Methods in Solving the Transportation Problem

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Abstract

Linear programming is the branch of mathematics which studies optimization problems dealing with linear functions. The transportation problem is a specific type of linear programming optimization problem aimed at finding the optimal transportation plan for the distribution of resources from a set of source locations to numerous destinations. The problem is to minimize the total transportation cost given supply constraints among the sources, demand requirements at the destinations, and the transportation costs of each potential route. Utilization of this mathematical model has become a major point of emphasis in operations research with practical applications across numerous industries such as public transit, manufacturing, and e-commerce, among others. In this paper, we discuss the history which leads to linear programming as a prominent area of study and explain that while the transportation problem can be formulated as a linear programming problem, because of its special structure, a more efficient solution method has been developed. We will introduce our sample example and work out the solution in two phases: 1) the initial phase which finds a feasible solution and 2) the optimization phase which starts with the feasible solution and proceeds to move from feasible solution to better feasible solution until the optimal solution has been found. For the initial phase, we offer three methods: the Northwest Corner Method, the Least Cost Method, and Vogel's Approximation Method. Then, we approach the optimization phase using the Stepping Stone Method. The special cases of unbalanced transportation problems and degeneracy are also briefly discussed. Finally, we suggest the possibility of further research into quantifying the true efficiencies of each method and the potential for discovering preferable methods in the future.

History and Motivation of the Transportation Problem

Human innovation in industry is driven by groundbreaking scientific discoveries, invention of new machinery, and improvements in technology that allow for greater efficiency in the production and distribution of goods. Yet, playing an equally significant role in transforming industrial processes are the improvements in logistical planning that have allowed us to best utilize the current resources and capabilities at our disposal. Today's colloquial idiom would be "getting the most bang for your buck", whether that be minimizing cost, maximizing profit, or, more generally, achieving the best possible result.

Linear programming is the branch of mathematics which studies optimization problems dealing with linear functions. Specifically, linear programming provides a mathematical approach to maximize or minimize a linear function, the objective function, while subject to given linear constraints. The methods and techniques utilized in linear programming can be used to solve problems where optimizing for a single factor is required when a multitude of permutations are possible, and thus have been employed by companies across many industries to guide their operational decision making. In this paper, we will study the Transportation Problem, a specific type of linear programming problem which finds the optimal transportation plan for the distribution of resources from a set of source locations to numerous destinations.

The emergence of linear programming as a prominent field of study occurred in the early to mid-twentieth century with the outbreak of World War II[6]. The war placed a heavy toll on competing nations that were dealing with armies stretched thin, economic stresses, and an unprecedented need to make the most of available resources. A growing focus emerged on finding ways to best utilize resources to help contribute to the war effort. Some of the most essential challenges included strategizing the most effective mobilization of troops and distribution of weapons, proper rationing of food, utilization of natural resources, maximizing production output, and efficiently distributing supplies. Manufacturing and retail corporations needed to keep up with demand despite workers being away at war. Furthermore, having the most efficient and cost effective transportation of personnel, weapons, goods, and supplies was critical in gaining an advantage over adversaries. Consequently, leading military strategists, economists, and mathematicians were tasked by their governments to find solutions to these complex problems, and the resulting work led to the development of new methods and techniques in optimization which paved the way for groundbreaking advancements in industrial planning and decision making.

Possibly the earliest published mathematical study of the transportation problem was conducted by A.N. Tolstoi, a Russian mathematician who published his

"Methods of Finding the Minimal Total Kilometrage in Cargo-Transportation Planning in Space" for the National Commissariat of Transportation of the Soviet Union in 1930[7]. In his article, he describes his methodology for formulating a transportation plan that minimized the total kilometers traveled to transport cargo across the Soviet railway system. His work considered a total of 10 sources, 68 destinations, and 155 connecting routes. Using techniques that he developed and named 'method of differences' and 'circle dependency', Tolstoi successfully determined an optimal transportation plan[5]. Although his exact methods are no longer used, his work demonstrated the significant impact that optimization planning can have on distribution efficiency.

Several years later, the Soviet mathematician and economist Leonid Kantorovich was commissioned by the Soviet government to work with the Laboratory of the Plywood Trust to optimize production in the plywood industry. Kantorovich's work focused on organizing the production process and determining the best strategies to maximize output efficiency. Kantorovich examined how to best optimize elements of the production process such as utilization of fuel sources, distribution of machinery workload, and minimization of scraps, among others. He concluded that all such optimization problems could be represented using the same mathematical model and devised a relatively simple and effective method to work out the solution. Kantorovich noted that more extensive research could prove his methods to be useful in solving problems even beyond his area of study, and that simpler, more effective methods could be discovered. In 1939, Kantorovich's findings were published by the Leningrad State University in a book titled "Mathematical Methods of Organizing and Planning Production", and his work became the foundation for what is now known as linear programming[4].

The current method of solving problems in linear programming was devised by American mathematician George Dantzig who, during World War II, put a pause on his education at the University of California, Berkeley to work for the United States Air Force. As head of the combat analysis branch of the statistical control division, Dantzig collected data used to prepare training schedules and deployment plans. At the time, logistical planning was done by hand through trial and error, a tremendous task that required the brightest minds to consider every potential permutation and determine the ideal result. After finishing his final semester at Berkeley and receiving his PhD in Mathematics, the Pentagon brought him back to mechanize the planning process. In 1947, Dantzig developed his simplex method*, an algorithm that efficiently solves optimization problems in linear programming by drastically reducing the number of permutations needed to be checked to reach optimality[2]. Then, in

*While Dantzig's simplex method is the primary method for solving all types of linear programming problems, this paper will discuss a different method specific to the optimization of transportation problems.

1951, he applied his simplex method to the transportation problem in an article titled "Application of the Simplex Method to a Transportation Problem" which was published in monograph 13 of the Cowles Commission for Research in Economics[1].

In the aftermath of World War II, linear programming and the simplex method gained rapid recognition and transformed the industrial world, becoming its own distinct area within the field of operations research. With a clear desired objective - such as maximizing space, profit, or output efficiency - and well-defined constraints - such as budget, time, and resources - companies started utilizing these new techniques in optimization to guide their logistical planning and decision-making process.

Defining the Transportation Problem

The first comprehensive formulation of the Transportation Problem is attributed to American mathematician and physicist Frank L. Hitchock in his "The Distribution of a Product from Several Sources to Numerous Localities" in 1941[3]. The following provides an explanation and mathematical definition of the problem:

Consider a collection of sources that need to transport materials to numerous destinations. Each source houses a fixed supply and each destination requires a specific demand. For now, we will assume that the total available supply at the sources and the total required demand by the destinations are equal. Thus, all available materials will need to be transported, allowing for sources to ship to multiple destinations and destinations to receive shipments from multiple sources. Given the costs of transporting one unit of material from each source to each destination, respectively, the Transportation Problem challenges us to find the optimal *transportation plan* (i.e. how many units of material should be shipped from each source to each destination) in order to minimize the total cost of transportation while meeting all supply and demand constraints. These constraints dictate that (1) the total sum of units of material shipped from a single source to each destination must equal the supply available at that source, (2) the total sum of units of material shipped from all sources to a single destination must equal the demand required by that destination, and (3) the number of units of material shipped from each source to each destination is a non-negative integer.

> Assume we have m sources $S_1, S_2, ..., S_m$ with supplies available of $s_1, s_2, ..., s_m$ and n destinations $D_1, D_2, ..., D_n$ with demands of $d_1, d_2, ..., d_n$, respectively. Let c_{ij} be the cost of transporting one unit of material from source i to destination j where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Let x_{ij} be the number of units of material to be transported from source i to destination j.
>
> Then our problem becomes:

Find all x_{ij} to minimize the total transportation cost $= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$.

$$\begin{aligned}
\text{s.t.} \quad & x_{11} + x_{12} + ... + x_{1n} = s_1 \\
& x_{21} + x_{22} + ... + x_{2n} = s_2 \qquad (1) \\
& ... \\
& x_{m1} + x_{m2} + ... + x_{mn} = s_m \\
\text{and} \quad & x_{11} + x_{21} + ... + x_{m1} = d_1 \\
& x_{12} + x_{22} + ... + x_{m2} = d_2 \qquad (2) \\
& ... \\
& x_{1n} + x_{2n} + ... + x_{mn} = d_n \\
\text{and} \quad & x_{ij} \geq 0 \ \forall \ i \in 1, 2, ..., m \text{ and } j \in 1, 2, ..., n \qquad (3)
\end{aligned}$$

Our Example

In this paper, we will study an example involving a bicycle manufacturing company that needs to distribute a total of 1,000 bicycle units from its factories to its stores.

