

RELATIONS, DIGRAPHS

SY Odd 2023-24

RELATIONS, DIGRAPHS (07)

- 3.1 Relations, Paths and Digraphs
- 3.2 Properties and types of binary relations
- 3.3 Manipulation of relations, Closures, Warshall's algorithm
- 3.4 Equivalence relations



INTRODUCTION

Definition: A binary relation from a set A to a set B is a subset $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B \}$

- Let A and B be nonempty sets. A relation R from A 'to' B is a subset of $A \times B$.
- If $R \subseteq A \times B$ and $(a, b) \in R$, we say that a 'is related to' b by R , and we also write $a R b$.
- If a is not related to b by R , we write $a \not R b$.
- Frequently, A and B are equal. In this case, we often say that $R \subseteq A \times A$ 'is a relation on' A , instead of a relation from A to A .



EXAMPLES

Ex. 1 :

Let $A = \{ 1, 2, 3 \}$ and $B = \{ r, s \}$

Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B.

Ex. 2 :

Let $A = \{ 1, 2, 3, 4, 5 \}$.

Define the following relation R (less than) on A :

$a R b$ if and only if $a < b$.

Then $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$



DEFINITIONS

- Let $\{ A_1, A_2, \dots, A_n \}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times \dots \times A_n$ is called an **n-ary relation** on A_1, A_2, \dots, A_n .
- If $R = \emptyset$, then R is called **void** or **empty relation**.
- If $R = A_1 \times A_2 \times \dots \times A_n$, then R is called the **universal relation**.
- If $A_i = A$ for all i , then R is called an ' n - ary relation on A '.
- If $n=1, 2$ or 3 , then R is called a **unary**, **binary** or **ternary** relation respectively.
- Among the relations, binary relations are the most important being widely used in various applications.



SET ARISING FROM RELATIONS

Domain of Relation R :

Let $R \subseteq A \times B$ be a relation from A to B . The **domain** of R , denoted by **Dom (R)**, is the set of elements in A that are related to some element in B . In other words, $\text{Dom } (R)$, a subset of A , is the set of all first elements in the pairs that make up R .

Range of relation R :

Similarly, we define the **range** of R , denoted by **Ran (R)**, to be the set of elements in B that are second elements of pairs in R , that is, all elements in B that are related to some element in A .



EXAMPLES

Ex. 1 :

Let $A = \{ 1, 2, 3 \}$, $B = \{ r, s \}$

and $R = \{(1, r), (2, s), (3, r)\}$

$\text{Dom } (R) = \{ 1, 2, 3 \} = A$

$\text{Ran } (R) = \{ r, s \} = B$

Ex. 2 :

Let $A = \{ 1, 2, 3, 4, 5 \}$, $B = \{ 1, 2, 3, 4, 5 \}$

$a R b$, if and only if $a < b$

$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

$\text{Dom } (R) = \{ 1, 2, 3, 4 \}$

$\text{Ran } (R) = \{ 2, 3, 4, 5 \}$



REPRESENTATION OF RELATION

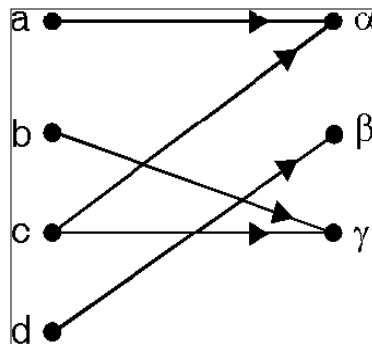
Graphical and tabular form :

For Example :

Let $A = \{a, b, c, d\}$, $B = \{\alpha, \beta, \zeta\}$
and R is a relation from A to B .

$$R = \{(a, \alpha), (b, \zeta), (c, \alpha), (c, \zeta), (d, \beta)\}$$

	α	β	ζ
a	✓		
b			✓
c	✓		✓
d		✓	



$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



DIAGRAPH

If A is a finite set and R is a relation on A , we can also represent R pictorially as follows :

(i) Draw a small circle for each element of A and label the circle with the corresponding element of A . These circles are called **vertices**.

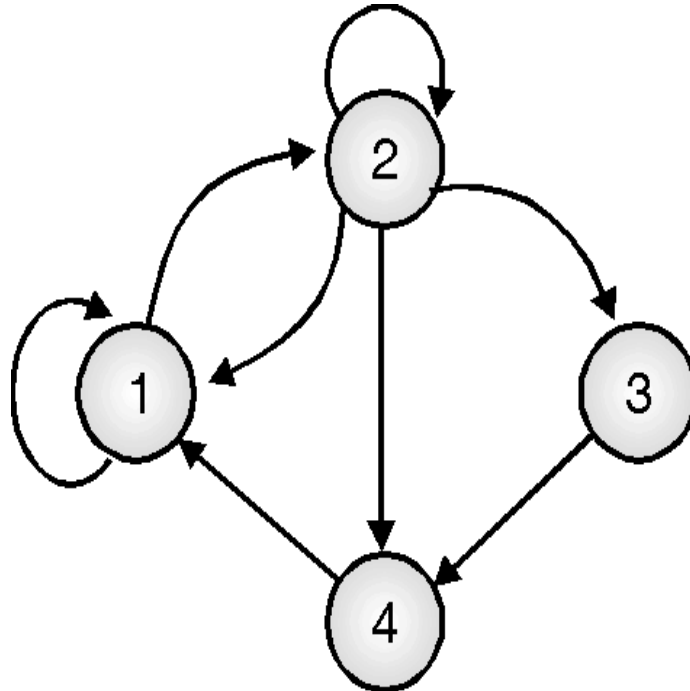
(ii) Draw an arrow, called an **edge**, from vertex a_i to vertex a_j if and only if $a_i R a_j$.

The resulting pictorial representation of R is called a **directed graph** or **digraph** of R .



