THEORETICAL PART

a) Discretization matrix A:

Equation: $-\Delta u(x,y) + k^2 . u(x,y) = f(x,y)$

Discretization of the above equation using finite Difference leads to

$$-\left[\frac{u(x-1,y) - 2u(x,y) + u(x+1,y)}{hx^2}\right] - \left[\frac{u(x,y-1) - 2u(x,y) + u(x,y+1)}{hy^2}\right] + k^2 u(x,y) = f(x,y)$$

Arranging the Co-efficients leads to,

$$\left(\frac{2}{hx^2} + \frac{2}{hy^2} + k^2\right) . u(x,y) - \frac{1}{hx^2}\left(u(x-1,y) + u(x+1,y)\right) - \frac{1}{hy^2}\left(u(x,y-1) + u(x,y+1)\right) = f(x,y)$$

Let $a_1 = \frac{2}{hx^2} + \frac{2}{hy^2} + k^2$; $b_1 = -\frac{1}{hx^2}$; $c_1 = -\frac{1}{hy^2}$.

Applying this discretized form to the given node formulation:

$$\underbrace{\begin{bmatrix} a_1 & b_1 & 0 & c_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_1 & 0 & c_1 & 0 & \cdots \\ 0 & b_1 & a_1 & 0 & 0 & c_1 & \cdots \\ c_1 & 0 & b_1 & a_1 & b_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \end{bmatrix}}_{A \in R^{N_x N_y \times N_x N_y}} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{x-2} \\ u_{x-1} \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{x-2} \\ f_{x-1} \end{bmatrix}}_{f}$$

It is a sparse matrix with only co-efficients appearing in a pattern. In order to find if it is strictly diagonally dominant matrix, we compared the magnitude of co-efficients.

Magnitude of non-diagonal entries : $\frac{2}{hx^2} + \frac{2}{hy^2}$.

Magnitude of diagonal entries : $\frac{2}{hx^2} + \frac{2}{hy^2} + k^2 > \frac{2}{hx^2} + \frac{2}{hy^2}$

It is a strictly diagonally dominant matrix.

(b) In part (a) we proved that matrix 'A' is strictly diagonal dominant.

For a strictly diagonal dominant matrix, the spectral radius is

$$\rho(A) \le \max \frac{1}{|a_{ii}|} \sum_{i \ne j} |a_{ij}|$$

$$\le \frac{2\left(|{-1/h_x^2}| + |{-1/h_y^2}| \right)}{\frac{2}{h_x^2} + \frac{2}{h_y^2} + k^2}$$

$$\le \frac{2/h_x^2 + 2/h_y^2}{2/h_x^2 + 2/h_y^2 + k^2} < 1$$

For a iterative method convergence will take place if and only if spectral radius of iteration A matrix satisfies $\rho(A) < 1$.

So, according to above calculation jacobi algorithm will converge with a boundary as,

$$\rho(A) \le \frac{1}{1 + \left(\frac{k^2}{\frac{2}{h_x^2} + \frac{2}{h_y^2}} \right)}$$

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