Composite functions

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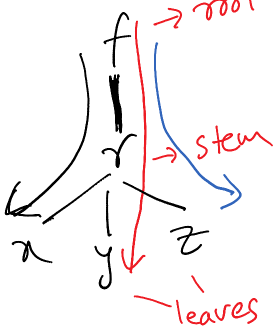
$$f = \log(x^2)$$

$$\frac{df}{dx} = \left(\frac{df}{du}\right) \times \left(\frac{du}{dx}\right)$$



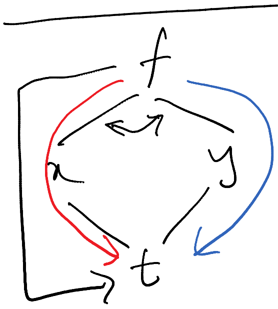
$f(r)$ $\quad r^2 = x^2 + y^2 + z^2$ | $f(r)$, $r = \phi(x, y, z)$

$$\boxed{\frac{df}{dr} = f'(r)}$$

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{df}{dr} \times \frac{\partial r}{\partial y}$$

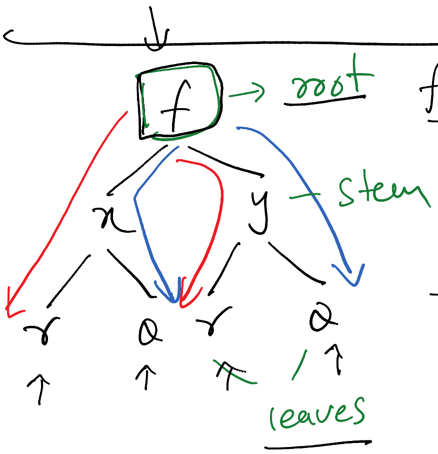
$$\frac{\partial f}{\partial z} = \frac{df}{dr} \times \frac{\partial r}{\partial z}$$



$f(x, y) \Rightarrow x^2 + 2xy$, $\boxed{x = 2at}$, $y = at^2$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}}$$

$$f'(t) = \frac{\partial f}{\partial x} \times x' + \frac{\partial f}{\partial y} \times y'$$

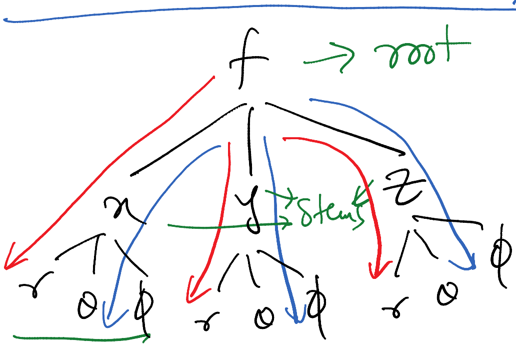


$f \Rightarrow (x^2 + y^2)$, $x = r\cos\theta$, $y = r\sin\theta$

$$f \rightarrow (r, \theta)$$

$$\boxed{\frac{\partial f}{\partial r}} = \boxed{\frac{\partial f}{\partial x}} \times \boxed{\frac{\partial x}{\partial r}} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$



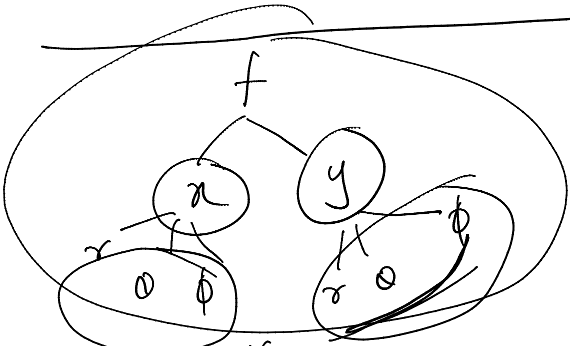
$$\rightarrow \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$\rightarrow \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}$$

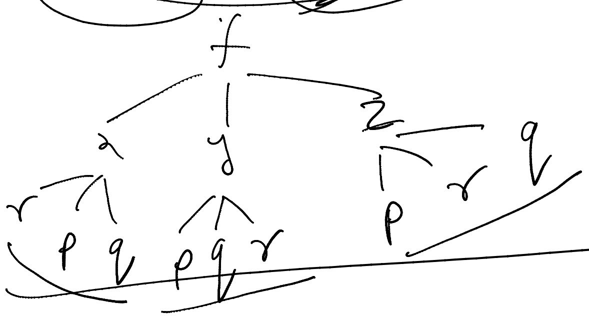
$$\rightarrow \frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$$

→ $\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$

leaves



3 $\begin{cases} \frac{\partial f}{\partial r} = \quad + \\ \frac{\partial f}{\partial \theta} = \quad + \\ \frac{\partial f}{\partial \phi} = \end{cases}$



