Function of Square matrix

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Case I } If Given Square matrix A is diagonalisable

Then $A \approx D$ or, $D = M^{-1} A M$

Premultiply by M & post multiply by M^{-1}

$$\boxed{A = M D M^{-1}} \quad \text{--- (1)}$$

for function of ~~matrix~~ A

$$\boxed{f(A) = M f(D) M^{-1}}$$

Suppose $A^n = (MDM^{-1})(MDM^{-1}) \ldots (MDM^{-1}) \ldots n$ times

$$\boxed{A^n} = M \boxed{D^n} M^{-1}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$f(A)$ ✓ $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$

✓ $\cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \ldots$

✓ $\tan A$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \ldots$$

$$\boxed{f(A) = M f(D) M^{-1}}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$e^A = M e^D M^{-1}$$

$$e^D = I + D + \frac{1}{2!} D^2 + \frac{1}{3!} D^3 + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 1^3 & 0 \\ 0 & 2^3 \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 2^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 1 & \\ & 2^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 1 + \frac{1}{2!}1^2 + \frac{1}{3!}1^3 & 0 \\ 0 & 1 + (2) + \frac{1}{2!}(2)^2 + \frac{1}{3!}(2)^3 \end{bmatrix}$$

$$\boxed{e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\cos D = \begin{bmatrix} \cos 1 & 0 \\ 0 & \cos 2 \end{bmatrix}$$

$$D \begin{bmatrix} 5 & & 0 \\ & -2 & \\ 0 & & 3 \end{bmatrix}$$

Then $\boxed{\cos D = \begin{bmatrix} \cos 5 & 0 & 0 \\ 0 & \cos(-2) & 0 \\ 0 & 0 & \cos 3 \end{bmatrix}}$

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1) If $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ Then Find A^{20} & e^A

Sol^n: (Method 1) Consider $Ax = \lambda x$, $[A - \lambda I]x = 0$

$$|A - \lambda I| = \begin{vmatrix} 3/2 - \lambda & 1/2 \\ 1/2 & 3/2 - \lambda \end{vmatrix} = 0$$

ch eq^n is $\boxed{\lambda^2 - 3\lambda + 2 = 0}$

eigenvalues are $\lambda = 1, 2$

1) for $\lambda = 1$, $[A - I]x = 0$

$$\begin{bmatrix} 1/2 & 1/2 \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$$R_2 - R_1 \quad \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x + y = 0 \Rightarrow y = -x$

$\therefore x = t,\ y = -t,\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} \cong \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is eigenvech for $\lambda = 1$

$AM = GM = 1$

2) for $\lambda = 2$, $[A - 2I]x = 0$

$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow R_2 + R_1 \quad \begin{bmatrix} -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -x + y = 0 \Rightarrow y = x$

If $x = t,\ y = t,\ \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} \cong \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = 2$

$AM = GM = 1$

$\therefore$ AM = GM for all eigenvalues $\therefore$ Matrix A is diagonalisable.

$\therefore \boxed{A = MDM^{-1}}$ where, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

also find $M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Also for $\boxed{f(A) = M\,f(D)\,M^{-1}}$

Also for $\boxed{f(A) = M\, f(D)\, M^{-1}}$

$\boxed{A}$ $\quad \underline{A^{20}} = M\, D^{20}\, M^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1^{20} & 0 \\ 0 & 2^{20} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1^{20} & -1^{20} \\ 2^{20} & 2^{20} \end{bmatrix}$$

$$\boxed{A^{20} = \frac{1}{2} \begin{bmatrix} 1+2^{20} & -1+2^{20} \\ -1+2^{20} & 1+2^{20} \end{bmatrix}}$$

$\boxed{B}$ $\quad \underline{e^{A} = M\, e^{D}\, M^{-1}}$

$$\boxed{e^{A} = \frac{1}{2} \begin{bmatrix} e^{1}+e^{2} & -e^{1}+e^{2} \\ -e^{1}+e^{2} & e^{1}+e^{2} \end{bmatrix}}$$

Method 4 Since above method can be used only in case of diagonalisable.

