While Stability Lasts: A Stochastic Model of Stablecoins

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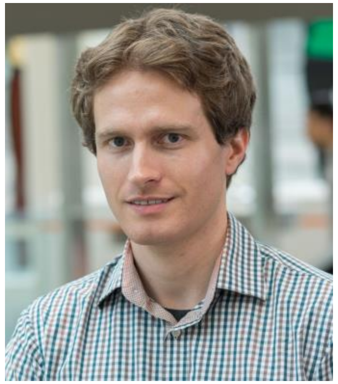


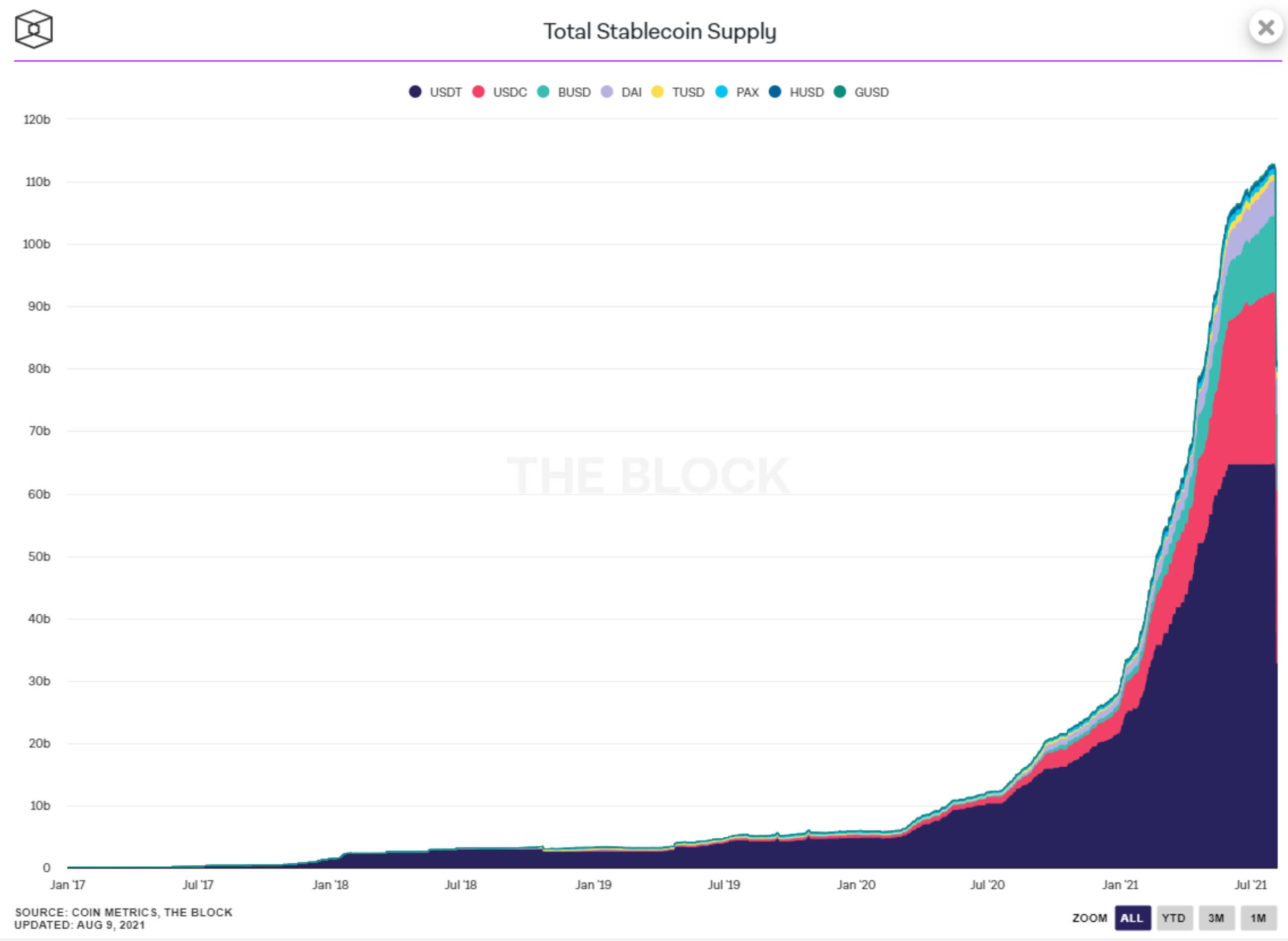
Cornell University

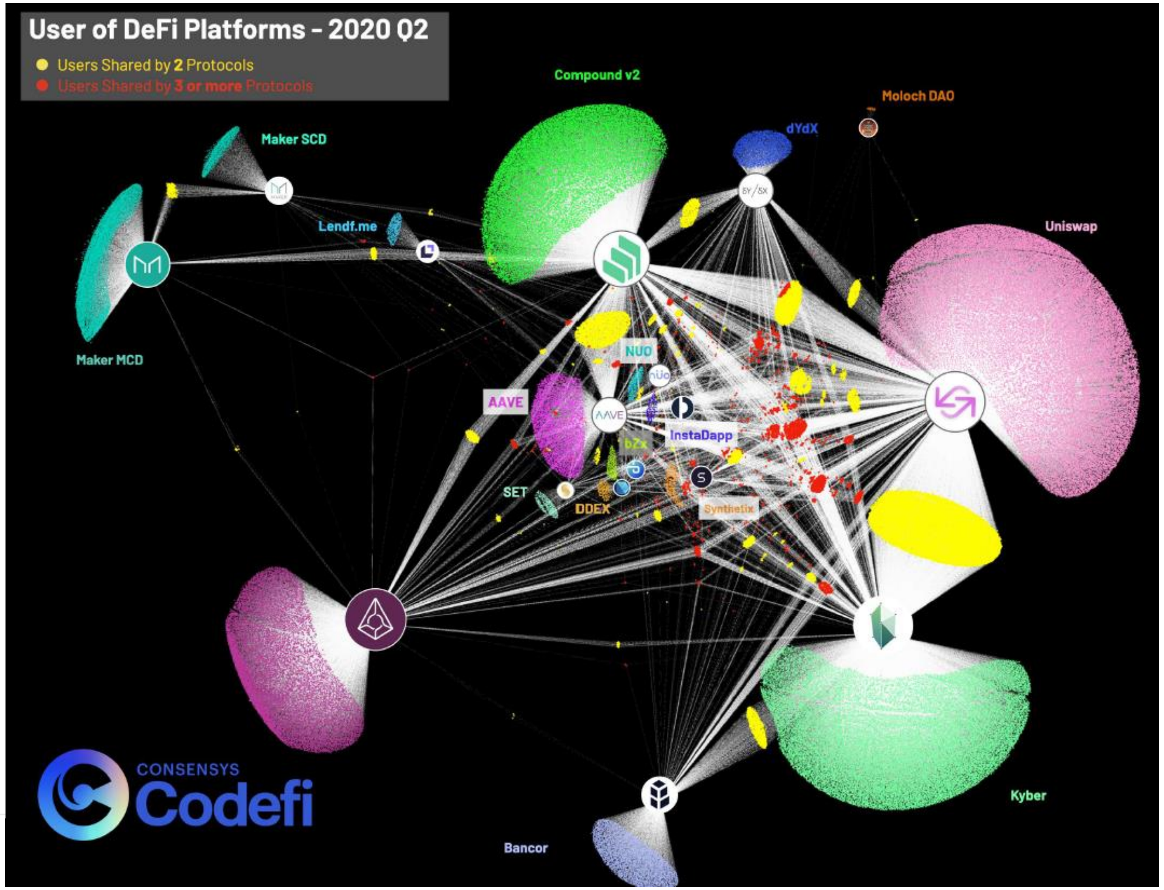
INFORMs, Oct. 2021

Intro to Cryptocurrency

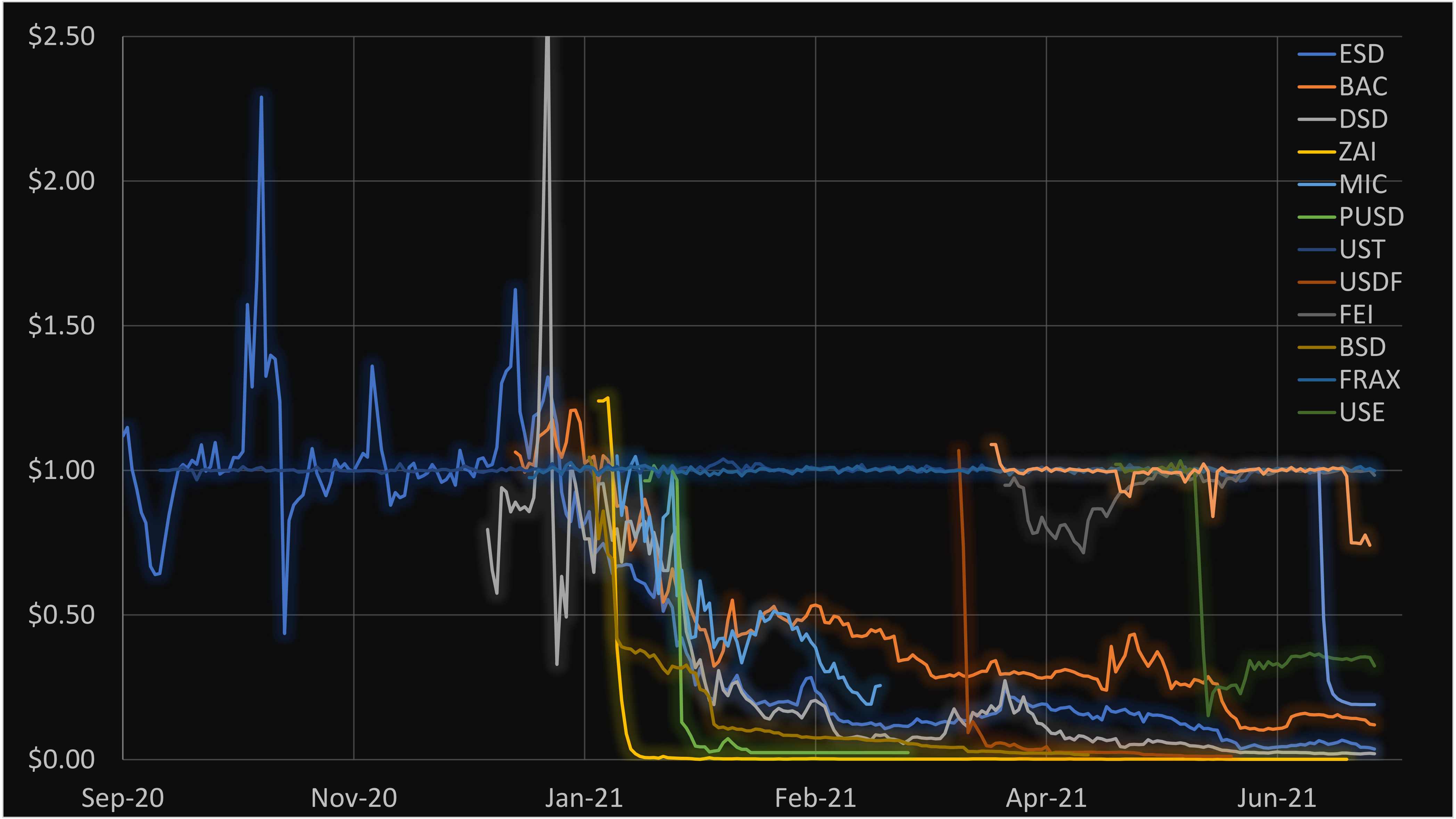
- **Blockchain:** new way for mistrusting agents to cooperate w/o trusted third parties
- **Cryptocurrency:** an asset native to a blockchain
  - Price usually volatile: network effects, technical progress, regulatory hurdles etc
- **Smart contracts:** programs that run on the blockchain computer
- **Stablecoins:** cryptocurrency with added economic structure that
  - Aim: stabilize price/purchasing power
  - Constructed using smart contracts

