Module :3
Matrices
Eigen Values & Eigen Vectors

Eigen Values & Eigen Vectors

- **Formal Definition:** Let A be an $n \times n$ matrix.
  - An ***eigenvector*** of A is a *nonzero* vector x in R^n such that $Ax = \lambda x$, for some scalar λ.
  - An ***eigenvalue*** of A is a scalar λ such that the equation $Ax = \lambda x$ has a *nontrivial* solution.
- **Definition 2:** Roots of Characteristic equation of a square matrix is called the characteristics roots / latent roots / characteristic values/ eigen values / proper values of the matrix.ie. Eigenvalues of matrix are roots of $|A - \lambda I| = 0$

If λ_1 is one of the eigenvalues of square matrix A then eigenvector (X) corresponding to λ_1 is given by $[A - \lambda_1 I]\, X = 0$.

Note

- Only square matrix possess eigenvalues.
- A square matrix of order n will have at the most n eigenvalues.
- Matrix of order n will have exactly n numbers of eigenvalues (may be distinct or repeated)
- Martix may have complex eigenvalues.

Question:

- What is the Geometrical Interpretation of eigenvalues and eigenvectors? (find out)

Calculate Eigenvalues & Eigenvectors

- **Steps to be followed:** A-square matrix of order n, I identity matrix of order n, λ any scalar (eigenvalue to be determined), X column vector of order n (eigenvector to be determined)
  - **Find Characteristic Matrix:** $A - \lambda I$
  - **Find Characteristic equation:** $|A - \lambda I| = 0$
  - **Solve Characteristic equation and find its roots.**

  (roots are called eigenvalues)
  - **For each eigenvalue,** solve $[\, A - \lambda I \,]\, X = 0$ to determine nonzero column vector X.

Find Eigenvalues and Eigenvector

Ex.1 Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial A-

Ch. eq $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow 3 - 4\lambda + \lambda^2 = 0$ **Find roots of Ch. eq.**

It has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

Eigen vector for λ=1

By Defination

Eigenvectors $\mathbf{x}$ of this transformation satisfy the equation,

$$A\mathbf{x} = \lambda \mathbf{x}$$

Rearrange this equation to obtain-

$$(A - \lambda I)\mathbf{x} = 0$$

For $\lambda = 1$, Equation becomes,

$$(A - I)\mathbf{x} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution, $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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n = no. of variable
 = 2
r = no. of distinct Equations/ Rank of echelon form
= 1
∴ n-r = 1 LI eigenvector

Method 1: Reduce to echelon form

R_2-R_1

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 = 0$$

$$\therefore x_1 = -x_2$$

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Put x_2= t $\therefore x_1 = -t$

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Put x_2= t $\therefore x_1 = -t$

Eigenvector X= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$

i.e. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigen vector for λ=1

Eigenvectors $\mathbf{x}$ of this transformation satisfy the equation,

$$A\mathbf{x} = \lambda \mathbf{x}$$

Rearrange this equation to obtain-

$$(A - \lambda I)\mathbf{x} = 0$$

For $\lambda = 1$, Equation becomes,

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$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Method 1: Reduce to echelon form

R_2-R_1

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Put x_2**= t** $\therefore x_1 = -t$

Eigenvector X= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$

Or $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Find for Eigen vector for λ=3

For $\lambda = 3$, Equation becomes,

$$(A - 3I)u = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution- $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find for Eigen vector for λ=3

For $\lambda = 3$, Equation becomes,

$$(A-3I)u=0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

n = no. of variable
= 2
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Method 2: Use of Algebraic Methods

Rewriting matrix form to algebraic form,

$$\therefore -u_1 + u_2 = 0$$
$$u_1 - u_2 = 0$$

Both equations are same.

$$\therefore u_1 = u_2$$

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Method 2: Use of Algebraic Methods

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Put u_2= t $\therefore u_1 = t$

Eigenvector u = $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$

Find for Eigen vector for λ=3

For $\lambda = 3$, Equation becomes,

$$(A-3I)u=0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution- $u=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Eigenvalues of A are 1 and 3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

Ex.2 Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

- $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\therefore (2-\lambda)[(1-\lambda)(-1-\lambda) - 3] + 2\,[1(-1-\lambda)\text{-}1]$

$+3[3\text{-}(1-\lambda)] = 0$

$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$\therefore \lambda = 1, 3, -2$ are the eigenvalues of A.

