K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 2 – Kinematics of Particles & Rigid Bodies

Module Section 2.1 – Kinematics of Particles

Class: FY BTech Division: C3
Professor: Aniket S. Patil Date: 06/04/2023

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Kinematics: It is concerned only with the study of motion of the body without consideration of the forces causing the motion.

Particle: In particle dynamics, we idealise the body being analysed as a particle. It does not mean that we are dealing with a very small object. But, it means that the size and shape of the body is not important for the analysis of the motion. For example, if a ship is traveling between two ports kilometres apart, then for a simple motion analysis the shape and size of the ship is not relevant in the calculations.

Whenever a body is treated as a particle, all the forces acting in the body are to be assumed to be concurrent at the mass centre of the body. Any rotation of the body is also neglected.

Reference Frame: For any motion analysis, we need to take a reference frame i.e. an origin with a set of co-ordinate axes, for the measurement of motion parameters. The reference frame could be fixed or moving. Newtonian frame of reference also known as inertial reference frame is a set of co-ordinates axes fixed or moving with uniform velocity. Newton's laws of motion are valid for these.

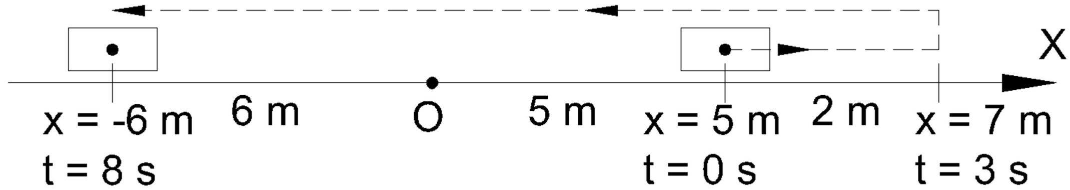
Rectilinear Motion: Motion of a particle in a straight line is known as a rectilinear motion. E.g., car moving on a straight highway, lift traveling in a vertical well, etc.

<u>Position (x)</u>, <u>Displacement (s)</u>, & <u>Distance (d)</u>:

Position means the location of a particle with respect to a fixed reference point, usually called origin O. The position is taken as positive on one side of the origin and negative on the other. It is labelled by 'x' in S.I. unit of metre (m).

Displacement is a change in position of the particle. It is a vector quantity. It is a straight line vector connecting the initial position to the final position and has no relation with the actual distance travelled. It is labelled by 's' in S.I. unit of metre (m).

Distance is the actual length of the total path traced by the particle during the period of motion. It is a scalar quantity. It is labelled by ‘d’ in S.I. unit of metre (m).



E.g., let a particle at t = 0 s, occupy a position x = 5 m. Then it moves 2 m in the +ve X-direction and occupies position x = 7 m at t = 3 s. Then it reverses itself in the –ve direction, and occupies position x = 6 m at t = 8 s.

So here, displacement, $s = \Delta x = x_8 - x_0 = (-6) - (5) = -11 \text{ m or } 11 \text{ m} \leftarrow$

And the distance, $d = |x_3 - x_0| + |x_8 - x_3| = |7 - 5| + |-6 - 7| = 15 \text{ m}$

<u>Velocity</u> & <u>Speed</u>:

Velocity is the rate of change of displacement with respect to time. It is a vector quantity. It is labelled by ‘v’ in S.I. unit of metre/second (m/s).

$$\text{Average velocity}, v_{av} = \frac{\Delta x}{\Delta t}$$

$$\text{Instantaneous velocity}, v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Speed is the rate of change of distance with respect to time. It is a scalar quantity. *The magnitude of velocity is also known as speed.*

$$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time interval}}$$

<u>Acceleration</u>:

Acceleration is the rate of change of velocity with respect to time. It is labelled by ‘a’ in S.I. unit of metre/second^2 (m/s^2).

$$\text{Average acceleration}, a_{av} = \frac{\Delta v}{\Delta t}$$

$$\text{Instantaneous acceleration}, a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Positive acceleration is simply called acceleration and negative acceleration is called retardation or deceleration. Positive acceleration means magnitude of velocity increase with time and particle is moving in the positive direction. Negative acceleration means the particle moves slowly in the positive direction or moves faster in the negative direction.

Types of Rectilinear Motions:

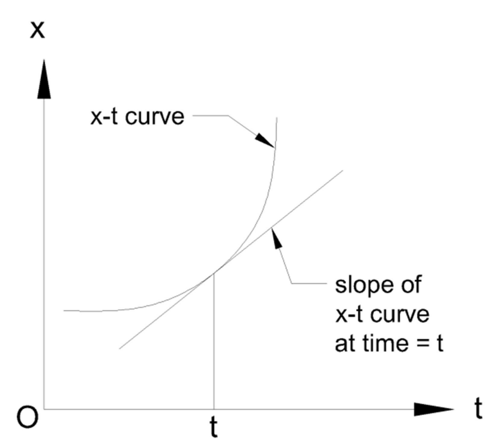
1. Uniform Velocity Motion
   If a particle's velocity remains constant throughout the motion, then it is said to be under motion with uniform velocity. E.g., motion of sound, package on a conveyor, etc.
$$v = \frac{s}{t}$$

2. Uniform Acceleration Motion
   If a particle's velocity changes at a constant rate throughout the motion, then it is said to be under motion with uniform acceleration. This means that its acceleration is constant. This gives us the equations of kinematics.
$$v = u + at$$
$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$
   Motion under gravity is a special case of this, with $a = g = 9.81\ m/s \downarrow$.

3. Variable Acceleration Motion
   If a particle's acceleration itself is changing throughout the motion, then it is said to be under the motion with variable acceleration. This motion is usually defined by acceleration written as a function of time or velocity or position. For the solution, we use calculus on the instantaneous relations between position, velocity, acceleration and time.
$$v = \frac{dx}{dt}\ \&\ a = \frac{dv}{dt}$$
$$\therefore \frac{a}{v} = \frac{dv/dt}{dx/dt} = \frac{dv}{dx} \rightarrow a = v\frac{dv}{dx}$$

Motion Curves: The motion of a particle along a straight line can be represented by motion curves. They are the graphical representation of position, displacement, velocity and acceleration with time.

1. Position-Time (x-t) curve
   Since,
$$v = \frac{dx}{dt}$$
   at any instant of time,
   the slope of x-t curve gives
   the velocity of the particle
   at that instant.
$$\therefore \boxed{v = [\text{slope } x - t \text{ curve}]_{\text{at time} = t}}$$



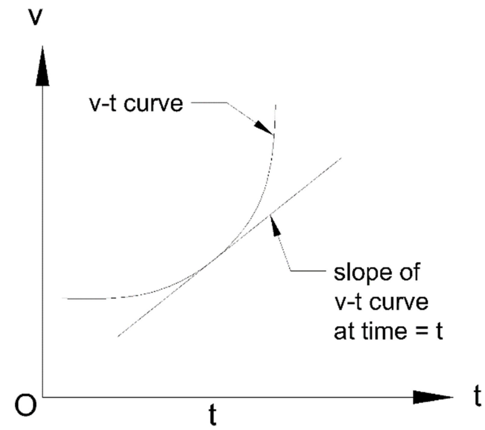
2. Velocity-Time (v-t) curve

Since,

$$a = \frac{dv}{dt}$$

at any instant of time,
the slope of v-t curve gives
the velocity of the particle
at that instant.

