MATRIX THEORY: RANK OF MATRIX
-
SYSTEM OF LINEAR EQUATIONS

FY BTECH SEM-I

MODULE-2

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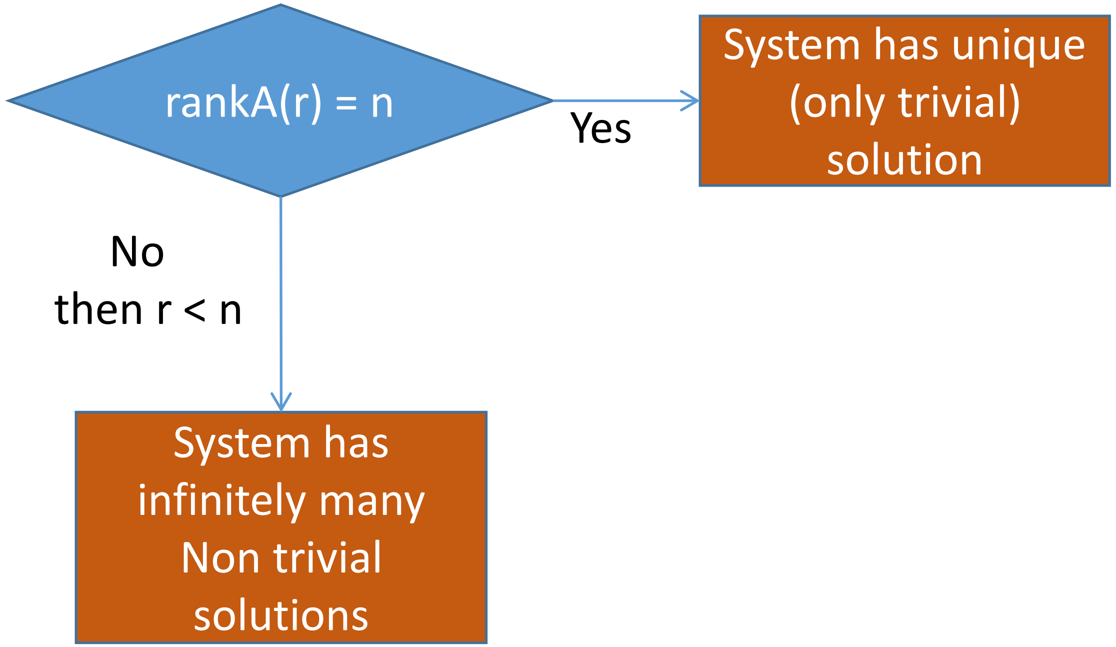
HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- **Solutions of $AX = 0$**
- The null column matrix is obviously a solution of $AX = 0$. The solution $X = 0$ is called the **trivial solution** or the **zero – solution**.
- **Note:** 1. These equations do not have constant term or intercept. Hence geometrically all these equations are passing through origin.
- 2. Since, they have at least one point of intersection. We will not have case of **No solution** here.
- If we could find a non-zero solution $X \neq 0$, then it is called **non – trivial solution.**
- If we have a Homogeneous system of m equations in n unknowns. Then, the matrix A will be $m \times n$.
- Let r be the rank of the matrix A.
- **Case I :** If $r = n$ i.e., rank (A) is equal to the number of unknowns then this is case of unique solution.
- $x_1 = x_2 \dots = x_n = 0$ i.e., only possible solution is **zero – solution** or **trivial solution**.
- **Case II:** If $r < n$ i.e., rank (A) is less than the number of unknowns then the system has Infinitely many
- **non – trivial solutions**. The no of independent solutions i.e parameters is equal to $n - r$

RULE TO SOLVE HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- **1.** Write the given system in the matrix form $AX = 0$.
- **2.** Apply **row transformations only** and reduce the coefficient matrix A to **row echelon form**.
- **3.** We know that the rank of a matrix in echelon form is equal to the number of non-zero rows.
- Determine the rank of A = r and find the solution by using following cases.



