ORIE 6590 Project

ADP using Fluid and Diffusion Approximations

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Chen et. al., *Approximate Dynamic Programming using Fluid and Diffusion Approximations with Applications to Power Management*, Proceedings of the 48h IEEE Conference on Decision and Control held jointly with 28th Chinese Control Conference (2009)

Question: How to determine a good feature space for approximate DP algorithms?

This paper: Use approximations from simplified versions of the problem to choose basis

Outline of this talk:

1. Speed scaling problem
2. Fluid approximation
3. Diffusion approximation
4. LSTD using fluid and diffusion basis

Idea behind fluid and diffusion models

Fluid model: *Scale time, state by n: analogous to SLLN*

$$\frac{1}{n}X(nt) \to \bar{X}(t)$$

Diffusion model: *Scale time by n, state by $\sqrt{n}$: analogous to CLT*

$$\sqrt{n}\left(\frac{1}{n}X(nt) - \bar{X}(t)\right) = \frac{X(nt) - \bar{X}(nt)}{\sqrt{n}} \to \hat{X}(t)$$

Then think of $\bar{X}(t) + \hat{X}(t)$ as an approximation to $X(t)$

Speed scaling problem

Problem: *Control computer processing speed to balance energy and job delay costs. Single server queue with controllable service rate.*

For $t = 0, 1, 2, \ldots,$

- $A(t) =$ i.i.d. job arrivals
- $X(t) =$ queue length (system state)
- $U(t) =$ service rate (the action to take).

States evolve as $X(t+1) = X(t) - U(t) + A(t+1)$

Cost function $c(x, u) = x + \beta\mathcal{P}(u)$ with $\beta > 0$.
For most part, use $c(x, u) = x + \frac{1}{2}u^2$

Average cost problem

For $h : \mathbb{R} \to \mathbb{R}$, **differential generator** $\mathcal{D}_u$ is

$$\mathcal{D}_u h(x) := \mathbf{E}\Big[h\Big(X(t+1)\Big) - h\Big(X(t)\Big) \mid X(t) = x, U(t) = u\Big]$$
$$= \sum_j P_{xj}(u)h(j) - h(x)$$

Average Cost Optimality Eq. *(ACOE)*

$$\min_u \Big(c(x,u) + \mathcal{D}_u h^*(x)\Big) = \eta^*$$

Countable state space constraint: $\forall u, x$, $\lim_{n\to\infty} \frac{1}{n}\mathbf{E}_x^u |h^*(x_n)| = 0$

Fluid model

$$\frac{d}{dt}x(t) = \bar{f}(x(t), u(t)) = \mathbf{E}[f(x, u, w(1))], \qquad x(0) \in X$$
$$= -u(t) + \alpha$$

$$\mathcal{D}_u^F h(x) = \frac{d}{dt} h(x(t))|_{t=0} = \nabla h(x) \cdot \bar{f}(x, u) \qquad u(0) = u, x(0) = x$$

Motivation: *Taylor series expansion. if J^* is smooth, then*

$$\begin{aligned}\mathcal{D}_u J^* &\approx \mathbf{E}\Big[\nabla J^*(X(0))(X(1) - X(0)) | X(0) = x, U(0) = u\Big] \\ &= \nabla J^*(x)\bar{f}(x, u) \\ &= \mathcal{D}_u^F J^*\end{aligned}$$

Fluid model

Total Cost Optimality Eq. *(TCOE)*

$$\min_u \left(c(x,u) + \mathcal{D}_u^F J^*(x) \right) = 0$$

Supposing J^* finite, solved with

$$J^*(x) = \inf_u \int_0^\infty c(x(t), u(t))dt, \qquad\qquad x(0) = x \in X$$

We want J^* to approximate the ACOE solution in a meaningful way

Fluid approximation

Consider cost functions $c(x, u) = x + \beta([u - \alpha]_+)^\varrho$, where

- $[\cdot]_+ = max(0, \cdot)$, $\beta > 0$,
- $\varrho > 0$ integer
- $c(0, \alpha) = 0$

Theorem

J^* approximately solves ACOE. In particular, there is a modified cost function $c^0 \approx c$ (with bounded error) and corresponding η^0 such that J^* satisfies ACOE

$$\min_u \left(c^0(x, u) + P_u J^*(x) \right) = J^*(x) + \eta^0$$

Fluid approximation

Proof Idea

Pick $\eta^0 > 0$ arbitrary and define error functions

$$\varepsilon(x,u) = c(x,u) - J^*(x) + P_u J^*(x),$$
$$\underline{\varepsilon}(x) = \min_u \varepsilon(x,u)$$

$c^0(x,u) := c(x,u) - \underline{\varepsilon}(x) + \eta^0$, *then* J^* *satisfies ACOE (easy check)*
Lower bound *on* $c^0 - c$*: use convexity of* J^*
Upper bound *on* $c^0 - c$*:*

