Advanced Electromagnetics

Chapter 1 – Maxwell's Equations

Jake W. Liu

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1.1 Formulation of Maxwell's Equations

1.1 Formulation of Maxwell's Equations

Faraday's law

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} - \vec{M} \tag{1.1.1}$$

Maxwell-Ampere's Law

$$\vec{\nabla} \times \vec{H} = \partial_t \vec{D} + \vec{J} \tag{1.1.2}$$

Continuity relation

$$\vec{\nabla} \cdot \vec{J} = -\partial_t \rho, \quad \vec{\nabla} \cdot \vec{M} = -\partial_t \varrho \tag{1.1.3}$$

*Lorentz equation of force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{1.1.4}$$

1.1 Formulation of Maxwell's Equations

- $\vec{E}$: electric field intensity (V/m)
- $\vec{H}$: magnetic field intensity (A/m)
- $\vec{D}$: electric flux density (C/m^2)
- $\vec{B}$: magnetic flux density (T)
- $\vec{J}$: volumetric electric current density (A/m^2)
- ρ: electric charge density (C/m^3)
- $\vec{M}$: volumetric magnetic current density (V/m^2)
- ϱ: magnetic charge density (Wb/m^3)

1.1 Formulation of Maxwell's Equations

Taking divergence of (1.1.2)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \partial_t \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \vec{J} \equiv 0 \qquad (1.1.5)$$

And from (1.1.3) we get

$$\partial_t (\vec{\nabla} \cdot \vec{D} - \rho) \equiv 0 \qquad (1.1.6)$$

This implies that

$$\vec{\nabla} \cdot \vec{D} - \rho = C(x, y, z) \qquad (1.1.7)$$

similarly

$$\vec{\nabla} \cdot \vec{B} - \varrho = C'(x, y, z) \qquad (1.1.8)$$

1.1 Formulation of Maxwell's Equations

In (1.1.7), if $C \neq 0$, it can be absorbed into ρ. The case is similar to (1.1.8). Thus, we can set $C = C' = 0$.

We get the Gauss's law

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{1.1.10}$$

$$\vec{\nabla} \cdot \vec{B} = \varrho \tag{1.1.11}$$

Note that (1.1.10) and (1.1.11) are not independent of (1.1.1)-(1.1.3).

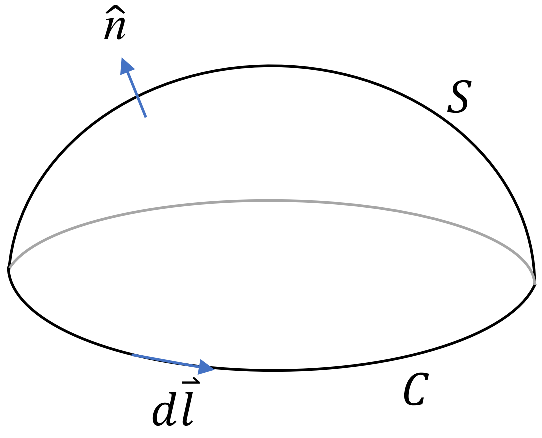
1.1 Formulation of Maxwell's Equations

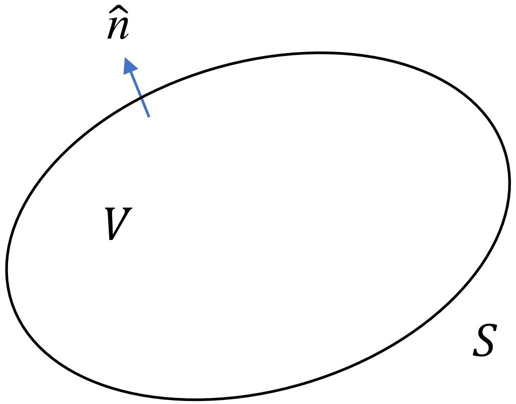
Stokes' theorem

$$\int_S \vec{\nabla} \times \vec{\Psi} \cdot d\vec{s} = \oint_C \vec{\Psi} \cdot d\vec{l} \qquad (1.1.12)$$

Divergence theorem

$$\int_V \vec{\nabla} \cdot \vec{\Psi}\, dv = \oint_S \vec{\Psi} \cdot d\vec{s} \qquad (1.1.13)$$





$d\vec{s} = \hat{n} ds$

1.1 Formulation of Maxwell's Equations

Applying (1.1.12) to (1.1.1) and (1.1.2), we get

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S (\partial_t \vec{B} + \vec{M}) \cdot d\vec{s} \tag{1.1.14}$$

