Computer Simulations of Stochastic Processes

Report II

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1. Estimation of stability parameter

A popular technique for designing estimators is to find formulas for the moments of a distribution in terms of the parameters of that distribution. This technique may not be directly applied to stable distributions since for α < 2 it is well-known that stable distributions have infinite variance and for α < 1 they have infinite mean. An alternative is to consider fractional lower order moments or FLOMs.

Let X be a stable vector with parameters (α,β,γ,0) than:

$$\begin{aligned}\mathbf{E}\left[X^{<p>}\right] &= \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma\left(1-p\right)}\left|\frac{\gamma}{\cos\theta}\right|^{\frac{p}{\alpha}}\frac{\sin\left(\frac{p\theta}{\alpha}\right)}{\sin\left(\frac{p\pi}{2}\right)},\ \text{for } p\in(-2,-1)\cup(-1,\alpha),\\ \mathbf{E}\left[|X|^{p}\right] &= \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma\left(1-p\right)}\left|\frac{\gamma}{\cos\theta}\right|^{\frac{p}{\alpha}}\frac{\cos\left(\frac{p\theta}{\alpha}\right)}{\cos\left(\frac{p\pi}{2}\right)},\ \text{for } p\in(-1,\alpha),\end{aligned}$$

$$\theta=\arctan\left(\beta\tan\frac{\alpha\pi}{2}\right).$$

One of the way of eliminating the gamma function from the FLOM formulas is to differentiate with respect to the moment order p. This results in the moments of the logarithm of the α-stable process. The advantage of the logarithmic estimators relative to the FLOM estimators is they do not require the inversion of a sinc function or the choice of a moment exponent p. Specifically, applying the result that:

$$\mathbf{E}\left[(\log|X|)^{n}\right]=\lim_{p\to 0}\frac{d^{n}}{dp^{n}}\mathbf{E}\left[|X|^{p}\right],\ n=1,2,3,\ldots$$

So we obtain the following moments:

$$\begin{aligned}L_1=\mathbf{E}\left[\log|X|\right] &= \psi_0(1-\frac{1}{\alpha})+\frac{1}{\alpha}\log\left|\frac{\gamma}{\cos\theta}\right|,\\ L_2=\mathbf{E}\left[(\log|X|-\mathbf{E}\left[\log|X|\right])^{2}\right] &= \psi_1(\frac{1}{2}+\frac{1}{\alpha^{2}})-\frac{\theta^{2}}{\alpha^{2}},\\ L_3=\mathbf{E}\left[(\log|X|-\mathbf{E}\left[\log|X|\right])^{3}\right] &= \psi_2(1-\frac{1}{\alpha^{3}}),\end{aligned}$$

Where

$$\psi_{k-1}=\frac{d^{k}}{dx^{k}}\log\Gamma\left(x\right)\bigg|_{x=1}$$

Applying centro-symmetrizatiton we can obtain an estimator for α in the following form:

$$\alpha=\left(\frac{L_2}{\psi_1}-\frac{1}{2}\right)^{-1/2}$$

The other method to estimate stability parameter is based on quantiles of the distribution. It idea and very deep analysis was resented by J. Huston McCuloch. This method works only for α in the range [0.6, 2].

Figure 1.1 presents the comparison of mentioned estimation methods. Simulations were performed for 100 vectors of length equal 1000. Green boxplots present FLOM method and blue boxplots present method based on quantiles. As we can see for alpha less than 0.6 quantile method does not work and FLOM method gives very similar results in each simulation(small variance) even though not very accurate. For alpha greater than 0.6 quantile method estimates rather accurate but variance of estimator increases with alpha. FLOM estimator getting less accurate with increasing alpha.

