Computer simulations of Stochastic Processes

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1 Introduction

In this report we are going to discuss and present some aspects regarding simulating stochastic processes. A couple of scripts has been attached to this document - they were used to produce plots and results in this report.

2 Stable distribution

In particular, we will discuss here stable distributions - a class of probability distributions, that allow skewness and heavy tails, and so have a lot of interesting mathematical properties, as well as many applications in finance, insurance mathematics, biology and many more. In general, an univariate stable random variables are characterized by four parameters:

- α - the most important one, called index of stability
- σ - scale parameter
- β - skewness parameter
- μ - shift parameter

Those parameters will be discussed later, here we can mention that if we want to receive Gaussian distribution, we put $\alpha = 2$, $\beta = 0$, and so μ will be mean and σ - standard deviation.

2.1 Definition

We define X as stable distribution if it holds the following property[1]

$$aX_1 + bX_2 \stackrel{d}{=} cX + d \tag{1}$$

where a, b and c are positive, $d \in R$ and X_1, X_2 are independent copies of X. However, there is also another representation of stable random variable; we can say, that random variable X is stable if and only if $X \stackrel{d}{=} aZ + b$ witch Z being random variable with characteristic function:

$$\mathbf{E}\exp(i\mu Z) = \begin{cases} \exp(|\mu|^{\alpha}[1 - i\beta\tan\frac{\pi\alpha}{2}(\text{sign}\mu)]) & \alpha \neq 1 \\ \exp(|\mu|[1 + i\beta\frac{2}{\pi}(\text{sign}\mu)\log|\mu|]) & \alpha = 1 \end{cases} \tag{2}$$

In this equation we have $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$, a must be different than 0 and b is a real number.
A more precise description and derivation of stable distribution can be found in dedicated literature[2]. Here we are going to discuss some aspect of simulations and properties.

[1] Nolan J. P., "Stable Distributions. Models for Heavy Tailed Data", http://academic2.american.edu/ jpnolan/stable/stable.html

[2] Samorodnitsky G, Taqqu M.S, "Stable Non-Gaussian Random Processes"

2.2 Simulation and properties

A simple function (for Matlab language) to simulate stable random variables[3] has been attached to this report. Crucial for us is to check if it works correctly and investigate properties of the simulated distribution. Fortunately, a there is a very well-known program[4], written by John P. Nolan, available on his website. With it's help, we will be able to test our function.

2.2.1 Testing for correctness of the function

Let's make four calls of the function to generate data, and then check if with Nolan's program:

```
gaussian = stable(2, 0, 1, 0, 100000);$
cauchy = stable(1, 0, 1, 0, 100000);$
levy = stable(0.5, 1.0, 1, 0, 100000);$
distr = stable(1.3, 0.3, 2, -5, 100000);$
```

As a result, we obtained four samples of stable distribution; now we can check them with Nolan's program. We choose each time maximum likelihood estimators of parameters method to get parameters based on sample.

For a variable *gaussian* the output is:

```
   Stable model with maximum likelihood estimator

Initial quantile estimate of S0 parameters
      alpha     beta     gamma        delta
   2.000000} 0.000000   1.00421      0.911811E-03
```

For a variable *cauchy* the output is:

```
   Stable model with maximum likelihood estimator

Initial quantile estimate of S0 parameters
      alpha      beta      gamma        delta
   0.995880 -0.017271   1.00109      0.190453E-02
```

For a variable *levy* the output is:

```
   Stable model with maximum likelihood estimator

Initial quantile estimate of S0 parameters
      alpha      beta      gamma        delta
   0.506668 0.815349   1.31190      1.17643
```

[3] Borak, Hardle, Weron, 2005

[4] http://academic2.american.edu/ jpnolan/stable/stablec.exe

For a variable *distr* the output is:

```
    Stable model with maximum likelihood estimator

 Initial quantile estimate of S0 parameters
       alpha      beta      gamma        delta
    1.306687 0.311575   1.99745      -6.18514
```

We can say here now, that estimation goes worse as α is going lower - especially for μ (or - in other sources σ) parameter; we should be careful when we are saying something about shift parameter, when we operating on the low-α samples.

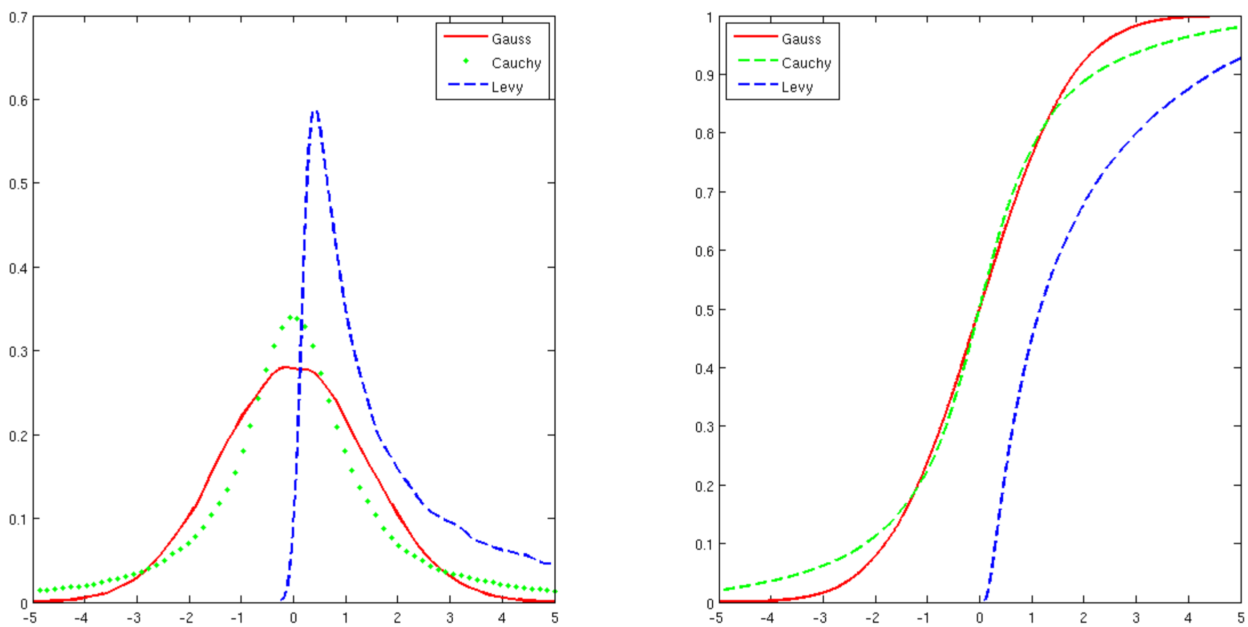

Figure 1: Simulated three important distributions with $\alpha = 2, 1, 0.5$

2.3 Description of the parameters

In this subsection we will discuss each of the parameters and how changing them, impacts obtained sample.

