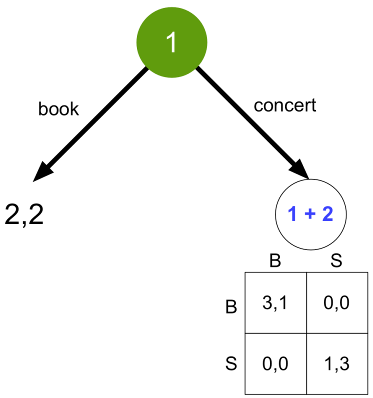


How to find the equilibrium?

	B	*S*
(concert, B)	3,1	0,0
(concert, S)	0,0	1,3
(book, B)	2,2	2,2
(book, S)	2,2	2,2

40-1

Simultaneous moves

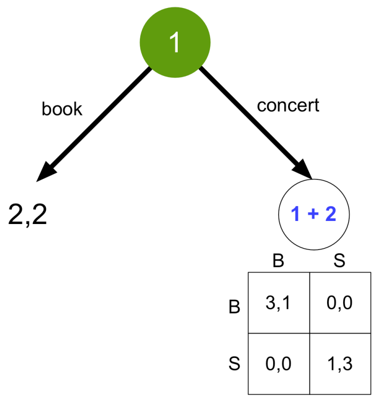


How to find the equilibrium?

	B	*S*
(concert, B)	3,1	0,0
(concert, S)	0,0	1,3
(book, B)	2,2	2,2
(book, S)	2,2	2,2

40-2

Simultaneous moves



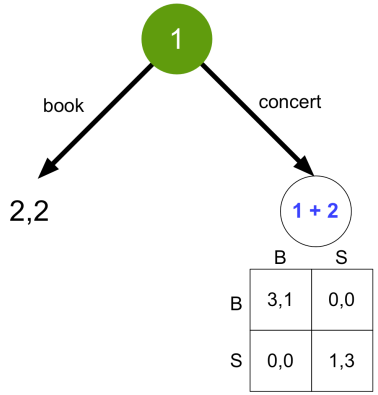
How to find the equilibrium?

	B	*S*
(concert, B)	3,1	0,0
(concert, S)	0,0	1,3
(book, B)	2,2	2,2
(book, S)	2,2	2,2

40-3

Simultaneous moves

How to find the equilibrium?

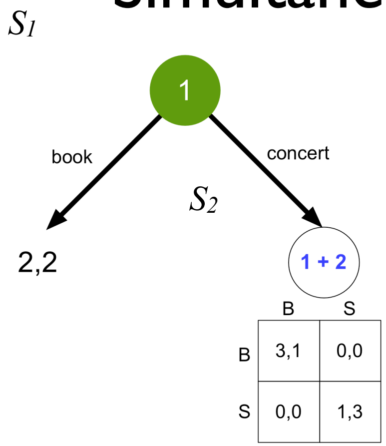


	B	S
(concert, B)	3 1	0,0
(concert, S)	0,0	1, 3
(book, B)	2, 2	2 2
(book, S)	2, 2	2 2

40-4

Simultaneous moves

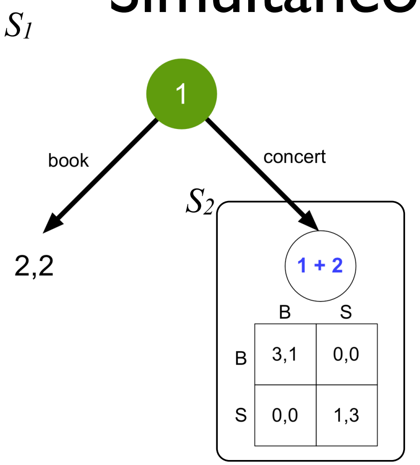
How to find the equilibrium?



41-1

Simultaneous moves

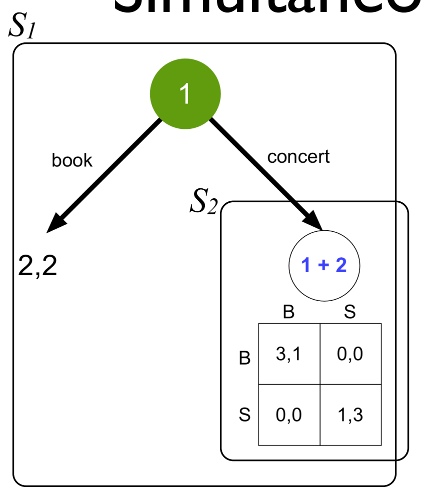
How to find the equilibrium?



