Practice problems

Monday, December 27, 2021
11:06 AM

2) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Find eigenvalues and eigenvectors for A

Sol^n: Consider $Ax = \lambda x$, $[A - \lambda I]x = 0$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

Characteristic eq^n is given by,

$$\lambda^3 - P\lambda^2 + Q\lambda - R = 0$$

$P = SOD = -2+1+0 = -1$

$Q = SOMD = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -12-3-6 = -21$

$R = |A| = 45$

Then ch eq^n $\Rightarrow$ $\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

$\rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

Then ~~roots of~~ ch eq^n are $\lambda = 5, \boxed{-3},$

Let us say $\underline{x}$ is third eigenvalue.

$5 + (-3) + x = -2+1+0 = -1$ | $5(-3)x = \underline{45}$

By solving $x = -3$,

$\therefore$ eigenvalues are $\underline{5, \boxed{-3, -3}}$

1. Consider $\lambda = -3$

$$[A - (-3)I]x = [A + 3I]x = 0$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1 \mid R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - 3z = 0$$

Here $r = 1$, $n = 3$, $\boxed{n - r} = 3 - 1 = 2$ free var

$\therefore$ $y = p$, $z = q$

$x = 3z - 2y = 3q - 2p$

$$\therefore \text{Sol}^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3q - 2p \\ p \\ q \end{pmatrix} = p \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{x_2} + q \underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{x_3}$$

$\therefore$ for $\lambda = -3$, corresponding eigenvectors are

$$x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ for $\lambda = -3$, $\boxed{A.M = 2, \quad GM = 2}$

$\lambda = 6$ $\quad$ $\lambda = -2$ $\quad$ -4

$$\boxed{\lambda = 5} \longrightarrow r = 2$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \frac{-y}{|\ |} \quad \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{matrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{matrix}$$

$$\frac{x}{\begin{vmatrix} -4 & -6 \\ -2 & -5 \end{vmatrix}} = \frac{+y}{\begin{vmatrix} 2 & -6 \\ -1 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -4 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x}{8} = \frac{+y}{+16} = \frac{z}{-8}$$

$$\begin{pmatrix} 8 \\ 16 \\ -8 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

3) $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Sol^n: Consider $Ax = \lambda x$, $[A - \lambda I]x = 0$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0 \quad \text{—} (1)$$

$\therefore$ Characteristic eq^n is given by $\lambda^3 - P\lambda^2 + Q\lambda - R = 0$

$P = \text{SOD} = 7$

$Q = \text{SOMD} = 16$ ✓

$R = |A| = 12$

Ch eq^n is $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

eigenvalues are $\lambda = \underline{3, 2, 2}$

$3 + 2 + x = 7$, $3(2)(x) = 12$, $\boxed{x = 2}$

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1) for $\underline{\lambda = 3}$, $[A - \lambda I]x = [A - 3I]x = 0$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

By cramer's Rule of 2 independent rows,

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x}{\begin{vmatrix} 10 & 5 \\ -6 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 5 \\ -2 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 10 \\ -2 & -6 \end{vmatrix}}$$

$$\frac{x}{-10} = \frac{-y}{10} = \frac{z}{20} = t$$

$$\therefore \text{Sol}^n \text{ is } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10t \\ -10t \\ 20t \end{pmatrix} \simeq \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$ is eigenvector for $\lambda = 3$

$\underline{\text{for } \lambda = 3}$, AM = 1 = GM

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for $\underline{\lambda = 2}$ $[A - 2I]x = 0$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ \ldots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} \cdots \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cdots \end{bmatrix}$$

$$\begin{array}{l} R_2 + 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + \frac{25}{15} R_2 \quad \begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank 2 case :

$$x + 10y + 5z = 0$$

$$5y + 2z = 0$$

$$\therefore \; z = t, \quad y = -\frac{2z}{5} = -\frac{2}{5}t$$

$$x = -10y - 5z = -t$$

$$= -\cancel{10}^{\,2}\left(-\frac{2}{\cancel{5}}t\right) - 5t$$

$$\text{sol}^n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ -\frac{2}{5}t \\ t \end{pmatrix} \cong \begin{pmatrix} -5\cancel{t} \\ -2\cancel{t} \\ 5\cancel{t} \end{pmatrix} \cong \underline{\begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix}} \; \text{OR} \; \underline{\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}}$$

Here for $\lambda = 2$, $AM = 2$, $GM = 1$

$$\boxed{AM \neq GM} \qquad AM = GM$$

order 3

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc} 4 & 1 & -1 \\ \downarrow & \downarrow & \downarrow \\ \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} & \begin{pmatrix} -6 \\ -2 \\ 7 \end{pmatrix} \end{array} \right]$$

$$= \begin{bmatrix} \cdots \\ -1 & -4 & -3 \end{bmatrix}$$

A^{100}

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \Rightarrow \begin{array}{ccc} 5 & -3 & -3 \\ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$n = 3$

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} \Rightarrow \begin{array}{ccc} 3 & 2 & 2 \\ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} & \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} & \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} \end{array}$$

M

AM $\neq$ GM

AM > GM

Properties of eigenvalues & vectors

1) Eigen vectors corresponds to distinct eigenvalues are always independent (AM = 1 = GM)

2) Sum of eigenvalues = sum of diag entries.

