Partial Differentiation

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Differentiation $y = f(x) \longrightarrow \frac{dy}{dx}$ $\quad \partial \quad \delta$

for function of more than single variable,

$u = f(x, y)$ | $u = f(x, y, z)$

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y, z) - f(x, y, z)}{\delta x}$$

$\frac{\partial u}{\partial x} =$

$\frac{\partial u}{\partial y} =$

$u_x \quad f_x$

$\rightarrow u_x \quad u_y \quad u_z$

$\rightarrow f_x \quad f_y \quad f_z$

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examples: 1) $u = x \log y$ $\quad$ 2) $u = \sin^{-1} x \, e^y$

3) $u = x^n y^m$ $\quad$ 4) $u = \sin^{-1}\left(\frac{x}{y}\right)$ $\quad$ 5) $u = x^y$

1) $\frac{\partial u}{\partial x} = u_x = (\log y)(1) = \log y$

$\frac{\partial u}{\partial y} = u_y = (x)\left(\frac{1}{y}\right) = \frac{x}{y}$

2) $\frac{\partial u}{\partial x} = u_x = \frac{e^y}{\sqrt{1-x^2}}$ | $u_y = \sin^{-1} x \, e^y$

3) $y = x^m y^n \Rightarrow u_x = m x^{m-1} \, y^n$

$u_y = x^m \, n y^{n-1}$

4) $u = \sin^{-1}\left(\frac{x}{y}\right) \Rightarrow u_x = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \left(\frac{1}{y}\right) = \frac{\cancel{y}}{\sqrt{y^2-x^2}} \frac{1}{\cancel{y}}$

$u_x = \frac{1}{\sqrt{y^2-x^2}}$

$$u_x = \frac{1}{\sqrt{y^2-x^2}}$$

$$\underline{u_y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right) = \frac{y}{\sqrt{y^2-x^2}}\left(-\frac{x}{y^2}\right)$$

$$u_y = -\frac{x}{y\sqrt{y^2-x^2}}$$

$$\underline{x^3} \to \underline{3x^2}$$

$$3^x$$

5)
$$\boxed{\begin{aligned} u &= \underline{x^y} \\ u_x &= \underline{y x^{y-1}} \\ u_y &= \underline{x^y \log x} \end{aligned}}$$

Standard rule of derivatives

$$\frac{\partial}{\partial x}(u \pm v) = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

Higher order:

$y = f(x)$

$$\frac{dy}{dx} = y' = f'$$

$$\frac{d^2y}{dx^2} = y'' = f''$$

$u = f(x,y)$

$\frac{\partial u}{\partial x} \mid u_x$ $\qquad$ $\frac{\partial u}{\partial y} \mid u_y$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ $\qquad$ $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

$\partial^2 u = \partial^2 u$ $\qquad$ $\partial^2 u$

$\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ $\frac{\partial^2 u}{\partial y^2}$

$\boxed{u_{xx} \quad u_{yx} \quad u_{xy} \quad u_{yy}}$

$u = f(x, y, z)$

1^{st} u_x u_y u_z

2^{nd}

3^{rd} order
How many

m^n

2^3 — 8

3^3 — 27

1) If $u = \cos(\sqrt{x} + \sqrt{y})$ Then prove that

$\rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \sin(\sqrt{x}+\sqrt{y})\frac{1}{2}(\sqrt{x}+\sqrt{y}) = 0$

Sol^n: $\frac{\partial u}{\partial x} = -\sin(\sqrt{x}+\sqrt{y})\left(\frac{1}{2\sqrt{x}} + 0\right)$ — (1)

$\frac{\partial u}{\partial y} = -\sin(\sqrt{x}+\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right)$ — (2)

(1) $\times x$ + (2) $\times y$

Consider, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\sqrt{x}\sin(\sqrt{x}+\sqrt{y})$

$-\frac{1}{2}\sqrt{y}\sin(\sqrt{x}+\sqrt{y})$

$= -\frac{1}{2}(\sqrt{x}+\sqrt{y})\sin(\sqrt{x}+\sqrt{y})$

$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x}+\sqrt{y})\sin(\sqrt{x}+\sqrt{y}) = 0}$

... order derivative ✓

$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x}+\sqrt{y})$ ...