Example 1

- Find the solution of the system given by
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$
- **Solution:** The system can be written as $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying elementary row transformations on A we will obtain the Echelon form.
- Applying $R_2 - R_1$ and $R_3 - 2R_1$, we have $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- i.e. the coefficient matrix is non – singular. Hence $\rho(A) = 3$, r = n Hence unique solution
- Therefore there exists a trivial solution $x_1 = x_2 = x_3 = 0$

Example 2

- Solve the following system of linear equations
  $x_1 + 2x_2 + 4x_3 + x_4 = 0$
  $2x_1 + x_2 + 5x_3 + 8x_4 = 0$
  $x_1 + 4x_2 + 6x_3 - 3x_4 = 0$
- **Solution:** The system can be written as

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 5 & 8 \\ 1 & 4 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_2 - 2R_1$ and $R_3 - R_1$, we have

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 + \frac{2}{3} R_2$, we have

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Hence $\rho(A) = 2 < 4$ (Number of unknowns).
- So the system has infinitely non – trivial solution.
- $(n - r) = (4 - 2) = 2$ free variables
- Then reduced from of system of equations is
- $x_1 + 2x_2 + 4x_3 + x_4 = 0$ ……………..(i)
  $- 3x_2 - 3x_3 + 6x_4 = 0$ …………….(ii)
- (ii) $\Rightarrow x_2 + x_3 - 2x_4 = 0 \quad \Rightarrow x_2 = -x_3 + 2x_4$
- Substituting in (i), we get $x_1 + 2(-x_3 + 2x_4) + 4x_3 + x_4 = 0 \quad \Rightarrow x_1 = -2x_3 - 5x_4$

Example 2 (contd…)

- Let $x_3 = k_1$ and $x_4 = k_2$, where k_1 and k_2 are some parameters.
- $\therefore$ We have $x_1 = -2k_1 - 5k_2 \ and \ x_2 = -k_1 + 2k_2$
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2k_1 - 5k_2 \\ -k_1 + 2k_2 \\ k_1 \\ k_2 \end{bmatrix}$ has infinite solutions as $k_1 \ and \ k_2$ vary.

Example 3

- Solve
$$\begin{aligned}3x_1 + 4x_2 - x_3 - 9x_4 &= 0\\ 2x_1 + 3x_2 + 2x_3 - 3x_4 &= 0\\ 2x_1 + x_2 - 14x_3 - 12x_4 &= 0\\ x_1 + 3x_2 + 13x_3 + 3x_4 &= 0\end{aligned}$$

- **Solution:** The system can be written as $AX = 0$

i.e., $\begin{bmatrix}3 & 4 & -1 & -9\\ 2 & 3 & 2 & -3\\ 2 & 1 & -14 & -12\\ 1 & 3 & 13 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

- Applying R_{14}, we get

$$\begin{bmatrix}1 & 3 & 13 & 3\\ 2 & 3 & 2 & -3\\ 2 & 1 & -14 & -12\\ 3 & 4 & -1 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

- Applying $R_2 - 2R_1, R_3 - 2R_1, R_4 - 3R_1$, we get

$$\begin{bmatrix}1 & 3 & 13 & 3\\ 0 & -3 & -24 & -9\\ 0 & -5 & -40 & -18\\ 0 & -5 & -40 & -18\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

- Applying $R_4 - R_3$, we get

$$\begin{bmatrix}1 & 3 & 13 & 3\\ 0 & -3 & -24 & -9\\ 0 & -5 & -40 & -18\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

- Applying $\frac{R_2}{-3}$, we get

$$\begin{bmatrix}1 & 3 & 13 & 3\\ 0 & 1 & 8 & 3\\ 0 & -5 & -40 & -18\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Example 3 (contd...)

- Applying $R_3 + 5R_2$, we get
$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
- Hence $\rho(A) = 3 < 4$ (number of variables).
- $\therefore$ the system has infinite non trivial solutions.
- The reduced form of system of equations can be written as
- $x_1 + 3x_2 + 13x_3 + 3x_4 = 0$ ....(i)
$x_2 + 8x_3 + 3x_4 = 0$ ....(ii)
$-3x_4 = 0$ ....(iii)
- Now, since $n = 4, r = 3$,
- $(n - r) = (4 - 3) = 1$ free variable
- (iii) $\Rightarrow x_4 = 0, \quad Let \ \ x_3 = k$ (arbitrary)
- $\therefore$ (ii) $\Rightarrow x_2 + 8k + 0 = 0 \ \Rightarrow \ x_2 = -8k$
- And (i) $\Rightarrow x_1 - 24k + 13k + 0 = 0 \Rightarrow x_1 = 11k$
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11k \\ -8k \\ k \\ 0 \end{bmatrix}$ will have infinite many solutions as k varies.