Stablecoins: A Growing DeFi Foundation





Over past year, many new types of stablecoins…



This Paper

Stablecoins = complex on-chain currencies

- Many similarities with traditional finance
- Also new risks that lack suitable models
- *Our focus:* leverage-based stablecoins

I. Conceptually, what is a stablecoin?
II. Model w/ Endogenous Price
III. Stochastic Analysis Results
IV. Design Insights

---Stablecoins---

Risk-based Overview

Stablecoin

Custodial

Risks
- Counterparty credit risk
- Censorship risk
- Traditional financial risks

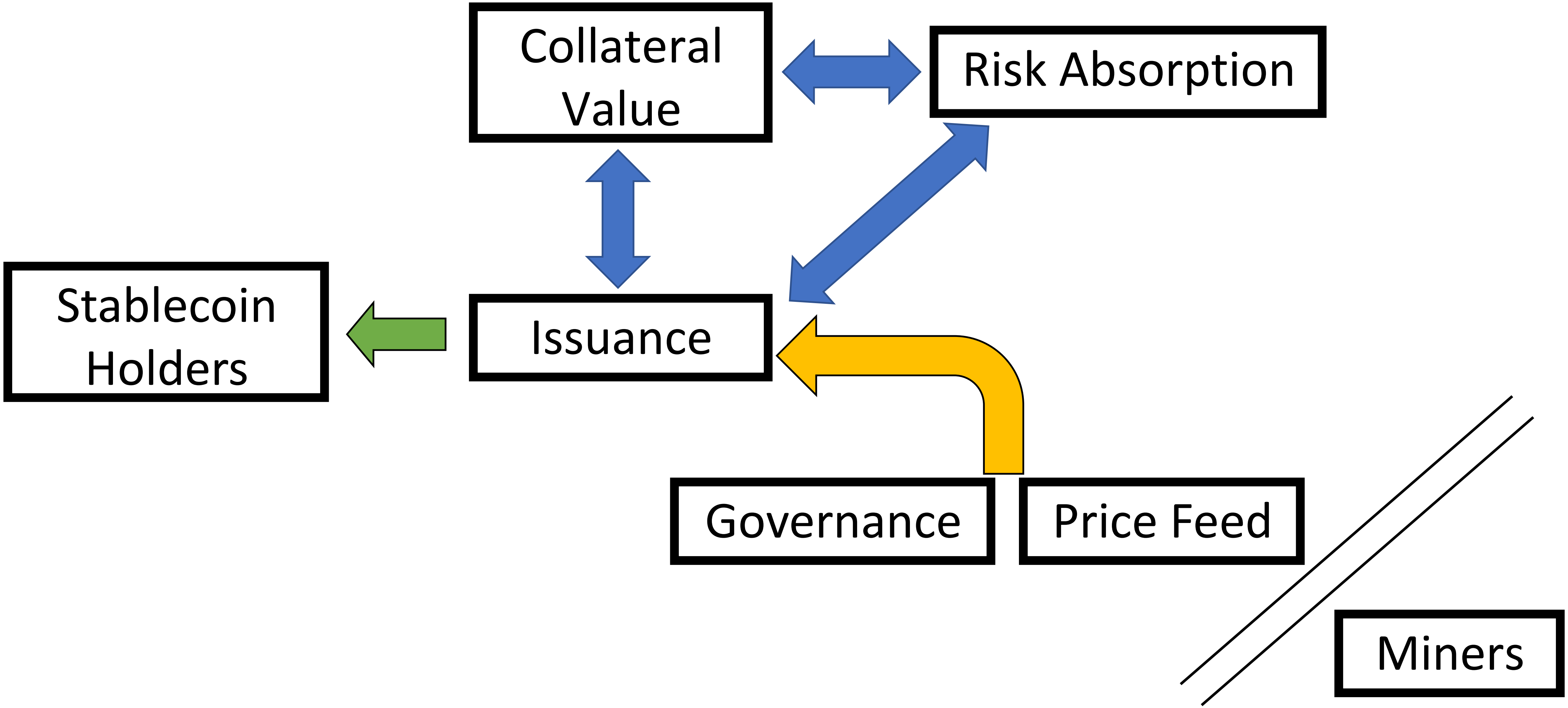
Well understood!

Non-Custodial

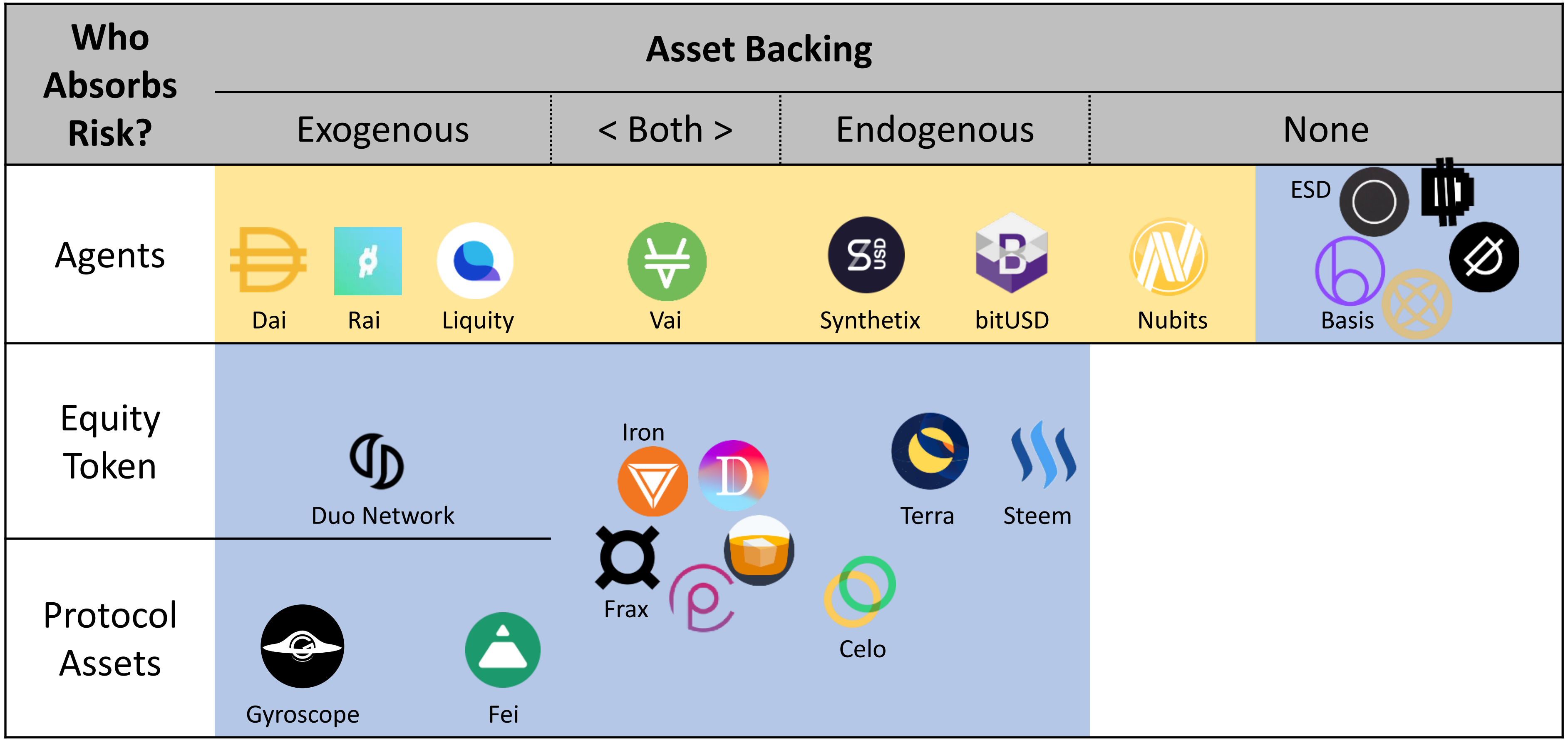
New Risks and attacks
- Deleveraging risks
- Price feeds, governance
- Miner extractable value
- Smart contract bugs