→ Continuous second order derivative ✓

$$z = x^y + y^x \rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

equality of Mixed Partial derivatives

$$\boxed{\frac{\partial z}{\partial x}} = \left(y\boxed{x^{y-1}} + \boxed{y^x}\boxed{\log y}\right) \quad \text{—(1)}$$

$$\boxed{\frac{\partial z}{\partial y}} = \left[x^y \log x\right] + \left[x y^{x-1}\right] \quad \text{—(2)}$$

Then find second order,

diff (1) w.r.to y

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = y\left[x^{(y-1)}\log x\,(1)\right] + \left(x^{y-1}\right)(1) + y^x\left(\frac{1}{y}\right) + \log y\left(x y^{x-1}\right)$$

$$= y x^{y-1}(\log x) + x^{y-1} + y^{x-1} + x y^{x-1}(\log y) \checkmark$$

diff (2) w.r.to x

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = y x^{y-1}(\log x) + \left(x^y \frac{1}{x}\right) + y^{x-1}(1) + x y^{x-1}\log y\,(1) \checkmark$$

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2) $u = e^{xyz}$ Prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = \left[1 + 3xyz + x^2y^2z^2\right]e^{xyz}$

HW

∇F Gradient

- If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Here consider L.H.S. $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\right]$

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz(1)}{x^3 + y^3 + z^3 - 3xyz} \quad \text{—(1)} \qquad \frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \quad \text{—(2)}$$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$ — (1) | $\dfrac{\partial u}{\partial y} = \dfrac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$

Observe here function $\boxed{u \text{ is symmetric in } x, y, z}$

Similarly $\dfrac{\partial u}{\partial z} = \dfrac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$ — (3)

$\therefore \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3x^2+3y^2+3z^2-3yz-3xz-3xy}{x^3+y^3+z^3-3xyz}$ (1) + (2) + (3)

$$= 3\left[\frac{\cancel{(x^2+y^2+z^2-yz-xz-xy)}}{(x+y+z)\cancel{(x^2+y^2+z^2-yz-xz-xy)}}\right]$$

$\left[x^3+y^3+z^3 = [x+y+z](x^2+y^2+z^2-yz-xz-xy)+3xyz\right]$

$$= \left(\frac{3}{x+y+z}\right)$$

Now L.H.S. $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u = \left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)\left[\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}\right]$

$$= \left(\qquad\qquad\right)\left(\frac{3}{x+y+z}\right)$$

(Symmetric f^n)

$\dfrac{1}{u} \to u'$

$$= \left[\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)\right]$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

(Since $\dfrac{3}{x+y+z}$ is Symmetric)

$$= -\frac{9}{(x+y+z)^2}$$

Def^n:

$u = f(x, y)$ is said to be symmetric if,

$u = f(x,y)$ is ...

$f(x,y) = f(y,x)$

If $u = f(x,y,z)$ Then $f(x,y,z) = f(y,z,x) = f(x,z,y)$
$= f(z,y,x)$

If $\theta = t^n e^{(-r^2/4t)}$ find n which will make $\frac{\partial\theta}{\partial t} = \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)}$.