Another method based upon C-H Th^{m} : $f(A) = 0$

Ch eq^{n} $\underset{=0}{\cancel{\lambda}}$

any Matrix of order 3 $\quad$ deg 3 ch polynomial $\equiv 0$

$\cos A \to F(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ ✓

eigenvalues will satisfy this eqⁿ

$\to F(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$

$F'(\lambda) = \alpha_2(2\lambda) + \alpha_1$ $\qquad 1, -1, -1$

repeated roots:

$F(1) = \alpha_2(1) + \alpha_1(1) + \alpha_0$ — ①

$F(-1) = \alpha_2(1) - \alpha_1 + \alpha_0$ — ②

$F'(-1) = \alpha_2(2(-1)) + \alpha_1$

$= -2\alpha_2 + \alpha_1$ — ③

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1) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ Find A^{50}

Solⁿ: Consider $Ax = \lambda x$, $[A - \lambda I]x = 0$

Then $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda+1)(\lambda-1) = 0$ is eqⁿ

$\Rightarrow \lambda = 1, 1, -1$

Here $\lambda = 1$ is repeated.

for order 3 matrix A, consider

$f(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ — ①

$\alpha_2 A^2 + \alpha_1 A + \alpha_0 I$

We Assume that, eigenvalues satisfies eq^n (1)

$\therefore \lambda^{50} = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ — (2)

putting $\lambda = 1$ & $\lambda = -1$ in (2)

$$1^{50} = \alpha_2 1^2 + \alpha_1 1 + \alpha_0$$

$$1 = \alpha_2 + \alpha_1 + \alpha_0 \quad \text{— (3)}$$

$$(-1)^{50} = \alpha_2(-1)^2 + \alpha_1(-1) + \alpha_0$$

$$1 = \alpha_2 - \alpha_1 + \alpha_0 \quad \text{— (4)}$$

Now diff (2) w.r.to λ, $50\lambda^{49} = 2\alpha_2\lambda + \alpha_1$

Putting $\lambda = 1$, Since only repeated root satisfied by derivative,

$$50(1)^{49} = 2\alpha_2(1) + \alpha_1$$

$$50 = 2\alpha_2 + \alpha_1 \quad \text{— (5)}$$

Solve simultaneously, (3), (4), (5)

$\therefore \alpha_2 = 25, \ \alpha_1 = 0, \ \alpha_0 = -24$

$\Rightarrow$ put in (1)

$$A^{50} = 25A^2 + 0 + (-24)I$$

$$A^{50} = 25A^2 - 24I$$

$$= 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}}$$

$\begin{bmatrix} 2 & & 0 \\ & 2 & \\ 0 & & 2 \end{bmatrix}$

$f(\lambda)$

$f(2)$

$2, 2, 2$

$f'(2) =$

$f''(2) =$

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2) $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ Show that $3 \boxed{\tan A} = A \tan 3$

Sol^n: $Ax = \lambda x$

$[A - \lambda I]x = 0$

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 9 = 0$$

$$\boxed{\lambda = \pm 3}$$

Here A is of order 2

$f(A) = \alpha_1 A + \alpha_0 I$

$\tan A = \alpha_1 A + \alpha_0 I$ — (1)

$\tan \lambda = \alpha_1 \lambda + \alpha_0$ — (2)

Put $\lambda = \pm 3$ $\tan 3 = \alpha_1 (+3) + \alpha_0$

$= 3\alpha_1 + \alpha_0$ — (3)

$\tan(-3) = -3\alpha_1 + \alpha_0$

$$-\tan 3 = -3\alpha_1 + \alpha_0 \quad \text{—} (4)$$

$$(3) + (4) \Rightarrow 2\alpha_0 = 0 \Rightarrow \boxed{\alpha_0 = 0}$$

put $\alpha_0 = 0$ in (3) $\quad (\tan 3) = 3\alpha_1 \Rightarrow \boxed{\alpha_1 = \frac{1}{3}(\tan 3)}$

put α_1 & α_0 in (1)

$$\tan A = \alpha_1 A + \alpha_0$$

$$\tan A = \frac{1}{3}(\tan 3)\, A + 0$$

$$\boxed{3 \tan A = A (\tan 3)}$$

H.W. $\begin{bmatrix} \pi/2 & \pi \\ 0 & 3\pi/2 \end{bmatrix}$ Then Find $\underline{\sin A}$

Ans. $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$