Not well understood

Anatomy of Non-custodial Stablecoins



Non-custodial Stablecoins in 3D



Issuance

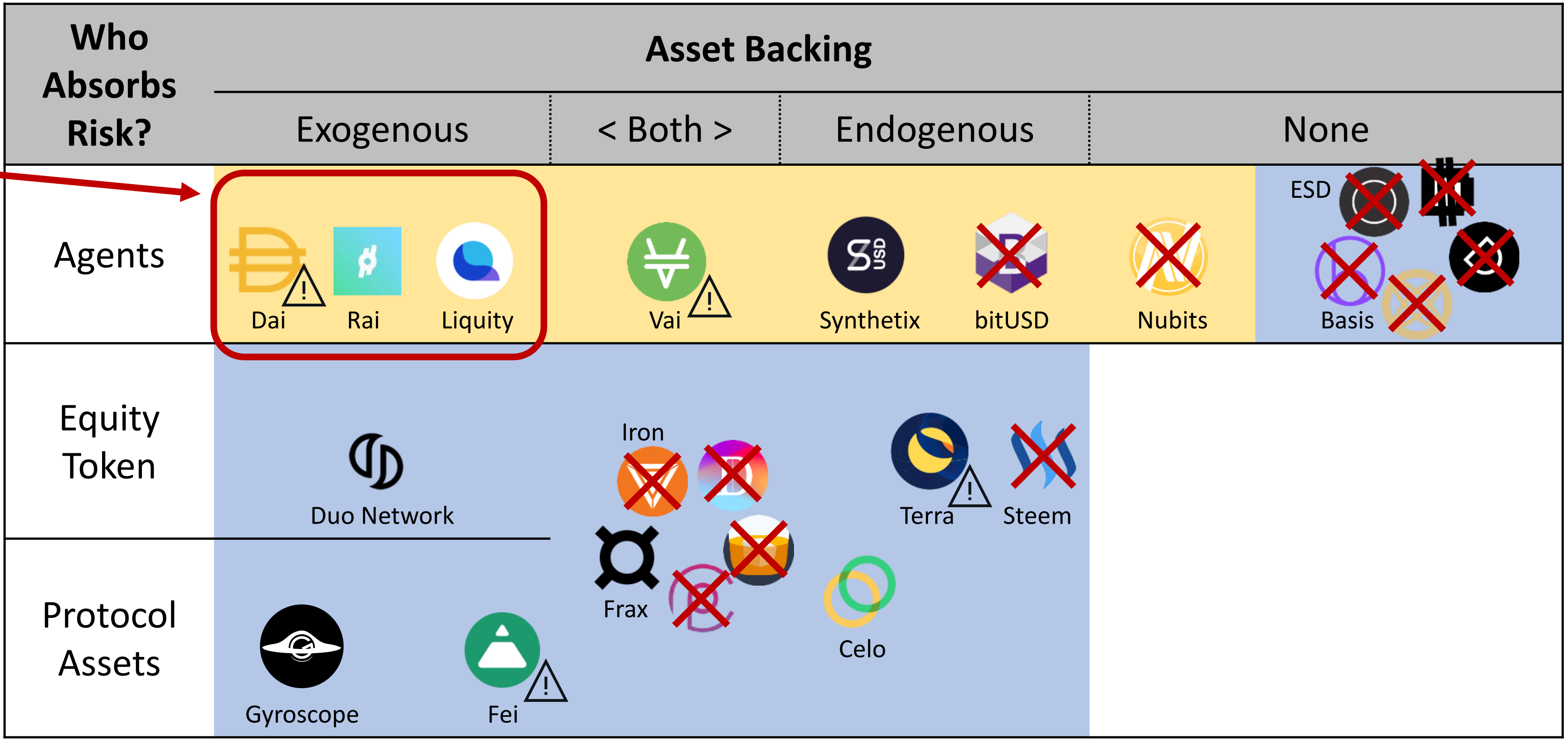
Agent

Algorithmic

Exogenous = asset price independent of protocol
Endogenous = asset price self-referential with protocol
Agent = speculative agents decide, as applicable, risk exposure or issuance

Non-custodial Stablecoins in 3D

Our direct focus



Issuance

Agent

Algorithmic

Exogenous = asset price independent of protocol

Endogenous = asset price self-referential with protocol

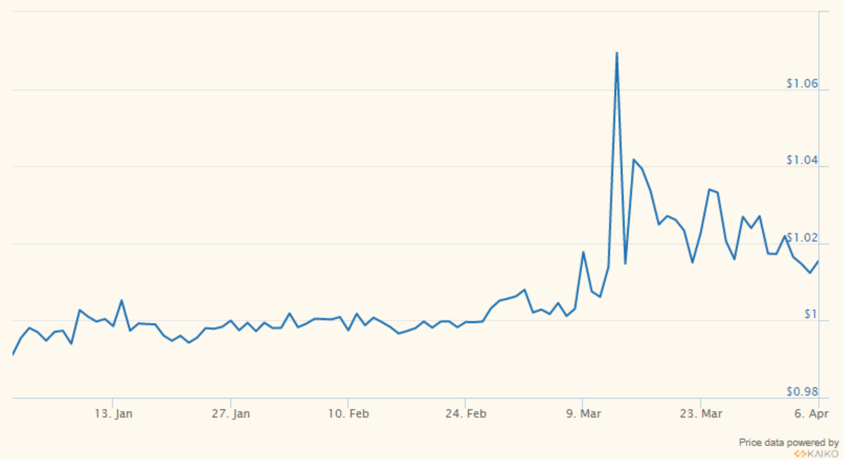
Agent = speculative agents decide, as applicable, risk exposure or issuance

⚠ = recent problems observed, X = broken

Black Thursday in Dai, March 2020

ETH price

DAI price



Black Thursday for MakerDAO: $8.32 million was liquidated for 0 DAI

Mempool Manipulation Enabled Theft of $8M in MakerDAO Collateral on Black Thursday: Report

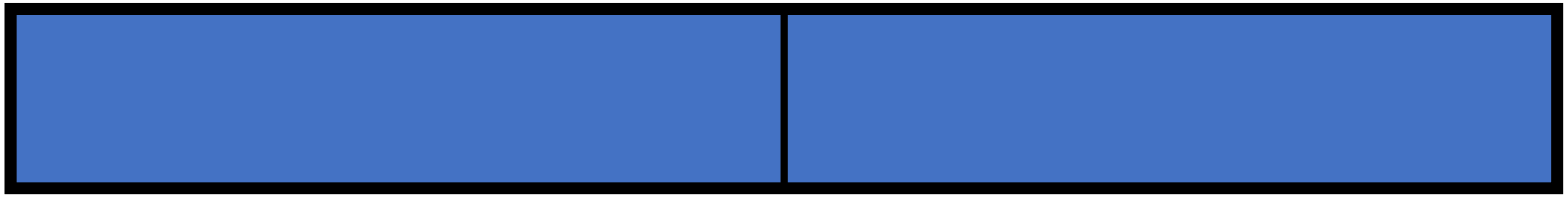
Jul 22, 2020 at 18:41 UTC · Updated Jul 28, 2020 at 19:04 UTC

