Computer Simulations of Stochastic Processes

Report I

Piotr Kobus 185677

Contents

1. Stable distribution

1.1. Definition of stable distribution

Let take two independent variable X_1 and X_2, which are copies of another variable X. If their linear combination has the same distribution up to location and scale parameter variable X is said to be stable. For some constant *a* and *b* we have:

$$aX_1 + bX_2 \overset{d}{=} cX + d.$$

In above equation c is positive and $d \in R$. What is more equality holds only for distributions i.e. both expressions have the same probability law.

Four parameters of stable random variable are $\alpha \in (0,2], \beta \in [-1,1], \sigma \in (0,\infty)\ and\ \mu \in R$. They give us compete information about α-stable distribution. In literature one can find instead of σ and μ, γ and δ respectively. In this report those will be used interchangeably just to let reader get used to both pair of parameters. First of parameters is called index of stability, the tail index, tail exponent or characteristic exponent. It determines the rate at which tails of the distribution taper off. For α = 2 occur Gaussian distribution. When α < 2 tails exhibit power-law behavior and variance is infinite. Convergence to power-law tail is slower for larger value of tail index. That can be easily saw on Figure 1.1. Mean of stable distribution exist if α > 1. In general, the p-th moment is finite only in p < α. Figure 1.2 presents probability density functions for different α and β = 0.5. Figure 1.3 presents cumulative distribution function for symmetric α-stable distribution. As we can see the greater value of α is, the heavier tail occurs. Figure 1.4 presents cumulative distribution function depending on α for skewed distribution (β = 0.5).

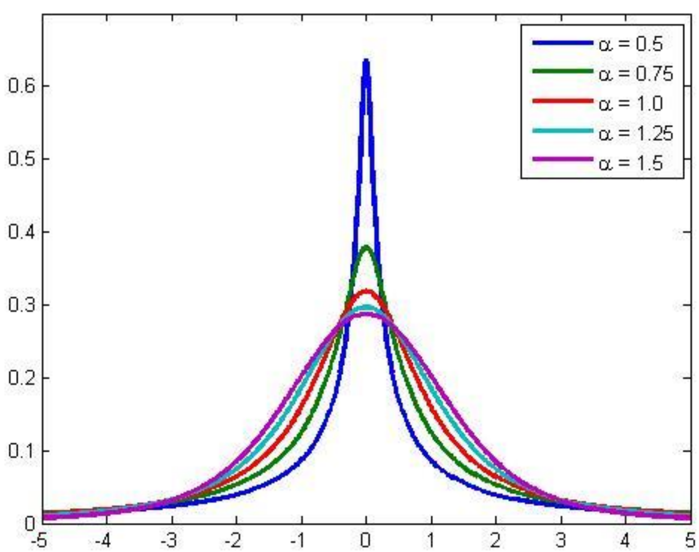

Figure 1.1 PDF for different alfa for symmetric distribution.

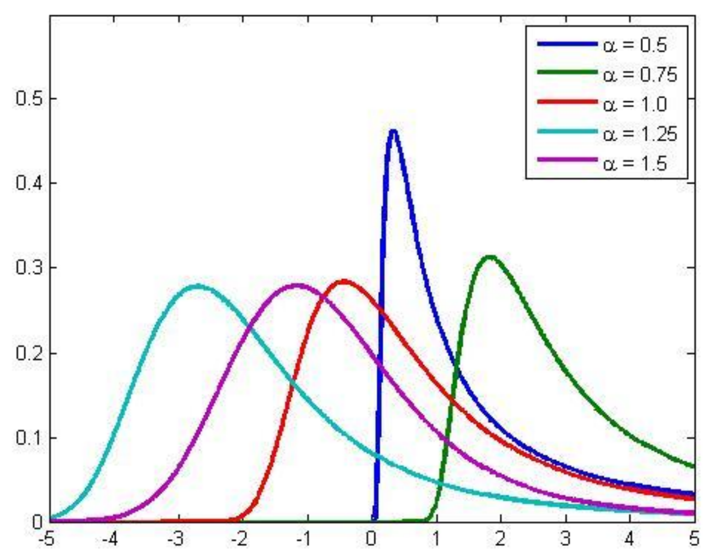

Figure 1.2 PDF for different alfa for skewed distribution.

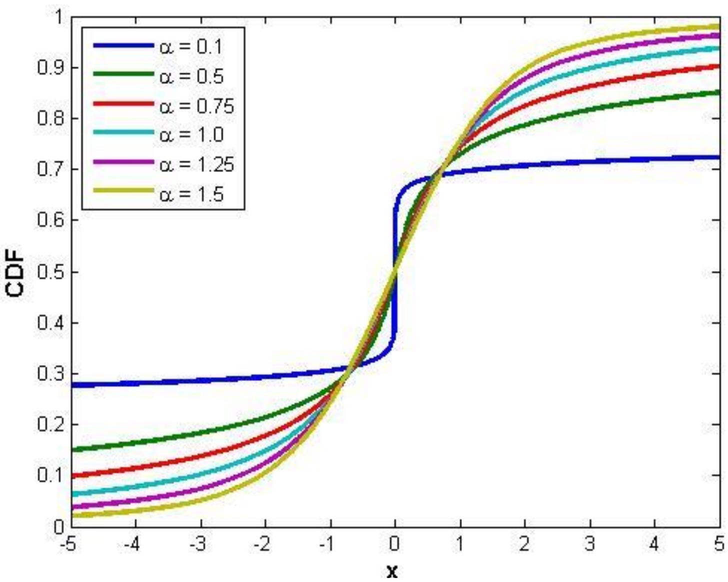

Figure 1.3 CDF for different alfa for symmetric distribution.

