SOMAIYA VIDYAVIHAR UNIVERSITY
K J Somaiya College of Engineering

Somaiya TRUST

# PARTIAL DIFFERENTIATION

## FYBTECH SEM-I

## MODULE-4

# Partial Derivatives of the first order

- Let $z = f(x, y)$ be a function of two independent variables $x$ and $y$.
- If we keep $y$ constant and allow only $x$ to vary then derivative, if it exists, so obtained is called the **partial derivative of $z$ with respect to $x$** and it is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x$.
- Thus, $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$
- Similarly, the derivative of $z$ with respect to $y$ keeping $x$ constant, if it exists is called the **partial derivative of $z$ with respect to $y$** and it is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y$.
- Thus, $\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$

# Partial Derivatives of Higher Order

- The partial derivatives of higher order, if they exist, can be obtained from partial derivatives of the first order by using the above definitions again.
- Thus, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ is the second order partial derivative of z w.r.t. x and is denoted by $\frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}$.
- Similarly, we have $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$ ,
- $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$
- And $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$

# Note

- **(1)** If $u = f(x, y)$ possesses continuous second-order partial derivatives $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$
- This is called commutative property
- **(2)** Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation

# Differentiation of a function of a function

- Let $z = f(u)$ and $u = \Phi(x, y)$ so that $z$ is function of $u$ and $u$ itself is a function of two independent variables $x$ and $y$.
- The two relations define $z$ as a function of $x$ and $y$.
- In such cases $z$ may be called a **function of a function** of $x$ and $y$.
- **e.g. (i)** $z = \frac{1}{u}$ and $u = \sqrt{x^2 + y^2}$
- **(ii)** $z = \tan u$ and $u = x^2 + y^2$

define $z$ as a function of a function of $x$ and $y$.

# Differentiation of a function of a function

If $z = f(u)$ is differentiable function of u and $u = \Phi(x, y)$possesses first order partial derivatives then,

- $\frac{\partial z}{\partial x} = \frac{dz}{du} . \frac{\partial u}{dx}$ i.e. $\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}$
- Similarly $\frac{\partial z}{\partial y} = \frac{dz}{du} . \frac{\partial u}{\partial y}$ i.e. $\frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y}$
- e.g. If $z = (ax + by)^n$ then
- $\frac{\partial z}{\partial x} = n(ax + by)^{n-1} . a$ and
- $\frac{\partial z}{\partial y} = n(ax + by)^{n-1} . b$

# EXAMPLE-1



- If $u = \cos\left(\sqrt{x} + \sqrt{y}\right)$, prove that
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)sin\left(\sqrt{x} + \sqrt{y}\right) = 0.$
- **Solution:** We have $\frac{\partial u}{\partial x} = -\sin\left(\sqrt{x} + \sqrt{y}\right)\frac{1}{2\sqrt{x}}$
- $\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x} + \sqrt{y}\right)\frac{1}{2\sqrt{y}}$
- $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x} + \sqrt{y}\right) \cdot \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)$$
- $$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sin\left(\sqrt{x} + \sqrt{y}\right) = 0$$

# EXAMPLE-2

- If $z(x+y) = x^2 + y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$
- **Solution:** Since $z = \frac{x^2+y^2}{x+y}$
- $\frac{\partial z}{\partial x} = \frac{(x+y)2x-(x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$
- $\frac{\partial z}{\partial y} = \frac{(x+y)2y-(x^2+y^2)}{(x+y)^2} = \frac{-x^2+2xy+y^2}{(x+y)^2}$
- $\therefore$ LHS $= \left[\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right]^2$
- $= \left[2 \cdot \frac{(x^2-y^2)}{(x+y)^2}\right]^2 = \left[2 \cdot \frac{(x-y)}{(x+y)}\right]^2 = 4\frac{(x-y)^2}{(x+y)^2}$
- Putting the values of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$
- RHS $= 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2}\right]$
- $= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = 4\frac{(x-y)^2}{(x+y)^2}$
- $\therefore$ LHS $=$ RHS

# EXAMPLE-3

- If $z = x^y + y^x$, verify that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$.
- **Solution:** Differentiating $z$ partially w.r.t. $y$ we get,
- $$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$
- Differentiating this partially w.r.t. $x$ we get,
- $$\frac{\partial^2 z}{\partial x\, \partial y} = yx^{y-1} \cdot \log x + x^y \cdot \frac{1}{x} + 1 \cdot y^{x-1} + xy^{x-1} \log y$$
- $$= yx^{y-1} \cdot \log x + x^{y-1} + y^{x-1} + xy^{x-1} \log y$$
- Now, differentiating $z$ partially w.r.t. $x$, we get,
- $$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$
- Differentiating this again partially w.r.t. $y$, we get,
- $$\frac{\partial^2 z}{\partial y\, \partial x} = x^{y-1} + y \cdot x^{y-1} \log x + \frac{y^x}{y} + xy^{x-1} \log y$$
- $$= yx^{y-1} \log x + x^{y-1} + y^{x-1} + xy^{x-1} \log y$$
- Hence, the result

