SUCCESSIVE DIFFERENTIATION
PRACTICE PROBLEMS

Type – I

Find the n^{th} derivatives of

1. $\frac{x}{x^2-a^2}$ **2.** $\frac{1}{x^4-a^4}$ **3.** $\frac{x^4}{(x-1)(x-2)}$

4. $\frac{x^2}{1-x^4}$ **5.** $\frac{1}{6x^2-5x+1}$ **6.** $\frac{x+1}{x^2-4}$

7. $\frac{x}{x^3-6x^2+11x-6}$ **8.** $\frac{x}{x^2+9}$ **9.** $\frac{x}{(x+1)^5}$

10. $\frac{1}{(3x-2)(x-3)^2}$ **11.** If $y = x\log\frac{(x-1)}{(x+1)}$, prove that $y_n = (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

ANSWERS

1. $\frac{(-1)^n n!}{2}\left[\frac{1}{(x+a)^{n+1}} + \frac{1}{(x-a)^{n+1}}\right]$

2. $\frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right] - \frac{(-1)^n n!}{4a^3 i}\left[\frac{1}{(x-a\,i)^{n+1}} - \frac{1}{(x+a\,i)^{n+1}}\right]$

3. $y = x^2 + 3x + 7 + \frac{16}{(x-2)} - \frac{1}{(x-1)}$; $y_n = (-1)^n n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right] for\ n \geq 3.$

4. $\frac{(-1)^n n!}{4}\left[\frac{1}{(1+x)^{n+1}} + \frac{1}{(1-x)^{n+1}}\right] - \frac{(-1)^n n!}{4i}\left[\frac{1}{(x-i)^{n+1}} - \frac{1}{(x+i)^{n+1}}\right]$

5. $(-1)^n n!\left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right]$ **6.** $(-1)^n n!\left[\frac{3}{4}.\frac{1}{(x-2)^{n+1}} + \frac{1}{4}.\frac{1}{(x+2)^{n+1}}\right]$

7. $(-1)n^n!\left[\frac{1}{2(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{3}{2(x-3)^{n+1}}\right]$ **8.** $\frac{(-1)^n n!}{2}\left[\frac{1}{(x+3i)^{n+1}} + \frac{1}{(x-3i)^{n+1}}\right]$

9. $\frac{(-1)^n(n+3)!}{4!(x+1)^{n+5}}(4x-n)$

10. $y = \frac{9}{49}.\frac{1}{3x-2} - \frac{3}{49}.\frac{1}{x-3} + \frac{1}{7}.\frac{1}{(x-3)^2}$, $y_n = \frac{9}{49}\frac{(-1)^n.n!3^n}{(3x-2)^{n+1}} - \frac{3}{49}.\frac{(-1)^n.n!}{(x-3)^{n+1}} + \frac{1}{7}\frac{(-1)^n.(n+1)!}{(x-3)^{n+2}}$

Type – II

Find n^{th} derivatives of the following

1. If $y = \sin r\,x + \cos r\,x, prove\ that\ y_n = r^n[1 + (-1)^n \sin 2\,r\,x]^{1/2}$

Find $y_8(\pi)\ where\ r = 1/4$.

2. $\sin x \cos 3x.$ **3.** $\sin 2x \sin 3x \cos 4x.$ **4.** $\sin 2x \sin 3x \sin 4x$

5. $sin^3 3x$ **6.** $sin^4 x$ **7.** $\sin^5 x$

8. $cos^2 x sin^3 x$ **9.** $sin^4 x cos^3 x$ **10.** $e^x \cos 2x \cos x$

11. $e^x sin^2 x \cos x$ **12.** $2^x sin^2 x \cos x$

13. $If\ y = \cosh 2x$ prove that $y_n = 2^n \sinh 2x$ if n is odd and $y_n = 2^n \cosh 2x$ if n is even

ANSWERS

1. $\left(\frac{1}{2}\right)^{31/2}$ **2.** $\frac{1}{2}\left[4^n sin\left(4x+\frac{n\,\pi}{2}\right)-2^n sin\left(2x+\frac{n\,\pi}{2}\right)\right]$

3. $\frac{1}{4}[5^n cos(5x+n\,\pi/2)+3^n cos(2x+n\,\pi/2)-9^n cos(9x+n\,\pi/2)-cos(x+n\,\pi/2)]$

