MATRIX THEORY: RANK OF MATRIX

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FY BTECH SEM-I

MODULE-2

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ELEMENTARY TRANFORMATIONS

(i) Interchanging any two rows or any two columns:

R_{ij} denotes the interchange of i^{th} and j^{th} rows and

C_{ij} denotes the interchange of i^{th} and j^{th} columns.

(ii) Multiplication of each element of i^{th} row by non zero k, i. e. kR_i

Multiplication of each element of i^{th} column by non zero k, kC_i

(iii) Adding a non zero multiple of any row (column) to some other row (column)

$(R_i + kR_j)$ or $(C_i + kC_j)$.

These are only valid transformations.

Two matrices A and B are said to be **Equivalent Matrices** if the matrix B is obtained by performing elementary transformations on the matrix A.

Denoted by, $A \sim B$ (A is equivalent to B).

RANK OF A MATRIX

- **Sub-matrix of order r** – If we select any r rows and r columns in Given m X n matrix then a matrix formed by these r rows and r columns is called a square sub-matrix of order r.
- **Determinant of this square sub-matrix of order r is called Minor of order r**
- **Definition of rank of 'A':** A number 'r' is said to be the rank of matrix A, if
- **(i)** There exists at least one sub – matrix of A of order r whose determinant is non – zero
- **(ii)** Every sub – matrix of A with order greater than r whose determinant, if it exists, should be zero.
- **In short,** the rank of matrix is the order of any highest non – vanishing (Non-zero) minor.
- The rank 'r' of a matrix A is denoted by $\rho(A)$.

RANK OF A MATRIX

Properties

(i) If A is a matrix of order $m \times n$, then $0 \leq \rho(A) \leq$min (m, n)

(ii) If A is a nonzero square matrix of order n, then $1 \leq \rho(A) \leq n$.

(iii) The rank of a null matrix is always zero.

(iv) Rank of a non – singular matrix is always equal to its order.

i.e. If $|A| \neq 0 \; then \;\; \rho(A) = n$

(v) Rank of a matrix is always unique.

RANK OF A MATRIX

Properties

(vi) $\rho(A) = \rho(A')$

(vii) $\rho(AB) \leq \rho(A)\ and\ \ \rho(AB) \leq \rho(B)$

(viii) Rank is invariant under elementary transformations.

i.e. If $A \sim B\ then\ \ \rho(A) = \rho(B)$

(ix) Rank of A = Rank of (kA), where k is any non zero scalar

(x) If $A_{n \times n}$ is non – singular i.e., $|A| \neq 0$ then rank of $A\ = \ n$ and rank of $A^2\ (Or\ A^k) = n$

Since $|A^2| = |A.A| = |A|.|A| \neq 0$

Examples
Determine the ranks of the following matrices

- 1) $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$
- Since it is a square matrix, first we will find $|A|$
- We have $|A| = 1(6-8) - 2(4-0) + 3(4-0)$
- $= -2 - 8 + 12 = 2 \neq 0$
- Thus A is non – singular matrix,
- i.e., $|A|$ is the highest order non – vanishing minor of order 3.
- Hence rank of A is 3.

Examples
Determine the ranks of the following matrices

- 2) $Let\ A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$
- Since it is a square matrix, first we will find $|A|$
- We have $|A| = 1(28 + 2) - (-2)(-14 - 1) + 3(-4 + 4) = 0$
- Here the only minor of order 3 is zero.
- So now we will find minors of order 2.
- Consider $\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0,$
- $but\ \begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = -10 \neq 0$
- i.e., at least one minor of order 2 is non – zero.
- Hence rank of A is 2.

Examples
Determine the ranks of the following matrices

- 3) $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$
- Since we have $|A| = 0$ i.e., the minor of order 3 is zero.
- All minors of order 2 are also zero.
- Minor of order one is not zero.
- Hence rank of A is 1.
- **Observation:** Here, observe that all rows are identical, so when all the rows of a given matrix are identical then rank of that matrix is always 1. (This problem can also solved by row reduction method as 2nd and 3rd rows will become zero)

Examples
Determine the ranks of the following matrices

- 4) $Let\ A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 1 & -1 & 0 & 3 \\ 3 & 5 & 1 & 6 \end{bmatrix}_{3\times4}$
- Here, A is the matrix of order 3×4.
- Therefore $1 \leq \rho(A) \leq \min(3,4)$
- So rank A can be maximum 3.
- Now, consider the 3×3 minor $\begin{vmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \\ 3 & 5 & 1 \end{vmatrix}$
- $= 2(-1-0) - 4(1-0) + 3(5+3)$
- $= -2 - 4 + 24 = 18 \neq 0$
- Hence rank of A is 3.

