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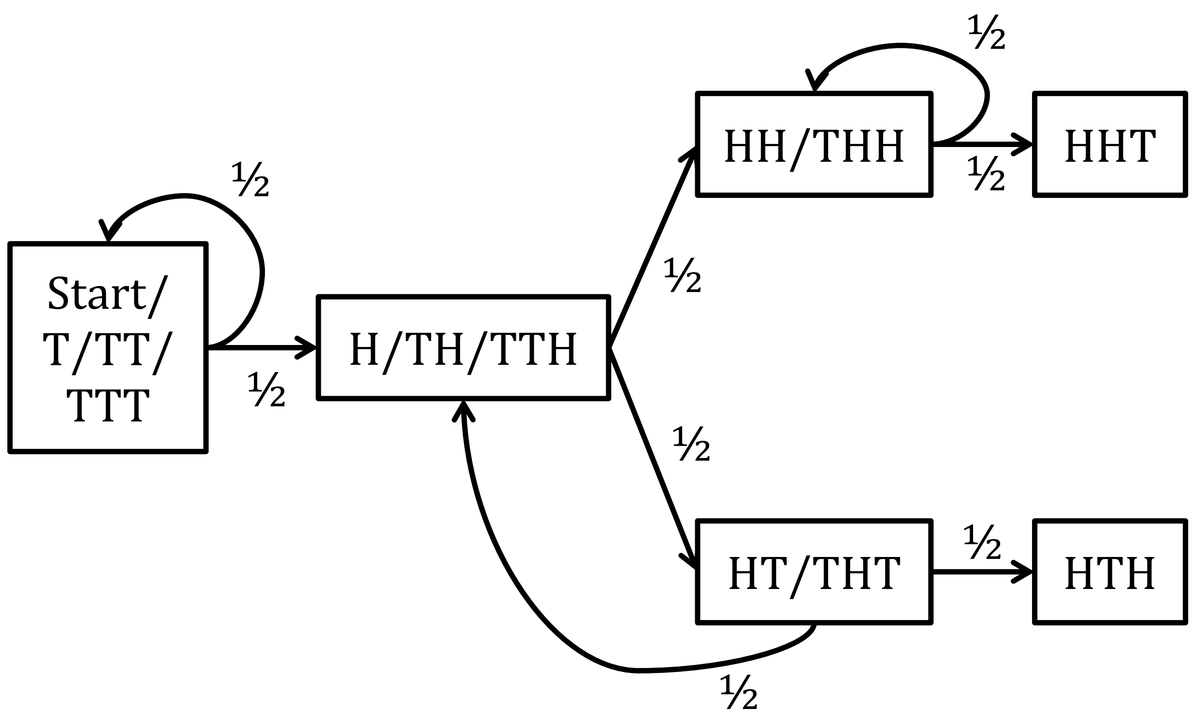
Sequences of Three Coin Flips – Markov Chains and Martingales

This is a cool stochastic process I encountered recently. I encountered it in the context of the first of the following questions (involving a Markov chain) and expanded the scope to include the second question regarding stopping times (involving martingales).

The process: suppose we have a game in which we repeatedly flip a fair coin. Person #1 wins the game if the sequence *HHT* occurs before Person #2's sequence of *HTH*.

Question 1: What is the probability that each person wins the game?

This game is a Markov chain with the following state space, where each state is the most recent three coin flips (or the first one or two coin flips if there have been less than three coin flips in the game).



In any game, we will reach the state *H/TH/TTH* before anyone wins, so we can calculate probabilities starting from that state. From the diagram, we can see that if *HH* occurs, then eventually *HHT* will win the game with probability 1. On the other hand, if *HT/THT* occurs, then there is a ½ chance that *HTH* will win on the next flip and a ½ chance that we return to the *H/TH/TTH* state. From this information, we can calculate probability that Person #1 wins as follows:

$$P(Person\ \#1\ wins) = \frac{1}{2} + \frac{1}{4} P(Person\ \#1\ wins) \quad => \quad P(Person\ \#1\ wins) = \frac{2}{3}.$$

Thus the probability that Person #1 wins is 2/3, and the probability that Person #2 wins is 1/3.

We can check that this makes sense by running the simple Monte Carlo simulation in simulate_game.py.

Question 2: What is the expected number of coin tosses before we observe the sequence *HHT*? The Sequence *HTH?* Before the game ends?

This is where things get pretty cool. We can construct a martingale around the game and use martingale theorems to answer the question.

Martingale method to calculate expected number of coin tosses before the sequence *HHT* occurs

Consider a fair coin toss gambling game (i.e., the expected gain from each bet is zero) run by a casino. A gambler betting $1 on the outcome of the next bet being *H* will lose with probability ½ and win $2= $1/(½) with probability ½.

