"Successive Over-Relaxation Solver for Linear Systems"

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Abstract

Almost anything in computer graphics, animation, computer vision, image processing or scientific computing, will involve extensive use of vectors and matrices (linear algebra) from simple things like representing spatial transformations and orientations, to very complex algorithms. These things used to be the domain of supercomputing, but now Linear Systems is implemented extensively. Thus, to operate on such applications and to bring about more functionality, Linear Systems need to be solved more efficiently.

This research involves the study of basic linear system solvers that can approximate the solution of a linear system of the form- **Ax = b**, **where b is a given n-dimensional vector, A is a n x n coefficient matrix and x is the unknown vector which needs to be approximated**. The aim of this project is to understand the numerical techniques behind the iterative procedures for approximating solution of a linear system of the form **Ax = b**. Linear systems of the aforementioned form can be solved using *direct* methods but these methods prove inefficient when the order of the coefficient matrix exceeds millions, analogous to the matrix of pixels in a computer screen. Thus iterative procedures/solvers/methods need to be used to carry out operations on linear systems more efficiently.

Three iterative methods were explored in this project, to approximate the unknown vector x –
1. Jacobi method,
2. Gaus-Seidal method,
3. Successive Over-Relaxation Method (SOR method).

The efficiency of the **Successive Over-Relaxation Method (SOR method)** method was compared with the other two. The research consists of Convergence Analysis which involves the use of a 3 × 3 coefficient matrix (diagonally dominant), used in approximating the unknown vector **x** by the Jacobi method. Using linear algebra, maximal norms and estimation and approximation, the **error e was deduced to be 0 at the end of the iterative method. The error e is the difference between the exact solution of the system and the approximate solution obtained at the end of each iteration.**

All the aforementioned iterative methods were used to solve a linear system with the coefficient matrix as the **Poisson matrix- a 100 × 100 diagonally dominant matrix** (all the coefficient matrices used to observe the working of the iterative methods, in this research, are diagonally dominant). The solution to the linear system was approximated using all the three iterative methods. In all cases, the iterations were stopped when the **residual error, 'r', was very small or 0. Residual error, r, is defined as the difference between, the product of the coefficient matrix and the approximated vector after each iteration (Ax_a), and the known vector b. Therefore, r = Ax_a – b.**

Scilab, a tool for numerical analysis, was used for developing algorithms to model the iterative methods and to run them. The results for approximating the solution of the aforementioned linear system (the one with the Poisson matrix) with all the iterative methods, were as follows-

- The Jacobi method converged after 411 iterations.
- The Gaus-Seidal method converged after 207 iterations.
- The Successive Over-Relaxation method converged in 64 iterations.

Graphs of residual error vs. iterations were plotted for all the iterative methods. **The above result for the SOR method was obtained with the parameter as 1.47 and the relaxation matrix as an identity matrix. The parameter, 1.47, was determined to be the ideal parameter for the linear system in consideration by another algorithm developed using Scilab.**

The results obtained above clearly indicate that the SOR method is far superior to the other two methods, and thus is the ideal method to approximate the solution to the linear system of the form Ax = b, where the coefficient matrix (A) is a diagonally dominant square matrix.

Key words: Computer graphics; Linear Algebra; Linear System Solvers; Simulations.