# Module :3
# Matrices
# Similarities of Matrices

# Similarities of Matrices

- **Definition :** If A and B are two square matrices of order n then B is said to be similar to A if there exists a non-singular matrix M much that

$$B = M^{-1}AM$$

- **Definition :** A square matrix A is said to be **diagonalisable** if it is similar to a diagonal matrix.

Combining the two definitions we see that A is diagonalisable if there exists a matrix M such that $M^{-1}AM = D$ where D is a diagonal matrix. In this case M is said to **diagonalise A** or **transform A** to diagonal form.

# Theorem



## If A is similar to B and B is similar to C, then A is similar to C.

**Proof :** Since A is similar to B, $\mathrm{A} = \mathrm{P}^{-1}\ \mathrm{BP}$ and since B is similar to C, B = $\mathrm{Q}^{-1}$ CQ.

$$\therefore \quad \mathrm{A} = P^{-1}\,(\mathrm{B})P$$

$$= P^{-1}\ (Q^{-1}\mathrm{CQ})\ \mathrm{P}$$

$$= (P^{-1}Q^{-1})C(QP)$$

$$= (QP)^{-1}C(QP)$$

A is similar to C.

# Theorem



## If A and B are similar matrices then $|A| = |B|$

**Proof :** Since A is similar to B, A = $P^{-1}$BP

$\therefore \det A = \det(P^{-1}BP)$

$= \det P^{-1} \cdot \det B \cdot \det P$

$= \det P^{-1} \cdot \det P \cdot \det B$

$= \det(P^{-1}P) \det B$

$= \det I \cdot \det B = \det B.$

# Theorem



**If A and B are two similar matrices then they have the same eigen values.**

**Proof :** Since A and B are similar matrices, there exists a non-singular matrix M such that $B = M^{-1}AM$

$$\therefore \quad B - \lambda I = M^{-1}AM - \lambda I$$

Because $M^{-1}(\lambda I)M = \lambda M^{-1}M = \lambda I$

$$\begin{aligned} \therefore \quad B - \lambda I &= M^{-1}AM - M^{-1}(\lambda I)M \\ &= M^{-1}(A - \lambda I)M \end{aligned}$$

$$\begin{aligned} \therefore \quad \det(B - \lambda I) &= det\ M^{-1} \cdot \det(A - \lambda I) \cdot \det M \\ &= detM^{-1} \cdot detM \cdot \det(A - \lambda I) \\ &= \det(M^{-1}M) \cdot \det(A - \lambda I) \\ &= 1 \cdot \det(A - \lambda I) \end{aligned}$$

Thus, the matrices A, B have the same characteristics polynomial and hence, they have same eigen values.

# Definitions

- **Algebraic Multiplicity (AM) of an eigen value.**

If $\lambda_1$ is an eigen value of the characteristic equation $|A - \lambda I| = 0$ repeated t times then t is called the algebraic multiplicity of $\lambda_1$.

- **Geometric Multiplicity(GM) of an eigen value.**

If s is the number of linearly independent eigen vectors corresponding to the eigen value $\lambda_1$ then s is called the geometric multiplicity of $\lambda_1$. So, the numbers of linearly independent solutions of $(A - \lambda_1 I)X = 0$ is s and the rank of the matrix $A - \lambda_1 I$ will be $n - s$.

- **A Matrix is diagonalizable if AM=GM for all eigenvalues**

# Example 1



**Show that the matrix A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable.**

**Find the transforming matrix and the diagonal matrix.**

**Sol.:** The characteristic equation is $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$=0

$\therefore\ (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+8]$
$+2[24-2(7-\lambda)]=0$

$\therefore\ \lambda^3-18\lambda^2+45\lambda=0$

$\therefore\ \lambda(\lambda^2-18\lambda+45)=0$

$\therefore\ \lambda(\lambda-3)(\lambda-15)=0$

$\therefore\ \lambda=0,3,15$ .

Since, all eigenvalues are distinct the matrix A is diagonalisable.