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S (\partial_t \vec{D} + \vec{J}\,) \cdot d\vec{s} \tag{1.1.15}$$

Applying (1.1.13) to (1.1.3), (1.1.10) and (1.1.11), we get

$$\oint_S \vec{J} \cdot d\vec{s} = -\partial_t \int_V \rho \, dv \tag{1.1.16}$$

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv \tag{1.1.17}$$

$$\oint_S \vec{B} \cdot d\vec{s} = \int_V \varrho \, dv \tag{1.1.18}$$

1.1 Formulation of Maxwell's Equations

Another theorem for the curl operator

$$\int_V \vec{\nabla} \times \vec{\Psi}\, dv = \oint_S \left(\hat{n} \times \vec{\Psi}\right) \cdot d\vec{s} \qquad (1.1.19)$$

Applying (1.1.19) to (1.1.1) and (1.1.2), we get

$$\oint_S \hat{n} \times \vec{E} \cdot d\vec{s} = -\int_V \left(\partial_t \vec{B} + \vec{M}\right) dv \qquad (1.1.20)$$

$$\oint_S \hat{n} \times \vec{H} \cdot d\vec{s} = \int_V \left(\partial_t \vec{D} + \vec{J}\right) dv \qquad (1.1.21)$$

1.2 Constitutive Relations

1.2 Constitutive Relations

Excluding $\vec{M}$ and ϱ (non-physical), there are 5 vectors and 1 scalar, resulting in 16 unknowns.

From (1.1.1)-(1.1.3), there are 7 scalar equations, which means extra 9 equations are needed to make the system determinate.

The constitutive relations relates $\overrightarrow{D}, \overrightarrow{B}, \overrightarrow{J}$ with $\vec{E}, \vec{H}$ by

$$\begin{cases} \overrightarrow{D} = \bar{\bar{C}}_1(\vec{E}, \partial_t \vec{E}, \partial_t^2 \vec{E}, \ldots \vec{H}, \partial_t \vec{H}, \partial_t^2 \vec{H}, \ldots) \\ \overrightarrow{H} = \bar{\bar{C}}_2(\vec{E}, \partial_t \vec{E}, \partial_t^2 \vec{E}, \ldots \vec{H}, \partial_t \vec{H}, \partial_t^2 \vec{H}, \ldots) \\ \overrightarrow{J} = \bar{\bar{C}}_3(\vec{E}, \partial_t \vec{E}, \partial_t^2 \vec{E}, \ldots \vec{H}, \partial_t \vec{H}, \partial_t^2 \vec{H}, \ldots) \end{cases} \tag{1.2.1}$$

In general, $\bar{\bar{C}}_j$ $(j = 1, 2, 3)$ are tensor functions of time.

1.2 Constitutive Relations

We confine our exposition on the material properties by the following restrictions :

- Stationary: $\bar{\bar{C}}_j$ are not functions of time
- Non-chiral: $\bar{\bar{C}}_1, \bar{\bar{C}}_3$ are related to $\vec{E}$ and $\bar{\bar{C}}_2$ is related to $\vec{H}$
- Linear: $\bar{\bar{C}}_j$ are related to $\vec{E}$ and $\vec{H}$ only (no higher derivatives)

Often simplifications can be made if the medium is

- Isotropic: $\bar{\bar{C}}_j$ are scalars, i.e., C_j (otherwise they are called anisotropic)
- Homogeneous: $\bar{\bar{C}}_j$ are not functions of space
- In this note, we call a medium simple if it is linear, isotropic and homogeneous.

1.2 Constitutive Relations

When a linear dielectric medium is perturbed by an electric field, the constitutive relation for $\vec{D}$ and $\vec{E}$ is

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{\mathcal{P}} \tag{1.2.2}$$

ε_0 is the electric permittivity of the vacuum (8.854×10^{-12} F/m), and $\vec{\mathcal{P}}$ is the electric polarization defined as

$$\vec{\mathcal{P}} = \bar{\bar{\chi}}_e \varepsilon_0 \vec{E} \tag{1.2.3}$$

$\bar{\bar{\chi}}_e$ is the electric susceptibility tensor. If the medium is isotropic, then $\bar{\bar{\chi}}_e$ becomes a scalar, and $\vec{\mathcal{P}}$ is parallel to $\vec{E}$. We can define

$$\vec{D} = \bar{\bar{\varepsilon}} \vec{E} \tag{1.2.4}$$

$$\bar{\bar{\varepsilon}} = \varepsilon_0 (\bar{\bar{I}} + \bar{\bar{\chi}}_e) \tag{1.2.5}$$

