Matrix Theory: Rank of Matrix

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Basic of Matrix

FY BTECH SEM-I

MODULE-2

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MATRIX: REVIEW OF BASIC CONCEPTS (Self-Learning)

- **Matrix** is a collection of information stored or arranged in an orderly fashion (Rectangular arrangement). It is denoted by Capital symbols and its elements are denoted by small letters with row and column indices. e.g. $A = [a_{ij}]$
- **Order of a Matrix:** (no. of rows) x (no. of columns), $A_{m \times n}$
- **Basic Types of Matrices:**

1. **Row or Column Matrix:** Matrix containing only one row is called **Row Matrix** and Matrix containing only one column is called **column Matrix.**
2. **Rectangular Matrix:** Matrix with unequal number of rows and columns.

Basic Types of Matrices

3. **Null Matrix:** Matrix with all entries as zero entries. Denoted by $O_{m\ X\ n}$

4. **Square Matrix:** Matrix with same number of rows and columns (m = n)

   e.g. $A = \begin{bmatrix} 5 & 7 & -3 \\ -2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Diagonal and Principal diagonal entries:

Diagonal entries a_{ii} (e.g. $a_{11},\ a_{22}, a_{33}$) are called Principal diagonal entries.

5. **Diagonal Matrix:** Matrix whose all non-diagonal elements are zero.

   e.g. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

6. **Scalar Matrix:** Diagonal matrix where all diagonal entries are equal to scalar k≠ 0.

7. **Unit / Identity Matrix:** Scalar matrix whose all diagonal entries are equal to scalar 1.

8. **Upper Triangular Matrix:** Square Matrix where below diagonal entries are zero.

9. **Lower Triangular Matrix:** Square Matrix where above diagonal entries are zero.

Basic operations on Matrices

- **Equality of two matrices**: Two matrices are equal if all corresponding entries of them are equal.
- They must be of same order.
- **No order relation.**
- **Addition/ subtraction**: Two matrices can be added (subtracted) by adding (subtracting) the corresponding elements of the two matrices. Both matrices must have same order (m x n).
- **Multiplication by scalar**: multiply each element by scalar k,
- e. g. $kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$
- **Product**: Two matrices can be multiplied together provided they are compatible with respect to their orders.
- The number of columns in the first matrix [A] must be equal to the number of rows in the second matrix [B]. The resulting matrix [C] will have the same number of rows as [A] and the same number of columns as [B].
- The entries are $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$ and The order relation is
- $C_{m\,X\,p} = A_{m\,X\,n}.B_{n\,X\,p}$

Basic operations on Matrices

- **Trace of a Matrix**: Sum of all principal diagonal entries of a square matrix is called trace of matrix. $Trace(A) = \sum_i a_{ii}$
- **Transpose of a Matrix**: The transpose A^T or A' of a m x n matrix A is a n x m matrix obtained by interchanging rows with columns of A
- **Properties of Transpose**

i. $(A^T)^T = A$

ii. $(A + B)^T = A^T + B^T$

iii. $(AB)^T = B^T A^T$

- **Determinant of a Matrix**: It is denoted by $|A|$
- **Properties of determinant**

  Det (AB) = Det (A).Det(B) ,

- $|A| = |A^T|$

- **Singular Matrix**: If $|A| = 0$
- **Non-singular Matrix**: If $|A| \neq 0$
- **Inverse of a Matrix**:
- It is denoted by A^{-1} and given by
- $A^{-1} = \frac{1}{|A|} adj\ A,$
- it exist only if $|A| \neq 0$
- **Properties of Inverse**

i. $(A^{-1})^{-1} = A$

ii. $(AB)^{-1} = B^{-1}A^{-1}$

iii. $AA^{-1} = I$

iv. $(A^T)^{-1} = (A^{-1})^T$

(not additive property)

Basic operations on Matrices

- **Conjugate of a Matrix**: Matrix obtained by replacing each element by its conjugate. (need not be square).
- It is denoted by $\bar{A}$
- **Example** $A = \begin{bmatrix} 2 & 2+3i & i \\ 7i & -3i & -i \end{bmatrix}$
- $Then\ \bar{A} = \begin{bmatrix} 2 & 2-3i & -i \\ -7i & 3i & i \end{bmatrix}$
- **Properties of Conjugate**

i. $\overline{(\bar{A})} = A$

ii. $\overline{(A+B)} = \bar{A} + \bar{B}$

iii. $\overline{(AB)} = \bar{A}.\bar{B}$ (check!!)

