Cauchy's Homogeneous Linear DE

Wednesday, March 10, 2021 10:46 AM

Eq^n of type, $\boxed{x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+7y = f(x)}$

are called Cauchy's Homogeneous LDE

It can be transformed into constant coeff LDE by substitution, $z = \log x$, $x = e^z$

$$\frac{dz}{dx} = \frac{1}{x}, \quad \boxed{\frac{dy}{dx}} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{x}\frac{d}{dz}(y)$$

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dz}\right]$$

$$= \frac{dy}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d}{dx}\left[\frac{dy}{dz}\right]$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{1}{x}\frac{d}{dz}\right)\left(\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Similarly, $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$

Put $\frac{d}{dz} \Rightarrow D$, we get

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (D^2 - D)y = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} = (D^3 - 3D^2 + 2D)y = D(D-1)(D-2)y$$

Problems on Cauchy's DE

Wednesday, March 10, 2021 12:02 PM

Problems:

1) $x^3\frac{d^3y}{dx^3}+6x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+2y=x^2+3x-4$ — (1)

This is Cauchy's DE, Putting $z=\log x$, $x=e^z$, $D=\frac{d}{dz}$

we get, $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$

$x^2\frac{d^2y}{dx^2}=D(D-1)y$, $x\frac{dy}{dx}=Dy$

put in (1) $\left[(D(D-1)(D-2)y)+6(D(D-1)y)+8Dy+2y\right]$

$=e^{2z}+3e^z-4$

$\therefore\left[(D^3-3D^2+2D)+6D^2-6D+8D+2\right]y=e^{2z}+3e^z-4$

$\therefore\left[D^3+3D^2+4D+2\right]y=e^{2z}+3e^z-4$

roots: $D=-1,\ -1\pm i$

$y_c=c_1e^{-z}+e^{-z}\left[c_2\cos z+c_3\sin z\right]$

Now, $y_p=\left(\frac{1}{D^3+3D^2+4D+2}\right)e^{2z}+3e^z-4e^{0z}$

$=\left[\frac{1}{F(D)}\right]e^{2z}+3\left[\frac{1}{F(D)}\right]e^z-4\left[\frac{1}{F(D)}\right]e^{0z}$

(Replace D by 2)

$=\frac{1}{30}e^{2z}+\frac{3}{10}e^z-\frac{4}{2}e^{0z}=$

$y_p=$

(2) $y_g=y_c+y_p=c_1e^{-z}+e^{-z}\left[c_2\cos z+c_3\sin z\right]+\frac{e^{2z}}{30}+\frac{3e^z}{10}-2$

(x) $y_g=c_1\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)\left[c_2\cos(\log x)+c_3\sin(\log x)\right]$

$+\frac{x^2}{30}+\frac{3x}{10}-2$

$+\left(\frac{1}{}\right)\frac{dy}{} = 12\log x$

2) $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

multiply by x^2 ✓

$x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 12\log x$ H.W.

$(D^2 - D)y + Dy = 12z \Rightarrow D^2 y = 12z$

3) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = (\log x)^2 + x\sin(\log x)$

This is in Cauchy DE form, $z = \log x$, $x = e^z$, $D = \dfrac{d}{dz}$

steps $\left\{ \right.$

$(D^2 - D)y + Dy + y = z^2 + e^z \sin z$

$(D^2+1)y = z^2 + e^z \sin z$

$y_c = c_1 \cos z + c_2 \sin z$

$y_p = \left[\dfrac{1}{D^2+1}\right](z^2 + e^z \sin z) = \left[\dfrac{1}{1+D^2}\right] z^2 + \left[\dfrac{1}{D^2+1}\right] e^z \sin z$

$= \left[1 - D^2\right] z^2 + e^z \left[\dfrac{1}{(D+1)^2+1}\right] \sin z$

(Replace D by $D+1$)

$= (z^2 - 2) + e^z \left[\dfrac{1}{D^2+2D+2}\right] \sin z$

(Replace D^2 by -1)

$— \| — \quad + e^z \left[\dfrac{1}{2D+1}\right] \sin z$

$+ e^z \left[\dfrac{(2D-1)}{(2D+1)(2D-1)}\right] \sin z$ → HW :

$y_p = z^2 - 2 - e^z (2\cos z - \sin z)$

$$y_p = z^2 - 2 - \frac{e^z}{5}(2\cos z - \sin z)$$

(a) $y_g = ?$

---

4) $\left(\frac{d}{dx} + \frac{1}{x}\right)^2 y = \frac{1}{x^4}$

$(a+b)^2 = a^2 + ab + ba + b^2$

$\frac{d}{dx}\left(\frac{1}{x}\right)$

$\frac{1}{x}\left(\frac{d}{dx}\right)$

? $\rightarrow$ $\left[\cancel{\frac{d^2}{dx^2}} + 2\frac{1}{x}\frac{d}{dx} + \frac{1}{x^2}\right] y = \frac{1}{x^4}$

Correct or wrong?

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x^2}\right] y = \frac{1}{x^4}$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{d}{dx}\left(\frac{1}{x}y\right) + \frac{y}{x^2} = \frac{1}{x^4}$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}\frac{dy}{dx} + y\left(-\cancel{\frac{1}{x^2}}\right) + \cancel{\frac{y}{x^2}} = \frac{1}{x^4}$$

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = \frac{1}{x^4}$$

multiply by x^2

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{1}{x^2}$$

H.W.

$$(D^2 - D)y + 2Dy = e^{-2z}$$

$$(D^2 + D)y = e^{-2z} \longrightarrow \text{Complete as HW}$$

---

5) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1+x)^2}$

6) displacement 'u' at distance 'r' is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + Kr = 0$$

Find displacement if $u=0$ when $r=0$ & $r=a$