41-2

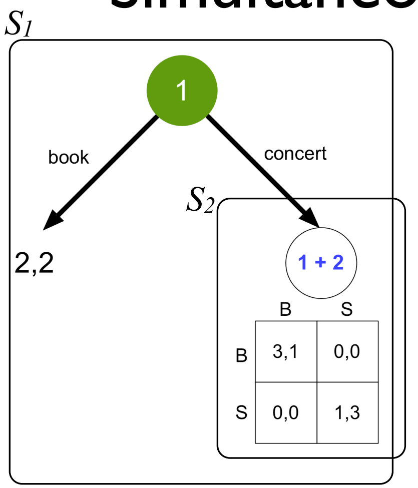
Simultaneous moves

How to find the equilibrium?



41-3

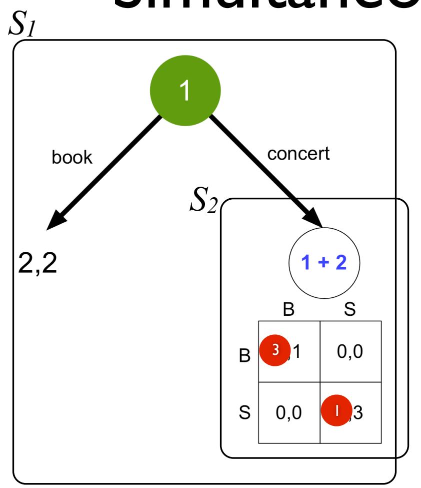
Simultaneous moves



How to find the equilibrium?

1. What are the pure strategy equilibria for the sub-game S_2

Simultaneous moves



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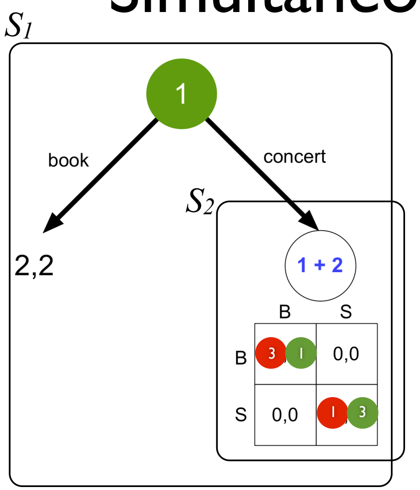
Simultaneous moves

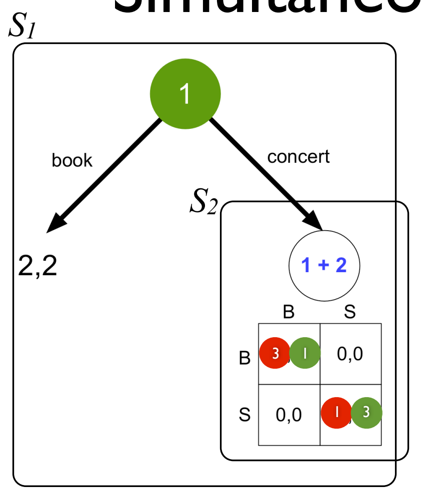


How to find the equilibrium?

1. What are the pure strategy equilibria for the sub-game S_2

(B,B) and (S,S)

Simultaneous moves



How to find the equilibrium?

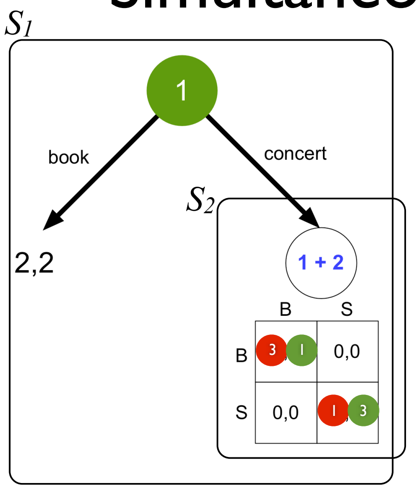
1. What are the pure strategy equilibria for the sub-game S_2

(B,B) and (S,S)

2. What is the optimal choice in S_1 of the the outcome of S_2 is (B,B)

3. What is the optimal choice in S_1 of the the outcome of S_2 is (S,S)

Simultaneous moves



How to find the equilibrium?

