Sub-Module :4.3 & 4.4
Maxima Minima
& Jacobian

Working rule

❖ To find maxima/minima (stationary values/ extreme values/turning values) of function of two variable $f(x, y)$

i) Find $f_x, f_y, f_{xx}, f_{yy} \& f_{xy}$.

ii) Solve equations $f_x = 0$ & $f_y = 0$ simultaneously for x & y. List all possible stationary points (x, y).

iii) At the above possible stationary points find $r = f_{xx}, s = f_{xy} \&$ $t = f_{yy}$.

- Check for the sign of $rt - s^2$ & r.

1. If $rt - s^2 > 0$ & $r < 0 \Rightarrow f(x, y)$is maximum
2. If $rt - s^2 > 0$ & $r > 0 \Rightarrow f(x, y)$is minimum
3. If $rt - s^2 > 0$ & $r = 0 \Rightarrow f(x, y)$has neither maxima or minimum
4. If $rt - s^2 < 0 \Rightarrow f(x, y)$has neither maxima or minimum
5. If $rt - s^2 = 0 \Rightarrow$ further investigation required

iv) Find the stationary value of the function at stationary points.

Example:1

Discuss the maxima and minima of

$$x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10.$$

- **Sol.:** We have $f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10.$
- **Step I :** $f_x = 3x^2 + y^2 - 24x + 21, \quad f_y = 2xy - 4y,$

$$f_{xx} = 6x - 24, \quad f_{xy} = 2y, \quad f_{yy} = 2x - 4.$$

- **Step II :** We now solve the equations $f_x = 0$, $f_y = 0$

$3x^2 + y^2 - 24x + 21 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$

and $2xy - 4y = 0 \Rightarrow 2y(x - 2) = 0 \Rightarrow x = 2$ or $y = 0.$

- When $x = 2$, (1) gives

$$12 + y^2 - 48 + 21 = 0$$

$$\therefore \quad y^2 - 15 = 0 \quad \therefore\ y^2 = 15 \quad \therefore\ y = \pm\sqrt{15}.$$

$\therefore$ The possible stationary points are $(2, \sqrt{15}), (2, -\sqrt{15})$

- When $y = 0$, (1) gives

$3x^2 - 24x + 21 = 0 \Rightarrow x^2 - 8x + 7 = 0$

$$\therefore \quad (x - 7)(x - 1) = 0 \quad \therefore \quad x = 1, 7.$$

The other possible stationary points are (1, 0), (7, 0).

- **Step III : (I) For x = 2, y = $\sqrt{15}$**

$$r = f_{xx} = 12 - 24 = -12, s = f_{xy} = 2\sqrt{15}, t = f_{yy} = 4 - 4 = 0$$

$$\therefore \quad rt - s^2 = 0 - 60 = -60 < 0.$$

∴ *f*(x, y) Is neither maximum nor minimum. we reject this pair

(II) For *x* = 2, y = $-\sqrt{15}$

$$r = f_{xx} = 12 - 24 = -12, s = f_{xy} = -2\sqrt{15}, t = f_{yy} = 4 - 4 = 0$$

$$\therefore \quad rt - s^2 = 0 - 60 = -60 < 0.$$

f(x, y) is neither maximum nor minimum. It is a saddle point. we reject this pair

- **(III) For *x* = 1, y = 0**

$$r = f_{xx} = 6 - 24 = -18, s = f_{xy} = 0, t = f_{yy} = 2 - 4 = -2$$

$\therefore \quad rt - s^2 = 36 - 0 = 36 > 0.$ And *r* = - 18 < 0 (negative),

∴ *f* has maxima at (1, 0).

∴ The maximum value = 1 + 0 -12- 0 + 21 + 10 = 20.

- **(iv) For *x* = 7, y = 0**

$$r = f_{xx} = 42 - 24 = 18, s = f_{xy} = 0, t = f_{yy} = 14 - 4 = 10$$

$\therefore \quad rt - s^2 = 180 - 0 = 180 > 0.$ And *r* = 18 > 0, (positive).

∴ (7, 0) is a minima.

∴ The minimum value = 343 + 0 – 588 – 0 + 147 + 10 = - 88.

