Cayley Hamilton's Theorem

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1:28 PM

function of Matrix

$y = f(x)$	$B = f(A)$
$x^2 + 2x + 3$	$A^2 + 2A + 3I$
e^x, a^x	e^A, a^A
$\log x$	$\log A$
$\sin x, \cos x$	$\sin A / \cos A$
$\tan^{-1} x$	

Statement: Every Square Matrix Satisfies its characteristic equation

means, If Given sq Matrix A, if its characteristic eq^n

$$\lambda^3 - 2\lambda^2 + 15\lambda - 25 = 0$$

Then, By Cayley-Hamilton's Theorem (C-H Th^m)

$$\boxed{A^3 - 2A^2 + 15A - 25I = 0}$$

validity eq^n

Verify C-H Theorem for Matrix A and Hence find A^4 & A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Sol^n: Consider $Ax = \lambda x$, $[A - \lambda I]x = 0$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$|A-\lambda I| = \begin{vmatrix} 2 & - & \\ 3 & 1 & -1-\lambda \end{vmatrix}$$

Characteristic eq^n is given by $\lambda^3 - P\lambda^2 + Q\lambda - R = 0$

$P = SOD = -1$

$Q = SOMD = -18$

$R = |A| = 40$

Ch eq^n is given by $\boxed{\lambda^3 + \lambda^2 - 18\lambda - 40 = 0}$

Then C-H Th^m say every square Matrix satisfies its characteristic eq^n

$\therefore \boxed{A^3 + A^2 - 18A - 40I = 0}$ — ① — $\boxed{2/3}$

To Verify, $A^2 = \begin{bmatrix} \checkmark \end{bmatrix}$ $A^3 = \begin{bmatrix} \checkmark \end{bmatrix}$

L.H.S. $= \boxed{A^3 + A^2 - 18A - 40I}$

$= \begin{bmatrix} \checkmark \end{bmatrix} + \begin{bmatrix} \checkmark \end{bmatrix} - 18\begin{bmatrix} \quad \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ (MATA × MATA ×) + (MATA ×)

Hence find A^4

multiply eq^n ① by A, $A^4 + A^3 - 18A^2 - 40A = 0$

$\Rightarrow \boxed{A^4 = 18A^2 + 40A - A^3}$

$= \begin{bmatrix} \checkmark \end{bmatrix}$

Similarly, multiply ① by A^{-1}

$^2 \quad - \quad A^{-1} = 0$

$$A^2 + A - 18I - 40A^{-1} = 0$$

$$\boxed{A^{-1} = \frac{1}{40}(A^2 + A - 18I)}$$

$$= [\ \ ]$$

2) Verify C-H Th^m for $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ & Hence find

Matrix $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$

Sol^n: $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$ | $P = 7$, $Q = 16$, $R = 12$

characteristic eq^n is given by, $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

Then By C-H Theorem, ( )

$$\therefore \boxed{A^3 - 7A^2 + 16A - 12I = 0}$$

(Dividend) = (Divisor) (Q) + R

A^{100}

$$\begin{array}{r} \lambda^3 + \lambda^2 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 \overline{)\lambda^6 - 6\lambda^5 + 9\lambda^4 + 4\lambda^3 - 12\lambda^2 + 2\lambda - I} \\ -\lambda^6 \mp 7\lambda^5 \pm 16\lambda^4 \mp 12\lambda^3 \\ \hline \lambda^5 - 7\lambda^4 + 16\lambda^3 - 12\lambda^2 \\ -\lambda^5 \mp 7\lambda^4 \pm 16\lambda^3 \mp 12\lambda^2 \\ \hline 2\lambda - I \end{array}$$

division $\Rightarrow$

Polynomial eq^n:

$(2\lambda - I)$

$\therefore$ Polynomial eq^n:

$$\left(\lambda^6 - 6\lambda^5 - - - \atop - - - - - \right) = (\lambda^3 - 7\lambda^2 + 16\lambda - 12)(\lambda^3 + \lambda^2) + (2\lambda - I)$$

$$\left(\qquad \right) = 2\lambda - I$$

We can write in matrix form

$$\left(A^6 - 5A^5 + 9A^4 + 4A^3 - 12A^2 \atop + 2A - I \right) = 2A - I$$

$$= 2\begin{bmatrix} & \\ & \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

H.W. Verify C-H

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Hence find A^{-1} if possible.