2.3.1 α parameter

Mentioned before, index of stability is often referred as the most important one. On the Figure 2 three samples have been presented, to illustrate how stable distribution behave with decreasing α. We can see that three things occur to the density: peak gets higher, the region flanking peak get lower, and - what makes stable distribution such important and interesting - the tails gets heavier. As shown in the next section, α affects heavily other parameters.

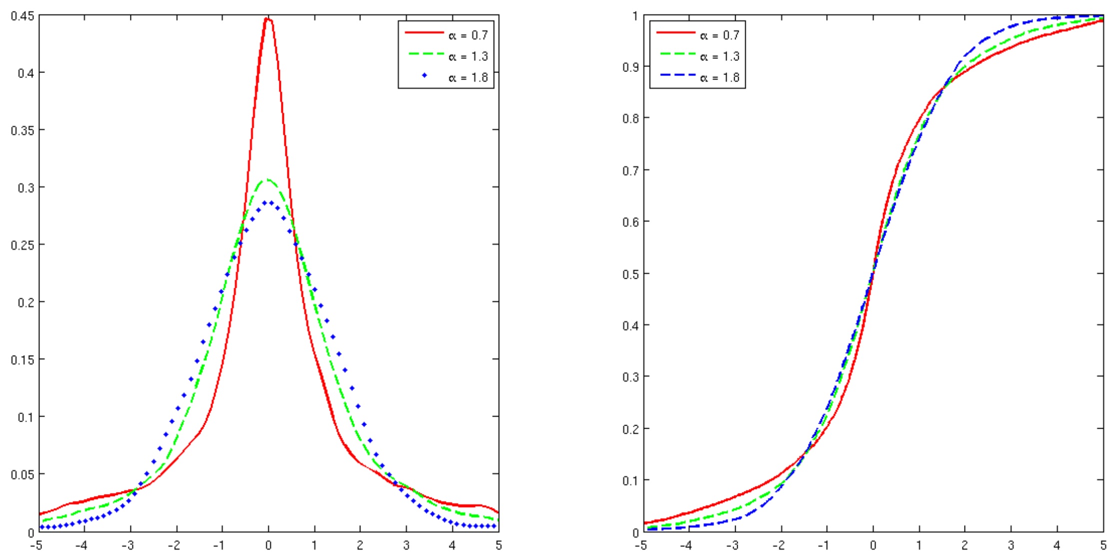

Figure 2: Stable distribution pdf and empirical cdf for $\alpha = 0.7, 1.3, 1.8$

2.3.2 β parameter

Three different plots for different simulated samples will be needed, to show how β influence our distribution.

Figure 3 allows us to see, that with increasing β, the distribution becomes more and more skewed to the right (in case of β negative, we would see similar behavior, but to the left side)

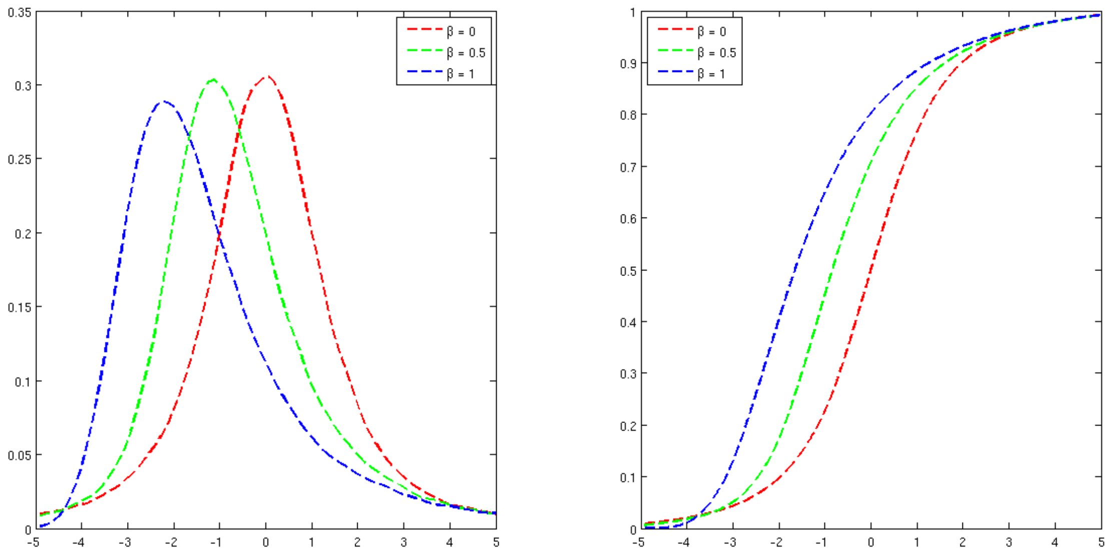

Figure 3: Stable distribution pdf and empirical cdf for $\alpha = 1.3$ and $beta = 0, 0.5, 1$

In Figures 4 and 5 β varying in the same way, but samples has been simulated for $\alpha = 1$ and $\alpha = 0.7$ respectively.

We see that with decreasing α, β becomes more and more significant, making distribution more and more skewed, until it reach 1 (-1) - then we say that distribution is *totally skewed to the right (left)*. It can also be shown, that for $\alpha < 1$ and $|\beta| = 1$ the whole weight is in one of the tails (depending if β is negative or positive).

