

APPLICATIONS OF DE MOIVRE'S THEOREM:

ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.

General values of $\cos \theta = \cos(2k\pi + \theta)$ and $\sin \theta = \sin(2k\pi + \theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos \theta + i \sin \theta$, we apply De Moivre's theorem

$$z = (\cos \theta + i \sin \theta)^{\frac{1}{n}} = \cos \frac{\theta}{n} + i \sin \frac{\theta}{n}$$

This shows that $\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos \theta + i \sin \theta$.

The other roots are obtained by expressing the number in the general form

$$z = \{\cos(2k\pi + \theta) + i \sin(2k\pi + \theta)\}^{\frac{1}{n}} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i \sin\left(\frac{2k\pi + \theta}{n}\right)$$

Taking $k = 0, 1, 2, \dots, (n-1)$. We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1 - \omega)^6 = -27$

Solution: Consider $x^3 = 1 \quad \therefore x = 1^{1/3}$

$$\therefore x = (\cos 0 + i \sin 0)^{1/3} = (\cos 2k\pi + i \sin 2k\pi)^{1/3} = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$$

Putting, $k = 0, 1, 2$, the cube roots of unity are

$$x_0 = 1, x_1 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \omega \text{ (say)}$$

$$\text{And } x_2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right]^2 = \omega^2$$

$$\begin{aligned} \text{Now, } 1 + \omega + \omega^2 &= 1 + \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) + \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) \\ &= 1 + \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2}\right) = 1 - 1 = 0 \end{aligned}$$

$$\therefore 1 + \omega^2 = -\omega \text{ and } \omega^3 = 1$$

$$\begin{aligned} \text{Now, } (1 - \omega)^6 &= [(1 - \omega)^2]^3 = (1 - 2\omega + \omega^2)^3 \\ &= (-\omega - 2\omega)^3 = (-3\omega)^3 = -27\omega^3 = -27 \end{aligned}$$

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

Solution: $\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)^{1/3}$$

$$= \left[\cos \left(2k\pi + \frac{\pi}{4}\right) + i \sin \left(2k\pi + \frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos \left((8k+1)\frac{\pi}{4}\right) + i \sin \left((8k+1)\frac{\pi}{4}\right)\right]^{1/3}$$

$$\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \cos \left((8k+1)\frac{\pi}{12}\right) + i \sin \left((8k+1)\frac{\pi}{12}\right)$$

$$\text{Similarly, } \sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = \cos \left((8k+1)\frac{\pi}{12}\right) - i \sin \left((8k+1)\frac{\pi}{12}\right)$$

$$\therefore \sqrt[3]{\frac{(1+i)}{\sqrt{2}}} + \sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = 2 \cos \left((8k+1)\frac{\pi}{12}\right)$$

$$\text{Putting } k = 0, 1, 2 \text{ we get the three roots as } 2 \cos \frac{\pi}{12}, 2 \cos \frac{9\pi}{12}, 2 \cos \frac{17\pi}{12}$$

$$\text{i.e., } 2 \cos \frac{r\pi}{12} \text{ where } r = 1, 9, 17$$

3. Find the cube roots of $(1 - \cos\theta - i \sin\theta)$.

Solution:
$$\begin{aligned} (1 - \cos\theta - i \sin\theta)^{1/3} &= \left[2 \sin^2 \left(\frac{\theta}{2}\right) - i 2 \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)\right]^{1/3} \\ &= \left[2 \sin \left(\frac{\theta}{2}\right) \left(\sin \left(\frac{\theta}{2}\right) - i \cos \left(\frac{\theta}{2}\right)\right)\right]^{1/3} \\ &= \left(2 \sin \left(\frac{\theta}{2}\right)\right)^{1/3} \left[\cos \left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i \sin \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]^{1/3} \\ &= \left(2 \sin \left(\frac{\theta}{2}\right)\right)^{1/3} \left[\cos \left(2k\pi - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) + i \sin \left(2k\pi - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)\right]^{1/3} \end{aligned}$$

$$= \left(2 \sin\left(\frac{\theta}{2}\right)\right)^{1/3} \left[\cos\left(\frac{(4k-1)\pi+\theta}{6}\right) + i \sin\left(\frac{(4k-1)\pi+\theta}{6}\right)\right]$$

