Homogeneous Functions

Tuesday, January 25, 2022
2:49 PM

Def^n: $u = f(x, y, z)$ is said to be homogeneous function of deg n If, by replacing x by xt, y by yt, z by zt

Then $\boxed{f(xt, yt, zt) = t^n f(x, y, z)}$

Explanation: 1) $f(x, y) = ax^2 + 2hxy + by^2$

Consider $f(xt, yt) = a(xt)^2 + 2h(xt)(yt) + b(yt)^2$

$= t^2\left[ax^2 + 2hxy + by^2\right]$

$$\boxed{f(xt, yt) = t^{2} f(x, y)}$$

This function is homogeneous of deg 2

---

2) $f(x,y,z) = \dfrac{x^2}{y^2} + \log\left(\dfrac{xz^2}{y^2 x}\right) + \sin^{-1}\left(\dfrac{y^2 + xy}{z^2}\right)$

Consider $f(xt, yt, zt) = \dfrac{x^2\cancel{t^2}}{y^2\cancel{t^2}} + \log\left(\dfrac{x\cancel{t}\, z^2\cancel{t^2}}{y^2\cancel{t^2}\, x\cancel{t}}\right) + \sin^{-1}\left(\dfrac{y^2\cancel{t^2} + xt\,y\cancel{t}}{z^2\cancel{t^2}}\right)$

$$\boxed{f(xt, yt, zt) = t^0 f(x, y, z)}$$

f is Homogeneous with deg zero.

If $\log\left(\dfrac{xz^2}{yx}\right) = \log\left(\dfrac{xt\, z^2t^2}{yt\, xt}\right) = \log\left(t\left(\dfrac{xz^2}{yx}\right)\right)$

---

3) $f(x, y) = \dfrac{x^2 + xy}{\sqrt{x} + \sqrt{y}}$

consider $f(xt, yt) = \dfrac{x^2t^2 + xtyt}{\ldots} = \dfrac{t^2\left[(x^2 + xy)\right.}{\ldots}$

consider $f(xt, yt) = \dfrac{x^2t^2 + xtyt}{\sqrt{xt} + \sqrt{y}\sqrt{t}} = \dfrac{t}{\sqrt{t}}\left(\dfrac{(x+xy)}{(\sqrt{x}+\sqrt{y})}\right)$

$$f(xt, yt) = t^{3/2}\left(f(x,y)\right)$$

Homogeneous with deg $\frac{3}{2}$

$$f(x,y) = \boxed{\tan^{-1}}\left(\frac{3y^2 + 7xy}{xy^2 - 3y^3}\right) \qquad t^{-1}$$

$$u = \tan^{-1}\left(\frac{t^2}{t^3}\left(\quad\right)\right) \qquad \frac{1}{t}$$

$\boxed{\tan u}$ ~~Not homogeneous~~

Euler's Theorem for homogeneous function.

If $u = f(x,y)$, u is homogeneous of deg n Then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu} \quad \text{—} (1)$$

This can be generalized for $u = f(x,y,z)$

i.e. $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu}$ — (2)

$u = f(x,y)$ & u is homogeneous of deg n

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u} \quad \text{—} (3)$$

$u = f(x,y,z)$ can be generalised.

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + 2xz\frac{\partial^2 u}{\partial x\partial z} + 2yz\frac{\partial^2 u}{\partial y\partial z}$$

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xz\frac{\partial^2 u}{\partial x\partial z}+2yz\frac{\partial^2 u}{\partial y\partial z} = n(n-1)u}$$

1) If $u=\dfrac{\sqrt{x}+\sqrt{y}}{x+y}$ Then Find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$

Sol^n: Let us show u is homogeneous.

$$f(xt,yt) = \frac{\sqrt{xt}+\sqrt{yt}}{xt+yt} = \frac{\sqrt{t}}{t}\left(\frac{\sqrt{x}+\sqrt{y}}{x+y}\right) = t^{-1/2} f(x,y)$$

$\therefore$ u is homogeneous of deg $n=-\frac{1}{2}$

Then By Euler's Theorem,

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu = -\frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{x+y}\right)}$$

2) If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\cos^{-1}\left(\dfrac{y}{z}\right)+\log\left(\dfrac{z}{x}\right)$

Then Find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$

Sol^n:

$$f(xt,yt,zt) = \sin^{-1}\left(\frac{x\cancel{t}}{y\cancel{t}}\right)+\cos^{-1}\left(\frac{y\cancel{t}}{z\cancel{t}}\right)+\log\left(\frac{z\cancel{t}}{x\cancel{t}}\right)$$

$$=1f(x,y,z) = t^0 f(x,y,z)$$

$\therefore$ u is homogeneous f^n of deg zero

$\therefore$ By Euler's Th^m, $\boxed{x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z} = nu = 0}$

3) If $u=\dfrac{x^2}{xy}+\dfrac{2y}{x}$ Then Find $\dfrac{\partial u/\partial x}{\partial u/\partial y}$

3) If $u = \frac{x^2}{xy} + \frac{2y}{x}$ ... $\frac{\quad}{\partial u/\partial y}$

Sol^n: $f(xt, yt) = \frac{x^2 \cancel{t^2}}{x\cancel{t}\, y\cancel{t}} + \frac{2y\cancel{t}}{x\cancel{t}} = f(x,y) = t^0 f(x,y)$

u is homogeneous f^n of deg zero.

