HYPERBOLIC FUNCTIONS

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

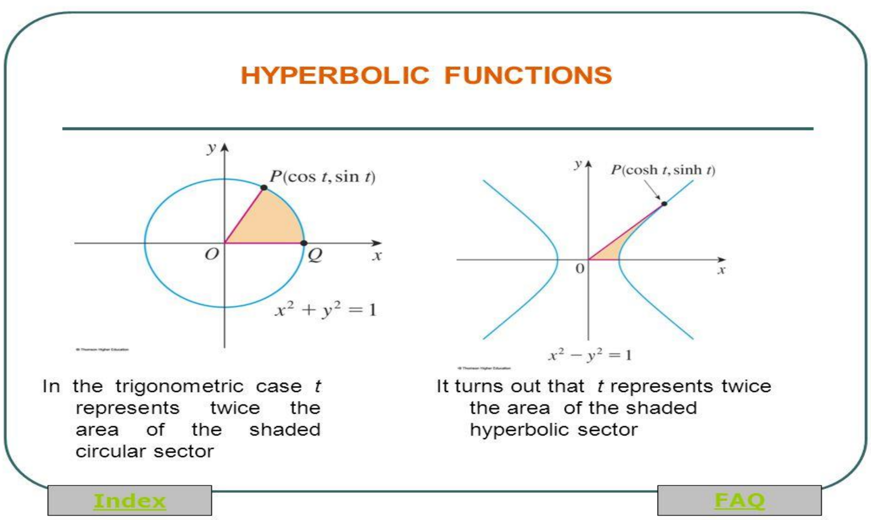
If x is real or complex, then sine hyperbolic of x is denoted by $sinh\, x$ and is given as, $\sinh x = \frac{e^{x}-e^{-x}}{2}$ and

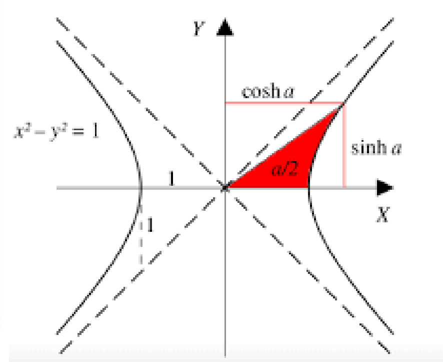
Cosine hyperbolic of x is denoted by $cosh\, x$ and is given as, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as $\tan \mathrm{hx} = \frac{\sinh \mathrm{x}}{\cosh \mathrm{x}} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$$\mathrm{cosechx} = \frac{1}{\sinh \mathrm{x}} = \frac{2}{e^{x}-e^{-x}}, \quad \mathrm{sech\,x} = \frac{1}{\cosh \mathrm{x}} = \frac{2}{e^{x}+e^{-x}}, \text{ and } \quad \coth \mathrm{x} = \frac{1}{\tanh \mathrm{x}} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

GRAPHICAL ILLUSTRATIONS AND APPLICATIONS:

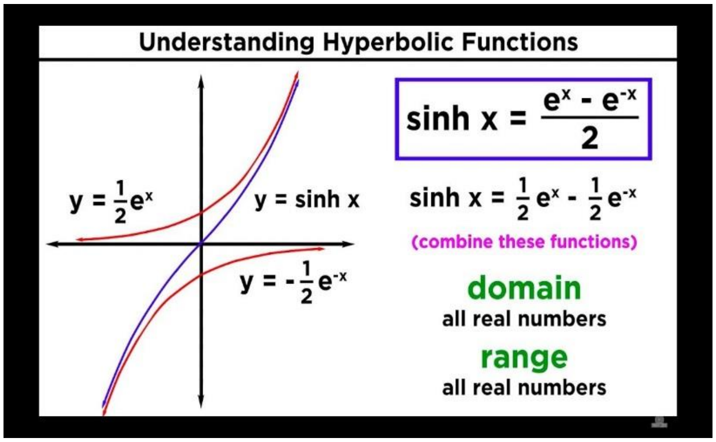




https://trigidentities.info

Hyperbolic functions arise as point on unit rectangular hyperbola with parameter as twice of area shaded in diagram.

