K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 2 – Kinematics of Particles & Rigid Bodies

Module Section 2.2 – Kinematics of Rigid Bodies

Class: FY BTech Division: C3

Professor: Aniket S. Patil Date: 09/04/2023

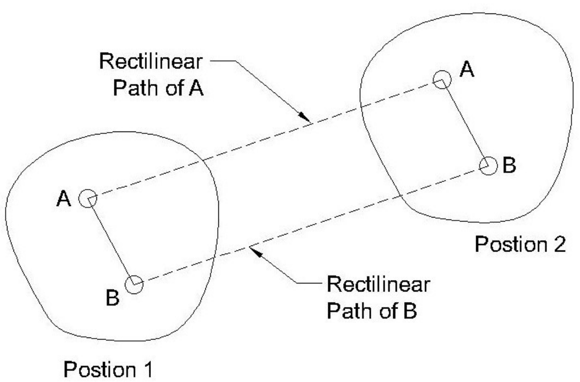
References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

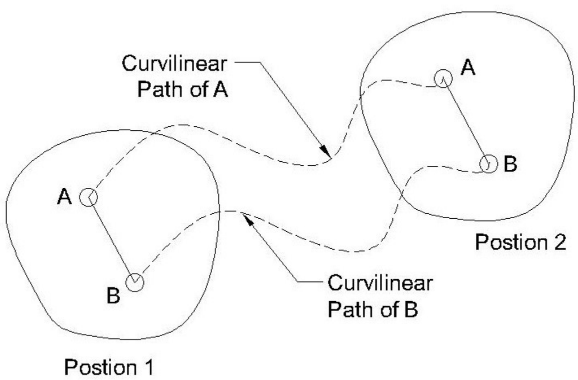
Types of Rigid Body Motion:

1. Translation Motion
2. Rotation about a fixed axis
3. General Plane Motion (More important for numericals)
4. Motion about a fixed point (not in syllabus)
5. General Motion (not in syllabus)

Translation Motion: In this, all the particles forming the body travel along parallel paths, and the orientation on the body does not change during the motion. The motion may be rectilinear or curvilinear.

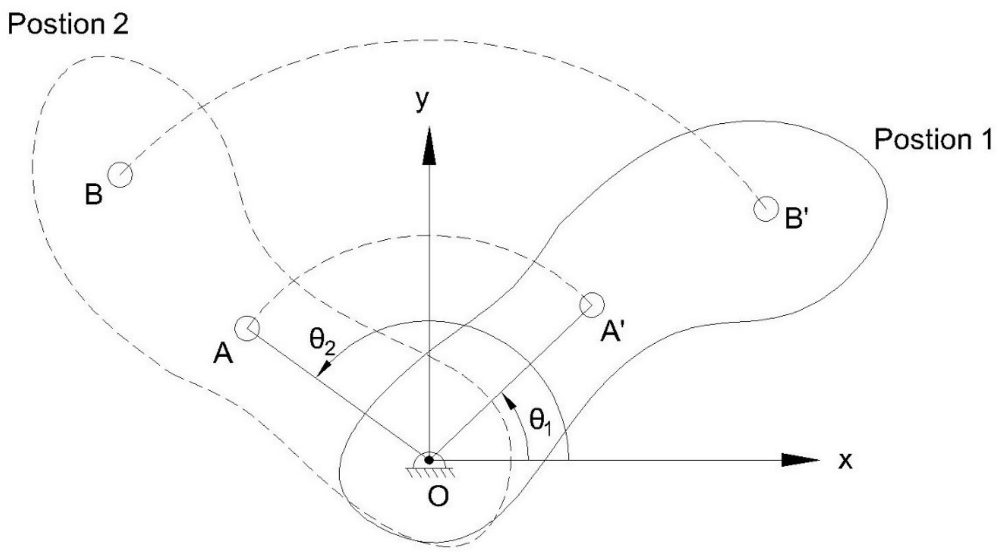
Let a body move from position 1 to position 2, with two points A & B labelled for reference. The line joining A & B maintains the same direction orientation in both positions. The path travelled by A is parallel to the path travelled by B, be it straight or curved.





At any given instant, in a translational motion, all particles of the body have the same displacement, same velocity and same acceleration. Hence, at its centre of gravity G, a rigid body is similar to a particle in translation motion.

Rotation about Fixed Axis: In this, all the particles of the body travel along concentric circular paths about a common centre of rotation. The axis of rotation is perpendicular to the plane of motion.