Finding rank by row Echelon method

- We know If $A \sim B$, then A and B have same rank.
- consider $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ whose rank is 2
- Now, we will obtain an equivalent matrix B of A by performing elementary transformations.
- Applying $R_2 + 2R_1$ and $R_3 + R_1$,
- we get $A \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix}$
- Again, applying $R_3 - 2R_2$,
- we get $A \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$
- $Let\ B = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ we have $|B| = 0$
- Consider the minor $\begin{vmatrix} -2 & 3 \\ 0 & 5 \end{vmatrix} = -10 \neq 0$
- Therefore, the rank of B is 2.
- Hence, $A \sim B$, and the rank of A = the rank of B.

ECHELON FORM OF A MATRIX

- **Definition:** If a matrix A is reduced to a matrix B by using elementary row transformations alone, then B is said to be row equivalent to A.
- **Defn:** The **Echelon form** or **Canonical form** of a matrix A is a row equivalent matrix of rank 'r' in which
- **(a)** One or more elements of each of the first r rows are non – zero while all other rows have only zero elements, (i.e all zero rows, if any, are placed at the bottom of the matrix so that the first r rows form an upper triangular matrix).
- **(b)** The number of zero before the first non – zero element in a row is less than the number of such zeros in the next row.
- **In short,** by performing only row transformations, a given matrix that is reduced to an **upper triangular form** is called its **Echelon form**.
- **Note:** Rank of a given matrix is equal to the number of non – zero rows in the Echelon form.

ECHELON FORM OF A MATRIX

- **For example**, the matrix $\begin{bmatrix} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ of order 4×5 is in the Echelon form.
- **(a)** First 2 rows contain at least one non – zero elements while other (i.e 3rd and 4th) rows have only zero elements.
- **(b)** The number of zeros before the first non – zero element in the first row is one while the number of zeros before the first non – zero element in the second row is two.
- Further, there are two non – zero rows in this Echelon form. Hence rank of the matrix is 2.

Example

- Reduce the following matrix to Echelon form and hence find it's rank.
- $A = \begin{bmatrix} 0 & 2 & -6 & -2 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$
- By R_{12}
- $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -6 & -2 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$
- By $R_3 - 3R_1, R_4 - R_1$
- $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -6 & -2 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$
- By $R_3 - \frac{1}{2}R_2\,, R_4 - \frac{1}{2}R_2$
- $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- This is the required echelon form.
- Number of non zero rows is 2.
- $\rho(A) = 2$

EXAMPLES
Find the ranks of the following matrices

- **(i)** $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$
- $R_4 - (R_1 + R_3)$, $\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $R_3 - (R_1 + R_2)$, $\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $\therefore$ Minor of order 4 is zero. All minors of order 3 are zero
- Consider the minor of order two $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$ Hence, the rank of matrix is 2.

EXAMPLES
Find the ranks of the following matrices

- **(ii)** $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$$\left.\begin{matrix} R_4 - R_1 \\ R_3 - R_1 \\ R_2 - R_1 \end{matrix}\right\} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 7 & 7 & 7 & 7 \end{bmatrix}$$

- $$\left.\begin{matrix} R_4 - 7R_2 \\ R_3 - 2R_2 \end{matrix}\right\} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $\therefore$ Minor of order 4 is zero. All minors of order 3 are zero
- Consider the minor of order two $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 3 = -1 \neq 0$ Hence, the rank of matrix is 2.

NORMAL FORM OF A MATRIX

- **Definition:** By performing elementary row and column transformations, every non – zero matrix can be reduced to one of the four forms, called the normal form of A:
- **(i)** $[\mathrm{I_r}]$ **(ii)** $[\mathrm{I_r} \quad \mathrm{O}]$ **(iii)** $\begin{bmatrix} \mathrm{I_r} \\ \mathrm{O} \end{bmatrix}$ **(iv)** $\begin{bmatrix} \mathrm{I_r} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{bmatrix}$
- **Note:** Rank of A = Rank of the normal form of $A = r$.

NORMAL FORM OF A MATRIX

- **Method to Reduce a Given Matrix to its Normal Form by Applying Elementary Transformations:**
- **Step 1:** Reduce the first diagonal element a_{11}, which is called a leading element (or a pivot), to 1 by applying any (row or column) transformation
- **Step 2:** Apply row – transformation to reduce all other elements in first column to zero.
- **Step: 3:** Apply column – transformation to reduce all other elements in first row to zero.
- **Step 4:** Reduce the second diagonal element a_{22}, which is then called the leading element, to 1 by applying any (row or column) transformation without disturbing the elements of the first row and first column.