Example:2

Find the stationary values of $\boldsymbol{x^3 + y^3 - 3a\,xy, a > 0}$

- **Sol.:** We have $f\,(x, y) = \boldsymbol{x^3 + y^3 - 3a\,xy}$
- **Step I :** $f_x = 3x^2 - 3ay,$ $f_y = 3y^2 - 3ax$

  $f_{xx} = 6x,$ $f_{xy} = -3a$ $f_{yy} = 6y$
- **Step II :** We now solve, $f_x = 0, \& \ f_y = 0.$

$$x^2 - ay = 0 \text{ and } y^2 - ax = 0$$

To eliminate y, we put $y = x^2/a$ in the second equation.

$\therefore \quad x^4 - a^3x = 0 \qquad \therefore \quad x(x^3 - a^3) = 0$

Hence, $x = 0$ or $x = a.$

- When $x = 0 \Rightarrow y = 0$ and when $x = a \Rightarrow y = a.$

$\therefore \quad (0, 0)$ and (a, a) are stationary points.

- **Step III :** (i) For $x = 0, y = 0,$

$r = f_{xx} = 0, \quad s = f_{xy} = -3a \; and \; t = f_{yy} = 0.$

$\therefore \; r\,t - s^2 = 0 - 9a^2 < 0$

$\therefore$ f *(x,* y) is neither maxima nor minima at (0,0).

- **(ii)** For $x = a, y = a,$

$r = f_{xx} = 6a, \quad s = f_{xy} = -3a, \quad t = f_{yy} = 6a$

$\therefore \; rt - s^2 = 36\,a^2 - 9a^2 = 27a^2 > 0$

$\therefore$ $f\,(x, y)$ is stationary at $x = a, y = a,$

And $r = f_{xx} = 6a > 0$, since $a > 0$

$\therefore$ $f\,(x, y)$ is minimum at $x = a, y = a.$

- Putting $x = a, y = a \; in \; x^3 + y^3 - 3a\,xy$ the minimum value of $f\,(x, y) = a^3 + a^3 - 3a^3 = -a^3$

Example: 3

Find the stationary values of $\sin x \cdot \sin y \cdot \sin (x+y)$.

❖ **Sol. :** We have $f(x,y) = \sin x \cdot \sin y \cdot \sin (x + y)$

Step I : $f_x = \sin y\,[\cos x \cdot \sin (x+y) + \sin x \cdot \cos (x+y)]$

$= \sin y \cdot \sin (2x + y)$

Similarly, $f_y = \sin x \cdot \sin (x + 2y)$

$f_{xx} = 2 \sin y \cdot \cos (2x + y)$

$f_{xy} = \cos y \cdot \sin (2x + y) + \sin y \cdot \cos (2x + y)$

$= \sin (2x + 2y)$

$f_{yy} = 2 \sin x \cdot \cos (x + 2y)$

❖ **Step II :** Now, we solve $f_x = 0$, $f_y = 0$.

$\therefore \quad \sin y \sin (2x + y) = 0$ and $\sin x \sin (x + 2y) = 0$

$\therefore,\ y = 0$ or $2x + y = 0$ or π $\quad x = 0$ or $x + 2y = 0$ or π

$$x = \frac{\pi}{3},\quad y = \frac{\pi}{3}$$

$\therefore$ (0, 0) and $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ are possible stationary points.

- **Step III : (i)** When $x = 0,\ y = 0;$

$$r = f_{xx} = 0, \qquad s = f_{xy} = 0, \qquad t = f_{yy} = 0$$

$$\therefore \qquad rt - s^2 = 0$$

$\therefore$ Our method fails. We reject this pair,

- **(ii)** When $x = \frac{\pi}{3},\ \ y = \frac{\pi}{3}$

$$r = f_{xx} = 2 \cdot \frac{\sqrt{3}}{2} \cdot (-1) = -\sqrt{3}, \qquad s = f_{xy} = -\frac{\sqrt{3}}{2}, t = f_{yy} = -\sqrt{3}$$

$$\therefore \qquad rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0 \text{ And } r = f_{xx} = -\sqrt{3} < 0$$

$\therefore$ $x = \frac{\pi}{3},\ \ y = \frac{\pi}{3}$ is a maxima.

