Nonlinear Inference of Unconventional Monetary Policy *

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Abstract

This paper explores U.S. business-cycle dynamics of the post-volcker period & investigates the consequences of monetary policy in a nonlinear DSGE model. Equipped with the particle filter algorithm and fed fund futures data, I estimate a regime switching model against U.S macro-data from 1990Q1-2019Q4. The novelty of this paper comes from two key components. First, I allow households in the economy to derive utility from wealth and consumption. Second, I enable agents to learn the regime switching equilibrium that arises when monetary policy is constrained at the zero lower bound. Through counterfactual analysis, I find more aggressive long term asset purchases during the Great recession and Dot-com recession lead to over 1% higher output growth each quarter, with a close to a 0% net change in Inflation. I find that investment and demand shocks make up over a 20% contribution in explaining U.S. GDP growth, while expectations of future wealth amplify the effects of business cycles. Lastly, I find for a given shock, output,inflation, and consumption are more negative on impact at the zlb, than the non-zlb, suggesting policy ought to guide expectations away from the zlb via bond purchase & interest rate policy.

*William Branch has given structure and guidance

1 Introduction

Over the last decade, central banks across the developed world have expanded to a broader set of policy tools. These actions, have been in part due to the ineffectiveness of reduced interest rates to stabilize output gap. A number of papers attribute this occurrence to notions of pessimism or uncertainty[1] and address the ZLB constraint by advocating measures like forward guidance[2] & large scale asset purchases[3]. Yet despite the existing literature, it is still not clear what the source of business cycles are or how best to conduct policy monetary policy given what could be an increasingly occurring Zero lower bound regime.[4] In this paper, I seek to answer two central questions: To what extent does monetary policy influence macro-variables in both the zlb and non-zlb regime? What does a New Keynesian model with learning and wage/investment/price frictions imply are the key sources of business cycle fluctuations?

In the standard New-Keneysian models, household savings decisions depend on the trade-off between the utility of consuming today versus tomorrow. However, in light of recent evidence[5], saving decisions not only depend on the trade off between present to future consumption, but the utility derived from wealth accumulation as well. Hence, asset demand depends not only on the utility of future consumption but the marginal utility of wealth accumulation as well. Thus, by appending the standard utility model, I examine the effects expected wealth accumulation on other variables such as inflation and output growth. Furthermore, most models that describe the economy often assume Rational Expectations, that is they assume agents form expectations with perfect foresight of how the economy evolves and with full information at time t when forming expectations of $t+1$ variables and onward. In light of contrary evidence[6], I take the view, in line with Evans and Honkapohja (2001),

[1] Benhabib et al.(2012) ;Planter et al.(2015)

[2] Woodford(2019);Swanson(2018)

[3] Chen(2011); Kim et al.(2020)

[4] Macroeconomics with Financial Frictions: A Survey(2011) & Brunnermeier. et. al. Forward Guidance, Monetary Policy Uncertainty, and the Term Premium Bundick et. al. (2021)

[5] A New Keynesian model with wealth in the utility function (Seaz & Michaillat 2018)

[6] See Milani(2006)

that agents learn about the economy over time and change their belief parameters in real time based on new observations each period. Equipped with this more realistic framework, I am able to incorporate how perceptions of future economic conditions and asset prices can exacerbate business cycles. When examining the zlb and non-zlb equillbria, perceptions of these equillbria are crucial toward explaining the effectiveness(or ineffectiveness) of policy to address slumps in output gap or inflation. Hence, in this paper, I take the approach that agents 'learn' about the economy over time via least squares learning, and update their beliefs over the zlb and non-zlb equillbria and the economy as new data is available to agents. The paper is outlined as follows. First, I describe the agents in the Economy and the corresponding equations governing the economy. Second, I discuss the empirical estimation procedure and highlight key differences in alternative hypotheses. Third, I compare the parameter results and model fit. Lastly, I discuss model results and what this means for policy and our understanding of business cycles moving forward.

2 Model Environment

We have the following agents in the economy: households, firms, fiscal authority, a labor-union, and a central bank. Households consume based on an anticipated utility framework[7] where they take into account expectations of their future wealth net of taxes. They earn labor income, and investment income from corporate & government issued debt. Often argued to more accurately reflect household saving decisions[8], households price assets based on both the utility of expected future consumption and wealth. Each period firms produce goods by employing labor and raising capital. Firms increase thier capital stock by issuing corporate debt to households who earn labor income. Firms determine their optimal labor demand and corporate debt supply by maximizing flow profits subject to an exogenous investment friction and productivity process. I assume a representative final goods firm aggregates the

[7]Cogley & Seargant (2005)

[8]A New Keynesian model with wealth in the utility function(Seaz & Michaillat 2018)

j intermediate firm's good via CES preferences & a competitive market. All j firms exist in a monopolistically competitive environment. Wages are set by unions and prices are set by intermediary firms such that they hold a degree of market power and are subject to a probabilities θ_w, θ_p, respectively, of being unable to optimize. The fiscal authority levies a lump sum tax on households, and issues both short & long term debt. Lastly, Monetary policy follows a lagged Taylor rule subject to a zero lower bound constraint.

2.1 Representative Household:

To get the traditional Euler equation for consumption and the anticipated utility model, I assume the household has additive preferences. I also assume the household derives utility from wealth through longer term issuance $b_t, b_t^k(j)$ and not from the one period short term government bond b_t^s. This set up is motivated in part to reflect the adjustment cost in scaling down long term debt for short term debt.

$$U(c_t, b_t, b_k^k, L_t) = \frac{c_t^{1-\delta}}{1-\delta} + \sum_j \frac{(q_t^k(j)b_t^k(j))^{1-\eta}}{1-\eta} + \frac{(q_t b_t)^{1-\eta}}{1-\eta} - \frac{L_t^{1+\psi}}{1+\psi} \tag{2.1}$$

$$\begin{aligned} c_t + b_t q_t + \sum_j b_t^k(j) q_t^k(j) + b_t^s = \pi_t^{-1}\{(1+\rho q_t)b_{t-1} + \sum_j (1+\rho_k q_t^k(j)) b_{t-1}^k(j)\} \\ + \pi_t^{-1} b_{t-1}^s (1 - R_{t-1}) + W_t(l)N_t(l) - T_t \end{aligned} \tag{2.2}$$

$$q_{t-1} = 1 + \rho q_t \tag{2.3}$$

$$q_{t-1}^k(j) = 1 + \rho^k q_t^k(j) \;\; \forall \text{j} \tag{2.4}$$

The household derives utility from consumption and bond wealth via CRRA preferences. They gain utility from the market value of $q_t^k(j)b_t^k(j)$ issued from each firm j, and long-term government issued bonds, $q_t b_t$. Equation (2.2), represents the household's budget constraint in which the total purchases of bonds must equal the return from the prior period plus labor income net of lump sump tax T_t levied by the fiscal authority. Note, b_t^s represents one period short term debt the household collects R_t^s from investing into. Households maximize the following:

$$\max E_t \sum_{k=0}^{\inf} \beta^k U_{t+k} \text{ s.t. (2.1-2.4)} \tag{2.5}$$

First order conditions yield:

$$\frac{\partial \mathcal{L}}{\partial c_t} : c_t^{-\delta} = \lambda_t \tag{2.6}$$

$$\frac{\partial \mathcal{L}}{\partial b_t^s} : -\lambda_t + \beta E_t \lambda_{t+1} \pi_{t+1}^{-1}(1 + R_t) = 0 \tag{2.7}$$

$$\frac{\partial \mathcal{L}}{\partial b_t} : (q_t b_t)^{-\nu} - \lambda_t + \beta E_t \lambda_{t+1} \pi_{t+1}^{-1}(1 + \rho E_t q_{t+1}) = 0 \tag{2.8}$$

$$\frac{\partial \mathcal{L}}{\partial b_t^k(j)} : (q_t^k(j) b_t^k(j))^{-\nu} - \lambda_t + \beta E_t \lambda_{t+1} \pi_{t+1}^{-1}(1 + \rho_k E_t q_{t+1}^k(j)) = 0 \tag{2.9}$$

$$b_t(i) = q_t^{-1}(i)\left(\frac{1}{\lambda_t - \beta E_t(\lambda_{t+1} \pi_{t+1}^{-1}(1 + \rho_i))}\right)^{\eta} \tag{2.10}$$

Above we see by rearranging (2.8-2.9), we obtain (2.10). Which tells us the household, all else held constant, will increase its bond holdings, if the expected discounted return is higher or if the price of purchasing the bond is lower.

After summing across the household's j different corporate bond purchases[9] and log linearizing around the steady state[10] , I obtain the following:

$$c_t = E_t c_{t+1} + \delta^{-1}(E_t \pi_{t+1} - R_t) \tag{2.11}$$

$$\nu b_t^d = -(d_1 + \nu) q_t + \delta d_1 c_t + d_2 \rho E_t q_{t+1} - d_2 E_t c_{t+1} - d_2 E_t \pi_{t+1} \tag{2.12}$$

$$\nu b_t^{d,k} = -(d_1 + \nu) q_t^k + \delta d_1 c_t + d_2 \rho E_t q_{t+1}^k - d_2 E_t c_{t+1} - d_2 E_t \pi_{t+1} \tag{2.13}$$

We see that consumption yields the standard Euler equation rule. However for long term government and corporate bond prices (2.12-2.13), the bond demand will decrease given an increase in the price. A standard supply/demand relationship one would naturally expect.

[9] See Appendix for Details

[10] $\bar{\pi} = 2\%, \bar{Y} = 3, \bar{C} = 6.5\%$

2.2 Monetary & Fiscal policy:

$$R_t = \max\{R^p_{t-1} Y_t^{(1-\rho)\phi_x} \Pi_t^{(1-\rho)\phi_\pi}, 0\} \tag{2.14}$$

$$T_t = (\frac{B_t + B^s_t}{B_{t-1} + B^s_{t-1}})^{\phi_b} \tag{2.15}$$

Log Linearizing the Policy Equations Yield:

$$R_t = \max\{\rho R_{t-1} + (1-\rho)(\phi_x y_t + \phi_\pi \pi_t) + \epsilon^i_t, 0\} \tag{2.16}$$

$$\tau_t = \phi_b(b_{t-1} + b^s_{t-1}) + \epsilon^\tau_t \tag{2.17}$$

The fiscal authority issues short and long maturity debt, B^s_t and B_t, respectively. Hence the fiscal authority's budget constraint evolves according to:

$$B_t + B^s_t = \pi_t^{-1}(R^T_t)(B_{t-1} + B^s_{t-1}) - T_t \tag{2.18}$$

The above equations state that the fiscal authority issues short B^s_t and long term B_t debt subject to constraint (2.18) and levies tax T_t based on (2.14). Assuming long term debt issuance is expressed as (2.23), the fiscal authority will issue short term debt based on the remaining outstanding balance obtained from (2.18-2.20). Log linearized, the fiscal authority's interest rate on total debt is:

$$\nu_t \equiv \frac{B_t}{B_t + B^s_t} \tag{2.19}$$

$$R^T_t \equiv \nu_{t-1}\{\frac{1 + \rho q_t}{q_{t-1}}\} + (1 - \nu_{t-1})(1 + R^s_t) \tag{2.20}$$

After linearizing the both policy rule and government budget constraint, I obtain:

$$b_t + b^s_t = b_{t-1} + b^s_{t-1} + (\frac{R^T \pi^{-1}}{\bar{B} + \bar{B}_s})(R^T_t - \pi_t) + (\frac{T}{\bar{B} + \bar{B}_s})\tau_t \tag{2.21}$$

I assume long term debt B_t issuance evolves according to:

$$q_t B_t = (q_{t-1} B_{t-1})^{\phi_s} S_t \tag{2.22}$$

Log Linearized, this becomes:

$$q_t + b_t = \phi_s(q_{t-1} + b_{t-1}) + s_t \tag{2.23}$$

$$\text{where: } s_t = \rho_s s_t + \epsilon_t^s \tag{2.24}$$

Note s_t represents the Federal reserve and Fiscal Authority's decision to swap short term to longer term debt. Throughout the paper, I denote this channel, Large Scale Asset Purchases, or LSAP.

A note on indifference of bonds:

One point of concern is that when we study an economy where we assume a no-abritrage condition, one would assume households have no risk-adjusted preference between long and short term government debt. And as a consequence, large scale asset purchases would be rendered irrelevant if households are able to costlessly shift from short to long term bonds, given a sudden swap on debt of differing maturities.[11] However, with similar reasoning to Chen(2012), government issued bonds of different maturities are not perfect substitutes. In equation (2.1) because the household values longer term bonds but not short term bonds in the utility function, if the household were to swap long for short term debt, doing so would incur a dis-utility wealth, and is unable to costlessly adjust asset holdings. Hence, for the purposes of evaluating the effectiveness of long term debt purchases, s_t, we avoid the irrelevance pitfall, given that the household cannot re-balance its portfolio of bond holdings without incurring a utility cost.

2.3 Firm Production Problem:

Each period, all j firms decide on their choice of investment via corporate bond issuance($b_t^k(j)$) and Labor to employ. When doing so, they face investment adjustment costs and a stochastic shock to Investment. Note: because all j firms are making similar decisions & face the same shock realizations, I drop the index j and obtain the aggregate law of motion for capital,

[11] Woodford & Eggertson(2003), Wallace(1981)

investment, and Tobin's Q_t.

The Firms have the following technology, with productivity A_t:

$$f_t(j) = e^{A_t} K_t^{\alpha}(j) L_t^{1-\alpha}(j) \tag{2.25}$$

Each period they carry capital net of depreciation from the previous period. During business cycles, it is often the case that investment becomes increasingly difficult and more costly to obtain.[12] Hence to account for this, I assume firms are subject to an exogenous capital investment depletion shock ζ_t that represents any event which restricts the investment formation process in the economy:

$$K_t(j) = (1-\delta)K_{t-1}(j) + I_t(j)e^{\zeta_t} \tag{2.26}$$

Investment is raised through the market value of corporate debt:

$$I_t(j) = q_t^k(j) b_t^k(j) \tag{2.27}$$

Firm Productivity follows:

$$A_t = \rho_a A_{t-1} + \epsilon_t^{\alpha} \tag{2.28}$$

The Capital investment shock follows:

$$\zeta_t = \rho_\zeta \zeta_{t-1} + \epsilon_t^{\zeta} \tag{2.29}$$

When changing the level of investment, firms often face adjustment costs either implicitly or explicitly to do so.[13] Hence, to account for such frictions, I assume the firm's investment adjustment cost is:

$$\Phi_t = \frac{\eta}{2}(\frac{I_t}{I_{t-1}} - \sigma)^2 \tag{2.30}$$

Firms discount future profits by the household marginal utility of consumption via:

$$\Lambda_t = c_t^{-\delta} \tag{2.31}$$

Lastly, Firms solve the following problem:

$$\begin{aligned}\max_{N_t, I_t} \sum_{\tau}^{\inf} \beta^\tau \Lambda_{t+\tau}\{ & f_{t+\tau} - w_{t+\tau}N_{t+\tau} - \pi_t^{-1}(1+\rho_k q_t^k)b_{t-1}^k \\ & - \Phi(I_{t+\tau}, I_{t+\tau-1})I_{t+\tau} + Q_{t+\tau}(I_{t+\tau}\zeta_{t+\tau} + (1-\delta)K_{t+\tau-1} - K_{t+\tau})\}\end{aligned} \tag{2.32}$$

[12] Justiniano et. al(2008)
[13] Groth and Khan(2007)

We see from (2.32), the firm seeks to maximize its discounted profits subject to the capital constraint and adjustment costs. Here the lagrange multiplier, $Q_{t+\tau}$ can be expressed in closed form when maximizing with respect to capital and investment.

First order conditions by the firm yield:

$$\frac{\partial \mathcal{L}}{\partial K_{t+\tau}} : E_t\{\Lambda_{t+\tau}\{f'_{t+k} - Q_{t+\tau}\} - \beta\Lambda_{t+\tau+1}\{(1-\delta)Q_{t+\tau+1}\}\} = 0 \tag{2.33}$$

The above equation represents the marginal profits a firm recieves from an additional unit of capital, or Tobin's Q[14]. Q_t in a sense represents a firms willingness to issue capital based on the current and expected economic conditions.

$$\frac{\partial \mathcal{L}}{\partial I_{t+\tau}} : E_t\{\Lambda_{t+\tau}\{-\pi_{t+1}^{-1}R_{t+\tau+1}^k - \Phi'_{t+\tau}I_{t+\tau} - \Phi_{t+\tau} + Q_{t+\tau}\zeta_{t+\tau}\} + \beta\Lambda_{t+\tau+1}\{-\Phi'_{t+\tau+1}I_{t+\tau+1}\}\} = 0 \tag{2.34}$$

$$\frac{\partial \mathcal{L}}{\partial N_{t+\tau}} : N_t^{\alpha} = A_t K_t^{\alpha}(1-\alpha)W_t^{-1}$$

(2.35)

Log-Linearized, this becomes:

$$\nu_1 E_t R_{t+1}^k - \nu_2\pi_{t+1} + (\nu_3 - \nu_4)I_{t-1} + \nu_5 Q_t + \nu_6\zeta_t + \delta(E_t c_{t+1} - c_t) = \{\frac{3\eta - 2\delta}{\eta - \delta} - \nu_4\}I_t \tag{2.36}$$

$$Q_t = \delta(a_t + (\alpha-1)k_t + (1-\alpha)n_t) + (1-\sigma)(\delta E_t c_{t+1} - c_t + E_t Q_{t+1}) \tag{2.37}$$

$$K_t = (1-\delta)K_{t-1} + \delta(q_t^k + b_t^k + \zeta_t) \tag{2.38}$$

$$\alpha N_t = a_t + \alpha K_t - w_t \tag{2.39}$$

After obtaining the linearized firm decision for labor and capital, we can see in equation (2.36), the level of investment depends on the future expected cost of borrowing, R_{t+1}^k. If

[14] Blundel et al.(1992)

raising capital is expected to become more costly, then firms are will raise more now to avoid paying higher costs in the future. Hence if we wish to understand perceptions of asset prices and their impact on the real economy, there is both a supply driven component from firms who raise capital, and demand driven component from households who consume based on expected wealth as we will see in the modified IS curve.