$\therefore \boxed{a = [\text{slope } v - t \text{ curve}]_{\text{at time} = t}}$



Now,

$$v = \frac{dx}{dt} \rightarrow dx = vdt$$

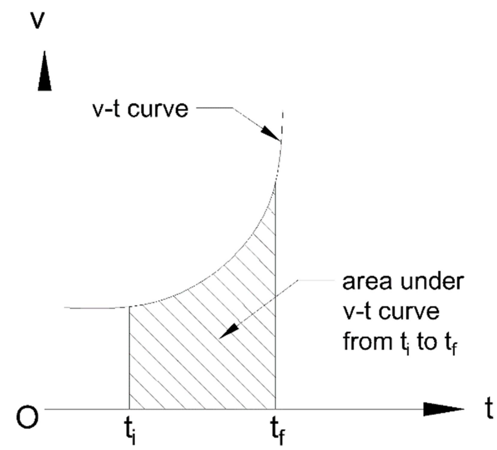
Integrating, $\int dx = \int vdt$

$\int vdt$ represents the area under
the v-t curve from t_i to t_f

$$\int_{x_i}^{x_f} dx = [AUC\ v - t]_{\text{from } t_i \text{ to } t_f}$$

$$x_f - x_i = [AUC\ v - t]_{t_i - t_f}$$

$\therefore \boxed{x_f = x_i + [AUC\ v - t]_{t_i - t_f}}$



3. Acceleration-Time (a-t) curve

Since,

$$a = \frac{dv}{dt} \rightarrow dv = adt$$

$$\rightarrow \int dv = \int adt$$

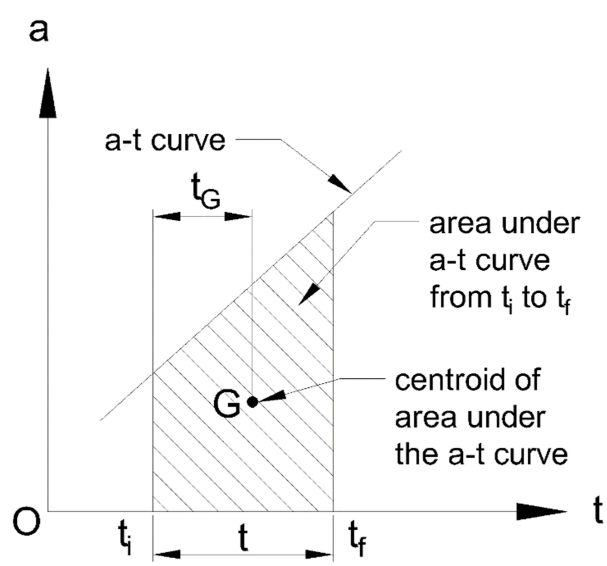
$\int vdt$ represents the area under
the a-t curve from t_i to t_f

$$\int_{v_i}^{v_f} dx = [AUC\ a - t]_{\text{from } t_i \text{ to } t_f}$$

$$v_f - v_i = [AUC\ a - t]_{t_i - t_f}$$

$\therefore \boxed{v_f = v_i + [AUC\ a - t]_{t_i - t_f}}$

From a-t curve particle's position
can also be known at an instant,
using area moment method.

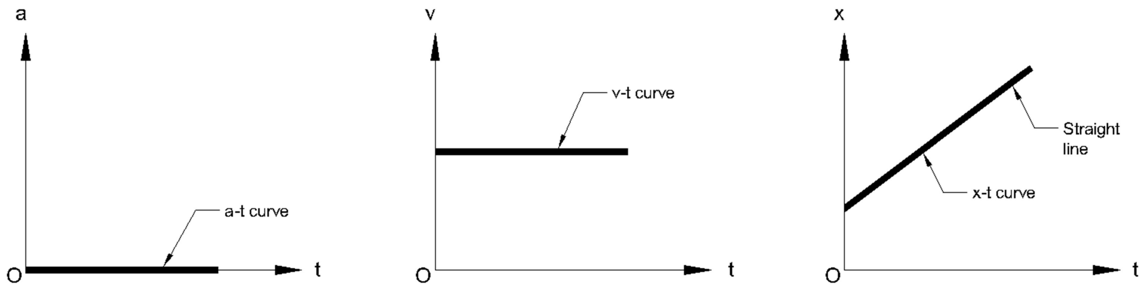


Here, $t = t_f - t_i$ and t_G is time from t_i to the centroid of AUC a-t.

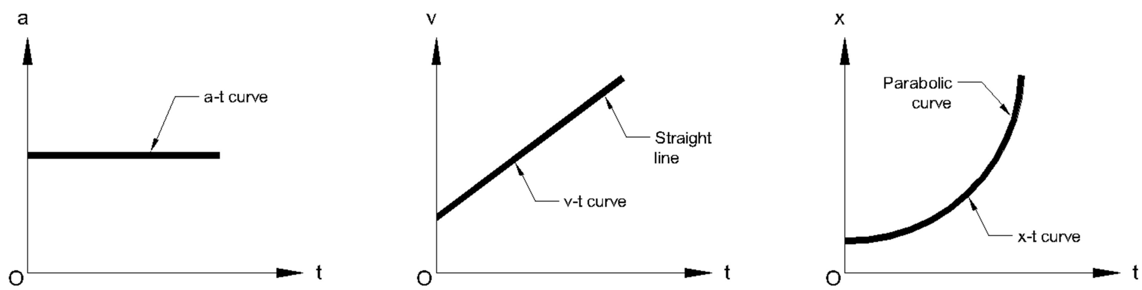
$$\boxed{x_f = x_i + v_i \times t + [AUC\ a - t]_{t_i - t_f} \times (t - t_G)}$$

Standard Motion Curves:

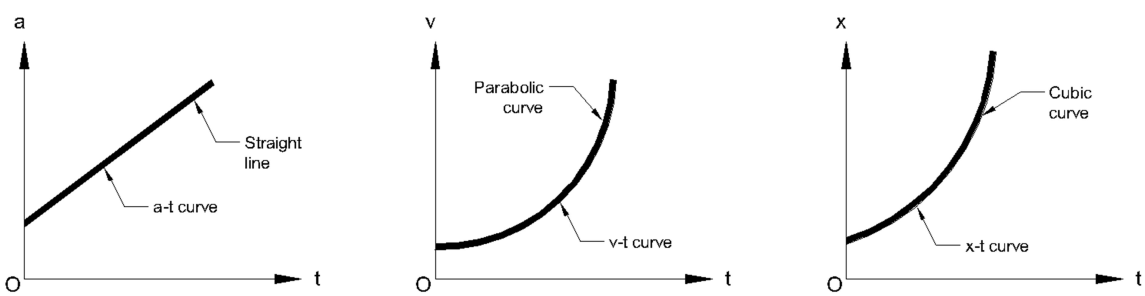
1. Uniform Velocity Motion Curves



2. Uniform Acceleration Motion Curves



3. Variable Acceleration (Linear Variation) Motion Curves



Curvilinear Motion: A particle which travels on a curved path is said to be performing curvilinear motion.

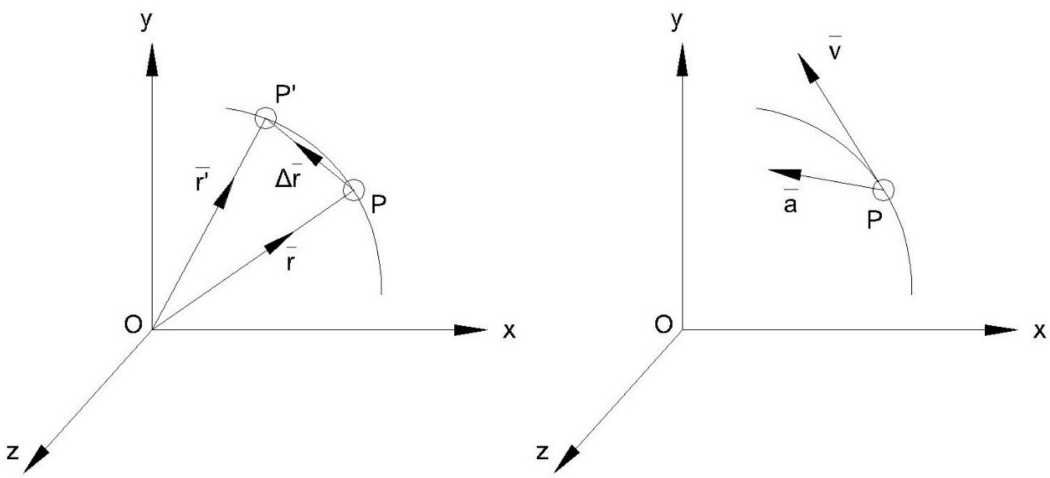
Position: It is represented by a position vector $\bar{r}$ with a starting point from the origin of the reference axis till the particle P. As the particle travels along the curved path, the value of $\bar{r}$ keeps changing.

Velocity: Suppose a particle changes position from P to P', i.e., the position vectors from $\bar{r}$ to $\bar{r}'$, in a time interval of Δt.

$$\text{Average velocity}, v_{av} = \frac{\Delta\bar{r}}{\Delta t}$$

$$\text{Instantaneous velocity}, \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta\bar{r}}{\Delta t} = \frac{d\bar{r}}{dt}$$

In curvilinear motion, instantaneous velocity of a particle is always tangent to the curved path at that instant.



