Projections onto subspaces

Projections

If we have a vector **b** and a line determined by a vector **a**, how do we find the point on the line that is closest to **b**?

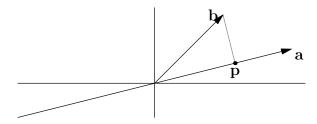


Figure 1: The point closest to **b** on the line determined by **a**.

We can see from Figure 1 that this closest point p is at the intersection formed by a line through b that is orthogonal to a. If we think of p as an approximation of b, then the length of e = b - p is the error in that approximation.

We could try to find **p** using trigonometry or calculus, but it's easier to use linear algebra. Since **p** lies on the line through **a**, we know $\mathbf{p} = x\mathbf{a}$ for some number x. We also know that **a** is perpendicular to $\mathbf{e} = \mathbf{b} - \mathbf{x}\mathbf{a}$:

$$\mathbf{a}^{T}(\mathbf{b} - x\mathbf{a}) = 0$$

$$x\mathbf{a}^{T}\mathbf{a} = \mathbf{a}^{T}\mathbf{b}$$

$$x = \frac{\mathbf{a}^{T}\mathbf{b}}{\mathbf{a}^{T}\mathbf{a}},$$

and $\mathbf{p} = \mathbf{a}x = \mathbf{a} \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}$. Doubling **b** doubles **p**. Doubling **a** does not affect **p**.

Projection matrix

We'd like to write this projection in terms of a projection matrix $P: \mathbf{p} = P\mathbf{b}$.

$$\mathbf{p} = \mathbf{x}\mathbf{a} = \frac{\mathbf{a}\mathbf{a}^T\mathbf{a}}{\mathbf{a}^T\mathbf{a}},$$

so the matrix is:

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

Note that $\mathbf{a}\mathbf{a}^T$ is a three by three matrix, not a number; matrix multiplication is not commutative.

The column space of P is spanned by **a** because for any **b**, P**b** lies on the line determined by **a**. The rank of P is 1. P is symmetric. P^2 **b** = P**b** because

the projection of a vector already on the line through **a** is just that vector. In general, projection matrices have the properties:

$$P^T = P$$
 and $P^2 = P$.

Why project?

As we know, the equation $A\mathbf{x} = \mathbf{b}$ may have no solution. The vector $A\mathbf{x}$ is always in the column space of A, and \mathbf{b} is unlikely to be in the column space. So, we project \mathbf{b} onto a vector \mathbf{p} in the column space of A and solve $A\hat{\mathbf{x}} = \mathbf{p}$.

Projection in higher dimensions

In \mathbb{R}^3 , how do we project a vector **b** onto the closest point **p** in a plane?

If \mathbf{a}_1 and \mathbf{a}_2 form a basis for the plane, then that plane is the column space of the matrix $A = [\begin{array}{cc} \mathbf{a}_1 & \mathbf{a}_2 \end{array}]$.

We know that $\mathbf{p} = \hat{x}_1 \mathbf{a}_1 + \hat{x}_2 \mathbf{a}_2 = A\hat{\mathbf{x}}$. We want to find $\hat{\mathbf{x}}$. There are many ways to show that $\mathbf{e} = \mathbf{b} - \mathbf{p} = \mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to the plane we're projecting onto, after which we can use the fact that \mathbf{e} is perpendicular to \mathbf{a}_1 and \mathbf{a}_2 :

$$\mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$$
 and $\mathbf{a}_2^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$.

In matrix form, $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$. When we were projecting onto a line, A only had one column and so this equation looked like: $a^T(\mathbf{b} - x\mathbf{a}) = \mathbf{0}$.

Note that $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$ is in the nullspace of A^T and so is in the left nullspace of A. We know that everything in the left nullspace of A is perpendicular to the column space of A, so this is another confirmation that our calculations are correct.

We can rewrite the equation $A^{T}(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$ as:

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}.$$

When projecting onto a line, A^TA was just a number; now it is a square matrix. So instead of dividing by $\mathbf{a}^T\mathbf{a}$ we now have to multiply by $(A^TA)^{-1}$

In *n* dimensions,

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$$

$$\mathbf{p} = A \hat{\mathbf{x}} = A (A^T A)^{-1} A^T \mathbf{b}$$

$$P = A (A^T A)^{-1} A^T.$$

It's tempting to try to simplify these expressions, but if A isn't a square matrix we can't say that $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$. If A does happen to be a square, invertible matrix then its column space is the whole space and contains **b**. In this case P is the identity, as we find when we simplify. It is still true that:

$$P^T = P$$
 and $P^2 = P$.

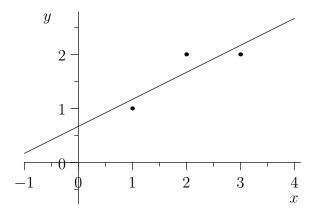


Figure 2: Three points and a line close to them.

Least Squares

Suppose we're given a collection of data points (t, b):

$$\{(1,1),(2,2),(3,2)\}$$

and we want to find the closest line b = C + Dt to that collection. If the line went through all three points, we'd have:

$$C+D = 1$$

$$C+2D = 2$$

$$C+3D = 2,$$

which is equivalent to:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

$$A \qquad \mathbf{x} \qquad \mathbf{b}$$

In our example the line does not go through all three points, so this equation is not solvable. Instead we'll solve:

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}.$$

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