Angular Position θ is measured in anticlockwise direction from x-axis in radians.

Angular Displacement is the change in angular position. It is also labelled with θ and measured in radians (rad) given by, θ or $\Delta\theta = \theta_2 - \theta_1$.

$$1 \text{ revolution} = 2\pi \text{ radians} = 360°$$

Angular Velocity is the rate of change of angular position with respect to time measured in radians per second (rad/s).

$$\omega = \frac{d\theta}{dt} \circlearrowleft +ve \qquad\qquad 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

Angular Acceleration is the rate of change of angular velocity with respect to time measured in radians per second squared (rad/s^2).

$$\alpha = \frac{d\omega}{dt}$$

Types of Rotational about Fixed Axis:

1. Uniform Angular Velocity Motion

$$\omega = \frac{\theta}{t}$$

2. Uniform Angular Acceleration Motion

$$\omega = \omega_0 + \alpha t$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

3. Variable Angular Acceleration Motion

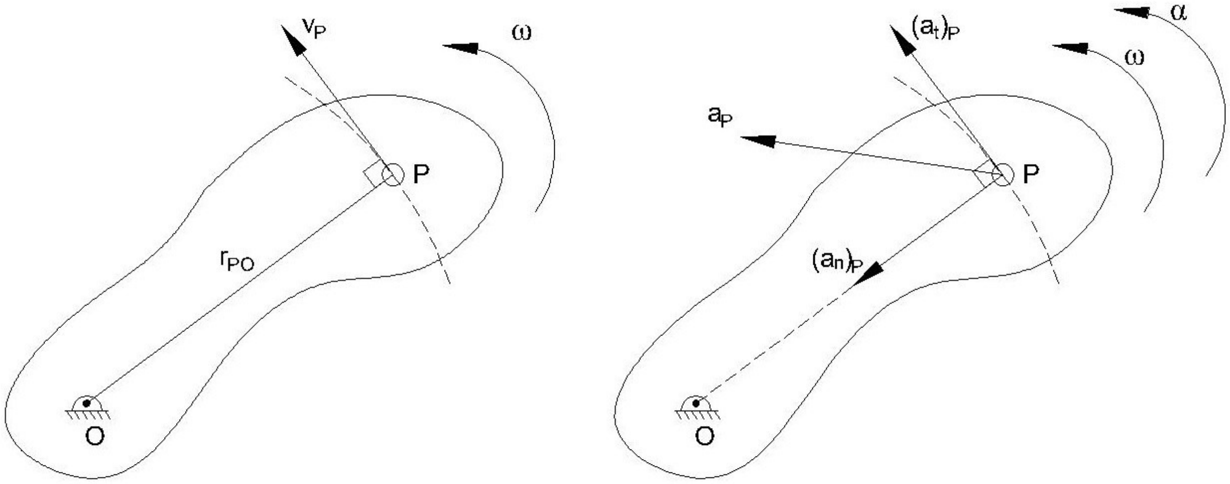
$$\omega = \frac{d\theta}{dt} \quad \& \quad \alpha = \frac{d\omega}{dt} \quad \rightarrow \quad \alpha = \omega\frac{d\omega}{d\theta}$$

Relations between Linear and Angular Parameters:

All particles in a rotating body will have the same angular velocity but different linear velocities. For a point P, if v_P is the linear velocity and r_{PO} is the radial distance from P to O, then $v_P = r_{PO} \times \omega$.

In general, for any particle with linear velocity v located at a radial distance of r from the axis of rotation with the body having an angular velocity of ω,

$$\mathbf{v = r\omega}$$



If the particle has a linear acceleration of a_P which can be resolved into normal component $(a_n)_P$ and tangential component $(a_t)_P$, and the body has an angular acceleration of α, then,

$$\because a_n = \frac{v^2}{\rho} \quad \Rightarrow \quad (a_n)_P = \frac{(v_P)^2}{r_{PO}} = \frac{(r_{PO} \times \omega)^2}{r_{PO}} = \frac{(r_{PO})^2(\omega)^2}{r_{PO}} = r_{PO} \times \omega^2$$

$$\because a_t = \frac{dv}{dt} \quad \Rightarrow \quad (a_t)_P = \frac{dv_P}{dt} = \frac{d(r_{PO} \times \omega)}{dt} = r_{PO}\frac{d\omega}{dt} = r_{PO} \times \alpha$$