Acceleration: As the direction of velocity continuously changes in a curvilinear motion, there exists acceleration at every instant of the motion.

$$\text{Average acceleration}, a_{av} = \frac{\Delta\bar{v}}{\Delta t}$$

$$\text{Instantaneous acceleration}, \bar{a} = \lim_{\Delta t \to 0} \frac{\Delta\bar{v}}{\Delta t} = \frac{d\bar{v}}{dt}$$

NOTE: In rectilinear motion, x, v, a are always along the path of the particle, whereas in curvilinear motion, $\bar{r}$, $\bar{v}$, $\bar{a}$ are always changing directions. Hence, we need to consider different component systems for its analysis.

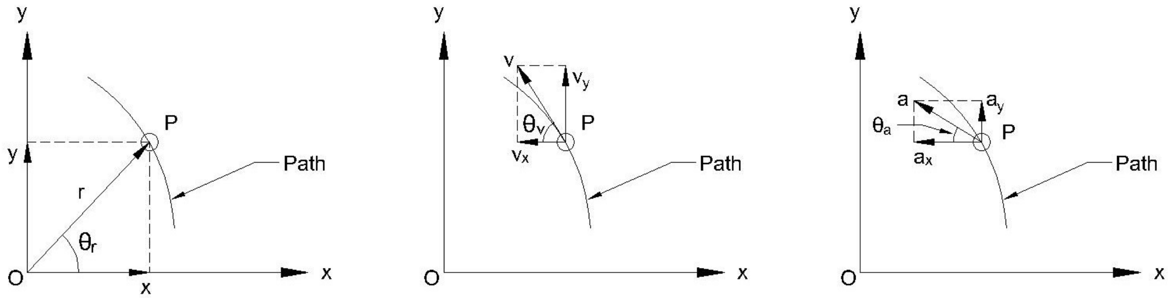
Curvilinear Motion by Rectangular Component System: Curvilinear motion can be split into motion along x, y, z directions, which can be independently considered as three rectilinear motions along those directions respectively.

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \& \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\bar{v} = \frac{d\bar{r}}{dt} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \quad \& \quad v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \quad \& \quad a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

For a particle in xy plane, the rectangular components will be as shown below:



$$\tan\theta_r = \frac{y}{x}; \qquad \tan\theta_v = \frac{v_y}{v_x}; \qquad \tan\theta_a = \frac{a_y}{a_x}$$

Curvilinear Motion by Tangential & Normal Component System (N-T System): Curvilinear motion can also be studied by splitting the acceleration along the tangent to the path and normal to the path.

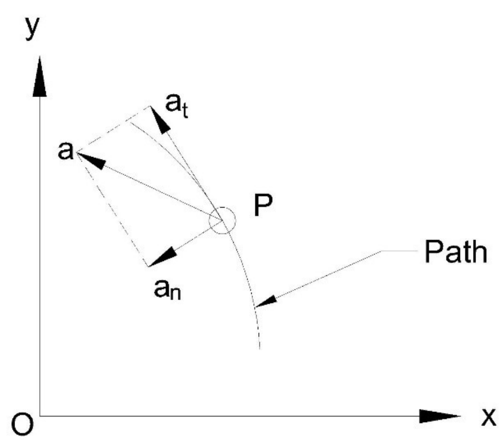
The velocity vector is always directed towards the tangential direction. But the net acceleration may be in any direction. So, it is convenient to express acceleration as *tangential acceleration (a_t)* and *normal acceleration (a_n)*.

$$\bar{a} = a_n\hat{e}_n + a_t\hat{e}_t$$

$$a = \sqrt{a_n^2 + a_t^2}$$

a_n represents the change in direction and is always directed towards the centre of curvature.

$$a_n = \frac{v^2}{\rho}$$



Where ρ is the radius of curvature and v is the velocity at a particular instant.

a_t represents the change in velocity and is given by, $$a_t = \frac{dv}{dt}$$

For curves which are defined as y = f(x),

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

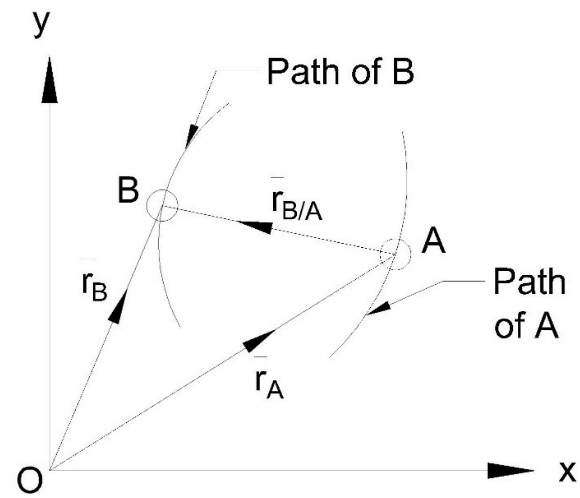
And, in terms of rectangular components,

$$\rho = \left|\frac{v^3}{a_x v_y - a_y v_x}\right|$$

Relative Motion: If motion analysis is done, not from a fixed reference frame, but from a moving reference, then such analysis comes under relative motion. E.g., person in a moving vehicle observing another moving vehicle, pilot of a fighter jet observing a moving target, etc.

In the figure, two particles move independent of each other, their position vectors measured from a fixed frame of reference xoy.

If now A observes B, then A will find B to be occupying the position $\bar{r}_{B/A}$. This is measured from the moving reference located at A.



Relative relations of B w.r.t A are:

$$\bar{r}_{B/A} = \bar{r}_B - \bar{r}_A$$

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

$$\bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$$

Numericals:

Part I – Rectilinear Motion:

N1: The velocity of a particle travelling in a straight line is given by the equation $v = (6t - 3t^2)$ m/s, where t is in seconds. If $s = 0$ when $t = 0$, determine the particle's deceleration and position when $t = 3$ s. How far has the particle travelled during the 3 second time interval and what is its average speed?

Soln: Given $v = 6t - 3t^2$

$$\therefore a = \frac{dv}{dt} = \frac{d(6t - 3t^2)}{dt} = (6 - 6t)\ m/s^2$$

$$\therefore \text{at } t = 3\text{ s}, \qquad a = 6 - 6(3) = -12\ m/s^2$$

$$\text{Now}, v = \frac{dx}{dt} \rightarrow dx = vdt$$

$$\therefore dx = (6t - 3t^2)dt$$

Integrating both sides going from $x = 0, t = 0$ (given) to unknown values x & t,

$$\int_0^x dx = \int_0^t (6t - 3t^2)dt$$

$$\therefore x = 3t^2 - t^3$$

$$\therefore \text{at } t = 3\text{ s}, \qquad x = 3(3)^2 - 3^3 = 0$$

$$\text{or } x_3 = 0$$

Now, to calculate distance, we must check whether the particle reverses its direction during the time interval of 3 seconds. For the particle to reverse, its velocity must become zero at the reversal point.

$$\therefore v = 0 \rightarrow 6t - 3t^2 = 0$$
$$3t(2 - t) = 0$$
$$\therefore t = 0 \text{ or } t = 2 \text{ s}$$

At $t = 0$ the particle has started from rest, and at $t = 2$ s the particle must have reversed its direction.

