PRACTICE PROBLEMS ON BETA FUNCTION

EXERCISE -1

Evaluate the following integrals. **(1 to 51)**

1. $\int_0^9 x^{3/2}\,(9-x)^{1/2}\,dx$

2. $\int_0^1 \left(1-\sqrt[5]{x}\right)^{3/2} dx$

3. $\int_0^2 x\,\sqrt[3]{8-x^3}\,dx$

4. $\int_0^1 \frac{x^2(4-x^2)}{\sqrt{1-x^2}}\,dx$

5. $\int_0^1 \sqrt{1-x^4}\;dx$

6. $\int_0^a x^6(a^4-x^4)^{1/4}\,dx$

7. $\int_0^1 \sqrt{\left[\sqrt{x}-x\right]}\,dx$

8. $\int_0^{2a} x^2\,\sqrt{2ax-x^2}\,dx$

9. $\int_0^2 x^7(16-x^4)^{10}\,dx$

10. $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$

11. $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}\;\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

12. $\int_0^1 \sqrt{1-\sqrt{x}}\;dx \cdot \int_0^{1/2} \sqrt{2y-4y^2}\;\;dy$

13. $\int_0^{\pi/4} sin^7\,2\theta\;d\theta$

14. $\int_0^{\pi/4} cos^7 2\theta\,d\theta$

15. $\int_0^{\pi/6}\; sin^6\,3\theta\;\cdot d\theta$

16. $\int_0^{\pi}\;(1-cos\,\theta)^6\;\cdot d\theta$

17. $\int_0^{\pi} \frac{sin^4\,\theta}{(1+cos\theta)^2}\cdot d\theta$

18. $\int_0^{\pi}\;(1-cos\,\theta)^3\;\cdot d\theta$

19. $\int_0^{\pi/4}\;(1+cos\,4\,\theta)^5\;\cdot d\theta$

20. $\int_0^{\pi/8} sin^4\,8\theta\,cos^2\,4\,\theta\;\,d\theta$

21. $\int_0^{\pi/6} cos^3 3\theta sin^2 6\theta\;\,d\theta$

22. $\int_0^{\pi/4} cos^3 2\theta sin^2 4\theta\;\,d\theta$

23. $\int_0^{\pi/4} cos^3 2x\,sin^4 4x\;\,dx$

24. $\int_{-\pi}^{\pi} sin^2 x\,cos^4 x\;\,dx$

25. $\int_0^{\pi} sin^2\theta(1+cos\theta)^4\;\,d\theta$

26. $\int_0^{2\pi} sin^2\theta(1+cos\theta)^4\;\,d\theta$

27. $\int_{-\pi/2}^{\pi/2} cos^3\theta(1+sin\theta)^2\;\,d\theta$

28. $\int_{-\pi/2}^{\pi/2} sin^4 x\,cos^2 x\;\,dx$

29. $\int_{-\pi}^{\pi} sin^4 x\,cos^2 x\;\,dx$

30. $\int_0^{\pi} xsin^7 x\,cos^4 x\;\,dx$

31. $\int_0^{\pi} xsin^5 x\,cos^6 x\;\,dx$

32. $\int_0^{\pi} xsin^4 x\,cos^6 x\;\,dx$

33. $\int_0^{\pi} xsin^5 x\,cos^4 x\;\,dx$

34. $\int_0^{\pi/2} \sqrt{tan\,\theta}\;\,d\theta$

35. $\int_0^{\pi/2} \sqrt{cot\theta}\;\,d\theta$

36. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{sin\theta}} \int_0^{\pi/2} \sqrt{sin\theta}\,d\theta$

37. $\int_0^{\pi/2} (sin\,2x)^{2t-1} dx$

38. $\int_0^{\pi/2} \frac{d\,\emptyset}{\sqrt{1-(1/2)sin^2\emptyset}}$

39. $\int_{-\pi/4}^{\pi/4} (sin\,\theta+cos\,\theta)^{1/3}\;\,d\theta$

40. $\int_{-\pi/6}^{\pi/3} \left(\sqrt{3}sin\,\theta+cos\,\theta\right)^{1/4}\;\,d\theta$