CDO Structure

A portfolio of underlying assets

CDO Structure

Split into 2 tranches

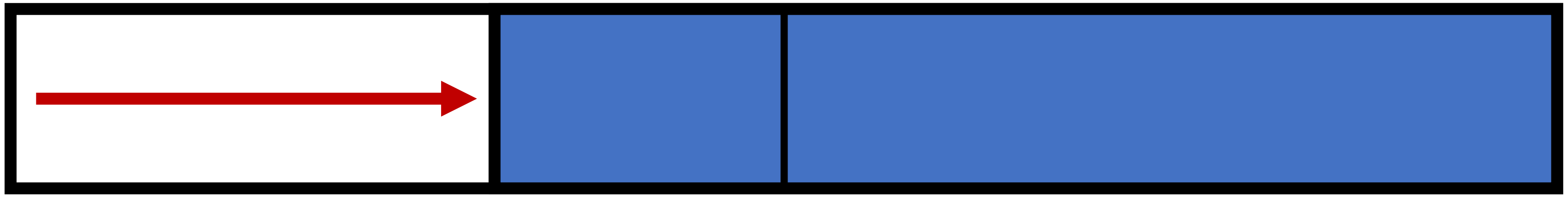


Junior tranche = more risky

Senior tranche = less risky

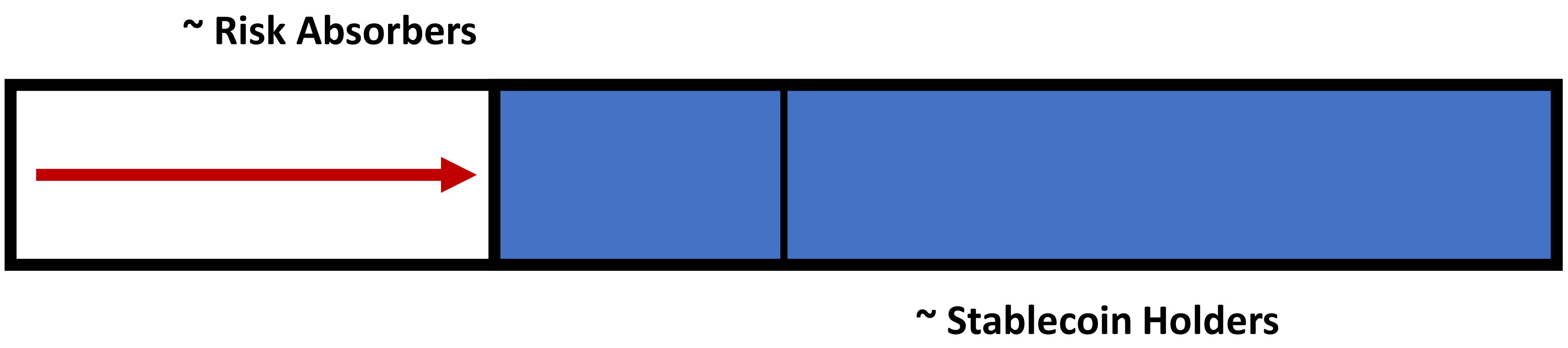
CDO Structure

Losses that occur are first borne by junior tranche

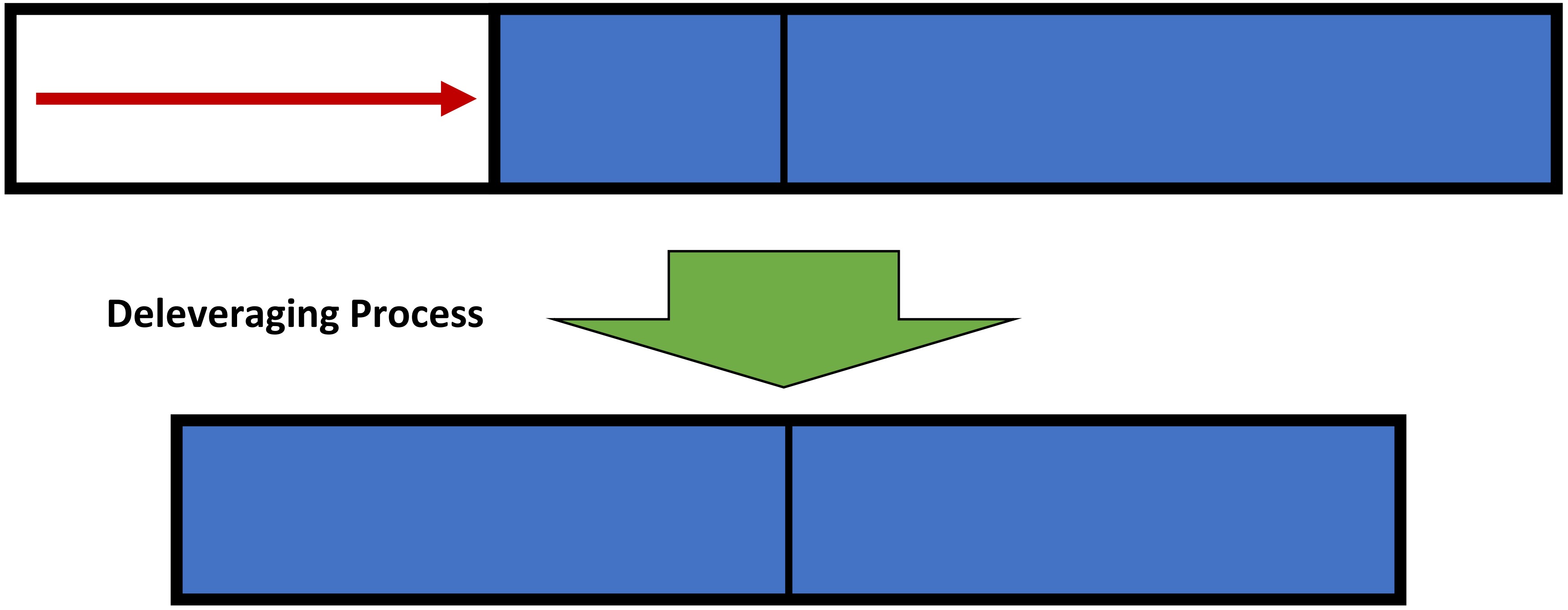


Senior tranche protected

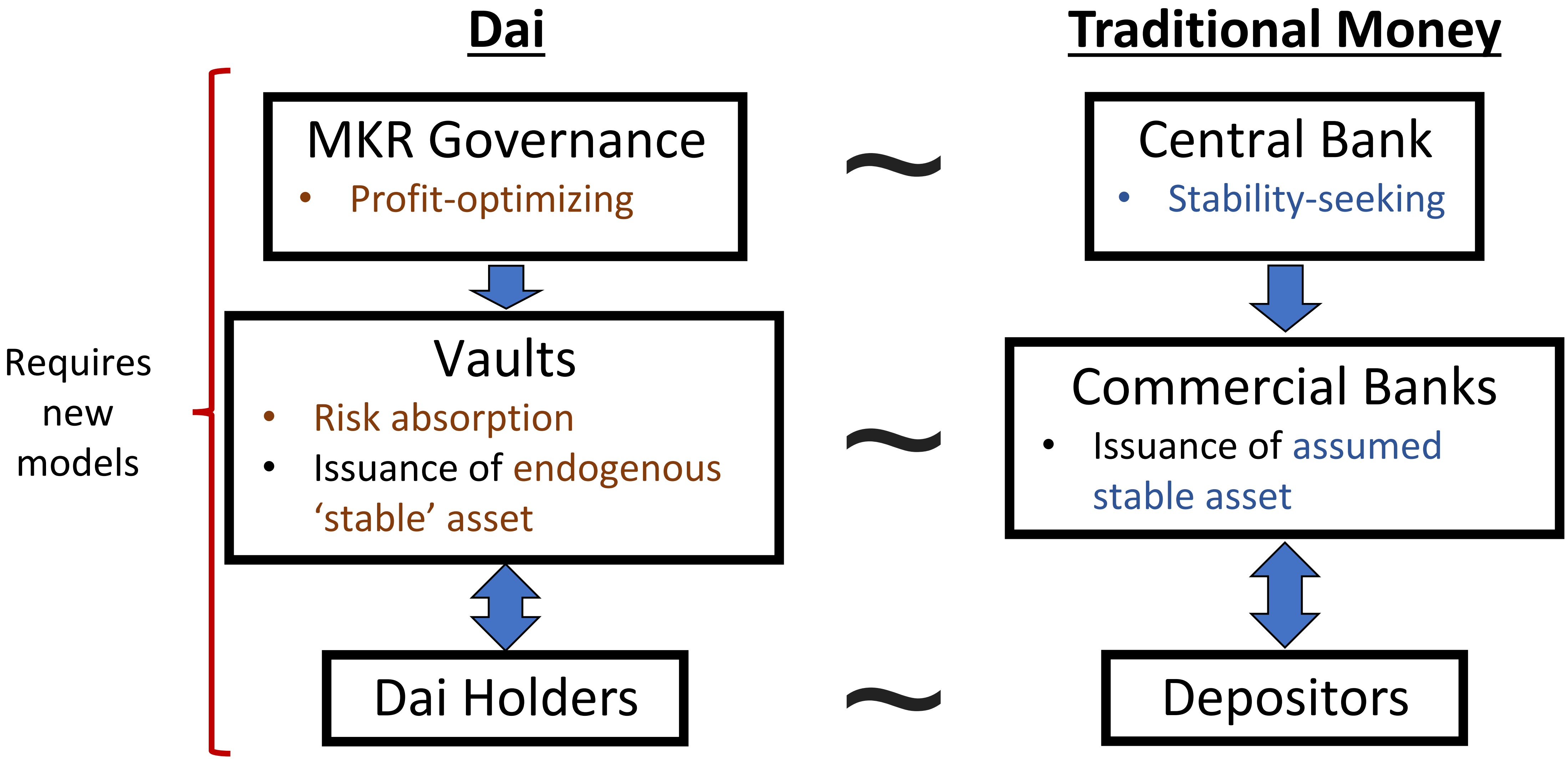
Stablecoin CDO-like Structure



Stablecoin CDO-like Structure



Parallels & Differences



---Model---

Modeling Price Dynamics

- **Currency peg models:** gov issuer mechanically committed to stability
- **Debt securities:** an asset that is assumed stable is borrowed against collateral, feedback effects on collateral asset liquidity

- **Dai-style stablecoins:** supply determined in leverage market
  - Created by speculator choosing to borrow against ETH (risky!)
  - Endogenous price, participation: supply needn't = demand at $1
  - Hope protocol well-designed and peg maintained through incentives

- **Our work:** stochastic model of endogenous stablecoin price
  - Deleveraging feedback effects → short squeeze effect, collateral drawdown
  - 'Stable' and 'unstable' regimes for stablecoins

Model

Agents

- **Stablecoin Holders** want stability, have imperfectly elastic demand
- **Speculator** decides supply of stablecoins secured by its collateral position

Assets

- **ETH:** risky asset with exogenous price
- **STBL** stablecoin with endogenous price over-collateralized in ETH

Stablecoin market clears by setting demand = supply in USD (target) terms

Model: Speculator

Collateral constraint: protocol requires over-collateralization

Price of ETH

Stablecoins "borrowed"

$$\bar{N}_t X_t \geq \beta L_t$$

Amount of ETH

Collateral factor

Model: Speculator

Decision: Change stablecoin supply to maximize next period expected returns subject to constraints (intended behavior)

$$\max_{\Delta_t} \quad \mathbb{E}[Y_{t+1} | \mathcal{F}_t]$$

$$\text{s.t.} \quad \bar{N}_t X_t \geq \beta L_t$$

$$Y_t = N_{t-1} X_t - L_{t-1} - \underbrace{\text{liquidation effect}}$$

Protocol can liquidate: costs and market effect

Model Details

We formalize the model as follows. We define the following *parameters*:

- $\mathcal{D}$ = STBL demand in dollar value (equivalent to constant unit price-elasticity)
- β = STBL collateral factor
- $\alpha \geq 1$ liquidation fee (representing 1+% fee)

The system is composed of the following *processes*:

- $(X_t)_{t\geq 0}$ = exogenous ETH price process
- $\mathcal{L}_t$ = stablecoin supply at time t that obeys

$$\mathcal{L}_t = \zeta + L_{t-1} + \Delta_t$$

  where $L_{t-1} > 0$ is the speculator's STBL liabilities from the previous period, Δ_t is the speculator's change in liabilities at time t (such that $L_t = L_{t-1} + \Delta_t$), and and ζ is a real number that modifies circulating supply
- N_t = speculator's ETH position at time t, including collateral
- $\bar{N}_t$ = speculator's locked ETH collateral at time t (and start of time $t+1$)
- $(Y_t)_{t\geq 0}$ = speculator's value process
- $Z_t = \frac{\mathcal{D}}{\mathcal{L}_t}$ defines the STBL price process

$$X_t$$

$$Y_{t+1} = \frac{\Delta_t \mathcal{D} X_{t+1}}{\mathcal{L}_t X_t} + (\bar{N}_t X_{t+1} - L_t)\,\mathbb{1}_{A_t \cup B_t} + \mathbb{1}_{B_t}(3L_t - 2\bar{N}_t X_{t+1})\left(1 - \frac{\alpha \mathcal{D}}{2\bar{N}_t X_{t+1} - 2L_t}\right)$$

$$\Delta_t^* = \begin{cases} \min\left(\arg\max_{\Delta_t} \mathbb{E}[Y_{t+1}|\mathcal{F}_t], \frac{\bar{N}_{t-1}X_t}{\beta} - L_{t-1}\right) & \text{if } X_t \geq \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ \min\left(\arg\max_{\Delta_t} \mathbb{E}[Y_{t+1}|\mathcal{F}_t], -(3\mathcal{L}_{t-1} - 2\bar{N}_{t-1}X_t)\right) & \text{if } X_t < \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \end{cases}$$

$$\mathcal{L}_t = \mathcal{L}_{t-1} + \Delta_t^*$$

$$N_t = \begin{cases} N_{t-1} + \Delta_t^* \frac{Z_t}{X_t} & \text{if } X_t \geq \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ N_{t-1} + \frac{Z_t}{X_t}(\Delta_t + (1-\alpha)(3\mathcal{L}_{t-1} - 2\bar{N}_{t-1}X_t)) & \text{if } X_t < \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \end{cases}$$

$$\bar{N}_t = \begin{cases} N_{t-1} & \text{if } X_t \geq \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ N_{t-1} - \alpha(3\mathcal{L}_{t-1} - 2\bar{N}_{t-1}X_t) & \text{if } X_t < \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \end{cases}$$

$$Z_t = \frac{\mathcal{D}}{\mathcal{L}_t}.$$

Worth recalling:

X_t = collateral price

Z_t = stablecoin price

$\mathcal{L}_t$ = stablecoin supply

---Results---

Model

Assumptions

- X_t is a submartingale (we'll see what happens when relaxed)
- Distributions appropriately nice, bounded moments
- Stablecoin demand has unit price elasticity (can be generalized somewhat)
- Ensure speculator's objective is concave (not much stronger than above, avoids model artifact)
- Simple form for liquidations: protocol specifies amount to deleverage fulfilled on endogenous market

Regions of Stability

Result: Bound large deviations in certain stopped process

THEOREM 1. *For* $m \geq Z_0$ *and* $\epsilon > 0$,

$$\mathbb{P}\left(\max_{n \leq \tau \wedge T_m} Z'_n > \epsilon\right) \leq 2\epsilon^{-1}\left(m - \frac{1}{\kappa r}\right)$$

where $Z'_t := |m - Z_t|$ with *m=1,* this is deviation from target

τ is the hitting time of $\mathbb{E}\left[\frac{1}{\mathcal{L}_{t+1}}|\mathcal{F}_t\right] > \frac{1}{\mathcal{L}_t}$,

T_m is the hitting time of $Z_t > m$, for $m \geq Z_0$,

Proof: Doob's inequality

Regions of Stability

Result: bounds probability of large quadratic variation (QV) in certain stopped process

THEOREM 2. *Suppose $m \geq Z_0$ and $\epsilon > 0$. Then*

$$\mathbb{P}\left(\sqrt{[Z']_{\tau \wedge T_m}} > \epsilon\right) \leq 6\epsilon^{-1}\left(m - \frac{1}{\kappa r}\right)$$

where $[Z']_t := \sum_{k=1}^{t} (Z'_k - Z'_{k-1})^2$ is QV

τ is the hitting time of $\mathbb{E}\left[\frac{1}{\mathcal{L}_{t+1}} | \mathcal{F}_t\right] > \frac{1}{\mathcal{L}_t}$,

T_m is the hitting time of $Z_t > m$, for $m \geq Z_0$,

Proof: Burkholder's inequality

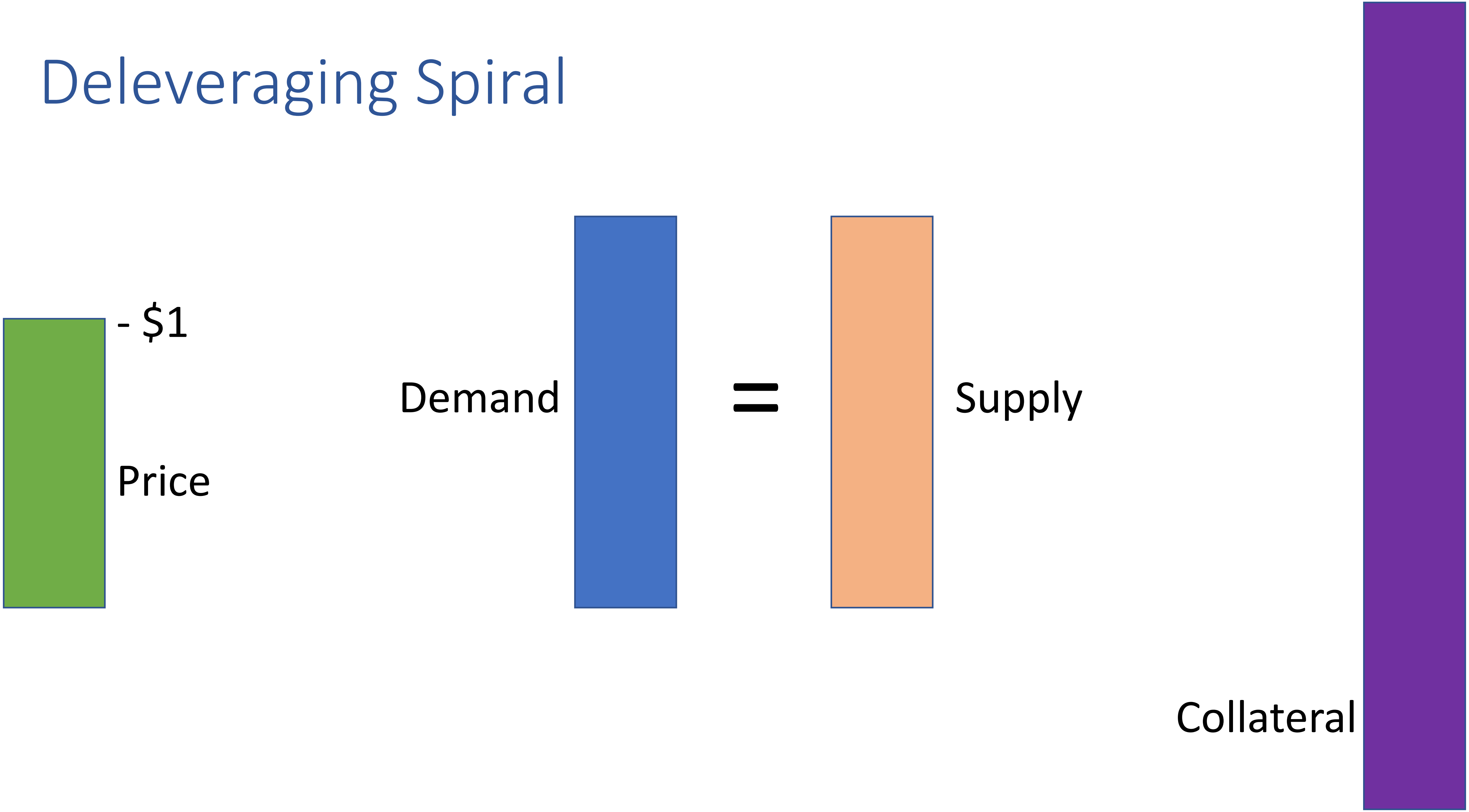
Regions of Instability

Result: Subject to different stopping conditions, stablecoin behaves as submartingale, depicting deleveraging spiral, akin to a short squeeze.

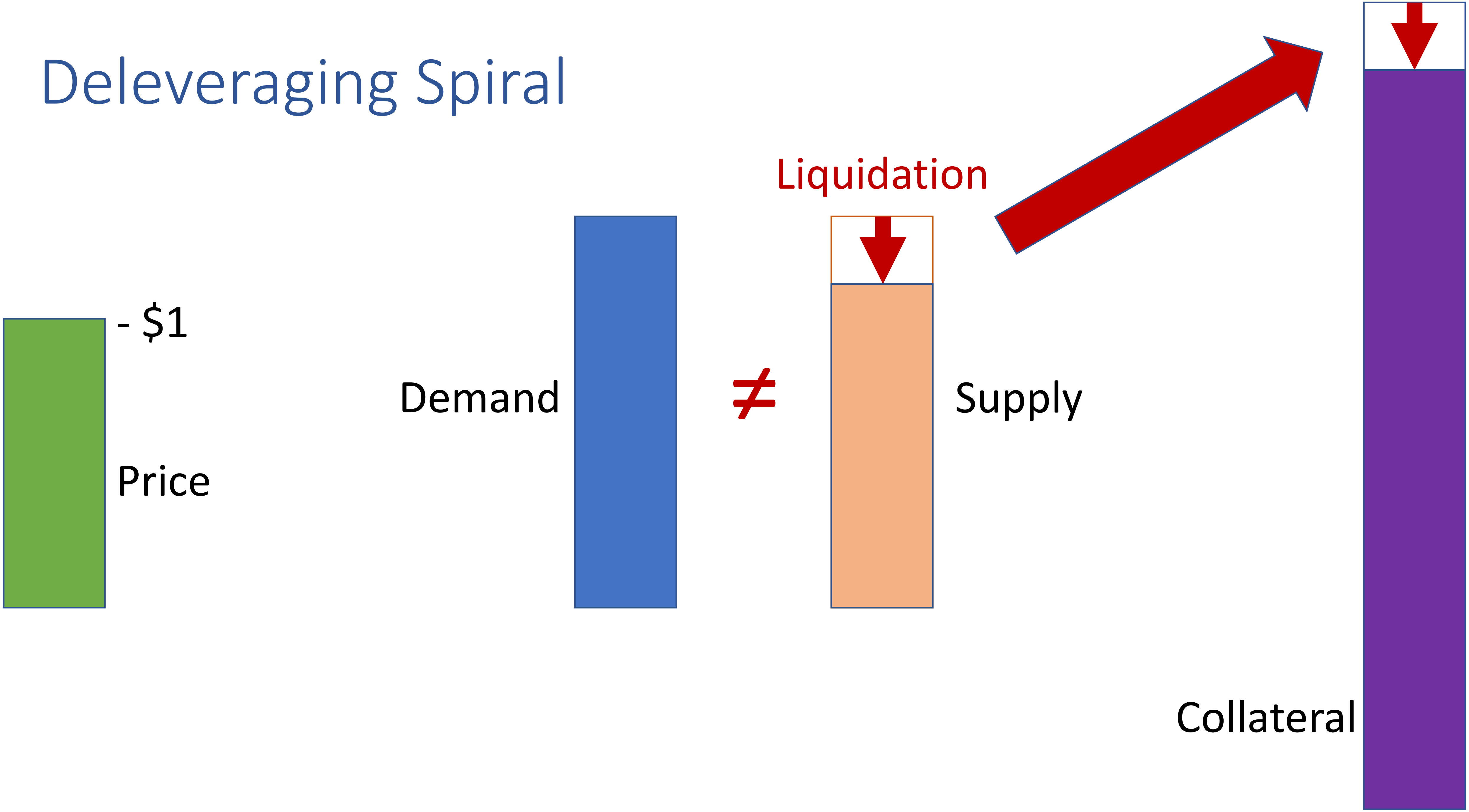
THEOREM 3. *Restarting the process at S_1, we have $(\mathcal{L}_{t\wedge S_2})$ is a supermartingale and $(Z_{t\wedge S_2})$ is a submartingale.*

where S_1 is the hitting time of $\mathbb{E}[\mathcal{L}_{t+1}|\mathcal{F}_t] < \mathcal{L}_t$,
S_2 is the hitting time of $\mathbb{E}[\mathcal{L}_{t+1}|\mathcal{F}_t] \geq \mathcal{L}_t$ such that $S_2 > S_1$.