Applications: Catenary, is a curve which describes the shape of a flexible hanging chain or cable, i.e. freely hanging string supported at two towers (fixed ends) and acted upon only by gravity. The graph of catenary is given by the function cosh x



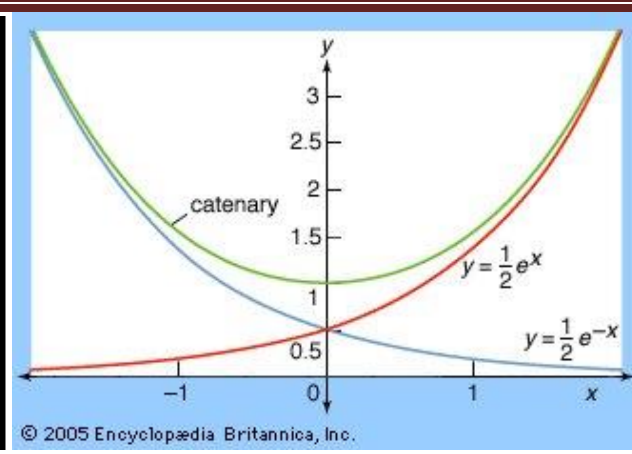
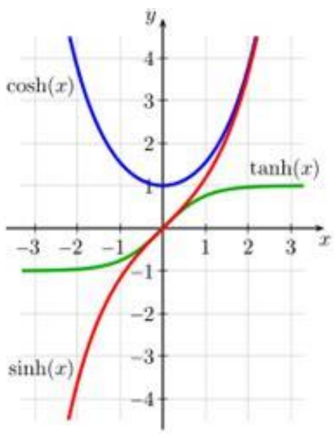
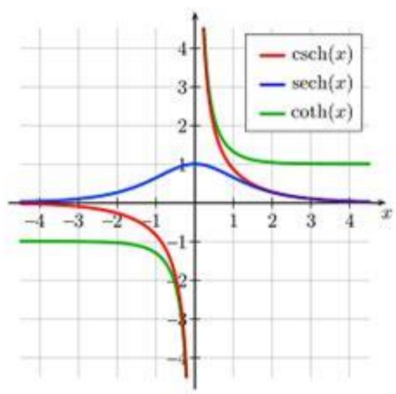


TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinh x, cosh x, tanh x, we can obtain the following values of hyperbolic function.





x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1,\ \tanh(0) = 0,\ \tanh(\infty) = 1$ $\therefore |\tanh x| \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	**HYPERBOLIC FUNCTIONS**
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x,$
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$
3	$e^{ix} = \cos x + i\sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$

5	$sin^2x + cos^2x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\text{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \text{cosec}^2 x$	$\coth^2 x - \text{cosech}^2 x = 1$
8	$\sin 2x = 2 \sin x \cos x$ $= \frac{2 \tan x}{1+ \tan^2 x}$	$\sinh 2x = 2 \sinh x \cosh x$ $= \frac{2 \tanh x}{1- \tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1- \tan^2 x}{1+ \tan^2 x}$	$\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1 + \tanh^2 x}{1- \tanh^2 x}$
10	$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$	$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$
11	$\sin 3x = 3 \sin x - 4 \sin^3 x$	$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3 \cos x$	$\cosh 3x = 4\cosh^3 x - 3 \cosh x$
13	$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$	$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$
14	$\sin(x \pm y) = \sin x \cos y \pm \cos x \, sin\, y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
15	$\cos(x \pm y) = \cos x \cos y \mp \sin x \, sin\, y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
16	$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tanh y}$	$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
17	$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \frac{-\coth x \coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$	$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}$
19	$\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$	$\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}$
20	$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$	$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}$
21	$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$	$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$
22	$2 \sin x \cos y = \sin(x+y) + \sin(x-y)$	$2 \sinh x \cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2 \cos x \sin y = \sin(x+y) - \sin(x-y)$	$2 \cosh x \sinh y = \sinh(x+y) - \sinh(x-y)$
24	$2 \cos x \cos y = \cos(x+y) + \cos(x-y)$	$2 \cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2 \sin x \sin y = \cos\,(x-y) - \cos(x+y)$	$2 \sinh x \sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$$\sinh(2\pi i + x) = \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x$$
$$= i\sin 2\pi \cosh x + \cos 2\pi \sinh x$$
$$= 0 + \sinh x$$
$$= \sinh x$$

Hence sinh x is a periodic function of period 2πi

Similarly we can prove that cosh x and tanh x are periodic functions of period 2πi and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$, $y = \frac{e^x - e^{-x}}{2}$ $\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\, x$, $y = \frac{e^x + e^{-x}}{2}$, $\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

If $y = cosh\, x$, $\frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$, $y = \frac{\sinh x}{\cosh x}$ $\therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$

If $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\, dx = \sinh x$, $\int \sinh x\, dx = \cosh x$, $\int sech^2 x dx = \tanh x$

SOME SOLVED EXAMPLES:

1. If tanh x $= \frac{1}{2}, find$ $\sinh 2x$ and $\cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$ $\therefore \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2}$ $\therefore 2\, e^{2x} - 2 = e^{2x} + 1$ $\therefore\ e^{2x} = 3$

Now, $sin\, h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3 - (1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3 + (1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\, x + 8sinh\, x = 1$ for real values of x.