# Example 1 ..



For $\lambda = 0$, $[ A - \lambda_1 I ] X = 0$

$$\therefore \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad 8x_1 - 6x_2 + 2x_3 = 0 \quad \& \quad -6x_1 + 7x_2 - 4x_3 = 0$$

By Crammer's rule

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \qquad \therefore \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = t$$

$$\therefore \quad x_1 = t, x_2 = 2t, x_3 = 2t$$

$$\therefore \quad X_1 = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \therefore \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Corresponding to eigen value 0 the eigen vector is $[1, 2, 2]'$.

# Example 1 ..



For $\lambda = 3$, $[A - \lambda I]X = 0$

$$\therefore \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad 5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$

By Crammer's rule,

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} \qquad \therefore \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = t$$

$$\therefore x_1 = 2t, x_2 = t, x_1 = -2t$$

$$\therefore \quad X_2 = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \qquad \therefore \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$\therefore$ Corresponding to eigen value 3, the eigen vector is $[2, 1, -2]'$.

# Example 1 ..



For $\lambda = 15$, $[A - \lambda_3 I] X = 0$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -7x_1 - 6x_2 + 2x_3 = 0 \,; -6x_1 - 8x_2 - 4x_3 = 0$$

*i.e.* $7x_1 + 6x_2 - 2x_3 = 0 \,; \quad 6x_1 + 8x_2 + 4x_3 = 0$

By crammer's rule,

$$\frac{x_1}{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 6 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & 6 \\ 6 & 8 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{40} = \frac{x_2}{40} = \frac{x_3}{20} \qquad \therefore \quad \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = t$$

$$\therefore \quad x_1 = 2t, x_2 = t, x_3 = -2t$$

$$\therefore \quad X_3 = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \qquad \therefore \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 15 the eigenvector is $[2, -2, 1]'$.

# Example 1 ..

Since, $M^{-1}AM = D$, the given matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ is diagonalized to

diagonal matrix $$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

by transforming matrix $$M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

# Example 2



**Show that the matrix A =** $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ **is diagonalisable. Find the diagonal form D and the diagonalising matrix M.**

**Sol.:** The characteristic equation is

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1+\lambda)(1+\lambda)(3-\lambda) = 0$$

$$\therefore \quad \lambda = -1, -1, 3$$

# Example 2..

**(i)** For $\lambda = -1$, $[\, A - \lambda_1 I \,]\, X = 0$ gives $\begin{bmatrix} -8 & 4 & 4 \\ -6 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ -(1/4)R_1 \end{matrix}$ $\begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \quad 2x_1 - x_2 - x_3 = 0$$

The rank of the coefficient matrix is r= 1. The number of unknowns is n=3. Hence, there are 3 - 1 = 2 linearly independent solutions.

Putting $x_2 = 2t$ and $x_3 = 2s$, we get

$$2x_1 = x_2 + x_3 = 2t + 2s \therefore x_1 = t + s$$

$$\therefore \quad x_1 = \begin{bmatrix} s + t \\ 0 + 2t \\ 2s + 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ Corresponding to the eigen values $-1$, we get the following two linearly independent eigen vectors.

$$X_1 = [\, 1, 0, 2 \,]' \text{ and } X_2 = [\, 1, 2, 0 \,]'.$$

# Example 2..

(ii) For $\lambda = 3$, $[\,A - \lambda_2 I\,]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} R_3 + R_2 \begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} -(1/4)R_1 \\ (1/4)R_2 \end{matrix} \begin{bmatrix} 3 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad 3x_1 - x_2 - x_3 = 0 \; and \; x_1 - x_2 = 0 \quad \therefore \quad x_1 = x_2.$$

Putting $x_2 = t$, we get $x_1 = x_2 = t$ and $x_3 = 3x_1 - x_2 = 3t - t = 2t$.

$$\therefore \quad X_3 = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \therefore \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

# Example 2..

Here the geometric multiplicity of each eigen value of A is equal to its algebraic multiplicity, A is diagonalisable.