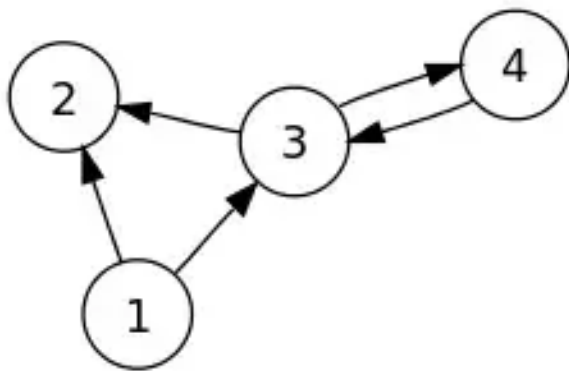
DIAGRAPH

Ex. 1 : Let $A = \{1, 2, 3, 4\}$, Let R is a relation from A to A .
 $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$



DEGREE OF VERTEX IN A DIRECTED GRAPH

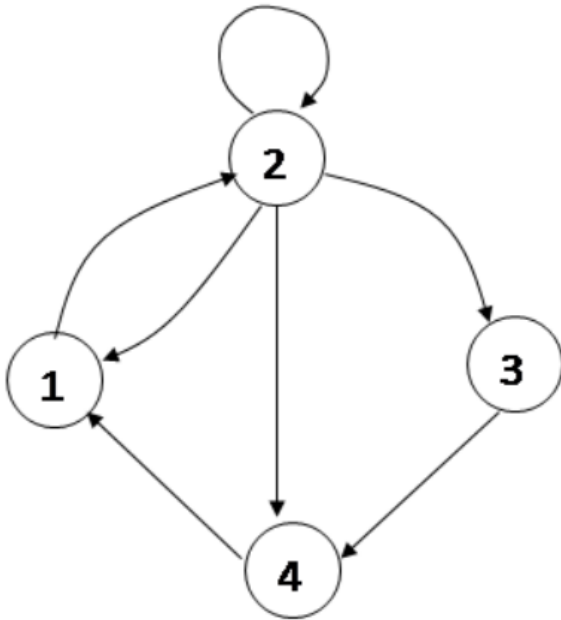
- A directed graph, each vertex has an **in-degree** and an **out-degree**.
- In-degree of a Graph-Number of edges which are coming into the vertex V .
- Out-degree of a Graph-Number of edges which are going out from the vertex V



VERTEX	1	2	3	4
In Degree	0	2	2	1
Out-degree	2	0	2	1



FIND OUT IN-DEGREE AND OUT-DEGREE



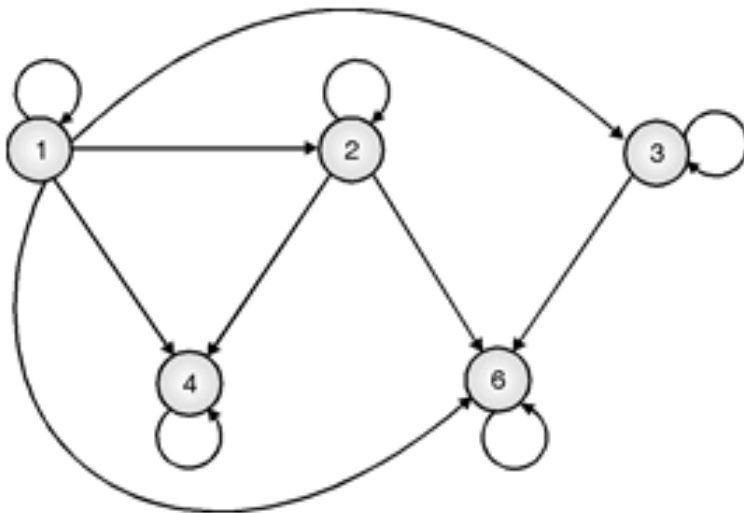
VERTEX	1	2	3	4
In Degree	2	2	1	2
Out-degree	1	4	1	1



EXAMPLE

Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R . Find Matrix of R . Find inverse relation of R .

Soln.: $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}$



Assume the rows and columns of M are each labelled 1, 2, 3, 4, 6, since R is relation from A to A , matrix M_R is square, i.e. M_R has the same number of rows as columns

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$R^{-1} = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (4,4), (6,6), (6,2), (3,3), (6,3), \boxed{}\}$$

EXAMPLE

Let $A = \{1, 2, 3, 4, 6\} = B$, $a R b$ if and only if a is a multiple of b . Find R and draw the digraph of R . Find Matrix of R . Find each of the following :

(i) $R(3)$ (ii) $R(6)$ (iii) $R(\{2, 4, 6\})$

Solution:

$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (6, 1), (6, 2), (6, 3), (6, 6)\}$

$\text{Dom}(R) = \{1, 2, 3, 4, 6\}$

$\text{Ran}(R) = \{1, 2, 3, 4, 6\}$

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

(i) $R(3) = \{1, 3\}$,

Since $(3, 1) \in R$ and $(3, 3) \in R$

(ii) $R(6) = \{1, 2, 3, 6\}$,

Since $(6, 1) \in R$, $(6, 2) \in R$, $(6, 3) \in R$ and $(6, 6) \in R$.

(iii) $R(\{2, 4, 6\}) = \{1, 2, 4, 3, 6\}$

Since $(2, 1) \in R$, $(4, 2) \in R$, $(6, 1) \in R$, $(6, 2) \in R$,
 $(6, 3) \in R$, $(6, 6) \in R$, $(4, 4) \in R$



PATHS IN RELATIONS AND DIGRAPHS

Suppose that R is a relation on a set A . A **path of length n** in R from a to b is a finite sequence $\pi : a, x_1, x_2, \dots, x_{n-1}, b$, beginning with a and ending with b , such that

$$a R x_1, x_1 R x_2, \dots, x_{n-1} R b$$

Note that a path of length n involves $n + 1$ elements of A , although they are not necessarily distinct.

The **length** of a path is the number of edges in the path, where the vertices need not all be distinct.

A path that begins and ends at the same vertex is called a **cycle**.