To construct a martingale, let there be a sequence of gamblers betting at the casino, each starting with $1. And let X_i be the outcome of the *i*-th game. Gambler *i* will bet $1 that $X_i = H$. If he wins, he bets his winnings ($1/(½) = $2) that $X_{i+1} = H$. And if he also wins that second bet, he bets all of his winnings ($1/(½*½) = $4) that $X_{i+2} = T$. If he wins this final bet, then he quits with a fortune of $1/(½*½*½) = $8.

Let M_n be the casino's winnings after *n* games. Then $\{M_n\}_{n\in\mathbb{N}}$ is a mean zero martingale (adapted to the natural filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$). Let τ be the stopping time as determined by first witnessing the pattern *HHT*. Then by Doob's Optional Stopping Theorem, since $E[\tau] < \infty$ and there exists $M < \infty$ such that

$$E[\,|M_{n+1} - M_n| \mid \mathcal{F}_n] < M, \qquad \forall n,$$

M_τ is integrable and $E[M_\tau] = E[M_0]$.

At stopping time τ:

- Gamblers 1, 2, …, $\tau - 3$ all lost $1
- Gambler $\tau - 2$ gained $8-1 = $7
- Gambler $\tau - 1$ lost $1
- Gambler τ lost $1

Then $M_\tau = (\tau - 3) - 7 + 2 = \tau - 8$. We have $E[M_\tau] = 0$, so

$$\boldsymbol{E[\tau] = 8}.$$

We can check that this makes sense by running the simple Monte Carlo simulation in simulate_HHT_stopping.py.

Using the same method to calculate the expected *HTH* stopping time:

- Gamblers 1, 2, …, $\tau - 3$ have all lost \$1
- Gambler $\tau - 2$ has gained \$8-1 = \$7
- Gambler $\tau - 1$ has lost \$1
- Gambler τ has gained \$2-1 = \$1

In this case, $M_\tau = (\tau - 3) - 7 + 1 - 1 = \tau - 10$. And so $\boldsymbol{E}[\boldsymbol{\tau}] = \mathbf{10}$.

We can check that this makes sense by running the simple Monte Carlo simulation in simulate_HTH_stopping.py.

Martingale method to calculate expected number of coin tosses before the game ends

This one is slightly more complicated, but we can follow the same general method as above. We use the same fair gambling game with τ gamblers. As above, gambler i will bet \$1 that $X_i = H$. If he wins, he bets his winnings (\$1/(½) = \$2) that $X_{i+1} = H$. And if he also wins that second bet, he bets all of his winnings (\$1/(½*½) = \$4) that $X_{i+2} = T$. If he wins this final bet, then he quits with fortune of \$1/(½*½*½) = \$8.

We again let M_n be the casino's winnings after n games and consider the martingale $\{M_n\}_{n\in\mathbb{N}}$. Let τ be the stopping time as determined by first witnessing either the pattern *HHT* or the pattern *HTH*. Again, using Doob's Optional Stopping Theorem, $E[M_\tau] = E[M_0] = 0$.

At stopping time τ there are two possible outcomes since the process could have stopped with *HHT* or *HTH* (with probabilities 2/3 and 1/3 respectively):

- Gamblers 1, 2, …, $\tau - 3$ all lost \$1 in either case
- Gambler $\tau - 2$ gained \$8-1 = \$7 with probability 2/3 and lost \$1 with probability 1/3
- Gambler $\tau - 1$ lost \$1
- Gambler τ lost \$1 with probability 2/3 and gained \$2-1 = \$1 with probability 1/3.

There are two possible states for M_τ, depending on the stopping sequence that occurred. We can calculate the expected value as

$$0 = E[M_\tau] = (E[\tau] - 3) + \frac{1}{3} - \frac{2}{3} * 7 + 1 + \frac{2}{3} - \frac{1}{3} = E[\tau] - 6.$$

And so $\boldsymbol{E}[\boldsymbol{\tau}] = \mathbf{6}$.

As before, we can check that this makes sense by running the simple Monte Carlo simulation in simulate_game_stopping.py.

Further Questions

These expected stopping times don't directly give you the probabilities of betting *HHT* vs. *HTH* winning the original game. It would be really cool to be able to calculate these probabilities from the stopping times themselves. One way to do this would be to somehow calculate the expected stopping time of the game (without using the probabilities themselves) and the conditional expectations $E[\tau_{HHT} \mid HHT\ happens\ first]$ and $E[\tau_{HTH} \mid HTH\ happens\ first]$. We could then calculate the probabilities from

$$E\left[\tau_{game}\right] = E[\tau_{HHT} \mid HHT\ first] * P(HHT\ first) + E[\tau_{HTH} \mid HTH\ first] * P(HTH\ first);$$

and the fact that $P(HTH\ first) = 1 - P(HHT\ first)$.