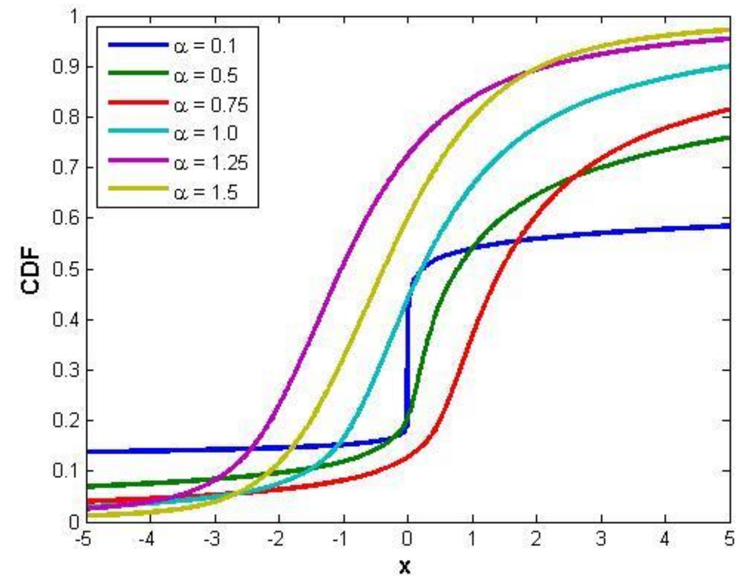

Figure 1.4 CDF for different alfa for skewed distribution.

Second parameter, namely β is called skewness parameter. It can take values from (-1,1). For β equal zero distribution is symmetric. For positive beta, distribution gets right skewed and for negative beta distribution is right skewed. This is very easy to observe in Figure 1.5.

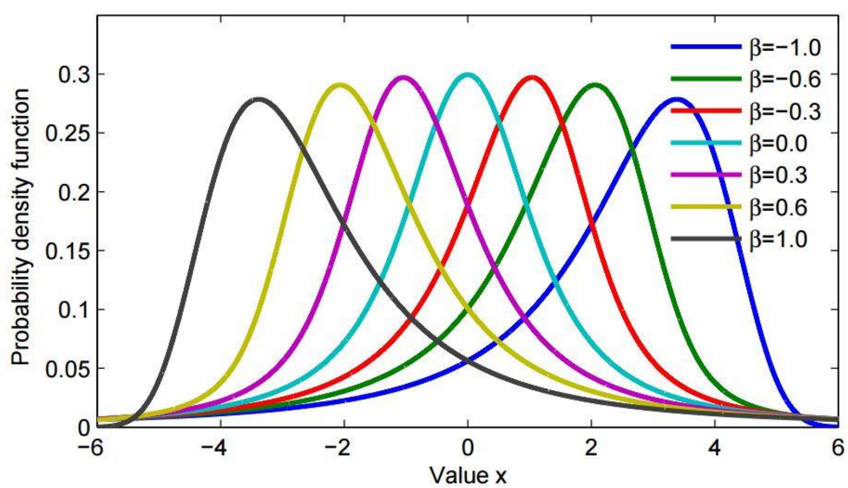

Figure 1.5 PDF of stable distribution for different β.

The last two parameters, σ and μ, are the usual scale and location parameters, i.e. σ determines the width and μ the shift of the peak of the distribution, which for symmetric distribution is just mean. Whereas σ, sometimes called thick parameter, can change the sharpness of density function. For larger value of σ density function becomes flatter. Figure 1.6 presents symmetric stable distribution for different value of parameter σ.

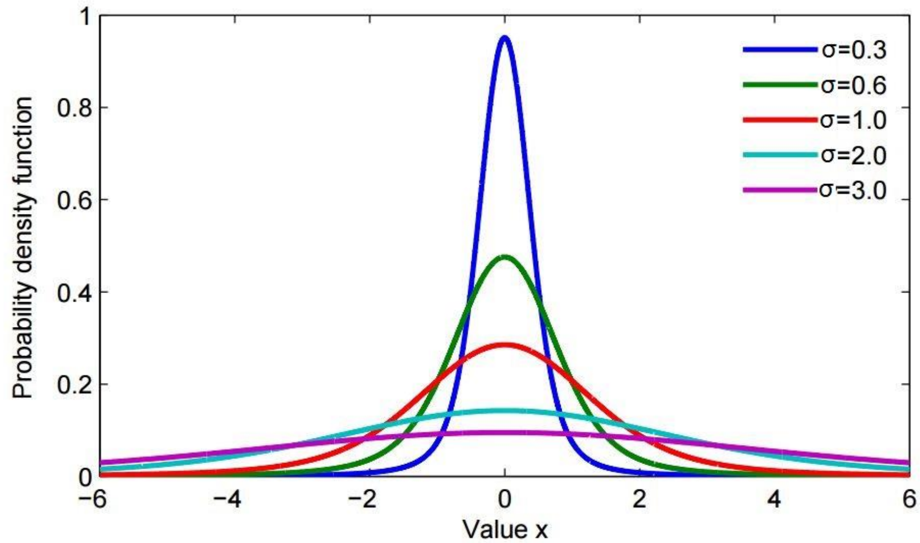

Figure 1.6 PDF of stable process for different σ.

