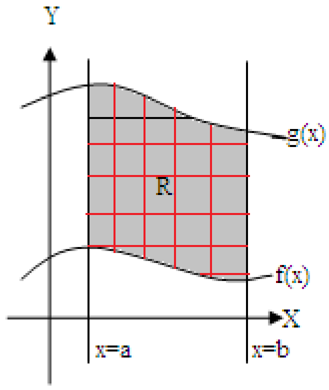
Mass of a lamina (plate)



Suppose that the lamina (*two dimensional planar closed surface with mass and density*) is the region R bounded by two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$. Suppose that $\rho(x, y)$ is the density at each point (x, y) of the lamina. Now, we can partition R into sub-rectangles with m of them in the x direction and n in the y-direction.Suppose that each sub-rectangle has width Δx and height Δy. Then a sub-rectangle containing the point (a, b) has approximate mass

$$\rho(a, b)\,\Delta x\,\Delta y$$

and the mass of R is approximately

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\rho(x_i, y_j)\,\Delta x_i\,\Delta y_j$$

where (x_i, y_j) is a point in the i, j-th subrectangle. Letting m and n go to infinity, we have

$$M = \text{Mass of } R = \rho\,A = \iint_R f(x, y)\mathrm{d}A$$

where A is the area of the region R.

Calculation of Mass

- **In Cartesian coordinates:** If A is the area of plane lamina and $\rho = f(x, y)$ be the density at each point $P(x, y)$. Then mass of plane lamina M is given by

$$M = \rho\,A = \iint_R f(x, y)\mathrm{d}x\,\mathrm{d}y$$

- **In polar coordinates:** If A is the area of plane lamina and $\rho = f(r, \theta)$ be the density at each point $P(r, \theta)$. Then mass of plane lamina M is given by

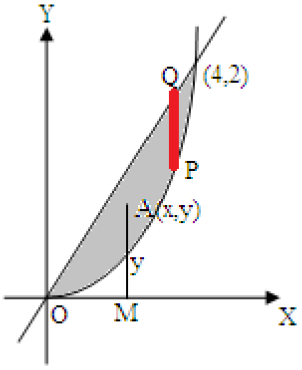
$$M = \rho\,A = \iint_R f(r, \theta)r\mathrm{d}r\,\mathrm{d}\theta$$

Example 1. Find mass distributed over the area bounded by $16y^2 = x^3$ and $2y = x$. If the density at any point of a lamina varies as the distance from x-axis.

Solution: First we shall find point of intersection of $16y^2 = x^3$ and $2y = x$. Putting $y = \dfrac{x}{2}$ in $16y^2 = x^3$, we get

$$16\frac{x^2}{4} = x^3 \Rightarrow x^3 - 4x^2 = 0 \Rightarrow x^2(x-4) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

When $x=0$, we get $y=0$ and for $x=4$ we get $y=2$. Thus $16y^2=x^3$ and $2y=x$ intersects at $(0,0)$ and $4,2)$. Now consider the region bounded by $16y^2=x^3$ and $2y=x$ as shown in the following figure.



Consider an integrating strip PQ parallel to y-axis as shown in above figure. The point P lies on $16y^2=x^3$ i.e. $y=\dfrac{x^{3/2}}{4}$ and Q lies on the curve $2y=x$ i.e. $y=\dfrac{x}{2}$. Therefore, y varies from $\dfrac{x^{3/2}}{4}$ to $\dfrac{x}{2}$ and x varies from 0 to 4.

Given that, density at any of a lamina varies with distance from x-axis. Let $A(x,y)$ be any point in region, then, we have

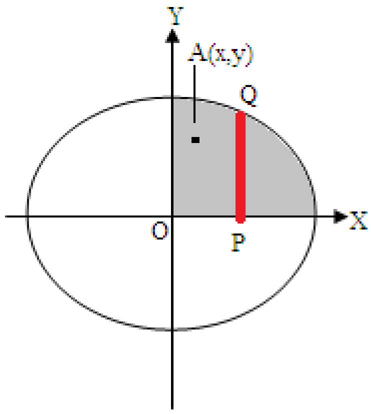
$$\rho \propto d(A,M) \Rightarrow \rho \propto y \Rightarrow \rho = ky, \text{where } k \text{ is proportional constant}$$