4. $\frac{1}{4}\left[5^n sin\left(5x+\frac{n\,\pi}{2}\right)+3^n sin\left(3x+\frac{n\,\pi}{2}\right)+sin\left(x+\frac{n\,\pi}{2}\right)-9^n sin\left(9x+\frac{n\,\pi}{2}\right)\right]$

5. $\frac{3}{4}.\,3^n sin(3x+n\ \pi/2)-\frac{1}{4}.\,9^n sin(9x+n\ \pi/2)$

6. $y=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$ $y_n=-\frac{1}{2}.\,2^n cos\left(2x+\frac{n\,\pi}{2}\right)+\frac{1}{8}4^n cos\left(4x+\frac{n\,\pi}{2}\right)$

7. $\mathrm{y_n}=\frac{1}{16}\left[5^n\sin\left(5x+\frac{n\,\pi}{2}\right)-5.3^n\sin\left(3x+\frac{n\,\pi}{2}\right)+10\sin\left(x+\frac{n\,\pi}{2}\right)\right]$

8. $\frac{1}{16}\left[2\ sin\left(x+\frac{n\,\pi}{2}\right)+3^n sin\left(3x+\frac{n\,\pi}{2}\right)-5^n sin\left(5x+\frac{n\,\pi}{2}\right)\right]$

9. $y_n=\frac{1}{64}\left[7^n cos\left(7x+\frac{n\pi}{2}\right)-cos\left(5x+\frac{n\,\pi}{2}\right)-3.3^n cos\left(3x+\frac{n\,\pi}{2}\right)+3\ cos\left(x+\frac{n\,\pi}{2}\right)\right]$

10. $\frac{1}{2}e^x\left[10^{n/2}cos(3x+n\ tan^{-1}3)+2^{n/2}cos(x+n\ \pi/4)\right]$

11. $-\frac{1}{4}(10)^{3/2}e^x cos(3x+n\ tan^{-1}3)+\frac{1}{4}2^{n/2}e^x cos(x+n\ tan^{-1}1)$

12. $-\frac{1}{4}{r_1}^n 2^x cos(3x+n\ \emptyset_1)+\frac{1}{4}{r_2}^n 2^x cos(x+n\ \emptyset_2)$

$r_1=\sqrt{(\log 2)^2+3^2},\emptyset_1=tan^{-1}(3/\log 2),\quad r_2=\sqrt{(\log 2)^2+1^2},\emptyset_2=tan^{-1}(1/\log 2)$

Type – III

1. If $\mathrm{y}=\frac{1}{x^2+1}$, prove that $\mathrm{y_n}=(-1)^n.\,n!\ \sin^{n+1}\theta\,\sin(n+1)\theta$ where $\theta=\tan^{-1}(1/x)$.

2. If $\mathrm{y}=\frac{x}{x^2+1}$,prove that $\mathrm{y_n}=(-1)^n.\,n!\ \sin^{n+1}\theta\,\cos(n+1)\theta$ where $\theta=\tan^{-1}(1/x)$

3. If $\mathrm{y}=\tan^{-1}(x/a)$, prove that $\mathrm{y_n}=(-1)^{n-1}(n-1)!\,a^{-n}\sin^n\theta\sin n\,\theta$, $\theta=\tan^{-1}(a/x)$

4. If $\mathrm{y}=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, prove that $\mathrm{y_n}=2.\,(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, $\theta=\tan^{-1}(1/x)$

Type – IV

Find the n^{th}derivative of y if

1. $y=x^3e^x$ **2.** $y=x^2a^x$ **3.** $y=x^3\sin 2x$

4. $y=(2x+3)^2e^x$ **5.** $y=(x+3)^3\sin 3x$

6. If $\mathrm{y}=\frac{\log x}{x}$, prove that $\mathrm{y_5}=\frac{5!}{x^6}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\log x\right]$

7. If $\mathrm{y}=x^2e^{2x}$, prove that at $x=0$, $\mathrm{y_n}=2^{n-2}n(n-1)$

ANSWERS

1. $y_n = e^x x^3 + 3ne^x . x^2 + 3n(n-1)e^x x + n(n-1)(n-2)e^x$
2. $y_n = a^x (\log a)^n x^2 + n. a^x . (\log a)^{n-1} . 2x + n(n-1). a^x (\log a)^{n-2}$
3. $y_n = 2^n sin\left(2x + \frac{n\pi}{2}\right) . x^3 + n. 3x^2 2^{n-1} sin\left(2x + (n-1)\frac{\pi}{2}\right)$