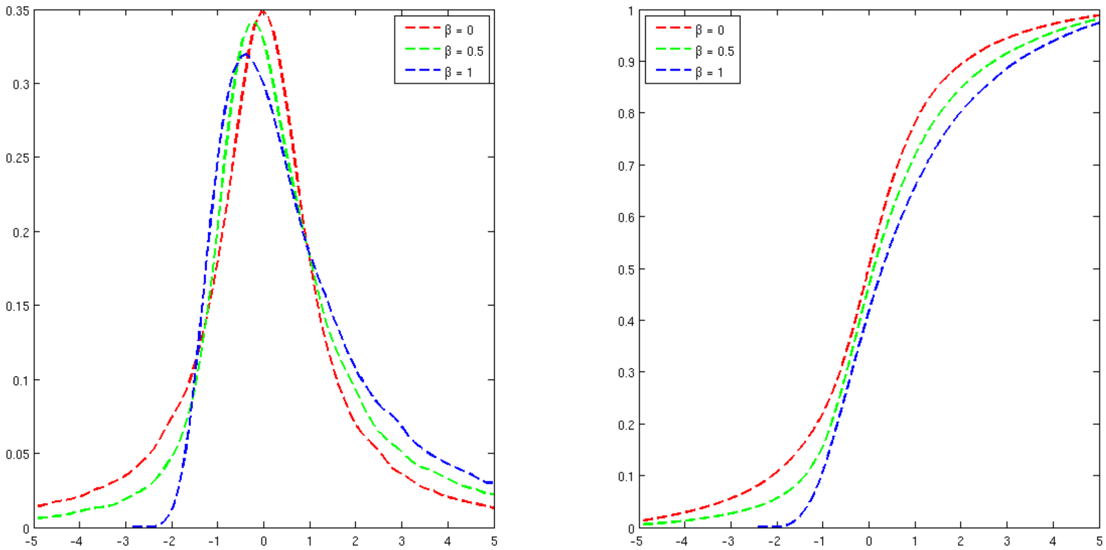

Figure 4: Stable distribution pdf and empirical cdf for $\alpha = 1$ and $beta = 0, 0.5, 1$

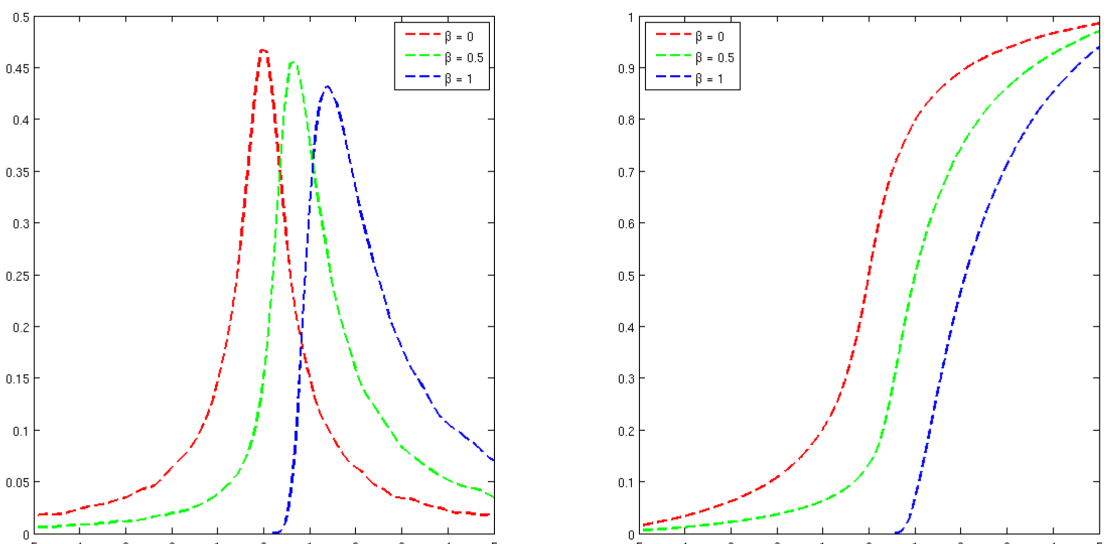

Figure 5: Stable distribution pdf and empirical cdf for $\alpha = 0.7$ and $beta = 0, 0.5, 1$

2.3.3 σ and μ parameters

Here we briefly take a look for remaining two parameters. σ tells us about how spread distribution is, while μ - where the "center" of the parameter is. For a Gaussian distribution we call them *variance* and *mean* respectively, however, we should generally avoid those terms, as they may not exist according to mathematical definitions.

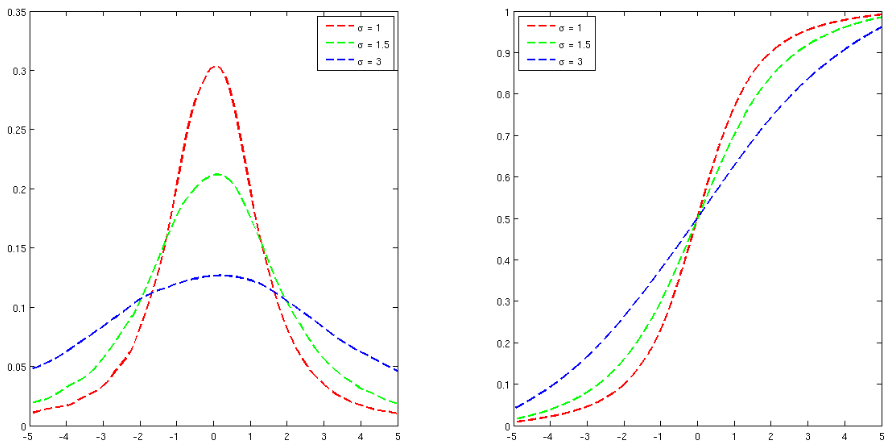

Figure 6: Stable distribution pdf and empirical cdf for $\alpha = 1.3$ and $sigma = 1, 1.5, 3$

