5) Finding PI when RHS is not any of above (General type)

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$$\left[\frac{1}{F(D)}\right] f(x) = \left[\frac{1}{(x-a)(x-b)\cancel{(x+di)(x+di)}}\right] f(x)$$

$$(x-(c+di))(x-(c-di))$$

$$\left(\frac{A}{x-a}\right) f(x) + \left(\frac{B}{x-b}\right) f(x)$$

Two methods 1) Applying linear factors on f(x) One by one

2) If complex factors, then Separate factors Using partial fraction and then operate on f(x)

Two formulae:
$$\left[\frac{1}{D}\right] f(x) = \int f(x)\, dx$$

$$\left[\frac{1}{D+a}\right] f(x) = e^{-ax} \int e^{ax} f(x)\, dx$$

e^{ax}, $e^{ax}\sin bx$

Composite : $\sin(e^{ax})$ ×

$e^{e^{x}}$ ×

Problems:

1) $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Sol^n: $y_c = c_1 e^{x} + c_2 e^{-x}$

then $y_p = \left[\frac{1}{D^2+1}\right]\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]$

$$= \left[\frac{1}{D-1}\right]\left\{\left[\frac{1}{D+1}\right]\left(e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right)\right\}$$

$$= \left[\frac{1}{D-1}\right]\left\{e^{-x}\int e^{x}\left[\underbrace{e^{-x}\sin(e^{-x})}_{f'(x)} + \underbrace{\cos(e^{-x})}_{f(x)}\right] dx\right\}$$

$$= \left[\frac{1}{D-1}\right]\left\{\cancel{e^{-x}}\,\cancel{e^{x}}\,\cos(e^{-x})\right\}$$

$\left[\text{Using } \int e^{x}[f(x)+f'(x)]\,dx = e^{x} f(x)\right]$

$= (D-1)\,l$ $\quad$ [Using $\int e^x[f(x)+f'(x)]\,dx = e^x f(x)$]

$= e^x \int e^{-x}\cos(e^{-x})\,dx$ $\quad$ [$e^{-x}=t$, $e^{-x}dx = -dt$]

$= e^x \int \cos t\,(-dt)$

$y_p = -e^x \sin(e^{-x})$

& $y_g = y_c + y_p =$

$\int e^{2x}[\quad]\,dx = e^{2x} f(x)$

$\int e^x[f(x)+f'(x)]\,dx = \int dt = t = e^x f(x)$

put $t = e^x f(x)$

$\frac{dt}{dx} = e^x f'(x) + e^x f(x)$

$dt = e^x[f'(x)+f(x)]\,dx$

$$\boxed{\int e^{ax}[f'(x)+a f(x)]\,dx = e^{ax} f(x)}$$

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2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Sol^n: $(D^2+3D+2)y = e^{e^x}$, $\quad y_c = c_1 e^{-2x} + c_2 e^{-x}$

$y_p = \left(\frac{1}{D^2+3D+2}\right)e^{e^x} = \left[\frac{1}{D+2}\right]\left[\frac{1}{D+1}\right]e^{e^x}$

$= \left[\frac{1}{D+2}\right] e^{-x}\int e^x e^{e^x}\,dx$ $\quad$ [$e^x = t$, $e^x dx = dt$]

$= \left[\frac{1}{D+2}\right] e^{-x}\int e^t\,dt$

$= \left[\frac{1}{D+2}\right](e^{-x} e^{e^x}) = e^{-2x}\int e^{2x}[e^{-x}e^{e^x}]\,dx$

$= e^{-2x}\int e^x e^{e^x}\,dx = e^{-2x} e^{e^x}\,dx$

$y_p = e^{-2x}$

$y_g = y_c + y_p =$

$y_g = y_c + y_p =$

HW
3) $(D^2 - 3D + 2)y = e^{x-e} + \cos(e^{-x})$

4) $(D^2 - 1)y = \dfrac{2}{1+e^x}$

5) $(D^2 + a^2)y = \operatorname{cosec} ax$

Sol^n: $D = \pm ai$, $y_c = c_1\cos ax + c_2 \sin ax$

$$y_p = \left[\frac{1}{D^2+a^2}\right]\operatorname{cosec} ax = \left[\frac{1}{D+ai}\right]\left[\frac{1}{D-ai}\right](\operatorname{cosec} ax)$$

~~Using~~

$$\frac{1}{(a+b)(a-b)} = \frac{1}{2b}\left[\frac{1}{a-b} - \frac{1}{a+b}\right]$$
$$= \frac{1}{2a}\left[\frac{1}{a-b} + \frac{1}{a+b}\right]$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right](\operatorname{cosec} ax)$$

$$= \frac{1}{2ai}\left\{\left(\frac{1}{D-ai}\right)\operatorname{cosec} ax - \frac{1}{(D+ai)}\operatorname{cosec} ax\right\}$$

$$= \frac{1}{2ai}\left\{e^{aix}\int e^{-iax}\operatorname{cosec} ax\,dx - e^{-iax}\int e^{iax}\operatorname{cosec} ax\,dx\right\}$$

$$= \frac{1}{2ai}\left\{e^{iax}\int(\cos ax - i\sin ax)\operatorname{cosec} ax\,dx - e^{-iax}\int(\cos ax + i\sin ax)\operatorname{cosec} ax\,dx\right\}$$