Hence, the positions at various key points is,

$$\begin{aligned} &\text{at } t = 0 \text{ s}, && x_0 = 0 \text{ (given)} \\ &\text{at } t = 2 \text{ s}, && x_2 = 3(2)^2 - 2^3 = 4 \text{ m} \\ &\text{at } t = 3 \text{ s}, && x_3 = 0 \text{ (found)} \end{aligned}$$

Hence, the total distance travelled in 3 seconds $= |x_2 - x_0| + |x_3 - x_2|$

$$d = 4 + 4 = 8 \text{ m}$$

So, the average speed,

$$v_{av} = \frac{d}{t} = \frac{8}{3} = 2.67 \text{ m/s}$$

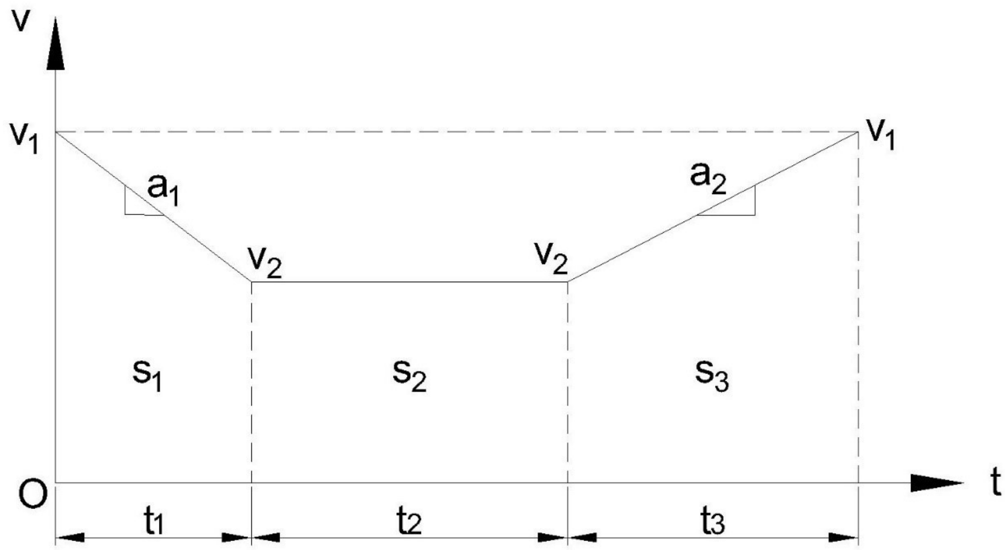
Hence, at $t = 3$ s, the particle's deceleration is 12 m/s^2 and its position is at origin. And, the particle has travelled 8 m during the 3 second time interval with an average speed of 2.67 m/s.

N2: A train travelling with a speed of 90 kmph slows down on account of work in progress, at a retardation of 1.8 kmph/s to 36 kmph. With this, it travels 600 m. Thereafter, it gains further speed with 0.9 kmph/s till getting the original speed. Find the delay caused.

Soln: Given data:

$$v_1 = 90 \text{ kmph} = 90 \times \frac{1000}{3600} \text{ m/s} = 25 \text{ m/s}$$
$$a_1 = -1.8 \text{ kmph/s} = -1.8 \times \frac{1000}{3600} \text{ m/s}^2 = -0.5 \text{ m/s}^2$$
$$v_2 = 36 \text{ kmph} = 36 \times \frac{1000}{3600} \text{ m/s} = 10 \text{ m/s}$$
$$a_2 = 0.9 \text{ kmph/s} = 0.9 \times \frac{1000}{3600} \text{ m/s}^2 = 0.25 \text{ m/s}^2$$
$$s_2 = 600 \text{ m}$$

We can solve this problem using the equations of kinematics or we can draw the v-t motion curve. Solving by v-t curve:



Section 1:

Acceleration is given by the slope of the v-t curve.

$$\therefore a_1 = [\text{Slope } v - t]_{t_1} = \frac{v_2 - v_1}{t_1}$$
$$\Rightarrow -0.5 = \frac{10 - 25}{t_1}$$
$$\therefore t_1 = \frac{10 - 25}{-0.5} = 30 \text{ s}$$

Distance is given by the area under the curve of the v-t curve.

$$\therefore s_1 = [\text{AUC } v - t]_{t_1} = \frac{1}{2} \times (v_1 + v_2) \times t_1$$
$$\Rightarrow s_1 = \frac{1}{2} \times (25 + 10) \times 30 = 525 \text{ m}$$

Section 2:

$$\because s_2 = [\text{AUC } v - t]_{t_2} = v_2 \times t_2$$
$$\Rightarrow 600 = 10 \times t_2$$
$$\therefore t_2 = \frac{600}{10} = 60 \text{ s}$$

Section 3:

$$\because a_2 = [\text{Slope } v - t]_{t_3} = \frac{v_1 - v_2}{t_3}$$
$$\Rightarrow 0.25 = \frac{25 - 10}{t_3}$$

$$\therefore t_3 = \frac{25-10}{0.25} = 60\ \text{s}$$

$$\because s_3 = [\text{AUC v}-\text{t}]_{t_3} = \frac{1}{2} \times (v_2 + v_1) \times t_3$$

$$\Rightarrow s_3 = \frac{1}{2} \times (10+25) \times 60 = 1050\ \text{m}$$

Hence, total distance, $d = s_1 + s_2 + s_3 = 525 + 600 + 1050 = 2175\ \text{m}$

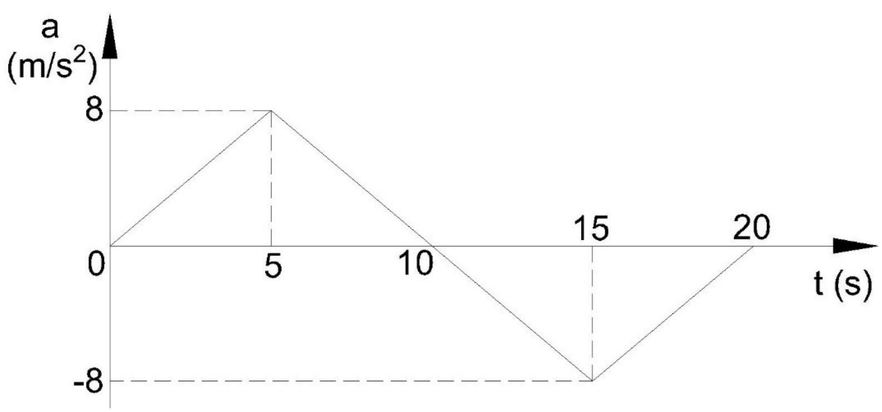
And, total time, $t = t_1 + t_2 + t_3 = 30 + 60 + 60 = 150\ \text{s}$

If there was no work being done, the train would have travelled the total distance at the constant speed of 25 m/s. The time required for such a scenario,

$$t' = \frac{d}{v_1} = \frac{2175}{25} = 87\ \text{s}$$

Hence, the delay caused $= t - t' = 150 - 87 = 63\ \text{s}$.

N3: The acceleration-time diagram for linear motion is shown below. Construct velocity-time and displacement-time diagrams for the motion assuming that the motion starts from the rest.



Soln: Velocity-Time Diagram:

$$v_f = v_i + [\text{AUC a}-\text{t}]_{t_i - t_f}$$

$$v_5 = v_0 + [\text{AUC a}-\text{t}]_{0-5} = 0 + \frac{1}{2} \times 5 \times 8 = 20\ \text{m/s}$$

$$v_{10} = v_5 + [\text{AUC a}-\text{t}]_{5-10} = 20 + \frac{1}{2} \times 5 \times 8 = 40\ \text{m/s}$$

$$v_{15} = v_{10} + [\text{AUC a}-\text{t}]_{10-15} = 40 - \frac{1}{2} \times 5 \times 8 = 20\ \text{m/s}$$

$$v_{20} = v_{15} + [\text{AUC a}-\text{t}]_{20-15} = 20 - \frac{1}{2} \times 5 \times 8 = 0\ \text{m/s}$$

Now, since a-t curve is a straight line with some slope, v-t curve will be parabolic.

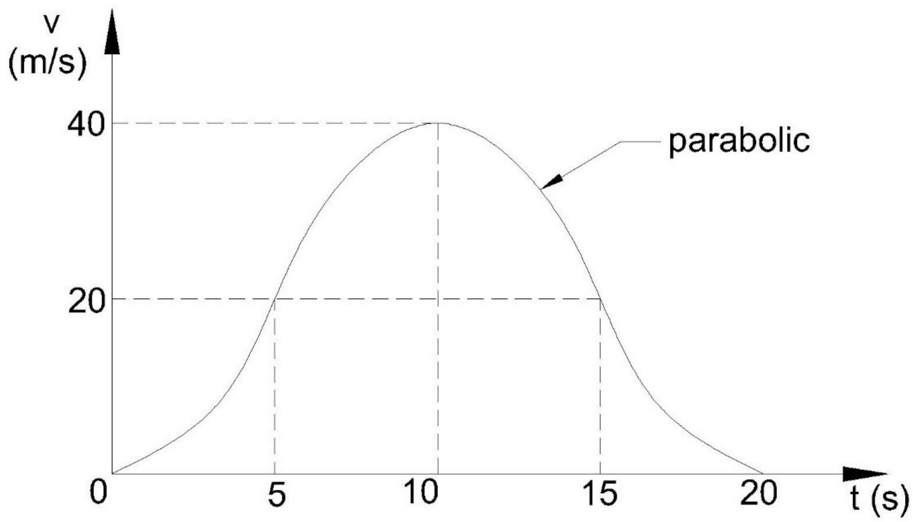
NOTE: Following statements are not required in exam, these are for reference.

