ITERATIVE METHODS

There is a class of methods of solving simultaneous equations called **iterative methods**. In these methods we start with certain assumptions as to the values of the variables. By applying a method of this type we get a better approximation. We repeat (iterate) this procedure as many times as we want till we arrive at a desired accuracy.

JOCOBI'S METHOD:

Consider the following system of equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$a_3x + b_3y + c_3z = d_3$$

When a_1, b_2, c_3 are large as compared to remaining coefficients , we write the equations as

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$
$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

We now start with the assumption that the roots of these equations are x = x_0 , y = y_0 , z = z_0.
Putting these values in (2) the first approximation is given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$
$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$
$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

We now assume that the roots of the equations are $x = x_1, y = y_1, z = z_1$.
Putting these values in (2), we get better approximation given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$
$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$
$$z_3 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

We repeat (iterate) the procedure as many times as we want, till we arrive at a desired accuracy

SOME SOLVED EXAMPLES:

1. Solve the following equations by Jacobi's method

$4x + y + 3z = 17, x + 5y + z = 14, 2x - y + 8z = 12$

Solution: We first write the equation as

$$x = \frac{1}{4}(17 - y - 3z)$$
$$y = \frac{1}{5}(14 - x - z) \qquad \ldots\ldots\ldots\ldots\ldots (1)$$
$$z = \frac{1}{8}(12 - 2x + y)$$

(i) First Iteration: We start with the approximation $x = 0, y = 0, z = 0$

$\therefore x_1 = \frac{17}{4} = 4.25, y_1 = \frac{14}{5} = 2.8, z_1 = \frac{12}{8} = 1.5$

(ii) Second Iteration: Putting these values on r.h.s. of (1), we get,

$$x_2 = \frac{1}{4}[17 - 2.8 - 3(1.5)] = 2.425$$
$$y_2 = \frac{1}{5}(14 - 4.25 - 1.5) = 1.65$$
$$z_2 = \frac{1}{8}[12 - 2(4.25) + 2.8] = 0.7875$$

(iii) Third Iteration: Putting these values on r.h.s. of (1), we get,

$x_3 = \frac{1}{4}[17 - 1.65 - 3(0.7875)] = 3.2469$

$y_3 = \frac{1}{5}(14 - 2.425 - 0.7875) = 2.1575$

$z_3 = \frac{1}{8}(12 - 2 \times (2.425) + 1.65) = 1.1$

(iv) Fourth Iteration: Putting these values on r.h.s. of (1)again, we get,

$x_4 = \frac{1}{4}[17 - 2.1575 - 3(1.1)] = 2.8856$

$y_4 = \frac{1}{5}(14 - 3.2469 - 1.1) = 1.9306$

$z_4 = \frac{1}{8}[12 - 2(3.2469) + 2.1575] = 0.9580$

(v) Fifth Iteration: Putting these values on r.h.s. of (1)again, we get,

$x_5 = \frac{1}{4}[17 - 1.9306 - 3(0.9580)] = 3.0488$

$y_5 = \frac{1}{5}(14 - 2.8856 - 0.9580) = 2.0313$

$z_5 = \frac{1}{8}[12 - 2(2.8856) + 1.9306] = 1.0199$

$\therefore x = 3, y = 2, z = 1$

We can write above solution in tabular form as follows:

We first write the equation as $x = \frac{1}{4}(17 - y - 3z)$

$y = \frac{1}{5}(14 - x - z)$ .... (1)

$z = \frac{1}{8}(12 - 2x + y)$

We start with the approximation $x = 0, y = 0, z = 0$

$x = \frac{1}{4}(17 - y - 3z)$	$y = \frac{1}{5}(14 - x - z)$	$z = \frac{1}{8}(12 - 2x + y)$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{17}{4} = 4.25$	$y_1 = \frac{14}{5} = 2.8$	$z_1 = \frac{12}{8} = 1.5$
$x_2 = \frac{1}{4}[17 - 2.8 - 3(1.5)] = 2.425$	$y_2 = \frac{1}{5}(14 - 4.25 - 1.5) = 1.65$	$z_2 = \frac{1}{8}[12 - 2(4.25) + 2.8]$ $= 0.7875$
$x_3 = \frac{1}{4}[17 - 1.65 - 3(0.7875)]$ $= 3.2469$	$y_3 = \frac{1}{5}(14 - 2.425 - 0.7875) = 2.1575$	$z_3 = \frac{1}{8}(12 - 2 \times (2.425) + 1.65)$ $= 1.1$
$x_4 = \frac{1}{4}[17 - 2.1575 - 3(1.1)]$ $= 2.8856$	$y_4 = \frac{1}{5}(14 - 3.2469 - 1.1) = 1.9306$	$z_4 = \frac{1}{8}[12 - 2(3.2469) + 2.1575]$ $= 0.9580$
$x_5 = \frac{1}{4}[17 - 1.9306 - 3(0.9580)]$ $= 3.0488$	$y_5 = \frac{1}{5}(14 - 2.8856 - 0.9580)$ $= 2.0313$	$z_5 = \frac{1}{8}[12 - 2(2.8856) + 1.9306]$ $= 1.0199$