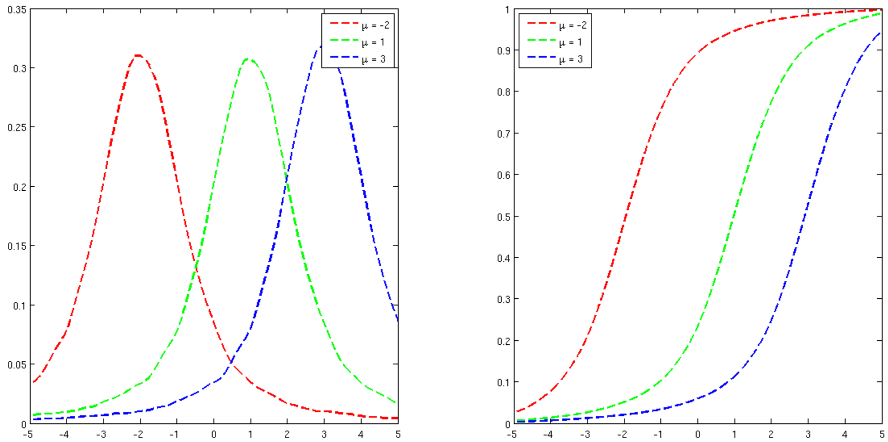

Figure 7: Stable distribution pdf and empirical cdf for $\alpha = 1.3$ and $mu = -2, 1, 3$

2.4 Properties

Here we discuss some of the commonly known properties[5] of the stable distributions.

2.4.1 Property 1 - sum of stable random variables

First, we are going to study the following property[6]:

Property 1 *Let X_1 and X_2 be independent stable random variables, $X_i \sim S_\alpha(\beta_i, \sigma_i, \mu_i)$, $i = 1, 2$. Then $X_1 + X_2 \sim S_\alpha(\beta, \sigma, \mu)$, with:*

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \qquad \mu = \mu_1 + \mu_2, \qquad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} \tag{3}$$

Following code has been ran to simulate two samples:

```
alpha = 1.8;
mu1 = 1; mu2 = 0.5;
sigma1 = 0.5; sigma2 = 0.3;
beta1 = 0.3; beta2 = 0.8;
sigma = ( sigma1^alpha + sigma2^alpha )^(1/alpha);
beta = ( beta1*(sigma1^alpha) + beta2*(sigma2^alpha) ) / ( sigma1^alpha
    + sigma2^alpha );
mu = mu1 + mu2;
```

[5] Samorodnitsky G, Taqqu M.S, "Stable Non-Gaussian Random Processes"
[6] Samorodnitsky G, Taqqu M.S, "Stable Non-Gaussian Random Processes"

```
X1 = stable(alpha, beta1, sigma1, mu1, 10000000);
X2 = stable(alpha, beta2, sigma2, mu2, 10000000);
X = X1 + X2;
save X.txt X -ascii;
```

Values computed according to formula above are as follows: $sigma = 0.6025, \beta = 0.4425, \mu = 1.5000$; output from Nolan's program, checking parameters:

```
 Initial quantile estimate of S0 parameters
        alpha     beta      gamma        delta
     1.800206 0.452978  0.602503       1.41195
```

The largest difference between theoretical and simulated value is at δ (or μ) parameter; however, taking consideration, that in the previous examples it was most variable parameter, we can say that property holds.

2.4.2 Property 2

Another property, that we can easily check is the following one

Property 2 For any $0 < \alpha < 2$,

$$X \sim S_\alpha(\beta, \sigma, \mu) \iff -X \sim S_\alpha(-\beta, \sigma, \mu) \tag{4}$$

```
alpha = 1.8; mu = 0; sigma = 1; beta = 0.7;
X = stable(alpha, beta, sigma, mu, 1000000);
X = -X; save X.txt X -ascii;
```

And again we check with the Nolan's program:

```
 Initial quantile estimate of S0 parameters
        alpha     beta      gamma        delta
     1.802983 -0.716207  1.00018      0.230816
```

We can say that property holds.

2.4.3 Property 3 - series representation

We are going to show now, that α-stable random variable can be represented as a convergent sum of random variables involving arrival times of a Poisson process.

We consider stable process for $0 < \alpha < 0$. Let's denote N_δ - Poisson r.v. with parameter δ. We need to introduce a new r.v. with distribution function:

$$P(Y_{\delta,k}) = \begin{cases} \delta^\alpha X^{-\alpha} & X > \delta \\ 1 & X < \delta \end{cases}$$

Now, if we look at the following variable:

$$X_\delta = \sum_{k>1}^{N_j} Y_{j,k}$$

We can say, that X_δ for δ going to 0 becomes α subordinator - stable r.v. with parameters $S(\alpha, \sigma, 1, 0)$ with:

$$\sigma^\alpha = \Gamma(1-\alpha)cos(\pi\frac{\alpha}{2})$$

Figure 8 shows how tails of the distribution converges to stable distribution (blue line is $S(\alpha, \sigma, 1, 0)$).