- **Tranjugate (Transposed conjugate) of a Matrix**: obtained by taking transpose of conjugate
- **Example** $A = \begin{bmatrix} 2 & 2+3i & i \\ 7i & -3i & -i \end{bmatrix}$
- $Then\ (\bar{A})^T = A^\theta = \begin{bmatrix} 2 & -7i \\ 2-3i & 3i \\ -i & i \end{bmatrix}$
- **Note:** need not be square matrix,
- Order of operation doesn't matter
- $(\bar{A})^T = \overline{(A^T)} = A^\theta$
- **Properties of Tranjugate**
- $\left(A^\theta\right)^\theta = A$
- $(A+B)^\theta = A^\theta + B^\theta$
- $(AB)^\theta = B^\theta A^\theta$ (check!!)

Comparative study of important Types: (self learning)

	Symmetric Matrix	Skew-symmetric Matrix
Defn	symmetric if i) A is square ii) $a_{ij} = a_{ji},\ \forall\, i,j$	Skew-symmetric if i) A is square ii) $a_{ij} = -a_{ji},\ \forall\, i,j$
condition	A is symmetric iff $A^T = A$	A is skew-symmetric iff $A^T = -A$
Diagonal elements	$a_{ii} = a_{ii}$ No specific condition on elements	Condition $a_{ii} = -a_{ii}$, true for zero Diagonal elements must be zero.
Example	$A = \begin{bmatrix} 5+2i & 1+i & 2-i \\ 1+i & 3 & 7i \\ 2-i & 7i & 0 \end{bmatrix}$	$A = \begin{bmatrix} 0 & -1-i & -2-i \\ 1+i & 0 & 7i \\ 2+i & -7i & 0 \end{bmatrix}$
observation	Diagonal can be any no and sign of real and img part of symmetric position no. are same	Diagonal is zero and sign of both real and img part of symmetric position no. are different

Comparative study of important Types:

	Hermitian Matrix	**Skew-Hermitian Matrix**
Defn	Hermitian if i) A is square ii) $a_{ij} = \overline{a_{ji}},\ \forall\, i,j$	Skew- Hermitian if i) A is square ii) $a_{ij} = \overline{-a_{ji}},\ \forall\, i,j$
condition	A is Hermitian iff $A^{\theta} = A$	A is skew- Hermitian iff $A^{\theta} = -A$
Diagonal elements	$a_{ii} = \overline{a_{ii}}$ Diagonal elements are real.	$a_{ii} = \overline{-a_{ii}},$ Diagonal elements are imaginary/ zero.
Example	$A = \begin{bmatrix} 5 & 1-i & 2+i \\ 1+i & -3 & -7i \\ 2-i & 7i & 0 \end{bmatrix}$	$A = \begin{bmatrix} 5i & -1-i & -2 \\ 1-i & -3i & 7i \\ 2 & 7i & 0 \end{bmatrix}$
observation	Diagonal entries are real and sign of only img part of symmetric position is different	Diagonal entries are img and sign of only real part of symmetric position is different

Properties of Matrices

- Hermitian Matrices with real entries are symmetric.
- Skew-Hermitian Matrices with real entries are skew-symmetric.
- If A is Hermitian then $\bar{A} = A^T$ (equivalent condition)

$$A^\theta = A \;\; Take\ transpose \;\; (\bar{A}^T)^T = A^T \;\; \Rightarrow \;\; \bar{A} = A^T$$

- If A is skew-Hermitian then $\bar{A} = -A^T$ (equivalent condition)
- If A is Hermitian then *iA is* skew-Hermitian.

$$(iA)^\theta = i^\theta A^\theta = -iA$$

- If A is skew-Hermitian then *iA is* Hermitian.