$\therefore x = 3, y = 2, z = 1$

2. Solve the following equations by Jacobi's method

$10x - 2y - 3z = 205, 2x - 10y + 2z = -154, 2x + y - 10z = -120$

Solution: We first write the equation as $x = \frac{1}{10}(205 + 2y + 3z)$

$y = -\frac{1}{10}(-154 - 2x - 2z)$

$z = -\frac{1}{10}(-120 - 2x - y)$

$$\left.\begin{aligned} x &= 20.5 + 0.2y + 0.3z \\ \text{Or } y &= 15.4 + 0.2x + 0.2z \\ z &= 12 + 0.2x + 0.1y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

We start with the approximation $x = 0, y = 0, z = 0$

$x = 20.5 + 0.2y + 0.3z$	$y = 15.4 + 0.2x + 0.2z$	$z = 12 + 0.2x + 0.1y$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 20.5$	$y_1 = 15.4$	$z_1 = 12$
$x_2 = (20.5) + (0.2)(15.4) + (0.3)(12) = 27.18$	$y_2 = (15.4) + (0.2)(20.5) + (0.2)(12) = 21.9$	$z_2 = (12) + (0.2)(20.5) +(0.1)(15.4) = 17.64$
$x_3 = (20.5) + (0.2)(21.9) +(0.3)(17.64) = 30.172$	$y_3 = (15.4) + (5.436) + (3.528) = 24.364$	$z_3 = (12) + (5.436) + (2.19) = 19.626$
$x_4 = (20.5) + (4.8728) + (5.8878) = 31.2606$	$y_4 = (15.4) + (6.0344) + (3.9252) = 25.3596$	$z_4 = (12) + (0.2)(30.172) +(0.1)(24.364) = 20.4708$
$x_5 = (20.5) + (5.07192) + (6.14124) = 31.71316$	$y_5 = (15.4) + (6.25212) + (4.09416) = 25.74628$	$z_5 = (12) + (6.25212) + (2.53596) = 20.78808$
$x_6 = (20.5) + (5.149256) +(6.236424) = 31.88568$	$y_6 = (15.4) + (6.342632) +(4.157616) = 25.900248$	$z_6 = (12) + (6.342632) +(2.574628) = 20.91726$
$x_7 = (20.5) + (5.1800496) +(6.275178) = 31.9552276$	$y_7 = (15.4) + (6.377136) +(4.183452) = 25.960588$	$z_7 = (12) + (6.377136) +(2.5900248) = 20.9671608$
$x_8 = (20.5) + (5.1921176) +(6.29014824) = 31.98226584$	$y_8 = (15.4) + (6.39104552) +(4.19343216) = 25.98447768$	$z_8 = (12) + (6.39104552) +(2.5960588) = 20.98710432$

The value of the 7th and 8th iteration being practically same, we can conclude that the solution of the given system
of equations is $\{x = 32, y = 26, z = 21\}$

GAUSS – SEIDEL METHOD:

This is a modification of Jacobi's method in which as soon as a new approximation of an unknown is obtained, it is
used immediately in the next calculation .

Consider as before the system of equations.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

When a_1, b_2, c_3 are large as compared to remaining coefficients , we write the equations as

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}$$

We now start with the assumption that the roots of these equations are x = x_0 , y = y_0 , z = z_0.