1. What are the pure strategy equilibria for the sub-game S_2

(B,B) and (S,S)

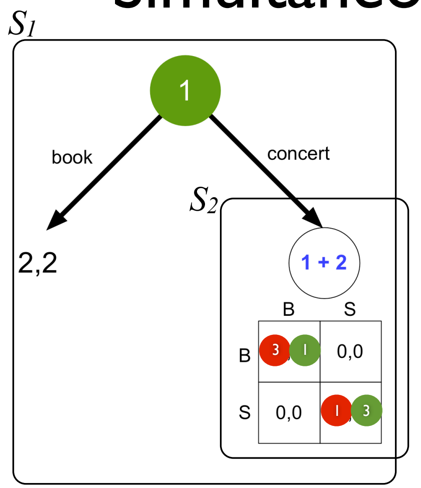
2. What is the optimal choice in S_1 of the the outcome of S_2 is (B,B)

Concert

3. What is the optimal choice in S_1 of the the outcome of S_2 is (S,S)

41-8

Simultaneous moves



How to find the equilibrium?

1. What are the pure strategy equilibria for the sub-game S_2

(B,B) and (S,S)

2. What is the optimal choice in S_1 of the the outcome of S_2 is (B,B)

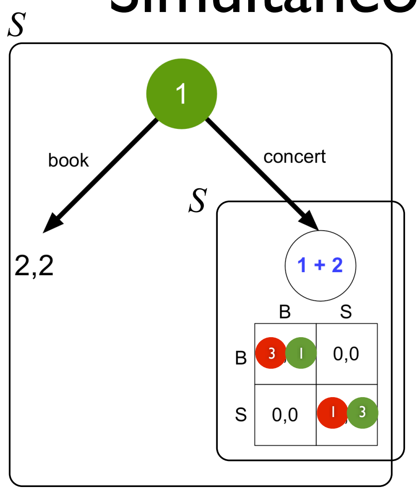
Concert

3. What is the optimal choice in S_1 of the the outcome of S_2 is (S,S)

book

41-9

Simultaneous moves



Thus the sub-game perfect equilibria are :

((concert,B),B) and ((book, S),S)

Note : BoS also has a mixed strategy Nash equilibrium, which we did not consider here

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Chance moves

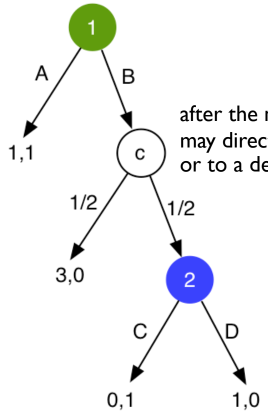
Remember that in the definition of extensive-form games, there is a function P that assigns a player to each history.

Here, one can also assign chance as opposed to a player

As a consequence, the preferences of the players become defined over the set of lotteries (probability distribution) over terminal histories

43

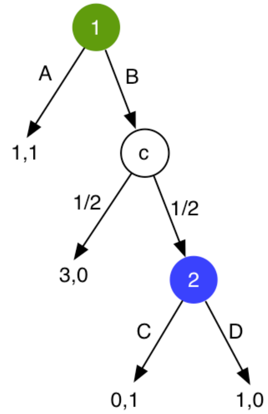
Chance moves



after the move of player 1, chance may direct you to a terminal history or to a decision of player 2

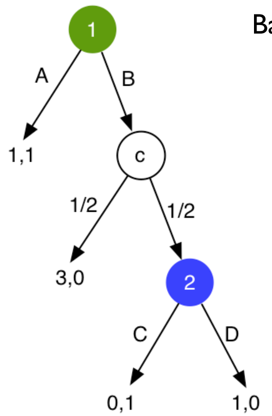
44-1

Chance moves



44-2

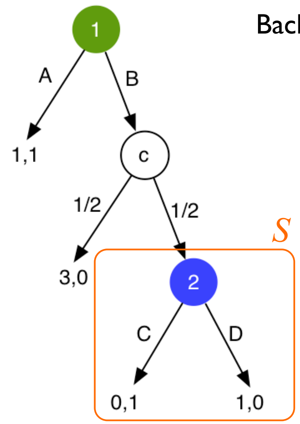
Chance moves



Backward induction:

44-3

Chance moves

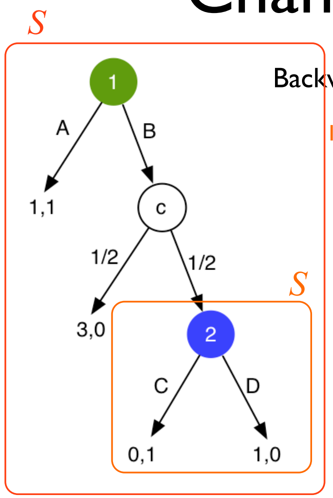


Backward induction:

In sub-game S_2, player 2 chooses C

44-4

Chance moves



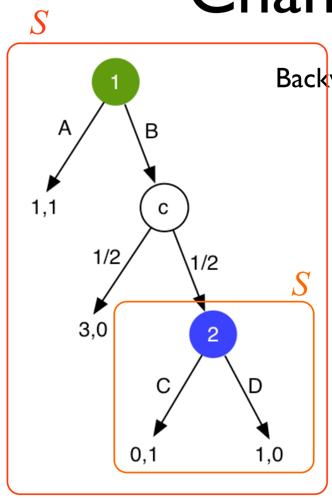
Backward induction:

In sub-game S_2, player 2 chooses C

In sub-game S_1, we consider the consequences of the actions of player 1

44-5

Chance moves



Backward induction:

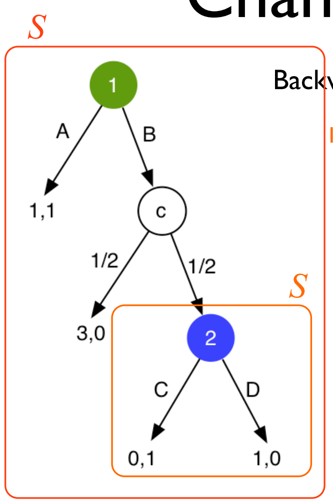
In sub-game S_2, player 2 chooses C

In sub-game S_1, we consider the consequences of the actions of player 1

If A is selected then 1 obtains a payoff of 1

44-6

Chance moves



Backward induction:

In sub-game S_2, player 2 chooses C

In sub-game S_1, we consider the consequences of the actions of player 1

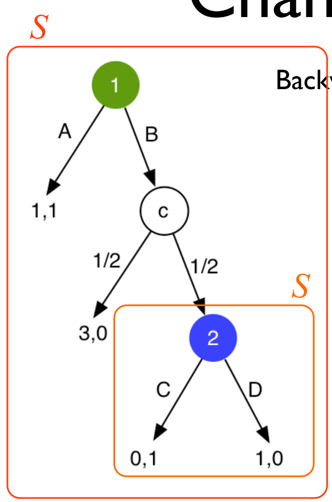
If A is selected then 1 obtains a payoff of 1

If B is selected then 1 obtains a payoff of 3 with probability 1/2 and a payoff of 0 with probability 1/2

Hence an expected payoff of 3/2

44-7

Chance moves



Backward induction:

In sub-game S_2, player 2 chooses C

In sub-game S_1, we consider the consequences of the actions of player 1

If A is selected then 1 obtains a payoff of 1

If B is selected then 1 obtains a payoff of 3 with probability 1/2 and a payoff of 0 with probability 1/2

Hence an expected payoff of 3/2

Thus the SPE is here is (B, C)

44-8

Chance moves

The duel from the good, the bad and the ugly (1966)

Chance moves

The duel from the good, the bad and the ugly (1966)

Chance moves

See the web-site for the first assignment

One of them will be this one ....

Sequential truel
Each of persons A, B, and C has a gun containing a single bullet. Each person, as long as she is alive, may shoot at any surviving person. First A can shoot, then B (if still alive), then C (if still alive).

Denote by p_i the probability that player i hits her intended target; assume that $0 < p_i < 1$. Assume that each player wish to maximize her probability of survival; among outcomes in which her survival probability is the same, she wants the danger posed by any other survivors to be as small as possible.

Model this situation as an extensive game with perfect information and chance moves. (Draw the diagram. Note that the sub-games following histories in which A misses her intended target are the same).
Find the subgame perfect equilibria of the game. (Consider only cases in which p_A, p_B, and p_C are all different.) Explain the logic behind A's equilibrium action. Show that "weakness is strength" for C: she is better off if $p_C < p_B$ than if $p_C > p_B$.

Imperfect information

- Often players do not know the preferences of their opponents
- or they may not know how well the opponent knows their preferences
- **Bayesian games** allows one to analyze any situation in which a player is not completely informed about an environmental aspect that may be relevant for her choice of action

Bayesian games

Consider another variant of the **battle of the sexes:**

player 1 is unsure whether player 2 prefers to go out with her or prefers to avoid her.