- Maximum value = $\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Example:4

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- **Sol.:** let three parts of the 90 are x, y & z.

$$\therefore x + y + z = 90$$

Function to be maximized $f\ (x, y) = xy + yz + zx$

$$= xy + y\,(90 - x - y) + x\,(90 - x - y)$$
$$= 90x + 90y - xy - x^2 - y^2$$

$f_x = 90 - y - 2x,\qquad f_y = 90 - x - 2y$

$f_{xx} = -2,\qquad f_{xy} = -1\,,\qquad f_{yy} = -2$

- Solving $f_x = 0$ &$f_y = 0$

$\therefore\quad 2x + y = 90\quad \&\quad x + 2y = 90$

$\therefore 3x = 90\quad \therefore\ x = 30, \&\ \ y = 30,$

at $(30,30), r = -2 < 0, t = -2, S = -1$,

$rt - s^2 > 0.$

Function has maxima at $(30,30)$.

$z = 90 - x - y = 30$

$\therefore$ required three parts of the 90 are $30,30$ &30.

Example:5

A rectangular box with open top has capacity of 32 cubiccms. Find the dimensions of the box such that the material required is minimum.

- **Sol.:** v = xyz = 32

$$f(x,y) = xy + 2yz + 2zx = xy + \frac{64}{x} + \frac{64}{y}$$

- $f_x = y - \frac{64}{x^2}$, $\quad f_y = x - \frac{64}{y^2}$

$f_{xx} = \frac{64*2}{x^3}$, $\quad f_{xy} = 1$, $\quad f_{yy} = \frac{64*2}{y^3}$

- If $f_x = 0 \quad \therefore \quad y - \frac{64}{x^2} = 0 \quad \therefore 64 = x^2 y \therefore \; y = \frac{64}{x^2}$ ......(1)

& $f_y = 0 \quad \therefore \quad x - \frac{64}{y^2} = 0 \quad \therefore \quad 64 = xy^2$ ....(2)

$$\therefore \quad 64 = x \cdot \frac{(64)^2}{x^4}$$

$$\therefore x^3 = 64 \quad \therefore \quad x = 4$$

For $x = 4$, $\quad y = \frac{64}{x^2} = 4$

For (4,4), $rt - s^2 > 0$, & $r > 0$, So, f has minima at x=4 & y=4 and $z = \frac{32}{xy} = 2$.

Example:6

Divide 24 into three parts such that the product of the first, square of the second and cube of the third is maximum.

- **Sol. :**let three parts of the 24 are x, y & z.

$$x + y + z = 24$$

$f(x, y) = x.y^2 \cdot z^3 = x \cdot y^2\,(24 - x - y)^3$ (try with replacing x)

$$f_x = y^2\,[1(24 - x - y)^3 + x \cdot 3(24 - x - y)^2 \cdot (-1)\}$$

$$f_y = x\,[2y(24 - x - y)^3 + y^2 \cdot 3(24 - x - y)^2 \cdot (-1)\}$$

- $f_x = 0 \Rightarrow y^2\,(24 - x - y)^2(24 - x - y - 3x) = 0$

$$f_y = 0 \Rightarrow xy\,(24 - x - y)^2\,[\,2\,(24 - x - y) - 3y] = 0$$

$$\therefore \quad x + y = 24, 4x + y = 24 \text{ and } 2x + 5y = 48$$

$$\therefore \quad \mathrm{y} = 8, \mathrm{x} = 4$$

- Find f_{xx}, f_{xy}, f_{yy} & hence r, t, s (HW)
- Check, $rt - s^2 > 0$, & $r < 0$ (HW)

Hence the Soln: $\mathrm{x} = 4, \mathrm{y} = 8, \mathrm{z} = 12.$

Jacobian

❖ If u & v are functions of two independent variables x & y, then the Jacobian of u, v with respect to x, y is denoted and defined by

$$\mathrm{J}\left(\frac{u,v}{x,y}\right)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}$$

❖ Similarly, If u, v &w are functions of three independent variables x, y & z, then the Jacobian of u, v, w with respect to x, y, z is denoted and defined by

$$\mathrm{J}\left(\frac{u,v,w}{x,y,z}\right)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

❖ If $\frac{\partial(u,v,w)}{\partial(x,y,z)}=J$ & $\frac{\partial(x,y,z)}{\partial(u,v,w)}=J'$ then $JJ'=1$.

