VOLUME OF SOLIDS

Wednesday, June 2, 2021 11:30 AM

(a) To express the volume of a solid as triple integral we note that the volume of an elementary cuboid with its faces parallel to the coordinate planes is $dx\,dy\,dz$ and therefore, the volume of the solid is given by

$V = \iiint dx\,dy\,dz$, where the limits of integration w.r.t z (if we integrate first w.r.t z) are z_1 and z_2 obtained from the equations of the top and the bottom of the given surface, and then the double integration is w.r.t x and y carried out over the area of projection of the given solid on the xy – plane.

(b) If the volume of an elementary cuboid in cylindrical polar system is $r\,d\theta\,dr\,dz$, then the volume of the solid is given by,

$$V = \iiint r\,d\theta\,dr\,dz$$

cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

(c) In spherical polar system, $V = \iiint r^2 \sin\theta\,dr\,d\theta\,d\phi$

$x = r\sin\theta\cos\phi$

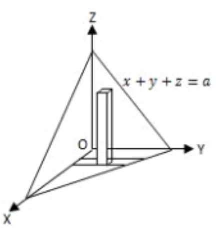
$y = r\sin\theta\sin\phi$

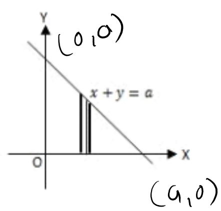
$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

1. Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a$.

Solution:





On the elementary cuboid z varies from $z = 0$ to $z = a - x - y$

y varies from $y = 0$ to $y = a - x$, x varies from $x = 0$ to $x = a$

$$\therefore V = \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dxdydz$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} (z)_0^{a-x-y}\, dy\,dz$$

$$= \int^{a}\int^{a-x} [(a-x) - y]\,dy\,dx$$

$$= \int_0 \int_0 (a-x) - y \,] \, dy \, dx$$

$$= \int_0^a \left[(a-x)y - \frac{y^2}{2} \right]_0^{a-x} dx$$

$$= \int_0^a \left[(a-x)^2 - \frac{(a-x)^2}{2} \right] dx$$

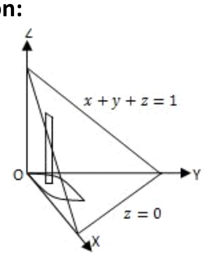
$$= \frac{1}{2} \int_0^a (a-x)^2 \, dx$$

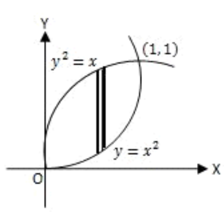
$$= \frac{1}{2} \left[\frac{(a-x)^3}{-3} \right]_0^a$$

$$= \frac{1}{2} \left[0 - \left(\frac{(a)^3}{-3} \right) \right] = \frac{a^3}{6}$$

2. Find the volume bounded by $y^2 = x, x^2 = y$ and the planes $z = 0$ and $x + y + z = 1.$

Solution:





$z = 0$ to $z = 1 - x - y$

The solid is bounded by the parabolas $y^2 = x$ and $x^2 = y$ in the xy –plane which is its base and by the plane $x + y + z = 1$ at the top

$$V = \iiint_R^{1-x-y} dz \, dy \, dx = \iint_R (1 - x - y) \, dx \, dy$$

$$V = \iint_R \int_0^{1-x-y} dz\, dy\, dx = \iint_R (1-x-y)\, dx\, dy$$

$\therefore V = \iint_R z\, dxdy = \iint_R (1-x-y)\, dxdy$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1-x-y)\, dy\, dx$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left((1-x)\sqrt{x} - \frac{x}{2} - (1-x)x^2 + \frac{x^4}{2} \right) dx$$

$$= \int_0^1 \left(\sqrt{x} - x^{3/2} - \frac{x}{2} - x^2 + x^3 + \frac{x^4}{2} \right) dx$$

