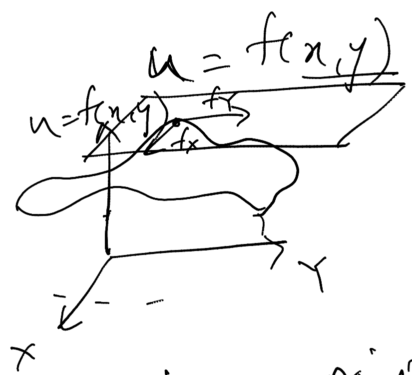
Applications of Partial Differentials

Thursday, January 20, 2022
10:33 AM

Application 1 Maxima / Minima (Extreme values, Stationary values) for a function $f(x,y)$

$u = f(x,y)$



① stationary points : Calculate

f_x, f_y,

$r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$,

② $f_x = 0$, $f_y = 0$ Simultaneous solution

(x_1, y_1), (x_2, y_2), ( ),

③ Hessian Matrix, Calculate at (x_i, y_i)

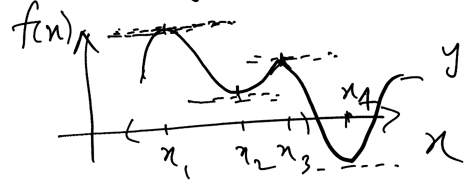
$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} r & s \\ s & t \end{vmatrix}$$

$\det(H) = rt - s^2 > 0$ ✓
accept the pair

If $\boxed{rt - s^2 \leq 0 \text{ Then reject the pair}}$

Check $r > 0 \rightarrow (x_i, y_i)$ is min

$y = f(x)$



① $f'(x) = 0$

To get points x_1, x_2, x_3 stationary

② check at x_i

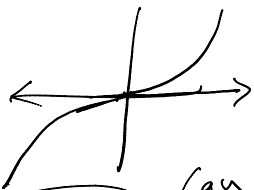
$f''(x_i) > 0$, x_i Min

$f''(x_i) < 0$, x_i Max

anamaly : Saddle point.

$f''(x_i) = 0$

→ neither max nor minima.



→ Constant, can not determine

→ many points satisfying cndn

at x_1 & x_3 max

✓ $\boxed{f(x_1)} > f(x_3) > f(x_n)$

globle max

is min

$r < 0 \to (x_i, y_i)$ is Max.