PATHS IN RELATIONS AND DIGRAPHS

$R = \{ (1, 2), (2, 3), (2, 4), (3, 3) \}$ is a relation on $A = \{1, 2, 3, 4\}$

$$R^1 = R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$$

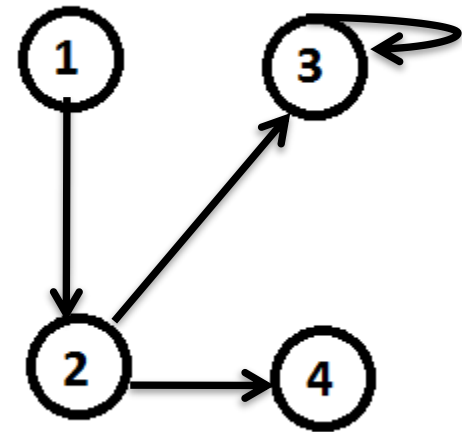
$$R^2 = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$$

$1 R^2 3$ Since $1 R 2$ and $2 R 3$

$1 R^2 4$ Since $1 R 2$ and $2 R 4$...

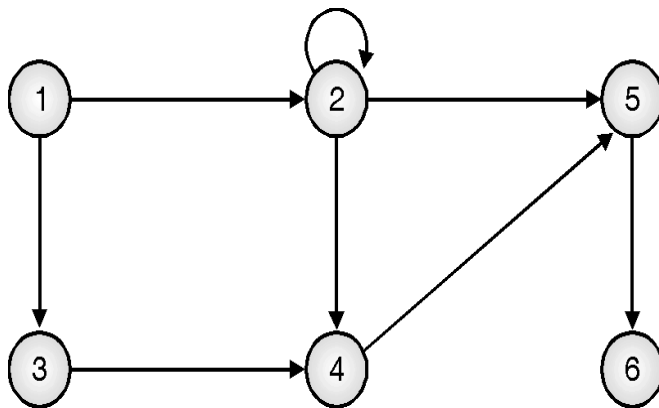
$$R^3 = \{ (1, 3), (2, 3), (3, 3) \}$$

$$R^4 = \{ (1, 3), (2, 3), (3, 3) \}$$

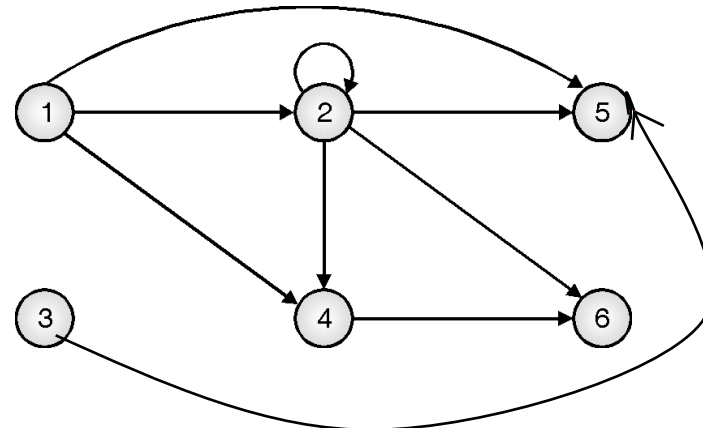


PATHS IN RELATIONS AND DIGRAPHS

Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be the relation whose digraph is shown in Fig. Find R^2 and draw digraph of the relation R^2 .



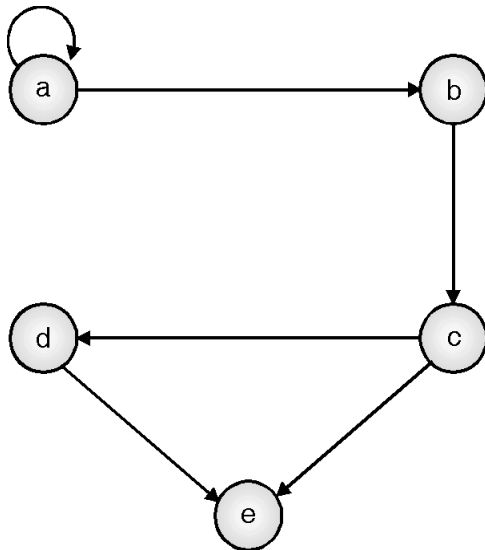
$1 R^2 2$	Since	$1 R 2$	and	$2 R 2$
$1 R^2 4$	Since	$1 R 2$	and	$2 R 4$
$1 R^2 5$	Since	$1 R 2$	and	$2 R 5$
$2 R^2 2$	Since	$2 R 2$	and	$2 R 2$
$2 R^2 4$	Since	$2 R 2$	and	$2 R 4$
$2 R^2 5$	Since	$2 R 2$	and	$2 R 5$
$2 R^2 6$	Since	$2 R 5$	and	$5 R 6$
$3 R^2 5$	Since	$3 R 4$	and	$4 R 5$
$4 R^2 6$	Since	$4 R 5$	and	$5 R 6$



PATHS IN RELATIONS AND DIGRAPHS

Let $A = \{a, b, c, d, e\}$
and $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$

Compute (i) R^2 (ii) R^∞



$a R^2 a$ Since $a R a$ and $a R a$

$a R^2 b$ Since $a R a$ and $a R b$

$a R^2 c$ Since $a R b$ and $b R c$

$b R^2 e$ Since $b R c$ and $c R e$

$b R^2 d$ Since $b R c$ and $c R d$

$c R^2 e$ Since $c R d$ and $d R e$

$R^2 = \{(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)\}$

$R^\infty = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e)\}$.

BOOLEAN PRODUCT

The 'Boolean product' of A and B, denoted $A \odot B$ is the $m \times n$ Boolean matrix.

$C = [C_{ij}]$ defined by

$$C_{ij} = \begin{cases} 1 & \text{if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some } k, 1 \leq k \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$M_R^2 = M_R \odot M_R$$

$$M_R^n = M_R \odot M_R \odot \dots \odot M_R \text{ (n factors)}$$



PATHS IN RELATIONS AND DIGRAPHS

Let $A = \{a, b, c, d, e\}$ and

$R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$

$$M_R^2 = M_R \odot M_R$$

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^2 = \{(a, a), (a, b), (a, c), (b, d), (b, e), (c, e)\}$$



PROPERTIES/TYPES OF RELATIONS

- Reflexive
- Symmetric
- Transitive
- Antisymmetric
- Asymmetric



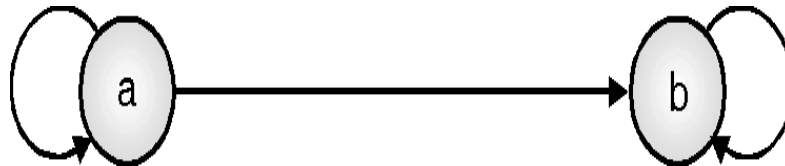
PROPERTIES: REFLEXIVITY

A relation R on a set A is **reflexive** if for 'every' element $a \in A$, $a R a$, i.e. $(a, a) \in R$.

