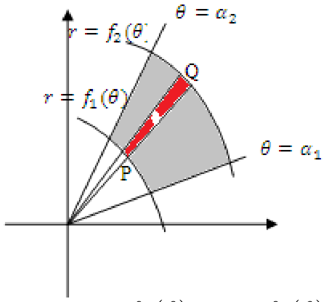
Area in Polar Coordinates

Let R be the region bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$. The area of region R is given by

$$A = \iint\limits_R r\mathrm{d}r\,\mathrm{d}\theta$$



Procedure to find area: To find area bounded by curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ we follow the steps given below.

step-a) Using given limits sketch the region of integration for area

step-b) Take integrating strip starting from origin.

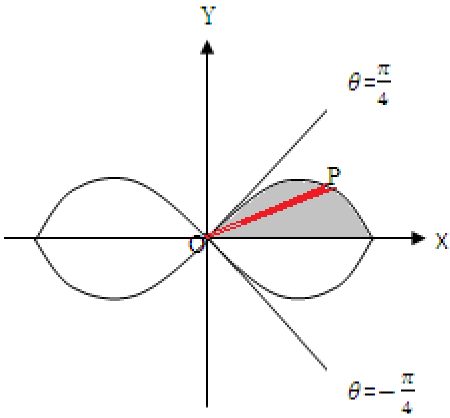
step-c) Find the integration limits for r and θ

step-d) Using the formula $A = \iint\limits_R r\mathrm{d}r\,\mathrm{d}theta = \int\limits_{\alpha}^{\beta}\int\limits_{f_1(\theta)}^{f_2(\theta)}$, find area of bounded region.

Example 1. Find the area of one loop of lemniscate $r^2 = a^2\cos 2\theta$.

Solution: The curve of $r^2 = a^2\cos 2\theta$ is symmetric about initial line i.e. x-axis. For the area of one loop of the curve first we will obtain area of loop above the x-axis and then area of one loop is equal to double of area of loop above the x-axis i.e. $A = 2\iint\limits_R r\mathrm{d}\theta\,\mathrm{d}\theta$.

Now consider integrating strip in the region above the x-axis starting from origin as shown in above figure. At origin, we have $r = 0$ and P lies on $r^2 = a^2\cos 2\theta$ i.e. $r = a\sqrt{\cos 2\theta}$.



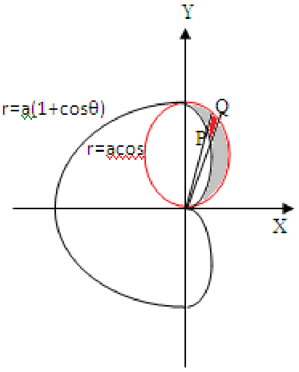
Therefore, r varies from 0 to $2\sqrt{\cos 2\theta}$ and θ varies from 0 to $\pi/4$. Therefore, the area of one loop is given by

$$A = 2\int\limits_0^{\pi/4}\int\limits_0^{a\sqrt{\cos 2\theta}} r\mathrm{d}r\,\mathrm{d}\theta = 2\int\limits_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}\mathrm{d}\theta = \int\limits_0^{\pi/4} a^2\cos 2\theta\,\mathrm{d}\theta = a^2\int\limits_0^{\pi/4}\cos 2\theta\,\mathrm{d}\theta$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{2}[1-0] = \frac{a^2}{2}$$

Example 2. Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioide $r = a(1-\cos\theta)$.

Solution: Given that $r = a\sin\theta$ is a circle. Using $y = r\sin\theta$, we get $r^2 = ay$ i.e. $x^2+y^2 = ay$. One can write it as $x^2+(y-a/2)^2 = a^2/4$. Thus $r = a\sin\theta$ is a circle with centre at $(o, a/2)$ and radius $a/2$. Consider the area lying inside the circle $r = a\sin\theta$ and outside the cardioide $r = a(1-\cos\theta)$ as shown in figure. Now, consider the integrating strip starting from origin as shown in figure. The point P lies on Cardioide $r = a(1-\cos\theta)$ and Q lies on circle $r = a\sin\theta$. Therefore r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\pi/2$