Putting these values in first equation of (2) , we get $x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$

Now, we put $x = x_1, z = z_0$ in the second equation of (2) and get, $y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$

We put $x = x_1, y = y_1$ and get $z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$

The process is continued till we get desired degree of accuracy

As soon as we obtain a new approximation, it is immediately used in the next calculation

SOME SOLVED EXAMPLES:

1. Solve the following equations by Gauss – Seidel method

$20x + y - 2z = 17, \quad 3x + 20y - z = -18, \quad 2x - 3y + 20z = 25$

Solution: We first write the equations as

$x = \frac{1}{20}[17 - y + 2z]$ .... (1)

$y = \frac{1}{20}[-18 - 3x + z]$ .... (2)

$z = \frac{1}{20}[25 - 2x + 3y]$ .... (3)

(i) **First Iteration:** We start with the approximation $y = 0, z = 0$ and then get from (1),

$\therefore x_1 = \frac{17}{20} = 0.85$

We use this approximation to find y i.e., we put $x = 0.85, z = 0$ in (2)

$\therefore y_1 = \frac{1}{20}[-18 - 3(0.85) - 0] = -1.0275$

We use these values of x_1 and y_1 to find z_1 i.e., we put $x = 0.85, y = -1.0275$ in (3)

$\therefore z_1 = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)] = 1.0109$

(ii) **Second Iteration:** We use latest values of y and z to find x i.e., we put $y = -1.0275$, $z = 1.0109$ in (1)

$\therefore x_2 = \frac{1}{20}[17 - (-1.0275) + 2(1.0109)] = 1.0025$

We put $x = 1.0025, z = 1.0109$ in (2)

$\therefore y_2 = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.9998$

We put $x = 1.0025, y = -0.9998$ in (3)

$\therefore z_2 = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$

(iii) **Third Iteration:** We use latest values of y and z to find x i.e., we put $y = -0.9998$, $z = 0.9998$ in (1)

$\therefore x_3 = \frac{1}{20}[17 - (-0.9998) + 2(0.9998)] = 1.000$

We put $x = 1.000, z = 0.9998$ in (2)

$\therefore y_3 = \frac{1}{20}[-18 - 3(1.000) + 0.9998] = -1.000$

We put $x = 1.000, y = -1.000$ in (3)

$\therefore z_3 = \frac{1}{20}[25 - 2(1.000) + 3(-1.000)] = 1$

Hence, we get $x = 1, y = -1, z = 1$

We can write above solution in tabular form as follows:

We first write the equations as

$x = \frac{1}{20}[17 - y + 2z]$ .... (1)

$y = \frac{1}{20}[-18 - 3x + z]$ .... (2)

$z = \frac{1}{20}[25 - 2x + 3y]$ .... (3)

We start with the approximation $x = 0, y = 0, z = 0$

$x = \frac{1}{20}[17 - y + 2z]$	$y = \frac{1}{20}[-18 - 3x + z]$	$z = \frac{1}{20}[25 - 2x + 3y]$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{17}{20} = 0.85$	$y_1 = \frac{1}{20}[-18 - 3(0.85) - 0]$ $= -1.0275$	$z_1 = \frac{1}{20}[25 - 2(0.85)$ $+3(-1.0275)] = 1.0109$
$x_2 = \frac{1}{20}[17 - (-1.0275)$ $+2(1.0109)] = 1.0025$	$y_2 = \frac{1}{20}[-18 - 3(1.0025) + 1.0109]$ $= -0.9998$	$z_2 = \frac{1}{20}[25 - 2(1.0025)$ $+3(-0.9998)] = 0.9998$
$x_3 = \frac{1}{20}[17 - (-0.9998)$ $+2(0.9998)] = 1.000$	$y_3 = \frac{1}{20}[-18 - 3(1.000) + 0.9998]$ $= -1.000$	$z_3 = \frac{1}{20}[25 - 2(1.000)$ $+3(-1.000)] = 1$

Hence, we get $x = 1, y = -1, z = 1$

2. Solve the following equations by Gauss – Seidel method

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$, $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$, $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$

Solution: We first write the equations as

$x_1 = \frac{1}{3}[7.85 + 0.1x_2 + 0.2x_3]$ .... (1)

$x_2 = \frac{1}{7}[-19.3 - 0.1x_1 + 0.3x_3]$ .... (2)

$x_3 = \frac{1}{10}[71.4 - 0.3x_1 + 0.2x_2]$ .... (3)

