Properties and results on Eigenvalues and vectors

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1:13 PM

If λ is eigenvalue & X is eigenvector for matrix A

11) Then λ^k is eigenvalue for A^k and eigenvector remains the same

$A^k x = \lambda^k x$

k positive integers

$Ax = \lambda x$

Then premultiply by A, $A(Ax) = A(\lambda x)$

$A^2 x = \lambda (Ax) = \lambda(\lambda x) = \lambda^2 x$

$\boxed{A^2 x = \lambda^2 x} \longrightarrow A^3 x = \lambda^3 x \to$ continue.

$\boxed{A^k x = \lambda^k x}$

12) Eigenvalues for A & A^T are same but their eigenvects are different.

$|A - \lambda I| = |A^T - \lambda I|$ $\to [A - \lambda I]x \neq [A^T - \lambda I]x$

13) $\bar{\lambda}$ is eigenvalue for $A^\theta = \overline{(A^t)}$ & eigenvector is differen

14) If λ is non zero eigenvalue of A then $\frac{1}{\lambda}$ is eigenvalue of A^{-1} & eigenvector is same.

$\lambda^{-1} \to \frac{1}{\lambda}$

15) If any of eigenvalue of A is zero then $|A| = 0$
A^{-1} does not exist.

16) Then $\boxed{F(\lambda)}$ is eigenvalue for $\boxed{F(A)}$ (polynomial) & eigenvector is same

$A^2 x - 3Ax + 5Ix - 3A^{-1}x$
$\lambda^2 x - 3\lambda x + 5(1) - 3\frac{1}{\lambda}x$

eigenvector is same ($Ax = \ldots$ $\lambda^2 x - 3\lambda x + 5(1) - 3\frac{1}{\lambda}x$)

<u>Question</u> $A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ Find eigenvalues & eigenvector for $F(A) = A^2 - 3A + 5I - 3A^{-1}$

$\boxed{3}$, $2, 2$

$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

Then we know $F(A)$ will have eigenval as $F(\lambda)$

$\therefore F(\lambda) = \lambda^2 - 3\lambda + 5 - 3(1/\lambda)$

$F(3) = 3^2 - 3(3) + 5 - \cancel{3}(1/\cancel{3})$
$= 4$

$F(2) = 2^2 - 3(2) + 5 - 3(1/2)$
$= 3/2$

$F(2) = 3/2$

$\therefore$ eigenvalue of $A^2 - 3A + 5I - 3A^{-1}$ is given by $4, 3/2, 3/2$ with corresponding eigenvector as $(1, -1, 1)$, $(2, 1, 0)$, $(-3, 0, 1)$

17) A is diag Then A^k is also diagonalisable.

$A = MDM^{-1} \Rightarrow A^k \approx D^k$

18) eigenvectors corresponding to distinct eigenvalues of real symmetric Matrix are Orthogonal.

$3 \quad -1 \quad 2 \quad M$

$\begin{bmatrix} 3 & 5 & -7 \\ & & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 5 & -7 \\ 5 & 2 & 2 \\ -7 & 2 & 1 \end{bmatrix}$$

orthogonal

19) Hermitian $\rightarrow$ $|A| \rightarrow$ real
$\rightarrow$ eigenvalues $\rightarrow$ real.

20) Skew Hermitian

Symmetric

skew

Orthogonal $\rightarrow$ $|A| = \pm 1$
$\rightarrow$ eigenvalues $= \pm 1$ $e^{i\alpha}$
= unit modulus

Unitary $\rightarrow$ $|A|$ = unit modulus
eigenvalues = unit modulus

Put all properties in tabular Form

	Matrix	eigenvalues	eigenvectors
1)	A	λ	×
2)	A^n	λ^n	×
3)	A^T	λ	different
4)	A^θ	$\bar{\lambda}$	different
5)	A^{-1}	$1/\lambda$	×
6)	KA	$K\lambda$	×

5)			
6)	kA	$k\lambda$	×
7)	I	1	$\hat{i}, \hat{j}, \hat{k}$
8)	Scalar kI	k	$\hat{i}, \hat{j}, \hat{k}$
9)	F(A)	$F(\lambda)$	×
10)	adj A	$\frac{\lvert A \rvert}{\lambda}$	×
11)	If $\lvert A \rvert = 0$	atleast one λ is zero	—
12)	Hermitian	all real	—
13)	Symmetric	all real	—
14)	<u>real symmetric</u>	(distinct)	orthogonal
15)	skew Hermitian	zero/pur img	—
16)	skew symmetric	zero/pur img	—
17)	Unitary	$\lvert\lambda\rvert = 1$	—
18)	Unitary	→ (distinct)	orthogonal
19)	orthogonal	± 1	—
20)	Triangular,	diag elements	—
21)	Diagonal	diag elements	$\hat{i}, \hat{j}, \hat{k}$

Problem: Find a symmetric Matrix $A_{3\times3}$ having the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 6$ & $\lambda_3 = 9$ with corresponding eigenvectors $X_1 = (1, 2, 2)'$ & $X_2 = (-2, 2, -1)$ & X_3

Sol^n:

AM = GM → Diagonalisable

$$A = M D M^{-1}$$

Sol^n: Since A is symmetric ∴ eigenvector corresponds to dinstinct eigenvalues are orthogonal.

∴ $X_1, X_2, \& X_3$ are mutually orthogonal.

Let $X_3 = (x, y, z)'$ be eigenvector corr to $\lambda = 9$

$$\therefore X_1 \cdot X_3 = 0 \qquad \boxed{x + 2y + 2z = 0} \quad \text{--- (1)}$$

$$\text{Similarly } X_2 \cdot X_3 = 0 \qquad \boxed{-2x + 2y - z = 0} \quad \text{--- (2)}$$

Use cramer's rule for two independent rows.

$$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \frac{x}{\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}} = t$$

$$x = -6t, \quad y = -3t, \quad z = 6t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6t \\ -3t \\ 6t \end{pmatrix} \cong \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6t \\ -3t \\ 6t \end{pmatrix} \simeq \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Since all eigenvalues are distinct $\therefore$ AM = GM

A is diagonalisable.

$\therefore$ $A = MDM^{-1}$ where $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 6 & 9 \end{bmatrix}$

& $M = \begin{bmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ $\quad M^{-1} = \frac{1}{|M|}(\text{adj} M)$

$A = MDM^{-1} =$

$$\longrightarrow \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$