NORMAL FORM OF A MATRIX

- **Step 5:** Applying row – transformation clear off all other non – zero elements of the second column and reduce them to zero without disturbing the first row.
- **Step 6:** Applying column – transformation clear off all other non – zero elements of the second row and reduce them to zero without disturbing the first column.
- Continuing the above procedure with the successive rows and columns, we can reduce a given matrix to its normal form.
- **Note:** Application of elementary transformation on any matrix A may differ but rank of A is unique.

EXAMPLES

- Reduce the following matrices to their normal form and hence obtain their ranks.
- **(i)** $\begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$
- $R_1 - R_2 \quad \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$
- $\left.\begin{matrix} R_2 - 3R_1 \\ R_3 - 7R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 7 & -4 & -6 \\ 0 & 14 & -8 & -12 \end{bmatrix}$
- $\left.\begin{matrix} C_2 + C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & -4 & -6 \\ 0 & 14 & -8 & -12 \end{bmatrix}$
- $\frac{C_2}{7} \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 2 & -8 & -12 \end{bmatrix}$
- $R_3 - 2R_1 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $\left.\begin{matrix} C_3 + 4C_2 \\ C_4 + 6C_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$
- Hence, the rank of matrix is 2.

EXAMPLES

- **(ii)** $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$
- R_{12} $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$
- $\left.\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 6R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 03 & 07 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$
- $\left.\begin{matrix} C_2 + C_1 \\ C_3 + 2C_1 \\ C_4 + 4C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$
- $R_2 - R_3$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$
- $\left.\begin{matrix} R_3 - 4R_2 \\ R_4 - 9R_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$
- $\left.\begin{matrix} C_3 + 6C_2 \\ C_4 + 3C_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$
- $\frac{C_3}{33}$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 2 & 44 \end{bmatrix}$

EXAMPLES

- $R_4 - 2R_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $C_4 - 22C_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$
- Hence, the rank of matrix is 3.

EXAMPLES

- If A and B are as given below, find the rank of A by reducing it to the normal form. Find 3A – B,hence or otherwise , show that $3A^2 - AB = 2A$ also find the rank of $3A^2 - AB$.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$

- **Solution:** $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix}$
- $$\left.\begin{matrix} R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$\left.\begin{matrix} C_2 - 2C_1 \\ C_3 - C_1 \\ C_4 - 2C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$R_3 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$\frac{C_2}{2} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$\left.\begin{matrix} C_3 - C_2 \\ C_4 - C_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

EXAMPLES

- $\therefore \rho(A) = 2.$
- $$3A - B = 3\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$
- $$= \begin{bmatrix} 3 & 6 & 3 & 6 \\ 0 & 6 & 3 & 3 \\ 6 & 18 & 9 & 15 \\ 6 & 12 & 6 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$
- $$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 2I$$
- $\therefore 3A^2 - AB = A(3A - B) = A(2I) = 2A$
- Since $\rho(A) = \rho(2A) = \rho(3A^2 - AB)$
- Hence $\rho(3A^2 - AB) = 2$

EXAMPLES

- Find the values of P for which the following matrix A will have **(i)** rank 1 **(ii)** rank 2 **(iii)** rank 3,
- where $A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$
- **Solution:** Let us first find the determinant of A.
- $|A| = \begin{vmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{vmatrix}$
- $= 3(9 - P^2) - P(3P - P^2) + P(P^2 - 3P)$
- $= 3(3 - P)(3 + P) - P^2(3 - P) + P^2(P - 3)$
- $= (3 - P)[3(3 + P) - P^2 - P^2]$
- $= (3 - P)[9 + 3P - 2P^2]$
- $= (3 - P)^2(3 + 2P)$
- If $|A| = 0$, i.e if $P = 3 \; or \; -3/2$,
- then the rank of A is either 1 or 2
- Consider, if $P = 3$, then $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ all minors of order 2 are zero.
- Hence rank of A is 1, when $P = 3$, …………(i)
- If $P = -3/2$, then $A = \begin{bmatrix} 3 & -3/2 & -3/2 \\ -3/2 & 3 & -3/2 \\ -3/2 & -3/2 & 3 \end{bmatrix}$
- Consider the minor of order of 2,
- $\begin{vmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & 3 \end{vmatrix} = 3 - \frac{9}{4} = \frac{3}{4} \neq 0$
- Hence rank of A is 2, when $P = -3/2$ ……(ii)
- For rank 3, $|A| \neq 0$. When P can take any value other than $3 \; or \; -3/2$ ……………(iii)

EXAMPLES

- If $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$ is the given square matrix of order 3, find the values of k for which rank of A is less than 3. Also find the ranks for those values of k.
- **Solution:** $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$
- R_{12}
- $\sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$
- $\left. \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \right\}$
- $\sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}$ ....(i)
- For the matrix A to be of rank less than 3, we must have $|A| = 0$
- i.e., $(k-8)(-k+2) - (-5k)(k-2) = 0$
- i.e., $-k^2 + 10k - 16 + 5k^2 - 10k = 0$
- i.e., $4k^2 - 16 = 0$
- i.e $k^2 = 4$
- i.e., $k = \pm 2$
- Now three cases arise.