R is not a reflexive relation if for 'some' element $a \in A$, $(a, a) \notin R$

Ex. 1 : Let $A = \{a, b\}$ and let $R = \{(a, a), (a, b), (b, b)\}$.

Then R is reflexive.



Ex. 2 : Let $A = \{1, 2\}$ and let $R = \{(1, 1), (1, 2)\}$.

R is not reflexive since $(2, 2) \notin R$.



PROPERTIES: SYMMETRY

A relation R on a set A is **symmetric** if whenever $a R b$, then $b R a$. It then follows that R is not symmetric if we have some a and $b \in A$ with $a R b$, but $b \not R a$.

Ex. 1 : $A = \{ 1, 2, 3 \}$, Is R symmetric ?

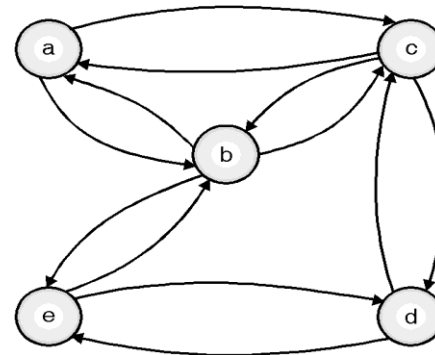
$R = \{ (1,2), (2,1), (2,3), (3,2), (1,1) \}$

Ex. 2 : $A = \{ 1, 2, 3, 4 \}$, Is R symmetric ?

$R = \{ (1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3) \}$

Ex. 3 : $A = \{a, b, c, d, e\}$

$R = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e), (c, d), (d, c)\}$



PROPERTIES: ASYMMETRIC RELATION

A relation R on a set A is **asymmetric** if whenever $(a,b) \in R$, then $(b,a) \notin R$. It then follows that R is not asymmetric if we have some a and $b \in A$ with both $(a,b) \in R$ and $(b,a) \in R$

Examples :

1. Let $A = \mathbb{R}$, the set of real numbers and let R be the relation ' $<$ '. If $a < b$, then $b \not< a$ (b is not less than a), so ' $<$ ' is asymmetric.

2. Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$. Then, R is not asymmetric, since $(2, 2) \in R$.

3. Let $A = \mathbb{Z}^+$, the set of positive integers, and let $R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$.

If $a = b = 3$, say then $a R b$ and $b R a$, so R is not asymmetric.



ANTISYMMETRIC RELATIONS

A relation R on a set A is **antisymmetric** if whenever $a R b$ and $b R a$, then $a = b$.

The contrapositive of this definition is that R is antisymmetric if whenever $a \neq b$, then $(a,b) \notin R$ or $(b,a) \notin R$.

It follows that R is **not antisymmetric** if we have a and b in A , $a \neq b$, and both $a R b$ and $b R a$.



SYMMETRY VERSUS ANTISYMMETRY

- In a symmetric relation $aRb \Leftrightarrow bRa$
- In an antisymmetric relation, if we have aRb and bRa hold only when $a=b$
- An antisymmetric relation is not necessarily a reflexive relation

$$A=\{1,2,3\}$$

$$R=\{(1,1),(2,2)\}$$

- A relation that is not symmetric is not necessarily asymmetric

$$A=\{1,2,3\}$$

$$R=\{(1,2),(2,2)\}$$



EXAMPLES

Ex. : Let $A = \mathbb{Z}$, the set of integers, and let $R = \{(a, b) \in A \times A \mid a < b\}$ Is R symmetric, asymmetric, or antisymmetric ?

Soln.:

Symmetry : If $a < b$, then it is not true that $b < a$, so R is **not symmetric**.

Asymmetry : If $a < b$, then $b \not< a$ (b is not less than a), so R is **asymmetric**.

Antisymmetry : If $a \neq b$, then either $a < b$ or $b < a$, so that R is **antisymmetric**.



EXAMPLES

Ex. : Let $A = \{1, 2, 3\}$ and let $R = \{(1, 2), (2, 1), (2, 3)\}$. Is R symmetric, asymmetric, or antisymmetric?

Soln.:

Symmetry : R is not symmetric either since $(2, 3) \in R$
but $(3, 2) \notin R$

Asymmetry : R is also not asymmetric since both $(1, 2)$ and $(2, 1) \in R$.

Antisymmetry : R is not antisymmetric since $(1, 2)$ and $(2, 1) \in R$.



EXAMPLES

Ex. : Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$. Is R symmetric, asymmetric, or antisymmetric?

Soln.:

Symmetry : R is not symmetric, since $(1, 2) \in R$, but $(2, 1) \notin R$.

Asymmetry : R is not asymmetric, since $(2, 2) \in R$.

Antisymmetry : R is antisymmetric, since if $a \neq b$, either $(a, b) \notin R$ or $(b, a) \notin R$.



PROPERTIES: TRANSITIVITY

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$.

$$\forall a,b,c \in A ((aRb) \wedge (bRc)) \Rightarrow aRc$$

Example

Let $A = \mathbb{Z}^+$, the set of positive integers, and let $R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$ Is R transitive?

Soln.: a divides b , aRb and b divides c , bRc
 a divides c , aRc . Thus, R is transitive.



SPECIAL CASES

1) Let $A = \{ 1, 2, 3, 4 \}$

$$R = \{ (1, 2), (1, 3), (4, 2) \}$$

Is R transitive?

Ans: YES

2) $R = \{ \}$

Ans: Yes

3) A relation that is symmetric and anti-symmetric

Ans: $R = \{(1,1), (2,2)\}$ on the set $A = \{1,2,3\}$



EXAMPLE

Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property.