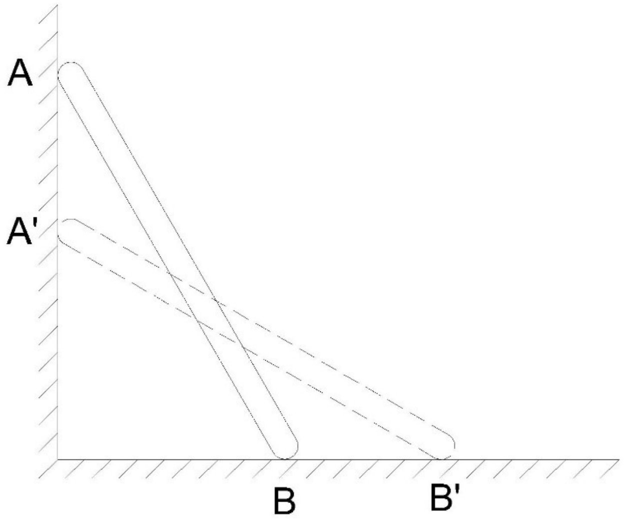
In general, for any particle with linear acceleration a located at a radial distance of r from the axis of rotation with the body having an angular velocity of ω,

$$\mathbf{a_n = r\omega^2} \quad \& \quad \mathbf{a_t = r\alpha}$$

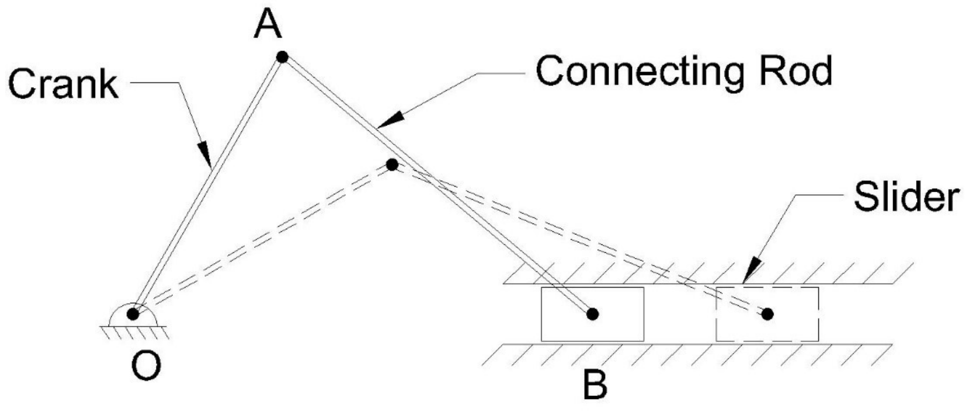
General Plane Motion: It is a combination of translation motion and rotational motion happening at the same time.

Example 1: Consider a ladder AB having the top end A on a vertical wall and bottom end B on the floor and its sliding. Hence, the velocity of A will be vertically down and that of B will be horizontally towards right.

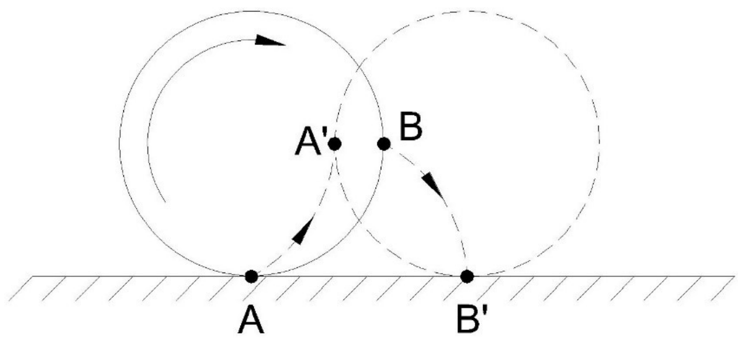
Here, since A and B are not moving in the same direction, it is not a translation motion, but it is not strictly rotation either even though there is some rotation involved. Hence, it is a combination of both, i.e., general plane (GP) motion.



Example 2: In a slider-crank mechanism, shown below, the crank undergoes rotational motion about a fixed hinge support and the slider undergoes back and forth translation motion. The connecting rod linking the crank and slider undergoes GP motion.