Properties of Matrices

- If A is Hermitian then $\bar{A}$ is also Hermitian.

$$(\bar{A})^{\theta} = \left(\bar{\bar{A}}\right)^{T} = A^{T} = \bar{A}$$

- If A is skew-Hermitian then $\bar{A}$ is also skew-Hermitian.
- If A is any square matrix then

i. $A + A^{\theta}$ is Hermitian

ii. $A - A^{\theta}$ is skew-Hermitian.

$$check\ (A - A^{\theta})^{\theta} = A^{\theta} - \left(A^{\theta}\right)^{\theta} = A^{\theta} - A = -\left(A - A^{\theta}\right)$$

$$hence\ skew - hermitian$$

Properties of Matrices (self learning)

- If A and B are symmetric then AB is symmetric if and only if A and B are square matrices of same order and AB = BA
  $Since\ A^T = A\ and\ B^T = B, consider\ \ (AB)^T = B^T A^T = BA = AB$
- If A and B are skew-symmetric then AB is symmetric if and only if A and B are square matrices of same order and AB = BA
- If A is skew symmetric of order n then $|A| = 0$, if n is odd.

$$Since\ A^T = -A,\ take \det both\ side\ \ |A^T| = |-A|$$
$$\therefore |A| = (-1)^n |A| \therefore |A| = 0\ if\ n\ is\ odd$$

Properties of Matrices (self learning)

- If A is skew symmetric and X is a column matrix then $X^T AX$ is null matrix.

Pf: Let $X^T AX = B$, taking transpose $(X^T AX)^T = B^T$

$\therefore\ X^T A^T X = B^T$ But $A^T = -A\ and$

$$X^T{}_{1\ X\ n} A_{n\ X\ n} X_{n\ X\ 1} = B_{1\ X\ 1} \quad \therefore\ B^T = B$$

$$\therefore\ -X^T AX = B \quad \therefore X^T AX = -B \quad \therefore B = -B \quad \therefore B = 0$$

- If A is any square matrix then

1. $A + A^T$ is symmetric
2. $A - A^T$ is skew-symmetric

Results on sum of Matrices

- **Show that every square matrix can be uniquely expressed as sum of Hermitian and skew Hermitian matrices.**
- **Proof:** Let A be any square matrix. Consider $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

Say, A = P + Q, Where, $P = \frac{1}{2}(A + A^{\theta})$ and $Q = \frac{1}{2}(A - A^{\theta})$

Part I] T.P.T P is Hermitian

- Consider $P^{\theta} = \left[\frac{1}{2}(A + A^{\theta})\right]^{\theta} = \frac{1}{2}\left(A^{\theta} + (A^{\theta})^{\theta}\right) = \frac{1}{2}(A^{\theta} + A) = P$

T.P.T Q is skew-Hermitian

- Consider $Q^{\theta} = \left[\frac{1}{2}(A - A^{\theta})\right]^{\theta} = \frac{1}{2}(A^{\theta} - A) = -\frac{1}{2}(A - A^{\theta}) = -Q$

Results on sum of Matrices

- **Proof Contd..**
- Part II] To prove uniqueness,
- Consider another representation, say A = R + S, where R is Hermitian and S is skew Hermitian
- Then $A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta} = R - S$ (since $R^{\theta} = R$, $S^{\theta} = -S$)
- Now consider, $P = \frac{1}{2}\left(A + A^{\theta}\right) = \frac{1}{2}(R + S + R - S) = R$
- and $Q = \frac{1}{2}\left(A - A^{\theta}\right) = \frac{1}{2}(R + S - (R - S)) = S$
- Thus we establish R is same as P and S is same as Q. Hence given representation is unique.

Results on sum of Matrices

- Show that every square matrix can be uniquely expressed as sum of symmetric and skew symmetric matrices. (self learning)
- Show that every square matrix can be uniquely expressed as P + iQ, where P and Q both are Hermitian matrices.
- **Proof:** Let A be any square matrix. Consider $A = \frac{1}{2}\left(A + A^{\theta}\right) + i\left[\frac{1}{2i}\left(A - A^{\theta}\right)\right]$
- Say, A = P + iQ   Where, $P = \frac{1}{2}\left(A + A^{\theta}\right)$ and $Q = \left[\frac{1}{2i}\left(A - A^{\theta}\right)\right]$
- Part I] T.P.T P is Hermitian
- Consider $P^{\theta} = \left[\frac{1}{2}\left(A + A^{\theta}\right)\right]^{\theta} = \frac{1}{2}\left(A^{\theta} + \left(A^{\theta}\right)^{\theta}\right) = \frac{1}{2}\left(A^{\theta} + A\right) = P$
- T.P.T Q is also Hermitian
- Consider $Q^{\theta} = \left[\frac{1}{2i}\left(A - A^{\theta}\right)\right]^{\theta} = \left(\frac{1}{2i}\right)^{\theta}\left(A^{\theta} - A\right) = -\frac{1}{2i}\left(A^{\theta} - A\right) = \frac{1}{2i}\left(A - A^{\theta}\right) = Q$

