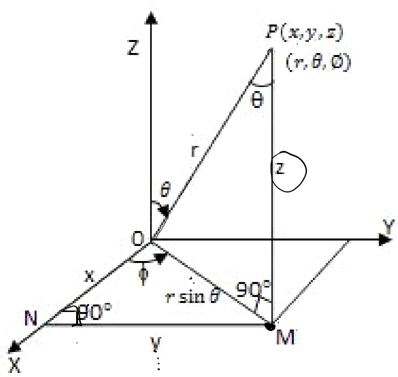
Change of Variable in Triple Integration

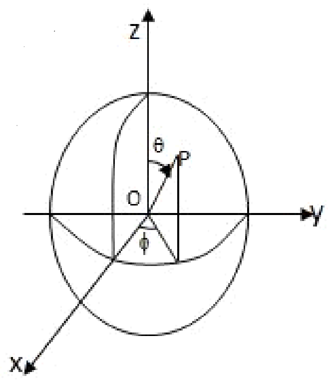
Thursday, May 20, 2021 11:40 AM

## CHANGE OF VARIABLES

Quite often, the evaluation of a double or triple integral is greatly simplified by a suitable change of variables.

### (I) TO CHANGE CARTESIAN COORDINATES $(x, y, z)$ TO SPHERICAL POLAR COORDINATES $(r, \theta, \emptyset)$




$x \to r\cos\theta$

$y \to r\sin\theta$

$dx\,dy$

$= r\,dr\,d\theta$

It is clear from the ΔOMP, OM = r sin θ and PM = r cos θ

∴ from ΔONM x = ON = r sin θ cos ϕ, y = MN = r sin θ sin ϕ and z = PM = r cos θ.

$$x^2+y^2+z^2 = r^2 sin^2\theta cos^2\emptyset + r^2 sin^2\theta sin^2\emptyset + r^2 cos^2\theta$$

$$= r^2\left[sin^2\theta\left(cos^2\emptyset + sin^2\emptyset\right) + cos^2\theta\right] = r^2$$

Hence $x^2+y^2+z^2 = r^2$

$$\therefore I = \iiint f(x,y,z)dx\,dy\,dz = \iiint f(r\sin\theta\cos\emptyset, r\sin\theta\, sin\emptyset, r\cos\theta)|J|dr d\theta\, d\emptyset$$

$|J| =$

$\frac{\partial(x,y)}{\partial(r,\theta)}$

$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

Where J $= \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)}$ is called the **Jacobian of transformation** from (x, y, z) to (r, θ , ϕ)

$$\text{Here}\quad J = \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix} = \begin{vmatrix} \sin\theta cos\emptyset & rcos\theta cos\emptyset & -r\sin\theta\, sin\emptyset \\ sin\theta\, sin\emptyset & rcos\theta sin\emptyset & r\sin\theta cos\emptyset \\ cos\theta & -rsin\theta & 0 \end{vmatrix} = r^2 sin\theta$$

$$I = \iiint f(rsin\,\theta\cos\emptyset, r\sin\theta\sin\emptyset, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\emptyset$$

i.e the element $dx\,dy\,dz$ will change to $r^2 sin\theta\, dr\, d\theta\, d\emptyset$

**FOR ELLIPSOID** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Substitutions: $x = ar\, sin\theta\ cos\emptyset$ , $y = br\, sin\theta\ sin\emptyset$ , $z = cr\, cos\theta$

$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = r^2 \qquad r^2 = 1 \Longrightarrow r = 1$

Also $dx\,dy\,dz = abc\, r^2\sin\theta\, d\theta\, dr\, d\emptyset$

$$\therefore I = \iiint f(r,\theta,\emptyset)abc\, r^2\sin\theta\, d\theta\, d\emptyset\, dr.$$

