Linear Differential Equations (LDE)

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Def^n: D.E. is said to be Linear if the Depedent variable and its Derivative appear as 1st Degree

Two types:

linear in y

Form: $\frac{dy}{dx} + Py = Q$

Where P and Q are purely function of x / constant

Then, I.F. $= e^{\int P\,dx}$

& sol^n is given by,

$$y\,e^{\int P\,dx} = \int \left[Q\,e^{\int P\,dx}\right] dx + C$$

linear in x

$\frac{dx}{dy} + P'x = Q'$

Where P' and Q' are purely function of y / constant

I.F. $= e^{\int P'\,dy}$

Then sol^n is given by

$$x\,e^{\int P'\,dy} = \int \left[Q\,e^{\int P'\,dy}\right] dy + C$$

Problems

$\rightarrow$ idetify linear in x/y

$\rightarrow$ Chk $\cancel{y\frac{dy}{dx}}$

$\frac{dy}{dx} + P\,y = Q$

$\rightarrow$ 3 separate terms.

$\rightarrow$ Coeff. of $\frac{dy}{dx}$ is 1

1) $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

Divide by $x(x-1)$

$$\frac{dy}{dx} - \frac{1}{x(x-1)}\,y = x(x-1)$$

compare with Fom $\frac{dy}{dx} + Py = Q$

$\therefore P = -\frac{1}{x(x-1)}$, $Q = x(x-1)$

Then,
$$\int P\,dx = \int -\frac{1}{x(x-1)}\,dx$$
$$= \int \left[\frac{1}{x} - \frac{1}{x-1}\right] dx = \log x - \log(x-1)$$
$$= \log\left(\frac{x}{x-1}\right)$$

Then I.F. $= e^{\int P\,dx} = e^{\log\left(\frac{x}{x-1}\right)} = \frac{x}{x-1}$

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$$y\, e^{\int P dx} = \int \left[Q\, e^{\int P dx} \right] dx + C$$

$$y \left(\frac{x}{x-1} \right) = \int x \cancel{(x-1)} \left[\frac{x}{\cancel{x-1}} \right] dx + C$$

$$\boxed{\frac{xy}{x-1} = \frac{x^3}{3} + C}$$

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2) $x \log x \dfrac{dy}{dx} + y = 2 \log_e x$

Divide by $x \log x$

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2 \cancel{\log_e x}}{x \cancel{\log_e x}} = \frac{2}{x}$$

Compare with $\dfrac{dy}{dx} + Py = Q$

$$\therefore \; P = \frac{1}{x \log x}, \quad Q = \frac{2}{x}$$

$t = \log x$

$dt = \frac{1}{x} dx$

Then $\displaystyle\int P\, dx = \int \frac{1}{x \log x} dx = \int \frac{1}{t} dt$

$$= \log t = \log(\log x)$$

$$I.F. = e^{\int P dx} = e^{\log_e(\log_e x)} = \log_e x$$

Then sol^n is given by,

$$y\, e^{\int P dx} = \int \left[Q\, e^{\int P dx} \right] dx + C$$

$$y \ln x = \int \frac{2}{x} (\log_e x)\, dx + C$$

$$y \log x = 2 \int t\, dt + C$$

$$y \log x = \cancel{2} \frac{t^2}{\cancel{2}} + C = (\log x)^2 + C$$

$$\boxed{y \ln x = (\ln x)^2 + C}$$

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3) $(1+y^2)\, dx = (\tan^{-1} y - x)\, dy$

3) $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$

$$(1+y^2)\frac{dx}{dy} = \tan^{-1} y - x$$

Divide by $(1+y^2)$

$$\frac{dx}{dy} = \left(\frac{1}{1+y^2}\right)\tan^{-1} y - \frac{x}{1+y^2}$$

$$\therefore \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1} y}{(1+y^2)} \quad \text{--- } (1)$$

$$\int P' dy = \tan^{-1} y$$

$$e^{\int P' dy} = e^{\tan^{-1} y}$$

Then Sol^n is given by,

$$x e^{\int P' dy} = \int \left[Q e^{\int P' dy}\right] dy + c$$

$$x e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y}\,dy + C$$

$$t = \tan^{-1} y, \quad dt = \frac{1}{1+y^2} dy$$

$$x e^t = \int (t \cdot e^t)\,dt + C$$

$$= t e^t - \int e^t dt + c$$

$$x e^t = t e^t - e^t + c$$

put back $t = \tan^{-1} y$

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y}\left[\tan^{-1} y - 1\right] + C$$

Problems continued

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3rd term?

4) $\boxed{(x+y+1)\frac{dy}{dx}} = 1$

H.W 5) $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$

6) $\sin 2y\,dx = (\tan y - x)\,dy$

H.W 7) $\cos^2 x \frac{dy}{dx} + y = \tan x$

5) Ans

$r\sin^2\theta = -\frac{1}{2}\sin^4\theta + c$

4) $x + (y+1) = \frac{dx}{dy}$

$\frac{dx}{dy} - x = (y+1)$

$P = -1 \quad | \quad Q = (y+1)$

$e^{\int P\,dy} = e^{\int -1\,dy} = e^{-y}$

$\cancel{e^{\int P\,dx}}$

Sol^n: $xe^{-y} = \int [(y+1)e^{-y}]\,dy + c$

$= (y+1)\left(\frac{e^{-y}}{-1}\right) - e^{-y} + c$

$xe^{-y} = e^{-y}(-y-2) + c$

$$\boxed{e^{-y}[x+y+2] = c}$$

6) $1\frac{dx}{dy} = \frac{\tan y - x}{\sin 2y}$

$\frac{dx}{dy} = -\frac{x}{\sin 2y} + \frac{\sin y/\cos y}{2\sin y\cos y}$

$$\boxed{\frac{dx}{dy} + (\operatorname{cosec} 2y)\,x = \frac{1}{2}\sec^2 y}$$

6) contd:

$\frac{dx}{dy} + \operatorname{cosec} 2y\, x = \frac{1}{2}\sec^2 y$

$e^{\int \operatorname{cosec}(2y)\,dy} = e^{\frac{\log(\operatorname{cosec}2y - \cot 2y)}{2}} = \sqrt{\tan y}$

$e^{\frac{\log(\tan\frac{2y}{2})}{2}}$

$\int \frac{1}{2}\sec^2 y\sqrt{\tan y}\;dy + c$

Sol^n: $x\left(\sqrt{\tan y}\right) = \int \frac{1}{2}\sec^2 y \sqrt{\tan y}\, dy + C$

put $t = \tan y$, $\sec^2 y\, dy = dt$

$$= \frac{1}{2}\int \sqrt{t}\, dt + C$$

$$x\sqrt{\tan y} = \frac{1}{\cancel{2}} \frac{t^{3/2}}{3/\cancel{2}} + C$$

$$\boxed{x\sqrt{\tan y} = \frac{1}{3}(\tan y)^{3/2} + C}$$

7) ans $\hookrightarrow$ $y e^{\tan x} = e^{\tan x}(\tan x - 1) + C$