I

$$\boxed{\lambda^2 - 4\lambda = 0} \qquad \boxed{A^2 - 4A = 0}$$

Minimal Polynomial / Minimal equation of Given Matrix (Sq)

Derogatory / Non Derogatory.

$C(x) \Rightarrow$ Characteristic Polynomial of $A_{n\times n}$

(... A has complexity 3)

3x3 $C(A) = 0$ (Matrix A has complexcity 3)

$\boxed{M(A) = 0}$ (complexcity is reduced)

$\deg(M(A)) = 2$

4, 1, −3

$(x-4)(x-1)(x+3)$

$(A-4I)(A-I)(A+3I)$

5, −3, −3

$(x-5)(x+3)(x+3)$

$x^2 - 2x - 15$

$(A^2 - 2A - 15I) \overset{?}{=} 0$

$\boxed{M(A)=0}$

$\deg(M(x)) \leq \deg(C(x))$

Annihilator of a Matrix :

If A is any square Matrix of order n & f(x) is polynomial such that $f(A) = 0$ Then we say that f(x) annihilates matrix A

f(x) is annihilator

e.g. By C-H Th^m, C(X) is always annihilator of A

Monic polynomial : Polynomial in which coefficient of highest power of A is 1 (unity) is called Monic polynomial.

e.g. C(X)

... l polynomial : The monic polynomial of lowest deg.

Minimal polynomial: The monic polynomial of lowest deg. that annilates matrix A is called min polynomial of A

denote it by $M(x)$, $M(A) = 0$ $\quad C(A) = 0$

$$\deg(M(A)) \le \deg(C(A))$$

Derogatory Matrix: If deg of Min Polynomial for order n Square Matrix is less than n then it is called derogatory

$$\boxed{\deg(M(A)) < \deg(C(A))}$$

Non derogatory Matrix: If Minimal polynomial is same as Characteristic polynomial then Matrix is called non-derogatory

Properties:

1) Each eigenvalue is also root of minimal polynomial
2) If eigenvalues are distinct then minimal poly is same as char poly (Non Derogatory)

$[(x-4)(x-5)](x+3)$ $\quad$ x $\quad 4 \quad 5 \quad -3$

$\quad 3 \quad 2 \quad 2$

Ch eq^n roots distinct Nonderogatory

$[(x-3)(x-2)](x-2)$ $\quad M(A)$

$x^2 - 5x + 6$

$\boxed{A^2 - 5A + 6I \neq 0}$?

$3, 3, 3$

$(x-3)$

$(A-3I) \overset{?}{=} 0 \quad \neq 0$

$(A-3I)(A-3I) \overset{?}{=} 0 \quad \neq 0$

$(A-3I)(A-3I)(A-3I) = 0$

$[A-3I]x = 0$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3 = 3$

AM = GM

GM = $n - r = 3 - 0 = 3$

no. of independent vector = 3

$r = 0$

Check whether Given Matrix A is derogatory or not

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Sol^n: Consider $Ax = \lambda x$, $[A - \lambda I]x = 0$

$\therefore$ ch eq^n $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

ch eq^n is $\lambda^3 - P\lambda^2 + Q\lambda^2 - R = 0 \Rightarrow$ $P = 5$, $Q = 8$, $R = 4$

$\rightarrow \lambda^3 - 5\lambda^2 + 8\lambda^2 - 4 = 0$

& roots of eq^n are $\lambda = 1, 2, 2$ ←

Then we know that If $f(x)$ is minimal polynomial then $(x-1)$ & $(x-2)$ are factors of $f(x)$

Consider, $f(x) = (x-1)(x-2) = x^2 - 3x + 2$

We check whether this annihilates A

$$\therefore A^2 - 3A + 2I = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(x) = x^2 - 3x + 2$ annihilates matrix A it is monic polynom

$\therefore$ Min Polynomial of A $= x^2 - 3x + 2$

$\deg(f(x)) = 2 <$ order of A

$\Rightarrow$ Matrix is derogatory

H.W.

2) $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$\hookrightarrow \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$\downarrow$

$\lambda = 1, -2, 3$

Since all the eigenvalues are distinct, which are also roots of Min polynomial $\Rightarrow$ Min polynomial is same as characteristic polynomial

$\therefore$ Matrix is non-derogatory.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \checkmark$$