

INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x , where x is real.

Similarly we can define $\cosh^{-1}x$, $\tanh^{-1}x$, $\coth^{-1}x$, $\operatorname{sech}^{-1}x$, $\operatorname{cosech}^{-1}x$.

The inverse hyperbolic functions are many valued but we will consider their principal value only.

Theorem: If x is real.

$$(i) \quad \sinh^{-1}x = \log (x + \sqrt{x^2 + 1})$$

$$(ii) \quad \cosh^{-1}x = \log (x + \sqrt{x^2 - 1})$$

$$(iii) \quad \tanh^{-1}x = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

Proof: (i) $\sinh^{-1} x = \log (x + \sqrt{x^2 + 1})$

$$\text{Let } \sinh^{-1} x = y$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y} - 1}{e^y}$$

$$e^{2y} - 2x e^y - 1 = 0$$

This equation is quadratic in e^y .

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$$y = \log (x \pm \sqrt{x^2 + 1})$$

But $x - \sqrt{x^2 + 1} < 0$ and $\log(-ve)$ is not defined.

$$\therefore y = \log (x + \sqrt{x^2 + 1})$$

$$\sinh^{-1}x = \log(x + \sqrt{x^2 + 1})$$

$$(ii) \quad \cosh^{-1}x = \log(x + \sqrt{x^2 - 1})$$

$$\text{Let } \cosh^{-1}x = y$$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y} + 1}{e^y}$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

$$y = \log (x \pm \sqrt{x^2 - 1}) \quad \dots\dots\dots(1)$$

$$\text{Consider, } y = \log (x - \sqrt{x^2 - 1}) \quad \dots\dots\dots(2)$$

$$e^y = x - \sqrt{x^2 - 1},$$

$$e^{-y} = \frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = \frac{x + \sqrt{x^2 - 1}}{x^2 - x^2 + 1} = x + \sqrt{x^2 - 1}$$

$$-y = \log(x + \sqrt{x^2 - 1})$$

$$y = -\log(x + \sqrt{x^2 - 1}) \quad \dots\dots\dots(3)$$

Equating equation (2) and (3), we get

$$\log(x - \sqrt{x^2 - 1}) = -\log(x + \sqrt{x^2 - 1}) \quad \dots\dots\dots(4)$$

From equation (1) and (4), we get

$$y = \pm \log(x + \sqrt{x^2 - 1})$$

$$\cosh^{-1}x = \pm \log(x + \sqrt{x^2 - 1})$$

$$\therefore x = \cosh\{\pm \log(x + \sqrt{x^2 - 1})\}$$

$$= \cosh\{\log(x + \sqrt{x^2 - 1})\}$$

$$\text{since } \cosh(-z) = \cosh z$$

$$\therefore \cosh^{-1}x = \log(x + \sqrt{x^2 - 1})$$

(iii) $\tanh^{-1}x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

Let $\tanh^{-1}x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo-dividendo

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$= \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh \log \sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log \sqrt{5/3} + \tanh \log \sqrt{7} = 1$

Solution: Let $\tanh \log \sqrt{x} = \alpha$

$$\log \sqrt{x} = \tanh^{-1} \alpha$$

$$\frac{1}{2} \log x = \frac{1}{2} \log\left(\frac{1+\alpha}{1-\alpha}\right)$$

$$x = \frac{1+\alpha}{1-\alpha}$$

$$\frac{x-1}{x+1} = \frac{(1+\alpha)-(1-\alpha)}{(1+\alpha)+(1-\alpha)} = \frac{2\alpha}{2} =$$

$$\therefore \tanh \log \sqrt{x} = \frac{x-1}{x+1}$$

Put $x = 5/3$ and $x = 7$ and add

$$\log h(\log \sqrt{5/3}) + \tanh (\log \sqrt{7}) = \frac{(5/3)-1}{(5/3)+1} + \frac{7-1}{7+1} = \frac{2}{8} + \frac{6}{8} = 1$$

2. (i) Prove that $\cosh^{-1}\sqrt{1+x^2} = \sinh^{-1}x$

Solution: Let $\cosh^{-1}\sqrt{1+x^2} = y \quad \therefore \sqrt{1+x^2} = \cosh y$

$$\therefore 1+x^2 = \cosh^2 y \quad \therefore x^2 = \cosh^2 y - 1 = \sinh^2 y$$

$$\therefore x = \sinh y \quad \therefore y = \sinh^{-1}x \quad \therefore \cosh^{-1}\sqrt{1+x^2} = \sinh^{-1}x$$

(ii) Prove that $\tanh^{-1}x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Solution: Let $\tanh^{-1}x = y \quad \therefore x = \tanh y$

$$\text{Now, } \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\cosh^2 y - \sinh^2 y / \cosh^2 y}} = \frac{\sinh y}{\cosh y} \times \frac{\cosh y}{1} = \sinh y$$

$$\therefore y = \sinh^{-1}\frac{x}{\sqrt{1-x^2}} \quad \therefore \tanh^{-1}x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$$