{From 0-5 s, acceleration is positive and increasing, that means the velocity is increasing quickly. So, the parabolic curve will be concave up.

From 5-10 s, acceleration is positive but decreasing, that means the velocity is increasing slowly. So, the parabolic curve will be concave down.

From 10-15 s, acceleration is negative and decreasing, that means the velocity is decreasing quickly. So, the parabolic curve will be concave down.

From 15-20 s, acceleration is negative but increasing, that means the velocity is decreasing slowly. So, the parabolic curve will be concave up.}



Displacement-Time Diagram (Or Position-Time Diagram):

Method I: Area under the v-t curve

$$x_f = x_i + [\mathrm{AUC}\ v - t]_{t_i - t_f}$$

$$x_5 = x_0 + [\mathrm{AUC}\ v - t]_{0-5} = 0 + \frac{1}{3} \times 5 \times 20 = 33.33\ \mathrm{m}$$

$$x_{10} = x_5 + [\mathrm{AUC}\ v - t]_{5-10} = 33.33 + 5 \times 20 + \frac{2}{3} \times 5 \times 20 = 200\ \mathrm{m}$$

$$x_{15} = x_{10} + [\mathrm{AUC}\ v - t]_{10-15} = 200 + 5 \times 20 + \frac{2}{3} \times 5 \times 20 = 366.67\ \mathrm{m}$$

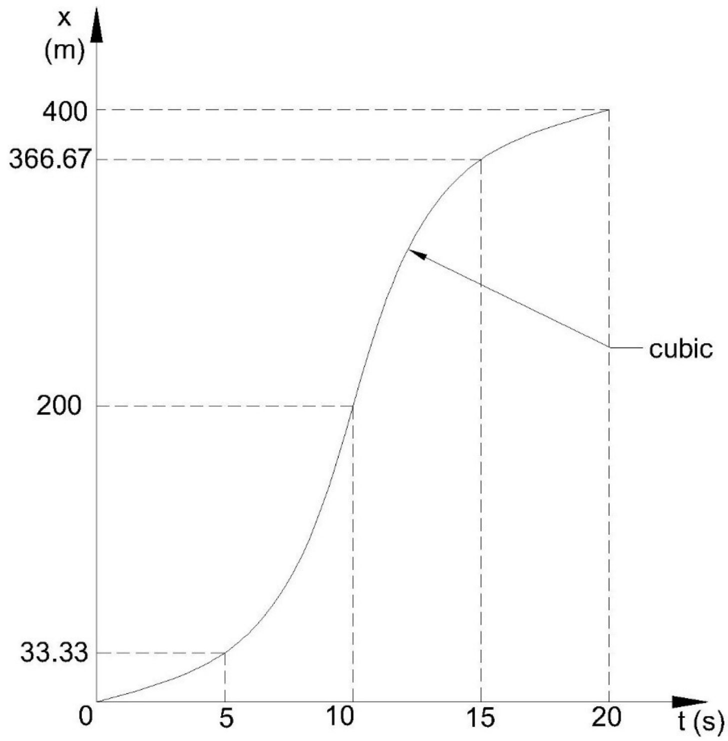
$$x_{20} = x_{15} + [\mathrm{AUC}\ v - t]_{20-15} = 366.67 + \frac{1}{3} \times 5 \times 20 = 0\ \mathrm{m}$$

[NOTE: Area under a concave up parabolic curve is given by $\frac{1}{3} \times$ base $\times$ height.

And for concave down parabolic curve is given by $\frac{2}{3} \times$ base $\times$ height.

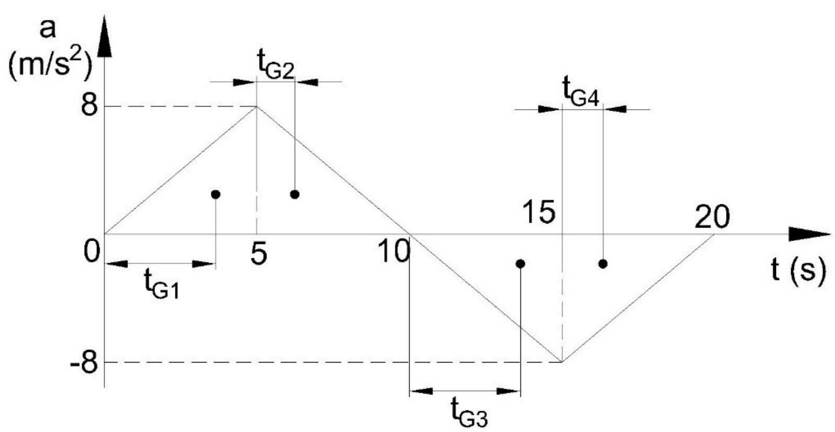
Don’t forget to add the rectangular section below the parabolic area.]

Now, since v-t curve is a parabolic curve, x-t curve will be cubic in nature.



Method II: Moment-area under a-t curve

$$x_f = x_i + v_i \times t + [\text{AUC a} - t]_{t_i - t_f} \times (t - t_G)$$



[NOTE: For a right-angled triangle, the centre of gravity is 1/3rd the length of base and 1/3rd the length of height, from the vertex having right angle.]

$$x_5 = x_0 + v_0 \times t + [\text{AUC a} - t]_{0-5} \times (t - t_{G1})$$

$$x_5 = 0 + 0 \times (5 - 0) + \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{2}{3} \times 5\right) = 33.33 \text{ m}$$

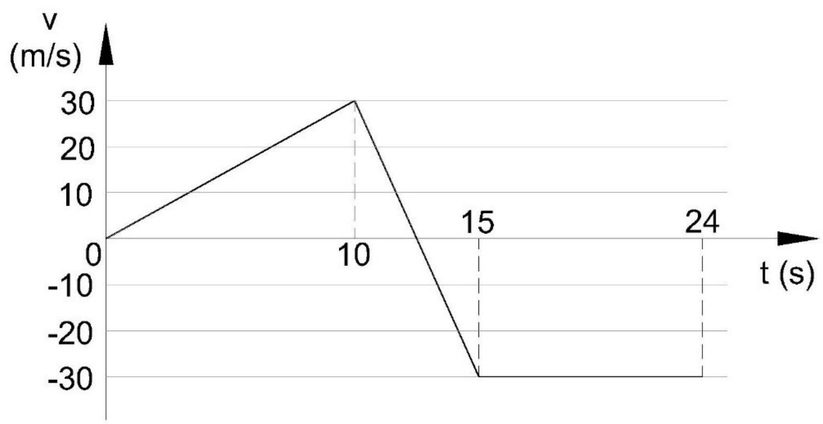
$$x_{10} = x_5 + v_5 \times t + [\text{AUC a} - t]_{5-10} \times (t - t_{G2})$$

$$x_{10} = 33.33 + 20 \times (10 - 5) + \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{1}{3} \times 5\right) = 200 \text{ m}$$

$$x_{15} = x_{10} + v_{10} \times t + [\text{AUC a} - t]_{10-15} \times (t - t_{G3})$$
$$x_{15} = 200 + 40 \times (15 - 10) - \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{2}{3} \times 5\right) = 366.67 \text{ m}$$

$$x_{20} = x_{15} + v_{15} \times t + [\text{AUC a} - t]_{15-20} \times (t - t_{G4})$$
$$x_{20} = 366.67 + 20 \times (20 - 15) - \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{1}{3} \times 5\right) = 400 \text{ m}$$

N4: A particle moves in a straight line with a velocity-time diagram as shown in figure. If $s = -25$ m at $t = 0$ s, draw displacement-time and acceleration-time diagrams for 0 to 24 seconds.