41. $\int_0^1 x^5 \sqrt{\left(\frac{1+x^2}{1-x^2}\right)}\,dx$

42. $\int_0^1 x^5 \sqrt{\left\{\frac{1-x^2}{1+x^2}\right\}}\,dx$

43. $\int_0^1 x^5\,sin^{-1}x\,dx$

44. $\int_0^1 x^4\,cos^{-1}x\,dx$

45. $\int_0^{\infty} \frac{t^6}{(1+t^2)^4}\,.dt$

46. $\int_0^{\infty} \frac{dx}{(1+x^2)^{9/2}}$

47. $\int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}}\,dx$

48. $\int_0^{\infty} \frac{x^3}{(1+x^8)^4}\,dx$

49. $\int_0^{\infty} \left(\frac{t}{1+t^2}\right)^4\;dt$

50. $\int_0^{\infty} \frac{dx}{1+x^4}$

51. $\int_0^{\infty} \left(\frac{t}{1+t^2}\right)^3\;dt$

52. Using Beta function, prove that $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$

53. Prove that $\int_0^{\infty} \frac{1}{(x^2+1)^{n+1}}\,dx = \frac{(2n)!}{2^{2n}\cdot(n!)^2}\,\frac{\pi}{2}$

54. Prove that $\int_0^{\pi/2}\; sin^{2n}\,x\,dx = \frac{1\,.3\,.5\;....(2n-1)}{2^n\,.n!}\cdot\frac{\pi}{2}$

55. Prove that $\int_0^{\pi/2}\; sin^p\,x\,dx \cdot \int_0^{\pi/2}\; sin^{p+1}\,x\,dx = \frac{1}{(p+1)}\cdot\frac{\pi}{2}$

56. Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2^{(2-n)/n}\left(\overline{|1/n}\right)^2}{n\overline{|2/n}}$

57. Prove that $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,dx = \frac{(2n)!}{2^{2n}(n!)^2}.\frac{\pi}{2}.$

58. Prove that $\int_0^a \frac{dx}{(a^n-x^n)^{1/n}} = \frac{\pi}{n} cosec\left(\frac{\pi}{n}\right)$

59. Prove that $\int_0^1 \left(\frac{1}{x}-1\right)^{1/4} dx = \frac{\pi}{2\sqrt{2}}$

ANSWERS

1. $\frac{729\pi}{16}$ **2.** $\frac{256}{3003}$ **3.** $\frac{16\pi}{9\sqrt{3}}$ **4.** $\frac{13\pi}{16}$ **5.** $\frac{\sqrt{\pi}}{6} \cdot \frac{\overline{|1/4}}{\overline{|3/4}}$

6. $\frac{3\sqrt{2}}{128}a^8\pi$ **7.** $\frac{\pi}{8}$ **8.** $\frac{5}{8}a^4 \cdot \pi$ **9.** $\frac{16^{11}}{33}$ **10.** $\frac{432}{35}\pi$

11. $\frac{\pi}{4}$ **12.** $\frac{\pi}{30}$ **13.** $\frac{8}{35}$ **14.** $\frac{8}{35}$ **15.** $\frac{5}{96}\pi$

16. $\frac{231}{16}\pi$ **17.** $\frac{3}{2}\pi$ **18.** $\frac{5}{2}\pi$ **19.** $\frac{63}{32}\pi$ **20.** $\frac{3\pi}{128}$

21. $\frac{32}{315}$ **22.** $\frac{16}{105}$ **23.** $\frac{128}{1155}$ **24.** $\frac{\pi}{8}$ **25.** $\frac{21\pi}{16}$

26. $\frac{21\pi}{8}$ **27.** $\frac{8}{5}$ **28.** $\frac{\pi}{16}$ **29.** $\frac{\pi}{8}$ **30.** $\frac{16\pi}{1155}$

31. $\frac{8\pi}{693}$ **32.** $\frac{3\pi^2}{512}$ **33.** $\frac{8\pi}{315}$ **34.** $\frac{\pi}{\sqrt{2}}$ **35.** $\frac{\pi}{\sqrt{2}}$

36. π **37.** $2^{2t-2}\frac{(\overline{|t})^2}{\overline{|2t}}$ **38.** $\frac{1}{4} \cdot \frac{(\overline{|1/4})^2}{\sqrt{\pi}}$ **39.** $\frac{1}{2^{5/6}}\frac{\overline{|2/3}}{\overline{|7/6}}\sqrt{\pi}$ **40.** $(2)^{\frac{3}{2}}\frac{(\overline{|5/8})^2}{\overline{|1/4}}$

41. $\frac{3\pi+8}{24}$ **42.** $\frac{\pi}{8}-\frac{1}{3}$ **43.** $\frac{11}{192}\pi$ **44.** $\frac{8}{75}$ **45.** $\frac{5}{32}\pi$

