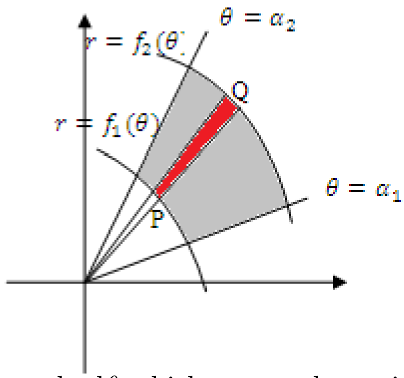
5 Double integrals in polar coordinated (r, θ)

Consider a function $f(r, \theta)$ of the polar coordinates r, θ which is to be integrated over a certain region R bounded by lines $\theta = \theta_1$, $\theta = \theta_2$ and curves $r = f_1(\theta)$, $r = f_2(\theta)$. That is to evaluate $\iint_R f(r, \theta) \mathrm{d}r \, \mathrm{d}\theta$, where R is the region bounded by $\theta = \theta_1$, $\theta = \theta_2$ and curves $r = f_1(\theta)$, $r = f_2(\theta)$.
Consider a region $ABCDA$ bounded by above lines and curves as shown in figure.



Now, consider a integrating strip *starting* from origin with a small angle $\mathrm{d}\theta$ which covers the region R. Here, $ABCDA$ is an integrating region therefore we take integrating strip PQ as shown in fig. The point P lies on $r = f_1(\theta)$ and Q lies on $r = f_2(\theta)$. Therefore r varies from $f_1(\theta)$ to $f_2(\theta)$. Now, if we rotate strip from $\theta = \theta_1$ to $\theta = \theta_2$, then it covers region $ABCDA$. Therefore θ varies from θ_1 to θ_2. Thus, we get

$$I = \iint_R f(r, \theta) \mathrm{d}r \, \mathrm{d}\theta = \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta) \mathrm{d}r \, \mathrm{d}\theta$$

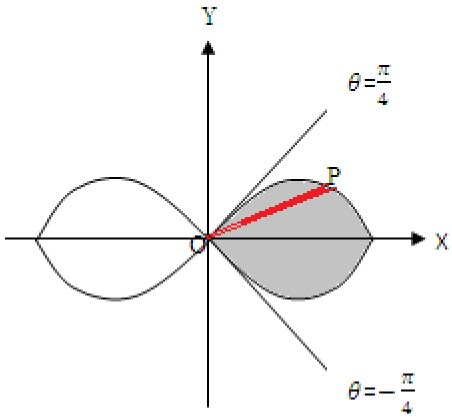
To evaluate above, we have to integrate with respect to r first taking θ constant and then integrate with respect to θ. Following are the examples:

Evaluation of double integral in polar coordinates over given region

Example 1. Evaluate $\iint_R \frac{r}{\sqrt{r^2 + a^2}} \mathrm{d}r \, \mathrm{d}\theta$ over one loop of lemniscate $r^2 = a^2 \cos 2\theta$.
Solution: Consider,

$$I = \iint_R \frac{r}{\sqrt{r^2 + a^2}} \mathrm{d}r \, \mathrm{d}\theta$$

Consider the one loop of Lemniscate which lies between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$ and integrating strip from origin as shown in the figure.



The point O lies on origin i.e. $r=0$ and P lies on $r^2=a^2\cos 2\theta$ i.e. $r=a\sqrt{\cos 2\theta}$. Therefore r varies from 0 to $a\sqrt{\cos 2\theta}$. To complete the loop we have vary θ from $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{4}$. Therefore,

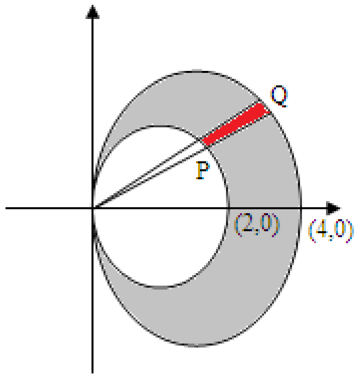
$$
\begin{aligned}
I &= \iint_R \frac{r}{\sqrt{r^2+a^2}}\mathrm{d}r\,\mathrm{d}\theta = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{r^2+a^2}}\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{2r}{\sqrt{r^2+a^2}}\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\left[2\sqrt{r^2+a^2}\right]_0^{2\sqrt{\cos 2\theta}}\mathrm{d}\theta \qquad \left(\because \int \frac{f'(r)}{\sqrt{f(r)}}\mathrm{d}r = 2\sqrt{f(r)}\right) \\
&= \int_{-\pi/4}^{\pi/4}[\sqrt{a^2+a^2\cos 2\theta}-\sqrt{a^2}]\mathrm{d}\theta \\
&= \int_{-\pi/4}^{\pi/4}[a\sqrt{2\cos^2\theta}-a]\mathrm{d}\theta \\
&= \int_{-\pi/4}^{\pi/4}[\sqrt{2}\cos\theta-1]\mathrm{d}\theta \\
&= 2a\int_0^{\pi/4}[\sqrt{2}\cos\theta-1]\mathrm{d}\theta \\
&= 2a[\sqrt{2}\sin\theta-\theta]_0^{\pi/4} \\
&= 2a\left[1-\frac{\pi}{4}\right] = \frac{a}{2}(4-\pi)
\end{aligned}
$$