Results on sum of Matrices

- **Proof Contd..**
- Part II] To prove uniqueness,
- Consider another representation, say $A = R + iS$, where R and S are Hermitian. $(R^{\theta} = R\ ,\ S^{\theta} = S\ )$
- Then $A^{\theta} = (R + iS)^{\theta} = R^{\theta} + i^{\theta} S^{\theta} = R - iS$
- Now consider, $P = \frac{1}{2}\left(A + A^{\theta}\right) = \frac{1}{2}(R + iS + R - iS) = R$
- and $Q = \frac{1}{2i}\left(A - A^{\theta}\right) = \frac{1}{2i}(R + iS - (R - iS)) = S$
- Thus we establish R is same as P and S is same as Q. Hence given representation is unique.

Results on sum of Matrices

- **Show that every Hermitian matrix can be uniquely expressed as P + iQ, where P is real symmetric and Q is real skew symmetric matrix.**
- **Proof:** Let A be any Hermitian matrix. Consider $A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$ Say, A = P + iQ
- Where, $P = \frac{1}{2}(A + \bar{A})$ and $Q = \left[\frac{1}{2i}(A - \bar{A})\right]$
- Part I] T.P.T P is Real symmetric, we show $\bar{P} = P$ and $P^T = P$
- Consider $\bar{P} = \overline{\frac{1}{2}(A + \bar{A})} = \frac{1}{2}\overline{(A + \bar{A})} = \frac{1}{2}(\bar{A} + A) = P$
- Consider $P^T = \left[\frac{1}{2}(A + \bar{A})\right]^T = \frac{1}{2}(A^T + (\bar{A})^T) = \frac{1}{2}\left(A^T + A^\theta\right)$ (Since A is Hermitian,
- $= \frac{1}{2}(\bar{A} + A) = P$ $A^\theta = A$ and $A^T = \bar{A}$)
- T.P.T Q is Real skew symmetric, we show $\bar{Q} = Q$ and $Q^T = -Q$
- Consider $\bar{Q} = \overline{\frac{1}{2i}(A - \bar{A})} = -\frac{1}{2i}\overline{(A - \bar{A})} = -\frac{1}{2i}(\bar{A} - A) = \frac{1}{2i}(A - \bar{A}) = Q$
- Consider $Q^T = \left[\frac{1}{2i}(A - \bar{A})\right]^T = \frac{1}{2i}(A^T - (\bar{A})^T) = \frac{1}{2i}\left(A^T - A^\theta\right)$ (Since A is Hermitian,
- $= \frac{1}{2i}(\bar{A} - A) = -Q$ $A^\theta = A$ and $A^T = \bar{A}$)

Results on sum of Matrices

- **Proof Contd..**
- Part II] To prove uniqueness, Consider another representation,
- say $A = R + iS$, where R is real symmetric and S is real skew symmetric.
- Then $\bar{A} = \overline{R + iS} = \bar{R} + \bar{i}\bar{S} = R - iS$ (since $\bar{R} = R$, $\bar{S} = S$)
- Now consider, $P = \frac{1}{2}(A + \bar{A}) = \frac{1}{2}(R + iS + R - iS) = R$
- and $Q = \frac{1}{2i}(A - \bar{A}) = \frac{1}{2i}(R + iS - (R - iS)) = S$
- Thus we establish R is same as P and S is same as Q. Hence given representation is unique.
- **Show that every skew Hermitian matrix can be uniquely expressed as P + iQ, where P is real skew symmetric and Q is real symmetric matrix.**
- (Try yourself)

Table of results on unique representation

Matrix	Expressed As	Unique Representation
Square	Symmetric + skew symmetric	$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$
Square	Hermitian + skew Hermitian	$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$
Square	P + iQ, P and Q both Hermitian	$A = \frac{1}{2}(A + A^\theta) + i\left[\frac{1}{2i}(A - A^\theta)\right]$
Hermitian	P + iQ, P real symmetric Q real skew symmetric	$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$
Skew Hermitian	P + iQ, P real skew symmetric Q real symmetric	$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

Examples based on representation results:

- Express following Matrix as sum of Hermitian and skew Hermitian Matrices. $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix}$
- **Solution:** As we know the unique representation, $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$ say, A = P + Q,
- Where, $P = \frac{1}{2}(A + A^\theta)$ and $Q = \frac{1}{2}(A - A^\theta)$ Now, Consider $A^\theta = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}$
- $\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} + \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & -i \\ 3+i & i & 0 \end{bmatrix}$
- Check here that P is Hermitian.
- $and \ \ Q = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} - \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 2-i \\ -1+3i & -2-i & 6i \end{bmatrix}$
- Check here that Q is skew Hermitian.
- Hence we get the unique expression,
- $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & -i \\ 3+i & i & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 2-i \\ -1+3i & -2-i & 6i \end{bmatrix}$

Example 2

- Express following skew Hermitian Matrix as P + iQ, where P is real skew symmetric and Q is real symmetric matrix. $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$
- **Solution:** As we know the unique representation, $A = \frac{1}{2}(A+\bar{A}) + i\left[\frac{1}{2i}(A-\bar{A})\right]$
- say, A = P + iQ, Where, $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$
- Now, Consider $\bar{A} = \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}$
- $\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} + \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & 2 \\ -6 & -2 & 0 \end{bmatrix}$
- Check here that P is real skew symmetric.

Example 2 (Contd..)

- $and\ \ Q = \frac{1}{2i}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} - \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}\right\}$
- Q = $\frac{1}{2i}\begin{bmatrix} 6i & 2i & -4i \\ 2i & -2i & 4i \\ -4i & 4i & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$
- Check here that Q is real symmetric.
- Hence we get the unique expression, A = P + iQ,
- $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$
- $= \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} + i\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$

ORTHOGONAL MATRIX

- **Definition:** A real square matrix A is called orthogonal

  if $AA^T = A^T A = I$

- **Properties:**
- If A is orthogonal matrix then $|A| = \pm 1$

Proof: Since A is orthogonal, $AA^T = I$ taking det on both sides

$$|AA^T| = |I| \qquad \therefore |A||A^T| = |I| \qquad \therefore |A||A| = 1$$

(Since $|AB| = |A||B|,\ |A^T| = |A|,\ |I| = 1$)

$$|A|^2 = 1 \qquad \therefore |A| = \pm 1$$

- If A is orthogonal then $A^{-1} = A^T$
- If A is orthogonal then A^{-1}, A^T are also orthogonal

Example 1

- Prove that following matrices is orthogonal and hence find A^{-1}, $A=\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}$

Soln: Consider $A^T=\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix}$ $\therefore\ AA^T=\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix}$

- $\therefore\ AA^T=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha & 0 & -\cos\alpha\sin\alpha+\cos\alpha\sin\alpha\\ 0 & 1 & 0\\ -\cos\alpha\sin\alpha+\cos\alpha\sin\alpha & 0 & \sin^2\alpha+\cos^2\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$
- Hence the given matrix is orthogonal.

For orthogonal matrix $A^{-1}=A^T=\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix}$

Example 2

- Is the following matrix orthogonal? If not, can it be converted into orthogonal matrix?
- $A = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$
- **Soln:** Consider $A^T = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$ $\therefore \; AA^T = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$
- $\therefore AA^T = \begin{bmatrix} 2+1+3 & 2-2 & 2+1-3 \\ 2-2 & 2+4 & 2-2 \\ 2+1-3 & 2-2 & 2+1+3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I \neq I$
- Thus Given matrix A is not orthogonal, But it can be converted into an orthogonal matrix as follow
- $\therefore AA^T = 6I \quad \therefore \frac{1}{6} AA^T = I \quad \therefore \left(\frac{1}{\sqrt{6}} A\right) . \left(\frac{1}{\sqrt{6}} A^T\right) = I$
- Hence $\frac{1}{\sqrt{6}} A = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ is the orthogonal matrix.

Example 3

- If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal, then find a, b, c
- **Soln:** Consider $A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$
- Since A is orthogonal we have, $AA^T = I$
- $\therefore AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$
- $= \frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ab & 5+b^2 & 2+bc \\ -2+ac & 2+bc & 8+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Comparing we get total 6 distinct equations,

Example 3(Contd..)