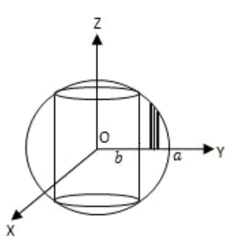
$$= \left(\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} - \frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right)_0^1$$

$$= \left(\frac{2}{3} - \frac{2}{5} - \frac{1}{4} - \frac{1}{3} + \frac{1}{4} + \frac{1}{10} \right)$$

$$V = \frac{1}{30}$$

3. A cylindrical hole of radius *b* is bored through a sphere of radius *a*. Find the volume of the remaining solid.

Solution:



Sphere $= x^2+y^2+z^2=a^2$

Cylinder $\to x^2+y^2=b^2$

$z \to$ xy plane to sphere

$z=0$ to $\sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2}$

$r \to$ cylinder to sphere

$r = b$ to $r = a$

$\theta \to 0$ to $\pi/2$ (in the first octant)

Let the equation of the sphere be $x^2 + y^2 + z^2 = a^2$.

Using cylindrical polar coordinates $x = r\cos\theta, y = r\sin\theta, z = z,$

we see that in the first octant z varies from $z = 0$ to $z = \sqrt{[a^2 - (x^2 + y^2)]} = \sqrt{[a^2 - r^2]}$

r varies from $r = b$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$

$$\therefore V = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dr d\theta dz$$

$$= 8\int_0^{\pi/2}\int_b^a r\cdot[z]_0^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= 8\int_0^{\pi/2}\int_b^a r\sqrt{a^2-r^2}\,dr\,d\theta$$

put $a^2 - r^2 = t \Rightarrow -2r\,dr = dt$

r	b	a
t	a^2-b^2	0

$$V = 8\int_0^{\pi/2}\int_{a^2-b^2}^{0}\sqrt{t}\left(\frac{-dt}{2}\right)d\theta$$

$$= \frac{8}{2}\int_0^{\pi/2}\int_0^{a^2-b^2}\sqrt{t}\,dt\,d\theta = 4\int_0^{\pi/2}\left(\frac{t^{3/2}}{3/2}\right)_0^{a^2-b^2}d\theta$$

$$= \frac{8}{3}(a^2-b^2)^{3/2}[\theta]_0^{\pi/2}$$

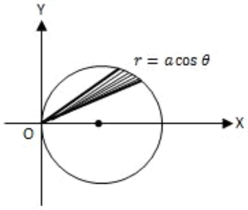
$$V = \frac{8}{3}(a^2-b^2)^{3/2}\cdot\frac{\pi}{2} = \frac{4\pi}{3}(a^2-b^2)^{3/2}$$

4. Show that the volume of the wedge intercepted between the cylinder $x^2 + y^2 = 2ax$ and planes $z = mx, z = nx$ is $\pi(n-m)\,a^3$.

Solution:

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta$



$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$x^2+y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$

$z \to nx$ to mx

$\to nr\cos\theta$ to $mr\cos\theta$

$r \to 0$ to $2a\cos\theta$

$\theta \to 0$ to $\pi/2$ (in the first quadrant)

We change to cylindrical polar coordinates by putting $x = r\cos\theta, y = r\sin\theta, z = z$

The equation $x^2 + y^2 = 2ax$ becomes $r^2 = 2ar\cos\theta$ i.e. $r = 2a\cos\theta$.

Hence, r varies from $r = 0$ to $r = 2a\cos\theta$, θ varies from $\theta = -\pi/2$ to $\theta = \pi/2$.

Since, volume is to be evaluated we can take into account symmetry and take the limits of θ from $\theta = 0$ to $\theta = \pi/2$, twice.