- (i) R is transitive but not symmetric.
- (ii) R is symmetric but not transitive.
- (iii) R is both symmetric and anti-symmetric.
- (iv) R is neither symmetric nor anti-symmetric.

Solution:

- i. $R = \{(1, 2), (2, 3), (1, 3)\}$
- ii. $R = \{(1, 2), (2, 1)\}$
- iii. $R = \{(1, 1), (2, 2)\}$
- iv. $R = \{(1, 2), (2, 3), (3, 2)\}$



EXAMPLE

Define a relation on the set $\{a, b, c, d\}$ that is

- (i) transitive, reflexive and symmetric,
- (ii) symmetric and transitive.

Solution:

- (i) Transitive, reflexive and symmetric,

$$A = \{a, b, c, d\}$$

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (a, c), (c, a), (a, d), (d, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c)\}$$

- (ii) Symmetric and transitive.

$$A = \{a, b, c, d\}$$

$$R = \{(a, b), (b, a), (c, d), (d, c), (a, a), (c, c)\}$$



IRREFLEXIVE RELATIONS

A relation R on a set A is **irreflexive** if a not related to a , i.e. $(a, a) \notin R$ for every $a \in A$. Thus, R is irreflexive if no element is related to itself.

Examples

1. Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$.

2. R is not reflexive $(1,1) (2,2) \notin R$

Then R is irreflexive since $(1, 1) (2, 2) \notin R$.

3. Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 2)\}$.

Then R is not irreflexive since $(2, 2) \in R$.

Note that R is not reflexive either; since $(1, 1) \notin R$.



IDENTITY RELATION

Identity relation I on set A is reflexive, transitive and symmetric.

Example:

$$A = \{1, 2, 3\}$$

$$R = \{(1, 1), (2, 2), (3, 3)\}$$



VOID RELATION

It is given by $R: A \rightarrow B$ such that $R = \emptyset (\subseteq A \times B)$ is a null relation.

Void Relation $R = \emptyset$ is **symmetric and transitive** but **not reflexive**.



UNIVERSAL RELATION

A relation $R: A \rightarrow B$ such that $R = A \times B (\subseteq A \times B)$ is a universal relation.

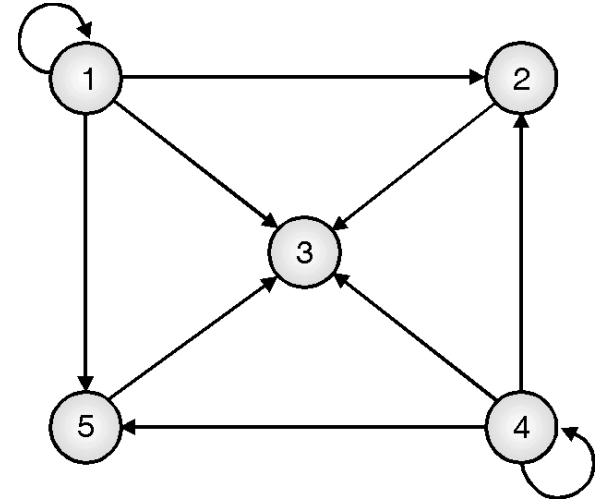
Universal Relation from $A \rightarrow B$ is **reflexive, symmetric and transitive.**



EXAMPLE

Let $A = \{1, 2, 3, 4, 5\}$

Determine whether the relation R whose digraph is given is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive.



R is not reflexive

R is not irreflexive

R is not symmetric

R is not asymmetric

R is antisymmetric

R is transitive



EXERCISE : PROPERTIES OF RELATIONS

State whether R satisfies property of reflexive ,
irreflexive , symmetry, asymmetry , antisymmetry ,
transitivity for $A=\{1,2,3,4\}$

$$R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,3),(3,4),(4,4)\}$$

$$R= \{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$$

$$R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4)\}$$

$$R=\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$$

$$R=\{(1,1),(2,2),(3,3),(4,4)\}$$



EQUIVALENCE RELATION

A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive.

Let $A = \{ a , b , c \}$ and

$R = \{ (a,a), (b,b), (b,c), (c,b), (c,c) \}$

is an equivalence relation since it is reflexive, symmetric, and transitive.



DETERMINE WHETHER R IS AN EQUIVALENCE RELATION

Let $A = \{a, b, c\}$ and let , $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Determine whether R is an equivalence relation.

Soln.: $R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$

R is reflexive since $(a, a), (b, b), (c, c) \in R$

R is symmetric since $(b, c) \in R \rightarrow (c, b) \in R$

R is transitive since,

(b, b)	and	$(b, c) \in R$	implies	$(b, c) \in R,$
(b, c)	and	$(c, b) \in R$	implies	$(b, b) \in R,$
(c, c)	and	$(c, b) \in R$	implies	$(c, b) \in R,$
(c, b)	and	$(b, b) \in R$	implies	$(c, b) \in R,$
(c, b)	and	$(b, c) \in R$	implies	$(c, c) \in R,$
(b, c)	and	$(c, c) \in R$	implies	$(b, c) \in R,$

Hence R is an equivalence relation.



DETERMINE WHETHER R IS AN EQUIVALENCE RELATION

Let $A = \mathbb{Z}$, the set of integers, and let R be defined by $a R b$ if and only if $a \leq b$. Is R an equivalence relation?

- Since $a \leq a$, R is reflexive.
- If $a \leq b$, it need not follow that $b \leq a$, so R is not symmetric.
- Incidentally, R is transitive, since $a \leq b$ and $b \leq c$ imply that $a \leq c$.
- We see that R is not an equivalence relation.



DETERMINE WHETHER R IS AN EQUIVALENCE RELATION

Let $A = \{1, 2, 3, 4\}$ and

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$

Determine whether the relation R on the set A is an equivalence relation.

Soln.:

R is reflexive since $(1, 1), (2, 2), (3, 3), (4, 4) \in R$. R is not symmetric since, $(4, 1) \in R$ but $(1, 4) \notin R$.