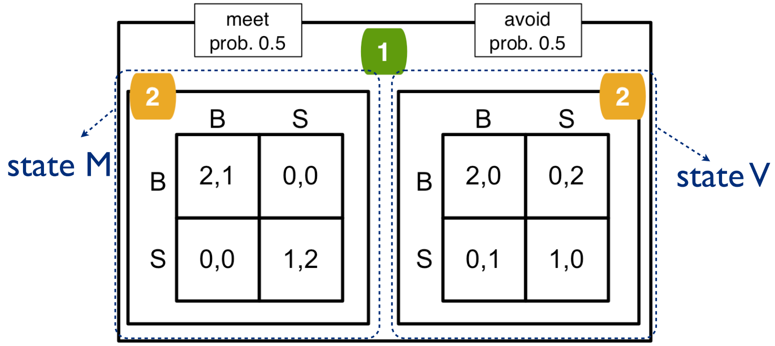
Let's assume that there is equal chance for both (which can be based on player 1's personal assessment)

So player 1 beliefs that with probability 1/2 she plays two different games

Player 2 knows which of the two games is being played.

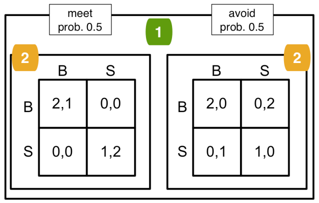
Bayesian games

These are the two potential games:



What are the equilibria of this kind of game?

Bayesian games



What are the equilibria of this kind of game?

Player 1 needs to form a belief about which kind of actions player 2 can take when he is either one of the types

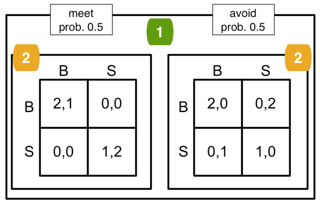
Belief of player 1; When being the left type player 2 will play B

When being the right type player 2 will play S

Then $\pi_B = 0.5 * 2 + 0.5 * 0 = 1$

$\pi_S = 0.5 * 0 + 0.5 * 1 = 0.5$

Bayesian games



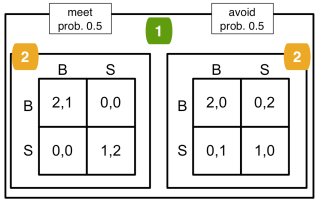
What are the equilibria of this kind of game?

We can now calculate the expected payoff for all possible combinations of player 2's types

action for both types

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0,5	0,5	1

Bayesian games



Definition a pure strategy Nash equilibrium is a triple of actions, with the property that

The **action of player 1 is optimal**, given both actions of the two player 2 types (and player 1's belief about the state)

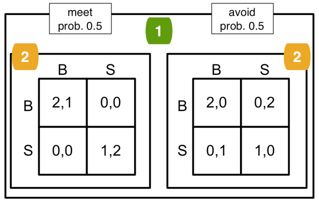
The **action of each player 2 type is optimal**, given the action of player 1

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0,5	0,5	1

First, player 1 has to determine his or her best response given the actions of both types

52-1

Bayesian games



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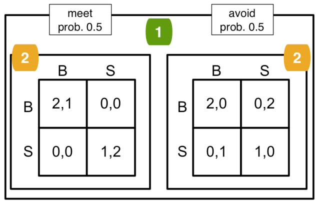
	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0,5	0,5	1

First, player 1 has to determine his or her best response given the actions of both types

52-2

Bayesian games



	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0,5	0,5	1

best response of player 1

Second, determine the best responses of player 2 against player 1 in both types

Type "meet"

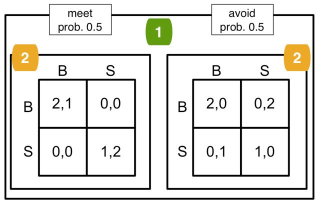
1	0
0	2

Type "avoid"

0	2
1	0

53-1

Bayesian games



	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0,5	0,5	1

best response of player 1

Second, determine the best responses of player 2 against player 1 in both types

Type "meet"

1	0
0	2

Type "avoid"

0	2
1	0

53-2

Bayesian games

meet prob. 0.5 | 1 | avoid prob. 0.5

2

	B	S
B	2,1	0,0
S	0,0	1,2

2

	B	S
B	2,0	0,2
S	0,1	1,0

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0,5	0,5	1

best response of player 1

Second, determine the best responses of player 2 against player 1 in both types

Type "meet"

1	0
0	2

Type "avoid"

0	2
1	0

Bayesian games

meet prob. 0.5 | 1 | avoid prob. 0.5

2

	B	S
B	2,1	0,0
S	0,0	1,2

2

	B	S
B	2,0	0,2
S	0,1	1,0

	(B,B)	(B,S)	(S,B)	(S,S)
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Bayesian games

