Sol^n: Observe that, u is not homogeneous

Write: $u = v + w$, where $v = \frac{x^2+xy}{y\sqrt{x}}$ & $w = \frac{1}{x^7}\sin^{-1}\left(\frac{x^2+xy}{y^2-x^2}\right)$

Then Consider v,

$$f(xt, yt) = \frac{x^2t^2 + xtyt}{yt\sqrt{xt}} = \frac{t^2}{t^{3/2}}\left(\frac{x^2+xy}{y\sqrt{x}}\right) = t^{1/2} f(x,y)$$

$\therefore v$ is homogeneous of deg $(n_1) = \frac{1}{2}$

consider w,

$$f(xt, yt) = \frac{1}{x^7t^7}\sin^{-1}\left(\frac{x^2\cancel{t^2} + xy\cancel{t^2}}{y^2\cancel{t^2} - x^2\cancel{t^2}}\right) = t^{-7}\left[f(x,y)\right]$$

w is homogeneous of deg $(n_2) = -7$

Now, By Eulers Theorem,

for v, $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n_1(n_1-1)v = \frac{1}{2}\left(-\frac{1}{2}\right)v = -\frac{1}{4}v \quad (1)$$

& $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n_1 v = \frac{1}{2}v \quad (2)$$

for w, $$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = n_2(n_2-1)w = (-7)(-8)w = 56w \quad (3)$$

& $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = n_2 W = -7w \quad (4)$$

Combining (1), (2), (3) & (4)

$$\therefore x^2\frac{\partial^2(v+w)}{\partial x^2} + 2xy\frac{\partial^2(v+w)}{\partial x\partial y} + y^2\frac{\partial^2(v+w)}{\partial y^2} + x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w)$$

$$= -\frac{1}{4}v + \frac{1}{2}v + 56w - 7w$$

$$= \frac{1}{4}v + 49w$$

$= \frac{1}{4}v.$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{1}{4}\,\frac{x^2+xy}{y\sqrt{x}} + 49\,\frac{1}{x^7}\,\sin^{-1}\left(\frac{x^2+xy}{y^2-x^2}\right)$$

put $x=1,\ y=2$

$$\text{expression} = \frac{1}{4}\left(\frac{1+2}{2}\right) + 49\,\sin^{-1}\left(\frac{1+2}{4-1}\right)$$

$$\boxed{= \frac{3}{8} + 49\left(\frac{\pi}{2}\right)}$$

If $\underline{u = \sin^{-1}}(\phi(x,y))$ / $\log(\phi(x,y))$ etc.

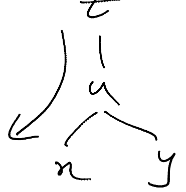
where $\boxed{\underline{u \text{ is not homogeneous}}}$ but $\phi(x,y)$ is homogeneous

$z = \boxed{\sin u} = \phi(x,y)$

is homogeneous.

1) $\underline{z = f(u)}$ $= \sin u,\ e^u$ etc. are homogeneous of deg n

Then $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}}$



explatⁿ: Since z is homogeneous,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Since $\underline{z = f(u)}$ $\therefore$ $\underline{\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}}$ & $\underline{\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}}$

$$x\underline{f'(u)}\frac{\partial u}{\partial x} + y\,\underline{f'(u)}\frac{\partial u}{\partial y} = nf(u)$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}}$$

2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]$

2) $$\boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]}$$

where $g(u)=n\dfrac{f(u)}{f'(u)}$

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1) If $u=\dfrac{x^2y^2z^2}{x^2+y^2+z^2}+\cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ Then

Find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$

Sol^n: Observe that u is not homogeneous.

write $u=v+w$, where $v=\dfrac{x^2y^2z^2}{x^2+y^2+z^2}$, $w=\cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$

$\therefore$ H.W.

v is homogeneous of deg $n_1=4$

By Euler's Th^m

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=n_1v=4v \quad (1)$$

H.W.

w is not homogeneous.