2.4 Firm Pricing:

I Assume firms follow Calvo pricing with probability θ_p of being unable to change prices. Hence, firms price each period by solving:

$$\max_{p_t^*} \quad E_t[\sum_{k=0}^{\inf}(\theta\beta)^k \Lambda_{t+k} A_{t+k}] \tag{2.40}$$

$$A_{t+k} \equiv (\frac{p_t^*}{P_{t+k}})^{-\epsilon} Y_{t+k} - (\frac{p_t^*}{P_{t+k}})^{-\epsilon} (\frac{\Phi_{t+k}}{P_{t+k}}) Y_{t+k} \tag{2.41}$$

Aggregate Price Identity:

$$P_t^{1-\epsilon} = [(1-\theta)(p_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}] \tag{2.42}$$

$$\pi_t = (1-\theta)(1-\beta\theta)(\phi_t + \alpha_\pi^{-1}\delta c_t + \beta\theta E_t \pi_{t+1}) \tag{2.43}$$

With the follow New-Keynesian Phillips Curve, we see inflation is driven by aggregate firm marginal cost of production along with consumption and expected inflation in the next period.

2.5 Labor Union Wage Setting:

To introduce wage stickiness in the model, I assume that there exists a labor union which aggregates labor via CES preferences and leases out labor to firms in a competitive market. Hence, the labor union faces the following problem:

$$\psi_w \equiv \frac{\epsilon_w}{\epsilon_w - 1} \tag{2.44}$$

$$\max_{N_t(l)} W_t \left(\int_0^1 N_t^{\psi_w}(l) dl \right)^{\psi_w^{-1}} - \int_0^1 W_t(l) N_t(l) dl \tag{2.45}$$

Solving this yields:

$$N_t(l) = (\frac{W_t(l)}{W_t})^{-\epsilon_w} N_t \tag{2.46}$$

When households decide their labor supply, they take the above result from the union as given. Consequently, each household solves for their optimal labor supply via their optimal wage. In order to induce frictions in the wage setting process, I assume the household has probability θ_w of keeping the wage they set in the previous period[15]. Hence, the household optimizes wage considering this friction and seeks to maximize their discounted utility via the following problem:

$$\max_{w_t^*} \sum \Lambda_{t+k} \theta_w^k B_{t+k} \tag{2.47}$$

$$B_{t+k} = (-\frac{N_{t+k}^{1+\psi}}{1+\psi})(\frac{W_t^*}{W_{t+k}})^{-\epsilon_w(1+\psi)} + \lambda_{t+k} W_{t+k}(l) N_{t+k}(l) \tag{2.48}$$

$$N_{t+k}(l) = (\frac{W_t^*}{W_t})^{-\epsilon_w} N_t) \tag{2.49}$$

$$\frac{\partial}{\partial W_t^*} : (W_t^*)^{1-\alpha_0-\epsilon_w} = \sum (\beta\theta_w)^k \psi_w \{ \frac{N_{t+k}^{1+\psi} W_{t+k}^{\alpha_0}}{W_t^{\epsilon_w} N_{t+k} \lambda_{t+k}} \} \tag{2.50}$$

Using the Expression for Aggregate Wages:

$$W_t^{1-\epsilon_w} = (1-\theta_w)(W_t^*)^{1-\epsilon_w} + \theta_w W_{t-1}^{1-\epsilon_w} \tag{2.51}$$

[15]See Calvo(1983)

And after Log-Linearizing, I obtain:

$$w_{t+1} = (1-\theta_w)\alpha_1 \sum(\beta\theta_w)^k E_t\{(O^1_{t+k} - O^2_{t+k})\} + \theta_w w_{t-1} + \theta_w w_t \tag{2.52}$$

$$O^1_t = (1+\psi)n_t + \alpha_0 w_t \tag{2.53}$$

$$\begin{aligned} O^2_t &= \epsilon_w w_t + n_t + \lambda_t(1+\theta_w+\epsilon_w-\alpha_0)w_t \\ &= (1-\theta_w)\alpha_1\psi n_t - (1-\theta_w)\alpha_1\lambda_t + E_t w_{t+1} + \theta_w w_{t-1} \end{aligned} \tag{2.54}$$

$$(1+\theta_w)w_t = E_t w_{+1} + O^1_t - O^2_t + \theta_w w_{t-1}$$

2.6 Deriving Marginal Cost:

In order to derive the aggregate marginal cost of all firms, I can re-express the firm production problem as a cost-minimization problem. Expressed below, firms can solve the following:

$$\min_{N_{t+k},K_{t+k}} \sum \Lambda_{t+k}\{W_{t+k}N_{t+k} + \Phi(I_{t+k}, I_{t+k-1})I_{t+k} + (1+\rho^k)q^k_t - \Phi_{t+k}(Y_{t+k} - A_{t+k}K^{\alpha}_{t+k}N^{1-\alpha}_{t+k})\} \tag{2.55}$$

$$\frac{\partial}{\partial N_{t+k}} : W_{t+k} = MC_{t+k}(1-\alpha)K^{\alpha}_t N^{-\alpha}_t A_t \tag{2.56}$$

Log-Linearized, this becomes(note, I express the marginal cost as ϕ_t):

$$\phi_t = (\alpha)n_t - \alpha K_t - a_t + w_t + u_t \tag{2.57}$$

$$u_t = \rho_u u_{t-1} + \epsilon^u_t \tag{2.58}$$

Where u_t represents an exogenous cost push shock.

2.7 Deriving IS-Curve:

With contributions from Woodford(2013) & Preston(2018), it is clear that when the rational expectations assumption is relaxed, the standard household model ought to be modified. More specifically, if households in the face of uncertainty form imperfect expectations, they can be shown to consume based on the path of future taxes and debt accumulation on the part of firms and the fiscal authority. Hence, I incorporate household perceptions of the fiscal authority's infinite horizon of real discounted revenue and debt into the optimal consumption decision. First, I express the infinite discounted wealth as a "composite variable" (as Woodford coins), and later use this in the consumption decision equation. By re-arranging the aggregate household budget constraint (2.2), I am able to express the following:

$$\begin{aligned} c_t = & -\eta_1(q_t + b_t) - \eta_2(q_t^k + b_t^k) - \eta_3 b_t^s + \eta_4(\rho q_t + b_{t-1}) \\ & + \eta_5(\rho_k q_t^k + b_{t-1}^k) + \eta_6(b_{t-1}^s + i_{t-1}) + \eta_7(w_t + N_t) - \eta_8\tau_t - \eta_9\pi_t \end{aligned} \tag{2.59}$$

Using the Euler Equation for consumption, I am able to express consumption as:

$$E_t\{\sum \beta^k c_{t+k}\} = (1-\beta)^{-1}c_t + \delta^{-1}(1-\beta)^{-1}\beta E_t\{\sum(\beta^k)(i_{t+k} - \pi_{t+k+1})\} \tag{2.60}$$

$$c_t = (1-\beta)\eta_6 b_{t-1}^s + v_t \tag{2.61}$$

$$\begin{aligned} v_t = E_t\{&\beta(\delta^{-1})(\pi_{t+1} - i_t) + (1-\beta)\{-\eta_1(q_t + b_t) - \eta_2(q_t^k + b_t^k)\eta_4(\rho q_t + b_{t-1}) \\ & + \eta_5(\rho_k q_t^k + b_{t-1}^k) + \eta_6 i_{t-1} + \eta_7(w_t + n_t)\} - \eta_8\tau_t + \beta v_{t+1}\} \end{aligned} \tag{2.62}$$

We see above that the household chooses its consumption based on the aggregate income flows conditional on the flow of not only future taxes/expenditures but also upon the perceived trajectory of asset prices, labor income, net of taxes. The flow of real wealth serves an important variable in amplifying business cycle dynamics for a given exogenous shock. Thus, I will henceforth label v_t the 'wealth channel'.

2.8 Market Clearing and Equilibrium:

Assuming all firms behave similarly, and that expectations are the same for each firm and household, the equilibrium can be defined as follows:

$w_t = \sum_j w_t(j)$, $n_t = \sum_j n_t(j)$, $c_t = \sum_j c_t(j)$, $q_t^k = \sum_j q_t^k(j)$, $q_t = \sum_j q_t(j)$, $b_t = \sum_j b_t(j)$, $b_t^k = \sum_j b_t^k(j)$, $b_t^s = \sum_j b_t^s(j)$, $\phi_t = \sum_j \phi_t(j)$, $Q_t = \sum_j Q_t(j)$, $P_t = \sum_j P_t(j)$, $K_t = \sum_j K_t(j)$

The final equation the describes aggregate output:

$$Y_t = C_t + I_t - \phi(\frac{I_t}{I_{t-1}}). \tag{2.63}$$

Log Linearized, this becomes:

$$y_t = c_1 c_t + c_2(I_t - \phi(\frac{I_t}{I_{t-1}})). \tag{2.64}$$

Where $c_1 = \frac{C}{Y}$ and $c_2 = \frac{I}{Y}$.

The equilibrium is the sequence of prices q_t, q_t^k, R_t^s, P_t & allocation of variables described above and their corresponding evolution such that agents are optimizing subject to the corresponding, policy, household, fiscal, and firm level constraints.

2.9 A Note on the Zero lower bound

Because the monetary policy rule departs from the traditional Taylor Rule[16], the model, given most combinations of structural parameters, will not admit a unique, or determinate, solution. Because, I am interested in the exploring a richer set of equillibria, throughout the analysis, I only impose uniqueness for the Non-zlb equilibrium, while allowing indeterminacy in the zlb equilibrium. Though for a given set of parameters, the non-zlb is unique, because the zlb-case may not be unique, it follows then that the regime switching equilibrium need not be unique. In this paper however, I abstract from such issues, and only impose uniqueness for the non-zlb case. For all results, the equillibria obtained for the zlb is the MSV[17] implied solution.

[16] See: Taylor 1992
[17] McCallum(2004)

3 Vector Representation of Equilibrium

After collecting all variables, I can now express the dynamics in the following state space form:

$$Z_t = AZ_{t-1} + BE_t Z_{t+1} + Q\bar{\epsilon}_t \tag{3.1}$$

$$Z_t = \begin{bmatrix} M_t \\ U_t \end{bmatrix} \tag{3.2}$$

$$M_t = \Big[c_t, \pi_t, q_t, q_t^k, b_t^k, v_t, i_t, y_t, k_t, \tau_t, w_t, b_t, b_t^s, \phi_t, Q_t, n_t\Big]' \tag{3.3}$$

Here, in equation (3.1), Z_t represents all variables previously described in the model. Where, M_t is a column vector that includes all endogenous variables. While U_t represents all exogenous AR(1) processes in the macro-economy. I solve for the Rational expectations equilibrium and obtain 2 equilibria: One when $i_t = 0$(Zero lower bound) and one when $i_t \geq 0$ (No Zero lower bound). Hence, after solving this becomes:

$$Z_t = CZ_{t-1} + D\bar{\epsilon}_t \tag{3.4}$$

Where:

$$M_t = C_{1,1}^j M_{t-1} + C_{1,2}^j U_{t-1} + D_1^j \bar{\epsilon}_t \tag{3.5}$$

$$U_t = RU_{t-1} + D_2^j \epsilon_t \tag{3.6}$$

for $j = \{z, n\}$

3.1 Adaptive Learning

We know from a body of literature [18] and in recent years, the difficulty of policy to meet expectations, and that the belief formation process is time varying. [19] Hence, it is imperative

[18] Branch and Evans(2005),Orphanides and Williams (2004)

[19] Evans and Honkapohja (2009)

to examine how beliefs depart from rational expectations, drive business cycles and asset prices. Therefore, I allow for the parameters of the Rational Expectations Equilibrium(REE) to change over time via Stochastic Gradient learning[20] . Ultimately, this framework provides a more realistic dimension[21] and allows for waves of pessimism or optimism to influence the dynamics over time. Below, I display the set of equations that govern the economy in the model and explain the expectations formation process:

I assume that agents form the following beliefs about the economy.

Perceived Law of Motion:

$$M_t = \Lambda_t^m M_{t-1} + \Lambda_t^u U_t + \nu_t^m \tag{3.7}$$

Thus when forming expectations, I obtain:

$$\hat{E}_t M_{t+1} = \Lambda_t^m \hat{E}_t M_t + \Lambda_t^u R U_t \tag{3.8}$$

$$\hat{E}_t M_t = \Lambda_t^z M_{t-1} + \Lambda_t^u U_t \tag{3.9}$$

$$\Lambda_t \equiv \begin{bmatrix} vec(\Lambda_t^z) \\ vec(\Lambda_t^u) \end{bmatrix} \; ; \tag{3.10}$$

$$\Lambda_t^z = \Lambda_{t-1} + g P_z V_t'(\upsilon_t) \text{ if } i_t < 0 \tag{3.11}$$

$$\Lambda_t^n = \Lambda_{t-1} + g P_n V_t'(\upsilon_t) \text{ if } i_t \geq 0 \tag{3.12}$$

$$\upsilon_t \equiv \begin{bmatrix} \hat{E}_t M_t \end{bmatrix} - \begin{bmatrix} M_t \end{bmatrix} ; \; V_t \equiv \begin{bmatrix} I \otimes M_{t-1}' \\ I \otimes U_t' \end{bmatrix} \tag{3.13}$$

after substituting (17) into (15), I obtain: Actual Law of Motion:

$$M_t = A_t M_{t-1} + B_t U_t + Q\bar{\epsilon}_t \tag{3.14}$$

[20] Evans(2010)
[21] Milani 2007

A note on constant gain learning:

In traditional models with learning, agents update beliefs in such a way that over time the changes in beliefs are stable and converge to the REE. This means agents dramatically update beliefs in earlier periods, and as time goes on, the changes to their belief parameters become incrementally smaller. In light of evidence by Branch and Evans(2005) as well as Orphanides and Williams (2004), I take seriously, the view that agents update beliefs each period with equal weight. This constant gain parameter g, determines the degree agents update beliefs based on forecast errors. Throughout the paper, this parameter is the subject of key interest in the proceeding sections.

3.2 Regime Switching Equillibria

To address both the zlb and non-ZLB equillibria, I equip agents with both the zlb and non-zlb parameters. Furthermore, I assume agents update their parameters by forming subjective probabilities. When doing so, I assume that the agent follows a binomial counting model a la' Cogley & Sargent (2005).[22] Hence, the model dynamics are expressed as follows:

Perceived Law of Motion

$$M_t = \Lambda_t^{m,*} M_{t-1} + \Lambda_t^{u,*} U_t + \nu_t^m \tag{3.15}$$

$$\Lambda_t^* \equiv \mu_t \Lambda_t^z + (1 - \mu_t)\Lambda_t^n \tag{3.16}$$

$$\Lambda_t^n = \Lambda_{t-1}^n + g P_n V_t (\upsilon_t)' \tag{3.17}$$

$$\Lambda_t^z = \Lambda_{t-1}^z + g P_z V_t (\upsilon_t)' \tag{3.18}$$

Note, I initialize the Learning Parameters for $t = 0$ as follows:

$$\Lambda_0^z = \begin{bmatrix} vec(C_{1,1}^z) \\ vec(C_{1,2}^z) \end{bmatrix} \tag{3.19}$$

[22]Anticipated Utility and Rational Expectations as Approximations of Bayesian Decision Making

$$\Lambda_0^n = \begin{bmatrix} vec(C_{1,1}^n) \\ vec(C_{1,2}^n) \end{bmatrix} \tag{3.20}$$

$n_{0,0}^0 = 1;\ n_{0,1}^0 = 1;\ n_{1,0}^0 = 1;\ n_{1,1}^0 = 200;$

$$\mu_t = \mathbb{P}(i_t = 0|\mathcal{I}_{t-1}) = \begin{cases} \frac{n_{1,0}^t}{n_{1,1}^t + n_{1,0}^t} & i_t \neq 0 \\ \frac{n_{0,0}^t}{n_{0,1}^t + n_{0,0}^t} & i_t = 0 \end{cases}$$

(3.21)

Hence, when computing the expected future variables, I obtain the following expression:

$$\hat{E}_t M_{t+1} = \Lambda_t^{m,*} \hat{E}_t M_t + \Lambda_t^{u,*} R U_t \tag{3.22}$$

$$\hat{E}_t M_t = \Lambda_t^{z,*} M_{t-1} + \Lambda_t^{u,*} U_t \tag{3.23}$$

<u>Augmented law of motion:</u>

$$M_t = \begin{cases} A^z M_{t-1} + B^z \hat{E}_t M_{t+1} + C^z U_t + D^z \bar{\epsilon}_t & i_t = 0 \\ A M_{t-1} + B \hat{E}_t M_{t+1} + C U_t + D \bar{\epsilon}_t & i_t \geq 0 \end{cases} \tag{3.24}$$

In the new model with zlb dynamics, the subjective probability u_t becomes an important feature driving values for $E_t M_{t+1}$. Consequently, each term $n_{i,j}^t$ represents a function which counts the number of times state i has moved to state j. Hence, when $i_t = 0$ we see this not only plays a role in driving the value of M_t through the non-expectation parameters seen in (3.5), but will also drive how $E_t M_{t+1}$ will evolve over time through Λ_t^*'s evolution.

