

MATRIX THEORY: RANK OF MATRIX - SYSTEM OF LINEAR EQUATIONS

FY BTECH SEM-I
MODULE-2

A SYSTEM OF LINEAR EQUATIONS

- Consider a system of **m linear equations in n unknowns**, say $x_1, x_2, x_3, \dots, x_n$
- $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
- $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$
- $a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3$
-
- $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$
- This system can be written compactly in matrix notation as **$AX = B$**
- $A = [a_{ij}]_{m \times n}$: matrix of coefficients
- $B = [b_1 \ b_2 \ b_3 \dots b_m]^T$ is the column vector of order $(m \times 1)$
- $X = [x_1 \ x_2 \ x_3 \dots x_n]^T$ is the column vector of order $(n \times 1)$
- Any vector U satisfying $AU = B$ is said to be a **solution** of $AX = B$.
- The matrix $[A, B]$ i.e., the matrix formed by the coefficients and the constants is called the **augmented matrix**.
- A system $AX = B$ is
 - (i) **Homogeneous** if $B = 0$ and
 - (ii) **Non – homogeneous** if $B \neq 0$

Different cases for solution

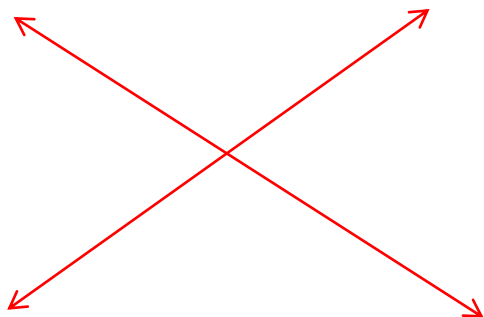
Consistent (has at least one solution)

• (i) Unique Solution:

• **Ex:** $4x_1 + 3x_2 = 11$,

• $4x_1 - 3x_2 = 5$

• Solving we get,
 $x_1 = 2$ and $x_2 = 1$



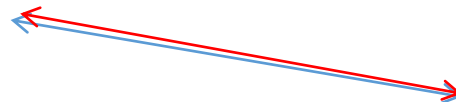
• (ii) Infinite Solutions:

• **Ex:** $4x_1 + 3x_2 = 11$,

• $8x_1 + 6x_2 = 22$

has more solutions, say $(2,1)^T$ or $(0,11/3)^T$.

$[k, \frac{11-4k}{3}]^T$ is a general solution for all k



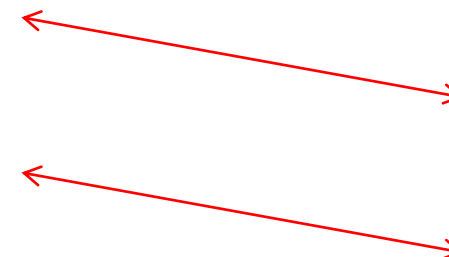
Inconsistent

• (iii) No Solution:

• **Ex:** $4x_1 + 3x_2 = 11$,

• $8x_1 + 6x_2 = 20$

• has no solution at all.



NON – HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

(i) n EQUATIONS IN n UNKNOWN:

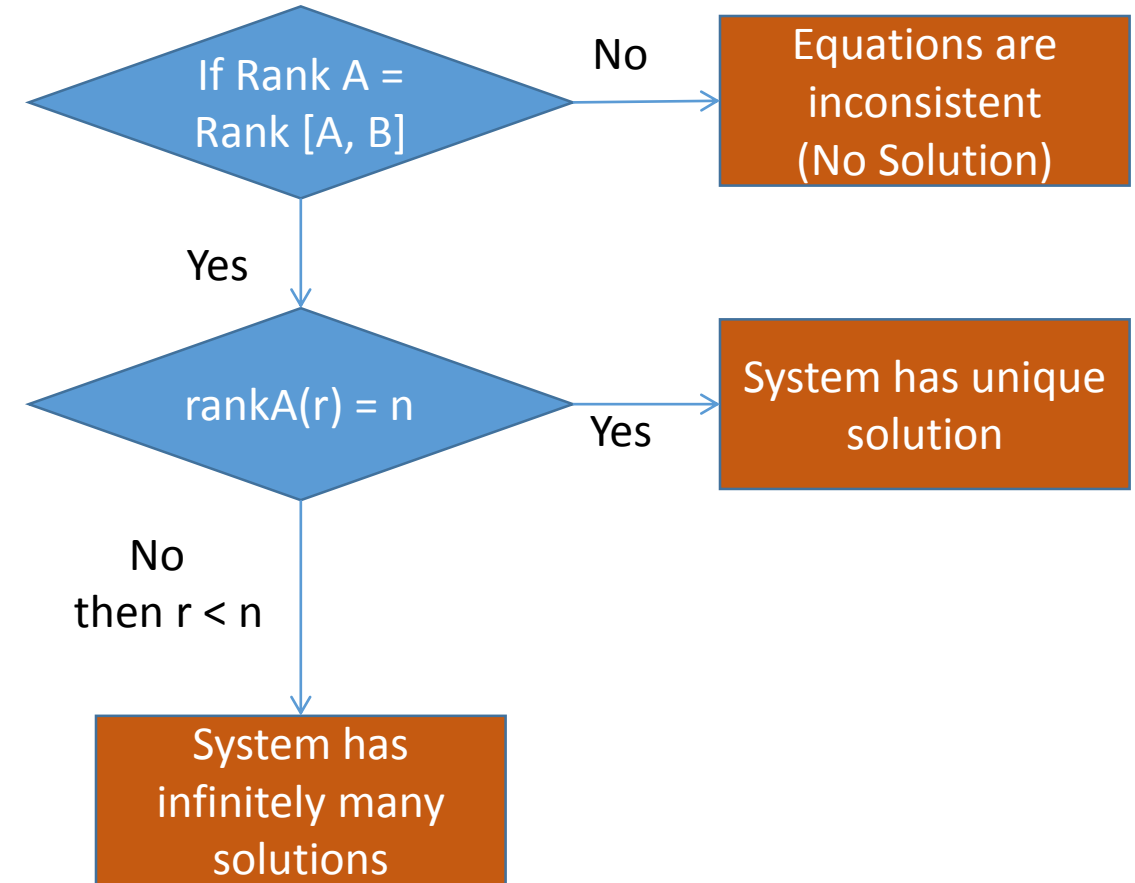
- Consider $AX = B$, where A is a non – singular $n \times n$ matrix, X is $n \times 1$ vector and B is $n \times 1$ matrix then the system has **unique solution**.
- Steps:
 - 1. Write $AX = B$
 - 2. Check that $|A| \neq 0$
 - 3. Find A^{-1} by any suitable method.
 - 4. Solution is given by $X = A^{-1}B$.
- **Note:** If A is singular matrix, then this inverse method fails. In that case the system may have infinitely many solutions or none at all.

NON – HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- **(ii) m EQUATIONS IN n UNKNOWN:**

- **Working Rule:**

- **1.** Write the given system in the matrix form $AX = B$.
- **2.** Apply row transformations on A as well as on the column matrix B i.e. on the augmented matrix $[A, B]$ till you get an row echelon form.
- **3.** We know that the rank of a matrix in echelon form is equal to the number of non-zero rows.
- Determine the rank of A and the rank of the augmented matrix $[A, B]$ and find the solution by using following cases.



