LOGARITHMS OF COMPLEX NUMBERS

Let $z = x + iy$ and also let $x = r\cos\theta$, $y = r\sin\theta$ so that $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos\theta + i\sin\theta)) = \log(r.e^{i\theta})$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$\therefore \quad \log(x + iy) = \log r + i\theta$

$\therefore \quad \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\,tan^{-1}\frac{y}{x}$ .... (1)

This is called **principal value** of log (x + iy)

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore \text{Log}\,(x + iy) = 2n\pi i + \log(x + iy)$

$\therefore \text{Log}\,(x + iy) = 2n\pi i + \frac{1}{2}\log(x^2 + y^2) + i\,tan^{-1}\frac{y}{x}$

$\text{Log}(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ .... (2)

Caution: $\theta = tan^{-1}y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta$, $y = r\sin\theta$, $-\pi \le \theta \le \pi$.

SOME SOLVED EXAMPLES:

1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\,\pi}{\log 2}$

Solution: Since $log(x + iy) = \frac{1}{2}log(x^2 + y^2) + i\,tan^{-1}\frac{y}{x}$

Putting $x = -3, y = 0$

we have $\log(-3) = \frac{1}{2}log(9) + i\,tan^{-1}\left(\frac{0}{-3}\right) = \frac{1}{2}\log 3^2 + i\pi = \log 3 + i\pi$

$log_2(-3) = \frac{log_e(-3)}{log_e 2} = \frac{\log 3 + i\pi}{\log 2}$

2. Find the general value of $\text{Log}(1 + i) + \text{Log}\,(1 - i)$

Solution: $\log(1 + i) = \frac{1}{2}\log 2 + i\,\frac{\pi}{4} = log\sqrt{2} + i\frac{\pi}{4}$

$\therefore \text{Log}(1 + i) = log\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right)$ (General value)

Changing the sign of i,

$\text{Log}(1 - i) = log\sqrt{2} - i\left(2n\pi + \frac{\pi}{4}\right)$

By addition, we get $\text{Log}(1 + i) + \text{Log}(1 - i) = 2\,log\sqrt{2} = 2.\frac{1}{2}\log 2 = \log 2$

3. Prove that $\log\left(1 + e^{2\,i\,\theta}\right) = \log(2\cos\theta) + i\,\theta$

Solution: $\log\left(1 + e^{2\,i\,\theta}\right) = \log(1 + \cos 2\theta + i\sin 2\theta)$

$$= \log(2\cos^2\theta + i2\sin\theta\cos\theta)$$

$$= \log\left(2\cos\theta\,(\cos\theta + i\sin\theta)\right)$$

$$= \log\left(2\cos\theta \cdot e^{i\theta}\right)$$
$$= \log(2\cos\theta) + \log\left(e^{i\theta}\right)$$
$$= \log(2\cos\theta) + i\theta$$

4. Find the value of log [sin(x + i y)]

Solution: We have, $\sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$

$$\therefore \log\sin(x+iy) = \frac{1}{2}\log(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y) + i\tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right)$$

Now, $\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = (1-\cos^2 x)\cosh^2 y + \cos^2 x(\cosh^2 y - 1)$

$$= \cosh^2 y - \cos^2 x$$
$$= \left(\frac{1+\cosh 2y}{2}\right) - \left(\frac{1+\cos 2x}{2}\right)$$
$$= \frac{1}{2}(\cosh 2y - \cos 2x)$$

$$\therefore \log\sin(x+iy) = \frac{1}{2}\log\left(\frac{\cosh 2y - \cos 2x}{2}\right) + i\tan^{-1}(\cot x \tanh y)$$

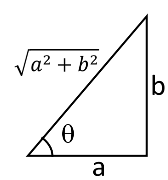
5. Show that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Solution: We have $\log(a-bi) = \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}$

$$\therefore \log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2i\tan^{-1}\frac{b}{a}$$

$$\therefore i\log\left(\frac{a-bi}{a+bi}\right) = -2i^2\tan^{-1}\frac{b}{a} = 2\tan^{-1}\frac{b}{a}$$



$$\therefore \tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan\left(2\tan^{-1}\frac{b}{a}\right)$$

$$\therefore \tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan 2\theta \quad \text{where } \tan^{-1}\frac{b}{a} = \theta$$

$$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(b/a)}{1-(b^2/a^2)} = \frac{2ab}{a^2-b^2}$$

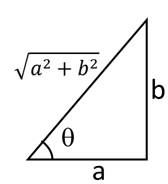
6. Prove that $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$