The company owns four factories (W, X, Y, and Z), our sources, which can produce 175, 200, 400, and 225 bikes, respectively. Additionally, the company sells its bikes at six stores throughout the city (A, B, C, D, E, and F), our destinations, which require 50, 175, 225, 175, 300, and 75 bikes, respectively, to stock their inventory. These supply and demand constraints, as well as the *unit transportation costs* from every factory to every store, are represented in the *solution matrix* found in Figure 1. For example, the cost of transporting one bicycle from Factory W to Store A is \$17; from Factory W to Store B is \$18; from Factory W to Store C is \$10; etc. The empty cells of the solution matrix will be for our *decision variables* indicating the number of bicycles that are to be transported from the corresponding factory to the corresponding store.

Certainly, there are many potential factors that can contribute to the cost of transportation such as labor, fuel, maintenance, customs fees, and distance traveled. In our example, we determine the cost of transportation solely based on the geographic distance between factory and store in Figure 2, with each grid unit representing a transportation cost of one dollar.†

†A strong understanding of the Transportation Problem definition and familiarity with this example and its solution matrix will lead to a better understanding of the rest of the paper.

	A	B	C	D	E	F	Supply
W	17	18	10	10	5	13	175
X	6	11	13	5	10	8	200
Y	9	10	4	4	3	5	400
Z	13	8	6	14	9	11	225
Demand	50	175	225	175	300	75	Total: 1,000

c_{ij}
x_{ij}

Figure 1: Structure of the *solution matrix* to our transportation problem. Rows indicate the supply at each factory. Columns indicate the demand at each store. Each cell in the matrix represents a transportation route from the corresponding factory to the corresponding store. The top left corner of each cell indicates the *unit transportation cost* of the corresponding route and the value in each cell will represent the number of units of material assigned to that route, as shown.

Finding an Initial Basic Feasible Solution

For any transportation problem with numerous sources and destinations, a multitude of allocation combinations of the materials among the routes could lead to a valid transportation plan. However, only one of these numerous possibilities will provide the cheapest total transportation cost. The set of all possible solutions to the transportation problem is known as the *feasible solution space*, with each individual transportation plan within the feasible solution space being a *basic feasible solution*. Each basic feasible solution can be represented by the set of *decision variables*, x_{ij}, which indicate how many units of material are allocated to each particular route in the given solution.

Solving the transportation problem requires two phases: 1) the initial phase and 2) the optimization phase. During the initial phase, we find an initial basic feasible solution to our transportation problem to serve as our starting point. Since the initial basic feasible solution we find will nearly never be optimal, the optimization phase is used to continuously modify our *current solution* in order to reduce the total transportation cost until we arrive at the *optimal solution*.

Numerous methods have been developed to obtain an initial basic feasible solution to the transportation problem. In this paper, we will introduce the three most common methods: the Northwest Corner Method, the Least Cost Method, and Vogel's Approximation Method. Each subsequent method will produce an initial basic feasible solution that is increasingly cheaper and thus closer to optimality.

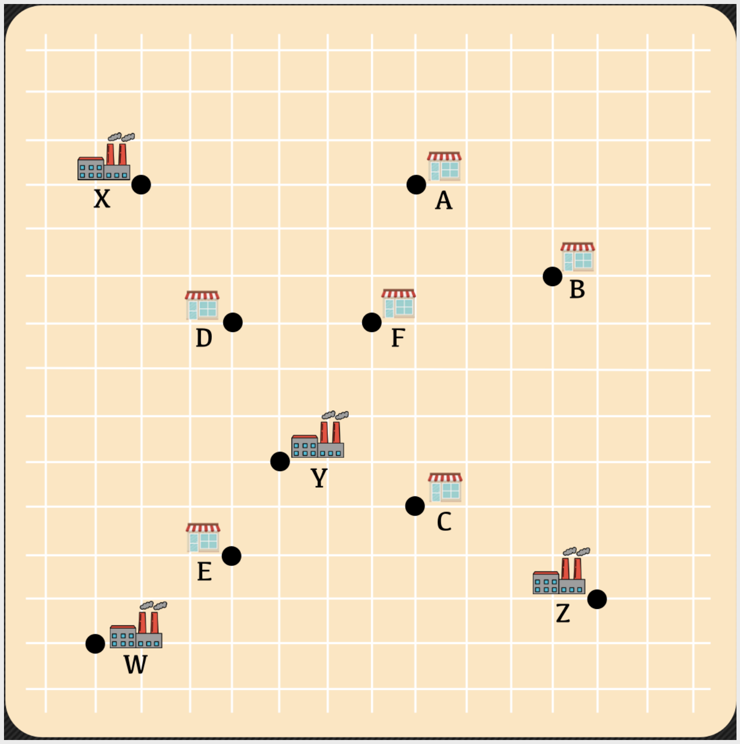

Figure 2: A map of the city in which our bicycle manufacturing company operates its 4 factories and supplies its 6 stores. Company transportation trucks may only travel along the streets of the grid. Each unit of distance traveled along the grid incurs a cost of one dollar for the company.

A. Northwest Corner Method

The Northwest Corner Method repeatedly assigns the maximum possible allocation to the top-left most available cell in our solution matrix. Considering each cell's allocation amount is limited by the supply and demand constraints of the cell's position in the matrix, the maximum possible allocation will be $\min(s_i, d_j)$ since we cannot transport more material than a source has available, and similarly, we cannot supply a destination with more material than it demands. Assigning the maximum possible allocation to a cell will ensure the fulfillment of a constraint by either exhausting the supply at the corresponding source or satisfying the demand of the corresponding destination. After each assignment, the allocated amount which has been fulfilled is subtracted from the associated source's supply and destination's demand and the rest of the exhausted row or column can be disregarded. This process is repeated with the top-left most available cell in our solution matrix until all requirements are satisfied.

The Northwest Corner Method is summarized by the following 5 steps:

1. Identify the top-left most available cell in the solution matrix.
2. Allocate the maximum number of materials under the supply and demand constraints.
3. Subtract the allocated amount from the cell's corresponding supply and demand.
4. Cross out the remaining cells of the exhausted row or column.
5. Repeat steps 1 - 4 until all materials have been allocated.

Applying the Northwest Corner Method to our bicycle example, the first route to be considered will be the number of bikes to be transported from Factory W to Store A. Since Factory W has a supply of 175 bikes and Store A requires only 50 bikes to fulfill its demand, we allocate a total of 50 bikes to be transported from Factory W to Store A. This satisfies Store A's demand and leaves Factory W with 125 bikes left to be transported. Having fulfilled the demand of Store A, we can eliminate the remaining cells in its column, since no more bicycles will need to be transported there from any other factory (Appendix A matrix a).

The next top-left most available route represents the number of bicycles to be transported from Factory W to Store B. In this case, Store B requires 175 bikes to fulfill its demand, however, Factory W only has 125 bikes left in its supply. Thus, we allocate all 125 bikes to be transported from Factory W to Store B. This exhausts the supply at Factory W and we can eliminate the remaining cells in its row, since no more bicycles are left to be transported from Factory W to any other store (Appendix A matrix b).

The third route to be assigned allocation is the number of bicycles to be transported from Factory X to Store B. Factory X has a supply of 200 bikes available and Store B still requires 50 bikes to fulfill its demand. Therefore, we allocate 50 bikes to be transported from Factory X to Store B, thereby fulfilling Store B's demand and allowing us to eliminate the remaining cells in its column (Appendix A matrix c).

Completing the remaining steps, shown in Appendix A, leads to a resulting transportation plan which the manufacturing company can successfully use to distribute its bicycles. The total cost of this initial basic feasible solution is $\$9,225$ (Figure 3).

The transportation plan obtained using the Northwest Corner Method is virtually never the optimal solution. This is because the order of allocation assignment fails to consider the costs of transportation. This method forced the use of Factory W to supply Store A and Store B despite Figure 2 clearly showing that it is the furthest factory from either store, incurring the greatest cost. Furthermore, some of Factory Z's supply is being sent to Store F while some of Factory X's supply is being sent to Store C. Simply swapping those assignments would certainly help in achieving a more optimal solution. In short, intuition tells us that utilizing routes with cheaper unit transportation costs will certainly lead to achieving a more cost efficient solution.