Example 4

- Determine the values of λ for which the following system of equations possess a non – trivial solution and obtain these solutions for each value of λ.

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

- **Solution:** The system can be written as $\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- System will possess non trivial solution if rank of coefficient matrix is less than number of variables
- i.e., $r < 3$ if $|A| = 0$
- $\therefore \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$
- $\therefore 3(-2\lambda + 12) - (4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$
- $\therefore \lambda^2 + 8\lambda - 9 = 0$ $\quad\quad \therefore (\lambda + 9)(\lambda - 1) = 0$
- $\therefore \lambda = -9 \;\; and \; \lambda = 1$ for which the system possesses a non – trivial solution.

Example 4 (contd…)

- **Case (i) For $\boldsymbol{\lambda = -9}$**
- the system can be written as
$$\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- Applying $R_2 - \frac{4}{3}R_1, R_3 + 6R_1$, we have
$$\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 10 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- Applying $R_3 + 3R_2,$ we have
$$\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- $\therefore$ The reduced form of system of equations is
- $3x_1 + x_2 + 9x_3 = 0$ and $-\left(\frac{10}{3}\right)x_2 - 15x_3 = 0$
- $\therefore x_2 = -(9/2)x_3$
- $\Rightarrow 3x_1 = (9/2)x_3 - 9x_3 = -(9/2)x_3$
- $\therefore x_1 = -(3/2)x_3$
- Let $x_3 = a$ (arbitrary)
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(3/2)a \\ -(9/2)a \\ a \end{bmatrix}$ has infinite solutions as 'a' varies

Example 4 (contd...)

- **Case (i) For $\lambda = 1$**
- System can be written as $\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_1 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 4R_1, R_3 - 2R_1$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- $\therefore$ The reduced form of system of equations is
- $x_1 - 3x_2 - 2x_3 = 0\ , \quad 10x_2 + 5x_3 = 0,$
- $\therefore x_2 = -(1/2)x_3$
- $\therefore x_1 - 3x_2 - 2x_3 = 0$
- $\Rightarrow x_1 = -(3/2)x_3 + 2x_3 = (1/2)x_3$
- $\therefore x_1 = (1/2)x_3$
- Let $x_3 = b$ (arbitrary)
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/2)b \\ -(1/2)b \\ b \end{bmatrix}$ has infinite solutions as ‘b’ varies.

Example 5

- Discuss for all values of k, the following system of equations possesses trivial and non – trivial solutions

- $$2x + 3ky + (3k+4)z = 0$$
$$x + (k+4)y + (4k+2)z = 0$$
$$x + 2(k+1)y + (3k+4)z = 0$$

- **Solution:** The given system of equations can be written as $AX = O$ i.e.,
$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying R_{12}, we have
$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_2 - 2R_1, R_3 - R_1$, we have

- $$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{........(1)}$$

- If the given system of equations is to posses non – trivial Solutions then the coefficient matrix A must be of rank less than 3. i.e., $|A|$ must be zero.

- i.e $(k-8)(-k+2) + 5k(k-2) = 0$

- i.e $-k^2 + 10k - 16 + 5k^2 - 10k = 0$

- i.e $4k^2 - 16 = 0$ i.e $k^2 = 4$ i.e $k = \pm 2$

- Now three cases arise:

- **Case (i):** If $k \neq \pm 2$,

- then given system of equations possesses only trivial solution

- i.e. $x = y = z = 0$ is the only solution.

Example 5 (contd...)

- **Case (ii):** If $k = 2,$
- then (1) $\Rightarrow \begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- The coefficient matrix is of rank 2, the given system of equations now possesses non – trivial solutions.
- The reduced form of the equations is
- $x + 6y + 10z = 0\,,\ 6y + 10z = 0$
- Let $z = c$ (arbitrary)
- Then we have $y = \frac{-5}{3} c \ and\ x = 0$
- Hence the general solution of the given system can be written as $x = 0, y = \frac{-5}{3} c, z = c$
- **Or** $x = 0, y = -5c', z = 3c'\,,\ c' = \frac{c}{3}$ is parameter