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

  Where S_1=Trace A

  S_2= Sum of Minors of Diagonal Elements

  $|A|$ = Determinant of A

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

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- For $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

  Where S_1=Trace A

  S_2= Sum of Minors of Diagonal Elements

  $|A|$ = Determinant of A

- For $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

  S_1=2

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

  Where S_1=Trace A

  S_2= Sum of Minors of Diagonal Elements

  $|A|$ = Determinant of A

- For $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

S_1=2

Minor of a_{11}=Minor of 2 = remove the raw and column in which it lies and find determinant

$=\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

  Where S_1=Trace A

  S_2= Sum of Minors of Diagonal Elements

  $|A|$ = Determinant of A

- For $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

  S_1=2

  S_2= $\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

  Where S_1=Trace A

  S_2= Sum of Minors of Diagonal Elements

  $|A|$ = Determinant of A

- For $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

  S_1=2

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  $= -5$

  $|A| = -6$

Other Method to find ch. eq. for 3*3 matrix

- $-\lambda^3 + S_1\lambda^2 - S_2\lambda + |A| = 0$

  Where S_1=Trace A

  S_2= Sum of Minors of Diagonal Elements

  $|A|$ = Determinant of A

- For $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

S_1=2

S_2= $\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$

$= -5$

$|A| = -6$

$\therefore -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$

$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

n = no. of variable
= 3
r = no. of distinct Equations/ Rank of echelon form
= 2
∴ n-r = 1 LI eigenvector

- $\lambda = 1,3,-2$ are the eigenvalues of A.

For Each Eigenvalue, find eigenvector using

$$[A - \lambda I]X = 0$$

For $\lambda = 1$

$$[A - I]X = 0$$

$$\therefore \begin{bmatrix} 2-1 & -2 & 3 \\ 1 & 1-1 & 1 \\ 1 & 3 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Method 3: Algebraic equation

Rewriting in Equation form:

$x_1 - 2x_2 + 3x_3 = 0$ $\quad \Rightarrow -2x_2 + 2x_3 = 0 \Rightarrow x_2 = x_3$

$x_1 + x_3 = 0$ $\quad \therefore$ Eigen vector corresponding to $\lambda = 1$

$x_1 + 3x_2 - 2x_3 = 0$ $\quad$ is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = -2$

$$[A - (-2)I]X = 0$$

$$\therefore \begin{bmatrix} 2-(-2) & -2 & 3 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Method 3: Algebraic equation (Crammer's Rule)

Rewriting in Equation form:

$$4x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$\therefore$ Eigen vector corresponding to$\lambda = 1$

is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}$

2 distinct equations available.
$\therefore$ Crammer's Rule applicable.

$$\frac{x_1}{\begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -2 \\ 1 & 3 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-11} = \frac{-x_2}{1} = \frac{x_3}{14}$$

- For $\lambda = 3$

$$[A - 3I]X = 0$$

$$\therefore \begin{bmatrix} 2-3 & -2 & 3 \\ 1 & 1-3 & 1 \\ 1 & 3 & -1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Use any method and find out eigenvector corresponding to $\lambda = 3$

Relation : $x_1 = x_2 = x_3$

$\therefore$ Eigen vector corresponding to $\lambda = 3$ is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

EXAMPLE

- Find the Eigen values and Eigen vectors of the matrix
- $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$
- **Solution:** The characteristic equation is $|A - \lambda I| = 0$
- $\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

$\therefore (8-\lambda)[(3+\lambda)(\lambda-1)-8] + 8[4-4\lambda+6] - 2[-16+9+3\lambda] = 0$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\therefore (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$