46. $\frac{16}{35}$ **47.** $\frac{8}{45}$ **48.** $\frac{5}{128}\pi$ **49.** $\frac{\pi}{32}$ **50.** $\frac{\pi}{2\sqrt{2}}$

51. $\frac{1}{4}$

EXERCISE -2

1. Prove that $\int_{-1}^1 (1+x)^m (1-x)^n \, dx = 2^{m+n+1}B(m+1,n+1)$ Hence, evaluate $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}dx$

2. Prove that $\int_0^n x^n(n-x)^p dx = n^{p+n+1}B(n+1,p+1)$

3. Evaluate the following integrals

(i) $\int_7^{11} \sqrt[4]{(x-7)(11-x)}\,dx$ **(ii)** $\int_3^7 \sqrt[4]{(x-3)(7-x)}\,dx$

4. Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx = \frac{1}{a^n b^m}B(m,n)$ and hence, evaluate

(i) $\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}dx$ **(ii)** $\int_0^\infty \frac{\sqrt{x}}{1+2x+x^2}dx$

5. Prove that $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = B(m,n)$

6. Evaluate the following integrals

(i) $\int_0^\infty \frac{\sqrt{x}}{a^2+2ax+x^2}dx$ **(ii)** $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$ **(iii)** $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$

(iv) $\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}dx$ **(v)** $\int_0^\infty \frac{x^6-x^3}{(1+x^3)^5}x^2dx$ **(vi)** $\int_0^\infty \frac{x^5}{(2+3x)^{16}}dx$

7. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx = \frac{B(m,n)}{(a+b)^m.a^n}$

8. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}dx = \frac{B(m,n)}{2^m}$ and hence, evaluate

(i) $\int_0^1 \frac{x^3-2x^4+x^5}{(1+x)^7}dx$ **(ii)** $\int_0^1 \frac{x-2x^2+x^3}{(1+x)^5}dx$

9. Prove that $\int_0^1 \frac{x^{-1/3}(1-x)^{-2/3}}{(1+2x)}dx = 3^{-2/3}\cdot B\left(\frac{2}{3},\frac{1}{3}\right)$

10. Prove that $\int_0^1 \frac{\left(1-x^4\right)^{3/4}}{(1+x^4)^2}dx = \frac{1}{2^{9/4}}B\left(\frac{7}{4},\frac{1}{4}\right) = \frac{3\pi}{2^{15/4}}$

11. Prove that **(i)** $yB(x+1,\ y) = xB(x,\ y+1)$ **(ii)** $B(x,x) = \frac{1}{2^{2x-1}}B\left(x,\frac{1}{2}\right)$

12. Prove that $B(m,n) = B(m,n+1) + B(m+1,n)$

13. Prove that $B(m,m).B\left(m+\frac{1}{2},m+\frac{1}{2}\right) = \frac{\pi}{m}.2^{1-4m}$

14. Prove that $B(m+1,n) = \frac{m}{m+n}B(m,n)$

15. Prove that $B(n,n) = \frac{\sqrt{\pi}}{2^{2n-1}}.\frac{\left|\overline{n}\right.}{\left|\overline{n+(1/2)}\right.}$.

16. Prove that $B(n,n) = 2\int_0^{1/2}(t-t^2)^{n-1}dt$.

17. Prove that $B(n,n+1) = \frac{1}{2}.\frac{\left(\left|\overline{n}\right.\right)^2}{\left|\overline{2n}\right.}$. Hence, deduce that $\int_0^{\pi/2}\left(\frac{1}{\sin^3\theta} - \frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta d\theta = \frac{\left(\left|\overline{1/4}\right.\right)^2}{2\sqrt{\pi}}$

18. If $B(n,3) = \frac{1}{105}$ and n is a positive integer, find n.

19. Prove that $B\left(n+\frac{1}{2},n+\frac{1}{2}\right) = \frac{1}{2^{2n}}.\frac{\left|\overline{n+(1/2)}\right.}{\left|\overline{n+1}\right.}.\sqrt{\pi}$. Hence, deduce that

$2^n\left|\overline{n+(1/2)}\right. = 1.3.5\ldots.(2n-1)\sqrt{\pi}$. Where n is positive integer

20. Given $\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin p\pi}$, prove that, $\left|\overline{p}\right.\left|\overline{1-p}\right. = \frac{\pi}{\sin p\pi}$ $(0<p<1)$ Hence, evaluate $\int_0^\infty \frac{dy}{1+y^4}$.