$$= \frac{1}{2ai}\left\{e^{iax}\int(\cot ax - i)\,dx - e^{-iax}\int(\cot ax + i)\,dx\right\}$$

$$= \frac{1}{2ai}\left\{e^{iax}\left(\frac{\log(\sin ax)}{a} - ix\right) - e^{-iax}\left(\frac{\log(\sin ax)}{a} + ix\right)\right\}$$

$$= \frac{1}{2ai}\left\{(\cos ax + i\sin ax)\left(\frac{\log(\sin ax)}{a} - ix\right) - (\cos ax - i\sin ax)\left(\frac{\log(\sin ax)}{a} + ix\right)\right\}$$

$$= \frac{1}{\cancel{2}a\cancel{i}}\left\{-\cancel{2i}x\cos ax + \cancel{2i}\,\frac{\sin ax\log(\sin ax)}{a}\right\}$$

$$y_p = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax\log(\sin ax)$$

Method of Variation of Parameter

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$(c_1 + c_2 x)e^{2x}$

$c_1 e^{2x} + c_2 x e^{2x}$

1) D.E. 2^{nd} order :

$y_c = C.F. = c_1 y_1 + c_2 y_2$

Then $y_p = u y_1 + v y_2$ (where u & v are parameters (function of x) to be determined. They vary.)

Consider Wronskian $(W) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\begin{matrix} + & - \\ - & + \end{matrix}$

Then Solving, $u = -\int \frac{y_2 f(x)}{W} dx$ | $v = \int \frac{y_1 f(x)}{W} dx$

Substitute back.

1) Apply method of variation of parameter to solve

$\frac{d^2y}{dx^2} + a^2 y = \sec ax$

Sol^n: $(D^2 + a^2) y = \underbrace{\sec ax}_{\to f(x)}$

$D = \pm ai$, $y_c = c_1 \cos ax + c_2 \sin ax = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos ax$, $y_2 = \sin ax$

$y_1' = -a \sin ax$, $y_2' = a \cos ax$

Then Assume, $y_p = u y_1 + v y_2$

& wronskian $(W) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a\cos^2 ax + a \sin^2 ax$

$= a$

Then $u = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{\sin ax (\sec ax)}{a} dx$

$u = -\frac{1}{a}\int \tan ax \, dx = -\frac{\log(\sec ax)}{a^2} = \frac{1}{a^2}\log(\cos ax)$

$v = \int \frac{y_1 f(x)}{W} = \int \frac{\cancel{\cos ax} \, \cancel{\sec ax}}{a} dx = \frac{x}{a}$

Substitute back,

$y_p = u y_1 + v y_2 = \frac{1}{a^2}\cos ax \log(\cos ax) + \frac{x}{a}\sin ax$

$\&\ y_g = y_c + y_p$

D.E. of 3^{rd} order :

$y_c = C.F. = C_1 y_1 + C_2 y_2 + C_3 y_3$

Assume $y_p = p y_1 + q y_2 + r y_3$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$p = \int \frac{+\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} f(x)}{W}\, dx \qquad q = \int \frac{-\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} f(x)}{W}\, dx$$

$$r = \int \frac{+\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} f(x)}{W}\, dx \qquad \underline{y_p = p y_1 + q y_2 + r y_3}$$

2) $(D^3 + D)\, y = \operatorname{cosec} x$

Sol^n: $D(D^2+1) = 0 \quad , \quad D = 0, \pm i$

$\therefore y_c = c_1 e^{0x} + c_2 \cos x + c_3 \sin x$ ✓

$y_1 = 1 \quad , \; y_2 = \cos x \quad , \; y_3 = \sin x$

$y_1' = 0 \quad , \; y_2' = -\sin x \quad , \; y_3' = \cos x$

$y_1'' = 0 \quad , \; y_2'' = -\cos x \quad , \; y_3'' = -\sin x$

Assume, $y_p = p y_1 + q y_2 + r y_3$ —— (1)

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \sin^2 x + \cos^2 x = \underline{1}$$

$$\&\ p = \int \frac{+\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} f(x)}{W}\, dx = \int \frac{(\cos^2 x + \sin^2 x) \operatorname{cosec} x\, dx}{1}$$

$$p = \log(\operatorname{cosec} x - \cot x)$$

$$\int \frac{-\begin{vmatrix} 1 & \sin x \\ \; & \; \end{vmatrix} \operatorname{cosec} x}{} dx = \int -\cos x \operatorname{cosec} x\, dx$$

$$p = \log(\mathrm{cosec} \ldots$$

$$q = \int - \frac{\begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix} \mathrm{cosec}\, x}{1}\, dx = \int -\cos x \,\mathrm{cosec}\, x \, dx$$

$$= -\int \cot x \, dx$$

$$= -\log(\sin x)$$

$$r = \int \frac{\begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} \mathrm{cosec}\, x}{1}\, dx = \int -\cancel{\sin x}\, \cancel{\mathrm{cosec}\, x}\, dx$$

$$= -x$$

$$y_p = p y_1 + q y_2 + r y_3$$

$$y_p = 1(\log(\mathrm{cosec}\, x - \cot x)) - \cos x \log(\sin x) - x \sin x$$

$$\& \quad y_g = y_c + y_p$$

H.W.

(3) $(D^3 - 6D^2 + 12D - 8)y = e^{2x}/x$

(4) $(D^2 - 3D + 2)y = \dfrac{e^x}{1+e^x}$