Let $f(u)=\boxed{\cos w}=\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$ & $\cos w$ is homogeneous of deg $\frac{1}{2}$

By corollary of Euler's Th^m

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=n_2\frac{f(u)}{f'(u)}=\frac{1}{2}\frac{\cos w}{(-\sin w)}$$

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=-\frac{1}{2}\cot w \quad (2)$$

add (1) & (2)

$$x\frac{\partial}{\partial x}(v+w)+y\frac{\partial}{\partial y}(v+w)+z\frac{\partial}{\partial z}(v+w)=4v-\frac{1}{2}\cot w$$

$$=4\left(\frac{x^2y^2z^2}{\ \ }\right)-\frac{1}{2}\cot\left[\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right]$$

$$= 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) - \frac{1}{2}\cot\left[\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right]$$

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2) $u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ Then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u)$$

Sol^n: u is not homogeneous.

$$f(u) = \operatorname{cosec} u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}} \quad g(x,y)$$

$$g(xt, yt) = \left(\frac{x^{1/2}t^{1/2}+y^{1/2}t^{1/2}}{x^{1/3}t^{1/3}+y^{1/3}t^{1/3}}\right)^{1/2} = t^{1/12}\left[\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$$

$f(u)$ is homogeneous function of deg. $n = \frac{1}{12}$

Then By corollary of Euler's Th^m, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]$$

where, $g(u) = n\frac{f(u)}{f'(u)} = \frac{1}{12}\frac{\cancel{\operatorname{cosec}} u}{-\cancel{\operatorname{cosec}} u \cot u} = -\frac{1}{12}\tan u$

$$\left[g'(u)-1\right] = \left[-\frac{1}{12}\sec^2 u - 1\right]$$

$$g(u)\left[g'(u)-1\right] = -\frac{1}{12}\tan u\left[-\frac{1}{12}\sec^2 u - 1\right]$$

$$= \frac{1}{144}\tan u\left[\sec^2 u + 12\right] = \frac{\tan u}{144}\left[1+\tan^2 u+12\right]$$

$$g(u)\left[g'(u)-1\right] = \frac{\tan u}{144}\left[13+\tan^2 u\right] \quad \text{— (1)}$$

$\therefore$ put back, $\therefore$ $\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left[13+\tan^2 u\right]}$

4) $x = e^u \tan v$, $y = e^u \sec v$ Then

Find $\left(x\boxed{\frac{\partial u}{\partial x}} + y\frac{\partial u}{\partial y}\right)\left(x\boxed{\frac{\partial v}{\partial x}} + y\frac{\partial v}{\partial y}\right)$

Sol^n: First we will express u & v as functions of x & y

Consider, $y^2 - x^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v = e^{2u}\left[\sec^2 v - \tan^2 v\right]$

$$y^2 - x^2 = e^{2u}$$

$$\therefore \boxed{u = \frac{1}{2}\log(y^2 - x^2)}$$

check, $\frac{x}{y} = \frac{\cancel{e^u}\tan v}{\cancel{e^u}\sec v} = \frac{\frac{\sin v}{\cancel{\cos v}}}{\frac{1}{\cancel{\cos v}}} = \sin v$

$$\therefore \boxed{v = \sin^{-1}\left(\frac{x}{y}\right)}$$

Since, v is homogeneous of deg zero

$\therefore$ By Euler's Th^m, $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0$

$$\therefore \boxed{\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0}$$

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5) $z = e^{x/y} + \log\left(\boxed{x^3 + y^3 - x^2y + xy^2}\right)$ Then Find

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

Sol^n: z is not homogeneous,

$z = v + w$, $v = e^{x/y}$, $w = \log(x^3 + y^3 - x^2y + xy^2)$

v is homogeneous of deg zero,

Euler. $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu = 0$ — (1)

Euler :

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu = 0$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)u = 0 \quad \text{— } (2)$$

Now consider w is not homogeneous,

$f(w) = e^w = \underline{(x^3+y^3 - x^2y + xy^2)}$ is homogeneous with $\underline{\text{deg } 3}$

By corollaries of Euler's th^m,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \overbrace{\boxed{n\frac{f(w)}{f'(w)}}}^{g(w)} = 3\frac{\cancel{e^w}}{\cancel{e^w}} = \underline{\underline{3}}$$

$$\& \quad x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = g(w)\left[g'(w) - 1\right]$$

$$= 3\left[0 - 1\right]$$

$$= -3$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 3 - 3 = 0 \quad \text{— } (3)$$

adding (1), (2) & (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

H.W. :

$$u = \log\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

Find $\underline{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}$