	A	B	C	D	E	F	Supply
W	17 50	18 125	10	10	5	13	175
X	6	11 50	13 150	5	10	8	200
Y	9	10	4 75	4 175	3 150	5	400
Z	13	8	6	14	9 150	11 75	225
Demand	50	175	225	175	300	75	Completed: 1,000

Bicycles transported from Factory W to Store A: $x_{wa} = 50$
Bicycles transported from Factory W to Store B: $x_{wb} = 125$
Bicycles transported from Factory X to Store B: $x_{xb} = 50$
Bicycles transported from Factory X to Store C: $x_{xc} = 150$
Bicycles transported from Factory Y to Store C: $x_{yc} = 75$
Bicycles transported from Factory Y to Store D: $x_{yd} = 175$
Bicycles transported from Factory Y to Store E: $x_{ye} = 150$
Bicycles transported from Factory Z to Store E: $x_{ze} = 150$
Bicycles transported from Factory Z to Store F: $x_{zf} = 75$

$$\text{Total Cost: } \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \$17(50) + \$18(125) + \$11(50) + \$13(150) + \$4(75) + \$4(175) + \$3(150) + \$9(150) + \$11(75) = \$9,225$$

Figure 3: *Initial basic feasible solution* obtained using the Northwest Corner Method. The transportation plan is summarized and its total cost is calculated.

B. Least Cost Method

The Least Cost Method was developed to account for the unit transportation costs of each route when deciding the order of allocation assignment. This method prioritizes the allocation of materials to available cells in ascending order of unit transportation cost. In the event that multiple cells share the lowest unit transportation cost, the cell to which we can allocate the most materials is chosen. After each allocation, we subtract the allocation amount from the associated supply and demand constraints, eliminate the remaining cells of the exhausted row or column, and assign the next value until all requirements are satisfied.

The Least Cost Method is summarized by the following 5 steps:

1. Identify the cell with the lowest unit transportation cost in the solution matrix. (tie goes to the cell that can take a larger allocation)

2. Allocate the maximum number of materials under the supply and demand constraints.

3. Subtract the allocated amount from the cell's corresponding supply and demand.

4. Cross out the remaining cells of the exhausted row or column.

5. Repeat steps 1 - 4 until all materials have been allocated.

The first route in our bicycle example to be considered using the Least Cost Method is the number of bicycles to be transported from Factory Y to Store E since this route has the cheapest unit transportation cost of \$3. Since Factory Y has a supply of 400 bikes and Store E requires only 300 bikes, we allocated 300 bikes to be transported from Factory Y to Store E. This fulfills the demand at Store E and leaves Factory Y with 100 bikes remaining in its supply (Appendix B matrix a).

Of the remaining available routes in our solution matrix, there are two routes that share the next lowest unit transportation cost at \$4. Since both options allow for a maximum of 100 bikes, we will arbitrarily select to allocate those 100 bikes to be transported from Factory Y to Store C. This exhausts the supply of Factory Y and leaves Store C still needing 125 bikes to fulfill its demand (Appendix B matrix b).

Next, with Factory X containing 200 bikes in its supply and Store D requiring 175 bikes in its demand, we allocate 175 bicycles to be transported from Factory X to store D at the next cheapest available unit transportation cost of \$5 (Appendix B matrix c).

The fourth step requires making a decision since two routes share the next lowest unit transportation cost at \$6. The chosen route will be the one that can transport the most bicycles. With a remaining supply of 25 bikes at Factory X and a demand of 50 bikes at Store A, the most bicycles that can be transported from Factory X to Store A is 25. With a supply of 225 bikes at Factory Z and a remaining demand of 125 bikes at Store C, the most bicycles that can be transported from Factory Z to Store C is 125. Thus, we allocate 125 bikes to be transported from Factory Z to store C, thereby fulfilling the demand at Store C (Appendix B matrix d).

Completing the remaining steps, shown in Appendix B, leads to another possible transportation plan which the manufacturing company can use to successfully distribute its bicycles. The total cost of this initial basic feasible solution is \$6,625 (Figure 4), which is 28.2% cheaper than the solution obtained by the Northwest Corner Method.

	A	B	C	D	E	F	Supply
W	17 25	18 75	10	10	5	13 75	175
X	6 25	11	13	5 175	10	8	200
Y	9	10	4 100	4	3 300	5	400
Z	13	8 100	6 125	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Bicycles transported from Factory W to Store A: $x_{wa} = 25$
Bicycles transported from Factory W to Store B: $x_{wb} = 75$
Bicycles transported from Factory W to Store F: $x_{wf} = 75$
Bicycles transported from Factory X to Store A: $x_{xa} = 25$
Bicycles transported from Factory X to Store D: $x_{xd} = 175$
Bicycles transported from Factory Y to Store C: $x_{yc} = 100$
Bicycles transported from Factory Y to Store E: $x_{ye} = 300$
Bicycles transported from Factory Z to Store B: $x_{zb} = 100$
Bicycles transported from Factory Z to Store C: $x_{zc} = 125$

Total Cost: $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \$17(25) + \$18(75) + \$13(75) + \$6(25) + \$5(175) + \$4(100) + \$3(300) + \$8(100) + \$6(125) = \$6,625$

Figure 4: *Initial basic feasible solution* obtained using the Least Cost Method. The transportation plan is summarized and its total cost is calculated.

C. Vogel's Approximation Method

Achieving an even more optimal initial basic feasible solution became possible with the creation of Vogel's Approximation Method which checks for the existence of similarly inexpensive alternative routes for each location before committing to the allocation of the absolute cheapest available route. The previous method of automatically assigning allocation to the route with the lowest unit transportation cost is flawed in that it is a greedy strategy, which selects the immediately best available option without concern for any future resulting consequences. This flaw is illustrated in our example by the automatic use of Factory Y to entirely supply Store E with its bicycles without ever considering nearby Factory W. While the \$3 route between Factory Y and Store E is certainly cheapest, Factory W can also supply Store E at a similarly low cost of \$5. The critical difference lies in the alternative options available to each factory. While Factory Y has several other cheap routes to alternative destinations, Factory W does not. In other words, the penalty of not using Factory W to supply Store E with its bicycles is much greater than the penalty of not using Factory Y, since only the latter can transport its supply to other stores at similarly low costs. The greedy strategy led to Factory W being forced to transport its remaining bicycles along the far and costly routes to supply Store A and Store B, a resulting consequence which is best avoided. Thus, Vogel's Approximation Method takes into consideration the best available alternative route for each location before making a decision in order to prioritize routes with the weakest alternative options. Each location's *penalty* is calculated by taking the difference between the two lowest unit transportation costs associated with that location. Then, allocation is assigned to the cheapest route in the row or column with the largest penalty.

Vogel's Approximation Method is summarized by the following 6 steps:

1. Calculate the penalty of each row and column.
2. Identify the cell with the lowest cost in the row/column with the highest penalty. (tie goes to the cell that can take a larger allocation)
3. Allocate the maximum number of materials under the supply and demand constraints.
4. Subtract the allocated amount from the cell's corresponding supply and demand.
5. Cross out the remaining cells of the exhausted row or column.
6. Repeat steps 1 - 5 until all materials have been allocated.

In order to apply Vogel's Approximation Method to our bicycle example, we add an extra row and column to our solution matrix for the penalties.

As previously noted, Factory W has the largest penalty in our example. Since the two cheapest routes associated with Factory W are the \$5 route to Store E and the \$10 routes to either Store C or Store D, failure to send the supply at Factory W to Store E will incur at least a \$5 penalty, more than any other location. Therefore, since Store E requires a demand of 300 bikes and Factory W has a supply of 175 bikes, we allocate all 175 bikes to be transported from Factory W to Store E. This exhausts the supply at Factory W and we can eliminate the rest of its row from consideration (Appendix C matrix a).

Having eliminated a row, we re-examine the penalty of each column. Now, Store E has the largest penalty of all remaining locations in our example. Since the two cheapest available routes associated with Store E are the \$3 route from Factory

Y and the $9 route from Factory Z, failure to supply Store E using Factory Y will incur at least a $6 penalty, more than any other remaining location. Therefore, since Factory Y has a supply of 400 bikes and Store E still requires 125 bikes to fulfill its demand, we allocate 125 bikes to be transported from Factory Y to Store E. This fulfills the demand of Store E and we can eliminate the rest of its column from consideration (Appendix C matrix b).