Solution: $7cosh\, x + 8sinh\, x = 1$

Putting the values of $coshx$ and $sin\, hx$, we get

$\therefore 7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$

$\therefore 7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$

$\therefore 15e^{x} - e^{-x} = 2$

$\therefore 15e^{2x} - 2e^{x} - 1 = 0$ Solving it as a quadratic equation in e^{x},

$e^{x} = \frac{2 \pm \sqrt{4-4(15)(-1)}}{2(15)} = \frac{2\pm 8}{30} = \frac{1}{3}\ \ or -\frac{1}{5}$

$\therefore x = \log\left(\frac{1}{3}\right)\ \ or\ x = \log\left(-\frac{1}{5}\right)$

Since x is real, $x = log\left(\frac{1}{3}\right) = -\log 3$

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Solution: Let $\sin h^{-1}\ a = \alpha,\ \sin h^{-1} b = \beta\ \ and\ \ \sin h^{-1}\ x = \gamma$

We are given $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ $\quad \therefore \alpha + \beta = \gamma$

$\therefore \sinh(\alpha+\beta) = \sinh\gamma$

$\therefore \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta = \sinh\gamma$ ....(A)

But $\sinh\alpha$ = a, $\sinh\beta = b$, $\sinh\gamma = x$

$\therefore \cos h\ \alpha = \sqrt{1+\sin h^2\alpha} = \sqrt{1+a^2}$ and $\cos h\ \beta = \sqrt{1+\sin h^2\beta} = \sqrt{1+b^2}$

Putting this values in (A), we get $a\sqrt{1+b^2} + b\sqrt{1+a^2} = x$

4. Prove that $16\ sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

Solution: LHS $= 16\sin^5 x$

$= 16\left(\frac{e^{x}-e^{-x}}{2}\right)^5$

$= \frac{16}{32}(e^{5x} - 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} - 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} - e^{-5x})$

$= \frac{1}{2}(e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x})$

$= \left(\frac{e^{5x}-e^{-5x}}{2}\right) - 5\left(\frac{e^{3x}-e^{-3x}}{2}\right) + 10\left(\frac{e^{x}-e^{-x}}{2}\right)$

$= \sinh 5x - 5\sinh 3x + 10\sinh x$

= RHS

5. Prove that $16cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$

Solution: $l.h.s = 16cosh^5 x$

$= 16\left(\frac{e^{x}+e^{-x}}{2}\right)^5$ [By Binomial Theorem]

$= \frac{16}{32}[e^{5x} + 5e^{4x}.e^{-x} + 10e^{3x}.e^{-2x} + 10e^{2x}.e^{-3x} + 5e^{x}.e^{-4x} + e^{-5x}]$

$= \frac{(e^{5x}+e^{-5x})}{2} + 5\frac{(e^{3x}+e^{-3x})}{2} + 10\frac{(e^{x}+e^{-x})}{2}$

$= \cos h\ 5\ x + 5\cos h\ 3\ x + 10\cos h\ x = r\ .h.s$

6. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1-cosh^2x}}} = cosh^2x$

Solution: $l.h.s = \frac{1}{1-\frac{1}{1-\frac{1}{-sinh^2x}}} = \frac{1}{1-\frac{1}{1+cosec\,h^2x}} = \frac{1}{1-\frac{1}{\cot h^2x}} = \frac{1}{1-\tan h^2x} = \frac{1}{1-\frac{sinh^2x}{\cos h^2x}} = \frac{\cos h^2x}{\cos h^2x-\sin h^2x} = \cos h^2x$

7. If $u = \log tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$

(iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Solution: **(i)** $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$\therefore e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1+\tan\theta/2}{1-\tan\theta/2}$