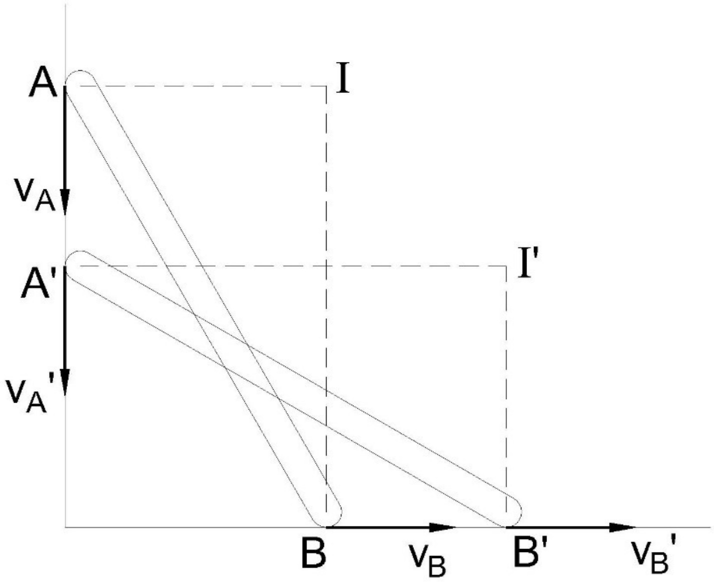
Example 3: When a wheel rolls without slipping on the ground, the wheel rotates as well as translates. Hence, it undergoes GP motion.

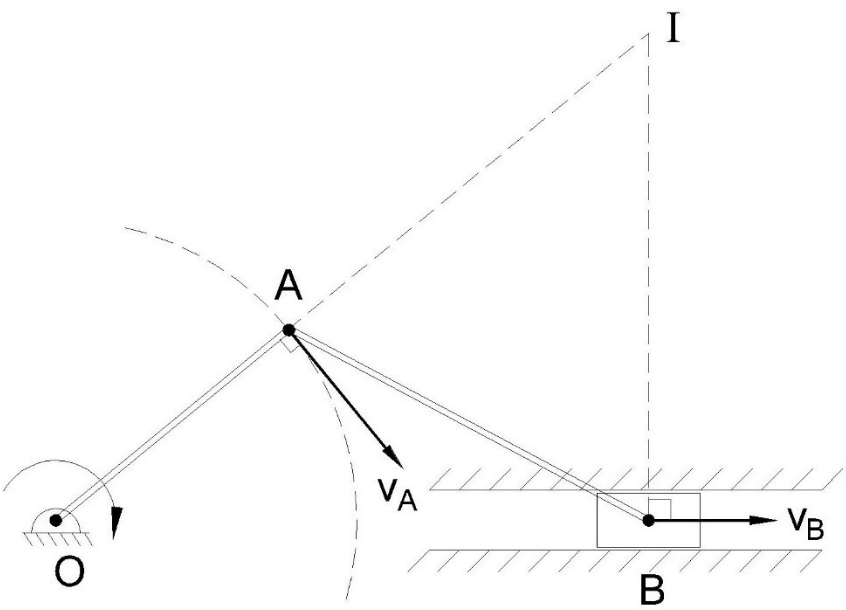


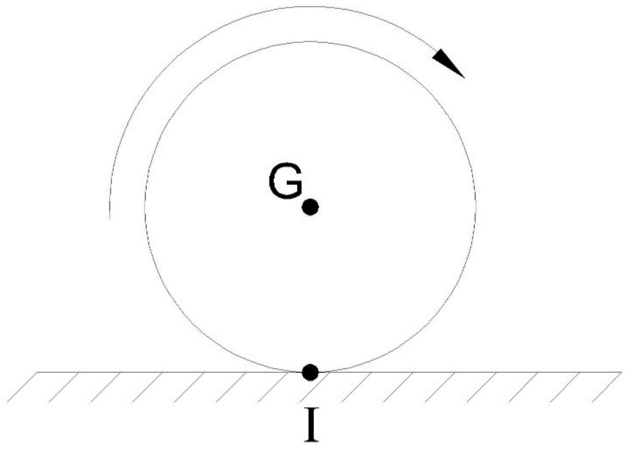
Instantaneous Centre of Rotation (I.C.R.): For general plane motion, at a particular instant, the body can be said to be rotating about a specific point. This point keeps changing as the body moves through the plane. This is called the instantaneous centre of rotation.

It is defined as the point about which a general plane moving body rotates at any given instant. The locus of the ICR's throughout the motion is known as centrode. ICR's are usually denoted by the letter I.

Instantaneous Centre Method: To find the angular velocity of a GP body, we use this method. Find the points on a GP body whose velocity is known. If we draw perpendiculars to the direction of velocity of those points, they will intersect at a certain point. This point is the I.C.R. and we can find the radial lengths to those points. Depending on the known quantities, we can use the relation v = rω to find the unknown quantities in a given problem.



