Module :3
Matrices
Cayley-Hamilton Theorem

Cayley-Hamilton Theorem

- **Statement**: Every Square Matrix satisfies its characteristic equation.

If A is given square matrix of order n, λ is an eigenvalue of A and I is an identity matrix of order n.

Then it's characteristic equation is given by

$$|A - \lambda I| = 0.$$

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda^1 + a_0 = 0$$

then by Cayley-Hamilton Theorem,

$$a_nA^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2A^2 + a_1A + a_0 = 0\,.$$

Ex. Verify Cayley-Hamilton Theorm for the matrix A, hence find

$$A^{-1} \,\&\, A^4 . A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Soln. The characteristic equation is given by $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

Using : $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

Where S_1=Trace A

S_2= Sum of Minors of Diagonal Elements

$|A| =$ Determinant of A

For given Matrix A,

S_1=6, S_2=-11, $|A| = -6$

$\therefore f(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$: The characteristic equation

By Cayley-Hamilton Theorem, A Should satisfy the characteristic equation.

Verification:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix}$$

$$\therefore f(A) = A^3 - 6A^2 + 11A - 6I$$

$$= \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} + 11\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} - 6I$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence , $\therefore f(A) = A^3 - 6A^2 + 11A - 6I = 0$..........(1)

Ie. A satisfies its characteristic equation. Hence Cayley-Hamilton theorem is verified.

To find A^{-1},

Pre-multiplying (1) by A^{-1}

$$\therefore A^2 - 6A + 11I - 6A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

To find A^4,

Pre-multiplying (1) by A

$$\therefore A^4 - 6A^3 + 11A^2 - 6A = 0$$

$$\therefore A^4 = \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$$

Ex. Verify Cayley-Hamilton Theorm for the matrix A, hence find A^{-1} for

$$A = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$ **. Also find eigenvalues for A.**

Soln. The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} cos\theta - \lambda & sin\theta \\ -sin\theta & cos\theta - \lambda \end{vmatrix} = 0$$

$$\therefore (cos\theta - \lambda)^2 - sin^2\theta = 0$$

$\therefore f(\lambda) = \lambda^2 - 2\lambda cos\theta + 1 = 0$: The characteristic equation

$\therefore$ roots of a characteristic equation are eigenvalues

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\therefore \lambda = \frac{2\cos\theta \pm 2i\sin\theta}{2}$$

$$\therefore \lambda = \cos\theta \pm i\sin\theta$$

By Cayley-Hamilton Theorem, A Should satisfies the characteristic equation $f(\lambda) = \lambda^2 - 2\lambda cos\theta + 1 = 0$.

Verification:

$$A^2 = \begin{bmatrix} cos^2\theta - sin^2\theta & 2cos\theta sin\theta \\ -2cos\theta sin\theta & cos^2\theta - sin^2\theta \end{bmatrix}$$

$$f(A) = A^2 - 2Acos\theta + I$$

$$= \begin{bmatrix} cos^2\theta - sin^2\theta & 2cos\theta sin\theta \\ -2cos\theta sin\theta & cos^2\theta - sin^2\theta \end{bmatrix} - 2cos\theta \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} + I$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore f(A) = A^2 - 2Acos\theta + I = 0 \quad ....1$$

Ie. A satisfies its characteristic equation. Hence Cayley-Hamilton theorem is verified.

Pre-multiplying (1) by A^{-1}

$$\therefore\ A^{-1}A^{2} - 2A^{-1}A\cos\theta + A^{-1}I = 0$$

$$\therefore A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Find Characteristic equation of the matrix A and hence find the matrix given by $\boldsymbol{A^7 - 4A^6 - 20A^5 - 34A^4 - 4A^3 - 20A^2 - 33A + I}$.

Where $\boldsymbol{A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}}$.

The characteristic equation is given by $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

Using : $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$ Where S_1=Trace A=4

S_2= Sum of Minors of Diagonal Elements= -20

$|A|$ = Determinant of A =35

$\therefore f(\lambda) = \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$: The characteristic equation

By Cayley-Hamilton Theorem, A satisfies the characteristic equation.

$\therefore f(A) = A^3 - 4A^2 - 20A - 35I = 0$ ………………1

Let $g(A) = A^7 - 4A^6 - 20A^5 - 34A^4 - 4A^3 - 20A^2 - 33A + I$

By division of two polynomial,

We get,

$$g(A) = A^7 - 4A^6 - 20A^5 - 34A^4 - 4A^3 - 20A^2 - 33A + I$$

$$g(A) = A^7 - 4A^6 - 20A^5 - 35A^4 + A^4 - 4A^3 - 20A^2 - 35A + 2A + I$$

$$= A^4(A^3 - 4A^2 - 20A - 35I) + A(A^3 - 4A^2 - 20A - 35I) + 2A + I$$

$$= 0 + 0 + 2A + I \qquad \text{(by (1))}$$

$$g(A) = \begin{bmatrix} 3 & 6 & 14 \\ 8 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}$$

Ex. Use Cayley Hamilton theorem to find $A^7 - 9A^2 + I$.

Where $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

The characteristic equation is given by $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)^2 - 4 = 0$$

$\therefore f(\lambda) = \lambda^2 - 2\lambda - 3 = 0$: The characteristic equation

$$\lambda = -1, 3$$

By Cayley-Hamilton Theorem, A satisfies the characteristic equation.

$$\therefore f(A) = A^2 - 2A - 3I = 0$$

Let $g(A) = A^7 - 9A^2 + I$

As coefficients of are $f(A)$ and $g(A)$ are not same/similar, we use division algorithm

$g(A) = f(A). q(A) + r(A)$; where $degree\ of\ f(A) < degree\ of\ r(A)$.

$\therefore g(A) = 0. q(A) + a_0 A + a_1 I$

$A^7 - 9A^2 + I = a_0 A + a_1 I$ 1

Eigenvalues of A satisfies this equation.

$\therefore \lambda^7 - 9\lambda^2 + I = a_0\ \lambda + a_1\ I$

For $\lambda = -1$

$-1 - 9 + 1 = 2107 = -a_0 + a_1$....................2

For $\lambda = 3$, $\quad 2187 - 9(9) + 1 = 2107 = 3a_0 + a_1$...........3

Solving (2) & (3),

We get, $a_0 = 529$ & $a_1 = 520$

By (1),

$$g(A) = A^7 - 9A^2 + I = 529A + 520I$$

Ex. Use Cayley Hamilton theorem to prove $A^8 = 625I$.

Where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Soln. The characteristic equation is given by $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$-(1-\lambda)(1+\lambda) - 4 = 0$

$\therefore f(\lambda) = \lambda^2 - 5 = 0$: The characteristic equation

$$\lambda = -1,3$$

By Cayley-Hamilton Theorem, A satisfies the characteristic equation.

$$\therefore f(A) = A^2 - 5I = 0$$

Ie.$A^2 = 5I$

Pre multiplying by A^2 on both sides

$$A^4 = 25I$$

Pre multiplying by A^4 on both sides

$$A^8 = 625I$$