Therefore,

$$\begin{aligned}\text{Mass} &= \iint_R \rho y \mathrm{d}x = \int_0^4 \int_{x^{3/2}/4}^{x/2} kyy\mathrm{d}x = k\int_0^4 \left[\frac{y^2}{2}\right]_{x^{3/2}/4}^{x/2} \mathrm{d}x \\ &= \frac{k}{2}\int_0^4 \left[\frac{x^2}{4}-\frac{x^3}{16}\right]\mathrm{d}x = \frac{k}{2}\left[\frac{x^3}{12}-\frac{x^4}{64}\right]_0^4 = \frac{k}{2}\left[\frac{64}{12}-4\right] \\ &= \frac{2k}{3}\end{aligned}$$

Example 2. Find mass of a lamina in the form of a quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, if the density at any point varies as the product of the coordinates of the point.

Solution: Consider the integrating strip PQ parallel to y-axis in the first quadrant as shown in the following figure.



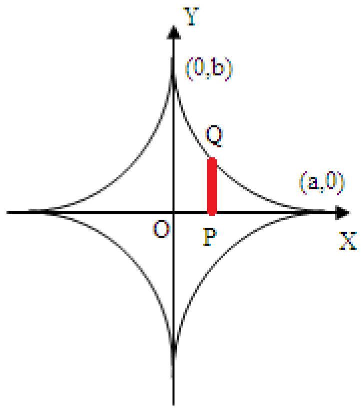
The point P lies on x-axis i.e. $y = 0$ and Q lies on ellipse i.e. $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Therefore y varies from 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$ and x varies from 0 to a.
Let $A(x, y)$ be any point of lamina. Given that density at any point varies as the product of the coordinates of the point. Therefore,

$$\rho \propto xy \Rightarrow \rho \propto kxy, \text{ where } k \text{ is proportional constant}$$

The mass of a quadrant is given by

$$\begin{aligned} M &= \iint_R \rho \mathrm{d}y\mathrm{d}x = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kxy\mathrm{d}y\,\mathrm{d}x = k\int_0^a x\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} \\ &= \frac{k}{2}\int_0^a x\left[\frac{b^2}{a^2}(a^2 - x^2)\right]\mathrm{d}x = \frac{kb^2}{2a^2}\int_0^a (a^2x - x^3)\mathrm{d}x \\ &= \frac{kb^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{a^4}{4}\right]_0^a = \frac{ka^2b^2}{8} \end{aligned}$$

Example 3. Find mass of a plate in the shape of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, the density being given by $\rho = \mu\, xy$
Solution: The given curve is symmetric about x and y axis. Therefore, the required mass M is four times the mass in the first quadrant.
Consider the integrating strip PQ in the first quadrant as shown in the following figure.



The point P lies on x-axis i.e. $y = 0$ and Q lies on $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ i.e. $y = b\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^{3/2} = y_1$ (say). Therefore, in first quadrant y varies from 0 to y_1 and x varies from 0 to b. Therefore,

$$\begin{aligned} M &= 4\int_0^a \int_0^{y_1} \rho\,\mathrm{d}y\,\mathrm{d}x = 4\int_0^a \int_0^{y_1} \mu\, xy\,\mathrm{d}y\,\mathrm{d}x = 4\mu\int_0^a x\left[\frac{y^2}{2}\right]_0^{y_1}\mathrm{d}x \\ &= 2\mu\int_0^a xb^2\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^3\mathrm{d}x \end{aligned}$$

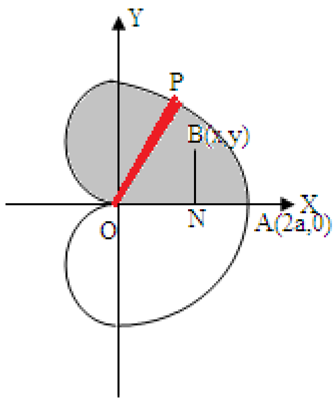
Put $x = a\sin^3\theta \therefore \mathrm{d}x = 3a\sin^2\theta\cos\theta\,\mathrm{d}\theta$ and when $x = 0$ we get $\theta = 0$ and for $x = a$ we get $\theta = \pi/2$. Therefore, above equation reduces to