21. State true or false with proper justification.

(i) If $m < n$ then $\left|\overline{m}\right. < \left|\overline{n}\right.$ **(ii)** $\left|\overline{1/6}\right.\left|\overline{2/6}\right.\left|\overline{3/6}\right.\left|\overline{4/6}\right.\left|\overline{5/6}\right. = 4\pi^2\sqrt{\frac{\pi}{3}}$

22. Prove that $\int_0^\infty \frac{x}{(1+x^4)^{5/4}}dx.\int_0^\infty \frac{1}{\sqrt{1+x^4}}dx = \frac{\pi}{2\sqrt{2}}$.

23. Prove that $\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}dx.\int_0^\infty \frac{1}{(1+x^4)^{1/2}}dx = \frac{\pi}{2\sqrt{2}}$

24. Show that $\left|\overline{p}\right.\left|\overline{\frac{1-p}{2}}\right. = \frac{\sqrt{\pi}\left|\overline{p/2}\right.}{2^{1-p}\cos(\pi p/2)}$.

25. Show that **(i)** $\left|\overline{x}\right.\left|\overline{-x}\right. = -\frac{\pi}{x\sin x\pi}$ **(ii)** $\left|\overline{\frac{1}{2}+x}\right.\left|\overline{\frac{1}{2}-x}\right. = \frac{\pi}{\cos\pi x}$

26. Show that $\left|\overline{\frac{3}{2}-x}\right.\left|\overline{\frac{3}{2}+x}\right. = \left(\frac{1}{4}-x^2\right)\pi\sec x\pi, (-1<2x<1)$.

27. Prove that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}$ Hence, evaluate **(i)** $\int_0^1 \frac{x^5+x^8}{(1+x)^{15}}dx$ **(ii)** $\int_0^1 \frac{x^2+x^3}{(1+x)^7}dx$

28. Show that $\int_0^{\pi/2} \frac{\cos^{2m-1}\theta.\ \sin^{2n-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{m+n}}d\theta = \frac{B(m,n)}{2.a^{2n}.b^{2m}}$.

29. Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}B\left(\frac{n}{2},\frac{n}{2}\right)$ & hence evaluate $\int_0^\infty sech^8 x\,dx$

30. Prove that $\int_0^\infty sech^6 x\,dx = \frac{8}{15}$

31. Prove that $\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}dx = \frac{1}{2}B(m+n, n-m)$

32. Prove that $\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q} = B(p+q, 1-q)$.

33. Show that $\int_0^{\pi/2} tan^n x dx = \frac{\pi}{2} sec\left(\frac{\pi n}{2}\right)$. Deduce that $\int_0^{\pi/2} cot^n x dx = \frac{\pi}{2} sec\left(\frac{\pi n}{2}\right)$

34. Prove that $\int_0^\pi \frac{sin^{n-1}x}{(a+bcos\,x)^n}dx = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B\left(\frac{n}{2}, \frac{n}{2}\right)$

35. Prove that $\int_0^\pi \frac{\sqrt{\sin x}}{(5+3\cos x)^{3/2}}dx = \frac{(\overline{|3/4})^2}{2\sqrt{2\pi}}$.

36. Given $\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin p\pi}, 0 < p < 1,$ prove that $\int_0^1 \frac{x^{n-1}}{(1+cx)(1-x)^n}dx = \frac{1}{(1+c)^n} \cdot \frac{\pi}{\sin(n\pi)}$

ANSWERS

1.	π	**3.** **(i)**	$\frac{2}{3} \cdot \frac{(\overline{|1/4})^2}{\sqrt{\pi}}$	**(ii)**	$\frac{2\,(\overline{|1/4})^2}{3\sqrt{\pi}}$
4. **(i)**	$\frac{\pi}{2\sqrt{2}}$	**(ii)**	$\frac{\pi}{2}$		
6. **(i)**	$\frac{\pi}{2\sqrt{a}}$	**(ii)**	0	**(iii)**	$\frac{1}{5005}$
(iv)	0	**(v)**	0	**(vi)**	$\frac{1}{2^{10}3^6} \cdot \frac{\overline{|6}\,\overline{|10}}{\overline{|16}}$
8. **(i)**	$\frac{1}{960}$	**(ii)**	$\frac{1}{48}$	**18.**	5
20.	$\frac{\pi}{2\sqrt{2}}$	**21.** **(i)**	false	**(ii)**	true
26. **(i)**	B(6, 9)	**(ii)**	B(3, 4)	**28.**	$\frac{16}{35}$