Solution: We have $\log(a-bi) = \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}$

$$\therefore \log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2i\tan^{-1}\frac{b}{a}$$

$$\therefore i\log\left(\frac{a-bi}{a+bi}\right) = -2i^2\tan^{-1}\frac{b}{a} = 2\tan^{-1}\frac{b}{a}$$



$$\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \cos\left(2\tan^{-1}\frac{b}{a}\right)$$

$$\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \cos 2\theta \quad \text{where } \tan^{-1}\frac{b}{a} = \theta$$

$$= \cos^2\theta - \sin^2\theta = \frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

7. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Solution: We have $\sqrt{i} = i^{1/2} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}$

Also $\sqrt{i} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \left(e^{i\,\pi/2}\right)^{1/2} = e^{i\,\pi/4}$

$\therefore \left(\sqrt{i}\right)^{\sqrt{i}} = \left\{e^{i\,\pi/4}\right\}^{\left(\frac{1}{\sqrt{2}}+i.\frac{1}{\sqrt{2}}\right)} = e^{i\,\pi/4\sqrt{2}-\pi/4\sqrt{2}} = e^{-\,\pi/4\sqrt{2}}.e^{i\,\pi/4\sqrt{2}}$

$= e^{-\,\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$

$\therefore$ Real Part $= e^{-\,\pi/4\sqrt{2}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$ & Imaginary Part $= e^{-\,\pi/4\sqrt{2}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$

8. Find the principal value of $(1+i)^{1-i}$

Solution: $z = (1+i)^{1-i}$

$\therefore \log z = (1-i)\log(1+i)$

$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + \mathrm{i}\tan^{-1}1\right]$

$= (1-i)\left[\frac{1}{2}\log 2 + i.\frac{\pi}{4}\right]$

$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}log2\right) = x + iy\ say$

$\therefore z = e^{x+iy} = e^{x}.e^{iy} = e^{x}(\cos y + i\sin y)$

$= e^{\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right]$

$= \sqrt{2}e^{\pi/4}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right]$ $\because e^{\frac{1}{2}\log 2} = e^{log\sqrt{2}} = \sqrt{2}$

9. Prove that the general value of $(1 + \mathrm{i}\tan\alpha)^{-\mathrm{i}}$ is $\mathrm{e}^{2\,\mathrm{m}\,\pi+\alpha}[\cos(\log\cos\alpha) + \mathrm{i}\sin(\log\cos\alpha)]$

Solution: Let $1 + i\tan\alpha = r\,e^{\,i\theta}$

$\therefore r^2 = 1 + tan^2\alpha = sec^2\alpha$ $\therefore r = \sec\alpha$

And $\theta = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \tan^{-1}(\tan\alpha) = \alpha$

Now, $Log\,(1 + i\ \tan\alpha) = log\left(r\,e^{\,i\theta}\right) = \log r + (2m\pi + \theta)i$

$= \log\sec\alpha + (2m\pi + \alpha)i$

$\therefore 1 + i\ \tan\alpha = e^{[\log\sec\alpha + (2m\pi+\alpha)i]}$

$\therefore (1 + i\ \tan\alpha)^{-i} = e^{-i[\log\sec\alpha + (2m\pi+\alpha)i]}$

$= e^{2m\pi+\alpha}.e^{-ilog\sec\alpha}$

$= e^{2m\pi+\alpha}.e^{i\,(log\cos\alpha)}$

$= e^{2m\pi+\alpha}.[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

10. Considering only principal value, if $(1 + i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{sec^2\beta}$

Solution: Let $z = (1 + i\ tan\alpha)^{1+i\tan\beta}$

Taking logarithms of both sides,

$\operatorname{Log} z = (1 + i\tan\beta) log(1 + i\ tan\alpha)$

$= (1 + i\tan\beta)\left[\frac{1}{2} log(1 + tan^2\alpha) + itan^{-1}tan\alpha\right]$

$= (1 + i\tan\beta)[\log\sec\alpha + i\alpha]$

$\therefore \log z = (\log\sec\alpha - \alpha\tan\beta) + i(\alpha + tan\beta\log\sec\alpha) = x + iy$ say

Where $x = \log\sec\alpha - \alpha\tan\beta$ $and\ y = \alpha + tan\beta\log secc\alpha$ ....(i)

Now, $z = e^{x+iy} = e^x . e^{iy} = e^x(cosy + i\sin y)$

Since by data z is real

$\therefore e^x \sin y = 0 \quad \therefore y = 0 \quad \therefore \cos y = 1$

$\therefore z = e^x \cos y = e^x = e^{\log sec\alpha - \alpha\tan\beta}$ from (i)