(iii) Prove that $\cosh^{-1}(\sqrt{1+x^2}) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Solution: Let $\cosh^{-1}\sqrt{1+x^2} = y \quad \therefore \sqrt{1+x^2} = \cosh y$

$$\therefore 1+x^2 = \cosh^2 y \quad \therefore x^2 = \cosh^2 y - 1 = \sinh^2 y \quad \therefore x = \sinh y$$

$$\therefore \tanh y = \frac{\sinh y}{\cosh y} = \frac{x}{\sqrt{1+x^2}} \quad \therefore y = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \cosh^{-1}(\sqrt{1+x^2}) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

(iv) Prove that $\cot h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2} \log \left(\frac{x+a}{x-a}\right)$

Solution: Let $\cot h^{-1}\left(\frac{x}{a}\right) = y \quad \therefore \frac{x}{a} = \cot hy \quad \therefore \tanh y = \frac{1}{\cot hy} = \frac{1}{x/a} = \frac{a}{x}$

$$\therefore y = \tanh^{-1}\left(\frac{a}{x}\right) = \frac{1}{2} \log \left(\frac{1+(a/x)}{1-(a/x)}\right) = \frac{1}{2} \log \left(\frac{x+a}{x-a}\right)$$

$$\therefore \cot h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2} \log \left(\frac{x+a}{x-a}\right)$$

(iii) Prove that $\operatorname{sech}^{-1}(\sin \theta) = \log \cot \frac{\theta}{2}$

Solution: Let $\operatorname{sech}^{-1}(\sin \theta) = x \quad \therefore \sin \theta = \operatorname{sech} x \quad \therefore \sin \theta = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}$

$$\therefore (\sin \theta)e^{2x} - 2e^x + \sin \theta = 0 \quad \text{This is a quadratic in } e^x$$

$$\therefore e^x = \frac{2 \pm \sqrt{4 - 4 \sin^2 \theta}}{2 \sin \theta} = \frac{1 \pm \cos \theta}{\sin \theta}$$

$$\therefore e^x = \frac{1 + \cos \theta}{\sin \theta} = \frac{2 \cos^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} = \cot \frac{\theta}{2}$$

$$\therefore x = \log \cot \left(\frac{\theta}{2} \right) \quad \therefore \sec h^{-1}(\sin \theta) = \log(\cot \theta/2)$$

3. Separate into real and imaginary parts $\cos^{-1} e^{i\theta}$ **or** $\cos^{-1}(\cos \theta + i \sin \theta)$

Solution: Let $\cos^{-1} e^{i\theta} = x + iy$, $e^{i\theta} = \cos(x + iy)$

$$\cos \theta + i \sin \theta = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y$$

$$\text{Equating real and imaginary parts } \cos \theta = \cos x \cosh y \text{ and } \sin \theta = -\sin x \sinh y$$

$$\text{Since } \cosh^2 y - \sinh^2 y = 1$$

$$\therefore \left(\frac{\cos \theta}{\cos x} \right)^2 - \left(\frac{-\sin \theta}{\sin x} \right)^2 = 1$$

$$\therefore \frac{\cos^2 \theta}{\cos^2 x} - \frac{\sin^2 \theta}{\sin^2 x} = 1$$

$$\therefore \frac{1 - \sin^2 \theta}{1 - \sin^2 x} - \frac{\sin^2 \theta}{\sin^2 x} = 1$$

$$\therefore \frac{(1 - \sin^2 \theta) \sin^2 x - \sin^2 \theta (1 - \sin^2 x)}{(1 - \sin^2 x) \sin^2 x} = 1$$

$$\therefore \sin^2 x - \sin^2 x \sin^2 \theta - \sin^2 \theta + \sin^2 x \sin^2 \theta = \sin^2 x - \sin^4 x$$

$$\therefore \sin^2 x - \sin^2 \theta = \sin^2 x - \sin^4 x$$

$$\therefore -\sin^2 \theta = -\sin^4 x$$

$$\therefore \sin^2 \theta = \sin^4 x$$

$$\therefore \sqrt{\sin \theta} = \sin x \quad \dots\dots\dots (1)$$

$$\therefore x = \sin^{-1} \sqrt{\sin \theta}$$

$$\text{Since } \sin \theta = -\sin x \sinh y$$

$$\sin \theta = -\sqrt{\sin \theta} \sinh y \quad \text{from (1)}$$

$$\therefore -\sqrt{\sin \theta} = \sinh y$$

$$\therefore y = \sinh^{-1}(\sqrt{\sin \theta}) = \log(-\sqrt{\sin \theta} + \sqrt{\sin \theta + 1})$$

$$\therefore y = \log(\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})$$

$$\therefore \cos^{-1} e^{i\theta} = x + iy = \sin^{-1} \sqrt{\sin \theta} + i \log(\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})$$

