Simulation and Scientific Computing

Assignment 3

Conjugate Gradient Method with MPI parallelization

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Theoretical Part:

a) Time taken for data transfer is given by,

Transfer time = $\alpha + (\kappa/\beta)$

Where α – Latency for one transfer.
κ - number of elements exchanged between two neighbor processes in one transfer.
β- Bandwidth for transfer between two neighbor processes.

Considering there are k elements to be transferred, there will be k-1 elements to be transferred after 1st element transfer and with X being the compute time for update of one element the computation time for n elements can be given by,

c.t = (k.(k-1)/2) * X (sec)

The total time for n iterations per process can be given by,

Total time = Computation time + Transfer time
= $\alpha + (\kappa/\beta)$ + (k.(k-1)/2) * X (sec)

The time taken per iteration per process can be determined by,

T = $(\alpha/k) + (1/\beta)$ + ((k-1)/2 * X) (sec)

b) To obtain k, we must minimize T, i.e. derivative of T with respect to k must be zero.

$$\partial T/\partial k = 0$$

Differentiating the above equation gives us,

$K^2 = 2\alpha / X$

Substituting α = 2ms and X = 0.2 m/s, K is obtained as 4.47 ≈ 4.

The value of K =4 is the best value that minimizes the parallel overhead.

The overhead for the fictional setup,

α = 2ms, X = 0.2 ms, and β = 30 elements/ms is calculated to be,

$T = (\alpha/k) + (1/\ \beta) + ((k-1)/2 * X)$ (sec)

$T = (2/4) + (1/30) + (3/2) *0.2 = 0.833$ ms

5.

Ans We know that

$$\left.\begin{aligned} x_i &= ih \\ y_j &= jh \end{aligned}\right\} \quad \text{--- (1)}$$

Stencil for A is

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

as we have $\lambda_{\nu,\mu} = \frac{4}{h^2}\left(\sin^2\left(\frac{\pi\nu h}{2}\right) + \sin^2\left(\frac{\pi\mu h}{2}\right)\right)$

&

$$v_{\nu,\mu} = \left(\sin(\pi\nu x_i).\sin(\pi\mu y_j)\right)_{(x_i,y_j)\in\Omega_h}$$

$$A v_{\nu,\mu} = \frac{4}{h^2} v_{\nu,\mu}^{i,j} - \frac{1}{h^2}\left(v_{\nu,\mu}^{i+1,j} + v_{\nu,\mu}^{i-1,j} + v_{\nu,\mu}^{i,j+1} + v_{\nu,\mu}^{i,j-1}\right)$$

$$\therefore \lambda_{\nu,\mu}.v_{\nu,\mu}^{i,j} = \frac{4}{h^2} v_{\nu,\mu}^{i,j} - \frac{1}{h^2}\left(v_{\nu,\mu}^{i+1,j} + v_{\nu,\mu}^{i-1,j} + v_{\nu,\mu}^{i,j+1} + v_{\nu,\mu}^{i,j-1}\right) \quad \text{--- (2)}$$

where $v^{i+1,j}$ — Right neighbour

$v^{i-1,j}$ — Left neighbour

$v^{i,j+1}$ — Top neighbour

$v^{i,j-1}$ — bottom neighbour.

Now

$$\lambda_{v,\mu} = \frac{4}{h^2}\left(\sin^2\left(\frac{\pi v h}{2}\right) + \sin^2\left(\frac{\pi \mu h}{2}\right)\right)$$

$$= \frac{4}{h^2}\left(\frac{1-\cos(\pi v h)}{2} + \frac{1-\cos(\pi \mu h)}{2}\right)$$

$$= \frac{2}{h^2}\left[2 - 2\cos(\pi v h) - 2\cos(\pi \mu h)\right]$$

$$= \frac{4 - 2\cos(\pi v h) - 2\cos(\pi \mu h)}{h^2}$$

$$\because \left.\begin{aligned} u_{v,\mu}^{i,j} &= \sin(\pi v x_i)\sin(\pi \mu y_j) \\ &= \sin(\pi v i h)\sin(\pi \mu j h) \end{aligned}\right\} \text{(3)} \ \left\{\text{from (ii)}\right\}$$