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Bayesian games

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	B	S
B	2,1	0,0
S	0,0	1,2

2

	B	S
B	2,0	0,2
S	0,1	1,0

	(B,B)	(B,S)	(S,B)	(S,S)
B	$\boxed{2}$	1	1	0
S	0	0,5	0,5	1

best response of player 1

The pure strategy Nash equilibria can now be determined by matching these results

Type "meet"

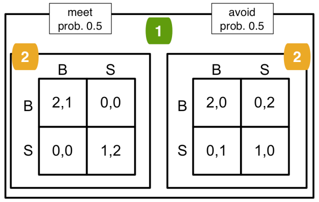
$\boxed{1}$	0
0	2

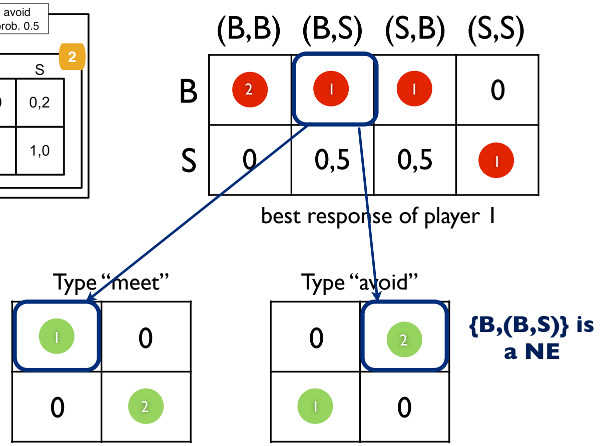
!

Type "avoid"

$\boxed{0}$	2
1	0

Bayesian games

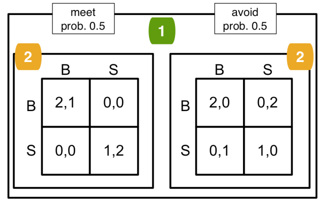


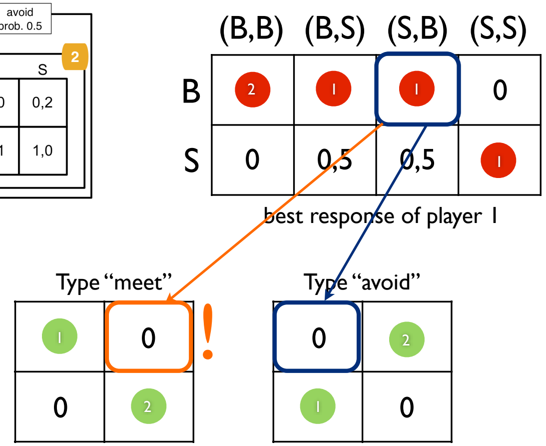


The pure strategy Nash equilibria can now be determined by matching these results

55

Bayesian games

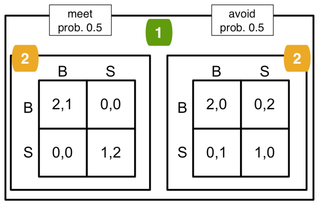


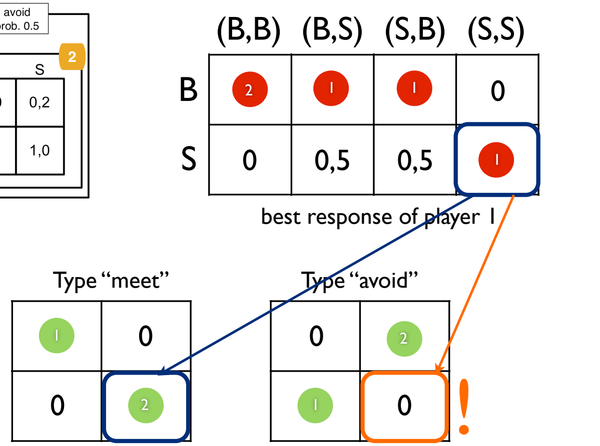


The pure strategy Nash equilibria can now be determined by matching these results