$$\theta = t^{n} e^{-r^2/4t}$$

$$\frac{\partial\theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(+\frac{r^2}{4t^2}\right)$$

$$L.H.S. = t^n e^{-r^2/4t}\left[\frac{n}{t} + \frac{r^2}{4t^2}\right] = \theta\left[\frac{n}{t} + \frac{r^2}{4t^2}\right] \quad \text{— (1)}$$

$$\boxed{\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\left(-\frac{2r}{4t}\right) = \theta\left(-\frac{r}{2t}\right)} \quad \text{— (2)}$$

$$r^2\frac{\partial\theta}{\partial r} = -\frac{\theta r^3}{2t}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2t}\left[\theta\left(\frac{d}{dr}r^3\right) + r^3\frac{\partial\theta}{\partial r}\right]$$

$$= -\frac{1}{2t}\left[3r^2\theta + r^3\left(-\frac{r\theta}{2t}\right)\right]$$

$$= \left[-\frac{3r^2}{2t} + \frac{r^4}{(2t)^2}\right]\theta$$

$$R.H.S. = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \left[-\frac{3}{2t} + \frac{r^2}{4t^2}\right]\theta \quad \text{— (3)}$$

By considering L.H.S. = R.H.S.

$$\theta\left[\frac{n}{t} + \frac{r^2}{4t^2}\right] = \left[-\frac{3}{2t} + \frac{r^2}{4t^2}\right]\theta$$

By comparing, $\boxed{n = -\frac{3}{2}}$

By comp u +

- If $u = f(r), r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

Sol^n: Consider $r^2 = x^2 + y^2 + z^2$ — (1)

diff (1) w.r.to x / (1) is symmetric in x, y, z

$$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly,

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}, \quad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}} \quad \text{— (2)}$$

Now for expression

$$\frac{\partial u}{\partial x} = \left(\frac{du}{dr}\right)\left(\frac{\partial r}{\partial x}\right) = f'(r)\left(\frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'(r)\left(\frac{1}{r}\right)x\right)$$

$$= \frac{\partial}{\partial x}(f'(r))\left(\frac{x}{r}\right) + f'(r)\, x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{f'(r)}{r}\frac{\partial}{\partial x}(x)$$

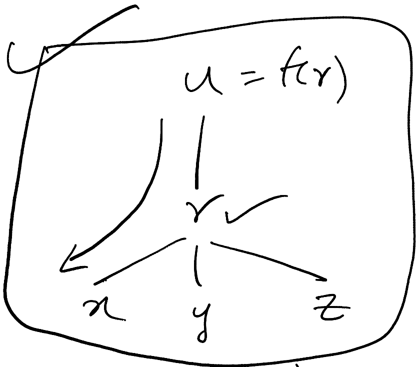
$$= \frac{d}{dr}(f'(r))\left(\frac{\partial r}{\partial x}\right)\left(\frac{x}{r}\right) + x f'(r)\left(-\frac{1}{r^2}\right)\left(\frac{\partial r}{\partial x}\right) + \frac{f'(r)}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{x^2}{r^2} - \frac{x^2}{r^3} f'(r) + \frac{f'(r)}{r} \quad \text{— (3)}$$

Similarly Since u is in term symmetric in x, y, z,

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} - \frac{y^2}{r^3} f'(r) + \frac{f'(r)}{r} \quad \text{— (4)}$$

$$\ldots \frac{z^2}{} - \frac{z^2}{} f'(r) + \frac{f'(r)}{} \quad \text{— (5)}$$



$$\frac{\partial u}{\partial x} = \left(\frac{du}{dr}\right)\left(\frac{\partial r}{\partial x}\right)$$

$$= f'(r)$$

$$\frac{d^2 u}{dr^2} = f''(r)$$

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\left(\frac{\partial r}{\partial x}\right)$$

$$\log y = x^2 \quad (y = f(x))$$

diff eq^n w.r.to x

$$\frac{1}{y}\frac{dy}{dx} = 2x$$

$$\frac{d}{dx}(uvwt)$$

$$= u'vwt + v'uwt + w'uvt + t'uvw$$

∂y^2

$$\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} - \frac{z^2}{r^3}f'(r) + \frac{f'(r)}{r} \quad \text{— (5)}$$