4 Estimation

In order to properly understand the dynamics of the model in the context to the U.S. economy, I take observable data from the U.S. and match this with the model generated nonlinear state space data. Using equations (3.1)-(3.24) The model is re-expressed as:

$$\begin{bmatrix} Y_t \\ M_t \end{bmatrix} = \tilde{K}_1(\Lambda_t^*(\mu_t), \Theta)M_{t-1} + \tilde{K}_2(\Lambda_t^*(\mu_t), \Theta)U_{t-1} + \tilde{K}_3(\Lambda_t^*(\mu_t), \Theta)\bar{\epsilon}_t \tag{4.1}$$

Above, we see that the variables of interest evolve in a nonlinear fashion. However, to properly empirically identify the likelihood, I must jointly estimate (3.5-3.6) with the learning equations (3.11-3.13) . Therefore the state transition equation and observation equations respectively, are:

$$V_t \equiv \begin{bmatrix} M_t \\ \Lambda_t^* \\ \mu_t \\ U_t \end{bmatrix} = \mathcal{F}(\Theta, Y_{t-1}, M_{t-1}, \Lambda_{t-1}^*, U_{t-1}, \bar{\epsilon}_t);\ \bar{\epsilon}_t \sim \mathcal{N}(0,\ \Psi(\Theta)\ ) \tag{4.2}$$

$$Y_t^{obs} = \bar{M}_1 V_t + \bar{M}_1 V_{t-1} + \epsilon_t^m; \tag{4.3}$$

$$\epsilon_t^m \sim \mathcal{N}(0, I) \tag{4.4}$$

$$\pi(Y_t^{obs}|\Theta, I_{t-1}) = N(Y_t^{obs} - \bar{M}_1 V_{t|t-1} - \bar{M}_1 V_{t-1|t-1}, I) \tag{4.5}$$

4.1 Particle Filter

In most models with learning, researchers hold the learning parameters fixed in order to utilize the Kalman filter[23]. This assumption has been shown to yield lower marginal & log likelihoods along with therefore inaccurate parameter estimates.[24] To overcome this

[23]See: Milani 2002, Hommes,MavroMatris,Ozden(2018)
[24]See: Kirpekar(2020)

challenge, I jointly estimate the learning parameters Λ_t^* and μ_t via a particle filter with re-sampling.[25]The importance of estimating the learning parameters as states can interpreted as follows: When agents form expectations of the future, the stochastic processes which drive macroeconomic variables are characterized by distributions that follow a mean and variance. Though the shocks are independent of the macro-economy, the distribution macroeconomic variables as well as the learning coefficients are endogenous to one another. By including the learning coefficients as part of the state variable, the researcher is in effect taking the stance that the distribution of macro variables are dependent on the distribution of beliefs and vice versa. Notice However, in the conditionally linear model, since we hold the learning parameters Λ_t^* and μ_t fixed, we assume that beliefs are driven by macro variables but not the reverse.

The particle filter simulates the structural error terms with a sufficient number of draws, places weights on the realized number states,and calculates the a weighted likelihood. Often this procedure places high weight on few particles and 0 weight on most particles which, in essence, induces a high variance of the Likelihood function.[26] In order to avoid this issue, I follow Rubio-Ramirez & Villaverde(2007) with the following algorithm:[27]

Step 1(Initialize): Set $e_t^j \sim \bar{\epsilon}_t$.

Step 2(Propagate): $V_t^j = \mathcal{F}(V_{t-1}^j, \epsilon_t^j)$.

Step 3(Evaluate): $w_t^j = \dfrac{P(Y_t|V_{t|t-1}^j, \Theta)}{\Sigma w_t^i}$.

Step 4(Re-sample): $q_i \sim \{w_t\}_{j=0}^J$

and set: $V_{t|t-1}^j = V_t^i$, for all $\{q_i\}_{i=0}^J$

[25]See:Rubio-Ramirez and Villaverde(2007) & Herbst and Schorfheide (2017)

[26]Kitigawa 1996

[27]For Convergence properties of the Likelihood, please refer to Rubio-Ramirez & Villaverde(2007) and Kitigawa(1996)

$$\text{Set } t = t+1; \text{and repeat till } t = T.$$

$$\textbf{Step 5}\text{(Calculate Likelihood): } P(Y^T|\theta) \approx \frac{1}{J}(\prod_{t=1}^{T}(\sum_{j=1}^{J} p(Y_t|w_t^j, V_{t|t-1}^j, Y^{t-1}, \theta))).$$

With the stated procedure, as the number of particles, J becomes greater, the Likelihood converges to the true distribution. However, because one must keep track of the states and their relative values across the sample, there exists a trade-off between and accuracy and computational time of the algorithm. In this paper, I find that the results of estimation do not change much between 1 - 10 thousand particles. Hence I assume J to be 5 thousand, a sufficient approximation to the true Likelihood function of interest in the proceeding results.

4.2 Data Description

In order to properly identify the model and come up with meaningful insights, I append Y_t^{obs} with Survey of Professional Forecasters' and Chicago Mercantile Exchange's Federal funds futures data. Equipped with expectations level data, the model is now better able to pin down equations (3) and (4). Furthermore, I take U.S. Macroeconomic data, notably including: Output gap, Core PCE inflation, FFR, and the level of Federal reserve balances as a measure of LSAPs.[28].

$$Y_t^{obs} = \left[Y_t^m, \Delta x_t^e, R_{t+1}^{e,k}, R_{t+1}^e, \Delta b_{t+1}^e, i_{t+1}^e \right] \tag{4.6}$$

$$Y_t^{obs} = M_1 V_t - M_1 V_{t-1} + M_2 \begin{bmatrix} \hat{E}_t[Y_{t+1}] \\ \hat{E}_t[M_{t+1}] \end{bmatrix} - M_2 \begin{bmatrix} \hat{E}_{t-1}[Y_t] \\ \hat{E}_{t-1}[M_t] \end{bmatrix} + f\{ \begin{bmatrix} \hat{E}_t[q_{t+1}^k] \\ \hat{E}_t[q_{t+1}] \\ \hat{E}_t[q_t] \\ \hat{E}_t[q_t^k] \end{bmatrix} \} + \epsilon_t^e \tag{4.7}$$

$$\epsilon_t^e \sim N(0, \Omega)$$

[28]Details of the Observation equations can be found in the Appendix section

4.3 A note on i_t^e

Because there is no current measure of the market expected FFR level, I follow Swanson & Piazzesi (2008) and claim:

$$i_{t+1} - f_t^n = -rx_{t+1} \tag{4.8}$$

$$i_{t+1}^e = f_t^n + rx_{t+1}^e \tag{4.9}$$

$$i_{t+1}^e = f_t^n + \alpha_n + \beta_n X_t \tag{4.10}$$

$$i_{t+1}^e = f_t^n + \epsilon_t^{e,i} \tag{4.11}$$

Equation (4.10) tells us the expected ffr is equal to the fed futures spot rate plus the expected risk premia. After expressing this in terms of risk factor $\alpha_n + \beta_n X_t$, where X_t represents monthly indicators of macroeconomic conditions(such as non-farm payrolls). One can attain (4.11) which generalizes the risk premium as a low variance measurement error. Because the CME futures data is a 1 month contract, the risk premium for the expected FFR is negligible. Hence, its use is an appropriate measure of FFR sentiment.[29]

The additional matrix, M_2 denotes entries that link changes in observed expectations to the model generated expectations data. furthermore, because the SPF data is for bond returns, I must compute the implied bond returns using asset prices and hence rely on a nonlinear function f for the observation equation. The strength from this specification comes from being able to extract the latent, or unobservable variables of interest. This includes not only $E_t[M_t]$, $E_t[Y_t]$, and μ_t , but additionally all of the stochastic processes that drive the business cycles in the model, i.e. U_t and $\hat{E}_t[k_t]$.

[29] Swanson & Piazzesi display evidence for low maturity futures exhibiting low risk premia.

4.4 Estimation Results

With the stated equations, I am now able to estimate the Likelihood function with non-linearity taken into account. Equipped with Priors given from the literature[30], I use a Tailored randomized block random walk metropolis-hastings MCMC algorithm.[31] This procedure allows for reduced simulations, while still sufficiently exploring the posterior distribution.[32]

Model Params.							
Para	Descr.	Prior Mean	Prior Std.	Post Mean	$5\%, 95\%$	Dist.	Bound
δ	CRRA	2.6	1	2.89	2.5,3.4	Γ	$0 < x$
ψ	Frisch.	$.3\bar{3}$	1	.83	.34,1.32	$\mathcal{N}$	$0 < x$
ϵ_w	Wage Markup	10	.5	11.0	10.8,11.2	Γ	$0 < \frac{x}{x-1} < 1$
ϵ_p	Price Markup	10	.5	11.1	10.9,11.2	Γ	$0 < \frac{x}{x-1} < 1$
θ	Calvo-prices	.2	.1	.33	.16,.35	Γ	$0 < x$
θ_w	Calvo-wages	.2	.1	.16	.03,.34	Γ	$0 < x$
n	Wealth Util.	2.6	1	2.6	2.4,2.7	Γ	$0 < x$

[30] Christiano et. al.
[31] See: Chib & Ramamurthy(2010)
[32] See Appendix for convergence results

Model Params Cont.							
ν	Cap. Adj. Cost	.	.	.39	.27,.63	.	$0 < x$
ρ	infl. pers.	.7	.1	.62	.57,.8	Γ	$0 < x$
ϕ_x	Output resp.	.125	.2	.07	2e-5,.39	Γ	$0 < x$
ϕ_π	Infl. resp.	1.3	.2	1.85	1.5,2.2	Γ	$0 < x$
ϕ_b	Fisc. resp.	1.2	.013	.013	8e-5,.17	Γ	$0 < x$
ψ_s	LSAP resp.	.	.	.15	.01,.51	.	$0 < x$
ρ_u	ar(1) u	.	.	.45	.1,.8	.	$0 < x$
ρ_r	ar(1) r	.	.	.16	.004,.96	.	$0 < x$
ρ_s	ar(1) s	.	.	.82	.4,.96	.	$0 < x$
ρ_e	ar(1) e	.	.	.54	.4,.7	.	$0 < x$
ρ_a	ar(1) a	.	.	.14	01,.49	.	$0 < x$
σ_u	std u	.5	1	.14	.09,.38	Γ^{-1}	$0 < x$
σ_r	std r	.5	1	.14	.01,.31	Γ^{-1}	$0 < x$
σ_s	std s	.5	1	.04	.03,.21	Γ^{-1}	$0 < x$
σ_e	std e	.5	1	.18	.07,1.06	Γ^{-1}	$0 < x$
σ_a	std a	.5	1	.19	.1,.34	Γ^{-1}	$0 < x$
σ_w	std w	.5	1	.1	.06,.21	Γ^{-1}	$0 < x$
σ_τ	std τ	.5	1	.07	.07,.51	Γ^{-1}	$0 < x$
σ_i	std i	.5	1	.01	.01,.18	Γ^{-1}	$0 < x$
g	rls gain.	.031	.022	.0283	.005,.071	Γ	$0 < x$

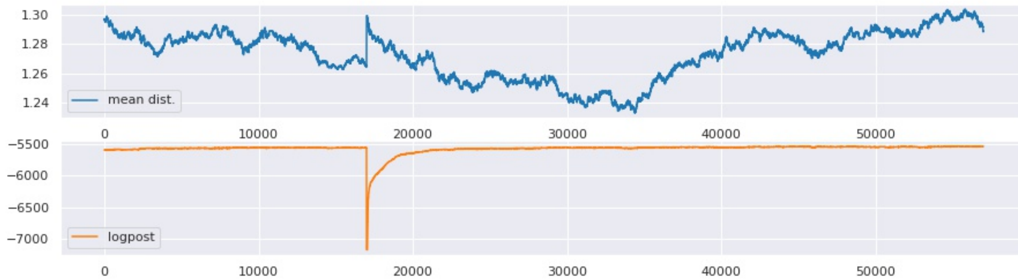


Figure 1: Mean parameter traceplot

Figure 1 displays the mean parameters for every draw in the simulation after 65% of the monte-carlo simulation. Because we see that the plots display stationary, this suggests strong evidence of model convergence. Likewise, the figure also shows the same but instead of the mean of each parameter draw, we see the log posterior exhibiting stationarity.

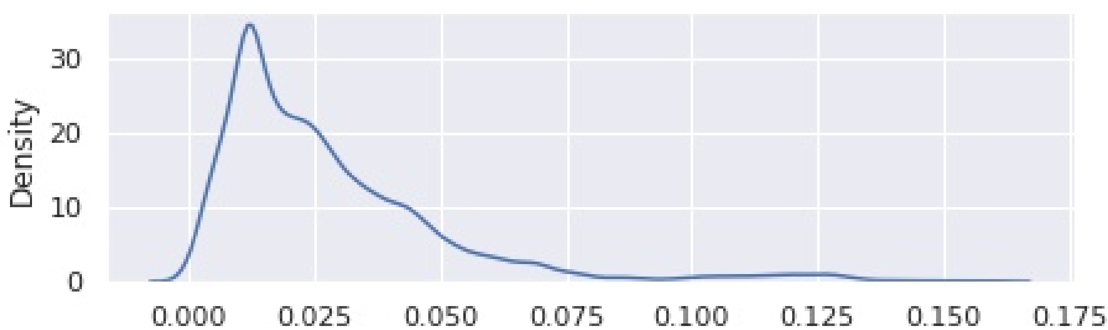


Figure 2: Posterior Dist. of RLS gain coeff.

Often in the learning literature, we see the RLS gain coefficient estimated to be close to .02. In line with this body of evidence, we see the posterior parameter distribution provides some evidence that although the mean is centered around to .02, there is a rightward skew. This suggests the coefficient could indeed be higher than suggested in the assessed literature.[33]

[33]See: Cole and Milani(2020) & Milani(2004), , and Kirpekar(2020)

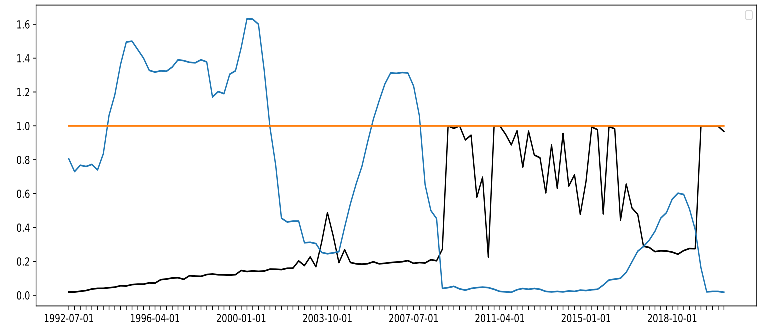


Figure 3: ZLB probability with fed funds futures data (μ_t)

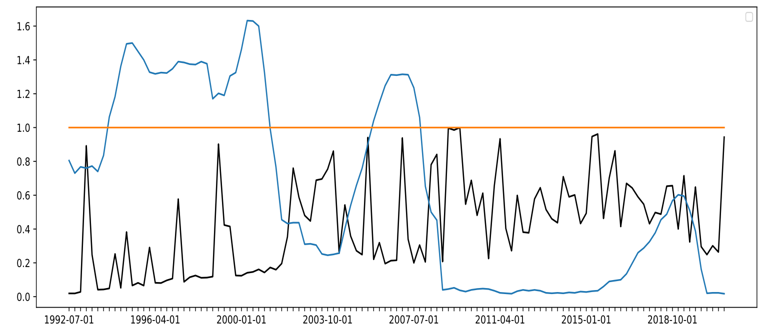


Figure 4: ZLB probability without fed funds futures data (μ_t)

In Figure 3 and 4 we see the implied subjective probability evaluated at the posterior mean. We can see in Figure 3, without utilizing the fed funds futures data, the subjective probability appears much less accurate and takes time to hit the zlb during the pandemic.