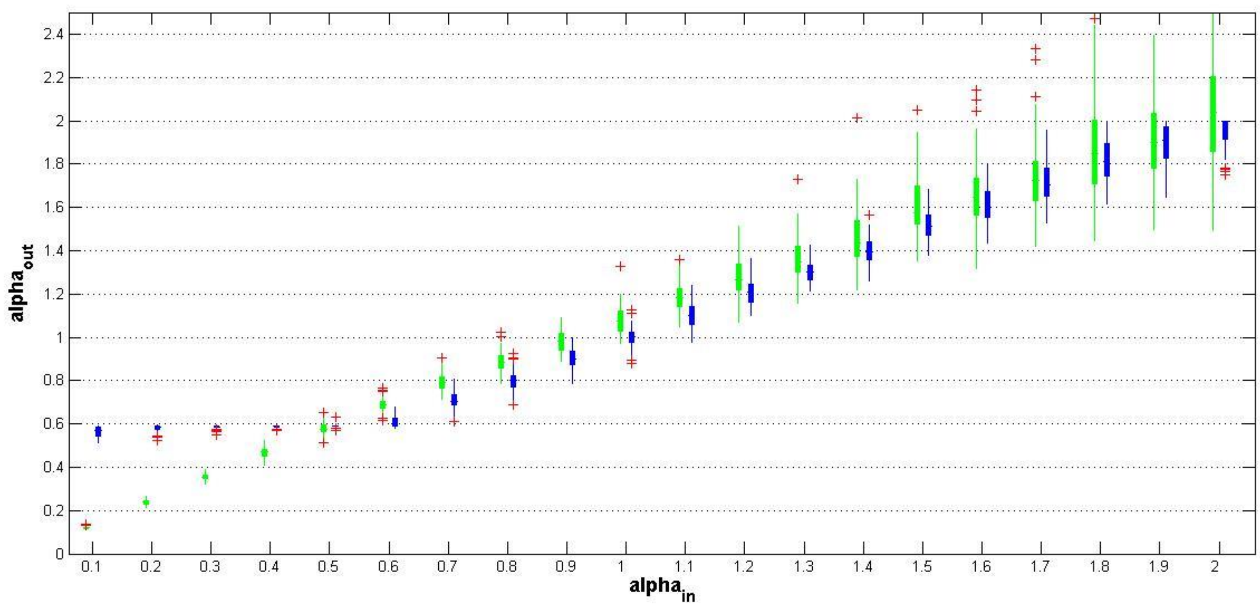

Figure 1.1. Boxplots of estimators of alpha using two methods.

Figures 1.2 and 1.3 presents mean square and mean relative error respectively. As we can easily see conclusions corresponds to those obtain from Figure 1.1.

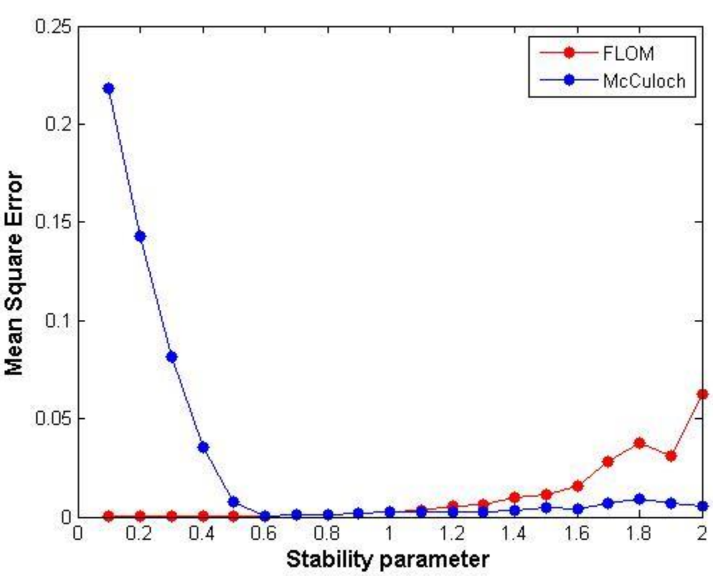

Figure 1.2. Mean square error of estimator of alpha for both methods.

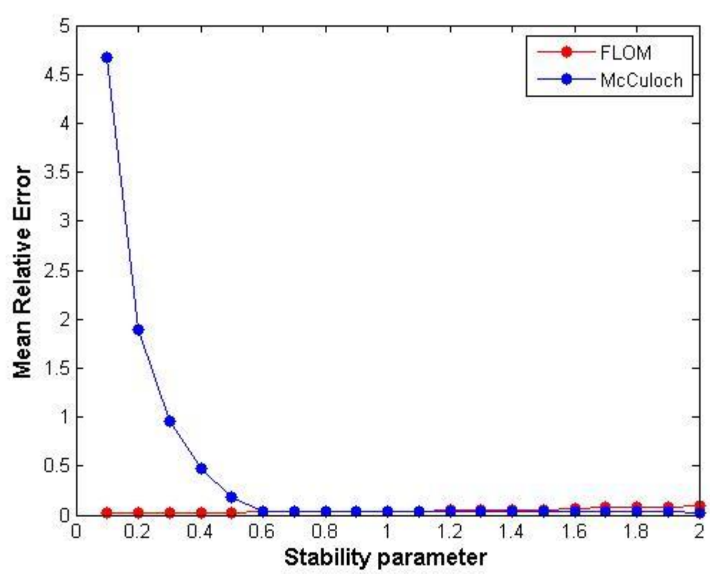

Figure 1.3. Mean relative error of estimator of alpha for both methods.