And z varies from $z = nx$ to $z = mx$ i.e. from $z = nr\cos\theta$ to $z = mr\cos\theta$

$$\therefore V = 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\int_{z=nr\cos\theta}^{mr\cos\theta} r\, dr d\theta dz$$

$$= 2\int_0^{\pi/2}\int_0^{2a\cos\theta} r\,[z]_{nr\cos\theta}^{mr\cos\theta}\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2a\cos\theta} r\cdot(m-n)\,r\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_0^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2a\cos\theta}\cos\theta\, d\theta$$

$$= \frac{16(m-n)}{3}a^3\int_0^{\pi/2}\cos^4\theta\, d\theta$$

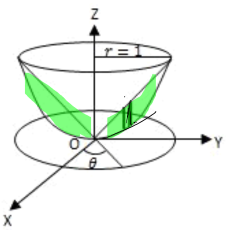
$$= \frac{16}{3}(m-n)a^3 \cdot \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= \frac{8}{3}(m-n)a^3 \cdot \frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{\Gamma 3}$$

$$V = (m-n)\pi a^3$$

5. Find the volume bounded by the cone $z^2 = x^2 + y^2$ and the paraboloid $z = x^2 + y^2$

Solution:



$x^2+y^2 = z^2 \qquad x^2+y^2 = z$

$z^2 = z \qquad z = 0$ or $z = 1$

limit for $z \rightarrow$ Paraboloid to cone

r to r^2

Projection of their intersection is circle with centre at origin and radius 1

$\therefore r \rightarrow 0$ to 1, $\theta \rightarrow 0$ to 2π

If we consider a section by a plane $z = k$ then on the cone we get, a circle $x^2 + y^2 = k^2$ and on the paraboloid we get, the circle $x^2 + y^2 = k$

If we use cylindrical coordinates then at the intersection of the two solids $x^2 + y^2 = r^2 = r$

i.e. $r^2 - r = 0 \ \therefore r(r-1) = 0 = 0 \ \therefore r = 0$ and 1

Hence, r varies from 0 to 1, θ varies from $\theta = 0$ to $\pi/2$ taken four times by symmetry,

z varies from r to r^2 (where $r^2 = x^2 + y^2$)

$\therefore V \ = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\int_{z=r}^{r^2} r\, dr d\theta dz$

$$= 4\int_0^{\pi/2}\int_0^1 r\,(z)_r^{r^2}\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 r(r^2 - r)\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 (r^3 - r^2)\, dr\, d\theta$$

$$= 4\int^{\pi/2}\left(\frac{r^4}{4} - \frac{r^3}{3}\right)_0^1 d\theta = 4\int^{\pi/2}\left(\frac{1}{4} - \frac{1}{3}\right) d\theta$$

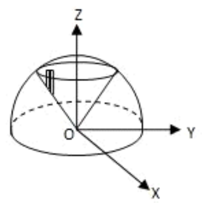
$$= 4\int_0 \left(\frac{r^4}{4} - \frac{r^3}{3}\right)_0 d\theta = 4\int_0 \left(\frac{1}{4} - \frac{1}{3}\right) d\theta$$

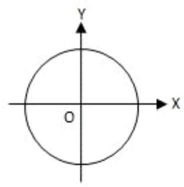
$$= 4\left(\frac{-1}{12}\right)\int_0^{\pi/2} d\theta = -\frac{1}{3}\left[\theta\right]_0^{\pi/2} = -\frac{\pi}{6}$$

$\therefore$ Volume $= \frac{\pi}{6}$.

6. Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$

Solution:





$x^2 + y^2 + z^2 = a^2$

$x^2 + y^2 = z^2$

$2z^2 = a^2 \qquad z^2 = \frac{a^2}{2}$

$\Rightarrow x^2 + y^2 = \frac{a^2}{2}$

projection of the intersection is a circle with centre $(0,0)$ and radius $a/\sqrt{2}$

$z \rightarrow$ cone to sphere

r to $\sqrt{a^2 - r^2}$

$r \rightarrow 0$ to $a/\sqrt{2}$

$\theta \rightarrow 0$ to 2π

Using triple integral $V = \iiint dz\, dx\, dy$

Consider the intersection of the sphere and the cone. On this intersection we have $x^2 + y^2 = a^2/2$.

In polar coordinates it is a circle $r = a/\sqrt{2}$.

