Module :3
Matrices
Properties:
Eigen Values & Eigen Vectors

❖ **Eigenvalues of the Transpose of a Matrix**

Statement: *Suppose* A *is a square matrix and* λ *is an eigenvalue of* A*. Then* λ *is an eigenvalue of the matrix* A^T*. i.e.* A *and* A^T *have eigenvalues.*

Proof: Suppose A has order n.

Now, $\det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$

$\therefore \det(A - \lambda I)^T = 0 \Leftrightarrow \det(A^T - \lambda I) = 0$

So A and A^T have the same characteristic polynomial and their eigenvalues are identical and have equal algebraic multiplicities.

❖ **Eigenvalues of a Scalar Multiple of a Matrix**

Statement: *Suppose* A *is a square matrix and λ is an eigenvalue of* A*. Then αλ is an eigenvalue of αA.*

Proof: Let x≠0 be one eigenvector of A for λ. Then

(αA)x=α(Ax)

    =α(λx)

    =(αλ)x

So x≠0 is an eigenvector of αA for the eigenvalue αλ.

❖ **Singular Matrices have Zero Eigenvalues.**

❖ **Eigenvalues Of Matrix Powers**

Statement: *Suppose* A *is a square matrix of order n and* λ *is an eigenvalue of* A*. Then for any integer* s≥0, $λ^s$ *is an eigenvalue of* A^s*.*

Proof: Let x≠0 be one eigenvector of A for λ.

We will prove the theorem by mathematical induction on s.

Step 1: For s=0,

$A^s x = A^0 x = I_n x = x$

$\qquad = 1x = λ^0 x = λ^s x$

∴The theorem is true for s=0.

Step 2: Assume the theorem is true for s, then we will prove its true for s+1.

$A^{s+1} x = A^s A x = A^s(λx) = λ(A^s x) = λ(λ^s x) = (λλ^s)x = λ^{s+1} x$

So, x≠0 is an eigenvector of A^{s+1} corresponding to eigenvalue $λ^{s+1}$

By Mathematical induction, Theorem is true for all s≥0.

❖ **Eigenvalues of the Polynomial of a Matrix**

Statement: *Suppose* A *is a square matrix and* λ *is an eigenvalue of* A*. Let* q(x) *be a polynomial in the variable x. Then* q(λ) *is an eigenvalue of the matrix* q(A)*.*

Proof : Let x≠0 be one eigenvector of A for λ, and write $q(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m$

Then $q(A)x=(a_0+a_1A+a_2A^2+\cdots+a_mA^m)x$

$=a_0(A^0x)+a_1(A^1x)+a_2(A^2x)+\cdots+a_m(A^mx)$

$=a_0(\lambda^0x)+a_1(\lambda^1x)+a_2(\lambda^2x)+\cdots+a_m(\lambda^mx)$

$=(a_0+a_1\lambda+a_2\lambda^2+\cdots+a_m\lambda^m)x$

$=q(\lambda)x$

So x≠0 is an eigenvector of q(A) for the eigenvalue q(λ).

❖ Eigenvalues of the Inverse of a Matrix

Statement: *Suppose* A *is a square nonsingular matrix and* λ *is an eigenvalue of* A. *Then* λ^{-1} *is an eigenvalue of the matrix* A^{-1}.

Proof: since A is assumed nonsingular, A^{-1} exists and $1/\lambda$ does not involve division by zero.

Let x≠0 be one eigenvector of A for λ. Suppose A has order n. Then

$$A^{-1} x = A^{-1} (1x) = A^{-1} \left(\frac{1}{\lambda} \lambda x\right)$$

$$= \frac{1}{\lambda} A^{-1} (\lambda x)$$

$$= \frac{1}{\lambda} A^{-1} (Ax) = \frac{1}{\lambda} (A^{-1} Ax)$$

$$= \frac{1}{\lambda} (I_n x) = \frac{1}{\lambda} x$$

So x≠0 is an eigenvector of A^{-1} for the eigenvalue $\frac{1}{\lambda}$.

❖ Eigenvalues of the Adjoint of a Matrix

Statement: *Suppose* A *is a square nonsingular matrix and* λ *is an eigenvalue of* A*. Then* $\frac{|A|}{\lambda}$ *is an eigenvalue of the matrix* Adj.(A).

Proof: We know that A adj.A=$|A|$I

Premultiplying by A^{-1}

A^{-1} A adj.A= A^{-1} $|A|$

∴ adj.A= A^{-1} $|A|$

Post multiplying by x, x is the eigenvector corresponding to λ .

(adj.A)x = $|A|$ A^{-1} x= $|A|$ $\frac{1}{\lambda}$ x =$\frac{|A|}{\lambda}$ x

∴ $\frac{|A|}{\lambda}$ *is an eigenvalue of the matrix* Adj.(A).

❖ The eigenvalues of a unitary matrix are of unit modulus.

Statement: *Suppose* A *is a Unitary matrix and* λ *is an eigenvalue of* A. *Then* $|\lambda| = 1$.

Proof: But A is Unitary. ∴ $AA^{\theta} = A^{\theta}A = I$

Let x≠0 be one eigenvector of A corresponding to eigenvalue λ.

