Eigen Values and Eigen Vectors

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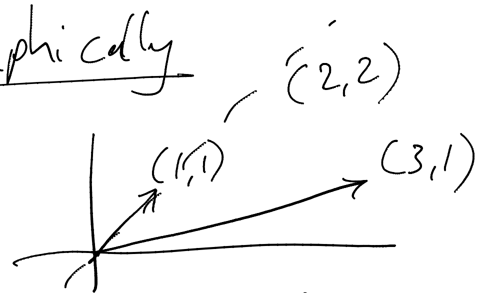
4:04 PM

$$\boxed{Ax = b}$$

Graphically

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\mathbb{R}^2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}^{\mathbb{R}^2}$$



$$\boxed{A(x) = y}$$

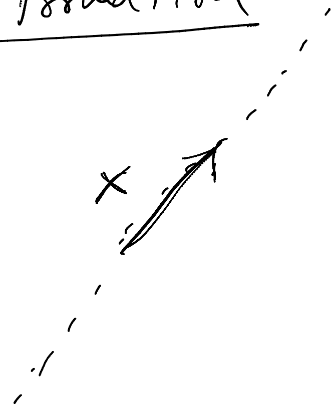
$f(x) = y$

$\boxed{Ax}$ is called as Transformation

The axis | pole of the Transformation

$|x| =$

$$\boxed{A\underline{x} = \lambda x}$$



$x \to$ eigenvector.

$\lambda \to$ eigenvalue.

(x, y) $\mathbb{R}^2$

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example: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

x_1

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

λ_1

$$A \, x_2 = \underline{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

λ_2

$$A x = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

square real/complex

$$\boxed{Ax = \lambda x}$$

$$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} \; \\ \; \end{bmatrix}}$$

$$\boxed{Ax = \lambda x}$$

$(2\times 2)\,(2\times 1) = (2\times 1)$

$$Ax - \lambda x = 0 \qquad [A - \lambda I] \rightarrow B$$

$$\boxed{[A - \lambda I]\,x = 0}$$

$$\boxed{Bx = 0} \longrightarrow \text{Homogeneous system}$$

Square

Unique	Infinite
$r = n$	$r < n$
$\lvert B \rvert \neq 0$	$\lvert B \rvert = 0$
$(0,0,0)$ only sol^n	(non-zero / nontrivial)

$$\lvert A - \lambda I \rvert = 0$$

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1) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ Find the eigenvalues & eigenvectors for A

sol^n: Consider $Ax = \lambda x \Rightarrow [A - \lambda I]\,x = 0$

$$[A - \lambda I] = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\lvert A - \lambda I \rvert = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = \boxed{\lambda^2 - 4\lambda - 5} = 0$$

Solving, we get $\underline{\lambda = 5, -1}$

I) for finding eigenvalue corresponding to $\lambda = -1$

$$[A - \lambda I]\, x = 0 \Rightarrow$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - 2R_1 \\ \frac{1}{2}R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y = 0 \checkmark \Rightarrow y = -x$$

take $x = t$, $y = -t$, Solution is $\underline{\begin{pmatrix} t \\ -t \end{pmatrix}}$

Smallest vector $\underline{(t = 1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is eigen vector corresponding to eigenvalue $\lambda = -1$

II) for $\lambda = 5$

$$[A - \lambda I]\, x = 0 \Rightarrow \begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 + R_1 \\ \frac{1}{2}R_1 \end{matrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x + y = 0 \Rightarrow y = 2x$$

Put $x = t$, $\underline{y = 2t}$, $Sol^n = \begin{bmatrix} t \\ 2t \end{bmatrix}$

put $x = t$, $y = 2t$, Sol^n = $\begin{bmatrix} t \\ 2t \end{bmatrix}$

Smallest vector = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is eigenvector for $\lambda = 5$

Square Matrix A, Then,

$Ax = \lambda x \rightarrow$ ✓ x nonzero vector (eigenvector)

✓ λ scaler (real/complex) (eigenvalue)

A is of order $n \Rightarrow$ polynomial of deg $n \Rightarrow$ n eigenvalues

$[A - \lambda I]\, x = 0$

$|A - \lambda I| = 0$ $(r<n)$ $\rightarrow$ Find values of λ

& corresponding eigenvector for λ_i is given by Solving Homogeneous system $(A - \lambda_i I)\, X_i = 0$

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2) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ Find eigenvalues and eigen vector for A

Sol^n: Consider $Ax = \lambda x \Rightarrow [A - \lambda I]\, x$

$$A - \lambda I = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix}$$

Check: $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 + 11\lambda - 4 = 0$$

$\Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\begin{vmatrix} 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} \Rightarrow \boxed{\lambda^3 - 4\lambda^2 - \lambda + 4 = 0}$$

$$1\lambda^3 - P\lambda^2 + Q\lambda - R = 0$$

where P = Trace of A = SOD

Q = Sum of Minors of diagonal (SOMD)