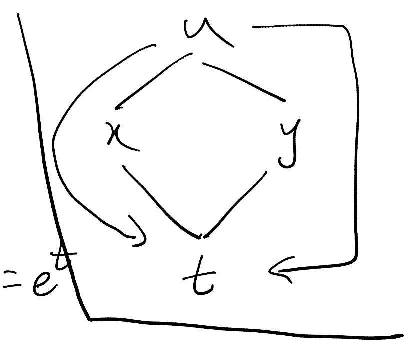
log r

$\frac{df}{dr} = \frac{1}{r}$

- If $u = x^2y^3, x = \log t, y = e^t$, find $\frac{du}{dt}$ (By using composite rule.) at $t=1$

Sol^n: formula for composite rule is,

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$



$\frac{\partial u}{\partial x} = 2xy^3, \quad \frac{\partial u}{\partial y} = 3x^2y^2, \quad \frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = e^t$

$$\frac{du}{dt} = (2xy^3)\left(\frac{1}{t}\right) + (3x^2y^2)(e^t)$$

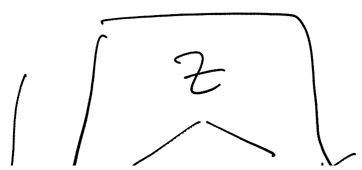
$$= 2(\log t)(e^{3t})\frac{1}{t} + 3(\log t)^2(e^{2t})e^t$$

$$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2 e^{3t}$$

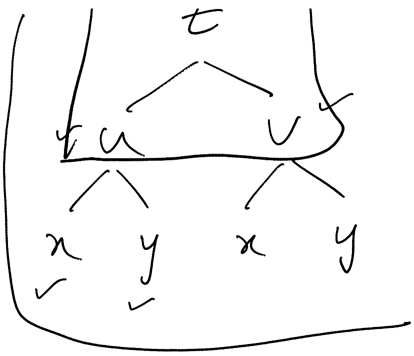
put $t=1$

- If z = f(u, v) and $u = \log(x^2+y^2), v = \frac{y}{x}$, prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$

Sol^n: for this problem, chain rule is



Sol^n: for this problem, chain rule is



$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\end{aligned}\right\} \quad (1)$$

$$u = \log(x^2+y^2) \Rightarrow \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

$$v = \frac{y}{x} \Rightarrow \frac{\partial v}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}$$

put in (1)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)$$

multiply by y

$$y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{2xy}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(-\frac{y^2}{x^2}\right) \quad — (2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{2y}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(\frac{1}{x}\right) \quad — (3)$$

$$x\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{2xy}{x^2+y^2}\right) + \frac{\partial z}{\partial v}(1)$$

(3) − (2)

$$\text{L.H.S.} = x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}\left(1+\frac{y^2}{x^2}\right)$$

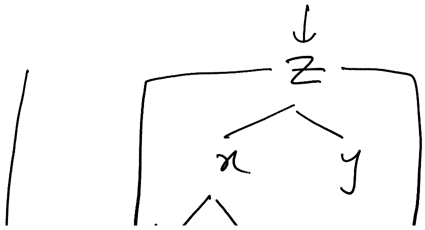
but $v = \frac{y}{x}$

$$\boxed{x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}(1+v^2)}$$

Hence Proved.

H.W.

If $x^2 = au + bv$, $y^2 = au - bv$ and $z = f(x,y)$, Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.



$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 2\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right).$

$x^2 = au + bv$

diff eqⁿ w.r.t. u

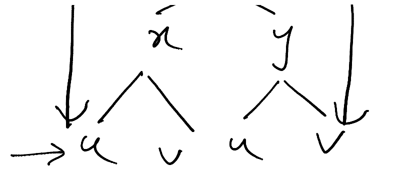
$2x\frac{\partial x}{\partial u} = a \Rightarrow \boxed{\frac{\partial x}{\partial u} = \frac{a}{2x}}$

diff eqⁿ w.r.t. v

$2x\frac{\partial x}{\partial v} = b \Rightarrow \boxed{\frac{\partial x}{\partial v} = \frac{b}{2x}}$

$\boxed{\frac{\partial y}{\partial u} = \frac{a}{2y}}$

$\boxed{\frac{\partial y}{\partial v} = -\frac{b}{2y}}$



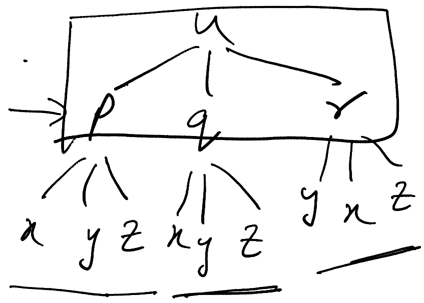
$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\boxed{\frac{\partial x}{\partial u}} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$

If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$

Dont Use (f') Symbol

Solⁿ: Let $p = x^2 - y^2,\ q = y^2 - z^2,\ r = z^2 - x^2$

$\Rightarrow \boxed{u = f(p, q, r)}$



formulae for chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$\frac{\partial p}{\partial x} = 2x, \quad \frac{\partial q}{\partial x} = 0, \quad \frac{\partial r}{\partial x} = -2x$