Having eliminated a column, we re-examine the penalty of each row. Now, both Store A and Store F share the largest penalty of all remaining locations in our example. The two cheapest available routes associated with Store A are the $6 route from Factory X and the $9 route from Factory Y, giving a penalty of $3 for failure to use Factory X to supply Store A. The two cheapest available routes associated with Store F are the $5 route from Factory Y and the $8 route from Factory X, also giving a penalty of $3 for failure to use Factory Y to supply Store F. In the case of a tie, we assign units to the route that can take the largest allocation. With a supply of 200 bikes at Factory X and a demand of 50 bikes at Store A, the route from Factory X to Store A can take a maximum allocation of 50 bikes. With a remaining supply of 275 bikes at Factory Y and a demand of 75 bikes at Store F, the route from Factory Y to Store F can take a maximum allocation of 75 bikes. Thus, we assign the larger allocation of 75 bikes to the transportation route from Factory Y to Store F. This fulfills the demand of Store F and we can eliminate the rest of its column from consideration (Appendix C matrix c).

Completing the remaining steps, shown in Appendix C, leads to a third possible transportation plan which the manufacturing company can use to successfully distribute its bicycles (Figure 5). The total cost of this initial basic feasible solution is $5,175. This transportation plan offers a 21.2% reduction in cost compared to the solution obtained by the Least Cost Method and a 43.9% reduction in cost compared to the solution obtained by the Northwest Corner Method.

	A	B	C	D	E	F	Supply
W	17	18	10	10	5 175	13	175
X	6 50	11	13	5 150	10	8	200
Y	9	10	4 175	4 25	3 125	5 75	400
Z	13	8 175	6 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Bicycles transported from Factory W to Store E: $x_{we} = 175$
Bicycles transported from Factory X to Store A: $x_{xa} = 50$
Bicycles transported from Factory X to Store D: $x_{xd} = 150$
Bicycles transported from Factory Y to Store C: $x_{xa} = 175$
Bicycles transported from Factory Y to Store D: $x_{xd} = 25$
Bicycles transported from Factory Y to Store E: $x_{yc} = 125$
Bicycles transported from Factory Y to Store F: $x_{ye} = 75$
Bicycles transported from Factory Z to Store B: $x_{zb} = 175$
Bicycles transported from Factory Z to Store C: $x_{zc} = 50$

Total Cost: $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \$5(175) + \$6(50) + \$5(150) + \$4(175) + \$4(25) + \$3(125) + \$5(75) + \$8(175) + \$6(50) = \$5,175$

Figure 5: *Initial basic feasible solution* obtained using Vogel's Approximation Method. The transportation plan is summarized and its total cost is calculated.

Optimization using the Stepping Stone Method

Given any transportation problem with m sources and n destinations, finding an initial basic feasible solution using any of the three aforementioned methods provides a valid transportation plan which successfully fulfills all supply and demand constraints. This transportation plan will necessarily utilize at most $m+n-1$ of the $m*n$ possible routes between our source and destination locations. The $m+n-1$ x_{ij} decision variables representing the allocation amount to these utilized routes are known as *basic variables*, since together they form a *basis* - the underlying support or foundation - which uniquely defines the transportation plan. All other x_{ij} decision variables are *non-basic variables* and will necessarily have a value of zero since their routes are not assigned any allocation in the transportation plan.

A solution to the transportation problem can only be declared optimal once it is no longer possible to lower the total transportation cost any further by introducing any of the non-basic routes into the basis. In other words, the optimal solution will be the transportation plan such that none of the unused routes can provide a more cost effective alternative. However, so long as there exists a non-basic route whose utilization can help lower the total transportation cost, such a transportation plan can still be optimized.

The optimization algorithm used to solve the transportation problem is known as the Stepping Stone Method. First, the method calculates the *opportunity cost* of each non-basic route, that is, we calculate how much money would be saved (or lost) in comparison to the current solution should one unit of material be assigned to that given route. Since the total number of units available in each transportation problem remains constant, changing the allocation of one route will necessarily require adjusting the allocation of other routes to comply with all supply and demand constraints. The addition of one unit of allocation to a non-basic route triggers a chain reaction, forcing the addition and removal of a unit of allocation to and from other routes until all constraints are once again satisfied. Every non-basic route will have exactly one such viable path, known as its *closed path*. By adding and subtracting the c_{ij} of each route along the closed path, the opportunity cost of each non-basic route can be calculated to determine if its introduction into the transportation plan will help lower the total transportation cost. Using the initial basic feasible solution obtained from the Least Cost Method, Figure 6 illustrates finding the closed path of the non-basic route from Factory W to Store C and calculating its opportunity cost. Assigning this allocation to x_{wc} adds to the total number of bicycles departing from Factory W, thereby exceeding its total available supply. This forces us to remove one bicycle from a different route departing from Factory W. We will remove one bicycle allocation from x_{wb}. Now, Store B is receiving one less bicycle than required by its demand. This forces us to add one bicycle to a different route delivering to Store B. We will add one bicycle allocation to x_{zb}. Now, the total number of bicycles departing from Factory Z exceeds its total available supply. This forces us to remove one bicycle from a different route departing from Factory Z. We will remove one bicycle allocation from x_{zc}. Although Store C now receives one less bicycle, this completes our cycle, as Store C was the original destination to which we assigned the extra bicycle allocation.[‡] Finally, we calculate the opportunity cost of x_{wc} by moving along its closed path, adding the c_{ij} of all x_{ij} where one bicycle unit of allocation was added and subtracting the c_{ij} of all x_{ij} where one bicycle unit of allocation was removed, as shown in Figure 6. Thus, the opportunity cost of utilizing the transportation route from Factory W to Store C is negative \$6. In other words, six dollars per bicycle can be saved compared to our current solution by including this route in our transportation plan. This process is repeated for every non-basic route, shown in Appendix D, and the route with the most negative opportunity cost is chosen to enter the basis. In this case, the route from Factory W to Store E, with an opportunity cost saving \$10 per bicycle as compared to our current solution, is selected to enter our basis.

[‡] As previously mentioned, only one possible closed path exists for every non-basic route. While we could have decided to remove one bicycle allocation from x_{wa} or x_{wf}, neither option would have lead to a complete cycle.

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17 25	18 − 75	10 +	10	5	13 75	175
X	6 25	11	13	5 175	10	8	200
Y	9	10	4 100	4	3 300	5	400
Z	13	8 + 100	6 − 125	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Opportunity Cost of x_{wc}: $c_{wc} - c_{wb} + c_{zb} - c_{zc} = \$10 - \$18 + \$8 - \$6 = -\$6$

Figure 6: The red arrow follows the *closed path* of the non-basic route x_{wc}. Addition '+' and subtraction '-' signs are added to indicate the changes in allocation to the routes along the path. The opportunity cost of x_{wc} is calculated: allocation is added to the \$10 route x_{wc}, allocation is subtracted from the \$18 route x_{wb}, allocation is added to the \$8 route x_{zb}, and allocation is subtracted from the \$6 route x_{zc}, netting a difference of negative \$6.

The next step requires determining how many units of material can be allocated to the non-basic route entering the basis of our current solution while still adhering to all constraints. Naturally, we want to allocate the maximum possible number of units to save the most money. Due to the constraint $\forall\ x_{ij} > 0$, and since assigning allocation to a new route necessarily requires subtracting that allocation amount from other routes along its closed path, the maximum possible number of units we can assign to the non-basic route entering the basis will be the minimum assigned allocation among all x_{ij} along the closed path from which we subtract units. Continuing with our example, Figure 7 shows the closed path of route x_{we} entering the basis. In order to allocate bicycles to x_{we}, we must subtract from routes x_{wb} transporting 75 bicycles, x_{zc} transporting 125 bicycles, and x_{ye} transporting 300 bicycles. Thus, the maximum possible number of bicycles we can assign to x_{we} is 75, and we adjust our current solution to reflect this change by adding and subtracting those 75 bicycles to the routes along the closed path (Figure 7). With each bicycle unit saving \$10, this iteration of the optimization lowers our total transportation cost by \$750. Each optimization iteration will save money equivalent to the opportunity cost of the non-basic route entering the basis multiplied by the number of units allocated to that non-basic route.

The Stepping Stone Method is repeated until the opportunity cost of every non-basic route is non-negative, indicating that no further money can be saved by editing the current solution to include an alternative route.