### (II) TO CHANGE CARTESIAN COORDINATES $(x, y, z)$ TO CYLINDRICAL COORDINATES $(r, \theta, z)$.

From the figure $x = MN = r\cos\theta$ $y = ON = r\sin\theta$ , $z = z$

**(II) TO CHANGE CARTESIAN COORDINATES $(x, y, z)$ TO CYLINDRICAL COORDINATES $(r, \theta, z)$.**

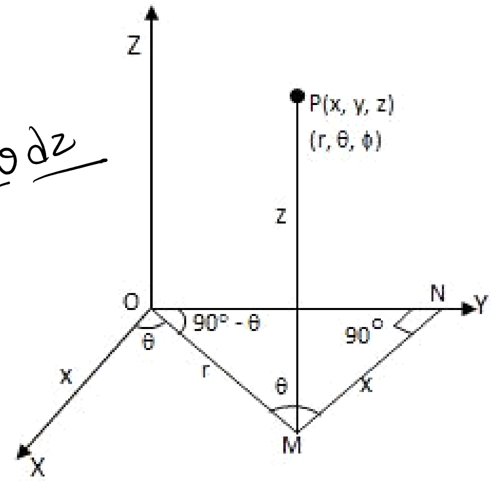
From the figure $x = MN = r\cos\theta$ $y = ON = r\sin\theta$ , $z = z$ .

And the Jacobian $J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

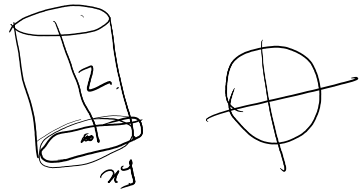
$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(cos^2\theta + sin^2\theta) = r$$

$dx\,dy\,dz$
$= r\,dr\,d\theta\,dz$




Hence $\iiint f(x, y, z)dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$.




**STANDARD LIMITS:**

| | | | | |
|---|---|---|---|---|
| **(i)** | For completes sphere $x^2 + y^2 + z^2 = a^2$ | $\theta \to 0\ to\ \pi$ | $\emptyset \to 0\ to\ 2\pi$ | $r \to 0\ to\ a$ |
| **(ii)** | For hemisphere $x^2 + y^2 + z^2 = a^2$ | $\theta \to 0\ to\ \frac{\pi}{2}$ | $\emptyset \to 0\ to\ 2\pi$ | $r \to 0\ to\ a$ |
| **(iii)** | For Positive octant of a sphere $x^2 + y^2 + z^2 = a^2$ | $\theta \to 0\ to\ \frac{\pi}{2}$ | $\emptyset \to 0\ to\ \frac{\pi}{2}$ | $r \to 0\ to\ a$ |
| **(iv)** | For ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ | $\theta \to 0\ to\ \pi$ | $\emptyset \to 0\ to\ 2\pi$ | $r \to 0\ to\ 1$ |
| **(v)** | for cylinder $x^2 + y^2 = a^2$ | $\theta \to 0\ to\ 2\pi$ | $z = z$ | $r \to 0\ to\ a$ |

## TYPE III : WHEN THE REGION OF INTEGRATION IS NOT BOUNDED BY PLANES, BUT BY SPHERE, ELLIPSOID ETC.

Evaluate the following integrals.

**1.** $\iiint_v \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$ where $V$ is the volume in the first octant.

Soln :- The first octant can be looked upon as $(\frac{1}{8})^{th}$ part of the sphere with infinite radius.

we first transform the integral to spherical coordinates by putting

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

Since we are in the first octant

$\theta \to 0$ to $\frac{\pi}{2}$, $\phi \to 0$ to $\frac{\pi}{2}$, $r \to 0$ to $\infty$

also $x^2+y^2+z^2 = r^2$

$$\therefore I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \frac{1}{(1+r^2)^2}\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^{\infty}\frac{r^2}{(1+r^2)^2}\,dr\right)$$

Ans :- $\frac{\pi^2}{8}$.