$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = 1, 2, 3$

EXAMPLE

(i) For $\lambda = 1, [A - \lambda_1 I]X = 0$ gives

- $$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- We may obtain the Eigen vector by solving simultaneous equations obtained from above matrix equation
- $\therefore$ From the first two rows, we get,
- $7x_1 - 8x_2 - 2x_3 = 0, 4x_1 - 4x_2 - 2x_3 = 0$
- Solving by Crammer's rule, $\dfrac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$
- $\therefore \dfrac{x_1}{8} = \dfrac{x_2}{6} = \dfrac{x_3}{4}$
- $\therefore \dfrac{x_1}{4} = \dfrac{x_2}{3} = \dfrac{x_3}{2}$
- Hence, corresponding to the Eigen value 1. We get the following Eigen vector
- $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ or $[4, \quad 3, \quad 2]'$

EXAMPLE

(ii) For $\lambda = 2, [A - \lambda_2 I]X = 0$ gives

- $$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- From the first two rows, we get,
- $6x_1 - 8x_2 - 2x_3 = 0, 4x_1 - 5x_2 - 2x_3 = 0$
- Solving by Crammer's rule, $\dfrac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$
- $\therefore \dfrac{x_1}{6} = \dfrac{x_2}{4} = \dfrac{x_3}{2}$ $\quad\quad$ $\therefore \dfrac{x_1}{3} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$
- Hence, corresponding to the Eigen value 2, we get the following
- Eigen vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ or $[3, \quad 2, \quad 1]'$

EXAMPLE

(iii) For $\lambda = 3, [A - \lambda_3 I]X = 0$ gives

- $\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- From the first two rows, we get,
- $5x_1 - 8x_2 - 2x_3 = 0, 4x_1 - 6x_2 - 2x_3 = 0$
- Solving by Crammer's rule, $\dfrac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$
- $\therefore \dfrac{x_1}{4} = \dfrac{x_2}{2} = \dfrac{x_3}{2}$
- $\therefore \dfrac{x_1}{2} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$
- Hence, corresponding to the Eigen value 3, we get the following
- Eigen vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ or $[2, \quad 1, \quad 1]'$

Practice Example

- $Eigen\ values\ of\ symmetric\ matrix\ are\ distinct$ and their corresponding eigenvectors are orthogonal.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Steps: 1. Find eigen values. (ans. 1,2,4)

2. Find corresponding eigenvectors (Say X_1, X_2, X_3).

Ans. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

3. Prove that $X_1, X_2, X_3$are orthogonal.

i.e. $X_1'X_2 = 0$, $X_3'X_2 = 0$ and $X_1'X_3 = 0$

Practice Example

- Find eigenvalues and eigenvectors of the matrix. Prove that eigenvectors are linearly independent.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Steps: 1. Find eigenvalues. (ans. 1,2,3)

2. Find corresponding eigenvectors (Say X_1, X_2, X_3).

Ans. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Prove that X_1, X_2, X_3 are LI.

i.e. $K_1X_1 + K_2X_2 + K_3X_3 = 0 \Rightarrow K_1 = K_2 = K_3 = 0$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

- $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

$\therefore \lambda = 1,1,7$ are the eigenvalues of A.

❖ For $\lambda = 1$

$$[A-(1)I]X = 0$$

$$\therefore \begin{bmatrix} 2-(1) & 1 & 1 \\ 2 & 3-(1) & 2 \\ 3 & 3 & 4-(1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

R2-2R1, R3-3R1$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore x_1 + x_2 + x_3 = 0$$

$$\therefore x_1 = -x_2 - x_3$$

n = no. of variable
= 3
r = no. of distinct Equations/ Rank of echelon form
= 1
∴ n-r = 2 LI eigenvector