5 Results

5.1 Variance Decomposition Evidence

Using the parameters of the model, I compute the generalized forecast error variance decomposition and thereby am able to obtain the overall contribution of variance from each exogenous shock the macro-variables of interest[34]. Following Lanne & Nyberg(2016) , I com-

[34]see: Pesaran et. al 1997

pute the following expression using the particle filter's forecast:

$Y_t = f(S_{t-1}, \epsilon_t),\ GI = E_t[Y_{t+h}|\epsilon^j_{t+1} = \sigma_j, I_{t-1}] - E[Y_{t+h}|I_{t-1}]$

$\lambda^j_t(h|e_{t+h}) = \frac{\sum_{l=1}^{h} GI(l|\epsilon^j_{t+h}=\sigma_j)}{\sum_{j=1}^{J}\sum_{l=1}^{h} GI(l|\epsilon_{t+1}=\sigma_j)^2}$

$FEVD^j_t \equiv \lambda^j_t / E[\lambda^j_t(h)] = \sum_{j=1}^{J} \lambda^j_t(h)$

We see from above, the Normalized measure of variance contribution relies on the information set I_{t-1} and hence is a function of the squared deviations of the mean forecast estimate. Below, I plot the relevant results for the contribution of Variance across the sample, and include only the top 5 shocks:

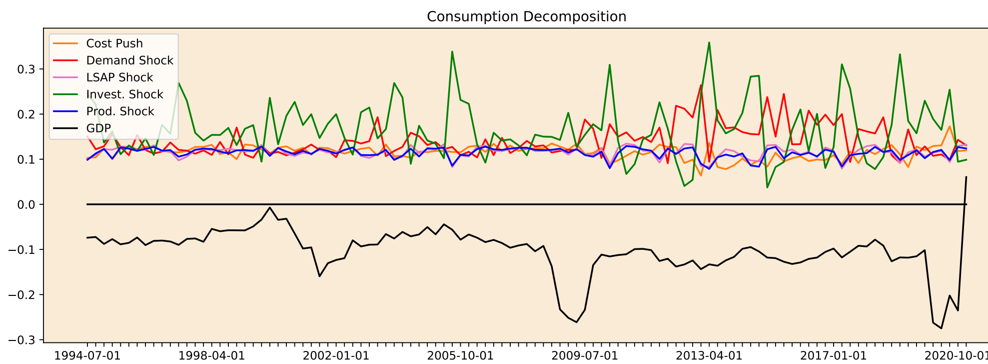


Figure 5: Shock decomposition for Consumption

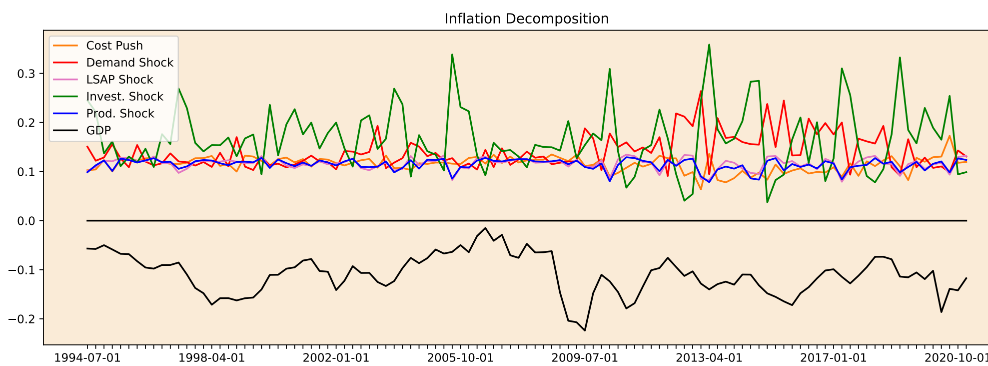


Figure 6: Shock decomposition for Inflation

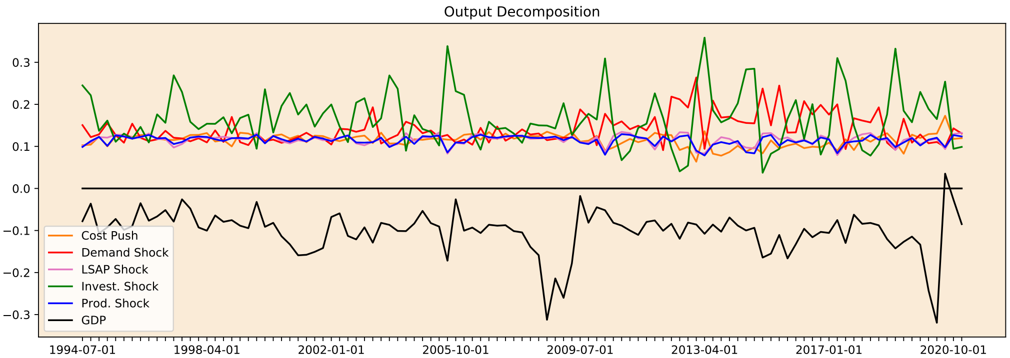


Figure 7: Shock decomposition for Output

With respect to GDP & Consumption Growth, we see the model strongly attributes the dot-com and Great-recession to Capital investments followed by Demand, Productivity, and Cost push shocks. While other shocks play a sizeable role in both the pandemic and Great-Recession, it is apparent that this explanation alone is not sufficient with respect to the exogenous shocks specified in the model. By summing both the blue and orange in Figures (5-7), the model confirms the root of the covid-19 recession, is primarily from productivity & cost push shocks to marginal cost.

With respect to Inflation, we see a similar story of shocks to capital investment playing a strong role in price growth. When a negative investment capital shock occurs, although the marginal cost may increase given equation (2.57). However, we also have a downward pressure imposed on the Phillip's curve coming from $\hat{E}_t c_t$. Because agents have some approximate knowledge that $c_t = c_1 y_t - c_2(I_t - \Phi(\frac{I_t}{I_{t-1}}))$ via (2.64) , the fall in $E_t c_t$ outweighs the potential rise in ϕ_t. Economically, this means that if firms have negative sentiment of consumption, they are more likely to lower prices in response to a shock to weak demand coming from this despite a low magnitude increase in marginal cost.

5.2 Impulse Responses

After presenting additional evidence, it is clear shocks to capital investment also play a key role in driving business cycles. Furthermore, other papers display differing economic outcomes at the zlb[35]. In accordance with this evidence, I present impulse responses that differ depending on the regime. For all plots, I compare the standard zlb regime with the aforementioned regime switching model(r-zlb). For each IRF, I plot the standard variables: consumption, output, and inflation given a σ_i shock. Additionally, because we are interested in understanding firm's willingness to invest, I plot Tobin's Q_t. I also plot v_t, which is the household's future expected discounted wealth and represents an overall measure of the household's wealth holdings.

Large Scale Asset Purchase Shock(s_t):

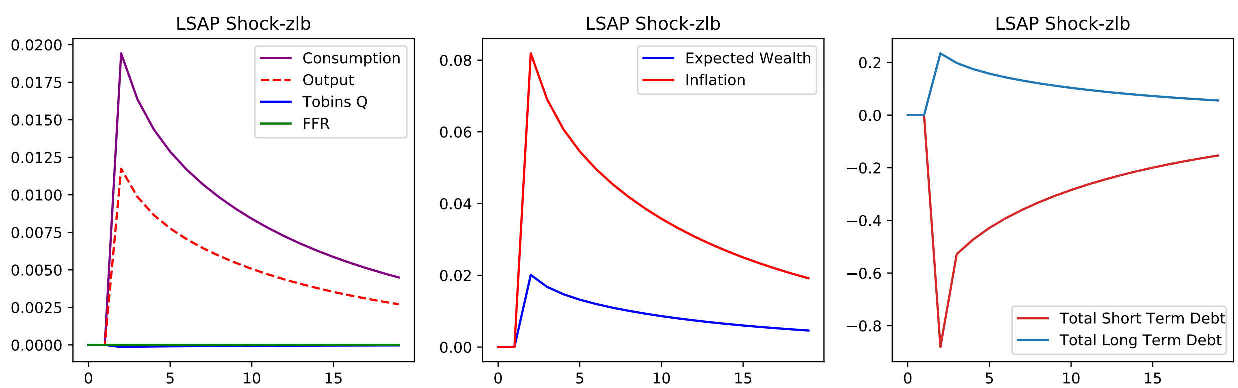


Figure 8: LSAP IRF zlb for s_t

[35]Ramey & Zubiary (2014), Bianchi et. al.(2017)

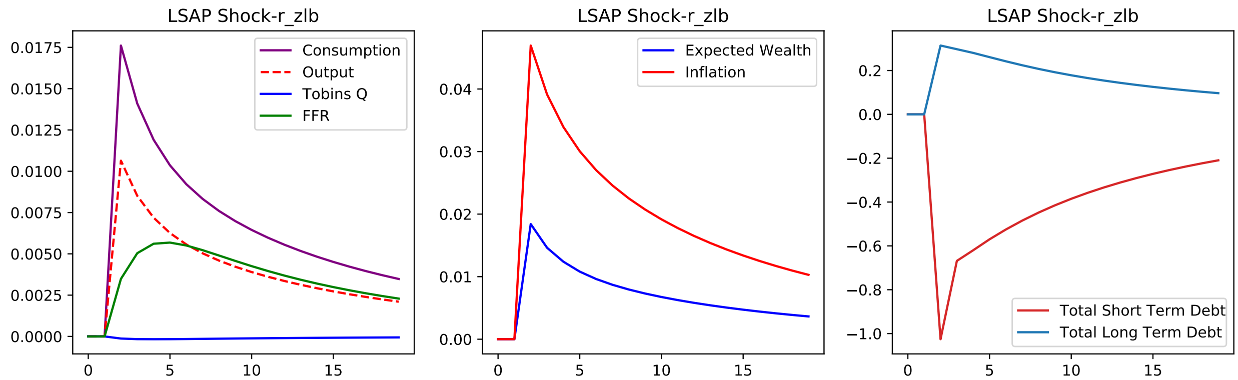


Figure 9: LSAP IRF r-zlb for s_t

Figure 8 & 9 tell us that a %90 exchange in short term term bond purchases yields a 5 % increase in inflation in the r-zlb case and an % 8 increase in the zlb case. Interestingly, we see slightly higher persistence in the effect of the LSAP-shock in the r-zlb case. This implies a benefit for policy to steer away from the zlb to maximize effectiveness of bond purchase programs. One explanation of higher inflation under the zlb regime is that agents in the economy know that monetary policy will continue on a strictly loose path of interest rates, while the overall macroeconomy remains in a depressed state. We also see for both impulses, LSAPs are able to raise expected wealth levels to about % 2, suggesting a positive impact on economic sentiment on the part of households coming from such a policy. This channel is important because we can see from (2.62) that this spurs a strong impact on consumption and thereby output. Given an increase in b_t, since agents know (2.24), $\hat{E}_t b_{t+1}$ increases, and hence, we have on balance, a rise in consumption and thereby output.

Capital Investment Shock(e_t)**:**

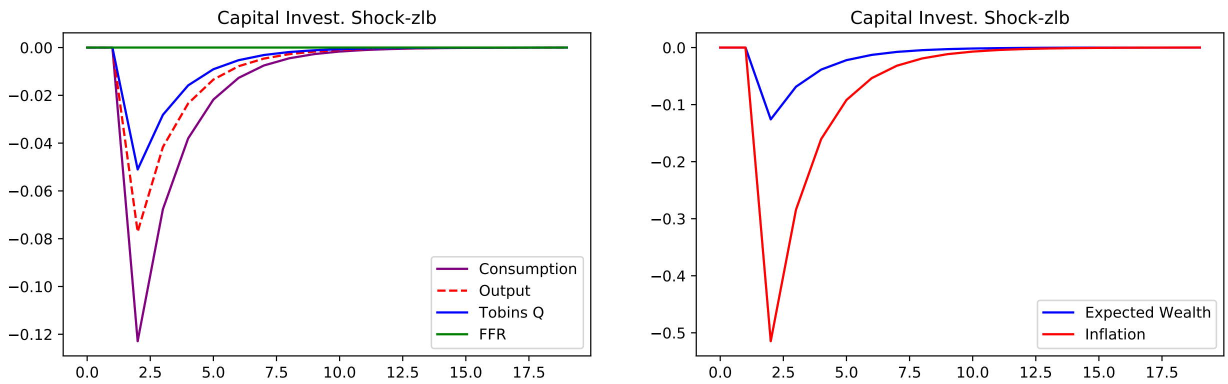


Figure 10: IRF zlb for ζ_t

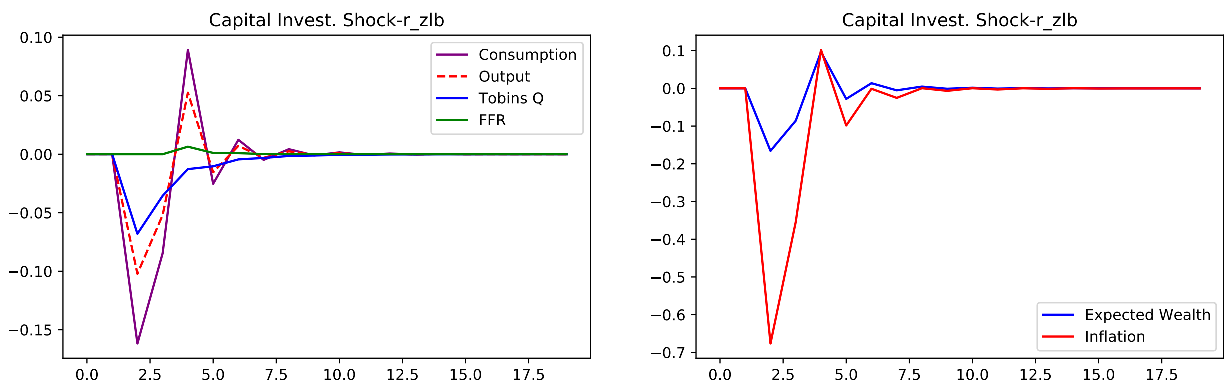


Figure 11: IRF r-zlb for for ζ_t

Figure 10 & 11 tell us that a negative capital investment shock is more negative on impact and without a 'rebound' period in the r-zlb regime. We see that for a standard deviation shock, consumption and output drop by more than % 2 in the zlb regime. Notice as well how firms' marginal gains from capital, Q_t is notably more affected. From equation (2.36) we can see that when c_t falls, Q_t falls, hence we obtain more depressed economic conditions for firms. When ζ_t is realized, firms are less willing to take on capital and hence, there is a cycle of low output and a subsequent drop in sentiment of future wealth, ultimately leading to lower consumption. Mechanically,both K_t via (2.38) declines and I_t declines, hence by the market clearing conditions, we see output decreases as well. Furthermore, because sentiment of future ζ_t's is known, the decline in macro-variables is excerbated via (2.37) and (2.11).

Demand Shock(r_t)**:**

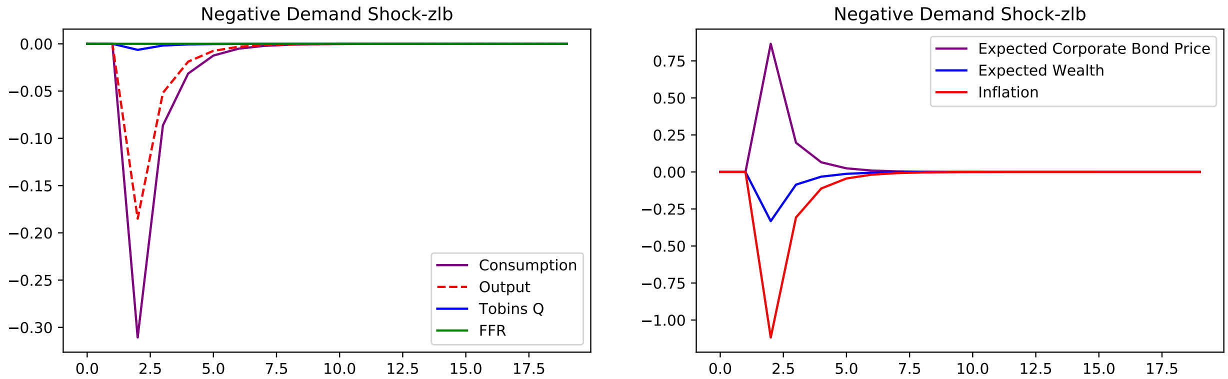


Figure 12: IRF zlb for r_t

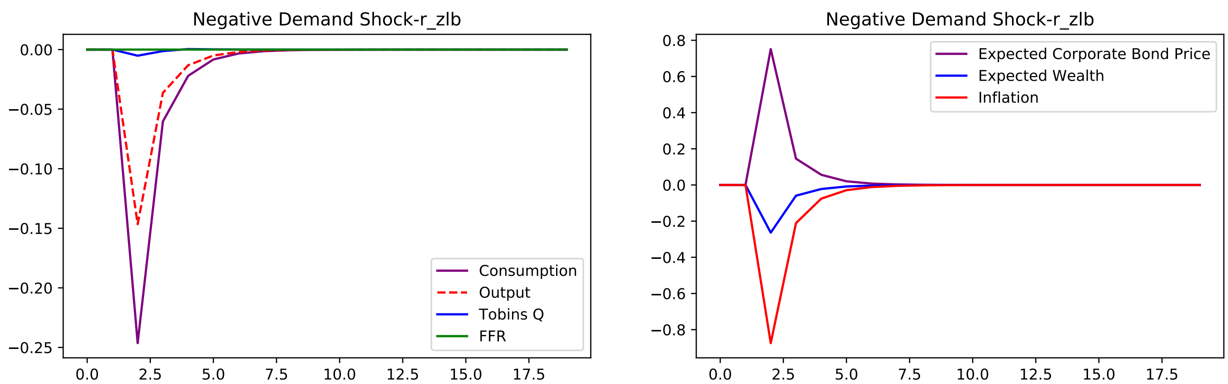


Figure 13: IRF r-zlb for for r_t

For a negative demand shock r_t, we see similar co-movement in the given variables to an a_t and e_t shock. However, without Q_t falling as drastically. One potential explanation for this lies in the expected bond price $\hat{E}q^k_{t+1}$. Given a demand shock r_t, $\hat{E}_t c_{t+1}$ declines, and hence the asset demand b^k_t increases as seen in equation (2.12). Hence, Q_t increases because the future ability to raise capital increases via iterating K_t forward(2.26). Intuitively, this means firms who face an sudden drop in demand, will still be willing to issue capital given the expectation of a higher cost of borrowing. Thus, the marginal benefit of raising capital remains largely unaffected due to perceptions of higher future borrowing costs.

