Computational Electromagnetics

Chapter 1 - Introduction

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Outline

1.1 Overview

1.2 Review of Vector Calculus

1.3 Review of Maxwell's Equations

1.1 Overview

Historical Notes, Maxwell’s Equations, Derivatives in Space and Time, Solution Techniques, Applications.

Historical Notes

- 1600-1700s: Calculus developed by Leibniz and Newton
- 1800s: Theory of differential equations
- 1900s: Various analytical methods
- 1960s: Computational/numerical methods

Maxwell’s Equations

- Maxwell’s time → 20 Eqns (1873), Heaviside - → 4 Eqns (1888)
- $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} - \vec{M}$, Faraday, 1843
- $\vec{\nabla} \times \vec{H} = \partial_t \vec{D} + \vec{J}$, Ampere (+ Maxwell), 1823 (1864)
- $\vec{\nabla} \cdot \vec{D} = \rho$, Coulomb, 1785
- $\vec{\nabla} \cdot \vec{B} = 0$, Gauss, 1841
- Maxwell’s equations are linear
- Optics vs. Electromagnetics
- Fourier techniques → time harmonic

Derivatives in Space and Time

- $\partial_\eta \sim 1/L$
- $\partial_t \sim \omega \sim c/\lambda$
- **Low frequency**: $L \ll \lambda$ → statics (uncoupled)
- **Mid frequency**: $L \sim \lambda$ → wave (coupled)
- **High frequency**: $L \gg \lambda$ → ray (optics)

Solution Techniques

- **Time** vs. **Frequency Domain**
  - **TD**: transient responses
  - **FD**: single frequency
- **Differential** vs. **Integral Equations**
  - **DE**: directly solves PDE
  - **I E**: applies divergence and Stokes theorem

Well-known Techniques

	I E	DE
TD	TDIE	FDTD
FD	MOM	FEM

Applications

- Antenna radiation modeling
- Device modeling (SIPI / EMC)
- Wave propagation and scattering
- RCS analysis
- Metamaterials
- nanostructures
- …

We select an appropriate CEM method for modeling based on the application requirements.

1. 2 Review of Vector Calculus

Chain Rule of Differentiation, Del Operator, Divergence and Stokes Theorem, Vector Identities

Chain Rule of Differentiation

- Consider a scalar function $f\ (x, y, z)$, we want to calculate a small change of f, i.e. df. From chain rule, we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

which is a dot product between $\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}_{\vec{\nabla} f}$ and $\underbrace{(dx, dy, dz)}_{d\vec{l}}$.

$$\boxed{df = \vec{\nabla} f \cdot d\vec{l}}$$

Chain Rule of Differentiation

- Total change from $\vec{a}$ to $\vec{b}$ is

$$\int_{\vec{a}}^{\vec{b}} df = \int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{a}) - f(\vec{b})$$

  which is path independent.

- Corollary: $\oint \vec{\nabla} f \cdot d\vec{l} = 0$

Del Operator

- $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \Rightarrow \text{define } \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$
- Three main operations:

gradient	**divergence**	**curl**
$\vec{\nabla} f$	$\vec{\nabla} \cdot \vec{F}$	$\vec{\nabla} \times \vec{F}$
scalar → vector	vector → scalar	vector → vector

Divergence and Stokes Theorem

- Divergence Theorem

$$\int_V \vec{\nabla} \cdot \vec{F} dv = \oint_S \vec{F} \cdot d\vec{s}$$

- Stokes Theorem

$$\int_S \vec{\nabla} \times \vec{F} d\vec{s} = \oint_c \vec{F} \cdot d\vec{l}, \qquad \int_V \vec{\nabla} \times \vec{F}\, dv = \oint_S (\hat{n} \times \vec{F}) \cdot d\vec{s}$$

- In CEM, these two theorems help reduce the dimensionality of the problem.