The area lying inside the circle $r = a\sin\theta$ and outside the cardioide $r = a(1-\cos\theta)$ is given by

$$A = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta}\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2}[a^2\sin^2\theta - a^2(1-\cos\theta)^2]\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\mathrm{d}\theta = \frac{a^2}{2}\int_0^{\pi/2}[2\cos\theta - 1 - \cos 2\theta]\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left[2\sin\theta - \theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{a^2}{2}\left[2 - \frac{\pi}{2} - 0\right]$$

$$= \frac{a^2}{4}[4-\pi]$$

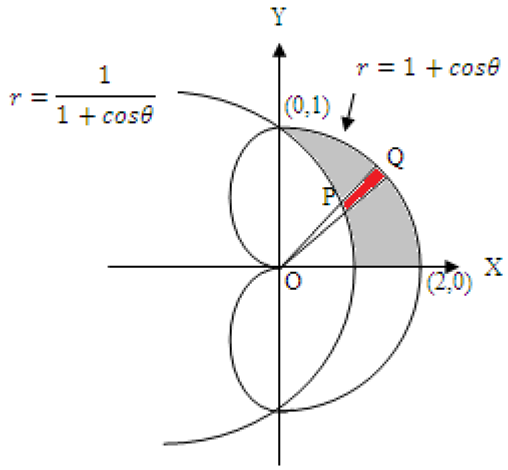
Example 3. Find the area inside the cardioide $r = 1+\cos\theta$ and outside the curve $r = \dfrac{1}{1+\cos\theta}$

Solution: First we shall convert the curve $r = \dfrac{1}{1+\cos\theta}$ into Cartesian coordinates. We have $x = r\cos\theta$, this gives $\cos\theta = \dfrac{x}{r}$. Putting in above equation, we get

$$r = \frac{1}{1+x/r} \Rightarrow r = \frac{r}{r+x} \Rightarrow r = 1-x \Rightarrow r^2 = (1-x)^2$$
$$\Rightarrow x^2+y^2 = 1-2x+x^2$$
$$\Rightarrow y^2 = -2\left(x-\frac{1}{2}\right)$$

Thus, $r = \dfrac{1}{1+\cos\theta}$ represents a parabola with vertex at $(1/2, 0)$ and symmetric about x-axis along -ve direction.

Now, consider the area inside the cardioide $r = 1+\cos\theta$ and outside the parabola $r = \dfrac{1}{1+\cos\theta}$ as shown in the following figure.



To find area of shaded region, consider a integrating strip starting from origin as shown in above figure. The point P lies on $r = \dfrac{1}{1+\cos\theta}$ and Q lies on $r = 1+\cos\theta$. Therefore r varies from $\dfrac{1}{1+\cos\theta}$ to $1+\cos\theta$ and θ varies from 0 to $\pi/2$. Since the area is symmeteric about x-axis. Therefore required area is equal to twice the area of shaded region. Therefore,

$$A = 2\int_0^{\pi/2}\int_{1/1+\cos\theta}^{1+\cos\theta} r\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} = \int_0^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]\mathrm{d}\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos^2\theta+2\cos\theta) - \frac{1}{4}\sec^4\left(\frac{\theta}{2}\right)\right]\mathrm{d}\theta = \int_0^{\pi/2}\left[(1+\frac{1+\cos 2\theta}{2}+2\cos\theta) - \frac{1}{4}\sec^2\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\theta}{2}\right)\right]\mathrm{d}\theta$$

$$= \int_0^{\pi/2}\left\{\frac{3}{2}+\frac{\cos 2\theta}{2}+2\cos\theta - \frac{1}{4}\sec^2\left(\frac{\theta}{2}\right)\left[1+\tan^2\left(\frac{\theta}{2}\right)\right]\right\}\mathrm{d}\theta$$

$$= \int_0^{\pi/2}\left\{\frac{3}{2}+\frac{\cos 2\theta}{2}+2\cos\theta - \frac{1}{4}\left[\sec^2\left(\frac{\theta}{2}\right)+\sec^2\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\theta}{2}\right)\right]\right\}\mathrm{d}\theta$$

$$= \left[\frac{3\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta\right]_0^{\pi/2} - \frac{1}{4}\left[2\tan\left(\frac{\theta}{2}\right)\right]_0^{\pi/2} - \frac{1}{4}\int_0^{\pi/2}\sec^2\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{3\pi}{4}+2-\frac{1}{2}-\frac{1}{4}\int_0^{\pi/2}\sec^2\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\theta}{2}\right)\mathrm{d}\theta$$