Productivity Shock(a_t)**:**

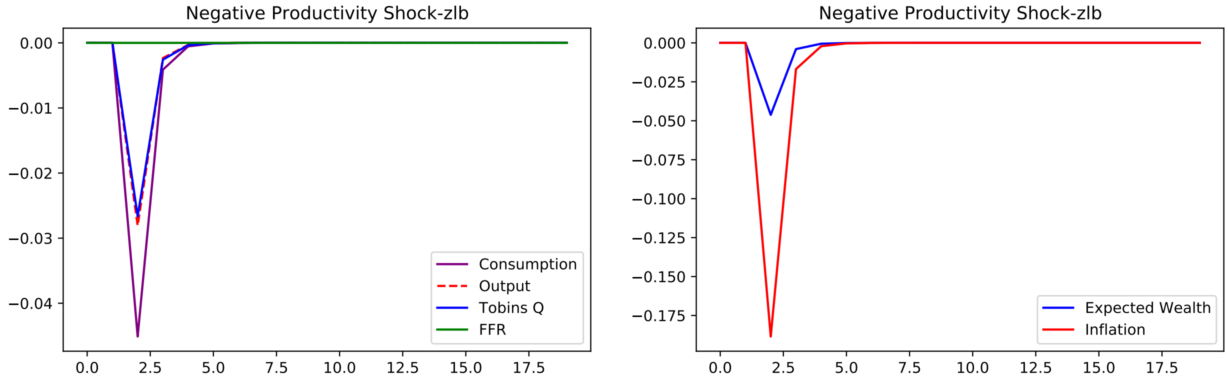


Figure 14: IRF zlb for for a_t

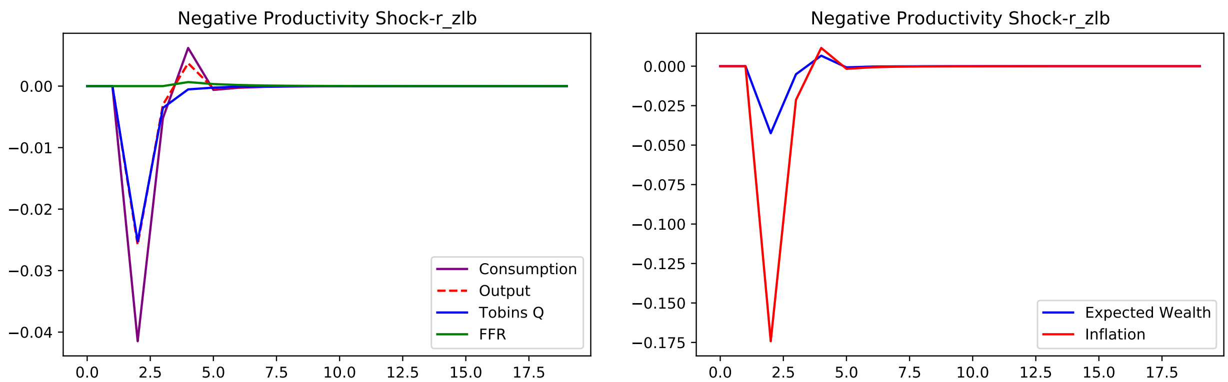


Figure 15: IRF r-zlb for for a_t

For a negative productivity shock a_t, we see a similar story play out, only with less of a 'rebound' period under the r-zlb regime. Furthermore, given a drop in productivity, we see a strong drop in inflation despite the upward pressure it has on the marginal cost. Economically, one reason for this has to do with the similar mechanics described for ζ_t in that firms more willing to lower prices given a drop in demand. However, we also see that when labor demand declines, n_t via (2.39), this impacts v_t and hence, the downward pressure on future labor income causes a lower c_t and hence y_t, while wages due to rigidity (2.53-2.54), remain relatively the same.

FFR shock (i_t)**:**

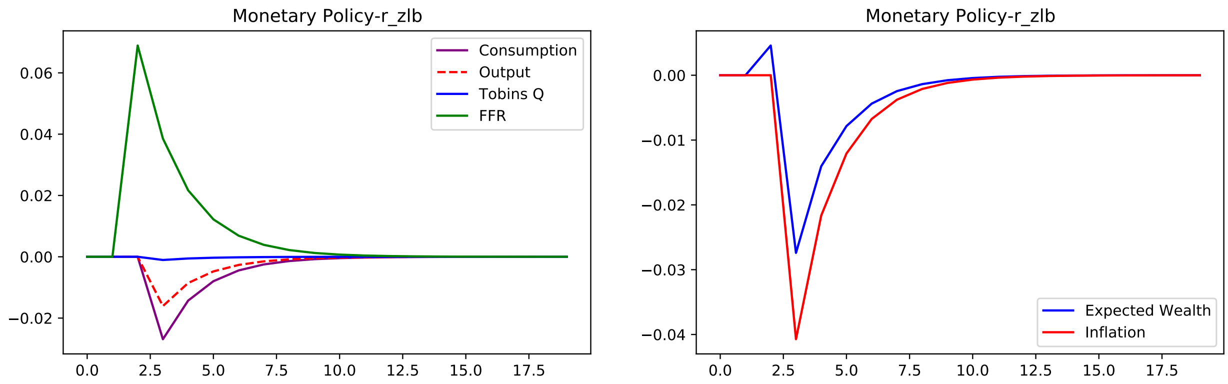


Figure 16: IRF r-zlb for for i_t

In IRF above, we see for a given increase in the federal funds rate, we obtain a decline in all macro-variables plotted above.

As previously mentioned, v_t plays a strong role in driving household consumption decisions. Hence, when simulating the impulse, I shock $\hat{E}_t v_t$, or the future percieved wealth flows, such that v_t yields a drop by % 200. Note that we see this shock enter (2.63) and thereby (2.60). Below I plot the corresponding results:

Wealth Shortfall $\hat{E}_t v_t$:

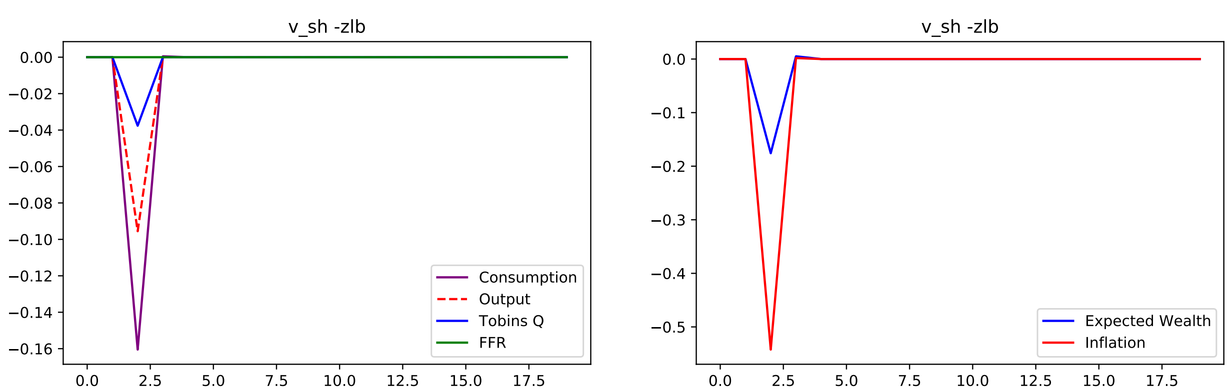


Figure 17: IRF zlb for $\hat{E}_t v_t$

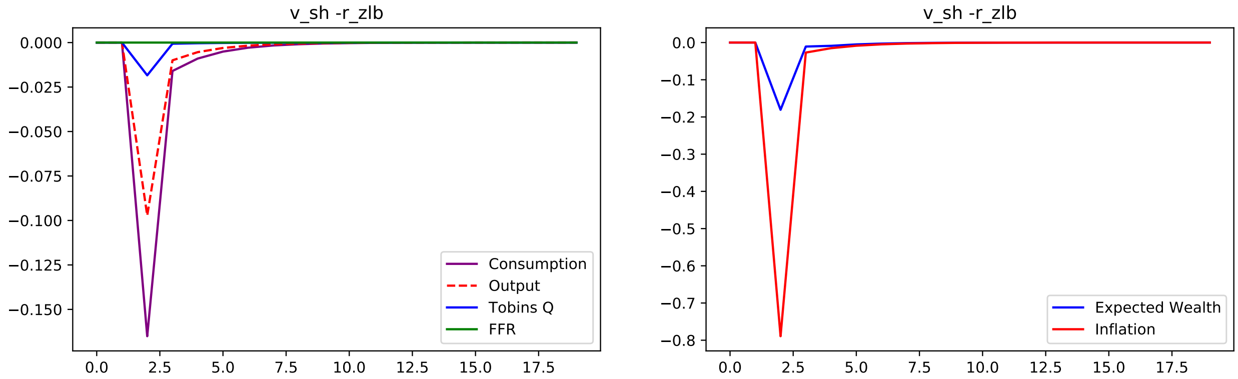


Figure 18: IRF r-zlb for $\hat{E}_t v_t$

We see from the above, a decrease in $E_t v_t$ causing a 200 % decrease in v_t leads to a % 8 reduction in output for the r-zlb case and a % 10 reduction in output for the zlb case. This result displays the importance of policy to stabilize asset price confidence due to the real time effect that subjective beliefs play in household consumption behavior. Thus in some sense, any exogenous shock will propagate changes macro-variables given a change in asset prices or bond holdings. Hence, when we think of policy response of LSAPs or the fed funds rate, it is important not only to think of the direct effect to output but also the amplification coming from the wealth channel.

5.3 Counterfactual Analysis

In light of the evidence presented, it is clear monetary policy's control over the federal funds rate and LSAPs have the potential to stabilize or worsen economic welfare. Hence, in similar spirit to Del Negro et. al.(2015), I produce 4 counterfactual experiments using the smoothed estimates from the particle filter. Before describing them, I will first detail mathematically how I obtain estimates of each macro-variable given a different policy scenario.

State Transition & Observation Equation:

$$S_{t|t-1} = f(\theta, S_{t-1}, \epsilon_t | I_{t-1}) \;\; = f(\theta, S_{t-1|t-1}, \epsilon_{t|t-1}) \tag{5.1}$$

$$Y_{t|t-1} = M_1 S_{t|t-1} + M_2 S_{t-1|t-1} + \epsilon_t^m \tag{5.2}$$

Counterfactual State Transition & Observation Equation:

$$S^*_{t|t-1} = f(\theta, S^*_{t-1|t-1}, \epsilon^*_{t|t-1}) \tag{5.3}$$

$$Y^*_{t|t-1} = M_1 S^*_{t|t-1} + M_2 S^*_{t-1|t-1} + \epsilon_t^m \tag{5.4}$$

Counterfactual Difference:

$$D_t = Y^*_{t|t-1} - Y_{t|t-1} \tag{5.5}$$

Above, I have expressed the empirical model described in the beginning sections as the state transition equation (5.1) and measurement equation in (5.2). Equations (5.3) and (5.4) are the same but with the counterfactual shock ϵ_t^*, rather than the true shock ϵ_t implied from the observable data. For each counterfactual experiment, I generate the model implied macro variable forecast. Next I produce the counterfactual forecast conditioned on a policy counterfactual expressed in terms of a shock. For example, If I am interested in understanding the effects of contractionary monetary policy, I generate the model forecast via the particle filter. Then I generate the counterfactual forecast via the filter, but conditioned on draws of the ϵ_t^i which have a positive mean σ_i rather than 0 in the base case. Then, by taking the difference of the two forecasts, I obtain the counterfactual effect. For each experiment, on the left, I plot both true forecast and the counterfactual variables. While on the right, I compute the difference between the two, D_t. Note: I leave out the predictive bands, because the % 95 and %5 percentile yield the same results as plotting the mean.

Counterfactual 1: How would the dot-com crash look like had the federal reserve raised rates earlier?

When conducting the following experiment, I set $\epsilon^*_{i,t} \sim \mathcal{N}(0, \sigma_i) + \sigma_i$, for:
01/2002< t <04/2006.

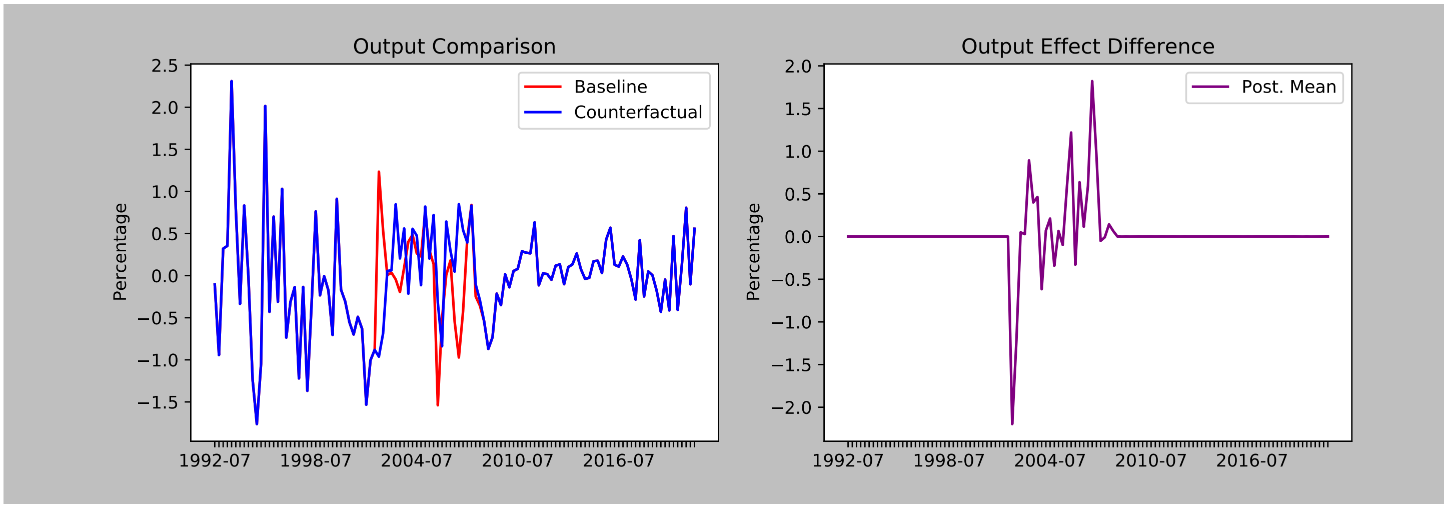


Figure 19: Percentage Effect of Output

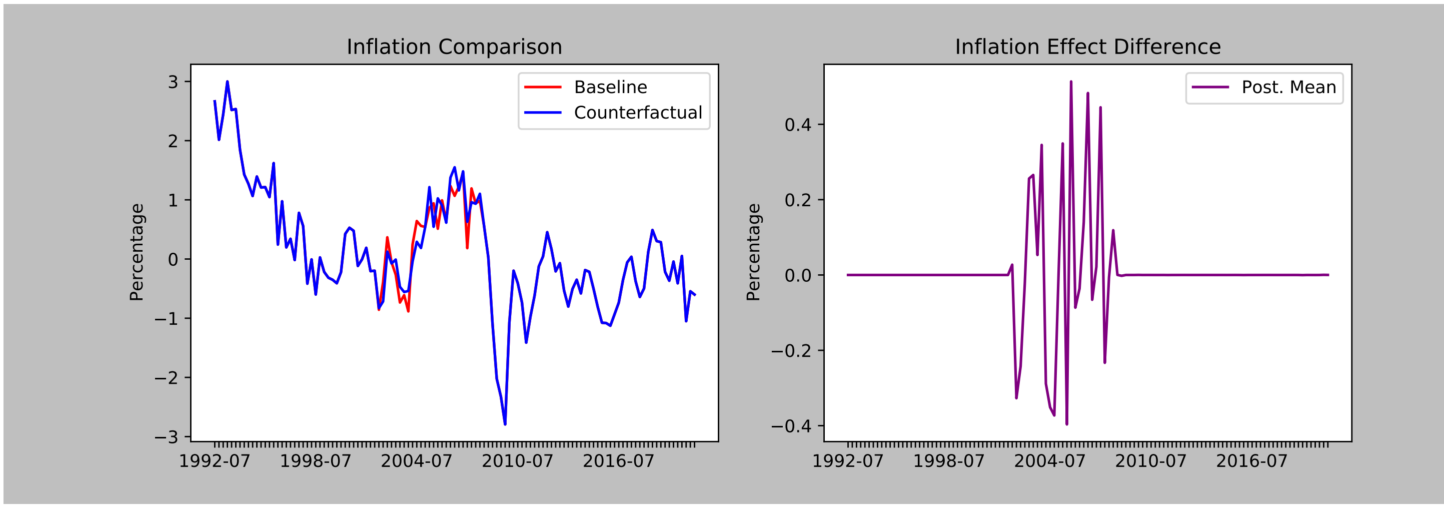


Figure 20: Percentage Effect of Inflation

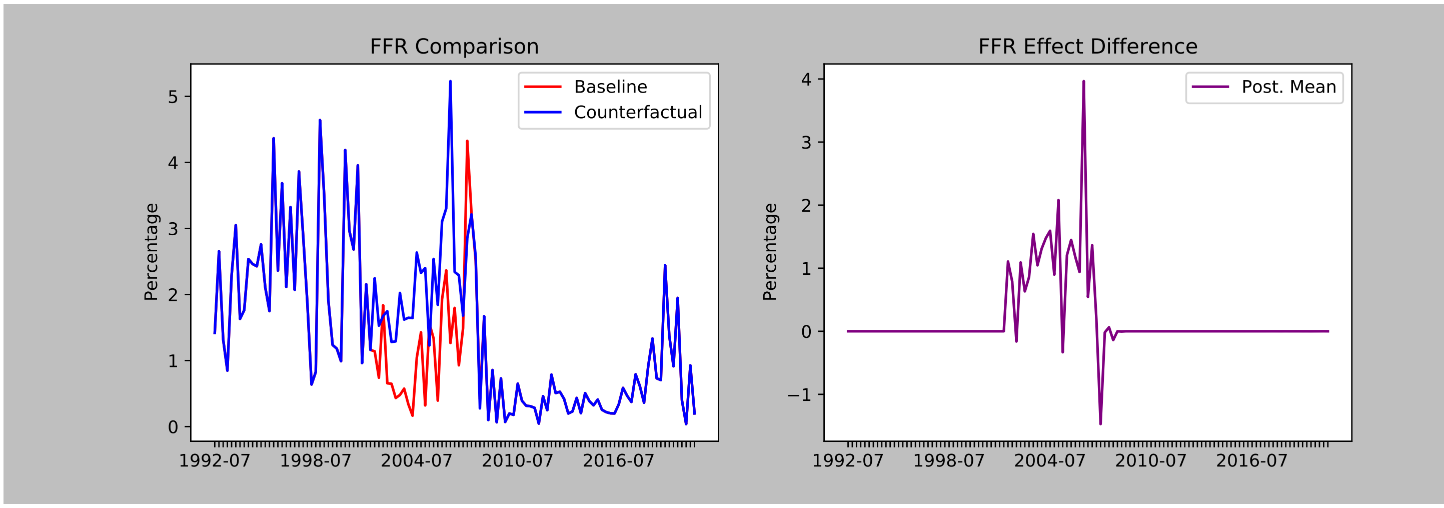


Figure 21: Percentage Effect of FFR

In the following Counterfactual, we see the model provides evidence that had Chairman Greenspan at the time raised rates earlier, this would have resulted in a lessening of the effects in the Great recession of 2008 as seen in the purple of output and consumption growth. Yet, it would have come at the cost of lower output growth during the Dot com recovery. We see the peak positive impact is observed during the middle of Great Recession in 2008. One possible reason for these results can be seen in the FFR plot. Initially the fed implements higher rates, as seen in the positive purple FFR plot. But as soon as the Great Recession starts, we see this sign reverse, which implies the fed is able to reduce rates more aggressively to further accommodate firm & household activity.