Since, $M^{-1}AM = D$, the matrix A = $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ will be diagonalised

to the diagonal matrix D = $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

by the transforming matrix M = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

# Example 3



**If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B == \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$, prove that both A and B are not diagonalisable but AB is diagonalisable.**

**Sol. :** The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda^2) = 0 \qquad \therefore \qquad \lambda = 1,1$$

For $\lambda = 1$, $[A - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix 1 and the number of variables is 2.

Hence, there is only one solution.

Now, the algebraic multiplicity of the eigen value 1 is 2 but its geometric multiplicity is 1. Hence, the matrix A is not diagonalizable.

# Example 3..



The characteristic equation of B is

$$\begin{vmatrix} 2-\lambda & 0 \\ 1/2 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda^2) = 0 \qquad \therefore \qquad \lambda = 2,2$$

For $\lambda$ = 2, [ A - $\lambda_2$I] X = 0 gives

$$\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix is 1 and the number of variables is 2. Hence, there is only one solution.

Now, the algebraic multiplicity of the eigen value 2 is 2 and its geometric multiplicity is one. Hence, the matrix B is not diagonalisable.

# Example 3..

Now, $C = AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1/2 & 2 \end{bmatrix}$

The characteristic equation of C is

$$\begin{vmatrix} 3-\lambda & 4 \\ 1/2 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (3-\lambda)(2-\lambda) - 2 = 0$$

$$\therefore \quad 4 - 5\lambda + \lambda^2 = 0$$

$\therefore \quad (\lambda - 4)(\lambda - 1) = 0 \qquad \therefore \lambda = 1,4$

**(i)** For $\lambda = 1$, $[A - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} 2 & 4 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 - \frac{1}{4}R_1 \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix is 1.

$$\therefore \quad 2x_1 + 4x_2 = 0.$$

There is only one solution. Hence, the algebraic multiplicity and geometric multiplicity of eigenvalue 1 are equal.

# Example 3..



**(ii)** For $\lambda = 1$, [ A - $\lambda_1$I] X = 0 gives

$$\begin{bmatrix} -1 & 4 \\ 1/2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 + \frac{1}{4} R_1 \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The rank of the matrix is 1.

$$\therefore \quad -x_1 + 4x_2 = 0$$

There is only one solution. Hence, the algebraic multiplicity and geometric multiplicity of eigen value 4 are equal.

Hence, the matrix C = AB is diagonalisable.

# Example 4

**Find the symmetric matrix $A$ having the eigen values $\lambda_1 = 0,\ \lambda_2 = 3\ and\ \lambda_3 = 15$ with the corresponding eigen vectors $X_1 = [1, 2, 2]', X_2 = [-2, -1, 2]'\ and\ X_3$.**

**Sol. :** Let $X_3 = [x_1, x_2, x_3]'$ be the third eigen vector corresponding to the eigen value 15.

Since the required matrix A is symmetric and all eigen values are distinct the three eigen vectors corresponding to the three eigen values are orthogonal.

$$\therefore \quad x_1 + 2x_2 + 2x_3 = 0; \qquad -2x_1 - x_2 + 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{6} = \frac{x_2}{-6} = \frac{x_3}{3} \qquad \therefore \quad \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\therefore \quad x_3 = [2, -2, 1]'$$

# Example 4..

Since A is symmetric it is orthogonally similar to a diagonal matrix D. There exists an orthogonal matrix P such that

$P^{-1}A\,P = D$ i.e. $A\ = PDP^{-1} = PDP'$ (Since P is orthogonal $P^{-1}\ =\ P'$)

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & 10 \\ 0 & -1 & -10 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

# Example 5



**Find $e^A$ and $4^A$ if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$.**

**Sol. :** The characteristic equation of A is

$$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$$

$\therefore \quad \left(\frac{3}{2} - \lambda\right)^2 - \frac{1}{4} = 0$

$\therefore \quad \frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$

$\therefore \quad \lambda^2 - 3\lambda + 2 = 0 \qquad \therefore \quad (\lambda - 1)(\lambda - 2) = 0$

$\therefore \quad \lambda = 1, 2$

# Example 5..



**(i)** For $\lambda = 1$, $[A - \lambda I]\,X = O$ gives $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} 2R_1 \\ 2R_2 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad x_1 + x_2 = 0$

Putting $x_2$ = - t, we get $x_1$ = t.

$$\therefore \quad X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \therefore \quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence, the eigen vector is [1, -1] '.