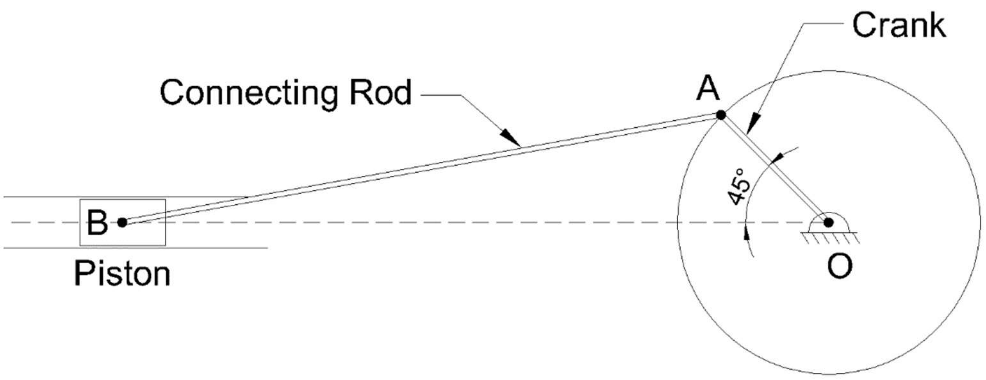




For a wheel which rolls without slipping (performing a general plane motion), the Instantaneous Centre of Rotation is the point of contact with the ground. This is because the centre of rotation should have zero velocity and the point in contact with the ground has velocity of the ground, which is zero.

Numericals:

N1: In a crank and connected rod mechanism, the length of crank and connecting rod are 300 mm and 1200 mm respectively. The crank is rotating at 180 rpm anticlockwise. Find the velocity of piston, when the crank is at an angle of 45° with the horizontal.



Soln: The crank OA performs rotation motion about fixed axis at O, the connecting rod AB performs General Plane Motion, while piston B performs translation motion.

Given:

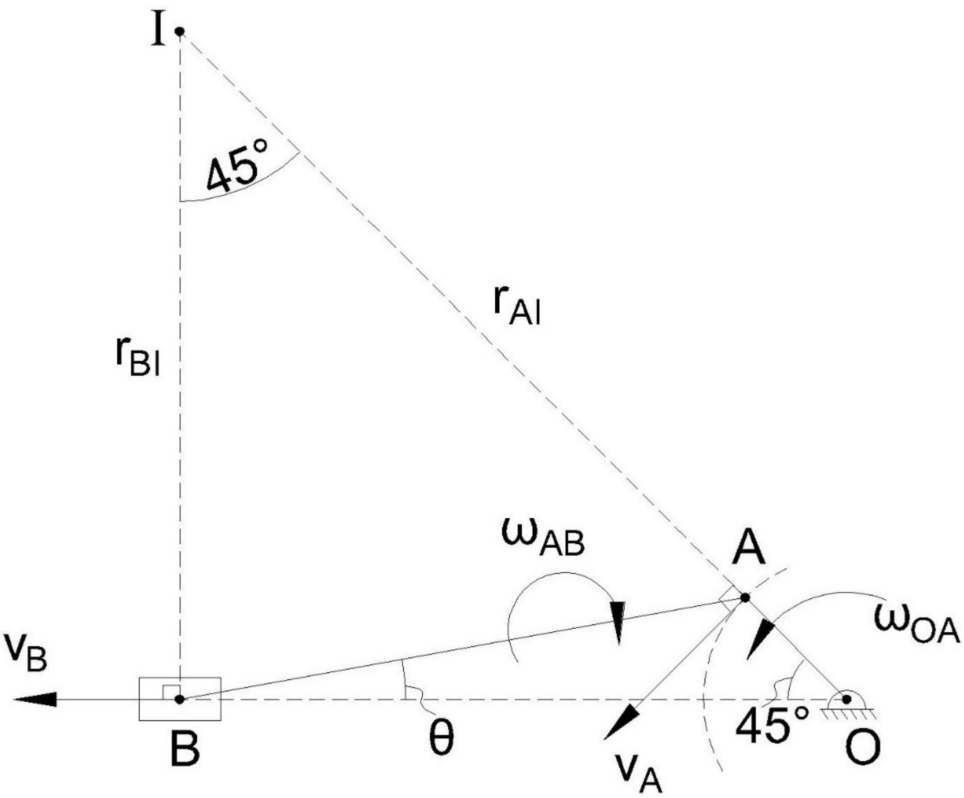
$$\mathrm{OA} = 300\ \mathrm{mm} = 0.3\ \mathrm{m}$$