The Stepping Stone Method is summarized by the following 5 steps:

1. Find an initial basic feasible solution.

	A	B	C	D	E	F	Supply
W	17 25	18 – 75	10	10	5 +	13 75	175
X	6 25	11	13	5 175	10	8	200
Y	9	10	4 + 100	4	3 – 300	5	400
Z	13	8 + 100	6 – 125	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Opportunity Cost of x_{we}: $5 − $18 + $8 − $6 + $4 − $3 = −$10

	A	B	C	D	E	F	Supply
W	17 25	18	10	10	5 75	13 75	175
X	6 25	11	13	5 175	10	8	200
Y	9	10	4 175	4	3 225	5	400
Z	13	8 175	6 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Money saved this iteration: (opportunity cost) * (units allocated) = $10 * 75 = $750

Figure 7: The red arrow follows the *closed path* of the non-basic route x_{we} which has the best *opportunity cost* and will therefore enter the basis. Addition '+' and subtraction '-' signs are added to indicate the changes in allocation to the routes along the path. Since allocation must be subtracted from x_{wb}, x_{zc}, and x_{ye} we can allocate at most 75 units to x_{we}. These 75 units are added and subtracted from the routes along the closed path to attain a new, more optimal transportation plan. The money saved in this iteration is calculated.

2. Calculate the opportunity cost of each non-basic route.
   (a) Find its closed path
   (b) Alternate labeling along the closed path with '+' and '-'
   (c) Add and subtract the costs along its closed path
3. If all opportunity costs are non-negative, you are done. Otherwise, identify the non-basic route with the lowest opportunity cost.
4. Edit its closed path.
   (a) Identify the minimum allocation value in a '-' labeled cell along the path
   (b) Add / subtract that value to each cell along the closed path accordingly
5. Return to step 2.

Completing the remaining iterations, shown in Appendix D, leads us to the optimal solution to our transportation problem, shown in Figure 8. This transportation plan amounts to a total of \$5,175 and is the most cost effective way for the bicycle manufacturing company to distribute its bicycles from its factories to its stores while fulfilling all requirements and constraints.[§]

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10	10	5 175	13	175
X	6 50	11	13	5 150	10	8	200
Y	9	10	4 175	4 25	3 125	5 75	400
Z	13	8 175	6 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Bicycles transported from Factory W to Store E: $x_{we} = 175$
Bicycles transported from Factory X to Store A: $x_{xa} = 50$
Bicycles transported from Factory X to Store D: $x_{xd} = 150$
Bicycles transported from Factory Y to Store C: $x_{xa} = 175$
Bicycles transported from Factory Y to Store D: $x_{xd} = 25$
Bicycles transported from Factory Y to Store E: $x_{yc} = 125$
Bicycles transported from Factory Y to Store F: $x_{ye} = 75$
Bicycles transported from Factory Z to Store B: $x_{zb} = 175$
Bicycles transported from Factory Z to Store C: $x_{zc} = 50$

Total Cost: $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \$5(175) + \$6(50) + \$5(150) + \$4(175) + \$4(25) + \$3(125) + \$5(75) + \$8(175) + \$6(50) = \$5,175$

Figure 8: Final *optimal solution* to our transportation problem. The transportation plan is summarized and its total cost is calculated. This is the best possible way for the bicycle manufacturing company to minimize the total cost of transportation while fulfilling all supply and demand constraints. The total cost is \$5,175.

Special Cases

A. Unbalanced Problems

Thus far, we have only considered *balanced transportation problems* where the total available supply equaled the total required demand. However, the majority of practical applications found across industry deal with *unbalanced transportation problems* in which supply exceeds demand or vice versa. The case of demand exceeding supply is trivially 'no solution', as the limited supply cannot meet the required demand. On the other hand, a solution certainly exists when there is a surplus in

[§]Note: the optimal solution for this transportation problem happens to be the same transportation plan obtained using Vogel's Approximation Method. However, this is just a coincidence and not always the case, especially in extremely large problems typically found in industry.

supply. In this case, the surplus number of units of material that should remain at each source must also be determined. This is done by adding an extra dummy destination to our transportation problem which will require a demand equal to the surplus. The unit transportation cost from each source to this dummy destination will be $0, and any allocation assigned to those "routes" in our final transportation plan will simply remain at the source location at no cost.

B. Degeneracy

Another situation we have yet to consider is the degenerate case. A solution is considered degenerate if it is not yet optimal and the number of allocated routes in its transportation plan is less than $m + n - 1$. During the initial phase of solving a transportation problem, degeneracy occurs if both a supply constraint and a demand constraint are satisfied simultaneously before the final allocation is assigned. During the optimization phase, degeneracy occurs if multiple routes from which we subtract units along a closed path share the minimum allocation amount. The degenerate case presents an issue for optimization via the Stepping Stone Method since not every non-basic route of a degenerate solution will have a complete closed path that can be used to determine its opportunity cost. This is resolved by artificially adding a non-basic route into the basis, assigning it a phantom allocation of zero. The chosen route should specifically be the non-basic route with the smallest unit transportation cost which cannot complete a closed path.

Conclusions

The formulation of the mathematical model of linear programming to represent optimization problems has undoubtedly allowed for massive improvements to many facets of the industrial process. Specifically, its application to the transportation problem has contributed heavily to the development of key business areas such as operations research and logistical management. The invention of algorithms to systematically and accurately determine the best distribution of resources has allowed for greater efficiency in industrial planning, enabling both exponential growth in production capabilities and major improvement in delivery service and customer satisfaction. Public transportation authorities employ this model to map out optimal bus routes and railway systems for commuters. It is used by airlines to schedule flight plans and shipping companies to organize international trade. Manufacturing companies and supply chain management personnel use it to minimize the cost of transporting everything from the produce at local supermarkets to the clothing sold at shopping malls to the essentials bought at Costco. Business-to-customer e-commerce companies rely on this model to provide exemplary delivery service. A noteworthy example of this would be the logistical planning required by Amazon to service the millions of orders received every day. This model can even be applied to the more altruistic effort of quickly mobilizing aid for disaster relief around the world. No longer must making these logistical decisions be done by hand through trial and error. Rather, mechanizing the process has allowed for optimal solutions to be acquired within minutes, with high certainty in achieving the best outcome.

As both markets and the general population continue to grow, these practical applications of the transportation problem are only going to increase in size and complexity. Even today, real-life applications deal with a plethora of factors and decision variables that make them orders of magnitude harder than anything a human could compute independently. Thus, developing fast algorithms to solve these problems and writing computer programs to execute them is essential to obtaining solutions in a timely manner. Further research ought to be devoted to quantifying the true efficiencies of each method for the initial phase. Can we produce a model to predict the average number of iterations required to reach the optimal solution based on the size of the problem? How does this relationship behave? Exactly how much better is one method over the other? Which method's coding implementation offers the best time complexity? Is an even better method yet to be found? Since its formulation during World War II, this form of optimization has already unimaginably transformed the industrial process. With implications that effect everyday life, investing more effort into further developing this field undoubtedly has the potential to both increase the efficiency of current processes and accelerate the timeline of future innovation.

APPENDIX A - Northwest Corner Method

	A	B	C	D	E	F	Supply
W	17 / 50	18	10	10	5	13	~~175~~ 125
X	6 / —	11	13	5	10	8	200
Y	9 / —	10	4	4	3	5	400
Z	13 / —	8	6	14	9	11	225
Demand	~~50~~ 0	175	225	175	300	75	Completed: 50

(a)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	~~125~~ 0
X	6 / —	11	13	5	10	8	200
Y	9 / —	10	4	4	3	5	400
Z	13 / —	8	6	14	9	11	225
Demand	0	~~175~~ 50	225	175	300	75	Completed: 175

(b)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13	5	10	8	~~200~~ 150
Y	9 / —	10 / —	4	4	3	5	400
Z	13 / —	8 / —	6	14	9	11	225
Demand	0	~~50~~ 0	225	175	300	75	Completed: 225

(c)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13 / 150	5 / —	10 / —	8 / —	~~150~~ 0
Y	9 / —	10 / —	4	4	3	5	400
Z	13 / —	8 / —	6	14	9	11	225
Demand	0	0	~~225~~ 75	175	300	75	Completed: 375

(d)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13 / 150	5 / —	10 / —	8 / —	0
Y	9 / —	10 / —	4 / 75	4	3	5	~~400~~ 325
Z	13 / —	8 / —	6 / —	14	9	11	225
Demand	0	0	~~75~~ 0	175	300	75	Completed: 450