1.2. Parametrizations

One can distinguish different parametrizations of stable process. To introduce them we will need additional random variable. Let call it Z and let it have the characteristic function of the following form:

$$\mathrm{E}\exp(\mathrm{itZ}) = \begin{cases} \exp\left(-\sigma^{\alpha}|\mathrm{t}|^{\alpha}\left(1-\beta\,\mathrm{sgn}(\mathrm{t})\tan\left(\frac{\alpha\pi}{2}\right)+\mathrm{it}\mu\right)\right) & \text{if } \alpha \neq 1 \\ \exp\left(-\sigma^{\alpha}|\mathrm{t}|\left(1-\mathrm{i}\beta\frac{2}{\pi}\mathrm{sgn}(\mathrm{t})\ln|\mathrm{t}|+\mathrm{it}\mu\right)\right) & \text{if } \alpha = 1. \end{cases}$$

Now we can present X~S(α,β,γ,δ,0), where the last parameter represents the parametrization. Namely this is zero-parametrization. Our α-stable random variable has following distribution:

$$X \stackrel{d}{=} \begin{cases} \gamma(Z-\beta\tan\frac{\pi\alpha}{2})+\delta & \alpha \neq 1 \\ \gamma Z+\delta & \alpha = 1 \end{cases},$$

and X has the characteristic function:

$$E\exp(iuX) = \begin{cases} \exp\left(-\gamma^{\alpha}|u|^{\alpha}\left[1+i\beta(\tan\frac{\pi\alpha}{2})(\operatorname{sign} u)(|\gamma u|^{1-\alpha}-1)\right]+i\delta u\right) & \alpha \neq 1 \\ \exp\left(-\gamma|u|\left[1+i\beta\frac{2}{\pi}(\operatorname{sign} u)\log(\gamma|u|)\right]+i\delta u\right) & \alpha = 1. \end{cases}$$

As another parametrization we take

$$X \stackrel{d}{=} \begin{cases} \gamma Z + \delta & \alpha \neq 1 \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \log \gamma) & \alpha = 1, \end{cases}$$

which has the characteristic function

$$E \exp(iuX) = \begin{cases} \exp\left(-\gamma^\alpha |u|^\alpha \left[1 - i\beta (\tan \frac{\pi\alpha}{2})(\operatorname{sign} u)\right] + i\delta u\right) & \alpha \neq 1 \\ \exp\left(-\gamma |u| \left[1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \log |u|\right] + i\delta u\right) & \alpha = 1. \end{cases}$$

The very important property of stable random variables is that their sum gives another stable random variable. Let take $X_1 \sim S_1(\alpha,\beta_1,\gamma_1,\delta_1,1)$ and $X_2 \sim S_2(\alpha,\beta_2,\gamma_2,\delta_2,1)$. If they are independent than $X_1 + X_2 = S(\alpha,\beta,\gamma,\delta,1)$ where:

$$\beta = \frac{\beta_1 \gamma_1^\alpha + \beta_2 \gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, \quad \gamma^\alpha = \gamma_1^\alpha + \gamma_2^\alpha, \quad \delta = \delta_1 + \delta_2.$$

1.3. Simulations

Let take two independent random variables, W and Θ. W is exponentially distributed with mean equal one. Formula for this random variable can be easily derived via e.g. inverse-transform method. Θ is uniform random variable distributed on $(-\pi/2, \pi/2)$. For simplicity we need one more parameter to be introduced namely $\theta_0 = \arctan(\beta \tan(\pi\alpha/2))/\alpha$. Now the random variable $Z \sim S(\alpha,\beta,1)$ if:

$$Z = \begin{cases} \dfrac{\sin \alpha(\theta_0 + \Theta)}{(\cos \alpha\theta_0 \cos \Theta)^{1/\alpha}} \left[\dfrac{\cos(\alpha\theta_0 + (\alpha - 1)\Theta)}{W}\right]^{(1-\alpha)/\alpha} & \alpha \neq 1 \\ \frac{2}{\pi}\left[(\frac{\pi}{2} + \beta\Theta)\tan\Theta - \beta \log\left(\dfrac{\frac{\pi}{2} W \cos\Theta}{\frac{\pi}{2} + \beta\Theta}\right)\right] & \alpha = 1. \end{cases}$$

Now we have explicit formula for stable random variable which can be easily implemented in any package.

2. Analysis of stable distribution

2.1. Tail of stable distribution

For α = 2 we are obtaining Gaussian distribution which tail represents asymptotic law. Let us consider only case when α < 2. Depending on β distribution can have either one or two tails. In that case tails decrease as power function. Let denote distribution function by F(x) and we can obtain the following power law:

$$1 - F(x) \approx x^{-\alpha}.$$

That leads to:

$$\frac{\log(1 - F(x))}{\log x} = -\alpha.$$

On the basis on above equation we can obtain α parameter of our distribution by plotting distribution function on double logarithmic scale and calculating the slope with negative sign. Figure 2.1 presents this procedure. Of course in order to obtain a correct value of α we should approximate only slope of tail which is representing by decreasing part of figure 2.1. Simulated distribution have following parameters S(1.5, 0, 30, 1). Result of regression corresponds to parameters used in simulation.

