Triple Integrals

The triple integral is a generalized form of a double integral. We know that the single integral and double integrals are defined for functions of single variable and two variables respectively. Similarly, triple integral is defined for function of three variables.

Consider the simplest case where the function of three independent variables x, y, z i.e. $f(x,y,z)$ is defined on a rectangular box (see fig-a):

$$B = \{(x,y,z) \mid a \leq x \leq b,\ c \leq y \leq d,\ p \leq z \leq q\}$$

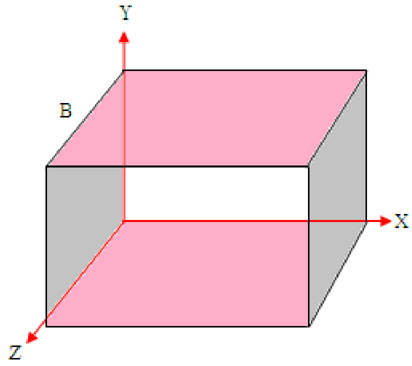

Fig-a

Now, we divide B into sub-boxes. we do this by dividing the interval $[a,b]$ into l sub-intervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c,d]$ into m sub-intervals $[y_{j-1}, y_j]$ of equal width Δy and $[p,q]$ into n sub-intervals $[z_{k-1}, z_k]$ of equal width Δz. Then B might look something like this (see fig-b)

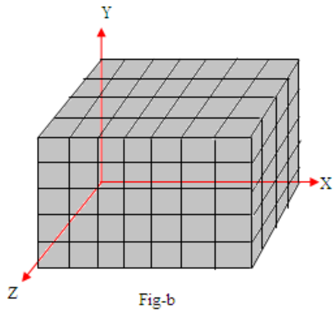

Fig-b

The volume of each small box is

$$\Delta V = \Delta x\, \Delta y\, \Delta z$$

In abobe figure the boxes are arranged in layers, with each layer arranged into rows and columns. Now, the box ijk refers to the box in the ith row, the jth column, and the kth layer. Suppose that the density of the box $ijk$is constant. The mass of the box ijk is its density times it volume.

$$f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$$

Now, we sum up these approximate masses to estimate the total mass of the box B. We obtain the Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$$

By analogy with the definition of a double integral, let $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ and the number of small boxes go to infinity, then the Riemann sum approaches the triple integral over the box B. Therefore,

$$\iiint_B f(x,y,z)\mathrm{d}V = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$$

if the limit exists.

Note: If the function $f(x,y,z)$ is continuous the triple integral always exists.

5.1 Rules to evaluate triple integrals

a) If the region B is specified by the inequalities, $a \leq x \leq b$, $c \leq y \leq d$ and $p \leq z \leq q$ and a, b, c, d, p and q are all constants, then evaluate triple integral with given order.

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_a^b \int_c^d \int_p^q f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_a^b \mathrm{d}z \int_c^d \mathrm{d}y \int_p^q f(x,y,z)\mathrm{d}x$$

i.e. To evaluate integrals of above type first integrate w.r.t x, then w.r.t. y and finally w.r.t. z.

b) If middle integral and inner integrals limits are functions of single variable, then first integrate w.r.t. a variable which is independent of middle and inner integral i.e. a variable which is not involved in middle and inner integral. Next, integrate w.r.t. a variable which is independent of middle integration limits i.e. w.r.t a variable which is not involved in middle integral. Finally, integrate w.r.t. remaining variable.

Ex.1) If $\iiint\limits_B f(x,y,z)\mathrm{d}V = \int\limits_a^b \int\limits_{f_1(z)}^{f_2(z)} \int\limits_{g_1(x)}^{g_2(x)} f(x,y,z)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$ where a and b are constants can be evaluated by integrating first w.r.t y, then w.r.t x and finally w.r.t z.

Ex.2) If $\iiint\limits_B f(x,y,z)\mathrm{d}V = \int\limits_a^b \int\limits_{f_1(y)}^{f_2(y)} \int\limits_{g_1(z)}^{g_2(z)} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ where a and b are constants. To evaluate, integrating first w.r.t x, then w.r.t z and finally w.r.t y. Therefore, we get

$$\iiint\limits_B f(x,y,z)\mathrm{d}V = \int\limits_a^b \int\limits_{f_1(y)}^{f_2(y)} \int\limits_{g_1(z)}^{g_2(z)} f(x,y,z)\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$$

c) If middle integral limits are functions of single variable and inner limits are functions of two variables, then first integrate w.r.t a variable which is not involved in inner integral, then w.r.t. a variable which is not involved in middle variable and finally integrate w.r.t a remaining variable.

Ex.1) If $\iiint\limits_B f(x,y,z)\mathrm{d}V = \int\limits_a^b \int\limits_{f_1(x)}^{f_2(x)} \int\limits_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ where a and b are constants can be evaluated by integrating first w.r.t z, then w.r.t y and finally w.r.t x.

Therefore, $\iiint\limits_B f(x,y,z)\mathrm{d}V = \int\limits_a^b \int\limits_{f_1(x)}^{f_2(x)} \int\limits_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

Ex.2) If $\iiint\limits_B f(x,y,z)\mathrm{d}V = \int\limits_a^b \int\limits_{f_1(z)}^{f_2(z)} \int\limits_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ where a and b are constants can be evaluated by integrating first w.r.t x, then w.r.t y and finally w.r.t z.