56

Bayesian games



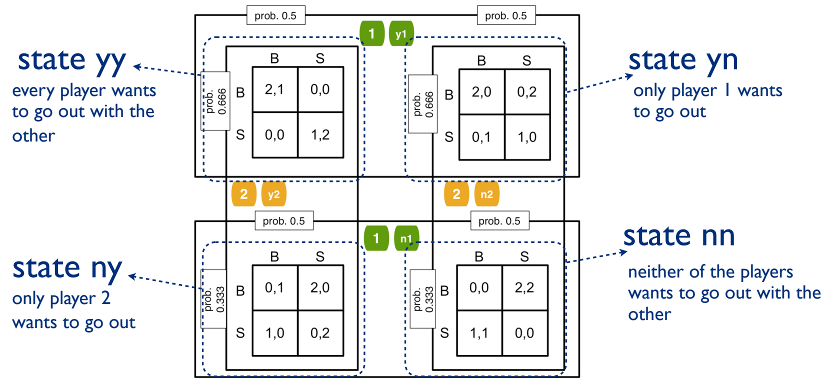


The pure strategy Nash equilibria can now be determined by matching these results

57

Bayesian games

We can make the game even more interesting when both players don't know whether the other one wants to meet or avoid the other one



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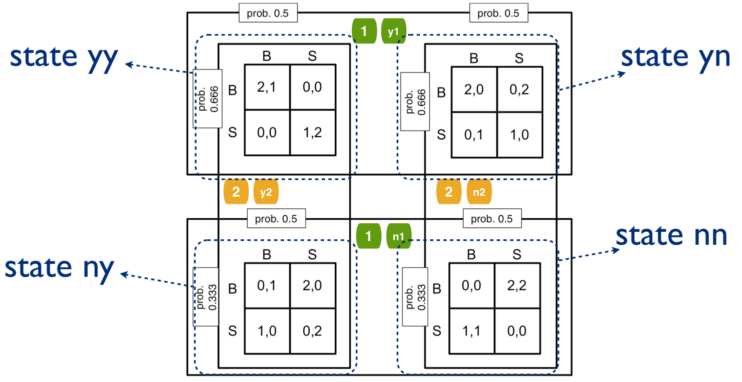
Bayesian games

Note that in this game, player 1 cannot distinguish between states yy and yn, and between ny and nn (vice versa for player 2)



59-1

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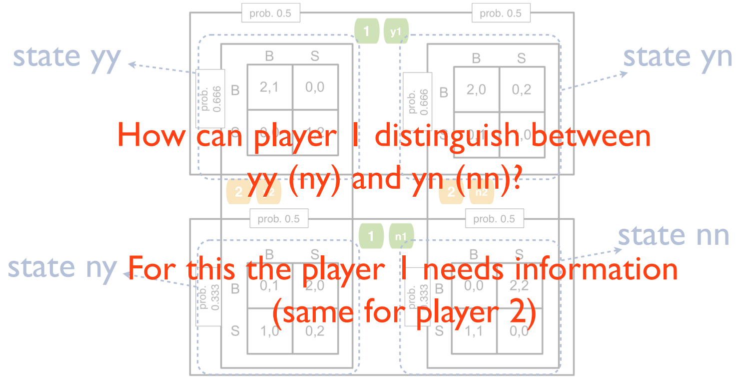
Bayesian games

Note that in this game, player 1 cannot distinguish between states yy and yn, and between ny and nn (vice versa for player 2)



How can player 1 distinguish between yy (ny) and yn (nn)?

For this the player 1 needs information (same for player 2)

59-2

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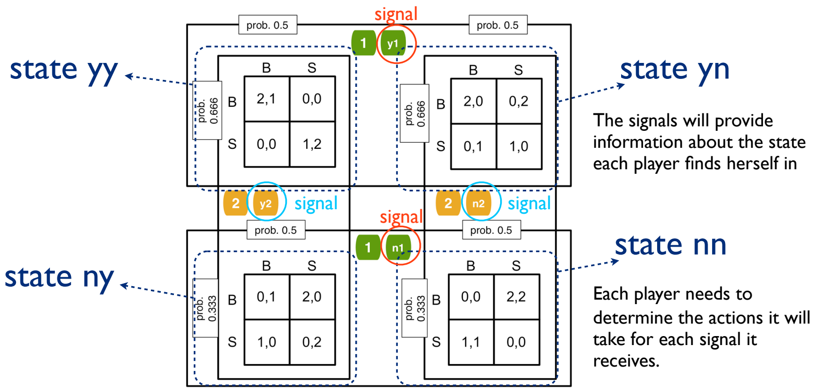
Bayesian games

We can make the game even more interesting when both players don't know whether the other one wants to meet or avoid the other one



The signals will provide information about the state each player finds herself in

Each player needs to determine the actions it will take for each signal it receives.