Vector Identities

- $\vec{\nabla} \times \vec{\nabla} f = 0$ for any scalar function f
- $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ for any vector function $\vec{F}$
- $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) - \nabla^2 \vec{F}$ (∇^2 is the Laplacian operator)

1.3 Review of Maxwell’s Equations

Maxwell’s Equations and Continuity Relation, Wave Equation Example, Time Harmonic Form, Boundary Conditions, Poynting Vector, Uniqueness Theorem, Volume Equivalence Theorem, Volume Equivalence Theorem.

Maxwell's Equations and Continuity Relation

- Faraday's law

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} - \vec{M}$$

- Maxwell-Ampere's Law

$$\vec{\nabla} \times \vec{H} = \partial_t \vec{D} + \vec{J}$$

- Continuity relation

$$\vec{\nabla} \cdot \vec{J} = -\partial_t \rho, \qquad \vec{\nabla} \cdot \vec{M} = -\partial_t \varrho$$

- Gauss's law

$$\vec{\nabla} \cdot \vec{D} = \rho$$
$$\vec{\nabla} \cdot \vec{B} = \varrho$$

$\vec{M}$ and ϱ are non-physical and are introduced only for symmetry of the equations.

Wave Equation Example

Consider a lossless and source free region

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\partial_t\left(\vec{\nabla} \times \mu \vec{H}\right)$$

$$\nabla^2 \vec{E} - v^{-2} \partial_t^2 \vec{E} = 0$$

$v = (\mu\varepsilon)^{-1/2}$. Take 1D as example ($\partial_y^2 = \partial_z^2 = 0$)

$$\partial_x^2 E_x - v^{-2} \partial_t^2 E_x = 0$$

The solution takes the form of

$$E_x = f^+(x - vt) + f^-(x + vt)$$

Time Harmonic Form

- Phasor: $\vec{E}(\vec{r},t) = \Re\left[\vec{E}(\vec{r})e^{i\omega t}\right]$
- $\vec{E}(\omega)$ is complex in general.

$$\boxed{\partial_t \Leftrightarrow i\omega}$$

$$\vec{\nabla} \times \vec{E} = -i\omega\vec{B} - \vec{M}$$

$$\vec{\nabla} \times \vec{H} = i\omega\vec{D} + \vec{J}$$

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = \varrho$$

$$\vec{\nabla} \cdot \vec{J} = -i\omega\rho$$

$$\vec{\nabla} \cdot \vec{M} = -i\omega\varrho$$

Constitutive Relations

$$\vec{D}(\vec{r}) = \varepsilon(\vec{r},\omega)\,\vec{E}(\vec{r})$$
$$\vec{B}(\vec{r}) = \mu(\vec{r},\omega)\,\vec{H}(\vec{r})$$
$$\vec{J}(\vec{r}) = \sigma(\vec{r},\omega)\,\vec{E}(\vec{r})$$

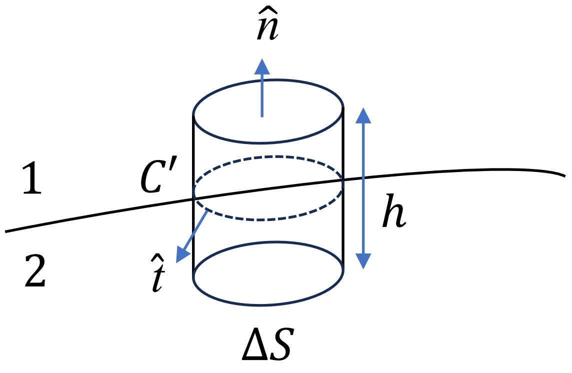
Boundary Conditions

To derive boundary conditions, integral form of Maxwell's equations are needed.

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2)\Delta S + \int_{side} \hat{t} \times \vec{E} \cdot d\vec{s} = -\Delta S \int_{-\frac{h}{2}}^{\frac{h}{2}} (\partial_t \vec{B} + \vec{M}) d\zeta$$

when $h \to 0$, we define the surface magnetic current density (V/m) as

$$\vec{M}_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{M} d\zeta$$



Boundary Conditions

we get

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = -\vec{M}_s$$

Similarly,

$$\vec{J}_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{J} d\zeta$$

we get

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s$$

Note that $\vec{J}_s$ exists only when one of the medium's $\sigma \to \infty$.