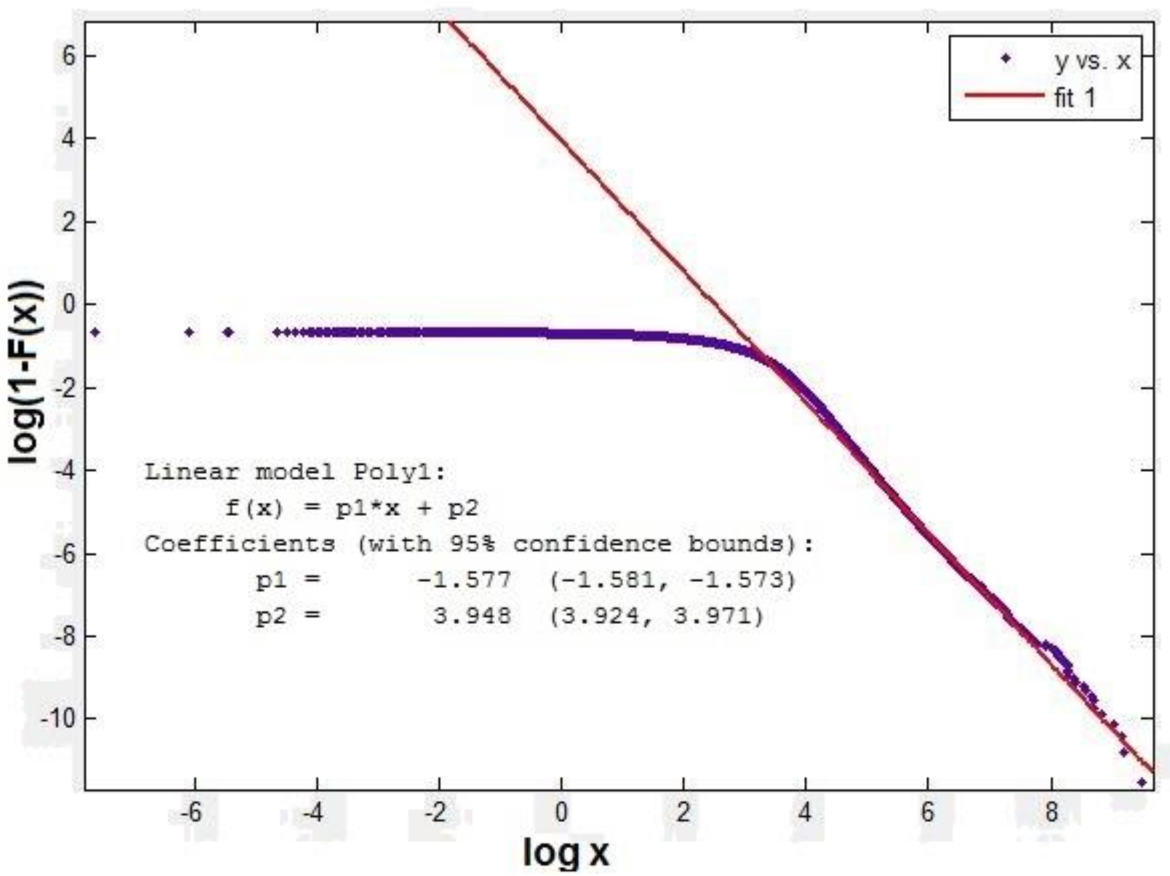

Figure 2.1 Tail of distribution on double logarithmic scale.

2.2. Sum of two stable distributions

All formulas for sum of stable distribution were presented in first chapter. Two stable distributions with following parameters were taken S(1.5, 0.2, 0.2, 1) and S(1.5, 1, 0.1, 0.9). In order to verify if above formulas work, all parameters were calculated using two approaches. First of them is based on analytical formulas mentioned in chapter 1. Second method uses program for approximation stable distribution parameters write by John Nolan. Below are presented the output of Nolan's program

```
Stable model with maximum likelihood estimator

 Initial quantile estimate of S0 parameters
       alpha       beta      gamma         delta
    1.508308  0.110621    1.05652       1.78345
```

Results of analytical calculations are as follows:

Alpha	Beta	Gamma	Delta
1.5	0.1082	0.7263	1.900

As we can see most of results obtained by approximations are similar to those obtained by analytical formula.

2.3. Series representation

Now we will consider only stable process with 0 <α <1. Let by N_δ denote Poisson random variable with parameter δ. Now let introduce a random variable with following distribution function:

$$P\left(Y_{\delta,k} > X\right) = \begin{cases} \delta^{\alpha} X^{-\alpha} & for X > \delta \\ 1 & for\ X < \delta \end{cases}.$$

And now let introduce another variable, namely:

$$X_\delta = \sum_{k>1}^{N_j} Y_{j,k}.$$

Variable X_δ for δ approaching 0 becomes so called alfa stable subordinator i.e. stable random variable with parameters S(α, σ, 1, 0), where:

$$\sigma^{\alpha} = \Gamma(1-\alpha)\cos\left(\pi\frac{\alpha}{2}\right).$$

Figure 2.2 presents tails of distribution for decreasing δ. Value of δ are presented in the legend. Thick blue line presents S (α, σ, 1, 0). As we can see the less value of δ the more closely X_δ is to S.