3) Product of eigenvalues is determinant.

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 & 5 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

since this is upper tri , eigenvalues are same as diag entries.

4) Diagonal Upper OR lower triangular, Scalar,

Identity Then eigenvalues are same as diag entries.

eigenvectors : :

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

Since Matrix is diagonal, eigenvalues are same as diagonal entries.

$$\boxed{\lambda = 3, 1, 2}$$

$\hat{i} \quad \hat{j} \quad \hat{k}$

1) for $\lambda = 3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2y = 0$ & $-z = 0$

x can take any value

Ans $\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$x = 0 = y$

z can take any value

$\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \cong \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda = 3, 3, 3$ $\qquad \boxed{AM = 3}$

$[A - \lambda I] \qquad [A - 3I]x = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

any vector of $\mathbb{R}^3$ is eigen vector of A

$\therefore$ we can choose any 3 independent vector as eigenvector

$\{\hat{i}, \hat{j}, \hat{k}\}$ $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix} \right\}$

HW

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \rightarrow \lambda \rightarrow AM = 3$$

Similarity of Matrices

Defn: If A and B are two square matrices of order n. then B is said to be similar to A if there exist a nonsingular Matrix M such that $\boxed{B = M^{-1} A M}$

It is also denoted by, $A \approx B$

This matrix M is called Modal / Transforming matrix

Properties: If $A \approx B$ Then

Power	$A^k \approx B^k$
det	$\lvert A \rvert = \lvert B \rvert$
rank	rank A = rank B
ch eq^n	Both A & B are same
eigenvalues	—"—

1) **Powers**: $A \approx B$, $A^2 \approx B^2$

Then $B = M^{-1} A M$

$$B^2 = B \cdot B = (M^{-1} A M)(M^{-1} A M) =$$

$$B^2 = M^{-1} A^2 M \Rightarrow A^2 \approx B^2$$

Continuing this way $\Rightarrow A^k \approx B^k$

2) **det**: $B = M^{-1} A M$

$$\det(B) = \det(M^{-1} A M) \qquad (\det(AB) = \det(A) \cdot \det(B))$$

$$= \det(M^{-1}) \cdot \det(A) \cdot \det(M)$$

$$= \det(M^{-1}) \det(M) \det(A)$$

$$= \det(M^{-1} M) \cdot \det(A)$$

$$\boxed{\det(B) = \det(A)}$$

Simplest form is diagonal Form

$$A = M^{-1} D M$$

$$D = \begin{bmatrix} 3 & & 0 \\ & 7_3 & \\ 0 & & -2 \end{bmatrix}$$

$\begin{pmatrix} 0 & 7_3 & -2 \end{pmatrix}$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1^3 & 0 \\ 0 & 2^3 \end{bmatrix}$$

$$D^{100} = \begin{bmatrix} 1^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \checkmark$$

If $\boxed{A \text{ is diagonalisable}}$ $\boxed{D = M^{-1} A M}$

Then

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \cdots & \lambda_k \end{bmatrix}, \quad M = \begin{bmatrix} \begin{bmatrix} X_1 \end{bmatrix} & \begin{bmatrix} X_2 \end{bmatrix} & \cdots & \begin{bmatrix} X_k \end{bmatrix} \end{bmatrix}$$

$\boxed{AM = GM \text{ for every } \lambda}$

Diagonalisable :

A square Matrix A is said to diagonalisable if it is similar to diagonal matrix D, $\boxed{D = M^{-1}AM}$ $A \approx D$

Condition : check $AM = GM$ for every eigen value of A

Then A is diagonalisable.

Conclusion: If A is diagonalisable, with $\lambda_1, \lambda_2, \lambda_3, \ldots$ as ... Then eigenvalues & $x_1, x_2, x_3, \dots$ as corresponding eigenvectors Then

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \ \ddots \end{bmatrix}, \quad M = \begin{bmatrix} x_1 & x_2 & x_3 & \dots \end{bmatrix}$$

$$D = M^{-1} A M$$

Q.: Check whether following Matrix is diagonalisable or not If yes then find the diagonal matrix D & Modal Matrix M

Sol^n: $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $\qquad \lambda = 5, \boxed{-3, -3}$

$\lambda = -3$ $\quad x_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ $\quad \boxed{AM = 2 = GM}$

$\lambda = 5$ $\quad x_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $\quad \boxed{AM = 1 = GM}$

$\therefore$ AM = GM for every eigenvalue Hence A is diagonalisable

$\therefore$ diagonal matrix $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\qquad M = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Check for diagonalisable

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & -1 \\ & & \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad D\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda = 3$

$[A - 3I]$

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2y + z = 0 \\ -z = 0 \end{cases}$$

$y = 0 = z$

x can take any value.

$\therefore \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

AM = 3 , GM = 1

AM $\neq$ GM

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

AM = 3

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2y + z = 0$

x can take any value.
$x = p$

y is free variable
$y = q$

$z = -2y = -2q$

$\therefore \begin{pmatrix} p \\ q \\ -2q \end{pmatrix} = p\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

AM = 3 , GM = 2

AM $\neq$ GM

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$D =$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

any 3 independen

$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right]$

GM = 3

$\boxed{\text{AM} = \text{GM}}$

A is diag