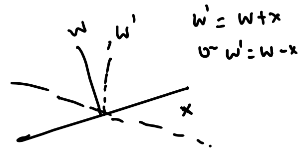
Optimization

Math for ML

→ Prob / Stat (Prob, Cond prob, hypothesisting, Est..)

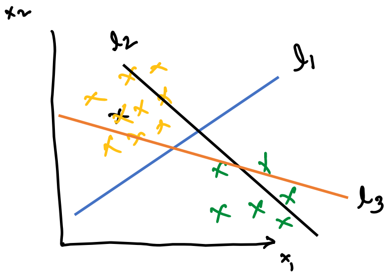
→ Linear Algebra / Coord Geometry - DB, Hyperplane, Dot. Products, distances

→ Optimization { Calculus, Gradient descent }



Why optimization?

What is a DB, How do we define (Math) a best 'DB' ?

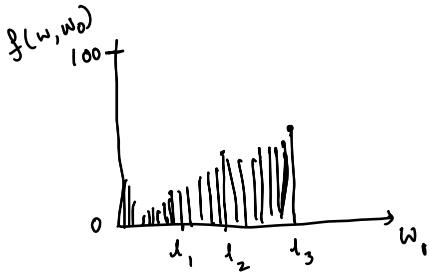


$$\max \uparrow f(w, w_0) = \frac{1}{n} \sum_{i=1}^{n} \frac{w^T \cdot x + w_0}{\|w\|} \cdot y_i$$

w, w_0 where $f(w, w_0)$ Max

Which is the best 'DB'

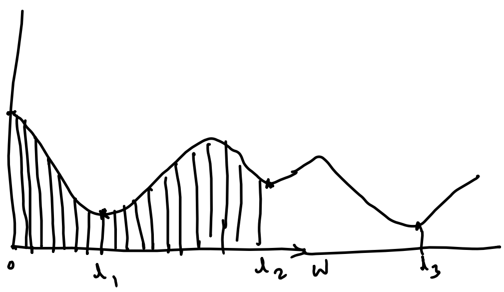
$-f(w, w_0)$. Min ? ✓



$w[w_1, w_2]$, w_0

Minimum Possible value

Cost function { loss function { error function



Data x_i y_i

(x_i, y_i) $\{x_i - R^d\}$

$x\{x_1, x_2\}$ $x_i - R^2$

Goal → $f(w, w_0)$ w, w_0 Minimum possible value.

$y - \{-1, 1\}$

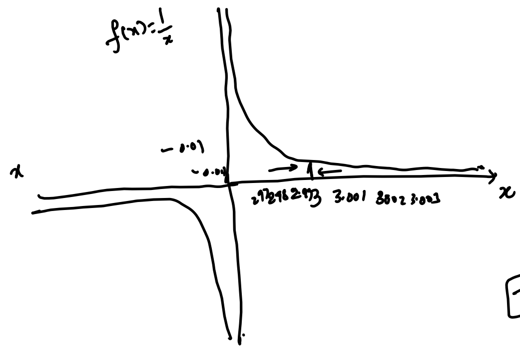
→ calculus → finding maxima or minima of a function.

w, w_0 at which $f(w, w_0)$ takes minimum possible value

Minima $\rightarrow$ derivative of a function $\rightarrow$ Limits and Continuity ?
(Partial derivative)

How do we find out if a function is Continuous? $(x \in R)$

$f(x) = \frac{1}{x}$



$0.01 \rightarrow 100$
$0.001 \rightarrow 1000$
$0.00001\ldots \rightarrow \infty$