<u>Counterfactual 2:</u> How would economy look like if the Fed pursued LSAPs during the dot-com crash?

When conducting the following experiment, I set $\epsilon^*_{s,t} \sim \mathcal{N}(0, \sigma_s) + \sigma_s$, for:

01/2002$<t<$10/2002.

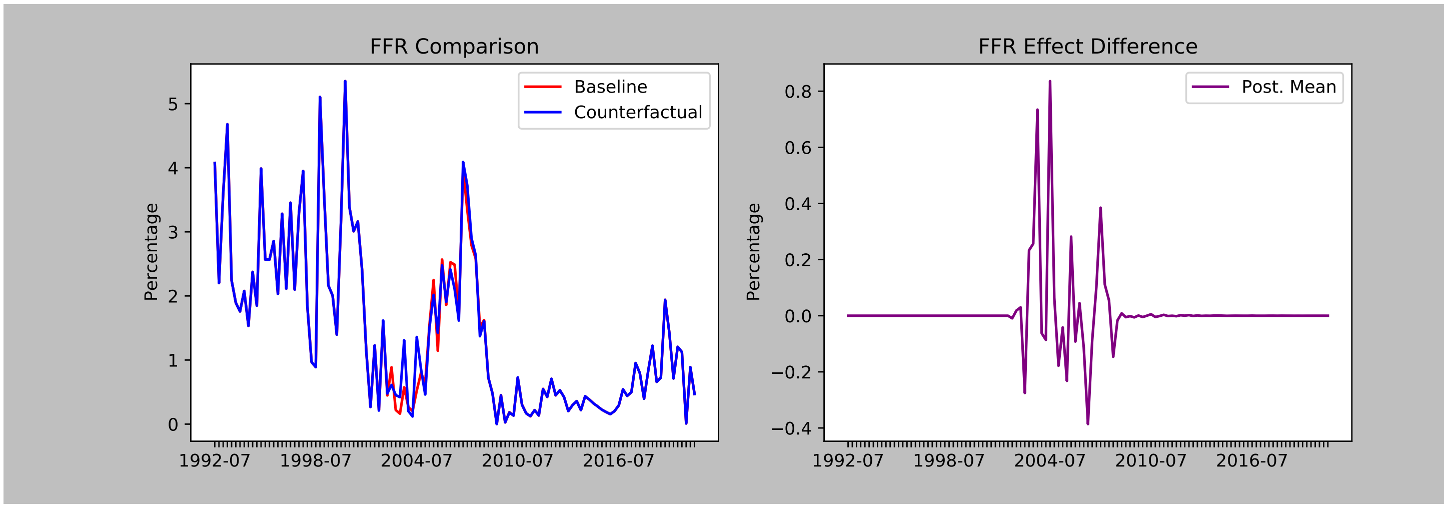


Figure 22: Percentage Effect of FFR

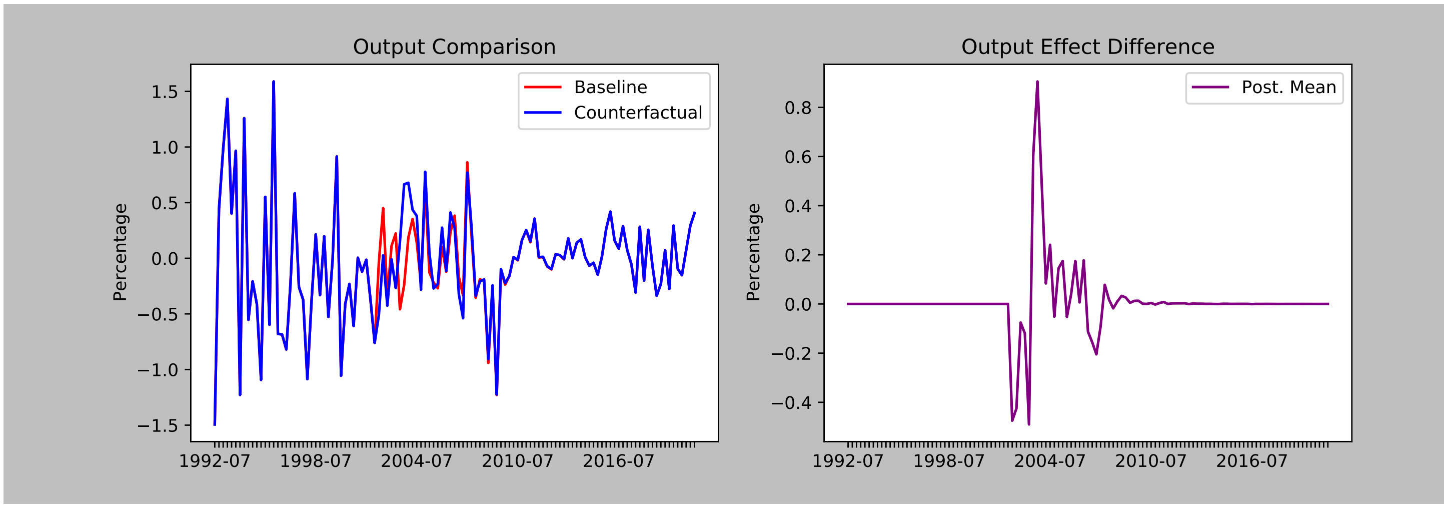


Figure 23: Percentage Effect of Output

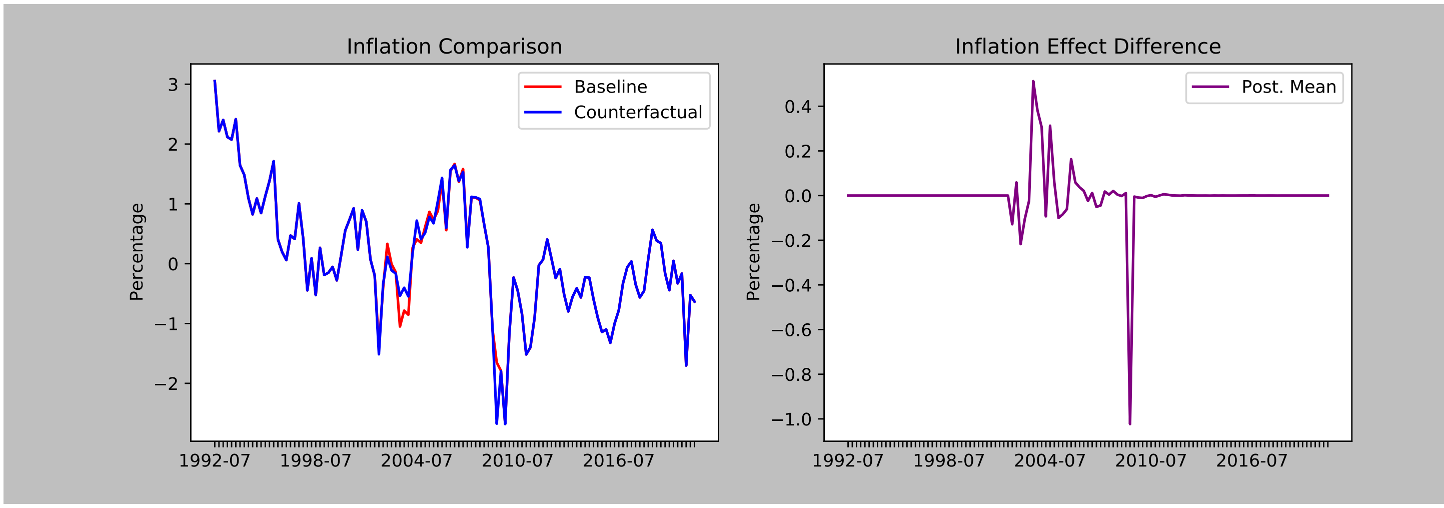


Figure 24: Percentage Effect of Inflation

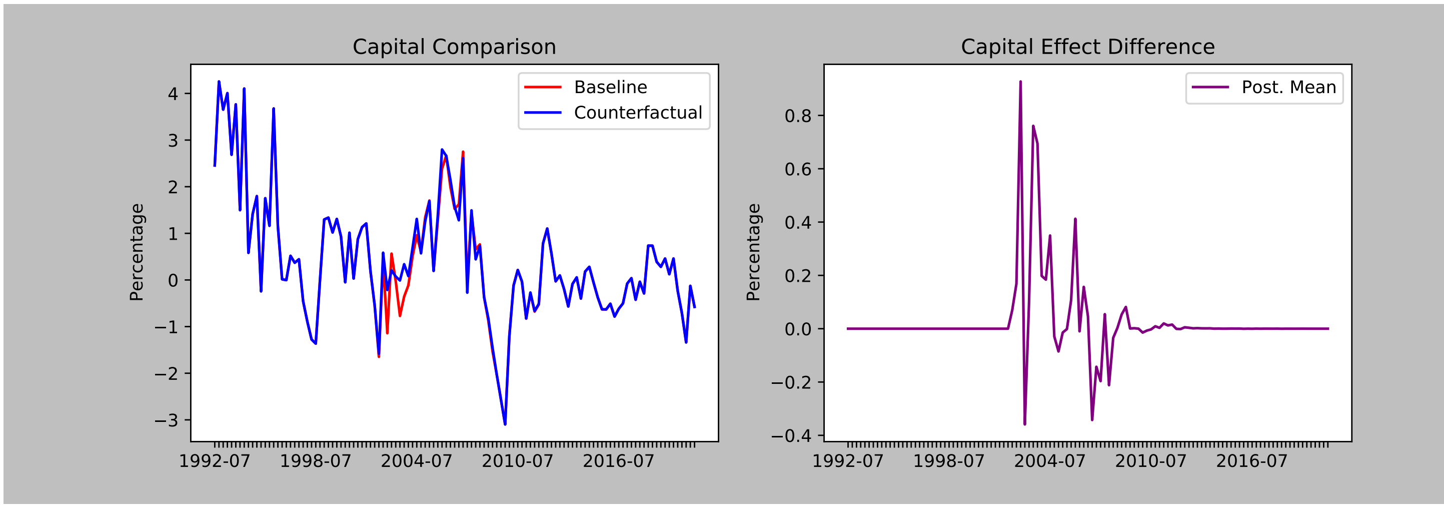
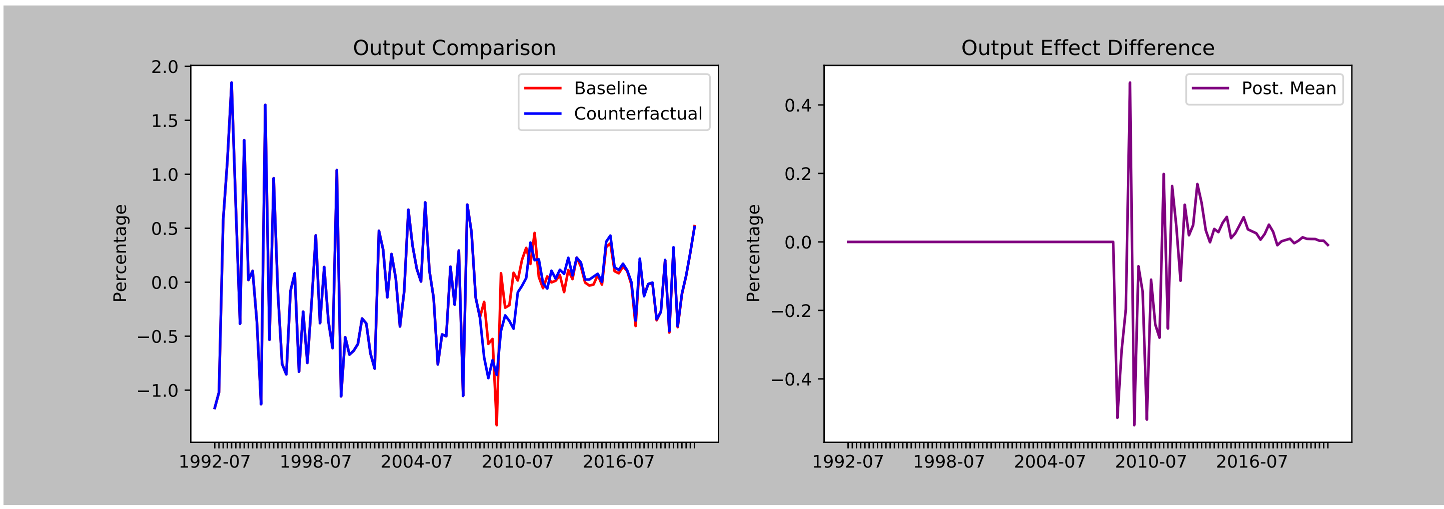


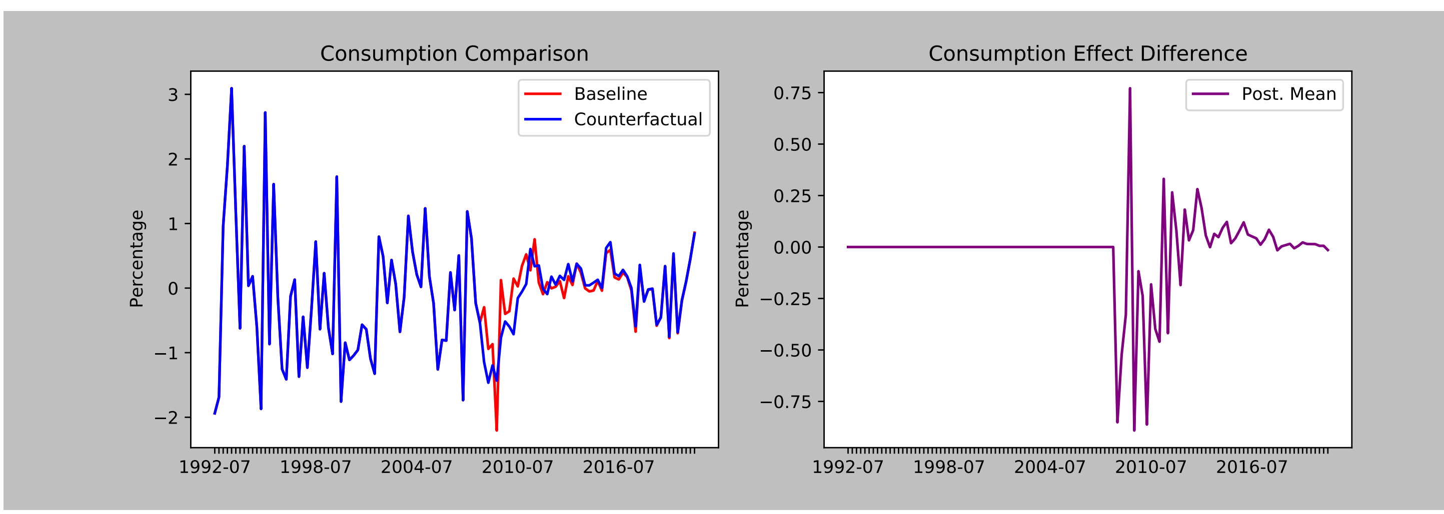
Figure 25: Percentage Effect of Investment