Put $x_2 = s;\ x_3 = t \Rightarrow \therefore x_1 = -s$ −t

∴ Eigen vector corresponding to$\lambda = 1$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 7$

$$[A-(7)I]X = 0$$

$$\therefore \begin{bmatrix} 2-(7) & 1 & 1 \\ 2 & 3-(7) & 2 \\ 3 & 3 & 4-(7) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

n = no. of variable
= 3
r = no. of distinct Equations/ Rank of echelon form
= 2
∴ n-r = 1 LI eigenvector

(½)R2, (1/3)R3⇒ $\begin{bmatrix} -5 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

R3-R2⇒ $\begin{bmatrix} -5 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore x_3 = \frac{3}{2}x_2 \;\; \& \; x_1 = \frac{1}{2}x_2$$

∴ Eigen vector corresponding to$\lambda = 7$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Practice Example

- Find eigenvalues and eigenvectors of

$$A=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

Prove that corresponding eigenvectors are Linearly Independent.

Ans: 1,1,-1 $\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix}$

Ex.4 Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

- $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -3-\lambda & -9 & -12 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - \lambda = 0$$

$\therefore \lambda = 0,0,1$ are the eigenvalues of A.

For $\lambda = 0$

$[A - 0I]X = 0$

$$\therefore \begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_3 = 0$

$x_1 + 3x_2 + 4x_3 = 0$

$\therefore x_1 + 3x_2 = 0 \Rightarrow x_1 = -3x_2$

n = no. of variable
= 3
r = no. of distinct Equations/ Rank of echelon form
= 2
$\therefore$ n-r = 1 LI eigenvector

$\therefore$ Eigen vector corresponding to λ =0 is $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 1$

Check: Eigen vector corresponding to λ =1 is $\begin{bmatrix} -12 \\ 4 \\ 1 \end{bmatrix}$

Practice Example

- Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

Prove that corresponding eigenvectors are Linearly Independent.

Ans: 1,2,2 $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

- $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 3\lambda^2 + 3\lambda + 1 = 0$$

$\therefore \lambda = 1,1,1$ are the eigenvalues of A.

For $\lambda = 1$ $[A - I]X = 0$

$$\therefore \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R3+R1 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R3-2R2 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_2 = x_3$ &

$\therefore x_1 = x_2$

n = no. of variable
= 3
r = no. of distinct Equations/ Rank of echelon form
= 2
∴ n-r = 1 LI eigenvector

$\therefore$ Eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} -12 \\ 4 \\ 1 \end{bmatrix}$

Practice Example

- Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Note:

- Eigenvalues of the diagonal matrix are its diagonal elements.
- Eigenvalues of the triangular matrix are its diagonal elements.

Ans. ???

Remember

- Eigenvectors are *non-zero column vectors*. Eigenvectors are Linearly independent. (check for the above example)
- Eigenvalues may be equal to zero.
- Eigenvalues are for the given matrix unique but not eigen vectors.
- If X is eigenvector of A corresponding to some eigenvalue λ, then any non-zero multiple of X is also an eigenvector for same λ.
- The Sum of Eigenvalues = Trace of Matrix
- Product of the eigenvalues= determinant of matrix

Ex.6 Find sum and product of eigenvalues of

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

- Sum of Eigen values = Trace of A

  = sum of diagonal elements

  = 18

- Product of Eigen values = determinant of A

  $=|A|$

  =8 (21-16) +6(-18+8) +2(24-14)

  =0

Ex.6 Two eigenvalues of a 3*3 matrix are -1,2 and if determinant of a matrix is 4, find its third eigenvalue.

- Let the third eigenvalue is x.

Product of Eigen values = determinant of A

$$(-1)(2)(x)=4$$

$$x=-2$$

❖ **Ex.** If A = $\begin{bmatrix} sinx & cosecx & 1 \\ secx & cosx & 1 \\ tanx & cotx & 1 \end{bmatrix}$ then there does not exists a rela value of x for which characteristic roots are -1,1&3

Open for discussion

Note:

- If a matrix A is singular then one of the eigenvalue of A must be zero.
- Eigenvalues of a triangular matrix are its diagonal elements.
- Eigenvalues of diagonal matrix are its diagonal elements.