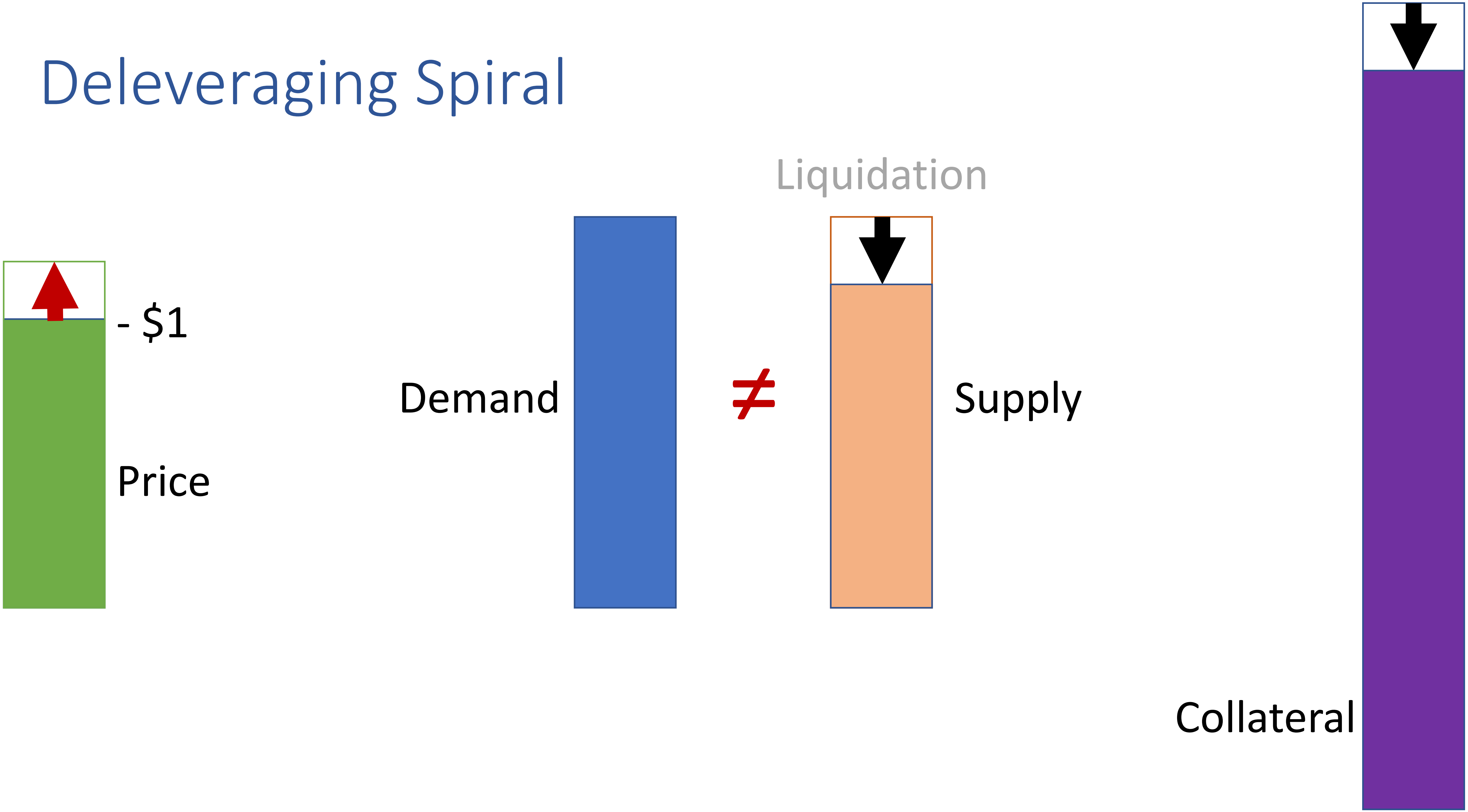
Deleveraging Spiral



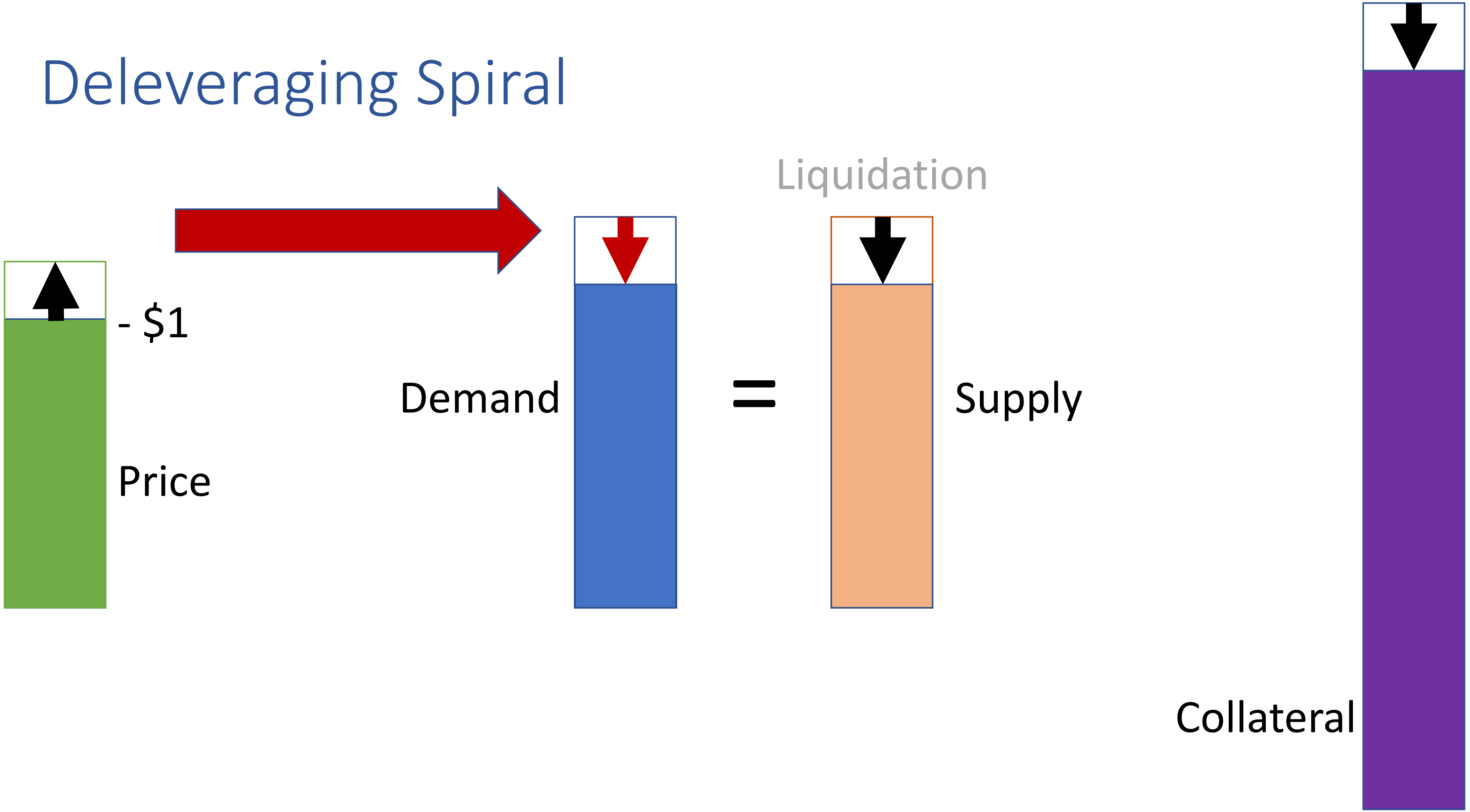
Deleveraging Spiral



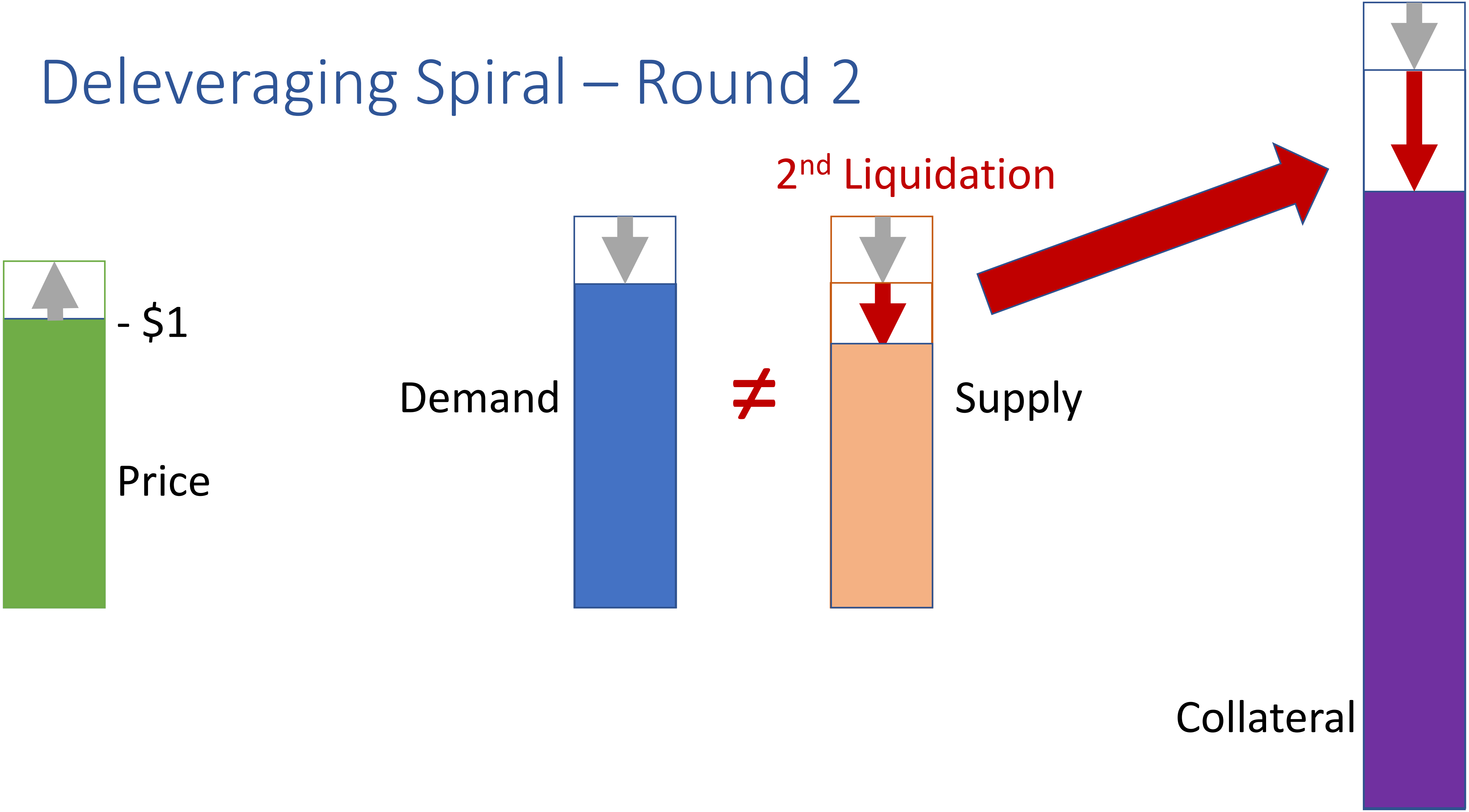
Deleveraging Spiral



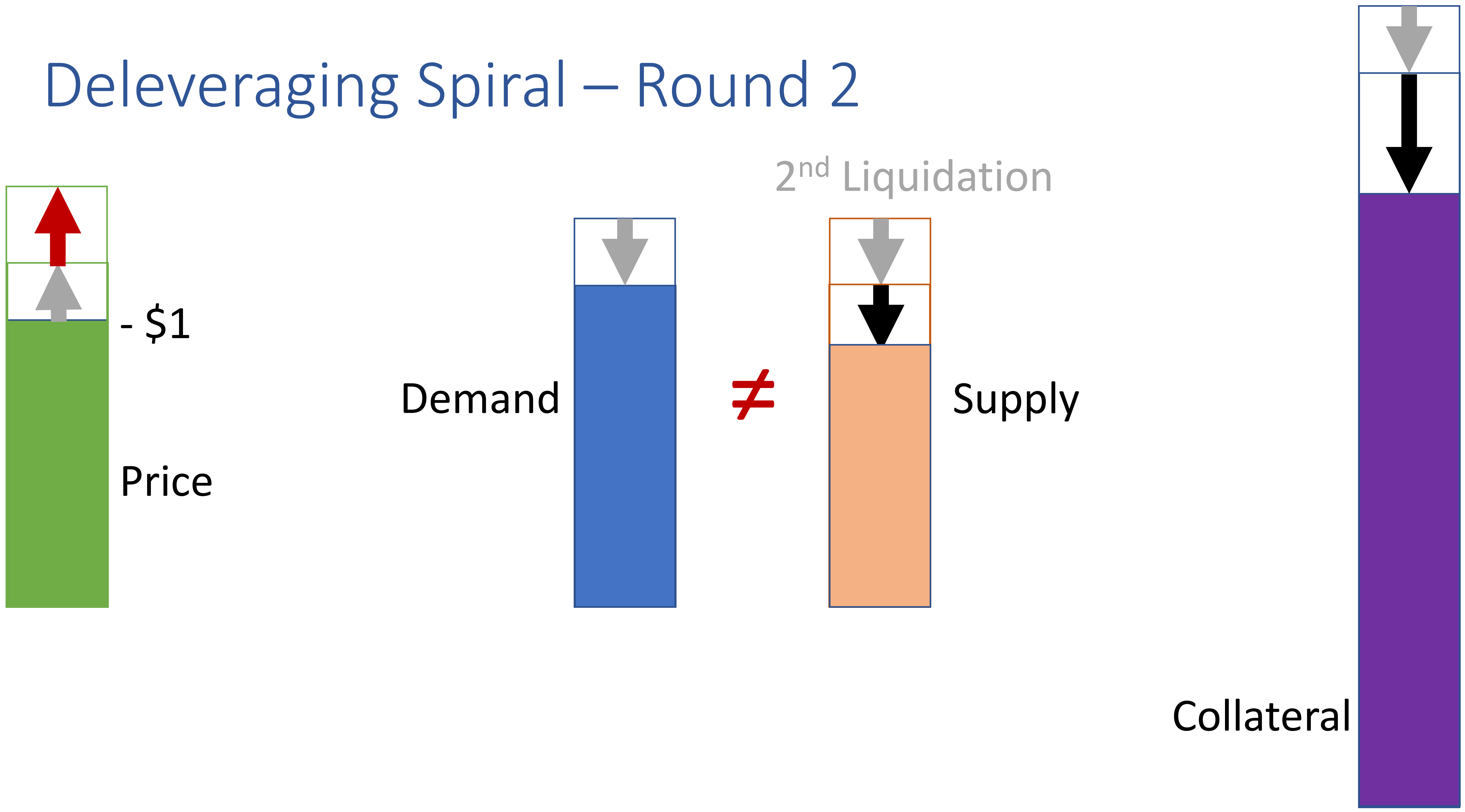
Deleveraging Spiral



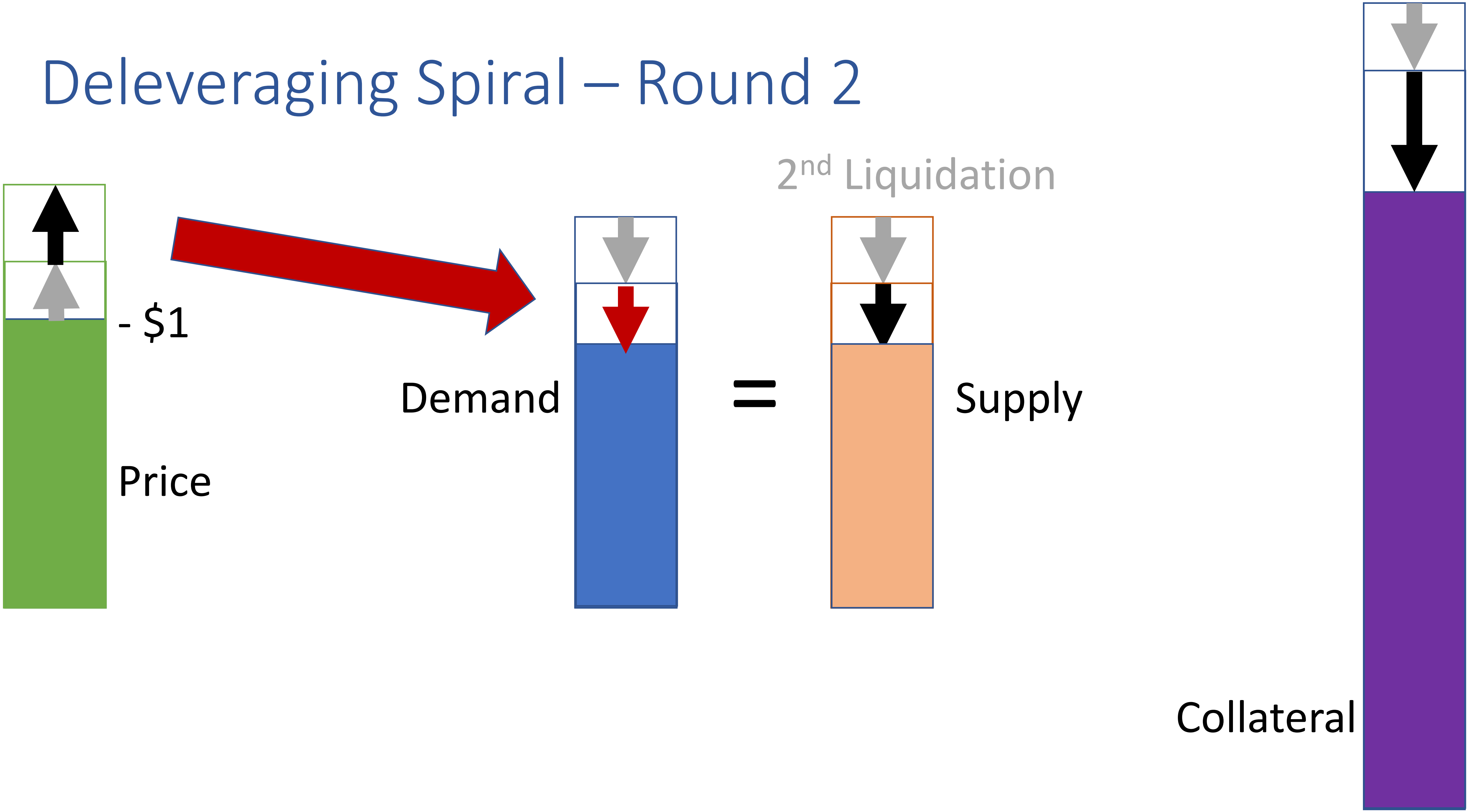
Deleveraging Spiral – Round 2



Deleveraging Spiral – Round 2



Deleveraging Spiral – Round 2



Regions of Instability

Remark 2. *(Estimating variances) Taylor approximations can be applied to estimate the variances of the stablecoin process. Consider $X_t = X_{t-1}R_t$ for return $R_t \geq 0$. For notational clarity, define*[10]

$$h(\rho, n) := \arg\max_{\mathcal{L}_t} \mathbb{E}[Y_{t+1}|\mathcal{F}_t] = \mathcal{L}_t,$$

where ρ, n are realizations of $R_t, \bar{N}_t$. Variance in stablecoin supply follows

$$Var(\mathcal{L}_t|\mathcal{F}_{t-1}) \approx h'\left(\mathbb{E}[R_t|\mathcal{F}_{t-1}], \bar{N}_t\right)^2 Var(R_t|\mathcal{F}_{t-1})$$

And the stablecoin price ***variance approximation*** *is*

$$Var(Z_t|\mathcal{F}_{t-1}) \approx \frac{\mathcal{D}h'(\mathbb{E}[R_t|\mathcal{F}_{t-1}], \bar{N}_t)^2}{\mathbb{E}[\mathcal{L}_t|\mathcal{F}_{t-1}]^4} Var(R_t|\mathcal{F}_{t-1}) \qquad (1)$$

Result 4: Variance approx. in Eq. (1) increases by order of $\frac{1}{R_t^2}$ in an ETH return shock and $\frac{1}{N_t^2}$ with different initial collateralization

Proof: Implicit Function Theorem

Regions of Instability

Result: Starting in the unstable regime, the stablecoin will always have higher forward-looking variance than in stable regime.