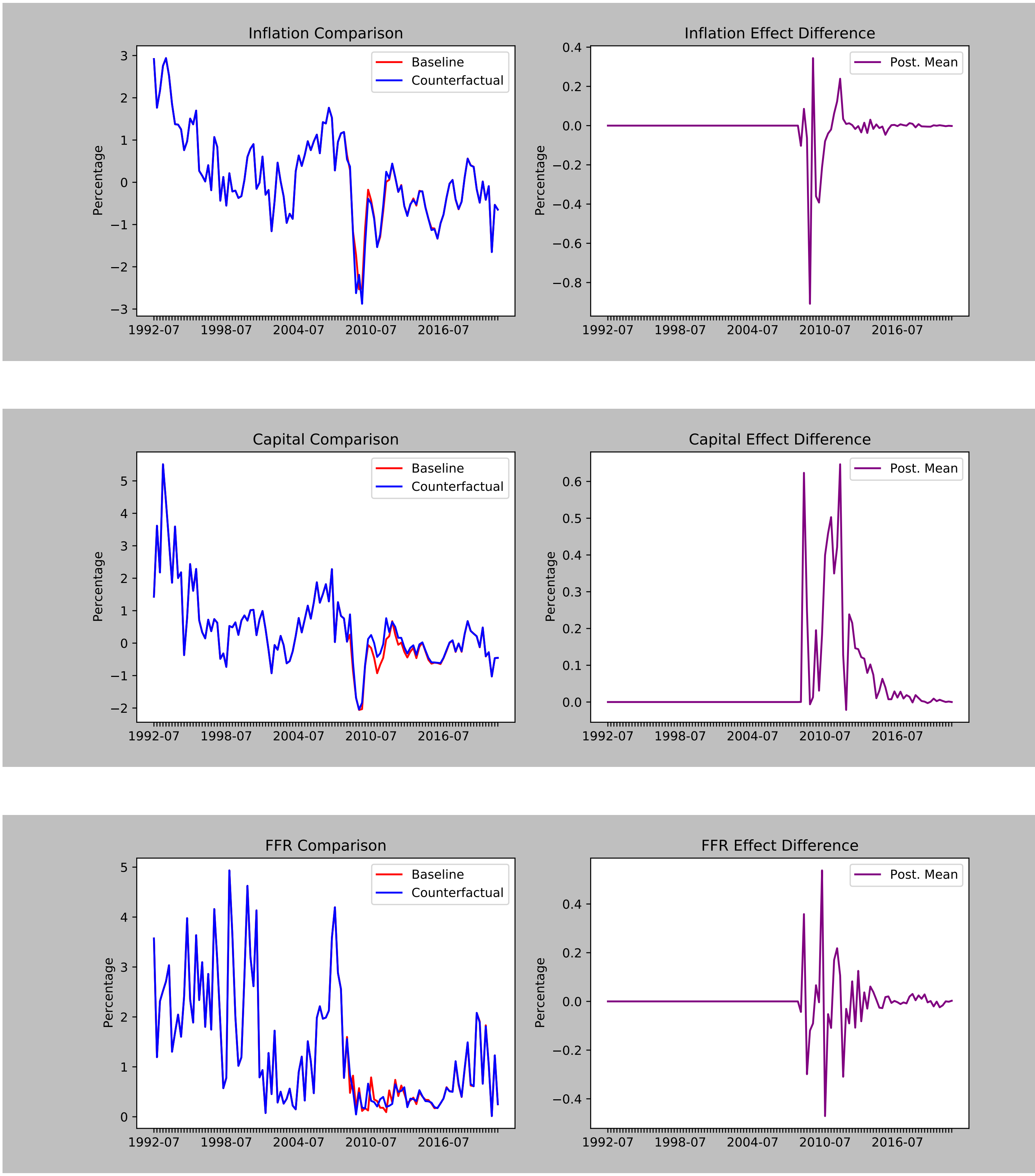
As indicated from the above results, had the fed engaged in LSAPs during the dot-com crash, output effects would initially decrease, and then continue with sustained positive effect. Interestingly, we can attribute the initial negative effect to the large drop in inflation following the LSAP. In line with the IRF for an s_t shock, agents expect a strong monetary policy tightening and hence a fall in the price level in the near term. We lastly turn to the marked increase in the capital stock of producers. This observation offers evidence that through LSAPs, policy is better able to prevent the sudden drop in investment activity in the presence of exogenous credit spillovers coming from bubbles in asset markets.

<u>Counterfactual 3</u>: What would the economy look like had the Federal reserve been more aggressive with sustained LSAPs during the early period of the Great Recession?:

When conducting the following experiment, I set experiment, I set $\epsilon^{*}_{s,t} \sim \mathcal{N}(0, \sigma_s) + \sigma_s$, for: 01/2002< t <04/2006.



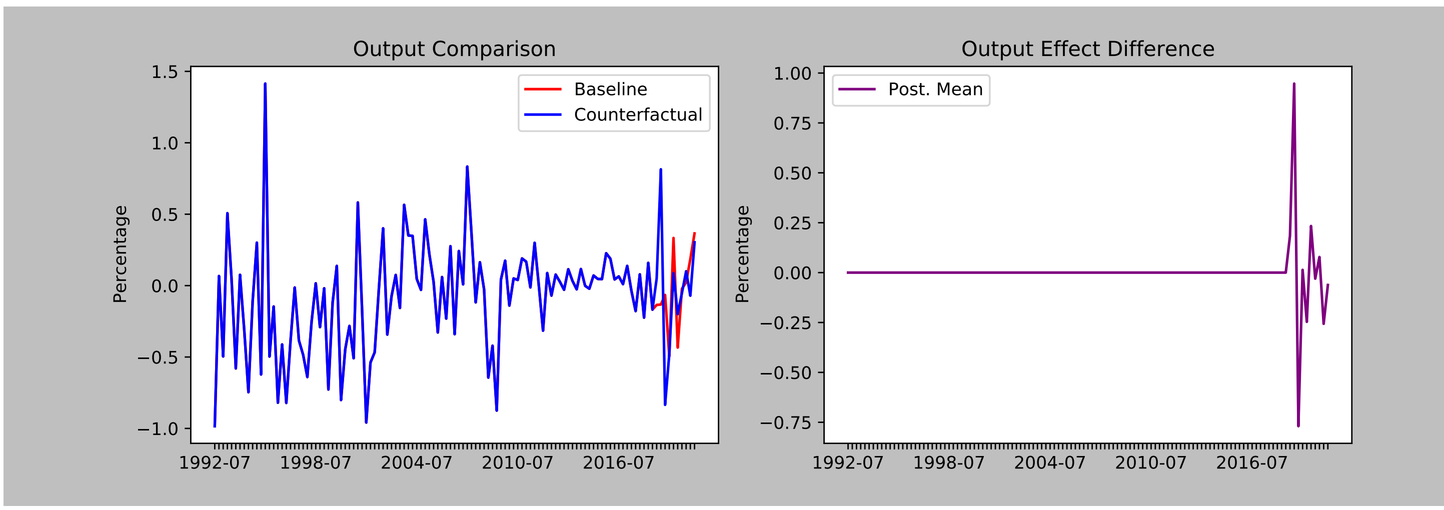




Similar to the previous counterfactual, we see nonlinearity in the effects of LSAPs. In particular, we see the same drop, decrease, and subsequent rise in output & consumption. Additionally, we also see a strong drop in inflation effects. This can be attributed to a sharp increase in the FFR measured in the counterfactual difference. In a sense, one can argue that policy should substitute aggressive drops in the FFR with increases in LSAPs in order to achieve both lower inflation and higher consumption & output. However, as seen in the dot-com recovery counterfactual, the FFR remains a vital channel towards achieving macroeconomic stability.

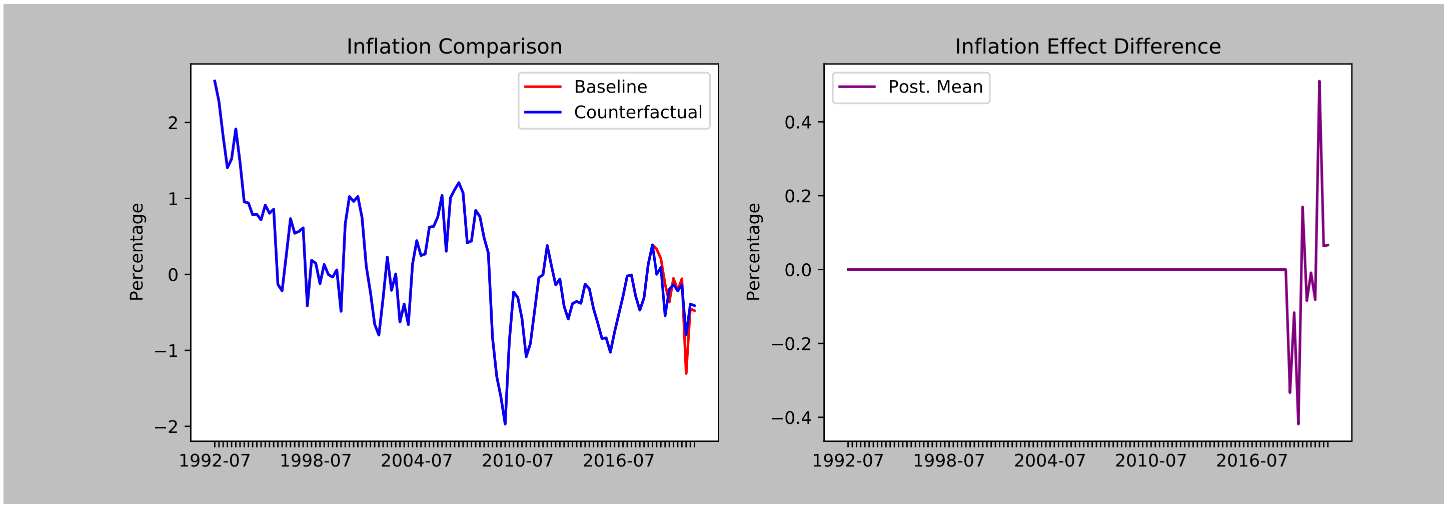
Counterfactual 4: What would the economy look like during the Covid-19 pandemic had the Economy been at steady-state?: [36]

When conducting the following experiment, I set experiment, I set $Y^*_{t|t} = Y^*_{ss}$, for:
$t = 04/2016$.

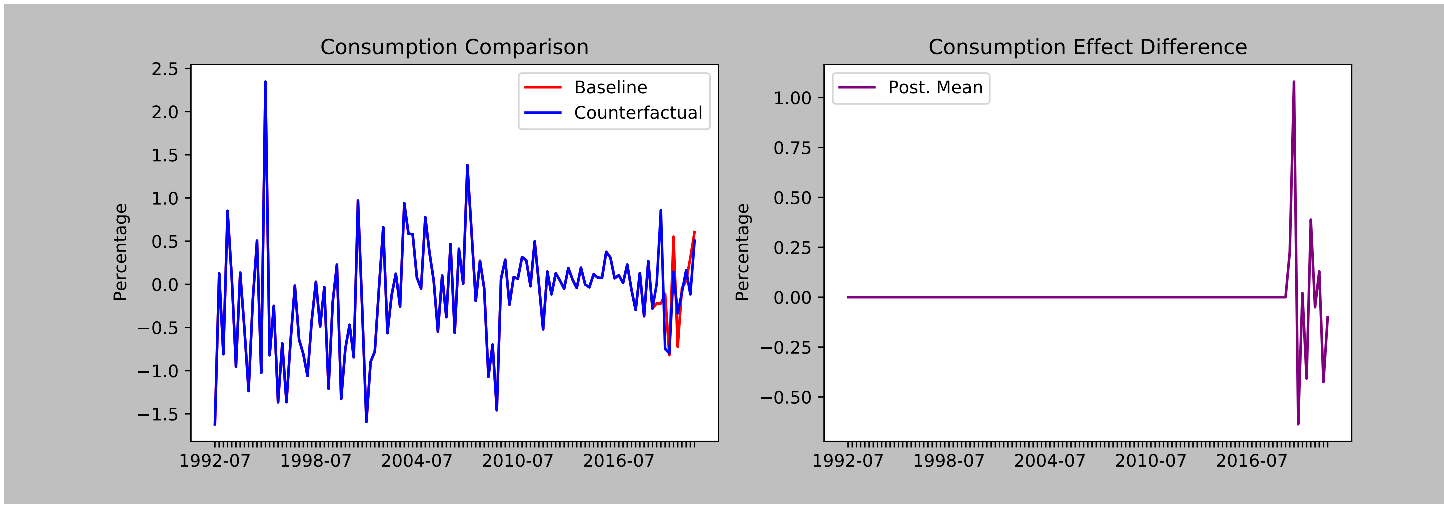


IRF r-zlb

[36]See Appendix for derivation of Steady State

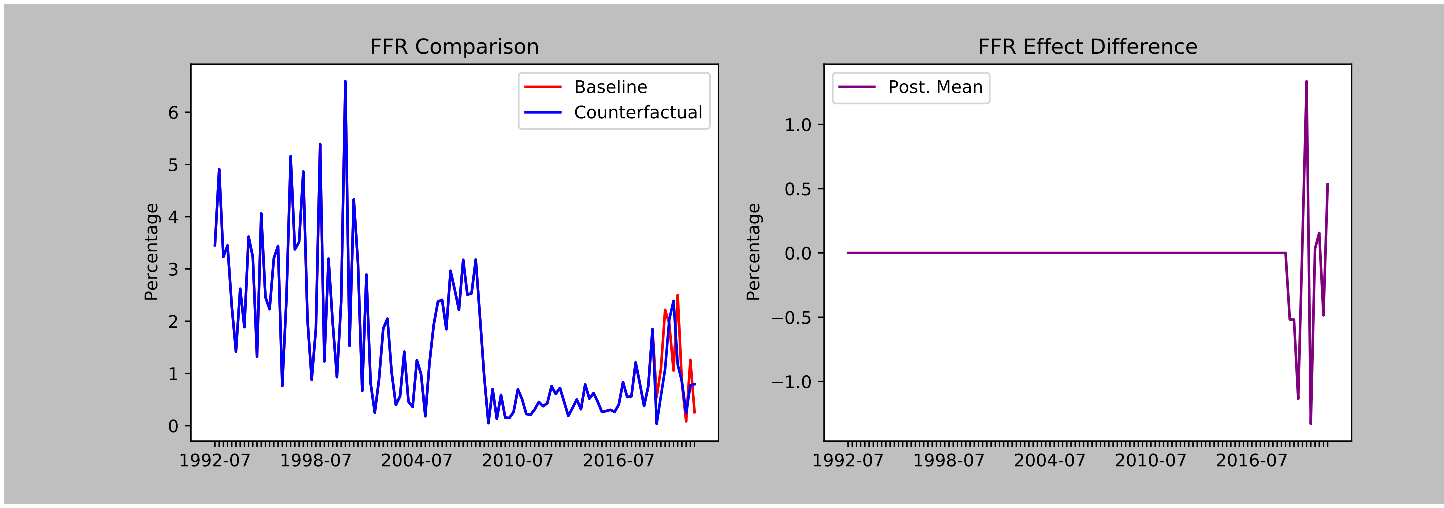


IRF r-zlb



IRF r-zlb

We see from the plots above, under the counterfactual where the economy is at steady state, Output and Consumption are substantially higher during the Covid-19 pandemic. One major driver of this comes from the FFR. It is clear that before Covid-19, the FOMC was raising the FFR towards the Steady state, however it still was not close to this value. A potential reason for the sizable difference in the counterfactual comes from the Fed's drastic lowering of the counterfactual FFR during the peak of the pandemic. We can see this visually via the below plot of the FFR:



IRF r-zlb

In summary, LSAPs add flexibility to monetary policy, seen in the results for its use in the dot-com bubble. However, it need not be considered a panacea for any and all economic sluggishness. This is clear from observing results in great recession counterfactual. One potential reasoning for the lack of strong results for this may come from monetary policy's tightening of i_t. It perhaps the case that the Δy_t would appear stronger, had the fed announced its commitment to keep i_t fixed while conducting LSAPs. However, I leave such hypotheses for future work.

6 Conclusion

Using a the Particle Filter via re-sampling , I model a medium Scale DSGE with regime switching behavior. I display the importance of capital investment shocks and wealth effects playing a strong role in reducing output, consumption, and inflation. Through impulse response and counterfactual analysis, I provide a cautionary tale for LSAP's. Though they carry weight in raising expected output, consumption, and inflation in a positive direction, they ought to coordinated with interest-rate policy. Otherwise contractionary monetary policy may nullify desired stabilization effects. Through impulse response analysis, I find strikingly differing equillibria derived from shocks to capital, productivity, and LSAPs. Thus implying the importance of policy to steer away from the zlb equilibrium. The paper ultimately seeks to address the importance of merging behavioral models with filtering methods to isolate sources of business cycles and generate accurate policy conclusions.

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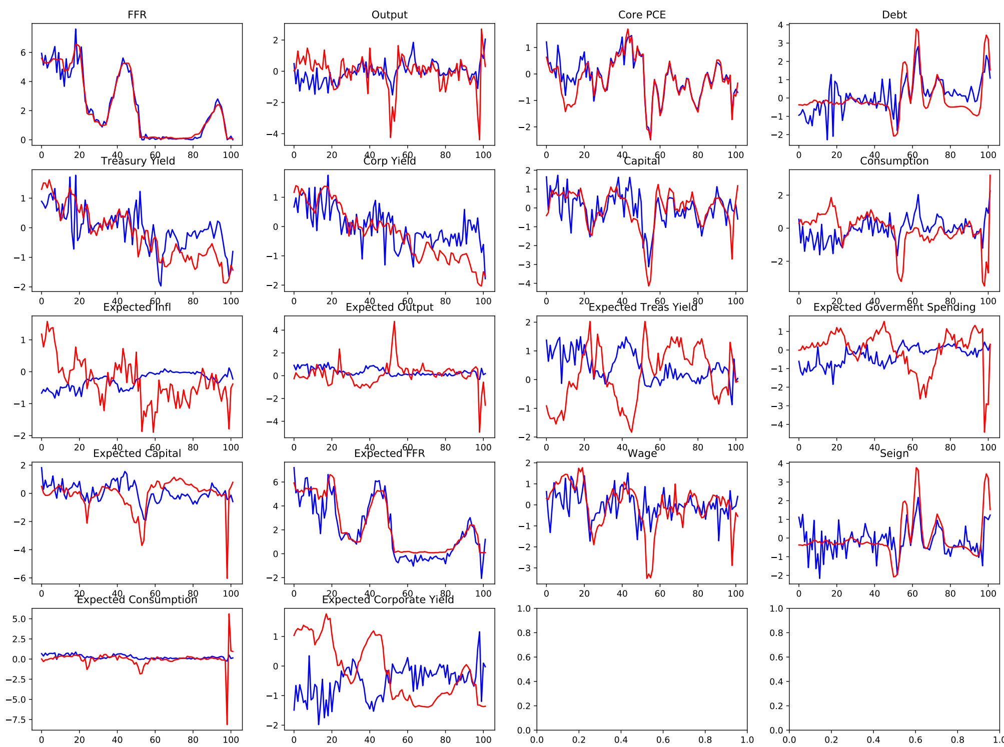
8 Plots



9 Appendix A: Model Linearization

9.1 Households:

The Representative Household maximizes the following Objective Function:[37]

$$\mathcal{L} = \max E_t \sum_{k=0}^{\inf} \beta^k \mathcal{Q}_{t+k}$$

$$\begin{aligned}\mathcal{Q}_t = U_t - \lambda_t (c_t + b_t q_t + \sum_j b_t^k q_t^k + b_t^s \\ - \pi_t^{-1}\{(1+\rho q_t) b_{t-1} + \sum_j (1+\rho_j q_t^k(j)) b_{t-1}^k(j) + b_{t-1}^s (1+R_{t-1}) + W_t(l) N_t(l) - T_t\})\end{aligned}$$

[37] each l household has a specific wage. I drop the l for all other FOCs since they are the same for each household.