Example 1

- Test the consistency of the equations
- $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6.$

- **Solution:** The system of equations can be written

$$\text{as } AX = B, \text{ i.e. } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

- \therefore the augmented matrix can be written as

$$[A: B] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}$$

- Applying $R_2 - R_1$ and $R_3 - R_1$ we get

$$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix}$$

- Applying $R_3 - 3R_2$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

- \therefore rank of $[A: B] = \text{rank of } A = 3.$
- \therefore The system has unique solution.
- The reduced form of equations can be written as
- $x + y + z = 3, \quad y + 2z = 1, \quad 2z = 0$
- \therefore (iii) $\Rightarrow z = 0$ substituting this value in (ii) $y = 1$
- \therefore (i) $\Rightarrow x + 1 + 0 = 3 \Rightarrow x = 2$

- Hence the solution set can be written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 + x_3 = 2$$

- Solve $2x_1 + 4x_2 + 3x_3 = 3$

$$3x_1 + 6x_2 + 5x_3 = 4$$

- The system of equations can be represented by the

matrix equation as,
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

- Where the augmented matrix is

- $$[A: B] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 \\ 3 & 6 & 5 & 4 \end{bmatrix}$$

- Applying elementary row transformations, the matrix $[A: B]$ can be reduced to Echelon form.

- Applying $R_2 - 2R_1$ and $R_3 - 3R_1$ we get

$$[A: B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Example 2

- Applying $R_3 - 2R_2$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Hence $\rho(A) = \rho[A: B]$, therefore the system is consistent.
- Further rank $r = 2 < 3$ (number of variables), therefore the system has infinite solutions.
- $\therefore (n - r) = 3 - 2 = 1$ (free variable)
- The reduced form of the linear equations can be written as, $x_1 + 2x_2 + x_3 = 2$, $x_3 = -1$
- Let $x_2 = k$, an arbitrary constant. $\therefore x_1 = 3 - 2k$
- Hence
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 2k \\ k \\ -1 \end{bmatrix}$$
 infinite solutions as k varies.

Example 3

- Are the following equations consistent?

$$2x + y + z = 4$$

$$x + y + z = 2$$

$$5x + 3y + 3z = 6$$

- Solution:** The system of linear equations can be written in the matrix form $AX = B$ i.e

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

- \therefore the augmented matrix can be written as $[A: B] =$

$$\begin{bmatrix} 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 2 \\ 5 & 3 & 3 & 6 \end{bmatrix}$$

- Applying R_{12} , we get

$$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 5 & 3 & 3 & 6 \end{bmatrix}$$

- Applying $R_2 - 2R_1$ and $R_3 - 5R_1$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & -4 \end{bmatrix}$$

- Applying $R_3 - 2R_2$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

- $\Rightarrow \text{rank of } A = 2$ and rank of $[A: B] = 3$.
- i.e $\rho(A) \neq \rho[A: B]$.
- Hence the given system of linear equation is inconsistent and therefore has no solution.

Example 4

- Solve the system of equations.

$$x_1 - x_2 + 2x_3 + x_4 = 2$$

$$3x_1 + 2x_2 + x_4 = 1$$

$$4x_1 + x_2 + 2x_3 + 2x_4 = 3$$

- Solution:** The system of equations can be represented by the matrix equation as,

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

- where the augmented matrix is

- $[A : B] = \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{bmatrix}$

- Applying $R_3 - (R_1 + R_2)$, we get

$$[A : B] \sim \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Applying $R_2 - 3R_1$, we get $[A :$

Example 4 contd..

• The reduced form of the linear equations is

• $x_1 - x_2 + 2x_3 + x_4 = 2$,

• $5x_2 - 6x_3 - 2x_4 = -5$

• Let $x_3 = p$ and $x_4 = q$, an arbitrary constant for free variables

• Substituting back, we get

• $x_2 = \frac{1}{5}(-5 + 6p + 2q)$ and

• $x_1 = \frac{1}{5}(5 - 4p - 3q)$

• Hence the infinite values of solution are given by,

•
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(5 - 4p - 3q) \\ \frac{1}{5}(-5 + 6p + 2q) \\ p \\ q \end{bmatrix} \text{ as } p \text{ and } q \text{ varies.}$$

Example 5

- Investigate for what values of a and b the following linear equations
- $x + 2y + 3z = 4, x + 3y + 4z = 5, x + 3y + az = b$, have
 (i) no solution, (ii) a unique solution,
 (iii) An infinite number of solutions.