Example:7

- If $u = \frac{x+y}{1-xy}, v = tan^{-1}x + tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2} \quad \& \quad \frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \quad \& \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

- $$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

Example:8

- **If** $u = x(1-y), v = xy(1-z), \; w = xyz,$ **find** $\frac{\partial(u,v,w)}{\partial(x,y,z)}$
- **Sol. :** $u = x - xy, \quad V = xy - xyz, \quad w = xyz$
- $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1-y & -x & 0 \\ y(1-z) & x(1-z) & -xy \\ yz & zx & xy \end{vmatrix}$$

$$= (1-y)[(x - xz)xy + xy\,zx] + x\,[(y-yz)xy + xy\,yz]$$

$$= (1-y)[x^2y - x^2yz + x^2yz] + x[xy^2 - xy^2z + xy^2z]$$

$$= (1-y)(x^2y) + x\,(xy^2)$$

$$= x^2y - x^2y^2 + x^2y^2 = x^2y$$

Example:9

- **If $x = rsin\theta cos\emptyset$, $y = rsin\theta sin\emptyset$ and $z = rcos\theta$ then evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)}$ and $\frac{\partial(r,\theta,\emptyset)}{\partial(x,y,z)}$.**

- $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix}$

$$= \begin{vmatrix} sin\theta cos\emptyset & rcos\theta cos\emptyset & -rsin\theta sin\emptyset \\ sin\theta sin\emptyset & rcos\theta sin\emptyset & rsin\theta cos\emptyset \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

$=r^2 sin\theta$

Since $JJ' = 1 \therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} \cdot \frac{\partial(r,\theta,\emptyset)}{\partial(x,y,z)} = 1$

$$\therefore \frac{\partial(r,\theta,\emptyset)}{\partial(x,y,z)} = \frac{1}{J} = \frac{1}{r^2 sin\theta}$$

Example:10

- If $x = e^u \cos v, \;\; y = e^u \sin v$, prove that $JJ' = 1$

- **Sol.:** $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$

$$= e^{2u} cos^2 v + e^{2u} sin^2 v = e^{2u}$$

$$Now, x^2 + y^2 = e^{2u} \; and \; \frac{x}{y} = \tan v \qquad \therefore \qquad 2u = \log(x^2 + y^2)$$

$$\therefore \quad u = \frac{1}{2}\log(x^2 + y^2) \qquad \text{and} \quad v = tan^{-1}\frac{y}{x}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} & \frac{1}{2} \cdot \frac{2y}{x^2 + y^2} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{e^{2u}}$$

$$\therefore \quad JJ' = \frac{\partial(x, \; y)}{\partial(u,v)} \cdot \frac{\partial(u, \; v)}{\partial(x,y)} = e^{2u} \cdot \frac{1}{e^{2u}} = 1$$

Example:11

❖ **If $x = u(1-v), y = uv$, prove that $JJ' = 1$**

❖ **Sol.:** $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$$= u - uv + uv = u = x + y \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, $\quad x = u - uv, \qquad y = uv$

$\therefore x + y = u$ and $v = \frac{y}{u}$ ie. $v = \frac{y}{x+y}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{(x+y)\cdot 1 - y\cdot 1}{(x+y)^2} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By (1) & (2) $JJ' = \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = (x+y) \times \frac{1}{x+y} = 1$

Example:12

- **If $x = uv, y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$**
- **Sol.:**

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{(u-v)1-(u+v)1}{(u-v)^2} & \frac{(u-v)1+(u+v)}{(u-v)^2} \end{vmatrix}$$

$$\begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2v}{(u-v)^2} \end{vmatrix} = \frac{2v}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}$$

$$\therefore \quad J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{(u-v)^2}{4uv}$$

- Since $(y^2 - 1) = \frac{(u+v)^2}{(u-v)^2} - 1 = \frac{4uv}{(u-v)^2}$
- $\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{y^2-1}$