$\therefore e^{-u} = \frac{1-\tan\theta/2}{1+\tan\theta/2}$

$\therefore \cosh u = \frac{e^u+e^{-u}}{2}$

$= \frac{1}{2}\left[\frac{(1+2\tan\theta/2+\tan^2\theta/2)+(1-2\tan\theta/2+\tan^2\theta/2)}{1-\tan^2\theta/2}\right]$

$= \frac{1}{2}\left(\frac{2+2\tan^2\theta/2}{1-\tan^2\theta/2}\right)$

$= \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1}{\cos\theta} = \sec\theta$

(ii) $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

(iii) $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

(iv) $\tan h\left(\frac{u}{2}\right) = \frac{\sin h(u/2)}{\cos h(u/2)} = \frac{2\sin h(u/2).\cos h(u/2)}{2\cos h(u/2)\cos h(u/2)} = \frac{\sin hu}{1+\cos hu} = \frac{\tan\theta}{1+\sec\theta}$ (By (i) and (ii)

$\therefore \tan h\left(\frac{u}{2}\right) = \frac{sin\theta/cos\theta}{(\cos\theta+1)/\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)} = \frac{sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Solution: **(i)** $\cosh x = \sec\theta$

$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta$ By definition $\cos hx = \frac{e^x+e^{-x}}{2}$

$\therefore e^x - 2\sec\theta + e^{-x} = 0$

$\therefore (e^x)^2 - 2e^x sec\theta + 1 = 0$

Solving the quadratic in e^x, $e^x = \frac{2sec\theta\pm2\sqrt{sec^2\theta-1}}{2} = \sec\theta \pm \tan\theta$

$\therefore x = \log(\sec\theta \pm \tan\theta) = \pm log(\sec\theta + \tan\theta)$

(we can prove that $log(\sec\theta - \tan\theta) = -log(\sec\theta + \tan\theta)$)

(ii) Let $tan^{-1}e^{-x} = \alpha \quad \therefore e^{-x} = \tan\alpha \quad \therefore e^{x} = \cot\alpha$

Now, by data $\sec\theta = \cos hx = \frac{e^x + e^{-x}}{2} = \frac{cot\alpha + tan\alpha}{2}$

$$2\,sec\theta = cot\alpha + tan\alpha = \frac{cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{cos\alpha} = \frac{2}{\sin 2\alpha}$$

$$\therefore cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$$

(iii) $\tan h\,\frac{x}{2} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^x - 1}{e^x + 1} = \frac{sec\theta + tan\theta - 1}{sec\theta + tan\theta + 1} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}$

$$= \frac{(1 - cos\theta) + \sin\theta}{(1 + cos\theta) + \sin\theta} = \frac{2sin^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)} = \frac{\sin(\theta/2)}{cos(\theta/2)} = tan\frac{\theta}{2}$$

SEPARATION OF REAL AND IMAGINARY PARTS:

Many a time we are required to separate real and imaginary parts of a given complex function, w = f(z) For this, we have to use identities of circular and hyperbolic functions.

w = f(z) $\therefore u + iv = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$ (but $\cos iy = \cosh y, \sin iy = i \sinh y$)

$$= \sin x \cosh y + i \cos x \sinh y$$

$$\therefore Real\ part\ (u) = \sin x \cosh y \qquad and\ \ Imaginary\ part\ (v) = \cos x \sinh y$$

Similarly we can separate real and imaginary parts for cos (x +iy) , cosh (x +iy), sinh (x +iy).

For tan (x+iy) = $\frac{\sin(x+iy)}{\cos(x+iy)}$ Multiply and divide by 2cos(x -iy)

$$= \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$$

$$\therefore Real\ part\ (u) = \frac{\sin 2x}{\cos 2x + \cosh 2y} \qquad and\ \ Imaginary\ part\ (v) = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$\therefore \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta) + \tan(\alpha-i\beta)}{1 - \tan(\alpha+i\beta).\ \tan(\alpha-i\beta)}$$

$$= \frac{(x+iy) + (x-iy)}{1 - (x+iy)(x-iy)} = \frac{2x}{1 - x^2 - y^2}$$

$\therefore 1 - x^2 - y^2 = 2x\cot 2\alpha \qquad \therefore x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta) - \tan(\alpha-i\beta)}{1 + \tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$

$$i \tanh 2\beta = \frac{(x+iy) - (x-iy)}{1 + (x+iy)(x-iy)} = \frac{2i\,y}{1 + x^2 + y^2}$$