(e)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13 / 150	5 / —	10 / —	8 / —	0
Y	9 / —	10 / —	4 / 75	4 / 175	3	5	~~325~~ 150
Z	13 / —	8 / —	6 / —	14 / —	9	11	225
Demand	0	0	0	~~175~~ 0	300	75	Completed: 625

(f)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13 / 150	5 / —	10 / —	8 / —	0
Y	9 / —	10 / —	4 / 75	4 / 175	3 / 150	5 / —	~~150~~ 0
Z	13 / —	8 / —	6 / —	14 / —	9	11	225
Demand	0	0	0	0	~~300~~ 150	75	Completed: 775

(g)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13 / 150	5 / —	10 / —	8 / —	0
Y	9 / —	10 / —	4 / 75	4 / 175	3 / 150	5 / —	0
Z	13 / —	8 / —	6 / —	14 / —	9 / 150	11	~~225~~ 75
Demand	0	0	0	0	~~150~~ 0	75	Completed: 925

(h)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10 / —	10 / —	5 / —	13 / —	0
X	6 / —	11 / 50	13 / 150	5 / —	10 / —	8 / —	0
Y	9 / —	10 / —	4 / 75	4 / 175	3 / 150	5 / —	0
Z	13 / —	8 / —	6 / —	14 / —	9 / 150	11 / 75	~~75~~ 0
Demand	0	0	0	0	0	~~75~~ 0	Completed: 1,000

(i)

	A	B	C	D	E	F	Supply
W	17 / 50	18 / 125	10	10	5	13	175
X	6	11 / 50	13 / 150	5	10	8	200
Y	9	10	4 / 75	4 / 175	3 / 150	5	400
Z	13	8	6	14	9 / 150	11 / 75	225
Demand	50	175	225	175	300	75	Completed: 1,000

(j)

APPENDIX B - Least Cost Method

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10	10	5 —	13	175
X	6	11	13	5	10 —	8	200
Y	9	10	4	4	3 300	5	~~400~~ 100
Z	13	8	6	14	9 —	11	225
Demand	50	175	225	175	~~300~~ 0	75	Completed: 300

(a)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10	10	5 —	13	175
X	6	11	13	5	10 —	8	200
Y	9 —	10 —	4 100	4 —	3 300	5 —	~~100~~ 0
Z	13	8	6	14	9 —	11	225
Demand	50	175	~~225~~ 125	175	0	75	Completed: 400

(b)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10	10 —	5 —	13	175
X	6	11	13	5 175	10 —	8	~~200~~ 25
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13	8	6	14 —	9 —	11	225
Demand	50	175	125	~~175~~ 0	0	75	Completed: 575

(c)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10 —	10 —	5 —	13	175
X	6	11	13 —	5 175	10 —	8	25
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13	8	6 125	14 —	9 —	11	~~225~~ 100
Demand	50	175	~~125~~ 0	0	0	75	Completed: 700

(d)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10 —	10 —	5 —	13	175
X	6 25	11 —	13 —	5 175	10 —	8 —	~~25~~ 0
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13	8	6 125	14 —	9 —	11	100
Demand	~~50~~ 25	175	0	0	0	75	Completed: 725

(e)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10 —	10 —	5 —	13	175
X	6 25	11 —	13 —	5 175	10 —	8 —	0
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13 —	8 100	6 125	14 —	9 —	11 —	~~100~~ 0
Demand	25	~~175~~ 75	0	0	0	75	Completed: 825

(f)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17	18	10 —	10 —	5 —	13 75	~~175~~ 100
X	6 25	11 —	13 —	5 175	10 —	8 —	0
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13 —	8 100	6 125	14 —	9 —	11 —	0
Demand	25	75	0	0	0	~~75~~ 0	Completed: 900

(g)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17 25	18	10 —	10 —	5 —	13 75	~~100~~ 75
X	6 25	11 —	13 —	5 175	10 —	8 —	0
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13 —	8 100	6 125	14 —	9 —	11 —	0
Demand	~~25~~ 0	75	0	0	0	0	Completed: 925

(h)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17 25	18 75	10 —	10 —	5 —	13 75	~~75~~ 0
X	6 25	11 —	13 —	5 175	10 —	8 —	0
Y	9 —	10 —	4 100	4 —	3 300	5 —	0
Z	13 —	8 100	6 125	14 —	9 —	11 —	0
Demand	0	~~75~~ 0	0	0	0	0	Completed: 1,000

(i)

	A	**B**	**C**	**D**	**E**	**F**	**Supply**
W	17 25	18 75	10	10	5	13 75	175
X	6 25	11	13	5 175	10	8	200
Y	9	10	4 100	4	3 300	5	400
Z	13	8 100	6 125	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

(j)

APPENDIX C - Vogel's Approximation Method

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	5	~~175~~ 0
X	6	11	13	5	10	8	1	200
Y	9	10	4	4	3	5	1	400
Z	13	8	6	14	9	11	2	225
Penalty	3	2	2	1	2	3		
Demand	50	175	225	175	~~300~~ 125	75		Completed: 175

(a)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6	11	13	5	10 —	8	1	200
Y	9	10	4	4	3 125	5	1	~~400~~ 275
Z	13	8	6	14	9 —	11	2	225
Penalty	3	2	2	1	6	3		
Demand	50	175	225	175	~~125~~ 0	75		Completed: 300

(b)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6	11	13	5	10 —	8 —	1	200
Y	9	10	4	4	3 125	5 75	0	~~275~~ 200
Z	13	8	6	14	9 —	11 —	2	225
Penalty	3	2	2	1	x	3		
Demand	50	175	225	175	0	~~75~~ 0		Completed: 375

(c)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6 50	11	13	5	10 —	8 —	1	~~200~~ 150
Y	9 —	10	4	4	3 125	5 75	0	200
Z	13 —	8	6	14	9 —	11 —	2	225
Penalty	3	2	2	1	x	x		
Demand	~~50~~ 0	175	225	175	0	0		Completed: 425

(d)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6 50	11 —	13 —	5 150	10 —	8 —	6	~~150~~ 0
Y	9 —	10	4	4	3 125	5 75	0	200
Z	13 —	8	6	14	9 —	11 —	2	225
Penalty	x	2	2	1	x	x		
Demand	0	175	225	~~175~~ 25	0	0		Completed: 575

(e)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6 50	11 —	13 —	5 150	10 —	8 —	x	0
Y	9 —	10	4	4 25	3 125	5 75	0	~~200~~ 175
Z	13 —	8	6	14 —	9 —	11 —	2	225
Penalty	x	2	2	10	x	x		
Demand	0	175	225	~~25~~ 0	0	0		Completed: 600

(f)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6 50	11 —	13 —	5 150	10 —	8 —	x	0
Y	9 —	10 —	4 175	4 25	3 125	5 75	6	~~175~~ 0
Z	13 —	8	6	14 —	9 —	11 —	2	225
Penalty	x	2	2	x	x	x		
Demand	0	175	~~225~~ 50	0	0	0		Completed: 775

(g)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6 50	11 —	13 —	5 150	10 —	8 —	x	0
Y	9 —	10 —	4 175	4 25	3 125	5 75	x	0
Z	13 —	8	6 50	14 —	9 —	11 —	2	~~225~~ 175
Penalty	x	0	0	x	x	x		
Demand	0	175	~~50~~ 0	0	0	0		Completed: 825

(h)

	A	B	C	D	E	F	P	Supply
W	17 —	18 —	10 —	10 —	5 175	13 —	x	0
X	6 50	11 —	13 —	5 150	10 —	8 —	x	0
Y	9 —	10 —	4 175	4 25	3 125	5 75	x	0
Z	13 —	8 175	6 50	14 —	9 —	11 —	0	~~175~~ 0
Penalty	x	0	x	x	x	x		
Demand	0	~~175~~ 0	0	0	0	0		Completed: 1,000

(i)

	A	B	C	D	E	F	Supply
W	17	18	10	10	5 175	13	175
X	6 50	11	13	5 150	10	8	200
Y	9	10	4 175	4 25	3 125	5 75	400
Z	13	8 175	6 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

(j)