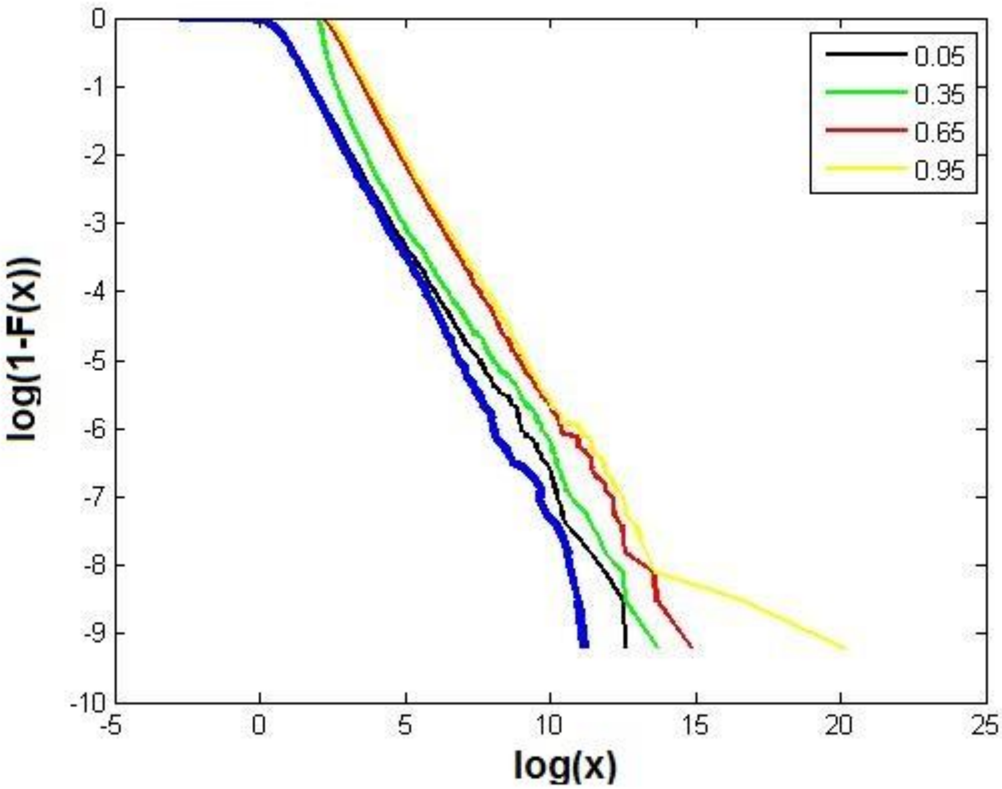

Figure 2.2. Tails of stable subordinator and r.v. given by series representation for different δ.

3. Multivariate stable distribution

Let by $X \sim S_{\alpha,d}(\Gamma,\mu^0)$ denote stable random vector. It can be described by a spectral measure Γ which is a finite Borel measure on unit sphere in R^d and μ^0 in R^d which is a shift vector. The characteristic function of X is:

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbf{E}\exp\{i < \mathbf{X}, \mathbf{t} >\} = \exp(-I_{\mathbf{X}}(\mathbf{t}) + i < \mu^0, t >),$$

where

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S_d} \psi_\alpha(< \mathbf{t}, \mathrm{s} >)\Gamma(ds).$$

Where $<t,s> = t_1s_1 + \ldots + t_ds_d$ is an inner product and

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha(1 - i\text{sign } (u)\tan\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ |u|(1 + i\frac{2}{\pi}\text{sign } (u)\ln|u|) & \alpha = 1. \end{cases}$$

In case of simulations we can consider only discrete world. That is why we have to represent spectral measure as a finite set of point masses i.e.,

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot),$$

Where γ_j's are weights, and δ_j's are point masses at the points s_j in S_d. Simulation is relatively easy. One just need to follow the rule below:

$$\mathbf{X} \stackrel{D}{=} \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j \mathbf{s}_j & \alpha \neq 1 \\ \sum_{j=1}^{n} \gamma_j (Z_j + \frac{2}{\pi}\ln\gamma_j)\mathbf{s}_j & \alpha = 1, \end{cases}$$

Where $Z_1,..,Z_n$ are i.i.d. totally skewed, standardized one dimensional α-stable random variables. Figure 3.1 presents densities of multivariate stable distribution with different point masses. On the left side masses were on two different axis and in right side of the figure masses are on the same axis.

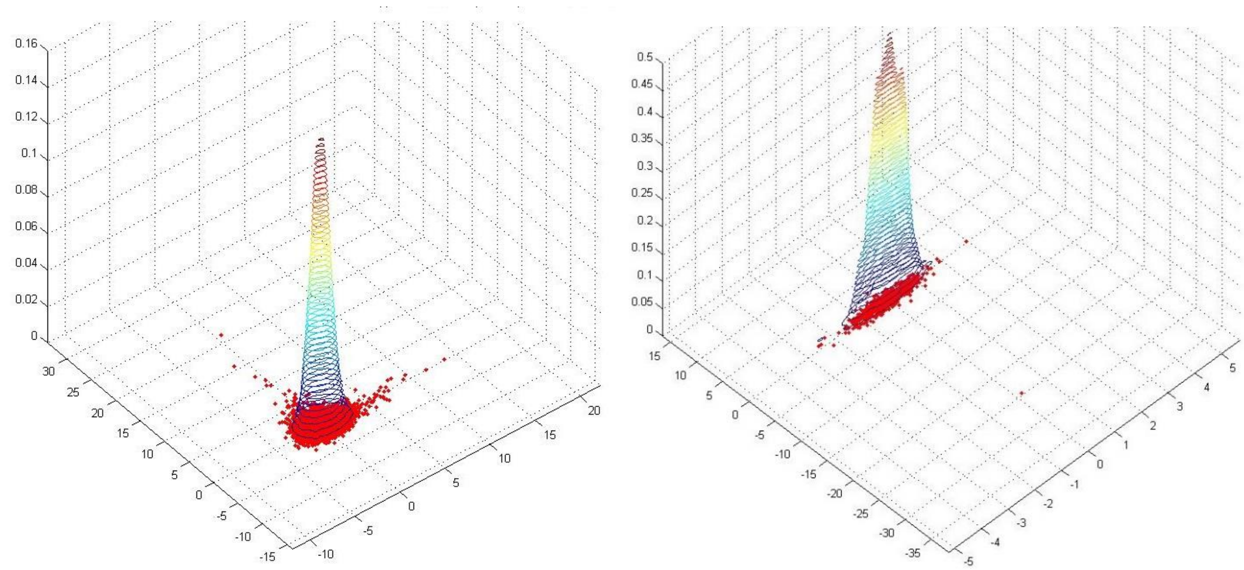

Figure 3.1. Stable density surface for masses on different (left) and the same (right) axis.

Figure 3.2 presents contour plots for above described distributions.