OR $\boxed{r = 0 \to \text{neither max nor min}}$
Saddle points

globle max

x_2 x_4

$f(x_2) > \boxed{f(x_4)}$

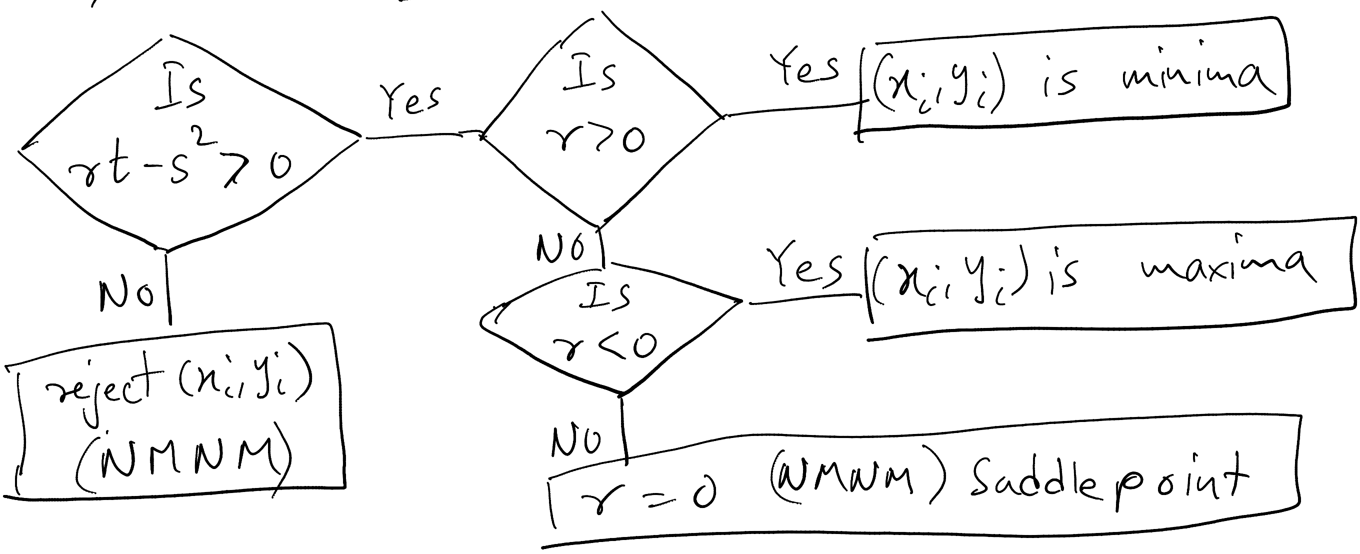
globle min

Steps: 1) Calculate $f_x, f_y, r = f_{xx}, s = f_{xy}, t = f_{yy}$

2) Solve $f_x = 0, f_y = 0$ simultaneously to get stationary points

$(x_1, y_1), (x_2, y_2), \ldots, (x_i, y_i)$

3) Calculate r, s, t at point (x_i, y_i) for all i



→ **Discuss the maxima and minima of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$.**

extreme

Solⁿ: Let $f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

① Then $f_x = 3x^2 + y^2 - 24x + 21$ ✓

$f_y = 2xy - 4y$

$r = f_{xx} = 6x - 24$

$s = f_{xy} = 2y$

$t = f_{yy} = 2x - 4$

(2) Solve $f_x = 0$ & $f_y = 0$ simultaneously — (2)

$3x^2 + y^2 - 24x + 21 = 0$ — (1) & $2xy - 4y = 0$

$y(x-2) = 0$

$\Rightarrow$ either $\boxed{y = 0}$ OR $x = 2$

put $y = 0$ in (1)

$3x^2 - 24x + 21 = 0 \Rightarrow x^2 - 8x + 7 = 0$

$x = 7, 1$

i.e. $(7,0)$ & $(1,0)$ are stationary point

put $x = 2$ in (1)

$12 + y^2 - 48 + 21 = 0 \Rightarrow y^2 - 15 = 0$

$\therefore \boxed{y = \pm\sqrt{15}}$

i.e. $(2,\sqrt{15})$ & $(2,-\sqrt{15})$ are stationary points.

(3) check the condition by tabulating data

points	r	s	t	Is $rt - s^2 > 0$	accept/reject	Check for r	Conclusion	Minimum/Maximum value
$(7,0)$	18	0	10	Yes	accept	$r > 0$	$(7,0)$ is minima	-88
$(1,0)$	-18	0	-2	Yes	accept	$r < 0$	$(1,0)$ is maxima	20
$(2,\sqrt{15})$	-12	$2\sqrt{15}$	0	No	reject	—	—	—
$(2,-\sqrt{15})$	-12	$-2\sqrt{15}$	0	No	reject	—	—	—

Find maximum/minimum Value

$f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10.$

at $(7,0)$, $f(7,0) = 7^3 + 0 - 12(7)^2 - 0 + 21(7) + 10$

$= -88$

at $(1,0)$, $f(1,0) = 1^3 + 0 - 12(1)^2 - 0 + 21(1) + 10$

$= 20$

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2) **Find the stationary values of $x^3 + y^3 - 3a\,xy, a > 0$**

Solⁿ: $f(x,y) = x^3 + y^3 - 3axy$

① $f_x = 3x^2 - 3ay$

$f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x$

$s = f_{xy} = -3a$

$t = f_{yy} = 6y$

② $f_x = 0$

$3x^2 - 3ay = 0$

$\therefore x^2 = ay$

$\boxed{\frac{x^2}{a} = y}$

$f_y = 0$

$3y^2 - 3ax = 0$

$y^2 - ax = 0$

$\left(\frac{x^2}{a}\right)^2 - ax = 0$

$\frac{x^4}{a^2} - ax = 0$

$x^4 - a^3x = 0$

$x(x^3 - a^3) = 0$

$x^3 - a^3 = 0$

$x = 0$ | $\boxed{x = a \text{ is only real root}}$

put $x = 0$, $y = \frac{x^2}{a} = 0$

put $x = a$, $y = \frac{x^2}{a} = \frac{a^2}{a} = a$ $\therefore$ $(0,0)$ (a,a) are possible real pairs.