Putting $k = 0, 1, 2$ we get the three roots

4. Find the continued product of all the value of $(-i)^{2/3}$

Solution: $(-i)^{2/3} = (0 + i(-1))^{2/3} = \left[\left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right)^2\right]^{1/3}$

$$= [\cos(2k\pi + \pi) - i \sin(2k\pi + \pi)]^{1/3}$$

$$= \cos\left((2k + 1)\frac{\pi}{3}\right) - i \sin\left((2k + 1)\frac{\pi}{3}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$$\left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right), (\cos\pi - i \sin\pi), \left(\cos\frac{5\pi}{3} - i \sin\frac{5\pi}{3}\right)$$

\therefore Continued product

$$= \left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right) (\cos\pi - i \sin\pi) \left(\cos\frac{5\pi}{3} - i \sin\frac{5\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{3\pi}{3} + \frac{5\pi}{3}\right) - i \sin\left(\frac{\pi}{3} + \frac{3\pi}{3} + \frac{5\pi}{3}\right)$$

$$= \cos 3\pi + i \sin 3\pi$$

$$= -1 - i(0)$$

$$= -1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

Solution: $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left\{\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)^3\right\}^{1/4}$

$$= (\cos\pi + i \sin\pi)^{1/4}$$

$$= [\cos(2k + 1)\pi + i \sin(2k + 1)\pi]^{1/4}$$

$$= \cos(2k + 1)\frac{\pi}{4} + i \sin(2k + 1)\frac{\pi}{4}$$

Putting $k = 0, 1, 2, 3$ we get the four roots as,

$$\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right), \left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right), \left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right), \left(\cos\frac{7\pi}{4} + i \sin\frac{7\pi}{4}\right)$$

$$\therefore \left(\cos \frac{r\pi}{4} + i \sin \frac{r\pi}{4} \right) \text{ where } r = 1, 3, 5, 7$$

$$\begin{aligned} \text{The required product} &= \cos \left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4} \right) + i \sin \left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4} \right) \\ &= \cos 4\pi + i \sin 4\pi = 1. \end{aligned}$$

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

Solution: $x^7 + x^4 + x^3 + 1 = 0$

$$\therefore x^4(x^3 + 1) + (x^3 + 1) = 0$$

$$\therefore (x^3 + 1)(x^4 + 1) = 0$$

$$\therefore x^3 = -1, x^4 = -1$$

Consider $x^3 = -1$

$$\begin{aligned} \therefore x &= (-1 + i0)^{1/3} = (\cos \pi + i \sin \pi)^{1/3} = [\cos(2k + 1)\pi - i \sin(2k + 1)\pi]^{1/3} \\ &= \cos(2k + 1)\frac{\pi}{3} + i \sin(2k + 1)\frac{\pi}{3} \end{aligned}$$

Putting $k = 0, 1, 2$ we get the three roots

Similarly from $x^4 = -1$ we get the remaining four roots as

$$x = \cos(2k + 1)\frac{\pi}{4} + i \sin(2k + 1)\frac{\pi}{4} \quad \text{where } k = 0, 1, 2, 3$$

7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

Solution: $x^4 + x^3 + x^2 + x + 1 = 0$

Multiplying the given equation by $x - 1$, we get $(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0$

$$\therefore \text{We have } x^5 - 1 = 0 \quad \therefore x^5 = 1 = \cos 0 + i \sin 0$$

$$\therefore x^5 = \cos(2k\pi) + i \sin(2k\pi)$$

$$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get the roots of the equation.

$$x_0 = \cos 0 + i \sin 0 = 1,$$

$$x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}, \quad x_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5},$$

$$x_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, \quad x_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$$