Another important thing which can be checked is the sensitivity of estimators to some random noise. Firstly, to the distribution was added normal noise with mean 0 and variance 3. For lower values of variance no visible changes occurred. Figure 1.4 presents boxplots of estimators of alpha obtained by both methods.

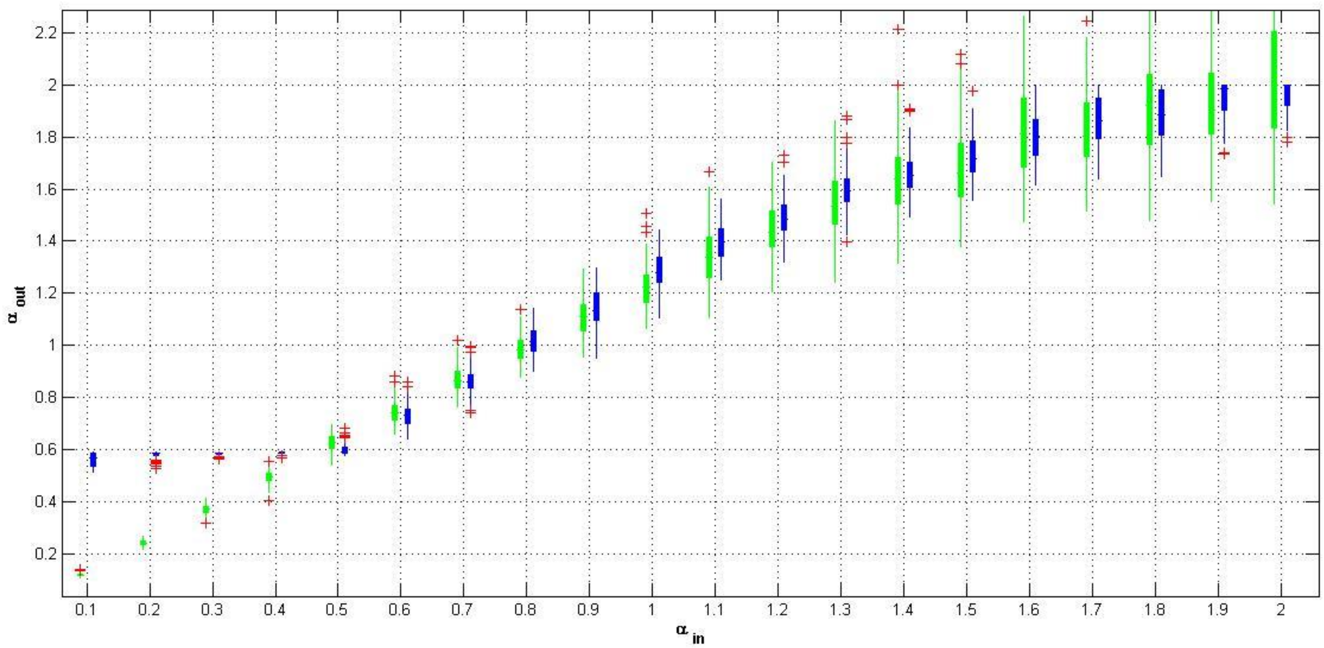

Figure 1.4. Boxplots of estimators of alpha using FLOM (green) and McCuloch (blue) methods with normal noise.

To easier obtain some conclusions mean square (Figure 1.5) and mean relative error (Figure 1.6) need to be presented. As we can see for increasing value of alpha FLOM methods get worst results but for alpha from interval 0.7 to 1.4 it is still better than McCuloch method. For alpha approaching 2 McCuloch methods gives better results than FLOM method but worst in comparison to unperturbed data.