After evaluating (2.7) at steady state, I obtain:

$$1 + \bar{R} = \bar{\pi}\beta^{-1}$$

Linearizing (2.6) & (2.7) becomes:

$$-\delta c_t = E_t(-\delta c_{t+1} - \pi_{t+1} + i_t)$$

Linearizing (2.8) & (2.9) becomes:

$$\partial log(q_t b_t)^{-\nu} = \partial log(\lambda_t - \beta E_t \lambda_{t+1} \pi_{t+1}^{-1}(1 + q_{t+1}))$$

$$(q_t + b_t)(-\nu) = \frac{\partial\lambda_t - \beta\partial E_t \lambda_{t+1}\pi_{t+1}^{-1}(1 + \rho q_{t+1}))}{\lambda q - \beta\lambda\bar{\pi}^{-1}(1+)}$$

$$(-\nu)(q_t + b_t) = d_1\{\delta c_t + q_t\} - d_2 E_t\{\delta c_{t+1} - \pi_{t+1} + \rho q_{t+1}\}$$

$$(\nu + c_1)q_t = -\nu b_t + d_1\delta c_t - d_2 E_t(-\delta c_{t+1} - \pi_{t+1} + \rho q_{t+1})$$

$$d_1 \equiv \frac{C^\delta q}{C^\delta q -^{-\delta} \bar{\pi}_{-1} q}$$

$$d_2 \equiv \frac{\beta C^\delta q}{C^\delta q -^{-\delta} \bar{\pi}_{-1} q}$$

9.2 Firm Level Equations:

$$\frac{\partial\Phi_{t+k}}{\partial I_t} = \eta(\frac{I_{t+1}}{I_t} - \sigma)\frac{I_{t+1}}{I_t} I_t^{-1}$$

$$\partial log(\frac{\partial\Phi_t}{\partial I_t}) = \partial log(\frac{I_{t+1}}{I_t} I_t^{-1}) + \partial log(\eta\frac{I_{t+1}}{I_t} - \sigma)$$

$$\partial log(\frac{\partial\Phi_t}{\partial I_t}) = \partial log(\Delta I_{t+1}) = (\frac{2-\sigma}{1-\sigma})\Delta I_{t+1} - I_t$$

After using the partial derivative of Φ_t, I Linearize Equation (2.34) and get the following expression:[38]

$$\partial log E_t\{-R_{t+1}^k - \eta I_t(\frac{I_t}{I_{t-1}} - \sigma) - (\frac{\eta}{2})(\frac{I_t}{I_{t-1}}^2) + Q_t\zeta_t\} = \partial log E_t\{(\beta\frac{C_t}{C_{t+1}})^\delta(\eta(\frac{I_{t+1}}{I_t}) - \sigma)\frac{I_{t+1}}{I_t}^2\}$$

$$\bar{Z} \equiv \bar{R}^k + \beta(\eta I(1-\sigma) - (\frac{\eta}{2})(1-\sigma)^2) + \bar{Q}$$

[38]Note: $R_{t+1}^k = \frac{1 + \rho^k q_{t+1}}{q_t}$

$$\nu_1 E_t R^k_{t+1} + \nu_2 I_{t-1} + \nu_3 I_t - \nu_3 I_{t-1} + \nu_4(Q_t + \zeta_t) = \delta(E_t c_{t+1} - c_t) + (\frac{3\eta - 2\sigma}{\eta - \sigma}) E_t I_{t+1}$$

$$\nu_1 \equiv \frac{\bar{R}^k}{\bar{Z}}$$

$$\nu_2 \equiv \frac{\eta I(1-\sigma)}{\bar{Z}}$$

$$\nu_3 \equiv \frac{\eta(1-\sigma)}{2\bar{Z}}$$

$$\nu_4 \equiv \frac{\bar{Q}}{\bar{Z}}$$

Linearizing (2.33) becomes:

$$\partial log(Q_t) = \frac{\partial(f'_t + (\beta(\frac{C_t}{C_{t+1}})^\delta (1-\sigma) E_t Q_{t+1}))}{\bar{Q}}$$

Where:

$$\partial log(f'_t) = a_t + (\alpha - 1)k_t + (1-\alpha)n_t$$

$$\bar{Q} = f' + (1-\sigma)\beta\bar{Q}$$

9.3 Wages:

Linearizing (2.51) becomes:

$$(1-\epsilon_w)w_t = \partial log\{(1-\theta_w)(w^*_t)^{1-\epsilon_w} + \theta_w (w_{t-1})^{1-\epsilon_w}\}$$

$$w_t = (1-\theta_w)w^*_t + \theta_w w_{t-1}$$

Linearizing (2.50) becomes:

$$(w^*_t)(1+\alpha_0 - \epsilon_w) = \partial log\{\sum_k (\beta\theta_w)^k (\frac{\epsilon_w}{\epsilon_w - 1}) \frac{N^{1+\psi}_{t+k} W^{\alpha_0}_{t+k}}{W^{\epsilon_w}_{t+k} N_{t+k} \lambda_{t+k}}\}$$

$$(w^*_t)(1+\alpha_0 - \epsilon_w) = \partial log\{\sum_k (\beta\theta_w)^k (\frac{\epsilon_w}{\epsilon_w - 1}) \frac{N^{1+\psi}_{t+k} W^{\alpha_0}_{t+k}}{W^{\epsilon_w}_{t+k} N_{t+k} \lambda_{t+k}}\}$$

$$\partial log(\sum (\theta_w \beta)^k O^1_{t+k}) = \frac{\partial \sum (\theta_w\beta)^k O^1_{t+k}}{O^1(1-\theta\beta)^{-1}}$$

$$\partial log(\sum (\theta_w \beta)^k O^2_{t+k}) = \frac{\partial \sum (\theta_w\beta)^k O^2_{t+k}}{O^2(1-\theta\beta)^{-1}}$$

$$w_t^* = (1-\epsilon_w\psi)^{-1}(1-\theta_w\beta)^{-1}\sum_k(\theta_w\beta)^k\{O^1_{t+k} - O^2_{t+k}\}$$

After some re-arranging, I obtain (2.54) where:

$$\alpha_1 \equiv ((1-\theta_w\beta)(1-\epsilon_w\psi))^{-1}$$

9.4 Phillips Curve

$$max_{p_t^*}\ E_t[\sum_{k=0}^{\inf}(\theta\beta)^k Q_{t+k}A_{t+k}]$$

$$A_{t+k} \equiv (\frac{p_t^*}{P_{t+k}})^{-\epsilon}Y_{t+k} - (\frac{p_t^*}{P_{t+k}})^{-\epsilon}(\frac{\Phi_{t+k}}{P_{t+k}})Y_{t+k}$$

$$\frac{\partial}{\partial p_t^*} := E_t[\sum_{k=0}^{\inf}(\beta\theta)^k\{C_{t+k}^{-\sigma}B_{t+k}\}] = 0$$

$$B_{t+k} \equiv (1-\epsilon)(p_t^*)^{-\epsilon}p_{t+k}^{\epsilon-1}Y_{t+k} + \epsilon(p_t^*)^{-\epsilon-1}p_{t+k}^{\epsilon}\Phi_{t+k}Y_{t+k}$$

$$\sum_{k=0}^{\inf}(\beta\theta)\bar{C}^{-\sigma}\bar{B} = 0$$

$$(1-\beta\theta)^{-1}\bar{C}^{-\sigma}\bar{B} = 0$$

Hence in Steady State:

$$\bar{B} = 0$$

$$\bar{B}_1 = -\bar{B}_2$$

Where:

$$\bar{B}_1 \equiv (1-\epsilon)pY$$

$$\bar{B}_2 \equiv -\epsilon p\Phi Y$$

$$\partial log(\sum(\beta\theta)^k C_{t+k}^{-\sigma}B_{t+k}) = 0$$

$$\frac{\partial(\sum(\beta\theta)^k C_{t+k}^{-\sigma}B_{t+k})}{(1-\beta\theta)^{-1}C^{-\sigma}\bar{B}} = 0$$

$$\sum(\beta\theta)^k\partial log(C_{t+k}^{-\sigma})\partial log(B_{t+k}) = 0$$

$$\partial log(B_{t+k}) = \partial log(B_{1,t+k} + B_{2,t+k})$$

$$\partial log(B_{t+k}) = \frac{\partial(\bar{B}_1 B_{1,t+k} - \bar{B}_2 B_{2,t+k})}{\bar{B}}$$

$$\partial log(B_{t+k}) = \frac{B_1\partial(B_{1,t+k} - B_{2,t+k})}{B_1 + B_2}$$

$$\partial log(B_{t+k}) = \alpha_\pi(\partial log(B_{1,t+k}) - \partial log(B_{2,t+k}))$$

$$\alpha_\pi \equiv \frac{B_1}{B_1 + B_2} = \frac{1}{1 + \dfrac{B_2}{B_1}} = \frac{1}{1 + (\frac{\epsilon}{\epsilon - 1})\bar{\phi}}$$

Using the following linearization, I now obtain the following result:

$$\sum_{k=0}^{\inf}(\beta\theta)^k p_t^* =_t \sum_{k=0}^{\inf}(\beta\theta)^k(p_{t+k} + \alpha_\pi^{-1}c_{t+k+1} + \phi_{t+k})$$

$$p_t^* = (1-\beta\theta)(p_t + \alpha_\pi^{-1}E_t c_{t+1} + \phi_t) + (1-\beta\theta)\sum_{k=1}^{\inf}(\beta\theta)^k(p_{t+k+1} + \alpha_\pi^{-1}\sigma c_{t+k+2} + \phi_{t+1})$$

$$p_t^* = (1-\beta\theta)(p_t + \delta\alpha_\pi^{-1}E_t c_{t+1} + \phi_t) + \beta\theta E_t p_{t+1}^*$$

Aggregate Price Identity:

$$P_t^{1-\epsilon} = [(1-\theta)(p_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}]$$

Log-Linearized, this becomes:

$$\frac{\pi_t}{1-\theta} = p_t^* - p_t$$

$$\pi_t = (1-\theta)(1-\beta\theta)(\phi_t + \alpha_\pi^{-1}\delta E_t c_{t+1} + \beta\theta E_t\pi_{t+1})$$

9.5 Anticipated Utility of Consumption:

Linearizing (2.2) yields:

$$c_t = -\eta_1(q_t+b_t) - \eta_2(b_t^k+q_t^k) - \eta_3 b_t^s - \eta_4(\rho q_t + b_{t-1}) + \eta_5(\rho_k q_t^k + b_{t-1}^k) + \eta_6(b_{t-1}+i_{t-1}) + \eta_7\{w_t+n_t\} - \eta_8\tau_t - \eta_9\pi_t$$

$$\eta_1 = \frac{qb}{c}$$

$$\eta_2 = \frac{q^k b^k}{c}$$

$$\eta_3 = \frac{(1+R^s)b_s}{\bar{\pi}C}$$

$$\eta_4 = \frac{(1+\rho_k q^k)b^k}{C\bar{\pi}}$$

$$\eta_5 = \frac{(1+\rho q)b}{C\bar{\pi}}$$

$$\eta_6 = \frac{(1+R^s)b_s}{C\bar{\pi}}$$

$$\eta_7 = \frac{\bar{W}\bar{N}}{C\bar{\pi}}$$
$$\eta_8 = \frac{\tau}{C\bar{\pi}}$$
$$\eta_9 = \eta_4 + \eta_5 + \eta_6$$

Using the the expression $1 + R^s = \bar{\pi}\beta^{-1}$, I obtain:

$$b^s_{t-1} = (\eta_6)^{-1}c_t + \tilde{\eta}_1(q_t + b_t) + \tilde{\eta}_2(q^k_t + b^k_t) - \tilde{\eta}_4(\rho q_t + b_{t-1}) + \beta b_t$$
$$-\tilde{\eta}_5(\rho_k q^k_t + b^k_{t-1}) + \tilde{\eta}_6(b^s_{t-1} + i_{t-1}) + \tilde{\eta}_7(w_t + N_t) - \tilde{\eta}_8\tau_t - \tilde{\eta}_9\pi_t$$

Iterating this equation forward, I obtain:

$$\begin{aligned} b^s_{t-1} = E_t \sum_k \beta^k \{ & \eta_6^{-1}c_{t+k} + \tilde{\eta}_1(q_{t+k} + b_{t+k}) + \tilde{\eta}_2(q^k_{t+k} + b^k_{t+k}) - \tilde{\eta}_4(\rho q_{t+k} + b_{t+k-1}) \\ & - \tilde{\eta}_5(\rho_k q^k_{t+k} + b^k_{t+k-1}) + \tilde{\eta}_6(b^s_{t+k-1} + i_{t+k-1}) + \tilde{\eta}_7(w_{t+k} + N_{t+k}) - \tilde{\eta}_8\tau_{t+k} - \tilde{\eta}_9\pi_{t+k}\} \end{aligned}$$

After plugging (2.6) into this result. I obtain (2.61-2.62)

9.6 Macaulay duration:

In order to properly evaluate the consol bond model to the observed expectations & real bond yield data, I follow a similar metholodology to Matveev(2016) to compute the Mcauley duration(evaluated at the steady state) as a function of the ρ_k & ρ I seek to use for estimation.

$$D^k_t = \sum_j (\frac{\beta\rho_k}{R})^j \frac{q^k_{t+j}}{q^k_t}$$
$$D^k_t = (\frac{\beta\rho_k}{R})^{-1} \sum_j (\frac{\beta\rho_k}{R})^{j+1} \frac{q^k_{t+j}}{q^k_t}$$
$$D^k_t = (\frac{\beta\rho_k}{R})^{-1} \frac{\partial}{\partial x} \sum_{j=0} ({\frac{\beta\rho_k}{R}}^j))$$

10 Appendix B: Stochastic Gradient Learning:

As described in Evans(2010). I use the Generalized Stochastic Gradient algorithm 1. This means I must derive a value for the inverse of the co-variance of regressors. I start with the perceived law of motion:

$$Z_t = M^t_m Z_{t-1} + M^t_u U_t$$

$$U_t = RU_{t-1} + \epsilon_t^u$$

Hence, the perceived law of motion can be characterized as:

$$H_t \equiv \begin{bmatrix} Z_t \\ U_t \end{bmatrix}$$

$$H_t = K_1 H_{t-1} + K_2 \epsilon_t^u + \eta_t^h$$

$$var(H^*) = K_1 var(H^*) K_1' + K_2 \epsilon_t^u K_2' + c_s^2 I$$

$$var(H^*) = (N_1 K_2)\Omega(K_2 N_1)' + c_s^2 (N_1 K_2)(K_2 N_1)'$$

Note in the above, I define $var(H^*)$ as the variance-covariance matrix of the regressor, H_t. The above perceived law of motion contains the term η_t^h that accounts for the uncertainty the agent perceives in her regression specification. I set the standard deviation of the forecast error, $c_s = 1e3$. I choose this number because this is the lowest value such that I can use a constant gain of .02 without yielding an explosive equillibria. Though this may be considered an ad-hoc choice, for empirical purposes, we are interested in an R matrix centered around the model equations such that they behave stable with a well supported constant gain learning parameter within the neighborhood of 0.02.

Because $Eh_t'h_t$ is equivalent to $var(h)$, I am able to define both R and R_z that go into the stochastic gradient algorithm as $R = var(H^*)^{-1}$ and $R_z = var(H_z^*)^{-1}$. Where:

$$var(H_z^*) = N_1^z K_2^z \Omega(K_2^z N_1^z)' + c_s^2 (N_1^z K_2^z)(K_2^z N_1^z)'$$

Note the superscript z denotes the entries of the matrix that come from the solution to the rational expectations equilibrium when $i_t = 0$. Hence, the parameter update follows the following process as described in Evans(2010):

$$\phi_t = \phi_t + gRh_{t-1}(y_{t-1} - \phi_{t-1}' h_{t-1})'$$

Because the agent has two equillbria to consider when forming expectations, I modify the learning rule in the following manner:

$$\tilde{\phi}_{t-1} = \mu_{t-1}\phi_{t-1}^z + (1 - \mu_{t-1})\phi_{t-1}$$

$$\phi_t = \phi_t + gRh_{t-1}(y_{t-1} - \tilde{\phi}'_{t-1}h_{t-1})'$$

$$\phi^z_t = \phi^z_{t-1} + gR_zh_{t-1}(y_{t-1} - \tilde{\phi}'_{t-1}h_{t-1})' \text{ if } t = 0$$

Note, in the model, I assume the agent updates ϕ^z_t only when $i_t = 0$. And I also assume ϕ_t updates at all times.