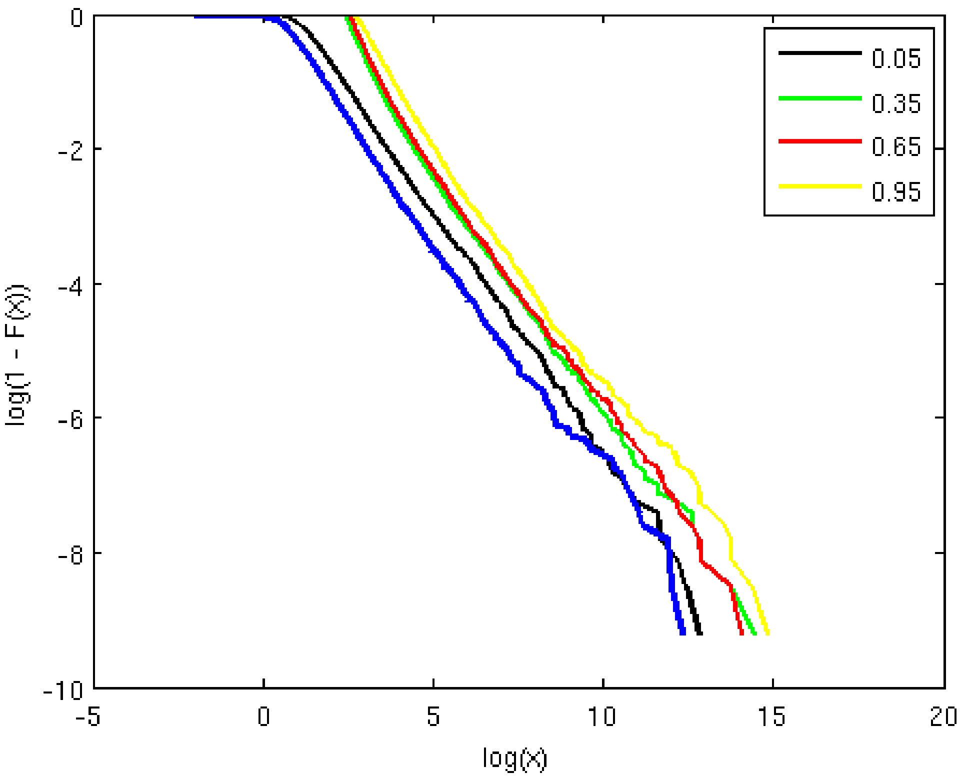

Figure 8: Tails of stable subordinators for different δ on double log scale; $\alpha = 0.7$ here.

3 Multivariate stable distributions

An logic step forward is to simulate and investigate multivariate stable distributions. We will briefly note important facts about them, and then present method of simulation and results.

3.1 Characterization

We use $\mathbf{X}$ to denote d-dimensional α-stable random vector determined by the spectral measure Γ (a finite Borel measure on S_d = unit sphere in $\mathbb{R}$) and a shift vector $\mu^0 \in \mathbb{R}^d$. The characteristic function of $\mathbf{X}$ is:

$$\psi_{\mathbf{X}}(\mathbf{t}) = \mathbf{E}\exp\{i < \mathbf{X}, \mathbf{t} >\} = \exp(-I_{\mathbf{X}}(\mathbf{t}) + i < \mu^0, t >) \tag{5}$$

Where $I_{\mathbf{X}}$ is:

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S_d} \psi_\alpha(< \mathbf{t}, \mathbf{s} >)\Gamma(ds) \tag{6}$$

and ψ_α:

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha(1 - i\text{sign}(u)\tan(\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ |u|(1 - i\frac{2}{\pi}\text{sign}(u)\log(|u|) & \alpha = 1 \end{cases} \tag{7}$$

3.2 Simulation

For simulation, we consider Γ as a discrete spectral measure with finite number of point masses:

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot) \tag{8}$$

Where γ_j's are weights and $\delta_{\mathbf{s_j}}$'s are point masses at the point $\mathbf{s_j} \in S_d$. With that knowledge, we can define method for simulating random vectors[7]:

$$\mathbf{X} \stackrel{D}{=} \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j \mathbf{s}_j & \alpha \neq 1 \\ \sum_{j=1}^{n} \gamma_j (Z_j + \frac{2}{\pi}\log\gamma_j)\mathbf{s}_j & \alpha = 1 \end{cases} \tag{9}$$

To test algorithm, we will try to retrieve contour plots from J. Nolan's paper; after setting the variables (whole procedure in *multi.m* file):

```
gam1 = 0.25; s1 = [1, 0];
gam2 = 0.125; s2 = [0.5, sqrt(3)/2];
gam3 = 0.25; s3 = [-0.5, sqrt(3)/2];
gam4 = 0.25; s4 = [-1, 0];
gam5 = 0.125; s5 = [-0.5, -sqrt(3)/2];
gam6 = 0.25; s6 = [0.5, -sqrt(3)/2];
```

And similar for second case, we obtain plots as in Figure 9 and 10.

[7] [3]

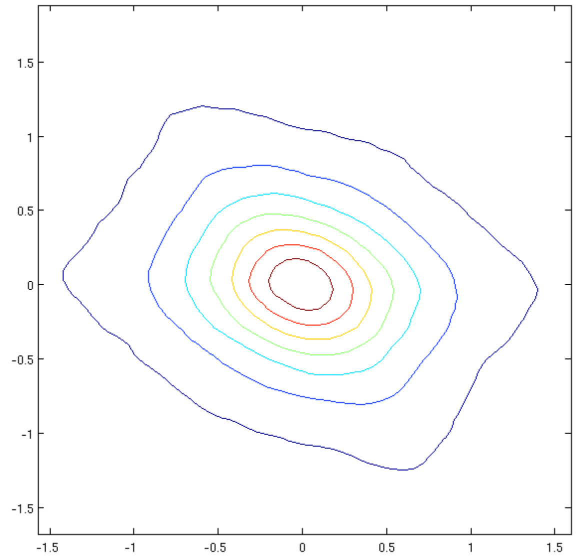

Figure 9: Two dimensional symmetric stable curves with $\alpha = 0.9$

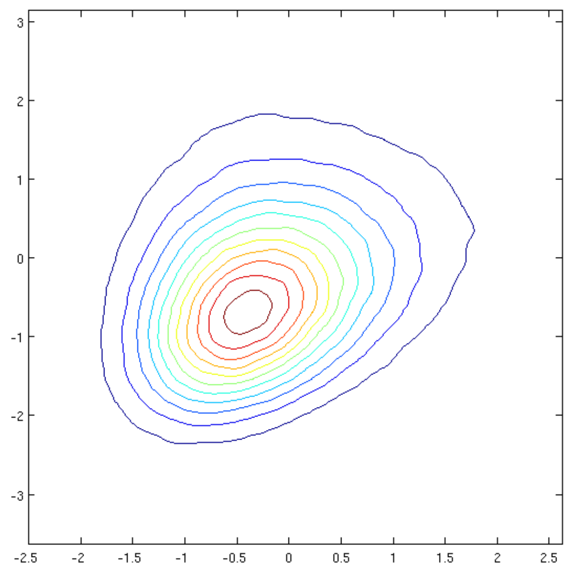

Figure 10: Two dimensional non-symmetric stable curves with $\alpha = 1.6$

4 Alpha-stable Levy motion

Define α-stable Levy motion as follows[8]:

Definition An *α-stable Levy motion* is a process $\{X(t), t \in \mathbb{R}\}$, starting from 0 with stationary independent increments having a α-stable distribution.