**2.** $\iiint(x^2+y^2+z^2)\,dx\,dy\,dz$ over the first octant of the sphere $x^2 + y^2 + z^2 = a^2$

Soln :- we put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$x^2+y^2+z^2 = r^2$

$r \to 0$ to $a$, $\theta = 0$ to $\frac{\pi}{2}$, $\phi = 0$ to $\frac{\pi}{2}$.

$$I = \int_0^{\pi/2}\int^{\pi/2}\int^{a} (r^2)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$I = \int_0 \int_0 \int_0 (r^2)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{\pi/2} \sin\theta \, d\theta\right)\left(\int_0^{a} r^4 \, dr\right) = \pi \frac{a^5}{10}$$

**3.** $\iiint xyz\, dx\, dy\, dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2 (H.W.)$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{a} (r \sin\theta \cos\phi)(r \sin\theta \sin\phi)(r\cos\theta)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{a^6}{48}$$

**4.** $\iiint xyz\,(x^2 + y^2 + z^2) dx\, dy\, dz$ over the first octant of the sphere $x^2 + y^2 + z^2 = a^2 (H.W.)$

Ans: $\dfrac{a^8}{64}$

**5.** $\iiint \frac{dx\, dy\, dz}{x^2+y^2+z^2}$ throughout the volume of the sphere $x^2 + y^2 + z^2 = a^2 . (H.W.)$

Ans: $4\pi a$

**6.** $\iiint \frac{z^2 dx\, dy\, dz}{x^2+y^2+z^2}$ over the volume of the sphere $x^2 + y^2 + z^2 = 2$

Soln: we first transform the integral to spherical coordinates by putting

$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$

$dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$

and $x^2+y^2+z^2 = r^2$

consider Integral in first octant

$r \to 0$ to $\sqrt{2}$, $\theta = 0$ to $\frac{\pi}{2}$, $\phi = 0$ to $\frac{\pi}{2}$

$\pi/2 \quad \pi/2 \quad \sqrt{2}$

$r \to 0$ to $\sqrt{2}$, $\theta = 0$ to $\frac{\pi}{2}$, $\phi = 0$ to $\frac{\pi}{2}$

Required Integral $= 8 \cdot \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{2}} \frac{r^2 \cos^2\theta}{r^2} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$

$$= 8 \cdot \left( \int_0^{\pi/2} d\phi \right) \left( \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta \right) \left( \int_0^{\sqrt{2}} r^2 \, dr \right)$$

$$= 8 \left( \frac{\pi}{2} \right) \frac{1}{2} B\left( \frac{3}{2}, 1 \right) \cdot \left( \frac{r^3}{3} \right)_0^{\sqrt{2}}$$

$$= 2\pi \cdot \frac{\Gamma\frac{3}{2}\,\Gamma 1}{\Gamma\frac{5}{2}} \cdot \frac{2\sqrt{2}}{3} = 2\pi \cdot \frac{\Gamma\frac{3}{2}}{\frac{3}{2}\Gamma\frac{3}{2}} \cdot \frac{2\sqrt{2}}{3} = \frac{8\sqrt{2}\pi}{9}$$

**7.** $\iiint_v \frac{dx\,dy\,dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ over the volume of sphere $x^2 + y^2 + z^2 = a^2$

$$I = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{a} \frac{r^2 \sin\theta \, dr\, d\theta\, d\phi}{\sqrt{a^2 - r^2}}$$

$$= 8 \left[ \int_0^{\pi/2} d\phi \right] \left[ \int_0^{\pi/2} \sin\theta \, d\theta \right] \left[ \int_0^{a} \frac{r^2}{\sqrt{a^2 - r^2}} dr \right]$$

To find the last integral,

we put $r = a \sin t$, $dr = a \cos t \, dt$

$$r=0 \;, t=0 \;;\; r=a \quad t=\frac{\pi}{2}$$

$$= 8\left(\frac{\pi}{2}\right)\left(-\cos\theta\right)_0^{\pi/2}\left[\int_0^{\pi/2}\frac{a^2\sin^2 t}{a\cos t}\cdot a\cos t\,dt\right]$$