- **Case (iii):** If $k = -2$
- then (1) $\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $\frac{R_2}{10}\ and\ \frac{R_3}{4}$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_3 - R_2,$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- The coefficient matrix is of rank 2, the given system of equations now possesses non – trivial solutions.
- The reduced form of the equations is
- $x + 2y - 6z = 0\,,\ \ -y + z = 0$
- $\Rightarrow y = z\ \ and\ x = 4z,$ Let $z = b,$
- then $x = 4b, y = b, z = b$ is the general soln

VECTORS

- **Definition:** An ordered set of n elements x_i is called n – dimensional vector or a vector of order n denoted by X.
- $X = [x_1 \ \ x_2 \ \ x_3 \ldots\ldots x_n]$ The elements $x_1, x_2, x_3, \ldots\ldots, x_n$ are called components of X.
- X is denoted by row matrix or column matrix.
- It is more convenient to denote it as column matrix
- $X = [x_1 \ \ x_2 \ \ x_3 \ldots \ldots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}$
- The vector, all of whose components are zero, is called a zero or null vector and is denoted by 0.

VECTORS

- LINEARLY DEPENDENT VECTORS
- **Definition:** The set of vectors $X_1,\ X_2, X_3, \dots . X_m$ is said to be **Linearly Dependent** if there exist m scalars $k_1, k_2, k_3 \dots . k_m$, not all zero, such that $k_1X_1 + k_2X_2 + k_3X_3 + \cdots . + k_mX_m = 0$
- If $k_1 \neq 0$, then
- $-k_1X_1 = k_2X_2 + k_3X_3 + \cdots . + k_mX_m$
- $\therefore X_1 = \mu_2X_2 + \mu_3X_3 + \cdots .. + \mu_mX_m$
- where $\mu_i = -\frac{k_i}{k_1}; \quad i = 2,3,4, \dots .., m$
- X_1 is expressed as linear combination of $X_2, \dots . X_m$
- **Note:** if the set of vectors $X_1, X_2, \dots \dots X_m$ is linearly dependent then any one vectors can be expressed as the linear combination of other vectors.

LINEARLY INDEPENDENT VECTORS

- **Definition:** The set of vectors $X_1, X_2, X_3, \dots . X_m$ is said to be **Linearly Independent** if they are not dependent.
- i.e., if $k_1X_1 + k_2X_2 + k_3X_3 + \cdots .. + k_mX_m = O$
- $\Rightarrow k_i = 0$ for all $i = 1,2, \dots .., m$
- then $X_1, X_2, X_3 \dots \dots X_m$ are said to be Linearly Independent.

VECTORS

- For any given set of vectors $X_1, X_2, \ldots \ldots X_m,$
- $k_1X_1 + k_2X_2 + k_3X_3 + \cdots .. + k_mX_m = O$ will form a homogeneous system of equation, say AX = 0. Here unknowns are $k_1, k_2, \ldots . k_m$ and coefficient matrix is made up of vectors $X_1, X_2, \ldots \ldots X_m$ as columns.
- Apply **row transformations only** on A and reduce the coefficient matrix A to **row echelon form**. Then find the rank A (r)
- **Case(i):** when rank A (r) is equal to number of variables then system has only trivial solution and vectors are independent.
- **Case(ii):** when rank A (r) is less than number of variables then system has infinite non-trivial solutions and they can be obtained by assigning $n - r$ variables as parameter and vectors are dependent.

Example 1

- Are the vector $X_1 = [1\ \ 3\ \ 4\ \ 2]$, $X_2 = [3\ \ -5\ \ 2\ \ 6], X_3 = [2\ \ -1\ \ 3\ \ 4]$ linearly dependent? If so, express X_1 as a linear combination of the others.
- **Solution:** Consider the matrix equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$ .... (i)
- $k_1[1\ 3\ 4\ 2] + k_2[3\ \ -5\ 2\ 6] + k_3[2\ \ -1\ 3\ 4] = [0\ 0\ 0\ 0]$
- $\therefore k_1 + 3k_2 + 2k_3 = 0, 3k_1 - 5k_2 - k_3 = 0,$
- $4k_1 + 2k_2 + 3k_3 = 0, 2k_1 + 6k_2 + 4k_3 = 0$
- which can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 6 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_2 - 3R_1, R_3 - 4R_1, R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $(-1/7)R_2, (-1/5)R_3$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 1 (contd..)