$$M = 2\mu b^2\int_0^a a\sin^3\theta(1 - \sin^2\theta)^3 3a\sin^2\theta\,\mathrm{d}\theta = 6\mu\, a^2b^2\int_0^{\pi/2}\sin^5\theta\,\cos^7\theta\,\mathrm{d}\theta$$

$$= 6\mu a^2 b^2 \frac{1}{2}\beta(3,4) = 3\mu\, a^2 b^2 \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}$$
$$= 3\mu\, a^2 b^2 \frac{2\times 6}{6\times 5\times 4\times 3\times 2\times 1}$$
$$= \frac{\mu\, a^2 b^2}{20}$$

Example 4. Find the mass of a lamina in the form of cardioide $r = a(1+\cos\theta)$ whose density at any point varies as the square of its distance from initial line.

Solution: The curve $r = a(1+\cos\theta)$ is symmetric about initial line. Therefore required mass be twice the mass above the initial line. Now consider the region above the initial line as shown in the following figure.



Take integrating strip OP starting from origin as shown in above figure. At origin, $r = 0$ and the point P lies $r = a(1+\cos\theta)$. Therefore r varies from 0 to $a(1+\cos\theta)$ and θ for the region above the initial line θ varies from 0 to π.

Let $A(x, y)$ be any point in cardioide $r = a(1+\cos\theta)$. Given that density varies as the square of the distance of the point from the initial line. Here, y i.e. $r\sin\theta$ (relation between Cartesian and polar) is the distance. Therefore,

$$\rho \propto d(BN)^2 \Rightarrow \rho \propto r^2\sin^2\theta \Rightarrow \rho = kr^2\sin^2\theta, \text{ (where } k \text{ is proporionality constant)}$$

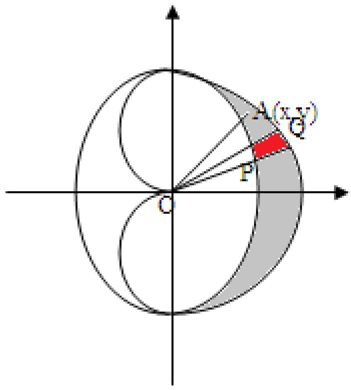
Therefore, required mass M is given by

$$M = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} \rho\, r\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin^2\theta\, r\mathrm{d}r\,\mathrm{d}\theta$$
$$= 2k\int_0^{\pi} \sin^2\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} \mathrm{d}r\,\mathrm{d}\theta = \frac{k}{2}\int_0^{\pi} \sin^2\theta[a^4(1+\cos\theta)^4]\mathrm{d}\theta$$
$$= \frac{a^4 k}{2}\int_0^{\pi} 4\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\left[2\cos^2\left(\frac{\theta}{2}\right)\right]^4 \mathrm{d}\theta$$
$$= 32a^4 k\int_0^{\pi} \sin^2\left(\frac{\theta}{2}\right)\cos^{10}\left(\frac{\theta}{2}\right)\mathrm{d}\theta$$

Put $\dfrac{\theta}{2} = t$. $\therefore$ $\mathrm{d}\theta = 2\mathrm{d}t$ and we get $t = 0$ for $\theta = 0$ and $t = \dfrac{\pi}{2}$ for $\theta = \pi$. Therefore, we get

$$\begin{aligned}
M &= 32a^4k \int_0^{\pi/2} \sin^2 t \, \cos^{10} t \, 2\mathrm{d}t = 64a^4k \int_0^{\pi/2} \sin^2 t \, \cos^{10} t \, \mathrm{d}t \\
&= 64a^4k \frac{1}{2} \beta\left(\frac{3}{2}, \frac{11}{2}\right) = 32a^4k \times \frac{\Gamma(3/2)\Gamma(11/2)}{7} \\
&= 32a^4k \times \frac{1/2 \times \pi \times 9/2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \pi}{6!} \\
&= \frac{21a^4k\pi}{32}
\end{aligned}$$

Example 5. Find the mass of area inside the cardioide $r = a(1 + \cos\theta)$ and outside $r = a$ if the density at any point is inversely proportional to distance from the origin.
Solution: Solving $r = a(1 + \cos\theta)$ and $r = a$, we get $a = a(1 + \cos\theta)$. This implies $\cos\theta = 0$. Therefore $\theta = \dfrac{\pi}{2}$. Thus cardioide and circle $r = a$ intersects at $(a, \pi/2)$ and $(a, -\pi/2)$. Now consider the region bounded by these curves as shown in following figure.