③ We analyse by tabulating the data

points	r	s	t	$rt - s^2 > 0$?	reject/accept	Conclusion for r	Comment	Value
$(0,0)$	0	$-3a$	0	No	reject	—	—	—
(a,a)	$6a$	$-3a$	$6a$	Yes	accept	$r > 0$	(a,a) is Minima	$-a^3$

Minimum value at (a,a),

$f(a,a) = x^3 + y^3 - 3a\,xy = a^3 + a^3 - 3aaa = -a^3$

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3) **Find the stationary values of $\sin x \cdot \sin y \cdot \sin(x+y)$.**

Sol^n: $f(x,y) = \sin x \cdot \sin y \, \sin(x+y)$

① $f_x = \sin y \left[\sin x \cos(x+y)(1) + \sin(x+y)\cos x\right]$

$= \sin y \left[\sin(2x+y)\right]$ ✓

$f_y = \sin x \sin(x+2y)$ ✓

$r = f_{xx} = 2\sin y \cos(2x+y)$ ✓

$s = f_{xy} = \sin y \cos(2x+y) + \cos y \sin(2x+y) = \sin(2x+2y)$ ✓

$t = f_{yy} = 2\sin x \cos(x+2y)$ ✓

② Solve $f_x = 0$

$\sin y \sin(2x+y) = 0$

$\sin y = 0$ $\quad$ $\sin(2x+y) = 0$

$y = 0, \pi, 2\pi, 3\pi$ | $2x+y = 0, \pi, 2\pi, \ldots$

$f_y = 0$

$\sin x \sin(x+2y) = 0$

$x = 0, \pi, 2\pi, \ldots$ $\quad$ $x+2y = 0, \pi, 2\pi, \ldots$

③ ⓐ $(0,0)$ ✓ (π,π) $(\pi, 3\pi)$ $(2\pi, 5\pi)$, $(k\pi, p\pi)$

$r = s = t = 0$ $\quad$ $\boxed{rt - s^2 = 0}$ $\quad$ we reject all such pair.

ⓑ $2x+y = \pi$ ✓ $\quad$ $x+2y = \pi$ ✓

$y = \pi - 2x \Rightarrow x + 2(\pi - 2x) = \pi$

$x - 4x = \pi - 2\pi$

$+3x = +\pi \Rightarrow \boxed{x = \frac{\pi}{3}}$

$y = \pi - 2\left(\frac{\pi}{3}\right)$

$\therefore$ $\boxed{y = \frac{\pi}{3}}$ $\quad$ $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ as stationary point

$r = 2\sin y \cos(2x+y) = 2\sin\frac{\pi}{3}\cos\pi$

$= \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)(-1) = \underline{-\sqrt{3}}$

$s = \sin(2x+2y) = \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$t = (\qquad\qquad) = -\sqrt{3}$

$\underline{rt - s^2} = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 = 9 - \frac{3}{4} = \underline{\frac{33}{4}} > 0$

we accept this pair, $r < 0$

$\underline{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)}$ is $\underline{\text{maxima}}$, & Since it is periodic f^n $\therefore \left(\frac{\pi}{3} + \underline{2k\pi}, \frac{\pi}{3} + \underline{2p\pi}\right)$ are Maxima

$\underline{\text{Maximum value}} = \sin x \cdot \sin y \cdot \sin(x+y) =$

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4)

Divide 90 into three parts such that the sum of their products taken two at a time is maximum