Soln: Position calculations: Given - $x_0 = -25$ m

$$x_f = x_i + [\text{AUC v} - t]_{t_i - t_f}$$

$$x_{10} = x_0 + [\text{AUC v} - t]_{0-10} = -25 + \frac{1}{2} \times 10 \times 30 = 125 \text{ m}$$

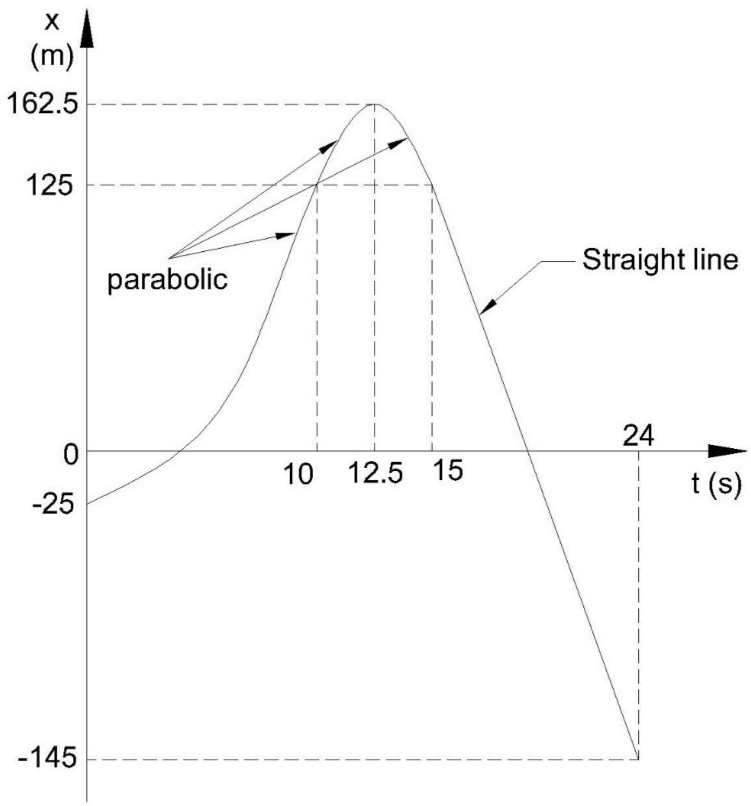
Somewhere between $t = 10$ s to $t = 15$ s, the particle has zero velocity and after that point it has negative velocity, which means at that point, particle reverses position. This point is at $t = 12.5$ s, considering the similarity of triangles.

$$x_{12.5} = x_{10} + [\text{AUC v} - t]_{10-12.5} = 125 + \frac{1}{2} \times 2.5 \times 30 = 162.5 \text{ m}$$

$$x_{15} = x_{12.5} + [\text{AUC v} - t]_{12.5-15} = 162.5 - \frac{1}{2} \times 2.5 \times 30 = 125 \text{ m}$$

$$x_{24} = x_{15} + [\text{AUC v} - t]_{15-24} = 125 - 9 \times 30 = -145 \text{ m}$$

{The x-t curves from 0 to 15 s will be parabolic since, v-t curves are straight lines with some slope. But from 15 to 24 s, x-t curve will be straight line since v-t curve is a horizontal line}



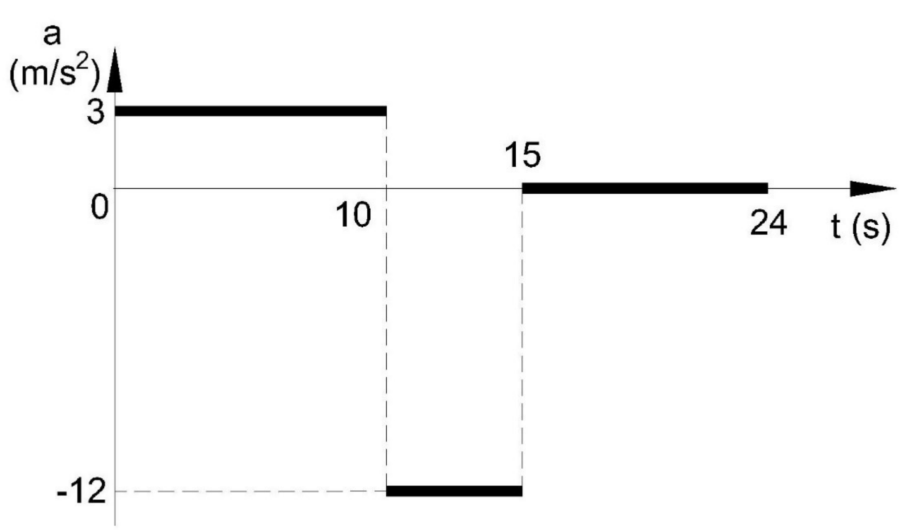
Acceleration calculations:

$$a = [\text{slope v} - \text{t curve}]_{\text{at time = t}}$$

$$a_{0-10} = \left[\frac{v_{10} - v_0}{\Delta t}\right]_{0-10} = \frac{30 - 0}{10 - 0} = 3\,\mathrm{m/s^2}$$

$$a_{10-15} = \left[\frac{v_{15} - v_{10}}{\Delta t}\right]_{10-15} = \frac{-30 - 30}{15 - 10} = -12\,\mathrm{m/s^2}$$

$$a_{15-2} = \left[\frac{v_{24} - v_{15}}{\Delta t}\right]_{15-24} = \frac{-30 - (-30)}{24 - 15} = 0$$



Part II – Curvilinear Motion:

N5: The position vector of a particle is given by $\bar{r} = \left({}^{1}\!/_{4}\, t^3\hat{\imath} + 3t^2\hat{\jmath} \right)$ m.

Determine at t = 2 s,

a) the radius of curvature of the path,
b) N-T components of acceleration.

Soln:

$$\text{Position vector, } \bar{r} = \left(\frac{1}{4}t^3\hat{\imath} + 3t^2\hat{\jmath}\right) \text{ m}$$
$$\therefore \text{Velocity vector, } \bar{v} = \frac{d\bar{r}}{dt} = \left(\frac{3}{4}t^2\hat{\imath} + 6t\hat{\jmath}\right) \text{ m/s}$$
$$\therefore \text{Acceleration vector, } \bar{a} = \frac{d\bar{v}}{dt} = \left(\frac{3}{2}t\hat{\imath} + 6\hat{\jmath}\right) \text{ m/s}^2$$
$$\therefore \text{at } t = 2 \text{ s,}$$
$$\bar{v} = \left(\frac{3}{4} \times 2^2\hat{\imath} + 6 \times 2\hat{\jmath}\right) = (3\hat{\imath} + 12\hat{\jmath}) \text{ m/s}$$
$$\Rightarrow v = 12.369 \text{ m/s}$$
$$\bar{a} = \left(\frac{3}{2} \times 2\hat{\imath} + 6\hat{\jmath}\right) = (3\hat{\imath} + 6\hat{\jmath}) \text{ m/s}^2$$
$$\Rightarrow a = 6.708 \text{ m/s}^2$$

Using the following equation for radius of curvature since rectangular components of velocity and acceleration are known,

$$\rho = \left|\frac{v^3}{a_x v_y - a_y v_x}\right|$$
$$\rho = \left|\frac{12.369^{\ 3}}{3 \times 12 - 6 \times 3}\right|$$
$$\boldsymbol{\rho = 105.1\ \mathrm{m}}$$

Now, for the N-T components of acceleration,

$$a_n = \frac{v^2}{\rho} = \frac{12.369^2}{105.1}$$
$$\boldsymbol{a_n = 1.456\ \mathrm{m/s^2}}$$

From $a = \sqrt{a_n^2 + a_t^2}$, we get,

$$a_t = \sqrt{a^2 - a_n^2}$$
$$a_t = \sqrt{6.708^2 - 1.456^2}$$
$$\boldsymbol{a_t = 6.548\ \mathrm{m/s^2}}$$

N6: The curvilinear motion of a particle is defined by $v_x = (25 - 8t)\ m/s$ and $y = (48 - 3t^2)$ m. Knowing at $t = 0, x = 0$, find at time $t = 4$ s, the position, velocity, and acceleration vectors. Also find corresponding magnitudes.