$\therefore z = e^{\log sec\alpha} . e^{-\alpha\tan\beta} = sec\alpha . e^{-\alpha\tan\beta}$ ....(ii)

But since $y = 0$, from (i) $\alpha + tan\beta\log\sec\alpha = 0$

$\therefore -\alpha = tan\beta\log sec\alpha$

$\therefore -\alpha\ tan\beta = tan^2\beta . \log sec\ \alpha = log(sec\ \alpha)^{tan^2\beta}$

$\therefore e^{-\alpha\tan\beta} = (sec\ \alpha)^{tan^2\beta}$

$\therefore$from (ii) $z = sec\alpha . (sec\ \alpha)^{tan^2\beta} = (sec\ \alpha)^{(1+tan^2\beta)} = (sec\ \alpha)^{sec^2\beta}$

11. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\ \beta$, find α and β

Solution: $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\ \beta$,

Taking logarithms of both sides, $log\left(\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}\right) = log(\alpha + i\ \beta)$

$\log(\alpha + i\beta) = (x + iy)\log(a + ib) - (x - iy)\log(a - ib)$

$log(\alpha + i\beta) = (x + iy)\left[\frac{1}{2} log(a^2 + b^2) + i\ tan^{-1}\left(\frac{b}{a}\right)\right] - (x - iy)\left[\frac{1}{2} log(a^2 + b^2) - i\ tan^{-1}\left(\frac{b}{a}\right)\right]$

$log(\alpha + i\beta) = 2i\left[xtan^{-1}\frac{b}{a} + \frac{y}{2} log(a^2 + b^2)\right]$

$= 2ik\ say \qquad where\ k = \left[xtan^{-1}\frac{b}{a} + \frac{y}{2} log(a^2 + b^2)\right]$

$\therefore (\alpha + i\beta) = e^{2ik} = cos2k + isin\ 2k$

$\therefore \alpha = cos2k,\ \beta = sin\ 2k \qquad where\ k = \left[xtan^{-1}\frac{b}{a} + \frac{y}{2} log(a^2 + b^2)\right]$

12. If $\mathrm{i}^{\alpha+\mathrm{i}\,\beta} = \alpha + \mathrm{i}\,\beta$ $\left(\text{or } \mathrm{i}^{\mathrm{i}^{\mathrm{i}\ldots\ldots\infty}} = \alpha + \mathrm{i}\,\beta\right)$, prove that $\alpha^2 + \beta^2 = \mathrm{e}^{-(4\mathrm{n}+1)\pi\beta}$ Where n is any positive integer

Solution: Since $i = \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$

we have $\mathrm{i}^{\alpha+\mathrm{i}\,\beta} = \alpha + \mathrm{i}\,\beta$

$\left[\cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{\alpha+i\beta} = \alpha + i\beta$

$\therefore e^{i\left(2n\pi+\frac{\pi}{2}\right)(\alpha+i\beta)} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta + i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$

$$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta} . e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$$

$$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\left[\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha\right] = \alpha + i\beta$$

Equating real and imaginary parts

$e^{-(4n+1)\frac{\pi}{2}\beta}\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha = \alpha$ and $e^{-(4n+1)\frac{\pi}{2}\beta}\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha = \beta$

Squaring and adding, we get $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$

13. Prove that $\log tan\left(\frac{\pi}{4} + i\,\frac{x}{2}\right) = i\ tan^{-1}(\sinh x)$.

Solution: $\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right) = log\left\{\frac{1+tan(ix/2)}{1-tan(ix/2)}\right\}$

$$= log\left\{\frac{1+\text{itan}\,h\ (x/2)}{1-\text{itan}\,h(x/2)}\right\}$$

$$= log[1 + i\tan h\ (x/2)] - log[1 - i\tan h(x/2)\ ]$$

$$= \left[\frac{1}{2}log\left(1 + tanh^2\left(\frac{x}{2}\right)\right) + itan^{-1}tanh\left(\frac{x}{2}\right)\right]$$

$$- \left[\frac{1}{2}log\left(1 + \tan h^2\left(\frac{x}{2}\right)\right) - i\ tan^{-1}\tan h\left(\frac{x}{2}\right)\right]$$

$$= 2i\ tan^{-1}\tan h\left(\frac{x}{2}\right) = i.tan^{-1}\left\{\frac{2\tan h(x/2)}{1-tanh^2(x/2)}\right\} = itan^{-1}(\sin hx)$$

$$\because 2tan^{-1}\alpha = tan^{-1}\left\{\frac{2\alpha}{1-\alpha^2}\right\}$$