Jacobian

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11:03 AM

def^n: consider $u = f(x,y)$ & $v = f(x,y)$, Then

Jacobian of u, v with respect to x & y,

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \boxed{\begin{array}{cc|l} x & y & \\ \hline u_x & u_y & \to u \\ v_x & v_y & \to v \end{array}}$$

Similarly we can generalise for 3 variables.,

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{array}{c|l} \begin{matrix} x & y & z \end{matrix} & \\ \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} & \begin{matrix} \to u \\ \to v \\ - w \end{matrix} \end{array}$$

Use

Transforming from one coordinate system to another

$\boxed{x,y} \to \boxed{r,\theta}$ $\quad \boxed{x = r\cos\theta}$. $\quad \boxed{y = r\sin\theta}$

$dx\,dz\,dy \to \boxed{ds} \to dx\,dy\,dz \to J\left(\frac{x,y,z}{r,\theta,\phi}\right) dr\, d\theta\, d\phi$

$J\left(\frac{r,\theta}{x,y}\right)$ $\quad$ $r \to x, y$; $\theta \to x, y$ $\quad$ $r = \sqrt{x^2+y^2}$ $\quad$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

If $J = \frac{\partial(x,y)}{\partial(r,\theta)}$ Then $J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{J}$

$$\boxed{J J' = 1}$$

1) **If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$, find $\boxed{\frac{\partial(u,v)}{\partial(x,y)}}$** $\quad J\left(\frac{u,v}{x,y}\right)$

$\underline{Sol^n}$: $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

$$u_x = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2} = \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$u_y = \frac{1+x^2}{(1-xy)^2} \quad \text{— (u is symmetric in } x \,\&\, y \text{)}$$

$$v_x = \frac{1}{1+x^2}, \quad v_y = \frac{1}{1+y^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = \underline{0}$$

$\underline{C.H.}$

2) • **If $u = x(1-y)$, $v = xy(1-z)$, $w = xyz$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$** $= \boxed{x^2 y}$

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3) • **If $x = e^u \cos v$, $y = e^u \sin v$, prove that $JJ' = 1$**

Given that $x \,\&\, y$ as function of $u \,\&\, v$

$$\underline{J\left(\frac{x,y}{u,v}\right)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$= e^{2u}\cos^2 v + e^{2u}\sin^2 v = e^{2u}(1)$$

for finding J', we will express $u \,\&\, v$ as f^n of $x \,\&\, y$

Squaring & adding, $x^2 + y^2 = e^{2u}\cos^2 v + e^{2u}\sin^2 v$

$$\boxed{x^2 + y^2 = e^{2u}}$$

$$\therefore \boxed{u = \frac{1}{2}\log(x^2+y^2)}$$

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Divide, $\frac{y}{x} = \frac{\cancel{e^u}\sin v}{\cancel{e^u}\cos v} = \tan v \quad \therefore \boxed{v = \tan^{-1}\left(\frac{y}{x}\right)}$

Then $J'\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{\cancel{2}}\frac{(\cancel{2}x)}{(x^2+y^2)} & \frac{y}{(x^2+y^2)} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \end{vmatrix}$

$$= \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}} =$$

$$= \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

$$\therefore \underline{JJ'} = e^{2u}\left(\frac{1}{e^{2u}}\right) = \underline{1}$$

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2) **If $x = u(1-v), y = uv$, prove that $JJ' = 1$**

$x = u - uv \quad , \quad y = uv$

$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv$

$= u$

To write u & v as function of x & y,

$\boxed{x+y = u}$, $\quad uv = y \quad \therefore \boxed{v = \frac{y}{u} = \frac{y}{x+y}}$

$J'\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$

$= \frac{x}{} + \frac{y}{} = \frac{\cancel{x+y}}{\cancel{x}} = \frac{1}{x+y}$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{\cancel{x+y}}{(x+y)^{\cancel{2}}} = \frac{1}{x+y}$$

$$= \frac{1}{u}$$

$$\boxed{JJ' = (u)\left(\frac{1}{u}\right) = 1}$$