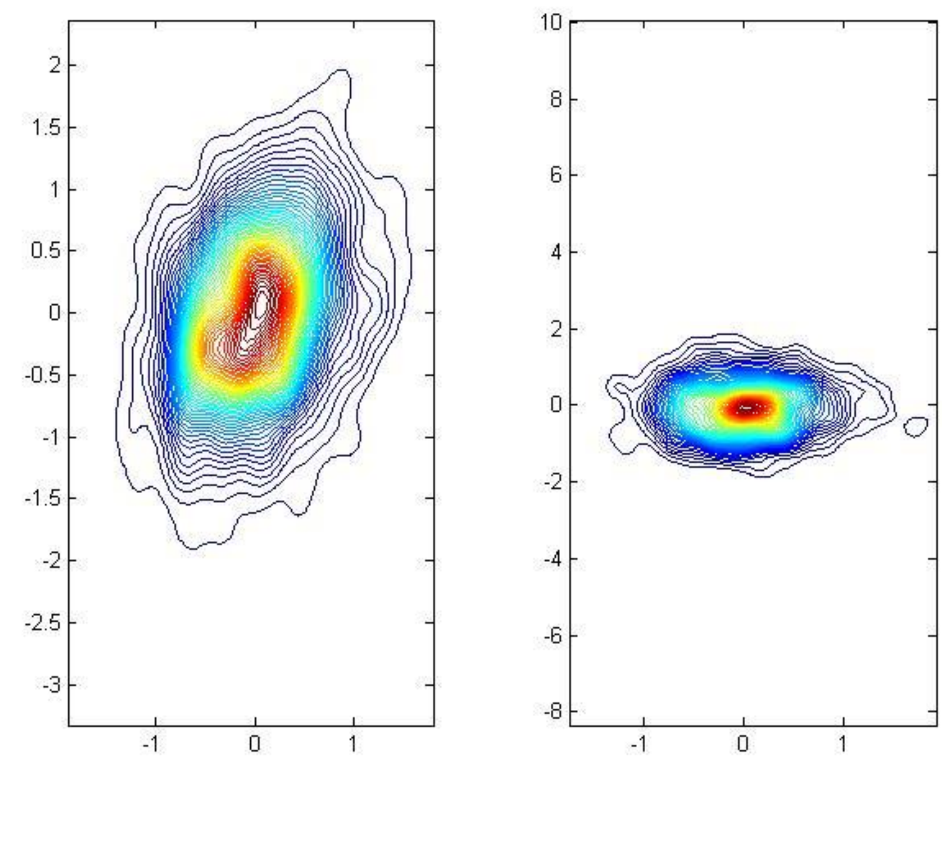

Figure 3.2. Stable density contour plots for masses on different (left) and the same (right) axis.

4. Self-similar processes

A process is called self-similar if for some H > 0 and every c > 0

$$X(ct) \overset{d}{=} c^{H} X(t).$$

Above equality holds only for distribution, not for values of a process. H is called self-similarity index, Hurst index or exponent of the self-similar process X. H index is a measure of long term memory. One of very well-known self-similar process is fractional Brownian motion for which 0 < H < 1. For H > 0.5 increments of the process are positively correlated which means that exhibits long range dependence and for H < 0.5 increments are negatively correlated. For H = 0.5 Brownian motion is obtained. Fractional Brownian process belongs to weightier class of self-similar process, namely Levy process.

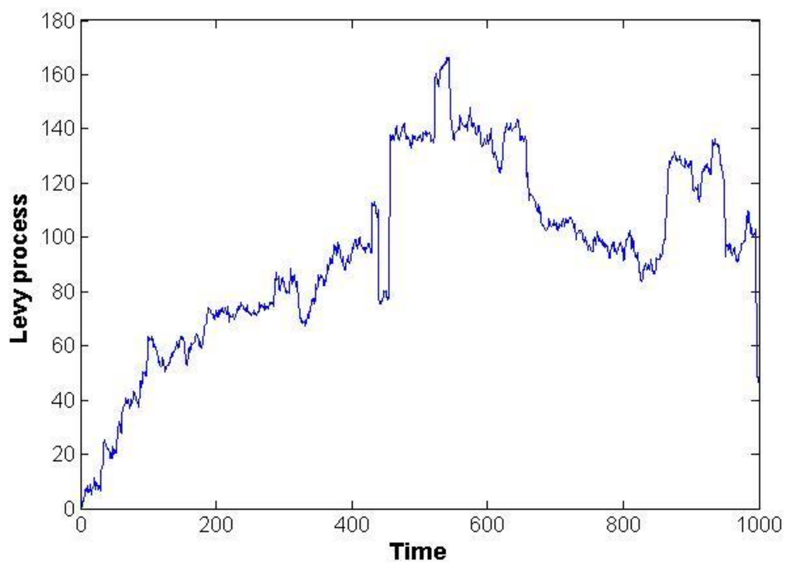

Figure 4.1. Sample path of Levy process

Figure 4.1. presents sample path of Levy motion with parameters alpha beta and miu equal 1.5; 0 and 0 respectively.

Figure 4.2 presents 0.1, 0.2, …, 0.9 quantile lines for Levy process with theoretical quantiles and one realization of the process. Observe that the obtained curves are not parallel and resemble a power function, which indicates nonstationary and self-similarity. For all self-similar processes undermentioned relation can be applied:

$$q_p(1) = t^{\frac{-1}{\alpha}} q_p(t).$$

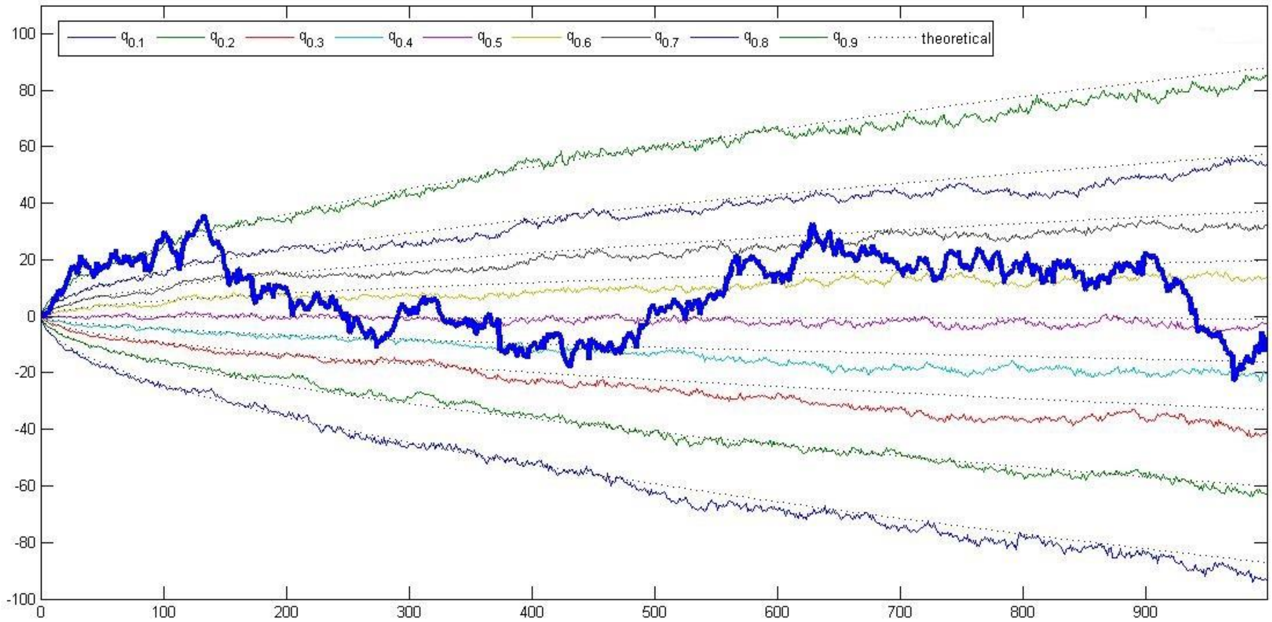

Figure 4.2. Empirical and theoretical quantile lines for Levy process with one sample path.

Lamperti introduced very important transformation (called Lamperti transformation) which changes stationary processes to the corresponding self-similar ones in the following way:

$$X(t) = t^H Y(\log t), \qquad for\ t > 0, \qquad X(0) = 0,$$

where Y(t) is a stationary process and H > 0. Than X(t) is H self-similar process. Transformation opposite way also exist. Let X(t) be H self-similar process.

$$Y(t) = e^{-tH} X(e^t), t \in R$$

than Y(t) is stationary.

This procedure can be used to simulate H self-similar process having trajectory of stable process or the other way around.

Figure 4.3 presents quantile lines of process obtain by Lamperti transformation of α-stable Levy process. As we can see, quantile lines are parallel to each other and constant*. Quantiles does not change in time and so does distribution! This proves that obtained process is stationary. Similar behavior is observed in Figure 4.4 which presents quantile lines of Orstein-Uhlenbeck process simulated via stochastic differential equation. As we can see in that case process is asymptotically stationary. The reason of this begavior is the approximation methods which force us to cut the interval of integration.

*Oscillate around some constant value.

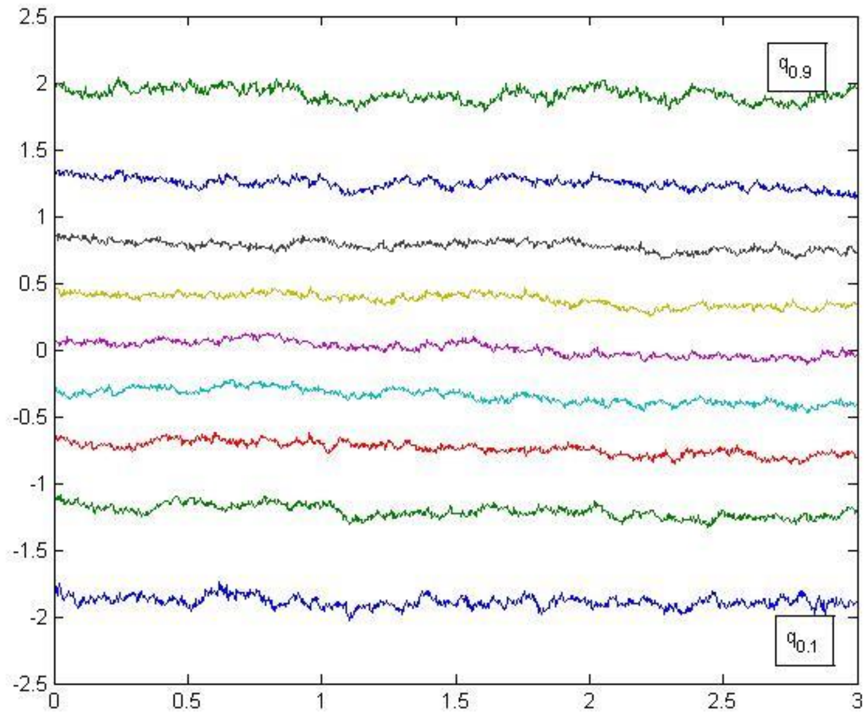

Figure 4.3. Quantile lines of so called stable Orstein-Uhlenbeck process.