- $k_1 + 3k_2 + 2k_3 = 0, \quad 2k_2 + k_3 = 0$
- if we put $k_3 = -2t$, we get $k_2 = t, k_1 = t$
- Now, from (i), we get, $tX_1 + tX_2 - 2tX_3 = 0$
  $$\therefore X_1 + X_2 - 2X_3 = 0$$
- Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent and $X_1 = -X_2 + 2X_3$

Example 2

- Examine whether the vectors

$X_1 = [3 \quad 1 \quad 1], X_2 = [2 \quad 0 \quad -1], X_3 = [4 \quad 2 \quad 1]$ are linearly dependent or independent.

- **Solution:** Consider the matrix equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$................(i)
- $\therefore 3k_1 + 2k_2 + 4k_3 = 0$, $\quad k_1 + 0k_2 + 2k_3 = 0,$ $k_1 - k_2 + k_3 = 0$
- This is a homogeneous system of equations

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- By $R_{13,}$ *we get*

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- By $R_2 - R_1,\ R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- By $R_3 - 5R_2$, we get $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- $\Rightarrow k_1 - k_2 + k_3 = 0$, $\quad k_2 + k_3 = 0,\ \ -4k_3 = 0$
- $\Rightarrow k_3 = 0$ and hence $k_2 = 0$ and $k_1 = 0$
- Thus non zero values of k_1, k_2, k_3 do not exist which can satisfy equation (i).
- Hence by definition the given system of vectors is linearly independent.

Example 3

- Show that the vectors X_1, X_2, X_3 are linearly independent and vector X_4 depends upon them, where, $X_1 = [1\ 2\ 4\ ]$, $X_2 = [2 - 1\ 3]$, $X_3 = [0\ 1\ \ 2]$, $X_4 = [-3\ 7\ 2]$
- **Solution:** consider $k_1X_1 + k_2X_2 + k_3X_3 = 0$
- $\therefore k_1[1\ \ 2\ \ 4] + k_2[2\ \ -1\ \ 3] + k_3[0\ \ 1\ \ 2] = [0\ \ 0\ \ 0]$
- $\therefore k_1 + 2k_3 + 0k_3 = 0,\quad 2k_1 - k_2 + k_3 = 0,$ $4k_1 + 3k_2 + 2k_3 = 0,$
- $\therefore \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 2R_1, R_3 - 4R_1$, we get

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 - R_2$ we get
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- $\therefore k_1 + 2k_2 = 0,\quad -5k_2 + k_3 = 0,\quad k_3 = 0$
- $\therefore k_3 = 0,\ \ k_2 = 0,\ \ k_1 = 0.$
- Since, the rank = 3 = the number of unknowns, $\therefore$ only trivial solution is possible.
- $\therefore X_1, X_2, X_3$ are linearly independent.

Example 3 (contd…)

- Now, consider the matrix equation
- $k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = 0$ …………..(i)
- $\therefore k_1[1\ 2\ 4] + k_2[2\ \ -1\ 3] + k_3[0\ 1\ 2] + k_4[-3\ \ 7\ \ 2] = 0$
- $\therefore k_1 + 2k_2 + 0k_3 - 3k_4 = 0, \quad 2k_1 - k_2 + k_3 + 7k_4 = 0, \quad 4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$
- $\therefore \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 2R_1, R_3 - 4R_1$, we get

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 - R_2$ we get

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\therefore k_1 + 2k_2 - 3k_4 = 0, \quad -5k_2 + k_3 + 13k_4 = 0$, $k_3 + k_4 = 0$ Let $k_4 = t \quad \therefore k_3 = -t$
- $\therefore -5k_2 - t + 13t = 0 \qquad \therefore k_2 = \frac{12}{5}t$
- $\therefore k_1 + \frac{24}{5}t - 3t = 0 \qquad \therefore k_1 = -\frac{9}{5}t$
- Putting the values of k_1, k_2, k_3, k_4 in (i) we get, $-\frac{9}{5}tX_1 + \frac{12}{5}tX_2 - tX_3 + tX_4 = 0$
- $\therefore 9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$
- Hence, X_1, X_2, X_3, X_4 are linearly dependent and $X_4 = \frac{9}{5}X_1 - \frac{12}{5}X_2 + X_3$