HOMOGENEOUS FUNCTIONS

Homogeneous Functions:

A function $u = f(x, y)$ is said to be homogeneous function of degree n, if $u = x^n f\left(\frac{y}{x}\right)$ where, n is real number

Note: Degree of a homogeneous function $u = f(x, y)$ can be obtained by replacing x by xt and y by yt and if $f(xt, yt) = t^n f(x, y) = t^n u$ then u is a homogeneous function of degree n.

Same method can be extended for a function of more than two variables

A function $f(x, y, z)$ is called a homogeneous function of degree n if by putting $X = xt, Y = yt, Z = zt$ the function $f(X, Y, Z)$ becomes $t^n f(x, y, z)$ i.e. $f(xt, yt, zt) = t^n f(x, y, z)$

e.g. $f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ is a homogeneous function of degree 2.

EULER'S THEOREM:

If u is a homogeneous function of two variables x and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Proof: Let $u = f(x, y)$ be the given function.

Since it is a homogeneous function of degree n, on putting $X = xt, \quad Y = yt$

$f(X, Y) = t^n f(x, y)$ .... (i)

Differentiating l. h. s. of (i) w. r. t. t,

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial t} = x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y}$

If we put $t = 1$, $i.e.$ $X = x$, $Y = y$, we get, $\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ .... (ii)

Differentiating r. h. s. of (i) w. r. t. t, $\frac{\partial f}{\partial t} = nt^{n-1} f(x, y)$

If we put t = 1, we get $\frac{\partial f}{\partial t} = nf(x, y)$ ....(iii)

From (ii) and (iii) we get, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$

i.e. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Alternate Proof:

$u = f(x, y)$ is a homogeneous function of degree n then

$\therefore u = x^n \emptyset\left(\frac{y}{x}\right)$ ....(i)

Differentiate partially w.r.t. x we get $\frac{\partial u}{\partial x} = nx^{n-1}\emptyset\left(\frac{y}{x}\right) + x^n \emptyset'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$

$\therefore x\frac{\partial u}{\partial x} = nx^n \emptyset\left(\frac{y}{x}\right) - yx^{n-1}\emptyset'\left(\frac{y}{x}\right)$ .... (ii)

Differentiate (i) partially w.r.t. y we get $\frac{\partial u}{\partial y} = x^n \emptyset'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1}\emptyset'\left(\frac{y}{x}\right)$

$\therefore y\frac{\partial u}{\partial y} = yx^{n-1}\emptyset'\left(\frac{y}{x}\right)$ .... (iii)

Adding (ii) & (iii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \emptyset\left(\frac{y}{x}\right) = nu$

EULER'S THEOREM FOR A FUNCTION OF THREE VARIABLES:

Theorem: If u is a homogeneous function of three variables x, y, z of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

Note: In general, if u is a homogeneous function of x, y, z ….. t of degree n then Euler's Theorem states that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \cdots + t\frac{\partial u}{\partial t} = nu$$

Corollary 1 If z is a homogeneous function of two variables x and y of degree n then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Proof: **By Euler's Theorem** we have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ …………….. (i)

Differentiating (i) partially w.r.t. x, we get $\left(x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.1\right) + y\frac{\partial^2 z}{\partial x\,\partial y} = n\frac{\partial z}{\partial x}$

$\therefore\ x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}$ …………….. (ii)

Differentiating (i) partially w.r.t. y, we get $x\frac{\partial^2 z}{\partial y\,\partial x} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n\frac{\partial z}{\partial y}$

$\therefore\ x\frac{\partial^2 z}{\partial y\,\partial x} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y}$ ……………. (iii)

multiplying (ii) by x and (iii) by y and adding, we get,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

$$= n(n-1)z \qquad \text{[by (i)]}$$

Corollary 2: If z is homogeneous function of degree n in x and y, and $z = f(u)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Proof: By Euler's Theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = nf(u)$ ……………(i)

Since $z = f(u)$

$\therefore\ \frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$

Putting these values in (i), we get, $xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nf(u)$

$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Corollary 3 If z is homogeneous function of degree n in x and y, and $z = f(u)$ then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \quad \text{where} \quad g(u) = n\frac{f(u)}{f'(u)}$$

Proof: By Corollary (2) we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = g(u)$ …………….. (i)

Differentiating (i) partially w.r.t. x, we get $\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.1\right) + y\frac{\partial^2 u}{\partial x\,\partial y} = g'(u)\frac{\partial u}{\partial x}$

$$\therefore \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (g'(u) - 1)\frac{\partial u}{\partial x} \qquad \text{................ (ii)}$$

Differentiating (i) partially w.r.t. y, we get $x\frac{\partial^2 u}{\partial y\,\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = g'(u)\frac{\partial u}{\partial y}$

$$\therefore \quad x\frac{\partial^2 u}{\partial y\,\partial x} + y\frac{\partial^2 u}{\partial y^2} = (g'(u) - 1)\frac{\partial u}{\partial y} \qquad \text{............... (iii)}$$

multiplying (ii) by x and (iii) by y and adding, we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (g'(u) - 1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= g(u)[g'(u) - 1] \qquad \text{[by (i)]}$$