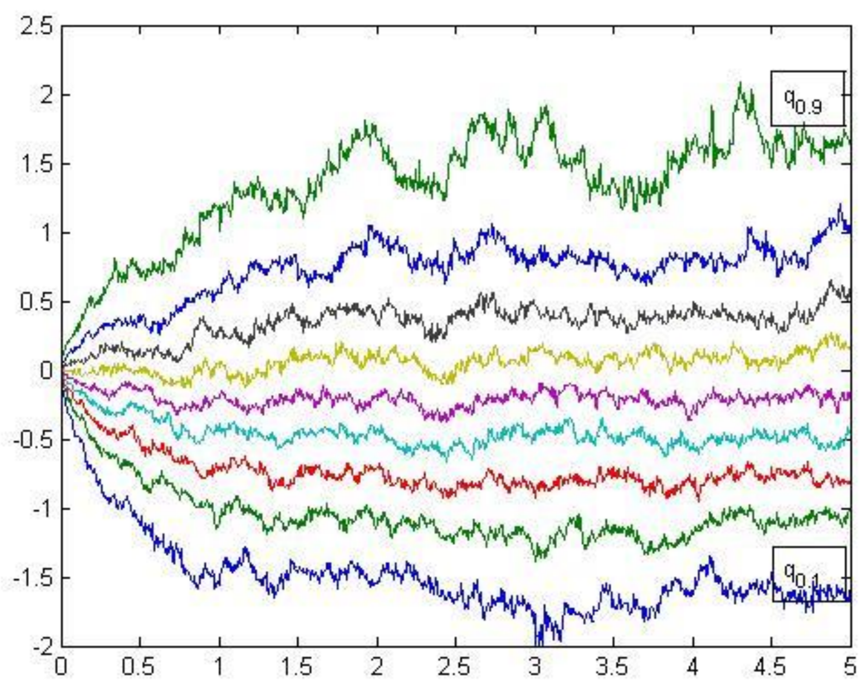

Figure 4.4. Quantile lines of Orstein-Uhlenbeck process.

As was mentioned before one of example of self-similar processes is fractional Brownian motion. It is a continuous-time Gaussian process which starts at zero, has expected value zero and has the following covariance function:

$$E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

Hurst exponent can take values from 0 to 1. As we can see for H equal 0.5 process becomes Brownian motion. For H greater than 0.5 increments of the process are positively correlated, and for H less than 0.5 increments are negatively correlated.

Long range dependence in stationary time series occurs when the autocorrelation function tends to zero like a power function. By contrast, one speaks of "short range dependence" if the autocorrelation function decreases at a geometric rate. But self-similarity does not imply long-range dependence in any way. Considered process (fractional Brownian motion) is self-similar and in this case there is a relationship between self-similarity and long range dependence. Increments of factional Brownian motion are stationary and are called fractional Gaussian noise. Those increments can display long range dependence. The intensity of long-range dependence is related to the scaling exponent of the self-similar process.

Figure 4.5 presents sample paths of fractional Brownian motion (left) and autocorrelations function of fractional Gaussian noise (right) for different H-index values. As we can see for H greater than 0.5 we are observing power decaying of autocorrelations function. According to literature for 0.5 < H < 1 we are observing long range dependence of increments of fractional Brownian motion. For 0 < H < 0.5 long range dependence is not observed and for H = 0.5, as was mentioned, Brownian motion is obtained.

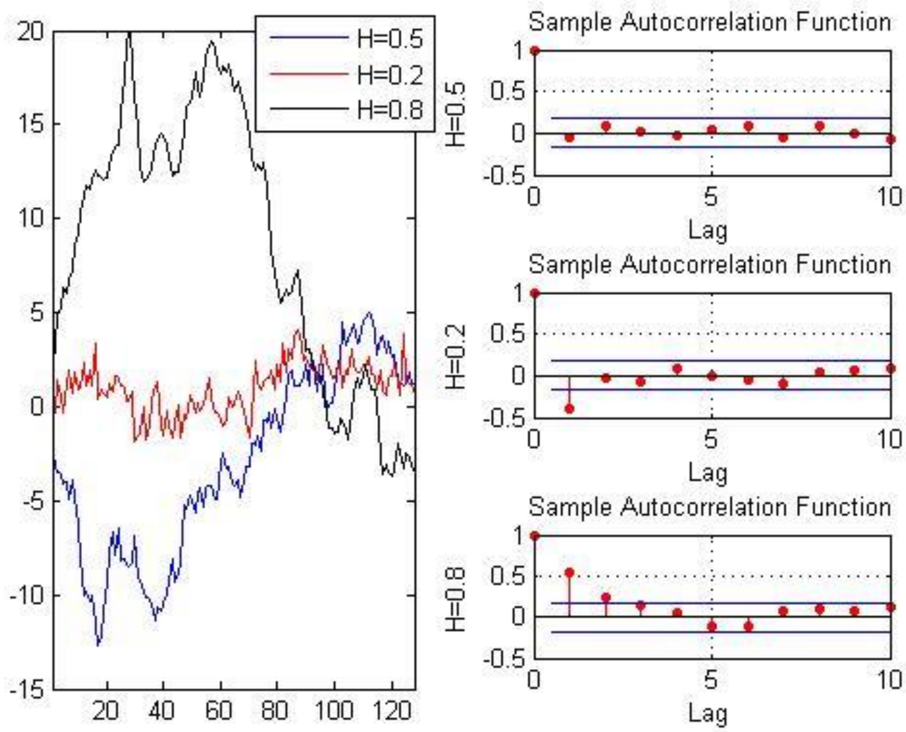

Figure 4.5. Sample paths of fBm and autocorrelation function of fGn.

Left part of figure 4.6 presents sample paths of fractional Brownian motion. As we can see the lower value of H-index the more process is oscillating around mean. For H = 1 we will obtain straight line with different slope for every simulation.

Another way to check weather process has long range dependence of not is checking the summability of correlations. If sum of correlations goes to infinity we are observing process with long memory. We can also check spectral domain to examine long memory property. For stationary process with finite variance and summable correlation the spectral density f satisfies.

$$\sigma^2 \rho_n = \int_0^{\pi} \cos(nx) f(x) dx.$$

For processes with long memory spectral density blows up at the origin. For fractional Brownian motion spectral density is behaves like:

$$f(x) \sim \frac{C}{2} x^{-(2H-1)}$$

for $x \to 0$, spectral density is continuous in case of $0 < H \leq \frac{1}{2}$, and has a pole at the origin if $\frac{1}{2} < H < 1$.

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Lecture and laboratory notes