On this circle r varies from 0 to $a/\sqrt{2}$ and θ varies from 0 to 2π

Consider, an elementary parallelopiped (as shown in the figure) and change $dxdy$ to $rd\theta dr$

$$\therefore V = \int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}} r\, d\theta dr dz$$

$$= \int_0^{2\pi}\int_0^{a/\sqrt{2}} \left[z\right]_r^{\sqrt{a^2-r^2}} r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a/\sqrt{2}} \left[\sqrt{a^2-r^2} - r\right] r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a/\sqrt{2}} \left[\sqrt{a^2-r^2} - r\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a/\sqrt{2}} \left[r\sqrt{a^2-r^2}\right] dr\,d\theta - \int_0^{2\pi}\int_0^{a/\sqrt{2}} r^2\,dr\,d\theta$$

put $a^2-r^2=t$

$-2r\,dr = dt$

r	0	$a/\sqrt{2}$
t	a^2	$a^2/2$

$$= \int_0^{2\pi}\int_{a^2}^{a^2/2} \sqrt{t}\left(-\frac{dt}{2}\right) d\theta - \int_0^{2\pi} \left(\frac{r^3}{3}\right)_0^{a/\sqrt{2}} d\theta$$

$$= \frac{-1}{2}\int_0^{2\pi} \frac{2}{3}\left[(t)^{3/2}\right]_{a^2}^{a^2/2} d\theta - \frac{a^3}{6\sqrt{2}}(\theta)_0^{2\pi}$$

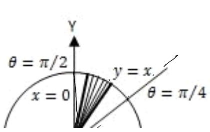
$$= \frac{-1}{3}\left[\frac{a^3}{2\sqrt{2}} - a^3\right]\left[\theta\right]_0^{2\pi} - \frac{a^3}{6\sqrt{2}}(2\pi)$$

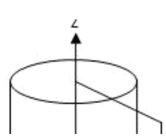
$$= \frac{1}{3}\left(a^3 - \frac{a^3}{2\sqrt{2}}\right)(2\pi) - \frac{a^3\pi}{3\sqrt{2}}$$

$$= \left[\frac{(2\sqrt{2}-1)}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}\right] a^3\pi = \left(\frac{2\sqrt{2}-2}{3\sqrt{2}}\right) a^3\pi = \frac{a^3\pi}{3}(2-\sqrt{2})$$

7. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x, z = 0$ and $x = 0$.

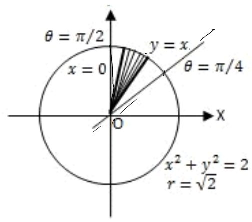
Solution:

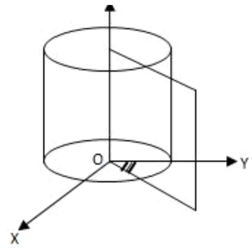




$x^2+y^2=2$

$x^2+y^2=2$





If we take projections on the xy −plane, the area is bounded by the circle $x^2 + y^2 = 2$, the line $y = x$ and the line $x = 0$ i.e. the y −axis

We change the coordinates to cylindircal polar by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Then the equation of the cylinder becomes $x^2 + y^2 = 2$ i.e. $r = \sqrt{2}$.

The line $y = x$ becomes, $r\sin\theta = r\cos\theta \;\therefore \theta = \pi/4$

The line $x = 0$ becomes, $r\cos\theta = 0 \;\therefore \theta = \pi/2$

Now, if we consider a radial strip in the projection, r varies from $r = 0$ to $r = \sqrt{2}$, θ varies from $\theta = \pi/4$ to $\theta = \pi/2$. Then z varies from $z = 0$ to $z = x + y = r(\cos\theta + \sin\theta)$

$$\therefore V = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}}\int_{z=0}^{r(\cos\theta+\sin\theta)} r\,dr\,d\theta\,dz$$

$$= \int_{\pi/4}^{\pi/2}\int_{0}^{\sqrt{2}} r\,[z]_0^{r(\cos\theta+\sin\theta)}\,dr\,d\theta$$