One of the most important properties of this process is *self-similarity*, namely:

$$X(ct) \stackrel{d}{=} c^{1/\alpha} X(t) \tag{10}$$

Where $\frac{1}{\alpha}$ is a H value, known as *self-similarity index*.

4.1 Simulation

Algorithm for simulating Levy motion in interval $[t, T]$ is as follows:

1. Set $I = T - t$ and $\tau = T/I$
2. Set $L(0) = 0$
3. Evaluate $L((i+1)\tau) = L_\alpha(i\tau) + \xi_i$, where $\xi_i \sim S_\alpha(\tau^{\frac{1}{\tau}}, \beta, u)$

On the Figure 11 we can see sample path of the Levy motion, simulated for $\alpha = 1.5$.

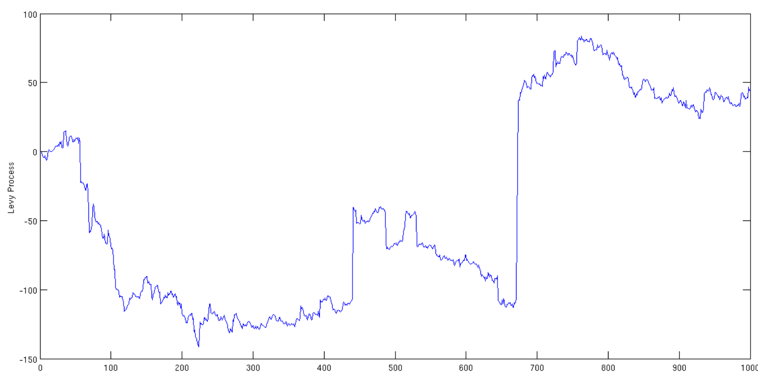

Figure 11: Path of the Levy motion for $\alpha = 1.5$

[8] [2]

To observe and interesting fact of the Levy motion, let's denote $q_p(t)$ as a quantile of order p. Then ($X(t)$ is Levy motion):

$$P(X(t) < q_p(t)) = p$$
$$P(X(1) < t^{\frac{-1}{\alpha}} q_p(t)) = p$$

We can see that:

$$q_p(1) = t^{\frac{-1}{\alpha}} q_p(t)$$

Figure 12 presents empirical and theoretical quantile lines for Levy process as well as sample path.

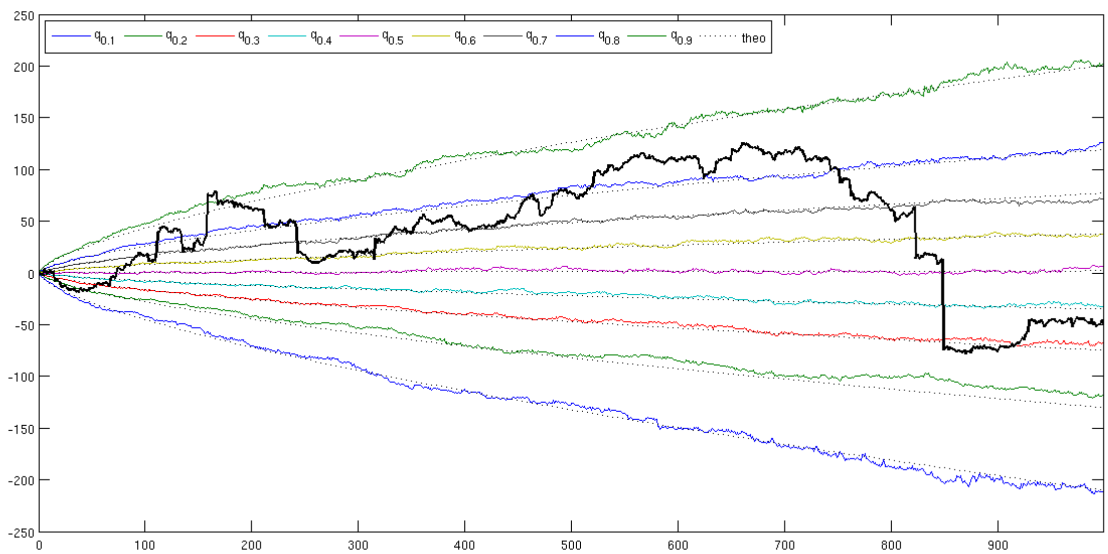

Figure 12: Quantile lines for Levy motion. Black line - sample path

4.2 Lamperti transformation

We would like to work on self-similar processes, as they are invariant under suitable translations of time and scale. To obtain such process, we use Lamperti transformation, that changes stationary process to the corresponding self-similar one in the following way[9]:

[9] [4]

Lamperti transformaton *If $Y = Y(t)$ is a stationary process and if for some $H > 0$:*

$$X(t) = t^H Y(log(t)), \qquad for \qquad t > 0, \qquad X(0) = 0$$

Then $X = X(t)$ is H-ss
Figure 13 presents quantile lines of Lamperti transformation of α-stable process; it is clear, that they are parallel - they do not change in time, as well as distribution.