It is clear that 1 is the root of $x^5 - 1 = 0$

and the remaining roots are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$

$$\text{i.e. } \cos \frac{2\pi}{5} \pm i \sin \frac{2\pi}{5}, \quad \cos \frac{4\pi}{5} \pm i \sin \frac{4\pi}{5}$$

8. SOLVE: $x^4 - x^2 + 1 = 0$

Solution: $x^4 - x^2 + 1 = 0$

Multiplying the given equation by $(x^2 + 1)$, we get, $(x^2 + 1)(x^4 - x^2 + 1) = 0$

$$\therefore (x^2)^3 + (1)^3 = 0 \quad \therefore x^6 + 1 = 0 \quad \therefore x^6 = -1$$

$$\begin{aligned} \therefore x &= (-1 + 0i)^{1/6} = (\cos \pi + i \sin \pi)^{1/6} \\ &= [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]^{1/6} \\ &= \cos(2k + 1)\frac{\pi}{6} + i \sin(2k + 1)\frac{\pi}{6} \end{aligned}$$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots of equation

$$x_0 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \quad x_1 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i(1) = i$$

$$x_2 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \quad x_3 = \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}$$

$$x_4 = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = 0 + i(-1) = -i \quad x_5 = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}$$

It is clear that i and $-i$ are the roots of $x^2 + 1 = 0$ and the remaining roots

x_0, x_2, x_3, x_5 are roots of $x^4 - x^2 + 1 = 0$

9. Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$.

Solution: Consider $x^4 + 1 = 0 \quad \therefore x^4 = -1$

$$x = (-1 + i0)^{1/4} = (\cos \pi + i \sin \pi)^{1/4} = [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]^{1/4}$$

$$x = \cos \left((2k + 1) \frac{\pi}{4} \right) + i \sin \left((2k + 1) \frac{\pi}{4} \right)$$

Putting $k = 0, 1, 2, 3$ we get the three roots as

$$x_0 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = 1 \quad x_1 = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$$

$$x_2 = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \quad x_3 = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = -\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

Now consider, $x^6 - i = 0 \quad \therefore x^6 = i$

$$x = (0 + 1i)^{1/6} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^{1/6} = \left[\cos \left(2k\pi + \frac{\pi}{2}\right) + i \sin \left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$$

$$= \cos \left((4k + 1) \frac{\pi}{12}\right) + i \sin \left((4k + 1) \frac{\pi}{12}\right)$$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots as

$$x_0 = \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \quad x_1 = \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}$$

$$x_2 = \cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12} = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$$

$$x_3 = \cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12}$$

$$x_4 = \cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12}$$

$$x_5 = \cos \frac{21\pi}{12} + i \sin \frac{21\pi}{12} = -\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$$

$$\therefore \text{common roots are } \pm \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$$

10. If $(1 + x)^6 + x^6 = 0$

show that $x = -\frac{1}{2} - \frac{i}{2} \cot \frac{\theta}{2}$ where $\theta = (2n + 1)\pi/6, n = 0, 1, 2, 3, 4, 5$.

Solution: $(1 + x)^6 + x^6 = 0 \quad \therefore \frac{(1+x)^6}{x^6} = -1$

$$\frac{1+x}{x} = (-1)^{1/6} = (\cos \pi + i \sin \pi)^{1/6} = [\cos(2k + 1)\pi + i \sin(2k + 1)\pi]^{1/6}$$

$$= \cos \left((2k + 1) \frac{\pi}{6}\right) + i \sin \left((2k + 1) \frac{\pi}{6}\right)$$

$$\frac{x+1-x}{x} = \cos \theta + i \sin \theta - 1$$

$$\frac{1}{x} = (\cos \theta - 1) + i \sin \theta$$

$$x = \frac{1}{(\cos \theta - 1) + i \sin \theta} \times \frac{(\cos \theta - 1) - i \sin \theta}{(\cos \theta - 1) - i \sin \theta} = \frac{(\cos \theta - 1) - i \sin \theta}{(\cos \theta - 1)^2 + \sin^2 \theta} = \frac{(\cos \theta - 1) - i \sin \theta}{2(1 - \cos \theta)}$$

$$= \frac{-2 \sin^2(\theta/2) - i 2 \sin(\theta/2) \cos(\theta/2)}{2(2 \sin^2(\theta/2))}$$

$$= -\frac{1}{2} - \frac{i}{2} \cot \left(\frac{\theta}{2}\right) \quad \text{where } \theta = (2k + 1) \frac{\pi}{6}$$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ is $1 + i$, find all other roots.