∴ $Ax = \lambda x$ ……(1)

Taking complex conjugate transpose on both sides

$(Ax)^{\theta} = (\lambda x)^{\theta}$

$x^{\theta} A^{\theta} = \bar{\lambda} x^{\theta}$ ……. (2)

Multiplying (1) & (2)

$(x^{\theta} A^{\theta})^{(}A x) = (\bar{\lambda} x^{\theta})(\lambda x)$

∴ $x^{\theta}(A^{\theta}A) x = \bar{\lambda} \lambda (x^{\theta}x)$

∴ $x^{\theta} x = \bar{\lambda} \lambda (x^{\theta}x)$

Since X is an eigen vector, $x^{\theta} x \neq 0$

∴ $\lambda \bar{\lambda} = \mathbf{1}$ ∴ $|\lambda| = 1$

❖ Hermitian Matrices have Real Eigenvalues

Statement: *Suppose that* A *is a Hermitian matrix and* λ *is an eigenvalue of* A. *Then* λ∈R.

Proof : Let x≠0 be one eigenvector of A corresponding to eigenvalue λ.

we know Ax= λx

Premultiplying by x^θ we get

$x^\theta Ax = x^\theta \lambda x = \lambda x^\theta x$ …….1

Taking complex congugate transpose on both sides

$(x^\theta Ax)^\theta = (\lambda x^\theta x)^\theta$

$x^\theta A^\theta (x^\theta)^\theta = \bar{\lambda} x^\theta (x^\theta)^\theta$ …….2

But A is Hermitian. ∴ A= A^θ

By 2, $x^\theta A x = \bar{\lambda} x^\theta x$ …..3

From (1) & (3), we get

$\lambda x^\theta x = \bar{\lambda} x^\theta x$

Since X is an eigen vector, $x^\theta x \neq 0$

∴λ = $\bar{\lambda}$

Corollaries

❖ **Determinent of Harmitian mtrix is real**

❖ **Real Symmetric Matrices have Real Eigenvalues**

❖ **Eigenvalues of a Skew-Hermitian Matrix are either purely imaginary or zero.**

❖ **The eigen values of a real skew-symmetric matrix are purely imaginary or zero.**

❖ **The eigenvalues of an Orthogonal Matrix are of unit modulus.**

Statement: *Suppose A is a Orthogonal matrix and λ is an eigenvalue of A. Then $|\lambda| = 1$.*

Hint: If the element of a unitary matrix A are all real, then A becomes the orthogonal Matrix.

Ex. Find the eigenvalues of Adj. A and inv. A Where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

❖ Eigenvalues of A are λ= 1,2,4,6

By properties,

Eigenvalues of Adj.A = $\dfrac{|A|}{\lambda}$ & $|A|$=Product of eigenvalues

∴ Eigenvalues of Adj.A = $\dfrac{48}{1}, \dfrac{48}{2}, \dfrac{48}{4}, \dfrac{48}{6}$

∴ Eigenvalues of Adj.A =48, 24,12, 8

Eigenvalues of Inv.A = $\dfrac{1}{\lambda} = $ 1, $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{6}$ …, …, …, …

❖Ex. 1 Find the eigenvalues and eigen vectors of

$6A^{-1}+A^3+2I$. Where A = $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

Eigenvalues of A are λ= -1, -1 and corresponding eigenvector is X= [1,0]' (Can be found by students)

By properties,

Eigenvalues of $6A^{-1}+A^3+2I$ = $6\dfrac{1}{\lambda} + \lambda^3 + 2(1)$

∴ Eigenvalues of $6\ A^{-1}+A^3+2I$ are $(-6)+(-1)^3+2 = 5, 5$ with and corresponding eigenvector is X= [1,0]'.

Ex. Find eigen values of A and prove they are of unit modulus.

$$A = \begin{bmatrix} \dfrac{1+i}{2} & -\dfrac{1-i}{2} \\ \dfrac{1+i}{2} & \dfrac{1-i}{2} \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} \dfrac{1+i}{2} - \lambda & -\dfrac{1-i}{2} \\ \dfrac{1+i}{2} & \dfrac{1-i}{2} - \lambda \end{vmatrix} = 0$$

$$\therefore \left(\frac{1+i}{2} - \lambda\right)\left(\frac{1-i}{2} - \lambda\right) + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) = 0$$

$$\therefore \left\{\left(\frac{1}{2} - \lambda\right)^2 - \left(\frac{i}{2}\right)^2\right\} + \left\{\left(\frac{1}{2}\right)^2 - \left(\frac{i}{2}\right)^2\right\} = 0$$

$$\therefore \lambda^2 - \lambda + 1 = 0$$

Eigen values are: $\lambda = \dfrac{1+i\sqrt{3}}{2}, \dfrac{1-i\sqrt{3}}{2}$ & their modulus are 1.

❖

Ex. If A = $\begin{bmatrix} sin\theta & cosec\theta & 1 \\ sec\theta & cos\theta & 1 \\ tan\theta & cot\theta & 1 \end{bmatrix}$ **then prove that there does not exists a real value of** θ **for which characteristic roots of A are -1,1 & 4.**

Soln. Sum of A= sum of diagonal elements

$= sin\theta + cos\theta + 1$ ....(1)

By properties, trace of A = sum of eigen values =4.....(2)

By (1) & (2),

$sin\theta + cos\theta + 1 = 4$ ....(3)

For no real value of θ equation (3) holds true.

Hence the proof.