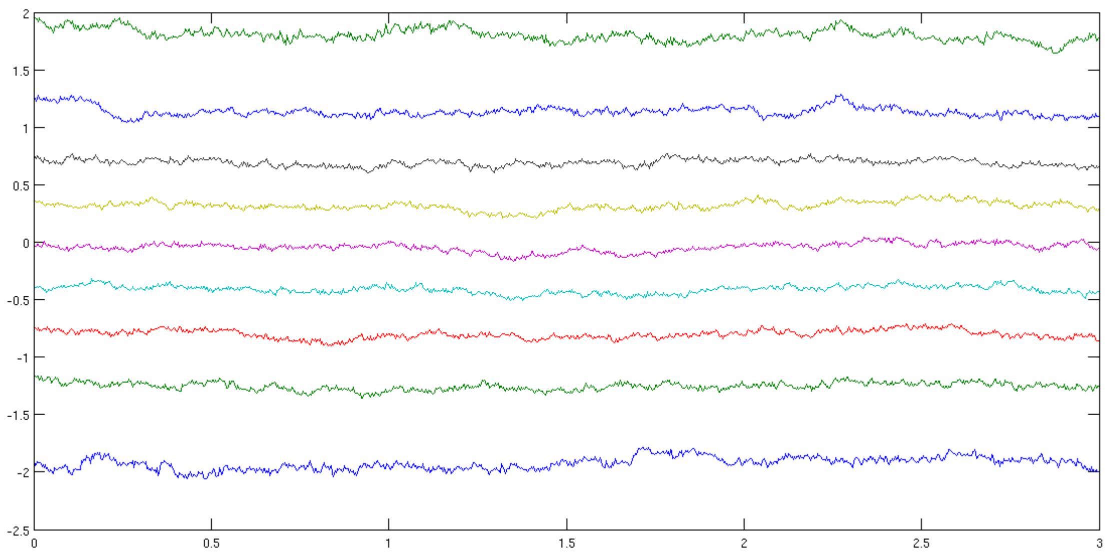

Figure 13: Quantile lines for Lamperti transformation

Next picture presents quantile lines for Orstein-Uhlenbeck process - we see that this process is asymptotically stationary.

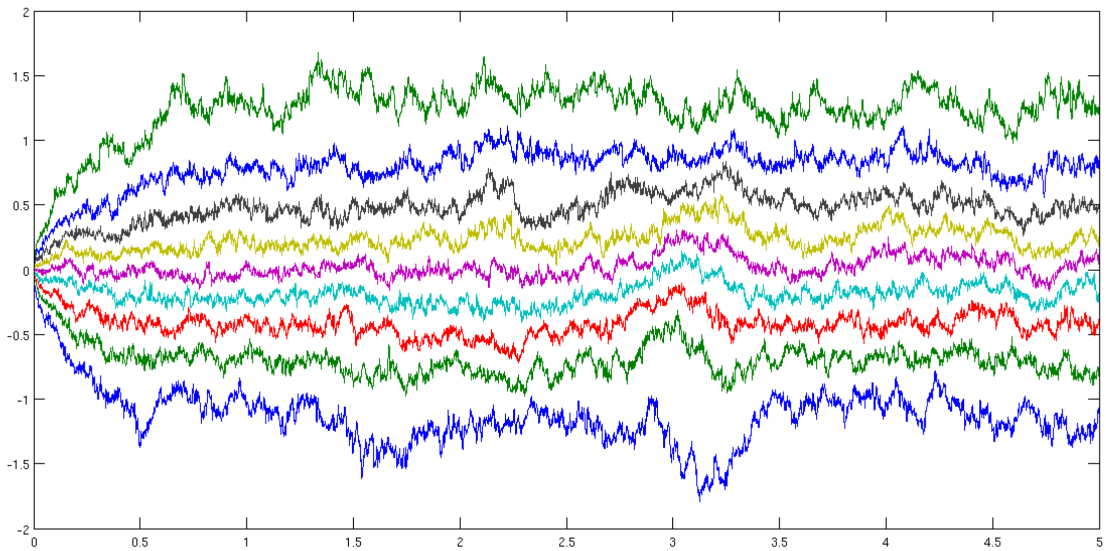

Figure 14: Quantile lines for Orstein-Uhlenbeck process

4.3 Fractional Brownian Motion

One of the most interesting self-similar processes is fractional Brownian motion (fBm). The most characteristic for it it it's covariance function:

$$E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

For $H = \frac{1}{2}$, it is a Brownian motion. Increments of the process are positively correlated for $H > \frac{1}{2}$ and negative correlated for $H < \frac{1}{2}$.
Increments of fBm, called Gaussian noise can have long range dependence (for $H > \frac{1}{2}$).
Figure 15 shows sample paths of fBm for different H; figure 16 shows corresponding autocorrelation functions.
As we can see from Figure 15, the lower value of H-index the more process is oscillating around mean.