# EXAMPLE-4



- If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.
- **Solution:** LHS $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$
- $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$ ............(i)
- Now, $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2 - 3yz)$ since u is symmetric
- Similarly, $\frac{\partial u}{\partial y} = \frac{3y^2-2zx}{x^3+y^3+z^3-3xyz}, \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$
- $\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\left(\frac{x^2+y^2+z^2-xy-yz-zx}{x^3+y^3+z^3-3xyz}\right) = \frac{3}{(x+y+z)}$
- $\{\because (x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z) = x^3 + y^3 + z^3 - 3xyz\}$
- Hence from (1), LHS $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot \frac{3}{(x+y+z)}$
- $= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right]$
- $= -\frac{9}{(x+y+z)^2} =$ RHS

# EXAMPLE-5

- If $u = e^{x^2+y^2+z^2}$, prove that $\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = 8\, xyzu$.
- **Solution:** $\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$
- $\frac{\partial^2 u}{\partial y\, \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)$
- $= 2z \cdot e^{x^2+y^2+z^2} \cdot 2y$
- $= 4yz \cdot e^{x^2+y^2+z^2}$
- $\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\, \partial z}\right)$
- $= 4yz \cdot e^{x^2+y^2+z^2} \cdot 2x$
- $= 8xyz \cdot e^{x^2+y^2+z^2}$
- $= 8xyzu$

# EXAMPLE-6



- If $\theta = t^n e^{-r^2/4t}$, find n which will make $\frac{\partial\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)$.
- **Solution:** $\frac{\partial\theta}{\partial t} = nt^{n-1}\cdot e^{-r^2/4t} + t^n e^{-r^2/4t}\cdot\left(\frac{r^2}{4t^2}\right)$
- $= \frac{n}{t}\theta + \frac{r^2}{4t^2}\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta$ ................(1)
- Also, $\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right) = -\frac{r\theta}{2t}$
- $\therefore r^2\frac{\partial\theta}{\partial r} = -\frac{r^3\theta}{2t}$
- $\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right) = -\frac{1}{2t}\frac{\partial}{\partial r}(r^3\theta)$
- $= -\frac{1}{2t}\left[r^3\frac{\partial\theta}{\partial r} + 3r^2\theta\right]$
- $= -\frac{1}{2t}\left[r^3\frac{-r\theta}{2t} + 3r^2\theta\right]$
- $= -\frac{1}{2t}\left[\frac{-r^4\theta}{2t} + 3r^2\theta\right]$
- $\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2t}\left[-\frac{r^2\theta}{2t} + 3\theta\right]$ ...............(2)
- $\therefore$ Equating (1) and (2), we get, $\frac{n}{t} = -\frac{3}{2t}$ $\quad\therefore n = -\frac{3}{2}$

# EXAMPLE-7

- If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ $and$ $a^2 + b^2 + c^2 = 1$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$
- **Solution:** $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$
- Differentiating $u$ partially w.r.t. $x$, $\frac{\partial u}{\partial x} = 6(ax + by + cz)a - 2x$
- Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. $x$, $\frac{\partial^2 u}{\partial x^2} = 6a \cdot a - 2 = 6a^2 - 2$
- Differentiating $u$ partially w.r.t. $y$, $\frac{\partial u}{\partial y} = 6(ax + by + cz)b - 2y$
- Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. $y$, $\frac{\partial^2 u}{\partial y^2} = 6b \cdot b - 2 = 6b^2 - 2$
- Differentiating $u$ partially w.r.t. $z$, $\frac{\partial u}{\partial z} = 6(ax + by + cz)c - 2z$
- Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. $z$, $\frac{\partial^2 u}{\partial z^2} = 6c \cdot c - 2 = 6c^2 - 2$
- Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(a^2 + b^2 + c^2) - 6$
- $= 6(1) - 6$ $[\because a^2 + b^2 + c^2 = 1]$
- $= 0$