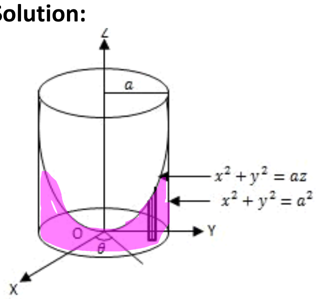
$$= \int_{\pi/4}^{\pi/2}\int_{0}^{\sqrt{2}} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\,d\theta \times \int_0^{\sqrt{2}} r^2\,dr$$

$$= (\sin\theta - \cos\theta)_{\pi/4}^{\pi/2} \times \left(\frac{r^3}{3}\right)_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

8. Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$.

Solution:



$z \to$ xy plane to paraboloid

$z=0$ to $z = \dfrac{r^2}{a}$

Projection $\to$ circle $x^2+y^2=a^2$

$r=a$.

The equations of the cylinder and the paraboloid in polar form are $r = a$ and $r^2 = az$

Now, z varies from $z = 0$ to $z = r^2/a$ and r varies from $r = 0$ to $r = a$ and

θ varies from $\theta = 0$ to $\theta = \pi/2$ taken 4 times

$$\therefore V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\int_{z=0}^{r^2/a} r\, dr\, d\theta\, dz$$

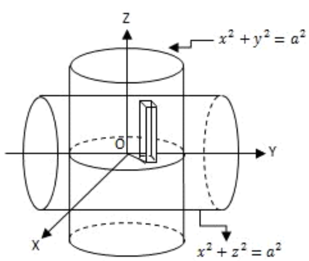
$$= 4\int_0^{\pi/2}\int_0^{a} r\,[z]_0^{r^2/a}\, dr\, d\theta$$

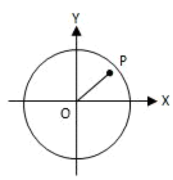
$$= 4\left[\int_0^{\pi/2} d\theta\right]\left[\int_0^{a} \frac{r^3}{a}\, dr\right]$$

$$= 4\left[\frac{\pi}{2}\right]\cdot\left(\frac{r^4}{4a}\right)_0^a = (2\pi)\cdot\frac{a^4}{4a} = \frac{\pi a^3}{2}$$

9. Find the volume common to the right circular cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$

Solution:





$z \to$ xy plane to the cylinder $x^2+z^2=a^2$

$z = 0$ to $\sqrt{a^2-x^2}$

By symmetry the required volume $=$ 8 volume in the first octant

$$\therefore V = 8\iiint dx\, dy\, dz$$

In the first octant z varies from 0 to $\sqrt{a^2 - x^2}$

$$\therefore V = 8\iint \sqrt{a^2 - x^2}\cdot dxdy$$

Now in the circle $x^2 + y^2 = a^2$, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a

$$V = 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= 8\int_0^{a}\int_0^{\sqrt{a^2-x^2}} \sqrt{\quad}\; dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\, dx$$

$$= 8\int_0^a \left(\sqrt{a^2-x^2}\right)\left(\sqrt{a^2-x^2}\right) dx$$

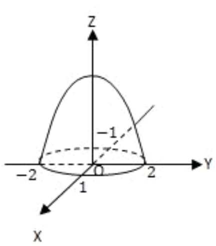
$$= 8\int_0^a (a^2-x^2)\, dx$$

$$= 8\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= 8\left[a^3 - \frac{a^3}{3}\right] = \frac{16a^3}{3}$$

10. Find the volume cut off from the paraboloid $x^2 + \frac{1}{4}y^2 + z = 1$ by the plane $z = 0$.

Solution:



$$x^2 + \frac{1}{4}y^2 = -z + 1$$

inverted paraboloid.