Solution: The given equation is $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

Since one of the root is $1 + i$

\therefore other root must be $1 - i$ (since roots always occurs as complex conjugate pairs)

$\therefore x = 1 \pm i$ are the two roots

$\therefore x - 1 = \pm i$

$\therefore (x - 1)^2 = (\pm i)^2$

$\therefore x^2 - 2x + 1 = -1$

$\therefore x^2 - 2x + 2 = 0$

Now we want to find other two remaining roots for that we divide

$x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 4x + 2$ and we obtain

$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = (x^2 - 4x + 2)(x^2 - 4x + 5)$

\therefore the remaining two roots are the roots of equation $x^2 - 4x + 5 = 0$

$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

\therefore The required remaining roots of given equation are $1 - i, 2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that

$(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5$.

Solution: We have $x^5 = 1 = \cos 0 + i \sin 0 \quad \therefore x^5 = \cos(2k\pi) + i \sin(2k\pi)$

$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as

$x_0 = \cos 0 + i \sin 0 = 1, \quad x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5},$

$x_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}, \quad x_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, \quad x_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5},$

Putting $x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \alpha$, we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$

\therefore the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$, and hence

$x^5 - 1 = (x - 1)(x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$

$$\therefore (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4) = \frac{x^5 - 1}{x - 1}$$

$$\therefore (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4) = x^4 + x^3 + x^2 + x + 1$$

$$\text{Putting } x = 1, \text{ we get } (1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5$$

13. Solve the equation $z^4 = i(z - 1)^4$ and show that the real part of all the roots is $1/2$.

Solution: We have $z^4 = i(z - 1)^4$

$$\therefore \left(\frac{z}{z-1}\right)^4 = i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = \cos \left(2n\pi + \frac{\pi}{2}\right) + i \sin \left(2n\pi + \frac{\pi}{2}\right)$$

$$\therefore \frac{z}{z-1} = \left[\cos \left(2n\pi + \frac{\pi}{2}\right) + i \sin \left(2n\pi + \frac{\pi}{2}\right) \right]^{1/4}$$

$$= \cos(4n + 1) \frac{\pi}{8} + i \sin(4n + 1) \frac{\pi}{8}$$

$$\therefore \frac{z}{z-1} = \cos \theta + i \sin \theta \quad \text{where } \theta = (4n + 1) \frac{\pi}{8}$$

$$\therefore \frac{z}{z-1-z} = \frac{z}{-1} = \frac{\cos \theta + i \sin \theta}{1 - \cos \theta - i \sin \theta} \quad \text{Simplifying as in the above example, we get}$$

$$\therefore \frac{z}{-1} = \frac{-\sin(\theta/2) + i \cos(\theta/2)}{2 \sin(\theta/2)}$$

$$\therefore -z = -\frac{1}{2} + \frac{i}{2} \cot \frac{\theta}{2}$$

$$\therefore z = \frac{1}{2} - \frac{i}{2} \cot \frac{\theta}{2}, \quad \text{where } \theta = (4n + 1) \frac{\pi}{8}$$

For, $n = 0, 1, 2$, we get three roots, All these roots have the real part $1/2$

14. If ω is a 7th root of unity, prove that

$$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$$

if n is a multiple of 7 and is equal to zero otherwise.