R is not transitive since,

$(2, 1), (1, 3) \in R$ but $(2, 3) \notin R$

Hence given relation R is not an equivalence relation.



Let R be a binary relation on the set of all positive integers such that,

$$R = \{(a, b) \mid a - b \text{ is an odd positive integer}\}$$

Is R reflexive ? Symmetric ? Transitive ? An equivalence relation ?

Soln.: R is not Reflexive since

$$a - a = 0 \neq \text{odd positive integer}$$

$$\therefore a \not R a$$

R is not symmetric also, since, as if $a R b$ then

$$a - b = 2n + 1 \text{ where } n = \text{integer number}$$

if $b R a$ then $b - a = -2n - 1$ where $n = \text{integer number}$

$$\therefore b - a \neq \text{odd positive integer}$$

R is not transitive since,

Let $a R b$ and $b R c$

$$\text{i.e. } a - b = 2n_1 + 1$$

$$b - c = 2n_2 + 1 \dots \text{i.e. odd positive integer}$$

$$a - c = (2n_1 + 1) + (2n_2 + 1) = 2(n_1 + n_2 + 1)$$

$$\neq \text{odd positive integer}$$

Hence R is not transitive.

Therefore R is not an equivalence relation.



EQUIVALENCE CLASS AND PARTITIONS

Let $A = \{ 1, 2, 3, 4 \}$ and consider the partition

$$P = \{ \{ 1, 2, 3 \}, \{ 4 \} \} \text{ of } A.$$

Find the equivalence relation R on A determined by P

“ Each element in a block is related to every other element in the same block and only to those elements”

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)\}$$



PROBLEMS

Find the equivalence relation on A by P

1) Let $A = \{a, b, c, d\}$ and $P = \{\{a, b\}, \{c\}, \{d\}\}$

$R = \{(a,a), (a,b), (b,b), (b,a), (c,c), (d,d)\}$

2) Let $A = \{1, 2, 3, 4, 5\}$ and $P = \{\{1, 2\}, \{3\}, \{4, 5\}\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4), (4,5), (5,5), (5,4)\}$

3) If $\{\{1, 3, 5\}, \{2, 4\}\}$ is a partition on the set $A = \{1, 2, 3, 4, 5\}$, determine the corresponding equivalence relation

$R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,4), (4,2), (4,4)\}$



EQUIVALENCE CLASS

Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be the equivalence relation on A defined by

$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$

Find the equivalence classes of R and find the partition of A induced by R



$$R = \{(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), \\ (3,6), (4,4), (5,1), (5,5), (6,2), (6,3), (6,6)\}$$

Equivalence Classes:

$$R(1) = \{1, 5\}$$

$$R(2) = \{2, 3, 6\}$$

$$R(3) = \{2, 3, 6\}$$

$$R(4) = \{4\}$$

$$R(5) = \{1, 5\}$$

$$R(6) = \{2, 3, 6\}$$

Therefore, the partition of A induced by R i.e

$$A | R = \{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$$

Rank R (Number of distinct equivalence classes)
= 3



PROBLEMS: FIND E-CLASSES, PARTITION AND RANK

1. Let $A=\{1,2,3\}$ and let $R=\{(1,1),(2,2),(1,3),(3,1),(3,3)\}$.

Find $A \mid R$.

Ans: $R(1)=\{1,3\}$ $R(2)=\{2\}$ $R(3)=\{1,3\}$

$A \mid R=\{\{1,3\},\{2\}\}$, Rank=2

2. Let $A=\{1,2,3,4\}$, and let $R=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$

Determine $A \mid R$.

Ans: $R(1)=\{1,2\}$ $R(2)=\{1,2\}$ $R(3)=\{3,4\}$ $R(4)=\{3,4\}$

$A \mid R=\{\{1,2\},\{3,4\}\}$ Rank=2

3. Let $A=\{1,2,3,4\}$, and let

$R=\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(2,3),(3,2),(3,3),(4,4)\}$ Show that R is an equivalence relation and determine the equivalence classes and hence find the rank of R

Ans: $R(1)=\{1,2,3\}$ $R(2)=\{1,2,3\}$ $R(3)=\{1,2,3\}$ $R(4)=\{4\}$

$A \mid R=\{\{1,2,3\},\{4\}\}$

Rank=2



COMBINING RELATIONS

- Relations are simply... sets (of ordered pairs); subsets of the Cartesian product of two sets
- Therefore, in order to combine relations to create new relations, it makes sense to use the usual set operations
 - Compliment \bar{R}
 - Intersection ($R_1 \cap R_2$)
 - Union ($R_1 \cup R_2$)
 - Set difference ($R_1 \setminus R_2$)
 - Inverse R^{-1}



EXAMPLES

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$ and

$R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$ and

$R_2 = \{(1, v), (3, u), (3, v)\}$

$R_1 \cup R_2 = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$

$R_1 \cap R_2 = \{(3, u)\}$

$R_1 - R_2 = \{(1, u), (2, u), (2, v)\}$

$R_2 - R_1 = \{(1, v), (3, v)\}$



EXAPLES

Let $A = \{ 1, 2, 3, 4 \}$ and $B = \{ a, b, c \}$ and let

$R = \{(1,a), (1,b), (2,b), (2,c), (3,b), (4,a)\}$ and

$S = \{(1,b), (2,c), (3,b), (4,b)\}$

Compute $R \cap S$, $R \cup S$, R^{-1}

$R \cap S = \{(1,b), (2,c), (3,b)\}$

$R \cup S = \{(1,a), (1,b), (2,b), (2,c), (3,b), (4,a), (4,b)\}$

$R^{-1} = \{(a,1), (b,1), (b,2), (c,2), (b,3), (a,4)\}$



COMBINING RELATIONS: EXAMPLE

Let

$$A=\{1,2,3,4\}$$

$$B=\{1,2,3,4\}$$

$$R_1=\{(1,2),(1,3),(1,4),(2,2),(3,4),(4,1),(4,2)\}$$

$$R_2=\{(1,1),(1,2),(1,3),(2,3)\}$$

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(3,4),(4,1),(4,2)\}$$

$$R_1 \cap R_2 = \{(1,2),(1,3)\}$$

$$R_1 - R_2 = \{(1,4),(2,2),(3,4),(4,1),(4,2)\}$$

$$R_2 - R_1 = \{(1,1),(2,3)\}$$



COMPOSITE OF RELATIONS

- **Definition:** Let R_1 be a relation from the set A to B and R_2 be a relation from B to C , i.e.