4. Separate into real and imaginary parts $\sinh^{-1}(ix)$

Solution: Let $\sinh^{-1}(ix) = \alpha + i\beta$

$$\therefore ix = \sinh(\alpha + i\beta) = \sinh \alpha \cosh(i\beta) + \cosh \alpha \sinh(i\beta)$$

$$= \sinh \alpha \cos \beta + i \cosh \alpha \sin \beta$$

$$\text{Equating real and imaginary parts } \sinh \alpha \cos \beta = 0$$

$$\therefore \cos \beta = 0 \quad \therefore \beta = \frac{\pi}{2}(2k + 1) \quad \therefore \sin \beta = 1$$

$$\text{Also } \cosh \alpha \sin \beta = x$$

$$\therefore \cosh \alpha = x \quad \left[\because \sin \frac{\pi}{2} = 1 \right]$$

$$\therefore \alpha = \cosh^{-1} x$$

$$\therefore \sinh^{-1}(ix) = \alpha + i\beta = \cosh^{-1} x + i \frac{\pi}{2} (2k + 1)$$

5. If $\tan z = \frac{i}{2}(1 - i)$, prove that $z = \frac{1}{2} \tan^{-1} 2 + \frac{i}{4} \log \left(\frac{1}{5} \right)$

Solution: $\tan z = \frac{i}{2}(1 - i)$

$$\tan z = \frac{1}{2}(i - i^2) = \frac{1}{2}i + \frac{1}{2}$$

Let $z = x + iy$ $\therefore \tan(x + iy) = \frac{1}{2} + \frac{i}{2}$, $\tan(x - iy) = \frac{1}{2} - \frac{i}{2}$

$$\therefore \tan(2x) = [(x + iy) + (x - iy)]$$

$$= \frac{\tan(x+iy) + \tan(x-iy)}{1 - \tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right) + \left(\frac{i}{2}\right)\right] + \left[\left(\frac{1}{2}\right) - \left(\frac{i}{2}\right)\right]}{1 - \left[\left(\frac{1}{2}\right) + \left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right) - \left(\frac{i}{2}\right)\right]} = \frac{1}{1 - \left[\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\right]} = \frac{1}{1/2} = 2$$

$$\therefore 2x = \tan^{-1} 2 \quad \therefore x = \frac{1}{2} \tan^{-1} 2$$

Now, $\tan(2iy) = \tan[(x + iy) - (x - iy)]$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1 + \tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right) + \left(\frac{i}{2}\right)\right] - \left[\left(\frac{1}{2}\right) - \left(\frac{i}{2}\right)\right]}{1 + \left[\left(\frac{1}{2}\right) + \left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right) - \left(\frac{i}{2}\right)\right]} = \frac{i}{1 + \left[\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\right]} = \frac{i}{1 + (1/2)} = \frac{2}{3}i$$

$$\therefore i \tanh 2y = \frac{2}{3}i \quad \therefore \tanh 2y = \frac{2}{3}$$

$$\therefore 2y = \tanh^{-1} \left(\frac{2}{3} \right) = \frac{1}{2} \log \left[\frac{1 + (2/3)}{1 - (2/3)} \right] = \frac{1}{2} \log 5 \quad \therefore y = \frac{1}{4} \log 5$$

$$\therefore z = x + iy = \frac{1}{2} \tan^{-1} 2 + \frac{i}{4} \log 5$$

6. Show that $\tan^{-1} \left[i \left(\frac{x-a}{x+a} \right) \right] = \frac{i}{2} \log \frac{x}{a}$

Solution: Let $\tan^{-1} \left[i \left(\frac{x-a}{x+a} \right) \right] = \theta$

$$\therefore i \left(\frac{x-a}{x+a} \right) = \tan \theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

$$\therefore \frac{x-a}{x+a} = \frac{e^{-i\theta} - e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \quad [\because i^2 = -1]$$

By componendo and dividendo $\frac{(x-a)+(x+a)}{(x-a)-(x+a)} = \frac{(e^{-i\theta} - e^{i\theta}) + (e^{i\theta} + e^{-i\theta})}{(e^{-i\theta} - e^{i\theta}) - (e^{i\theta} + e^{-i\theta})}$

$$\therefore \frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}} = e^{-2i\theta} \quad \therefore \frac{x}{a} = e^{-2i\theta} \quad \therefore -2i\theta = \log \frac{x}{a}$$

Multiply by i throughout, $2\theta = i \log \frac{x}{a} \quad \therefore \theta = \frac{i}{2} \log \left(\frac{x}{a} \right)$

$$\tan^{-1} \left[i \left(\frac{x-a}{x+a} \right) \right] = \frac{i}{2} \log \frac{x}{a}$$

(All these problems can be solved by alternate method also, do look for other methods)