   $+n(n-1). 3x\, 2^{n-2} sin\left(2x + (n-2)\frac{\pi}{2}\right) + n(n-1)(n-2)\, 2^{n-3} sin\left(2x + (n-3). \frac{\pi}{2}\right)$
4. $y_n = e^x (2x+3)^2 + ne^x . 4(2x+3) + n(n-1)4e^x$
5. $y_n = 3^n sin\left(3x + \frac{n\pi}{2}\right)(x+3)^3 + n. 3^n sin\left(3x + (n-1)\frac{\pi}{2}\right)(x+3)^2$

   $+n(n-1)3^{n-1} sin\left(3x + (n-2)\frac{\pi}{2}\right)(x+3) + n(n-1)(n-2)3^{n-3} sin\left(3x + (n-3). \frac{\pi}{2}\right)$

Type – V

1. If $y = sin^{-1}x$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. Also find $y_9(0)\ and\ y_{10}(0)$.
2. If $y = tan^{-1}x$, prove that, $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$.
   Hence deduce that $y_n(0) = 0$ if n is even and $y_n(0) = (n-1)!$ if n is odd.
3. If $y = \left(x + \sqrt{a^2+x^2}\right)^2$, prove that $(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-4)y_n = 0$.
4. If $y = a\cos\log x + b\sin\log x$, prove that, $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.
5. If $\mathrm{y} = \cos(\mathrm{m}\sin^{-1}\mathrm{x})$, prove that $(1-\mathrm{x}^2)\,\mathrm{y}_{\mathrm{n}+2} - (2\mathrm{n}+1)\mathrm{x}\mathrm{y}_{\mathrm{n}+1} + (\mathrm{m}^2-\mathrm{n}^2)\mathrm{y}_\mathrm{n} = 0$.
   Hence, obtain $\mathrm{y}_\mathrm{n}(0)$.
6. If $\mathrm{y} = \mathrm{e}^{\mathrm{m}\sin^{-1}\mathrm{x}}\left(\textbf{or}\ \mathrm{x} = \sin\left(\frac{1}{\mathrm{m}}\log\mathrm{y}\right)\right)$, prove that $(1-\mathrm{x}^2)\mathrm{y}_{\mathrm{n}+2} - (2\mathrm{n}+1)\mathrm{x}\,\mathrm{y}_{\mathrm{n}+1} - (\mathrm{n}^2+\mathrm{m}^2)\mathrm{y}_\mathrm{n} = 0$.
7. If $\mathrm{x} = \tan\log\mathrm{y}$ **or** $\mathrm{y} = \mathrm{e}^{\tan^{-1}\mathrm{x}}$, prove that $(1+\mathrm{x}^2)\mathrm{y}_{\mathrm{n}+1} + (2\mathrm{n}\mathrm{x}-1)\mathrm{y}_\mathrm{n} + \mathrm{n}(\mathrm{n}-1)\mathrm{y}_{\mathrm{n}-1} = 0$.
8. If $cos^{-1}\left(\frac{y}{b}\right) = log\left(\frac{x}{n}\right)^n$, prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$.
9. If $y^{1/m} + y^{-1/m} = 2x$, prove that, $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.
10. If $x = \cosh\left(\frac{1}{m}\log y\right)$, prove that, $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.
11. If $y = log\left(x + \sqrt{x^2+a^2}\right)^2$, prove that $(x^2+a^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$.
12. If $y = \left[log\left(x + \sqrt{1+x^2}\right)\right]^2$, prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$.
    Hence, deduce that $y_{n+2}(0) = -n^2 y_n(0)$.
13. If $x = \sin\theta\ and\ y = \cos m\,\theta$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

ANSWERS

1. $y_9(0) = 1^2 . 3^2 . 5^2 . 7^2 . and\ y_{10}(0) = 0$
5. $y_n(0) = 0$ if n is odd , $\mathrm{y}_\mathrm{n}(0) = ((\mathrm{n}-2)^2 - \mathrm{m}^2) \ldots (4^2-\mathrm{m}^2)(2^2-\mathrm{m}^2)(-\mathrm{m}^2)$ if n is even