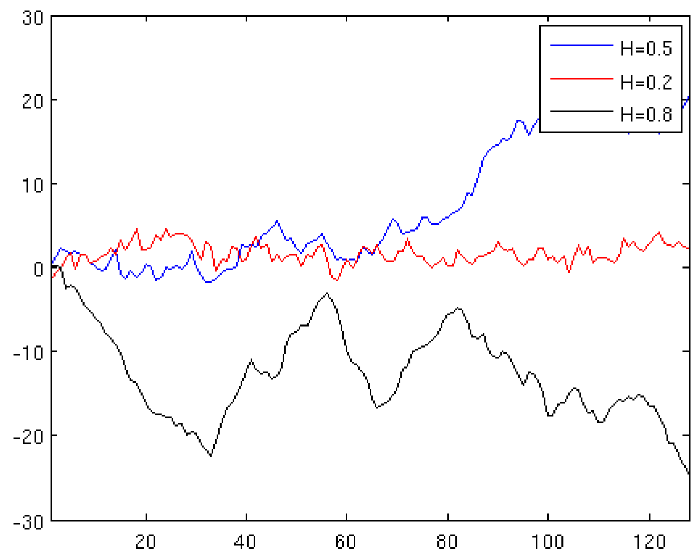

Figure 15: Sample paths of fBm

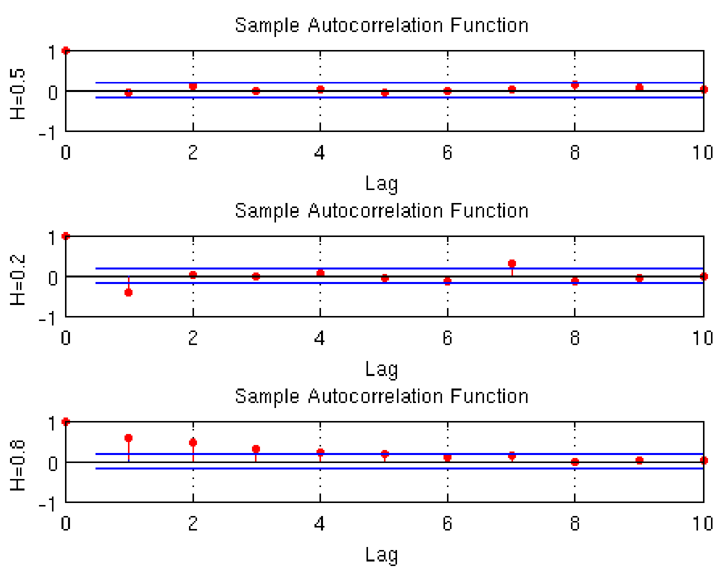

Figure 16: Autocorrelation functions

5 Estimation of parameters

In this section we will estimate sample parameters and compare to theoretical ones.

5.1 α parameters

There are many method to estimate α parameter. Here we will use FLOM[10], as it is simple and easy to understand.
Whole derivation of formulas can be found in proper document. Here we are interested with following ones:

$$L_2 = \mathbf{E}[(log|X| - \mathbf{E}[log|X|)^2]]$$

$$\psi_1 = \frac{\pi^2}{6}$$

$$\alpha = ((\frac{L_2}{\psi_1} - \frac{1}{2}) - \psi_1)^{\frac{1}{2}}$$

Using this, we can check how this method estimate α from our sample. For each value of alpha, 100 vectors of stable random variables of length 1000 has generated. With formulas above, 100 estimators for each value of α has been obtained. Below we can see boxplots of that, as well as Mean Squared Error (MSE). We can see, that for large value (greater than 1.5) of α, error becomes very significant - variance becomes very large.

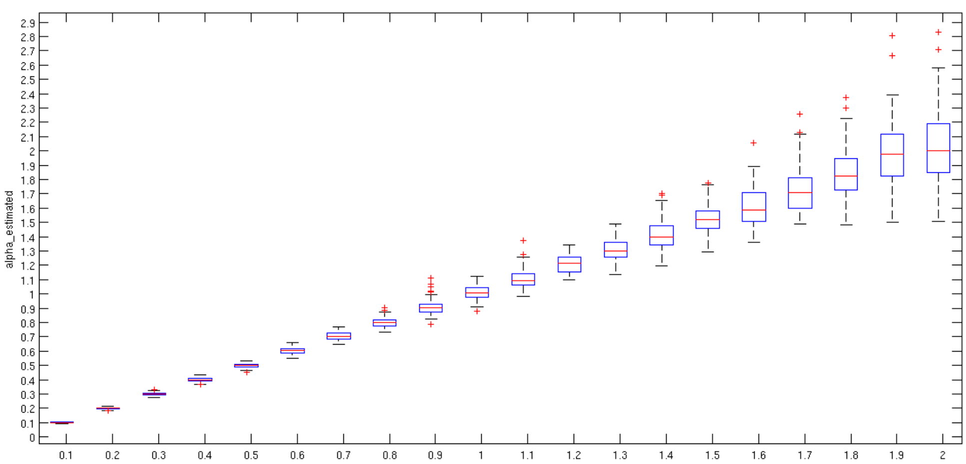

Figure 17: Boxplots of α

[10] [5]

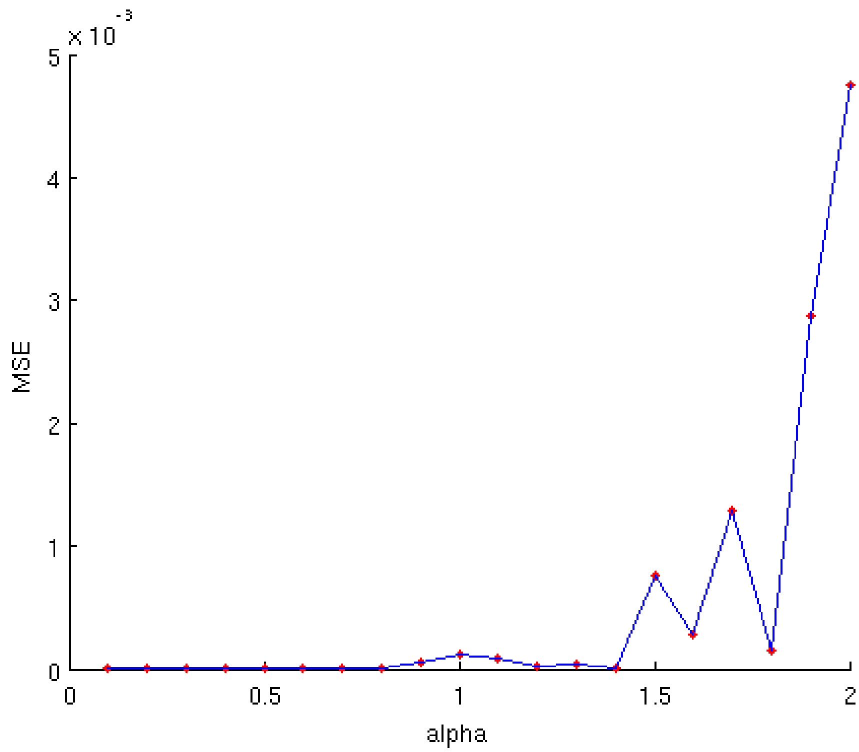

Figure 18: MSE of our parameter

6 Bibliography

[1] Stable Distributions, Models for Heavy Tailed Data; John P. Nolan; 2014

[2] Stable Non-Gaussian Random Processes; G. Samorodnitsky, M.S. Taqqu; 1994

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[4] The Lamperti Transformation for Self-Similar Processes; K. Burnecki, M. Maejima, A. Weron; 1997

[5] Estimation of the Parameters of Skewed *alpha*-Stable Distributions; Ch. R. Dance; 1999