$\frac{\partial p}{\partial y} = -2y, \quad \frac{\partial q}{\partial y} = 2y, \quad \frac{\partial r}{\partial y} = 0$

$\frac{\partial p}{\partial z} = 0, \quad \frac{\partial q}{\partial z} = -2z, \quad \frac{\partial r}{\partial z} = 2z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(2x) - \frac{\partial u}{\partial r}(2x)$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r} \quad \text{—(1)}$$

$$\frac{1}{y}\frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial q} - 2\frac{\partial u}{\partial p} \quad \text{—(2)}$$

$$\frac{1}{z}\frac{\partial u}{\partial z} = 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial q} \quad \text{—(3)}$$

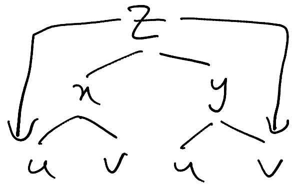
$$\frac{1}{y}\frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial q} - \frac{\partial u}{\partial p}\Big|_{②} z \qquad — ③$$

adding ① ② & ③ $\quad \frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

H.W.

- If $x = e^u cosec\, v, y = e^u \cot v$ and $\boxed{z \text{ is a function of x and y}}$, prove that
- $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left|\left(\frac{\partial z}{\partial u}\right)^2 - sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right|$

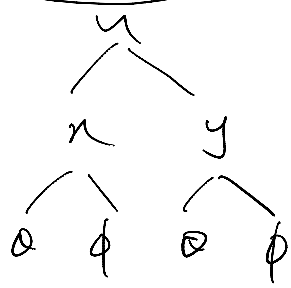
$\frac{\partial z}{\partial u}$



$$\frac{\partial z}{\partial u} = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right) \longrightarrow$$

- If $x + y = 2e^\theta \cos\Phi$ ①, $x - y = 2\, i\, e^\theta \sin\Phi$ ②, show that $\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial\Phi^2} = 4xy\frac{\partial^2 u}{\partial x\partial y}$

$u = f(?)$



Sol^n: ① + ②

$$\cancel{2}x = 2e^\theta[\cos\phi + i\sin\phi] = \cancel{2}e^\theta e^{i\phi}$$

$$\Rightarrow \boxed{x = e^{\theta + i\phi}}$$

① − ②

$$\cancel{2}y = 2e^\theta[\cos\phi - i\sin\phi] = \cancel{2}e^\theta e^{-i\phi}$$

$$\boxed{y = e^{\theta - i\phi}}$$

Then chain rule,

$$\frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta}$$

$$\frac{\partial u}{\partial\phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\phi} \qquad ③$$

$$\frac{\partial x}{\partial\theta} = e^{\theta + i\phi}(1) = x, \quad \frac{\partial y}{\partial\theta} = e^{\theta - i\phi}(1) = y$$

$$\frac{\partial x}{\partial\phi} = e^{\theta + i\phi}(i) = ix, \quad \frac{\partial y}{\partial\phi} = -iy$$

put in ③

$$\therefore \frac{\partial u}{\partial\theta} = \boxed{\frac{\partial u}{\partial x}(x) + \frac{\partial u}{\partial y}(y)} \qquad — ④$$

$$\left[\frac{\partial}{\partial\theta}\right](u) = \left[x\frac{\partial}{\partial x} + y\frac{\partial u}{\partial y}\right](u) \qquad — ⑤$$

$$\ldots \quad \frac{\partial}{\partial\theta}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

Consider $\dfrac{\partial^2 u}{\partial \theta^2} = \left(\dfrac{\partial}{\partial \theta}\right)\left[\dfrac{\partial u}{\partial \theta}\right] = \left[x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right]\left[x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right]$

$$\frac{\partial^2 u}{\partial \theta^2} = x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$= x\left[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(1)\right] + xy\frac{\partial^2 u}{\partial x \partial y} + yx\frac{\partial^2 u}{\partial y \partial x}$$

(equality of mixed partial derivative)

$$+ y\left[y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}(1)\right]$$

$$\boxed{\frac{\partial^2 u}{\partial \theta^2} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}} \quad \text{— (6)}$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}(ix) + \frac{\partial u}{\partial y}(-iy)$$

$$\frac{\partial u}{\partial \phi} = i\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right] \quad \text{— (7)}$$

$$\text{Operator } \frac{\partial}{\partial \phi} = i\left[x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right] \quad \text{— (8)}$$

$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left[\frac{\partial u}{\partial \phi}\right] = i\left[x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right]\left\{i\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right]\right\}$$

$$= i^2\left[x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - xy\frac{\partial^2 u}{\partial x \partial y} - xy\frac{\partial^2 u}{\partial x \partial y} + \left(y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y}\right)\right]$$

$$= -\left[x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] \quad \text{— (9)}$$

$(6) + (9)$

$$\boxed{\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}}$$ — Hence proved.