APPENDIX D - Stepping Stone Method

Iteration 1

	A	B	C	D	E	F	Supply
W	$\boxed{17}$ 25	$\boxed{18}-$ 75	$\boxed{10}$	$\boxed{10}$	$\boxed{5}+$	$\boxed{13}$ 75	175
X	$\boxed{6}$ 25	$\boxed{11}$	$\boxed{13}$	$\boxed{5}$ 175	$\boxed{10}$	$\boxed{8}$	200
Y	$\boxed{9}$	$\boxed{10}$	$\boxed{4}+$ 100	$\boxed{4}$	$\boxed{3}-$ 300	$\boxed{5}$	400
Z	$\boxed{13}$	$\boxed{8}+$ 100	$\boxed{6}-$ 125	$\boxed{14}$	$\boxed{9}$	$\boxed{11}$	225
Demand	50	175	225	175	300	75	Completed: 1,000

	A	B	C	D	E	F	Supply
W	$\boxed{17}$ 25	$\boxed{18}$	$\boxed{10}$	$\boxed{10}$	$\boxed{5}$ 75	$\boxed{13}$ 75	175
X	$\boxed{6}$ 25	$\boxed{11}$	$\boxed{13}$	$\boxed{5}$ 175	$\boxed{10}$	$\boxed{8}$	200
Y	$\boxed{9}$	$\boxed{10}$	$\boxed{4}$ 175	$\boxed{4}$	$\boxed{3}$ 225	$\boxed{5}$	400
Z	$\boxed{13}$	$\boxed{8}$ 175	$\boxed{6}$ 50	$\boxed{14}$	$\boxed{9}$	$\boxed{11}$	225
Demand	50	175	225	175	300	75	Completed: 1,000

Opportunity Costs

$$
\begin{array}{lll}
x_{wc} \rightarrow c_{wc} - c_{wb} + c_{zb} - c_{zc} & = \$10 - \$18 + \$8 - \$6 & = -\$6 \\
x_{wd} \rightarrow c_{wd} - c_{wa} + c_{xa} - c_{xd} & = \$10 - \$17 + \$6 - \$5 & = -\$6 \\
x_{we} \rightarrow c_{we} - c_{wb} + c_{zb} - c_{zc} + c_{yc} - c_{ye} & = \$5 - \$18 + \$8 - \$6 + \$4 - \$3 & = -\$10 \leftarrow \\
x_{xb} \rightarrow c_{xb} - c_{wb} + c_{wa} - c_{xa} & = \$11 - \$18 + \$17 - \$6 & = \$4 \\
x_{xc} \rightarrow c_{xc} - c_{xa} + c_{wa} - c_{wb} + c_{zb} - c_{zc} & = \$13 - \$6 + \$17 - \$18 + \$8 - \$6 & = \$8 \\
x_{xe} \rightarrow c_{xe} - c_{xa} + c_{wa} - c_{wb} + c_{zb} - c_{zc} + c_{yc} - c_{ye} & = \$10 - \$6 + \$17 - \$18 + \$8 - \$6 + \$4 - \$3 & = \$6 \\
x_{xf} \rightarrow c_{xf} - c_{wf} + c_{wa} - c_{xa} & = \$8 - \$13 + \$17 - \$6 & = \$6 \\
x_{ya} \rightarrow c_{ya} - c_{wa} + c_{wb} - c_{zb} + c_{zc} - c_{yc} & = \$9 - \$17 + \$18 - \$8 + \$6 - \$4 & = \$4 \\
x_{yb} \rightarrow c_{yb} - c_{zb} + c_{zc} - c_{yc} & = \$10 - \$8 + \$8 - \$4 & = \$4 \\
x_{yd} \rightarrow c_{yd} - c_{xd} + c_{xa} - c_{wa} + c_{wb} - c_{zb} + c_{zc} - c_{yc} & = \$4 - \$5 + \$6 - \$17 + \$18 - \$8 + \$6 - \$4 & = \$0 \\
x_{yf} \rightarrow c_{yf} - c_{wf} + c_{wb} - c_{zb} + c_{zc} - c_{yc} & = \$5 - \$13 + \$18 - \$8 + \$6 - \$4 & = \$4 \\
x_{za} \rightarrow c_{za} - c_{zb} + c_{wb} - c_{wa} & = \$13 - \$8 + \$18 - \$17 & = \$6 \\
x_{zd} \rightarrow c_{zd} - c_{xd} + c_{xa} - c_{wa} + c_{wb} - c_{zb} & = \$14 - \$5 + \$6 - \$17 + \$18 - \$8 & = \$8 \\
x_{ze} \rightarrow c_{ze} - c_{ye} + c_{yc} - c_{zc} & = \$9 - \$3 + \$4 - \$6 & = \$4 \\
x_{zf} \rightarrow c_{zf} - c_{wf} + c_{wb} - c_{zb} & = \$11 - \$13 + \$18 - \$8 & = \$8
\end{array}
$$

The best non-basic route to enter into the basis is x_{we} at an opportunity cost of $10 for every bicycle reassigned to this route. Its closed path allows for the reallocation of at most 75 bicycles to this route. With this iteration we save $\$10 * 75 = \$750$.

Iteration 2

	A	B	C	D	E	F	Supply
W	$\boxed{17}-$ 25	$\boxed{18}$	$\boxed{10}$	$\boxed{10}$	$\boxed{5}+$ 75	$\boxed{13}$ 75	175
X	$\boxed{6}+$ 25	$\boxed{11}$	$\boxed{13}$	$\boxed{5}-$ 175	$\boxed{10}$	$\boxed{8}$	200
Y	$\boxed{9}$	$\boxed{10}$	$\boxed{4}$ 175	$\boxed{4}+$	$\boxed{3}-$ 225	$\boxed{5}$	400
Z	$\boxed{13}$	$\boxed{8}$ 175	$\boxed{6}$ 50	$\boxed{14}$	$\boxed{9}$	$\boxed{11}$	225
Demand	50	175	225	175	300	75	Completed: 1,000

	A	B	C	D	E	F	Supply
W	$\boxed{17}$	$\boxed{18}$	$\boxed{10}$	$\boxed{10}$	$\boxed{5}$ 100	$\boxed{13}$ 75	175
X	$\boxed{6}$ 50	$\boxed{11}$	$\boxed{13}$	$\boxed{5}$ 150	$\boxed{10}$	$\boxed{8}$	200
Y	$\boxed{9}$	$\boxed{10}$	$\boxed{4}$ 175	$\boxed{4}$ 25	$\boxed{3}$ 200	$\boxed{5}$	400
Z	$\boxed{13}$	$\boxed{8}$ 175	$\boxed{6}$ 50	$\boxed{14}$	$\boxed{9}$	$\boxed{11}$	225
Demand	50	175	225	175	300	75	Completed: 1,000

Opportunity Costs

$$
\begin{array}{lll}
x_{wb} \rightarrow c_{wb} - c_{zb} + c_{zc} - c_{yc} + c_{ye} - c_{we} & = \$18 - \$8 + \$6 - \$4 + \$3 - \$5 & = \$10 \\
x_{wc} \rightarrow c_{wc} - c_{yc} + c_{ye} - c_{we} & = \$10 - \$4 + \$3 - \$5 & = \$4 \\
x_{wd} \rightarrow c_{wd} - c_{wa} + c_{xa} - c_{xd} & = \$10 - \$17 + \$6 - \$5 & = -\$6 \\
x_{xb} \rightarrow c_{xb} - c_{zb} + c_{zc} - c_{yc} + c_{ye} - c_{we} + c_{wa} - c_{xa} & = \$11 - \$8 + \$6 - \$4 + \$3 - \$5 + \$17 - \$6 & = \$14 \\
x_{xc} \rightarrow c_{xc} - c_{yc} + c_{ye} - c_{we} + c_{wa} - c_{xa} & = \$13 - \$4 + \$3 - \$5 + \$17 - \$6 & = \$18 \\
x_{xe} \rightarrow c_{xe} - c_{we} + c_{wa} - c_{xa} & = \$10 - \$5 + \$17 - \$6 & = \$16 \\
x_{xf} \rightarrow c_{xf} - c_{wf} + c_{wa} - c_{xa} & = \$8 - \$13 + \$17 - \$6 & = \$6 \\
x_{ya} \rightarrow c_{ya} - c_{ye} + c_{we} - c_{wa} & = \$9 - \$3 + \$5 - \$17 & = -\$6 \\
x_{yb} \rightarrow c_{yb} - c_{zb} + c_{zc} - c_{yc} & = \$10 - \$8 + \$6 - \$4 & = \$4 \\
x_{yd} \rightarrow c_{yd} - c_{ye} + c_{we} - c_{wa} + c_{xa} - c_{xd} & = \$4 - \$3 + \$5 - \$17 + \$6 - \$5 & = -\$10 \leftarrow \\
x_{yf} \rightarrow c_{yf} - c_{wf} + c_{we} - c_{ye} & = \$5 - \$13 + \$5 - \$3 & = -\$6 \\
x_{za} \rightarrow c_{za} - c_{zc} + c_{yc} - c_{ye} + c_{we} - c_{wa} & = \$13 - \$6 + \$4 - \$3 + \$5 - \$17 & = -\$4 \\
x_{zd} \rightarrow c_{zd} - c_{zc} + c_{yc} - c_{ye} + c_{we} - c_{wa} + c_{xa} - c_{xd} & = \$14 - \$6 + \$4 - \$3 + \$5 - \$17 + \$6 - \$5 & = -\$2 \\
x_{ze} \rightarrow c_{ze} - c_{ye} + c_{yc} - c_{zc} & = \$9 - \$3 + \$4 - \$6 & = \$4 \\
x_{zf} \rightarrow c_{zf} - c_{wf} + c_{we} - c_{ye} + c_{yc} - c_{zc} & = \$11 - \$13 + \$5 - \$3 + \$4 - \$6 & = -\$2
\end{array}
$$