- $\frac{5+a^2}{9} = 1 \quad \therefore a^2 = 9 - 5 = 4 \quad \therefore a = \pm 2,$
- $\frac{5+b^2}{9} = 1 \quad \therefore b^2 = 9 - 5 = 4 \quad \therefore b = \pm 2$
- $\frac{8+c^2}{9} = 1 \quad \therefore c^2 = 9 - 8 = 1 \quad \therefore c = \pm 1$
- $4 + ab = 0 \quad \therefore ab = -4 \quad \rightarrow when\ a = +2, \quad b = -2$
- $and\ when\ a = -2, \quad b = +2$
- Also, $-2 + ac = 0 \quad \therefore ac = 2 \quad \rightarrow when\ a = +2, \quad c = +1$
- $and\ when\ a = -2, \quad c = -1$
- Hence (2, -2, 1) and (-2, 2, -1) are the required pairs.

- **(Note Observation)** For orthogonal matrix, column vectors are orthonormal to each other

UNITARY MATRIX

- **Definition:** A square matrix A is called unitary if $AA^{\theta} = A^{\theta}A = I$
- **Properties:**
- If A is Unitary then $A^{-1} = A^{\theta}$
- If A and B are unitary matrices of order n then $A^{-1}, A^{T}, A^{\theta}, AB \; and \; BA$ are also unitary.

Proof for $\boldsymbol{A^T}$: Since A is unitary, $A^{\theta}A = I$

taking transpose on both sides, $\left(A^{\theta}A\right)^{T} = I^{T} \quad A^{T}\left(A^{\theta}\right)^{T} = I$

$\therefore (A^{T})(A^{T})^{\theta} = I$ Hence, A^{T} is unitary

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UNITARY MATRIX

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- If A is unitary matrix then Its determinant is of unit modulus
- **Proof:** Since A is Unitary, $AA^{\theta} = I$ taking det on both sides
- $|AA^{\theta}| = |I|$ $\therefore |A||A^{\theta}| = |I|$
- $\therefore |A||(\bar{A})^{T}| = |I|$ $\therefore |A||\bar{A}| = 1$ $\therefore |A|\overline{|A|} = 1$
- (Since $|AB| = |A||B|$, $|A^{T}| = |A|$, $|I| = 1$ Also check that, $|\bar{A}| = \overline{|A|}$)
- Now, we know that for complex number z, $z\bar{z} = (mod\ z)^{2}$
- Hence, $(mod\ |A|)^{2} = 1$ $\therefore mod\ |A| = \pm 1$,
- But modulus is never negative
- $\therefore mod\ |A| = 1$ i.e. determinant of unitary matrix is of unit modulus.

Example 1

Show matrix A is unitary and hence find A^{-1} where $A = \begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

- **Soln:** T. S. T. A is unitary, we have to show $AA^{\theta} = I$
- Consider $A = \frac{1}{3}\begin{bmatrix} 2+i & 2i \\ 2i & 2-i \end{bmatrix}$ $\therefore A^{\theta} = \frac{1}{3}\begin{bmatrix} 2-i & -2i \\ -2i & 2+i \end{bmatrix}$
- $AA^{\theta} = \frac{1}{9}\begin{bmatrix} 2+i & 2i \\ 2i & 2-i \end{bmatrix}\begin{bmatrix} 2-i & -2i \\ -2i & 2+i \end{bmatrix}$
- $= \frac{1}{9}\begin{bmatrix} (2+i)(2-i)-(2i)(2i) & -2i(2+i)+2i(2+i) \\ 2i(2+i)-2i(2+i) & -(2i)(2i)+(2-i)(2+i) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5+4 & 0 \\ 0 & 5+4 \end{bmatrix}$
- $= I$
- Hence the given matrix is unitary. For unitary matrix $A^{-1} = A^{\theta} = \frac{1}{3}\begin{bmatrix} 2-i & -2i \\ -2i & 2+i \end{bmatrix}$

Example 2

Show matrix A is unitary and hence find A^{-1} where, $A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Soln: $A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = A$ and

$$AA^{\theta} = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix} 4 & -2i+2i & 0 \\ 2i-2i & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = I$$

- Hence the given matrix is unitary.
- For unitary matrix $A^{-1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

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Example 3

Show that $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

- **Soln:** Since, $A^\theta = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$ Let us check,
- $AA^\theta = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$

$$= \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & (\alpha + i\gamma)(\beta - i\delta) - (\beta - i\delta)(\alpha + i\gamma) \\ (\beta + i\delta)(\alpha - i\gamma) - (\beta + i\delta)(\alpha - i\gamma) & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- By comparing, we get the required condition,

  that A is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$