1.2 Constitutive Relations

Similarly in magnetic medium, we have

$$\vec{B} = \mu_0 \vec{H} + \overrightarrow{\mathcal{M}} \tag{1.2.6}$$

μ_0 is the magnetic permeability of the vacuum ($4\pi \times 10^{-7}$ H/m), and $\overrightarrow{\mathcal{M}}$ is the magnetic polarization defined as

$$\overrightarrow{\mathcal{M}} = \bar{\bar{\chi}}_m \mu_0 \vec{H} \tag{1.2.7}$$

$\bar{\bar{\chi}}_m$ is the magnetic susceptibility tensor. We can define

$$\vec{B} = \bar{\bar{\mu}} \vec{H} \tag{1.2.8}$$

$$\bar{\bar{\mu}} = \mu_0 (\bar{\bar{I}} + \bar{\bar{\chi}}_m) \tag{1.2.9}$$

1.2 Constitutive Relations

The constitutive relation between $\vec{J}$ and $\vec{E}$ is

$$\vec{J} = \bar{\bar{\sigma}}\,\vec{E} \tag{1.2.10}$$

$\bar{\bar{\sigma}}$ is the conductivity of the medium. The conductivity in vacuum is 0 (S/m).

For a simple medium, we can simplify the tensors $\bar{\bar{\varepsilon}}$, $\bar{\bar{\mu}}$ and $\bar{\bar{\sigma}}$ to ε, μ and σ, respectively. (1.1.1) and (1.1.2) can be rewritten as

$$\vec{\nabla} \times \vec{E} = -\mu\partial_t\vec{H} \tag{1.2.11}$$

$$\vec{\nabla} \times \vec{H} = (\sigma + \varepsilon\partial_t)\vec{E} \tag{1.2.12}$$

1.3 Boundary Conditions

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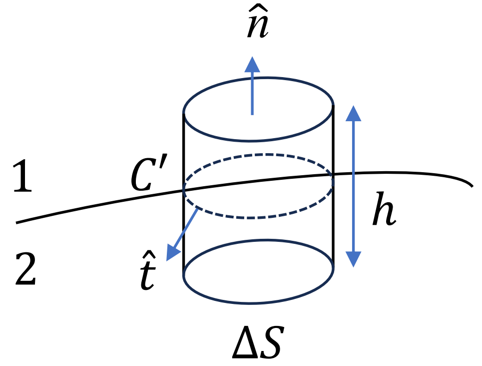
To derive boundary conditions, integral form of Maxwell's equations are needed.

Applying (1.1.20) to the pillbox

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right)\Delta S + \int_{side} \hat{t} \times \vec{E} \cdot d\vec{s} = -\Delta S \int_{-h/2}^{h/2} (\partial_t \vec{B} + \vec{M}) d\zeta \quad (1.3.1)$$

when $h \to 0$, we define the surface magnetic current density (V/m) as

$$\vec{M}_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{M} d\zeta \quad (1.3.2)$$



1.3 Boundary Conditions

we get

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = -\vec{M}_s \tag{1.3.3}$$

Similarly, applying (1.1.21) to the pillbox, when $h \to 0$, we define the surface electric current density (A/m) as

$$\vec{J}_s = \lim_{h\to 0} \int_{-h/2}^{h/2} \vec{J} d\zeta \tag{1.3.4}$$

we get

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s \tag{1.3.5}$$

Note that $\vec{J}_s$ exists only when one of the medium's $\sigma \to \infty$.

1.3 Boundary Conditions

Applying (1.1.17) to the pillbox

$$\hat{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)\Delta S+\int_{side}\hat{t}\cdot\vec{D}\,ds=\Delta S\int_{-h/2}^{h/2}\rho d\zeta \tag{1.3.6}$$

when $h\to 0$, we define the surface charge density (C/m^2) as

$$\rho_s=\lim_{h\to 0}\int_{-h/2}^{h/2}\rho d\zeta \tag{1.3.7}$$

we get

$$\hat{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\rho_s \tag{1.3.8}$$

Similarly, from (1.1.18) we get

$$\hat{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=\varrho_s \tag{1.3.9}$$