$=$

Ans :- $a^2\pi^2$

**8.** $\iiint(x^2y^2+y^2z^2+z^2x^2)dx\,dy\,dz$ over the volume of the sphere $x^2 + y^2 + z^2 = a^2(H.W.)$

Ans :- $\frac{4a^7\pi}{35}$

**9.** Evaluate $\iiint e^{(x^2+y^2+z^2)^{3/2}}\,dV$ throughout the volume of the unit sphere

$r \to 0$ to $1$ , $\theta = 0$ to $\frac{\pi}{2}$ , $\phi = 0$ to $\frac{\pi}{2}$

$$I = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{(r^2)^{3/2}}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\left[\int_0^{\pi/2}d\phi\right]\left[\int_0^{\pi/2}\sin\theta\,d\theta\right]\left[\int_0^1 e^{r^3}r^2\,dr\right]$$

$$= 8\cdot\left(\frac{\pi}{2}\right)(1)\left[\frac{e^{r^3}}{3}\right]_0^1$$

$$= \frac{4\pi}{3}(e-1)$$

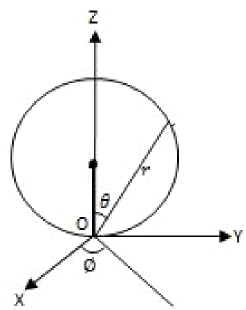
**10.** $\iiint \frac{z^2}{\;}\,dx\,dy\,dz$ where V is the volume bounded by the sphere $x^2 + y^2 + z^2 = z$

**10.** $\iiint_v \frac{z^2}{x^2+y^2+z^2}\, dx\, dy\, dz$ where V is the volume bounded by the sphere $x^2 + y^2 + z^2 = z$

$x^2+y^2+z^2 = z$ can be written as

$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

This is a sphere with centre at $(0,0,\frac{1}{2})$ and radius $\frac{1}{2}$.




we transform the integral to spherical coordinates by putting

$x = r\sin\theta\sin\phi,\ y = r\sin\theta\cos\phi,$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

eqn of the sphere becomes

$x^2+y^2+z^2 = z$

$r^2 = r\cos\theta \rightarrow r = \cos\theta$

Now, $r$ varies from $0$ to $\cos\theta$

$\theta$ varies from $0$ to $\frac{\pi}{2}$

$\phi$ varies from $0$ to $2\pi$

$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\theta} \frac{r^2\cos^2\theta}{r^2}\cdot r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int^{2\pi}\int^{\pi/2}\int^{\cos\theta}$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\theta} r^2 \cos^2\theta \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \cos^2\theta \sin\theta \left(\frac{r^3}{3}\right)_0^{\cos\theta} d\theta \, d\phi$$

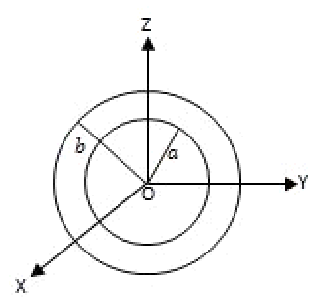
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \cos^5\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{3}\left[\int_0^{2\pi} d\phi\right]\left[\int_0^{\pi/2} \cos^5\theta \sin\theta \, d\theta\right]$$

$$I = \frac{\pi}{9}$$

**11.** $\iiint_v \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$ where V is the volume bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2, (b > a)$

$$I = \int_{\theta=0}^{\pi}\int_{\phi}^{2\pi}\int_a^b \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{(r^2)^{3/2}}$$




Ans :- $4\pi \log\left(\frac{b}{a}\right)$

**12.** $\iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Since the volume is that of a ellipsoid, we shall use elliptical coordinates

$x = ar\sin\theta\cos\phi$, $y = br\sin\theta\sin\phi$, $z = cr\cos\theta$

$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \Rightarrow r^2$

$\therefore I = \iiint \sqrt{1-r^2}\cdot abc\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$

Now $r \to 0$ to $1$, $\theta = 0$ to $\frac{\pi}{2}$, $\phi = 0$ to $\frac{\pi}{2}$

$I = 8\left[\int_0^{\pi/2} d\phi\right]\left[\int_0^{\pi/2}\sin\theta\,d\theta\right]\left[\int_0^1\sqrt{1-r^2}\cdot r^2\,dr\right]$

put $r = \sin t$

Ans :- $\frac{\pi^2}{4}abc$.