$$\mathrm{AB} = 1200\ \mathrm{mm} = 1.2\ \mathrm{m}$$

$$\omega_{\mathrm{OA}} = 180\ \mathrm{rpm}\ \circlearrowleft = 180 \times \frac{2\pi}{60}\ \mathrm{rad/s}$$

$$\omega_{\mathrm{OA}} = 18.849\ \mathrm{rad/s}$$

$$\angle \mathrm{AOB} = 45°$$



$$\because v = r\omega \rightarrow v_A = r_{OA} \times \omega_{OA}$$
$$v_A = 0.3 \times 18.849 = 5.655 \ \ m/s \ \swarrow$$

Drawing perpendiculars from the velocity of A (↙) and the velocity of B (←), we can locate their intersection point I.

In $\Delta OBI, \because \angle BOI = 45° \ \& \ \angle IBO = 90° \Rightarrow \angle BIO = 180° - 90° - 45° = 45°$

In $\Delta OAB, \angle AOB = 45° \ \ \& \ \ \text{let} \ \angle ABO = \theta$

$\therefore$ using sine rule, $\dfrac{0.3}{\sin\theta} = \dfrac{1.2}{\sin 45°} \Rightarrow \sin\theta = \dfrac{0.3 \times 0.707}{1.2} \Rightarrow \theta = 10.18°$

$\therefore$ In $\Delta ABI, \qquad \angle ABI = 90° - \theta = 90° - 10.18° = 79.82°$

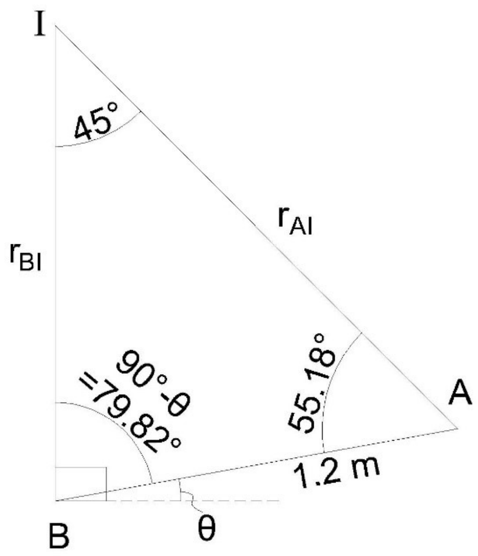
$\Rightarrow \angle IAB = 180° - 79.82° - 45° = 55.18°$

$\therefore$ using sine rule in ΔABI,

$$\frac{1.2}{\sin 45°} = \frac{r_{AI}}{\sin 79.82°} = \frac{r_{BI}}{\sin 55.18°}$$

$$\therefore r_{AI} = \frac{1.2 \times \sin 79.82°}{\sin 45°} = 1.6703 \text{ m}$$

$$\therefore r_{BI} = \frac{1.2 \times \sin 55.18°}{\sin 45°} = 1.39 \text{ m}$$



"I" is the instantaneous centre of rotation for connecting rod AB which is undergoing GP Motion. If AB is rotating about I with an angular velocity of ω_{AB},

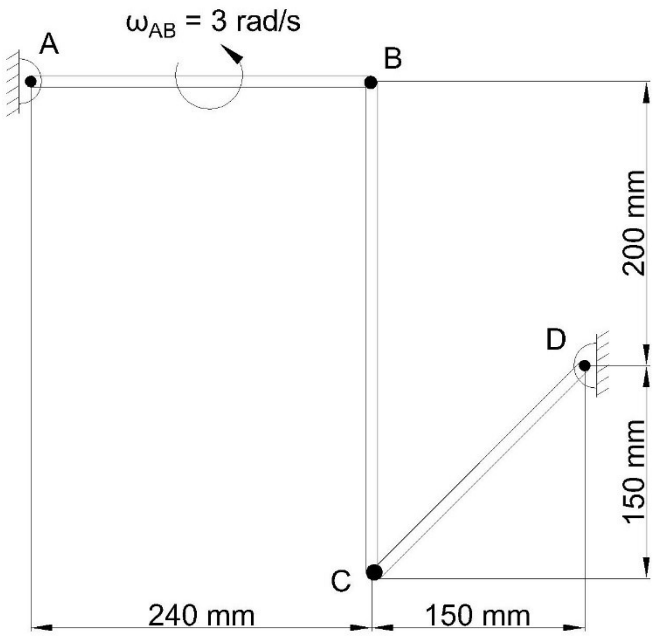
For point A, $v_A = r_{AI} \times \omega_{AB} \Rightarrow 5.655 = 1.6703 \times \omega_{AB}$