# Example 5..



**(ii)** For $\lambda = 2$, $[A - \lambda I]\, X = 0$ gives

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} 2R_1 \\ 2R_2 \end{matrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad -x_1 + x_2 = 0 \qquad \therefore \quad x_1 = x_2$

Putting $x_2 = t$, we get $x_1 = t$.

$$\therefore \quad X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence, the eigenvector is [1, 1] '.

# Example 5..



$$\therefore \quad M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \therefore \quad |M| = 2$$

$$M^{-1} = \frac{adj.M}{|M|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Now D = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

If $\mathrm{f(A)} = \mathrm{e}^{A}, \mathrm{f(D)} = \mathrm{e}^{D} = \begin{bmatrix} \mathrm{e}^{1} & 0 \\ 0 & \mathrm{e}^{2} \end{bmatrix}$

$$\therefore \quad \mathrm{e}^{A} = Mf(D)\mathrm{M}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^{2} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e & e^{2} \\ -e & e^{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad e^{A} = \frac{1}{2}\begin{bmatrix} e + e^{2} & -e + e^{2} \\ -e + e^{2} & e + e^{2} \end{bmatrix}$$

# Example 5..



If $f(A) = 4^A, f(D) = 4^D = \begin{bmatrix} 4^1 & 0 \\ 0 & 4^2 \end{bmatrix}$

Similarly,

$$\therefore \quad 4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

# Example 6



**If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that 3 tan A= A tan 3.**

**Sol. :** The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1+\lambda)(1-\lambda) - 8 = 0 \qquad \therefore \lambda^2 - 9 = 0$$

$\therefore \qquad \lambda = 3, -3.$

**(i)** For $\lambda = 3$, $[A - \lambda I]\,X = 0$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\quad R_2 + \frac{1}{2} R_1 \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad -4x_1 + 4x_2 = 0 \qquad \therefore \quad x_1 - x_2 = 0$

*Putting* $x_2 = t$, we get $x_1 = t$.

$X_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ The eigen vector is $[1, -1]$ '.

**(ii)** For $\lambda$ = - 3, [ A - $\lambda$ $I$] X = 0 gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$ $\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad 2x_1 + 4x_2 = 0 \qquad \therefore \quad x_1 + 2x_2 = 0$

Putting $x_2 = -t$, we $get$ $x_1 = -2x_2 = 2t.$

$$\therefore \quad X_2 = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \therefore \; X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector is $[2, -1]'$

$\therefore \quad M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $|M| = -3$

$$M^{-1} = \frac{adj.M}{|M|} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

Now D = $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

$$\because \quad f(A) = \tan A$$

$$f(D) = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \begin{bmatrix} \tan 3 & 0 \\ 0 & -\tan 3 \end{bmatrix}$$

$$\therefore \tan A = Mf(D)M^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} \left(-\frac{1}{3}\right) \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} \tan 3 & -2\tan 3 \\ \tan 3 & -\tan 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} \tan 3 & -2\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

$$\therefore \quad 3\tan A \begin{bmatrix} -\tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix} = \tan 3 \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$$

# Example 7



**If A = $\begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix}$, find cos A.**

**Sol.:** The characteristic equation is

$$\begin{vmatrix} \pi-\lambda & \dfrac{\pi}{4} \\ 0 & \left(\dfrac{\pi}{2}\right)-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (\pi-\lambda)\left(\frac{\pi}{2}-\lambda\right)=0 \quad \therefore \quad \lambda=\frac{\pi}{2},\pi$$

Let $\Phi(A) = \cos A = \alpha_1 A + \alpha_0 I$ ...(1)