EXAMPLES

- **Case (i)** If $k \neq \pm 2$ then A has $rank = 3.$
- **Case (ii)** If $k = 2$, then (i) $\Longrightarrow$
- $A \sim \begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix}$
- $\left.\frac{C_2}{6}, \frac{C_3}{10}\right\} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
- $\left.\begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
- $(-1)R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $C_3 - C_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$
- $\therefore \rho(A) = 2$

- **Case (iii)** If $k = -2$, then (i) $\Longrightarrow$

  $A \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix}$
- $\left.\frac{R_2}{-10}, \frac{R_3}{-4}\right\} \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
- $R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
- $\left.\begin{matrix} C_2 - 2C_1 \\ C_3 + 6C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
- $C_3 + C_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

  Hence $\rho(A) = 2$

REDUCTION OF A MATRIX A TO NORMAL FORM PAQ

- **Theorem:** If A is a matrix of rank r, then there exist non – singular matrices P and Q such that PAQ is in the normal form i.e $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$
- To obtain the matrices P and Q use the following procedure.
- **Working Rule:**
- **1.** If A an $m \times n$ matrix, write
- $$A = I_m A I_n$$
- **2.** Apply row transformations of A on l.h.s. and the same row transformations on the pre-factor I_m.
- **3.** Apply column transformations on A on l.h.s. and the same column transformations on the post-factor I_n.
- So that, A on the l.h.s is reduced to normal form.
- **Remark:**
- **(i)** No transformations are applied on A on the r.h.s.
- **(ii)** The matrices P and Q thus obtained are not unique. They depend upon the transformations used.

EXAMPLES

- Find non – singular matrices P and Q such that PAQ is in normal form, Hence obtain rank of A where A is $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$
- Since A is the matrix of order 3×4, we write $A = I_3 . A_{3\times4} . I_4$
- Thus $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- To find non – singular matrices P and Q, we reduce the matrix A on the left hand side to normal form by applying suitable elementary transformations. Every row operation will also be applied to the pre – factor of the product on the right hand side and every column operation to the post factor.
- Applying $R_2 - 2R_1,\ R_3 - 3\ R_1$ $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

EXAMPLES

- Applying $C_2 - 2C_1,\ C_3 - 3C_1,\ C_4 + 2C_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Applying $R_3 - R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Applying $\frac{C_2}{-6}, \frac{C_3}{-5}, \frac{C_4}{7}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 3/5 & 2/7 \\ 0 & -1/6 & 0 & 0 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$
- Applying $C_3 - C_2,\ C_4 - C_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$

EXAMPLES

- Thus, $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ Where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ $and\ Q = \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$

- Hence rank of A is 2.

EXAMPLES

- Find non – singular matrices P and Q such that PAQ is in normal form. Hence find
- **(i)** rank of A, **(ii)** A^{-1} , where A is $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$
- **Solution:** Since A is a square matrix of order 4, we write $A = I_4 . A . I_4$
- i.e., $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Applying $R_2 - 2R_1, R_3 + R_1, R_4 - 2R_1$ $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

EXAMPLES

Applying $C_2 - 2C_1, C_3 + 2C_1, C_4 - 3C_1$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Applying $R_3 + R_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Applying $\frac{R_4}{3}$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 0 & 0 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Applying C_{34}, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 0 & 0 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

EXAMPLES

- Thus, we have $[I_4] = PAQ$ is the required normal form.
- Where $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 0 & 0 & 1/3 \end{bmatrix}$ $and\ Q = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- Hence rank of A is 4. Since $|A| \neq 0$, therefore A^{-1} exists

EXAMPLES

- To find A^{-1}, we have $PAQ = I$
- $\therefore (PAQ)^{-1} = I^{-1}$
- $\therefore Q^{-1}A^{-1}P^{-1} = I \quad \{\because I^{-1} = I\}$
- $\therefore QQ^{-1}A^{-1}P^{-1} = QI$
- $\therefore I\, A^{-1}P^{-1} = Q$
- $\therefore A^{-1}P^{-1}P = QP$
- $\therefore A^{-1}I = QP$
- $\therefore A^{-1} = QP$

$$= \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 0 & 0 & 1/3 \end{bmatrix}$$

- $= \begin{bmatrix} \frac{20}{3} & -5 & -3 & \frac{2}{3} \\ -2 & 1 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ -1 & 1 & 1 & 0 \end{bmatrix}$