$$\therefore \omega_{AB} = 3.392 \text{ rad/s} \circlearrowright$$

For point B, $v_B = r_{BI} \times \omega_{AB} = 1.39 \times 3.392$

$$\therefore \mathbf{v_B = 4.714 \ m/s} \leftarrow$$

N2: In the position shown, bar AB has a constant angular velocity of 3 rad/s anticlockwise. Determine the angular velocity of bar CD.



Soln: Rods AB and CD perform rotational motion and rod BC performs GP motion.

Given:

$$AB = 240 \text{ mm} = 0.24 \text{ m}$$

$$BC = 350 \text{ mm} = 0.35 \text{ m}$$

$$CD = \sqrt{150^2 + 150^2} = 212.132 \text{ mm} = 0.212 \text{ m}$$

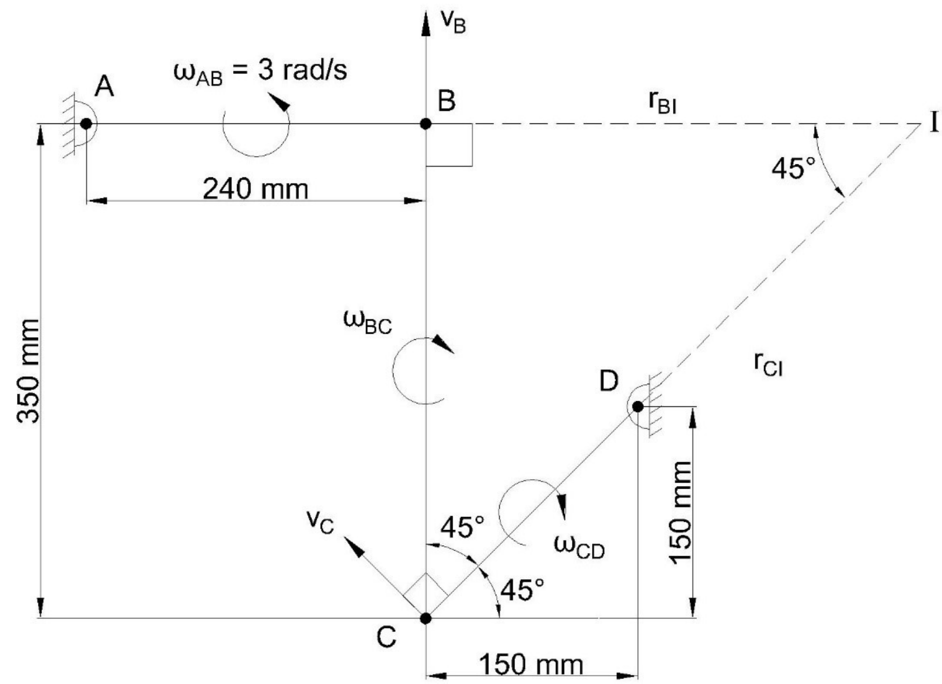
$$\omega_{AB} = 3 \text{ rad/s} \circlearrowleft$$

Velocity of B will be perpendicular to the rod AB in the upward direction.

$$v_B = r_{AB} \times \omega_{AB} = 0.24 \times 3$$
$$\therefore v_B = 0.72 \text{ m/s} \uparrow$$

Velocity of C will be perpendicular to the rod CD, up and to the left (↖).

We draw perpendiculars from the velocity of B (↑) and the velocity of C (↖), to find their intersection point I, which is the instantaneous centre of rotation for rod BC.



ΔBCI is a right-angles isosceles triangle.

$\therefore\ r_{BI} = BC = 0.35\text{ m} \quad \& \quad r_{CI} = \sqrt{0.35^2 + 0.35^2} = 0.495\text{ m}$

The rod BC performs GP motion about centre of rotation I.

