PRACTICE PROBLEMS ON DUIS

EXERCISE

Prove the following **(1 to 34):**

1. $\int_0^1 \frac{x^a - x^b}{\log x} dx = log\left(\frac{a+1}{b+1}\right)$ Hence, evaluate $\int_0^1 \frac{x^7 - x^3}{\log x} dx$

2. $\int_0^\infty \frac{e^{-x}}{x}\left(e^{-ax} - e^{-bx}\right) dx = log\left(\frac{1+b}{1+a}\right)$

3. $\int_0^\infty e^{-xy} dx = \frac{1}{y}, y > 0$ and hence, deduce that $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = log\left(\frac{b}{a}\right), a > 0, b > 0$

4. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin mx \, dx = tan^{-1}\left(\frac{m}{a}\right) - tan^{-1}\left(\frac{m}{b}\right)$

5. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin mx \, dx = tan^{-1}\left(\frac{b}{m}\right) - tan^{-1}\left(\frac{a}{m}\right)$

6. $\int_0^\infty \frac{e^{-\beta x} \sin \alpha x}{x} dx = \tan^{-1}\frac{\alpha}{\beta}$ Deduce that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

7. $\int_0^\infty \frac{cos\lambda x}{x}\left(e^{-ax} - e^{-bx}\right) dx = \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right)$. $(a > 0, b > 0)$

8. $\int_0^\infty \frac{e^{-x} - e^{-\alpha x}}{x \sec x} dx = \frac{1}{2} log\left(\frac{\alpha^2+1}{2}\right)$

9. $\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x \sec x} dx = \frac{1}{2} log\left(\frac{\beta^2+1}{\alpha^2+1}\right)$

10. $\int_0^\infty \frac{log(1+ax^2)}{x^2} dx = \pi\sqrt{a}, (a > 0)$. Also, deduce that $\int_0^\infty \frac{\log(1+x^2)}{x^2} dx = \pi$

11. $\int_0^\pi \frac{\log(1+\sin\alpha\cos x)}{\cos x} dx = \pi\alpha$

12. $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x} dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$

13. $\int_0^\pi \log(1 - a\cos x) dx = \pi log\left[\frac{1+\sqrt{1-a^2}}{2}\right], |a| < 1$

14. $\int_0^\pi log(1 - cos\alpha \cos x) dx = \pi log\left[\frac{1+sin\alpha}{2}\right]$

15. $\int_0^\infty e^{-x^2} \cos 2ax \, dx = \frac{\sqrt{\pi}}{2} e^{-a^2}$, Given: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

16. $\int_0^\infty e^{-bx^2} \cos 2ax \, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{b}\right)} . e^{-a^2/b}, b > 0$ Given: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

17. $\int_0^\infty \frac{1-\cos mx}{x} e^{-x} dx = \frac{1}{2} log(m^2 + 1)$

18. $\int_0^\infty \frac{1-\cos ax}{x^2} dx = \frac{\pi a}{2}$. Given: $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

19. $\int_0^\infty e^{-(x^2+a^2/x^2)} dx = \frac{\sqrt{\pi}}{2} e^{-2a}, a > 0, Given$: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

20. $\int_0^\pi \frac{dx}{a - \cos x} = \frac{\pi}{\sqrt{a^2-1}}, (a > 0)$. Deduce that

(i) $\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{\pi a}{(a^2-1)^{3/2}}$ **(ii)** $\int_0^{\pi} \frac{dx}{(2-\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$

21. $\int_0^{\infty} \frac{tan^{-1}ax - tan^{-1}bx}{x} dx = \frac{\pi}{2} log\left(\frac{a}{b}\right)$

22. $\int_0^{\pi/2} log\left[\frac{a+b\sin x}{a-b\sin x}\right]\frac{dx}{\sin x} = \pi\, sin^{-1}\left(\frac{b}{a}\right)$

23. $\int_0^{\pi/2} log(a^2cos^2\theta + b^2sin^2\theta)d\theta = \pi log\left(\frac{a+b}{2}\right)$

24. Verify the rule of differentiation under the integral sign for $\int_0^{\infty} e^{-at}\sin bt\, dt$, Where a is a parameter.

25. Verify the rule of differentiation under the integral sign for $\int_0^{\infty} e^{-at}\cos bt\, dt$

26. Evaluate $\int_0^{\pi/2} \frac{dx}{1+acos^2x}$ and hence, deduce that $\int_0^{\pi/2} \frac{cos^2x}{(3+cos^2x)^2}dx = \frac{\pi\sqrt{3}}{96}$

27. Evaluate $\int_0^{\pi/2} \frac{dx}{1+asin^2x}$ and deduce that $\int_0^{\pi/2} \frac{sin^2x}{(3+sin^2x)^2}dx = \frac{\pi\sqrt{3}}{96}$

28. Evaluate $\int_0^{\pi} \frac{dx}{a+bcos\, x}, a > 0, b > 0$ and deduce that $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$

And $\int_0^{\pi} \frac{\cos x\, dx}{(a+b\, cosx)^2} = -\frac{\pi b}{(a^2-b^2)^{3/2}}$

29. Evaluate $\int_0^{\pi} \frac{dx}{a+b\cos x}, a > 0, |b| < a$. Hence, deduce that $\int_0^{\pi} \frac{dx}{(5+4\cos x)^3} = \frac{11\pi}{81}$

30. Assuming $\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{a^2-b^2}} cos^{-1}\left(\frac{b}{a}\right), a > b$. Show that $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{cosx}dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$

31. Prove that $\int_0^1 x^a(\log x)^n\, dx = \frac{(-1)^n n!}{(a+1)^{n+1}}$ ⊛

32. By differentiating $\int_0^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ w.r.t a under the integral sign successively, prove that

$\int_0^{\infty} \frac{dx}{(x^2+a^2)^{n+1}} = \frac{(2n)!\pi}{2^{2n+1}.(n!)^2a^{2n+1}}$

ANSWERS

1. $log\ 2$

26. $\frac{\pi}{2\sqrt{1+a}}$

27. $\frac{\pi}{2\sqrt{1+a}}$

28. $\frac{\pi}{\sqrt{a^2-b^2}}$

29. $\frac{\pi}{\sqrt{a^2-b^2}}$