Since $\lambda$ satisfies the above equation, we have

$\cos\lambda = \alpha_1\lambda + \alpha_0$ ...(2)

Putting $\lambda = \frac{\pi}{2}$, we get

$\cos\frac{\pi}{2} = \alpha_1 \cdot \lambda + \alpha_0$

$\therefore \quad 0 = \alpha_1 \cdot \frac{\pi}{2} + \alpha_2$ ...(3)

$$\cos\pi = \alpha_1 \cdot \pi + \alpha_0$$

$$\therefore \quad -1 = \alpha_1 \cdot \pi + \alpha_0 \qquad \ldots(4)$$

From (iii) and (iv), we get

$$\alpha_1 \cdot \frac{\pi}{2} = -1 \quad \therefore \quad \alpha_2 = -\frac{2}{\pi} \quad \therefore \quad \alpha_1 = -1 + 2$$
$$= 1$$

Putting these values in (1), we get

$$\cos A = -\frac{2}{\pi}\begin{bmatrix} \pi & \frac{\pi}{4} \\ 0 & \frac{\pi}{2} \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

# Example 8

If A = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find $A^{50}$.

**Sol. :** The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1-\lambda)\,[(-\lambda)\,(-\lambda) - 1] = 0$

$\therefore \quad (1-\lambda)\,(\lambda^2 - 1) = 0$

$\therefore \quad \lambda = 1, 1, -1.$

Since, the matrix is of order 3, we consider

$$\Phi(A) = A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I \qquad ...(1)$$

X satisfies this equation.

$$\therefore \qquad \lambda^{50} = \alpha_2 \lambda + \alpha_1 \lambda + \alpha_0 \qquad ...(2)$$

Putting $\lambda$ = 1, $\lambda$ = -1, we get

$$1 = \alpha_2 + \alpha_1 + \alpha_0 \qquad ...(3)$$

$$1 = \alpha_2 - \alpha_1 + \alpha_0 \qquad ...(4)$$

Differentiating (2), w.r.t. X, we get

$$50\lambda^{49} = 2\alpha_2 \lambda + \alpha_1$$

Putting $\lambda = 1$, we get

$$50 = 2\alpha_2 + \alpha_1 \qquad \text{...(5)}$$

Solving (iii), (iv) and (v), we get

$\alpha_2 = 25$, $\alpha_1 = 0$ and $\alpha_0 = -24$.

Putting these values in (1), we get $A^{50} = 25A^2 - 24I$

$$\text{But } A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore\ A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

# Definitions

- A polynomial in x with the coefficient of highest power of x is unity is called monic polynomial.
- The monic polynomial of the lowest degree that annihilates a matrix A is called minimal polynomial of A.
- The square matrix A of order n is said to be derogatory if degree of minimal polynomial of A is less than order of A(ie. n).
- The square matrix A of order n is said to be non-derogatory if degree of minimal polynomial of A is greater than or equal to order of A(ie. n).

# Example 9



**Show that A =** $\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ **is derogatory.**

**Sol. :** The characteristic equation of A is

$$\begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (7-\lambda)\,[(7-\lambda)\,(4-\lambda)-4]-4\,[4\,(4-\lambda)-4]-1[-16 + 4(7-\lambda)] = 0$$

$$\therefore \quad (7-\lambda)[24-11\,\lambda\ +\ \lambda^2]-4[12-4\,\lambda]-[12-4\,\lambda] = 0$$

$$\therefore \quad \lambda^3-18\lambda^2+18\lambda-108 = 0$$

$$\therefore \quad (\lambda-3)(\lambda^2-15\lambda+36) = 0$$

$$(\lambda-3)\,(\lambda-12)\,(\lambda-3) = 0$$

Hence, the roots of $|A-\lambda I\,|$= 0 are 3, 3, 12.

Let us now find the minimal polynomial of A. We know that each characteristic root of A is also a root of the minimal polynomial of A.

So if $f(x)$ is the minimal polynomial of A then $x - 3$ and $x - 12$ are the factors of $f(x)$.