**13.** $\iiint\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}\,dx\,dy\,dz$ over the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 (H.W.)$ Ans :- $\pi abc$

## TYPE IV : WHEN THE REGION OF INTEGRATION IS BOUNDED BY A CONE OR A CYLINDER OR A PARABOLOID.

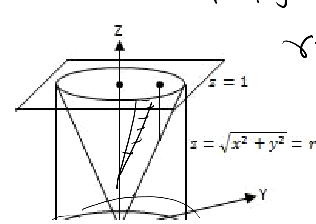
**1.** $\iiint\sqrt{x^2+y^2}\,dx\,dy\,dz$ over the volume bounded by the right circular cone $x^2 + y^2 = z^2, z > 0$ and the planes $z = 0$ and $z = 1$.

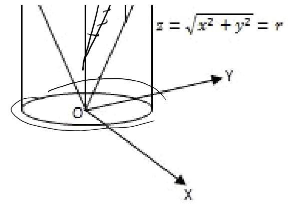
we transform the given integral to cylindrical polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$x^2+y^2 = 1$

$r^2 = 1$

$r = 1$






by putting $x = r\cos\theta$, $y = r\sin\theta$,

$z = z$

$dx\,dx\,dy = r\,dr\,d\theta\,dz$

$x^2 + y^2 = z^2 \qquad r^2 = z^2 \Rightarrow r = z$

z varies from r to 1

θ varies from 0 to 2π

r varies from 0 to 1

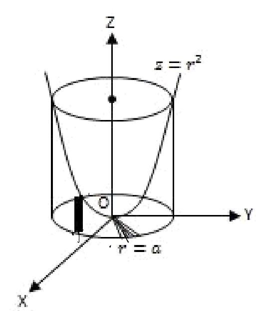
$$I = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=r}^{1} r \cdot r\,dr\,d\theta\,dz$$

Ans :- $\frac{\pi}{6}$.

**2.** $\iiint z^2\,dx\,dy\,dz$ over the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the paraboloid $x^2 + y^2 = z$ and the plane $z = 0$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$




Cylinder : $x^2 + y^2 = a^2 \longrightarrow r = a$.

paraboloid : $x^2 + y^2 = z \longrightarrow r^2 = z$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{r^2} z^2\,r\,dr\,d\theta\,dz$$

$= \pi a^8$

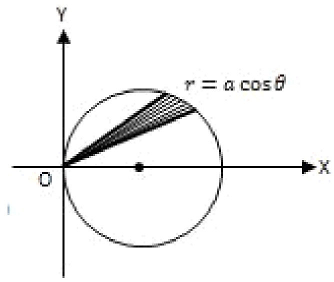
$\overline{12}$

**3.** $\iiint z^2\,dx\,dy\,dz$ over the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2+z^2=ax$.

we use cylindrical coordinates

The Sphere becomes

$$r^2+z^2=a^2$$




Cylinder becomes $r^2 = ar\cos\theta$

ie $r = a\cos\theta$

The volume of integration is bounded by sphere and cylinder

$z$ varies from from $-\sqrt{a^2-r^2}$ to $\sqrt{a^2-r^2}$

$r$ varies from $0$ to $a\cos\theta$

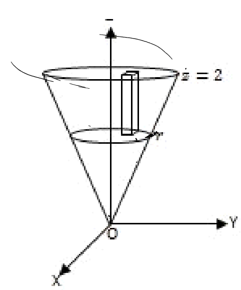
$\theta$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

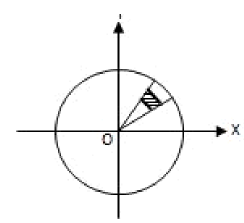
$$I = \int_{-\pi/2}^{\pi/2}\int_{0}^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} z^2\,dz\,r\,dr\,d\theta$$

Ans: $\dfrac{2a^5\pi}{15}$.

**4.** Evaluate $\iiint_V (x^2+y^2)\,dV$ where $V$ is the solid bounded by the surface $x^2+y^2=z^2$ and the planes $z=0, z=2$

Using cylindrical coordinates




$z$ varies from $r$ to $2$

$r$ varies from $0$ to $2$

and $\theta$ varies from $0$ to $2\pi$

and $\theta$ varies from 0 to $2\pi$

$$I = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 \cdot r \, dz \, dr \, d\theta$$

$$= \frac{16\pi}{5}.$$