Soln: Considering the factors in x-direction:

$$\because v_x = (25 - 8t)\ m/s$$
$$\therefore v_x = \frac{dx}{dt} = (25 - 8t)$$
$$dx = (25 - 8t)dt$$

Integrating both sided from $x = 0, t = 0$ till some unknown x & t, we get,

$$\int_0^x dx = \int_0^t (25 - 8t)dt$$
$$x = (25t - 4t^2)\ m$$

Also, for acceleration in x-direction,

$$a_x = \frac{dv_x}{dt} = \frac{d(25 - 8t)}{dt}$$
$$a_x = -8\ m/s^2$$

Considering the factors in y-direction:

$$\because y = (48 - 3t^2)\ m$$
$$\therefore v_y = \frac{dy}{dt} = -6t\ m/s$$
$$\therefore a_y = \frac{dv_y}{dt} = -6\ m/s^2$$

Position vector at $t = 4$ s,

$$x = (25(4) - 4(4)^2) = 36\ m$$
$$y = (48 - 3(4)^2) = 0\ m$$
$$\mathbf{\bar{r} = (36\hat{i} + 0\hat{j})\ m \rightarrow r = 36\ m}$$

Velocity vector at $t = 4$ s,

$$v_x = \left(25 - 8(4)\right) = -7\ m/s$$
$$v_y = -6(4) = -24\ m/s$$
$$\mathbf{\bar{v} = (-7\hat{i} - 24\hat{j})\ m/s \rightarrow v = 25\ m/s}$$

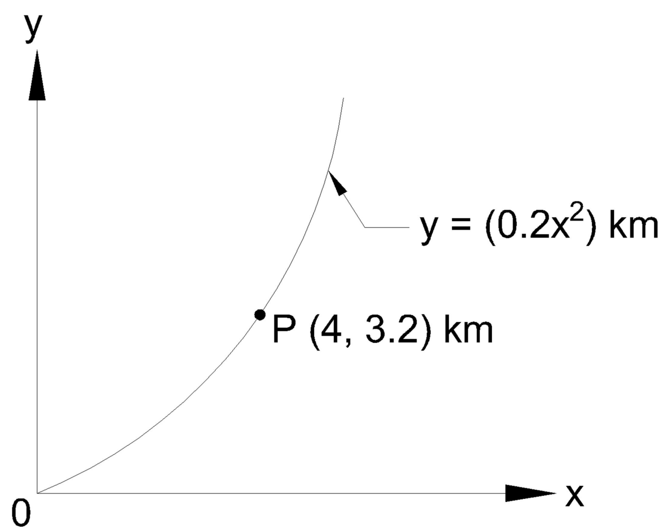
Acceleration vector at $t = 4$ s,

$$a_x = -8\ m/s^2$$
$$a_y = -6\ m/s^2$$
$$\mathbf{\bar{a} = (-8\hat{i} - 6\hat{j})\ m/s^2 \rightarrow a = 10\ m/s^2}$$

N7: An airplane travels on a curved path. At P it has speed of 360 kmph which is increasing at a rate of 0.5 m/s^2. Figure shows more details. Determine at P:

a) the magnitude of total acceleration

b) angle made by the acceleration vector with the positive x-axis.



Soln: Since speed (or velocity) given is along the path of the curve, it is tangential,

$$v = 360 \text{ kmph} = 360 \times \frac{1000}{3600} \text{ m/s} = 100 \text{ m/s}$$

Acceleration given is also along the direction of velocity, hence it is also tangential,

$$a_t = 0.5 \text{ m/s}^2$$

Equation of the path of airplane is given as $y = 0.2x^2$

$$\therefore \frac{dy}{dx} = 0.4x \Rightarrow \left[\frac{dy}{dx}\right]_{x=4\text{ km}} = 1.6$$

$$\therefore \frac{d^2y}{dx^2} = 0.4 \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=4\text{ km}} = 0.4$$

Now, using the following relation of radius of curvature in terms of derivatives,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{[1 + (1.6)^2]^{\frac{3}{2}}}{0.4}$$

$$\rho = 16.792486 \text{ km} = 16792.486 \text{ m}$$

Now, using the relations for acceleration,

$$a_n = \frac{v^2}{\rho} = \frac{100^2}{16792.486} = 0.5955 \text{ m/s}^2$$

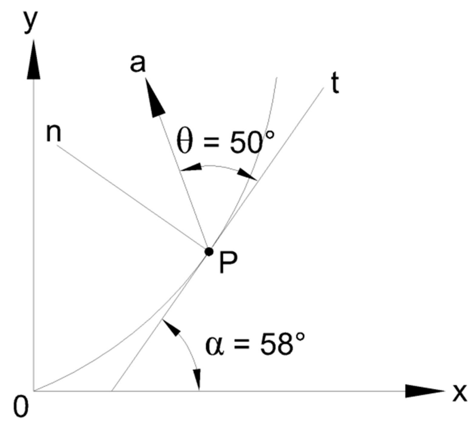
$$\therefore a = \sqrt{a_n^2 + a_t^2} = \sqrt{0.5955^2 + 0.5^2}$$
$$\therefore \mathbf{a = 0.777\ m/s^2}$$

Let θ be the angle made by the acceleration vector with the tangent to the curved path at x = 4 km.

$$\tan\theta = \frac{a_n}{a_t} = \frac{0.5955}{0.5}$$
$$\therefore \theta = 49.98° \approx 50°$$

Let α be the angle made by the tangent of the path at x = 4 km with the x-axis.

$$\tan\alpha = \frac{dy}{dx} = 1.6$$
$$\therefore \alpha = 57.99° \approx 58°$$



Hence, the angle made by the acceleration vector with the x-axis is given by,

$$\theta + \alpha = 50° + 58° = \mathbf{108°}$$

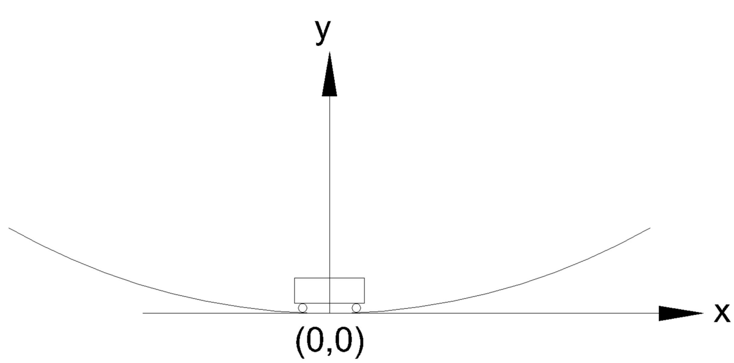
N8: A car travels along a vertical curve on a road, the equation of the curve being $x^2 = 200y$ (x-horizontal and y-vertical in m). The speed of the car is constant and equal to 72 kmph. (i) Find its acceleration when the car is at the deepest point on the curve, (ii) What is the radius of curvature of the curve at this point?

Soln: Given:

$$x^2 = 200y \rightarrow y = \frac{x^2}{200}$$
$$v = 72\ \text{kmph} = 20\ \ m/s$$

Since, the velocity is constant; the acceleration along the tangential direction is zero.

$$\therefore a_t = 0\ m/s^2$$



The deepest point of the curve will be at (0,0) since x-axis is to be taken as horizontal and y-axis is to be taken as vertical (given).

$$\therefore \frac{dy}{dx} = \frac{1}{200}(2x) = \frac{x}{100} \Rightarrow \left[\frac{dy}{dx}\right]_{x=0} = 0$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{100} \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{100}$$

Now, using the following relation of radius of curvature in terms of derivatives,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{[1 + (0)^2]^{\frac{3}{2}}}{\frac{1}{100}}$$

$$\boldsymbol{\rho = 100\ m}$$

Now, using the relations for acceleration,

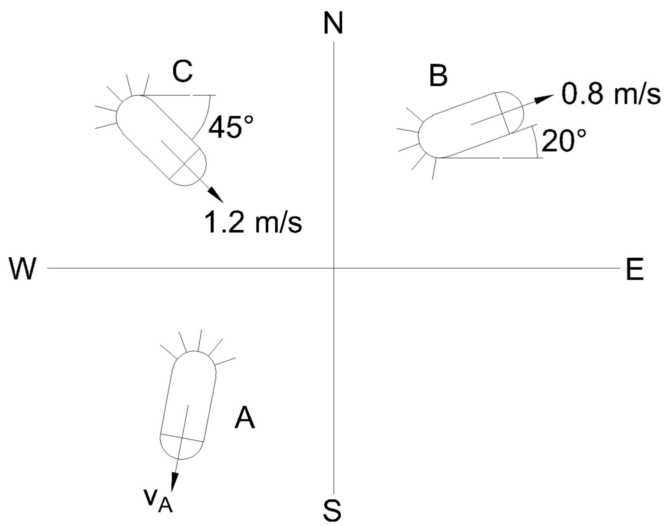
$$a_n = \frac{v^2}{\rho} = \frac{20^2}{100}$$

$$\boldsymbol{a_n = 4\ m/s^2}$$

<u>Part III – Relative Motion</u>:

<u>N9</u>: Three ships sail in different directions as shown. If the captain of ship C observes ship A, he finds ship A sailing at 3 m/s at $\theta = 60° \swarrow$. Find

a) True velocity of ship A,
b) Velocity of B as observed by A
c) Velocity of C as observed by B.