intersection with xy plane

$$x^2 + \frac{1}{4}y^2 + z = 1 \quad \text{and} \quad z = 0$$

$$x^2 + \frac{1}{4}y^2 = 1 \rightarrow \text{ellipse}$$

$$z \rightarrow 0 \quad \text{to} \quad 1 - x^2 - \frac{1}{4}y^2$$

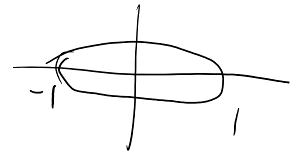
$$V = \iiint dz\, dy\, dx$$

$$= \iint_R \int_0^{1-x^2-\frac{1}{4}y^2} dz\, dy\, dx$$

$$\iint_R \quad \int_0$$

$$= \iint_R \left(1 - x^2 - \frac{1}{4}y^2\right) dy\, dx$$

The xy −plane cuts the paraboloid in the ellipse $x^2 + \frac{y^2}{4} = 1$



Hence, the volume $V = \iint_R z\, dx\, dy$

$$= \iint_R \left(1 - x^2 - \frac{y^2}{4}\right) dxdy$$

where R is the area of the ellipse

$\therefore V \; = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{+2\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{4}\right) dxdy$

$$= 4 \int_0^1 \int_0^{2\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{4}\right) dy dx$$

$$= 4 \int_0^1 \left[(1-x^2)y - \frac{y^3}{12}\right]_0^{2\sqrt{1-x^2}} dx$$

$$= 4 \int_0^1 \frac{4}{3}(1-x^2)^{3/2} dx$$

put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

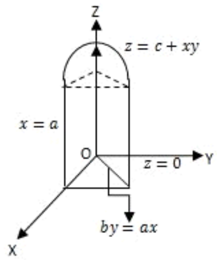
x	0	1
θ	0	$\pi/2$

$$V = \frac{16}{3} \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{16}{3} \cdot \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$V = \frac{8}{3} \cdot \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} = \pi$$

11. Find the volume of the triangular prism formed by the planes $ay = bx, y = 0, x = a$ from $z = 0$ to $z = c + xy$

Solution:



$V = \iiint dxdydz$

$$= \int_{x=0}^{a}\int_{y=0}^{bx/a}\int_{z=0}^{c+xy} dxdydz$$

$$= \int_0^a \int_0^{bx/a} \left[z\right]_0^{c+xy} dy\,dx = \int_0^a \int_0^{bx/a} (c+xy)\,dy\,dx$$

$$= \int_0^a \left(cy + x\frac{y^2}{2}\right)_0^{bx/a} dx$$

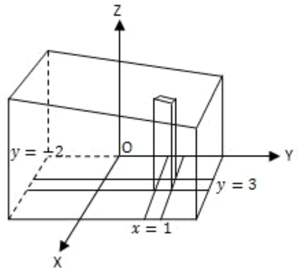
$$= \int_0^a \frac{cbx}{a} + \frac{x}{2}\cdot\frac{b^2x^2}{a^2}\,dx$$

$$= \frac{cb}{a}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a^2}\left(\frac{x^4}{4}\right)_0^a$$

$$= \frac{cb}{a}\left(\frac{a^2}{2}\right) + \frac{b^2}{2a^2}\left(\frac{a^4}{4}\right) = \frac{abc}{2} + \frac{a^2b^2}{8}$$

12. Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R\{(x,y)|0 \le x \le 1, -2 \le y < 3\}$

Solution:



Consider an elementary cuboid as shown in the figure

On this cuboid z varies from $z = 0$ to $z = 12 - 3x - 2y$,
y varies from $y = -2$ to $y = 3$ and x varies from $x = 0$ to $x = 1$

Hence, the volume is given by

$\therefore V \quad = \int_{x=0}^{1} \int_{y=-2}^{3} \int_{z=0}^{12-3x-2y} dz dy dx$

$$= \int_0^1 \int_{-2}^{3} (12 - 3x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[12y - 3xy - y^2 \right]_{-2}^{3} dx$$

$$= \int_0^1 (36 - 9x - 9) - (-24 + 6x - 4)\, dx$$

$$= \int_0^1 (55 - 15x)\, dx \quad = \left(55x - \frac{15x^2}{2}\right)_0^1$$

$$= 55 - \frac{15}{2}$$

$$V = \frac{95}{2}$$