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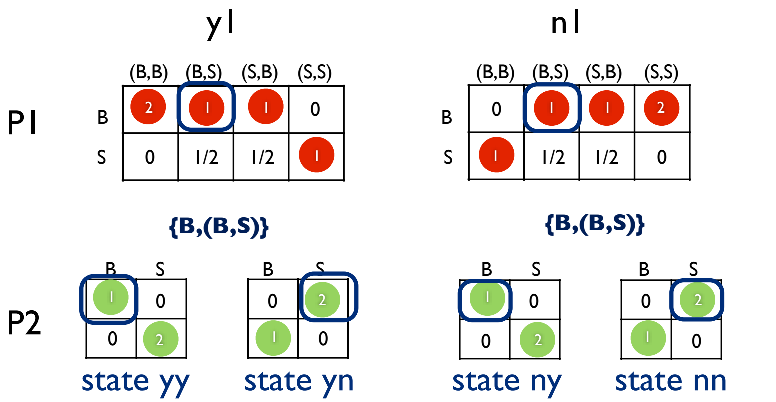
Bayesian games

Depending on the signal, each player has to determine what he can expect in terms of payoff, given the probabilities of each state.



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Bayesian games

Depending on the signal, each player has to determine what he can expect in terms of payoff, given the probabilities of each state.

P1

state yy

B	2	0
S	0	1

state ny

B	0	2
S	1	0

{(B,S),B} and {(S,B),S}

state yn

B	2	0
S	0	1

state nn

B	0	2
S	1	0

{(S,B),S}

P2

y2

	(B,B)	(B,S)	(S,B)	(S,S)
B	1	2/3	1/3	0
S	0	2/3	4/3	2

n2

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	1/3	2/3	1
S	2	4/3	2/3	0

Bayesian games

Putting things together, one can see that there 2 pure strategy Nash equilibria for this bayesian game

	y1	n1	
P1	{B,(B,S)}	{B,(B,S)}	→ **{(B,B),(B,S)}**
P2	{(B,S),B} and {(S,B),S}	{(S,B),S}	→ **{(S,B),(S,S)}**
	y2	n2	

Bayesian games

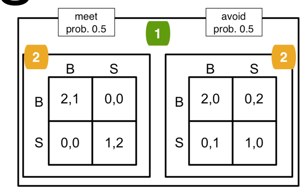
Definition: A Bayesian game consists of

A set of **players**
A set of **states**

And for each player

A set of **actions**
A set of **signals** that she may receive
A **signal function** that associates a signal with each state
for each signal
a **belief** about the states consistent with the signal
A **payoff function over pairs (a, ω)** where a is an action profile and is a ω state

Bayesian games



Players: 1 and 2

States: meet and avoid

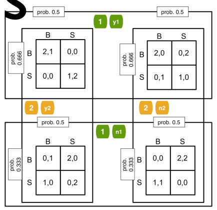
Actions: for each player {B,S}

Signals:
For an assumed signal z, the signal function of 1 is τ_1(meet)=τ_1(avoid)=z
Player 2 receives one of two signals, m or v. The signal function is τ_2(meet)= m and τ_2(avoid)=v

Beliefs:
Player 1 assigns a probability of 1/2 to each state when receiving the signal z
Player 2 assigns a probability 1 to state meet when receiving signal m and a probability of 1 to state avoid when receiving signal v

payoffs:
the payoff $u_i(a,$ meet$)$ for each player are given by the first matrix in the figure and the payoff $u_i(a,$ avoid$)$ for each player are given by the second matrix in the same figure

Bayesian games



Players: 1 and 2

States: {yy, yn, ny, nn}

Actions: for each player {B,S}

Signals:
Player 1 receives two signals, y_1 or n_1; the signal function is $\tau_1(yy)=\tau_1(yn)=y_1$ and $\tau_1(ny)=\tau_1(nn)=n_1$
Player 2 receives two signals, y_2 or n_2; The signal function is $\tau_2(yy)=\tau_2(ny)=y_2$ and $\tau_2(yn)=\tau_2(nn)=n_2$

Beliefs:
Player 1 assigns a probability of 1/2 to the states yy and yn when receiving the signal y_1 and the probability 1/2 to the states ny and nn when receiving n_1
Player 2 assigns a probability 2/3 to state yy and 1/3 to state ny when receiving signal y_2 and the probability of 2/3 to state yn and 1/3 to state nn when receiving signal n_2

payoffs:
the payoff $u_i(a, \omega)$ for each player i for all possible action pairs and states as provided by the figure above.

Bayesian games

- These utility function can now be used to formally define the notion of a pure strategy Nash equilibrium
- This can be further extended towards extensive form games, signaling games, ...
- maybe next year :)