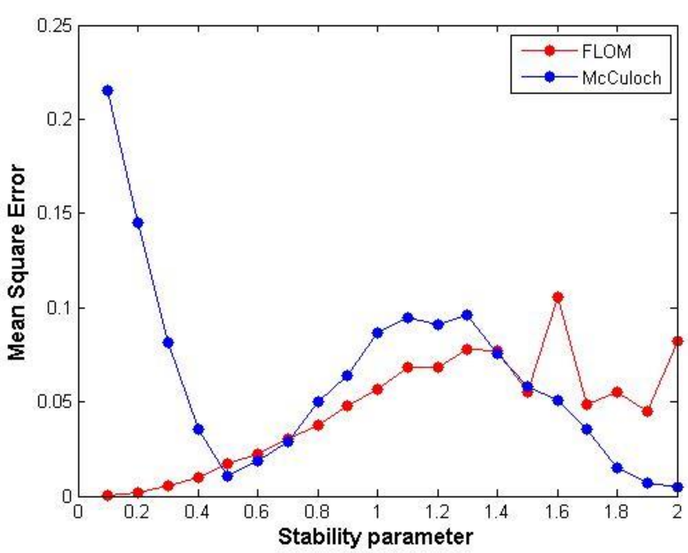

Figure 1.5. Mean square error of alpha estimator with normal noise for both methods.

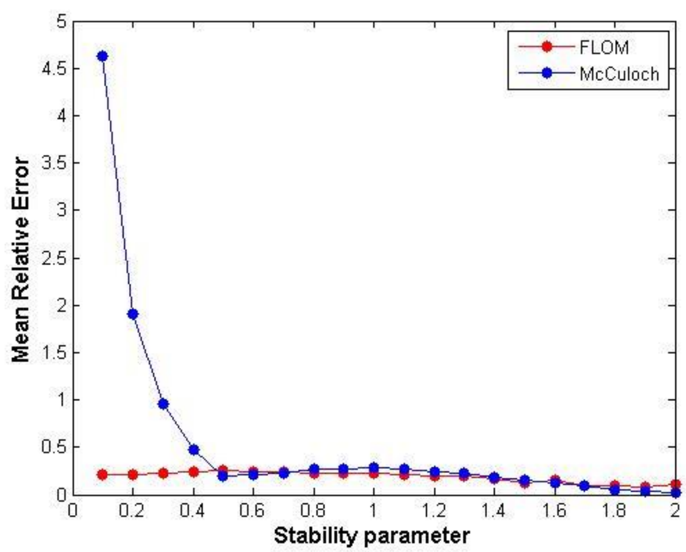

Figure 1.6. Mean relative error of alpha estimator with normal noise for both methods.

As we know, α-stable distribution for α different than 2 has infinite variance. That is why another choice as noise was Pareto distribution, with scale parameter 1 and shape parameter equal 1.1, which also does not have finite variance. Figure 1.7 presents boxplots of estimators of alpha. As we can see for increasing values of alpha both methods estimate with worst accuracy. What more was observed that for increasing values of shape parameter McCuloch methods gives better approximation than FLOM method but we should have in mind that if shape parameter is greater than 2 Pareto distribution have finite variance.