R = det of A

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \Rightarrow (a-\lambda)(d-\lambda) - bc$$

$$\lambda^2 - \underset{\text{Trace}}{(a+d)}\lambda + \underset{\text{det}}{(ad-bc)}$$

$$\begin{bmatrix} \boxed{4} & 6 & 6 \\ 1 & \boxed{3} & 2 \\ -1 & -4 & \boxed{-3} \end{bmatrix}$$

$$P = \text{SOD} = 4 + 3 - 3 = 4$$

$$Q = \text{SOMD} = \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= -1 - \cancel{6} + \cancel{6} = -1$$

$$R = \det(A) = -4$$

$$\Rightarrow \lambda^3 - P\lambda^2 + Q\lambda - R = 0$$

Ch eq^n is $\boxed{\lambda^3 - 4\lambda^2 - \lambda + 4 = 0}$

and Roots of characteristic eq^n are,

$= 4, 1, -1$

and Roots of characteristic eq

$\lambda = 4, 1, -1$ ✓

(1) Let us find eigenvector for $\lambda = 4$

for $\lambda = 4$
AM = 1
GM = 1

$[A - 4I]\, x = 0$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & 6 \\ -1 & -4 & -7 \end{bmatrix}$

$R_3 + R_1$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & 6 \\ 0 & -5 & -5 \end{bmatrix}$

$R_3 + \frac{5}{6} R_2$ | $\frac{1}{6} R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\to x - y + 2z = 0$

$\to y + z = 0$

$z = t, \quad y = -t, \quad x = -t - 2t = -3t$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3t \\ -t \\ t \end{pmatrix}$, eigenvector $= \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$ for $\lambda = 4$

AM =

GM =

Shortcut for rank 2 case

(Cramer's Rule for 2 independent rows)

Two independent rows

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \end{bmatrix}$$

$6y + 6z = 0$

$x - y + 2z = 0$

$\dfrac{x}{\quad} = \dfrac{-y}{\quad} = \dfrac{z}{\quad} = t$

Then $\dfrac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\dfrac{y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}} = t$

$$\frac{x}{18} = + \frac{y}{(+6)} = \frac{z}{(-6)} = t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18t \\ 6t \\ -6t \end{pmatrix}, \text{ eigenvector} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{bmatrix} 0 & 6 & 6 \\ -1 & -4 & -7 \end{bmatrix}$$

$$\Rightarrow \frac{x}{\begin{vmatrix} 6 & 6 \\ -4 & -7 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 6 \\ -1 & -7 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ -1 & -4 \end{vmatrix}} = t$$

$$\Rightarrow \frac{x}{-18} = -\frac{y}{+6} = \frac{z}{6} = t$$

$$\begin{aligned} x &= -18t \\ y &= -6t \\ z &= 6t \end{aligned} \Rightarrow \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix} \qquad \begin{array}{ccc} x & y & z \\ 0 & 0 & 0 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{pmatrix}$$

$$\lambda = 1$$

for $\lambda = 1$
AM $= 1$
GM $= 1$

$[A - 1I]$

$$\checkmark \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve this by Cramer's Rule for 2 independent rows.

Consider 2^{nd} & 3^{rd} row

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = t$$

$$\underline{x} = \frac{-y}{-2} = \frac{z}{-2} = t$$

$$\frac{r}{0} \qquad -2$$

$$\begin{aligned} x &= 0 \\ y &= 2t \\ z &= -2t \end{aligned} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda = -1 \Rightarrow \boxed{\begin{pmatrix} -6 \\ -2 \\ 7 \end{pmatrix}}$$

Def^n: AM (Algebraic Multiplicity of λ_i)

AM for λ_i is number of times λ_i is repeated.

Geometric Multiplicity of λ_i (GM for λ_i)

GM for λ_i is number of independent eigenvectors which corresponds to eigenvalue λ_i

$$[A - \lambda_i I] X = 0$$

r

$$\boxed{\text{number of independent eigenvector (GM)}} = n - r$$

$r = 1$ & $n = 3$, $\boxed{n - r = 3 - 1 = 2}$

$r = 1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x + 2y - z = 0$$

$y = p$, $z = q$

$x = z - 2y = q - 2p$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} q-2p \\ p \\ q \end{pmatrix} = p\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$1\lambda^3 - P\lambda^2 + Q\lambda - R = 0$$

$\rightarrow ax^2 + bx + c = 0$

Suppose α & β are roots

$\alpha + \beta = -b/a$

$\alpha\beta = c/a$

$\rightarrow$ Sum of eigenvalue $= P$ $= SOD$

Product of eigenvalue $= R$ $= |A|$

Property :