So our LHS is

$$LHS = \lambda_{v,\mu}\, u_{v,\mu}^{i,j}$$

$$= \left[\frac{4}{h^2} - \frac{2}{h^2}\cos(\pi v h) - \frac{2}{h^2}\cos(\pi \mu h)\right]$$

$$\times \sin(\pi v i h) \cdot \sin(\pi \mu j h)$$

$$\left[\begin{aligned} LHS = {} & \frac{4}{h^2}\sin(\pi v i h)\cdot\sin(\pi \mu j h) \\ & - \frac{2}{h^2}\sin(\pi v i h)\cdot\sin(\pi \mu j h)\cdot\cos(\pi v h) \\ & - \frac{2}{h^2}\sin(\pi v i h)\cdot\sin(\pi \mu j h)\cdot\cos(\pi \mu h) \end{aligned}\right.$$

Our RHS

$$RHS = A v_{\nu,\mu} = \frac{4}{h^2} v_{\nu,\mu}^{i,j} - \frac{1}{h^2}\left[v_{\nu,\mu}^{i+1,j} + v_{\nu,\mu}^{i-1,j} + v_{\nu,\mu}^{i,j+1} + v_{\nu,\mu}^{i,j-1} \right]$$

$$= \frac{4}{h^2}\left(\sin(\pi\nu i h) \cdot \sin(\pi\mu j h)\right)$$

$$- \frac{1}{h^2}\Big[\sin(\pi\nu(i-1)h)\sin(\pi\mu j h) + \sin(\pi\nu(i+1)h)\sin(\pi\mu j h) + \sin(\pi\nu i h)\cdot\sin(\pi\mu(j+1)h) + \sin(\pi\nu i h)\cdot\sin(\pi\mu(j-1)h) \Big]$$

$$= \frac{4}{h^2}\left(\sin(\pi\nu i h) \cdot \sin(\pi\mu j h)\right)$$

$$- \frac{1}{2h^2}\Big[\cos(\pi\nu i h - \pi\nu h - \pi\mu j h) - \cos(\pi\nu i h - \pi\nu h + \pi\mu j h)$$
$$+ \cos(\pi\nu i h + \pi\nu h - \pi\mu j h) - \cos(\pi\nu i h + \pi\nu h + \pi\mu j h)$$
$$+ \cos(\pi\nu i h - \pi\mu j h - \pi\mu h) - \cos(\pi\nu i h + \pi\mu j h + \pi\mu h)$$
$$+ \cos(\pi\nu i h - \pi\mu j h + \pi\mu h) - \cos(\pi\nu i h + \pi\mu j h - \pi\mu h) \Big]$$

$$= \frac{4}{h^2}\left(\sin(\pi\nu i h) \cdot \sin(\pi\mu j h)\right)$$

$$- \frac{1}{h^2}\Big[\sin(\pi\mu j h)\left[\sin(\pi\nu(i-1)h) + \sin(\pi\nu(i+1)h)\right] + \sin(\pi\nu i h)\left[\sin(\pi\mu(j+1)h) + \sin(\pi\mu(j-1)h)\right] \Big]$$

using

$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$$

$$= \frac{4}{h^2} \sin(\pi v_i h) . \sin(\pi \mu_j h)$$

$$-\frac{1}{h^2}\left[\sin(\pi \mu_j h) . 2 \sin(\pi v_i h) . \cos(-\pi v h) + \sin(\pi v_i h) . 2 \sin(\pi \mu_j h) . \cos(\pi \mu h) \right]$$

$$\because \cos(-a) = \cos a .$$

So

$$RHS = \frac{4}{h^2} \sin(\pi v_i h) . \sin(\pi \mu_j h)$$

$$- \frac{2}{h^2} \sin(\pi v_i h) \sin(\pi \mu_j h) . \cos(\pi v h)$$

$$- \frac{2}{h^2} \sin(\pi v_i h) \sin(\pi \mu_j h) . \cos(\pi \mu h)$$

So LHS = RHS II