$\therefore$ By Euler's Th^m, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0$

$$x\frac{\partial u}{\partial x} = -y\frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{-y}{x}}$$

---

4) If $u = \frac{f(\theta)}{r}$ & $x = r\cos\theta$, $y = r\sin\theta$ Then

Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -u$

Sol^n: Since differentiation is w.r.to x & y

$\therefore$ we will express r & θ in terms of x & y

$\therefore$ $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$u = \frac{f(\theta)}{r} = \frac{f\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}$$ now put $x = xt$, $y = yt$

$$\frac{f\left(\tan^{-1}\left(\frac{y\cancel{t}}{x\cancel{t}}\right)\right)}{\sqrt{x^2t^2+y^2t^2}} = \frac{1}{t}\frac{f(\theta)}{r} = t^{-1}$$

$\therefore$ u is homogeneous of deg $\boxed{n = -1}$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -u}$$

Type I

1) Verify Euler's theorem for $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$

Sol^n: (I) $\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}$, $\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}}$, $\frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}}$

consider, $x\frac{\partial u}{\partial x} = \frac{\sqrt{x}}{2}$, $y\frac{\partial u}{\partial y} = \frac{\sqrt{y}}{2}$, $z\frac{\partial u}{\partial z} = \frac{\sqrt{z}}{2}$

add all these,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}} = \frac{\sqrt{x}}{2} + \frac{\sqrt{y}}{2} + \frac{\sqrt{z}}{2} = \frac{1}{2}(\sqrt{x}+\sqrt{y}+\sqrt{z})$$

$$= \frac{1}{2}u \quad \text{(I)}$$

(II) Consider

$f(xt, yt) = \sqrt{xt} + \sqrt{yt} + \sqrt{zt} = \sqrt{t}\left[\sqrt{x}+\sqrt{y}+\sqrt{z}\right]$

$= t^{1/2} f(x,y)$

$\therefore u$ is homogeneous of deg $n = \frac{1}{2}$

Then By Euler's Theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = \frac{1}{2}u$ — (II)

$\therefore$ from (I) & (II), Euler's Theorem is verified.

HW: Verify Euler's theorem for

2) $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u$

---

5) If $u = \frac{x^2y^3z}{x^2+y^2+z^2} + \sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right)$ Then Find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Sol^n: Check that u is not homogeneous.

Sol^n: Check that u is not homogeneous.

But $u = v + w$, where $v = \dfrac{x^2y^3z}{x^2+y^2+z^2}$, $w = \sin^{-1}\left(\dfrac{xy+yz}{y^2+z^2}\right)$

Consider v, $f(xt, yt, zt) = \dfrac{x^2t^2y^3t^3zt}{x^2t^2+y^2t^2+z^2t^2} = \dfrac{t^6}{t^2}\left(\dfrac{x^2y^3z}{x^2+y^2+z^2}\right)$

$= t^4 f(x,y,z)$

v is homogeneous of deg $n_1 = 4$

$\therefore$ By Euler's Th^m, $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = n_1 v = 4v$ —— (1)

Consider w, $f(xt, yt, zt) =$ Complete

w is homogeneous of deg $n_2 = 0$

$\therefore$ By Euler's Th^m, $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = n_2 w = 0$ —— (2)

(1) + (2)

$\therefore$ $x\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\right) + y\left(\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial y}\right) = 4v + 0$

$x\dfrac{\partial}{\partial x}(v+w) + y\dfrac{\partial}{\partial y}(v+w) = 4v$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4\,\frac{x^2y^3z}{x^2+y^2+z^2}}$$

H.W.:

$u = \log(x^2+y^2) + \left(\dfrac{x^2+y^2}{\sqrt{2xy}}\right) - 2\log(x+y)$

u is not homogeneous $\quad || \quad \log(x^2+y^2) - \log(x+y)^2$

u is not homogeneous

u is not piecewise homogeneous

$$\log(x^2+y^2) - \log(x+y)^2$$

$$\log\left(\frac{x^2+y^2}{(x+y)^2}\right)$$

$$u = \frac{x^2+y^2}{\sqrt{2xy}} + \log\left(\frac{x^2+y^2}{(x+y)^2}\right)$$

---

1) If $u = \dfrac{x^3y + y^3x}{3x}$ Then Find $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$

Sol^n: $f(xt, yt) = \dfrac{x^3t^3yt + y^3t^3xt}{3xt} = \dfrac{t^4}{t}\left[f(x,y)\right]$

$$= t^3 f(x,y)$$

$\therefore$ u is homogeneous of deg $n = 3$

By Euler's Theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$= 3(2)u = 6u$$

$$= 6\left[\frac{x^3y + y^3x}{3x}\right]$$

---

2) If $\boxed{u = \dfrac{x^2+xy}{y\sqrt{x}} + \dfrac{1}{x^7}\sin^{-1}\left(\dfrac{x^2+xy}{y^2-x^2}\right)}$ then

Find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

at $x = 1$ & $y = 2$