$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$

$\therefore 1 + x^2 + y^2 = 2y \coth 2\beta$ i. e., $x^2 + y^2 - 2y \coth 2\beta + 1 = 0$

SOME SOLVED EXAMPLES:

1. Separate into real and imaginary parts $tan^{-1}(e^{i\,\theta})$

Solution: Let $tan^{-1}e^{i\theta} = x + iy$ $\therefore e^{i\theta} = \tan(x + iy)$ $\therefore cos\theta + i \sin\theta = \tan(x + iy)$

Similarly, $cos\theta - i \sin\theta = \tan(x - iy)$

Now, $tan\ 2x = \ tan\ [\ (x + iy) + \ (x - iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)+(cos\theta-i\sin\theta)}{1-(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ cos\theta}{1-(cos^2\theta+sin^2\theta)} = \frac{2\ cos\theta}{1-1} = \frac{2\ cos\theta}{0} = \infty$$

$\therefore 2x = \frac{\pi}{2}$ $\therefore x = \frac{\pi}{4}$

Also $\tan 2\ iy = \tan[(x + iy) - (x - iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)-(cos\theta-i\sin\theta)}{1+(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ isin\theta}{1+(cos^2\theta+sin^2\theta)} = \frac{2isin\theta}{2}$$

$\therefore i \tan h\ 2y = i \sin\theta$ $\therefore \tan h\ 2y = \sin\theta$

$\therefore 2y = tanh^{-1} \sin\theta$ $\therefore y = \frac{1}{2}\tan h^{-1} \sin\theta$

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Solution: $\sin(\alpha - i\,\beta) = x + i\,y$

$\therefore \sin\alpha \cos h\,\beta - i \cos\alpha \sin h\,\beta = x + iy$

Equating real and imaginary parts, we get, $\sin\alpha \cos h\,\beta = x$ and $-\cos\alpha \sin h\,\beta = y$

$\therefore \frac{x^2}{\cos h^2\beta} + \frac{y^2}{\sin h^2\beta} = sin^2\alpha + cos^2\alpha = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = \cos h^2\beta - \sin h^2\beta = 1$

3. If $cos(x + i\,y) = \cos\alpha + i\sin\alpha\,, prove\ that$

(i) $\sin\alpha = \ \pm sin^2x = \pm\ \sin h^2y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Solution: $\cos(x + iy) = \cos\alpha + i\sin\alpha$

$\cos x \cos(iy) - \sin x \sin(iy) = \cos\alpha + i\sin\alpha$

$\cos x \cosh y - i \sin x \sinh y = \cos\alpha + i\sin\alpha$

Equating real and imaginary parts, we get,

$\cos x \cosh y = \cos\alpha$ and $-\sin x \sinh y = \sin\alpha$

(i) Since $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$

$\sin^2 x \sinh^2 y + (1 - \sin^2 x)(1 + \sinh^2 y) = 1$

$\sin^2 x \sinh^2 y + 1 + \sinh^2 y - \sin^2 x - \sin^2 x \sinh^2 y = 1$

$1 + \sinh^2 y - \sin^2 x = 1$

$\sinh^2 y - \sin^2 x = 0$

$\therefore \sinh^2 y = \sin^2 x$ ....(i)

$\therefore \sinh y = \pm \sin x$

$\therefore \sin\alpha = -\sin x \sinh y = -\sin x\,(\pm \sin x) = \pm \sin^2 x$

(ii) $\cos 2x + \cosh 2y = 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sin^2 x$ .... from (i)

$= 2$

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Solution: We have to separate real part $\pi/6$ and imaginary part α

$\because tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$ $\quad\therefore tan\left(\frac{\pi}{6} - i\alpha\right) = x - iy$

$\therefore tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{tan\left(\frac{\pi}{6}+i\alpha\right)+\tan\left(\frac{\pi}{6}-i\alpha\right)}{1-\tan\left(\frac{\pi}{6}+i\alpha\right).\tan\left(\frac{\pi}{6}-i\alpha\right)}$

$\therefore tan\frac{\pi}{3} = \frac{(x+iy)+(x-iy)}{1-(x+iy).(x-iy)}$

$\therefore \sqrt{3} = \frac{2x}{1-x^2-y^2}$

$\therefore 1 - x^2 - y^2 = \frac{2x}{\sqrt{3}}$

$\therefore x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1.$

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$. (You can use alternate method)