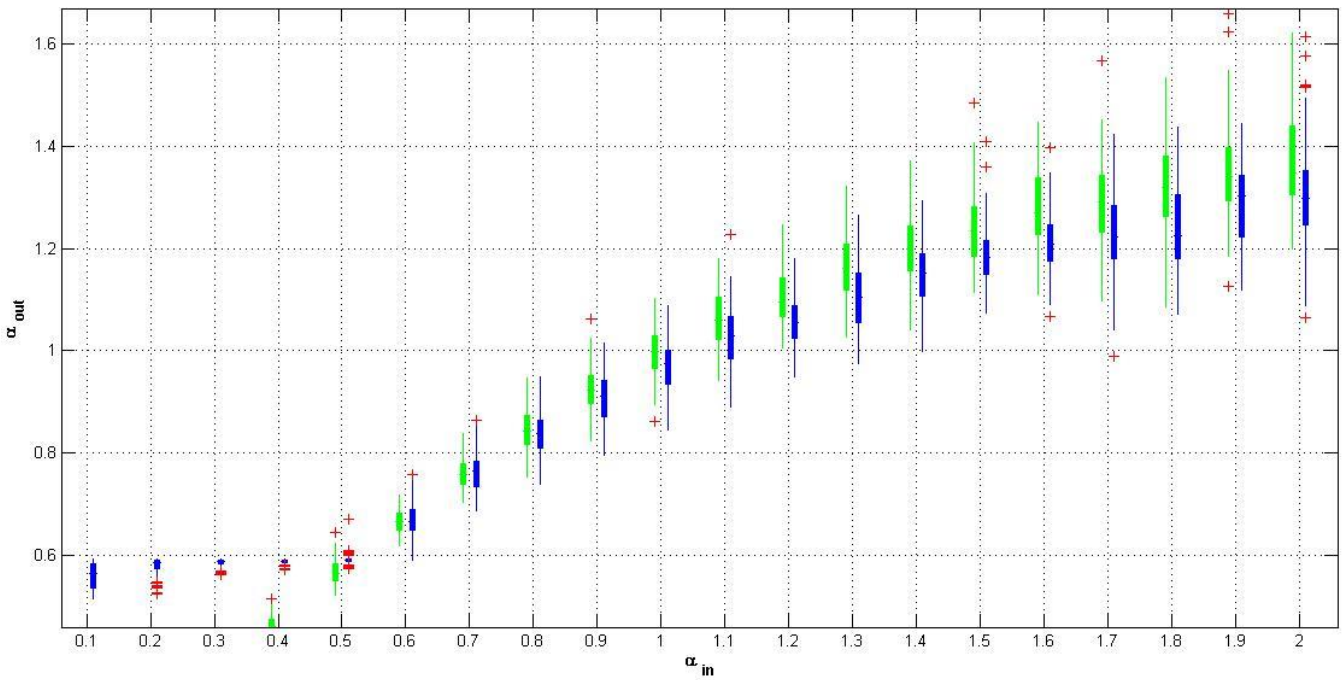

Figure 1.7. Boxplots of estimators of alpha using FLOM (green) and McCuloch (blue) methods with Pareto noise.

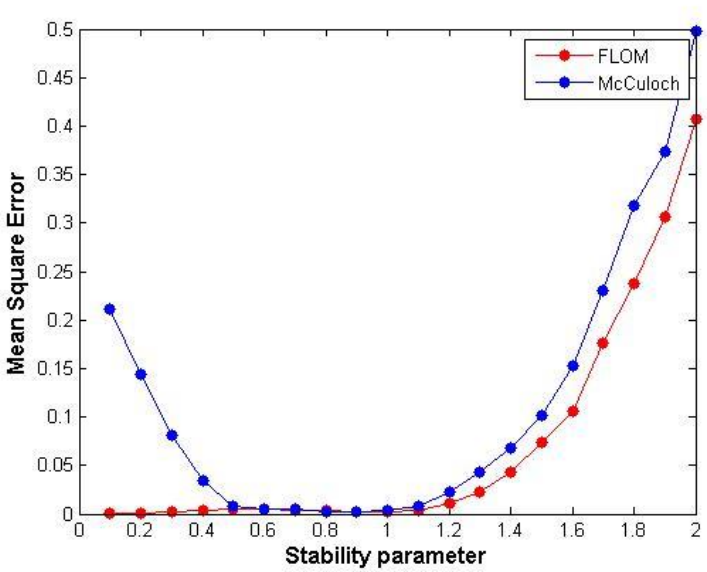

Figure 1.8. Mean square error of alpha estimator with Pareto noise for both methods.

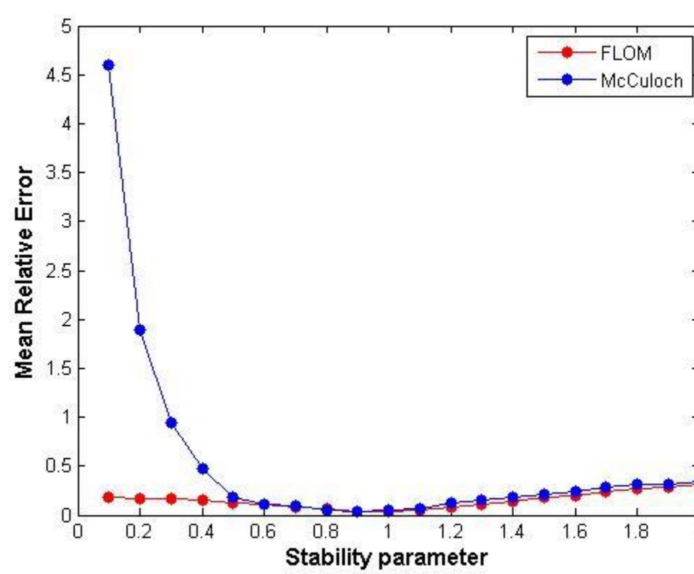

Figure 1.9. Mean relative error of alpha estimator with Pareto noise for both methods.

Bibliography

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