Solution: We have $x = 1^{\frac{1}{7}} = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{7}}$

$$= \cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7}, \quad \text{where } n = 0, 1, 2, 3, 4, 5, 6$$

$$\text{Let } \omega = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$$

$$\therefore \omega^7 = \left(\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} \right)^7 = \cos 2\pi + i \sin 2\pi = 1 \therefore \omega^{7n} = 1^n = 1$$

If n is not a multiple of 7, $\therefore \omega^n \neq 1$

$$\begin{aligned} \text{Now, } S &= 1 + \omega^n + \omega^{2n} + \omega^{3n} + \dots + \omega^{6n} = \frac{1 - \omega^{7n}}{1 - \omega^n} \quad \text{sum of 7 terms of G.P} \\ &= \frac{1 - 1}{1 - \omega^n} = \frac{0}{1 - \omega^n} = 0 \end{aligned}$$

If n is a multiple of 7, say $n = 7k$

$$\begin{aligned} \text{Then, } S &= 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} \\ &= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + (\omega^7)^{4k} + (\omega^7)^{5k} + (\omega^7)^{6k} \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7 \end{aligned}$$

15. Prove that $\sqrt{1 + \sec(\theta/2)} = (1 + e^{i\theta})^{-1/2} + (1 + e^{-i\theta})^{-1/2}$

Solution: We have to show that $\sqrt{1 + \sec(\theta/2)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

$$\text{Squaring both sides, we get, } 1 + \sec \frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$$

We shall prove this result

$$\begin{aligned} \text{Now, r. h. s} &= \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}} \\ &= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{-i\theta}+e^{i\theta}+1}} \\ &= 1 + \frac{2}{\sqrt{2+(e^{i\theta}+e^{-i\theta})}} = 1 + \frac{2}{\sqrt{2+2\cos\theta}} \\ &= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}} = 1 + \frac{2}{\sqrt{4\cos^2(\theta/2)}} \\ &= 1 + \frac{2}{2\cos(\theta/2)} = 1 + \sec \frac{\theta}{2} = l. h. s \end{aligned}$$

SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1 + \omega + \omega^2 = 0$

(ii) $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$

2. Prove that the n n th roots of unity are in geometric progression.
3. Show that the sum of the n n th roots of unity is zero.
4. Prove that the product of n n th roots of unity is $(-1)^{n-1}$
5. Find all the values of the following :
 - (i) $(-1)^{1/5}$
 - (ii) $(-i)^{1/3}$
 - (ix) $(1 - i\sqrt{3})^{1/4}$
6. Find the continued product of all the values of $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{3/4}$
7. Find all the value of $(1 + i)^{2/3}$ and find the continued product of these values.
8. Solve the equations
 - (i) $x^9 + 8x^6 + x^3 + 8 = 0$
 - (ii) $x^4 - x^3 + x^2 - x + 1 = 0$
 - (iii) $(x + 1)^8 + x^8 = 0$
9. If $(x + 1)^6 = x^6$, show that $x = -\frac{1}{2} - i \cot \frac{\theta}{2}$ where $\theta = \frac{2k\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$.
10. Show that the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm i \cot \frac{r\pi}{7}$, $r = 1, 2, 3$.
11. If $\alpha, \alpha^2, \alpha^3, \dots, \alpha^6$ are the roots of $x^7 - 1 = 0$, find them and prove that $(1 - \alpha)(1 - \alpha^2) \dots (1 - \alpha^6) = 7$.
12. Prove that $x^5 - 1 = (x - 1) \left(x^2 + 2x \cos \frac{\pi}{5} + 1\right) \left(x^2 + 2x \cos \frac{3\pi}{5} + 1\right) = 0$.
13. Solve the equation $z^n = (z + 1)^n$ and show that the real part of all the roots is $-1/2$.
14. If $a = e^{i 2\pi/7}$ and $b = a + a^2 + a^4$, $c = a^3 + a^5 + a^6$. then prove that b & c are roots of quadratic equation $x^2 + x + 2 = 0$.
15. Prove that
 - (i) $\sqrt{1 - \cos \theta} = (1 - e^{i\theta})^{-1/2} - (1 - e^{-i\theta})^{-1/2}$
 - (iv) $\sqrt{1 + \cos \theta} = (1 + e^{i\theta})^{-1/2} - (1 + e^{-i\theta})^{-1/2}$
16. If $1 + 2i$ is a root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.