# EXAMPLE-8

- If $u = f(r), r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.
- **Solution:** $u = f(r)$
- Differentiating $u$ partially w.r.t. $x$, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = \frac{\partial}{\partial r} f(r) \cdot \frac{\partial r}{\partial x}$
- $= f'(r) \cdot \frac{\partial r}{\partial x}$ ............... (1)
- But $r^2 = x^2 + y^2 + z^2$
- Differentiating $r^2$ partially w.r.t. $x$, $2r \frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$
- Substituting in Eq. (1), $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$
- Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. $x$, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r) \cdot \frac{x}{r}\right]$
- $= f''(r) \frac{\partial r}{\partial x} \cdot \frac{x}{r} + \frac{f'(r)}{r} + x f'(r)\left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}$
- $= f''(r) \frac{x}{r} \frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2} f'(r) \cdot \frac{x}{r}$
- $= f''(r) \frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3} f'(r)$ .............. (2)

# EXAMPLE-8



- Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r)$ .............. (3)
- and $\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r)$ .............. (4)

- Adding Eqs (2), (3) and (4),
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$
- $= \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2)}{r^3}f'(r)$
- $= \frac{f''(r)}{r^2} \cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3}f'(r)$
- $= f''(r) + \frac{2f'(r)}{r}$

# EXAMPLE-9

- If $z = u(x,y)\, e^{ax+by}$ where $u(x,y)$ is such that $\frac{\partial^2 u}{\partial x \partial y} = 0$, find the constants a, b
- such that $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0.$
- **Solution:** We have, from $z = u(x,y)e^{ax+by}$ ............... (1)
- $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \cdot e^{ax+by} + u \cdot e^{ax+by} \cdot a = e^{ax+by}\left(\frac{\partial u}{\partial x} + au\right)$ ............... (2)
- $\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \cdot e^{ax+by} + u \cdot e^{ax+by} \cdot b = e^{ax+by}\left(\frac{\partial u}{\partial y} + bu\right)$ ............... (3)
- Differentiating (3) partially w.r.t. $x$,
- $\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by} \cdot a \cdot \left(\frac{\partial u}{\partial y} + bu\right) + e^{ax+by}\left(\frac{\partial^2 u}{\partial x \partial y} + b \cdot \frac{\partial u}{\partial x}\right)$ ............... (4)
- But since by data $\frac{\partial^2 u}{\partial x \partial y} = 0$, we get,
- $\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\left(a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu\right)$ ............... (5)

# EXAMPLE-9



Further by data $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$ ............... (6)

- Putting the values from (1), (2), (3) and (5) in (6), we get,
- $$e^{ax+by}\left[a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu - \frac{\partial u}{\partial x} - au - \frac{\partial u}{\partial y} - bu + u\right] = 0$$
- $\therefore e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + au(b-1) - u(b-1)\right] = 0$
- $\therefore e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (b-1) \cdot u(a-1)\right] = 0$

- Since $u \neq 0, \frac{\partial u}{\partial x} \neq 0$ and $\frac{\partial u}{\partial y} \neq 0$
- We should have $a - 1 = 0, b - 1 = 0$ i.e., $a = 1, b = 1$

# COMPOSITE FUNCTIONS



- **(a)** Let $z = f(x, y)$ and $x = \Phi(t), y = \Psi(\mathrm{t})$ so that $z$ is function of $x$ , $y$ and $x$ , $y$ are function of third variable $t$.
- The three relations define $z$ as a function of $t$. In such cases $z$ is called a **composite function of *t***.
- **e.g. (i)** $z = x^2 + y^2 , x = at^2 , y = 2at$
- **(ii)** $z = \mathrm{x}^2\mathrm{y} + \mathrm{xy}^2 \,,\; x = acost \,,\; y = bsint$

  define $z$ as a composite function of $t$
- **Differentiation:** Let $z = f(x, y)$ posses continuous first order partial derivatives and $x = \Phi(t), y = \Psi(t)$ posses continuous first order derivatives then,

  $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

# EXAMPLE-11

- If $u = x^2y^3, x = \log t\,, y = e^t$, find $\frac{du}{dt}$
- **Solution:** $u = x^2y^3, x = \log t\,, y = e^t$
- $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$
- $$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$
- Substituting $x$ and $y$,
- $$\frac{du}{dt} = 2(\log t)e^{3t} \cdot \frac{1}{t} + 3(\log t)^2e^{2t} \cdot e^t$$
- $$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2e^{3t}$$

# EXAMPLE-12

- If $u = xy + yz + zx$ where $x = \frac{1}{t}, y = e^t, z = e^{-t}$, find $\frac{du}{dt}$
- **Solution:** $u = xy + yz + zx, x = \frac{1}{t}, y = e^t, z = e^{-t}$
- $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$
- $= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$
- Substituting $x, y$ and $z$,
- $\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$
- $= -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$