1.3 Boundary Conditions

Lastly, applying (1.1.16) to the pillbox

$$\hat{n} \cdot \left(\vec{J}_1 - \vec{J}_2\right)\Delta S + \int_{side} \hat{t} \cdot \vec{J}\, ds = -\Delta S \int_{-h/2}^{h/2} \partial_t \rho d\zeta \tag{1.3.10}$$

when $h \rightarrow 0$, we can express the side integral as

$$\oint_{C'} \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{J} d\zeta \cdot \hat{t} dl = \oint_{C'} \vec{J}_s \cdot \hat{t} dl = \int_{S'} \vec{\nabla}_s \cdot \vec{J}_s\, ds \tag{1.3.11}$$

we get

$$\left(\vec{J}_1 - \vec{J}_2\right) + \vec{\nabla}_s \cdot \vec{J}_s = -\partial_t \rho_s \tag{1.3.12}$$

Similarly

$$\left(\vec{M}_1 - \vec{M}_2\right) + \vec{\nabla}_s \cdot \vec{M}_s = -\partial_t \rho_s \tag{1.3.13}$$

1.4 Wave Equations

1.4 Wave Equations

Consider a lossless simple medium, taking the curl of (1.1.1)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\partial_t\left(\vec{\nabla} \times \mu\vec{H}\right) \qquad (1.4.1)$$

$$\nabla^2 \vec{E} - \mu\varepsilon\partial_t^2 \vec{E} = \mu\partial_t \vec{J} + \vec{\nabla}(\rho/\varepsilon) + \vec{\nabla} \times \vec{M} \qquad (1.4.2)$$

Similarly, for the magnetic field, we have

$$\nabla^2 \vec{H} - \mu\varepsilon\partial_t^2 \vec{H} = -\vec{\nabla} \times \vec{J} + \vec{\nabla}(\varrho/\mu) + \varepsilon\partial_t \vec{M} \qquad (1.4.3)$$

1.4 Wave Equations

(1.4.2) and (1.4.3) form a set of coupled inhomogeneous DE

$$\Box \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} \mu \partial_t \vec{J} + \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) + \vec{\nabla} \times \vec{M} \\ -\vec{\nabla} \times \vec{J} + \vec{\nabla}\left(\frac{\varrho}{\mu}\right) + \varepsilon \partial_t \vec{M} \end{pmatrix} \tag{1.4.4}$$

$\Box \equiv \nabla^2 - v^{-2}\partial_t^2$ is the d'Alembert operator with $v = (\mu\varepsilon)^{-1/2}$.

1.5 Energy Flow

1.5 Energy Flow

Considering the identity

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) \equiv \vec{H} \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot \vec{\nabla} \times \vec{H} \tag{1.5.1}$$

We define the instantaneous Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ (W/m^2)

$$\vec{\nabla} \cdot \vec{S} = \vec{H} \cdot \left(-\partial_t \vec{B} - \vec{M}\right) - \vec{E} \cdot \left(\partial_t \vec{D} + \vec{J}\right) =$$

$$-\partial_t \left(\frac{\mu}{2} \vec{H} \cdot \vec{H}\right) - \partial_t \left(\frac{\varepsilon}{2} \vec{E} \cdot \vec{E}\right) - \vec{H} \cdot \vec{M} - \vec{E} \cdot \vec{J} \tag{1.5.2}$$

1.5 Energy Flow

The term $\frac{\varepsilon}{2}\vec{E}\cdot\vec{E} = w_e$ and $\frac{\mu}{2}\vec{H}\cdot\vec{H} = w_m$ denote the electric and magnetic energy density, and $\vec{H}\cdot\vec{M} + \vec{E}\cdot\vec{J} = p_l$ denotes the power loss/supply per unit volume. We rewrite the Poynting theorem as

$$\vec{\nabla}\cdot\vec{S} = -\partial_t(w_e + w_m) - p_l \qquad (1.5.3)$$

and conduct integration over a finite volume V by applying the divergence theorem:

$$\oint_S \hat{n}\cdot\vec{S}ds = -\partial_t \int_V (w_e + w_m)dv - \int_V p_l dv \qquad (1.5.4)$$

1.6 Time Harmonic Form

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A time domain signal can be decomposed into a spectrum of time harmonic components

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega \tag{1.6.1}$$

with

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}d\omega \tag{1.6.2}$$

The Fourier and inverse Fourier transform relationship is

$$g(\omega) = \mathfrak{F}[f(t)]$$

$$f(t) = \mathfrak{F}^{-1}[g(\omega)] \tag{1.6.3}$$

1.6 Time Harmonic Form

Now we consider the harmonic electric field at angular frequency ω. We consider its phasor form

$$\vec{E}(\vec{r},t) = \Re\left[\vec{E}(\vec{r},\omega)e^{i\omega t}\right] \tag{1.6.4}$$

Note that we use the same notation for both the time domain field and frequency domain field, and we will suppress its dependency when there is no ambiguity.