We start with the approximation $x = 0, y = 0, z = 0$

$x_1 = \frac{1}{3}[7.85 + 0.1x_2 + 0.2x_3]$	$x_2 = \frac{1}{7}[-19.3 - 0.1x_1 + 0.3x_3]$	$x_3 = \frac{1}{10}[71.4 - 0.3x_1 + 0.2x_2]$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_{1,1} = \frac{7.85}{3} = 2.6167$	$x_{2,1} = \frac{1}{7}[-19.3 - 0.1(2.6167) +$ $0.3(0)] = -2.7945$	$x_{3,1} = \frac{1}{10}[71.4 - 0.3(2.6167)$ $+0.2(-2.7945)] = 7.0056$
$x_{1,2} = \frac{1}{3}[7.85 + 0.1(-2.7945)$ $+0.2(7.0056)] = 2.9906$	$x_{2,2} = \frac{1}{7}[-19.3 - 0.1(2.9906)$ $-0.3(7.0056)] = -2.4996$	$x_{3,2} = \frac{1}{10}[71.4 - 0.3(2.9906)$ $+0.2(-3.1001)] = 6.98828$
$x_{1,3} = \frac{1}{3}[7.85 - 0.1(-3.1001)$ $[+0.2(6.98828)] = 3.000$	$x_{2,3} = \frac{1}{7}[-19.3 - 0.1(3.000)$ $+0.3(6.98828)] = -2.500$	$x_{3,3} = \frac{1}{10}[71.4 - 0.3(3.000)$ $+0.2(-2.500)] = 7.000$

Hence, the values are $x_1 = 3, x_2 = -2.5, x_3 = 7$

3. Solve the following equations by Gauss – Seidel method

$28x + 4y - z = 32$, $2x + 17y + 4z = 35$, $x + 3y + 10z = 24$

Solution: We first write the equations as $x = \frac{1}{28}(32 - 4y + z)$ .... (1)

$y = \frac{1}{17}(35 - 2x - 4z)$ .... (2)

$z = \frac{1}{10}(24 - x - 3y)$ .... (3)

We start with the approximation $x = 0, y = 0, z = 0$

$x = \frac{1}{28}(32 - 4y + z)$	$y = \frac{1}{17}(35 - 2x - 4z)$	$z = \frac{1}{10}(24 - x - 3y)$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{32}{28} = 1.1429$	$y_1 = \frac{1}{17}[35 - 2(1.1429)] = 1.9244$	$z_1 = \frac{1}{10}[24 - 1.1429 - 3(1.9244)]$ $= 1.7084$
$x_2 = \frac{1}{28}[32 - 4(1.9244) + 1.7084]$ $= 0.9289$	$y_2 = \frac{1}{17}[35 - 2(0.9289) - 4(1.7084)]$ $= 1.5476$	$z_2 = \frac{1}{10}[24 - 0.9289 - 3(1.5476)]$ $= 1.8428$

$x_3 = \frac{1}{28}[32 - 4(1.5476) + 1.8428]$ $= 0.9876$	$y_3 = \frac{1}{17}[35 - 2(0.9876) - 4(1.8428)]$ $= 1.5090$	$z_3 = \frac{1}{10}[24 - 0.9876 - 3(1.5090)]$ $= 1.8485$
$x_4 = \frac{1}{28}[32 - 4(1.5090) + 1.8485]$ $= 0.9933$	$y_4 = \frac{1}{17}[35 - 2(0.9933) - 4(1.8485)]$ $= 1.5070$	$z_4 = \frac{1}{10}[24 - (0.9933) - 3(1.5070)]$ $= 1.8486$

Since, the third and fourth iterations give the same value upto two places of decimals, we get after rounding

$x = 0.99, y = 1.51, z = 1.85$

EXERCISE

1. Solve the following equations by Jacobi's method

(i) $15x + y - z = 14$, $x + 20y + z = 23$, $2x - 3y + 18z = 37$

(ii) $12x + 2y + z = 27$, $2x + 15y - 3z = 16$, $2x - 3y + 25z = 23$

(iii) $14x - y + 3z = 18$, $2x - 14y + 3z = 19$, $x - 3y + 16z = 20$

2. Solve the following equations by Gauss – Seidel method

(i) $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$

(ii) $10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 2x_2 + 10x_3 = 14$

(iii) $5x + y - z = 10$, $2x + 4y + z = 14$, $x + y + 8z = 20$

ANSWERS

(These are actual answers, please write numerical answers you obtained as final solutions)

1. **(i)** $x = 1, y = 1, z = 2$ **(ii)** $x = 2, y = 1, z = 1$

(iii) $x = 1, y = -1, z = 1$

2. **(i)** $x = 2.43, \; y = 3.57, \; z = 1.93$ **(ii)** $x_1 = 1, x_2 = 1, x_3 = 1$

(iii) $x = 2, y = 2, z = 2.$