# EXAMPLE-13



- If $z = e^{xy}, x = t\cos t\,, y = t\sin t\,$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$
- **Solution:** $z = e^{xy}, x = t\cos t\,, y = t\sin t$
- $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$
- $= e^{xy} y(\cos t - t\sin t) + e^{xy} x(\sin t + t\cos t)$
- At $t = \frac{\pi}{2}, x = 0, y = \frac{\pi}{2}$
- Hence, $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^{0}\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right] = -\frac{\pi^2}{4}$`

# COMPOSITE FUNCTIONS

- **(b)** Let $z = f(x, y)$ and $x = \Phi(u, v)$, $y = \Psi(\mathrm{u}, \mathrm{v})$ so that z is function of $x$ , $y$ and $x$ , $y$ are function of $u$ , $v$.
- The three relations define $z$ as a function of $u$ , $v$. In such cases $z$ is called a **composite function of $\boldsymbol{u}$ , $\boldsymbol{v}$.**
- **e.g. (i)** $z = xy$ , $x = e^{u} + e^{-v}$ , $y = e^{-u} + e^{v}$
- **(ii)** $z = \mathrm{x}^2 - \mathrm{y}^2$ , $x = 2u - 3v$ , $y = 3u + 2v$
- define z as a composite function of $u$ and $v$

- **Differentiation:** Let $z = f(x, y)$ possess continuous first order partial derivatives and $x = \Phi(u, v), y = \Psi(u, v)$ possess continuous first order partial derivatives then,
- $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial v}$

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# EXAMPLE-14

If $x^2 = au + bv,\ y^2 = au - bv\ and\ z = f(x,y)$, Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.

- **Solution:** $z = f(x,y),\ \ x^2 = au + bv,\ y^2 = au - bv$
- $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$
- $= \frac{\partial z}{\partial x}\cdot\frac{a}{2x} + \frac{\partial z}{\partial y}\cdot\frac{a}{2y}$
- $u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y}$ ............. (1)
- $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$
- $= \frac{\partial z}{\partial x}\cdot\frac{b}{2x} + \frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$
- $v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y}$ ............. (2)

# EXAMPLE-14



- Adding Eqs. (1) and (2),
- $u\dfrac{\partial z}{\partial u}+v\dfrac{\partial z}{\partial v}$
- $=\dfrac{\partial z}{\partial x}\cdot\dfrac{au}{2x}+\dfrac{\partial z}{\partial y}\cdot\dfrac{au}{2y}+\dfrac{\partial z}{\partial x}\cdot\dfrac{bv}{2x}-\dfrac{\partial z}{\partial y}\cdot\dfrac{bv}{2y}$
- $=\dfrac{\partial z}{\partial x}\left(\dfrac{au+bv}{2x}\right)+\dfrac{\partial z}{\partial y}\left(\dfrac{au-bv}{2y}\right)$
- $=\dfrac{\partial z}{\partial x}\left(\dfrac{x^2}{2x}\right)+\dfrac{\partial z}{\partial y}\left(\dfrac{y^2}{2y}\right)$
- $=\dfrac{1}{2}\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)$

# EXAMPLE-15



- If z = f(u, v) and $u = \log(x^2 + y^2)\,, v = \frac{y}{x}$, prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}$
- **Solution:** $z = f(u, v), u = \log(x^2 + y^2)\,, v = \frac{y}{x},$
- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$
- $= \frac{\partial z}{\partial u} \cdot \frac{1}{x^2+y^2} \cdot 2x + \frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)$
- $y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2} \cdot \frac{\partial z}{\partial u} - \frac{y^2}{x^2} \cdot \frac{\partial z}{\partial v}$ ................. (1)
- $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$
- $= \frac{\partial z}{\partial u} \cdot \frac{2y}{x^2+y^2} + \frac{\partial z}{\partial v} \cdot \frac{1}{x}$
- $x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2} \cdot \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ ................. (2)
- Subtracting Eq. (1) from Eq. (2),
- Hence, $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} = (1 + v^2)\frac{\partial z}{\partial v}$