$$R_1 \subseteq A \times B \text{ and } R_2 \subseteq B \times C$$

the composite of R_1 and R_2 is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_2 \circ R_1$$



COMPOSITE OF RELATIONS

Ex: Let $A = \{1,2,3\}$, $B = \{0,1,2\}$ and $C = \{a,b\}$

$R = \{(1,0),(1,2),(3,1),(3,2)\}$

$S = \{(0,b),(1,a),(2,b)\}$

$S \circ R = ?$

$\{(1,b), (3,a), (3,b)\}$

Since $(1,0) \in R$ and $(0,b) \in S$, $\therefore (1,b) \in S \circ R$

Since $(1,2) \in R$ and $(2,b) \in S$, $\therefore (1,b) \in S \circ R$

Since $(3,1) \in R$ and $(1,a) \in S$, $\therefore (3,a) \in S \circ R$

Since $(3,2) \in R$ and $(2,b) \in S$, $\therefore (3,b) \in S \circ R$



PROBLEMS

Let $A=\{1,2,3\}$ and let

$R=\{(1,1),(1,3),(2,1),(2,2),(2,3),(3,2)\}$ and

$S=\{(1,1),(2,2),(2,3),(3,1),(3,3)\}$.

Find M_{SoR}

$\text{SoR}=\{(1,1),(1,3),(2,1),(2,2),(2,3),(3,2),(3,3)\}$

2. Let $A=\{1,2,3,4\}$

$R=\{(1,1),(1,2),(2,3),(2,4),(3,4),(4,1),(4,2)\}$

$S=\{(3,1),(4,4),(2,3),(2,4),(1,1),(1,4)\}$

Compute $\text{SoR}, \text{RoS}, \text{RoR}, \text{SoS}$

$\text{SoR}=\{(1,1),(1,3),(2,1),(2,4),(3,4),(4,1),(4,4),(1,4)\}$

$\text{RoS}=\{(3,1),(3,2),(4,1),(4,2),(2,4),(2,1),(2,2),(1,1),(1,2)\}$

$\text{RoR} = ?$

$\text{SoS} = ?$



CLOSURES

The 'smallest' relation R_1 on A that contains R and possesses the property we desire. Sometimes R_1 does not exist. If a relation such as R_1 does exist, we call it the 'closure' of R with respect to the property in question.



REFLEXIVE CLOSURE

Let R be a relation on a set A , and R is not reflexive (i.e. some pairs of the diagonal relation Δ are not in R).

A relation $R_1 = R \cup \Delta$ is the reflexive closure of the relation R if $R \cup \Delta$ is the smallest relation containing R which is reflexive.

$$R_1 = R \cup \Delta$$

where Δ is the set of elements of the type (a, a)
where $a \in A$.



EXAMPLE

$A = \{1, 2, 3\}$ and the relation R is given by

$R = \{(1, 1), (1, 2), (2, 3)\}$ then

$R_1 = R \cup \Delta$ where

$\Delta = \{(1, 1), (2, 2), (3, 3)\}$

$R \cup \Delta = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$

Reflexive closure is,

$$R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$$



SYMMETRIC CLOSURE

Suppose that R is a relation on A that is not symmetric. Then there must exist pairs (x, y) in R such that (y, x) is not in R . Of course, $(y, x) \in R^{-1}$, so if R is to be symmetric, we must add all pairs from R^{-1} ; that is, we must enlarge R to $R \cup R^{-1}$. Clearly $(R \cup R^{-1})^{-1} = R \cup R^{-1}$, So $R \cup R^{-1}$ is the smallest symmetric relation containing R ; that is $R \cup R^{-1}$ is the 'symmetric closure' of R .



EXAMPLE

$A = \{a, b, c, d\}$ and

$R = \{(a, b), (b, c), (a, c), (c, d)\}$ then

$R^{-1} = \{(b, a), (c, b), (c, a), (d, c)\}$

so the symmetric closure of R is

$R \cup R^{-1} = \{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a), (c, d), (d, c)\}$



TRANSITIVE CLOSURE

Let R be a relation on a set A . Then the 'transitive closure' of a relation R is the smallest transitive relation containing R . The transitive closure of R is just the connectivity relation R^∞ .

$R^* = \text{Transitive closure of } R$

$$= R \cup \{(a, c), \text{ if and only if } (a, b), (b, c) \in R\}$$



EXAMPLE

Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph shown

Soln.:

$R = \{(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$

Here transitive closure of R is

$= R \cup \{(a, c) \mid \text{if } (a, b), (b, c) \in R\}$

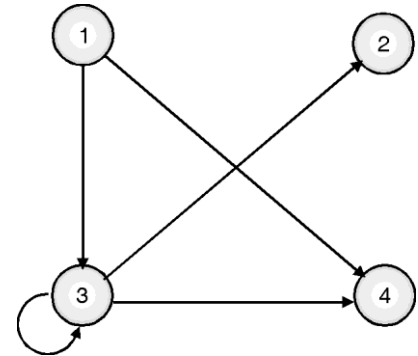
To find transitive closure

$(1, 3) \in R$ and $(3, 4) \in R$, hence add $(1, 4)$ in R

$(1, 3) \in R$ and $(3, 3) \in R$, hence add $(1, 3)$ in R

$(1, 3) \in R$ and $(3, 2) \in R$, hence add $(1, 2)$ in R

Transitive closure of $R = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$



MATRIX METHOD

Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. Find the transitive closure of R . The matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)_\odot^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (M_R)_\odot^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)_\odot^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^2 = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$$

$$R^3 = \{(1, 2), (1, 4), (2, 1), (2, 3)\}$$

$$R^4 = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$$

$$R^\infty = R \cup R^2 \cup R^3 \cup R^4$$

$$R^\infty = \{(1, 1), (1, 2), (1, 3), (1, 4), \\ (2, 1), (2, 2), (2, 3), \\ (2, 4), (3, 4)\}$$

$$M_R^\infty = M_R \vee (M_R)_\odot^2 \vee (M_R)_\odot^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

WARSHALL'S ALGORITHM

Ex. 1 : Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$.