Boundary Conditions

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2)\Delta S + \int_{side} \hat{t} \cdot \vec{D}\, ds = \Delta S \int_{-h/2}^{h/2} \rho d\zeta$$

when $h \to 0$, we define the surface charge density (C/m²) as

$$\rho_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \rho d\zeta$$

we get

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

Similarly,

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = \varrho_s$$

Poynting Vector

Instantaneous Poynting vector

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)$$

From $\vec{E}(\vec{r},t) = \Re\left[\vec{E}(\vec{r})e^{i\omega t}\right] = \left(\vec{E}(\vec{r})e^{i\omega t} + \vec{E}^*(\vec{r})e^{-i\omega t}\right)/2$, we get

$$\vec{S}(\vec{r},t) = \frac{\left(\Re\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right] + \Re\left[\vec{E}(\vec{r}) \times \vec{H}(\vec{r})e^{i2\omega t}\right]\right)}{2}$$

$$\boxed{\vec{S}_{\mathrm{av}} = \Re\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right]/2}$$

Uniqueness Theorem

- The fields $\vec{E}, \vec{H}$ created by sources $\vec{J}$ and $\vec{M}$ in a lossy volume V are unique if any one of the following are true
  - Tangential $\vec{E}$ over S is known
  - Tangential $\vec{H}$ over S is known
  - Tangential $\vec{E}$ is known on some part of S,
    and tangential $\vec{H}$ is known on the remaining part of S

Volume Equivalence Theorem

- Sources in vacuum

$$\vec{\nabla} \times \vec{E}_0 = -i\omega\mu_0\vec{H}_0 - \vec{M}$$

$$\vec{\nabla} \times \vec{H}_0 = i\omega\varepsilon_0\vec{E}_0 + \vec{J}$$

- Obstacle presented in the medium

$$\vec{\nabla} \times \vec{E} = -i\omega\mu\,\vec{H} - \vec{M}$$

$$\vec{\nabla} \times \vec{H} = i\omega\varepsilon\,\vec{E} + \vec{J}$$

- Subtract the two set of Eqns, we obtain the Eqns for **scattered field**

$$\vec{\nabla} \times \underbrace{\left(\vec{E} - \vec{E}_0\right)}_{\boxed{\vec{E}_s}} = -i\omega\left(\mu\,\vec{H} - \mu_0\vec{H}_0\right), \qquad \vec{\nabla} \times \underbrace{\left(\vec{H} - \vec{H}_0\right)}_{\boxed{\vec{H}_s}} = i\omega\left(\varepsilon\,\vec{E} - \varepsilon_0\vec{E}_0\right)$$

Volume Equivalence Theorem

$$\vec{\nabla} \times \vec{E}_s = -i\omega(\mu \vec{H} - \mu_0 \vec{H}_0) = -i\omega\left(\mu \vec{H} - \mu_0(\vec{H} - \vec{H}_s)\right)$$

$$= -i\omega\left((\mu - \mu_0)\vec{H} + \mu_0 \vec{H}_s\right) = -i\omega\mu_0 \vec{H}_s - \vec{M}_{eq}$$

Similarly, $\vec{\nabla} \times \vec{H}_s = i\omega\varepsilon_0 \vec{E}_s + \vec{J}_{eq}$

$$\boxed{\begin{aligned} \vec{M}_{eq} &= i\omega(\mu - \mu_0)\vec{H} \\ \vec{J}_{eq} &= i\omega(\varepsilon - \varepsilon_0)\vec{E} \end{aligned}}$$

Replace obstacles with equivalent sources

Surface Equivalence Theorem

From $\hat{n} \times (\vec{E}_1 - \vec{E}_2) = -\vec{M}_s$ and $\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$, we can set the fields in V_2 to be zero, and place surface currents on the imaginary surface.