- *From MVT, for some* $\bar{X}$ *with value between* x *and* $x - u + A(1)$*,*

$$\begin{aligned}\mathcal{D}_u J^*(x) &:= \mathbf{E}\Big[J^*(X(1)) - J^*(X(0)) | X(0) = x, U(0) = u\Big] \\ &= \nabla J^*(x)\cdot(-u+\alpha) + \frac{1}{2}\mathbf{E}\Big[\nabla^2 J^*(\bar{X})\cdot(-u + A(1))^2\Big].\end{aligned}$$

- *Use error from fluid optimal policy and non-increasing* $\nabla^2 J^*$

Diffusion model

$$dX(t) = \bar{f}\Big(X(t), U(t)\Big)dt + \sigma\Big(U(t)\Big)dN(t) + dI(t),$$

where N is BM, I is reflection process

$\mathcal{D}_u$ defined over C^2 fns. $g : \mathbb{R}_+ \to \mathbb{R}_+$
restricted to $\nabla g(x)|_{x=0} = 0 \implies I$ term vanishes in Ito's Lemma

$$\mathcal{D}_u g(x) = \nabla g(x) \cdot \bar{f}(x, u) + \frac{1}{2}\sigma^2(u)\nabla^2 g(x)$$

Motivation: *2nd order Taylor series approximation (previous slide), which also suggests*

$$\sigma^2(u) = \mathbf{E}\Big[(u - A(1))^2\Big] = u^2 - 2\alpha u + m_A^2$$

Diffusion approximation

Under $c(x,u) = x + \frac{1}{2}u^2$, optimal policy is $u^*(x) := \frac{\nabla h^*(x) + \alpha \nabla^2 h^*(x)}{1 + \nabla^2 h^*(x)}$
$u^* \geq 0$ (valid policy) by h^* convexity and boundary condition

Fluid TCOE has solution $J^*(x) = \alpha x + \frac{1}{3}\left((2x + \alpha^2)^{3/2} - \alpha^3\right)$

Theorem

$h^0(x) = J^*(x) + \frac{1}{2}x$ approximately solves the diffusion ACOE. In particular, there is a modified cost function $c^0 \approx c$ (with bounded error) such that h^0 solves the ACOE exactly.

Diffusion approximation

Proof Idea

Let $c^0(x,u) = c(x,u) + \frac{1}{8}\left(\frac{y}{y+1} - 4\frac{\sigma_A^2}{y}\right) + \eta^0$*, where* σ_A^2 *variance of* A*,* $y := (2x+\alpha^2)^{1/2}$*,* $\eta^0 > 0$ *arbitrary constant*

Optimal average cost of c^0 *is* η^0 *(fixed point equation)*

$|c^0(x,u) - c(x,u)|$ *is uniformly bounded over* x, u

Issue: h^0 *doesn't satisfy boundary condition* $\nabla h^0(x)|_{x=0} = 0$
Resolution: *decaying exponential perturbation for some* $\nu > 0$

$$h^{00}(x) = h^0(x) - \left(2\alpha + \frac{1}{2}\right)\nu e^{-x/\nu},$$

h^{00} *solves diffusion ACOE with some* c^{00} *that retains the uniform boundedness of* $c^{00}(x,u) - c(x,u)$

LSTD

In summary:

- J^* *approximates* h^* *(+caveats)*
- $h^0(x) = J^*(x) + \frac{1}{2}x$ *approximates* h^* *for diffusion*

Together, this suggests a basis for h^*:

$$\phi^1(x) = J^*(x), \qquad \phi^2(x) = x$$

Computational setup

$A(t) = \Delta_A G(t)$ for $t \geq 1$ where

- G geometrically distributed with $p_A = 0.96$
- $\Delta_A = \frac{1}{24}$

$X = \{\Delta_A m | m = 0, \ldots, 480\}$

$X(t+1) = \Big[X(t) - U(t) + A(t+1)\Big]$ projected onto interval $[0, 20]$.

$U(t)$ restricted to non-negative integer multiples of Δ_A

$c(x, u) = x + \frac{1}{2}u^2$

Computational results: value iteration

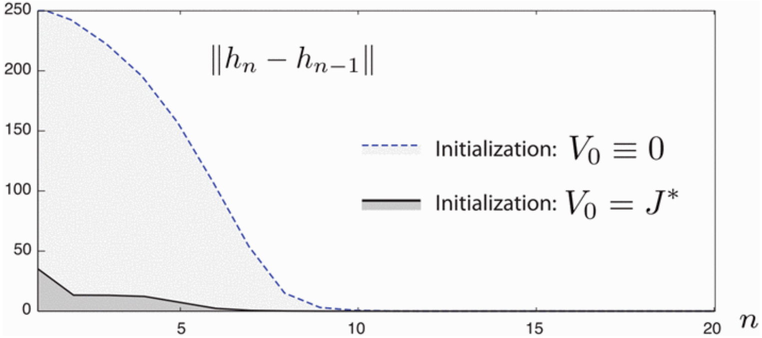

Figure: Convergence of Value Iteration. J^* is good initial guess.