Solution: We have $x + iy = c\cot(u + iv)$ $\quad\therefore\; x - iy = c\cot(u - iv)$

$\therefore 2x = c[\cot(u + iv) + \cot(u - iv)]$

$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$

$= c\,\frac{[\cos(u+iv)\sin(u-iv)+\sin(u+iv)\cos(u-iv)]}{\sin(u+iv)\sin(u-iv)}$

$\therefore 2x = \frac{c sin[(u-iv)+(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u-iv-u+iv)]/2}$

$\therefore x = \frac{c\sin 2u}{-[cos 2u-\cos 2\,iv]} = \frac{c\sin 2u}{\cos h\;2v-\cos 2u}$ ....(1)

Now, $2iy = c[\cot(u + iv) - \cot(u - iv)]$

$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} - \frac{\cos(u-iv)}{\sin(u-iv)}\right]$

$= c\left[\frac{\cos(u+iv)\sin(u-iv)-\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$

$\therefore 2iy = \frac{c\sin[(u-iv)-(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u+iv-u+iv)]/2}$

$\therefore\; iy = \frac{c\sin(-2iv)}{-[\cos 2u-\cos 2iv]} = -\frac{i\,c\sin h\;2v}{\cos h\;2v-\cos 2u}$

$\therefore\; y = \frac{-\,c\sin h\;2v}{\cos h\;2v-\cos 2u}$ ....(2)

From (1) & (2) $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$

6. If $u + i\,v = cosec\left(\frac{\pi}{4} + i\,x\right), prove\ that\ (u^2 + v^2)^2 = 2(u^2 - v^2)$

Solution: We have $\frac{1}{\sin[(\pi/4)+ix]} = u + iv$

$\therefore \sin\left(\frac{\pi}{4} + ix\right) = \frac{1}{u+iv} = \frac{1}{u+iv}.\frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$

$\therefore \sin\frac{\pi}{4}\cos i\,x + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$

$\frac{1}{\sqrt{2}}\cos h\,x + i\frac{1}{\sqrt{2}}\sin hx = \frac{u-iv}{u^2+v^2}$

Equating real and imaginary parts $\cos hx = \sqrt{2}.\left(\frac{u}{u^2+v^2}\right)$; $\sin hx = -\sqrt{2}.\left(\frac{v}{u^2+v^2}\right)$

But $cosh^2x - sinh^2x = 1$

$\therefore 2\left(\frac{u^2}{(u^2+v^2)^2}\right) - 2\left(\frac{v^2}{(u^2+v^2)^2}\right) = 1$

$\therefore 2(u^2 - v^2) = (u^2 + v^2)^2$

7. If $x + iy = \cos(\alpha + i\,\beta)$ or if $\cos^{-1}(\mathrm{x} + \mathrm{iy}) = \alpha + \mathrm{i}\,\beta$ express x and y in terms of α and β.

Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$

Solution: We have $\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = x + iy$

$\therefore \cos\alpha\cos h\,\beta - i\sin\alpha\sin h\beta = x + iy$

Equating real and imaginary parts $\cos\alpha\cos h\,\beta = x$ *and* $\sin\alpha\sin h\beta = -y$

We know that, in terms of the roots, the quadratic equation is given by

$\lambda^2 - (sum\ of\ the\ roots)\lambda + (product\ of\ the\ roots) = 0$

Hence the equation whose roots are $\cos^2\alpha$ and $\cosh^2\beta$ is

$\lambda^2 - (cos^2\alpha + cos^2\beta)\lambda + (cos^2\alpha.cos^2\beta) = 0$

This means we have to prove that $x^2 + y^2 + 1 = cos^2\alpha + cos^2\beta$ and $x^2 = cos^2\alpha + cos^2\beta$

Now, $x^2 + y^2 + 1 = cos^2\alpha\cos h^2\beta + sin^2\alpha\sin h^2\beta + 1$

$= cos^2\alpha\cos h^2\beta + (1 - cos^2\alpha)(\cos h^2\beta - 1) + 1$

$= cos^2\alpha\cos h^2\beta + \cos h^2\beta - 1 - cos^2\alpha\cos h^2\beta + cos^2\alpha + 1$

$= cos^2\alpha + \cos h^2\beta = sum\ of\ the\ roots$

And $x^2 = cos^2\alpha\cos h^2\beta$ = Product of the roots

Hence the equation whose roots are $cos^2\alpha$, $\cos h^2\beta$ is $\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$