# EXAMPLE-16

If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

- **Solution:** Let $l = x^2 - y^2, m = y^2 - z^2, n = z^2 - x^2$
- $\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -2x$
- $\frac{\partial l}{\partial y} = -2y, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial n}{\partial y} = 0$
- $\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -2z, \quad \frac{\partial n}{\partial z} = 2z$
- $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) = f(l, m, n)$
- $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$
- $= \frac{\partial u}{\partial l} \cdot 2x + \frac{\partial u}{\partial m} \cdot 0 + \frac{\partial u}{\partial n} \cdot (-2x)$
- $\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n}$ ........... (1)

# EXAMPLE-16



$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$=\frac{\partial u}{\partial l}(-2y)+\frac{\partial u}{\partial m}(2y)+\frac{\partial u}{\partial n}(0)$$

$$\frac{1}{y}\frac{\partial u}{\partial y}=-2\frac{\partial u}{\partial l}+2\frac{\partial u}{\partial m} \qquad \text{..........} (2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$=\frac{\partial u}{\partial l}\cdot 0+\frac{\partial u}{\partial m}(-2z)+\frac{\partial u}{\partial n}(2z)$$

$$\frac{1}{z}\frac{\partial u}{\partial z}=-2\frac{\partial u}{\partial m}+2\frac{\partial u}{\partial n} \qquad \text{..........} (3)$$

Adding Eqs (1), (2) and (3),

$$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$$

# EXAMPLE-17



- If $x = e^u cosec\ v, y = e^u \cot v\ and\ z$ is a function of x and y, prove that
- $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - sin^2 v \left(\frac{\partial z}{\partial v}\right)^2\right]$
- **Solution:** $z = f(x, y), x = e^u cosec\ v, y = e^u \cot v$
- $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$
- $= \frac{\partial z}{\partial x} e^u cosec\ v + \frac{\partial z}{\partial y} e^u \cot v$
- $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$
- $= \frac{\partial z}{\partial x}(-e^u cosec\ v \cot v) + \frac{\partial z}{\partial y}(-e^u cosec^2 v)$

# EXAMPLE-17



- R.H.S = $e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$
- $=$
  $e^{-2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 e^{2u} cosec^2\, v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} e^{2u} cosec\, v\cot v\right.$
- $+(-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2 (e^{2u} cosec^2\, v\cot^2 v) +$
- $\left.(-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\,\mathrm{cosec}^4\, v + (-\sin^2 v)2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} e^{2u} cosec^3\, v\cot v\right]$
- $= \left(\frac{\partial z}{\partial x}\right)^2 (cosec^2 v - \cot^2 v) + \left(\frac{\partial z}{\partial y}\right)^2 (\cot^2 v - cosec^2\, v)$
- $= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$ = L.H.S.

# EXAMPLE-18

- If $x+y=2e^{\theta}\cos\Phi\,, x-y=2\,i\,e^{\theta}\sin\Phi$, show that $\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\Phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$
- **Solution:** Adding the given results, $2x=2e^{\theta}(\cos\Phi+i\sin\Phi)$
- $\therefore x=e^{\theta}\cdot e^{i\Phi}=e^{\theta+i\Phi}$
- and subtracting results, $2y=2e^{\theta}(\cos\Phi-i\sin\Phi)$
- $\therefore y=e^{\theta-i\Phi}$
- Now, $u$ is a function of $x, y$ and $x, y$ are functions of $\theta$ and $\Phi$
- $\therefore \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$
- $=\frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}+\frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}=\frac{\partial u}{\partial x}\cdot x+\frac{\partial u}{\partial y}\cdot y$ ............ (1)
- $\therefore \frac{\partial}{\partial\theta}\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ ............ (2)
- $\therefore \frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$ ................. [From (1)]
- $=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$ ................. [From (2)]
- $=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ ............ (3)

# EXAMPLE-18

- Also, $\frac{\partial u}{\partial \Phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \Phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \Phi}$
- $= \frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}\cdot i + \frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}\cdot(-i)$
- $= \frac{\partial u}{\partial x}\cdot ix - i\frac{\partial u}{\partial y}\cdot y$ ………… (4)
- $\therefore \frac{\partial}{\partial \Phi} \equiv i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$ ………… (5)
- $\therefore \frac{\partial^2 u}{\partial \Phi^2} = \frac{\partial}{\partial \Phi}\left(\frac{\partial u}{\partial \Phi}\right) = \frac{\partial}{\partial \Phi}\left[i\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right)\right]$ ……………… [From (4)]
- $= i\left[i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right)\right]$ ……………… [From (5)]
- $= -\left[x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$
- $= -x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} - y^2\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$ ………… (6)
- $\therefore$ Adding the two results, (5) and (6) we get,
- $\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \Phi^2} = 4xy\frac{\partial^2 u}{\partial x\partial y}$