The best non-basic route to enter into the basis is x_{yd} at an opportunity cost of $10 for every bicycle reassigned to this route. Its closed path allows for the reallocation of at most 25 bicycles to this route. With this iteration we save $\$10 * 25 = \$250$.

Iteration 3

	A	B	C	D	E	F	Supply
W	17	18	10	10	5 + / 100	13 − / 75	175
X	6 / 50	11	13	5 / 150	10	8	200
Y	9	10	4 / 175	4 / 25	3 − / 200	5 +	400
Z	13	8 / 175	6 / 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

	A	B	C	D	E	F	Supply
W	17	18	10	10	5 / 175	13	175
X	6 / 50	11	13	5 / 150	10	8	200
Y	9	10	4 / 175	4 / 25	3 / 125	5 / 75	400
Z	13	8 / 175	6 / 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Opportunity Costs

$x_{wa} \rightarrow c_{wa} - c_{xa} + c_{xd} - c_{yd} + c_{ye} - c_{we} = \$17 - \$6 + \$5 - \$4 + \$3 - \$5 = \$10$

$x_{wb} \rightarrow c_{wb} - c_{zb} + c_{zc} - c_{yc} + c_{ye} - c_{we} = \$18 - \$8 + \$6 - \$4 + \$3 - \$5 = \$10$

$x_{wc} \rightarrow c_{wc} - c_{yc} + c_{ye} - c_{we} = \$10 - \$4 + \$3 - \$5 = \$4$

$x_{wd} \rightarrow c_{wd} - c_{yd} + c_{ye} - c_{we} = \$10 - \$4 + \$3 - \$5 = \$4$

$x_{xb} \rightarrow c_{xb} - c_{zb} + c_{zc} - c_{yc} + c_{yd} - c_{xd} = \$11 - \$8 + \$6 - \$4 + \$4 - \$5 = \$4$

$x_{xc} \rightarrow c_{xc} - c_{yc} + c_{yd} - c_{xd} = \$13 - \$4 + \$4 - \$5 = \$8$

$x_{xe} \rightarrow c_{xe} - c_{xd} + c_{yd} - c_{ye} = \$10 - \$5 + \$4 - \$3 = \$6$

$x_{xf} \rightarrow c_{xf} - c_{wf} + c_{we} - c_{ye} + c_{yd} - c_{xd} = \$8 - \$13 + \$5 - \$3 + \$4 - \$5 = -\$4$

$x_{ya} \rightarrow c_{ya} - c_{yd} + c_{xd} - c_{xa} = \$9 - \$4 + \$5 - \$6 = \$4$

$x_{yb} \rightarrow c_{yb} - c_{zb} + c_{zc} - c_{yc} = \$10 - \$8 + \$6 - \$4 = \$4$

$x_{yf} \rightarrow c_{yf} - c_{wf} + c_{we} - c_{ye} = \$5 - \$13 + \$5 - \$3 = -\$6 \leftarrow$

$x_{za} \rightarrow c_{za} - c_{zc} + c_{yc} - c_{yd} + c_{xd} - c_{xa} = \$13 - \$6 + \$4 - \$4 + \$5 - \$6 = \$6$

$x_{zd} \rightarrow c_{zd} - c_{yd} + c_{yc} - c_{zc} = \$14 - \$4 + \$4 - \$6 = \$8$

$x_{ze} \rightarrow c_{ze} - c_{ye} + c_{yc} - c_{zc} = \$9 - \$3 + \$4 - \$6 = \$4$

$x_{zf} \rightarrow c_{zf} - c_{wf} + c_{we} - c_{ye} + c_{yc} - c_{zc} = \$11 - \$13 + \$5 - \$3 + \$4 - \$6 = -\$2$

The best non-basic route to enter into the basis is x_{yf} at an opportunity cost of \$6 for every bicycle reassigned to this route. Its closed path allows for the reallocation of at most 75 bicycles to this route. With this iteration we save $\$6 * 75 = \$450$.

Iteration 4

	A	B	C	D	E	F	Supply
W	17	18	10	10	5 / 175	13	175
X	6 / 50	11	13	5 / 150	10	8	200
Y	9	10	4 / 175	4 / 25	3 / 125	5 / 75	400
Z	13	8 / 175	6 / 50	14	9	11	225
Demand	50	175	225	175	300	75	Completed: 1,000

Opportunity Costs

$x_{wa} \rightarrow c_{wa} - c_{xa} + c_{xd} - c_{yd} + c_{ye} - c_{we} = \$17 - \$6 + \$5 - \$4 + \$3 - \$5 = \$10$

$x_{wb} \rightarrow c_{wb} - c_{zb} + c_{zc} - c_{yc} + c_{ye} - c_{we} = \$18 - \$8 + \$6 - \$4 + \$3 - \$5 = \$10$

$x_{wc} \rightarrow c_{wc} - c_{yc} + c_{ye} - c_{we} = \$10 - \$4 + \$3 - \$5 = \$4$

$x_{wd} \rightarrow c_{wd} - c_{yd} + c_{ye} - c_{we} = \$10 - \$4 + \$3 - \$5 = \$4$

$x_{wf} \rightarrow c_{wf} - c_{we} + c_{ye} - c_{yf} = \$13 - \$5 + \$3 - \$5 = \$6$

$x_{xb} \rightarrow c_{xb} - c_{zb} + c_{zc} - c_{yc} + c_{yd} - c_{xd} = \$11 - \$8 + \$6 - \$4 + \$4 - \$5 = \$4$

$x_{xc} \rightarrow c_{xc} - c_{yc} + c_{yd} - c_{xd} = \$13 - \$4 + \$4 - \$5 = \$8$

$x_{xe} \rightarrow c_{xe} - c_{xd} + c_{yd} - c_{ye} = \$10 - \$5 + \$4 - \$3 = \$6$

$x_{xf} \rightarrow c_{xf} - c_{xd} + c_{yd} - c_{yf} = \$8 - \$5 + \$4 - \$5 = \$2$

$x_{ya} \rightarrow c_{ya} - c_{yd} + c_{xd} - c_{xa} = \$9 - \$4 + \$5 - \$6 = \$4$

$x_{yb} \rightarrow c_{yb} - c_{zb} + c_{zc} - c_{yc} = \$10 - \$8 + \$6 - \$4 = \$4$

$x_{za} \rightarrow c_{za} - c_{zc} + c_{yc} - c_{yd} + c_{xd} - c_{xa} = \$13 - \$6 + \$4 - \$4 + \$5 - \$6 = \$6$

$x_{zd} \rightarrow c_{zd} - c_{yd} + c_{yc} - c_{zc} = \$14 - \$4 + \$4 - \$6 = \$8$

$x_{ze} \rightarrow c_{ze} - c_{ye} + c_{yc} - c_{zc} = \$9 - \$3 + \$4 - \$6 = \$4$

$x_{zf} \rightarrow c_{zf} - c_{yf} + c_{yc} - c_{zc} = \$11 - \$5 + \$4 - \$6 = \$4$

This is the optimal solution since every opportunity cost is ≥ 0.

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