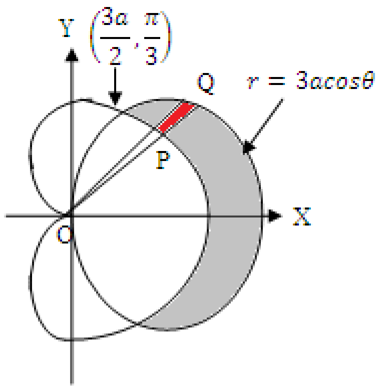
$$= \frac{3\pi}{4}+2-\frac{1}{2}-\frac{1}{4}\int_0^1 t^2 2\mathrm{d}t \quad (\text{put} \quad \tan(\theta/2)=t)$$

$$= \frac{3\pi}{4}+2-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{3}\right) = \frac{3\pi}{4}+\frac{4}{3}$$

Example 4. Find the area which is inside the circle $r = 2a\cos\theta$ and outside the cardioide $r = a(1+\cos\theta)$.

Solution: We have $x = r\cos\theta$. Then $r = 3a\cos\theta$ reduces to $r^2 = 3ax$ i.e. $x^2+y^2 = 3ax$. One can rewrite as $\left(x-\dfrac{3a}{2}\right)^2 + y^2 = \dfrac{9a^2}{4}$. Thus $r = 3a\cos\theta$ is a circle with center at $\left(\dfrac{3a}{2},0\right)$ and radius

$\frac{3a}{2}$. Solving $r = 3a\cos\theta$ and $r = a(1+\cos\theta)$ we get, $3a\cos\theta = a(1+\cos\theta)$. This gives $2\cos\theta = 1$. Thus, $\theta = \frac{\pi}{3}$. Therefore, the circle $r = 3a\cos\theta$ and the cardioide $r = a(1+\cos\theta)$ intersects at $\left(\frac{3a}{2}, \frac{\pi}{3}\right)$. Consider the area inside the circle $r = 2a\cos\theta$ and outside the cardioide $r = a(1+\cos\theta)$ as shown in the following figure.



Since both the curves are symmetric about x-axis, therefore we shall find area above the x-axis first abd then the required area will be double of this. Now, consider integrating strip starting from origin as shown in above figure. The point P lies on cardioide and Q lies on the circle. Therefore r varies from $a(1+\cos\theta)$ to $3a\cos\theta$ and θ varies from 0 to $\frac{\pi}{3}$. Therefore, the required area of shaded portion is

$$A = 2\int_0^{\pi/3}\int_{a(1+\cos\theta)}^{3a\cos\theta} r\mathrm{d}r\mathrm{d}\theta = 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_{a(1+\cos\theta)}^{3a\cos\theta}\mathrm{d}\theta$$

$$= \int_0^{\pi/3}\left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right]\mathrm{d}\theta = a^2\int_0^{\pi/3}[9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta]\mathrm{d}\theta$$

$$= a^2\int_0^{\pi/3}[8\cos^2\theta - 1 - 2\cos\theta]\mathrm{d}\theta = a^2\int_0^{\pi/3}[4 + 4\cos 2\theta - 1 - 2\cos\theta]\mathrm{d}\theta$$

$$= a^2\int_0^{\pi/3}[3 + 4\cos 2\theta - 2\cos\theta]\mathrm{d}\theta = a^2\left[3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right]_0^{\pi/3}$$

$$= a^2\left[\frac{3\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) - 2\sin\left(\frac{\pi}{3}\right) - 0\right]$$

$$= a^2\pi$$