For point B, $v_B = r_{BI} \times \omega_{BC} \Rightarrow 0.72 = 0.35 \times \omega_{BC}$

$$\therefore\ \omega_{BC} = 2.057\ \text{rad/s} \circlearrowright$$

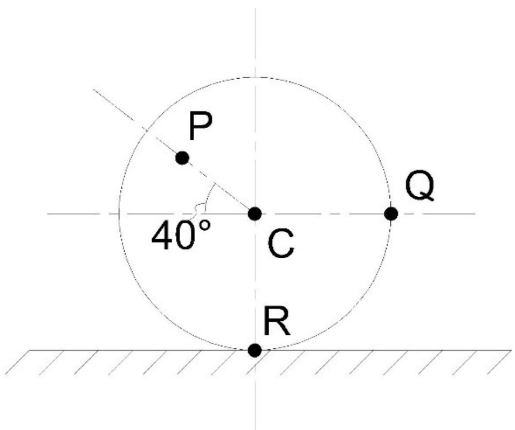
For point C, $v_C = r_{CI} \times \omega_{BC} = 0.495 \times 2.057$

$$\therefore\ v_C = 1.018\ \text{m/s} \nwarrow$$

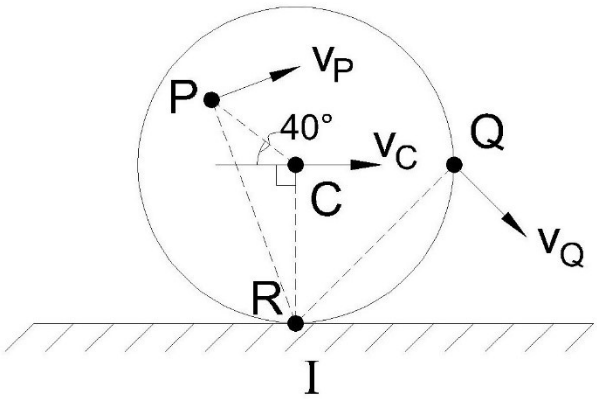
Rod CD is rotating about D, $v_C = r_{CD} \times \omega_{CD} \Rightarrow 1.018 = 0.212 \times \omega_{BC}$

$$\therefore\ \boldsymbol{\omega_{CD} = 4.8\ \text{rad/s}} \circlearrowright$$

N3: A 0.4 m diameter wheel rolls on a horizontal plane without slip, such that its centre has a velocity of 10 m/s towards right. Find angular velocity of the wheel and also velocities of points P, Q, and R shown on the wheel. Given l(CP) = 0.15 m.



Soln: For a wheel which rolls without slipping (performing a general plane motion), the Instantaneous Centre of Rotation is the point of contact with the ground. This is because the centre of rotation should have zero velocity and the point in contact with the ground has velocity of the ground, which is zero. Hence, point R is the instantaneous centre of rotation I.



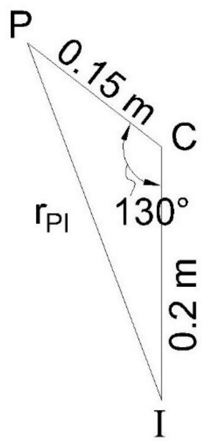
Given:

$$\text{Diameter} = 0.4\text{ m} \Rightarrow \text{CI} = 0.2\text{ m}$$
$$v_C = 10\ \text{m/s}$$
$$\text{CP} = 0.15\text{ m}$$

Let ω be the angular velocity of the wheel (all points will have the same ω as it is one single body moving together)

For point C, $v_C = r_{CI} \times \omega \Rightarrow 10 = 0.2 \times \omega$

$$\therefore \omega = 50\ \text{rad/s} \circlearrowright$$



$\therefore$ In ΔCPI, $\angle PCI = 90° + 40° = 130°$

$\therefore$ using cosine rule,

$r_{PI}^2 = 0.15^2 + 0.2^2 - 2(0.15)(0.2)\cos 130°$

$r_{PI} = 0.3179\text{ m}$

For point P, $v_P = r_{PI} \times \omega = 0.3179 \times 50 \Rightarrow$

$$\therefore \mathbf{v_P = 15.895\ m/s} \nearrow$$

$\therefore$ In right angled ΔICQ, $r_{PI} = \sqrt{0.2^2 + 0.2^2} = 0.2828\text{ m}$

For point Q, $v_Q = r_{QI} \times \omega = 0.2828 \times 50 \Rightarrow$

$$\therefore \mathbf{v_Q = 14.14\ m/s} \searrow$$

Since, R coincides with the instantaneous centre I, velocity of point R is zero because ICR has zero velocity. $\therefore \mathbf{v_R = 0}$