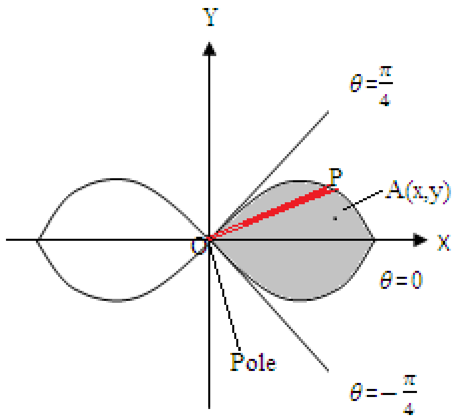


Consider integrating strip OQ starting from origin as shown in above figure. At origin, we have $r = 0$. The point Q lies on $r = a(1 + \cos\theta)$. Therefore r varies from 0 to $a(1 + \cos\theta)$. For the region above the initial line, θ varies from 0 to $\pi/2$. As the shaded area is symmetrical about initial line, required mass be twice the mass above the initial line. Let $A(x, y)$ be any point in the region. Given that, $\rho \propto \dfrac{1}{d(OA)}$. Therefore, $\rho = \dfrac{k}{r}$, where k is proportionality constant. Then required mass is

$$\begin{aligned}
M &= 2 \iint\limits_R \rho \, r\mathrm{d}r \, \mathrm{d}\theta = 2 \int_0^{\pi/2} \int_0^{a(1+\cos\theta)} \frac{k}{r} r\mathrm{d}r \, \mathrm{d}\theta \\
&= 2k \int_0^{\pi/2} [r]_0^{a(1+\cos\theta)} \, \mathrm{d}\theta = 2k \int_0^{\pi/2} [a(1 + \cos\theta) - a]\mathrm{d}\theta \\
&= 2ak \int_0^{\pi/2} \cos\theta \mathrm{d}\theta \\
&= 2ak[\sin\theta]_0^{\pi/2} \\
&= 2ak
\end{aligned}$$

Example 6. Find the mass of a plate in the form of one loop of lemniscate $r^2 = a^2\cos 2\theta$, if the density at any point varies as the square of its distance from the pole.

Solution: The curve $r^2 = a^2\cos 2\theta$ is symmetric about initial line i.e. x-axis. Therefore mass of one loop be twice the mass above the initial line. Now consider the integrating strip OP starting from origin as shown in the following figure.



At origin, we have, $r = 0$. The point P lies on $r^2 = a^2\cos 2\theta$ i.e. $r = a\sqrt{\cos 2\theta}$. For region above the initial line θ varies from 0 to $\dfrac{\pi}{4}$.

Let $A(x, y)$ be any point in the region above the initial line. Given that density at $A(x, y)$ varies with the square of distance of A from origin. Therefore,

$$\begin{aligned}\rho \propto [d(OA)]^2 &\Rightarrow \rho = k[d(OA)]^2\\ &\Rightarrow \rho = k(x^2+y^2) = kr^2, \text{ where } k \text{ is proprtional constant}\end{aligned}$$

The required mass M is given by,

$$\begin{aligned}M &= 2\iint\limits_R \rho\, r\mathrm{d}r\mathrm{d}\theta = 2\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} kr^2\, r\mathrm{d}r\mathrm{d}\theta\\
&= 2k\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{\cos 2\theta}}\mathrm{d}\theta = \frac{a^4k}{2}\int_0^{\pi/4}\cos^2 2\theta\mathrm{d}\theta\\
&= \frac{a^4k}{2}\int_0^{\pi/4}\frac{1+\cos 4\theta}{2}\theta\mathrm{d}\theta = \frac{a^4k}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4}\\
&= \frac{a^4k}{4}\left[\frac{\pi}{4}+0-0\right]\\
&= \frac{a^4k\pi}{16}\end{aligned}$$