(3) + (4) + (5)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{\cancel{r^2}}\cancel{(x^2+y^2+z^2)}^{r^2} - \cancel{(x^2+y^2+z^2)}^{r^2}\frac{f'(r)}{\cancel{r^3}} + 3\frac{f'(r)}{r}$$

$$= f''(r) - \frac{f'(r)}{r} + 3\frac{f'(r)}{r} = \boxed{f''(r) + \frac{2f'(r)}{r}}$$

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6) $u = \log(\tan x + \tan y + \tan z)$

Prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Sol^n: Consider $\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$

u is symmetric f^n in x, y & z

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2\sin x \cancel{\cos x}\left(\frac{1}{\cos^2 x}\right)}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2\tan x}{\tan x + \tan y + \tan z} \quad \text{— (1)}$$

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z} \quad \text{— (2)} \quad \& \quad \sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} \quad \text{— (3)}$$

(1) + (2) + (3)

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z}$$

$$= 2$$

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8) If $u = Ae^{-gx}\sin(nt - gx)$ where n, g, A are constants

Then Prove that, $g = \sqrt{\frac{n}{2}}$ If $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Sol^n: $\frac{\partial u}{\partial x} = A\left[e^{-gx}(-g)\sin(nt-gx) + e^{-gx}\cos(nt-gx)(-g)\right]$

$= -gA\, e^{-gx}\left[\sin(nt-gx) + \cos(nt-gx)\right]$

diff again w.r.to x

$\frac{\partial^2 u}{\partial x^2} = -gA\left[e^{-gx}(-g)\left(\sin(nt-gx) + \cos(nt-gx)\right) + e^{-gx}\left[\cos(nt-gx)(-g) - \sin(nt-gx)(-g)\right]\right]$

$= g^2 A\, e^{-gx}\left[\cancel{\sin(nt-gx)} + \cos(nt-gx) + \cos(nt-gx) - \cancel{\sin(nt-gx)}\right]$

$\frac{\partial^2 u}{\partial x^2} = g^2 A\, e^{-gx}\, 2\cos(nt-gx)$ —— 1

$\frac{\partial u}{\partial t} = A e^{-gx}\cos(nt-gx)(n)$ —— 2

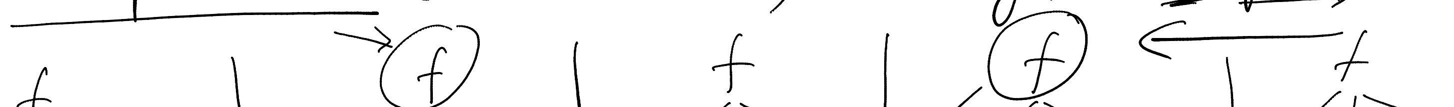
Comparing 1 & 2

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \cancel{A} g^2 \cancel{e^{-gx}}\, 2\cancel{\cos(nt-gx)} = n\cancel{A}\cancel{e^{-gx}}\cancel{\cos(nt-gx)}$

$2g^2 = n \Rightarrow g^2 = \frac{n}{2}$

$\boxed{g = \sqrt{\frac{n}{2}}}$

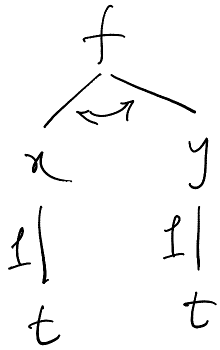
Composite f^n (function of f^n) $f\circ g(x) = f(g(x))$



$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$f(r)$

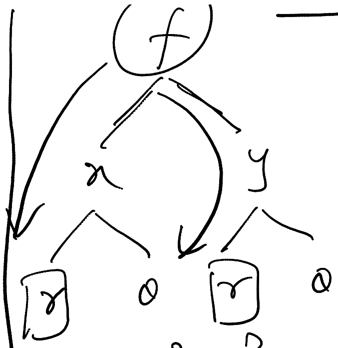
$r^2 = x^2 + y^2$



$y^2 = 4ax$

$x = 2at$

$y = at^2$



$\Rightarrow x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$