Let us see whether

$(x - 3)(x - 12) = x^2 - 15x + 36$ annihilates A.

$$\text{Now, } A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$\therefore \quad A^2 - 15A + 36I$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad f(x) = x^2 - 15x + 36$ annihilates A.

Thus, $f(x)$ is the monic polynomial of lowest degree that annihilates A. Hence, $f(x)$ is the minimal polynomial of A.

Since, degree of $f(x)$ is less than the order of A, A is derogatory.

# Example 10



**Show that the matrix A =** $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ **is non-derogarory.**

**Sol. :** The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (2-\lambda)[-(1-\lambda)(1+\lambda)-3] + 2[-(1+\lambda)-1] + 3[3-(1-\lambda)] = 0$

$\therefore \quad (2-\lambda)(-4+\lambda^2) - 2(2+\lambda) + 3(2+\lambda) = 0$

$\therefore \quad \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$\therefore \quad \lambda^3 - \lambda^2 - \lambda^2 + \lambda - 6\lambda + 6 = 0$

$\therefore \quad (\lambda-1)(\lambda^2-\lambda-6) = 0$

$\therefore \quad (\lambda-1)(\lambda-3)(\lambda+2) = 0 \quad \therefore \quad \lambda = 1, -2, 3$

Since, all the roots of the characteristic equation are distinct,

$$f(x) = (x-1)(x+2)(x-3)$$

It the minimal polynomial. Hence, the matrix is non-derogatory.

# Example 11

**Find eigen values and eigen vectors of $A^3$ where**

**$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ Is A derogatory ?**

**Sol. :** The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)[(2-\lambda)(3-\lambda)-2] - 0\,[1(3-\lambda)-2]$

$-1\,[2-2\,(2-\lambda)] = 0$

$\therefore \quad (1-\lambda)[6-5\lambda+\lambda^2-2] - 0\,[4-5\lambda+\lambda^2] + 2\,[1-\lambda] = 0$

$\therefore \quad (1-\lambda)\,(\lambda^2 - 5\,\lambda + 6) = 0$

$\therefore \quad (1-\lambda)\,(\lambda-2)\,(\lambda-3) = 0 \therefore \lambda = 1, 2, 3.$

$\therefore$ Eigen values of $A^3$ are $1^3$, $2^3$, $3^3$.

**(i)** For $\lambda = 1$, $[A - \lambda_1 I]\lambda = 0$ gives

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2$
$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_3 = 0, x_1 + x_2 + x_3 = 0,$$

$$\therefore \quad x_1 + x_2 = 0$$

Let $x_2 = -1, \ x_1 = 1$

$\therefore \quad X = [1, -1, 0]'$

**(ii)** For $\lambda = 2, [A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 + R_1 \\ R_2 + 2R_1 \end{matrix} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -x_1 - x_3 = 0, \qquad 2x_2 - x_3 = 0$$

$$Let\ x_2 = 1, x_3 = 2 \qquad \therefore \quad x_1 = -x_3 = -2$$

$$\therefore \quad x_2 = [-2, 1, 2]'.$$

**(iii)** For $\lambda = 3$, $[A - \lambda_3 I]X = 0$ gives

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_1 + 2R_1 \\ R_3 + R_1 \end{matrix}\begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -2x_1 + x_3 = 0, \qquad x_1 - x_2 + x_3 = 0$$

$$Let\ x_2 = 1, x_3 = 2 \qquad \therefore \quad x_1 - 1 + 2 = 0$$

$$\therefore \quad x_1 = -1$$

$$\therefore \quad x_3 = [-1, 1, 2]'$$

Now, if $AX = \lambda X$, then $A^n X = \lambda^n X$.

Hence, eigenvalues of $A^3$ are 1, 8, 27.

Eigen vectors of $A^3$ are $X_1 = [\ 1, -1, 0]'$, $X_2 = [\ -2, 1, 2]'$,

$X_3 = [\ -1, 1, 2\ ]'$

Since, eigenvalues of A are all distinct, A is derogatory.