Find transitive closure of R using Warshall's algorithm.

Solution:

$$W_0 = M_R = \begin{bmatrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

First we find W_1 , so that $k = 1$. W_0 has 1's in location 2 of column 1 i.e. (2, 1) and location 2 of row 1 i.e. (1, 2)

$i \quad j$
 $p_1: (2, 1)$

$i \quad j$
 $q_1: (1, 2)$

add (p_i, q_j) i.e. (2, 2) in W_k

Thus, W_1 is just W_0 with a new 1 in position (2, 2)

$$W_1 = \begin{bmatrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$



$$W_1 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Matrix W_1 has 1's at row 1 and 2 of column 2 and columns 1, 2, and 3 of row 2. i.e.

$$\begin{array}{ll} \begin{matrix} i & j \\ p_1 & : (1, 2) \end{matrix} & \begin{matrix} i & j \\ p_2 & : (2, 2) \end{matrix} \\ \begin{matrix} i & j \\ q_1 & : (2, 1) \end{matrix} & \begin{matrix} i & j \\ q_2 & : (2, 2) \end{matrix} & \begin{matrix} i & 1 \\ q_3 & : (2, 3) \end{matrix} \end{array}$$

We must put 1's in positions (p_i, q_j) i.e. $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$ and $(2, 3)$ of matrix W_1 (if 1's are not already there).

$$W_2 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} i & j & & i & j \\ p_1 & : (1, 3) & & p_2 & : (2, 3) \\ & & i & j \\ q_1 & : (3, 4) & & (1, 4) \text{ and } (2, 4) \end{matrix}$$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, W_3 has 1's in locations 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and $MR_\infty = W_4 = W_3$.



THE SMALLEST EQUIVALENCE RELATION CONTAINING UNION OF TWO EQUIVALENCE RELATION

- If R and S are two equivalent relations defined on the set A , then the smallest equivalence relation that contains both R and S is the transitive closure of $R \cup S$ i.e. $(R \cup S)^\infty$.
- Find transitive closure using Warshall's Algorithms



EXAMPLE:

Let $A = \{1,2,3,4,5\}$ $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$ and $S = \{(1,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$. The reader may verify that both R and S are equivalence relations. The partition $A \mid R$ of A corresponding to R is $\{\{1,2\}, \{3,4\}, \{5\}\}$, and the partition $A \mid S$ of A corresponding to S is $\{\{1\}, \{2\}, \{3\}, \{4,5\}\}$. Find the smallest equivalence relation containing R and S , and compute the partition of A that it produces.



$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{So } M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We now compute $M_{(R \cup S)^\infty}$ by Warshall's algorithm. First, $W_0 = M_{R \cup S}$. We next compute W_1 so $k = 1$. Since W_0 has 1's in locations 1 and 2 of column 1 and in locations 1 and 2 of row 1, we find that no new 1's must be adjoined to W_1 . Thus

$$W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

K=1

$$\begin{array}{cc} i & j \\ p_1 : & (1, 1) \quad p_2 : & (2, 1) \end{array}$$

$$\begin{array}{cc} i & j \\ q_1 : & (1, 1) \quad q_2 : & (1, 2) \end{array}$$

To obtain W_1 , we must put 1's in positions (1, 1), (1, 2), (2, 1) and (2, 2). We see that

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_1 = W_0$

We now compute W_2 , so $k = 2$.

Since W_1 has 1's in locations 1 and 2 : of column 2 and in locations 1 and 2 of row 2, we find that no new 1's must be added to W_1 . That is,

$$\begin{array}{cc} i & j \\ p_1 & : (1, 2) \end{array} \quad \begin{array}{cc} i & j \\ p_2 & : (2, 2) \end{array}$$

$$\begin{array}{cc} i & j \\ q_1 & : (2, 1) \end{array} \quad \begin{array}{cc} i & j \\ q_2 & : (2, 2) \end{array}$$

To obtain W_2 , we must put 1's in positions (1, 1), (1, 2), (2, 1), (2, 2). We see that

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_2 = W_1$

We next compute W_3 , so $k = 3$. Since W_2 has 1's in locations 3 and 4 of column 3 and in locations 3 and 4 of row 3, we find that no new 1's must be added to W_2 . That is

$$\begin{array}{cc} i & j \\ p_1 & : (3, 3) \end{array} \quad \begin{array}{cc} i & j \\ p_2 & : (4, 3) \end{array}$$

$$\begin{array}{cc} i & j \\ q_1 & : (3, 3) \end{array} \quad \begin{array}{cc} i & j \\ q_2 & : (3, 4) \end{array}$$

To obtain W_3 , we must put 1's in position (3, 3), (3, 4), (4, 3), (4, 4). We see that

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$



Thus $W_3 = W_2$

Things change when we now compute W_4 . Since W_3 has 1's in locations 3, 4, and 5 of column 4 and in locations 3, 4 and 5 of column 4, and in locations 3, 4 and 5 of row 4 we must add new 1's to W_3 in positions 3, 5, and 5, 3, i.e.

$$\begin{array}{ccc} \begin{array}{c} i \quad j \\ p_1 : (3, 4) \end{array} & \begin{array}{c} i \quad j \\ p_2 : (4, 4) \end{array} & \begin{array}{c} i \quad j \\ p_3 : (5, 4) \end{array} \\ \begin{array}{c} i \quad j \\ q_1 : (4, 3) \end{array} & \begin{array}{c} i \quad j \\ q_2 : (4, 4) \end{array} & \begin{array}{c} i \quad j \\ q_3 : (4, 5) \end{array} \end{array}$$

To obtain W_4 , we must put 1's in positions (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5). We see that,

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

You may verify that $W_5 = W_4$ and thus

$$(R \cup S)^\infty = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$$