- Solution:** The system of linear equations can be written in the matrix form as $AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix}$$

- where the augmented matrix is $[A: B] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 3 & a & b \end{bmatrix}$

- Applying $R_2 - R_1$ and $R_3 - R_1$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & a-3 & b-4 \end{bmatrix}$$

- Applying $R_3 - R_2$, $[A: B] \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-4 & b-5 \end{bmatrix}$

- (i) **For no solution:**

- If $a = 4$ and $b \neq 5$,
- Then we have $\rho(A) = 2$ and $\rho[A: B] = 3$.
- In this case $\rho(A) \neq \rho[A: B]$.

- (ii) **For unique solution:**

- If $a \neq 4$, we have $\rho(A) = \rho[A: B] = 3$
- In this case the system is consistent. Further, Since $\rho(A) =$ number of unknowns, therefore the system possesses unique solution if $a \neq 4$ and for any value of b .

- (iii) **For infinite number of solutions:**

- If $a = 4$ and $b = 5$, we get $\rho[A: B] = \rho(A) = 2 < 3$, the number of unknowns,
- therefore system of equations is consistent and possesses an infinite number of solutions.

Example 6

- For which values of λ following set of equations is consistent? Find and solve equations for those values
- $x + 2y + z = 3, x + y + z = \lambda, 3x + y + 3z = \lambda^2$

• **Solution:** consider $[A: B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{bmatrix}$

- Applying $R_2 - R_1$ and $R_3 - 3R_1$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & -5 & 0 & \lambda^2 - 9 \end{bmatrix}$$

- Applying $R_3 - 5R_2$, we get

$$[A: B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{bmatrix}$$

- For consistency of the equation, the rank of A and rank of $[A: B]$ must be the same.

From above reduced form of $[A: B]$ it is clear that the rank of $A = 2$. To have the rank of $[A: B] = 2$,

- Consider $\lambda^2 - 5\lambda + 6 = 0$ i.e., $(\lambda - 2)(\lambda - 3) = 0$
 $\Rightarrow \lambda = 2$ and $\lambda = 3$

- (i) **Solution for $\lambda = 2$**

- The reduced Echelon form of $[A: B]$ is

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{bmatrix}$$

- Substituting $\lambda = 2$, we have $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- The reduced form of linear equations is

- $x + 2y + z = 3$ and $y = 1$

Example 6 (contd...)

- Let $z = k$, an arbitrary constant, $\therefore x = 1 - k$
- Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - k \\ 1 \\ k \end{bmatrix}$ has infinite values as k varies
- **(ii) Solution for $\lambda = 3$:**
- The reduced Echelon form of $[A: B]$
- By substituting $\lambda = 3$, we have $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- The reduced form of linear equations is
- $x + 2y + z = 3$ and $y = 0$
- Let $z = c$, an arbitrary constant. $\therefore x = 3 - c$
- Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - c \\ 0 \\ c \end{bmatrix}$ has infinite solutions as c varies.

Example 7

- For what values of λ the equations $3x - 2y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$, will have no unique solution? Will the equations have any solutions for this value of λ .
- Solution:** (taking the equations in reverse order)
- $$\begin{bmatrix} 1 & 2 & -\lambda \\ 2 & 1 & 1 \\ 3 & -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$
- Applying $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & -3 & 1 + 2\lambda \\ 0 & -8 & 4\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix} \dots\dots\dots(1)$$
- The equations have unique solutions if the coefficient matrix is non-singular.
- $\therefore -12\lambda + 8 + 16\lambda \neq 0$, $4\lambda \neq -8 \quad \therefore \lambda \neq -2$
- \therefore The equations have unique solutions if $\lambda \neq -2$
- and they have no unique solutions if $\lambda = -2$
- Further, if $\lambda = -2$, we have from (1)

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -8 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$$
- Applying $R_3 - \frac{8}{3}R_2$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -20/3 \end{bmatrix}$$
- $\therefore 0x + 0y + 0z = -20/3$ which is absurd
- Also the rank of $A = 2 < \text{the rank of } [A, B] = 3$
- \therefore The equations are inconsistent, For $\lambda = -2$ there is no solution.