$\boxed{x=3}$ at $x=0$? undefined

Mathematical Limits

$\lim_{x \to k} f(x)$

$\lim_{x \to 3} f(x) = \frac{1}{3}$ $\qquad f(x) = \frac{1}{x}$

$\lim_{x \to 0} f(x) \rightarrow$ Left hand side limit / Right hand side limit

$\lim_{x \to 3^+} f(x)$
$\lim_{x \to 3^-} f(x)$

$\lim_{x \to 3}$ RHL

$\frac{1}{3.01} = 0.33---$

$\frac{1}{3.001} = 0.33.---$

$\frac{1}{3.0001} = 0.33---$

$\lim_{x \to 3}$ LHL

$\frac{1}{2.90} = 0.32----$

$\frac{1}{2.99} = 0.32--$

$\frac{1}{2.9999} = 0.3.2---$

$f(x)$
$x=3$ $\quad \frac{1}{3} = 0.33$

LHL = RHL = f(a) ✓ $\quad a=3$

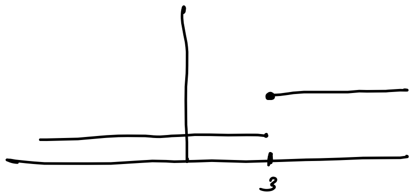
$a=0$

$\lim_{x \to 0} f(x) = \frac{1}{x}$

LHL		RHL		f(a)
$\frac{1}{-0.01}$	-100	$\frac{1}{0.01}$	100	
$\frac{1}{-0.001}$	-1000	$\frac{1}{0.001}$	1000	undefined.
$\frac{1}{-0.00001}$	-100000	$\frac{1}{0.0001}$	10000	
$\vdots$	$-\infty$	$\vdots$	$+\infty$	

at '0' LHL = RHL = f(a) ?

e.g. $\boxed{f(x) = \begin{cases} 2 & x < 3 \\ 5 & x \geq 3 \end{cases}}$



$\lim_{x \to 2} f(x) = 2$ ✓

$f(3) = 5$?

$\lim_{x \to 3} f(x) <$ LHL / RHL

LHL	f(x)	f(x)	RHL
2.9 →	2	5	← 3.01
2.99 →	2	5	← 3.001
2.999 →	2	5	← 3.0001

LHL = 2, RHL = 5, $f(a) = 5$?

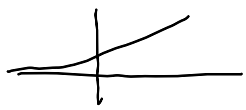
Continuity:-
a function $f(x)$ is Cont. at a if LHL = RHL = $f(a)$ ✓

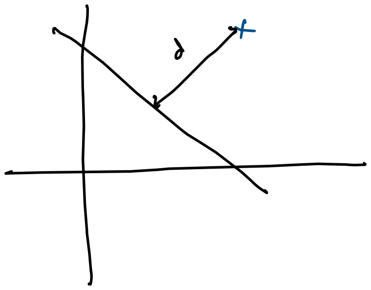
Cont. function:-
if it is Cont. for all values it can take

all values a function can take is called 'domain'

(exclusive
[inclusive)

function	Domain	Range (output)
$\frac{1}{x}$	$(-\infty, \infty)$ R	$(-\infty, \infty)$ R
e^x	R	$(0, \infty)$
$\log x$	$(0, \infty)$	R
$\frac{1}{1+e^{-x}}$ (Sigmoid)	R	$(0, 1)$ ← prob





d → Sigmoid → (0 - 1)

Examples

Here are a few example problems. Using the definition above, try to determine if they are continuous or not.

EXAMPLE

Is the function $f(x) = \begin{cases} 2x+1 \ (x < 3) \\ 3x-2 \ (x \geq 3) \end{cases}$ continuous for all $x \in \mathbb{R}$?

We know that the graphs of $y = 2x + 1$ and $y = 3x - 2$ are continuous, so we only need to see if the function is continuous at $x = 3$. The procedure is simply using the definition above, as follows:

(i) Since $f(3) = 3 \times 3 - 2 = 7$, $f(3)$ exists.

(ii) In order to see whether the limit exists or not, we have to check the limit from both sides. The left-hand and right-hand limits are

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x+1) = 2 \times 3 + 1 = 7 \quad \text{and} \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3x-2) = 3 \times 3 - 2 = 7,$$

respectively. Because the limits from both sides are equal, $\lim_{x \to 3} f(x)$ exists.

(iii) Now from (i) and (ii), we have $\lim_{x \to 3} f(x) = f(3) = 7$, so the function is continuous at $x = 3$. $\square$