In general, the phasor field $\vec{E}(\vec{r},\omega)$ is a complex number.

1.6 Time Harmonic Form

The time harmonic Maxwell's equation takes the form

$$\vec{\nabla} \times \vec{E} = -i\omega\vec{B} - \vec{M} \tag{1.6.5}$$

$$\vec{\nabla} \times \vec{H} = i\omega\vec{D} + \vec{J} \tag{1.6.6}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{1.6.7}$$

$$\vec{\nabla} \cdot \vec{B} = \varrho \tag{1.6.8}$$

The method to translate the time-domain equation to frequency domain is to replace ∂_t with $i\omega$, and vice versa.

1.6 Time Harmonic Form

Analyzing (1.6.6), with $\vec{D} = \varepsilon\vec{E}$ and $\vec{J} = \vec{J}_c + \vec{J}_i = \sigma\vec{E} + \vec{J}_i$, we get

$$\vec{\nabla} \times \vec{H} = i\omega\left(\varepsilon - i\frac{\sigma}{\omega}\right)\vec{E} + \vec{J}_i \qquad (1.6.9)$$

We define the complex permittivity as

$$\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega} \qquad (1.6.10)$$

In general $\varepsilon = \varepsilon' - i\varepsilon''$. We define the loss tangent of the medium as

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} + \frac{\sigma}{\omega\varepsilon'} \qquad (1.6.11)$$

Medium with $\tan\delta \ll 1$ is characterized as good dielectric, and with $\tan\delta \gg 1$ is characterized as good conductor.

1.6 Time Harmonic Form

The d'Alembert operator in (1.4.4) in frequency domain becomes

$$\nabla^2 - \frac{(i\omega)^2}{1/\mu\varepsilon_c} = \nabla^2 + k^2 \tag{1.6.12}$$

where

$$k = \omega\sqrt{\mu\varepsilon_c} = k_R + ik_I \tag{1.6.13}$$

is the wavenumber. The square root of ε_c is chosen so that k_I relates to the physical attenuation of the wave propagation.

Several literatures uses $\gamma^2 = (\alpha + i\beta)^2 = -\omega^2\mu\varepsilon_c = -k^2$. The relationship between the real and imaginary parts are $\alpha = -k_I$ and $\beta = k_R$.

1.7 Complex Poynting Theorem

1.7 Complex Poynting Theorem

From $\vec{E}(\vec{r},t) = \Re\left[\vec{E}(\vec{r})e^{i\omega t}\right] = \left(\vec{E}(\vec{r})e^{i\omega t} + \vec{E}^*(\vec{r})e^{-i\omega t}\right)/2$, and

$$\vec{S}(\vec{r},t) = \left(\Re\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right] + \Re\left[\vec{E}(\vec{r}) \times \vec{H}(\vec{r})e^{i2\omega t}\right]\right)/2 \quad (1.7.1)$$

we have

$$\vec{S}_{\mathrm{av}} = \Re\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})/2\right] \quad (1.7.2)$$

Define the vector

$$\vec{S}(\vec{r}) = \vec{E} \times \vec{H}^*/2 \quad (1.7.3)$$

as the complex Poynting vector.

1.7 Complex Poynting Theorem

From the identity (1.5.1)

$$\vec{\nabla} \cdot \vec{S} = \vec{\nabla} \cdot \left(\frac{\vec{E} \times \vec{H}^*}{2} \right) = -i2\omega \left(\frac{\mu \left| \vec{H} \right|^2}{4} - \frac{\varepsilon \left| \vec{E} \right|^2}{4} \right) - \frac{\vec{E} \cdot \vec{J}^*}{2} - \frac{\vec{H}^* \cdot \vec{M}}{2} = -i2\omega (w_m - w_e) - p_l \tag{1.7.4}$$

is the complex Poynting theorem, where $w_m = \mu \left| \vec{H} \right|^2 / 4$, $w_e = \varepsilon \left| \vec{E} \right|^2 / 4$ and $p_l = \left(\vec{E} \cdot \vec{J}^* + \vec{H}^* \cdot \vec{M} \right) / 2$ are the time average electric and magnetic energy density and the time average power loss/supply.