$\underline{\text{Sol}^n}$: Assume x, y, z are required parts of 90

& $\boxed{x+y+z = 90}$ — (1) $\therefore z = 90 - x - y$

We are asked to maximise the function $\underline{(xy + yz + zx)}$

$\underline{f(x,y)} = xy + y(90-x-y) + (90-x-y)x$

$= 90x + 90y - xy - x^2 - y^2$

$f_x = \underline{90} - \underline{y} - 2x$

$f_y = 90 - x - 2y$

$$\boxed{\begin{aligned} r &= f_{xx} = -2 \\ s &= f_{xy} = -1 \\ t &= f_{yy} = -2 \end{aligned}}$$ ✓

$f_x = 0$

$\underline{90 - y - 2x = 0}$

✓ $2x + y = 90$

$y = 90 - 2x$

$f_y = 0$

$\underline{90 - x - 2y = 0}$

✓ $\underline{x + 2y = 90}$

$\boxed{x = 30, \; y = 30}$

$\therefore (30, 30)$

at $(30,30)$

$r = -2, \; s = -1, \; t = -2, \quad rt - s^2 = 4 - 1 = 3 > 0$

we accept the pair

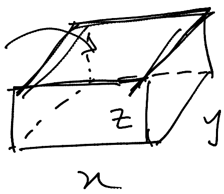
$r < 0 \quad \therefore (30, 30)$ is Maximum point as required.

$\therefore \; x = 30, \; y = 30, \quad z = 90 - x - y = 30$ are the required parts of 90 which satisfies given condition.

5) **A rectangular box with open top has capacity of 32 cubiccms. Find the dimensions of the box such that the material required is minimum.**

Soln : Let x, y, z be the dimentions of box such that

$\rightarrow$ x is length, y is breadth, z is height



Then given condition $\Rightarrow \underset{\text{(volume)}}{xyz} = 32$

The function which we have to minimise is,

(Surface area for opentop) $= xy + 2yz + 2zx$

$= xy + 2\left(\frac{32}{x}\right) + 2\left(\frac{32}{y}\right)$

$$\boxed{f(x,y) = xy + \frac{64}{x} + \frac{64}{y}}$$

① $f_x = y - \frac{64}{x^2}, \quad f_y = x - \frac{64}{y^2}$

$r = f_{xx} = \frac{64 \times 2}{x^3}, \quad t = f_{yy} = \frac{64 \times 2}{y^3}, \quad s = f_{xy} = 1$

② Solve, $f_x = 0$ & $f_y = 0$

$y - \frac{64}{x^2} = 0 \qquad x - \frac{64}{y^2} = 0$

$y = \frac{64}{x^2} \qquad xy^2 - 64 = 0$

$y = \frac{64}{x^2}$

$$xy^2 - 64 = 0$$
$$xy^2 = 64 \Rightarrow x\left(\frac{64}{x^2}\right)^2 = 64$$
$$\therefore \frac{64}{x^3} = 1 \qquad \boxed{\therefore x^3 = 64}$$

Consider only real root $x = 4$

$y = \frac{64}{x^2} = \frac{64}{16} = 4 \qquad \therefore (4,4)$ is required point.

(3) ~~at (4,4)~~, $r = \frac{128}{x^3} = 2$, $s = 1$, $t = \frac{128}{y^3} = 2$

$\therefore rt - s^2 = 4 - 1 = 3 > 0$, $r > 0$

$\therefore$ We can conclude that $x=4, y=4$ is minimum point

$$\therefore z = \frac{32}{xy} = \frac{32}{4\times4} = 2$$

$\therefore x = 4$ cm, $y = 4$ and $z = 2$ cm are dimension of the box such that material required is minimum

Divide 24 into three parts such that the product of the first, square of the second cube of the third is maximum.

Sol^n: $x + y + z = 24 \qquad x = 24 - y - z$

f^n to maximize is $\boxed{xy^2z^3}$

$$f(y,z) = (24 - y - z)\, y^2 z^3$$

$f_y =$

$f_z =$

$f_z =$

$r = f_{yy} =$

$s = f_{yz} =$

$t = f_{zz} =$