Soln: Given: Using cosine and sine for x & y components of velocity, we get,

$$v_B = 0.8\ m/s, \theta_B = 20° \nearrow \quad \Rightarrow \quad \bar{v}_B = (0.752\hat{\imath} + 0.274\hat{\jmath})\ m/s$$

$$v_C = 1.2\ m/s, \theta_C = 45° \searrow \quad \Rightarrow \quad \bar{v}_c = (0.848\hat{\imath} - 0.848\hat{\jmath})\ m/s$$

$$v_{A/C} = 3\ m/s, \theta_{A/C} = 60° \swarrow \quad \Rightarrow \quad \bar{v}_{A/C} = (-1.5\hat{\imath} - 2.6\hat{\jmath})\ m/s$$

We know, relative velocity is given by,

$$\bar{v}_{A/C} = \bar{v}_A - \bar{v}_C$$

$$(-1.5\hat{\imath} - 2.6\hat{\jmath}) = \bar{v}_A - (0.848\hat{\imath} - 0.848\hat{\jmath})$$

$$\bar{v}_A = (-0.652\hat{\imath} - 3.45\hat{\jmath})\ \mathrm{m/s}$$

$$\mathbf{v_A = 3.51\ m/s\,, \theta_A = 79.3°} \swarrow$$

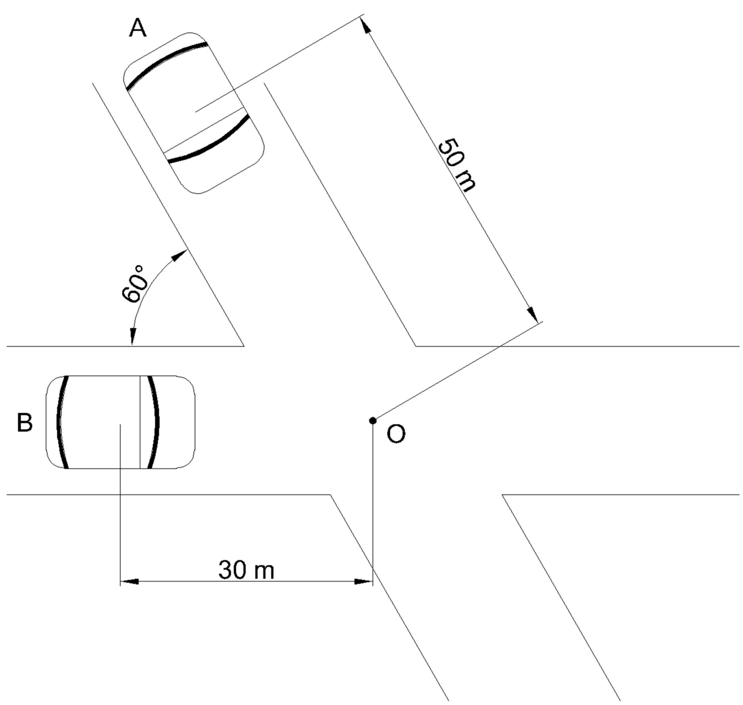
Now, for velocity of ship B w.r.t. A,

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$
$$\bar{v}_{B/A} = (0.752\hat{\imath} + 0.274\hat{\jmath}) - (-0.652\hat{\imath} - 3.45\hat{\jmath})$$
$$\bar{v}_{B/A} = (1.404\hat{\imath} + 3.724\hat{\jmath})\ \mathrm{m/s}$$
$$\mathbf{v_{B/A} = 3.98\ m/s\,, \theta_{B/A} = 69.34°} \nearrow$$

Similarly, for velocity of ship C w.r.t. B,

$$\bar{v}_{C/B} = \bar{v}_C - \bar{v}_B$$
$$\bar{v}_{C/B} = (0.848\hat{\imath} - 0.848\hat{\jmath}) - (0.752\hat{\imath} + 0.274\hat{\jmath})$$
$$\bar{v}_{C/B} = (0.096\hat{\imath} - 1.122\hat{\jmath})\ \mathrm{m/s}$$
$$\mathbf{v_{C/B} = 1.126\ m/s\,, \theta_{C/B} = 85.1°} \searrow$$

<u>N10</u>: Figure shows the location of cars A and B at t = 0. Car A starts from rest and travels towards the intersection at a uniform rate of 2 m/s^2. Car B travels towards the intersection at a constant speed of 8 m/s. Determine the relative velocity and acceleration of car B w.r.t. car A at t = 6 s.



Soln: Uniform Acceleration Motion of Car A:

Given, $u = 0, a = 2\ m/s^2, t = 6\ s$

$$v = u + at = 0 + 2 \times 6 = 12\ m/s$$

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 2 \times 6^2 = 36\ m$$

Uniform Velocity Motion of Car B:

Given, $v = 8\ m/s, t = 6\ s$

$$\text{From } v = \frac{s}{t} \rightarrow s = v \times t = 8 \times 6 = 48\ m$$

Car A has travelled 36 m, that means it is $50 - 36 = 14$ m away from intersection.

$$r_A = 14\ m, \qquad \theta_r = 60° \nwarrow \quad \Rightarrow \quad \bar{r}_A = (-7\hat{\imath} - 12.12\hat{\jmath})\ m$$

Car B has travelled 48 m, that means it is $48 - 30 = 18$ m past the intersection.

$$r_B = 18\ m, \qquad \theta_r = 0° \longrightarrow \quad \Rightarrow \quad \bar{r}_A = (18\hat{\imath})\ m$$

Now, relative position of B w.r.t A at t = 6 s is given by,

$$\bar{r}_{B/A} = \bar{r}_B - \bar{r}_A = (18\hat{\imath}) - (-7\hat{\imath} - 12.12\hat{\jmath})$$
$$\bar{r}_{B/A} = (25\hat{\imath} - 12.12\hat{\jmath})\ m$$
$$\mathbf{r_{B/A} = 27.78\ m, \theta_r = 25.86° \searrow}$$

Velocity vector of A is, $v_A = 12\ m/s,\ \theta_v = 60° \searrow \quad \Rightarrow \quad \bar{v}_A = (6\hat{\imath} - 10.39\hat{\jmath})\ m/s$

Velocity vector of B is, $v_B = 8\ m/s,\ \theta_v = 0° \longrightarrow \quad \Rightarrow \quad \bar{v}_B = (8\hat{\imath})\ m/s$

Now, relative velocity of B w.r.t A at t = 6 s is given by,

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$
$$\bar{v}_{B/A} = (8\hat{\imath}) - (6\hat{\imath} - 10.39\hat{\jmath})$$
$$\bar{v}_{B/A} = (2\hat{\imath} + 10.39\hat{\jmath})\ m/s$$
$$\mathbf{v_{B/A} = 10.58\ m/s, \theta_v = 79.1° \nearrow}$$

Acceleration vector of A, $a_A = 2\ m/s^2,\ \theta_a = 60° \searrow \Rightarrow \bar{a}_A = (1\hat{\imath} - 1.732\hat{\jmath})\ m/s^2$

Acceleration vector of B, $a_B = 0 \rightarrow \bar{a}_B = 0$

Now, relative acceleration of B w.r.t A at t = 6 s is given by,

$$\bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$$
$$\bar{a}_{B/A} = 0 - (1\hat{\imath} - 1.732\hat{\jmath})$$
$$\bar{a}_{B/A} = (-1\hat{\imath} + 1.732\hat{\jmath})\ m/s^2$$
$$\mathbf{a_{B/A} = 2\ m/s^2, \theta_a = 60° \nwarrow}$$