Computational results: fluid approximation

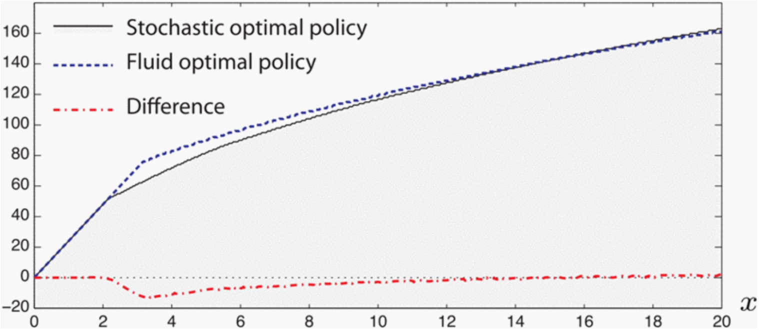

Figure: Optimal policy vs. fluid optimal policy

Computational results: LSTD approximation

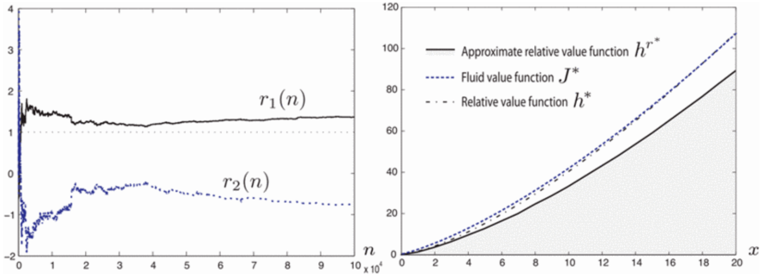

Figure: Simulation results. Left shows coefficients for basis functions. Right shows final approximations of h^*.

100,000 iterations of LSTD

Initial condition $r(0) = (0, 0)^T$

Computational results: LSTD approximation

Now use different cost function: $\varrho = 15$, $\beta = 1/\varrho$.

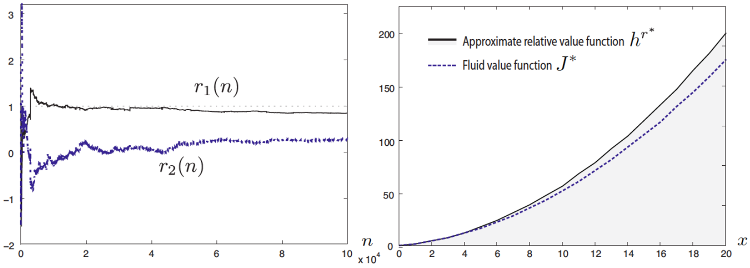

Figure: Simulation results using $\varrho = 15$ cost function. Left shows coefficients for basis functions. Right shows final approximations of h^*.

Computational results: PIA convergence

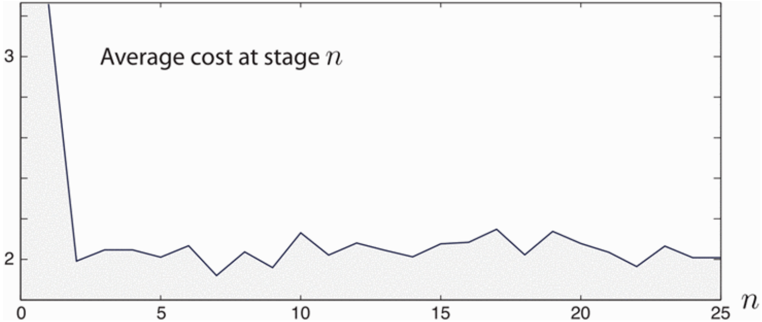

Figure: Estimated average cost from LSTD policy iteration with quadratic cost function. Fits idea that LSTD tends to converge quickly if we have a good basis.

Initial policy $u^0(x) = \min(x, 1)$ for $x \geq 0$ for policy iteration with LSTD

Discussion

What's good:

- Fluid-Diffusion basis works well in computations
- Some theoretical justification for why fluid-diffusion approximations are reasonable
- Extend some theoretical results to other cost functions (e.g., more general polynomials and exponentials)

What's questionable/unaddressed:

- Caveats on cost function in fluid model theoretical results
- Are the basis functions counterintuitive, not something would have tried anyway?
- Does method like this generalize well to other problems?
- The computational problem considered isn't that difficult to solve. Is there a difficult problem that this method helps us to solve?