- ‘Stable’ and ‘unstable’ regimes well-interpreted

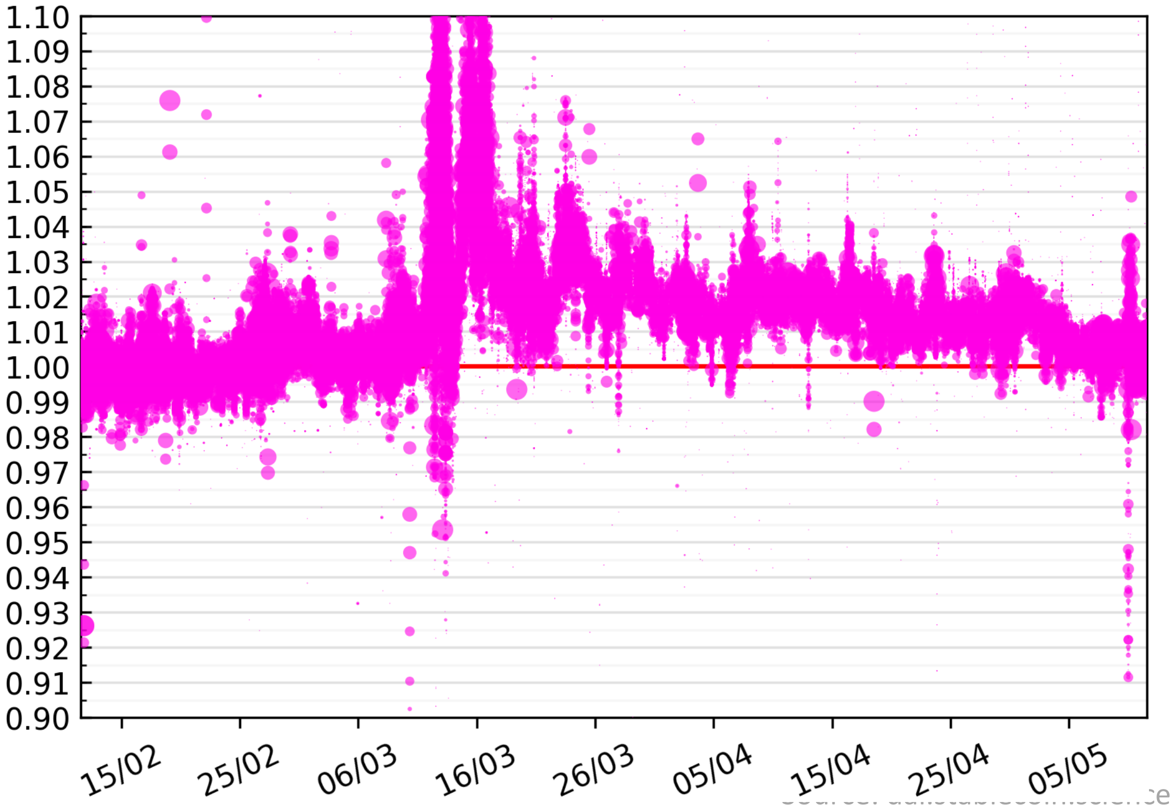
THEOREM 5. *In addition to the previous assumptions, suppose* $X_t \geq b(L_{t-1}) + \epsilon$ *for some* $\epsilon > 0$ *(the pre-decision collateral constraint is exceeded by* ϵ*, which restricts the ranges of both* X_t *and* $\bar{N}_{t-1}$*). Consider two possible states* s *and* u *of the stablecoin at time* t *that differ only in collateral amounts* $\bar{N}^s_{t-1} > N^u_{t-1}$ *and evolve driven by the common price process* (X_t)*. Then the forward-looking price variances satisfy*

$$Var(Z^s_t|\mathcal{F}_{t-1}) < Var(Z^u_t|\mathcal{F}_{t-1}).$$

Proof: inequalities on variances of convex functions of RVs

Black Thursday in Dai, March 2020

~50% ETH price crash



Liquidation price effect on Dai DEX trades

More detailed empirical analysis further validates results (Kjäer et al. 2021)

Model Extensions

- **Idealized settings**
  - ➢ Perfectly elastic demand or unlimited speculator supply -> perfect stability
  - ➢ But still no stability if not submartingale

- **Model extensions**
  - ➢ Generalized STBL demand parameterized by price elasticity
  - ➢ Endogeneity of collateral prices defined by price impact function

----Design Insights---

Non-custodial Complications

- No stable region when X_t is not ~ submartingale (positive expectations)

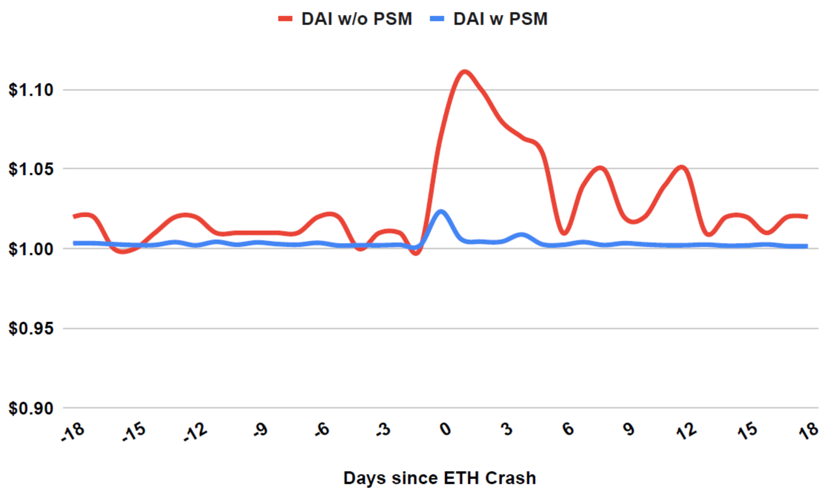
- *Seeming contradiction:* goal to make decentralized stablecoin, but can only be fully stabilized by adding uncorrelated assets, which are currently custodial

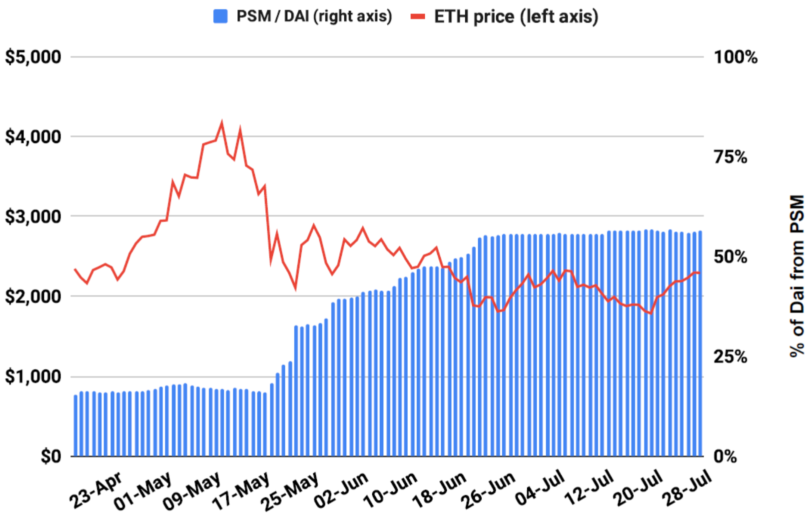
- Patching this has been major topic since Black Thursday

Non-custodial Complications

Solutions:

- **Maker:** Since Black Thursday has tethered to USDC (+ custodial risks)
  - Maintaining exchangeability via USDC reserve ("PSM")



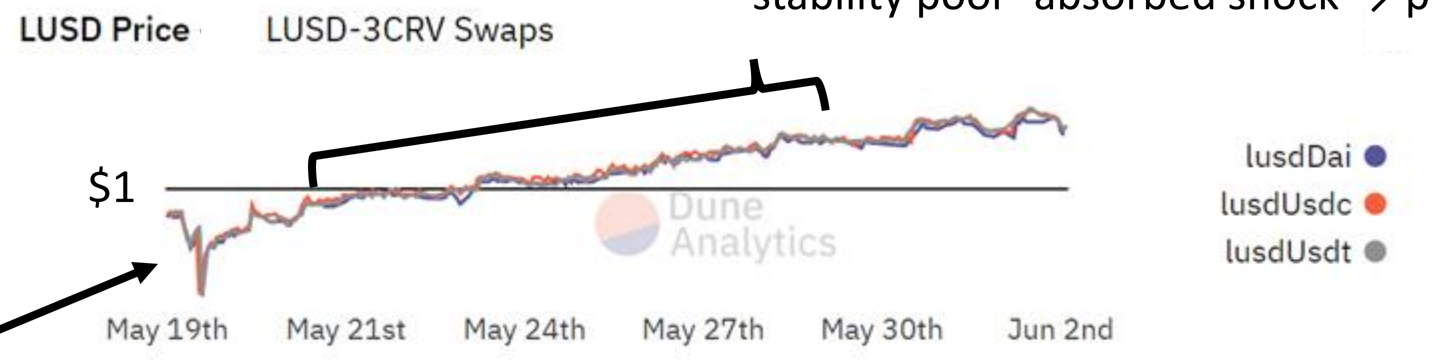


Non-custodial Complications

Solutions:

- **Maker:** Since Black Thursday has tethered to USDC (+ custodial risks)
  - Maintaining exchangeability via USDC reserve ("PSM")
- **Rai:** negative rates during crises (equilibrium participation, liquidity?)
- **Liquity (and proposed in this paper):** Dedicated liquidity pools for crises

"stability pool" absorbed shock → postponed, smoothed effect



Redemptions at start of crisis

Non-custodial Complications

Solutions:

- **Maker:** Since Black Thursday has tethered to USDC (+ custodial risks)
  - ➢ Maintaining exchangeability via USDC reserve (“PSM”)
- **Rai:** negative rates during crises (equilibrium participation, liquidity?)
- **Liquity (and proposed in this paper):** Dedicated liquidity pools for crises
- **Reserve-backed primary markets:** anchors price to $1 worth of assets

Conclusion: Paper available on arXiv

Stablecoins = complex on-chain currencies

- Many similarities with traditional finance
- Also new risks that lack suitable models

Key takeaways

- Stochastic analysis results in endogenous price model:
  - Stable and unstable regimes
  - Deleveraging spirals as submartingale
- Design insights, new mechanisms now in use