Example 2. Evaluate $\iint_R r^3\mathrm{d}r\,\mathrm{d}\theta$ over the region between the circles $r=2\cos\theta$ and $r=4\cos\theta$.
Solution: Consider,

$$I=\iint_R r^3\mathrm{d}r\,\mathrm{d}\theta$$

We have $x=r\cos\theta$. This gives $\cos\theta=\dfrac{x}{r}$. Putting this value in $r=2\cos\theta$, we get $r=2\dfrac{x}{r}$. This gives $r^2=2x \;\Rightarrow\; x^2+y^2=2x$ i.e. $(x-1)^2+y^2=1$. This shows that $r=2\cos\theta$ is a circle with radius 1 and center at $(1,0)$. Similarly, $r=4cos\theta$ represents a circle with radius 2 and center at $(2,0)$.

Now, consider the region bounded by $r = 2\cos\theta$ and $r = 4\cos\theta$ and the integrating strip from origin as shown in the following figure.



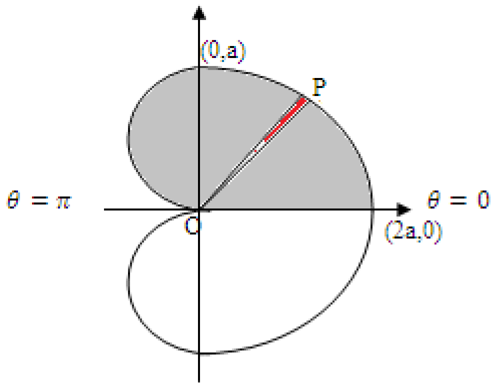
Here the point P lies on $r = 2\cos\theta$ and point Q lies on $r = 4\cos\theta$ and the whole region is covered by moving strip from $\theta = -\pi/2$ to $\pi/2$. Therefore, r varies from $2\cos\theta$ to $4\cos\theta$ and θ varies from $-\pi/2$ to $\pi/2$. Hence,

$$\begin{aligned}
I &= \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3 \mathrm{d}r\ \mathrm{d}\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta}\mathrm{d}\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\left[256\cos^4\theta - 16\cos^4\theta\right]\mathrm{d}\theta \\
&= \frac{240}{4}\int_{-\pi/2}^{\pi/2}\cos^4\theta \mathrm{d}\theta = 60\times 2\int_0^{\pi/2}\cos^4\theta\,\mathrm{d}\theta \qquad (\because \text{even function}) \\
&= 120\frac{1}{2}\beta\left(\frac{1}{2},\frac{5}{2}\right) \qquad \left(\because \int_0^{\pi/2}\sin^p x\cos^q x\mathrm{d}x = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)\right) \\
&= \frac{60\Gamma(1/2)\Gamma(5/2)}{\Gamma(3)} \\
&= \frac{60\sqrt{\pi}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{2} \\
&= \frac{45\pi}{2}
\end{aligned}$$

Example 3. Evaluate $\iint_R r\sin\theta\,\mathrm{d}r\mathrm{d}\theta$ over cardioide $r = a(1+\cos\theta)$ above initial line.
Solution: Consider,

$$I = \iint_R r\sin\theta\,\mathrm{d}r\mathrm{d}\theta$$

Consider the region of integration above initial line bounded by cardioide and the integrating strip as shown in the figure.



The point O of integrating strip lies on origin i.e. $r = 0$ and P lies on $r = a(1 + \cos\theta)$. Therefore, r varies from 0 to $a(1 + \cos\theta)$. The region above the initial line is covered by rotating integrating strip from 0 to π. i.e. θ varies from 0 to π. Therefore,

$$
\begin{aligned}
I &= \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \,\mathrm{d}r\mathrm{d}\theta = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi} \sin\theta[a^2(1+\cos\theta)^2]\mathrm{d}\theta \\
&= \frac{a^2}{2}\int_0^{\pi} \sin\theta[(1+\cos\theta)^2]\mathrm{d}\theta \\
&= \frac{a^2}{2}\int_2^{0} [t^2](-\mathrm{d}t) \qquad (\text{put} 1+\cos\theta = t \Rightarrow \sin\theta\mathrm{d}\theta = -\mathrm{d}t \\
&= -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_2^0 = -\frac{a^2}{2}\left[0 - \frac{8}{3}\right] = \frac{4a^2}{3}
\end{aligned}
$$